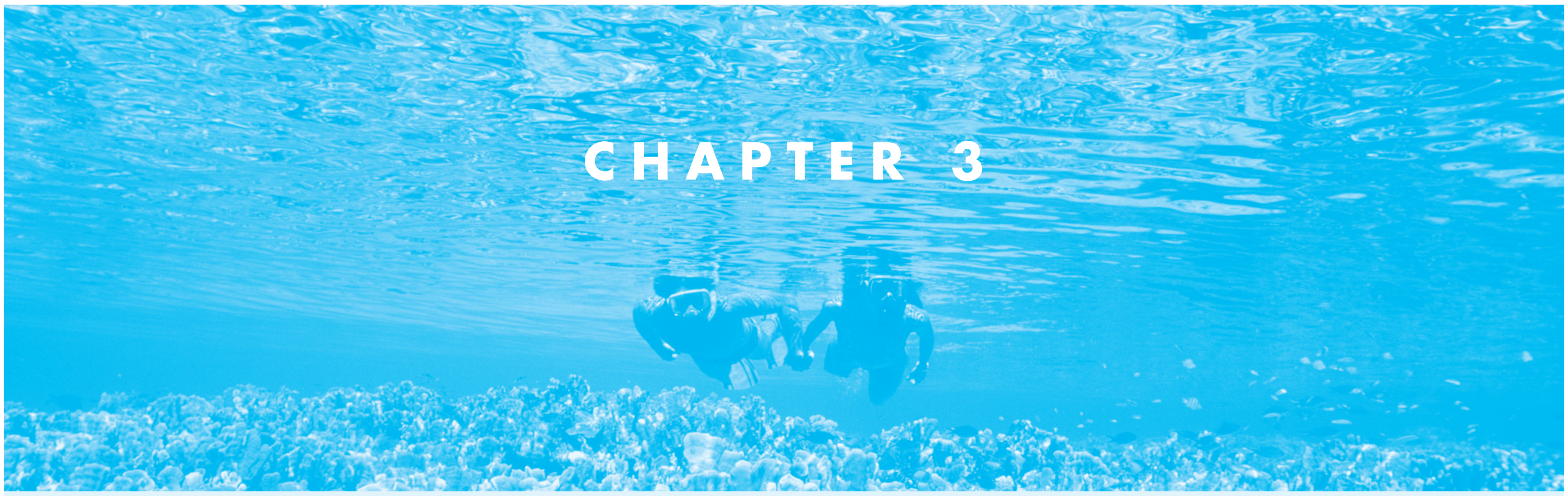


WHEN LINES MEET: LINEAR SYSTEMS

OVERVIEW

When is solar heating cheaper than conventional heating? Will you pay more tax under a flat tax or a graduated tax plan? We can answer such questions using a system of linear equations or inequalities.

After reading this chapter, you should be able to

- construct, graph, and interpret:
  - systems of linear equations
  - systems of linear inequalities
  - piecewise linear functions
- find a solution for:
  - a system of two linear equations
  - a system of two linear inequalities

3.1 *Systems of Linear Equations*

An Economic Comparison of Solar vs. Conventional Heating Systems

On a planet with limited fuel resources, heating decisions involve both monetary and ecological considerations. Typical costs for three different kinds of heating systems for a small one-bedroom housing unit are given in Table 3.1.

Typical Costs for Three Heating Systems

Type of System	Installation Cost ($)	Operating Cost ($/yr)
Electric	5,000	1,100
Gas	12,000	700
Solar	30,000	150

Table 3.1

Solar heating is clearly the most costly to install and the least expensive to run. Electric heating, conversely, is the cheapest to install and the most expensive to run.

Setting up a system

By converting the information in Table 3.1 into equations, we can find out when the solar heating system begins to pay back its initially higher cost. If no allowance is made for inflation or changes in fuel price,[1] the general equation for the total cost, C, is

$$C = \text{installation cost} + (\text{annual operating cost})(\text{years of operation})$$

If we let n equal the number of years of operation and use the data from Table 3.1, we can construct the following linear equations:

$$C_{\text{electric}} = 5000 + 1100n$$
$$C_{\text{gas}} = 12{,}000 + 700n$$
$$C_{\text{solar}} = 30{,}000 + 150n$$

Together they form a *system of linear equations*. Table 3.2 gives the cost data at 5-year intervals, and Figure 3.1 shows the costs over a 40-year period for the three heating systems.

Heating System Total Costs

Year	Electric ($)	Gas ($)	Solar ($)
0	5,000	12,000	30,000
5	10,500	15,500	30,750
10	16,000	19,000	31,500
15	21,500	22,500	32,250
20	27,000	26,000	33,000
25	32,500	29,500	33,750
30	38,000	33,000	34,500
35	43,500	36,500	35,250
40	49,000	40,000	36,000

Table 3.2

[1] A more sophisticated model might include many other factors, such as interest, repair costs, the cost of depleting fuel resources, risks of generating nuclear power, and what economists call opportunity costs.

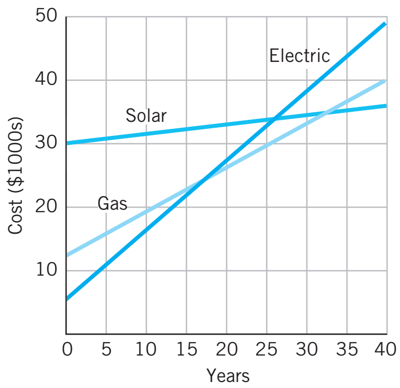


Figure 3.1 Comparison of home heating costs.

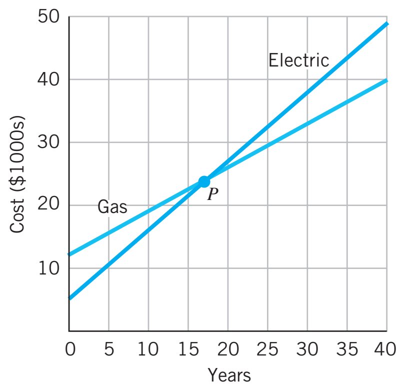


Figure 3.2 Gas versus electric.

Comparing costs

Using Graphs Let's compare the costs for gas and electric heat. Figure 3.2 shows the graphs of the equations for these two heating systems. The point of intersection, P, shows where the lines predict the *same* total cost for both gas and electricity, given a certain number of years of operation. From the graph, we can estimate the coordinates of the point P:

$$\begin{aligned} P &= (\text{number of years of operation, cost}) \\ &\approx (17, \$24{,}000) \end{aligned}$$

The total cost of operation is about \$24,000 for both gas and electric after about 17 years of operation. We can compare the relative costs of each system to the left and right of the point of intersection. Figure 3.2 shows that gas is less expensive than electricity to the right of the intersection point and more expensive than electricity to the left of the intersection point.

Using Equations Where the gas and electric lines intersect, the coordinates satisfy both equations. At the point of intersection, the total cost of electric heat, C_{electric}, equals the total cost of gas heat, C_{gas}. Thus, the two expressions for the total cost can be set equal to each other to find the exact values for the coordinates of the intersection point:

$$C_{\text{electric}} = 5000 + 1100n \qquad (1)$$

$$C_{\text{gas}} = 12{,}000 + 700n \qquad (2)$$

Set (1) equal to (2)	$C_{\text{electric}} = C_{\text{gas}}$
substitute	$5000 + 1100n = 12{,}000 + 700n$
subtract 5000 from each side	$1100n = 7000 + 700n$
subtract $700n$ from each side	$400n = 7000$
divide each side by 400	$n = 17.5$ years

When $n = 17.5$ years, the total cost for electric or gas heating is the same. The total cost can be found by substituting this value for n in Equation (1) or (2):

Substitute 17.5 for n in Equation (1)

$$\begin{aligned} C_{\text{electric}} &= 5000 + 1100(17.5) \\ &= 5000 + 19{,}250 \\ C_{\text{electric}} &= \$24{,}250 \end{aligned}$$

Since we claim that the pair of values (17.5, $24,250) satisfies both equations, we need to check, when $n = 17.5$ years, that C_{gas} is also $24,250:

Substitute 17.5 for n in Equation (2)

$$C_{gas} = 12{,}000 + 700(17.5)$$
$$= 12{,}000 + 12{,}250$$
$$C_{gas} = \$24{,}250$$

The coordinates (17.5, $24,250) satisfy both equations.

Solutions to a system

When $n = 17.5$ years, then $C_{electric} = C_{gas} = \$24{,}250$. The point (17.5, $24,250) is called a *solution* to the system of these two equations. After 17.5 years, a total of $24,250 could have been spent on heat for either an electric or a gas heating system.

A Solution to a System of Equations

A pair of real numbers is a *solution* to a system of equations in two variables if and only if the pair of numbers is a solution to each equation in the system.

EXAMPLE 1 **Estimating solutions to systems from graphs**

Using the graphs in Figures 3.3 and 3.4, estimate when the cost will be the same for each of the following systems:

a. Electric vs. solar heating

b. Gas vs. solar heating

You will be asked to find more accurate solutions in the exercises.

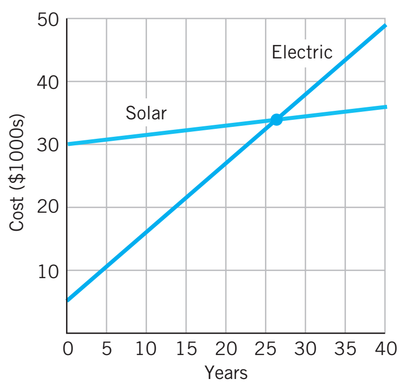


Figure 3.3 Electric versus solar heating.

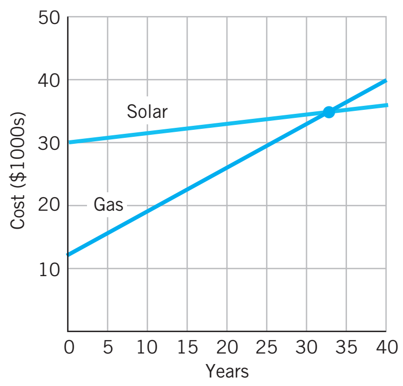


Figure 3.4 Gas versus solar heating.

SOLUTION

a. The graphs of electric and solar costs intersect at approximately (26, \$34,000). That means the costs of electric and solar heating are both approximately \$34,000 at about 26 years of operation.

b. Gas and solar heating costs are equivalent at approximately (33, \$35,000), or in 33 years at a cost of \$35,000.

Algebra Aerobics 3.1

1. Our model tells us that electric systems are cheaper than gas for the first 17.5 years of operation. Using Figure 3.1, estimate the interval over which gas is the cheapest of the three heating systems. When does solar heating become the cheapest system compared with the other two heating systems?

2. For the system of equations

$$4x + 3y = 9$$
$$5x + 2y = 13$$

   a. Determine whether $(3, -1)$ is a solution.

   b. Show why $(1, 4)$ is not a solution for this system.

3. **a.** Show that the following equations are equivalent:

$$4x = 6 + 3y$$
$$12x - 9y = 18$$

   b. How many solutions are there for the system of equations in part (a)?

4. Estimate the solution(s) for each of the systems of equations in Figure 3.5.

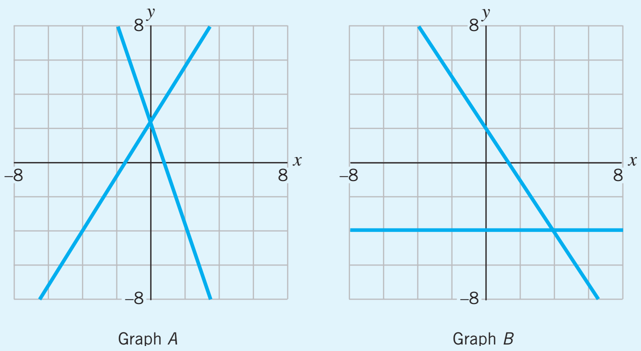


Figure 3.5 Two graphs of systems of linear equations.

Exercises for Section 3.1

1. **a.** Determine whether $(5, -10)$ is a solution for the following system of equations:

$$4x - 3y = 50$$
$$2x + 2y = 5$$

   b. Explain why $(-10, 5)$ is not a solution for the system in part (a).

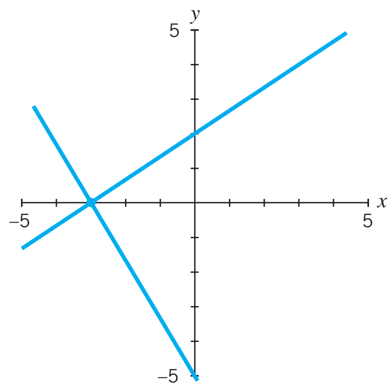
2. **a.** Determine whether $(-2, 3)$ is a solution for the following system of equations:

$$3x + y = -3$$
$$x + 2y = 4$$

   b. Explain why $(3, -2)$ is not a solution for the system in part (a).

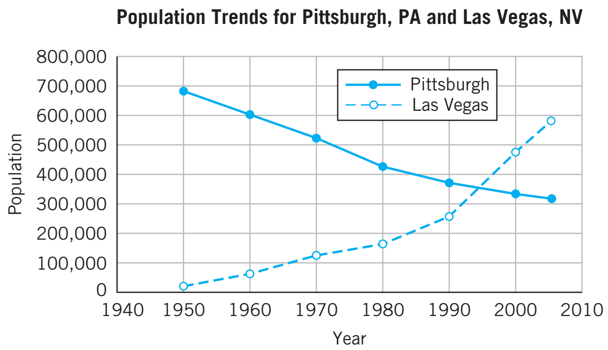
3. Explain what is meant by "a solution to a system of equations."

4. Estimate the solution to the system of linear equations graphed in the accompanying figure.

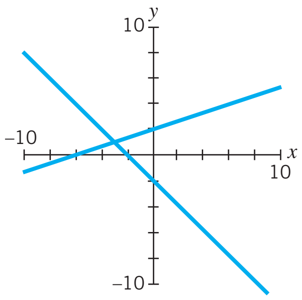


5. a. Estimate the coordinates of the point of intersection on the accompanying graph.

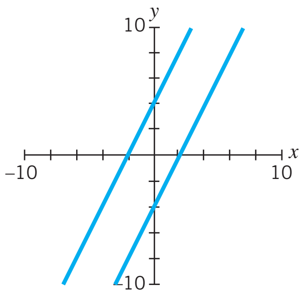
b. Describe what happens to the population of Pittsburgh in relation to the population of Las Vegas to the right and to the left of this intersection point.



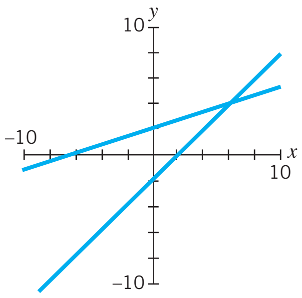
6. a. Match each system of linear equations with the graph of the system.



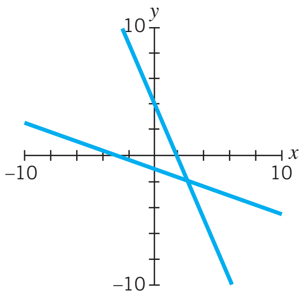
Graph *A*



Graph *C*



Graph *B*



Graph *D*

i.	$y = -x - 2$	$y = \frac{1}{3}x + 2$
ii.	$y - 2x = 4$	$y - 2x = -4$
iii.	$y - x = -2$	$3y - x = 6$
iv.	$3y + x = -3$	$y = -2x + 4$

b. Estimate the point of intersection if there is one.

c. Verify that the point of intersection satisfies both equations.

7. Construct a sketch of each system by hand and then estimate the solution(s) to the system (if any).

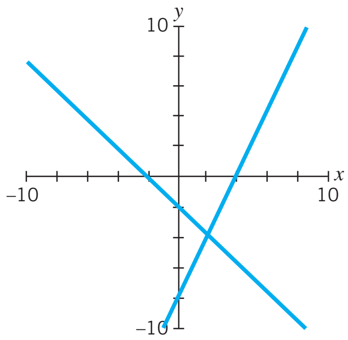
a. $x + 2y = 1$, $x + 4y = 3$ **b.** $x + y = 9$, $2x - 3y = -2$

8. For the linear system $\begin{cases} x - y = 5 \\ 2x + y = 1 \end{cases}$

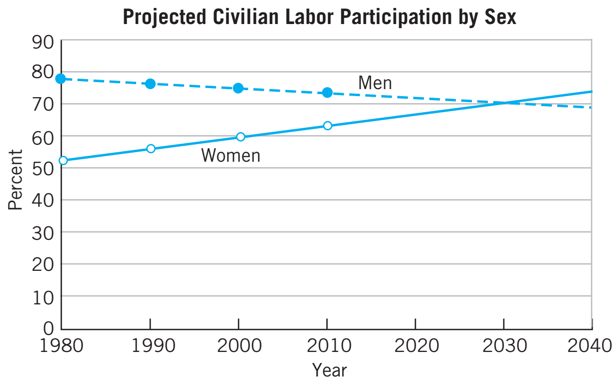
a. Graph the system. Estimate the solution for the system and then find the exact solution.

b. Check that your solution satisfies both of the original equations.

9. Create the system of equations that produced the accompanying graph. Estimate the solution for the system from the graph and then confirm using your equations.



10. The U.S. Bureau of Labor Statistics reports on the percent of all men in the civilian work force, and the corresponding percent of all women. Lines of best-fit on the accompanying graph are used to make predictions about the future of the labor force.



a. Estimate the point of intersection.

b. Describe the meaning of the point of intersection.

3.2 *Finding Solutions to Systems of Linear Equations*

In the heating example, we found a solution for a system of two equations by finding the point where the graphs of the two equations intersect. At the point of intersection, the equations have the same value for the independent variable (input) and the same value for the dependent variable (output).

Visualizing Solutions

For a single linear equation, the graph of its solutions is one line, and every point on that line is a solution. So, one linear equation has an unlimited number of solutions. In a system of two linear equations, a solution must satisfy both equations. We can easily visualize what might happen. If we graph two different straight lines and the lines intersect (Figure 3.6), there is only one solution—at the intersection point. The coordinates of the point of intersection can be estimated by inspecting the graph. For example, in Figure 3.6 the two lines appear to cross near the point $(-1, 1.5)$. If the lines are parallel, they never intersect and there are no solutions (Figure 3.7).

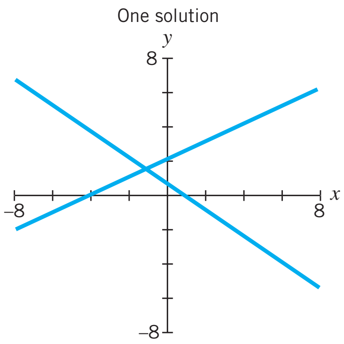


Figure 3.6 Lines intersect at a single point.

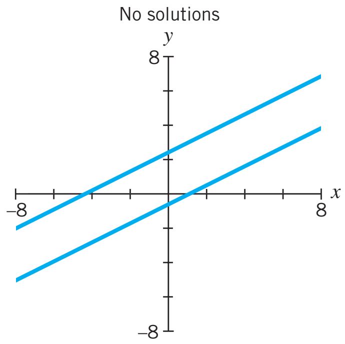


Figure 3.7 Parallel lines never intersect.

