# Project Risk and Uncertainty 

Oil Forecasts Are a Roll of the Dice ${ }^{\text {' You may know as much }}$ as the oil experts. That is, you know that a barrel of oil is pricey and getting pricier. Beyond that, nobody-not even those who get paid to prognosticate-has a real handle on the push and pull that goes into figuring how much oil people need, how much can be pumped, and how much can be refined.

The unreliable data and forecasts have plagued the industry for decades. They became more of a problem once demand and prices


[^0]starting climbing two years ago, because the substantial margins of error in these numbers are even larger than the oil industry's shrinking margin of spare pumping capacity.

Put another way, even a small error in predicting oil consumption can cause energy markets to gyrate if demand turns out higher than the market assumed, because the industry lacks the ability it had in the 1990s to gin up extra oil on the fly to meet a surge in buying. Yet traders, companies, and consumers have no choice but to rely on the numbers that are out there. One problem anyone who factors oil into an investing decision faces is that accurate oil data come only with a time lag. Oil data on actual demand and supply bounce around because of bad weather, accidents such as pipeline ruptures, or political shocks such as terrorist strikes. Amplifying the fuzziness, meanwhile, is that projections of economic growth, a critical factor in assessing energy needs, are forever changing.

| Crude-Oil Price Forecast <br> West Texas Intermediate, Spot Price. USD/bbl. Average of Month |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Dec | Jan | Feb | Mar | Apr | May |
|  | 2005 | 2006 | 2006 | 2006 | 2006 | 2006 |
| Forecast |  |  |  |  |  |  |
| Value | 59.3 | 58.7 | 61.8 | 61.6 | 62.1 | 63.8 |
| Standard |  |  |  |  |  |  |
| Deviation | 0.9 | 1.2 | 1.5 | 1.8 | 2.2 | 2.6 |
| Correlation |  |  |  |  |  |  |
| Coefficient | 0.9925 | 0.9912 | 0.9899 | 0.9886 | 0.9873 | 0.9860 |

Suppose that your business depends on the price of oil. For example, the airline industry, UPS, FedEx, and rental companies are all heavily affected by the price of fuel. Now, if your proposed project also depends on the price of crude oil, how would you factor the fluctuation and uncertainty into the analysis?

Risk: The chance that an investment's actual return will be different than expected.

In previous chapters, cash flows from projects were assumed to be known with complete certainty; our analysis was concerned with measuring the economic worth of projects and selecting the best ones to invest in. Although that type of analysis can provide a reasonable basis for decision making in many investment situations, we should certainly consider the more usual uncertainty. In this type of situation, management rarely has precise expectations about the future cash flows to be derived from a particular project. In fact, the best that a firm can reasonably expect to do is to estimate the range of possible future costs and benefits and the relative chances of achieving a reasonable return on the investment. We use the term risk to describe an investment project whose cash flow is not known in advance with absolute certainty, but for which an array of alternative outcomes and their probabilities (odds) are known. We will also use the term project risk to refer to variability in a project's NPW. A greater project risk usually means a greater variability in a project's NPW, or simply that the risk is the potential for loss. This chapter begins by exploring the origins of project risk.

## CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- How to describe the nature of project risk.
- How to conduct a sensitivity analysis of key input variables.
- How to conduct a break-even analysis.
- How to develop a net-present-worth probability distribution.
- How to compare mutually exclusive risky alternatives.
- How to develop a risk simulation model.
- How to make a sequential investment decision with a decision tree.


## 12. Origins of Project Risk

The decision to make a major capital investment such as introducing a new product requires information about cash flow over the life of a project. The profitability estimate of an investment depends on cash flow estimations, which are generally uncertain. The factors to be estimated include the total market for the product; the market share that the firm can attain; the growth in the market; the cost of producing the product, including labor and materials; the selling price; the life of the product; the cost and life of the equipment needed; and the effective tax rates. Many of these factors are subject to substantial uncertainty. A common approach is to make single-number "best estimates" for each of the uncertain factors and then to calculate measures of profitability, such as the NPW or rate of return for the project. This approach, however, has two drawbacks:

1. No guarantee can ever ensure that the "best estimates" will match actual values.
2. No provision is made to measure the risk associated with an investment, or the project risk. In particular, managers have no way of determining either the probability that a project will lose money or the probability that it will generate large profits.
Because cash flows can be so difficult to estimate accurately, project managers frequently consider a range of possible values for cash flow elements. If a range of values
for individual cash flows is possible, it follows that a range of values for the NPW of a given project is also possible. Clearly, the analyst will want to gauge the probability and reliability of individual cash flows and, consequently, the level of certainty about the overall project worth.

### 12.2 Methods of Describing Project Risk

We may begin analyzing project risk by first determining the uncertainty inherent in a project's cash flows. We can do this analysis in a number of ways, which range from making informal judgments to calculating complex economic and statistical quantities. In this section, we will introduce three methods of describing project risk: (1) sensitivity analysis, (2) break-even analysis, and (3) scenario analysis. Each method will be explained with reference to a single example, involving the Boston Metal Company.

### 12.2.I Sensitivity Analysis

One way to glean a sense of the possible outcomes of an investment is to perform a sensitivity analysis. This kind of analysis determines the effect on the NPW of variations in the input variables (such as revenues, operating cost, and salvage value) used to estimate after-tax cash flows. A sensitivity analysis reveals how much the NPW will change in response to a given change in an input variable. In calculating cash flows, some items have a greater influence on the final result than others. In some problems, the most significant item may be easily identified. For example, the estimate of sales volume is often a major factor in a problem in which the quantity sold varies with the alternatives. In other problems, we may want to locate the items that have an important influence on the final results so that they can be subjected to special scrutiny.

Sensitivity analysis is sometimes called "what-if" analysis, because it answers questions such as "What if incremental sales are only 1,000 units, rather than 2,000 units? Then what will the NPW be?" Sensitivity analysis begins with a base-case situation, which is developed by using the most likely values for each input. We then change the specific variable of interest by several specified percentage points above and below the most likely value, while holding other variables constant. Next, we calculate a new NPW for each of the values we obtained. A convenient and useful way to present the results of a sensitivity analysis is to plot sensitivity graphs. The slopes of the lines show how sensitive the NPW is to changes in each of the inputs: The steeper the slope, the more sensitive the NPW is to a change in a particular variable. Sensitivity graphs identify the crucial variables that affect the final outcome most. Example 12.1 illustrates the concept of sensitivity analysis.

## EXAMPLE 12.| Sensitivity Analysis

Boston Metal Company (BMC), a small manufacturer of fabricated metal parts, must decide whether to enter the competition to become the supplier of transmission housings for Gulf Electric, a company that produces the housings in its own in-house manufacturing facility, but that has almost reached its maximum production capacity. Therefore, Gulf is looking for an outside supplier. To compete, BMC must design a new fixture for
the production process and purchase a new forge. The available details for this purchase are as follows:

- The new forge would cost $\$ 125,000$. This total includes retooling costs for the transmission housings.
- If BMC gets the order, it may be able to sell as many as 2,000 units per year to Gulf Electric for $\$ 50$ each, in which case variable production costs, ${ }^{2}$ such as direct labor and direct material costs, will be $\$ 15$ per unit. The increase in fixed costs, ${ }^{3}$ other than depreciation, will amount to $\$ 10,000$ per year.
- The firm expects that the proposed transmission-housings project will have about a five-year product life. The firm also estimates that the amount ordered by Gulf Electric in the first year will be ordered in each of the subsequent four years. (Due to the nature of contracted production, the annual demand and unit price would remain the same over the project after the contract is signed.)
- The initial investment can be depreciated on a MACRS basis over a seven-year period, and the marginal income tax rate is expected to remain at $40 \%$. At the end of five years, the forge is expected to retain a market value of about $32 \%$ of the original investment.
- On the basis of this information, the engineering and marketing staffs of BMC have prepared the cash flow forecasts shown in Table 12.1. Since the NPW is positive $(\$ 40,168)$ at the $15 \%$ opportunity cost of capital (MARR), the project appears to be worth undertaking.
What Makes BMC Managers Worry: BMC's managers are uneasy about this project, because too many uncertain elements have not been considered in the analysis:
- If it decided to take on the project, BMC would have to invest in the forging machine to provide Gulf Electric with some samples as a part of the bidding process. If Gulf Electric were not to like BMC's sample, BMC would stand to lose its entire investment in the forging machine.
- If Gulf were to like BMC's sample, then if it was overpriced, BMC would be under pressure to bring the price in line with competing firms. Even the possibility that BMC would get a smaller order must be considered, as Gulf may utilize its overtime capacity to produce some extra units. Finally, BMC is not certain about its projections of variable and fixed costs.
Recognizing these uncertainties, the managers want to assess the various possible future outcomes before making a final decision. Put yourself in BMC's management position, and describe how you may resolve the uncertainty associated with the project. In doing so, perform a sensitivity analysis for each variable and develop a sensitivity graph.

DISCUSSION: Table 12.1 shows BMC's expected cash flows-but that they will indeed materialize cannot be assumed. In particular, BMC is not very confident in its revenue forecasts. The managers think that if competing firms enter the market,

[^1]
## TABLE I 2. I After-Tax Cash Flow for BMC's Transmission-Housings Project (Example 12.1)

|  | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Revenues: |  |  |  |  |  |  |
| Unit price |  | \$ 50 | \$ 50 | \$ 50 | \$ 50 | \$ 50 |
| Demand (units) |  | 2,000 | 2,000 | 2,000 | 2,000 | 2,000 |
| Sales revenue |  | \$100,000 | \$100,000 | \$100,000 | \$100,000 | \$100,000 |
| Expenses: |  |  |  |  |  |  |
| Unit variable cost |  | \$ 15 | \$ 15 | \$ 15 | \$ 15 | \$ 15 |
| Variable cost |  | 30,000 | 30,000 | 30,000 | 30,000 | 30,000 |
| Fixed cost |  | 10,000 | 10,000 | 10,000 | 10,000 | 10,000 |
| Depreciation |  | 17,863 | 30,613 | 21,863 | 15,613 | 5,575 |
| Taxable income |  | \$ 42,137 | \$ 29,387 | \$ 38,137 | \$ 44,387 | \$ 54,425 |
| Income taxes (40\%) |  | 16,855 | 11,755 | 15,255 | 17,755 | 21,770 |
| Net income |  | \$ 25,282 | \$ 17,632 | \$ 22,882 | \$ 26,632 | \$ 32,655 |
| Cash flow statement: |  |  |  |  |  |  |
| Operating activities: |  |  |  |  |  |  |
| Net income |  | 25,282 | 17,632 | 22,882 | 26,632 | 32,655 |
| Depreciation |  | 17,863 | 30,613 | 21,863 | 15,613 | 5,575 |
| Investment activities: |  |  |  |  |  |  |
| Investment | $(125,000)$ |  |  |  |  |  |
| Salvage |  |  |  |  |  | 40,000 |
| Gains tax |  |  |  |  |  | $(2,611)$ |
| Net cash flow | \$ 125,500 ) | \$ 43,145 | \$ 48,245 | \$ 44,745 | \$ 42,245 | \$ 75,619 |

BMC will lose a substantial portion of the projected revenues by not being able to increase its bidding price. Before undertaking the project, the company needs to identify the key variables that will determine whether it will succeed or fail. The marketing department has estimated revenue as follows:

$$
\begin{aligned}
\text { Annual revenue } & =(\text { Product demand })(\text { unit price }) \\
& =(2,000)(\$ 50)=\$ 100,000
\end{aligned}
$$

The engineering department has estimated variable costs, such as those of labor and materials, at $\$ 15$ per unit. Since the projected sales volume is 2,000 units per year, the total variable cost is $\$ 30,000$.

After first defining the unit sales, unit price, unit variable cost, fixed cost, and salvage value, we conduct a sensitivity analysis with respect to these key input variables. This is done by varying each of the estimates by a given percentage and determining
what effect the variation in that item will have on the final results. If the effect is large, the result is sensitive to that item. Our objective is to locate the most sensitive item(s).

## SOLUTION

Sensitivity analysis: We begin the sensitivity analysis with a consideration of the base-case situation, which reflects the best estimate (expected value) for each input variable. In developing Table 12.2, we changed a given variable by $20 \%$ in $5 \%$ increments, above and below the base-case value, and calculated new NPWs, while other variables were held constant. The values for both sales and operating costs were the expected, or base-case, values, and the resulting $\$ 40,169$ is the base-case NPW. Now we ask a series of "what-if" questions: What if sales are $20 \%$ below the expected level? What if operating costs rise? What if the unit price drops from $\$ 50$ to $\$ 45$ ? Table 12.2 summarizes the results of varying the values of the key input variables.
Sensitivity graph: Next, we construct a sensitivity graph for five of the transmission project's key input variables. (See Figure 12.1.) We plot the base-case NPW on the


Figure 12.I Sensitivity graph for BMC's transmission-housings project (Example 12.1).

## TABLE \| 2.2 Sensitivity Analysis for Five Key Input Variables (Example 12.1)

| Deviation | $\mathbf{- 2 0 \%}$ | $\mathbf{- 1 5 \%}$ | $\mathbf{- 1 0 \%}$ | $\mathbf{- 5 \%}$ | $\mathbf{0 \%}$ | $\mathbf{5 \%}$ | $\mathbf{1 0 \%}$ | $\mathbf{1 5 \%}$ | $\mathbf{2 0 \%}$ |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Unit price | $\$(57)$ | $\$ 9,999$ | $\$ 20,055$ | $\$ 30,111$ | $\$ 40,169$ | $\$ 50,225$ | $\$ 60,281$ | $\$ 70,337$ | $\$ 80,393$ |
| Demand | 12,010 | 19,049 | 26,088 | 33,130 | 40,169 | 47,208 | 54,247 | 61,286 | 68,325 |
| Variable cost | 52,236 | 49,219 | 46,202 | 43,186 | 40,169 | 37,152 | 34,135 | 31,118 | 28,101 |
| Fixed cost | 44,191 | 43,185 | 42,179 | 41,175 | 40,169 | 39,163 | 38,157 | 37,151 | 36,145 |
| Salvage value | 37,782 | 38,378 | 38,974 | 39,573 | 40,169 | 40,765 | 41,361 | 41,957 | 42,553 |

ordinate of the graph at the value 1.0 on the abscissa (or $0 \%$ deviation). Then we reduce the value of product demand to 0.95 of its base-case value and recompute the NPW with all other variables held at their base-case value. We repeat the process by either decreasing or increasing the relative deviation from the base case. The lines for the variable unit price, variable unit cost, fixed cost, and salvage value are obtained in the same manner. In Figure 12.1, we see that the project's NPW is (1) very sensitive to changes in product demand and unit price, (2) fairly sensitive to changes in variable costs, and (3) relatively insensitive to changes in the fixed cost and the salvage value.

Graphic displays such as the one in Figure 12.1 provide a useful means to communicate the relative sensitivities of the different variables to the corresponding NPW value. However, sensitivity graphs do not explain any interactions among the variables or the likelihood of realizing any specific deviation from the base case. Certainly, it is conceivable that an answer might not be very sensitive to changes in either of two items, but very sensitive to combined changes in them.

### 12.2.2 Break-Even Analysis

When we perform a sensitivity analysis of a project, we are asking how serious the effect of lower revenues or higher costs will be on the project's profitability. Managers sometimes prefer to ask instead how much sales can decrease below forecasts before the project begins to lose money. This type of analysis is known as break-even analysis. In other words, break-even analysis is a technique for studying the effect of variations in output on a firm's NPW (or other measures). We will present an approach to break-even analysis based on the project's cash flows.

To illustrate the procedure of break-even analysis based on NPW, we use the generalized cash flow approach we discussed in Section 10.4. We compute the PW of cash inflows as a function of an unknown variable (say, $x$ ), perhaps annual sales. For example,

$$
\text { PW of cash inflows }=f(x)_{1} .
$$

Next, we compute the PW of cash outflows as a function of $x$ :

$$
\text { PW of cash outflows }=f(x)_{2} .
$$

NPW is, of course, the difference between these two numbers. Accordingly, we look for the break-even value of $x$ that makes

$$
f(x)_{1}=f(x)_{2}
$$

Note that this break-even value is similar to that used to calculate the internal rate of return when we want to find the interest rate that makes the NPW equal zero. The break-even value is also used to calculate many other similar "cutoff values" at which a choice changes.

## EXAMPLE 12.2 Break-Even Analysis

Through the sensitivity analysis in Example 12.1, BMC's managers become convinced that the NPW is most sensitive to changes in annual sales volumes. Determine the break-even NPW value as a function of that variable.

## SOLUTION

The analysis is shown in Table 12.3, in which the revenues and costs of the BMC transmission-housings project are set out in terms of an unknown amount of annual sales $X$.

We calculate the PWs of cash inflows and outflows as follows:

- PW of cash inflows:

$$
\begin{aligned}
\mathrm{PW}(15 \%)_{\text {Inflow }}= & (\mathrm{PW} \text { of after-tax net revenue }) \\
& +(\mathrm{PW} \text { of net salvage value }) \\
& +(\mathrm{PW} \text { of tax savings from depreciation }) \\
= & 30 X(P / A, 15 \%, 5)+\$ 37,389(P / F, 15 \%, 5)
\end{aligned}
$$

TABLE \| 2.3 Break-Even Analysis with Unknown Annual Sales (Example 12.2)

| 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- |

Cash inflow:
Net salvage
Revenue:
$X(1-0.4)(\$ 50) \quad 30 X \quad 30 X \quad 30 X \quad 30 X \quad 30 X$

Depreciation credit

| 0.4 (depreciation) | 7,145 | 12,245 | 8,745 | 6,245 | 2,230 |
| :--- | :--- | :--- | :--- | :--- | :--- |

Cash outflow:

| Investment | $-125,000$ |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Variable cost: |  |  |  |  |  |
| $-X(1-0.4)(\$ 15)$ | $-9 X$ | $-9 X$ | $-9 X$ | $-9 X$ | $-9 X$ |

Fixed cost:


$$
\begin{aligned}
& +\$ 7,145(P / F, 15 \%, 1)+\$ 12,245(P / F, 15 \%, 2) \\
& +\$ 8,745(P / F, 15 \%, 3)+\$ 6,245(P / F, 15 \%, 4) \\
& +\$ 2,230(P / F, 15 \%, 5) \\
= & 30 X(P / A, 15 \%, 5)+\$ 44,490 \\
= & 100.5650 X+\$ 44,490 .
\end{aligned}
$$

- PW of cash outflows:

$$
\begin{aligned}
\mathrm{PW}(15 \%)_{\text {Outflow }}= & (\mathrm{PW} \text { of capital expenditure }) \\
& +(\mathrm{PW} \text { of after-tax expenses }) \\
= & \$ 125,000+(9 X+\$ 6,000)(P / A, 15 \%, 5) \\
= & 30.1694 X+\$ 145,113
\end{aligned}
$$

The NPW of cash flows for the BMC is thus

$$
\begin{aligned}
\operatorname{PW}(15 \%) & =100.5650 X+\$ 44,490 \\
& =-(30.1694 X+\$ 145,113) \\
& =70.3956 X-\$ 100,623 .
\end{aligned}
$$

In Table 12.4, we compute the PW of the inflows and the PW of the outflows as a function of demand ( $X$ ).

The NPW will be just slightly positive if the company sells 1,430 units. Precisely calculated, the zero-NPW point (break-even volume) is 1,429.43 units:

$$
\begin{aligned}
\operatorname{PW}(15 \%) & =70.3956 X-\$ 100,623 \\
& =0, \\
X_{b} & =1,430 \text { units. }
\end{aligned}
$$

TABLE I 2.4 Determination of Break-Even Volume Based on Project's NPW (Example 12.3)

|  | PW of Inflow <br> $(\mathbf{1 0 0 . 5 6 5 0 \boldsymbol { X }}$ | PW of Outflow <br> $\mathbf{( 3 0 . 1 6 9 4 \boldsymbol { X } +}$ <br> $\mathbf{\$ 4 4 , 4 9 0 )}$ | NPW <br> $\mathbf{( 7 0 . 3 9 5 6} \boldsymbol{X}-$ <br> $\mathbf{\$ 1 0 0 , 6 2 3 )}$ |
| :---: | :---: | :---: | :---: |
| 0 | $\$ 44,490$ | $\$ 145,113$ | $(100,623)$ |
| 500 | 94,773 | 160,198 | $(65,425)$ |
| 1,000 | 145,055 | 175,282 | $(30,227)$ |
| 1,429 | 188,197 | 188,225 | $(28)$ |
| 1,430 | 188,298 | 188,255 | 43 |
| 1,500 | 195,338 | 190,367 | 4,970 |
| 2,000 | 245,620 | 205,452 | 40,168 |
| 2,500 | 295,903 | 220,537 | 75,366 |

Break-even volume $=1,430$ units.


