



Crystal Clear Electronics

The development of the Crystal Clear Electronics curriculum was supported by the European Commission in the framework of the Erasmus + programme in connection with the “Developing an innovative electronics curriculum for school education” project under “2018-1-HU01-KA201-047718” project number.



Erasmus+

The project was implemented by an international partnership of the following 5 institutions:

- Xtalin Engineering Ltd. – Budapest
- ELTE Bolyai János Practice Primary and Secondary Grammar School – Szombathely
- Bolyai Farkas High School – Târgu Mureş
- Selye János High School – Komárno
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Contact <http://crystalclearelectronics.eu/en/>
info@kristalytisztaelektronika.hu

01 - Revision of Basic Mathematics

Written by Zsolt Verasztó

English translation by Xtalín Engineering Ltd.

Revised by Szabolcs Veréb, Ádám Szabó

In the following few pages, we will revise the basic mathematics that are required to understand the further parts of the teaching material. It's not necessary to remember all the small details, but if you are not familiar with one of the definitions, take your time understanding it.

MANAGING THE ORDERS OF MAGNITUDE

The physical quantities, which are represented by real numbers, can be really big or really small, in terms of scale. Look at the following example illustrating the problem:

The charge of a proton is: $q_e = 0.000000000000000001602$ [C]

The output power of the Paks Nuclear Power Plant is: $P_{Paks} = 2000000000$ [W]

As you can see, it's uncomfortable to work with such numbers, as it's hard to pronounce them and it's easy to make mistakes while writing them down. This chapter will discuss two solutions, one of them is the scientific notation, the other one is using prefixes. Both are based on the simplified notation of the powers of ten.

SCIENTIFIC NOTATION

When using the scientific notation the number a is written like this:

$$a = \alpha \cdot 10^\beta$$

However, this definition is not complete, as there are endless α and β numbers which satisfy the equation; therefore, we must make further restrictions, which can be different for every discipline. In science it's widespread to choose α the following way:

$$1 \leq \alpha < 10 \quad \text{or} \quad -10 < \alpha \leq -1$$

In engineering we usually use the engineering or the technical normal form, where β is chosen to be a multiple of 3.

Let's see how we can write the previous examples with the introduced normal forms.

- The charge of the proton with normal form is: $q_e = 1.602 \cdot 10^{-19}$ [C]
- The charge of the proton with technical normal form is: $q_e = 0.1602 \cdot 10^{-18}$ [C]



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- The power output of the Paks Nuclear Power Plant with both scientific notations is: $P_{Paks} = 2 \cdot 10^9$ [W]

How to write zero?

Based on the previous definition of scientific notation the form of zero is not defined, as it cannot be expressed in any of the normal forms we have seen. Representing zero is only possible by simply writing '0', a special normal form notation is not defined.

PREFIXES

When quantities are used and described with units, like gram, meter, litre, second; prefixes can be used to easily represent the order of magnitude of the unit. You can see the prefixes standardized by the SI measurement system in the table below:

SI prefixes		
Name	Notation	Value
yotta	Y	10^{24}
zetta	Z	10^{21}
exa	E	10^{18}
peta	P	10^{15}
tera	T	10^{12}
giga	G	10^9
mega	M	10^6
kilo	k	10^3
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}
zepto	z	10^{-21}
yocto	y	10^{-24}

In practice, we rarely use prefix notation for numbers smaller than 10^{-15} and bigger than 10^{15} . In those cases, the scientific notation is the usual representation. Of course, like the scientific notation, the prefix notation is not fixed. Let's see the previous quantities in prefix notation:



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- The charge of the proton is: $q_e = 0.1602 \cdot 10^{-18} \text{ [C]} = 0.1602 \text{ [aC]} = 160.2 \text{ [zC]}$
- The power output of Paks Nuclear Power Plant is: $P_{Paks} = 2 \cdot 10^9 \text{ [W]} = 2 \text{ [GW]} = 2000 \text{ [MW]}$

In certain areas, some units have prefix notations that are so widespread that we would rarely use any other prefix. This can be true for very small or very large numbers as well. Mass is a good example for this, for which the SI unit is the [kg] and not [g]. For example:

- The mass of proton: $m_p = 1.6726 \cdot 10^{-27} \text{ [kg]}$
- The mass of Earth: $m_{Earth} = 5.9736 \cdot 10^{24} \text{ [kg]}$

NUMERAL SYSTEMS

In both scientific and everyday life, numbers are mainly represented by digits. To understand the basic idea of numeric representation, let's take n dots. (this figure shows the case for $n=113$).

