

## GROWTH AND DECAY: AN INTRODUCTION TO EXPONENTIAL FUNCTIONS

## OVERVIEW

Exponential and linear functions are used to describe quantities that change over time. Exponential functions represent quantities that are multiplied by a constant factor during each time period. Linear functions represent quantities to which a fixed amount is added (or subtracted) during each time period. Exponential functions can model such diverse phenomena as bacteria growth, radioactive decay, compound interest rates, inflation, musical pitch, and family trees.

## After reading this chapter you should be able to

recognize the properties of exponential functions and their graphs understand the differences between exponential and linear growth model growth and decay phenomena with exponential functions represent exponential functions using percentages, factors, or rates use semi-log plots to determine if data can be modeled by an exponential function

### 5.1 Exponential Growth

The program "E2: Exponential Growth \& Decay" in Exponential \& Log Functions offers a dramatic visualization of growth and decay.

Growth of E. coli Bacteria

| Time Periods (of 20 minutes each) | Number of <br> E. coli <br> Bacteria |
| :---: | :---: |
| 0 | 100 |
| 1 | 200 |
| 2 | 400 |
| 3 | 800 |
| 4 | 1,600 |
| 5 | 3,200 |
| 6 | 6,400 |
| 7 | 12,800 |
| 8 | 25,600 |
| 9 | 51,200 |
| 10 | 102,400 |
| 11 | 204,800 |
| 12 | 409,600 |
| 13 | 819,200 |
| 14 | 1,638,400 |
| 15 | 3,276,800 |
| 16 | 6,553,600 |
| 17 | 13,107,200 |
| 18 | 26,214,400 |
| 19 | 52,428,800 |
| 20 | 104,857,600 |
| 21 | 209,715,200 |
| 22 | 419,430,400 |
| 23 | 838,860,800 |
| 24 | 1,677,721,600 |

Table 5.1

## The Growth of E. coli Bacteria

Measuring and predicting growth is of concern to population biologists, ecologists, demographers, economists, and politicians alike. The growth of bacteria provides a simple model that scientists can generalize to describe the growth of other phenomena such as cells, countries, or capital.

Bacteria are very tiny, single-celled organisms that are by far the most numerous organisms on Earth. One of the most frequently studied bacteria is $E$. coli, a rod-shaped bacterium approximately $10^{-6}$ meter (or 1 micrometer) long that inhabits the intestinal tracts of humans and other mammals. ${ }^{1}$ The cells of E. coli reproduce by a process called fission: The cell splits in half, forming two "daughter cells."

Under ideal conditions $E$. coli divide every 20 minutes. If we start with an initial population of 100 E. coli bacteria that doubles every 20 -minute time period, we generate the data in Table 5.1. The initial 100 bacteria double to become 200 bacteria at the end of the first time period, double again to become 400 at the end of the second time period, and so on. At the end of the twenty-fourth time period (at $24 \cdot 20$ minutes $=480$ minutes, or 8 hours , the initial 100 bacteria in our model have grown to over 1.6 billion bacteria!

Because the numbers become astronomically large so quickly, we run into the problems we saw in Chapter 4 when graphing numbers of widely different sizes. Figure 5.1 shows a graph of the data in Table 5.1 for only the first ten time periods. We can see from the graph that the relationship between number of bacteria and time is not linear. The number of bacteria seems to be increasing more and more rapidly over time.


Figure 5.1 Growth of E. coli bacteria.

## A mathematical model for E. coli growth

Table 5.1 shows us that the initial number of 100 bacteria repeatedly doubles. If we record in a third column (Table 5.2) the number of times we multiply 2 times the original value of 100 , we begin to see a pattern emerge.

[^0]Pattern in E. coli Growth

| Number of <br> Time Periods | Number of E. coli <br> Bacteria | Generalized Expression |
| :---: | ---: | ---: |
| 0 | 100 | $100=100 \cdot 2^{0}$ |
| 1 | 200 | $100 \cdot 2=100 \cdot 2^{1}$ |
| 2 | 400 | $100 \cdot 2 \cdot 2=100 \cdot 2^{2}$ |
| 3 | 800 | $100 \cdot 2 \cdot 2 \cdot 2=100 \cdot 2^{3}$ |
| 4 | 1,600 | $100 \cdot 2 \cdot 2 \cdot 2 \cdot 2=100 \cdot 2^{4}$ |
| 5 | 3,200 | $100 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2=100 \cdot 2^{5}$ |
| 6 | 6,400 | $100 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2=100 \cdot 2^{6}$ |
| 7 | 12,800 | $100 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2=100 \cdot 2^{7}$ |
| 8 | 25,600 | $100 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2=100 \cdot 2^{9}$ |
| 9 | 51,200 | $100 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2=100 \cdot 2^{10}$ |
| 10 | 102,400 |  |

Table 5.2

Remembering that $2^{0}$ equals 1 by definition, we can describe the relationship by

$$
\text { number of } E \text {. coli bacteria }=100 \cdot 2^{\text {number of time periods }}
$$

If we let $N=$ number of bacteria and $t=$ number of time periods, we can write the equation more compactly as

$$
N=100 \cdot 2^{t}
$$

Since each value of $t$ determines one and only one value for $N$, the equation represents $N$ as a function of $t$. The number 100 is the initial bacteria population. This function is called exponential since the input or independent variable, $t$, occurs in the exponent of the base 2 . The base 2 is the growth factor, or the multiple by which the population grows during each time period. The bacteria double every time period, which means that the amount of increase equals the previous amount. Thus it increases at a rate of $100 \%$ during each time period. If $E$. coli grew unchecked at this pace, the offspring from one cell would cover Earth with a layer a foot deep in less than 36 hours!

## The General Exponential Growth Function

The E. coli growth equation

$$
N=100 \cdot 2^{t}
$$

is in the form

$$
\text { output }=(\text { initial quantity }) \cdot(\text { growth factor })^{\text {input }}
$$

Such an equation, with the input in the exponent and a growth factor $>1$, describes an exponential growth function.

## Exponential Growth Function

An exponential growth function can be represented by an equation of the form

$$
y=C a^{x} \quad(a>1, C>0) \quad \text { where }
$$

$a$, the base, is the growth factor, the amount by which $y$ is multiplied when $x$ increases by 1 .
$C$ is the initial value or $y$-intercept.

## E X A M P L E 1 Constructing an exponential function

If you start with 200 cells that triple during every time period $t$, an equation to model its growth is

$$
N=200 \cdot 3^{t}
$$

E X A M P L E 2 Interpreting an exponential equation Interpret the numbers in the following equation:

$$
Q=\left(4 \cdot 10^{6}\right) \cdot 2.5^{T}
$$

SOLUTION $4 \cdot 10^{6}$, or $4,000,000$, represents an initial population that grows by a factor of 2.5 each time period, $T$. This means the initial population is multiplied by 2.5 each time period, $T$.

EXAMPLE3 Take a piece of ordinary paper, about 0.1 mm in thickness. Now start folding it in half, then in half again, and so on. Assume you could continue indefinitely.
a. How would you model the growth in thickness of the folded paper?
b. How many folds would it take to reach:
i. The height of a human?
ii. The height of the Matterhorn, a famous mountain in Switzerland?
iii. The sun?
iv. The edge of the known universe?
a. Each time we fold the paper, we double the thickness. If $N=$ number of folds, the initial thickness $=0.1 \mathrm{~mm}$, and $T=$ thickness of the folded paper (in mm), then

$$
T=(0.1) \cdot 2^{N}
$$

b. If you start plugging in values for $N$ (this is where technology is useful), it takes 14 folds to reach the height of an average person (well, a short person). At 26 folds, the paper is higher than the Matterhorn. At 51 folds, the paper would reach the sun, and at 54 folds the edge of our solar system. It would take only 84 folds to reach the limits of the Milky Way. A mere 100 folds takes you to the edge of the known universe!

## Looking at Real Growth Data for E. coli Bacteria

Figure 5.2 shows a plot of real $E$. coli growth over 24 time periods ( 8 hours). The growth appears to be exponential for the first 12 time periods ( 4 hours). We can use technology to generate a best-fit exponential function for that section of the curve. The function is

$$
N=\left(1.37 \cdot 10^{7}\right) \cdot 1.5^{t}
$$

where $N=$ number of cells per milliliter and $t=$ the number of 20-minute periods. So the initial quantity is $1.37 \cdot 10^{7}$ (or 13.7 million) cells. More important, the growth factor is 1.5 . Every 20 minutes the number of cells is multiplied by 1.5 (a $50 \%$ increase), as opposed to the doubling (a $100 \%$ increase) in our first, idealized example.

The bacterial growth rate in the laboratory is half that of the idealized data. But even this rate can't be sustained for long. Conditions are rarely ideal; bacteria die, nutrients are used up, space to grow is limited. In the real world, growth that starts out as exponential must eventually slow down (see Figure 5.2). The curve flattens out as the number of bacteria reaches its maximum size, called the carrying capacity. The overall shape is called sigmoid. The arithmetic of exponentials leads to the inevitable conclusion that in the long term-for bacteria, mosquitoes, or humans-the rate of growth for populations must approach zero.


Figure 5.2 Sigmoid curve of real E. coli growth over 24 time periods and best-fit exponential for 12 time periods

## Algebra Aerobics 5.1

A calculator that can evaluate powers is recommended for Problems 3 and 4.

1. Identify the initial value and the growth factor in each of the following exponential growth functions:
a. $y=350 \cdot 5^{x}$
b. $Q=25,000 \cdot 1.5$
c. $P=\left(7 \cdot 10^{3}\right) \cdot 4^{t}$
d. $N=5000 \cdot 1.025^{t}$
2. Write an equation for an exponential growth function where:
a. The initial population is 3000 and the growth factor is 3 .
b. The initial population is $4 \cdot 10^{7}$ and the growth factor is 1.3 .
c. The initial population is 75 and the population quadruples during each time period.
d. The initial amount is $\$ 30,000$ and the growth factor is 1.12 .
3. The population growth of a small country is described by the function $P=28 \cdot 1.065^{t}$, where $t$ is in years and $P$ is in millions.
a. Determine $P$ when $t=0$. What does that quantity represent?
b. Determine $P$ when $t=10,20$, and 30 .
c. Estimate the value of $t$ that would result in a doubling of the population to $56,000,000$. If available, use technology to check your estimate.
4. Fill in a table of values for the amount $A=80 \cdot 1.06^{t}$ for values of the time period $t$ of $0,1,10,15$, and 20 , and then complete the following statements.
a. The initial amount is $\qquad$ -.
b. During the first time period, the amount grew from 80 to $\qquad$
c. Based on this table, the amount doubles between the values $t=$ $\qquad$ and $t=$
d. Estimate the number of time periods it will take the amount to double.

## Exercises for Section 5.1

1. The following exponential functions represent population growth. Identify the initial population and the growth factor.
a. $Q=275 \cdot 3^{T}$
b. $P=15,000 \cdot 1.04^{t}$
c. $y=\left(6 \cdot 10^{8}\right) \cdot 5^{x}$
d. $A=25(1.18)^{t}$
e. $P(t)=8000(2.718)^{t}$
f. $f(x)=4 \cdot 10^{5}(2.5)^{x}$
2. Write the equation of each exponential growth function in the form $y=C a^{x}$ where:
a. The initial population is 350 and the growth factor is $\frac{4}{3}$.
b. The initial population is $5 \cdot 10^{9}$ and the growth factor is 1.25 .
c. The initial population is 150 and the population triples during each time period.
d. The initial population of 2 quadruples every time period.
3. Fill in the missing parts of the table.

| Initial <br> Value $C$ | Growth Factor $a$ | Exponential <br> Function $y=C a^{x}$ |
| :---: | :---: | :---: |
| 1600 | 1.05 | $y=6.2 \cdot 10^{5}(2.07)^{x}$ <br> $\quad 3.25$ | | $y=1400(\ldots)^{x}$ |
| :--- |

4. The populations of four towns for time $t$, in years, are given by:

$$
\begin{aligned}
& P_{1}(t)=12,000(1.05)^{t} \\
& P_{2}(t)=6000(1.07)^{t} \\
& P_{3}(t)=100,000(1.01)^{t} \\
& P_{4}(t)=1000(1.9)^{t}
\end{aligned}
$$

a. Which town has the largest initial population?
b. Which town has the largest growth factor?
c. At the end of 10 years, which town would have the largest population?
5. Find $C$ and $a$ such that the function $f(x)=C a^{x}$ satisfies the given conditions.
a. $f(0)=6$ and for each unit increase in $x$, the output is multiplied by 1.2 .
b. $f(0)=10$ and for each unit increase in $x$, the output is multiplied by 2.5 .
6. In the United States during the decade of the 1990s, live births to unmarried mothers, $B$, grew according to the exponential model $B=1.165 \cdot 10^{6}(1.013)^{t}$, where $t$ is the number of years after 1990.
a. What does the model give as the number of live births to unwed mothers in 1990 ?
b. What was the growth factor?
c. What does the model predict for the number of live births to unwed mothers in 1995? In 2000 ?
7. A cancer patient's white blood cell count grew exponentially after she had completed chemotherapy treatments. The equation $C=63(1.17)^{d}$ describes $C$, her white blood cell count per milliliter, $d$ days after the treatment was completed.
a. What is the white blood cell count growth factor?
b. What was the initial white blood cell count?
c. Create a table of values that shows the white blood cell counts from day 0 to day 10 after the chemotherapy.
d. From the table of values, approximate when the number of white blood cells doubled.
8. Match each function with its graph.
a. $y=2(1.5)^{x}$
b. $y=3(1.5)^{x}$
c. $y=2(3)^{x}$
d. $y=4(2)^{x}$


9. National health care expenditures in 2005 were approximately $\$ 2016$ billion and are expected to increase by a factor of 1.076 per year. In 5 years what would be the predicted expenditures?
10. The per-capita consumption of bottled water was 8 gallons in 1990 and has been increasing yearly by a factor of 1.088 . What was the per capita consumption of bottled water 12 years later?
11. A tuberculosis culture increases by a factor of 1.185 each hour.
a. If the initial concentration is $5 \cdot 10^{3}$ cells $/ \mathrm{ml}$, construct an exponential function to describe its growth over time.
b. What will the concentration be after 8 hours?
12. An ancient king of Persia was said to have been so grateful to one of his subjects that he allowed the subject to select his own reward. The clever subject asked for a grain of rice on the first square of a chessboard, two grains on the second square, four on the next, and so on.
a. Construct a function that describes the number of grains of rice, $G$, as a function of the square, $n$, on the chessboard. (Note: There are 64 squares.)
b. Construct a table recording the numbers of grains of rice on the first ten squares.
c. Sketch your function.
d. How many grains of rice would the king have to provide for the 64th (and last) square?
13. A new species of fish is introduced into a pond. The size of the fish population can be modeled by the accompanying graph of the function $P(t)$, where $t$ is time in months.

a. What is the initial population of the fish?
b. How long did it take for the initial fish population to double?
c. What was the sustainable fish population (carrying capacity) of the pond?
d. Estimate the number of months it took for the fish population to reach its sustainable size.
14. The Northern Wildlife Prairie Research Center in Jamestown, North Dakota, measured the weights of three duckling
species (wild mallard, gadwall, and blue-winged teal). Each one-day-old duckling weighed about 32 grams. Weights were tracked and are graphed on the following chart. The weight curves appear sigmoidal, each approaching a maximum sustainable (or mature) weight.


Source: http://www.npwrc.usgs.gov.
a. Estimate the mature weight for the female ducklings in each species.
b. Estimate the mature weight for the male ducklings in each species.
c. Approximately how long did it take for each species of ducklings to attain the mature weight?

### 5.2 Linear vs. Exponential Growth Functions

## Linear vs. Exponential Growth

Linear growth is intrinsically additive. Linear growth means that for each unit increase in the input, we must $a d d$ a fixed amount (the slope or rate of change) to the value of the output. For example, in the linear function $N=100+2 t$, each time we increase $t$ by 1 , we add 2 to the value of $N$. After $t$ time periods, we will have added $2 t$ to the initial value of $N$. Assuming $t$ is a positive integer, we can write this linear function as

$$
N=100+\underbrace{2+2+\cdots+2}_{t \text { times }}
$$

Exponential growth is multiplicative, which means that for each unit increase in the input, we multiply the value of the output by a fixed number (the growth factor). For example, in the exponential function $N=100 \cdot 2^{t}$, each time we increase $t$ by 1 , we multiply the value of $N$ by 2 . After $t$ time periods have elapsed, we will have multiplied the value of $N$ by $2^{t}$. If $t$ is a positive integer, we can write this exponential function as

$$
N=100 \cdot \underbrace{2 \cdot 2 \cdots \cdot 2}_{t \text { times }}
$$

## Comparing the general equations

There are both similarities and differences in the general forms of the equations for linear and exponential functions.


Figure 5.3 Graph comparing linear and exponential growth.

If $y$ is a linear function of $x$, then

$$
\begin{aligned}
& y=b+m x \\
& y=y \text {-intercept }+(\text { slope }) \cdot x \\
& y=b+\underbrace{(m+m+m+\cdots+m)}_{x \text { times }}
\end{aligned}
$$

If $y$ is an exponential function of $x$, then

$$
\text { and if } x \text { is a positive integer } \begin{aligned}
& y=C \cdot a^{x} \\
& y=y \text {-intercept } \cdot(\text { base })^{x} \\
& y=C \cdot \underbrace{(a \cdot a \cdot a \cdot a \cdots \cdot a)}_{x \text { times }}
\end{aligned}
$$

Linear functions involve repeated additions, whereas exponential functions involve repeated products. In each case, $x$ determines the number of repetitions. For both linear and exponential equations, the vertical intercept gives the initial or starting value.

Linear vs. Exponential Functions
A linear function represents a quantity to which a constant amount is added for each unit increase in the input.

An exponential function represents a quantity that is multiplied by a constant factor for each unit increase in the input.

## In the long run, exponential growth will always outpace linear growth

Exponential growth is more rapid as time goes on, as can be seen in Table 5.3. For both the linear function $N=100+2 t$ and the exponential function $N=100 \cdot 2^{t}$, when $t=0$, then $N=100$. After ten time periods, the values for $N$ are strikingly different: 102,400 for the exponential function versus 120 for the linear function. The initial value of 100 has been multiplied by 2 ten times in the exponential function to get $100 \cdot 2^{10}$. In the linear function, 2 has been added to 100 ten times, to get $100+2 \cdot 10$.

Comparing the Additive Pattern of Linear Growth to the Multiplicative Pattern of Exponential Growth

| Time, $t$ | Linear Function, $N=100+2 t$ Pattern | Exponential Function, $N=100 \cdot 2^{t}$ Pattern |  |  |
| :---: | :---: | :---: | :---: | :---: |
| 0 | $100+2 \cdot 0=100$ | $100 \cdot 2^{0}$ | = | 100 |
| 1 | $100+2 \cdot 1=102$ | $100 \cdot 2^{1}$ | = | 200 |
| 2 | $100+2 \cdot 2=104$ | $100 \cdot 2^{2}$ | = | 400 |
| 3 | $100+2 \cdot 3=106$ | $100 \cdot 2^{3}$ | $=$ | 800 |
| 4 | $100+2 \cdot 4=108$ | $100 \cdot 2^{4}$ | $=$ | 1,600 |
| 5 | $100+2 \cdot 5=110$ | $100 \cdot 2^{5}$ | = | 3,200 |
| 6 | $100+2 \cdot 6=112$ | $100 \cdot 2^{6}$ | $=$ | 6,400 |
| 7 | $100+2 \cdot 7=114$ | $100 \cdot 2^{7}$ | $=$ | 12,800 |
| 8 | $100+2 \cdot 8=116$ | $100 \cdot 2^{8}$ | = | 25,600 |
| 9 | $100+2 \cdot 9=118$ | $100 \cdot 2^{9}$ | $=$ | 51,200 |
| 10 | $100+2 \cdot 10=120$ | $100 \cdot 2^{10}$ | = | 102,400 |

Table 5.3

If we compare any linear growth function (whose graph will be a straight line with a positive slope) to any exponential growth function (whose graph will curve upward),
we see that sooner or later the exponential curve will permanently lie above the linear graph and continue to grow faster and faster (Figure 5.3). The exponential function eventually dominates the linear function.

## Comparing the Average Rates of Change

Another way to compare linear and exponential functions is to examine average rates of change. Recall that if $N$ is a function of $t$, then

$$
\text { average rate of change }=\frac{\text { change in } N}{\text { change in } t}=\frac{\Delta N}{\Delta t}
$$

Examine Table 5.4, which contains average rates of change for both the linear and exponential functions, where $\Delta t=1$. For all linear functions, we know that the average rate of change is constant. For the linear function $N=100+2 t$, we can tell from the equation, Table 5.4, and Figure 5.4(a) that the (average) rate of change is constant at two units. Average rates of change for the exponential function $N=100 \cdot 2^{t}$ are calculated in Table 5.4 and then graphed in Figure 5.4(b). These suggest that the average rates of change of an exponential growth function grow exponentially.

## Comparing Average-Rate-of-Change Calculations

|  | Linear Function |  |  | Exponential Function |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $N=100+2 t$ | Average Rate of <br> Change (between <br> $t$ |  | $N=100 \cdot 2^{t}$ | Average Rate of <br> Change (between |
| $t$ | $N$ | $t-1$ and $t)$ |  | $N$ | $t-1$ and $t$ ) |

Table 5.4


Figure 5.4 (a) Graph of the average rates of change between several pairs of points that satisfy the linear function $N=100+2 t$. (b) Graph of the average rates of change between several pairs of points that satisfy the exponential function $N=100 \cdot 2^{t}$.