Figure I 2.2 Break-even analysis based on net cash flow (Example 12.2).

In Figure 12.2, we have plotted the PWs of the inflows and outflows under various assumptions about annual sales. The two lines cross when sales are 1,430 units, the point at which the project has a zero NPW. Again we see that, as long as sales are greater or equal to 1,430 , the project has a positive NPW.

### 12.2.3 Scenario Analysis

Although both sensitivity and break-even analyses are useful, they have limitations. Often, it is difficult to specify precisely the relationship between a particular variable and the NPW. The relationship is further complicated by interdependencies among the variables. Holding operating costs constant while varying unit sales may ease the analysis, but in reality, operating costs do not behave in this manner. Yet, it may complicate the analysis too much to permit movement in more than one variable at a time.

Scenario analysis is a technique that considers the sensitivity of NPW both to changes in key variables and to the range of likely values of those variables. For example, the decision maker may examine two extreme cases: a "worst-case" scenario (low unit sales, low unit price, high variable cost per unit, high fixed cost, and so on) and a "best-case" scenario. The NPWs under the worst and the best conditions are then calculated and compared with the expected, or base-case, NPW. Example 12.3 illustrates a plausible scenario analysis for BMC's transmission-housings project.

## EXAMPLE 12.3 Scenario Analysis

Consider again BMC's transmission-housings project first presented in Example 12.1. Assume that the company's managers are fairly confident of their estimates of all the project's cash flow variables, except the estimates of unit sales. Assume further that they regard a drop in unit sales below 1,600 or a rise above 2,400 as extremely unlikely. Thus, a decremental annual sale of 400 units defines the lower bound, or the worst-case scenario, whereas an incremental annual sale of 400 units defines the upper bound, or the best-case scenario. (Remember that the most likely value was 2,000 in annual unit sales.) Discuss the worst- and best-case scenarios, assuming that the unit sales for all five years are equal.

DISCUSSION: To carry out the scenario analysis, we ask the marketing and engineering staffs to give optimistic (best-case) and pessimistic (worst-case) estimates for the key variables. Then we use the worst-case variable values to obtain the worst-case NPW and the best-case variable values to obtain the best-case NPW.

## SOLUTION

The results of our analysis are summarized as follows:

| Variable <br> Considered | Worst-Case <br> Scenario | Most-Likely-Case <br> Scenario | Best-Case <br> Scenario |
| :--- | :---: | :---: | ---: |
| Unit demand | 1,600 | 2,000 | 2,400 |
| Unit price (\$) | 48 | 50 | 53 |
| Variable cost (\$) | 17 | 15 | 12 |
| Fixed cost (\$) | 11,000 | 10,000 | 8,000 |
| Salvage value (\$) | 30,000 | 40,000 | 50,000 |
| PW $(15 \%)$ | $-\$ 5,856$ | $\$ 40,169$ | $\$ 104,295$ |

We see that the base case produces a positive NPW, the worst case produces a negative NPW, and the best case produces a large positive NPW. Still, by just looking at the results in the table, it is not easy to interpret the scenario analysis or to make a decision based on it. For example, we could say that there is a chance of losing money on the project, but we do not yet have a specific probability for that possibility. Clearly, we need estimates of the probabilities of occurrence of the worst case, the best case, the base (most likely) case, and all the other possibilities.

The need to estimate probabilities leads us directly to our next step: developing a probability distribution (or, put another way, the probability that the variable in question takes on a certain value). If we can predict the effects on the NPW of variations in the parameters, why should we not assign a probability distribution to the possible outcomes of each parameter and combine these distributions in some way to produce a probability distribution for the possible outcomes of the NPW? We shall consider this issue in the next two sections.

Risk analysis is a technique to identify and assess factors that may jeopardize the success of a project.

## [12.3 Probability Concepts for Investment Decisions

In this section, we shall assume that the analyst has available the probabilities (likelihoods) of future events from either previous experience with a similar project or a market survey. The use of probability information can provide management with a range of possible outcomes and the likelihood of achieving different goals under each investment alternative.

### 12.3.I Assessment of Probabilities

We begin by defining terms related to probability, such as random variable, probability distribution, and cumulative probability distribution. Quantitative statements about risk are given as numerical probabilities or as likelihoods (odds) of occurrence. Probabilities are given as decimal fractions in the interval from 0.0 to 1.0 . An event or outcome that is certain to occur has a probability of 1.0 . As the probability of an event approaches 0 , the event becomes increasingly less likely to occur. The assignment of probabilities to the various outcomes of an investment project is generally called risk analysis.

## Random Variables

A random variable is a parameter or variable that can have more than one possible value (though not simultaneously). The value of a random variable at any one time is unknown until the event occurs, but the probability that the random variable will have a specific value is known in advance. In other words, associated with each possible value of the random variable is a likelihood, or probability, of occurrence. For example, when your college team plays a football game, only two events regarding the outcome of the game are possible: win or lose. The outcome is a random variable, dictated largely by the strength of your opponent.

To indicate random variables, we will adopt the convention of a capital italic letter (for example, $X$ ). To denote the situation in which the random variable takes a specific value, we will use a lowercase italic letter (for example, $x$ ). Random variables are classified as either discrete or continuous:

- Any random variables that take on only isolated (countable) values are discrete random variables.
- Continuous random variables may have any value within a certain interval.

For example, the outcome of a game should be a discrete random variable. By contrast, suppose you are interested in the amount of beverage sold on a given day that the game is played. The quantity (or volume) of beverage sold will depend on the weather conditions, the number of people attending the game, and other factors. In this case, the quantity is a continuous random variable-a random variable that takes a value from a continuous set of values.

## Probability Distributions

For a discrete random variable, the probability of each random event needs to be assessed. For a continuous random variable, the probability function needs to be assessed, as the event takes place over a continuous domain. In either case, a range of probabilities for each feasible outcome exists. Together, these probabilities make up a probability distribution.


Figure I2.3 A triangular probability distribution: (a) Probability function and (b) cumulative probability distribution.

Probability assessments may be based on past observations or historical data if the trends that were characteristic of the past are expected to prevail in the future. Forecasting weather or predicting the outcome of a game in many professional sports is done on the basis of compiled statistical data. Any probability assessments based on objective data are called objective probabilities. However, in many real investment situations, no objective data are available to consider. In these situations, we assign subjective probabilities that we think are appropriate to the possible states of nature. As long as we act consistently with our beliefs about the possible events, we may reasonably account for the economic consequences of those events in our profitability analysis.

For a continuous random variable, we usually try to establish a range of values; that is, we try to determine a minimum value $(L)$ and a maximum value $(H)$. Next, we determine whether any value within these limits might be more likely to occur than the other values; in other words, does the distribution have a mode $\left(M_{O}\right)$, or a most frequently occurring value?

- If the distribution does have a mode, we can represent the variable by a triangular distribution, such as that shown in Figure 12.3.
- If we have no reason to assume that one value is any more likely to occur than any other, perhaps the best we can do is to describe the variable as a uniform distribution, as shown in Figure 12.4.
These two distributions are frequently used to represent the variability of a random variable when the only information we have is its minimum, its maximum, and whether the distribution has a mode. For example, suppose the best judgment of the analyst was that the sales revenue could vary anywhere from $\$ 2,000$ to $\$ 5,000$ per day, and any value within the range is equally likely. This judgment about the variability of sales revenue could be represented by a uniform distribution.

For BMC's transmission-housings project, we can think of the discrete random variables ( $X$ and $Y$ ) as variables whose values cannot be predicted with certainty at the time of decision making. Let us assume the probability distributions in Table 12.5. We


Figure I2.4 A uniform probability distribution: (a) Probability function and (b) cumulative probability distribution.

| Product Demand ( $X$ ) |  | Unit Sale Price ( $Y$ ) |  |
| :---: | :---: | :---: | :---: |
| Units ( $x$ ) | $\boldsymbol{P}(\boldsymbol{X}=x)$ | Unit Price ( y ) | $P(Y=y)$ |
| 1,600 | 0.20 | \$48 | 0.30 |
| 2,000 | 0.60 | 50 | 0.50 |
| 2,400 | 0.20 | 53 | 0.20 |

see that the product demand with the highest probability is 2,000 units, whereas the unit sale price with the highest probability is $\$ 50$. These, therefore, are the most likely values. We also see a substantial probability that a unit demand other than 2,000 units will be realized. When we use only the most likely values in an economic analysis, we are in fact ignoring these other outcomes.

## Cumulative Distribution

As we have observed in the previous section, the probability distribution provides information regarding the probability that a random variable will assume some value $x$. We can use this information, in turn, to define the cumulative distribution function. The cumulative distribution function gives the probability that the random variable will attain a value smaller than or equal to some value $x$. A common notation for the cumulative distribution is

$$
F(x)=P(X \leq x)= \begin{cases}\sum_{j=1}^{j} p_{j} & (\text { for a discrete random variable }) \\ \int_{L}^{x} f(x) d x & (\text { for a continuous random variable })\end{cases}
$$

where $p_{j}$ is the probability of occurrence of the $x_{j}$ th value of the discrete random variable and $f(x)$ is a probability function for a continuous variable. With respect to a continuous random variable, the cumulative distribution rises continuously in a smooth (rather than stairwise) fashion.

Example 12.4 reveals the method by which probabilistic information can be incorporated into our analysis. Again, BMC's transmission-housings project will be used. In the next section, we will show you how to compute some composite statistics with the use of all the data.

## EXAMPLE 12.4 Cumulative Probability Distributions

Suppose that the only parameters subject to risk are the number of unit sales $(X)$ to Gulf Electric each year and the unit sales price ( $Y$ ). From experience in the market, BMC assesses the probabilities of outcomes for each variable as shown in Table 12.5. Determine the cumulative probability distributions for these random variables.

## SOLUTION

Consider the demand probability distribution ( $X$ ) given in Table 12.5 for BMC's transmission-housings project:

| Unit Demand $(\boldsymbol{X})$ | Probability, $\boldsymbol{P}(\boldsymbol{X}=\boldsymbol{x})$ |
| :---: | :---: |
| 1,600 | 0.2 |
| 2,000 | 0.6 |
| 2,400 | 0.2 |

If we want to know the probability that demand will be less than or equal to any particular value, we can use the following cumulative probability function:

$$
F(x)=P(X \leq x)= \begin{cases}0.2 & x \leq 1,600 \\ 0.8 & x \leq 2,000 \\ 1.0 & x \leq 2,400\end{cases}
$$

For example, if we want to know the probability that the demand will be less than or equal to 2,000 , we can examine the appropriate interval ( $x \leq 2,000$ ), and we shall find that the probability is $80 \%$.

We can find the cumulative distribution for $Y$ in a similar fashion. Figures 12.5 and 12.6 illustrate probability and cumulative probability distributions for $X$ and $Y$, respectively.


Figure I2.5 Probability and cumulative probability distributions for random variable $X$ (annual sales).


Figure I2.6 Probability and cumulative probability distributions for random variable $Y$ from Table 12.5.

## I2.3.2 Summary of Probabilistic Information

Although knowledge of the probability distribution of a random variable allows us to make a specific probability statement, a single value that may characterize the random variable and its probability distribution is often desirable. Such a quantity is the expected value of the random variable. We also want to know something about how the values of the random variable are dispersed about the expected value (i.e., the variance). In investment analysis, this dispersion information is interpreted as the degree of project risk. The expected value indicates the weighted average of the random variable, and the variance captures the variability of the random variable.

## Measure of Expectation

The expected value (also called the mean) is a weighted average value of the random variable, where the weighting factors are the probabilities of occurrence. All distributions (discrete and continuous) have an expected value. We will use $E[X]$ (or $\mu_{x}$ ) to denote the expected value of random variable $X$. For a random variable $X$ that has either discrete or continuous values, we compute the expected value with the formula

$$
E[X]=\mu_{x}= \begin{cases}\sum_{j=1}^{J}\left(p_{j}\right) x_{j} & (\text { discrete case })  \tag{12.1}\\ \int_{L}^{H} x f(x) d x & \text { (continuous case) }\end{cases}
$$

where $J$ is the number of discrete events and $L$ and $H$ are the lower and upper bounds of the continuous probability distribution.

The expected value of a distribution gives us important information about the "average" value of a random variable, such as the NPW, but it does not tell us anything about the variability on either side of the expected value. Will the range of possible values of the random variable be very small, and will all the values be located at or near the expected value? For example, the following represents the temperatures recorded on a typical winter day for two cities in the United States:

| Location | Low | High | Average |
| :--- | :---: | :---: | :---: |
| Atlanta | $15^{\circ} \mathrm{F}$ | $67^{\circ} \mathrm{F}$ | $41^{\circ} \mathrm{F}$ |
| Seattle | $36^{\circ} \mathrm{F}$ | $48^{\circ} \mathrm{F}$ | $42^{\circ} \mathrm{F}$ |

Even though both cities had almost identical mean (average) temperatures on that particular day, they had different variations in extreme temperatures. We shall examine this variability issue next.

## Measure of Variation

Another measure needed when we are analyzing probabilistic situations is a measure of the risk due to the variability of the outcomes. Among the various measures of the variation of a set of numbers that are used in statistical analysis are the range, the variance, and the standard deviation. The variance and the standard deviation are used most commonly in the analysis of risk. We will use $\operatorname{Var}[X]$ or $\sigma_{x}^{2}$ to denote the variance, and $\sigma_{x}$ to denote the standard deviation, of random variable $X$. (If there is only one random variable in an analysis, we normally omit the subscript.)

Expected value represents the average amount one "expects" to win per bet if bets with identifical odds are repeated many times.

Variance of a random variable is a measure of its statistical dispersion, indicating how far from the expected value its values typically are.

The variance tells us the degree of spread, or dispersion, of the distribution on either side of the mean value. As the variance increases, the spread of the distribution increases; the smaller the variance, the narrower is the spread about the expected value.

To determine the variance, we first calculate the deviation of each possible outcome $x_{j}$ from the expected value $\left(x_{j}-\mu\right)$. Then we raise each result to the second power and multiply it by the probability of $x_{j}$ occurring (i.e., $p_{j}$ ). The summation of all these products serves as a measure of the distribution's variability. For a random variable that has only discrete values, the equation for computing the variance is ${ }^{4}$

$$
\begin{equation*}
\operatorname{Var}[X]=\sigma_{x}^{2}=\sum_{j=1}^{J}\left(x_{j}-\mu\right)^{2}\left(p_{j}\right) \tag{12.2}
\end{equation*}
$$

where $p_{j}$ is the probability of occurrence of the $j$ th value of the random variable $\left(x_{j}\right)$, and $\mu$ is as defined by Eq. (12.1). To be most useful, any measure of risk should have a definite value (unit). One such measure is the standard deviation, which we may calculate by taking the positive square root of $\operatorname{Var}[X]$, measured in the same units as $X$ :

$$
\begin{equation*}
\sigma_{x}=\sqrt{\operatorname{Var}[X]} \tag{12.3}
\end{equation*}
$$

The standard deviation is a probability-weighted deviation (more precisely, the square root of the sum of squared deviations) from the expected value. Thus, it gives us an idea of how far above or below the expected value the actual value is likely to be. For most probability distributions, the actual value will be observed within the $\pm 3 \sigma$ range.

In practice, the calculation of the variance is somewhat easier if we use the formula

$$
\begin{align*}
\operatorname{Var}[X] & =\sum p_{j} x_{j}^{2}-\left(\sum p_{j} x_{j}\right)^{2} \\
& =E\left[X^{2}\right]-(E[X])^{2} \tag{12.4}
\end{align*}
$$

The term $E\left[X^{2}\right]$ in Eq. (12.4) is interpreted as the mean value of the squares of the random variable (i.e., the actual values squared). The second term is simply the mean value squared. Example 12.5 illustrates how we compute measures of variation.

## EXAMPLE 12.5 Calculation of Mean and Variance

Consider once more BMC's transmission-housings project. Unit sales $(X)$ and unit price $(Y)$ are estimated as in Table 12.5. Compute the means, variances, and standard deviations for the random variables $X$ and $Y$.

## SOLUTION

For the product demand variable $(X)$, we have

| $\boldsymbol{x}_{\boldsymbol{j}}$ | $\boldsymbol{p}_{\boldsymbol{j}}$ | $\boldsymbol{x}_{\boldsymbol{j}}\left(\boldsymbol{p}_{\boldsymbol{j}}\right)$ | $\left(\mathbf{x}_{\boldsymbol{j}}-\mathbf{E}[X]\right)^{\mathbf{2}}$ | $\left(\mathbf{x}_{\boldsymbol{j}}-\boldsymbol{E}[X]\right)^{\mathbf{2}}\left(\boldsymbol{p}_{\boldsymbol{j}}\right)$ |
| :---: | :---: | :---: | :---: | ---: |
| 1,600 | 0.20 | 320 | $(-400)^{2}$ | 32,000 |
| 2,000 | 0.60 | 1,200 | 0 | 0 |
| 2,400 | 0.20 | 480 | $(400)^{2}$ | 32,000 |
|  |  | $E[X]=2,000$ |  | $\operatorname{Var}[X]=64,000$ |
|  |  | $\sigma_{x}=252.98$ |  |  |

${ }^{4}$ For a continuous random variable, we compute the variance as follows: $\operatorname{Var}[X]=\int_{L}^{H}(x-\mu)^{2} f(x) d x$.

For the variable unit price $(Y)$, we have

| $\boldsymbol{y}_{\boldsymbol{j}}$ | $\boldsymbol{p}_{\boldsymbol{j}}$ | $\boldsymbol{y}_{\boldsymbol{j}}\left(\boldsymbol{p}_{\boldsymbol{j}}\right)$ | $\left(\boldsymbol{y}_{\boldsymbol{j}}-\mathbf{E}[\boldsymbol{Y}]\right)^{\mathbf{2}}$ | $\left(\boldsymbol{y}_{\boldsymbol{j}}-\mathbf{E}[\boldsymbol{Y}]\right)^{\mathbf{2}}\left(\boldsymbol{p}_{\boldsymbol{j}}\right)$ |
| ---: | ---: | ---: | ---: | ---: |
| $\$ 48$ | 0.30 | $\$ 14.40$ | $(-2)^{2}$ | 1.20 |
| 50 | 0.50 | 25.00 | $(0)^{2}$ | 0 |
| 53 | 0.20 | 10.60 | $(3)^{2}$ | 1.80 |
|  |  | $E[Y]=50.00$ |  | $\operatorname{Var}[Y]=3.00$ |
|  |  |  | $\sigma_{y}=1.73$ |  |

## I2.3.3 Joint and Conditional Probabilities

Thus far, we have not looked at how the values of some variables can influence the values of others. It is, however, entirely possible-indeed, it is likely-that the values of some parameters will be dependent on the values of others. We commonly express these dependencies in terms of conditional probabilities. An example is product demand, which will probably be influenced by unit price.

We define a joint probability as

$$
\begin{equation*}
P(x, y)=P(X=x \mid Y=y) P(Y=y) \tag{12.5}
\end{equation*}
$$

where $P(X=x \mid Y=y)$ is the conditional probability of observing $x$, given $Y=y$, and $P(Y=y)$ is the marginal probability of observing $Y=y$. Certainly, important cases exist in which a knowledge of the occurrence of event $X$ does not change the probability of an event $Y$. That is, if $X$ and $Y$ are independent, then the joint probability is simply

$$
\begin{equation*}
P(x, y)=P(x) P(y) . \tag{12.6}
\end{equation*}
$$

The concepts of joint, marginal, and conditional distributions are best illustrated by numerical examples.

Suppose that BMC's marketing staff estimates that, for a given unit price of $\$ 48$, the conditional probability that the company can sell 1,600 units is 0.10 . The probability of this joint event (unit sales $=1,600$ and unit sales price $=\$ 48$ ) is

$$
\begin{aligned}
P(x, y) & =P(1,600, \$ 48) \\
& =P(x=1,600 \mid y=\$ 48) P(y=\$ 48) \\
& =(0.10)(0.30) \\
& =0.03 .
\end{aligned}
$$

We can obtain the probabilities of other joint events in a similar fashion; several are presented in Table 12.6.