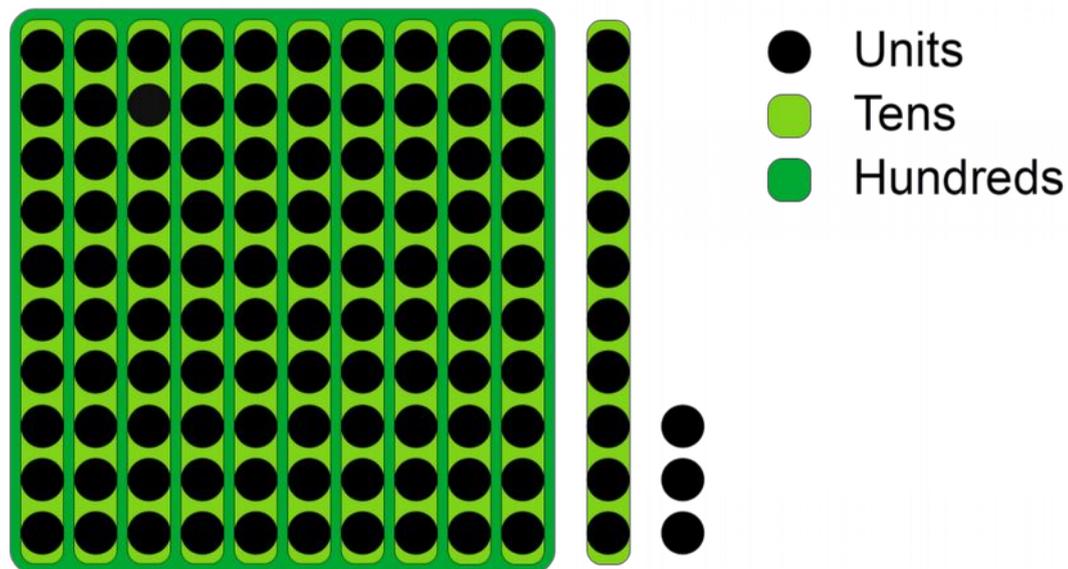


Figure 1 - Numeric representation

Let's arrange the dots into groups of ten, as in the picture. Then, we can arrange the groups of ten into other groups of ten, so that way, we get groups of hundred, which correspond the second power of ten.

$$n = 113 = 11 \cdot 10 + 3 \cdot 1 = (10 + 1) \cdot 10 + 3 \cdot 1 = 1 \cdot 100 + 1 \cdot 10 + 3 \cdot 1$$

Continuing like this gives us groups with increasing powers of ten. If you think about it, any number of dots can be divided into groups if we introduce more and more groups (while increasing the power of ten). Numbers are written in the following way: we count how many times a group is present and write the numbers after each other. In our case it is 1, 1, 3, however, without any doubt it could be 3, 1, 1 as well. To be specific, which number applies to which group, the concept of place-value has been introduced. According to the accepted writing mode, the place-values are written in decreasing order from left to right. The number of digits shows exactly which of place-values are in the given number. In this example, we have three place-values for our three-digit number: hundreds (10^2), tens (10^1) and



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ones (10^0). So, place-values are nothing more than the powers of the base number of the numeral system. In practice, this won't give anybody a hard time, as we're raised to write the numbers using the place-value notation from an early age in primary school.

Now, we have reached the most used numeral system in everyday life, the base-10 or decimal numeral system. To identify every single number, we need 10 different symbols. You can write zero, one, two, ..., nine to each place-value. If you would have ten or more from a single place-value, that can be regrouped by increasing the place-value. For example, when you have 12 from the hundreds, you can think of it that you actually have 1 from the thousands and 2 from the hundreds. The required symbols are the well-known digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Consecutive digits of any number give the increasing powers of ten from right to left. You may wonder why we started to arrange the numbers into groups of ten. The reason for that is not complicated: humans have 10 fingers, and it is very easy for us to represent any decimal number with our fingers. So, children learn counting this way, therefore it is natural to arrange numbers by groups of ten.

BINARY NUMERAL SYSTEM

It comes naturally, that we can choose different numbers as a base number, therefore, similarly to the decimal numeral system, let's introduce the base-2, also known as the binary numeral system, which is very important for digital electronics. In the binary numeral system, we organise the units into groups of two instead of ten. Two groups of two make one group of four, two groups of four make a group of eight, and so on. In this case, we work with two different symbols, ones and zeros. This brings us to the simplest possible numerical notation. You can see the grouping of the last example's number 113 in binary numeral system.

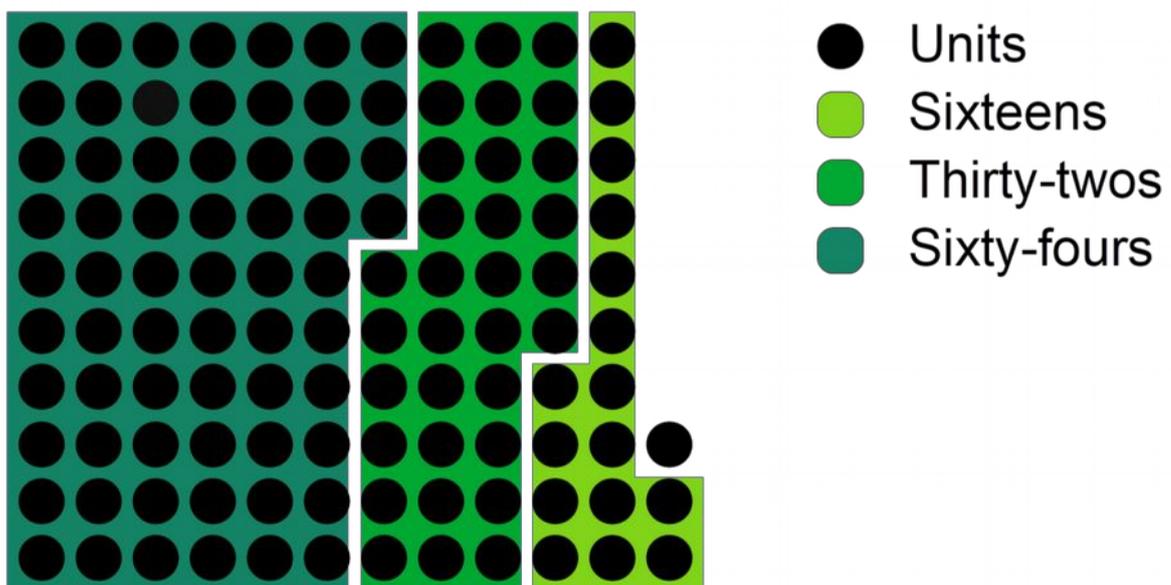


Figure 2 – Grouping in the binary numeral system

The binary numeral system has several practical uses apart from its theoretical significance. In electrical engineering (digital electronics) and IT applications it is way more widespread than the decimal system.