## A Linear vs. an Exponential Model through Two Points

A town's population increased from 20,000 to 24,000 over a 5 -year period. The town council is concerned about the rapid population growth and wants to predict future population size. The functions most commonly used to predict growth patterns over time are linear and exponential.

If we let $t=$ number of years and $P=$ population size, then when $t=0$, $P=20,000$ and when $t=5, P=24,000$. This gives us two points, $(0,20000)$ and ( 5,24000 ), which we will use to construct our models.

## Constructing a linear model: Adding a constant amount each year

Assuming linear growth, our function will be of the form $P=b+m t$. The initial population, $b$, is 20,000 . The average rate of change or slope, $m$, through the two points is

$$
m=\frac{\text { change in population }}{\text { change in time }}=\frac{24,000-20,000}{5-0}=\frac{4000}{5}=800 \text { people } / \mathrm{yr}
$$

So the equation is

$$
P=20,000+800 t
$$

This linear model says that the original population of 20,000 is increasing by the constant amount of 800 people each year.

## Constructing an exponential model: Multiplying by a constant factor each year

Assuming exponential growth and an initial population of 20,000, our function will be of the form

$$
P=20,000 \cdot a^{t}
$$

where $P$ is the population of the town and $t$ the number of years. To complete our model, we need a value for $a$, the annual growth factor.

Finding the Growth Factor from Two Data Points. We have enough information to find the 5 -year growth factor. The inputs 0 and 5 years give us two outputs, 20,000 and 24,000 people, respectively. Exponential growth is multiplicative, so for each additional year, the initial population is multiplied by the growth factor, $a$. After 5 years, the initial population of 20,000 is multiplied by $a^{5}$ to get 24,000 .

$$
\begin{array}{lrl}
\text { Given } & 20,000 \cdot a^{5} & =24,000 \\
\text { divide by } 20,000 & a^{5} & =\frac{24,000}{20,000}=1.2
\end{array}
$$

So the 5 -year growth factor, $a^{5}$, is 1.2. Using a calculator, we can find the annual growth factor, $a$.

Take the fifth root of both sides

$$
\begin{aligned}
\left(a^{5}\right)^{1 / 5} & =(1.2)^{1 / 5} \\
a^{1} & \approx 1.0371
\end{aligned}
$$

Now we can represent our exponential function as

$$
P=20,000 \cdot(1.0371)^{t}
$$

where $t=$ number of years and $P=$ population size. Our exponential model tells us the initial population of 20,000 is multiplied by 1.0371 each year.

## Making predictions with our models

We can use the models not only to describe past behavior, but also to predict future population sizes (see Table 5.5 and Figure 5.5). The population size is the same in both models for year 0 and year 5. For year 10, the linear and exponential predictions are

Town Population

| Year | Linear Model <br> $P=20,000+800 t$ | Exponential Model <br> $P=20,000(1.0371)^{t}$ |
| :---: | :---: | :---: |
| 0 | 20,000 | 20,000 |
| 5 | 24,000 | 24,000 |
| 10 | 28,000 | 28,790 |
| 15 | 32,000 | 34,540 |
| 20 | 36,000 | 41,440 |
| 25 | 40,000 | 49,720 |

Table 5.5


Figure 5.5 Predictions for the town population using linear and exponential models.
fairly close: 28,000 versus approximately 28,790 . As we move further beyond year 10 , the exponential predictions exceed the linear by an increasingly greater amount. By year 25 , the linear model predicts a population of 40,000 whereas the exponential model predicts almost 50,000.

Both models should be considered as generating only crude future estimates, particularly since we constructed the models using only two data points. Clearly, the further out we try to predict, the more unreliable the estimates become.

## Identifying Linear vs. Exponential Functions in a Data Table

Table 5.5 shows a linear function side by side with an exponential function. Every 5 years, the linear model added 800 people five times, adding a total of 4000 to the population. Every 5 years, the exponential model multiplied the population size by the annual growth factor 1.0371 five times; that is, the population size is multiplied by $1.0371^{5} \approx 1.2$. The 1 -year growth factor $=1.0371$ and the 5 -year growth factor $=1.0371^{5} \approx 1.2$.

If $P_{1}$ is the starting population size and $P_{2}$ is the population size 5 years later, then

| In the linear model | $P_{1}+4000=P_{2}$ |
| :--- | :--- |
| or equivalently | $4000=P_{2}-P_{1}$ |
| In the exponential model | $P_{1} \cdot 1.2=P_{2}$ |
| or equivalently | $1.2=P_{2} / P_{1}$ |

Using these equations we can translate our statements about addition (for linear) and multiplication (for exponential) into statements about differences and ratios.

In general, given any data table of the form $(x, y)$ where $\Delta x$ is constant, then the data are linear if the difference between successive $y$ values is constant. The data are exponential if the ratio of successive $y$ values is constant. We can test whether a data table represents a linear or exponential function by using differences or ratios, as long as the $\Delta x$ values are constant.

EXAMPLE1 Determine whether each function in the table is linear or exponential.

| $x$ | 0 | 10 | 20 | 30 | 40 |
| :---: | ---: | :---: | :---: | :---: | :---: |
| $f(x)$ | 5 | 25 | 45 | 65 | 85 |
| $g(x)$ | 20 | 30 | 45 | 67.5 | 101.25 |

Table 5.6

SOLUTIO N The consecutive $x$ values or input in Table 5.6 are ten units apart, so $\Delta x=10$. Now we compare the output values. The difference between successive values of $f(x)$ is always 20 , so $f(x)$ is a linear function. The difference between successive values of $g(x)$ is not constant. For example,

$$
\begin{aligned}
g(10)-g(0) & =30-20=10 \\
\text { but } \quad g(20)-g(10) & =45-30=15
\end{aligned}
$$

So $g(x)$ cannot be linear. To determine if $g(x)$ is exponential, we calculate the ratios of successive values of $g(x)$. For example,

$$
\frac{30}{20}=\frac{45}{30}=\frac{67.5}{45}=\frac{101.25}{67.5}=1.5
$$

The ratios of consecutive $y$ values are constant at 1.5 when $\Delta x=10$ (this ratio represents the 10 -year growth factor). Hence the function is exponential.

For a table describing $y$ as a function of $x$, where $\Delta x$ is constant: If the difference between consecutive $y$ values is constant, the function is linear.

If the ratio of consecutive $y$ values is constant, the function is exponential.

## Algebra Aerobics 5.2

1. Fill in the following table and sketch the graph of each function.

| $t$ | $N=10+3 t$ | $N=10 \cdot 3^{t}$ |
| :--- | :--- | :--- |
| 0 |  |  |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |

2. a. Create a table of values (from $t=0$ to $t=5$ ) for the functions $f(t)$, with a vertical intercept at 200 and a constant rate of change of 20 , and $g(t)$, with a vertical intercept at 200 and a growth factor of 1.20 .
b. Sketch the graphs of $g$ and $f$ on the same grid.
c. Compare the graphs.
3. Given the values in Table 5.7, determine which functions (if any) are linear and which are approximately exponential. Justify your answer.

| $T$ | $f(t)$ | $g(t)$ | $h(t)$ | $p(t)$ | $r(t)$ |
| :---: | :---: | :---: | :--- | ---: | :--- |
| 0 | 10 | 10 | 10 | 10 | 10 |
| 1 | 16 | 10 | 15 | 100 | 10.4 |
| 2 | 22 | 10 | 22.5 | 190 | 10.82 |
| 3 | 28 | 10 | 33.75 | 280 | 11.25 |
| 4 | 34 | 10 | 50.63 | 370 | 11.70 |
| 5 | 40 | 10 | 75.94 | 460 | 12.17 |

Table 5.7
4. Write an equation for the linear function and the exponential function that pass through the given points.
a. $(0,500)$ and $(1,620)$
b. $(0,3)$ and $(1,3.2)$
5. In each of the following situations, assume growth is exponential, find the growth factor, and then construct an exponential function that models the situation.
a. The initial population, $P$, is 1500 and one time period, $t$, later, the population grew to 3750 .
b. The initial amount, $A$, is $\$ 80,000$ and one time period, $t$, later, the amount grew by $\$ 2300$.
c. A quantity, $Q$, grew from 30 mg to 32.7 mg after one time period, $t$.
6. Assume in Table 5.8 that $Q$ is an exponential function of $t$.

| $t$ (years) | $Q$ |
| :---: | :---: |
| 0 | 10 |
| 2 | 20 |
| 4 |  |
| 6 |  |
| 8 |  |

Table 5.8
a. By what factor is $Q$ multiplied when $t$ increases by 2 years? This is the 2 -year growth factor.
b. Fill in the rest of the $Q$ values in the table.
c. What is the annual growth factor?
d. Construct an equation to model the relationship between $Q$ and $t$.
b. If the size of the herd increases by a factor of 1.8 each year, find the formula for the deer population, $Q(t)$, over time.
c. For each model create a table of values for the deer population for a 10-year period.
d. Using the table, estimate when the two models predict the same population size.
5. Construct both a linear and an exponential function that go through the points $(0,6)$ and $(1,9)$.
6. Construct both a linear and an exponential function that go through the points $(0,200)$ and $(10,500)$.
7. Find the equation of the linear function and the exponential function that are sketched through two points on each of the following graphs.


8. Match the description to any appropriate graph(s) shown below.
a. When $x$ increases by $10, y$ increases by 5 .
b. When $x$ increases by $10, y$ is multiplied by a factor of 5 .

9. Each of the following tables contains values representing either linear or exponential functions. Find the equation for each function.

a. \begin{tabular}{cccccc}
\hline$x$ \& -2 \& -1 \& 0 \& 1 \& 2 <br>

b. |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $f(x)$ | 1.12 | 2.8 | 7 | 17.5 | 43.75 |
|  | -2 | -1 | 0 | 1 | 2 |
|  | $g(x)$ | 0.1 | 0.3 | 0.5 | 0.7 | <br>

\hline
\end{tabular}

10. Each table has values representing either linear or exponential functions. Find the equation for each function.

a. | $x$ | -2 | -1 | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $h(x)$ | 160 | 180 | 200 | 220 | 240 |
|  |  |  |  |  |  |
| $x$ | 0 | 10 | 20 | 30 | 40 |
| $j(x)$ | 200 | 230 | 264.5 | 304.17 | 349.8 |

11. (Graphing program recommended.) Create a table of values for the following functions, then graph the functions.
a. $f(x)=6+1.5 x$
b. $g(x)=6(1.5)^{x}$
c. $h(x)=1.5(6)^{x}$
12. Suppose you are given a table of values of the form $(x, y)$ where $\Delta x$, the distance between two consecutive $x$ values, is constant. Why is calculating $y_{2}-y_{1}$, the distance between two consecutive $y$ values, equivalent to calculating the average rate of change between consecutive points?
13. Mute swans were imported from Europe in the nineteenth century to grace ponds. Now there is concern that their population is growing too rapidly, edging out native species. Their population along the Atlantic coast has grown from 5800 in 1986 to 14,313 in 2002. The increase is most acute in the mid-Atlantic region, but Massachusetts has also seen a jump, with 2939 mute swans counted in 2002 as compared with 585 in 1986.
a. Compare the growth factor in the mute swan population for the entire Atlantic coast with that for Massachusetts.
b. Compare the average rate of change in the mute swan population for the entire Atlantic coast with that for Massachusetts.
c. Construct both a linear and an exponential model for the mute swan population in Massachusetts since 1986.
d. Compare the projected populations of mute swans in Massachusetts by the year 2010 as predicted by your linear and exponential models.
14. The price of a home in Medford was $\$ 100,000$ in 1985 and rose to $\$ 200,000$ in 2005.
a. Create two models, $f(t)$ assuming linear growth and $g(t)$ assuming exponential growth, where $t=$ number of years after 1985.
b. Fill in the following table representing linear growth and exponential growth for $t$ years after 1985.

|  | Linear Growth <br> $f(t)=$ Price | Exponential Growth <br> $g(t)=$ Price <br> (in thousands of dollars) |
| ---: | :---: | :---: |
| $t$ | (in thousands of dollars) | 100 |
| 0 | 100 |  |
| 20 |  | 200 |
| 30 | 200 |  |
| 40 |  |  |
| 50 |  |  |

c. Which model do you think is more realistic?
15. The Mass Media, a student publication at the University of Massachusetts-Boston, reported on a proposed parking fee increase. The university administration recommended gradually increasing the daily parking fee on this campus to $\$ 7.00$ by the year 2004, followed by an increase of 5\% every year after that. Call this plan A. Several other plans were also proposed; one of them, plan B, recommended that every year after 2004 the rate be increased by 50 cents each year.
a. Let $t=0$ for year 2004 and fill in the chart for parking fees under plans A and B.

| Years <br> after 2004 | Parking Fee under <br> Plan A | Parking Fee under <br> Plan B |
| :---: | :---: | :---: |
| 0 | $\$ 7.00$ | $\$ 7.00$ |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |

b. Write an equation for parking fees $F_{A}$ as a function of $t$ (years since 2004) for plan A and an equation $F_{B}$ for plan B.
c. What will the daily parking fee be by the year 2025 under each plan? (Show your calculations.)
d. Imagine that you are the student representative to the UMass Board of Trustees. Which plan would you recommend for adoption? Explain your reasons and support your position using quantitative arguments.

### 5.3 Exponential Decay

An exponential growth function is of the form $y=C a^{x}$, where $a>1$. Each time $x$ increases by 1 , the population is multiplied by $a$, a number greater than 1 , so the population increases. But what if $a$ were positive and less than 1 , that is, $0<a<1$ ? Then each time $x$ increases by 1 and the population is multiplied by $a$, the population size will be reduced. We would have exponential decay.

## The Decay of lodine-131

Iodine-131 is one of the radioactive isotopes of iodine that is used in nuclear medicine for diagnosis and treatment. It can be used to test how well the thyroid gland is functioning. An exponential function can be used to describe the decay of iodine-131. After being generated by a fission reaction, iodine-131 decays into a nontoxic, stable isotope. Every 8 days the amount of iodine-131 remaining is cut in half. For example, we show in Table 5.9 and Figure 5.6 how 160 milligrams (mg) of iodine-131 decays over four time periods (or $4 \cdot 8=32$ days).

| Time Periods <br> (8 days each) | Amount of <br> Iodine-131 (mg) |  | General Expression |
| :---: | :---: | :---: | :---: |

Table 5.9


Figure 5.6 Exponential decay of iodine-131.

We can describe the decay of iodine-131 with the equation

$$
Q=160 \cdot\left(\frac{1}{2}\right)^{T}
$$

where $Q=$ quantity of iodine-131 (in mg ) and $T=$ number of time periods (of 8 days each). The number 160 is the initial amount of iodine-131 and $\frac{1}{2}$ is the decay factor.

## The General Exponential Decay Function

The equation for $Q$ is in the form

$$
\text { output }=(\text { initial quantity }) \cdot(\text { decay factor })^{\text {input }}
$$

This is the standard format for an exponential decay function.

## Exponential Decay Function

An exponential decay function $y=f(x)$ can be represented by an equation of the form

$$
y=C a^{x} \quad(0<a<1 \text { and } C>0) \quad \text { where }
$$

$a$ is the decay factor, the amount by which $y$ is multiplied when $x$ increases by 1 . $C$ is the initial value or $y$-intercept.

E X A M PLE 1 a. How would you model the decay of 500 mg of iodine-131?
b. How many milligrams would be left after four time periods (or 32 days)?

SOLUTION a. The initial value is 500 mg , and we know from the previous discussion that the decay factor is $\frac{1}{2}$ for each time period $T$ (of 8 days). So the decay function is $Q=500 \cdot\left(\frac{1}{2}\right)^{T}$.
b. When $T=4$, then $Q=500 \cdot\left(\frac{1}{2}\right)^{4}=500 \cdot\left(\frac{1}{16}\right)=31.25$. So after four time periods ( 32 days) there would be 31.25 mg of iodine-131 remaining.

EXAMPLE 2 Identify the initial value and the decay factor for the function $N=\left(3 \cdot 10^{4}\right) \cdot(0.25)^{T}$. Interpret the decay factor.

SOLUTION The initial value is $3 \cdot 10^{4}$ (or 30,000 ) and the decay factor is 0.25 (or $1 / 4$ ). So each time $T$ increases by 1 , the value for $N$ is one-fourth of its previous value.

EXAMPLE3 Which of the following exponential functions represent growth and which represent decay? Identify the growth or decay factor.
a. $P(t)=2000(1.05)^{t}$
b. $Q(t)=25(0.75)^{t}$
c. $N(t)=16\left(\frac{2}{3}\right)^{t}$
d. $f(x)=5(4)^{-x}$

SOLUTION a. $P(t)$ represents exponential growth because $1.05>1$; the term 1.05 is the growth factor.
b. $Q(t)$ represents exponential decay because $0<0.75<1$; the term 0.75 is the decay factor.
c. $N(t)$ represents exponential decay because $0<\frac{2}{3}<1$; the term $\frac{2}{3}$ is the decay factor.
d. We can rewrite the function as $f(x)=5\left(\frac{1}{4}\right)^{x}$. Thus $f(x)$ represents exponential decay because $0<\frac{1}{4}<1$; the term $\frac{1}{4}$ is the decay factor.

E X M P L E 4 Finding an exponential decay function through two points: Measuring caffeine levels

When you drink an 8-ounce cup of coffee, virtually all the caffeine is absorbed in your gut and passes through the liver and into your bloodstream, acting as a stimulant. The peak blood levels of caffeine are reached in about 30 minutes-and then they begin to fall exponentially. Five hours after the peak, your blood contains 60 milligrams of caffeine. Fifteen hours after the peak your blood contains 15 milligrams of caffeine.
a. Construct a function to model the caffeine decrease in your bloodstream over time.
b. How much caffeine was in the original cup?
c. Generate a sketch of the caffeine levels in your bloodstream over time.
a. The maximum caffeine blood levels occur about 30 minutes after drinking the coffee. Since the decrease in caffeine is exponential from that point on, we can model the decline with an exponential decay function. Let $t=$ number of hours after reaching maximum caffeine blood level and $C=$ amount of caffeine in your blood (mg). Five hours after reaching your maximum level, your caffeine level is 60 mg . Fifteen hours after the maximum level, your caffeine level is 15 mg . Assuming $C$ is a function of $t$, the two points $(5,60)$ and $(15,15)$ should satisfy that function.

If we let $a=$ the hourly decay factor, then we can find the 10 -hour decay factor, $a^{10}$, using the ratio of the two caffeine levels.

$$
a^{10}=\frac{\text { caffeine at } t=15 \mathrm{hr}}{\text { caffeine at } t=5 \mathrm{hr}}=\frac{15 \mathrm{mg}}{60 \mathrm{mg}}=0.25
$$

So the 10-hour decay factor is $a^{10}=0.25$
Taking the 10th root
$\left(a^{10}\right)^{1 / 10}=(0.25)^{1 / 10}$
Using a calculator
$a \approx 0.871, \quad$ the hourly decay factor

So the exponential decay function is of the form

$$
C=C_{0} \cdot(0.871)^{t}
$$

where $C_{0}=$ the peak amount of caffeine and $t=$ number of hours since the peak caffeine level.
b. To find $C_{0}$ we can substitute either of the two points $(5,60)$ and $(15,15)$ known to satisfy our function.

| Given | $C$ | $=C_{0} \cdot(0.871)^{t}$ |
| :--- | ---: | :--- |
| Substitute $(5,60)$ | 60 | $=C_{0} \cdot(0.871)^{5}$ |
|  | Use a calculator | 60 |
|  | $\approx C_{0} \cdot 0.501$ |  |
| Solve for $C_{0}$ | $C_{0}$ | $\approx \frac{60}{0.501} \approx 120 \mathrm{mg}$ |

So the peak amount of caffeine in your blood was approximately 120 mg . Since virtually all the caffeine in the coffee is passed into the bloodstream, your original cup contained about 120 mg of caffeine. ${ }^{2}$
c.


Figure 5.7 Blood caffeine levels after drinking 8 oz of coffee.
${ }^{2}$ Note to coffee drinkers: Consumption of 250 mg of caffeine ( 2 to 3 cups) is considered a moderate amount; over 800 mg of caffeine is considered excessive. Various factors can influence the rate of caffeine decay. For a pregnant woman or a woman on oral contraceptives, the decrease in caffeine slows down considerably; a little caffeine takes a long time to pass through the body. For smokers, the decrease in caffeine is speeded up, so they can actually drink more coffee without feeling its side effects.

Translating the Growth (or Decay) Factor from $n$ Time Units to One Time Unit If $a$ is the growth (or decay) factor for $n$ time units, then
$a^{1 / n}$
is the growth (or decay) factor for one time unit.
Examples: If 2.3 is the 10 -year growth factor, then $2.3^{1 / 10} \approx 1.0869$ is the annual growth factor.
If 0.7 is the 5 -month decay factor, then $0.7^{1 / 5} \approx 0.9311$ is the one-month decay factor.