## TABLE I 2.6 Assessments of Conditional and Joint Probabilities

| Unit Price $Y$ | Probability | Conditional <br> Unit Sales $X$ | Joint Probability | Probability |
| :---: | :---: | :---: | :---: | :---: |
| \$48 | 0.30 | 1,600 | 0.10 | 0.03 |
|  |  | 2,000 | 0.40 | 0.12 |
|  |  | 2,400 | 0.50 | 0.15 |
| 50 | 0.50 | 1,600 | 0.10 | 0.05 |
|  |  | 2,000 | 0.64 | 0.32 |
|  |  | 2,400 | 0.26 | 0.13 |
| 53 | $0.20$ | 1,600 | 0.50 | 0.10 |
|  |  | 2,000 | 0.40 | 0.08 |
|  |  | 2,400 | 0.10 | 0.02 |

From Table 12.6, we can see that the unit demand $(X)$ ranges from 1,600 to 2,400 units, the unit price $(Y)$ ranges from $\$ 48$ to $\$ 53$, and nine joint events are possible. These joint probabilities must sum to unity:

| Joint Event $(\boldsymbol{x y})$ | $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})$ |
| :---: | ---: |
| $(1,600, \$ 48)$ | 0.03 |
| $(2,000, \$ 48)$ | 0.12 |
| $(2,400, \$ 48)$ | 0.15 |
| $(1,600, \$ 50)$ | 0.05 |
| $(2,000, \$ 50)$ | 0.32 |
| $(2,400, \$ 50)$ | 0.13 |
| $(1,600, \$ 53)$ | 0.10 |
| $(2,000, \$ 53)$ | 0.08 |
| $(2,400, \$ 53)$ | $\underline{0.02}$ |
|  | $S u m=1.00$ |

The marginal distribution for $x$ can be developed from the joint event by fixing $x$ and summing over $y$ :

| $\boldsymbol{x}_{\boldsymbol{j}}$ | $\boldsymbol{P}\left(\boldsymbol{x}_{\boldsymbol{j}}\right)=\sum_{\boldsymbol{y}} \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})$ |
| :---: | :---: |
| 1,600 | $P(1,600, \$ 48)+P(1,600, \$ 50)+P(1,600, \$ 53)=0.18$ |
| 2,000 | $P(2,000, \$ 48)+P(2,000, \$ 50)+P(2,000, \$ 53)=0.52$ |
| 2,400 | $P(2,400, \$ 48)+P(2,400, \$ 50)+P(2,400, \$ 53)=0.30$ |

This marginal distribution tells us that $52 \%$ of the time we can expect to have a demand of 2,000 units and $18 \%$ and $30 \%$ of the time we can expect to have a demand of 1,600 and 2,400, respectively.

### 12.3.4 Covariance and Coefficient of Correlation

When two random variables are not independent, we need some measure of their dependence on each other. The parameter that tells the degree to which two variables $(X, Y)$ are related is the covariance $\operatorname{Cov}(X, Y)$, denoted by $\sigma_{x y}$. Mathematically, we define

$$
\begin{align*}
\operatorname{Cov}(X, Y) & =\sigma_{x y} \\
& =E\{(X-E[X])(Y-E[Y])\} \\
& =E(X Y)-E(X) E(Y) \\
& =\rho_{x y} \sigma_{x} \sigma_{y}, \tag{12.7}
\end{align*}
$$

where $\rho_{x y}$ is the coefficient of correlation between $X$ and $Y$. It is clear that if $X$ tends to exceed its mean whenever $Y$ exceeds its mean, $\operatorname{Cov}(X, Y)$ will be positive. If $X$ tends to fall below its mean whenever $Y$ exceeds its mean, then $\operatorname{Cov}(X, Y)$ will be negative. The sign of $\operatorname{Cov}(X, Y)$, therefore, reveals whether $X$ and $Y$ vary directly or inversely with one another. We can rewrite Eq. (12.7) in terms of $\rho_{x y}$ :

$$
\begin{equation*}
\rho_{x y}=\frac{\operatorname{Cov}(X, Y)}{\sigma_{x} \sigma_{y}} . \tag{12.8}
\end{equation*}
$$

The value of $\rho_{x y}$ can vary within the range of -1 and +1 , with $\rho_{x y}=0$ indicating no correlation between the two random variables. As shown in Table 12.7, the coefficient of correlation between the product demand $(X)$ and the unit price $(Y)$ is negatively correlated, $\rho_{x y}=-0.439$, indicating that as the firm lowers the unit price, it tends to generate a higher demand.

### 12.4 Probability Distribution of NPW

After we have identified the random variables in a project and assessed the probabilities of the possible events, the next step is to develop the project's NPW distribution.

### 12.4.I Procedure for Developing an NPW Distribution

We will consider the situation in which all the random variables used in calculating the NPW are independent. To develop the NPW distribution, we may follow these steps:

- Express the NPW as functions of unknown random variables.
- Determine the probability distribution for each random variable.
- Determine the joint events and their probabilities.
- Evaluate the NPW equation at these joint events.
- Rank the NPW values in increasing order of NPW.

These steps are illustrated in Example 12.6.
TABLE I 2.7 Calculating the Correlation Coefficient between Two Random Variables $X$ and $Y$

| $(\boldsymbol{x}, \boldsymbol{y})$ | $(\boldsymbol{x}-\boldsymbol{E}[\boldsymbol{X}])$ | $(\boldsymbol{y}-\boldsymbol{E}[\boldsymbol{Y}])$ | $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ | $(\boldsymbol{x}-\boldsymbol{E}[\boldsymbol{X}])(\boldsymbol{y}-\boldsymbol{E}[\boldsymbol{Y}])$ | $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}) \times(\boldsymbol{x}-\boldsymbol{E}[\boldsymbol{X}])(\boldsymbol{y}-\boldsymbol{E}[\boldsymbol{Y}])$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $(1,600,48)$ | $(1,600-2,000)$ | $(48-50)$ | 0.03 | 800 | 24 |
| $(2,000,48)$ | $(2,000-2,000)$ | $(48-50)$ | 0.12 | 0 | 0 |
| $(2,400,48)$ | $(2,400-2,000)$ | $(48-50)$ | 0.15 | -800 | -120 |
| $(1,600,50)$ | $(1,600-2,000)$ | $(50-50)$ | 0.05 | 0 | 0 |
| $(2,000,50)$ | $(2,000-2,000)$ | $(50-50)$ | 0.32 | 0 | 0 |
| $(2,400,50)$ | $(2,400-2,000)$ | $(50-50)$ | 0.13 | 0 | 0 |
| $(1,600,53)$ | $(1,600-2,000)$ | $(53-50)$ | 0.10 | $-1,200$ | -120 |
| $(2,000,53)$ | $(2,000-2,000)$ | $(53-50)$ | 0.08 | 0 | 0 |
| $(2,400,53)$ | $(2,400-2,000)$ | $(53-50)$ | 0.02 |  | $\operatorname{Cov}^{2}(X, Y)=-1900$ |
|  |  |  |  | $\rho_{x y}=\frac{\operatorname{Cov}(X, Y)}{\sigma_{x} \sigma_{y}}$ |  |
|  |  |  |  | $=\frac{-192}{(\sqrt{64,000} \times \sqrt{3.00})}$ |  |
|  |  |  |  | -0.439 |  |
|  |  |  |  |  |  |

## EXAMPLE 12.6 Procedure for Developing an NPW Distribution

Consider again BMC's transmission-housings project first set forth in Example 12.1. Use the unit demand $(X)$ and price $(Y)$ given in Table 12.5, and develop the NPW distribution for the BMC project. Then calculate the mean and variance of the NPW distribution.

## SOLUTION

Table 12.8 summarizes the after-tax cash flow for the BMC's transmission-housings project as functions of random variables $X$ and $Y$. From the table, we can compute the PW of cash inflows as follows:

$$
\begin{aligned}
\operatorname{PW}(15 \%) & =0.6 X Y(P / A, 15 \%, 5)+\$ 44,490 \\
& =2.0113 X Y+\$ 44,490 .
\end{aligned}
$$

The PW of cash outflows is

$$
\begin{aligned}
\operatorname{PW}(15 \%) & =\$ 125,000+(9 X+\$ 6,000)(P / A, 15 \%, 5) \\
& =30.1694 X+\$ 145,113 .
\end{aligned}
$$

Thus, the NPW is

$$
\operatorname{PW}(15 \%)=2.0113 X(Y-\$ 15)-\$ 100,623 .
$$

If the product demand $X$ and the unit price $Y$ are independent random variables, then PW ( $15 \%$ ) will also be a random variable. To determine the NPW distribution, we need to consider all the combinations of possible outcomes. ${ }^{5}$ The first possibility is the event in which $x=1,600$ and $y=\$ 48$. Since $X$ and $Y$ are considered to be independent random variables, the probability of this joint event is

$$
\begin{aligned}
P(x=1,600, y=\$ 48) & =P(x=1,600) P(y=\$ 48) \\
& =(0.20)(0.30) \\
& =0.06
\end{aligned}
$$

With these values as input, we compute the possible NPW outcome as follows:

$$
\begin{aligned}
\operatorname{PW}(15 \%) & =2.0113 X(Y-\$ 15)-\$ 100,623 \\
& =2.0113(1,600)(\$ 48-\$ 15)-\$ 100,623 \\
& =\$ 5,574 .
\end{aligned}
$$

Eight other outcomes are possible; they are summarized with their joint probabilities in Table 12.9 and depicted in Figure 12.7.

The NPW probability distribution in Table 12.9 indicates that the project's NPW varies between $\$ 5,574$ and $\$ 82,808$, but that no loss occurs under any of the circumstances examined. On the one hand, from the cumulative distribution, we further observe that there is a 0.38 probability that the project would realize an NPW less than that forecast for the base case $(\$ 40,168)$. On the other hand, there is a 0.32 probability that the NPW will be greater than this value. Certainly, the probability distribution provides much more information on the likelihood of each possible event than does the scenario analysis presented in Section 12.2.3.

[^2]TABLE I 2.8 After-Tax Cash Flow as a Function of Unknown Unit Demand $(X)$ and Unit Price ( $Y$ )

| Item | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Cash inflow: |  |  |  |  |  |  |
| Net salvage |  |  |  |  |  | 37,389 |
| Revenue: |  |  |  |  |  |  |
| $X(1-0.4) Y$ |  | $0.6 X Y$ | $0.6 X Y$ | 0.6XY | 0.6 XY | $0.6 X Y$ |
| Depreciation credit: |  |  |  |  |  |  |
| 0.4 (depreciation) |  | 7,145 | 12,245 | 8,745 | 6,245 | 2,230 |
| Cash outflow: |  |  |  |  |  |  |
| Investment | -125,000 |  |  |  |  |  |
| Variable cost: |  |  |  |  |  |  |
| $-X(1-0.4)(\$ 15)$ |  | $-9 X$ | $-9 X$ | $-9 X$ | $-9 X$ | $-9 X$ |
| Fixed cost: |  |  |  |  |  |  |
| $-(1-0.4)(\$ 10,000)$ |  | -6,000 | -6,000 | -6,000 | -6,000 | -6,000 |
| Net cash flow | -125,000 | $0.6 X(Y-15)$ | $0.6 X(Y-15)$ | $0.6 X(Y-15)$ | $0.6 X(Y-15)$ | $0.6 X(Y-15)$ |
|  |  | +1,145 | +6,245 | +2,745 | +245 | +33,617 |

## TABLE I 2.9 The NPW Probability Distribution with Independent Random Variables

| Event No. | $\boldsymbol{x}$ | $\boldsymbol{y}$ | $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})$ | Cumulative Joint Probability | $\boldsymbol{N P W}$ |
| :---: | :---: | ---: | :---: | :---: | :---: |
| 1 | 1,600 | $\$ 48.00$ | 0.06 | 0.06 | $\$ 5,574$ |
| 2 | 1,600 | 50.00 | 0.10 | 0.16 | 12,010 |
| 3 | 1,600 | 53.00 | 0.04 | 0.20 | 21,664 |
| 4 | 2,000 | 48.00 | 0.18 | 0.38 | 32,123 |
| 5 | 2,000 | 50.00 | 0.30 | 0.68 | 40,168 |
| 6 | 2,000 | 53.00 | 0.12 | 0.80 | 52,236 |
| 7 | 2,400 | 48.00 | 0.06 | 0.86 | 58,672 |
| 8 | 2,400 | 50.00 | 0.10 | 0.96 | 68,326 |
| 9 | 2,400 | 53.00 | 0.04 | 1.00 | 82,808 |

We have developed a probability distribution for the NPW by considering random cash flows. As we observed, a probability distribution helps us to see what the data imply in terms of project risk. Now we can learn how to summarize the probabilistic information-the mean and the variance:

- For BMC's transmission-housings project, we compute the expected value of the NPW distribution as shown in Table 12.10. Note that this expected value is the same as the most likely value of the NPW distribution. This equality was expected because both $X$ and $Y$ have a symmetrical probability distribution.
- We obtain the variance of the NPW distribution, assuming independence between $X$ and $Y$ and using Eq. (12.2), as shown in Table 12.11. We could obtain the same result more easily by using Eq. (12.4).


Figure I 2.7 The NPW probability distribution when $X$ and $Y$ are independent (Example 12.6).

## TABLE I2.| 0 Calculation of the Mean of the NPW Distribution

| Event No. | $\boldsymbol{x}$ | $\boldsymbol{y}$ | $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})$ | Cumulative <br> Joint Probability | Weighted <br> NPW | NPW |
| :---: | :---: | :---: | :---: | :---: | :---: | ---: |
| 1 | 1,600 | $\$ 48.00$ | 0.06 | 0.06 | $\$ 5,574$ | $\$ 334$ |
| 2 | 1,600 | 50.00 | 0.10 | 0.16 | 12,010 | 1,201 |
| 3 | 1,600 | 53.00 | 0.04 | 0.20 | 21,664 | 867 |
| 4 | 2,000 | 48.00 | 0.18 | 0.38 | 32,123 | 5,782 |
| 5 | 2,000 | 50.00 | 0.30 | 0.68 | 40,168 | 12,050 |
| 6 | 2,000 | 53.00 | 0.12 | 0.80 | 52,236 | 6,268 |
| 7 | 2,400 | 48.00 | 0.06 | 0.86 | 58,672 | 3,520 |
| 8 | 2,400 | 50.00 | 0.10 | 0.96 | 68,326 | 6,833 |
| 9 | 2,400 | 53.00 | 0.04 | 1.00 | 82,808 | 3,312 |
|  |  |  |  |  | $E[N P W]=\$ 40,168$ |  |

## TABLE |2.|| Calculation of the Variance of the NPW Distribution

| Event <br> No. | $\boldsymbol{x}$ | $\boldsymbol{y}$ | $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})$ | $\mathbf{N P W}$ | $(\mathbf{N P W}-\boldsymbol{E}[\mathbf{N P W}])^{2}$ | Weighted <br> $(\mathbf{N P W}-\boldsymbol{E}[\mathbf{N P W}])^{2}$ |
| :--- | :---: | :---: | ---: | :---: | :---: | :---: |
| 1 | 1,600 | $\$ 48.00$ | 0.06 | $\$ 5,574$ | $1,196,769,744$ | $\$ 71,806,185$ |
| 2 | 1,600 | 50.00 | 0.10 | 12,010 | $792,884,227$ | $79,288,423$ |
| 3 | 1,600 | 53.00 | 0.04 | 21,664 | $342,396,536$ | $13,695,861$ |
| 4 | 2,000 | 48.00 | 0.18 | 32,123 | $64,725,243$ | $11,650,544$ |
| 5 | 2,000 | 50.00 | 0.30 | 40,168 | 0 | 0 |
| 6 | 2,000 | 53.00 | 0.12 | 52,236 | $145,631,797$ | $17,475,816$ |
| 7 | 2,400 | 48.00 | 0.06 | 58,672 | $342,396,536$ | $20,543,792$ |
| 8 | 2,400 | 50.00 | 0.10 | 68,326 | $792,884,227$ | $79,288,423$ |
| 9 | 2,400 | 53.00 | 0.04 | 82,808 | $1,818,132,077$ | $72,725,283$ |
|  |  |  |  |  |  | Var[NPW]=366,474,326 |
|  |  |  |  |  |  | $\sigma=\$ 19,144$ |

COMMENTS: We can obtain the mean and variance of the NPW analytically by using the properties of the product of random variables. Let $W=X Y$, where $X$ and $Y$ are random variables with known means and variances $\left(\mu_{x}, \sigma_{x}^{2}\right)$ and $\left(\mu_{y}, \sigma_{y}^{2}\right)$, respectively. If $X$ and $Y$ are independent of each other, then

$$
\begin{equation*}
E(W)=E(X Y)=\mu_{x} \mu_{y} \tag{12.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Var}(W)=\operatorname{Var}(X Y)=\mu_{x}^{2} \sigma_{y}^{2}+\mu_{y}^{2} \sigma_{x}^{2}+\sigma_{x}^{2} \sigma_{y}^{2} \tag{12.10}
\end{equation*}
$$

The expected net present worth is then

$$
\begin{aligned}
E[\mathrm{PW}(15 \%)] & =2.0113 E(X Y)-(2.0113)(15) E(X)-100,623 \\
& =2.0113(2,000)(50)-(2.0113)(15)(2,000)-100,623 \\
& =\$ 40,168
\end{aligned}
$$

Now let $Z=Y-15$. Then $E(Z)=E(Y)-15=\mu_{y}-15=50-15=35$ and $\operatorname{Var}[Z]=\operatorname{Var}[Y]=\sigma_{y}^{2}=3$. So

$$
\begin{aligned}
\operatorname{Var}[\operatorname{PW}(15 \%)] & =\operatorname{Var}[2.0113 X(Y-15)-100,623] \\
& =\operatorname{Var}[2.0113 X Z] \\
& =(2.0113)^{2} \operatorname{Var}[X Y] \\
& =(2.0113)^{2}\left(\mu_{x}^{2} \sigma_{y}^{2}+\mu_{y}^{2} \sigma_{x}^{2}+\sigma_{x}^{2} \sigma_{y}^{2}\right) \\
& =(2.0113)^{2}\left[\left(2,000^{2}\right)(3)+\left(50^{2}\right)(64,000)+(64,000)(3)\right] \\
& =366,474,326
\end{aligned}
$$

Note that the mean and variance thus calculated are exactly the same as in Tables 12.10 and 12.11 , respectively. Note also that if $X$ and $Y$ are correlated with each other, then Eq. (12.10) cannot be used. ${ }^{6}$

### 12.4.2 Aggregating Risk over Time

In the previous section, we developed an NPW equation by aggregating all cash flow components over time. Another approach to estimating the amount of risk present in a particular investment opportunity is to determine the mean and variance of cash flows in each period; then we may be able to aggregate the risk over the project life in terms of net present worth. We have two cases:

- Independent random variables. In this case,

$$
\begin{equation*}
E[\mathrm{PW}(i)]=\sum_{n=0}^{N} \frac{E\left(A_{n}\right)}{(1+i)^{n}} \tag{12.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Var}[\mathrm{PW}(i)]=\sum_{n=0}^{N} \frac{\operatorname{Var}\left(A_{n}\right)}{(1+i)^{2 n}} \tag{12.12}
\end{equation*}
$$

where

$$
\begin{aligned}
& A_{n}=\text { cash flow in period } n \\
& E\left(A_{n}\right)=\text { expected cash flow in period } n, \\
& \operatorname{Var}\left(A_{n}\right)=\text { variance of the cash flow in period } n
\end{aligned}
$$

[^3]In defining Eq. (12.12), we are also assuming the independence of cash flows, meaning that knowledge of what will happen to one particular period's cash flow will not allow us to predict what will happen to cash flows in other periods.

- Dependent random variables. In case we cannot assume a statistical independence among cash flows, we need to consider the degree of dependence among them. The expected-value calculation will not be affected by this dependence, but the project variance will be calculated as

$$
\begin{equation*}
\operatorname{Var}[\mathrm{PW}(i)]=\sum_{n=0}^{N} \frac{\operatorname{Var}\left(A_{n}\right)}{(1+i)^{2 n}}+2 \sum_{n=0}^{N-1} \sum_{s=n+1}^{N} \frac{\rho_{n s} \sigma_{n} \sigma_{s}}{(1+i)^{n+s}}, \tag{12.13}
\end{equation*}
$$

where $\rho_{n s}=$ correlation coefficient (degree of dependence) between $A_{n}$ and $A_{s}$. The value of $\rho_{n s}$ can vary within the range from -1 to +1 , with $\rho_{n s}=-1$ indicating perfect negative correlation and $\rho_{n s}=+1$ indicating perfect positive correlation. The result $\rho_{n s}=0$ implies that no correlation exists between $A_{n}$ and $A_{s}$. If $\rho_{n s}>0$, we can say that $A_{n}$ and $A_{s}$ are positively correlated, meaning that if the actual realization of cash flow for $A_{n}$ is higher than its expected value, it is likely that you will also observe a higher cash flow than its expected value for $A_{s}$. If $\rho_{n s}<0$, the opposite relation exists.
Example 12.7 illustrates the mechanics involved in calculating the mean and variance of a project's net present worth.