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Using a number larger than ten as base number would be uncomfortable, because we need more and more symbols. In practice, however the base-16 or hexadecimal numeral system is used especially in the IT sector. In case of hexadecimal numbers, we need 16 different symbols to describe every value, therefore, the numbers from 0...9 need to be supplemented by additional six symbols. These are the capital letters of English alphabet, A, B, C, D, E and F.

For further discussion, let's introduce the notation $a_{(b)}$, which means that a given a number is defined in the numeral system of b . For simplicity, decimal numbers usually have no extra labeling. Let's see an example:

$$113 = 113_{(10)} = 71_{(16)} = 1110001_{(2)}$$

CONVERSION BETWEEN NUMERAL SYSTEMS

In practice, the binary, decimal and hexadecimal numeral systems are the most commonly used, we will show how to convert numbers between these systems. Converting a decimal value to any other numeral system leads back to a simple operation: division with remainder, which give us two values, a quotient and a residue. For example, dividing 16 by 3, the quotient is 5 and the residue is 1, or dividing 113 by 5 the quotient is 22, the residue is 3. In this case, we say that „the residue of 16 divided by three is 1”, or similarly „the residue of 113 divided by five is 3”.

For easier calculation, it's recommended to make all conversion through the decimal numeral system. So, if you want to convert a base-3 number to a base-7 number, it is easier to convert the base-3 number to decimal, and then convert the decimal number to base-7. You can convert any decimal number to other numeral system with the following method. We will convert 113 to binary (base-2), so we have to divide our number with 2, and write down the residues:

113	1	Write down the 113 and its residue divided by two next to it!	$\Rightarrow 1110001_{(2)}$
56	0	Dividing 113 by two: the quotient is 56, the residue is 0.	
28	0	56/2=28, residue is 0.	
14	0	28/2=14, residue 0.	
7	1	14/2=7, residue 1.	
3	1	7/2=3, residue 1.	
1	1	3/2=1, residue 1.	
0		Reading the residues from below, we get the final result.	

For practice, let's convert the same number to base-3, base-5 and base-8 (octal) numeral system. Of course, in these cases, division with remainder must be made with three, five and eight.

$10 \rightarrow 3$	113	2	$\Rightarrow 11012_{(3)}$	$10 \rightarrow 5$	113	3	$\Rightarrow 423_{(5)}$	$10 \rightarrow 8$	113	1	$\Rightarrow 161_{(8)}$
	37	1			22	2			14	6	
	12	0			4	4			1	1	
	4	1			0				0		
	1	1									
	0										



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To convert a number to decimal from any numeral system, each of the digits must be multiplied by their place-value, then you have to sum the numbers. See the example below:

$$\begin{aligned} 11012_{(3)} &= 1 \cdot 3^4 + 1 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0 \\ &= 1 \cdot 81 + 1 \cdot 27 + 0 \cdot 9 + 1 \cdot 3 + 2 \cdot 1 = 113 \end{aligned}$$

$$423_{(5)} = 4 \cdot 5^2 + 2 \cdot 5^1 + 3 \cdot 5^0 = 4 \cdot 25 + 2 \cdot 5 + 3 \cdot 1 = 113$$

We don't have to use the decimal numeral system, if the conversion is done between two numeral systems, where one of the base numbers is the power of the other one. For example, if we have to convert a number from the base-2 numeral system to base-4, base-8 or base-16 numeral systems.

In this case, using the binary form of the number is very handy to convert it to a larger base numeral system. We just need to arrange its digits into groups to do that. Two consecutive digits of a number in the binary numeral system can be 00, 01, 10 or 11 variations, which clearly correspond to a digit of a four digit number 0, 1, 2 or 3. It's similarly reasonable, that arranging the digits into groups of three, gives the base-8 numeral system format. Arranging the digits into groups of four, gives the hexadecimal numeral system format. To present the process, consider the following number 1100011110100000000101101101111₍₂₎. Watch out! Grouping must be started with the lowest place-value from the right. If needed, zeros can be added in front of the highest place-value group.

Creating the groups of two and converting the groups one by one:

$$\underbrace{11}_3 \underbrace{00}_0 \underbrace{01}_1 \underbrace{11}_3 \underbrace{10}_2 \underbrace{10}_2 \underbrace{00}_0 \underbrace{00}_0 \underbrace{00}_0 \underbrace{01}_1 \underbrace{01}_1 \underbrace{10}_2 \underbrace{11}_3 \underbrace{01}_1 \underbrace{11}_3 \underbrace{11}_3$$

$$1100011110100000000101101101111_{(2)} = 3013220001123133_{(4)}$$

Similarly, in the base-8 numeral system:

$$\underbrace{011}_3 \underbrace{000}_0 \underbrace{111}_7 \underbrace{101}_5 \underbrace{000}_0 \underbrace{000}_0 \underbrace{001}_1 \underbrace{011}_3 \underbrace{011}_3 \underbrace{011}_3 \underbrace{111}_7$$

$$1100011110100000000101101101111_{(2)} = 30750013337_{(8)}$$

As you can see, here, in the highest group, there would be only two digits, we had to add a zero in the front.