## Algebra Aerobics 5.3

1. Which of the following exponential functions represent growth and which represent decay?
a. $y=100 \cdot 3^{x}$
b. $f(t)=75 \cdot\left(\frac{2}{3}\right)^{t}$
c. $w=250 \cdot(0.95)^{r}$
d. $g(r)=\left(2 \cdot 10^{6}\right) \cdot(1.15)^{r}$
e. $y=\left(7 \cdot 10^{9}\right) \cdot(0.20)^{2}$
f. $h(x)=150 \cdot\left(\frac{5}{2}\right)^{x}$
2. Write an equation for an exponential decay function where:
a. The initial population is 2300 and the decay factor is $\frac{1}{3}$.
b. The initial population is $3 \cdot 10^{9}$ and the decay factor is 0.35 .
c. The initial population is 375 and the population drops to one-tenth its previous size during each time period.
3. Does the exponential function $y=12 \cdot(5)^{-x}$ represent growth or decay? (Hint: Rewrite the function in the standard form $y=C a^{x}$.)
4. Rewrite each of the following using the general form and indicate whether the function represents growth or decay.
a. $y=23(2.4)^{-x}$
b. $f(x)=8000(0.5)^{-x}$
c. $P=52,000(1.075)^{-t}$
5. Given $f(x)=100(0.9)^{x}$ and $g(x)=100(0.7)^{x}$,
a. Which function decreases more rapidly?
b. By what percentage does each function decrease each time period?

## Exercises for Section 5.3

A graphing program is recommended for Exercise 15.

1. Identify and interpret the decay factor for each of the following functions:
a. $P=450(0.43)^{t}$
b. $f(t)=3500(0.95)^{t}$
c. $y=21(3)^{-x}$
2. Which of the following exponential functions represent growth and which decay?
a. $N=50 \cdot 2.5^{T}$
b. $y=264(5 / 2)^{x}$
c. $R=745(1.001)^{t}$
d. $g(z)=\left(3 \cdot 10^{5}\right) \cdot(0.8)^{z}$
e. $f(T)=\left(1.5 \cdot 10^{11}\right) \cdot(0.35)^{T}$
f. $h(x)=2000\left(\frac{2}{3}\right)^{x}$
3. Write an equation for an exponential decay function where:
a. The initial population is 10,000 and the decay factor is $\frac{2}{5}$.
b. The initial population is $2.7 \cdot 10^{13}$ and the decay factor is 0.27 .
c. The initial population is 219 and the population drops to one-tenth its previous size during each time period.
4. The accompanying tables here and on the next page show approximate values for the four exponential functions: $\quad f(x)=5\left(2^{x}\right), g(x)=5\left(0.7^{x}\right), h(x)=6\left(1.7^{x}\right)$, and $j(x)=6\left(0.6^{x}\right)$. Which table is associated with each function?

| Function $\boldsymbol{A}$ |  |
| ---: | ---: |
| $x$ | $y$ |
| -2 | 16.67 |
| -1 | 10.00 |
| 0 | 6.00 |
| 1 | 3.60 |
| 2 | 2.16 |

Function B

| $x$ | $y$ |
| ---: | ---: |
| -2 | 10.2 |
| -1 | 7.1 |
| 0 | 5.0 |
| 1 | 3.5 |
| 2 | 2.5 |

Function $C$

| $x$ | $y$ |
| ---: | ---: |
| -2 | 1.25 |
| -1 | 2.50 |
| 0 | 5.00 |
| 1 | 10.00 |
| 2 | 20.00 |

Function $D$

| $x$ | $y$ |
| ---: | ---: |
| -2 | 2.1 |
| -1 | 3.5 |
| 0 | 6.0 |
| 1 | 10.0 |
| 2 | 17.3 |

5. Determine which of the following functions are exponential. For each exponential function, identify the growth or decay factor and the vertical intercept.
a. $y=5\left(x^{2}\right)$
b. $y=100 \cdot 2^{-x}$
c. $P=1000(0.999)^{t}$
6. Determine which of the following functions are exponential. Identify each exponential function as representing growth or decay and find the vertical intercept.
a. $A=100\left(1.02^{t}\right)$
b. $f(x)=4\left(3^{x}\right)$
c. $g(x)=0.3\left(10^{x}\right)$
d. $y=100 x+3$
e. $M=2^{p}$
f. $y=x^{2}$
7. Fill in the missing parts of the table.

| Initial <br> Value, $C$ | Decay Factor, $a$ | Exponential <br> Function <br> $y=C a^{x}$ |
| :---: | :---: | :---: |
| 500 | 0.95 | $y=\left(1.72 \cdot 10^{6}\right)(0.75)^{x}$ |
|  | 0.25 | $y=1600(\ldots)^{x}$ |

8. a. Complete the following table for the exponential function $y=20(0.75)^{x}$.

|  |  | Difference | Ratio <br> $y_{2}-y_{1}$ |
| :--- | :--- | :---: | :---: |
| $y$ | $\frac{y_{2}}{y_{1}}$ |  |  |
| 0 | 20 |  |  |
| 1 | 15 |  |  |
| 2 | 11.25 |  |  |
| 3 | 8.4375 |  |  |
| 4 | 6.328125 |  |  |

b. Choose the correct word in each italicized pair to describe the function:
For the exponential function $y=20(0.75)^{x}$, the differences are constant/decreasing in magnitude and the ratios are constant/decreasing in magnitude.
9. Match each function with its graph.
a. $y=100(0.8)^{x}$

b. $y=100(0.5)^{x}$

10. The per capita (per person) consumption of milk was 27.6 gallons in 1980 and has been steadily decreasing by an annual decay factor of 0.99 .
a. Form an exponential function for per capita milk consumption $M(t)$ for year $t$ after 1980.
b. According to your function, what was the per capita consumption of milk in 2000? If available, use the Internet to check your predictions.
11. The U.S. Department of Agriculture's data on per capita food commodity consumption for 1980 are listed in the following table.
a. Using the data in the following table, construct exponential functions for each food category. Then evaluate each function for the year 2000. Assume $t$ is the number of years since 1980 .

|  | Per Capita <br> Consumption <br> (pounds) in | Yearly <br> Growth/Decay <br> Factor | Exponential <br> Function |
| :--- | :---: | :---: | :---: |
| Beef | 7980 | 0.994 | $B(t)=$ |
| Chicken | 32.1 | 1.024 | $C(t)=$ |
| Pork | 52.1 | 0.996 | $P(t)=$ |
| Fish | 12.4 | 1.010 | $F(t)=$ |

b. Which commodities showed exponential growth? Which showed exponential decay?
c. Write a 60 -second summary about the consumption of meat, chicken, and fish from 1980 to 2000.
12. Find the formula for the exponential function that satisfies the given conditions.
a. $f(0)=4$ and $f(1)=2$
b. $g(0)=6$ and $g(1)=3$
c. $h(0)=100$ and $h(1)=75$
d. $k(0)=12$ and $k(1)=7.2$
13. a. A linear function $f(t)=b+m t$ has a slope of -4 and a vertical intercept of 20 . Find its equation.
b. An exponential function $g(t)=C a^{t}$ has a decay factor of $1 / 4$ and an initial value of 20 . Find its equation.
c. Plot both functions on the same grid.
14. Which of the following functions (if any) are equivalent? Explain your answer.
a. $f(x)=40(0.625)^{x}$
b. $g(x)=40\left(\frac{5}{8}\right)^{x}$
c. $h(x)=40\left(\frac{8}{5}\right)^{-x}$
15. (Graphing program recommended.) Which of the following functions declines more rapidly? Graph the functions on the same grid and check your answer.
a. $f(x)=25(5)^{-x}$
b. $g(x)=25(0.5)^{x}$
16. Find the equation of the exponential function through the indicated points in graphs $A$ and $B$.


17. Plutonium- 238 is used in bombs and power plants but is dangerously radioactive. It decays very slowly into nonradioactive materials. If you started with 100 grams today, a year from now you would have 99.2 grams.
a. Construct an exponential function to describe the decay of plutonium-238 over time.
b. How much of the original 100 grams of plutonium-238 would be left after 50 years? After 500 years?
18. It takes 1.31 billion years for radioactive potassium- 40 to drop to half its original size
a. Construct a function to describe the decay of potassium-40.
b. Approximately what amount of the original potassium-40 would be left after 4 billion years? Justify your answer.

### 5.4 Visualizing Exponential Functions

We can summarize what we've learned so far about exponential functions:

## Exponential Functions

Exponential functions can be represented by equations of the form

$$
y=C a^{x} \quad(a>0, a \neq 1) \quad \text { where }
$$

$C$ is the initial value and $a$ is the base
Assuming $C>0$, then if
$a>1$, the function represents growth and $a$ is called the growth factor.
$0<a<1$, the function represents decay and $a$ is called the decay factor.

## The Effect of the Base a

## Exponential growth: $a>1$

Given an exponential function $y=C a^{x}$, when $a>1$ (and $C>0$ ), the function represents growth. The graph of an exponential growth function is concave up and
increasing. Figure 5.8 shows how the value of $a$ affects the steepness of the graphs of the following functions:

$$
y=100 \cdot 1.2^{x} \quad y=100 \cdot 1.3^{x} \quad y=100 \cdot 1.4^{x}
$$

Each function has the same initial value of $C=100$ but different values of $a$, that is, $1.2,1.3$, and 1.4 , respectively. When $a>1$, the larger the value of $a$, the more rapid the growth and the more rapidly the graph rises. So as the values of $x$ increase, the values of $y$ not only increase, but increase at an increasing rate.


Figure 5.8 Three exponential growth functions.


Figure 5.9 Three exponential decay functions.

## Exponential decay: $0<a<1$

In an exponential function $y=C a^{x}$, when $a$ is positive and less than 1 (and $C>0$ ), we have decay. The graph of an exponential decay function is concave up but decreasing.

The smaller the value of $a$, the more rapid the decay. For example, if $a=0.7$, after each unit increase in $x$, the value of $y$ would be multiplied by 0.7 . So $y$ would drop to $70 \%$ of its previous size-a loss of $30 \%$. But if $a=0.5$, then when $x$ increases by $1, y$ would be multiplied by 0.5 , equivalent to dropping to $50 \%$ of its previous size-a loss of $50 \%$.

Figure 5.9 shows how the value of $a$ affects the steepness of the graph. Each function has the same initial population of 1000 but different values of $a$, that is, 0.3 , 0.5 , and 0.7 , respectively. When $0<a<1$, the smaller the value of $a$, the more rapid the decay and the more rapidly the graph falls.

## The Effect of the Initial Value C

In the exponential function $y=C a^{x}$, the initial value $C$ is the vertical intercept. When $x=0$, then

$$
\begin{aligned}
y & =C a^{0} \\
& =C \cdot 1 \\
& =C
\end{aligned}
$$

Figure 5.10 compares the graphs of three exponential functions with the same base $a$ of 1.1 , but with different $C$ values of 50,100 , and 250 , respectively.


Figure 5.10 Three exponential functions with the same growth factor but different $y$-intercepts, or initial values.

In Exploration 5.1, you can examine what happens when $C<0$.

Changing the value of $C$ changes where the graph of the function will cross the vertical axis.

E X A M PLE 1 Match each of the graphs $(A$ to $D)$ in Figure 5.11 to one of the following equations and explain your answers.
a. $f(x)=1.5(2)^{x}$
b. $g(x)=1.5(3)^{x}$
c. $j(x)=5(0.6)^{x}$
d. $k(x)=5(0.8)^{x}$


Figure 5.11 Graphs of four exponential functions.

SOLUTION $A$ is the graph of $k(x)=5(0.8)^{x}$. $B$ is the graph of $j(x)=5(0.6)^{x}$.
Reasoning: Graphs $A$ and $B$ both have a vertical intercept of 5 and represent exponential decay. The steeper graph $(B)$ must have the smaller decay factor (in this case 0.6 ). $C$ is the graph of $g(x)=1.5(3)^{x} . D$ is the graph of $f(x)=1.5(2)^{x}$.

Reasoning: Graphs $C$ and $D$ both have a vertical intercept of 1.5 and both represent exponential growth. The steeper graph $(C)$ must have the larger growth factor (in this case 3 ).

## Horizontal Asymptotes

In the exponential decay function $y=1000 \cdot\left(\frac{1}{2}\right)^{x}$, graphed in Figure 5.12, the initial population of 1000 is cut in half each time $x$ increases by 1 . So when $x=0,1,2,3,4,5$, $6, \ldots$, the corresponding $y$ values are $1000,500,250,125,62.5,31.25,15.625, \ldots$ As $x$ gets larger and larger, the $y$ values come closer and closer to, but never reach, zero. Thus as $x$ approaches positive infinity $(x \rightarrow+\infty), y$ approaches zero $(y \rightarrow 0)$. The graph of the function $y=1000\left(\frac{1}{2}\right)^{x}$ is said to be asymptotic to the $x$-axis, or we say the $x$-axis is a horizontal asymptote to the graph. In general, a horizontal asymptote is a horizontal line that the graph of a function approaches for extreme values of the input or independent variable.


Figure 5.12 The $x$-axis is a horizontal asymptote for both exponential growth and decay functions of the form $y=C a^{x}$.

Exponential decay functions of the form $y=C a^{x}$ are asymptotic to the $x$-axis. Similarly, for exponential growth functions, as $x$ approaches negative infinity $(x \rightarrow-\infty)$, $y$ approaches zero $(y \rightarrow 0)$. Examine, for instance, the exponential growth function $y=1000 \cdot 2^{x}$, also graphed in Figure 5.12. When $x=0,-1,-2$, $-3, \ldots$, the corresponding $y$ values are $1000,1000 \cdot 2^{-1}=500,1000 \cdot 2^{-2}=250$, $1000 \cdot 2^{-3}=125, \ldots$ So as $x \rightarrow-\infty$, the $y$ values come closer and closer to but never reach zero. So exponential growth functions are also asymptotic to the $x$-axis.

## Graphs of Exponential Functions

For functions in the form $y=C a^{x}$,
The value of $C$ tells us where the graph crosses the $y$-axis.
The value of $a$ affects the steepness of the graph.
Exponential growth $(a>1, C>0)$. The larger the value of $a$, the more rapid the growth and the more rapidly the graph rises.

Exponential decay $(0<a<1, C>0)$. The smaller the value of $a$, the more rapid the decay and the more rapidly the graph falls.

The graphs are asymptotic to the $x$-axis.
Exponential growth: As $x \rightarrow-\infty, y \rightarrow 0$
Exponential decay: As $x \rightarrow+\infty, y \rightarrow 0$

## Algebra Aerobics 5.4

1. a. Draw a rough sketch of each of the following functions all on the same graph:

$$
\begin{aligned}
& y=1000(1.5)^{x} \\
& y=1000(1.1)^{x} \\
& y=1000(1.8)^{x}
\end{aligned}
$$

b. Do the three curves intersect? If so, where?
c. In the first quadrant (where $x>0$ and $y>0$ ), which curve should be on the top? Which in the middle? Which on the bottom?
d. In the second quadrant (where $x<0$ and $y>0$ ), which curve should be on the top? Which in the middle? Which on the bottom?
2. a. Draw a rough sketch of each of the following functions all on the same graph:
$Q=250(0.6)^{t} \quad Q=250(0.3)^{t} \quad Q=250(0.2)^{t}$
b. Do the three curves intersect? If so, where?
c. In the first quadrant, which curve should be on the top? Which in the middle? Which on the bottom?
d. In the second quadrant, which curve should be on the top? Which in the middle? Which on the bottom?
3. a. Draw a rough sketch of each of the following functions on the same graph:

$$
P=50 \cdot 3^{t} \quad \text { and } \quad P=150 \cdot 3^{t}
$$

b. Do the curves intersect anywhere?
c. Describe when and where one curve lies above the other.
4. Identify any horizontal asymptotes for the following functions:
a. $Q=100 \cdot 2^{t}$
c. $y=100-15 x$
b. $g(r)=\left(6 \cdot 10^{7}\right) \cdot(0.95)^{r}$
5. Which of these functions has the most rapid growth? Which the most rapid decay?
a. $f(t)=100 \cdot 1.06^{t}$
b. $g(t)=25 \cdot 0.89^{t}$
c. $h(t)=4000 \cdot 1.23^{t}$
d. $r(t)=45.9 \cdot 0.956^{t}$
e. $P(t)=32,000 \cdot 1.092^{t}$
6. Examine the graphs of the exponential growth functions $f(x)$ and $g(x)$ in Figure 5.13.


Figure 5.13 Graphs of $f(x)$ and $g(x)$.
a. Which graph has the larger initial value?
b. Which graph has the larger growth factor?
c. As $x \rightarrow-\infty$, which graph approaches zero more rapidly?
d. Approximate the point of intersection. After the point of intersection, as $x \rightarrow+\infty$, which function has larger values?

## Exercises for Section 5.4

A graphing program is required for Exercises 9 and 10, and recommended for Exercises 11, 15, and 16.

1. Each of the following three exponential functions is in the standard form $y=C \cdot a^{x}$.

$$
y=2^{x} \quad y=5^{x} \quad y=10^{x}
$$

a. In each case identify $C$ and $a$.
b. Specify whether each function represents growth or decay. In particular, for each unit increase in $x$, what happens to $y$ ?
c. Do all three curves intersect? If so, where?
d. In the first quadrant, which curve should be on top? Which in the middle? Which on the bottom?
e. Describe any horizontal asymptotes.
f. For each function, generate a small table of values.
g. Graph the three functions on the same grid and verify that your predictions in part (d) are correct.
2. Repeat Exercise 1 for the functions

$$
y=(0.5) 2^{x}, y=2 \cdot 2^{x}, \text { and } y=5 \cdot 2^{x}
$$

3. Repeat Exercise 1 for the functions

$$
y=3^{x}, \quad y=\left(\frac{1}{3}\right)^{x}, \text { and } y=3 \cdot\left(\frac{1}{3}\right)^{x}
$$

4. Match each equation with its graph (at the top of the next page).

$$
\begin{array}{ll}
f(x)=30 \cdot 2^{x} & h(x)=100 \cdot 2^{x} \\
g(x)=30 \cdot(0.5)^{x} & j(x)=50 \cdot(0.5)^{x}
\end{array}
$$


5. Below are graphs of four exponential functions. Match each function with its graph.
$P=5 \cdot(0.7)^{x}$
$R=10 \cdot(1.8)^{x}$
$Q=5 \cdot(0.4)^{x}$
$S=5 \cdot(3)^{x}$




6. Examine the accompanying graphs of three exponential growth functions.

a. Order the functions from smallest growth factor to largest.
b. What point do all of the functions have in common? Will they share any other points?
c. As $x \rightarrow-\infty$, will the function with the largest growth factor approach zero more slowly or more quickly than the other functions?
d. For $x>0$, the graph of which function remains on top?
7. Examine the accompanying graphs of three exponential functions.

a. Order the functions from smallest decay factor to largest.
b. What point do all of the functions have in common? Will they share any other points?
c. As $x \rightarrow+\infty$, will the function with the largest decay factor approach zero more slowly or more quickly than the other functions?
d. When $x<0$, the graph of which function remains on top?
8. Generate quick sketches of each of the following functions, without the aid of technology.

$$
\begin{array}{ll}
f(x)=4(3.5)^{x} & g(x)=4(0.6)^{x} \\
h(x)=4+3 x & k(x)=4-6 x
\end{array}
$$

a. As $x \rightarrow+\infty$, which function(s) approach $+\infty$ ?
b. As $x \rightarrow+\infty$, which function(s) approach 0 ?
c. As $x \rightarrow-\infty$, which function(s) approach $-\infty$ ?
d. As $x \rightarrow-\infty$, which function(s) approach 0 ?
9. (Graphing program required.) Graph the functions $f(x)=30+5 x$ and $g(x)=3(1.6)^{x}$ on the same grid. Supply the symbol $<$ or $>$ in the blank that would make the statement true.
a. $f(0)$ $\qquad$ $g(0) \quad$ e. $f(-6)$ $\qquad$ $g(-6)$
b. $f(6)$ $\qquad$ $g(6)$ f. As $x \rightarrow+\infty, \quad f(x)$ $\qquad$ $g(x)$
c. $f(7)$ $\qquad$ $g(7)$ g. As $x \rightarrow-\infty, \quad f(x)$ $\qquad$ $g(x)$
d. $f(-5)$ $\qquad$ $g(-5)$
10. (Graphing program required.) Graph the functions $f(x)=6(0.7)^{x}$ and $g(x)=6(1.3)^{x}$ on the same grid. Supply the symbol $<,>$, or $=$ in the blank that would make the statement true.
a. $f(0)$ $\qquad$ $g(0)$
b. $f(5)$ $\qquad$ $g(5)$
c. $f(-5)$ $\qquad$ $g(-5)$
d. As $x \rightarrow+\infty, f(x)$ $\qquad$ $g(x)$
e. As $x \rightarrow-\infty, f(x)$ $\qquad$ $g(x)$
11. (Graphing program recommended.)
a. As $x \rightarrow+\infty$, which function will dominate, $f(x)=100+500 x$ or $g(x)=2(1.005)^{x}$ ?
b. Determine over which $x$ interval(s) $g(x)>f(x)$.
12. Two cities each have a population of 1.2 million people. City A is growing by a factor of 1.15 every 10 years, while city $B$ is decaying by a factor of 0.85 every 10 years.
a. Write an exponential function for each city's population $P_{A}(t)$ and $P_{B}(t)$ after $t$ years.
b. For each city's population function generate a table of values for $x=0$ to $x=50$, using 10-year intervals, then sketch a graph of each town's population on the same grid.
13. Which function has the steepest graph?

$$
\begin{aligned}
& F(x)=100(1.2)^{x} \\
& G(x)=100(0.8)^{x} \\
& H(x)=100(1.2)^{-x}
\end{aligned}
$$

14. Examine the graphs of the following functions.

time, the initial number of $E$. coli is multiplied by 1.5 . How can we translate this into a statement involving percentages?