## EXAMPLE 12.7 Aggregation of Risk over Time

Consider the following financial data for an investment project, where only random components are the operating expenses in each period:

- Investment required $=\$ 10,000$
- Project life $=3$ years
- Expected salvage value $=\$ 0$
- Annual operating revenue $=\$ 20,000$
- Annual operating expenses are random variables with the following means and variances:

| $\boldsymbol{n}$ | $\boldsymbol{X}_{\mathbf{I}}$ | $\boldsymbol{X}_{\mathbf{2}}$ | $\boldsymbol{X}_{\mathbf{3}}$ |
| :--- | ---: | :---: | ---: |
| Mean | $\$ 9,000$ | $\$ 13,000$ | $\$ 15,000$ |
| Variance | 250,000 | 490,000 | $1,000,000$ |

- Depreciation method is alternative MACRS (3 years):

| $\boldsymbol{n}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\mathbf{3}$ |
| :---: | :---: | :---: | :---: |
| $D_{n}$ | $\$ 1,667$ | $\$ 3,333$ | $\$ 1,667$ |

- Tax rate $=40 \%$
- Discount rate (or MARR) $=12 \%$

Determine the expected net present worth and the variance of the NPW assuming the following two situations: (a) $X_{i}$ are independent random variables and (b) $X_{i}$ are dependent random variables.

## SOLUTION

Step 1: Calculate the net proceeds from disposal of the equipment at the end of the project life:
Salvage value $=0$,
Book value $=10,000-\sum_{n=1}^{3} D_{n}=\$ 3,333$,
Taxable gain or (loss) $=0-\$ 3,333=(\$ 3,333)$,
Loss credit $=0.4(\$ 3,333)=\$ 1,333$,
Net proceeds from sale $=$ Salvage value + Loss credit $=\$ 1,333$.
Step 2: Construct a generalized cash flow table as a function of the random variable $X_{n}$ :

|  | Cash Flow |  |  |  |
| :--- | :---: | ---: | ---: | ---: |
| Description | $\mathbf{0}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\mathbf{3}$ |
| Investment | $-10,000$ |  |  |  |
| Revenue $\times(0.6)$ |  | 12,000 | 12,000 | 12,000 |
| $-\mathrm{O} \& \mathrm{M} \times(0.6)$ |  | $-0.6 X_{1}$ | $-0.6 X_{2}$ | $-0.6 X_{3}$ |
| $D_{n} \times(0.4)$ | 667 | 1,334 | 667 |  |
| Net proceeds from sale |  |  |  | 1,333 |
| Net cash flow | $-10,000$ | 12,667 | 13,334 | 14,000 |
|  |  | $-0.6 X_{1}$ | $-0.6 X_{2}$ | $-0.6 X_{3}$ |

$$
\begin{array}{rlrl}
E\left[X_{1}\right]=9,000, & E\left[X_{2}\right] & =13,000, \quad E\left[X_{3}\right]=15,000 \\
\operatorname{Var}\left[X_{1}\right]=250,000, & \operatorname{Var}\left[X_{2}\right] & =490,000, & \operatorname{Var}\left[X_{3}\right]=1,000,000
\end{array}
$$

The net present worth is then

$$
\begin{aligned}
\operatorname{PW}(12 \%)= & {\left[-10,000+\frac{12,667}{1.12}+\frac{13,334}{1.12^{2}}+\frac{14,000}{1.12^{3}}\right] } \\
& -0.6\left[\frac{X_{1}}{1.12}+\frac{X_{2}}{1.12^{2}}+\frac{X_{3}}{1.12^{3}}\right] \\
= & 21,905-0.6\left[\frac{X_{1}}{1.12}+\frac{X_{2}}{1.12^{2}}+\frac{X_{3}}{1.12^{3}}\right] .
\end{aligned}
$$

- Case 1. Independent Cash Flows: Using Eqs. (12.10) and (12.12), we obtain

$$
\begin{aligned}
& E[\mathrm{PW}(12 \%)]= 21,905-0.6\left[\frac{E\left[X_{1}\right]}{1.12}+\frac{E\left[X_{2}\right]}{1.12^{2}}+\frac{E\left[X_{3}\right]}{1.12^{3}}\right] \\
&= 21,905-17,446 \\
&= \$ 4,459 \\
& \begin{aligned}
\operatorname{Var}[\mathrm{PW}(12 \%)]= & \left(\frac{-0.6}{1.12}\right)^{2} \operatorname{Var}\left[X_{1}\right]+\left(\frac{-0.6}{1.12^{2}}\right)^{2} \operatorname{Var}\left[X_{2}\right] \\
& +\left(\frac{-0.6}{1.12^{3}}\right)^{2} \operatorname{Var}\left[X_{3}\right] \\
= & 71,747.45+112,103.39+182,387.20 \\
= & 366,240, \\
\sigma[\mathrm{PW}(12 \%)]= & \$ 605.18 .
\end{aligned}
\end{aligned}
$$

- Case 2. Dependent Cash Flows: If the random variables $X_{1}, X_{2}$, and $X_{3}$ are partially correlated with the correlation coefficients $\rho_{12}=0.3, \rho_{13}=0.5$, and $\rho_{23}=0.4$, respectively (see accompanying diagram), then the expected value and the variance of the NPW can be calculated with Eq. (12.13):

$$
\begin{aligned}
E[\mathrm{PW}(12 \%)]= & 21,905-0.6\left[\frac{E\left[X_{1}\right]}{1.12}+\frac{E\left[X_{2}\right]}{1.12^{2}}+\frac{E\left[X_{3}\right]}{1.12^{3}}\right] \\
= & 21,905-17,446 \\
= & \$ 4,459 \\
\operatorname{Var}[\mathrm{PW}(12 \%)]= & (\text { Original variance })+(\text { Covariance terms }) \\
= & 366,240 \\
& +2\left(\frac{-0.6}{1.12}\right)\left(\frac{-0.6}{1.12^{2}}\right) \rho_{12} \sigma_{1} \sigma_{2} \\
& +2\left(\frac{-0.6}{1.12}\right)\left(\frac{-0.6}{1.12^{3}}\right) \rho_{13} \sigma_{1} \sigma_{3} \\
& +2\left(\frac{-0.6}{1.12^{2}}\right)\left(\frac{-0.6}{1.12^{3}}\right) \rho_{23} \sigma_{2} \sigma_{3} \\
= & 366,240 \\
& +2\left(\frac{0.6^{2}}{1.12^{3}}\right)(0.3)(500)(700) \\
& +2\left(\frac{0.6^{2}}{1.12^{4}}\right)(0.5)(500)(1,000) \\
& +2\left(\frac{0.6^{2}}{1.12^{5}}\right)(0.4)(700)(1,000)
\end{aligned}
$$

$$
\begin{aligned}
& =366,240+53,810.59+114,393.25+114,393.25 \\
& =648,837 \\
\sigma[\mathrm{PW}(12 \%)] & =\$ 805.50
\end{aligned}
$$



Note that the variance increases significantly when at least some of the random variables are positively correlated.

COMMENTS: How is information such as the preceding used in decision making? Most probability distributions are completely described by six standard deviationsthree above, and three below, the mean. As shown in Figure 12.8, the NPW of this project would almost certainly fall between $\$ 2,643$ and $\$ 6,275$ for the independent case and between $\$ 2,043$ and $\$ 6,876$ for the dependent case. In either situation, the NPW below $3 \sigma$ from the mean is still positive, so we may say that the project is quite safe. If that figure were negative, it would then be up to the decision maker to determine whether it is worth investing in the project, given its mean and standard deviation.


Figure 12.8 NPW distributions with $\pm 3 \sigma$ : (a) Independent case and (b) dependent case.

## Laws of large

 numbers imply that the average of a random sample from a large population is likely to be close to the mean of the whole population.
## I2.4.3 Decision Rules for Comparing Mutually Exclusive Risky Alternatives

Once the expected value has been located from the NPW distribution, it can be used to make an accept-reject decision in much the same way that a single NPW is used when a single possible outcome for an investment project is considered.

## Expected-Value Criterion

The decision rule is called the expected-value criterion, and using it, we may accept a single project if its expected NPW value is positive. In the case of mutually exclusive alternatives, we select the one with the highest expected NPW. The use of the expected NPW has an advantage over the use of a point estimate, such as the likely value, because it includes all possible cash flow events and their probabilities.

The justification for the use of the expected-value criterion is based on the law of large numbers, which states that if many repetitions of an experiment are performed, the average outcome will tend toward the expected value. This justification may seem to negate the usefulness of the criterion, since, in project evaluation, we are most often concerned with a single, nonrepeatable "experiment" (i.e., an investment alternative). However, if a firm adopts the expected-value criterion as a standard decision rule for all of its investment alternatives, then, over the long term, the law of large numbers predicts that accepted projects tend to meet their expected values. Individual projects may succeed or fail, but the average project tends to meet the firm's standard for economic success.

## Mean-and-Variance Criterion

The expected-value criterion is simple and straightforward to use, but it fails to reflect the variability of the outcome of an investment. Certainly, we can enrich our decision by incorporating information on variability along with the expected value. Since the variance represents the dispersion of the distribution, it is desirable to minimize it. In other words, the smaller the variance, the less the variability (the potential for loss) associated with the NPW. Therefore, when we compare mutually exclusive projects, we may select the alternative with the smaller variance if its expected value is the same as or larger than those of other alternatives.

In cases where preferences are not clear cut, the ultimate choice depends on the decision maker's trade-offs-how far he or she is willing to take the variability to achieve a higher expected value. In other words, the challenge is to decide what level of risk you are willing to accept and then, having decided on your tolerance for risk, to understand the implications of that choice. Example 12.8 illustrates some of the critical issues that need to be considered in evaluating mutually exclusive risky projects.

## EXAMPLE 12.8 Comparing Risky Mutually Exclusive Projects

With ever-growing concerns about air pollution, the greenhouse effect, and increased dependence on oil imports in the United States, Green Engineering has developed a prototype conversion unit that allows a motorist to switch from gasoline to compressed natural gas (CNG) or vice versa. Driving a car equipped with Green's conversion kit is not much different from driving a conventional model. A small dial switch on the dashboard controls which fuel is to be used. Four different
configurations are available, according to the type of vehicle: compact size, midsize, large size, and trucks. In the past, Green has built a few prototype vehicles powered by alternative fuels other than gasoline, but has been reluctant to go into higher volume production without more evidence of public demand.

As a result, Green Engineering initially would like to target one market segment (one configuration model) in offering the conversion kit. Green Engineering's marketing group has compiled the potential NPW distribution for each different configuration when marketed independently.

| Event <br> (NPW) <br> (unit: thousands) | Probabilities |  |  |  |
| :---: | :--- | :--- | :--- | :--- |
|  | 0.35 | 0.10 | 0.40 | 0.20 |
| 1,500 | 0 | 0.45 | 0 | 0.40 |
| 2,000 | 0.40 | 0 | 0.25 | 0 |
| 2,500 | 0 | 0.35 | 0 | 0.30 |
| 3,000 | 0.20 | 0 | 0.20 | 0 |
| 3,500 | 0 | 0 | 0 | 0 |
| 4,000 | 0.05 | 0 | 0.15 | 0 |
| 4,500 | 0 | 0.10 | 0 | 0.10 |

Evaluate the expected return and risk for each model configuration, and recommend which one, if any, should be selected.

## SOLUTION

For model 1, we calculate the mean and variance of the NPW distribution as follows:

$$
\begin{aligned}
E[\mathrm{NPW}]_{1}= & \$ 1,000(0.35)+\$ 2,000(0.40) \\
& +\$ 3,000(0.20)+\$ 4,000(0.05) \\
= & \$ 1,950 \\
\operatorname{Var}[\mathrm{NPW}]_{1}= & 1,000^{2}(0.35)+2,000^{2}(0.40) \\
& +3,000^{2}(0.20)+4,000^{2}(0.05)-(1,950)^{2} \\
= & 747,500 .
\end{aligned}
$$

In a similar manner, we compute the other values as follows:

| Configuration | E[NPW] | Var[NPW] |
| :--- | :---: | ---: |
| Model 1 | $\$ 1,950$ | 747,500 |
| Model 2 | 2,100 | 915,000 |
| Model 3 | 2,100 | $1,190,000$ |
| Model 4 | 2,000 | $1,000,000$ |

## Risk-Return

 Tradeoffs: The greater the amount of risk that an investor is willing to take on, the greater the potential return.

Figure $\mathbf{1 2 . 9}$ Mean-variance chart showing project dominance. Both model 3 and model 4 are dominated by model 2 .

The results are plotted in Figure 12.9. If we make a choice solely on the basis of the expected value, we may select either model 2 or model 3, because they have the highest expected NPW.

If we consider the variability along with the expected NPW, however, the correct choice is not obvious. We will first eliminate those alternatives which are clearly inferior to others:

- Model 2 versus Model 3. We see that models 2 and 3 have the same mean of $\$ 2,100$, but model 2 has a much lower variance. In other words, model 2 dominates model 3, so we eliminate model 3 from further consideration.
- Model 2 versus Model 4. Similarly, we see that model 2 is preferred over model 4 , because model 2 has a higher expected value and a smaller variance; model 2 again dominates model 4 , so we eliminate model 4 . In other words, model 2 dominates both models 3 and 4 as we consider the variability of NPW.
- Model 1 versus Model 2. Even though the mean-and-variance rule has enabled us to narrow down our choices to only two models (models 1 and 2), it does not indicate what the ultimate choice should be. In other words, comparing models 1 and 2 , we see that $E[\mathrm{NPW}]$ increases from $\$ 1,950$ to $\$ 2,100$ at the price of a higher $\operatorname{Var}[\mathrm{NPW}]$, which increases from 747,500 to 915,000 . The choice, then, will depend on the decision maker's trade-offs between the incremental expected return $(\$ 150)$ and the incremental risk $(167,500)$. We cannot choose between the two simply on the basis of mean and variance, so we must resort to other probabilistic information. ${ }^{7}$


### 2.5 Risk Simulation

In the previous sections, we examined analytical methods of determining the NPW distributions and computing their means and variances. As we saw in Section 12.4.1, the NPW distribution offers numerous options for graphically presenting to the decision maker probabilistic information, such as the range and likelihoods of occurrence of possible levels

[^4]of NPW. Whenever we can adequately evaluate the risky investment problem by analytical methods, it is generally preferable to do so. However, many investment situations are such that we cannot solve them easily by analytical methods, especially when many random variables are involved. In these situations, we may develop the NPW distribution through computer simulation.

### 12.5.I Computer Simulation

Before we examine the details of risk simulation, let us consider a situation in which we wish to train a new astronaut for a future space mission. Several approaches exist for training this astronaut. One (somewhat unlikely) possibility is to place the trainee in an actual space shuttle and to launch him or her into space. This approach certainly would be expensive; it also would be extremely risky, because any human error made by the trainee would have tragic consequences. As an alternative, we can place the trainee in a flight simulator designed to mimic the behavior of the actual space shuttle in flight. The advantage of this approach is that the astronaut trainee learns all the essential functions of space operation in a simulated space environment. The flight simulator generates test conditions approximating operational conditions, and any human errors made during training cause no harm to the astronaut or to the equipment being used.

The use of computer simulation is not restricted to simulating a physical phenomenon such as the flight simulator. In recent years, techniques for testing the results of some investment decisions before they are actually executed have been developed. As a result, many phases of business investment decisions have been simulated with considerable success. In fact, we can analyze BMC's transmission-housings project by building a simulation model. The general approach is to assign a subjective (or objective) probability distribution to each unknown factor and to combine all these distributions into a probability distribution for the project's profitability as a whole. The essential idea is that if we can simulate the actual state of nature for unknown investment variables on a computer, we may be able to obtain the resulting NPW distribution.

The unit demand ( $X$ ) in our BMC's transmission-housing project was one of the random variables in the problem. We can know the exact value for this random variable only after the project is implemented. Is there any way to predict the actual value before we make any decision about the project?

The following logical steps are often suggested for a computer program that simulates investment scenarios:
Step 1. Identify all the variables that affect the measure of investment worth (e.g., NPW after taxes).
Step 2. Identify the relationships among all the variables. The relationships of interest here are expressed by the equations or the series of numerical computations by which we compute the NPW of an investment project. These equations make up the model we are trying to analyze.
Step 3. Classify the variables into two groups: the parameters whose values are known with certainty and the random variables for which exact values cannot be specified at the time of decision making.
Step 4. Define distributions for all the random variables.
Step 5. Perform Monte Carlo sampling (see Section 12.5.3) and describe the resulting NPW distribution.
Step 6. Compute the distribution parameters and prepare graphic displays of the results of the simulation.

Monte Carlo methods are a class of computational algorithms for simulating the behavior of various physical and mathematical system.


Figure 12.10 Logical steps involved in simulating a risky investment.
Figure 12.10 illustrates the logical steps involved in simulating a risky investment project. The risk simulation process has two important advantages compared with the analytical approach discussed in Section 12.4:

1. The number of variables that can be considered is practically unlimited, and the distributions used to define the possible values of each random variable can be of any type and any shape. The distributions can be based on statistical data if they are available or, more commonly, on subjective judgment.
2. The method lends itself to sensitivity analyses. By defining some factors that have the most significant effect on the resulting NPW values and using different distributions (in terms of either shape or range) for each variable, we can observe the extent to which the NPW distribution is changed.

## I2.5.2 Model Building

In this section, we shall present some of the procedural details related to the first three steps (model building) outlined in Section 12.5.1. To illustrate the typical procedure involved, we shall work with the investment setting for BMC's transmission-housings project described in Example 12.6.

The initial step is to define the measure of investment worth and the factors which affect that measure. For our presentation, we choose the measure of investment worth as an after-tax NPW computed at a given interest rate $i$. In fact, we are free to choose any measure of worth, such as annual worth or future worth. In the second step, we must divide into two groups all the variables that we listed in Step 1 as affecting the NPW. One group consists of all the parameters for which values are known. The second group includes all remaining parameters for which we do not know exact values at the time of analysis. The third step is to define the relationships that tie together all the variables. These relationships may take the form of a single equation or several equations.

## EXAMPLE 12.9 Developing a Simulation Model

Consider again BMC's transmission-housings project, as set forth in Example 12.6. Identify the input factors related to the project and develop the simulation model for the NPW distribution.

DISCUSSION: For the BMC project, the variables that affect the NPW are the investment required, unit price, demand, variable production cost, fixed production cost, tax rate, and depreciation expenses, as well as the firm's interest rate. Some of the parameters that might be included in the known group are the investment cost and interest rate (MARR). If we have already purchased the equipment or have received a price quote, then we also know the depreciation amount. In addition, assuming that we are operating in a stable economy, we would probably know the tax rates for computing income taxes due.

The group of parameters with unknown values would usually include all the variables relating to costs and future operating expense and to future demand and sales prices. These are the random variables for which we must assess the probability distributions.

For simplicity, we classify the input parameters or variables for BMC's trans-mission-housings project as follows:

| Assumed to Be <br> Known Parameters | Assumed to Be <br> Unknown Parameters |
| :--- | :--- |
| MARR | Unit price |
| Tax rate | Demand |
| Depreciation amount | Salvage value |
| Investment amount |  |
| Project life |  |
| Fixed production cost |  |
| Variable production cost |  |

Note that, unlike the situation in Example 12.6, here we treat the salvage value as a random variable. With these assumptions, we are now ready to build the NPW equation for the BMC project.