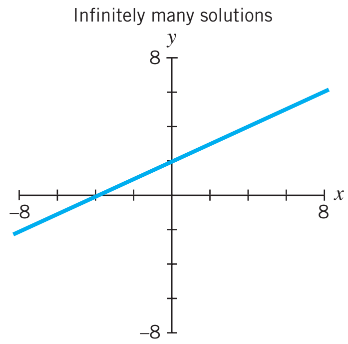


Figure 3.8 The two equations represent the same line. Every point on the line is a solution to both equations.

How else might the graphs of two lines be related? The two equations could represent the same line. Consider the following two equations:

$$y = 0.5x + 2 \qquad (1)$$

$$3y = 1.5x + 6 \qquad (2)$$

If we multiply each side of Equation (1) by 3, we obtain Equation (2). The two equations are *equivalent*, since any solution of one equation is also a solution of the other equation. The two equations represent the same line. There are an infinite number of points on that line, and they are all solutions to both equations (Figure 3.8).

> **The Number of Solutions for a Linear System**
>
> On the graph of a system of two linear equations, a *solution* is a point where the two lines intersect. There can be
>
> One solution, if the lines intersect once
> No solution, if the lines are parallel and distinct
> Infinitely many solutions, if the two lines are the same

Strategies for Finding Solutions

A system of two linear equations can be solved in several ways, each of which will provide the same result. In each case a solution, if it exists, consists of either one or infinitely many pairs of numbers of the form (x, y). The form of the equations can help determine the most efficient strategy. Two of the most common methods are *substitution* and *elimination*.

Substitution method

When at least one of the equations is in (or can easily be converted to) function form, $y = b + mx$, we can use the substitution method. The point of intersection is a solution to both equations, so at that point the two equations have the same value for x and the same value for y. To find values for the intersection point, we can start by substituting the expression for y from one equation into the other equation.

EXAMPLE 1 **When both equations are in function form**

A long-distance phone plan seen on TV costs \$0.03 per minute plus a fixed charge of \$0.39 per call. Your current service charges \$0.05 per minute plus a fixed charge of \$0.20 per call. During one call, after how many minutes would the cost be the same under the two plans?

SOLUTION If we let $n =$ the number of minutes in one call, then

$C_1 = 0.39 + 0.03n$ models the cost of one call on the TV plan

$C_2 = 0.20 + 0.05n$ models the cost of one call on your current plan

This is the simplest case of substitution, where both equations are already in function form. To find out when the costs of the two plans are equal, that is, when $C_1 = C_2$, we can substitute C_2 for C_1.

Given the equation for C_1: $C_1 = 0.39 + 0.03n$

substitute C_2 for C_1: $0.20 + 0.05n = 0.39 + 0.03n$

solve for n: $0.05n - 0.03n = 0.39 - 0.20$

$$0.02n = 0.19$$

$$n = 0.19/0.02 = 9.5 \text{ minutes}$$

When $n = 9.5$, the costs of the two plans are equal. To find the cost, we can substitute $n = 9.5$ into the equation for C_1 or C_2. Using C_1, we have

$$\begin{aligned} C_1 &= 0.39 + 0.03(9.5) \\ &= 0.39 + 0.285 \\ &= \$0.675 \end{aligned}$$

So for a call lasting 9.5 minutes, the cost on either plan is approximately 68 cents.

Double-Checking the Solution. We can check our answer by substituting $n = 9.5$ into C_2.

$$\begin{aligned} C_2 &= 0.20 + 0.05(9.5) \\ &= 0.20 + 0.475 \\ &= \$0.675 \end{aligned}$$

This confirms our original calculation.

EXAMPLE 2 **When only one equation is in function form**

a. Find the point (if any) where the graphs of the following two linear equations intersect:

$$6x + 7y = 25 \qquad (1)$$

$$y = 15 + 2x \qquad (2)$$

b. Graph the two equations on the same grid, labeling their intersection point.

SOLUTION

a. In Equation (1) substitute the expression for y from Equation (2):

$$6x + 7(15 + 2x) = 25$$

Simplify

$$6x + 105 + 14x = 25$$
$$20x = -80$$
$$x = -4$$

We can use one of the original equations to find the value for y when $x = -4$. Using Equation (2),

$$y = 15 + 2x \quad (2)$$

Substitute -4 for x $\quad y = 15 + 2(-4)$

multiply $\quad y = 15 - 8$

$$y = 7$$

Try double-checking your answer in Equation (1).

b. Figure 3.9 shows a graph of the two equations and the intersection point at $(-4, 7)$.

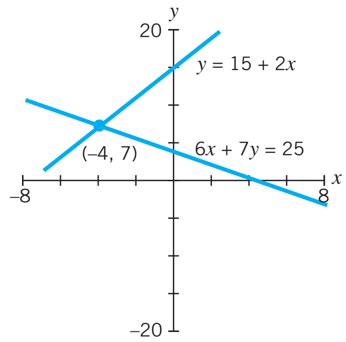


Figure 3.9 Graphs of $6x + 7y = 25$ and $y = 15 + 2x$.

Algebra Aerobics 3.2a

1. Solve for the indicated variable.

a. $2x + y = 7$ for y

b. $3x + 5y = 6$ for y

c. $x - 2y = -1$ for x

2. Determine the number of solutions without solving the system. Justify your answer.

a. $y = 3x - 5$, $y = 3x + 8$

b. $y = 2x - 4$, $y = 3x - 4$

3. Solve the following systems of equations using the substitution method.

a. $y = x + 4$, $y = -2x + 7$

b. $y = -1700 + 2100x$, $y = 4700 + 1300x$

c. $F = C$, $F = 32 + \frac{9}{5}C$

[Part (c) was a question on the TV program *Who Wants to Be a Millionaire?*]

4. Solve the following systems of equations using the substitution method.

a. $y = x + 3$, $5y - 2x = 21$

b. $z = 3w + 1$, $9w + 4z = 11$

c. $x = 2y - 5$, $4y - 3x = 9$

d. $r - 2s = 5$, $3r - 10s = 13$

Elimination method

Another method, called *elimination,* can be useful when neither equation is in function form. The strategy is to modify the equations (through multiplication or rearrangement) so that adding (or subtracting) the modified equations eliminates one variable.

EXAMPLE 3 Assume you have \$10,000 to invest in an "up market" when the economy is booming. You want to split your investment between conservative bonds and riskier stocks. The bonds will stay fixed in value but return a guaranteed 7% per year in dividends. The stocks pay no dividends, but your return is from the increase in stock value, predicted to be 14% per year. Overall you want a 12% or \$1200 return on your \$10,000 at the end of one year.

a. How much should you invest in bonds and how much in stocks?

b. What if the economy has a drastic downturn, as it did between 1999 and 2002? Assuming you split your investment as recommended in part (a), what will your return be after one year if your stock value decreased by 10%?

SOLUTION **a.** If B = \$ invested in bonds and S = \$ invested in stocks, then

$$B + S = \$10{,}000 \qquad (1)$$

The expected return on your investments after one year is

$$(7\% \text{ of } B) + (14\% \text{ of } S) = \$1200$$

or

$$0.07B + 0.14S = \$1200 \qquad (2)$$

We can solve the system by eliminating one variable, in this case B, from both equations. Given the two equations

$$B + S = \$10{,}000 \qquad (1)$$

$$0.07B + 0.14S = \$1200 \qquad (2)$$

If we multiply both sides of Equation (1) by 0.07 we get an equivalent Equation (1)*, which has the same coefficient for B as Equation (2). We do this so that we can subtract the equations and eliminate B.

Given	$0.07B + 0.07S = \$700$	(1)*
subtract Equation (2)	$-(0.07B + 0.14S = \$1200)$	(2)
to eliminate B	$-0.07S = -\$500$	

Dividing both sides by -0.07, we have $S \approx \$7143$.

So to achieve your goal of a \$1200 return on \$10,000 you would need to invest \$7143 in stocks and \$10,000 − \$7143 = \$2857 in bonds.

b. If the economy turns sour at the end of the year and the value of your stock drops 10%, then

$$\begin{aligned}\text{your return} &= (7\% \text{ of } \$2857 \text{ from bonds}) - (10\% \text{ of } \$7143 \text{ from stocks}) \\ &\approx \$200 - \$714 \\ &= -\$514\end{aligned}$$

So you would lose over \$500 on your \$10,000 investment that year.

How can you tell if your system has no or infinitely many intersection points?

As we remarked at the beginning of this section, two parallel lines will never intersect and duplicate lines will intersect everywhere, creating infinitely many intersection points. How can you tell whether or not your system of equations falls into either category?

EXAMPLE 4 **A system with no solution: Parallel lines**

a. Solve the following system of two linear equations:

$$y = 20{,}000 + 1500x \qquad (1)$$

$$2y - 3000x = 50{,}000 \qquad (2)$$

b. Graph your results.

SOLUTION

a. If we assume the two lines intersect, then the y values for Equations (1) and (2) are the same at some point. We can try to find this value for y by substituting the expression for y from Equation (1) into Equation (2):

$$2(20{,}000 + 1500x) - 3000x = 50{,}000$$

Simplify $$40{,}000 + 3000x - 3000x = 50{,}000$$

false statement $$40{,}000 = 50{,}000 \text{ (???)}$$

What could this possibly mean? Where did we go wrong? If we return to the original set of equations and solve Equation (2) for y in terms of x, we can see why there is no solution for this system of equations.

$$2y - 3000x = 50{,}000 \qquad (2)$$

Add $3000x$ to both sides $$2y = 50{,}000 + 3000x$$

divide by 2 $$y = 25{,}000 + 1500x \qquad (2)^*$$

Equations (1) and (2)* (rewritten form of the original Equation (2)) have the same slope of 1500 but different y-intercepts.

$$y = 20{,}000 + 1500x \qquad (1)$$
$$y = 25{,}000 + 1500x \qquad (2)^*$$

Written in this form, we can see that we have two parallel lines, so the lines never intersect (see Figure 3.10). Our initial premise, that the two lines intersected and thus the two y-values were equal at some point, was incorrect.

b. Figure 3.10 shows the graphs of the two lines.

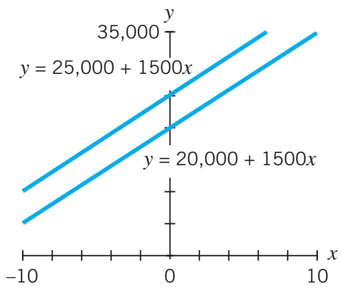


Figure 3.10 Graphs of $y = 25{,}000 + 1500x$ and $y = 20{,}000 + 1500x$.

EXAMPLE 5 **A system with infinitely many solutions: Equivalent equations**

Solve the following system:

$$45x = -y + 33 \qquad (1)$$
$$2y + 90x = 66 \qquad (2)$$

SOLUTION

As always, there are multiple ways of solving the system. One strategy is to put both equations in function form:

Solve Equation (1) for y	$45x = -y + 33$	(1)
add y to both sides	$y + 45x = 33$	
add $-45x$ to both sides	$y = -45x + 33$	
Solve Equation (2) for y	$2y + 90x = 66$	(2)
add $-90x$ to both sides	$2y = -90x + 66$	
divide by 2	$y = -45x + 33$	

The two original equations really represent the same line, $y = -45x + 33$, so any of the infinitely many points on the line is a solution to the system (Figure 3.11).

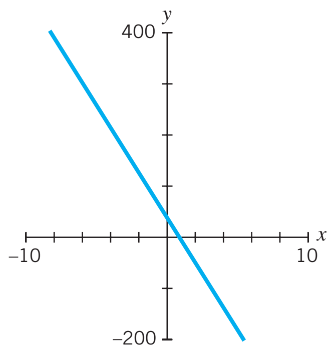


Figure 3.11 Graph of $y = -45x + 33$.

Linear Systems in Economics: Supply and Demand

Economists study the relationship between the price p of an item and the quantity q, the number of items produced. Economists traditionally place quantity q on the horizontal axis and price p on the vertical axis.[2] From the consumer's point of view, an increase in price decreases the quantity demanded. So the consumer's *demand curve* would slope downward (see Figure 3.12).

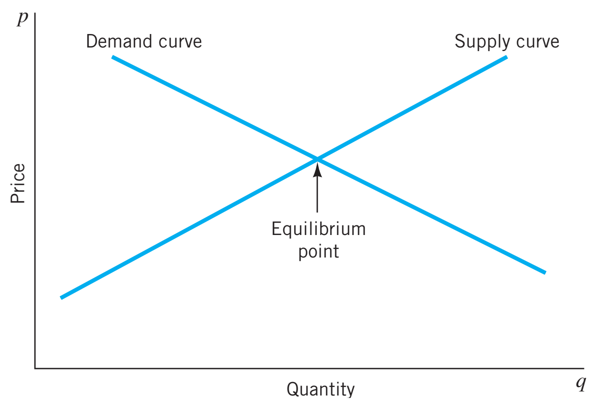


Figure 3.12 Supply and demand curves.

From the manufacturer's (or supplier's) point of view, an increase in price is linked with an increase in the quantity they are willing to supply. So the manufacturer's *supply curve* slopes upward. The intersection point between the demand and supply curves is called the *equilibrium point*. At this point supply equals demand, so both suppliers and consumers are happy with the quantity produced and the price charged.

[2]This can be confusing since we usually think of quantity as a function of price.

EXAMPLE 6 **Milk supply and demand curves**

Loren Tauer[3] studied the U.S. supply and demand curves for milk. If q = billions of pounds of milk and p = dollars per cwt (where 1 cwt = 100 lb), he estimated that the demand function for milk is $p = 55.9867 - 0.2882q$ and the supply function is $p = 0.0865q$.

a. Find the equilibrium point.

b. What will happen if the price of milk is higher than the equilibrium price?

SOLUTION **a.** The equilibrium point occurs where

	supply = demand
substituting for p	$0.0865q = 55.9867 - 0.2882q$
solving for q	$0.3747q = 55.9867$
we have	$q \approx 149.42$ billions of pounds of milk

If we use the supply function to find p, we have

the supply function	$p = 0.0865q$
when $q = 149.42$	$p = 0.0865 \cdot 149.42 \approx \$12.92$ per cwt

If we use the demand function to find p we would also get

the demand function	$p = 55.9867 - 0.2882q$
when $q = 149.42$	$p = 55.9867 - (0.2882 \cdot 149.42) \approx \$12.92$

So the equilibrium point is (149.42, \$12.92); that is, when the price is \$12.92 per cwt, manufacturers are willing to produce and consumers are willing to buy 149.42 billion pounds of milk.

b. If the price of milk rises above \$12.92 per cwt to, say, p_1, then, as shown in Figure 3.13, consumers would buy less than 149.42 billion pounds (amount q_1) while manufacturers would be willing to produce more than 149.42 billion pounds (amount q_2).

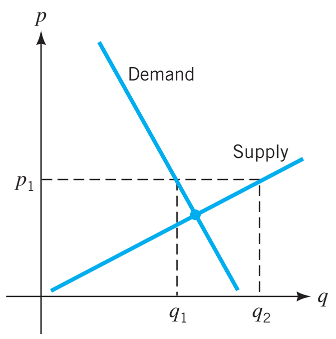


Figure 3.13 At price p_1, consumers would buy q_1 billion pounds of milk, but producers would manufacture q_2 billion pounds.

So there would be a surplus of $q_2 - q_1$ billion pounds of milk, which would drive down the price of milk toward the equilibrium point.

[3]Loren W. Tauer, "The value of segmenting the milk market into bST-produced and non-bST produced milk," *Agribusiness 10*(1): 3–12 as quoted in Edmond C. Tomastik, *Calculus: Applications and Technology,* 3rd ed. (Belmont, CA: Thomson Brooks/Cole, 2004).

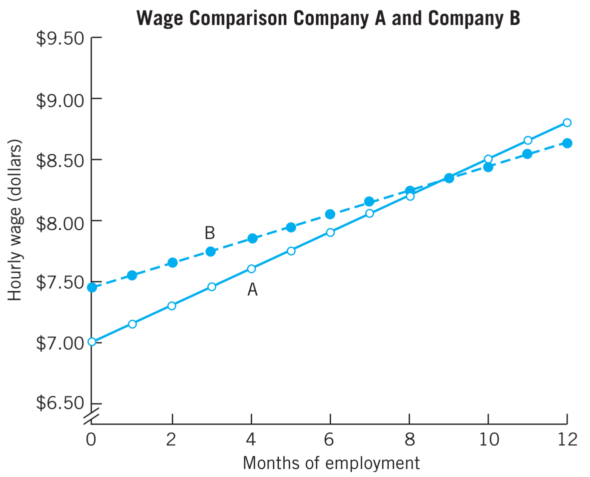
Algebra Aerobics 3.2b

1. Solve each system of equations using the method you think is most efficient.
   - **a.** $2y - 5x = -1$, $3y + 5x = 11$
   - **b.** $3x + 2y = 16$, $2x - 3y = -11$
   - **c.** $t = 3r - 4$, $4t + 6 = 7r$
   - **d.** $z = 2000 + 0.4(x - 10{,}000)$, $z = 800 + 0.2x$
2. Solve each system of equations. If technology is available, check your answers by graphing each system.
   - **a.** $y = 2x + 4$, $y = -x + 4$
   - **b.** $5y + 30x = 20$, $y = -6x + 4$
   - **c.** $y = 1500 + 350x$, $2y = 700x + 3500$
3. Construct a system of two linear equations in two unknowns that has no solution.
4. Determine the number of solutions without solving the system. Explain your reasoning.
   - **a.** $2x + 5y = 7$, $3x - 8y = -1$
   - **b.** $3x + y = 6$, $6x + 2y = 5$
   - **c.** $2x + 3y = 1$, $4x + 6y = 2$
   - **d.** $3x + y = 8$, $3x + 2y = 8$
5. Solve each of the following systems of equations.
   - **a.** $y = x + 4$, $\frac{x}{2} + \frac{y}{3} = 3$
   - **b.** $0.5x + 0.7y = 10$, $30x + 50y = 1000$
6. A small paint dealer has determined that the demand function for interior white paint is $4p + 3q = 240$, where p = dollars/gallon of paint and q = number of gallons.
   - **a.** Find the demand for white paint when the price is $39.00 per gallon.
   - **b.** If consumer demand is for 20 gallons of paint, what price would these consumers be willing to pay?
   - **c.** Sketch the demand function, placing q on the horizontal axis and p on the vertical axis.
   - **d.** The supply function for interior white paint is $p = 0.85q$. Sketch the supply curve on the same graph as the demand curve.
   - **e.** Find the equilibrium point and interpret its meaning.
   - **f.** At a price of $39.00 per gallon of paint, is there a surplus or shortage of supply?
7. Fill in the missing coefficient of x such that there will be an infinite number of solutions to the system of equations:

$$y = 2x + 4$$
$$??\,x = -2y + 8$$

Exercises for Section 3.2

1. Two companies offer starting employees incentives to stay with the company after they are trained for their new jobs. Company A offers an initial hourly wage of $7.00, then increases the hourly wage by $0.15 per month. Company B offers an initial hourly wage of $7.45, then increases the hourly wage by $0.10 per month.



   - **a.** Examine the accompanying graph. After how many months does it appear that the hourly wage will be the same for both companies?
   - **b.** Estimate that hourly wage.
   - **c.** Form two linear functions for the hourly wages in dollars of $W_A(m)$ for company A and $W_B(m)$ for company B after m months of employment.
   - **d.** Does your estimated solution from part (a) satisfy both equations? If not, find the exact solution.
   - **e.** What is the exact hourly wage when the two companies offer the same wage?
   - **f.** Describe the circumstances under which you would rather work for company A. For company B.