If $Q$ is the number of bacteria at any point in time, then one time period later,

| Rewrite 1.5 as a sum | $=Q(1+0.5)$ |
| :--- | :--- |
| Use the distributive property | $=1 \cdot Q+(0.5) \cdot Q$ |
| Rewrite 0.5 as a percentage | $=Q+50 \%$ of $Q$ |

So for each unit increase in time, the number of bacteria increases by $50 \%$. The term $50 \%$ is called the growth rate in percentage form and 0.5 , the equivalent in decimal form, is called the growth rate in decimal form.

A quantity that increases by a constant percent over each time interval represents exponential growth. We can represent this growth in terms of growth factors or growth rates. Exponential growth requires a growth factor $>1$. If the exponential growth factor $=1$, the object would never grow, but would stay constant through time. So,

$$
\begin{aligned}
& \text { growth factor }=1+\text { growth rate } \\
& \text { growth factor }=1+r
\end{aligned}
$$

where $r$ is the growth rate in decimal form.

E X A M P L 1 Converting the growth rate to a growth factor
The U.S. Bureau of the Census has made a number of projections for the first half of this century. For each projection, convert the growth rate into a growth factor.
a. Disposable income is projected to grow at an annual rate of $2.9 \%$.
b. Nonagriculture employment is projected to grow at $1.0 \%$ per year.
c. Employment in manufacturing is projected to grow at $0.2 \%$ per year.

SOLUTION a. A growth rate of $2.9 \%$ is 0.029 in decimal form. The growth factor is $1+0.029=1.029$.
b. A growth rate of $1.0 \%$ is 0.01 in decimal form. The growth factor is $1+0.01=1.01$.
c. A growth rate of $0.2 \%$ is 0.002 in decimal form. The growth factor is $1+0.002=$ 1.002.

EXAMPLE 2 According to UN statistics, in the year 2005 there were an estimated 3.2 billion people living in urban areas. The number of people living in urban areas is projected to grow at a rate of $1.7 \%$ per year. If this projection is accurate, how many people will be living in urban areas in 2030?

SOLUTION If the urban population is increasing by a constant percent each year, the growth is exponential. To construct an exponential function modeling this growth, we need to find the growth factor. We know that the growth rate $=1.7 \%$ or 0.017 . So,

$$
\text { growth factor }=1+0.017=1.017
$$

Given an initial population of 3.2 billion in 2005 , our model is

$$
U(n)=3.2 \cdot(1.017)^{n}
$$

where $n=$ number of years since 2005 and $U(n)=$ urban population (in billions).
Using our model, the urban population in $2030, U(25) \approx 4.9$ billion people.

## Exponential Decay: Decreasing by a Constant Percent

In Section 5.3, we used the exponential function $C=C_{0} \cdot(0.871)^{t}$ (where $C_{0}=$ the peak amount of caffeine and $t=$ number of hours since peak caffeine level) to model the
amount of caffeine in the body after drinking a cup of coffee. The decay factor, 0.871 , tells us that for each unit increase in time, the amount of caffeine is multiplied by 0.871 . As with exponential growth, we can translate this statement into one that involves percentages.

If $Q$ is the amount of caffeine at any point in time, then one time period later,

| amount of caffeine | $=Q \cdot(0.871)$ |
| :--- | :--- |
| Rewrite 0.871 as a difference | $=Q \cdot(1-0.129)$ |
| Use the distributive property | $=1 \cdot Q-(0.129) \cdot Q$ |
| Rewrite 0.129 as a percentage | $=Q-12.9 \%$ of $Q$ |

So for each unit increase in time, the amount of caffeine decreases by $12.9 \%$. The term $12.9 \%$ is called the decay rate in percentage form and the equivalent decimal form, 0.129 , is called the decay rate in decimal form.

A quantity that decreases by a constant percent over each time interval represents exponential decay. We can represent this decay in terms of decay factors or decay rates. Exponential decay requires a decay factor between 0 and 1 .

$$
\begin{aligned}
\text { decay factor } & =1-\text { decay rate } \\
\text { decay factor } & =1-r
\end{aligned}
$$

where $r$ is the decay rate in decimal form.

## Factors, Rates, and Percentages

For an exponential function in the form $y=C a^{x}$, where $C>0$,
The growth factor $a>1$ can be represented as

$$
a=1+r
$$

where $r$ is the growth rate, the decimal representation of the percent rate of change.
The decay factor $a$, where $0<a<1$, can be represented as

$$
a=1-r
$$

where $r$ is the decay rate, the decimal representation of the percent rate of change.

Examples: If a quantity is growing by $3 \%$ per month, the growth rate $r$ in decimal form is 0.03 , so the growth factor $a=1+0.03=1.03$.

If a quantity is decaying by $3 \%$ per month, the decay rate $r$ in decimal form is 0.03 , so the decay factor $a=1-0.03=0.97$.

E X A M P L E 3 Constructing an exponential decay function
Sea ice that survives the summer and remains year round, called perennial sea ice, is melting at the alarming rate of $9 \%$ per decade, according to a 2006 report by the National Oceanic and Atmospheric Administration. Assuming the percent decrease per decade in sea ice remains constant, construct a function that represents this decline.

SOLUTION If the percent decrease remains constant, then the melting of sea ice represents an exponential function with a decay rate of 0.09 in decimal form. So,

$$
\text { decay factor }=1-0.09=0.91
$$

If $d=$ number of decades since 2006 and $A=$ amount of sea ice (in millions of acres) in 2006, then we can model the amount of sea ice, $S(d)$, by the exponential function:

$$
S(d)=A(0.91)^{d}
$$

E X A M P L E 4 Growth or decay?
In the following functions, identify the growth or decay factors and the corresponding growth or decay rates in both percentage and decimal form.
a. $f(x)=100(0.01)^{x}$
b. $g(x)=230(3.02)^{x}$
c. $p=5.34(0.015)^{n}$
d. $y=8.75(2.35)^{x}$

SOLUTION a. The decay factor is 0.01 and if $r$ is the decay rate in decimal form, then

$$
\begin{aligned}
\text { decay factor } & =1-\text { decay rate } \\
0.01 & =1-r \\
r & =0.99
\end{aligned}
$$

The decay rate is 0.99 , or $99 \%$ as a percentage.
b. The growth factor is 3.02 and if $r$ is the growth rate in decimal form, then

$$
\begin{aligned}
\text { growth factor } & =1+\text { growth rate } \\
3.02 & =1+r \\
3.02-1 & =r \\
r & =2.02
\end{aligned}
$$

The growth rate in decimal form is 2.02 , or $202 \%$ as a percentage.
c. The decay factor is 0.015 and the decay rate is 0.985 in decimal form, or $98.5 \%$ as a percentage.
d. The growth factor is 2.35 and the growth rate is 1.35 in decimal form, or $135 \%$ as a percentage.

## Revisiting Linear vs. Exponential Functions

In Section 5.2 we compared linear vs. exponential functions as additive versus multiplicative. We now have another way to compare them.

Linear vs. Exponential Functions
Linear functions represent quantities that increase or decrease by a constant amount. Exponential functions represent quantities that increase or decrease by a constant percent.

E X A P L E 5 Identifying linear vs. exponential growth
According to industry sources, U.S.wireless data service revenues are growing by an amazing $75 \%$ annually. India is the biggest growth market, increasing by about 6 million cell phones every month. Which of these two statements represents linear and which represents exponential growth? Justify your answers.

SOLUTION Quantities that increase by a constant amount represent linear growth. So, an increase of 6 million cell phones every month describes linear growth. Quantities that increase by a constant percent represent exponential growth. So, an increase of $75 \%$ annually describes exponential growth

E X A M P L E 6 Construct an equation for each description of the population of four different cities.
a. In 1950 , city A had a population of 123,000 people. Each year since 1950 , the population decreased by $0.8 \%$.
b. In 1950, city B had a population of $4,500,000$ people. Since 1950 , the population has grown by approximately 2000 people per year.
c. In 1950, city C had a population of 625,000 . Since 1950 , the population has declined by about 5000 people each decade.
d. In 1950, city D had a population of 2.1 million people. Each decade since 1950, the population has increased by $15 \%$.

SOLUTION For each of the following equations, let $P=$ population (in thousands) and $t=$ years since 1950.
a. $P_{A}=123(0.992)^{t}$
b. $P_{B}=4500+2 t$
where the unit for rate of change is people (in thousands) per year.
c. $P_{C}=625-0.5 t$
where the unit for rate of change is people (in thousands) per year. The rate of change is -0.5 people (in thousands) per year, since the population declined by 5000 people per decade, or 500 people per year.
d. growth rate per decade $=0.15$
growth factor per decade $=1+0.15=1.15$
growth factor per year $=(1.15)^{1 / 10} \approx 1.014$
$P_{D}=2100(1.014)^{t}$

E X A M PLE 7 You are offered a job with a salary of $\$ 30,000$ a year and annual raises of $5 \%$ for the first 10 years of employment. Show why the average rate of change of your salary in dollars per year is not constant.

SOLUTION Let $n=$ number of years since you signed the contract and $S(n)=$ your salary in dollars after $n$ years. When $n=0$, your salary is represented by $S(0)=\$ 30,000$. At the end of your first year of employment, your salary increases by $5 \%$, so

$$
\begin{aligned}
S(1) & =30,000+5 \% \text { of } 30,000 \\
& =30,000+0.05(30,000) \\
& =30,000+1500 \\
& =\$ 31,500
\end{aligned}
$$

At the end of your second year, your salary increases again by $5 \%$, so

$$
\begin{aligned}
S(2) & =31,500+5 \% \text { of } 31,500 \\
& =31,500+0.05(31,500) \\
& =31,500+1575 \\
& =\$ 33,075
\end{aligned}
$$

The average rate of change between years 0 and 1 is

$$
\frac{S(1)-S(0)}{1}=\$ 31,500-\$ 30,000=\$ 1500
$$

The average rate of change between years 1 and 2 is

$$
\frac{S(2)-S(1)}{1}=\$ 33,075-\$ 31,500=\$ 1575
$$

So, the average rates of change differ. This is to be expected, since in the first year the $5 \%$ raise applies only to the initial salary of $\$ 30,000$, whereas in the second year the $5 \%$ raise applies to both the initial $\$ 30,000$ salary and to the $\$ 1500$ raise from the first year.

## Algebra Aerobics 5.5

A calculator that can evaluate powers is recommended for Problem 4.

1. Complete the following statements.
a. A growth factor of 1.22 corresponds to a growth rate of $\qquad$ .
b. A growth rate of $6.7 \%$ corresponds to a growth factor of $\qquad$
c. A decay factor of 0.972 corresponds to a decay rate of $\qquad$
d. A decay rate of $12.3 \%$ corresponds to a decay factor of $\qquad$ -.
2. Fill in the table below.
3. Determine the growth or decay factor and the growth or decay rate in percentage form.
a. $f(t)=100 \cdot 1.06^{t}$
b. $g(t)=25 \cdot 0.89^{t}$
c. $h(t)=4000 \cdot 1.23^{t}$
d. $r(t)=45.9 \cdot 0.956^{t}$
e. $P(t)=32,000 \cdot 1.092^{t}$
4. Find the percent increase (or decrease) if:
a. Hourly wages grew from $\$ 1.65$ per hour to $\$ 6.00$ per hour in 30 years.
b. A player's batting average dropped from 0.299 to 0.167 in one season.
c. The number of new AIDS cases grew from 17 last month to 23 this month

| Exponential <br> Function | Initial Value | Growth or <br> Decay? | Growth or <br> Decay Factor | Growth or <br> Decay Rate (as \%) |
| :---: | :---: | :---: | :---: | :---: |
| $A=4(1.03)^{t}$ |  |  |  |  |
| $A=10(0.98)^{t}$ |  |  | 1.005 |  |
|  | 1000 |  | 0.96 |  |
|  | 30 | Growth |  | $7.05 \%$ |
|  | $\$ 50,000$ | Decay |  | $49 \%$ |

## Exercises for Section 5.5

Access to the internet and a graphing program is required for Exercise 26 part (e). A graphing program is recommended for Exercises 16, 24, and 25

1. Assume that you start with 1000 units of some quantity $Q$. Construct an exponential function that will describe the value of $Q$ over time $T$ if, for each unit increase in $T, Q$ increases by:
a. $300 \%$
b. $30 \%$
c. $3 \%$
d. $0.3 \%$
2. Given each of the following exponential growth functions, identify the growth rate in percentage form:
a. $Q=10,000 \cdot 1.5^{T}$
b. $Q=10,000 \cdot 1.05^{T}$
c. $Q=10,000 \cdot 1.005^{T}$
d. $Q=10,000 \cdot 2^{T}$
e. $Q=10,000 \cdot 2.5^{T}$
f. $Q=10,000 \cdot 3^{T}$
3. Given the following exponential decay functions, identify the decay rate in percentage form.
a. $Q=400(0.95)^{t}$
b. $A=600(0.82)^{t}$
c. $P=70,000(0.45)^{t}$
d. $y=200(0.655)^{x}$
e. $A=10(0.996)^{T}$
f. $N=82(0.725)^{T}$
4. What is the growth or decay factor for each given time period?
a. Weight increases by $0.2 \%$ every 5 days
b. Mass decreases by $6.3 \%$ every year.
c. Population increases $23 \%$ per decade.
d. Profit increases $300 \%$ per year.
e. Blood alcohol level decreases $35 \%$ per hour.
5. Fill in the following chart and then construct exponential functions for each part (a) to (g).

|  | Initial |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| Value | Growth or <br> Decay? | Growth or <br> Decay <br> Factor | Growth or <br> Decay Rate <br> (\% form) |  |
| a. | 600 |  | 2.06 |  |
| b. | 1200 |  |  | $200 \%$ |
| c. | 6000 | Decay |  | $75 \%$ |
| d. | 1.5 million | Decay |  | $25 \%$ |
| e. | 1.5 million | Growth |  | $25 \%$ |
| f. | 7 |  | 4.35 |  |
| g. | 60 |  | 0.35 |  |

6. Match the statements (a) through (d) with the correct exponential function in (e) through (h). Assume time $t$ is measured in the unit indicated.
a. Radon- 222 decays by $50 \%$ every $t$ days.
b. Money in a savings account increases by $2.5 \%$ per year.
c. The population increases by $25 \%$ per decade.
d. The pollution in a stream decreases by $25 \%$ every year.
e. $A=1000(1.025)^{t}$
f. $A=1000(0.75)^{t}$
g. $A=1000\left(\frac{1}{2}\right)^{t}$
h. $A=1000(1.25)^{t}$
7. Each of two towns had a population of 12,000 in 1990. By 2000 the population of town A had increased by $12 \%$ while the population of town B had decreased by $12 \%$. Assume these growth and decay rates continued.
a. Write two exponential population models $A(T)$ and $B(T)$ for towns A and B, respectively, where $T$ is the number of decades since 1990.
b. Write two new exponential models $a(t)$ and $b(t)$ for towns A and B, where $t$ is the number of years since 1990.
c. Now find $A(2), B(2), a(20)$, and $b(20)$ and explain what you have found.
8. On November 25, 2003, National Public Radio did a report on Under Armour, a sports clothing company, stating that their "profits have increased by $1200 \%$ in the last 5 years."
a. Let $P(t)$ represent the profit of the company during every 5-year period, with $A_{0}$ the initial amount. Write the exponential model for the company's profit.
b. Assuming an initial profit of $\$ 100,000$, what would be the profit in year 5? Year 10?
c. Determine the annual growth rate for Under Armour.

Each of the tables in Exercises 9-12 represents an exponential function. Construct that function and then identify the corresponding growth or decay rate in percentage form.

13. Generate equations that represent the pollution levels, $P(t)$, as a function of time, $t$ (in years), such that $P(0)=150$ and:
a. $P(t)$ triples each year.
b. $P(t)$ decreases by twelve units each year.
c. $P(t)$ decreases by $7 \%$ each year.
d. The annual average rate of change of $P(t)$ with respect to $t$ is constant at 1 .
14. Given an initial value of 50 units for parts (a)-(d) below, in each case construct a function that represents $Q$ as a function of time $t$. Assume that when $t$ increases by 1 :
a. $Q(t)$ doubles
b. $Q(t)$ increases by $5 \%$
c. $Q(t)$ increases by ten units
d. $Q(t)$ is multiplied by 2.5
15. Between 1970 and 2000, the United States grew from about 200 million to 280 million, an increase of approximately $40 \%$ in this 30 -year period. If the population continues to expand by $40 \%$ every 30 years, what will the U.S. population be in the year 2030? In 2060? Use technology to estimate when the population of the United States will reach 1 billion.
16. (Graphing program recommended.) According to Mexico's National Institute of Geography, Information and Statistics, in 2005 in Mexico's Quintana Roo state, where the tourist industry in Cancun has created a boom economy, the population was about 1.14 million and growing at a rate of $5.3 \%$ per year. In 2005 in Mexico's Baja California state, where many labor-intensive industries are located next to the California border, the population was about 2.8 million and growing at a rate of $3.3 \%$ per year. The nation's capital, Mexico City, had 19.2 million inhabitants as of 2005 and the population was declining at a rate of $0.012 \%$ per year.
a. Construct three exponential functions to model the growth or decay of Quintana Roo, Baja California, and Mexico City. Identify your variables and their units.
b. Use your functions to predict the populations of Quintana Roo, Baja California, and Mexico City in 2010.
17. A pollutant was dumped into a lake, and each year its amount in the lake is reduced by $25 \%$.
a. Construct a general formula to describe the amount of pollutant after $n$ years if the original amount is $A_{0}$.
b. How long will it take before the pollution is reduced to below $1 \%$ of its original level? Justify your answer.
18. A swimming pool is initially shocked with chlorine to bring the chlorine concentration to 3 ppm (parts per million). Chlorine dissipates in reaction to bacteria and sun at a rate of about $15 \%$ per day. Above a chlorine concentration of 2 ppm , swimmers experience burning eyes, and below a concentration of 1 ppm , bacteria and algae start to proliferate in the pool environment.
a. Construct an exponential decay function that describes the chlorine concentration (in parts per million) over time.
b. Construct a table of values that corresponds to monitoring chlorine concentration for at least a 2 -week period.
c. How many days will it take for the chlorine to reach a level tolerable for swimmers? How many days before bacteria and algae will start to grow and you will need to add more chlorine? Justify your answers.
19. a. If the inflation rate is $0.7 \%$ a month, what is it per year?
b. If the inflation rate is 5\% a year, what is it per month?
20. Rewrite each expression so that no fraction appears in the exponent and each expression is in the form $a^{x}$.
a. $3^{x / 4}$
b. $2^{x / 3}$
c. $\left(\frac{1}{2}\right)^{x / 4}$
d. $\left(\frac{1}{4}\right)^{x / 2}$

Hint: Remember that $a^{x / n}=\left(a^{l / n}\right)^{x}$
21. a. Show that the following two functions are roughly equivalent.

$$
f(x)=15,000(1.2)^{x / 10} \quad \text { and } \quad g(x)=15,000(1.0184)^{x}
$$

b. Create a table of values for each function for $0<x \leq 25$.
c. Would it be correct to say that a $20 \%$ increase over 10 years represents a $1.84 \%$ annual increase? Explain your answer.
22. The exponential function $Q(T)=600(1.35)^{T}$ represents the growth of a species of fish in a lake, where $T$ is measured in 5-year intervals.
a. Determine $Q(1), Q(2)$, and $Q(3)$.
b. Find another function $q(t)$, where $t$ is measured in years.
c. Determine $q(5), q(10)$, and $q(15)$.
d. Compare your answers in parts (a) and (c) and describe your results.
23. Blood alcohol content (BAC) is the amount of alcohol present in your blood as you drink. It is calculated by determining how many grams of alcohol are present in 100 milliliters ( 1 deciliter) of blood. So if a person has 0.08 grams of alcohol per 100 milliliters of blood, the BAC is $0.08 \mathrm{~g} / \mathrm{dl}$. After a person has stopped drinking, the BAC declines over time as his or her liver metabolizes the alcohol. Metabolism proceeds at a steady rate and is impossible to speed up. For instance, an average ( $150-\mathrm{lb}$ ) male metabolizes about 8 to 12 grams of alcohol an hour (the amount in one bottle of beer ${ }^{3}$ ). The behavioral effects of alcohol are closely related to the blood alcohol content. For example, if an average $150-\mathrm{lb}$ male drank two bottles of beer within an hour, he would have a BAC level of 0.05 and could suffer from euphoria, inhibition, loss of motor coordination, and overfriendliness. The same male after drinking four bottles of beer in an hour would be legally drunk with a BAC of 0.10 . He would likely suffer from impaired motor function and decision making, drowsiness, and slurred speech. After drinking twelve beers in one hour, he will have attained the dosage for stupor (0.30) and possibly death (0.40).