## SOLUTION

Recall that the basic investment parameters assumed for BMC's five-year project in Example 12.6 were as follows:

- Investment $=\$ 125,000$
- Marginal tax rate $=0.40$
- Annual fixed cost $=\$ 10,000$
- Variable unit production cost $=\$ 15 / \mathrm{unit}$
- $\operatorname{MARR}(i)=15 \%$
- Annual depreciation amounts:

| $\boldsymbol{n}$ | $\mathbf{D}_{\boldsymbol{n}}$ |
| :---: | ---: |
| 1 | $\$ 17,863$ |
| 2 | 30,613 |
| 3 | 21,863 |
| 4 | 15,613 |
| 5 | 5,575 |

The after-tax annual revenue is expressed in terms of functions of product demand $(X)$ and unit price $(Y)$ :

$$
R_{n}=X Y\left(1-t_{m}\right)=0.6 X Y
$$

The after-tax annual expenses excluding depreciation are also expressed as a function of product demand ( $X$ ):

$$
\begin{aligned}
E_{n} & =(\text { Fixed cost }+ \text { variable cost })\left(1-t_{m}\right) \\
& =(\$ 10,000+15 X)(0.60) \\
& =\$ 6,000+9 X .
\end{aligned}
$$

Then the net after-tax cash revenue is

$$
\begin{aligned}
V_{n} & =R_{n}-E_{n} \\
& =0.6 X Y-9 X-\$ 6,000 .
\end{aligned}
$$

The present worth of the net after-tax cash inflow from revenue is

$$
\begin{aligned}
\sum_{n=1}^{5} V_{n}(P / F, 15 \%, n) & =[0.6 X(Y-15)-\$ 6,000](P / A, 15 \%, 5) \\
& =0.6 X(Y-15)(3.3522)-\$ 20,113
\end{aligned}
$$

The present worth of the total depreciation credits is

$$
\begin{aligned}
\sum_{n=1}^{5} D_{n} t_{m}(P / F, i, n)= & 0.40[\$ 17,863(P / F, 15 \%, 1)+\$ 30,613(P / F, 15 \%, 2) \\
& +\$ 21,863(P / F, 15 \%, 3)+\$ 15,613(P / F, 15 \%, 4) \\
& +\$ 5,575(P / F, 15 \%, 5)] \\
= & \$ 25,901
\end{aligned}
$$

Since the total depreciation amount is $\$ 91,527$, the book value at the end of year 5 is $\$ 33,473$ ( $\$ 125,000-\$ 91,527)$. Any salvage value greater than this book value is treated as a gain taxed at the rate $t_{m}$. In our example, the salvage value is considered to be a random variable. Thus, the amount of taxable gains (losses) also becomes a random variable. Therefore, the net salvage value after tax adjustment is

$$
\begin{aligned}
S-(S-\$ 33,473) t_{m} & =S\left(1-t_{m}\right)+33,473 t_{m} \\
& =0.6 S+\$ 13,389
\end{aligned}
$$

Then the equivalent present worth of this amount is

$$
(0.6 S+\$ 13,389)(P / F, 15 \%, 5)=(0.6 S+\$ 13,389)(0.4972)
$$

Now the NPW equation can be summarized as

$$
\begin{aligned}
\operatorname{PW}(15 \%)= & -\$ 125,000+0.6 X(Y-15)(3.3522)-\$ 20,113+\$ 25,901 \\
& +(0.6 S+\$ 13,389)(0.4972) \\
= & -\$ 112,555+2.0113 X(Y-15)+0.2983 S
\end{aligned}
$$

Note that the NPW function is now expressed in terms of the three random variables $X, Y$, and $S$.

### 12.5.3 Monte Carlo Sampling

With some variables, we may base the probability distribution on objective evidence gleaned from the past if the decision maker feels that the same trend will continue to operate in the future. If not, we may use subjective probabilities as discussed in Section 12.3.1. Once we specify a distribution for a random variable, we need to determine how to generate samples from that distribution. Monte Carlo sampling is a simulation method in which a random sample of outcomes is generated for a specified probability distribution. In this section, we shall discuss the Monte Carlo sampling procedure for an independent random variable.

## Random Numbers

The sampling process is the key part of the analysis. It must be done such that the sequence of values sampled will be distributed in the same way as the original distribution. To accomplish this objective, we need a source of independent, identically distributed uniform random numbers between 0 and 1 . Toward that end, we can use a table of random numbers, but most digital computers have programs available to generate "equally likely (uniform)" random decimals between 0 and 1 . We will use $U(0,1)$ to denote such a statistically reliable uniform random-number generator, and we will use $U_{1}, U_{2}, U_{3}, \ldots$ to represent uniform random numbers generated by this routine. (In Microsoft Excel, the RAND function can be used to generate such a sequence of random numbers.)

## Sampling Procedure

For any given random numbers, the question is, How are they used to sample a distribution in a simulation analysis? The first task is to convert the distribution into its corresponding cumulative frequency distribution. Then the random number generated is set equal to its numerically equivalent percentile and is used as the entry point on the $F(x)$ axis of the cumulative-frequency graph. The sampled value of the random variable is the $x$ value corresponding to this cumulative percentile entry point.

This method of generating random values works because choosing a random decimal between 0 and 1 is equivalent to choosing a random percentile of the distribution. Then the random value is used to convert the random percentile to a particular value. The method is general and can be used for any cumulative probability distribution, either continuous or discrete.

## EXAMPLE 12. 10 Monte Carlo Sampling

In Example 12.9, we developed an NPW equation for BMC's transmission-housings project as a function of three random variables-demand $(X)$, unit price $(Y)$, and salvage value ( $S$ ):

$$
\operatorname{PW}(15 \%)=-\$ 112,555+2.0113 X(Y-15)+0.2983 S
$$

- For random variable $X$, we will assume the same discrete distribution as defined in Table 12.5.
- For random variable $Y$, we will assume a triangular distribution with $L=\$ 48$, $H=\$ 53$, and $M_{O}=\$ 50$.
- For random variable $S$, we will assume a uniform distribution with $L=\$ 30,000$ and $H=\$ 50,000$.

With the random variables ( $X, Y$, and $S$ ) distributed as just set forth, and assuming that these random variables are mutually independent of each other, we need three uniform random numbers to sample one realization from each random variable. We determine the NPW distribution on the basis of 200 iterations.

DISCUSSION: As outlined in Figure 12.11, a simulation analysis consists of a series of repetitive computations of the NPW. To perform the sequence of repeated simulation trials, we generate a sample observation for each random variable in the model and


Figure 12.1I A logical sequence of a Monte Carlo simulation to obtain the NPW distribution for BMC's transmission-housings project.
substitute all the values into the NPW equation. Each trial requires that we use a different random number in the sequence to sample each distribution. Thus, if three random variables affect the NPW, we need three random numbers for each trial. After each trial, the computed NPW is stored in the computer. Each value of the NPW computed in this manner represents one state of nature. The trials are continued until a sufficient number of NPW values is available to define the NPW distribution.

## SOLUTION

Suppose the following three uniform random numbers are generated for the first iteration: $U_{1}=0.12135$ for $X, U_{2}=0.82592$ for $Y$, and $U_{3}=0.86886$ for $S$.

- Demand ( $\boldsymbol{X}$ ). The cumulative distribution for $X$ is already given in Example 12.4. To generate one sample (observation) from this discrete distribution, we first find the cumulative probability function, as depicted in Figure 12.12(a).


Figure 12.12 Illustration of a sampling scheme for discrete and continuous random variables.

In a given trial, suppose the computer gives the random number 0.12135 . We then enter the vertical axis at the 12.135 th percentile (the percentile numerically equivalent to the random number), read across to the cumulative curve, and then read down to the $x$-axis to find the corresponding value of the random variable $X$; this value is 1,600 , and it is the value of $x$ that we use in the NPW equation. In the next trial, we sample another value of $x$ by obtaining another random number, entering the ordinate at the numerically equivalent percentile and reading the corresponding value of $x$ from the $x$-axis.

- Price ( $\boldsymbol{Y}$ ). Assuming that the unit-price random variable can be estimated by the three parameters, its probability distribution is shown in Figure 12.12(b). Note that $Y$ takes a continuous value (unlike the assumption of discreteness in Table 12.5). The sampling procedure is again similar to that in the discrete situation. Using the random number $U=0.82592$, we can approximate $y=\$ 51.38$ by performing a linear interpolation.
- Salvage ( $\boldsymbol{S}$ ). With the salvage value $(S)$ distributed uniformly between $\$ 30,000$ and $\$ 50,000$, and a random number $U=0.86886$, the sample value is $s=$ $\$ 30,000+(50,000-30,000) 0.86886$, or $s=\$ 47,377$. The sampling scheme is shown in Figure 12.12(c).

Now we can compute the NPW equation with these sample values, yielding

$$
\begin{aligned}
\operatorname{PW}(15 \%)= & -\$ 112,555+2.0113(1,600)(\$ 51.3841-\$ 15) \\
& +0.2983(\$ 47,377) \\
= & \$ 18,665 .
\end{aligned}
$$

This result completes the first iteration of $\mathrm{NPW}_{1}$ computation.
For the second iteration, we need to generate another set of three uniform random numbers (suppose they are $0.72976,0.79885$, and 0.41879 ), to generate the respective sample from each distribution, and to compute $\mathrm{NPW}_{2}=\$ 44,752$. If we repeat this process for 200 iterations, we obtain the NPW values listed in Table 12.12.

By ordering the observed data by increasing NPW value and tabulating the ordered NPW values, we obtain the frequency distribution shown in Table 12.13. Such a tabulation results from dividing the entire range of computed NPWs into a series of subranges ( 20 in this case) and then counting the number of computed values that fall in each of the 20 intervals. Note that the sum of all the observed frequencies (column 3) is the total number of trials.

Column 4 simply expresses the frequencies of column 3 as a fraction of the total number of trials. At this point, all we have done is arrange the 200 numerical values of NPW into a table of relative frequencies.

TABLE I 2. 12 Observed NPW Values (\$) for BMC's Simulation Project



## I2.5.4 Simulation Output Analysis

After a sufficient number of repetitive simulation trials has been run, the analysis is essentially completed. The only remaining tasks are to tabulate the computed NPW values in order to determine the expected value and to make various graphic displays that will be useful to management.

## Interpretation of Simulation Results

Once we obtain an NPW frequency distribution (such as that shown in Table 12.13), we need to make the assumption that the actual relative frequencies of column 4 in the table are representative of the probability of having an NPW in each range. That is, we assume that the relative frequencies we observed in the sampling are representative of the proportions we would have obtained had we examined all the possible combinations.

This sampling is analogous to polling the opinions of voters about a candidate for public office. We could speak to every registered voter if we had the time and resources, but a simpler procedure would be to interview a smaller group of persons selected with an unbiased sampling procedure. If $60 \%$ of this scientifically selected sample supports the candidate, it probably would be safe to assume that $60 \%$ of all
registered voters support the candidate. Conceptually, we do the same thing with simulation. As long as we ensure that a sufficient number of representative trials has been made, we can rely on the simulation results.

Once we have obtained the probability distribution of the NPW, we face the crucial question, How do we use this distribution in decision making? Recall that the probability distribution provides information regarding the probability that a random variable will attain some value $x$. We can use this information, in turn, to define the cumulative distribution, which expresses the probability that the random variable will attain a value smaller than or equal to some $x$ [i.e., $F(x)=P(X \leq x)$ ]. Thus, if the NPW distribution is known, we can also compute the probability that the NPW of a project will be negative. We use this probabilistic information in judging the profitability of the project.

With the assurance that 200 trials was a sufficient number for the BMC project, we may interpret the relative frequencies in column 4 of Table 12.13 as probabilities. The NPW values range between $\$ 5,944$ and $\$ 81,532$, thereby indicating no loss for any situation. The NPW distribution has an expected value of $\$ 44,245$ and a standard deviation of $\$ 18,585$.

## Creation of Graphic Displays

Using the output data (the relative frequency) in Table 12.13, we can create the distribution in Figure 12.13(a). A picture such as this can give the decision maker a feel for the


Figure I2.13 Simulation result for BMC's transmission-housings project based on 200 iterations.
ranges of possible NPWs, for the relative likelihoods of loss versus gain, for the range of NPWs that are most probable, and so on.

Another useful display is the conversion of the NPW distribution to the equivalent cumulative frequency, as shown in Figure 12.13(b). Usually, a decision maker is concerned with the likelihood of attaining at least a given level of NPW. Therefore, we construct the cumulative distribution by accumulating the areas under the distribution as the NPW decreases. The decision maker can use Figure 12.13(b) to answer many questions-for example, what is the likelihood of making at least a $15 \%$ return on investment (i.e., the likelihood that the NPW will be at least 0)? In our example, this probability is virtually $100 \%$.

## Dependent Random Variables

All our simulation examples have involved independent random variables. We must recognize that some of the random variables affecting the NPW may be related to one another. If they are, we need to sample from distributions of the random variables in a manner that accounts for any such dependencies. For example, in the BMC project, both the demand and the unit price are not known with certainty. Each of these parameters would be on our list of variables for which we need to describe distributions, but they could be related inversely. When we describe distributions for these two parameters, we have to account for that dependency. This issue can be critical, as the results obtained from a simulation analysis can be misleading if the analysis does not account for dependencies. The sampling techniques for these dependent random variables are beyond the scope of this text, but can be found in many textbooks on simulation.

### 12.5.5 Risk Simulation with @RISK ${ }^{8}$

One practical way to conduct a risk simulation is to use an Excel-based program such as $@$ RISK. All we have to do is to develop a worksheet for a project's cash flow statement. Then we identify the random variables in the cash flow elements. With @RISK, we have a variety of probability distributions to choose from to describe our beliefs about the random variables of interest. We illustrate how to conduct a risk simulation on @RISK, using the same financial data described in Example 12.9. @RISK uses Monte Carlo simulation to show you all possible outcomes. Running an analysis with @RISK involves three simple steps.

## 1. Create a Cash Flow Statement with Excel

@RISK is an add-in to Microsoft Excel. As an add-in, @RISK becomes seamlessly integrated with your spreadsheet, adding risk analysis to your existing models. Table 12.14 is an Excel spreadsheet in which the cash flow entries are a function of the input variables. Here what we are looking for is the NPW of the project when we assigned specific values to the random variables. As in Example 12.9 , we treat the unit price $(Y)$, the demand $(X)$, and the salvage value $(S)$ as random variables.

[^5]TABLE I 2.| 4 An Excel Worksheet of the Transmission-Housing Project Prepared for @RISK


## 2. Define Uncertainty

We start by replacing uncertain values in our spreadsheet model with @RISK probability distribution functions. These @RISK functions simply represent a range of different possible values that cell could take instead of limiting it to just one value.

| Random Variable | Probability Distribution | Probability Functions Provided by @RISK | Cell |
| :---: | :---: | :---: | :---: |
| Unit price | Triangular (48, 50, 53) | $=$ RiskTriang (48,50,53) | C6 |
| Demand | Event Probability | $=\operatorname{RiskDiscrete}(\{1600,2000,2400\},\{0.2,0.6,0.2\})$ | C8 |
|  | $\begin{array}{ll}1,600 & 0.2 \\ 2,000 & 0.6 \\ 2,400 & 0.2\end{array}$ |  |  |
| Salvage | Uniform $(30,000,50,000)$ | =RiskUniform(30000,50000) | C10 |

As shown in Figure 12.14, the probability functions are found from the "Define Distribution" menu in the tool bar.

## 3. Pick Your Bottom Line

We select our output cells, the "bottom line" cells whose values we are interested in. In our case, this is the net present value for the transmission-housing project, located at cell G13. To designate cell G13 as the output cell, we move the cursor to cell G13 and select the "Add Output" command in the tool bar. Then the cell formula in G13 will look like RiskOutput("Output (NPW)") + NPV(C12, D43:H43) + C43.


## 4. Simulate

Now we are ready to simulate. Click the "Start Simulation" button and watch. @RISK recalculates our spreadsheet model hundreds or thousands of times. We can specify the number of iterations through the "Simulation Settings" command. There is no limit to the number of different scenarios we can look at in our simulations. Each time, @RISK samples random values from the @RISK functions we entered and records the resulting outcome. The result: a look at a whole range of possible outcomes, including the probabilities that they will occur! Almost instantly, you will see what critical situations to seek out—or avoid! With @RISK, we can answer questions like, "What is the probability of NPW exceeding $\$ 50,000$ ?" or "What are the chances of losing money on this investment?"


Figure 12.14 @RISK comes with RISKview, a built-in distribution viewer that lets you preview various distributions before selecting them. You can choose distributions from a gallery of thumbnail distribution pictures, and then watch as @RISK builds a graph of the distribution for you while you enter your parameters.

## 5. Analyzing the Simulation Result Screen

@RISK provides a wide range of graphing options for interpreting and presenting the simulation results. @RISK also gives us a full statistical report on our simulations, as well as access to all the data generated. Quick Reports include cumulative graphs, Tornado charts for sensitivity analysis, histograms, and summary statistics. As shown in Figure 12.15, based on 200 iterations, we find that the NPW ranges between $\$ 6,813$ and $\$ 80,380$, with the mean value of $\$ 41,468$ and the standard deviation of $\$ 18,361$. These results are comparable to those obtained in Example 12.9.

### 2.6 Decision Trees and Sequential Investment Decisions

Most investment problems that we have discussed so far involved only a single decision at the time of investment (i.e., an investment is accepted or rejected), or a single decision that entails a different decision option, such as "make a product in-house" or "farm out," and so on. Once the decision is made, there are no later contingencies or decision options to follow up on. However, certain classes of investment decisions cannot be analyzed in a single step. As an oil driller, for example, you have an opportunity to bid on an offshore lease. Your first decision is whether to bid. But if you decide to bid and eventually win the


Figure I2.15 The NPW distribution created by @RISK based on 200 iterations. @RISK lets you use the output data to accurately describe what the NPW distribution will look like using the BestFit command, @RISK's built-in distribution fitting tool.
contract, you will have another decision regarding whether to drill immediately or run more seismic (drilling) tests. If you drill a test well that turns out to be dry, you have another decision whether to drill another well or drop the entire drilling project. In a sequential decision problem such as this, in which the actions taken at one stage depend on actions taken in earlier stages, the evaluation of investment alternatives can be quite complex. Certainly, all these future options must be considered when one is evaluating the feasibility of bidding at the initial stage. In this section, we describe a general approach for dealing with more complex decisions that is useful both for structuring the decision problems and for finding a solution to them. The approach utilizes a decision tree-a graphic tool for describing the actions available to the decision maker, the events that can occur, and the relationship between the actions and events.

### 12.6.I Structuring a Decision-Tree Diagram

To illustrate the basic decision-tree concept, let's consider an investor, named Bill Henderson, who wants to invest some money in the financial market. He wants to choose between a highly speculative growth stock $\left(d_{1}\right)$ and a very safe U.S. Treasury bond $\left(d_{2}\right)$.

Figure 12.16 illustrates this situation, with the decision point represented by a square box $(\square)$ or decision node. The alternatives are represented as branches emanating from the node. Suppose Bill were to select some particular alternative, say, "invest in the stock." There are three chance events that can happen, each representing a potential return on investment. These events are shown in Figure 12.16 as branches emanating from a circular node $(O)$. Note that the branches represent uncertain events over which Bill has no control. However, Bill can assign a probability to each chance event and enter it beside each branch in the decision tree. At the end of each branch is the conditional monetary transaction associated with the selected action and given event.

## Relevant Net After-Tax Cash Flow

Once the structure of the decision tree is determined, the next step is to find the relevant cash flow (monetary value) associated with each of the alternatives and the possible chance outcomes. As we have emphasized throughout this book, the decision has to be made on an after-tax basis. Therefore, the relevant monetary value should be on that basis. Since the costs and revenues occur at different points in time over the study period, we also need to convert the various amounts on the tree's branches to their equivalent lump-sum amounts,


Figure I2.16 Structure of a typical decision tree (Bill's investment problem).
say, net present value. The conditional net present value thus represents the profit associated with the decisions and events along the path from the first part of the tree to the end.

## Rollback Procedure

To analyze a decision tree, we begin at the end of the tree and work backwards-the rollback procedure. In other words, starting at the tips of the decision tree's branches and working toward the initial node, we use the following two rules:

1. For each chance node, we calculate the expected monetary value (EMV). This is done by multiplying probabilities by conditional profits associated with branches emanating from that node and summing these conditional profits. We then place the EMV in the node to indicate that it is the expected value calculated over all branches emanating from that node.
2. For each decision node, we select the one with the highest EMV (or minimum cost). Then those decision alternatives which are not selected are eliminated from further consideration. On the decision-tree diagram, we draw a mark across the nonoptimal decision branches, indicating that they are not to be followed.
Example 12.11 illustrates how Bill could transform his investment problem into a tree format using numerical values.