2. In the text the following cost equations were given for gas and solar heating:

$$C_{gas} = 12{,}000 + 700n$$
$$C_{solar} = 30{,}000 + 150n$$

where n represents the number of years since installation and the cost represents the total accumulated costs up to and including year n.

   - **a.** Sketch the graph of this system of equations.
   - **b.** What do the coefficients 700 and 150 represent on the graph, and what do they represent in terms of heating costs?
   - **c.** What do the constant terms 12,000 and 30,000 represent on the graph? What does the difference between 12,000 and 30,000 say about the costs of gas vs. solar heating?

d. Label the point on the graph where the total accumulated gas and solar heating costs are equal. Make a visual estimate of the coordinates, and interpret what the coordinates mean in terms of heating costs.

e. Use the equations to find a better estimate for the intersection point. To simplify the computations, you may want to round values to two decimal places. Show your work.

f. When is the total cost of solar heating more expensive than gas? When is the total cost of gas heating more expensive than solar?

3. Answer the questions in Exercise 2 (with suitable changes in wording) for the following cost equations for electric and solar heating:

$$C_{\text{electric}} = 5000 + 1100n$$
$$C_{\text{solar}} = 30{,}000 + 150n$$

4. Consider the following job offers. At Acme Corporation, you are offered a starting salary of $20,000 per year, with raises of $2500 annually. At Boca Corporation, you are offered $25,000 to start and raises of $2000 annually.

a. Find an equation to represent your salary, $S_A(n)$, after n years of employment with Acme.

b. Find an equation to represent your salary, $S_B(n)$, after n years of employment with Boca.

c. Create a table of values showing your salary at each of these corporations for integer values of n up to 12 years.

d. In what year of employment would the two corporations pay you the same salary?

5. a. Solve the following system algebraically:

$$S = 20{,}000 + 2500n$$
$$S = 25{,}000 + 2000n$$

b. Graph the system in part (a) and use the graph to estimate the solution to the system. Check your estimate with your answer in part (a).

6. Predict the number of solutions to each of the following systems. Give reasons for your answer. You don't need to find any actual solutions.

a. $y = 20{,}000 + 700x$ $\quad y = 15{,}000 + 800x$

b. $y = 20{,}000 + 700x$ $\quad y = 15{,}000 + 700x$

c. $y = 20{,}000 + 700x$ $\quad y = 20{,}000 + 800x$

7. For each system:

a. Indicate whether the substitution or elimination method might be easier for finding a solution to the system of equations.

i. $y = \frac{1}{3}x + 6$, $y = \frac{1}{3}x - 4$

ii. $2x - y = 5$, $5x + 2y = 8$

iii. $3x + 2y = 2$, $x = 7y - 30$

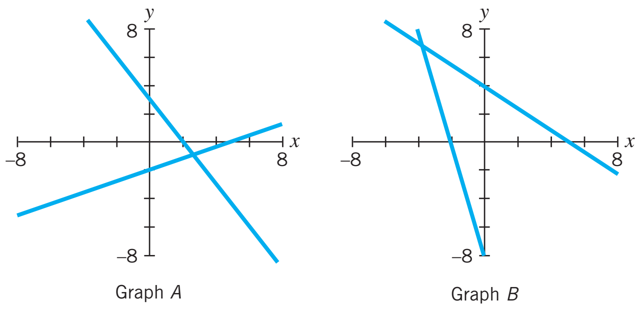
iv. $y = 2x - 3$, $4y - 8x = -12$

v. $-3x + y = 4$, $-3x + y = -2$

vi. $3y = 9$, $x + 2y = 11$

b. Using your chosen method, find the solution(s), if any, of each system.

8. For each graph, construct the equations for each of the two lines in the system, and then solve the system using your equations.



9. a. Solve the following system algebraically:

$$x + 3y = 6$$
$$5x + 3y = -6$$

b. Graph the system of equations in part (a) and estimate the solution to the system. Check your estimate with your answers in part (a).

10. Calculate the solution(s), if any, to each of the following systems of equations. Use any method you like.

a. $y = -1 - 2x$, $y = 13 - 2x$

b. $t = -3 + 4w$, $-12w + 3t + 9 = 0$

c. $y = 2200x - 700$, $y = 1300x + 4700$

d. $3x = 5y$, $4y - 3x = -3$

In some of the following examples you may wish to round off your answers:

e. $y = 2200x - 1800$, $y = 1300x - 4700$

f. $y = 4.2 - 1.62x$, $1.48x - 2y + 4.36 = 0$

g. $4r + 5s = 10$, $2r - 4s = -3$

h. $2x + 3y = 13$, $3x + 5y = 21$

i. $xy = 1$, $x^2y + 3x = 2$ (A nonlinear system! *Hint:* Solve $xy = 1$ for y and use substitution.)

11. Assume you have $2000 to invest for 1 year. You can make a safe investment that yields 4% interest a year or a risky investment that yields 8% a year. If you want to combine safe and risky investments to make $100 a year, how much of the $2000 should you invest at the 4% interest? How much at the 8% interest? (*Hint:* Set up a system of two equations in two variables, where one equation represents the total amount of money you have to invest and the other equation represents the total amount of money you want to make on your investments.)

12. Two investments in high-technology companies total $1000. If one investment earns 10% annual interest and the other earns 20%, find the amount of each investment if the total interest earned is $140 for the year (clearly in dot com days).

13. Solve the following systems:

a. $\frac{x}{3} + \frac{y}{2} = 1$, $x - y = \frac{4}{3}$

b. $\frac{x}{4} + y = 9$, $y = \frac{x}{2}$

14. For each of the following systems of equations, describe the graph of the system and determine if there is no solution, an infinite number of solutions, or exactly one solution.

a. $2x + 5y = -10$
$y = -0.4x - 2$

b. $3x + 4y = 5$
$3x - 2y = 5$

c. $2x - y = 5$
$6x - 3y = 4$

15. If $y = b + mx$, solve for values for m and b by constructing two linear equations in m and b for the given sets of ordered pairs.

a. When $x = 2$, $y = -2$ and when $x = -3$, $y = 13$.

b. When $x = 10$, $y = 38$ and when $x = 1.5$, $y = -4.5$.

16. The following are formulas predicting future raises for four different groups of union employees. N represents the number of years from the start date of all the contracts. Each equation represents the salary that will be earned after N years.

Group A: Salary = 30,000 + 1500N
Group B: Salary = 30,000 + 1800N
Group C: Salary = 27,000 + 1500N
Group D: Salary = 21,000 + 2100N

a. Will group A ever earn more per year than group B? Explain.

b. Will group C ever catch up to group A? Explain.

c. Which group will be making the highest yearly salary in 5 years? How much will that salary be?

d. Will group D ever catch up to group C? If so, after how many years and at what salary?

e. How much total salary would an individual in each group have earned 3 years after the contract?

17. A small T-shirt company created the following cost and revenue equations for a line of T-shirts, where cost C is in dollars for producing x units and revenue R is in dollars from selling x units:

$$C = 12.5x + 360 \quad \text{and} \quad R = 15.5x$$

a. What does 12.5 represent?

b. What does 15.5 represent?

c. Find the equilibrium point.

d. What is the cost of producing x units at the equilibrium point? The revenue at the equilibrium point?

18. A large wholesale nursery sells shrubs to retail stores. The cost $C(x)$ and revenue $R(x)$ equations (in dollars) for x shrubs are

$$C(x) = 15x + 12{,}000 \quad \text{and} \quad R(x) = 18x$$

a. Find the equilibrium point.

b. Explain the meaning of the coordinates for the equilibrium point.

19. The supply and demand equations for a particular bicycle model relate price per bicycle, p (in dollars) and q, the number of units (in thousands). The two equations are

$p = 250 + 40q$ Supply
$p = 510 - 25q$ Demand

a. Sketch both equations on the same graph. On your graph identify the supply equation and the demand equation.

b. Find the equilibrium point and interpret its meaning.

20. For a certain model of DVD-VCR combo player, the following supply and demand equations relate price per player, p (in dollars) and number of players, q (in thousands).

$p = 80 + 2q$ Supply
$p = 185 - 5q$ Demand

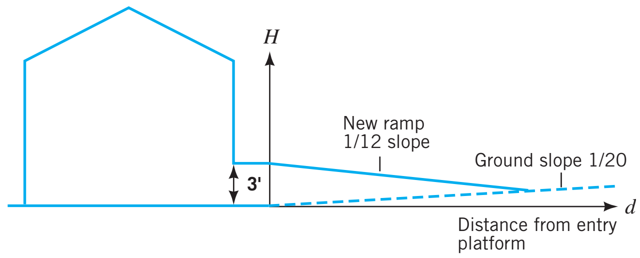
a. Find the point of equilibrium.

b. Interpret this result.

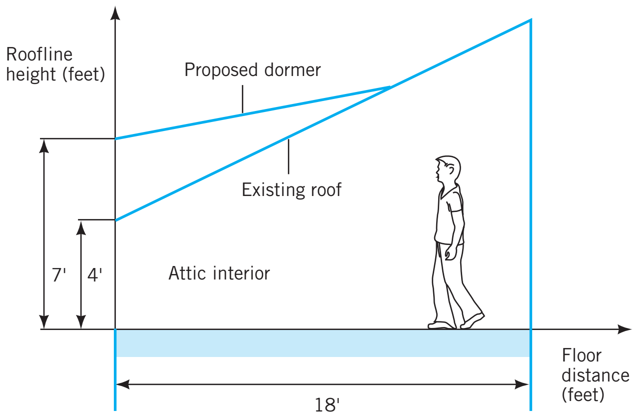
21. Explain what is meant by "two equivalent equations." Give an example of two equivalent equations.

22. Construct a problem not found in this text that involves supply and demand where the situation can be modeled with a system of linear equations. Solve your system and verify your results by graphing the system.

23. A restaurant is located on ground that slopes up 1 foot for every 20 horizontal feet. The restaurant is required to build a wheelchair ramp starting from an entry platform that is 3 feet above ground. Current regulations require a wheelchair ramp to rise up 1 foot for every 12 horizontal feet, (See accompanying figure where H = height in feet, d = distance from entry in feet, and the origin is where the H-axis meets the ground.) Where will the new ramp intersect the ground?



24. A house attic as shown has a roofline with a slope of 5″ up for every 12″ of horizontal run; this is a slope of 5/12. Since the roofline starts at 4′ above the floor, it is not possible to stand in much of the attic space. The owner wants to add a dormer with a 2/12 slope, starting at a 7′ height, to increase the usable space.



a. With height and floor distance coordinates as shown on the house sketch, find a formula for the original roofline, using R for roof height and F for floor distance from the wall. Also find a formula for the dormer roofline, using D for dormer roof height and F for floor distance from the wall.

b. Make a graph showing the R and D roof height lines for floor distances F from 0′ to 18′.

c. At what height and floor distance will the dormer roofline intersect the existing roofline?

d. How far do you need to measure along the horizontal floor distance to give 6′6″ of head room in the original roofline? What percent of the horizontal floor distance of 18 ft allows less than 6′6″ of head room?

25. Solve the following system of three equations in three variables, using the steps outlined below:

$$2x + 3y - z = 11 \qquad (1)$$
$$5x - 2y + 3z = 35 \qquad (2)$$
$$x - 5y + 4z = 18 \qquad (3)$$

a. Use Equations (1) and (2) to eliminate one variable, creating a new Equation (4) in two variables.

b. Use Equations (1) and (3) to eliminate the same variable as in part (a). You should end up with a new Equation (5) that has the same variables as Equation (4).

c. Equations (4) and (5) represent a system of two equations in two variables. Solve the system.

d. Find the corresponding value for the variable eliminated in part (a).

e. Check your work by making sure your solution works in all three original equations.

26. Using the strategy described in Exercise 25, solve the following system:

$$2a - 3b + c = 4.5 \qquad (1)$$
$$a - 2b + 2c = 0 \qquad (2)$$
$$3a - b + 2c = 0.5 \qquad (3)$$

27. a. Construct a system of linear equations in two variables that has no solution.

b. Construct a system of linear equations in two variables that has exactly one solution.

c. Solve the system of equations you constructed in part (b) by using two different algebraic strategies and by graphing the system of equations. Do your answers all agree?

28. Nenuphar wants to invest a total of $30,000 into two savings accounts, one paying 6% per year in interest and the other paying 9% per year in interest (a more risky investment). If after 1 year she wants the total interest from both accounts to be $2100, how much should she invest in each account?

29. When will the following system of equations have no solution? Justify your answer.

$$y = m_1x + b_1$$
$$y = m_2x + b_2$$

30. While totally solar energy–powered home energy systems are quite expensive to install, passive solar systems are much more economical. Many passive solar features can be incorporated at the time of construction with a small additional initial cost to a conventional system. These features enable energy costs to be one-half to one-third of the costs in conventional homes.

Below is the cost analysis from the case study Esperanza del Sol.[4]

Cost of installation of a conventional system:	$10,000
Additional cost to install passive solar features:	$150
Annual energy costs for conventional system:	$740
Annual energy costs of hybrid system with additional passive solar features:	$540

a. Write the cost equation for the conventional system.

b. Write the cost equation for the passive hybrid solar system.

c. When would the total cost of the passive hybrid solar system be the same as the conventional system?

d. After 5 years, what would be the total energy cost of the passive hybrid solar system? What would be the total cost of the conventional system?

31. a. Construct a system of linear equations where both of the following conditions are met:

The coordinates of the point of intersection are (2, 5).

One of the lines has a slope of -4 and the other line has a slope of 3.5.

b. Graph the system of equations you found in part (a). Verify that the coordinates of the point of intersection are the same as the coordinates specified in part (a).

32. A husband drives a heavily loaded truck that can go only 55 mph on a 650-mile turnpike trip. His wife leaves on the same trip 2 hours later in the family car averaging 70 mph. Recall that distance traveled = speed · time traveled.

a. Derive an expression for the distance, D_h, the husband travels in t hours since he started.

b. How many hours has the wife been traveling if the husband has traveled t hours ($t \geq 2$)?

c. Derive an expression for the distance, D_w, that the wife will have traveled while the husband has been traveling for t hours ($t \geq 2$).

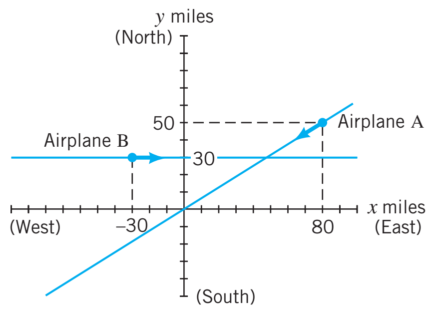
d. Graph distance vs. time for husband and wife on the same axes.

e. Calculate when and where the wife will overtake the husband.

f. Suppose the husband and wife wanted to arrive at a restaurant at the same time, and the restaurant is 325 miles from home. How much later should she leave, assuming he still travels at 55 mph and she at 70 mph?

[4]Adapted from *Buildings for a Sustainable America: Case Studies,* American Solar Energy Society, Boulder, CO.

33. Life-and-death travel problems are dealt with by air traffic computers and controllers who are trying to prevent collisions of planes traveling at various speeds in three-dimensional space. To get a taste of what is involved, consider this situation: Airplanes A and B are traveling at the same altitude on the paths shown on the position plot.

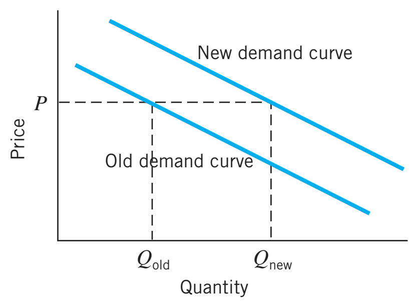


a. Construct two equations that describe the positions of airplanes A and B in x- and y-coordinates. Use y_A and y_B to denote the north/south coordinates of airplanes A and B, respectively.

b. What are the coordinates of the intersection of the airplanes' paths?

c. Airplane A travels at 2 miles/minute and airplane B travels at 6 miles/minute. Clearly, their paths will intersect if they each continue on the same course, but will they arrive at the intersection point at the same time? How far does plane A have to travel to the intersection point? How many minutes will it take to get there? How far does plane B have to travel to the intersection point? How many minutes will it take to get there? (*Hint:* Recall the rule of Pythagoras for finding the hypotenuse of a right triangle: $a^2 + b^2 = c^2$, where c is the hypotenuse and a and b are the other sides.) Is this a safe situation?

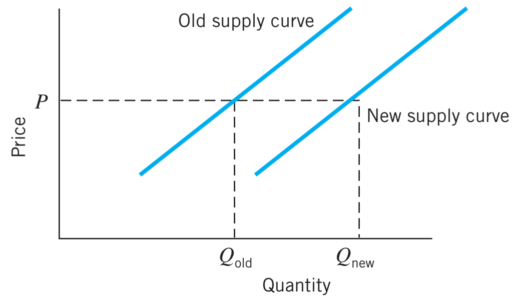
34. a. Examine the accompanying figure, where the demand curve has been moved to the right. Does the new demand curve represent an increase or decrease in demand. Why? (*Hint:* Pick an arbitrary price, and see if consumers would want to buy more or less at that price.)



b. Sketch in a possible supply curve identifying the old and new equilibrium points. What does the shift from the old equilibrium point to the new mean for both consumers and suppliers?

35. a. Examine the accompanying figure. Does the new supply curve represent an increase or decrease in supply. Why? (Again, try picking an arbitrary price and see if at that price, the supplier would want to increase or decrease production.)

b. Sketch in a possible demand curve, and label the old and new equilibrium points. What does the shift from the old to the new equilibrium point mean for both consumers and suppliers?



36. In studying populations (human or otherwise), the two primary factors affecting population size are the birth rate and the death rate. There is abundant evidence that, other things being equal, as the population density increases, the birth rate tends to decrease and the death rate tends to increase.[5]

a. Generate a rough sketch showing birth rate as a function of population density. Note that the units for population density on the horizontal axis are the number of individuals for a given area. The units on the vertical axis represent a rate, such as the number of individuals per 1000 people. Now add to your graph a rough sketch of the relationship between death rate and population density. In both cases assume the relationship is linear.

b. At the intersection point of the two lines the growth of the population is zero. Why? (*Note:* We are ignoring all other factors, such as immigration.)

The intersection point is called the *equilibrium point*. At this point the population is said to have stabilized, and the size of the population that corresponds to this point is called the *equilibrium number*.

c. What happens to the equilibrium point if the overall death rate decreases, that is, at each value for population density the death rate is lower? Sketch a graph showing the birth rate and both the original and the changed death rates. Label the graph carefully. Describe the shift in the equilibrium point.

d. What happens to the equilibrium point if the overall death rate increases? Analyze as in part (c).

37. Use the information in Exercise 36 to answer the following questions:

a. What if the overall birth rate increases (that is, if at each population density level the birth rate is higher)? Sketch a

[5]See E. O. Wilson and W. H. Bossert, *A Primer of Population Biology.* Sunderland, MA: Sinauer Associates, 1971, p. 104.

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graph showing the death rate and both the original and the changed birth rates. Be sure to label the graph carefully. Describe the shift in the equilibrium point.

b. What happens if the overall birth rate decreases? Analyze as in part (a).