The following table gives the BAC of an initially legally drunk person over time (assuming he doesn't drink any additional alcohol).

| Time (hours) | 0 | 1 | 2 | 3 | 4 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| BAC (g/dl) | 0.100 | 0.067 | 0.045 | 0.030 | 0.020 |

a. Graph the data from the table (be sure to carefully label the axes).
b. Justify the use of an exponential function to model the data. Then construct the function where $B(t)$ is the BAC for time $t$ in hours.
c. By what percentage does the BAC decrease every hour?
d. What would be a reasonable domain for your function? What would be a reasonable range?
e. Assuming the person drinks no more alcohol, when does the BAC reach $0.005 \mathrm{~g} / \mathrm{dl}$ ?
${ }^{3}$ One drink is equal to $1 \frac{1}{4} \mathrm{oz}$ of 80 -proof liquor, 12 oz of beer, or 4 oz of table wine.
24. (Graphing program recommended.) If you have a heart attack and your heart stops beating, the amount of time it takes paramedics to restart your heart with a defibrillator is critical. According to a medical report on the evening news, each minute that passes decreases your chance of survival by $10 \%$. From this wording it is not clear whether the decrease is linear or exponential. Assume that the survival rate is $100 \%$ if the defibrillator is used immediately.
a. Construct and graph a linear function that describes your chances of survival. After how many minutes would your chance of survival be $50 \%$ or less?
b. Construct and graph an exponential function that describes your chances of survival. Now after how many minutes would your chance of survival be $50 \%$ or less?
25. (Graphing program recommended.) The infant mortality rate (the number of deaths per 1000 live births) fell in the United States from 7.2 in 1996 to 6.4 in 2006.
a. Assume that the infant mortality rate is declining linearly over time. Construct an equation modeling the relationship between infant mortality rate and time, where time is measured in years since 1996. Make sure you have clearly identified your variables.
b. Assuming that the infant mortality rate is declining exponentially over time, construct an equation modeling the relationship, where time is measured in years since 1996.
c. Graph both of your models on the same grid.
d. What would each of your models predict for the infant mortality rate in 2010?
26. [Part (e) requires use of the Internet and technology to find a best-fit function.] A "rule of thumb" used by car dealers is that the trade-in value of a car decreases by $30 \%$ each year.
a. Is this decline linear or exponential?
b. Construct a function that would express the value of the car as a function of years owned.
c. Suppose you purchase a car for $\$ 15,000$. What would its value be after 2 years?
d. Explain how many years it would take for the car in part (c) to be worth less than $\$ 1000$. Explain how you arrived at your answer.
e. Internet search: Go to the Internet site for the Kelley Blue Book (www.kbb.com).
i. Enter the information about your current car or a car you would like to own. Specify the actual age and mileage of the car. What is the Blue Book value?
ii. Keeping everything else the same, assume the car is 1 year older and increase the mileage by 10,000 . What is the new value?
iii. Find a best-fit exponential function to model the value of your car as a function of years owned. What is the annual decay rate?
iv. According to this function, what will the value of your car be 5 years from now?

### 5.6 Examples of Exponential Growth and Decay

Exponential and linear behaviors are ones you will frequently (though perhaps unknowingly) encounter throughout life. Here we'll take a look at some examples showing the wide range of applications of exponential functions.

E X A M P LE 1 Fitting a curve to data: Medicare costs
The costs for almost every aspect of health care in America have risen dramatically over the last 30 years. One of the central issues is the amount of dollars spent on Medicare, a federal program that provides health care for nearly all people age 65 and over. Estimate the average annual percentage increase in Medicare expenses.

SOLUTION Table 5.10 shows Medicare expenses from 1970 to 2005 as reported by the federal government. In column 2, the years are reinitialized to become the years since 1970.

|  | Medicare Expenses |  |  |  | Medicare Expenses (in billions of \$) |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Year | Years since 1970 | Medicare Expenses (billions of dollars) |  | 1 |
|  | 1970 | 0 | 7.7 |  |  |
| - | 1975 | 5 | 16.3 |  |  |
| Tee Excel or graph link file | 1980 | 10 | 37.1 |  | - |
| MEDICARE. | 1985 | 15 | 71.5 |  |  |
|  | 1990 | 20 | 109.5 |  |  |
|  | 1995 | 25 | 184.4 |  |  |
|  | 2000 | 30 | 224.3 |  |  |
|  | 2005 | 35 | 342.0 |  | Years since 1970 |

Table 5.10
Source: www.census.gov.

Figure 5.14 Medicare expenses (in billions) with best-fit exponential model.

A curve-fitting program gives a best-fit exponential function as

$$
C(n)=10.6 \cdot(1.11)^{n}
$$

where $n=$ number of years since 1970 and $C(n)$ is the associated cost in billions of dollars. The data and best-fit exponential curve are plotted in Figure 5.14.

In our model the initial value (in year 0 or in 1970) is $\$ 10.6$ billion. This differs from the actual value in 1970 of $\$ 7.7$ billion, since the model is the best fit to all the data and does not necessarily include any of the original data points. More important, the growth factor is 1.11 , so the growth rate is 0.11 in decimal form. So our model estimates that between 1970 and 2005 Medicare expenses were increasing by $11 \%$ each year, an amount that far exceeds the general inflation rate.

E X A M PLE 2 Changing the time unit: Credit card debt
In 2007 the average American household owed a balance of almost $\$ 9200$ on credit cards. ${ }^{4}$ For late payments, credit card companies typically charge a high annual percentage rate (APR) that could be anywhere from $10 \%$ to $20 \%$.
a. Assuming you have a $\$ 500$ balance on a credit card with a $20 \%$ APR, construct a model for your debt over time.
b. What is your monthly interest rate?
c. If you made no payments and no charges for 6 months, how much would you owe?

SOLUTION a. Since the debt is increasing by a constant percent, the growth is exponential. Let $T=$ number of years and $D=$ credit card debt in dollars. An APR of $20 \%$ means that
the annual growth rate in decimal form is 0.20 , and the annual growth factor is 1.20 . So given an initial value of $\$ 500$, an exponential model would be

$$
D=500 \cdot(1.20)^{T}
$$

b. We know that the yearly growth factor is 1.20 . If $a$ stands for the monthly growth factor, then

$$
a^{12}=1.20
$$

To find the value for $a$ we can take the twelfth root of both sides.
or

$$
\begin{aligned}
\left(a^{12}\right)^{1 / 12} & =(1.20)^{1 / 12} \\
a & \approx 1.015
\end{aligned}
$$

Since the monthly growth factor is 1.015 , the monthly growth rate is 0.015 in decimal form. So a yearly interest rate of $20 \%$ translates into a monthly interest rate of about $1.5 \%$.
c. To find your debt after 6 months, we need the monthly exponential growth function. If we let $t=$ number of months and 1.015 is the monthly growth factor, then

$$
D \approx 500 \cdot(1.015)^{t}
$$

Note that we have transformed debt $(D)$ as a function of number of years $(T)$ into a function of number of months $(t)$. Assuming you made no payments, after 6 months you would owe

$$
D \approx 500 \cdot(1.015)^{6}=\$ 547
$$

So you would owe $\$ 47$ in interest (almost $10 \%$ of the initial amount) for the "privilege" of borrowing $\$ 500$ for 6 months.

## Half-Life and Doubling Time

Every exponential growth function has a fixed doubling time, and every exponential decay function has a fixed half-life.

## Doubling Time and Half-Life

The doubling time of an exponentially growing quantity is the time required for the quantity to double in size.

The half-life of an exponentially decaying quantity is the time required for one-half of the quantity to decay.

E X A M LE 3 Constructing an exponential function given the doubling time or half-life
In each situation construct an exponential function.
a. The doubling time is 3 years; the initial quantity is 1000 .
b. The half-life is 5 days; the initial quantity is 50 .

SOLUTION a. A doubling time of 3 years describes an exponentially growing quantity that doubles every 3 years. We can construct an exponential function,

$$
\begin{align*}
& P(t)=1000 a^{t} \begin{array}{l}
\text { where } a \text { is the yearly } \\
\\
\\
\\
\\
\\
t=\text { nowth factor and }
\end{array}  \tag{1}\\
& \text { number of years. }
\end{align*}
$$

Since the doubling time is 3 years, then

$$
a^{3}=2
$$

Taking the cube root of each side $a=2^{1 / 3}$

Substituting in (1)
Rule 3 for exponents

$$
\begin{aligned}
& P(t)=1000 \cdot\left(2^{1 / 3}\right)^{t} \\
& P(t)=1000 \cdot 2^{t / 3}
\end{aligned}
$$

b. A half-life of 5 days means an exponentially decaying quantity that decreases by half every 5 days. We can construct an exponential decay function,

$$
\begin{array}{ll}
Q(t)=50 a^{t} & \begin{array}{l}
\text { where } a \text { is the daily } \\
\text { decay factor and }
\end{array}  \tag{2}\\
& t=\text { number of days. }
\end{array}
$$

Since the half-life is 5 days, then

$$
a^{5}=\frac{1}{2}
$$

Taking the fifth root of each side $a=\left(\frac{1}{2}\right)^{1 / 5}$
Substituting in (2) and
Using Rule 3 for exponents $\quad Q(t)=50 \cdot\left(\frac{1}{2}\right)^{t / 5}$

Writing an exponential function using the doubling time (or half-life) If an exponential growth function $G(t)=C a^{t}$ has a doubling time of $n$, then

| $a^{n}$ | $=2$ |
| ---: | :--- |
| $a$ | $=2^{1 / n}$ |
| so, $\quad G(t)$ | $=C \cdot 2^{t / n}$ |

If an exponential decay function $D(t)=C a^{t}$ has a half-life of $n$, then

$$
\begin{gathered}
a^{n}=\frac{1}{2} \\
a=\left(\frac{1}{2}\right)^{1 / n} \\
D(t)=C \cdot\left(\frac{1}{2}\right)^{t / n}
\end{gathered}
$$

so,
For example, if $G(t)=500 a^{t}$ has a doubling time of 6 hours, then $G(t)=500 \cdot 2^{t / 6}$, where $t=$ number of hours.

If $D(t)=25 a^{t}$ has a half-life of 3 minutes, then $D(t)=25 \cdot\left(\frac{1}{2}\right)^{t / 3}$,
where $t=$ number of minutes.

E X A M P L E 4 Half-life: Radioactive decay
One of the toxic radioactive by-products of nuclear fission is strontium-90. A nuclear accident, like the one in Chernobyl, can release clouds of gas containing strontium-90. The clouds deposit the strontium- 90 onto vegetation eaten by cows, and humans ingest strontium-90 from the cows' milk. The strontium-90 then replaces calcium in bones, causing cancer and birth defects. Strontium-90 is particularly insidious because it has a half-life of approximately 28 years. That means that every 28 years about half (or $50 \%$ ) of the existing strontium-90 has decayed into nontoxic, stable zirconium-90, but the other half still remains.
a. Construct a model for the decay of 100 mg of strontium-90 as a function of 28-year time periods.
b. Construct a new model that describes the decay as a function of the number of years. Generate a corresponding table and graph.

SOLUTION a. Strontium-90 decays by $50 \%$ every 28 years. If we define our basic time period $T$ as 28 years, then the decay rate is 0.5 and the decay factor is $1-0.5=0.5$ (or $1 / 2$ ). Assuming an initial value of 100 mg , the function

$$
S=100 \cdot(0.5)^{T}
$$

gives $S$, the remaining milligrams of strontium-90 after $T$ time periods (of 28 years).
b. The number 0.5 is the decay factor over a 28 -year period. If we let $a=$ annual decay factor, then

$$
a^{28}=0.5
$$

To find the value for $a$ we can take the 28th root of both sides.

$$
\begin{aligned}
\left(a^{28}\right)^{1 / 28} & =(0.5)^{1 / 28} \\
a & \approx 0.976
\end{aligned}
$$

So if $t=$ number of years, the yearly decay factor is 0.976 , making our function

$$
S \approx 100 \cdot(0.976)^{t}
$$

We now have an equation that describes the amount of strontium-90 as a function of years, $t$. The decay factor is 0.976 , so the decay rate is $1-0.976=0.024$, or $2.4 \%$. So each year there is $2.4 \%$ less strontium- 90 than the year before.

| Decay of Strontium-90 |  |  |
| :---: | :---: | :---: |
| Time $T$ |  |  |
| (28-year <br> time periods) | Time $t$ <br> (years) | Strontium-90 <br> $(\mathrm{mg})$ |
| 0 | 0 | 100 |
| 1 | 28 | 50 |
| 2 | 56 | 25 |
| 3 | 84 | 12.5 |
| 4 | 112 | 6.25 |
| 5 | 140 | 3.125 |

Table 5.11


Figure 5.15 Radioactive decay of strontium-90.

Table 5.11 and Figure 5.15 show the decay of 100 mg of strontium- 90 over time. After 28 years, half ( 50 mg ) of the original 100 mg still remains. It takes an additional 28 years for the remaining amount to halve again, still leaving 25 grams out of the original 100 after 56 years.

E X A M P L E 5 Identifying the doubling time or half-life
Assuming $t$ is in years, identify the doubling time or half-life for each of the following exponential functions.
a. $Q(t)=80(2)^{t / 12}$
b. $P(t)=5 \cdot 10^{6}\left(\frac{1}{2}\right)^{4 t}$
c. $R(t)=130(0.5)^{3 t}$

SOLUTION a. When $t=12$ years, the quantity $Q(12)=80 \cdot 2^{12 / 12}=80 \cdot 2^{1}=160$. So the quantity has doubled from the initial value of 80 and the doubling time is 12 years.
b. When $t=1 / 4$, the quantity $P(1 / 4)=\left(5 \cdot 10^{6}\right)(1 / 2)^{4(1 / 4)}=\left(5 \cdot 10^{6}\right)(1 / 2)^{1}$. So the quantity is half the initial value of $5 \cdot 10^{6}$ and the half-life is $1 / 4$ years or 3 months.
c. When $t=1 / 3$, the quantity $R(t)=130(0.5)^{3(1 / 3)}=130(0.5)^{1}=65$. So the quantity is half the initial value of 130 and the half-life is $1 / 3$ year or 4 months.

## The "rule of 70": A rule of thumb for calculating doubling or halving times

A simple way to understand the significance of constant percent growth rates is to compute the doubling time. A rule of thumb is that a quantity growing at $R \%$ per year

2SOMETHING TO THINK ABOUT

Using a graphing calculator or a spreadsheet, test the rule of 70. Can you find a value for $R$ for which this rule does not work so well?
has a doubling time of approximately $70 / R$ years. If the quantity is growing at $R \%$ per month, then $70 / R$ gives its doubling time in months. The same reasoning holds for any unit of time.

The rule of 70 is easy to apply. For example, if a quantity is growing at a rate of $2 \%$ a year, then the doubling time is about $70 / 2$ or approximately 35 years. The rule of 70 also applies when $R$ represents the percentage at which some quantity is decaying. In these cases, $70 / R$ equals the half-life.

For now we'll take the rule of 70 on faith. It provides good approximations, especially for smaller values of $R$ (those under $10 \%$ ). When we return to logarithms in Chapter 6, we'll find out why this rule works.

E X A M P L E 6 Suppose that at age 23 you invest $\$ 1000$ in a retirement account that grows at $5 \%$ per year.
a. Roughly how long will it take your investment to double?
b. If you retire at age 65, approximately how much will you have in your account?

SOLUTION a. Using the rule of 70 , we get $70 / 5=14$, so your investment will double approximately every 14 years.
b. If you retire at 65 , then 42 years or three doubling periods will have elapsed $(3 \cdot 14=42)$. So your $\$ 1000$ investment (disregarding inflation) will have increased by a factor of $2^{3}$ and be worth about $\$ 8000$.

E X M P L E 7 According to Mexico's National Institute of Geography, Information and Statistics (at www.inegi.gob.mx/difusion/ingles), in 1997 Mexico's population reached 93.7 million. The annual growth rate is listed as approximately $1.4 \%$. The website states that "if this rate persists, the Mexican population will double in 49.9 years." Does that time period seem about right?

SOLUTION Using the rule of 70, the approximate doubling time for a $1.4 \%$ annual growth rate would be $70 / 1.4=50$ years. So 49.9 is a reasonable value for the doubling time.

## The Rule of 70

The rule of 70 states that if a quantity is growing (or decaying) at $R \%$ per time period, then the doubling time (or half-life) is approximately

$$
\frac{70}{R}
$$

provided $R$ is not much bigger than 10 .
Examples: If a quantity is growing at $7 \%$ per year, the doubling time is approximately $70 / 7=10$ years.
If a quantity is decaying at $5 \%$ per minute, the half-life is approximately $70 / 5=14$ minutes .

E X A M P L E 8 Plutonium, the fuel for atomic weapons, has an extraordinarily long half-life, about 24,400 years. Once the radioactive element plutonium is created from uranium, 24,400 years later half the original amount will still remain. You can see why there is concern over stored caches of atomic weapons. Use the rule of 70 to estimate plutonium's annual decay rate.

## SOLUTION

SOMETHING TO THINK ABOUT
An atomic weapon is usually designed with a $1 \%$ mass margin. That is, it will remain functional until the original fuel has decayed by more than 1\%, leaving less than $99 \%$ of the original amount. Estimate how many years a plutonium bomb would remain functional.

|  | The rule of 70 says | $70 / R$ | $=24,400$ |
| ---: | :--- | ---: | :--- |
|  | Multiply by $R$ | 70 | $=24,400 R$ |
|  | Divide and simplify | $R$ | $=70 / 24,400$ |
|  | $\approx 0.003$ |  |  |

Remember that $R$ is already in percentage, not decimal, form. So the annual decay rate is a tiny $0.003 \%$, or three-thousandths of $1 \%$.

## Algebra Aerobics 5.6a

A calculator that can evaluate powers is recommended for Problem 2.

1. For each of the following exponential functions (with $t$ in days) determine whether the function represents exponential growth or decay, then estimate the doubling time or half-life. For each function find the initial amount and the amount after one day, one month, and one year. (Note: To make calculations easier, use 360 days in one year.)
a. $f(t)=300 \cdot 2^{t / 30}$
b. $g(t)=32 \cdot 0.5^{t / 2}$
c. $P=32,000 \cdot 0.5^{t}$
d. $h(t)=40,000 \cdot 2^{t / 360}$
2. Estimate the doubling time using the rule of 70 .
a. $g(x)=100(1.02)^{x}$, where $x$ is in years.
b. $M=10,000(1.005)^{t}$, where $t$ is in months.
c. The annual growth rate is $8.1 \%$.
d. The annual growth factor is 1.065 .
3. Use the rule of 70 to approximate the growth rate when the doubling time is:
a. 10 years
b. 5 minutes
c. 25 seconds
4. Estimate the time it will take an initial quantity to drop to half its value when:
a. $h(x)=10(0.95)^{x}$, where $x$ is in months.
b. $K=1000(0.75)^{t}$, where $t$ is in seconds.
c. The annual decay rate is $35 \%$.

## E X A M P L E 9 White Blood Cell Counts

On September 27 a patient was admitted to Brigham and Women's Hospital in Boston for a bone marrow transplant. The transplant was needed to cure myelodysplastic syndrome, in which the patient's own marrow fails to produce enough white blood cells to fight infection.

The patient's bone marrow was intentionally destroyed using chemotherapy and radiation, and on October 3 the donated marrow was injected. Each day, the hospital carefully monitored the patient's white blood cell count to detect when the new marrow became active. The patient's counts are plotted in Figure 5.16. Normal counts for a healthy individual are between 4000 and 10,000 cells per milliliter.

celcount.


Figure 5.16 White blood cell count.

What was the daily percent increase and estimated doubling time during the period of exponential growth?

SOLUTIONClearly, the whole data set does not represent exponential growth, but we can reasonably model the data between, say, October 15 and 31 with an exponential function. Figure 5.17 shows a plot of this subset of the original data, along with a computer-generated best-fit exponential function, where $t=$ number of days after October 15 .

$$
W(t)=105(1.32)^{t}
$$

The fit looks quite good.


Figure 5.17 White blood cell counts with a best-fit exponential function between October 15 and October 31.

The initial quantity of 105 is the white blood cell count (per milliliter) for the model, not the actual white blood cell count of 70 measured by the hospital. This discrepancy is not unusual; remember that a best-fit function may not necessarily pass through any of the specific data points.

The growth factor is 1.32 , so the growth rate is 0.32 . So between October 15 and 31, the number of white blood cells was increasing at a rate of $32 \%$ a day. According to the rule of 70 , the number of white blood cells was doubling roughly every $70 / 32=2.2$ days!

## Compound Interest Rates

Exponential functions occur frequently in finance. Probably the most common application is compound interest. For example, suppose you have $\$ 100$ in a passbook savings account that returns $1 \%$ compounded annually. That means that at the end of the first year, you earn $1 \%$ in interest on the initial $\$ 100$, called the principal. From then on you earn $1 \%$ not only on your principal, but also on the interest that you have already earned. This is called compounding and represents exponential growth.

If the growth rate is $1 \%$, or 0.01 in decimal form, then the growth factor is $1+0.01=1.01$. So each subsequent year, the current value of your savings account will be multiplied by 1.01 . The following function models the growth:

$$
\begin{array}{cl}
\left.\begin{array}{cl}
P_{1} & = \\
& =100 \\
100 & \cdot(1.01)^{t} \\
\text { Value of account } & =(1+0.01)^{t} \\
\text { original } \\
\text { investment }
\end{array}\right) \cdot(1+\text { interest rate })^{\text {no. of years }}
\end{array}
$$

## Calculating Compound Interest

If

$$
\begin{aligned}
P_{0} & =\text { original investment } \\
r & =\text { interest rate (in decimal form) } \\
t & =\text { time periods at which the interest rate is compounded }
\end{aligned}
$$

the resulting value, $P_{r}$, of the investment after $t$ time periods is given by the formula

$$
P_{r}=P_{0} \cdot(1+r)^{t}
$$

E X A M PLE 10 Comparing investments
Suppose you have $\$ 10,000$ that you could put in either a checking account that earns no interest, a mutual fund that earns 5\% a year, or a risky stock investment that you hope will return $15 \%$ a year.
a. Model the potential growth of each investment.
b. Compare the investments after 10, 20, 30, and 40 years.