## EXAMPLE 12.|| Bill's Investment Problem in a Decision-Tree Format

Suppose Bill has $\$ 50,000$ to invest in the financial market for one year. His choices have been narrowed to two options:

- Option 1. Buy 1,000 shares of a technology stock at $\$ 50$ per share that will be held for one year. Since this is a new initial public offering (IPO), there is not much research information available on the stock; hence, there will be a brokerage fee of $\$ 100$ for this size of transaction (for either buying or selling stocks). For simplicity, assume that the stock is expected to provide a return at any one of three different levels: a high level (A) with a $50 \%$ return ( $\$ 25,000$ ), a medium level (B) with a $9 \%$ return $(\$ 4,500)$, or a low level (C) with a $30 \%$ loss $(-\$ 15,000)$. Assume also that the probabilities of these occurrences are assessed at $0.25,0.40$, and 0.35 , respectively. No stock dividend is anticipated for such a growth-oriented company.
- Option 2. Purchase a $\$ 50,000$ U.S. Treasury bond, which pays interest at an effective annual rate of $7.5 \%(\$ 3,750)$. The interest earned from the Treasury bond is nontaxable income. However, there is a $\$ 150$ transaction fee for either buying or selling the bond.

Bill's dilemma is which alternative to choose to maximize his financial gain. At this point, Bill is not concerned about seeking some professional advice on the stock before making a decision. We will assume that any long-term capital gains will be taxed at $20 \%$. Bill's minimum attractive rate of return is known to be $5 \%$ after taxes. Determine the payoff amount at the tip of each branch.

## SOLUTION

Figure 12.17(a) shows the costs and revenues on the branches, transformed to their present equivalents.


Figure 12.17 Decision tree for Bill's investment problem: (a) Relevant cash flows (before tax) and (b) net present worth for each decision path.

## - Option 1

1. With a $50 \%$ return (a real winner) over a one-year holding period,

- The net cash flow associated with this event at Period 0 will include the amount of investment and the brokerage fee. Thus, we have

$$
\text { Period 0: }(-\$ 50,000-\$ 100)=-\$ 50,100
$$

- When you sell stock, you need to pay another brokerage fee. However, any investment expenses, such as brokerage commissions, must be included in determining the cost basis for investment. The taxable capital gains will be calculated as $(\$ 75,000-\$ 50,000-\$ 100-\$ 100)=\$ 24,800$. Therefore, the net cash flow at Period 1 would be

Period 1: $(+\$ 75,000-\$ 100)-0.20(\$ 24,800)=\$ 69,940$.

- Then the conditional net present value of this stock transaction is

$$
\operatorname{PW}(5 \%)=-\$ 50,100+\$ 69,940(P / F, 5 \%, 1)=\$ 16,510 .
$$

This amount of $\$ 16,510$ is entered at the tip of the corresponding branch. This procedure is repeated for each possible branch, and the resulting amounts are shown in Figure 12.17(b).
2. With a $9 \%$ return, we have

- Period 0: - \$50,100
- Period 1: \$53,540
- $\operatorname{PW}(5 \%)=-50,100+53,540(P / F, 5 \%, 1)=\$ 890$

3. With a $30 \%$ loss, we have

- Period 0: - \$50,100
- Period 1: \$37,940
- $\operatorname{PW}(5 \%)=-\$ 50,100+\$ 37,940(P / F, 5 \%, 1)=-\$ 13,967$
- Option 2. Since the interest income on the U.S. government bond will not be taxed, there will be no capital-gains tax. Considering only the brokerage commission, we find that the relevant cash flows would be as follows:
- Period 0: $(-\$ 50,000-\$ 150)=\$ 50,150$
- Period 1: $(+\$ 53,750-\$ 150)=\$ 53,600$
- $\operatorname{PW}(5 \%)=-\$ 50,150+\$ 53,600(P / F, 5 \%, 1)=\$ 898$

Figure 12.17(b) shows the complete decision tree for Bill's investment problem. Now Bill can calculate the expected monetary value (EMV) at each chance node. The EMV of Option 1 represents the sum of the products of the probabilities of high, medium, and low returns and the respective conditional profits (or losses):

$$
\mathrm{EMV}=\$ 16,510(0.25)+\$ 890(0.40)-\$ 13,967(0.35)=-\$ 405
$$

For Option 2, the EMV is simply

$$
\mathrm{EMV}=\$ 898
$$

In Figure 12.17, the expected monetary values are shown in the event nodes. Bill must choose which action to take, and this would be the one with the highest EMV, namely, Option 2, with EMV $=\$ 898$. This expected value is indicated in the tree by putting $\$ 898$ in the decision node at the beginning of the tree. Note that the decision tree uses the idea of maximizing expected monetary value developed in the previous section. In addition, the mark \|is drawn across the nonoptimal decision branch (Option 1), indicating that it is not to be followed. In this simple example, the benefit of using a decision tree may not be evident. However, as the decision problem becomes more complex, the decision tree becomes more useful in organizing the information flow needed to make the decision. This is true in particular if Bill must make a sequence of decisions, rather than a single decision, as we next illustrate.

### 12.6.2 Worth of Obtaining Additional Information

In this section, we introduce a general method for evaluating whether it is worthwhile to seek more information. Most of the information that we can obtain is imperfect, in the sense that it will not tell us exactly which event will occur. Such imperfect information may have some value if it improves the chances of making a correct decision-that is, if it improves the expected monetary value. The problem is whether the reduced uncertainty is valuable enough to offset its cost. The gain is in the improved efficiency of decisions that may become possible with better information.

We use the term experiment in a broad sense in what follows. An experiment may be a market survey to predict sales volume for a typical consumer product, a statistical sampling of the quality of a product, or a seismic test to give a well-drilling firm some indications of the presence of oil.

## The Value of Perfect Information

Let us take Bill's prior decision to "Purchase U.S. Treasury bonds" as a starting point. How do we determine whether further strategic steps would be profitable? We could do more to obtain additional information about the potential return on an investment in stock, but such steps cost money. Thus, we have to balance the monetary value of reducing uncertainty with the cost of obtaining additional information. In general, we can evaluate the worth of a given experiment only if we can estimate the reliability of the resulting information. In our stock investment problem, an expert's opinion may be helpful in deciding whether to abandon the idea of investing in a stock. This can be of value, however, only if Bill can say beforehand how closely the expert can estimate the performance of the stock.

The best place to start a decision improvement process is with a determination of how much we might improve our incremental profit by removing uncertainty. Although we probably couldn't ever really obtain such uncertainty, the value of perfect information is worth computing as an upper bound to the value of additional information.

We can easily calculate the value of perfect information. First we merely note the mathematical difference of the incremental profit from an optimal decision based on perfect information and the original decision to "Purchase Treasury bonds" made without foreknowledge of the actual return on the stock. This difference is called opportunity loss and must be computed for each potential level of return on the stock. Then we compute the expected opportunity loss by weighting each potential loss by the assigned probability associated with that event. What turns out in this expected opportunity loss is exactly the expected value of perfect information (EVPI). In other words, if we had perfect information about the performance of the stock, we should have made a correct decision about each situation, resulting in no regrets (i.e., no opportunity losses). Example 12.12 illustrates the concept of opportunity loss and how it is related to the value of perfect information.

## EXAMPLE 12.12 Expected Value of Perfect Information

Reluctant to give up a chance to make a larger profit with the stock option, Bill may wonder whether to obtain further information before acting. As with the decision to "Purchase U.S. Treasury bonds," Bill needs a single figure to represent the expected value of perfect information (EVPI) about investing in the stock. Calculate the EVPI on the basis of the financial data in Example 12.11 and Figure 12.17(b).

TABLE I2.|5 Opportunity Loss Cost Associated with Investing in Bonds


## SOLUTION

As regards stock, the decision whether to purchase may hinge on the return on the stock during the next year-the only unknown, subject to a probability distribution. The opportunity loss associated with the decision to purchase bonds is shown in Table 12.15.

- For example, the conditional net present value of $\$ 16,510$ is the net profit if the return is high. Also, recall that, without Bill's having received any new information, the indicated action (the prior optimal decision) was to select Option 2 ("Purchase Treasury bond"), with an expected net present value of $\$ 898$.
- However, if we know that the stock will be a definite winner (yield a high return), then the prior decision to "Purchase Treasury bond" is inferior to "Invest in stock" to the extent of $\$ 16,510-\$ 898=\$ 15,612$. With either a medium or low return, however, there is no opportunity loss, as a decision strategy with and without perfect information is the same.

Again, an average weighted by the assigned chances is used, but this time weights are applied to the set of opportunity losses (regrets) in Table 12.15. The result is

$$
\mathrm{EVPI}=(0.25)(\$ 15,612)+(0.40)(\$ 0)+(0.35)(\$ 0)=\$ 3,903
$$

This EVPI, representing the maximum expected amount that could be gained in incremental profit from having perfect information, places an upper limit on the sum Bill would be willing to pay for additional information.

## Updating Conditional Profit (or Loss) after Paying a Fee to the Expert

With a relatively large EVPI in Example 12.12, there is a huge potential that Bill can benefit by obtaining some additional information on the stock. Bill can seek an expert's advice by receiving a detailed research report on the stock under consideration. This report, available at a nominal fee, say, $\$ 200$, will suggest either of two possible outcomes: (1) The report may come up with a "buy" rating on the stock after concluding that business conditions were favorable, or (2) the report may not recommend any


Figure I2.18 Decision tree for Bill's investment problem with an option of getting professional advice.
action or may take a neutral stance after concluding that business conditions were unfavorable or that the business model on which the decision is based was not well received in the financial marketplace. Bill's alternatives are either to pay for this service before making any further decision or simply to make his own decision without seeing the report.

If Bill seeks advice before acting, he can base his decision upon the stock report. This problem can be expressed in terms of a decision tree, as shown in Figure 12.18. The upper part of the tree shows the decision process if no advice is sought. This is the same as Figure 12.17, with probabilities of $0.25,0.40$, and 0.35 for high, medium, and low returns, respectively, and an expected loss of $\$ 405$ for investing in the stock and an expected profit of $\$ 898$ from purchasing Treasury bonds.

The lower part of the tree, following the branch "Seek advice," displays the results and the subsequent decision possibilities. Using the analyst's research report, Bill can expect two possible events: the expert's recommendation and the actual performance of the stock. After reading either of the two possible report outcomes, Bill must make a decision about whether to invest in the stock. Since Bill has to pay a nominal fee $(\$ 200)$ to get the report, the profit or loss figures at the tips of the branches must also be updated. For the stock investment option, if the actual return proves to be very high (50\%), the resulting cash flows would be as follows:

$$
\begin{aligned}
\text { Period } 0: & (-\$ 50,000-\$ 100-\$ 200)=-\$ 50,300 \\
\text { Period 1: } & (+\$ 75,000-\$ 100)-(0.20)(\$ 25,000-\$ 400)=\$ 69,980 \\
& \operatorname{PW}(5 \%)=-\$ 50,300+\$ 69,980(P / F, 5 \%, 1)=\$ 16,348
\end{aligned}
$$

Since the cost of obtaining the report ( $\$ 200$ ) will be a part of the total investment expenses, along with the brokerage commissions, the taxable capital gains would be adjusted accordingly. With a medium return, the equivalent net present value would be $\$ 728$. With a low return, there would be a net loss in the amount of $\$ 14,129$.

Similarly, Bill can compute the net profit for Option 2 as follows:

$$
\begin{aligned}
\text { Period } 0: & (-\$ 50,000-\$ 150-\$ 200)=-\$ 50,350 \\
\text { Period 1: } & (+\$ 53,750-\$ 150)=\$ 53,600 \\
& P W(5 \%)=-\$ 50,350+\$ 53,600(P / F, 5 \%, 1)=\$ 698 .
\end{aligned}
$$

Once again, note that gains earned from the U.S. government bond will not be taxed.

## I 2.6.3 Decision Making after Having Imperfect Information

In Figure 12.16, the analyst's recommendation precedes the actual investment decision. This allows Bill to change his prior probabilistic assessments regarding the performance of the stock. In other words, if the analyst's report has any bearing on Bill's reassessments of the stock's performance, then Bill must change his prior probabilistic assessments accordingly. But what types of information should be available and how should the prior probabilistic assessments be updated?

## Conditional Probabilities of the Expert's Predictions, Given a Potential Return on the Stock

Suppose that Bill knows an expert (stock analyst) to whom he can provide his research report on the stock described under Option 1. The expert will charge a fee in the amount of $\$ 200$ to provide Bill with a report on the stock that is either favorable or unfavorable. This research report is not infallible, but it is known to be pretty reliable. From past experience, Bill estimates that when the stock under consideration is relatively highly profitable (A), there is an $80 \%$ chance that the report will be favorable; when the stock return is medium (B), there is a $65 \%$ chance that the report will be favorable; and when the stock return is relatively unprofitable (C), there is a $20 \%$ chance that the report will be favorable. Bill's estimates are summarized in Table 12.16.

These conditional probabilities can be expressed in a tree format as shown in Figure 12.19. For example, if the stock indeed turned out to be a real winner, the probability

TABLE I2.| 6 Conditional Probabilities of the Expert's Predictions

|  | Given Level of Stock Performance |  |  |
| :--- | :---: | :---: | :---: |
| What the Report Will Say | High <br> (A) | Medium <br> $(B)$ | Low <br> $(\mathbf{C})$ |
| Favorable (F) | 0.80 | 0.65 | 0.20 |
| Unfavorable (UF) | 0.20 | 0.35 | 0.80 |



Figure I2.19 Conditional probabilities and joint probabilities expressed in a tree format.
that the report would have predicted the same (with a favorable recommendation) is 0.80 , and the probability that the report would have issued an "unfavorable recommendation" is 0.20 . Such probabilities would reflect past experience with the analyst's service of this type, modified perhaps by Bill's own judgment. In fact, these probabilities represent the reliability or accuracy of the expert opinion. It is a question of counting the number of times the expert's forecast was "on target" in the past and the number of times the expert "missed the mark," for each of the three actual levels of return on the stock. With this past history of the expert's forecasts, Bill can evaluate the economic worth of the service. Without such a history, no specific value can be attached to seeking the advice.

## Joint and Marginal Probabilities

Suppose now that, before seeing the research report, Bill has some critical information on (1) all the probabilities for the three levels of stock performance (prior probabilities) and (2) with what probability the report will say that the return on the stock is favorable or unfavorable, given the three levels of the stock performance in the future. A glance at the sequence of events in Figure 12.18, however, shows that these probabilities are not the ones required to find "expected" values of various strategies in the

Marginal probability is the probability of one event, regardless of the other event. decision path. What is really needed are the probabilities pertaining to what the report will say about the stock-favorable (a recommendation to buy) or unfavorable (no recommendation or a neutral stance) -and the conditional probabilities of each of the three potential returns after the report is seen. In other words, the probabilities of Table 12.16 are not directly useful in Figure 12.18. Rather, we need the marginal probabilities of the "favorable" and "unfavorable" recommendations.

To remedy this situation, the probabilities must be put in a different form: We must construct a joint probabilities table. To start with, we have the original (prior) probabilities assessed by Bill before seeing the report: a 0.25 chance that the stock will result in a high return (A), a 0.40 chance of a medium return (B), and a 0.35 chance of a loss (C). From these and the conditional probabilities of Table 12.16, the joint probabilities of Figure 12.19 can be calculated. Thus, the joint probability of both a prediction of a favorable stock market ( F ) and a high return on the stock ( A ) is calculated by multiplying the conditional probability of a favorable prediction, given a high return on the stock (which is 0.80 from Table 12.16 ), by the probability of a high return (A):

$$
P(\mathrm{~A}, \mathrm{~F})=P(\mathrm{~F} \mid \mathrm{A}) P(\mathrm{~A})=(0.80)(0.25)=0.20
$$

Similarly,

$$
\begin{gathered}
P(\mathrm{~A}, \mathrm{UF})=P(\mathrm{UF} \mid \mathrm{A}) P(\mathrm{~A})=(0.20)(0.25)=0.05 \\
P(\mathrm{~B}, \mathrm{~F})=P(\mathrm{~F} \mid \mathrm{B}) P(\mathrm{~B})=(0.65)(0.40)=0.26 \\
P(\mathrm{~B}, \mathrm{UF})=P(\mathrm{UF} \mid \mathrm{B}) P(\mathrm{~B})=(0.35)(0.40)=0.14
\end{gathered}
$$

and so on.
The marginal probabilities of return in Table 12.17 are obtained by summing the values across the columns. Note that these are precisely the original probabilities of a high, medium, and low return, and they are designated prior probabilities because they were assessed before any information from the report was obtained.

In understanding Table 12.17, it is useful to think of it as representing the results of 100 past situations identical to the one under consideration. The probabilities then represent the frequency with which the various outcomes occurred. For example, in 25 of the 100 cases, the actual return on the stock turned out to be high; and in these 25 high-return cases, the report predicted a favorable condition in 20 instances [that is $P(\mathrm{~A}, \mathrm{~F})=0.20$ ] and predicted an unfavorable one in 5 instances.

## TABLE I2.| 7 Joint as Well as Marginal Probabilities

| When <br> Potential <br> Level of Return Is Given | What Report Will Say |  | Marginal Probabilities of Return |
| :---: | :---: | :---: | :---: |
|  | Joint Probabilities |  |  |
|  | Favorable (F) | Unfavorable (UF) |  |
| High (A) | 0.20 | 0.05 | 0.25 |
| Medium (B) | 0.26 | 0.14 | 0.40 |
| Low (C) | 0.07 | 0.28 | 0.35 |
| Marginal probabilities | 0.53 | 0.47 | 1.00 |

Similarly, the marginal probabilities of prediction in the bottom row of Table 12.17 can be interpreted as the relative frequency with which the report predicted favorable and unfavorable conditions. For example, the survey predicted favorable conditions 53 out of 100 times, 20 of these times when stock returns actually were high, 26 times when returns were medium, and 7 times when returns were low. These marginal probabilities of what the report will say are important to our analysis, for they give us the probabilities associated with the information received by Bill before the decision to invest in the stock is made. The probabilities are entered beside the appropriate branches in Figure 12.18.

## Determining Revised Probabilities

We still need to calculate the probabilities for the branches that represent "high return," "medium return," and "low return" in the lower part of Figure 12.18. We cannot use the values $0.25,0.40$, and 0.35 for these events, as we did in the upper part of the tree, because those probabilities were calculated prior to seeking the advice. At this point on the decision tree, Bill will have received information from the analyst, and the probabilities should reflect that information. (If Bill's judgment has not changed even after seeing the report, then there will be no changes in the probabilities.) The required probabilities are the conditional probabilities for the various levels of return on the stock, given the expert's opinion. Thus, using the data from Table 12.17, we can compute the probability $P(\mathrm{AIF})$ of seeing a high return (A), given the expert's buy recommendation $(\mathrm{F})$, directly from the definition of conditional probability:

$$
P(\mathrm{AlF})=P(\mathrm{~A}, \mathrm{~F}) / P(\mathrm{~F})=0.20 / 0.53=0.38
$$

The probabilities of a medium and a low return, given a buy recommendation, are, respectively,

$$
P(\mathrm{BIF})=P(\mathrm{~B}, \mathrm{~F}) / P(\mathrm{~F})=0.26 / 0.53=0.49
$$

and

$$
P(\mathrm{C} \mid \mathrm{F})=P(\mathrm{C}, \mathrm{~F}) / P(\mathrm{~F})=0.07 / 0.53=0.13 .
$$

These probabilities are called revised (or posterior) probabilities, since they come after the inclusion of the information received from the report. To understand the meaning of the preceding calculations, think again of Table 12.17 as representing 100 past identical situations. Then, in 53 cases [since $P(\mathrm{~F})=0.53$ ], 20 actually resulted in a high return. Hence, the posterior probability of a high return is $20 / 53=0.38$, as calculated.

The posterior probabilities of other outcomes after seeing an unfavorable recommendation can be calculated similarly:

$$
\begin{gathered}
P(\mathrm{AlUF})=P(\mathrm{~A}, \mathrm{UF}) / P(\mathrm{UF})=0.05 / 0.47=0.11 \\
P(\mathrm{BIUF})=P(\mathrm{~B}, \mathrm{UF}) / P(\mathrm{UF})=0.14 / 0.47=0.30 \\
P(\mathrm{ClUF})=P(\mathrm{C}, \mathrm{UF}) / P(\mathrm{UF})=0.28 / 0.47=0.59
\end{gathered}
$$

These values are also shown in Figure 12.20 at the appropriate points in the decision tree.


Figure I 2.20 Decision tree for Bill's investment problem with an option of having professional advice.

## EXAMPLE 12.13 Decision Making after Getting Some Professional Advice

On the basis of Figure 12.16, determine how much the research report is worth to Bill. Should Bill seek an expert's advice (research report)?