38. In Section 2.7, Exercise 19, you read of a math professor who purchased her condominium in Cambridge, MA, for \$70,000 in 1977. Its assessed value has climbed at a steady rate so that it was worth \$850,000 as of 2007. Alas, one of her colleagues has not been so fortunate. He bought a house in that same year for \$160,000. Not long after his family moved in, rumors began to circulate that the housing complex had been built on the site of a former toxic dump. Although never substantiated, the rumors adversely affected the value of his home, which has steadily decreased in value over the years and in 2007 was worth a meager \$40,000. In what year would the two homes have been assessed at the same value?

3.3 *Reading between the Lines: Linear Inequalities*

Above and Below the Line

Sometimes we are concerned with values that lie above or below a line, or that lie between two lines. To describe these regions we need some mathematical conventions.

Terminology for describing regions

The two linear functions $y_1 = 1 + 3x$ and $y_2 = 5 - x$ are graphed in Figure 3.14. How would you describe the various striped regions?

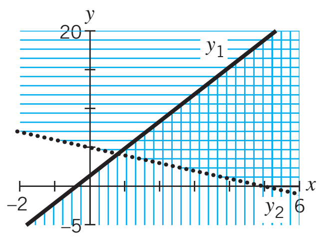


Figure 3.14 Regions bounded by the lines $y_1 = 1 + 3x$ and $y_2 = 5 - x$.

A solid line indicates that the points on the line are included in the area. A dotted line indicates that the points on the line are *not* included in the area. So the vertical-striped region below the solid line y_1 can be described as all points (x, y) that satisfy the inequality

$$y \leq y_1 \qquad \text{Condition (1)}$$

$$\text{or equivalently} \qquad y \leq 1 + 3x$$

The equation $y_1 = 1 + 3x$ is a *boundary line* that is included in the region.

The horizontally striped region above the dotted line y_2 can be described as all points (x, y) that satisfy the inequality

$$y_2 < y \qquad \text{Condition (2)}$$

$$\text{or} \qquad 5 - x < y$$

So $y_2 = 5 - x$ is a boundary line that is not included in the region.

In the cross-hatched region, the y values must satisfy both conditions (1) and (2); that is, we must have

$$y_2 < y \leq y_1$$

$$\text{or equivalently} \qquad 5 - x < y \leq 1 + 3x$$

This is called a *compound inequality*. We could describe this inequality by saying that y is greater than $5 - x$ and less than or equal to $1 + 3x$. So the region can be described as all points (x, y) that satisfy the compound inequality

$$5 - x < y \leq 1 + 3x$$

Manipulating Inequalities

Recall that any term may be added to or subtracted from both sides of an inequality without changing the direction of the inequality. The same holds for multiplying or dividing by a positive number. However, multiplying or dividing by a negative number reverses the inequality. For example,

Given the previous inequality	$5 - x < y$
if we wanted to solve for x we could subtract 5 from both sides	$-x < y - 5$
then multiply both sides by -1	$x > -y + 5$

Note that subtracting 5 (or equivalently adding -5) preserved the inequality, but multiplying by -1 reversed the inequality. So "<" in the first two inequalities became ">" in the last inequality.

EXAMPLE 1

The U.S. Army recommends that sleeping bags, which will be used in temperatures between $-40°$ and $+40°$ Fahrenheit, have a thickness of 2.5 inches minus 0.025 times the number of degrees Fahrenheit.

a. Construct a linear equation that models the recommended sleeping bag thickness as a function of degrees Fahrenheit. Identify the variables and domain of your function.

b. Graph the model (displaying it over its full domain) and shade in the area where the sleeping bag is not warm enough for the given temperature.

c. What symbolic expressions would describe the shaded area?

d. If a manufacturer submitted a sleeping bag to the Army with a thickness of 2.75 inches, would it be suitable for

i. $-20°$ Fahrenheit? **ii.** $0°$ Fahrenheit? **iii.** $+20°$ Fahrenheit?

SOLUTION

a. If we let F = number of degrees Fahrenheit and T = thickness of the sleeping bag in inches, then

$$T = 2.5 - 0.025F \qquad \text{where the domain is } -40° \leq F \leq 40°$$

b. See Figure 3.15.

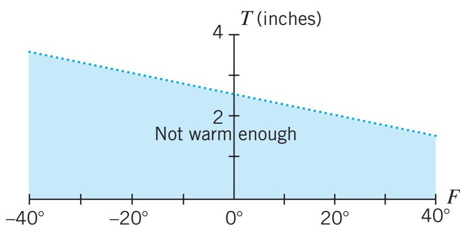


Figure 3.15 *T*, sleeping bag thickness as a function of *F*, degrees Fahrenheit.

When the thickness of the sleeping bag, *T*, is less than the recommended thickness, then the bag will not be warm enough.

c. The shaded region can be described as

$$0 < \text{thickness of sleeping bag} < \text{recommended thickness}$$
$$0 < T < 2.5 - 0.025F$$

where $-40° \le F \le 40°$. We could rephrase this to say that the region is bounded by four lines: $T = 0$, $T = 2.5 - 0.025F$, $F = -40°$, and $F = 40°$. Note that the dotted line indicates that the line itself is not included.

d. If the sleeping bag thickness T is 2.75, we can find the corresponding recommended temperature by solving our equation for F.

Substitute 2.75 for T	$2.75 = 2.5 - 0.025F$
simplify to get	$0.25 = -0.025F$
or	$F = -10°$

So the sleeping bag would not be thick enough for $-20°$F, since the point $(-20°, 2.75'')$ lies in the shaded area below the Army's recommended values. It would be more than thick enough for 0°F or 20°F since the points $(0°, 2.75'')$ and $(20°, 2.75'')$ both lie above the shaded area.

Reading between the Lines

EXAMPLE 2 The U.S. Department of Agriculture recommends healthy weight zones for adults based on their height. For men between 60 and 84 inches tall, the recommended lowest weight, W_{lo} (in lb), is

$$W_{lo} = 105 + 4.0H$$

and the recommended highest weight, W_{hi} (in lb), is

$$W_{hi} = 125.4 + 4.6H$$

where H is the number of inches above 60 inches (5 feet).

a. Graph and label the two boundary equations and indicate the underweight, healthy, and overweight zones.

b. Give a mathematical description of the healthy weight zone for men.

c. Two men each weigh 180 lb. One is 5′ 10″ tall and the other 6′ 1″ tall. Is either within the healthy weight zone?

d. A 5′ 11″ man weighs 135 lb. If he gains 2 lb a week, how long will it take him to reach the healthy range?

SOLUTION **a.** See Figure 3.16.

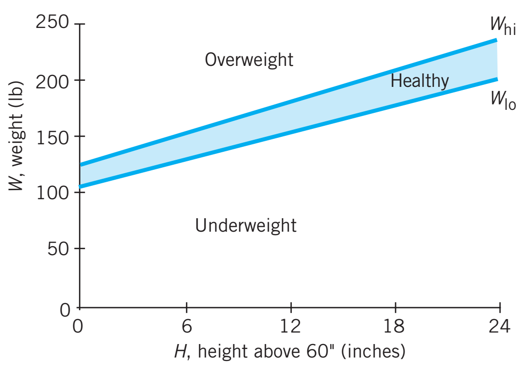


Figure 3.16 Graph of men's healthy weight zone between recommended low (W_{lo}) and high (W_{hi}) weights.

b. Assuming that men's heights generally run between 5 feet (60″) and 7 feet (84″), then H (the height in inches above 60 inches) is bounded by

$$0'' \le H \le 24''$$

The recommendations say that a man's weight W (in lb) should be more than or equal to (W_{lo}) and should be less than or equal to W_{hi}. So we have the computed inequality

$$W_{lo} \le W \le W_{hi}$$

or

$$105 + 4.0H \le W \le 125.4 + 4.6H$$

c. For a man who is 5′ 10″ (or 70″) tall, $H = 10''$ and his maximum recommended weight, W_{hi}, is $125.4 + (4.6 \cdot 10) = 171.4$ lb. So if he weighs 180 lb, he is overweight.

For a man who is 6′ 1″ (or 73″) tall, $H = 13''$. His minimum recommended weight, W_{lo}, is $105 + (4.0 \cdot 13) = 157$ lb. His maximum recommended weight, W_{hi}, is $125.4 + (4.6 \cdot 13) = 185.2$ lb. So his weight of 180 lb would place him in the healthy zone.

d. If a man is 5′ 11″, his recommended minimum weight, W_{lo} is $105 + (4.0 \cdot 11) = 149$ lb. If he currently weighs 135 lb, he would need to gain at least $149 - 135 = 14$ lb. If he gained 2 lb a week, it would take him 7 weeks to reach the minimum recommended weight of 149 lb.

EXAMPLE 3 Given the inequalities $2x - 3y \le 12$ and $x + 2y < 4$:

a. Solve each for y.

b. On the same graph, plot the boundary line for each inequality (indicating whether it is solid or dotted) and then shade the region described by each inequality.

c. Write a compound inequality describing the overlapping region.

SOLUTION **a.** To solve for y in the first inequality $\quad 2x - 3y \le 12$

add $-2x$ to both sides $\quad -3y \le -2x + 12$

divide both sides by -3, reversing the inequality symbol $\quad \dfrac{-3y}{-3} \ge \dfrac{-2x}{-3} + \dfrac{12}{-3}$

then simplify $\quad y \ge \dfrac{2}{3}x - 4$

Solving for y in the second inequality $\quad x + 2y < 4$

add $-x$ to both sides $\quad 2y < -x + 4$

divide both sides by 2 $\quad \dfrac{2y}{2} < \dfrac{-x}{2} + \dfrac{4}{2}$

then simplify $\quad y < -\dfrac{1}{2}x + 2$

b. See Figure 3.17.

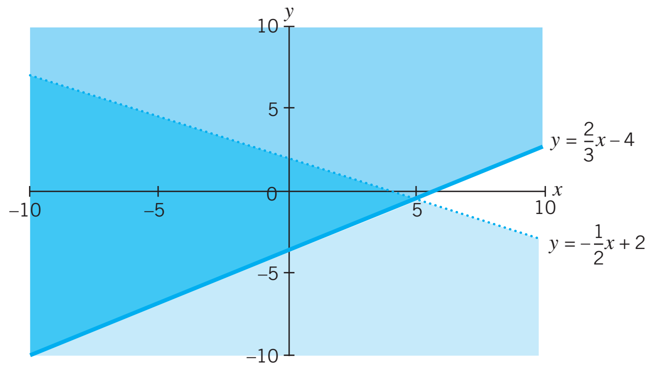


Figure 3.17 The darkest shaded region lies between $y < -\frac{1}{2}x + 2$ and $y \geq \frac{2}{3}x - 4$.

c. The overlapping region consists of all ordered pairs (x, y) such that $\frac{2}{3}x - 4 \leq y < -\frac{1}{2}x + 2$. We could describe the compound inequality by saying "y is greater than or equal to $\frac{2}{3}x - 4$ and less than $-\frac{1}{2}x + 2$."

EXAMPLE 4 Describe the shaded region in Figure 3.18.

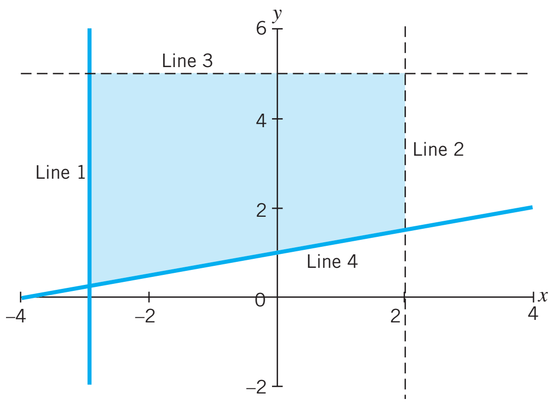


Figure 3.18 A shaded region with four boundary lines.

SOLUTION We need to find the equation for each of the four boundary lines. The two vertical lines are the easiest: Line 1 is the line $x = -3$ (a solid line, so included in the region); Line 2 is the line $x = 2$ (dotted, so excluded from the region). Line 3 is the horizontal line $y = 5$ (dotted, so it is excluded). Line 4 has a vertical intercept of 1. It passes through (0, 1) and $(-4, 0)$ so its slope is $(0 - 1)/(-4 - 0) = 1/4$ or 0.25. So the equation for line 4 is $y = 1 + 0.25x$. The region can be described as all pairs (x, y) that satisfy both compound inequalities:

$$-3 \leq x < 2 \quad \text{and} \quad 1 + 0.25x \leq y < 5$$

Breakeven Points: Regions of Profit or Loss

A simple model for the total cost C to a company producing n units of a product is

$$C = \text{fixed costs} + (\text{cost per unit}) \cdot n$$

A corresponding model for the total revenue R is

$$R = (\text{price per unit}) \cdot n$$

The breakeven point occurs when $C = R$, or the total cost is equal to the total revenue. When $R > C$, or total revenues exceed total cost, the company makes a profit and when $R < C$, the company loses money.

EXAMPLE 5 **Cost versus revenue**

In 1996 two professors from Purdue University reported on their study of fertilizer plants in Indiana.[6] They estimated that for a large-sized fertilizer manufacturing plant the fixed costs were about \$450,000 and the additional cost to produce each ton of fertilizer was about \$210. The fertilizer was sold at \$270 per ton.

a. Construct and graph two equations, one representing the total cost $C(n)$, and the other the revenue $R(n)$, where n is the number of tons of fertilizer produced.

b. What is the breakeven point, where costs equal revenue?

c. Shade between the lines to the right of the breakeven point. What does the region represent?

SOLUTION **a.** $C(n) = 450{,}000 + 210n$ and $R(n) = 270n$. See Figure 3.19.

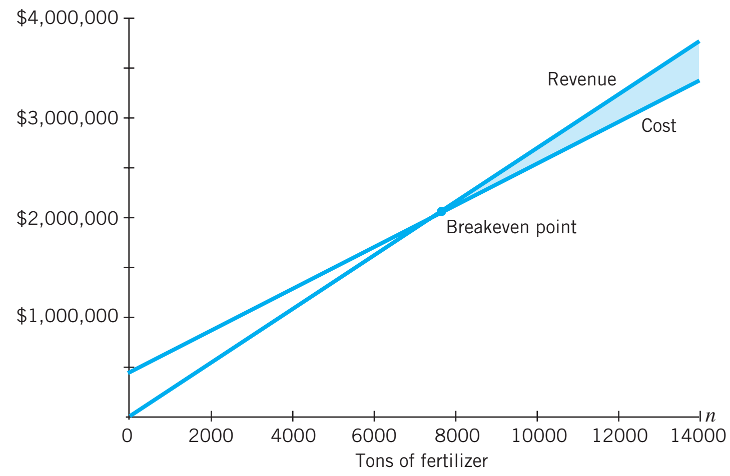


Figure 3.19 Graph of revenue vs. cost with breakeven point.

b. The breakeven point is where

The breakeven point is where	cost = revenue
or	$C(n) = R(n)$
substituting for $C(n)$ and $R(n)$	$450{,}000 + 210n = 270n$
solving for n	$450{,}000 = 60n$
we get	$n = 7500$ tons of fertilizer

Substituting 7500 tons into the revenue equation, we have

$$R(7500) = 270 \cdot 7500 = \$2{,}025{,}000$$

We could have substituted 7500 tons into the cost equation to get the same value.

$$C(7500) = 450{,}000 + 210 \cdot 7500 = \$2{,}025{,}000$$

[6]Duane S. Rogers and Jay T. Aldridge, "Economic impact of storage and handling regulation on retail fertilizer and pesticide plants," *Agribusiness 12*(4): 327–337, as quoted in Edmond C. Tomastik, *Calculus: Applications and Technology,* 3rd ed. (Belmont CA: Thomson Brooks/Cole, 2004).

So the breakeven point is (7500, \$2,025,000). At this point the cost of producing 7500 tons of fertilizer and the revenue from selling 7500 tons both equal \$2,025,000.

c. In the shaded area between $R(n)$ and $C(n)$ to the right of the breakeven point, $R(n) > C(n)$, so revenue exceeds costs, and the manufacturers are making money. Economists call this the *region of profit*.

Algebra Aerobics 3.3

1. Solve graphically each set of conditions.

a. $y \geq 2x - 1$
$y \geq 4 - x$

b. $y \geq 2x - 1$
$y \leq 3 - x$

c. $y \geq 400 + 10x$
$y \geq 200 + 20x$
$x \geq 0$
$y \geq 0$

d. $y \geq 0$
$y < -0.5x + 2$
$y < 0.5x + 2$

e. $y \leq 0$
$x \geq 0$
$y \geq -100 + x$

f. $0 \leq x \leq 200$
$y > 2x - 400$
$y < 100 - 1.5x$

2. Determine which of the following points (if any) satisfy the system of inequalities $y > 2x - 3$ and $y \leq 3x + 8$

a. (2, 3) **c.** (0, 8) **e.** (20, −8)
b. (−4, 7) **d.** (−4, −6) **f.** (1, −1)

3. Determine each inequality that has the given graph in Figure 3.20.

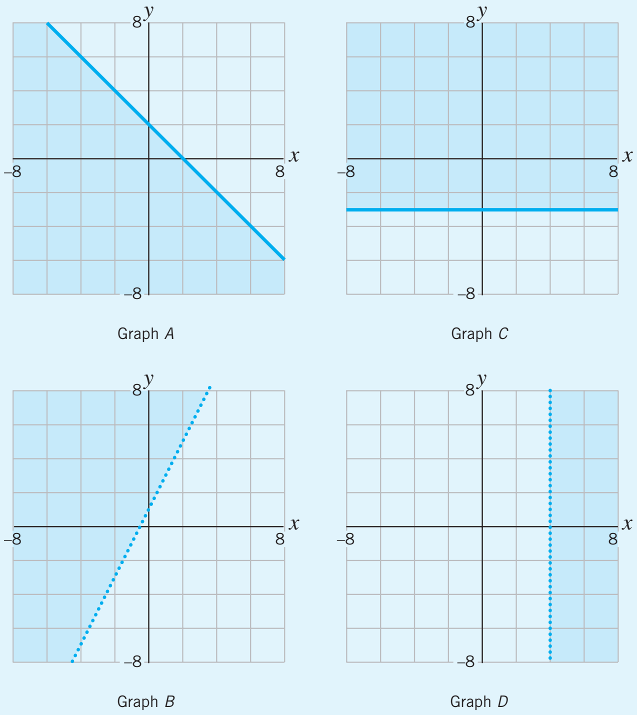


Figure 3.20 Graphs of four linear inequalities.

4. A small company sells dulcimer[7] music books on the Internet. Examine the graph in Figure 3.21 of the cost (C) and revenue (R) equations for selling n books.