SOLUTION a. Assuming you make no withdrawals, the money in the checking account will remain constant at $\$ 10,000$. For both the mutual fund and the stock investment you are expecting a constant annual percent rate, so you can think of them as compounding annually. The mutual fund's predicted annual growth rate is $5 \%$, so its growth factor would be 1.05 . The stock's annual growth rate (you hope) will be $15 \%$, so its growth factor would be 1.15 . The initial value is $\$ 10,000$, so after $t$ years

$$
\begin{array}{ll}
\text { value of mutual fund } & =10,000 \cdot(1.05)^{t} \\
\text { value of stock investment } & =10,000 \cdot(1.15)^{t}
\end{array}
$$

b. Table 5.12 shows the value of each investment over time.

| Number of <br> Years | Checking Account <br> $(0 \%$ per year $)$ | Mutual Fund <br> $(5 \%$ per year $)$ | Stock <br> $(15 \%$ per year $)$ |
| :---: | :---: | :---: | :---: |
| 0 | $\$ 10,000$ | $\$ 10,000$ | $\$ 10,000$ |
| 10 | $\$ 10,000$ | $\$ 16,289$ | $\$ 40,456$ |
| 20 | $\$ 10,000$ | $\$ 26,533$ | $\$ 163,665$ |
| 30 | $\$ 10,000$ | $\$ 43,219$ | $\$ 662,118$ |
| 40 | $\$ 10,000$ | $\$ 70,400$ | $\$ 2,678,635$ |

Table 5.12
Forty years from now, your checking account will still have $\$ 10,000$, but your mutual fund will be worth $\$ 70,400$. Should you have been lucky enough to have invested in the next Microsoft, 40 years from now, with a $15 \%$ interest rate compounded annually, your $\$ 10,000$ would have become well over $\$ 2.5$ million.

## Inflation and the diminishing dollar

Compound calculations are the same whether you are dealing with inflation or investments. For example, a 5\% annual inflation rate would mean that what cost $\$ 1$ today would cost $\$ 1.05$ one year from today. If we think of the percentages in Table 5.14 as representing inflation rates, then something that costs $\$ 10,000$ today in 10 years will cost $\$ 10,000, \$ 16,289$, or $\$ 40,456$ if the annual inflation rate is, respectively, $0 \%, 5 \%$, or $15 \%$.

How much will your money be worth in 20 years?
Suppose you have a $\$ 100,000$ nest egg hidden under your bed, awaiting your retirement. In 20 years, how much will your nest egg be worth in today's dollars if the annual inflation rate is:
a. $3 \%$ ?
b. $7 \%$ ?

SOLUTION a. Given a $3 \%$ inflation rate, what costs $\$ 1$ today will cost $\$ 1.03$ next year. Using a ratio to compare costs, we get

Rewriting

$$
\begin{aligned}
\frac{\text { cost now }}{\text { cost in } 1 \text { year }} & =\frac{\$ 1.00}{\$ 1.03} \approx 0.971 \\
\text { cost now } & \approx 0.971 \cdot(\text { cost in } 1 \text { year })
\end{aligned}
$$

Our equation shows that next year's dollars need to be multiplied by 0.971 to reduce them to the equivalent value in today's dollars. An annual inflation rate of $3 \%$ means that in one year, $\$ 1$ will be worth only $0.971 \cdot(\$ 1)=\$ 0.971$, or about 97 cents in today's dollars. Each additional year reduces the value of your nest egg by a factor of 0.971 ; in other words, 0.971 is the decay factor. So the value of your nest egg in real dollars is given by

$$
V_{3 \%}(n)=\$ 100,000 \cdot 0.971^{n}
$$

where $V_{3 \%}(n)$ is the real value (measured in today's dollars) of your nest egg after $n$ years with $3 \%$ annual inflation. When $n=20$, we have

$$
V_{3 \%}(20)=\$ 100,000 \cdot 0.971^{20} \approx \$ 100,000 \cdot 0.555=\$ 55,500
$$

b. Similarly, since $\$ 1.00 / \$ 1.07 \approx 0.935$, then

$$
V_{7 \%}(n)=\$ 100,000 \cdot 0.935^{n}
$$

where $V_{7 \%}(n)$ is the real value (measured in today's dollars) of your nest egg after $n$ years with $7 \%$ annual inflation. When $n=20$, we have

$$
V_{7 \%}(20)=\$ 100,000 \cdot 0.935^{20} \approx \$ 100,000 \cdot 0.261=\$ 26,100
$$

So assuming $3 \%$ annual inflation, after 20 years your nest egg of \$100,000 in real dollars would be almost cut in half. With a $7 \%$ inflation your nest egg in real dollars would only be about one-quarter of the original amount.

Real or Constant Dollars To make meaningful comparisons among dollar values in different years, economists use real or constant dollars, dollars that are adjusted for inflation. For example, the U.S. Census Bureau reported that in 2004 the median income of households was $\$ 44,389$ and in 1998 only $\$ 38,885$ in 1998 dollars. This suggests an increase of about $\$ 5500$. But measured in 2004 dollars, the median income in 1998 was $\$ 45,003$. So in real (inflation adjusted) dollars, there was a decrease in median household income.

## Algebra Aerobics 5.6b

A calculator that can evaluate powers is recommended for Problem 4.

1. Approximately how long would it take for your money to double if the interest rate, compounded annually, were:
a. $3 \%$ ?
b. $5 \%$ ?
c. $7 \%$ ?
2. Suppose you are planning to invest a sum of money. Estimate the rate that you need so that your investment doubles in:
a. 5 years
b. 10 years
c. 7 years
3. Construct a function that represents the resulting value if you invested $\$ 1000$ for $n$ years at an annually compounded interest rate of:
a. $4 \%$
b. $11 \%$
c. $110 \%$
4. In the early 1980s Brazil's inflation was running rampant at about $10 \%$ per month. ${ }^{5}$ Assuming this inflation rate continued unchecked, construct a function to describe the purchasing power of 100 cruzeiros after $n$ months. (A cruzeiro is a Brazilian monetary unit.) What would 100 cruzeiros be worth after 3 months? After 6 months? After a year?

[^1]5. For each of the following equations, determine the initial investment, the growth factor, and the growth rate, and estimate the time it will take to double the investment.

| Function | Initial <br> Investment | Growth <br> Factor | Growth <br> Rate | Amount 1 <br> Year Later | Doubling <br> Time |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\$ 50,000$ |  | $7.2 \%$ |  |  |
|  | $\$ 100,000$ |  |  | $\$ .8 \%$ | $\$ 106,700$ |
|  | $\$ 3,000$ | 1.13 |  |  |  |
|  |  |  |  |  |  |

Table 5.13

There is an exponential relationship between musical octaves and vibration frequency. The vibration frequency of the note A above middle C is 440 cycles per second (or 440 hertz). The vibration frequency doubles at each octave.
a. Construct a function that gives the vibration frequency as a function of octaves. Construct a corresponding table and graph.
b. Rewrite the function as a function of individual note frequencies. (Note: There are twelve notes to an octave.)

SOLUTION
a. Let $F$ be the vibration frequency in hertz $(\mathrm{Hz})$ and $N$ the number of octaves above or below the chosen note A. Since the frequency doubles over each octave, the growth factor is 2 . If we set the initial frequency at 440 Hz , then we have

$$
F=440 \cdot 2^{N}
$$

Table 5.14 and the graph in Figure 5.18 show a few of the values for the vibration frequency.

| Musical Octaves |  |
| :---: | :---: |
| Number of Octaves | Vibration |
| above or below A | Frequency (Hz), |
| (at 440 Hz), $N$ | $F$ |
| -3 | 55 |
| -2 | 110 |
| -1 | 220 |
| 0 | 440 |
| 1 | 880 |
| 2 | 1760 |
| 3 | 3520 |

Table 5.14


Figure 5.18 Octaves versus frequency.
b. There are twelve notes in each octave. If we let $n=$ number of notes above or below A, then we have $n=12 N$. Solving for $N$, we have $N=n / 12$. By substituting this expression for $N$, we can define $F$ as a function of $n$.

$$
\begin{aligned}
F & =440 \cdot 2^{n / 12} \\
& =440 \cdot 2^{(1 / 12) \cdot n} \\
& =440 \cdot\left(2^{1 / 12}\right)^{n} \\
& \approx 440 \cdot(1.059)^{n}
\end{aligned}
$$

So each note on the "even-tempered" scale has a frequency 1.059 times the frequency of the preceding note.

## The Malthusian Dilemma

The most famous attempt to predict growth mathematically was made by a British economist and clergyman, Thomas Robert Malthus, in an essay published in 1798. He argued that the growth of the human population would overtake the growth of food supplies, because the population size was multiplied by a fixed amount each year, whereas food production only increased by adding a fixed amount each year. In other words, he assumed populations grew exponentially and food supplies grew linearly as functions of time. He concluded that humans are condemned always to breed to the point of misery and starvation, unless the population is reduced by other means, such as war or disease.

E X A M PLE 13 Constructing the Malthusian equations
Malthus believed that the population of Great Britain, then about $7,000,000$, was growing by $2.8 \%$ per year. He counted food supply in units that he defined to be enough food for one person for a year. At that time the food supply was adequate, so he assumed that Britons were producing $7,000,000$ food units. He predicted that food production would increase by about 280,000 units a year.
a. Construct functions modeling Britain's population size and the amount of food units over time.
b. Plot the functions from part (a) on the same grid. Estimate when the population would exceed the food supply.
a. Since the population is assumed to increase by a constant percent each year, an exponential model is appropriate. Given an initial size of 7,000,000 and an annual growth rate of $2.8 \%$, the exponential function

$$
P(t)=7,000,000 \cdot(1.028)^{t}
$$

describes the population growth over time, where $t$ is the number of years and $P(t)$ is the corresponding population size.

Since the food units are assumed to grow by a constant amount each year, a linear model is appropriate. Given an initial value of $7,000,000$ food units and an annual constant rate of change of 280,000 , the linear function

$$
F(t)=7,000,000+280,000 t
$$

describes the food unit increase over time, where $t=$ number of years and $F(t)=$ number of food units in year $t$.
Figure 5.19 reveals that if the formulas were good models, then after about 25 years the population would start to exceed the food supply, and some people would starve.


Figure 5.19 Malthus's predictions for population and food.

The two centuries since Malthus published his famous essay have not been kind to his theory. The population of Great Britain in 2002 was about 58 million, whereas Malthus's model predicted over 100 million people before the year 1900. Improved food production techniques and the opening of new lands to agriculture have kept food production in general growing faster than the population. The distribution of food is a problem and famines still occur with unfortunate regularity in parts of the world, but the mass starvation Malthus predicted has not come to pass.

## Forming a Fractal Tree

Tree structures offer another useful way of visualizing exponential growth. A computer program can generate a tree by drawing two branches at the end of a trunk, then two smaller branches at the ends of each of those branches, and two smaller branches at the ends of the previous branches, and so on until a branch reaches twig size. This kind of structure produced from self-similar repeating scaled graphic operations is called a fractal (Figure 5.20). There are many examples of fractal structures in nature, such as ferns, coastlines, and human lungs.


Figure 5.20 A fractal tree.

The fractal tree shown is drawn in successive levels (Figure 5.21):

- At level 0 the program draws the trunk, one line.
- At level 1 it draws two branches on the previous one, for a total of two new lines.
- At level 2 it draws two branches on each of the previous two, for a total of four new lines.
- At level 3 it draws two branches on each of the previous four, for a total of eight new lines.


Figure 5.21 Forming a fractal tree.

If you look back at your family tree forty generations ago (roughly 800 years), you had $2^{40}$ ancestors. This number is larger than all the people that ever lived on the surface of the earth. How can that be?

Table 5.15 shows the relationship between the level, $L$, and the number of new lines, $N$, at each level. The formula

$$
N=2^{L}
$$

describes the relationship between level, $L$, and the number of new lines, $N$. For example, at the fifth level, there would be $2^{5}=32$ new lines.

E X A M PLE 14 You can think of Figure 5.20 as depicting your family tree. Each level represents a generation. The trunk is you (at level 0). The first two branches are your parents (at level 1). The next four branches are your grandparents (at level 2), and so on. How many ancestors do you have ten generations back?

SOLUTION The answer is $2^{10}=1024$; that is, you have 1024 great-great-great-great-great-great-great-great-grandparents.

EXAMPLE15 An information system that is similar to this process is an emergency phone tree, in which one person calls two others, each of whom calls two others, until everyone in the organization has been called. How many levels of phone calls would be needed to reach an organization with 8000 people?

SOLUTION If we think of Figure 5.20 and Table 5.15 as representing this phone tree, then each new line (or branch) represents a person. We need to count not just the number of new people $N$ at each level $L$, but also all the previous people called.

At level 0 , there is the one person who originates the phone calls. At level 1 , there is the original person plus the two he or she called, for a total of $1+2=3$ people. At level 2 , there are $1+2+4=7$ people, etc. At level 11, there are $1+2+4+8+16+32+64+128+256+512+1024+2048=4095$ people who have been called. At level 12, there would be $2^{12}=4096$ new people called, for a total of $4095+4096=8191$ people called. So it would take twelve levels of the phone tree to reach 8000 people.

## Algebra Aerobics 5.6c

1. Identify the value of $x$ that would make each of the following equations a true statement.
a. $2^{x}=32$
b. $2^{x}=256$
c. $2^{x}=1024$
d. $2^{x}=2$
e. $2^{x}=1$
f. $2^{x}=\frac{1}{2}$
g. $2^{x}=\frac{1}{8}$
h. $2^{x}=\sqrt{2}$
2. Assume the tree-drawing process was changed to draw three branches at each level.
a. Draw a trunk and at least two levels of the tree.
b. What would the general formula be for $N$, the number of new lines, as a function of $L$, the level?
3. In an emergency phone tree in which one person calls three others, each of whom calls three others, and so on, until everyone in the organization has been called, how many levels of phone calls are required for this phone tree to reach an organization of 8000 people?

## Exercises for Section 5.6

Technology for finding a best-fit function is required for Exercises 7, 9, 17, and 18. Internet access is required for Exercises 9 part (c), 29 part (c), and 32 part (e).

1. Which of the following functions have a fixed doubling time? A fixed half-life?
a. $y=6(2)^{x}$
b. $y=5+2 x$
c. $Q=300\left(\frac{1}{2}\right)^{T}$
d. $A=10(2)^{t / 5}$
e. $P=500-\frac{1}{2} T$
f. $N=50\left(\frac{1}{2}\right)^{t / 20}$
2. Identify the doubling time or half-life of each of the following exponential functions. Assume $t$ is in years. [Hint: What value of $t$ would give you a growth (or decay) factor of 2 (or $1 / 2$ )?]
a. $Q=70(2)^{t}$
b. $Q=1000(2)^{t / 50}$
c. $Q=300\left(\frac{1}{2}\right)^{t}$
d. $Q=100\left(\frac{1}{2}\right)^{t / 250}$
e. $N=550(2)^{t / 10}$
f. $N=50\left(\frac{1}{2}\right)^{t / 20}$
3. Fill in the following chart. (The first column is done for you.)

|  | a. | b. | c. | d. |
| :--- | :---: | :---: | :---: | :---: |
| Initial Value | 50 | 1000 | 4 | 5000 |
| Doubling time | days | 7 <br> years | 25 <br> minutes | 18 <br> months |
| Exponential <br> function <br> $\left(f(t)=C a^{t / n}\right)$ | $f(t)=50(2)^{t / 30}$ |  |  |  |
| Growth factor <br> per unit <br> of time | $2^{1 / 30}=1.0234$ |  |  |  |
| Growth rate per day <br> unit of time <br> (in percentage <br> form) | per day |  |  |  |

4. Make a table of values for the dotted points on each graph. Using your table, determine if each graph has a fixed doubling time or a fixed half-life. If so, create a function formula for that graph.




5. Insert the symbol $>,<$, or $\approx$ (approximately equal) to make the statement true. Assume $x>0$.
a. $3(2)^{x / 5}$ $\qquad$ $3(1.225)^{x}$
b. $50\left(\frac{1}{2}\right)^{x / 20}$ $\qquad$ $50(0.9659)^{x}$
c. $200(2)^{x / 8}$ $\qquad$ $200(1.0905)^{x}$
d. $750\left(\frac{1}{2}\right)^{x / 165}$ $\qquad$ $750(0.911)^{x}$
6. Lead-206 is not radioactive, so it does not spontaneously decay into lighter elements. Radioactive elements heavier than lead undergo a series of decays, each time changing from a heavier element into a lighter or more stable one. Eventually, the element decays into lead-206 and the process stops. So, over billions of years, the amount of lead in the universe has increased because of the decay of numerous radioactive elements produced by supernova explosions.

Radioactive uranium- 238 decays sequentially into thirteen other lighter elements until it stabilizes at lead-206. The half-lives of the fifteen different elements in this decay chain vary from 0.000164 seconds (from polonium-214 to lead-210) all the way up to 4.47 billion years (from uranium- 238 to thorium-234).
a. Find the decay rate per billion years for uranium-238 to decay into thorium-234.
b. Find the decay rate per second for polonium-214 to decay into lead-210.
7. (Requires technology to find a best-fit function.) We have seen the accompanying table and graph of the U.S. population at the beginning of Chapter 2.

Population of the United States, 1790-2000

| Year | Millions | Year | Millions |
| :---: | :---: | :---: | :---: |
| 1790 | 3.9 | 1900 | 76.2 |
| 1800 | 5.3 | 1910 | 92.2 |
| 1810 | 7.2 | 1920 | 106.0 |
| 1820 | 9.6 | 1930 | 123.2 |
| 1830 | 12.9 | 1940 | 132.2 |
| 1840 | 17.1 | 1950 | 151.3 |
| 1850 | 23.2 | 1960 | 179.3 |
| 1860 | 31.4 | 1970 | 203.3 |
| 1870 | 39.8 | 1980 | 226.5 |
| 1880 | 50.2 | 1990 | 248.7 |
| 1890 | 63.0 | 2000 | 281.4 |

Source: U.S. Bureau of the Census, www.census.gov.

a. Find a best-fit exponential function. (You may want to set 1790 as year 0 .) Be sure to clearly identify the variables and their units for time and population. What is the annual growth factor? The growth rate? The estimated initial population?
b. Graph your function and the actual U.S. population data on the same grid. Describe how the estimated population size differs from the actual population size. In what ways is this exponential function a good model for the data? In what ways is it flawed?
c. What would your model predict the population to be in the year 2010? In 2025?
8. (Requires results from Exercise 7.) According to a letter published in the Ann Landers column in the Boston Globe on Friday, December 10, 1999, "When Elvis Presley died in 1977, there were 48 professional Elvis impersonators. In 1996, there were 7,328 . If this rate of growth continues, by the year 2012, one person in every four will be an Elvis impersonator."
a. What was the growth factor in the number of Elvis impersonators for the 19 years between 1977 and 1996?
b. What would be the annual growth factor in number of Elvis impersonators between 1977 and 1996?
c. Construct an exponential function that describes the growth in number of Elvis impersonators since 1977.
d. Use your function to estimate the number of Elvis impersonators in 2012.
e. Use your model for the U.S. population from Exercise 7 to determine if in 2012 one person out of every four will be an Elvis impersonator. Explain your reasoning.
9. (Requires technology to find a best-fit function.) Reliable data on Internet use are hard to find, but World Telecommunications Indicators cites estimates of 3 million U.S. users in 1991, 30 million in 1996, 166 million in 2002, 199 million in 2004 and 232 million in 2007.
a. Use technology to plot the data, and generate a best-fit linear and a best-fit exponential function for the data. Which do you think is the better model?
b. What would the linear model predict for Internet usage in 2010? What would the exponential model predict?
c. Internet use: Go online and see if you can find the number of current internet users in the U.S. Which of your models turned out to be more accurate?
10. China is the most populous country in the world. In 2000 it had about 1.262 billion people. By 2005 the population had grown to 1.306 billion. Use this information to construct models predicting the size of China's population in the future.
a. Identify your variables and units.
b. Construct a linear model.
c. Construct an exponential model.
d. What will China's population be in 2050 according to each of your models?
11. Tritium, the heaviest form of hydrogen, is a critical element in a hydrogen bomb. It decays exponentially with a half-life of about 12.3 years. Any nation wishing to maintain a viable hydrogen bomb has to replenish its tritium supply roughly every 3 years, so world tritium supplies are closely watched.