## SOLUTION

All the necessary information is now available, and Figure 12.20 can be analyzed, starting from the right and working backward. The expected values are shown in the circles. For example, follow the branches "Seek Advice $\left(e_{1}\right)$," "Favorable (Buy recommendation)," and "Invest in the Stock $\left(d_{1}\right)$." The expected value of $\$ 4,732$ shown in the circle at the end of these branches is calculated as

$$
0.38(\$ 16,348)+0.49(\$ 728)+0.13(-\$ 14,129)=\$ 4,732
$$

Since Bill can expect a profit of $\$ 4,732$ from investing in the stock after receiving a buy recommendation, the investment in Treasury bonds (\$698) becomes a less profitable
alternative. Thus, it can be marked out $(\neq)$ of the "Purchase Treasury bonds" branch to indicate that it is not optimal.

There is an expected loss of $\$ 6,531$ if the report gives an unfavorable recommendation, but Bill goes ahead with buying the stock. In this situation, the "Treasury bonds" option becomes a better choice, and the "Buy stock" branch is marked out. The part of the decision tree relating to seeking the advice is now reduced to two chance events (earning \$4,732 and earning \$698), and the expected value in the circular node is calculated as

$$
0.53(\$ 4,732)+0.47(\$ 698)=\$ 2,836
$$

Thus, if the expert's opinion is taken and Bill acts on the basis of the information received, the expected profit is $\$ 2,836$. Since this amount is greater than the $\$ 898$ profit that would be obtained in the absence of the expert's advice, we conclude that it is worth spending $\$ 200$ to receive some additional information from the expert.

In sum, the optimal decision is as follows: (1) Bill should seek professional advice at the outset. (2) If the expert's report indicates a buy recommendation, go ahead and invest in the stock. (3) If the report says otherwise, invest in the Treasury bonds.

## SUMMARY

Often, cash flow amounts and other aspects of an investment project analysis are uncertain. Whenever such uncertainty exists, we are faced with the difficulty of project risk: the possibility that an investment project will not meet our minimum requirements for acceptability and success.
Three of the most basic tools for assessing project risk are as follows:

1. Sensitivity analysis-a means of identifying those project variables which, when varied, have the greatest effect on the acceptability of the project.
2. Break-even analysis-a means of identifying the value of a particular project variable that causes the project to exactly break even.
3. Scenario analysis-a means of comparing a "base-case" or expected project measurement (such as the NPW) with one or more additional scenarios, such as the best case and the worst case, to identify the extreme and most likely project outcomes.
Sensitivity, break-even, and scenario analyses are reasonably simple to apply, but also somewhat simplistic and imprecise in cases where we must deal with multifaceted project uncertainty. Probability concepts allow us to further refine the analysis of project risk by assigning numerical values to the likelihood that project variables will have certain values.
The end goal of a probabilistic analysis of project variables is to produce an NPW distribution from which we can extract such useful information as (1) the expected NPW value, (2) the extent to which other NPW values vary from, or are clustered around, the expected value (the variance), and (3) the best- and worst-case NPWs.
Our real task is not try to find "risk-free" projects; they don't exist in real life. The challenge is to decide what level of risk we are willing to assume and then, having decided on our tolerance for risk, to understand the implications of our choice.
Risk simulation, in general, is the process of modeling reality to observe and weigh the likelihood of the possible outcomes of a risky undertaking.

Monte Carlo sampling is a specific type of randomized sampling method in which a random sample of outcomes is generated for a specified probability distribution. Because Monte Carlo sampling and other simulation techniques often rely on generating a significant number of outcomes, they can be more conveniently performed on the computer than manually.
The decision tree is another technique that can facilitate investment decision making when uncertainty prevails, especially when the problem involves a sequence of decisions. Decision-tree analysis involves the choice of a decision criterion-say, to maximize expected profit. If possible and feasible, an experiment is conducted, and the prior probabilities of the states of nature are revised on the basis of the experimental results. The expected profit associated with each possible decision is computed, and the act with the highest expected profit is chosen as the optimum action.

## PROBLEMS

## Sensitivity Analysis

12.1 Ford Construction Company is considering acquiring a new earthmover. The mover's basic price is $\$ 90,000$, and it will cost another $\$ 18,000$ to modify it for special use by the company. This earthmover falls into the MACRS five-year class. It will be sold after four years for $\$ 30,000$. The purchase of the earthmover will have no effect on revenues, but it is expected to save the firm $\$ 35,000$ per year in before-tax operating costs, mainly labor. The firm's marginal tax rate (federal plus state) is $40 \%$, and its MARR is $10 \%$.
(a) Is this project acceptable, based on the most likely estimates given?
(b) Suppose that the project will require an increase in net working capital (spareparts inventory) of $\$ 5,000$, which will be recovered at the end of year 5. Taking this new requirement into account, would the project still be acceptable?
(c) If the firm's MARR is increased to $18 \%$, what would be the required savings in labor so that the project remains profitable?
12.2 Minnesota Metal Forming Company has just invested \$500,000 of fixed capital in a manufacturing process that is estimated to generate an after-tax annual cash flow of $\$ 200,000$ in each of the next five years. At the end of year 5, no further market for the product and no salvage value for the manufacturing process is expected. If a manufacturing problem delays the start-up of the plant for one year (leaving only four years of process life), what additional after-tax cash flow will be needed to maintain the same internal rate of return as would be experienced if no delay occurred?
12.3 A real-estate developer seeks to determine the most economical height for a new office building, which will be sold after five years. The relevant net annual revenues and salvage values are as follows:

|  | Height |  |  |  |
| :--- | ---: | ---: | ---: | ---: |
|  | 2 Floors | 3 Floors | 4 Floors | 5 Floors |
| First cost <br> (net after tax) | $\$ 500,000$ | $\$ 750,000$ | $\$ 1,250,000$ | $\$ 2,000,000$ |
| Lease revenue | 199,100 | 169,200 | 149,200 | 378,150 |
| Net resale value <br> (after tax) | 600,000 | 900,000 | $2,000,000$ | $3,000,000$ |

(a) The developer is uncertain about the interest rate $(i)$ to use, but is certain that it is in the range from $5 \%$ to $30 \%$. For each building height, find the range of values of $i$ for which that building height is the most economical.
(b) Suppose that the developer's interest rate is known to be $15 \%$. What would be the cost, in terms of net present value, of a $10 \%$ overestimation of the resale value? (In other words, the true value was $10 \%$ lower than that of the original estimate.)
12.4 A special-purpose milling machine was purchased 4 years ago for $\$ 20,000$. It was estimated at that time that this machine would have a life of 10 years and a salvage value of $\$ 1,000$, with a cost of removal of $\$ 1,500$. These estimates are still good. This machine has annual operating costs of $\$ 2,000$, and its current book value is $\$ 13,000$. If the machine is retained for its entire 10 -year life, the remaining annual depreciation schedule would be $\$ 2,000$ for years 5 through 10 . A new machine that is more efficient will reduce operating costs to $\$ 1,000$, but will require an investment of $\$ 12,000$. The life of the new machine is estimated to be 6 years, with a salvage value of $\$ 2,000$. The new machine would fall into the 5 -year MACRS property class. An offer of $\$ 6,000$ for the old machine has been made, and the purchaser would pay for removal of the machine. The firm's marginal tax rate is $40 \%$, and its required minimum rate of return is $10 \%$.
(a) What incremental cash flows will occur at the end of years 0 through 6 as a result of replacing the old machine? Should the old machine be replaced now?
(b) Suppose that the annual operating costs for the old milling machine would increase at an annual rate of $5 \%$ over the remaining service life of the machine. With this change in future operating costs for the old machine, would the answer in (a) change?
(c) What is the minimum trade-in value for the old machine so that both alternatives are economically equivalent?
12.5 A local telephone company is considering installing a new phone line for a new row of apartment complexes. Two types of cables are being examined: conventional copper wire and fiber optics. Transmission by copper wire cables, although cumbersome, involves much less complicated and less expensive support hardware than does fiber optics. The local company may use five different types of copper wire cables: 100 pairs, 200 pairs, 300 pairs, 600 pairs, and 900 pairs per cable. In calculating the cost per foot of cable, the following equation is used:

$$
\begin{aligned}
\text { Cost per length }= & {[\text { Cost per foot }+} \\
& \text { cost per pair ( number of pairs })](\text { length }),
\end{aligned}
$$

where
22-gauge copper wire $=\$ 1.692$ per foot
and

$$
\text { Cost per pair }=\$ 0.013 \text { per pair. }
$$

The annual cost of the cable as a percentage of the initial cost is $18.4 \%$. The life of the system is 30 years.

In fiber optics, a cable is referred to as a ribbon. One ribbon contains 12 fibers, grouped in fours; therefore, one ribbon contains three groups of 4 fibers. Each group can produce 672 lines (equivalent to 672 pairs of wires), and since each ribbon contains three groups, the total capacity of the ribbon is 2,016 lines. To transmit signals via fiber optics, many modulators, wave guides, and terminators are needed to convert the signals from electric currents to modulated light waves. Fiber-optic ribbon costs $\$ 15,000$ per mile. At each end of the ribbon, three terminators are needed, one for each group of 4 fibers, at a cost of $\$ 30,000$ per terminator. Twenty-one modulating systems are needed at each end of the ribbon, at a cost of $\$ 12,092$ for a unit in the central office and $\$ 21,217$ for a unit in the field. Every 22,000 feet, a repeater is required to keep the modulated light waves in the ribbon at an intensity that is intelligible for detection. The unit cost of this repeater is $\$ 15,000$. The annual cost, including income taxes for the 21 modulating systems, is $12.5 \%$ of the initial cost of the units. The annual cost of the ribbon itself is $17.8 \%$ initially. The life of the whole system is 30 years. (All figures represent after-tax costs.)
(a) Suppose that the apartments are located 5 miles from the phone company's central switching system and that about 2,000 telephones will be required. This would require either 2,000 pairs of copper wire or one fiber-optic ribbon and related hardware. If the telephone company's interest rate is $15 \%$, which option is more economical?
(b) In (a), suppose that the apartments are located 10 miles or 25 miles from the phone company's central switching system. Which option is more economically attractive under each scenario?
12.6 A small manufacturing firm is considering purchasing a new boring machine to modernize one of its production lines. Two types of boring machine are available on the market. The lives of machine A and machine B are 8 years and 10 years, respectively. The machines have the following receipts and disbursements:

| Item | Machine A | Machine B |
| :--- | :---: | :---: |
| First cost | $\$ 6,000$ | $\$ 8,500$ |
| Service life | 8 years | 10 years |
| Salvage value | $\$ 500$ | $\$ 1,000$ |
| Annual O\&M <br> costs | $\$ 700$ | $\$ 520$ |
| Depreciation <br> (MACRS) | 7 years | 7 years |

Use a MARR (after tax) of $10 \%$ and a marginal tax rate of $30 \%$, and answer the following questions:
(a) Which machine would be most economical to purchase under an infinite planning horizon? Explain any assumption that you need to make about future alternatives.
(b) Determine the break-even annual O\&M costs for machine A so that the present worth of machine A is the same as that of machine B .
(c) Suppose that the required service life of the machine is only 5 years. The salvage values at the end of the required service period are estimated to be $\$ 3,000$ for machine A and \$3,500 for machine B. Which machine is more economical?
12.7 The management of Langdale Mill is considering replacing a number of old looms in the mill's weave room. The looms to be replaced are two 86-inch President looms, sixteen 54-inch President looms, and twenty-two 72-inch Draper X-P2 looms. The company may either replace the old looms with new ones of the same kind or buy 21 new shutterless Pignone looms. The first alternative requires purchasing 40 new President and Draper looms and scrapping the old looms. The second alternative involves scrapping the 40 old looms, relocating 12 Picanol looms, and constructing a concrete floor, plus purchasing the 21 Pignone looms and various related equipment.

| Description | Alternative 1 | Alternative 2 |
| :--- | ---: | ---: |
| Machinery/related <br> equipment | $\$ 2,119,170$ | $\$ 1,071,240$ |
| Removal cost of <br> old looms/site <br> preparation | 26,866 | 49,002 |
| Salvage value of <br> old looms | 62,000 | 62,000 |
| Annual sales <br> increase with <br> new looms | $7,915,748$ | $7,455,084$ |
| Annual labor | 261,040 | 422,080 |
| Annual O\&M | $1,092,000$ | $1,560,000$ |
| Depreciation <br> (MACRS) | 7 years | 7 years |
| Project life | 8 years | 8 years |
| Salvage value | 169,000 | 54,000 |

The firm's MARR is $18 \%$, set by corporate executives, who feel that various investment opportunities available for the mills will guarantee a rate of return on investment of at least $18 \%$. The mill's marginal tax rate is $40 \%$.
(a) Perform a sensitivity analysis on the project's data, varying the operating revenue, labor cost, annual maintenance cost, and MARR. Assume that each of these variables can deviate from its base-case expected value by $\pm 10 \%$, by $\pm 20 \%$, and by $\pm 30 \%$.
(b) From the results of part (a), prepare sensitivity diagrams and interpret the results.
12.8 Mike Lazenby, an industrial engineer at Energy Conservation Service, has found that the anticipated profitability of a newly developed water-heater temperature control device can be measured by present worth with the formula

$$
\mathrm{NPW}=4.028 \mathrm{~V}(2 X-\$ 11)-77,860,
$$

where $V$ is the number of units produced and sold and $X$ is the sales price per unit. Mike also has found that the value of the parameter $V$ could occur anywhere over the range from 1,000 to 6,000 units and that of the parameter $X$ anywhere between $\$ 20$ and $\$ 45$ per unit. Develop a sensitivity graph as a function of the number of units produced and the sales price per unit.
12.9 A local U.S. Postal Service office is considering purchasing a 4,000-pound forklift truck, which will be used primarily for processing incoming as well as outgoing postal packages. Forklift trucks traditionally have been fueled by either gasoline, liquid propane gas (LPG), or diesel fuel. Battery-powered electric forklifts, however, are increasingly popular in many industrial sectors due to the economic and environmental benefits that accrue from their use. Therefore, the postal service is interested in comparing forklifts that use the four different types of fuel. The purchase costs as well as annual operating and maintenance costs are provided by a local utility company and the Lead Industries Association. Annual fuel and maintenance costs are measured in terms of number of shifts per year, where one shift is equivalent to 8 hours of operation.

|  | Electrical Power | LPG | Gasoline | Diesel Fuel |
| :--- | ---: | ---: | ---: | ---: |
| Life expectancy | 7 years | 7 years | 7 years | 7 years |
| Initial cost | $\$ 29,739$ | $\$ 21,200$ | $\$ 20,107$ | $\$ 22,263$ |
| Salvage value | $\$ 3,000$ | $\$ 2,000$ | $\$ 2,000$ | $\$ 2,200$ |
| Maximum shifts |  |  |  |  |
| per year | 260 | 260 | 260 | 260 |
| Fuel consumption/shift | 31.25 kWh | 11 gal | 11.1 gal | 7.2 gal |
| Fuel cost/unit | $\$ 0.10 / \mathrm{kWh}$ | $\$ 3.50 / \mathrm{gal}$ | $\$ 2.24 / \mathrm{gal}$ | $\$ 2.45 / \mathrm{gal}$ |
| Fuel cost per shift | $\$ 3.125$ | $\$ 38.50$ | $\$ 24.86$ | $\$ 17.64$ |
| Annual maintenance cost |  | $\$ 500$ | $\$ 1,000$ | $\$ 1,000$ |
| $\quad$ Fixed cost | $\$ 4.5$ | $\$ 7$ | $\$ 7$ | $\$ 1,000$ |
| Variable cost/shift |  |  | $\$ 7$ |  |

The postal service is unsure of the number of shifts per year, but it expects it should be somewhere between 200 and 260 shifts. Since the U.S. Postal Service does not pay income taxes, no depreciation or tax information is required. The U.S. government uses $10 \%$ as an interest rate for any project evaluation of this nature. Develop a sensitivity graph that shows how the choice of alternatives changes as a function of number of shifts per year.

## Break-Even Analysis

12.10 Susan Campbell is thinking about going into the motel business near Disney World in Orlando. The cost to build a motel is $\$ 2,200,000$. The lot costs $\$ 600,000$. Furniture and furnishings cost $\$ 400,000$ and should be recovered in 7 years (7-year MACRS property), while the motel building should be recovered in 39 years
(39-year MACRS real property placed in service on January 1). The land will appreciate at an annual rate of $5 \%$ over the project period, but the building will have a zero salvage value after 25 years. When the motel is full ( $100 \%$ capacity), it takes in (receipts) \$4,000 per day, 365 days per year. Exclusive of depreciation, the motel has fixed operating expenses of $\$ 230,000$ per year. The variable operating expenses are $\$ 170,000$ at $100 \%$ capacity, and these vary directly with percent capacity down to zero at $0 \%$ capacity. If the interest is $10 \%$ compounded annually, at what percent capacity over 25 years must the motel operate in order for Susan to break even? (Assume that Susan's tax rate is $31 \%$.)
12.11 A plant engineer wishes to know which of two types of lightbulbs should be used to light a warehouse. The bulbs that are currently used cost $\$ 45.90$ per bulb and last 14,600 hours before burning out. The new bulb (at $\$ 60$ per bulb) provides the same amount of light and consumes the same amount of energy, but lasts twice as long. The labor cost to change a bulb is $\$ 16.00$. The lights are on 19 hours a day, 365 days a year. If the firm's MARR is $15 \%$, what is the maximum price (per bulb) the engineer should be willing to pay to switch to the new bulb? (Assume that the firm's marginal tax rate is $40 \%$.)
12.12 Robert Cooper is considering purchasing a piece of business rental property containing stores and offices at a cost of $\$ 250,000$. Cooper estimates that annual receipts from rentals will be $\$ 35,000$ and that annual disbursements, other than income taxes, will be about $\$ 12,000$. The property is expected to appreciate at the annual rate of $5 \%$. Cooper expects to retain the property for 20 years once it is acquired. Then it will be depreciated as a 39 -year real-property class (MACRS), assuming that the property will be placed in service on January 1. Cooper's marginal tax rate is $30 \%$ and his MARR is $15 \%$. What would be the minimum annual total of rental receipts that would make the investment break even?
12.13 Two different methods of solving a production problem are under consideration. Both methods are expected to be obsolete in six years. Method A would cost $\$ 80,000$ initially and have annual operating costs of $\$ 22,000$ a year. Method B would cost $\$ 52,000$ and costs $\$ 17,000$ a year to operate. The salvage value realized would be $\$ 20,000$ with Method A and $\$ 15,000$ with Method B. Method A would generate $\$ 16,000$ revenue income a year more than Method B. Investments in both methods are subject to a five-year MACRS property class. The firm's marginal income tax rate is $40 \%$. The firm's MARR is $20 \%$. What would be the required additional annual revenue for Method A such that an engineer would be indifferent to choosing one method over the other?
12.14 Rocky Mountain Publishing Company is considering introducing a new morning newspaper in Denver. Its direct competitor charges $\$ 0.25$ at retail, with $\$ 0.05$ going to the retailer. For the level of news coverage the company desires, it determines the fixed cost of editors, reporters, rent, press-room expenses, and wire service charges to be $\$ 300,000$ per month. The variable cost of ink and paper is $\$ 0.10$ per copy, but advertising revenues of $\$ 0.05$ per paper will be generated. To print the morning paper, the publisher has to purchase a new printing press, which will cost $\$ 600,000$. The press machine will be depreciated according to a 7 -year MACRS class. The press machine will be used for 10 years, at which time its salvage value would be about $\$ 100,000$. Assume 20 weekdays in a month, a $40 \%$ tax rate, and a $13 \%$ MARR. How many copies per day must be sold to break even at a retail selling price of $\$ 0.25$ per paper?