By making groups of four, we get the hexadecimal numeral system format:

$$\underbrace{1100}_C \underbrace{0111}_7 \underbrace{1010}_A \underbrace{0000}_0 \underbrace{0001}_1 \underbrace{0110}_6 \underbrace{1101}_D \underbrace{1111}_F$$

$$1100011110100000000101101101111_{(2)} = C7A016DF_{(16)}$$

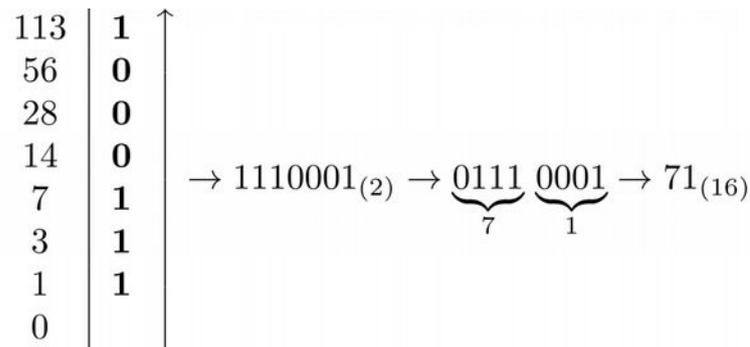
Finally, let's see how the number 113₍₁₀₎ can be converted to a hexadecimal number by using the binary system. As in the previous examples, the highest place-value group has to be supplemented to four digits as well.



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BOOLEAN ALGEBRA

Boolean algebra is the mathematics of statements, that are clearly classified as „true” or „false”. In the following cases, the false statements will be marked with a digit 0, true statements will be marked with a digit 1, so the value of the logical variables is either 0 or 1.

This way, everything that can be written as answers for a series of decidable questions (yes/no), can be encoded using boolean algebra. This code can be immediately written with digits as well, which is the same as the binary numeral system. If we remember the essence of local representation, we shall realize the following: binary numbers can be interpreted as answers to a series of decidable questions.

Take the number 13₍₁₀₎ as an example, then convert it to binary format!

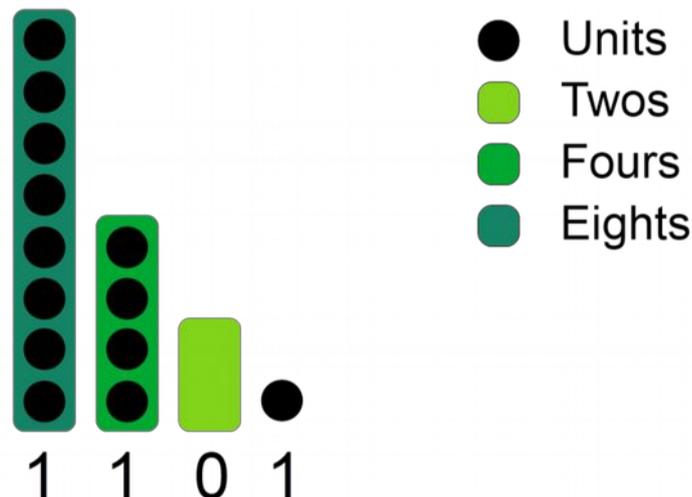


Figure 3 - Connection of the second numeral system and the Boole algebra

The first decidable question could be the following: „Is there any group, which contains exactly one element?”. Based on the figure above, we have a single unit, so the answer is true, coded as 1. The second question: „Is there any group which contains exactly two elements?”. You can see that there isn’t any, so the answer is false, a 0. The answer for the question „Is there any group, which contains exactly four elements?” is true as well, so it’s 1. For the final question „Is there any group which contains exactly eight elements?” we can write a solution of one again. Arranging the place-values in appropriate row, the binary format of the number is produced, 1101₍₂₎.



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LOGICAL OPERATIONS

We can define logical variables which have a truth value (true/false). With these logical variables, mathematical operations can be done. The simplest of such operations is the denial or in other words negation (in English NOT), which assigns false to true, and true to false. An A logical variable's negation is called „ A negated” in speech, or simply we call it „not A ”, or “ A not”, its sign is \underline{A} . The easiest way to illustrate logical operations, is to use truth tables, which shows the value returned by the operation in addition to the possible values of the variables.

negation (NOT)	
A	\underline{A}
0	1
1	0

The first operation to be described, that works with two logical variables, is the „or” operation. The appropriate technical term is disjunction, the sign of the operation is “+” (plus). The $A + B$ statement is true only, if either variable A , or variable B values are true. In the OR operation's truth table, the operation's result will be 0 only if both variables are 0. The OR operation's truth table:

disjunction (OR)		
A	B	$A + B$
0	0	0
0	1	1
1	0	1
1	1	1

The next bivariable operation is the „and”, the technical term is conjunction, its sign is “ \cdot ” (dot). $A \cdot B$ statement is true only, if both variables are true. This can be seen in the truth table of the „and” operation:

conjunction (AND)		
A	B	$A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1



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This kind of notation of the „and” and „or” connections isn’t usual in mathematics, however it is used in digital electronics applications.