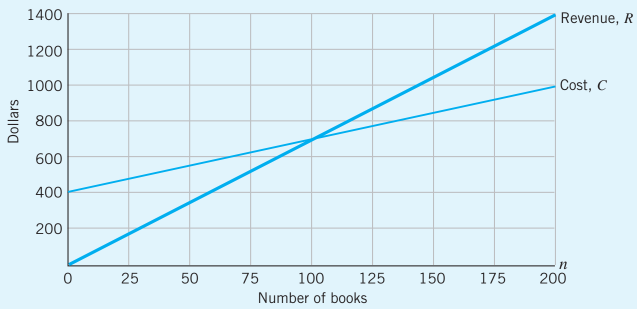


Figure 3.21 Cost and revenue for dulcimer books.

a. Estimate the breakeven point and interpret its meaning.

b. Shade in the region corresponding to losses for the company.

c. What are the fixed costs for selling dulcimer music books?

d. Another company buys dulcimer music books for \$3.00 each and sells them for \$7.00 each. The fixed cost for this company is \$400. Form a system of inequalities that represents when this company would make a profit. Use C_1 for cost and R_1 for revenue for n books.

5. Suppose that the two professors from Example 5 in this section estimated 6 years later that for a large-sized fertilizer manufacturing plant the fixed costs were about \$50,000 and the additional cost to produce each ton of fertilizer was about \$235. However, market conditions and competition caused the company to continue to sell the fertilizer at \$270 per ton.

a. Form the cost function $C(n)$ and revenue function $R(n)$ for n tons of fertilizer.

b. Find the breakeven point and interpret its meaning.

c. Graph the two functions and shade in the region corresponding to profits for the company.

[7]A mountain dulcimer is an Appalachian string instrument, usually with four strings, commonly played on the lap by strumming or plucking.

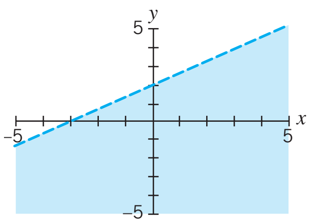
Exercises for Section 3.3

A graphing program is required for Exercises 19 and 20 and recommended for Exercises 15, 16 & 17. Access to the Internet is required in Exercise 14(c).

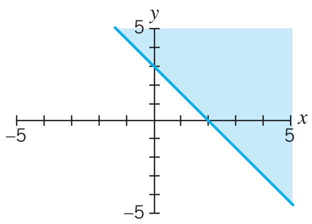
1. On different grids, graph and shade in the areas described by the following linear inequalities.
   - **a.** $y < 2x + 2$ **c.** $y < 4$
   - **b.** $y \geq -3x - 3$ **d.** $y \geq \frac{2}{3}x - 4$

2. On different grids, graph and shade in the areas described by the following linear inequalities.
   - **a.** $x - y < 0$ **c.** $3x + 2y > 6$
   - **b.** $x \leq -2$ **d.** $5x - 2y \leq 10$

3. Use inequalities to describe each shaded region.

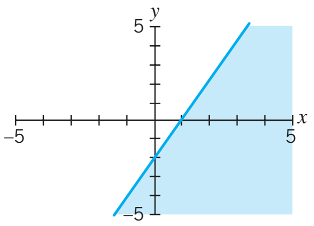


Graph *A*

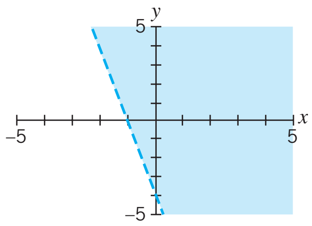


Graph *B*

4. Use inequalities to describe each shaded region.



Graph *A*



Graph *B*

5. On different grids, graph each inequality (shading in the appropriate area) and then determine whether or not the origin, the point (0, 0), satisfies the inequality.
   - **a.** $-2x + 6y < 4$ **c.** $y > 3x - 7$
   - **b.** $x \geq 3$ **d.** $y - 3 > x + 2$

6. Determine whether or not the point (−1, 3) satisfies the inequality.
   - **a.** $x - 3y > 6$ **c.** $y \leq 3$
   - **b.** $x < 3$ **d.** $y \leq -\frac{1}{2}x + 3$

7. Explain how you can tell if the region described by the inequality $3x - 5y < 15$ is above or below the boundary line of $3x - 5y = 15$.

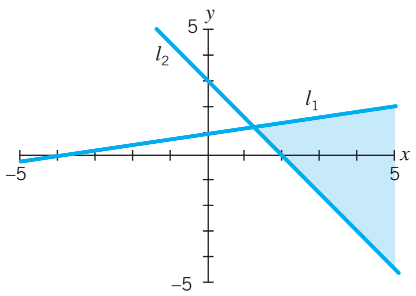
8. Shade the region bounded by the inequalities

$$x + 3y \leq 15$$
$$2x + y \leq 15$$
$$x \geq 0$$
$$y \geq 0$$

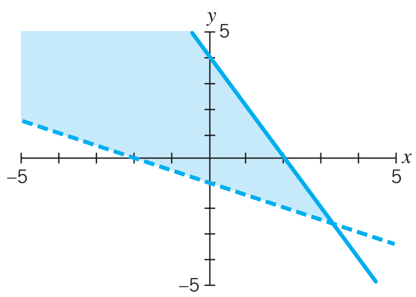
9. Match each description in parts (a) to (e) with the appropriate compound inequality in parts (f) to (j).
   - **a.** y is greater than 4 and less than $x - 3$.
   - **b.** y is greater than or equal to $x - 3$ and less than -6.
   - **c.** y is less than $2x + 5$ and greater than -6.
   - **d.** y is greater than or equal to $2x + 5$ and less than -6.
   - **e.** y is less than or equal to $x - 3$ and greater than $2x + 5$.
   - **f.** $2x + 5 \leq y < -6$
   - **g.** $4 < y < x - 3$
   - **h.** $2x + 5 < y \leq x - 3$
   - **i.** $x - 3 \leq y < -6$
   - **j.** $-6 < y < 2x + 5$

10. For the inequalities $y > 4x - 3$ and $y \leq -3x + 4$:
    - **a.** Graph the two boundary lines and indicate with different stripes the two regions that satisfy the individual inequalities.
    - **b.** Write the compound inequality for y. Indicate the double-hatched region on the graph that satisfies both inequalities.
    - **c.** What is the point of intersection for the boundary lines?
    - **d.** If $x = 3$, are there any corresponding y values in the region defined in part (b)?
    - **e.** Is the point (1, 4) part of the double-hatched region?
    - **f.** Is the point (−1, 4) part of the double-hatched region?

11. Examine the shaded region in the graph.
    - **a.** Create equations for the boundary lines l_1 and l_2 using y as a function of x.
    - **b.** Determine the compound inequality that created the shaded region.



12. Examine the shaded region in the graph. Determine the compound inequality of y in terms of x that created the shaded region.



13. The Food and Drug Administration labels suntan products with a sun protection factor (SPF) typically between 2 and 45. Multiplying the SPF by the number of unprotected minutes you can stay in the sun without burning, you are supposed to get the increased number of safe sun minutes. For example, if you can stay unprotected in the sun for 30 minutes without burning and you apply a product with a SPF of 10, then supposedly you can sun safely for $30 \cdot 10 = 300$ minutes or 5 hours.

Assume that you can stay unprotected in the sun for 20 minutes without burning.

a. Write an equation that gives the maximum safe sun time T as a function of S, the sun protection factor (SPF).

b. Graph your equation. What is the suggested domain for S?

c. Write an inequality that suggests times that would be unsafe to stay out in the sun.

d. Shade in and label regions on the graph that indicate safe and unsafe regions. (Use two different shadings and remember to include boundaries.)

e. How would the graph change if you could stay unprotected in the sun for 40 minutes?

Note that you should be cautious about spending too much time in the sun. Factors such as water, wind, and sun intensity can diminish the effect of SPF products.

14. (Access to Internet required for part (c).) Doctors measure two kinds of cholesterol in the body: low-density lipoproteins, LDL, called "bad cholesterol" and high-density lipoproteins, HDL, called "good cholesterol" because it helps to remove the bad cholesterol from the body. Rather than just measuring the total cholesterol, TC, many doctors use the ratio of TC/HDL to help control heart disease. General guidelines have suggested that men should have TC/HDL of 4.5 or below, and women should have TC/HDL of 4 or below.

On the following graphs place HDL on the horizontal and TC on the vertical axis.

a. Construct an equation for men that describes TC as a function of HDL assuming that the ratio of the two numbers is at the recommended maximum for men. Graph the function, using HDL values up to 75. Label on the graph higher-risk and lower-risk areas for heart disease for men.

b. Construct a similar equation for women that describes TC as a function of HDL assuming that the ratio of the two numbers is at the recommended maximum for women. Graph the function, using HDL values up to 75. Label on the graph higher-risk and lower-risk areas for heart disease for women.

c. Use the Internet to find the most current cholesterol guidelines.

15. (Graphing program recommended.) The blood alcohol concentration (BAC) limits for drivers vary from state to state, but for drivers under the age of 21 it is commonly set at 0.02. This level (depending upon weight and medication levels) may be exceeded after drinking only one 12-oz can of beer. The formula

$$N = 6.4 + 0.0625(W - 100)$$

gives the number of ounces of beer, N, that will produce a BAC legal limit of 0.02 for an average person of weight W. The formula works best for drivers weighing between 100 and 200 lb.

a. Write an inequality that describes the condition of too much blood alcohol for drivers under 21 to legally drive.

b. Graph the corresponding equation and label the areas that represent legally safe to drive, and not legally safe to drive conditions.

c. How many ounces of beer is it legally safe for a 100-lb person to consume? A 150-lb person? A 200-lb person?

d. Simplify your formula in part (b) to the standard $y = mx + b$ form.

e. Would you say that "6 oz of beer + 1 oz for every 20 lb over 100 lb" is a legally safe rule to follow?

16. (Graphing program recommended.) The Ontario Association of Sport and Exercise Sciences recommends the minimum and maximum pulse rates P during aerobic activities, based on age A. The maximum recommended rate, P_{max}, is $0.87(220 - A)$. The minimum recommended pulse rate, P_{min}, is $0.72(220 - A)$.

a. Convert these formulas to the $y = mx + b$ form.

b. Graph the formulas for ages 20 to 80 years. Label the regions of the graph that represent too high a pulse rate, the recommended pulse rate, and too low a pulse rate.

c. What is the maximum pulse rate recommended for a 20-year-old? The minimum for an 80-year-old?

d. Construct an inequality that describes too low a pulse rate for effective aerobic activity.

e. Construct an inequality that describes the recommended pulse range.

17. (Graphing program recommended.) We saw in this section the U.S. Department of Agriculture recommendations for healthy weight zones for males based on height. There are comparable recommendations for women between 5′ (or 60″) and 6′3″ (or 75″) tall. For women the recommended lowest weight W_{lo} (in lb) is

$$W_{lo} = 100 + 3.5H$$

and the recommended highest weight W_{hi} (in lb) is

$$W_{hi} = 118.2 + 4.2H$$

where H is the number of inches above 60″ (or 5 feet).

a. Graph and label the two equations and indicate the underweight, healthy weight, and overweight zones.

b. Give a mathematical description of the healthy weight zone for women.

c. Two women have the same weight of 130 lb. One is 5′2″ and the other is 5′5″. Does either one lie within the healthy weight zone? Why?

d. A 5′4″ woman weighs 165 lb. If her doctor puts her on a weight-loss diet of 1.5 lb per week, how many weeks would it take for her to reach the healthy range?

18. Cotton and wool fabrics, unless they have been preshrunk, will shrink when washed and dried at high temperatures. If washed in cold water and drip-dried they will shrink a lot less, and if dry-cleaned, they will not shrink at all. A particular cotton fabric has been found to shrink 8% with a hot wash/dry, and 3% with a cold wash/drip dry.

a. Find a formula to express the hot wash length, *H*, as a function of the original length, *L*. Then find a formula for the cold wash length, *C*, as a function of the original length, *L*. What formula expresses dry-clean length, *D*, as a function of the original length?

b. Make a graph with original length, *L*, on the horizontal axis, up to 60 inches. Plot three lines showing cold wash length, *C*, hot wash length, *H*, and dry-clean length, *D*. Label the lines.

c. If you buy trousers with an original inseam length of 32″, how long will the inseam be after a hot wash? A cold wash?

d. If you need 3 yards of fabric (a yard is 3′) to make a well-fitted garment, how much would you have to buy if you plan to hot-wash the garment?

19. (Graphing program required.) Two professors from Purdue University reported that for a typical small-sized fertilizer plant in Indiana the fixed costs were $235,487 and it cost $206.68 to produce each ton of fertilizer.

a. If the company planned to sell the fertilizer at $266.67 per ton, find the cost, *C*, and revenue, *R*, equations for *x* tons of fertilizer.

b. Graph the cost and revenue equations on the same graph and calculate and interpret the breakeven point.

c. Indicate the region where the company would make a profit and create the inequality to describe the profit region.

20. (Graphing program required.) A company manufactures a particular model of DVD player that sells to retailers for $85. It costs $55 to manufacture each DVD player, and the fixed manufacturing costs are $326,000.

a. Create the revenue function $R(x)$ for selling *x* number of DVD players.

b. Create the cost function $C(x)$ for manufacturing *x* DVD players.

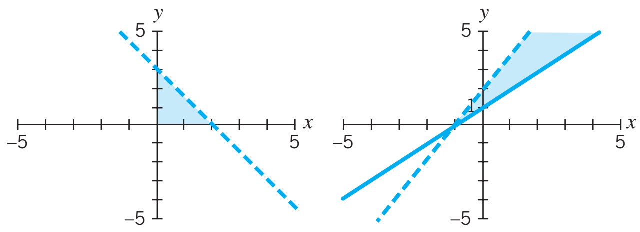
c. Plot the cost and revenue functions on the same graph. Estimate and interpret the breakeven point.

d. Shade in the region where the company would make a profit.

e. Shade in the region where the company would experience a loss.

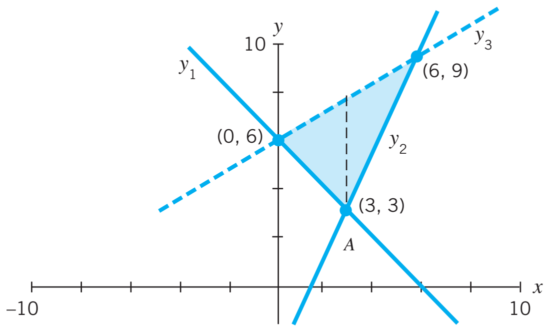
f. What is the inequality that represents the profit region?

21. Describe the shaded region in each graph with the appropriate inequalities.



Graph *A* Graph *B*

22. Describe the shaded region. (*Hint:* Create a vertical line through point *A*, dividing the shaded region into two smaller triangular regions.)



23. A financial advisor has up to $30,000 to invest, with the stipulation that at least $5000 is to be placed in Treasury bonds and at most $15,000 in corporate bonds.

a. Construct a set of inequalities that describes the relationship between buying corporate vs. Treasury bonds where the total amount invested must be less than or equal to $30,000. (Let *C* be the amount of money invested in corporate bonds, and *T* the amount invested in Treasury bonds.).

b. Construct a feasible region of investment; that is, shade in the area on a graph that satisfies the spending constraints on both corporate and Treasury bonds. Label the horizontal axis "Amount invested in Treasury bonds" and the vertical axis "Amount invested in corporate bonds."

c. Find all of the intersection points (corner points) of the bounded investment feasibility region and interpret their meanings.

24. A Texas oil supplier sends out at most 10,000 barrels of oil per week. Two distributors need oil. Southern Oil needs at least 2000 barrels of oil per week and Regional Oil needs at least 5000 barrels of oil per week.

a. Let *S* be the number of barrels of oil sent to Southern Oil and let *R* be the number of barrels sent to Regional Oil per week. Create a system of inequalities that describes all of the conditions.

b. Graph the feasible region of the system.

c. Choose a point inside the region that would satisfy the conditions and describe its meaning.

3.4 *Systems with Piecewise Linear Functions: Tax Plans*

Graduated vs. Flat Income Tax

Income taxes may be based on either a flat or a graduated tax rate. With a flat tax rate, no matter what the income level, everyone is taxed at the same percentage. Flat taxes are often said to be unfair to those with lower incomes, because paying a fixed percentage of income is considered more burdensome to someone with a low income than to someone with a high income.

A graduated tax rate means that people with higher incomes are taxed at higher rates. Such a tax is called *progressive* and is generally less popular with those who have high incomes. Whenever the issue appears on the ballot, the pros and cons of the graduated vs. flat tax rate are hotly debated in the news media and paid political broadcasting. Of the forty-one states with a broad-based income tax, thirty-five had a graduated income tax in 2004.

The *New York Times* article "How a Flat Tax Would Work for You and for Them" discusses the trade-offs in using a flat tax.

The taxpayer

For the taxpayer there are two primary questions in comparing the effects of flat and graduated tax schemes. For what income level will the taxes be the same under both plans? And, given a certain income level, how will taxes differ under the two plans?

Taxes are influenced by many factors, such as filing status, exemptions, source of income, and deductions. For our comparisons of flat and graduated income tax plans, we examine one filing status and assume that exemptions and deductions have already been subtracted from income.

The *New York Times* article "A Taxation Policy to Make John Stuart Mill Weep" discusses the growing trend by Congress of taxing different sources of income differently.

EXAMPLE 1

Match each graph in Figure 3.22 with the appropriate description.

a. Income taxes are a flat rate of 5% of your income.

b. Income taxes are graduated, with a rate of 5% for first \$100,000 of income and a rate of 8% for any additional income $>$ \$100,000.

c. Sales taxes are 5% for $0 \leq$ purchases $\leq$ \$100,000 and a flat fee of \$5000 for purchases $>$ \$100,000.

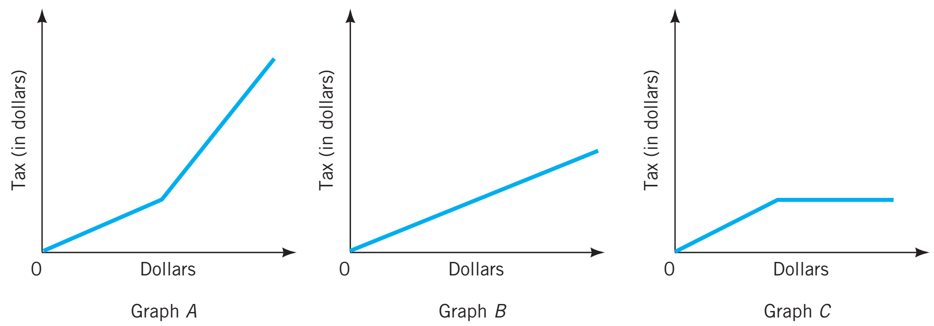


Figure 3.22 Graphs of different tax plans.

SOLUTION

Graph *A* matches description (b).
Graph *B* matches description (a).
Graph *C* matches description (c).

A flat tax model

Flat taxes are a fixed percentage of income. If the flat tax rate is 15% (or 0.15 in decimal form), then flat taxes can be represented as

$$f(i) = 0.15i$$

where i = income. This flat tax plan is represented in Figure 3.23.

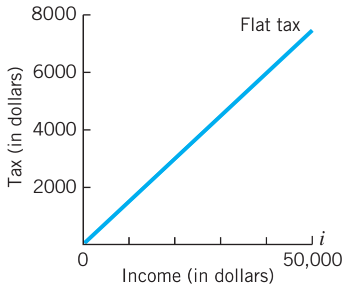


Figure 3.23 Flat tax at a rate of 15%.