Construct an exponential function that shows the remaining amount of tritium as a function of time as 100 grams of tritium decays (about the amount needed for an average size bomb). Be sure to identify the units for your variables.
12. (Graphing program recommended.) Cosmic ray bombardment of the atmosphere produces neutrons, which in turn react with nitrogen to produce radioactive carbon-14. Radioactive carbon-14 enters all living tissue through carbon dioxide (via plants). As long as a plant or animal is alive, carbon-14 is maintained in the organism at a constant level. Once the organism dies, however, carbon-14 decays exponentially into carbon-12. By comparing the amount of carbon-14 to the amount of carbon-12, one can determine approximately how long ago the organism died. Willard Libby won a Nobel Prize for developing this technique for use in dating archaeological specimens. The half-life of carbon-14 is about 5730 years. In answering the following questions, assume that the initial quantity of carbon- 14 is 500 milligrams.
a. Construct an exponential function that describes the relationship between $A$, the amount of carbon-14 in milligrams, and $t$, the number of 5730-year time periods.
b. Generate a table of values and plot the function. Choose a reasonable set of values for the domain. Remember that the objects we are dating may be up to 50,000 years old.
c. From your graph or table, estimate how many milligrams are left after 15,000 years and after 45,000 years.
d. Now construct an exponential function that describes the relationship between $A$ and $T$, where $T$ is measured in years. What is the annual decay factor? The annual decay rate?
e. Use your function in part (d) to calculate the number of milligrams that would be left after 15,000 years and after 45,000 years.
13. The body eliminates drugs by metabolism and excretion. To predict how frequently a patient should receive a drug dosage, the physician must determine how long the drug will remain in the body. This is usually done by measuring the half-life of the drug, the time required for the total amount of drug in the body to diminish by one-half.
a. Most drugs are considered eliminated from the body after five half-lives, because the amount remaining is probably too low to cause any beneficial or harmful effects. After five half-lives, what percentage of the original dose is left in the body?
b. The accompanying graph shows a drug's concentration in the body over time, starting with 100 milligrams.


Use the given graph to answer the following questions.
i. Estimate the half-life of the drug.
ii. Construct an equation that approximates the curve. Specify the units of your variables.
iii. How long would it take for five half-lives to occur? Approximately how many milligrams of the original dose would be left then?
iv. Write a paragraph describing your results to a prospective buyer of the drug.
14. Estimate the doubling time using the rule of 70 when:
a. $P=2.1(1.0475)^{t}$, where $t$ is in years
b. $Q=2.1(1.00475)^{T}$, where $T$ is in years
15. Use the rule of 70 to approximate the growth rate when the doubling time is:
a. 5730 years
c. 5 seconds
b. 11,460 years
d. 10 seconds
16. Estimate the time it will take an initial quantity to drop to half its value when:
a. $P=3.02(0.998)^{t}$, with $t$ in years
b. $Q=12(0.75)^{T}$, with $T$ in decades
17. (Requires technology to find a best-fit function.) Estimates for world population vary, but the data in the accompanying table are reasonable estimates. WORLDPOP

## World Population

| Year | Total Population <br> (millions) |
| :---: | :---: |
| 1800 | 980 |
| 1850 | 1260 |
| 1900 | 1650 |
| 1950 | 2520 |
| 1970 | 3700 |
| 1980 | 4440 |
| 1990 | 5270 |
| 2000 | 6080 |
| 2005 | 6480 |

Source: United Nations Population
Division, www.undp.org/popin.
a. Enter the data table into a graphing program (you may wish to enter 1800 as 0,1850 as 50 , etc.) or use the data file WORLDPOP in Excel or in graph link form.
b. Generate a best-fit exponential function.
c. Interpret each term in the function, and specify the domain and range of the function.
d. What does your model give for the growth rate?
e. Using the graph of your function, estimate the following:
i. The world population in $1750,1920,2025$, and 2050
ii. The approximate number of years in which world population attained or will attain 1 billion (i.e., 1000 million), 4 billion, and 8 billion
f. Estimate the length of time your model predicts it takes for the population to double from 4 billion to 8 billion people.
18. (Requires technology to find a best-fit function.) In 1911, reindeer were introduced to St. Paul Island, one of the Pribilof Islands, off the coast of Alaska in the Bering Sea. There was plenty of food and no


REINDEER hunting or reindeer predators. The size of the reindeer herd grew rapidly for a number of years, as given in the accompanying table.

## Population of Reindeer Herd

| Year | Population <br> Size | Year | Population <br> Size |
| :---: | :---: | :---: | :---: |
| 1911 | 17 | 1925 | 246 |
| 1912 | 20 | 1926 | 254 |
| 1913 | 42 | 1927 | 254 |
| 1914 | 76 | 1928 | 314 |
| 1915 | 93 | 1929 | 339 |
| 1916 | 110 | 1930 | 415 |
| 1917 | 136 | 1931 | 466 |
| 1918 | 153 | 1932 | 525 |
| 1919 | 170 | 1933 | 670 |
| 1920 | 203 | 1934 | 831 |
| 1921 | 280 | 1935 | 1186 |
| 1922 | 229 | 1936 | 1415 |
| 1923 | 161 | 1937 | 1737 |
| 1924 | 212 | 1938 | 2034 |

Source: V.B. Scheffer, "The rise and fall of a reindeer herd," Scientific Monthly, 73:356-362, 1951.
a. Use the reindeer data file (in Excel or graph link form) to plot the data.
b. Find a best-fit exponential function.
c. How does the predicted population from part (b) differ from the observed ones?
d. Does your answer in part (c) give you any insights into why the model does not fit the observed data perfectly?
e. Estimate the doubling time of this population.
19. In medicine and biological research, radioactive substances are often used for treatment and tests. In the laboratories of a large East Coast university and medical center, any waste containing radioactive material with a half-life under 65 days must be stored for 10 half-lives before it can be disposed of with the non-radioactive trash.
a. By how much does this policy reduce the radioactivity of the waste?
b. Fill out the accompanying chart and develop a general formula for the amount of radioactive pollution at any period, given an initial amount, $A_{0}$.

| Number of Half-Life <br> Periods | Pollution Amount |
| :--- | :--- |
| 0 | $A_{0}$, original amount |
| 1 | $A_{1}=0.5 A_{0}$ |
| 2 | $A_{2}$ |
| 3 |  |
| 4 |  |
| Period $n$ |  |

20. Belgrade, Yugoslavia (from USA Today, September 21, 1993):

A 10 billion dinar note hit the streets today. . . . With inflation at $20 \%$ per day, the note will soon be as worthless as the 1 billion dinar note issued last month. A year ago the biggest note was 5,000 dinars. . . . In addition to soaring inflation, which doubles prices every 5 days, unemployment is at $50 \%$.

In the excerpt, inflation is described in two very different ways. Identify these two descriptions in the text and determine whether they are equivalent. Justify your answer.
21. It takes 3 months for a malignant lung tumor to double in size. At the time a lung tumor was detected in a patient, its mass was 10 grams.
a. If untreated, determine the size in grams of the tumor at each of the listed times in the table. Find a formula to express the tumor mass $M$ (in grams) at any time $t$ (in months).

| $t$, Time (months) | $M$, Mass $(\mathrm{g})$ |
| :---: | :---: |
| 0 | 10 |
| 3 |  |
| 6 |  |
| 9 |  |
| 12 |  |

b. Lung cancer is fatal when a tumor reaches a mass of 2000 grams. If a patient diagnosed with lung cancer went untreated, estimate how long he or she would survive after the diagnosis.
c. By what percentage of its original size has the 10 -gram tumor grown when it reaches 2000 grams?
22. It is now recognized that prolonged exposure to very loud noise can damage hearing. The accompanying table gives the permissible daily exposure hours to very loud noises as recommended by OSHA, the Occupational Safety and Health Administration.

| Sound Level, <br> $D$ (decibels) | Maximum Duration, <br> $H$ (hours) |
| :---: | :---: |
| 120 | 0 |
| 115 | 0.25 |
| 110 | 0.5 |
| 105 | 1 |
| 100 | 2 |
| 95 | 4 |
| 90 | 8 |

a. Examine the data for patterns. How is $D$ progressing? How is $H$ progressing? Do the data represent a growth or a decay phenomenon? Explain your answer.
b. Find a formula for $H$ as a function of $D$. Fit the data as closely as possible. Graph your formula and the data on the same grid.
23. a. Construct a function that would represent the resulting value if you invested $\$ 5000$ for $n$ years at an annually compounded interest rate of:
i. $3.5 \%$
ii. $6.75 \%$
iii. $12.5 \%$
b. If you make three different $\$ 5000$ investments today at the three different interest rates listed in part (a), how much will each investment be worth in 40 years?
24. A bank compounds interest annually at $4 \%$.
a. Write an equation for the value $V$ of $\$ 100$ in $t$ years.
b. Write an equation for the value $V$ of $\$ 1000$ in $t$ years.
c. After 20 years will the total interest earned on $\$ 1000$ be ten times the total interest earned on $\$ 100$ ? Why or why not?
25. (Graphing program recommended.) You have a chance to invest money in a risky investment at $6 \%$ interest compounded annually. Or you can invest your money in a safe investment at $3 \%$ interest compounded annually.
a. Write an equation that describes the value of your investment after $n$ years if you invest $\$ 100$ at $6 \%$ compounded annually. Plot the function. Estimate how long it would take to double your money.
b. Write an equation that describes the value of your investment after $n$ years if you invest $\$ 200$ at $3 \%$ compounded annually. Plot the function on the same grid as in part (a). Estimate the time needed to double your investment.
c. Looking at your graph, indicate whether the amount in the first investment in part (a) will ever exceed the amount in the second account in part (b). If so, approximately when?
26. According to the Arkansas Democrat Gazette (February 27, 1994):

Jonathan Holdeen thought up a way to end taxes forever. It was disarmingly simple. He would merely set aside some money in trust for the government and leave it there for 500 or 1000 years. Just a penny, Holdeen calculated, could grow to trillions of dollars in that time. But the stash he had in mind would grow much bigger-to quadrillions or quintillions-so big that the government, one day, could pay for all its operations simply from the income. Then taxes could be abolished. And everyone would be better off.
a. Holdeen died in 1967 , leaving a trust of $\$ 2.8$ million that is being managed by his daughter, Janet Adams. In 1994, the trust was worth $\$ 21.6$ million. The trust was debated in Philadelphia Orphans' Court. Some lawyers who were trying to break the trust said that it is dangerous to let it go on, because "it would sponge up all the money in the world." Is this possible?
b. After 500 years, how much would the trust be worth? Would this be enough to pay off the current national debt (over $\$ 7$ trillion in 2004)? What about after 1000 years? Describe the model you used to make your predictions.
27. Describe how a $6 \%$ inflation rate will erode the value of a dollar over time. Approximately when would a dollar be worth only 50 cents? This is the half-life of the dollar's buying power under $6 \%$ inflation.
28. The future value $V$ of a savings plan, where regular payments $P$ are made $n$ times to an account in which the interest rate, $i$, is compounded each payment period, can be calculated using the formula

$$
V=P \cdot \frac{(1+i)^{n}-1}{i}
$$

The total number of payments, $n$, equals the number of payments per year, $m$, times the number of years, $t$, so

$$
n=m \cdot t
$$

The interest rate per compounding period, $i$, equals the annual interest rate, $r$, divided by the number of compounding periods a year, $m$, so

$$
i=r / m
$$

a. Substitute $n=m \cdot t$ and $i=r / m$ in the formula for $V$, getting an expression for $V$ in terms of $m, t$, and $r$.
b. If a parent plans to build a college fund by putting $\$ 50$ a month into an account compounded monthly with a $4 \%$ annual interest rate, what will be the value of the account in 17 years?
c. Solve the original formula for $P$ as a function of $V, i$, and $n$.
d. Now you are able to find how much must be paid in every month to meet a particular final goal. If you estimate the child will need $\$ 100,000$ for college, what monthly payment must the parent make if the interest rate is the same as in part (b)?
29. [Part (c) requires use of the Internet, and technology to find a best-fit function is recommended.] The following data show the total government debt for the United States from 1950 to 2006.

| Year | Debt (\$ billions) |
| :---: | :---: |
| 1950 | 257 |
| 1955 | 274 |
| 1960 | 291 |
| 1965 | 322 |
| 1970 | 381 |
| 1975 | 542 |
| 1980 | 909 |
| 1985 | 1818 |
| 1990 | 3207 |
| 1995 | 4921 |
| 2000 | 5674 |
| 2005 | 7933 |
| 2006 | 8507 |

a. By hand or with technology, plot the data in the accompanying table and sketch a curve that approximates the data.
b. Construct an exponential function that models the data. What would your model predict for the current debt?
c. Use the "debt clock" at www.brillig.com/debt_clock to find the current debt. How accurate was your prediction in part (b)?
30. The average female adult Ixodes scapularis (a deer tick that can carry Lyme disease) lives only a year but can lay up to 10,000 eggs right before she dies. Assume that the ticks all live to adulthood, and half are females that reproduce at the same rate and half the eggs are male.
a. Describe a formula that will tell you how many female ticks there will be in $n$ years, if you start with one impregnated tick.
b. How many male ticks will there be in $n$ years? How many total ticks in $n$ years?
c. If the surface of Earth is approximately $5.089 \cdot 10^{14}$ square meters and an adult tick takes up 0.5 square centimeters of land, approximately how long would it take before the total number of ticks stemming from one generation of ticks would cover the surface of Earth?
31. MCI, a phone company that provides long-distance service, introduced a marketing strategy called "Friends and Family." Each person who signed up received a discounted calling rate to ten specified individuals. The catch was that the ten people also had to join the "Friends and Family" program.
a. Assume that one individual agrees to join the "Friends and Family" program and that this individual recruits ten new members, who in turn each recruit ten new members, and so on. Write a function to describe the number of new people who have signed up for "Friends and Family" at the $n$th round of recruiting.
b. Now write a function that would describe the total number of people (including the originator) signed up after $n$ rounds of recruiting.
c. How many "Friends and Family" members, stemming from this one person, will there be after five rounds of recruiting? After ten rounds?
d. Write a 60 -second summary of the pros and cons of this recruiting strategy. Why will this strategy eventually collapse?
32. In a chain letter one person writes a letter to a number of other people, $N$, who are each requested to send the letter to $N$ other people, and so on. In a simple case with $N=2$, let's assume person A1 starts the process.


A1 sends to B1 and B2; B1 sends to C1 and C2; B2 sends to C 3 and C 4 ; and so on. A typical letter has listed in order the chain of senders who sent the letters. So D7 receives a letter that has A1, B2, and C4 listed.
If these letters request money, they are illegal. A typical request looks like this:

- When you receive this letter, send $\$ 10$ to the person on the top of the list.
- Copy this letter, but add your name to the bottom of the list and leave off the name at the top of the list.
- Send a copy to two friends within 3 days.

For this problem, assume that all of the above conditions hold.
a. Construct a mathematical model for the number of new people receiving letters at each level $L$, assuming $N=2$ as shown in the above tree.
b. If the chain is not broken, how much money should an individual receive?
c. Suppose A1 sent out letters with two additional phony names on the list (say A1a and Alb) with P.O. box addresses she owns. So both B1 and B2 would receive a letter with the list A1, A1a, Alb. If the chain isn't broken, how much money would A1 receive?
d. If the chain continued as described in part (a), how many new people would receive letters at level 25 ?
e. Internet search: Chain letters are an example of a "pyramid growth" scheme. A similar business strategy is multilevel marketing. This marketing method uses the customers to sell the product by giving them a financial incentive to promote the product to potential customers or potential salespeople for the product. (See Exercise 31.) Sometimes the distinction between multilevel marketing and chain letters gets blurred. Search the U.S. Postal Service website (www.usps.gov) for "pyramid schemes" to find information about what is legal and what is not Report what you find.

### 5.7 Semi-Log Plots of Exponential Functions

With exponential growth functions, we often face the same problem that we did in Chapter 4 when we tried to compare the size of an atom with the size of the solar system. The numbers go from very small to very large. In our E. coli model, for example, the number of bacteria started at 100 and grew to over 1 billion in twenty-four time periods (see Table 5.1). It is virtually impossible to display the entire data set on a standard graph. Whenever we need to graph numbers of widely varying sizes, we turn to a logarithmic scale.

Previously, using standard linear scales on both axes, we could graph only a subset of the E. coli data in order to create a useful graph (see Figure 5.1). However, if we convert the vertical axis to a logarithmic scale, we can plot the entire data set (Figure 5.22). When one axis uses a logarithmic scale and the other a linear scale, the graph is called a semi-log (or log-linear) plot.


Figure 5.22 Semi-log plot of E. coli models data over twentyfour time periods.

Why Does the Graph Appear as a Straight Line? On a semi-log plot, moving $n$ units horizontally to the right is equivalent to adding $n$ units, but moving up vertically $n$ units is equivalent to multiplying by a factor of $n$. To stay on a line with slope $m$, we need to move vertically $m$ units for each unit we move horizontally. So the line tells us that each time we increase the time period by 1 , the number of $E$. coli is multiplied by a constant (namely 2). That's precisely the definition of an exponential function.

In general, the graph of any exponential function on a semi-log plot will be a straight line. We'll take a closer look at why this is true in Chapter 6. Since most graphing software easily converts standard linear plots to log-linear or semi-log plots, this is one of the simplest and most reliable ways to recognize exponential growth in a data set.

When an exponential function is plotted using a standard linear scale on the horizontal axis and a logarithmic scale on the vertical axis, its graph is a straight line. This type of graph is called a log-linear or semi-log plot.

E X A M P L E 1 Moore's law
In 1965 Gordon Moore, cofounder of Intel, made his famous prediction that the number of transistors per integrated circuit would increase exponentially over time (doubling every 18 months). This became known as Moore's Law. Figure 5.23 shows the actual increase in the number of transistors over time. Do the data justify his claim of exponential growth?


Figure 5.23 The semi-log plot of number of transistors (per circuit board) over time. Source: Intel's website at www.intel.com.

SOLUTION Figure 5.23 shows a semi-log plot of the number of transistors (per integrated circuit) over time. The plot looks basically linear, indicating that the data are exponential in nature.

## Algebra Aerobics 5.7

1. a. Using the graph in Figure 5.22, estimate the time it takes for the $E$. coli to increase by a factor of 10 .
b. From the original expression for the population, $N=100 \cdot 2^{t}$, when does the population increase by a factor of 8 ?
c. By a factor of 16 ?
d. Are these three answers consistent with each other?
2. What would you expect the graph of $y=25 \cdot 10^{x}$ to look like on a semi-log plot? Construct a table of values for $x$ equal to $0,1,2,3,4$ and 5 . Plot the graph of this function on the accompanying semi-log grid. Does your graph match your prediction?
3. Use your graph for Problem 2 to estimate the values of $y$ when $x=3.5$ and when $x=7$.


## Exercises for Section 5.7

Technology for finding a best-fit function is required for Exercises 5 and 6.

1. (Graphing program recommended.) Below is a table of values for $y=500(3)^{x}$ and for $\log y$.

| $x$ | $y$ | $\log y$ |
| ---: | ---: | ---: |
| 0 | 500 | 2.699 |
| 5 | 121,500 | 5.085 |
| 10 | $29,524,500$ | 7.470 |
| 15 | $7.17 \cdot 10^{9}$ | 9.856 |
| 20 | $1.74 \cdot 10^{12}$ | 12.241 |
| 25 | $4.24 \cdot 10^{14}$ | 14.627 |
| 30 | $1.03 \cdot 10^{17}$ | 17.013 |

a. Plot $y$ vs. $x$ on a linear scale. Remember to identify the largest number you will need to plot before setting up axis scales.
b. Plot $\log y$ vs. $x$ on a semi-log plot with a $\log$ scale on the vertical axis and a linear scale on the horizontal axis.
c. Rewrite the $y$-values as powers of 10 . How do these values relate to $\log y$ ?
2. Match each function with its semi-log plot.
a. $y=200(1.5)^{x}$
b. $y=200(2.5)^{x}$
c. $y=200(0.9)^{x}$
d. $y=200(0.5)^{x}$




3. The three accompanying graphs are all of the same function, $y=1000(1.5)^{x}$.



a. Which graph uses a linear scale for $y$ on the vertical axis? A power-of-10 scale on the vertical axis? Logarithms on the vertical axis?
b. Why do graphs $B$ and $C$ look the same?
c. For graphs $A$ and $B$, estimate the number of units needed on the horizontal scale for the value of $y=1000$ to increase by a factor of 10 .
d. On which graph is it easier to determine when the function has increased by a factor of 10 ?
e. On which graph is it easier to determine when the function has doubled?
4. According to Rubin and Farber's Pathology, "Smoking tobacco is the single largest preventable cause of death in the United States, with direct health costs to the economy of tens of billions of dollars a year. Over 400,000 deaths a yearabout one sixth of the total mortality in the United Statesoccur prematurely because of smoking." The accompanying graph compares the risk of dying for smokers, ex-smokers, and nonsmokers. It shows that individuals who have smoked for 2 years are twice as likely to die as a nonsmoker. Someone who has smoked for 14 years is three times more likely to die than a nonsmoker.