Although any kind of logical statement can be constructed using just these operations, in practice, primarily in digital electronics applications, additional operations are introduced. The three most common ones are the „not and”, the „not or” and the „exclusive or”. The „not and” and the „not or” operations are the negated form of the “and” and the “or” functions respectively. Truth tables look like the following:

not or (NOR)		
A	B	$\underline{A + B}$
0	0	1
0	1	0
1	0	0
1	1	0

not and (NAND)		
A	B	$\underline{A \cdot B}$
0	0	1
0	1	1
1	0	1
1	1	0

The „exclusive or” operation is different from the „or” operation in a way that it is only true, if exclusively only one variable is true. Let’s try to write this logical connection down with the help of our previously introduced functions. This function is true only, if A is true and B is not true, or if A is not true and B is true, i.e. $A \cdot \underline{B} + \underline{A} \cdot B$. This logical connection is marked with a \oplus symbol, that is $A \cdot \underline{B} + \underline{A} \cdot B = A \oplus B$. Let’s check the truth table:

exclusive or (XOR)		
A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

LOGIC FORMULAS

In case of the „exclusive or” it was easily solvable how to write down the statement with the help of our basic operations. Let’s try two similar tasks, write down the „not and” and the „not or” functions’ equivalents with the use of „and”, „or” and „negation” operations.

As the „and” operation’s output was false, if A was not true, or B was not true, therefore the „not and” will be true just in these cases, so according to our conjecture $\underline{A \cdot B} = \underline{A} + \underline{B}$



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With similar reasoning an „or” connection is false, if A is not true and B is not true, so the „not or” is true in this same case, thus $\underline{A + B} = \underline{A} \cdot \underline{B}$. These statements - which can be easily checked using their truth tables as well - are the so-called **de Morgan’s laws**.

not or (NOR)			
A	B	$\underline{A + B}$	$\underline{A} \cdot \underline{B}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

not and (NAND)			
A	B	$\underline{A \cdot B}$	$\underline{A + B}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

SYSTEM OF LINEAR EQUATIONS WITH TWO VARIABLES

A system of linear equations with two variables can be generally written down in the

$$ax + by = c$$

$$dx + ey = f$$

form, where we would like to solve the equation for x and y variables. Solving this kind of equations, we can get three different cases, which cases will be presented using examples.

Comment

This form isn’t completely general, it is only the “System of Linear Equations with Two Variable and Two Equations”.

There are two simple method for the solution. On the one hand, we can try to express one variable from one of the equations, then we can replace the expressed variable in the other equation. This gives us a single variable equation for the second variable, which can be solved.

The other method: we can multiply an equation with a number, so in both of the equations one of the variables’ coefficient will to be the same or opposite. By adding or subtracting the equations, we get an equation with one variable again. Let’s take a look at some examples!

SINGLE UNIQUE SOLUTION

Let’s solve the

$$-3x - 2y = 10$$

$$2x + 8y = 20$$

system of equations!



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Multiply the first equation by four, so we get the following:

$$-12x - 8y = 40$$

$$2x + 8y = 20$$

Let's add up the two equations, the expression gets simplified:

$$-10x = 60$$

Its solution is $x = -6$. Let's substitute this into the second equation,

$$2 \cdot (-6) + 8y = 20$$

Its solution is $y = 4$. So, the solution of the equation system is $x = -6, y = 4$

Solve the same system of equations using the other method as well! Express the x variable from the second equation. Arrange the equation for the x , then divide it with the multiplier of x , i.e. with two.

$$2x = 20 - 8y$$

$$x = 10 - 4y$$

Substitute this into the first equation, and solve it for y :

$$(-3) \cdot (10 - 4y) - 2y = 10$$

$$-30 + 12y - 2y = 10$$

$$10y = 40$$

$$y = 4$$

Substituting this back to the $x = 10 - 4y$ equation, we get the solution $x = -6, y = 4$. It is not surprising that this is the same as the solution of the other method.

NO SOLUTION

Let's solve the

$$-3x + 2y = 12$$

$$6x - 4y = -20$$

system of equations! Multiply the first equation by two:

$$-6x + 4y = 24$$

$$6x - 4y = -20$$

Add the equations together

$$0 = 4$$

We get a contradiction; the equation system does not have a solution.



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INFINITELY MANY SOLUTIONS

Solve the

$$-3x + 2y = 10$$

$$6x - 4y = -20$$

system of equations! Multiply the first equation by two and add them up together:

$$-6x + 4y = 20$$

$$6x - 4y = -20$$

$$0x + 0y = 0$$

The two equations differed from each other in a constant multiplier, the two equations are equivalent. In this case, actually, we have one equation for two unknown variables. Leaving one of the equations accordingly, it gives that

$$-3x + 2y = 10$$

Every pair of number that fulfils the equation is solution for the system of equations too. This equation equivalent to the function of a line, so the solutions on the x, y plain are located on a straight line.

FUNCTIONS

CARTESIAN COORDINATE SYSTEM

In engineering practice, it's very hard to create formulas for certain connections often, therefore, we rather represent them graphically. In a right-angled coordinate system, the most common representing form practically consists of two number lines, which are called axis. We often call the horizontal axis as x axis, the vertical one as y axis. In this coordinate system, we can mark points, every single point has an x and y coordinate which describes exactly the points' place in the coordinate system. In the figure below we marked the A point, of which x coordinate is $A_x = 1$ and y coordinate is $A_y = 2$, in short $A(1,2)$.