A graduated tax model

A Piecewise Linear Function Under a graduated income tax, the tax rate changes for different income levels. We can use a piecewise linear function to model a graduated tax. Let's consider a graduated tax where the first \$10,000 of income is taxed at 10% and any income over \$10,000 is taxed at 20%. For example, if your income after deductions is \$30,000, then your taxes under this plan are

$$\begin{aligned}\text{graduated tax} &= (10\% \text{ of } \$10{,}000) + (20\% \text{ of income over } \$10{,}000)\\ &= 0.10(\$10{,}000) + 0.20(\$30{,}000 - \$10{,}000)\\ &= 0.10(\$10{,}000) + 0.20(\$20{,}000)\\ &= \$1000 + \$4000\\ &= \$5000\end{aligned}$$

If an income, i, is over \$10,000, then under this plan,

$$\begin{aligned}\text{graduated tax} &= 0.10(\$10{,}000) + 0.20(i - \$10{,}000)\\ &= \$1000 + 0.20(i - \$10{,}000)\end{aligned}$$

The Graph This graduated tax plan is represented in Table 3.3 and Figure 3.24. The graph of the graduated tax is the result of piecing together two different line segments that represent the two different formulas used to find taxes. The short segment represents taxes for low incomes between \$0 and \$10,000, and the longer, steeper segment represents taxes for higher incomes that are greater than \$10,000.

Taxes under Graduated Tax Plan

Income after Deductions (\$)	Taxes (\$)
0	0
5,000	500
10,000	1,000
20,000	1,000 + 2,000 = 3,000
30,000	1,000 + 4,000 = 5,000
40,000	1,000 + 6,000 = 7,000
50,000	1,000 + 8,000 = 9,000

Table 3.3

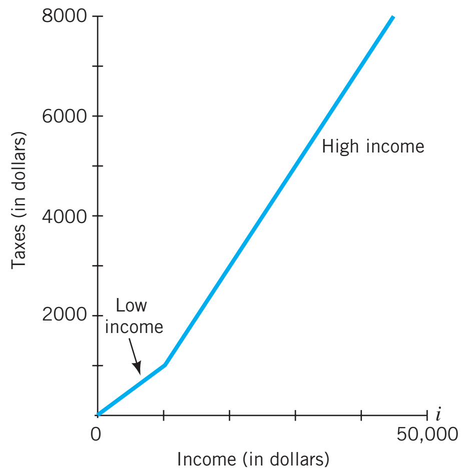


Figure 3.24 Graduated tax.

The Equations To find an algebraic expression for the graduated tax, we need to use different formulas for different levels of income. Functions that use different formulas for different intervals of the domain are called *piecewise defined*. Since each income determines a unique tax, we can define the graduated tax as a piecewise function, g, of income i as

$$g(i) = \begin{cases} 0.10i & \text{for } i \leq \$10{,}000 \\ 1000 + 0.20(i - 10{,}000) & \text{for } i > \$10{,}000 \end{cases}$$

The value of i (the input or independent variable) determines which formula to use to evaluate the function. This function is called a *piecewise linear function,* since each piece consists of a different linear formula. The formula for incomes equal to or below \$10,000 is different from the formula for incomes above \$10,000.

Evaluating Piecewise Functions To find $g(\$8000)$, the value of $g(i)$ when $i = \$8000$, use the first formula in the definition since income, i, in this case is less than \$10,000:

$$\begin{aligned} &\text{For } i \leq \$10{,}000 & g(i) &= 0.10i \\ &\text{substituting } \$8000 \text{ for } i & g(\$8000) &= 0.10(\$8000) \\ & & &= \$800 \end{aligned}$$

To find $g(\$40{,}000)$, we use the second formula in the definition, since in this case income is greater than \$10,000:

$$\begin{aligned} &\text{For } i > \$10{,}000 & g(i) &= \$1000 + 0.20(i - \$10{,}000) \\ &\text{substituting } \$40{,}000 \text{ for } i & g(\$40{,}000) &= \$1000 + 0.20(\$40{,}000 - \$10{,}000) \\ & & &= \$1000 + 0.20(\$30{,}000) \\ & & &= \$1000 + \$6000 \\ & & &= \$7000 \end{aligned}$$

Comparing the Two Tax Models

Using graphs

In Figure 3.25 we compare the different tax plans by plotting the flat and graduated tax equations on the same graph.

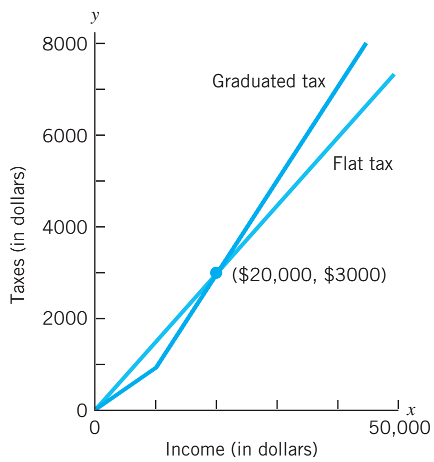


Figure 3.25 Flat tax versus graduated tax.

The intersection points indicate the incomes at which the amount of tax is the same under both plans. From the graph, we can estimate the coordinates of the two points as (0, 0) and (\$20,000, \$3000). That is, under both plans, with \$0 income you pay \$0 taxes, and with approximately \$20,000 in income you would pay approximately \$3000 in taxes. Individual voters want to know what impact these different plans will have on their taxes. From the graph in Figure 3.25, we can see that to the left of the intersection point at ($20,000, $3000), the flat tax is *greater* than the graduated tax for the same income. To the right of this intersection point, the flat tax is *less* than the graduated tax for the same income. So for incomes *less than* $20,000, taxes are *greater* under the flat tax plan, and for incomes *greater than* $20,000, taxes will be *less* under the flat tax plan.

Using equations

To verify the accuracy of our estimates for the coordinates of the point(s) of intersection, we can set $f(i) = g(i)$. We know that $f(i) = 0.15i$. Which of the two expressions do we use for $g(i)$? The answer depends upon what value of income, i, we consider. For $i \le \$10{,}000$, we have $g(i) = 0.10i$.

If	$f(i) = g(i)$
and $i \le \$10{,}000$, then	$0.15i = 0.10i$
This can only happen when	$i = 0$

If $i = 0$, both $f(i)$ and $g(i)$ equal 0; therefore, one intersection point is indeed (0, 0).
For $i > \$10{,}000$, we use $g(i) = \$1000 + 0.20(i - \$10{,}000)$ and again set $f(i) = g(i)$.

If	$f(i) = g(i)$
and $i > \$10{,}000$, then	$0.15i = \$1000 + 0.20(i - \$10{,}000)$
apply the distributive property	$0.15i = \$1000 + 0.20i - (0.20)(\$10{,}000)$
multiply	$0.15i = \$1000 + 0.20i - \$2000$
combine terms	$0.15i = -\$1000 + 0.20i$
add $-0.20i$ to each side	$-0.05i = -\$1000$
divide by -0.05	$i = -\$1000/(-0.05)$
	$i = \$20{,}000$

SOMETHING TO THINK ABOUT

We have assumed that deductions are the same under both plans. If we drop this assumption, how could we make a flat tax plan less burdensome for people with low incomes?

So each plan results in the same tax for an income of $20,000. How much tax is required? We can substitute $20,000 for i into either the flat tax formula or the graduated tax formula for incomes over $10,000 and solve for the tax. Given the flat tax function,

	$f(i) = 0.15i$
if $i = \$20{,}000$, then	$f(i) = (0.15)(\$20{,}000)$
	$= \$3000$

We can check to make sure that when $i = \$20{,}000$, the graduated tax, $g(i)$, will also be $3000:

If $i > \$10{,}000$, then	$g(i) = \$1000 + 0.20(i - \$10{,}000)$
so, if $i = \$20{,}000$, then	$g(i) = \$1000 + 0.20(\$20{,}000 - \$10{,}000)$
perform operations	$= \$3000$

These calculations confirm that the other intersection point is, as we estimated, ($20,000, $3000).

The Case of Massachusetts

In the state of Massachusetts there has been an ongoing debate about whether or not to change from a flat tax to a graduated income tax[8]. In 1994 Massachusetts voters considered a proposal called Proposition 7. Proposition 7 would have replaced the flat tax rate (at the time of the proposal, 5.95%) with graduated income tax rates (called *marginal* rates) as shown in Table 3.4.

[8]For a state-by-state comparison of current income tax rates see *www.taxadmin.org/fta/rate/ind_inc.html*. For a comparison of the recent state tax proposals in Tennessee see *www.yourtax.org*.

In Exploration 3.1 you can analyze the impact of Proposition 7 on people using different filing statuses.

Massachusetts Graduated Income Tax Proposal

	Marginal Rate		
Filing Status	5.5%	8.8%	9.8%
Married/joint	<$81,000	$81,000–$150,000	$150,000+
Married/separate	<$40,500	$40,500–$ 75,000	$ 75,000+
Single	<$50,200	$50,200–$ 90,000	$ 90,000+
Head of household	<$60,100	$60,100–$120,000	$120,000+

Table 3.4

Source: Office of the Secretary of State, Michael J. Connolly, Boston, 1994.

Questions for the taxpayer

For what income and filing status would the taxes be equal under both plans? Who will pay less tax and who will pay more under the graduated income tax plan? We analyze the tax for a single person and leave the analyses of the other filing categories for you to do.

Finding out who pays what

The proposed graduated income tax is designed to tax at higher rates that portion of the individual's income that exceeds a certain threshold. For example, for those who file as single people, the graduated tax rate means that earned income under $50,200 would be taxed at a rate of 5.5%. Any income between $50,200 and $90,000 would be taxed at 8.8%, and any income over $90,000 would be taxed at 9.8%.

Using Equations. The tax for a single person earning $100,000 would be the sum of three different dollar amounts.

5.5% on the first $50,200 = (0.055)($50,200)
= $2761

8.8% on the next $39,800 (the portion of income between $50,200 and $90,000)
= (0.088)($39,800)
≈ $3502

9.8% on the remaining $10,000 = (0.098)($10,000)
= $980

The total graduated tax would be $2761 + $3502 + $980 = $7243. Under the flat tax rate the same individual would pay 5.95% of $100,000 = 0.0595($100,000), or $5950.

Table 3.5 shows the differences between the flat tax and the proposed graduated tax plan for single people at several different income levels. We can represent flat taxes for single people as a linear function f of income i, where

$$f(i) = 0.0595i$$

Massachusetts Taxes: Flat Rate vs. Graduated Rate for Single People

Income after Exemptions and Deductions ($)	Current Flat Tax at 5.95% ($)	Graduated Tax under Proposition 7 ($)
0	0	0
25,000	1488	1375
50,000	2975	2750
75,000	4463	4943
100,000	5950	7243

Table 3.5

We can write the graduated tax for single people as a piecewise linear function g of income i, where

$$g(i) = \begin{cases} 0.055i & \text{for } 0 \leq i < \$50{,}200 \\ \$2761 + 0.088(i - \$50{,}200) & \text{for } \$50{,}200 \leq i \leq \$90{,}000 \\ \$6263 + 0.098(i - \$90{,}000) & \text{for } i > \$90{,}000 \end{cases}$$

Note that \$2761 in the second formula of the definition is the tax on the first \$50,200 of income (5.5% of \$50,200), and \$6263 in the third formula of the definition is the sum of the taxes on the first \$50,200 and the next \$39,800 of income (5.5% of \$50,200 + 8.8% of \$39,800 ≈ \$2761 + \$3502 = \$6263).

Using Graphs. For what income would single people pay the same tax under both plans? The flat tax and the graduated income tax for single people are compared in Figure 3.26. An intersection point on the graph indicates where taxes are equal. One intersection point occurs at (0, 0). That makes sense, since under either plan if you have zero income, you pay zero taxes. The second intersection point is at approximately (\$58,000, \$3500). That means for an income of approximately \$58,000, the taxes are the same and are approximately \$3500.

In the Algebra Aerobics you are asked to describe what happens to the right and to the left of the intersection point.

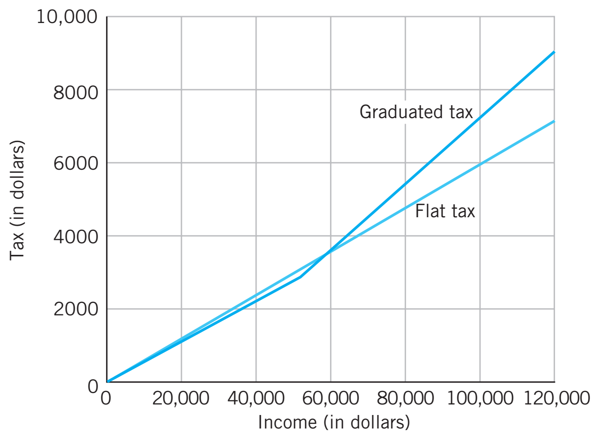


Figure 3.26 Massachusetts graduated tax vs. flat tax.

Algebra Aerobics 3.4

1. Graph the following piecewise functions.

   a. $f(x) = \begin{cases} -x - 1 & \text{for } x \leq 0 \\ \frac{1}{2}x - 1 & \text{for } x > 0 \end{cases}$

   b. $g(x) = \begin{cases} 4 & \text{for } 0 \leq x \leq 5 \\ 2x - 6 & \text{for } x > 5 \end{cases}$

   c. $k(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq 10 \\ 3x - 30 & \text{for } x > 10 \end{cases}$

2. Use equations to describe the piecewise linear function on each graph in Figure 3.27.

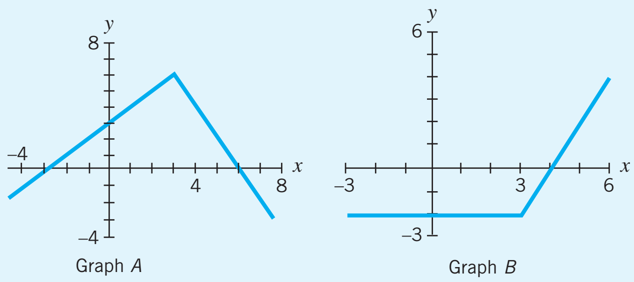


Figure 3.27 Two piecewise linear functions.

3. Evaluate each of the following piecewise defined functions at $x = -5, 0, 2,$ and 10.

a. $P(x) = \begin{cases} 3 & \text{for } x \leq 1 \\ 1 - 2x & \text{for } x > 1 \end{cases}$

b. $W(x) = \begin{cases} x - 4 & \text{for } x < 2 \\ x + 4 & \text{for } x \geq 2 \end{cases}$

4. Construct a new graduated tax function, if:

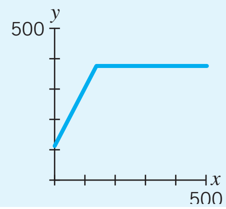
a. The tax is 5% on the first \$50,000 of income and 8% on any income in excess of \$50,000.

b. The tax is 6% on the first \$30,000 of income and 9% on any income in excess of \$30,000.

5. Use the graph in Figure 3.26 to predict which tax is larger for each of the following incomes and by approximately how much. Then use the equations defining f and g on page 197 and 198 to check your predictions.

a. \$30,000 **b.** \$60,000 **c.** \$120,000

6. Use equations to describe the following graph.



Exercises for Section 3.4

Graphing program recommended for Exercise 12.

1. Consider the following function:

$$f(x) = \begin{cases} 2x + 1 & x \leq 0 \\ 3x & x > 0 \end{cases}$$

Evaluate $f(-10), f(-2), f(0), f(2),$ and $f(4)$.

2. Consider the following function:

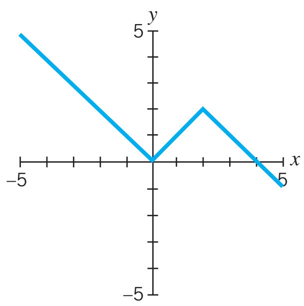
$$g(x) = \begin{cases} 3 & x \leq 1 \\ 4 + 2x & x > 1 \end{cases}$$

Evaluate $g(-5), g(-2), g(0), g(1), g(1.1), g(2),$ and $g(10)$.

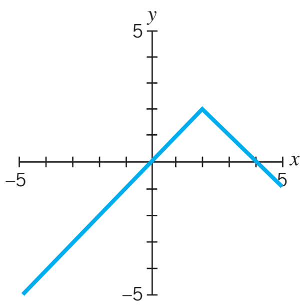
3. Match each function with its graph.

a. $f(x) = \begin{cases} x & \text{if } x \leq 2 \\ -x + 4 & \text{if } x > 2 \end{cases}$

b. $f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 2 \\ -x + 4 & \text{if } x > 2 \end{cases}$



Graph *A*



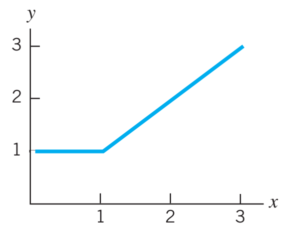
Graph *B*

4. Construct a small table of values and graph the following piecewise linear functions. In each case specify the domain.

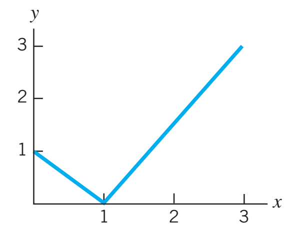
a. $f(x) = \begin{cases} 5 & \text{for } x < 10 \\ -15 + 2x & \text{for } x \geq 10 \end{cases}$

b. $g(t) = \begin{cases} 1 - t & \text{for } -10 \leq t \leq 1 \\ t & \text{for } 1 < t < 10 \end{cases}$

5. Construct a piecewise linear function for each of the accompanying graphs.



Graph *A*



Graph *B*

6. Create a graph for each piecewise function:

a. $f(x) = \begin{cases} 2x + 1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$

b. $g(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

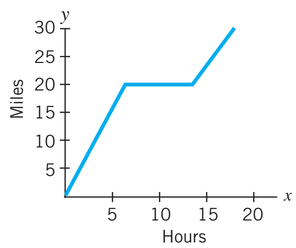
7. Create a graph for each piecewise function:

a. $h(x) = \begin{cases} 20 - 2x & \text{if } 0 \leq x < 5 \\ 10 & \text{if } 5 \leq x \leq 10 \\ 10 + 2(x - 10) & \text{if } x > 10 \end{cases}$

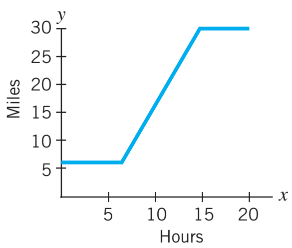
b. $k(x) =$

$$\begin{cases} 0.15x & \text{if } 0 \leq x \leq 20{,}000 \\ 3000 + 0.18(x - 20{,}000) & \text{if } 20{,}000 < x \leq 50{,}000 \\ 8400 + 0.23(x - 50{,}000) & \text{if } x > 50{,}000 \end{cases}$$

8. Each of the following graphs shows the activity of a hiker relative to her base campsite. Describe her actions first with words and then with a piecewise function.



Graph *A*



Graph *B*

9. Find out what kind of income tax (if any) your state has. Is it flat or graduated? Construct and graph a function that describes your state's income tax for one filing status. Compute the income tax for various income levels.

10. a. Construct a graduated tax function where the tax is 10% on the first $30,000 of income, then 20% on any income in excess of $30,000.

b. Construct a flat tax function where the tax is 15% of income.

c. Calculate the tax for both the flat tax function from part (b) and the graduated tax function from part (a) for each of the following incomes: $10,000, $20,000, $30,000, $40,000, and $50,000.

d. Graph the graduated and flat tax functions on the same grid and estimate the coordinates of the points of intersection. Interpret the points of intersection.