## Probabilistic Analysis

12.15 A corporation is trying to decide whether to buy the patent for a product designed by another company. The decision to buy will mean an investment of $\$ 8$ million, and the demand for the product is not known. If demand is light, the company expects a return of $\$ 1.3$ million each year for three years. If demand is moderate, the return will be $\$ 2.5$ million each year for four years, and high demand means a return of $\$ 4$ million each year for four years. It is estimated the probability of a high demand is 0.4 and the probability of a light demand is 0.2 . The firm's (risk-free) interest rate is $12 \%$. Calculate the expected present worth of the patent. On this basis, should the company make the investment? (All figures represent after-tax values.)
12.16 Juan Carlos is considering two investment projects whose present values are described as follows:

- Project 1. PW $(10 \%)=20 X+8 X Y$, where $X$ and $Y$ are statistically independent discrete random variables with the following distributions:

| $\boldsymbol{X}$ |  |  | $\boldsymbol{Y}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| Event | Probability |  | Event | Probability |
| $\$ 20$ | 0.6 |  | $\$ 10$ | 0.4 |
| 40 | 0.4 | 20 | 0.6 |  |

## - Project 2.

| PW(10\%) | Probability |
| :---: | :---: |
| $\$ 0$ | 0.24 |
| 400 | 0.20 |
| 1,600 | 0.36 |
| 2,400 | 0.20 |

(Note: Cash flows between the two projects are also assumed to be statistically independent.)
(a) Develop the NPW distribution for Project 1.
(b) Compute the mean and variance of the NPW for Project 1.
(c) Compute the mean and variance of the NPW for Project 2.
(d) Suppose that Projects 1 and 2 are mutually exclusive. Which project would you select?
12.17 A financial investor has an investment portfolio worth $\$ 350,000$. A bond in the portfolio will mature next month and provide him with $\$ 25,000$ to reinvest. The choices have been narrowed down to the following two options:

- Option 1. Reinvest in a foreign bond that will mature in one year. This will entail a brokerage fee of $\$ 150$. For simplicity, assume that the bond will provide interest of $\$ 2,450, \$ 2,000$, or $\$ 1,675$ over the one-year period and that the probabilities of these occurrences are assessed to be $0.25,0.45$, and 0.30 , respectively.
- Option 2. Reinvest in a $\$ 25,000$ certificate with a savings-and-loan association. Assume that this certificate has an effective annual rate of 7.5\%.
(a) Which form of reinvestment should the investor choose in order to maximize his expected financial gain?
(b) If the investor can obtain professional investment advice from Salomon Brothers, Inc., what would be the maximum amount the investor should pay for this service?
12.18 Kellogg Company is considering the following investment project and has estimated all cost and revenues in constant dollars. The project requires the purchase of a $\$ 9,000$ asset, which will be used for only two years (the project life).
- The salvage value of this asset at the end of two years is expected to be $\$ 4,000$.
- The project requires an investment of $\$ 2,000$ in working capital, and this amount will be fully recovered at the end of the project year.
- The annual revenue, as well as general inflation, are discrete random variables, but can be described by the following probability distributions (both random variables are statistically independent):

| Annual <br> Revenue <br> $(\boldsymbol{X})$ | Probability | General <br> Inflation <br> Rate <br> $(\boldsymbol{Y})$ | Probability |
| :---: | :---: | :---: | :---: |
| $\$ 10,000$ | 0.30 | $3 \%$ | 0.25 |
| 20,000 | 0.40 | $5 \%$ | 0.50 |
| 30,000 | 0.30 | $7 \%$ | 0.25 |

- The investment will be classified as a three-year MACRS property (tax life).
- It is assumed that the revenues, salvage value, and working capital are responsive to the general inflation rate.
- The revenue and inflation rate dictated during the first year will prevail over the remainder of the project period.
- The marginal income tax rate for the firm is $40 \%$. The firm's inflation-free interest rate ( $i^{\prime}$ ) is $10 \%$.
(a) Determine the NPW as a function of $X$.
(b) In (a), compute the expected NPW of this investment.
(c) In (a), compute the variance of the NPW of the investment.


## Comparing Risky Projects

12.19 A manufacturing firm is considering two mutually exclusive projects, both of which have an economic service life of one year with no salvage value. The initial cost and the net year-end revenue for each project are given in Table P12.19.

We assume that both projects are statistically independent of each other.
(a) If you are an expected-value maximizer, which project would you select?
(b) If you also consider the variance of the project, which project would you select?

TABLE I2.|9 Comparison of Mutually Exclusive Projects

| First Cost | Project 1 <br> $(\$ 1,000)$ |  |  | Project 2 <br> $(\$ 800)$ |  |
| :--- | :---: | ---: | :--- | :--- | :--- | ---: |
|  | Probability | Revenue |  |  |  |
| Probability | Revenue |  |  |  |  |
| Net revenue, |  |  |  |  |  |
| given in PW | 0.2 | $\$ 2,000$ |  | 0.3 | $\$ 1,000$ |
|  | 0.6 | 3,000 |  | 0.4 | 2,500 |
|  | 0.2 | 3,500 |  | 0.3 | 4,500 |

12.20 A business executive is trying to decide whether to undertake one of two contracts or neither one. He has simplified the situation somewhat and feels that it is sufficient to imagine that the contracts provide alternatives as follows:

| Contract A |  |  | Contract B |  |
| ---: | :---: | :---: | :---: | :---: |
| NPW | Probability |  | NPW | Probability |
| $\$ 100,000$ | 0.2 |  | $\$ 40,000$ | 0.3 |
| 50,000 | 0.4 |  | 10,000 | 0.4 |
| 0 | 0.4 |  | $-10,000$ | 0.3 |

(a) Should the executive undertake either one of the contracts? If so, which one? What would he do if he made decisions with an eye toward maximizing his expected NPW?
(b) What would be the probability that Contract A would result in a larger profit than that of Contract B?
12.21 Two alternative machines are being considered for a cost-reduction project.

- Machine A has a first cost of \$60,000 and a salvage value (after tax) of \$22,000 at the end of 6 years of service life. The probabilities of annual after-tax operating costs of this machine are estimated as follows:

| Annual O\&M Costs | Probability |
| :---: | :---: |
| $\$ 5,000$ | 0.20 |
| 8,000 | 0.30 |
| 10,000 | 0.30 |
| 12,000 | 0.20 |

- Machine B has an initial cost of $\$ 35,000$, and its estimated salvage value (after tax) at the end of 4 years of service is negligible. The annual after-tax operating costs are estimated to be as follows:

| Annual O\&M Costs | Probability |
| :---: | :---: |
| $\$ 8,000$ | 0.10 |
| 10,000 | 0.30 |
| 12,000 | 0.40 |
| 14,000 | 0.20 |

The MARR on this project is $10 \%$. The required service period of these machines is estimated to be 12 years, and no technological advance in either machine is expected.
(a) Assuming independence, calculate the mean and variance for the equivalent annual cost of operating each machine.
(b) From the results of part (a), calculate the probability that the annual cost of operating Machine A will exceed the cost of operating Machine B.
12.22 Two mutually exclusive investment projects are under consideration. It is assumed that the cash flows are statistically independent random variables with means and variances estimated as follows:

| End of <br> Year | Project A |  |  | Project B |  |
| :---: | ---: | ---: | ---: | ---: | ---: |
|  | Mean | Variance | Mean | Variance |  |
| 0 | $-\$ 5,000$ | $1,000^{2}$ |  | $\$ 10,000$ | $2,000^{2}$ |
| 1 | 4,000 | $1,000^{2}$ |  | 6,000 | $1,500^{2}$ |
| 2 | 4,000 | $1,500^{2}$ |  | 8,000 | $2,000^{2}$ |

(a) For each project, determine the mean and standard deviation of the NPW, using an interest rate of $15 \%$.
(b) On the basis of the results of part (a), which project would you recommend?

## Decision-Tree Analysis

12.23 Delta College's campus police are quite concerned with ever-growing weekend parties taking place at the various dormitories on the campus, where alcohol is commonly served to underage college students. According to reliable information, on a given Saturday night, one may observe a party to take place $60 \%$ of the time. Police Lieutenant Shark usually receives a tip regarding student drinking that is to take place in one of the residence halls the next weekend. According to Officer Shark, this tipster has been correct $40 \%$ of the time when the party is planned. The tipster has been also correct $80 \%$ of the time when the party does not take place. (That is, the tipster says that no party is planned.) If Officer Shark does not raid the residence hall in question at the time of the supposed party, he loses 10 career progress points. (The police chief gets complete information on whether there was a party only after the weekend.) If he leads a raid and the tip is false, he loses 50 career progress points, whereas if the tip is correct, he earns 100 points.
(a) What is the probability that no party is actually planned even though the tipster says that there will be a party?
(b) If the lieutenant wishes to maximize his expected career progress points, what should he do?
(c) What is the EVPI (in terms of career points)?
12.24 As a plant manager of a firm, you are trying to decide whether to open a new factory outlet store, which would cost about $\$ 500,000$. Success of the outlet store depends on demand in the new region. If demand is high, you expect to gain $\$ 1$ million per year; if demand is average, the gain is $\$ 500,000$; and if demand is low, you lose $\$ 80,000$. From your knowledge of the region and your product, you feel that the chances are 0.4 that sales will be average and equally likely that they will be high or low ( 0.3 , respectively). Assume that the firm's MARR is known to be $15 \%$, and the marginal tax rate will be $40 \%$. Also, assume that the salvage value of the store at the end of 15 years will be about $\$ 100,000$. The store will be depreciated under a 39-year property class.
(a) If the new outlet store will be in business for 15 years, should you open it? How much would you be willing to pay to know the true state of nature?
(b) Suppose a market survey is available at $\$ 1,000$, with the following reliability (the values shown were obtained from past experience, where actual demand was compared with predictions made by the survey):

| Actual <br> Demand | Survey Prediction |  |  |
| :--- | :---: | :---: | :---: |
|  | Low | Medium | High |
| Low | 0.75 | 0.20 | 0.05 |
| Medium | 0.20 | 0.60 | 0.20 |
| High | 0.05 | 0.25 | 0.70 |

Determine the strategy that maximizes the expected payoff after taking the market survey. In doing so, compute the EVPI after taking the survey. What is the true worth of the sample information?

## Short Case Studies

ST12.1 In Virginia's six-number lottery, or lotto, players pick six numbers from 1 to 44. The winning combination is determined by a machine that looks like a popcorn machine, except that it is filled with numbered table-tennis balls. On February 15, 1992, the Virginia lottery drawing offered the prizes shown in Table ST12.1, assuming that the first prize is not shared.
Common among regular lottery players is this dream: waiting until the jackpot reaches an astronomical sum and then buying every possible number, thereby guaranteeing a winner. Sure, it would cost millions of dollars, but the payoff would be much greater. Is it worth trying? How do the odds of winning the first prize change as you increase the number of tickets purchased?

## TABLE STI2.I Virginia Lottery Prizes

| Number <br> of Prizes | Prize Category | Total Amount |
| ---: | :--- | ---: |
| 1 | First prize | $\$ 27,007,364$ |
| 228 | Second prizes $(\$ 899$ each $)$ | 204,972 |
| 10,552 | Third prizes (\$51 each) | 538,152 |
| 168,073 | Fourth prizes (\$1 each) | 168,073 |
|  | Total winnings | $\$ 27,918,561$ |

ST12.2 Mount Manufacturing Company produces industrial and public safety shirts. As is done in most apparel-manufacturing factories, the cloth must be cut into shirt parts by marking sheets of paper in a particular way. At present, these sheet markings are done manually, and their annual labor cost is running around $\$ 103,718$. Mount has the option of purchasing one of two automated marking systems: the Lectra System 305 and the Tex Corporation Marking System. The comparative characteristics of the two systems are as follows:

|  | Most Likely Estimates |  |
| :--- | :--- | :--- |
|  | Lectra <br> System | Tex <br> System |
| Annual labor cost | $\$ 51,609$ | $\$ 51,609$ |
| Annual material savings | $\$ 230,000$ | $\$ 274,000$ |
| Investment cost | $\$ 136,150$ | $\$ 195,500$ |
| Estimated life | 6 years | 6 years |
| Salvage value | $\$ 20,000$ | $\$ 15,000$ |
| Depreciation <br> method (MACRS) | 5 years | 5 years |

The firm's marginal tax rate is $40 \%$, and the interest rate used for project evaluation is $12 \%$ after taxes.
(a) Based on the most likely estimates, which alternative is the best?
(b) Suppose that the company estimates the material savings during the first year for each system on the basis of the following probability distribution:

| Lectra System |  |
| :---: | :---: |
| Material Savings | Probability |
| $\$ 150,000$ | 0.25 |
| 230,000 | 0.40 |
| 270,000 | 0.35 |


| Tex System |  |
| :---: | :---: |
| Material Savings | Probability |
| $\$ 200,000$ | 0.30 |
| 274,000 | 0.50 |
| 312,000 | 0.20 |

Suppose further that the annual material savings for both Lectra and Tex are statistically independent. Compute the mean and variance for the equivalent annual value of operating each system.
(c) In part (b), calculate the probability that the annual benefit of operating Lectra will exceed the annual benefit of operating Tex. (Use @RISK or Excel to answer this question.)
ST12.3 The city of Opelika was having a problem locating land for a new sanitary landfill when the Alabama Energy Extension Service offered the solution of burning the solid waste to generate steam. At the same time, Uniroyal Tire Company seemed to be having a similar problem disposing of solid waste in the form of rubber tires. It was determined that there would be about 200 tons per day of waste to be burned, including municipal and industrial waste. The city is considering building a waste-fired steam plant, which would cost $\$ 6,688,800$. To finance the construction cost, the city will issue resource recovery revenue bonds in the amount of $\$ 7,000,000$ at an interest rate of $11.5 \%$. Bond interest is payable annually. The differential amount between the actual construction costs and the amount of bond financing ( $\$ 7,000,000-\$ 6,688,800=\$ 311,200$ ) will be used to settle the bond discount and expenses associated with the bond financing. The expected life of the steam plant is 20 years. The expected salvage value is estimated to be about $\$ 300,000$. The expected labor costs are $\$ 335,000$ per year. The annual operating and maintenance costs (including fuel, electricity, maintenance, and water) are expected to be $\$ 175,000$. The plant would generate 9,360 pounds of waste, along with 7,200 pounds of waste after incineration, which will have to be disposed of as landfill. At the present rate of $\$ 19.45$ per pound, this will cost the city a total of $\$ 322,000$ per year. The revenues for the steam plant will come from two sources: (1) sales of steam and (2) tipping fees for disposal. The city expects $20 \%$ downtime per year for the waste-fired steam plant. With an input of 200 tons per day and 3.01 pounds of steam per pound of refuse, a maximum of $1,327,453$ pounds of steam can be produced per day. However, with $20 \%$ downtime, the actual output would be $1,061,962$ pounds of steam per day. The initial steam charge will be approximately $\$ 4.00$ per thousand pounds, which will bring in $\$ 1,550,520$ in steam revenue the first year. The tipping fee is used in conjunction with the sale of steam to offset the total plant cost. It is the goal of the Opelika steam plant to phase out the tipping fee as soon as possible. The tipping fee will be $\$ 20.85$ per ton in the first year of plant operation and will be phased out at the end of the eighth year. The scheduled tipping fee assessment is as follows:

| Year | Tipping Fee |
| :---: | :---: |
| 1 | $\$ 976,114$ |
| 2 | 895,723 |
| 3 | 800,275 |
| 4 | 687,153 |
| 5 | 553,301 |
| 6 | 395,161 |
| 7 | 208,585 |

(a) At an interest rate of $10 \%$, would the steam plant generate sufficient revenue to recover the initial investment?
(b) At an interest rate of $10 \%$, what would be the minimum charge (per thousand pounds) for steam sales to make the project break even?
(c) Perform a sensitivity analysis to determine the input variable of the plant's downtime.

ST12.4 Burlington Motor Carriers, a trucking company, is considering installing a twoway mobile satellite messaging service on its 2,000 trucks. On the basis of tests done last year on 120 trucks, the company found that satellite messaging could cut $60 \%$ from its $\$ 5$ million bill for long-distance communication with truck drivers. More important, the drivers reduced the number of "deadhead" milesthose driven with nonpaying loads-by $0.5 \%$. Applying that improvement to all 230 million miles covered by the Burlington fleet each year would produce an extra $\$ 1.25$ million savings.

Equipping all 2,000 trucks with the satellite hookup will require an investment of $\$ 8$ million and the construction of a message-relaying system costing $\$ 2$ million. The equipment and onboard devices will have a service life of eight years and negligible salvage value; they will be depreciated under the five-year MACRS class. Burlington's marginal tax rate is about $38 \%$, and its required minimum attractive rate of return is $18 \%$.
(a) Determine the annual net cash flows from the project.
(b) Perform a sensitivity analysis on the project's data, varying savings in telephone bills and savings in deadhead miles. Assume that each of these variables can deviate from its base-case expected value by $\pm 10 \%, \pm 20 \%$, and $\pm 30 \%$.
(c) Prepare sensitivity diagrams and interpret the results.

ST12.5 The following is a comparison of the cost structure of conventional manufacturing technology (CMT) with a flexible manufacturing system (FMS) at one U.S. firm.

|  | Most Likely Estimates |  |
| :--- | ---: | ---: |
|  | CMT | FMS |
| Number of part types | 3,000 | 3,000 |
| Number of pieces produced/year | 544,000 | 544,000 |
| Variable labor cost/part | $\$ 2.15$ | $\$ 1.30$ |
| Variable material cost/part | $\$ 1.53$ | $\$ 1.10$ |
| Total variable cost/part | $\$ 3.68$ | $\$ 2.40$ |
| Annual overhead | $\$ 3.15 \mathrm{M}$ | $\$ 1.95 \mathrm{M}$ |
| Annual tooling costs | $\$ 470,000$ | $\$ 300,000$ |
| Annual inventory costs | $\$ 141,000$ | $\$ 31,500$ |
| Total annual fixed operating costs | $\$ 3.76 \mathrm{M}$ | $\$ 2.28 \mathrm{M}$ |
| Investment | $\$ 3.5 \mathrm{M}$ | $\$ 10 \mathrm{M}$ |
| Salvage value | $\$ 0.5 \mathrm{M}$ | $\$ 1 \mathrm{M}$ |
| Service life | 10 years | 10 years |
| Depreciation method (MACRS) | 7 years | 7 years |

(a) The firm's marginal tax rate and MARR are $40 \%$ and $15 \%$, respectively. Determine the incremental cash flow (FMS - CMT), based on the most likely estimates.
(b) Management feels confident about all input estimates in CMT. How.ever, the firm does not have any previous experience operating an FMS. Therefore, many input estimates, except the investment and salvage value, are subject to variation. Perform a sensitivity analysis on the project's data, varying the elements of operating costs. Assume that each of these variables can deviate from its base-case expected value by $\pm 10 \%, \pm 20 \%$, and $\pm 30 \%$.
(c) Prepare sensitivity diagrams and interpret the results.
(d) Suppose that probabilities of the variable material cost and the annual inventory cost for the FMS are estimated as follows:

| Material Cost |  |
| :---: | :---: |
| Cost per Part | Probability |
| $\$ 1.00$ | 0.25 |
| 1.10 | 0.30 |
| 1.20 | 0.20 |
| 1.30 | 0.20 |
| 1.40 | 0.05 |


| Inventory Cost |  |
| :---: | :---: |
| Annual Inventory Cost | Probability |
| $\$ 25,000$ | 0.10 |
| 31,000 | 0.30 |
| 50,000 | 0.20 |
| 80,000 | 0.20 |
| 100,000 | 0.20 |

What are the best and the worst cases of incremental NPW?
(e) In part (d), assuming that the random variables of the cost per part and the annual inventory cost are statistically independent, find the mean and variance of the NPW for the incremental cash flows.
(f) In parts (d) and (e), what is the probability that the FMS will be the more expensive of the two investment options? (Use @RISK or Excel to answer this question.)


[^0]:    1 "Oil Forecasts Are a Roll of the Dice," Bhusan Bahree, The Wall Street Journal, Tuesday, August
    2, 2005, Section C1.

[^1]:    ${ }^{2}$ These are expenses that change in direct proportion to the change in volume of sales or production, as defined in Section 8.3.
    ${ }^{3}$ These are expenses that do not vary with the volume of sales or production. For example, property taxes, insurance, depreciation, and rent are usually fixed expenses.

[^2]:    ${ }^{5}$ If $X$ and $Y$ are dependent random variables, the joint probabilities developed in Table 12.6 should be used.

[^3]:    ${ }^{6}$ Analytical treatment for the products of random variables including the correlated case is given by Chan S . Park and Gunter Sharp-Bette, Advanced Engineering Economics, New York, John Wiley, 1990 (Chapter 10).

[^4]:    ${ }^{7}$ As we seek further refinement in our decision under risk, we may consider the expected-utility theory, or stochastic dominance rules, which is beyond the scope of our text. (See Park, C. S., and Sharp-Bette, G. P., Advanced Engineering Economics (New York: John Wiley, 1990), Chapters 10 and 11.)

[^5]:    ${ }^{8}$ A risk simulation software developed by Palisade Corporation, Ithaca, NY 14850.