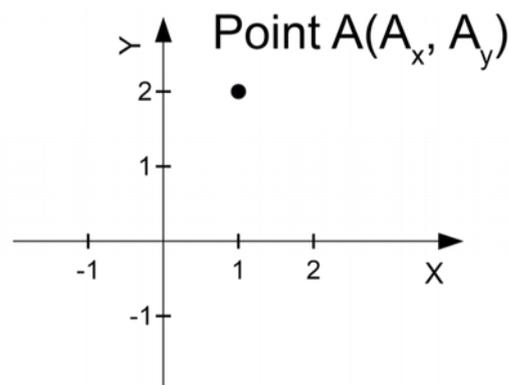


Figure 4 - Cartesian coordinate system

Suppose that, somebody gives us two golden coin for every single step we do. This could be written down as an equation the following way:



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$$\text{golden coin} = 2 \cdot \text{number of steps}$$

So, as you don't want to write down the expressions „golden coin” and „number of steps” every single time, let's use letters as markings

$$y = 2 \cdot x$$

Instead of x and y we could have used other characters as well. Such expressions are usually called functions, in this example y depends on x . Usually this connection is marked by the $y(x)$ sign:

$$y(x) = 2 \cdot x$$

In a Cartesian/perpendicular coordinate system, the relation looks like this:

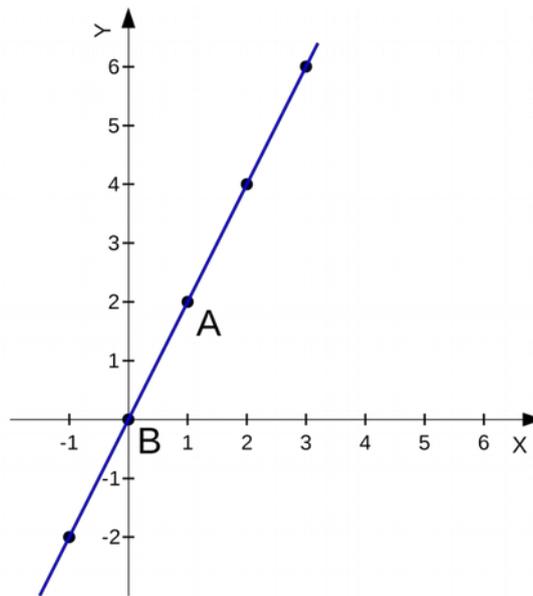


Figure 5 - Line in the Cartesian coordinate system

The graphical representation can be imagined as the following: if we substitute the horizontal axis's values one by one into the x , we get a y value for every single x value. Then, we represent the coherent x - y pairs in the coordinate system with a single point. For example, when $x = 1$ then $y = 2 \cdot 1 = 2$ (Point A), when $x = 0$ then $y = 2 \cdot 0 = 0$ (Point B). If we take all the points, then the full function can be drawn.

TRIGONOMETRIC FUNCTIONS

An angle can be usually given in two different units, in degrees and in radians. The symbol of degree is a $[\circ]$ character, the angle of a full circle, is 360° . The symbol of radians is $[\text{rad}]$, angle of a full circle measuring it in radians, is 2π $[\text{rad}]$. Out of this information, the value of a radian can be calculated in degrees:

$$1 [\text{rad}] = \frac{360 [\circ]}{2\pi} = 57.2958 [\circ]$$

The knowledge of trigonometric functions is considered to be known in right-angled triangles and we take over the method for defining sine and cosine functions for angles greater than 90° , i.e. angles greater than $\pi/2$ radians.



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During the definition of trigonometric functions, the so-called “unit circle” is a great help for us. Its radius is one unit long, drawn in the origin of the usual Cartesian/perpendicular coordinate system. Take a line, which has a φ angle measured from the positive side of the x axis. According to the definition, the coordinates of the intersection point between the line and the unit circle, are $(\cos(\varphi), \sin(\varphi))$. In the following, we consider these as the definitions of the sine and cosine functions. It’s visible that for angles $\varphi \leq \frac{\pi}{2}$ this definition is the same as in primary trigonometry. Because the hypotenuse (the side opposite the right angle) of the right-angled triangle is unit long, the length of the legs are just $\sin(\varphi)$ and $\cos(\varphi)$. Advantage of the definition is that we visibly get functions that are interpreted for any real number.

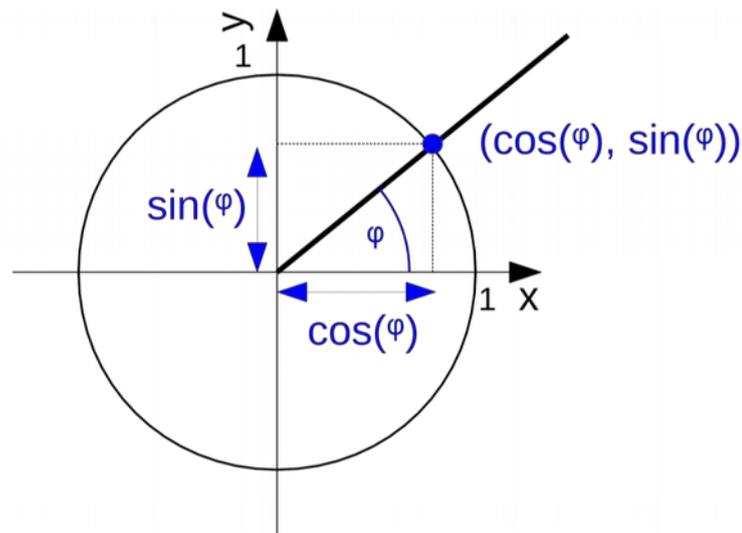


Figure 6 - Trigonometric functions on the unit radius circle

PERIODIC FUNCTIONS

The introduced sine and cosine functions are periodic, in common sense a finite section is repeated. More precisely, it’s true for them that it is possible to find a $p > 0$ real number, of which it’s true that $f(\varphi + p) = f(\varphi)$. The smallest p number like this, is called the period of the function. This value for the sine and cosine functions is $p = 2\pi$ [rad].