11. Fines for a particular speeding ticket are defined by the following piecewise function, where s is the speed in mph and $F(s)$ is the fine in dollars.

$$F(s) = \begin{cases} 0 & \text{if } \quad s \leq 45 \\ 50 + 5(s - 45) & \text{if } 45 < s \leq 55 \\ 100 + 10(s - 55) & \text{if } 55 < s \leq 65 \\ 200 + 20(s - 65) & \text{if } \quad s > 65 \end{cases}$$

a. What is the implied posted speed limit for this situation?

b. Create a table of values for the fine, beginning at 40 mph and incrementing by 5-mph steps up to 80 mph, and then graph $F(s)$.

c. Describe in words how a speeding fine is calculated.

d. Explain what 5, 10, and 20 in the formulas for the respective sections of the piecewise function represent.

e. Find $F(30)$, $F(57)$, and $F(67)$.

f. Graph $F(s)$.

12. (Graphing program recommended). In 2004, Missouri had a graduated tax plan but it was considering adopting a flat-rate tax of 4% on income after deductions. The tax rate for single people under the graduated plan is shown in the accompanying table. For what income levels would a single person pay less tax under the flat tax plan than under the graduated tax plan?

Missouri State Tax for a Single Person

Income after Deductions ($)	Marginal Tax Rate (%)
≤1000	1.50
1001–2000	2.00
2001–3000	2.50
3001–4000	3.00
4001–5000	3.50
5001–6000	4.00
6001–7000	4.50
7001–8000	5.00
8001–9000	5.50
>9000	6.00

13. You are thinking about replacing your long-distance telephone service. A cell phone company charges a monthly fee of $40 for the first 450 minutes and then charges $0.45 for every minute after 450. Every call is considered a long-distance call. Your local phone company charges you a fee of $60 per month and then $0.05 per minute for every long-distance call.

a. Assume you will be making only long-distance calls. Create two functions, $C(x)$ for the cell phone plan and $L(x)$ for the local telephone plan, where x is the number of long-distance minutes.

After how many minutes would the two plans cost the same amount?

b. Describe when it is more advantageous to use your cell phone for long-distance calls.

c. Describe when it is more advantageous to use your local phone company to make long-distance calls.

14. The accompanying table, taken from a pediatrics text, provides a set of formulas for the approximate "average" weight and height of normal infants and children.

Age	Weight (lb)
At birth	7
3–12 months	(age in months) + 11
1–5 years	5 · (age in years) + 17
6–12 years	7 · (age in years) + 5

Age	Height (in)
At birth	20
1 year	30
2–12 years	2.5 · (age in years) + 30

Source: R. E. Behrman and V. C. Vaughan (eds.), *Nelson Textbook of Pediatrics,* 12th ed. (Philadelphia: W. B. Saunders, 1983), p. 19.

For children from birth to 12 years of age, construct and graph a piecewise linear function for each of the following (assuming age is the independent variable):

a. Weight in pounds (How does this model compare to the model for female infants in Chapter 2, Section 5?)

b. Height in inches

Note that there are certain gaps in the table that need to be resolved in order to construct piecewise linear functions. For example, you will need to decide which weight formula to use for a child who is $5\frac{1}{2}$ years old and which height formula to use for a child who is $1\frac{1}{2}$ years old.

15. Heart health is a prime concern, because heart disease is the leading cause of death in the United States. Aerobic activities such as walking, jogging, and running are recommended for cardiovascular fitness, because they increase the heart's strength and stamina.

a. A typical training recommendation for a beginner is to walk at a moderate pace of about 3.5 miles/hour (or approximately 0.0583 miles/minute) for 20 minutes. Construct a function that describes the distance traveled

D_{beginner}, in miles, as a function of time, T, in minutes, for someone maintaining this pace. Construct a small table of values and graph the function using a reasonable domain.

b. A more advanced training routine is to walk at a pace of 3.75 miles/hour (or 0.0625 miles/minute) for 10 minutes and then jog at 5.25 miles/hour (or 0.0875 miles/minute) for 10 minutes. Construct a piecewise linear function that gives the total distance, D_{advanced}, as a function of time. T, in minutes. Generate a small table of values and plot the graph of this function on your graph in part (a).

c. Do these two graphs intersect? If so, what do the intersection point(s) represent?

16. A graduated income tax is proposed in Borduria to replace an existing flat rate of 8% on all income. The new proposal states that persons will pay no tax on their first \$20,000 of income, 5% on income over \$20,000 and less than or equal to \$100,000, and 10% on their income over \$100,000. (*Note:* Borduria is a fictional totalitarian state in the Balkans that figures in the adventures of Tintin.)

a. Construct a table of values that shows how much tax persons will pay under both the existing 8% flat tax and the proposed new tax for each of the following incomes: \$0, \$20,000, \$50,000, \$100,000, \$150,000, \$200,000.

b. Construct a graph of tax dollars vs. income for the 8% flat tax.

c. On the same graph plot tax dollars vs. income for the proposed new graduated tax.

d. Construct a function that describes tax dollars under the existing 8% tax as a function of income.

e. Construct a piecewise function that describes tax dollars under the proposed new graduated tax rates as a function of income.

f. Use your graph to estimate the income level for which the taxes are the same under both plans. What plan would benefit people with incomes below your estimate? What plan would benefit people with incomes above your estimate?

g. Use your equations to find the coordinates that represent the point at which the taxes are the same for both plans. Label this point on your graph.

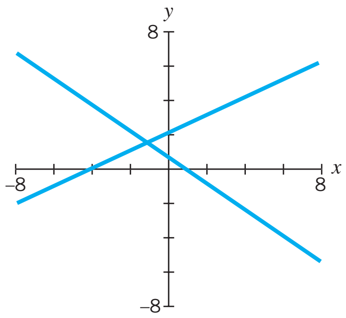
h. If the median income in the state is \$27,000 and the mean income is \$35,000, do you think the new graduated tax would be voted in by the people? Explain your answer.

CHAPTER SUMMARY

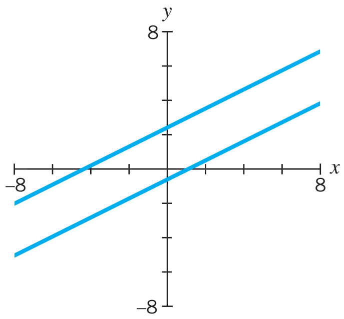
Systems of Linear Equations

A pair of real numbers is a *solution* to a system of linear equations in two variables if and only if the pair of numbers is a solution of each equation. A system of two linear equations in two variables may have one solution, no solutions, or infinitely many solutions.

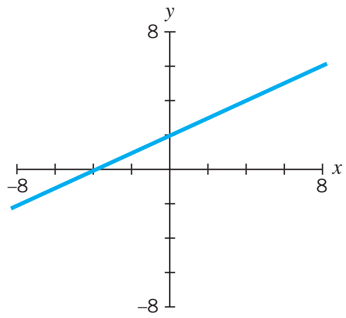
One solution



No solutions



Infinitely many solutions

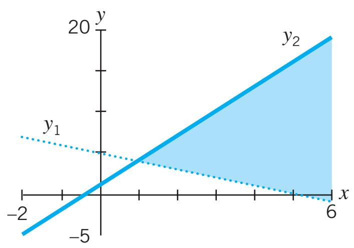


Systems of Linear Inequalities

The solutions to a system of linear inequalities in two variables are pairs of real numbers that satisfy all the inequalities. The solutions typically form a region of the plane and may be represented by a compound inequality such as

$$y_1 < y \leq y_2$$
$$5 - x < y \leq 1 + 3x$$

The shaded area in the figure represents the solution area for this system.

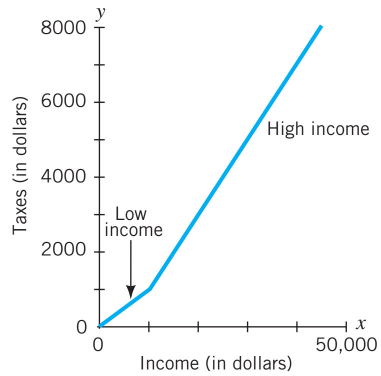


Piecewise Linear Functions

Functions that use different formulas for different intervals of the domain are called *piecewise functions*. Functions constructed out of pieces of several different linear functions are called *piecewise linear functions*.

The accompanying figure shows the graph of a linear piecewise function g of income i, where

$$g(i) = \begin{cases} 0.10i & \text{for } i \le \$10{,}000 \\ 1000 + 0.20(i - 10{,}000) & \text{for } i > \$10{,}000 \end{cases}$$



CHECK YOUR UNDERSTANDING

I. Are the statements in Problems 1–29 true or false? Give an explanation for your answer.

1. Assuming y is a function of x, the number pair $(-5, 3)$ is a solution to the following system of equations:

$$\begin{cases} 2x - 3y = 21 \\ x = 2y + 3 \end{cases}.$$

2. Assuming y is a function of x, the number pair $\left(\frac{16}{5}, \frac{2}{5}\right)$ is a solution to the following system of equations:

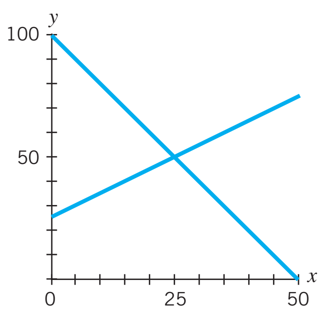
$$\begin{cases} y = 10 - 3x \\ 2x - y = 6 \end{cases}$$

3. The system $\begin{cases} x - \dfrac{5}{y} = 2 \\ \dfrac{x}{3} + 2y = 1 \end{cases}$ is not a linear system of equations.

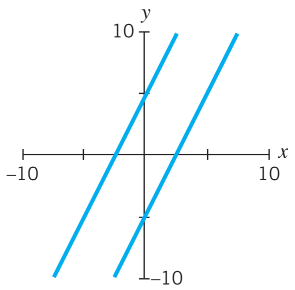
4. The system of linear equations $\begin{cases} 2y - \dfrac{x}{3} = -1 \\ \dfrac{6y}{5} - \dfrac{x}{5} = \dfrac{-3}{5} \end{cases}$ has no solution(s).

5. A system of linear equations in two variables either has a pair of numbers that is a unique solution or has no solution.

6. A pair of numbers can be a solution to a system of linear equations in two variables if the pair is a solution to at least one of the equations.



System *A*



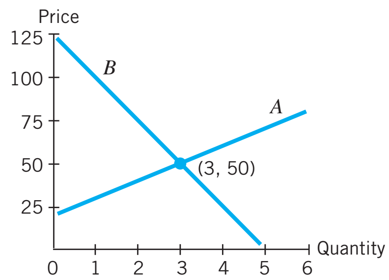
System *B*

Questions 7 and 8 refer to the accompanying graphs of systems of two equations in two variables.

7. System *A* has a unique solution at approximately (50, 25).

8. System *B* appears to have no solution.

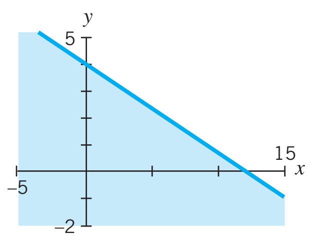
Questions 9 and 10 refer to the supply and demand curves in the accompanying graph.



9. Graph *B* is the demand curve because as price increases, the quantity demanded decreases.

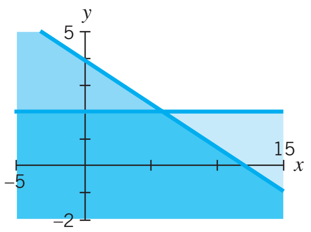
10. If three items are produced and sold at a price of \$50, the quantity supplied will be equal to the quantity demanded.

11. The shaded region in the accompanying graph appears to satisfy the linear inequality $12 - x < 3y$.



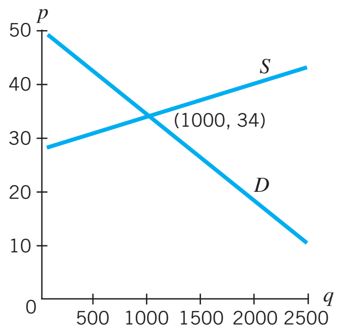
12. The darkest shaded region in the graph on the next page appears to satisfy the system of linear inequalities

$$\begin{cases} y \le 4 - \dfrac{x}{3} \\ y \le 2 \end{cases}.$$



13. The linear inequalities $y \leq 5x - 3$ and $y < 5x - 3$ have exactly the same solutions.

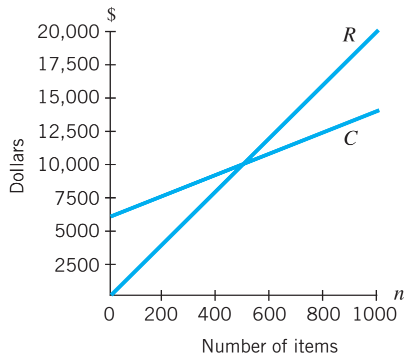
Questions 14 and 15 refer to the accompanying graph of supply and demand, where q = quantity and p = price:



14. The equilibrium point (1000, 34) means that at a price of \$1000, suppliers will supply 34 items and consumers will demand 34 items.

15. If the price is less than the equilibrium price, the quantity supplied is less than the quantity demanded and therefore competition will cause the price to increase toward the equilibrium price.

Questions 16–20 refer to the accompanying graph, where R represents total revenue and C represents total costs.



16. The estimated breakeven point is approximately (500, \$10,000).

17. The profit is zero at the breakeven point.

18. If 200 units are produced and sold, the total revenue is larger than the total cost.

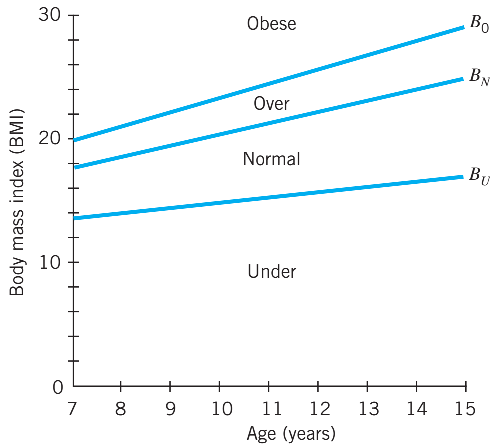
19. If the revenue per unit increases, and hence, R, the revenue line, becomes steeper, the breakeven point moves to the right.

20. If 700 units are produced and sold, the company is making about a \$5000 profit.

Questions 21–23 refer to the accompanying graph of linear approximations for the body mass index (BMI) of girls age 7–15. B_U gives the minimal recommended BMI and B_N the maximum recommended BMI. B_0 is the dividing line between being somewhat overweight and being obese.

The National Center For Health Statistics calculates the BMI as

$$\text{BMI} = \left[\frac{\text{weight (lb)}}{\text{height (in)}}\right] \cdot \left[\frac{703}{\text{height (in)}}\right]$$



Sources: National Center for Health Statistics, National Center for Chronic Disease Prevention and Health Promotion (2000), *www.cdc.gov/growthcharts.*

21. A 15-year-old girl who weighs 105 pounds and is 55 inches tall has a BMI that is within the normal zone.

22. A 10-year-old girl with BMI = 12 is underweight for her height and age.

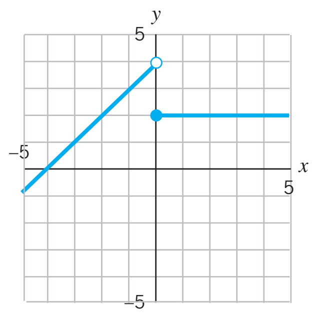
23. A mathematical description of the normal BMI zone for girls age 7–15 is $B_U \leq \text{BMI} \leq B_N$.

Questions 24 and 25 refer to the following system of linear inequalities: $\begin{cases} 2x + 3 \leq y \\ y < 5 + x \end{cases}$

24. The boundary line $y = 5 + x$ is not included in the solution region.

25. The pair of numbers (0, 0) is a solution to the system.

Questions 26 and 27 refer to the accompanying graph of the piecewise function $y = f(x)$.



26. $f(-2) = 2$

27. $f(0) = 4$

Questions 28 and 29 refer to the piecewise function

$$f(x) = \begin{cases} 4 - 2x & \text{for } -3 \le x < 2 \\ 0 & \text{for } 2 \le x \le 3 \\ x + 1 & \text{for } 3 < x \le 5 \end{cases}$$

28. $f(-1) = 6$

29. $f(3) = 0$

II. In Problems 30–36, give an example of a function or functions with the specified properties. Express your answer using formulas, and specify the independent and dependent variables.

30. A system of two linear equations in two variables that has no solution.

31. A system of two linear equations in two variables that has an infinite number of solutions, including the pairs (2, 3) and $(-1, 9)$.

32. A system of two linear inequalities in two variables that has no solution.

33. A system of two linear equations in variables c and r with the ordered pair $(-1, 0)$ as a solution, where c is a function of r.

34. A cost and revenue function that has a breakeven point at (100, $5000).

35. A demand equation (downward sloping) that, with the supply function $p = q + 20$, has an equilibrium point at (20 units, $40).

36. A system of linear inequalities that has as its solution all of quadrant IV (the region where $x > 0$ and $y < 0$).

III. Is each of the statements in Problems 37–40 true or false? If a statement is true, explain how you know. If a statement is false, give a counterexample.

37. Linear systems of equations in two variables always have exactly one pair of numbers as a solution.

38. A linear system of equations in two variables can have exactly two pairs of numbers that are solutions to the system.

39. Any solution(s) to a linear system of equations in two variables is either a point or a line.

40. Solutions to a linear system of inequalities in two variables can be a region in the plane.

CHAPTER 3 REVIEW: PUTTING IT ALL TOGETHER

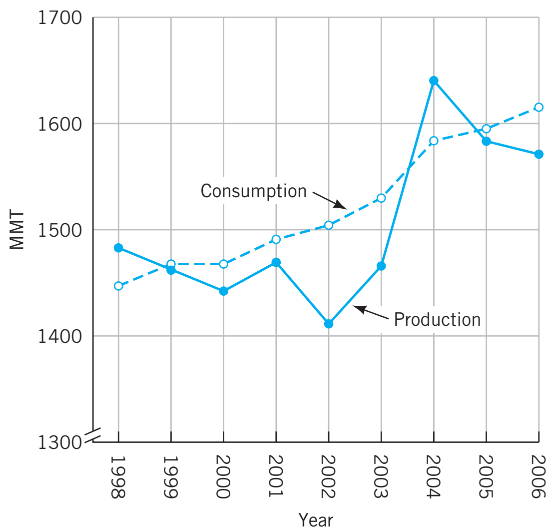
1. The following graph shows worldwide production and consumption of grain in millions of metric tons (MMT). It is based on data from 1998 to May 2006 and includes projections to the end of 2006.

a. Estimate the maximum production of grain in this time period. In what year did it occur? What was the minimum production and in what year did it occur?

b. Explain the meaning of the intersection point(s) in this context.

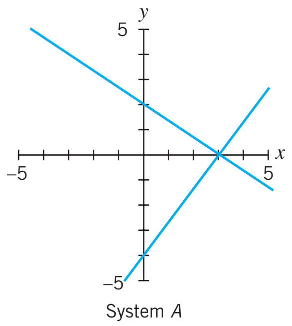
c. In what year was the difference between consumption and production the largest? Was there a surplus or deficit?