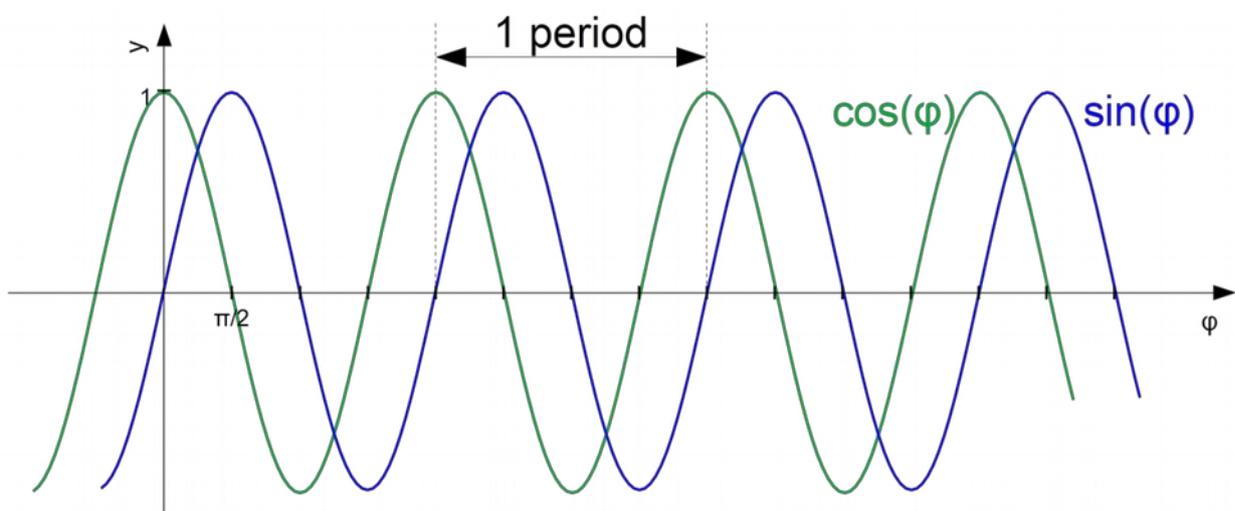


Figure 7 - Representation of trigonometrical functions



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The square, triangle and the sawtooth waves are periodic functions which occur often in electronic applications as different waveforms. They are hard to describe with an equation, while it is easy to represent them graphically.

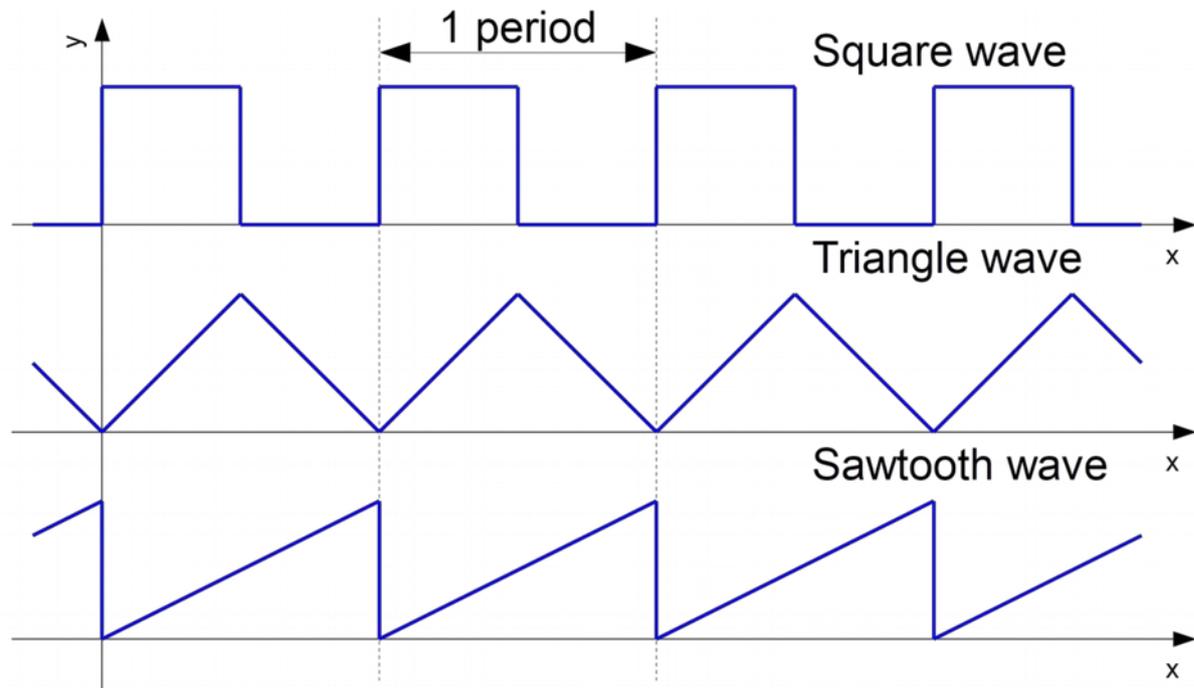


Figure 8 – Typical periodic functions

TRIGONOMETRIC FUNCTIONS AS PERIODIC FUNCTIONS IN TERMS OF ENGINEERING

Previously, we saw the definitions of the primary trigonometric functions. We frequently use the sine and cosine functions for modelling any function that is periodic and changes over time. If we consider the time as variable, then the primary transformations of these functions can be given generally in the following shape:

$$A \cdot \sin(\omega \cdot t + \varphi) + A_0$$

In this form, the A_0 is a shift to the positive direction of the vertical axis which is called the **mean value** in applications. The A multiplier, which is an A long provision on the vertical axis, is the **amplitude** of the signal. The φ_0 , which is a shift to the negative direction of the horizontal axis with the value of φ_0 , is the **phase** of the function. The ω parameter, which stretches, or shrinks in the direction of the horizontal axis, is called **angular frequency**.

As it was introduced in the previous part, the $\sin(\varphi)$ function is periodic with a period of $p = 2\pi$. Now let's see what the upper functions' **periods** are, i.e. the periods depend on the t variable. Mark this with a T , so



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$$A \cdot \sin(\omega \cdot (t + T) + \varphi_0) + A_0 = A \cdot \sin(\omega \cdot t + \varphi) + A_0$$

Subtracting A_0 from both sides, distributed by A , then opening the parentheses

$$\sin(\omega t + \omega T + \varphi_0) = \sin(\omega t + \varphi)$$

Since the period of the $\sin(\varphi)$ function is $p = 2\pi$, it is visible, that $\omega T = p = 2\pi$, so the period is:

$$T = \frac{2\pi}{\omega}$$

Decomposing the sine function

The above transformation of the sine function is identical to the sum of two sine and cosine functions in the same phase (so the values of φ_0 are the same), but with different amplitude. Let's explain this, after all it leads us to an educational equation system with two unknown parameters. The statement, which we would like to prove is

$$A \cdot \sin(\omega t + \varphi_0) + A_0 = A_s \sin(\omega t) + A_c \cos(\omega t) + A_0$$

The statement is satisfied for every t with a proper choice for the constants A and φ_0 . To do this, subtract A_0 from both sides and separate the left side by using the relevant addition law ($\sin(x + y) = \sin x \cos y + \sin y \cdot \cos x$),

$$A \cdot \sin(\omega t) \cdot \cos(\varphi_0) + A \cdot \sin(\varphi_0) \cdot \cos(\omega t) = A_s \sin(\omega t) + A_c \cos(\omega t)$$

The right and the left side can be equal only if the coefficients of the $\sin(\omega t)$ and $\cos(\omega t)$ members on both sides are equal. From this, we get an equation system which consists of one or two equations,

$$A \cdot \cos(\varphi_0) = A_s$$

$$A \cdot \sin(\varphi_0) = A_c$$

By squaring and adding the two equations together we get this

$$A_s^2 + A_c^2 = A^2(\sin^2(\varphi_0) + \cos^2(\varphi_0)) = A^2$$

So, the appropriate A is:

$$A = \sqrt{A_s^2 + A_c^2}$$

If the equations, which form the equation systems are divided with each other (right side with right, left side with left), then we get

$$\frac{A \cdot \sin(\varphi_0)}{A \cdot \cos(\varphi_0)} = \text{tg}(\varphi_0) = \frac{A_c}{A_s}$$



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the following equation. Out of this, the value of φ_0 can be easily calculated using calculator, so we really have found any proper A and φ_0 values for $A_s \neq 0$, $A_c \neq 0$ amplitudes. (If one or both of amplitudes are zero, the equation system can be trivially solved.)

Calculation of the sine and cosine functions at consecutive points

The question is rightly raised, how we can calculate the values of sine and cosine functions at a given point. For reasons of scale, we cannot give full answer for this, however we accept that without explaining the method, these can be calculated with any accuracy at a very high cost. Many times, it's enough to know the values of the sine and the cosine functions at finite many points. A good application example for this, when we would like to produce a signal, which is sinusoidal in time, refreshing the value at the same intervals.

We know from the definitions, that $\sin(0) = 0$ and $\cos(0) = 1$. Apart from these values, accept the fact that, we already defined the values of the trigonometric functions at another point. For the sake of the example, let's suppose $\varphi_0 = 0.01$. Now we already know the values of trigonometric functions at the 0 and the 0.01 points. The task is to calculate the values of the functions at the 0.02, 0.03, 0.04 etc. points without further complicated calculations. Let's look at these

$$\sin(0.02) = \sin(0.01 + 0.01) \text{ and}$$

$$\cos(0.02) = \cos(0.01 + 0.01)$$

expressions. Using the laws of addition, it's visible, that

$$\sin(0.01 + 0.01) = \sin(0.01) \cdot \cos(0.01) + \sin(0.01) \cdot \cos(0.01)$$

$$\cos(0.01 + 0.01) = \cos(0.01) \cdot \cos(0.01) - \sin(0.01) \cdot \sin(0.01)$$

Since the values of $\sin(0.01)$ and $\cos(0.01)$ were already known, values of $\sin(0.02)$ and $\cos(0.02)$ could be determined with few elemental actions, so now these are known. Continuing the proceeding

$$\sin(0.03) = \sin(0.02 + 0.01) = \sin(0.02) \cdot \cos(0.01) + \sin(0.01) \cdot \cos(0.02)$$

$$\cos(0.03) = \cos(0.02 + 0.01) = \cos(0.02) \cdot \cos(0.01) - \sin(0.02) \cdot \sin(0.01)$$

It's visible that, again, we only need the values of the trigonometric functions at already known points. Further points' values of trigonometric functions can be determined with the help of doing four multiplications and two summations.



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