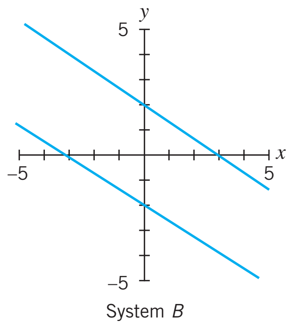
d. Create a title for the graph.



Source: Foreign Agricultural Service Circular Series, FG 05-06, May 2006, *www.fas.usda.gov/grain/circular/2006/05-06/graintoc.htm.*

2. The accompanying graphs show two systems of linear equations.





For each system:

a. Determine the number of solutions using the graph of the system.

b. Construct the equations of the lines.

c. Solve the system using your equations from part (b). Explain why your answer makes sense.

3. a. Construct a system of linear equations where both of the following conditions are met:

i. The coordinates of the point of intersection are (4, 10).

ii. The two lines are perpendicular to each other and one of the lines has a slope of –4.

b. Graph the system of equations you found in part (a). Estimate the coordinates of the point of intersection on your graph. Does your estimate confirm your answer for part (a)?

4. New York City taxi fares are as follows: initial fare \$2.00, \$0.30 per 1/5 mile, and night surcharge (8 p.m.–6 a.m.) of \$0.50.[9]

a. Create and graph the equation for C_d, the cost of a daytime taxi ride for *m* miles.

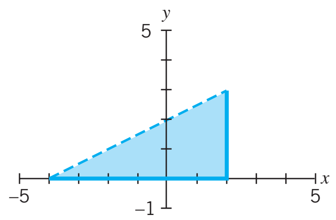
b. Create and graph the equation for C_n, the cost of a night taxi ride for *m* miles.

c. How do the two graphs compare?

5. Graph and shade the region bounded by the following inequalities:

$$y < 2$$
$$y \geq -2x + 3$$
$$x \leq 5$$

6. Determine the inequalities that describe the shaded region in the following graph.



7. A musician produces and sells CDs on her website. She estimates fixed costs of \$10,000, with an additional cost of \$7 to produce each CD. She currently sells the CDs for \$12 each.

a. Create the cost equation, *C*, and revenue equation *R*, in terms of *x* number of CDs produced and sold.

b. Find the breakeven point.

c. By how much should the musician charge for each CD if she wants to break even when producing and selling 1600 CDs, assuming other costs remain the same?

d. By how much would she need to reduce fixed costs if she wants to break even when producing and selling 1600 CDs, assuming other costs and prices remain the same as originally stated?

8. Data from the U.S. Department of Energy show that gasoline prices vary immensely for different countries. These prices include taxes and are in U.S. dollars/gallon of unleaded regular gasoline, standardized on the U.S. gallon.

Year	Germany	Japan	United States
1990	2.65	3.16	1.16
1992	3.26	3.59	1.13
1994	3.52	4.39	1.11
1996	3.94	3.65	1.23
1998	3.34	2.83	1.06
2000	3.45	3.65	1.51
2002	3.67	3.15	1.36
2004	5.24	3.93	1.88

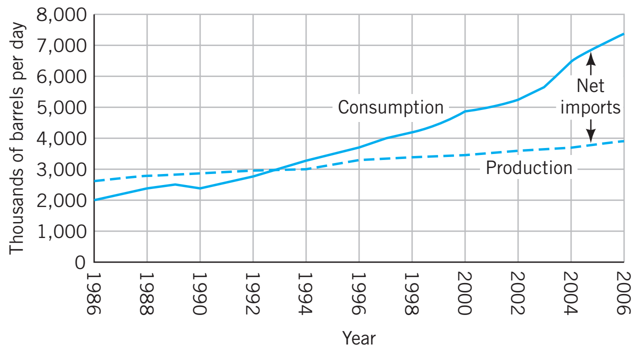
a. Compute the average annual rate of change of gas prices for each country using data for 1990 and 2004.

b. Compute the average (mean) gas price from 1990 to 2004 for each country.

c. Write a paragraph comparing U.S. gas prices in this 14-year period with prices in Japan and Germany.

9. In 2006 China was the world's second largest consumer of oil, behind the United States.

China's Oil Production and Consumption, 1986–2006



Source: EIA International Petroleum Monthly. Data include projections for last four months of 2006.

a. Estimate the amount of oil consumption and production in China in 1990. What is the net difference between production and consumption? What does this mean?

b. Estimate the year when oil consumption equaled oil production. Explain the meaning of this condition.

c. Estimate the amount of oil consumption and oil production in China in 2006. What is the net difference? What does this mean?

10. You keep track of how much gas your car uses and estimate that it gets 32 miles/gallon.

a. If you start out with a full tank of 14 gallons, write a formula for how many gallons of gas *G* are left in the tank after you have gone *M* miles.

b. You consider borrowing your friend's larger car, which gets 18 miles/gallon and holds 20 gallons. Write a formula for how many gallons of gas G_F are left in the tank after you have gone *M* miles. Generate a data table and graph the gas remaining (vertical axis) versus miles (horizontal axis) for both cars.

c. Which car has the longer mileage range?

d. Is there a distance at which they both have the same amount of gas left? If so, what is it?

11. a. Solve algebraically each of the following systems of linear equations.

i. $4x + 21y = -5$
$3x + 7y = -10$

ii. $2.5a - b = 7$
$b = \frac{a}{6}$

b. Create a system of linear equations for which there is no solution.

[9] *http://www.ny.com/transportation/taxis/* gives more detailed information on NYC taxi fares.

The accompanying table is for problems 12 and 13.

Activity	Calories Burned per Minute
Running (moderate pace)	10
Swimming laps	8

Let r and s represent the number of minutes spent running and swimming, respectively.

12. Use the table to answer the following questions. Your goal is to run and swim so that you burn at least 800 calories but no more than 1200 calories.

a. Create a system of inequalities that represents the set of combinations of minutes running and minutes swimming that meet your goal. (*Note:* $r \geq 0$ and $s \geq 0$.)

b. Graph the solution to the system of inequalities using s on the horizontal axis and r on the vertical axis.

c. Give an example of one combination of running and swimming times that is in the solution set and one that is not in the solution set.

d. How would your solution set change if your goal were to burn at least 600 calories but not more than 1000 calories?

13. Use the table to answer the following questions. Your goal is to run and swim so that you spend no more than 60 minutes exercising but burn at least 560 calories.

a. Create a system of inequalities that represents the set of combinations of minutes running and minutes swimming that meet your goal. (*Note:* $r \geq 0$ and $s \geq 0$.)

b. Graph the solution to the system of inequalities with s on the horizontal axis and r on the vertical axis.

c. Give an example of one combination of running and swimming times that is in the solution set and one that is not in the solution set.

d. How would your solution set change if you spend no more than 70 minutes exercising but still burn at least 560 calories?

14. In a 400-meter relay swim, each team has four swimmers. In sequence, the swimmers each swim 100 meters. The total time (cumulative) and rates for team A are provided in the accompanying table.

Swimmer	Total Time, Cumulative (seconds)	Rate (seconds/meter), Rounded	Total Distance, Cumulative (meters)
1	99	0.99	100
2	201	1.02	200
3	279	0.78	300
4	341	0.62	400

a. Which swimmer is the fastest? The slowest?

b. The total swim time function can be written as

$$t(m) = \begin{cases} 0.99m & \text{for } 0 \leq m \leq 100 \\ 99 + 1.02(m - 100) & \text{for } 100 \leq m \leq 200 \\ 201 + 0.78(m - 200) & \text{for } 200 < m \leq 300 \\ 279 + 0.62(m - 300) & \text{for } 300 < m \leq 400 \end{cases}$$

Find and interpret the following: $t(50)$, $t(125)$, $t(250)$, $t(400)$.

15. The accompanying table lists the monthly charge, the number of minutes allowed, and the charge per additional minute for three wireless phone plans.

Cell Phone Plan	Monthly Charge	Number of Minutes	Overtime Charge/Minute
A	\$39.99	450	\$0.45
B	\$59.99	900	\$0.40
C	\$79.99	1350	\$0.35

Let m be the number of minutes per month, $0 \leq m \leq 2500$.

a. Construct a piecewise function for the cost $A(m)$ for plan A.

b. Construct a piecewise function for the cost $B(m)$ for plan B.

c. Construct a piecewise function for the cost $C(m)$ for plan C.

d. Complete the following table to determine the best plan for each of the estimated number of minutes per month.

Number of Minutes/Month	Cost		
	Plan A	Plan B	Plan C
500			
800			
1000			

16. It is often said that 1 year of a dog's life is equivalent to 7 years of a human's life. A more accurate veterinarian's estimate is that for the first 2 years of a dog's life, each dog year is equivalent to 10.5 years of a human's life, and after 2 years each dog year is equivalent to 4 years of a human's life.

a. Write formulas to describe human-equivalent years as a function of dog years for the two methods of relating dog years to human years. Use H and D for the popular formula, and H_v and D_v for the veterinary method.

b. At what dog age do the 2 methods give the same human years, and what human age is that?

17. The time series at the top of the next page shows the price per barrel of crude oil from 1861 to 2006 in both dollars actually spent (*nominal*) and dollars adjusted for inflation (*real* 2006 dollars). Write a 60-second summary about crude oil prices.

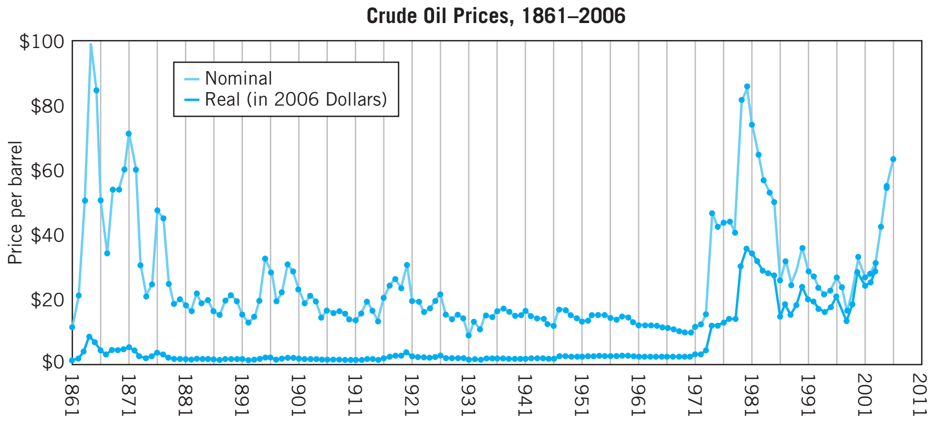
18. Suppose a flat tax amounts to 10% of income. Suppose a graduated tax is a fixed \$1000 for any income $\leq$ \$20,000 plus 20% of any income over \$20,000.

a. Construct functions for the flat tax and the graduated tax.

b. Construct a small table of values:

Income	Flat Tax	Graduated Tax
\$0		
\$10,000		
\$20,000		
\$30,000		
\$40,000		

c. Graph both tax plans and estimate any point(s) at which the two plans would be equal.



Source: http://en.wikipedia.org/wiki/Image:Oil_Prices_1861_2006.jpg Graph created by Michael Ströck, 2006. Released under the GFDL.

d. Use the function definitions to calculate any point(s) at which the two plans would be equal.

e. For what levels of income would the flat tax be more than the graduated tax? For what levels of income would the graduated tax be more than the flat tax.

f. Are there any conditions in which an individual might have negative income under either of the above plans, that is, the amount of taxes would exceed an individual's income? This is not as strange as it sounds. For example, many states impose a minimum corporate tax on a company, even if it is a small, one-person operation with no income in that year.

19. You check around for the best deal on your prescription medicine. At your local pharmacy it costs \$4.39 a bottle; by mail-order catalog it costs \$3.85 a bottle, but there is a flat shipping charge of \$4.00 for any size order; by a source found on the Internet it costs \$3.99 a bottle and shipping costs \$1.00 for each bottle, but for orders of ten or more bottles it costs \$3.79 a bottle, and handling is \$2.50 per order plus shipping costs of \$1.00 for each bottle.

a. Find a formula to express each of these costs if N is the number of bottles purchased and C_p, C_c, and C_I are the respective costs for ordering from the pharmacy, catalog, and Internet.

b. Graph the costs for purchases up to thirty bottles at a time. Which is the cheapest source if you buy fewer than ten bottles at a time? If you buy more than ten bottles at a time? Explain.

20. Older toilets use about 7 gallons per flush. Since using this much pure water to transport human waste is especially undesirable in areas of the country where water is scarce, new toilets now must meet a water conservation standard and are designed to use 1.6 gallons or less per flush.

a. An old toilet that leaks about 2 cups of water an hour from the tank to the bowl and uses 7 gallons/flush is replaced with a new toilet, which does not leak and uses 1.6 gallons/flush. There are 16 cups in a gallon. For each toilet write an equation for daily water loss, W (gallons), as a function of number of flushes per day, F.

b. Graph the equations in part (a) on the same plot with F on the horizontal axis, where $0 \le F \le 25$. From your graph, estimate the amount of water used by each system for 20 flushes a day, and check your estimates with your equations. What is the net difference? What does this mean?

c. If a family flushes the toilet an average of 10 times a day, how much water do they save every day by replacing their leaky old toilet with a new water saver toilet? How much water would be saved over a year?

d. In England during World War II the citizens were asked to flush their toilets only once every five times the toilet was used, in order to save water. A pencil was hung near the toilet, and each user made a vertical mark on the wall, the fifth user made a horizontal line through the last four marks and then flushed the toilet. If this method were used today for the family with the old leaky toilet, would the amount saved be greater than the amount saved with the new toilet if the new toilet is flushed after every use?

21. Regular aerobic exercise at a target heart rate is recommended for maintaining health. Someone starting an exercise program might begin at an intensity level of 60% of the target rate and work up to 70%; athletes need to work at 85% or higher. The American College of Sports Medicine method to compute target heart rate, H (in beats per minute), is based on maximum heart rate, H_{max}, age in years, A, and exercise intensity level, I, where

$$H_{max} = 220 - A$$

and $$H = I \cdot H_{max}$$

thus $$H = I \cdot (220 - A)$$

a. Write a formula for beginners' target heart rate H_b if the intensity level is 60%, that is $I = 0.60$.

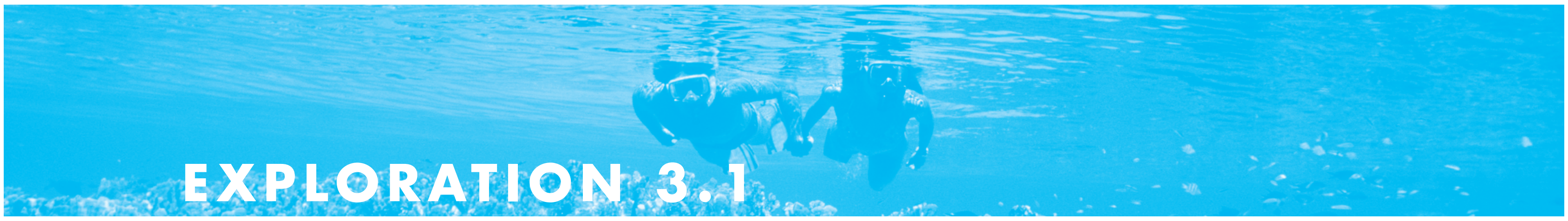
b. Write a formula for intermediate target heart rate H_i if the intensity level is 70%, that is $I = 0.70$.

c. Write a formula for athletes' target heart rate H_a if the intensity level is 85%, that is $I = 0.85$.

d. Construct a graph showing all three heart rate levels, H_b, H_i, H_a, and the maximum rate, H_{max}, for ages 15 to 75 years. Put age on the horizontal axis. Mark the zone in which athletes should work.

e. Compute the target heart rate for a 20-year-old to work at all three levels. What is H_{max} for a 20-year-old?

f. If a 65-year-old is working at a heart rate of 134 beats/minute, what is her intensity level? Has she exceeded the maximum heart rate for her age? If her 45-year-old son is working at the same heart rate, what is his intensity level? If her 25-year-old granddaughter is also working at the same heart rate, what is her intensity level?



EXPLORATION 3.1

Flat vs. Graduated Income Tax: Who Benefits?

Objectives

- compare the effects of different tax plans on individuals in different income brackets
- interpret intersection points

Material/Equipment

- spreadsheet or graphing calculator (optional). If using a graphing calculator, see examples on graphing piecewise functions in the Graphing Calculator Manual.
- graph paper

Related Readings

"How a Flat Tax Would Work, for You and for Them," *The New York Times,* Jan. 21, 1996.
"Flat Tax Goes from 'Snake Oil' to G.O.P. Tonic," *The New York Times,* Nov. 14, 1999.
"A Taxation Policy to Make John Stuart Mill Weep," *The New York Times,* April 18, 2004.
"Your Real Tax Rate," *www.msnmoney.com*, Feb. 21, 2007.

Procedure

A variety of tax plans were debated in all recent elections for president. (See related readings.) One plan recommended a flat tax of 19% on income after exemptions and deductions. In this exploration we examine who benefits from this flat tax as opposed to the current graduated tax plan. The questions we explore are

For what income will taxes be equal under both plans?
Who will benefit under the graduated income tax plan compared with the flat tax plan?
Who will pay more taxes under the graduated plan compared with the flat tax plan?

In a Small Group or with a Partner

The accompanying table gives the 2006 federal graduated tax rates on income after deductions for single people.

1. Construct a function for flat taxes of 19%, where income, i, is income after deductions.
2. Construct a piecewise linear function for the graduated federal tax for single people in 2006, where income, i, is income after deductions.
3. Graph your two functions on the same grid. Estimate from your graph any intersection points for the two functions.
4. Use your equations to calculate more accurate values for the points of intersection.
5. (Extra credit.) Use your results to make changes in the tax plans. Decide on a different income for which taxes will be equal under both plans. You can use what you know about the distribution of income in the United States from the FAM1000 data to make your decision. Alter one or both of the original functions such that both tax plans will generate the same tax given the income you have chosen.

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2006 Tax Rate Schedule for Single Persons

Schedule X—Use if your filing status is **Single**

If the amount on Form 1040, line 40, is: *Over—*	*But not over—*	Enter on Form 1040, line 41	*of the amount over—*
$0	$7,550	10%	$0
$7,550	$30,650	$755+ 15%	$7,550
$30,650	$74,200	$4,220+ 25%	$30,650
$74,200	$154,800	$15,107.50+ 28%	$74,200
$154,800	$336,550	$37,675.50+ 33%	$154,800
$336,550	No limit	$97,653.00+ 35%	$336,550

Source: www.irs.gov.

Analysis

- Interpret your findings. Assume deductions are treated the same under both the flat tax plan and the graduated tax plan. What do the intersection points tell you about the differences between the tax plans?
- What information would be useful in deciding on the merits of each of the plans?
- What if the amount of deductions that most people can take under the flat tax is less than the graduated tax plan? How will your analysis be affected?

Exploration-Linked Homework

Reporting Your Results

Take a stance for or against a flat federal income tax. Using supportive quantitative evidence, write a 60-second summary for a voters' pamphlet advocating your position. Present your arguments to the class.