Source: E. Rubin and J. L. Farber, Pathology, 3rd ed. (Philadelphia: Lippincott-Raven, 1998), p. 310. Copyright © 1998 by LippincottRaven. Reprinted by permission.
a. The graphs for the smokers (one to two packs per day), ex-smokers, and nonsmokers all appear roughly as straight lines on this semi-log plot. What, then, would be appropriate functions to use to model the increased probability of dying over time for all three groups?
b. The plots for smokers and ex-smokers appear roughly as two straight lines that start at the same point, but the graph for smokers has a steeper slope. How would their two function models be the same and how would they be different?
c. The plots of ex-smokers and nonsmokers appear as two straight lines that are roughly equidistant. How would their two function models be the same and how would they be different?
5. (Requires technology to find a best-fit function.)
a. Load the file CELCOUNT, which contains all of
 the white blood cell counts for the bone marrow transplant patient. Now graph the data on a semi-log plot. Which section(s) of the curve represent exponential growth or decay? Explain how you can tell.
b. Load the file ECOLI, which contains the E. coli counts for twenty-four time periods. Graph the E. coli data on a semi-log plot. Which section of this curve represents exponential growth or decay? Explain your answer.
6. (Requires technology to find a best-fit function.) The accompanying table shows the U.S. international trade in goods and services.

| U.S. International Trade (Billions of Dollars) |  |  |
| :---: | :---: | ---: |
|  | Total <br> Exports | Total <br> Imports |
| Year | 25.9 | 22.4 |
| 1960 | 35.3 | 30.6 |
| 1965 | 56.6 | 54.4 |
| 1970 | 132.6 | 120.2 |
| 1975 | 271.8 | 291.2 |
| 1980 | 288.8 | 410.9 |
| 1985 | 537.2 | 618.4 |
| 1990 | 793.5 | 891.0 |
| 1995 | 1070.6 | 1448.2 |
| 2000 | 1275.2 | 1992.0 |

Source: U.S. Department of Commerce, Bureau of Economic Analysis, U.S. Bureau of the Census, Statistical Abstract of the United States: 2006.
a. U.S. imports and exports both expanded rapidly between 1960 and 2005. Use technology to plot the total U.S. exports and total U.S. imports over time on the same graph.
b. Now change the vertical axis to a logarithmic scale and generate a semi-log plot of the same data as in part (a). What is the shape of the data now, and what does this suggest would be an appropriate function type to model U.S. exports and imports?
c. Construct appropriate function models for total U.S. imports and for total exports.
d. The difference between the values of exports and imports is called the trade balance. If the balance is negative, it is called a trade deficit. The balance of trade has been an object of much concern lately. Calculate the trade balance for each year and plot it over time. Describe the overall pattern.
e. We have a trade deficit that has been increasing rapidly in recent years. But for quantities that are growing exponentially, the "relative difference" is much more meaningful than the simple difference. In this case the relative difference is

$$
\frac{\text { exports }- \text { imports }}{\text { exports }}
$$

This gives the trade balance as a fraction (or if you multiply by 100 , as a percentage) of exports.
Calculate the relative difference for each year in the above table and graph it as a function of time. Does this present a more or less worrisome picture? That is, in particular over the last decade, has the relative difference remained stable or is it also rapidly increasing in magnitude?
7. A Fidelity Investments report included the graph, on the following page, illustrating how $\$ 10,000$ invested in a Fidelity Fund © created on December 1992 would have grown over 10 years. The graph also includes the Standard \& Poor's 500 Index ${ }^{6}$ (S\&P 500) for comparison.
a. What sort of plot is this?
b. The growth from 1993 to 2000 in both the Fidelity Fund and the S\&P 500 Index appears roughly linear. What does that tell you?
${ }^{6}$ The Standard \& Poor's 500 Index is an index of 500 stocks that is used to measure the performance of the entire U.S. domestic stock market.
c. Between 2000 and 2002, the values of the Fund and 500 Index have a roughly linear decline. What does that tell you?
d. Give at least two reasons why Fidelity would publish this graph.
\$10,000 Over 10 Years
Let's say hypothetically that \$10,000 was invested in Fidelity Fund on December 31, 1992. The chart shows how the value of your investment would have grown, and also shows how the Standard \& Poor's 500 Index did over the same period.


## CHAPTER SUMMARY

## Exponential Functions

The general form of the equation for an exponential function is

$$
y=C a^{x} \quad(a>0 \text { and } a \neq 1), \quad \text { where }
$$

$C$ is the initial value or $y$-intercept $a$ is the base and is called the growth (or decay) factor If $C>0$ and
$a>1$, the function represents growth $0<a<1$, the function represents decay

## Linear vs. Exponential Growth

Exponential growth is multiplicative, whereas linear growth is additive. Exponential growth involves multiplication by a constant factor for each unit increase in input. Linear growth involves adding a fixed amount for each unit increase in input.

In the long run, any exponential growth function will eventually dominate any linear growth function.


## Graphs of Exponential Functions

For functions in the form $y=C a^{x}$ :
The value of $C$ tells us where the graph crosses the $y$-axis.
The value of $a$ affects the steepness of the graph.
Exponential growth ( $a>1, C>0$ ). The larger the value of $a$, the more rapid the growth and the more rapidly the graph rises.
Exponential decay $(0<a<1, C>0)$. The smaller the value of $a$, the more rapid the decay and the more rapidly the graph falls.

The graphs of both exponential growth and decay functions are asymptotic to the $x$-axis.

Exponential growth: As $x \rightarrow-\infty, y \rightarrow 0$.
Exponential decay: As $x \rightarrow+\infty, y \rightarrow 0$.


## Factors, Rates, and Percents

An exponential function can be represented as a constant percent change. Growth and decay factors can be translated into constant percent increases or decreases, called growth or decay rates. For an exponential function in the form $y=C a^{x}$,

$$
\begin{aligned}
\text { growth factor } & =1+\text { growth rate } \\
a & =1+r
\end{aligned}
$$

where $r$ is the growth rate in decimal form.

$$
\begin{aligned}
\text { decay factor } & =1-\text { decay rate } \\
a & =1-r
\end{aligned}
$$

where $r$ is the decay rate in decimal form.

## Properties of Exponential Functions

The doubling time of an exponentially growing quantity is the time required for the quantity to double in size. The half-life of an exponentially decaying quantity is the time required for one-half of the quantity to decay.

The rule of 70 offers a simple way to estimate the doubling time or the half-life. If a quantity is growing at $R \%$ per year, then its doubling time is approximately $70 / R$ years. If a quantity is decaying at $R \%$ per month, then $70 / R$ gives its half-life in months.

## Semi-Log Plots of Exponential Functions

When an exponential function is plotted using a standard scale on the horizontal axis and a logarithmic scale on the vertical axis, its graph is a straight line. This is called a $\log$ linear or semi-log plot.

## CHECK YOUR UNDERSTANDING

I. Is each of the statements in Problems 1-24 true or false? Give an explanation for your answer.

1. If $y=f(x)$ is an exponential function and if increasing $x$ by 1 increases $y$ by a factor of 3 , then increasing $x$ by 2 increases $y$ by a factor of 6 .
2. If $y=f(x)$ is an exponential function and if increasing $x$ by 1 increases $y$ by $20 \%$, then increasing $x$ by 3 increases $y$ by about $73 \%$.
3. $y=x^{3}$ is an exponential function.
4. The graph of $y=10 \cdot 2^{x}$ is decreasing.
5. The average rate of change between any two points on the graph of $y=32.5(1.06)^{x}$ is constant.
6. Of the three exponential functions $y_{1}=5.4(0.8)^{x}$, $y_{2}=5.4(0.7)^{x}$, and $y_{3}=5.4(0.3)^{x}$, $y_{3}$ decays the most rapidly.
7. The graph of the exponential function $y=1.02^{x}$ lies below the graph of the line $y=x$ for $x>5$.
8. If $y=100(0.976)^{x}$, then as $x$ increases by $1, y$ decreases by $97.6 \%$.
9. The function in the accompanying figure represents exponential decay with an initial population of 150 and decay factor of 0.8 .

10. If the exponential function that models federal budget expenditures ( $E$ in billions of dollars) for a particular department is $E=134(1.041)^{t}$, where $t=$ number of
years since 1990, then in 1990 expenditures were about $\$ 134$ billion and were increasing by $4.1 \%$ per year.
11. Increasing $\$ 1000$ by $\$ 100$ per year for 10 years gives you more than increasing $\$ 1000$ by $10 \%$ per year for 10 years.
12. The value of the dollar with inflation at $2 \%$ per month is the same as the value of the dollar with inflation at $24 \%$ per year.
13. If $M$ dollars is invested at $6.25 \%$ compounded annually, the amount of money, $A$, in 14 years is $A=M(0.0625)^{14}$.
Problems 14 and 15 refer to the following graph.

14. Of the functions plotted in the figure, graph $C$ best describes the exponential function $f(x)=20(3)^{x}$.
15. Of the functions graphed in the figure, graph $B$ best describes the exponential function $g(x)=20(3)^{-x}$.
16. If $B=100(0.4)^{x}$, then as $x$ increases by $1, B$ decreases by $60 \%$.
17. If $y=f(x)$ is an exponential function and if increasing $x$ by 1 increases $y$ by 2, then increasing $x$ by 5 increases $y$ by 10 .

For Problems 18 and 19, refer to the following table.

| Year | 1990 | 1995 | 2000 | 2005 |
| :--- | :---: | :---: | :---: | :---: |
| Population $(000 \mathrm{~s})$ | 100 | 127.6 | 162.9 | 207.9 |

18. An exponential function would be a reasonable model for the population data.
19. Assuming this population grows exponentially, the annual growth rate is $6.29 \%$ because

$$
\frac{\text { population in year } 2000}{\text { population in year } 1990}=\frac{162.9}{100}=1.629
$$

a growth of $62.9 \%$ in 10 years, or $62.9 \% / 10$ years $=$ $6.29 \%$ per year.
20. After 30 years, the amount of interest earned on $\$ 5000$ invested at 5\% interest compounded annually will be half as much as the amount of interest earned on $\$ 10,000$ invested at the same rate.
21. If a population behaves exponentially over time and if $\frac{\text { population in year } 3}{\text { population in year } 0}=0.98$, then after 3 years the population will have decreased by $2 \%$.
22. If the half-life of a substance is 10 years, then three halflives of the substance would be 30 years.
23. If the doubling time of $\$ 100$ invested at an interest rate $r$ compounded annually is 9 years, then in 27 years the amount of the investment will be $\$ 600$.
24. The doubling time for the function in the accompanying graph is approximately 10 years.

II. In Problems 25-33, give an example of a function with the specified properties. Express your answer using formulas, and specify the independent and dependent variables.
25. An exponential function that has an initial population of 2.2 million people and increases $0.5 \%$ per year.
26. An exponential function that has an initial population of 2.2 million people and increases $0.5 \%$ per quarter.
27. An exponential function that has an initial value of $\$ 1.4$ billion and decreases $2.3 \%$ per decade.
28. An exponential function that passes through $(2,125)$ and $(4,5)$.
29. Two exponential decay functions with the same initial population but with the first decaying at twice the rate of the second.
30. Two functions, one linear and one exponential, that pass through the data points $(0,5)$ and $(3,0.625)$.
31. An exponential function that describes the value of a dollar over time with annual inflation of $3 \%$.
32. An exponential function that has a doubling time of 14 years.
33. A function that describes the amount remaining of an initial amount of 200 mg of a substance with a half-life of 20 years.

IIII. Is each of the statements in Problems 34-43 true or false? If a statement is false, give a counter-example.
34. Exponential functions are multiplicative and linear functions are additive.
35. All exponential functions are increasing.
36. If $P=C a^{t}$ describes a population $P$ as a function of $t$ years since 2000, then $C$ is the initial population in the year 2000.
37. Exponential functions $y=C a^{t}$ have a fixed or constant percent change per year.
38. Quantities that increase by a constant amount represent linear growth, whereas quantities that increase by a constant percent represent exponential growth.
39. The graph of an exponential growth function plotted on a semi-log plot is concave upward.
40. Eventually, the graph of every exponential function meets the horizontal axis.
41. For exponential functions, the growth rate is the same as the growth factor.
42. Exponential functions of the form $y=C \cdot a^{x}$ (where $a>0, a \neq 1$ ) are always asymptotic to the horizontal axis.
43. Of the two functions in the accompanying figure, only $A$ is decreasing at a constant percent.


## CHAPTER 5 REVIEW: PUTTING IT ALL TOGETHER

Internet access is required for Exercise 9 part (c). A calculator that can evaluate powers is recommended for Exercises 18, 25 part (c), and 27.

1. In each case, generate equations that represent the population, $P$, as a function of time, $t$ (in years), such that when $t=0, P=150$ and:
a. $P$ doubles each year.
b. $P$ decreases by twelve units each year.
c. $P$ increases by $5 \%$ each year.
d. The annual average rate of change of $P$ with respect to $t$ is constant at 12 .
2. Describe in words the following functions by identifying the type of function, the initial value, and what happens with each unit increase in $x$.
a. $f(x)=100(4)^{x}$
b. $g(x)=100-4 x$
3. Match each of the following functions with the appropriate graph.
a. $y=2^{x}$
b. $y=3^{x}$
c. $y=4^{x}$

4. Sketch graphs of the following functions on the same grid:
a. $y=0.1^{x}$
b. $y=0.2^{x}$
c. $y=0.3^{x}$
5. Studies have shown that lung cancer is directly correlated with smoking. The accompanying graph shows for each year after a smoker has stopped smoking his or her relative risk of lung cancer. The relative risk is the number of times more likely a former smoker is to get lung cancer than someone who has never smoked. For example, if the relative risk is 4.5 , then that patient is 4.5 times more likely to get lung cancer than a life-long non-smoker. Time is the measured in number of years since a smoker stopped smoking.
a. According to the graph, how many times more likely (the relative risk) is a male who just stopped smoking to get lung cancer than a lifelong nonsmoker? How many times more likely is a female who stopped smoking 12 years ago to get lung cancer than someone who never smoked?
b. Why is it reasonable that the relative risk is declining?
c. What does it mean if the relative risk is 1 ? Would you expect the relative risk to go below 1 ? Explain.

6. A polluter dumped 25,000 grams of a pollutant into a lake. Each year the amount of pollutant in the lake is reduced by $4 \%$. Construct an equation to describe the amount of pollutant after $n$ years.
7. You have a sore throat, so your doctor takes a culture of the bacteria in your throat with a swab and lets the bacteria grow in a Petri dish. If there were originally 500 bacteria in the dish and they grow by $50 \%$ per day, describe the relationship between $G(t)$, the number of bacteria in the dish and $t$, the time since the culture was taken (in days).
8. Which of the following functions are exponential and which are linear? Justify your answers.

| $x$ | $f(x)$ | $g(x)$ | $h(x)$ | $j(x)$ |
| :---: | :---: | :---: | :---: | :--- |
| 0 | -500 | 1 | 10.25 | 1 |
| 1 | -200 | 2.5 | 8.25 | 0.5 |
| 2 | 100 | 6.25 | 6.25 | 0.25 |
| 3 | 400 | 15.625 | 4.25 | 0.125 |
| 4 | 700 | 39.0625 | 2.25 | 0.0625 |
| 5 | 1000 | 97.65625 | 0.25 | 0.03125 |

9. The United Nations Department of Economic and Social Affairs reported in 1999 that "the world's population stands at 6 billion and is growing at $1.3 \%$ per year, for an annual net addition of 78 million people."
a. This statement actually contains two contradictory descriptions for predicting world population growth. What are they?
b. In order to decide which of the statements is more accurate, use the information provided to construct a linear model and an exponential model for world population growth.
c. The U.S. Census Bureau has a website that gives estimates for the current world population at http://www.census.gov/ ipc/www/popclockworld.html. Look up the current population. Which of your models is a more accurate predictor of the current world population?
10. Professional photographers consider exposure value (a combination of shutter speed and aperture) in relation to luminance (the amount of light that falls on a certain region) in setting camera controls to produce a desired effect. The data shown in the table give the relationship of the exposure value, $E_{v}$, to the luminance, $L$, for a particular type of film. Find a formula giving $L$ as a function of $E_{v}$.

| $E_{v}$, Exposure Value | $L$, Luminance <br> (candelas/sq. meter) |
| :---: | :---: |
| 0 | 0.125 |
| 1 | 0.25 |
| 2 | 0.5 |
| 3 | 1 |
| 4 | 2 |
| 5 | 4 |
| 6 | 8 |
| 7 | 16 |
| 8 | 32 |
| 9 | 64 |
| 10 | 128 |
| 11 | 256 |
| 12 | 512 |

11. According to Optimum Population Trust, the number of motor vehicles in the United Kingdom has nearly doubled every 25 years since 1925 . Use the "rule of 70 " to estimate the annual percent increase.
12. Mirex, a pesticide used to control fire ants, is rather longlived in the environment. It takes 12 years for half of the original amount to break down into harmless products. Use the "rule of 70" to estimate the annual percent decrease.
13. The U.S. Department of Human Services reported, "Health care spending in the United States is projected to grow 7.4 percent and surpass $\$ 2$ trillion in 2005 , down from the 7.9 percent growth experienced in 2004." The Department also published the following data about national health care expenditures and projections. Find the annual growth factor and the annual percent growth rate for health care since 2003. Show your calculations, and check to see if they agree with percent growth reported by the U.S. Department of Human Services.

|  |  |  | Projected |  |
| :---: | :---: | :---: | :---: | :---: |
| Year | 2003 | 2004 | 2005 | 2006 |
| Health care <br> expenditures <br> (billions) | $\$ 1,740.6$ | $\$ 1,877.6$ | $\$ 2,016.0$ | $\$ 2,169.5$ |

Source: http://www.cms.hhs.gov/NationalHealthExpendData/.
14. The U.S. Department of Human Services projected that in 2010 national health care expenditures would be approximately $\$ 2,887.3$ billion. Use your results from Problem 13 to estimate expenditures in 2010, assuming the initial value is $\$ 2,169.5$ billion (the projected estimate for 2006) and the annual percent growth rate is equal to:
a. The annual percent growth rate from 2005 to 2006.
b. The annual percent growth rate from 2004 to 2005.
c. How close is each projection for 2010 from parts (a) and (b) to the DHS projection? From your calculations, estimate the percent growth rate used by DHS. Assume

DHS used an annual percent growth rate from the projected estimate for the initial value in 2006.
15. The Energy Information Administration has made the following projections for energy consumption for the world:

## World Energy Consumption

| Year | 2015 | 2020 | 2025 | 2030 |
| :--- | :---: | :---: | :---: | :---: |
| Quadrillion Btu | 563 | 613 | 665 | 722 |

Source: www.eia.doe.gov/iea/.
Would a linear or an exponential function be an appropriate model for these data? Construct a function that models the data.
16. a. Generate a series of numbers $N$ using $N=2^{0.5 n}$, for integer values of $n$ from 0 to 10 . Use the value of $2^{0.5} \approx 1.414$ and the rules of exponents to calculate values for $N$.
b. Rewrite the $N$ formula using a square root sign.
17. NUA.com estimated that there were 605 million Internet users worldwide in 2002. Useit.com estimated that the number of Internet users will reach 2 billion in 2015 and 3 billion in 2040.
a. Given NUA's 2002 estimate of 605 million users, what annual growth rate would give 2 billion users in 2015?
b. Given Useit's estimate of 2 billion users in 2015 , what annual growth rate would give 3 billion users in 2040?
18. (A calculator that can evaluate powers is recommended.) The following graph shows median house prices in the United States from 1968 to 2004.
a. Estimate from the graph the median price paid for a home in 1968 and in 1993. Use these data points to construct a linear and an exponential model to represent the growth in median price from 1968 to 1993.
b. Which model is a better predictor of the national median price of a home in 2004 ? Would you use this model to make predictions for next year? Why or why not?


Source: National Association of Realtors www.realtor.org.
19. Radioactive iodine (I-131), used to test for thyroid problems, has a half-life of 8 days. If you start with 20 grams of radioactive iodine, describe the relationship between $A(t)$, the amount of I-131 (in grams), and $t$, the time (in days) since you first measured the sample.
20. Xenon gas (Xe-133) is used in medical imaging to study blood flow in the heart and brain. When inhaled, it is quickly absorbed into the bloodstream, and then gradually eliminated from the body (through exhalation). About 5 minutes after inhalation, there is half as much xenon left in the body (i.e., Xe-133 has a biological half-life of about 5 minutes). If you originally breathe in and absorb 3 ml . of xenon gas, describe the relationship between $X$, the amount of xenon gas in your body (in ml), and $t$, the time (in minutes) since you inhaled it.
21. In the accompanying chart, which graphs approximate exponential growth?


Source: http://www.useit.com/alertbox/9509.html.
22. In 2005 the Government Accountability Office issued a study of textbook prices. The report noted that in the preceding two decades textbook prices had been rising at the rate of $6 \%$ per year, roughly double the annual inflation rate of about $3 \%$ a year.
a. If a textbook cost $\$ 30$ in 1985, what would it cost in 2005?
b. Assuming textbook prices continue to rise at $6 \%$ per year, if a textbook costs $\$ 80$ in 2005, what will it cost in 2015?
c. A textbook that was first published in 1990 costs $\$ 120$ in 2007. Since 1990 its price has increased at the rate of $6 \%$ per year. What was its price in 1990 ?
23. You may have noticed that when you take a branch from certain trees, the branch looks like a miniature version of the tree. When you break off a piece of the branch, that looks like the tree too. Mathematicians call this property self-similarity. The village of Bourton-on-the-Water in southwest England has a wonderful example of selfsimilarity: it has a $1 / 10$ scale model of itself. Because the
$1 / 10$ scale model is a complete model of the town, it must contain a model of itself (that is, a $1 / 100$ scale model of Bourton), and because the $1 / 100$ scale model is also a complete model of Bourton, it also contains a scale model (that is, a $1 / 1000$ scale model of Bourton.)
a. If $A_{o}$ is the area of the actual village, how does the area of the first $1 / 10$ scale model (where each linear dimension is one-tenth of the actual size) compare with $A_{o}$ ?
b. If the scale models are made of the same materials as the actual village, how does the weight of the building materials in the $1 / 100$ scale model church compare with the weight $W_{o}$ of the original church?
c. If the actual village is called a level 0 model, the $1 / 10$ model is level 1 , the $1 / 100$ model is level 2 , and so on, find a formula for the area $A_{n}$ at any level $n$ of the model as a function of $A_{o}$ and for the weight $W_{n}$ at any level $n$ of the model as a function of $W_{o}$.
24. Computer worms are annoying and potentially very destructive. They are self-reproducing programs that run independently and travel across network connections. For some worms, such as SoBig, if you are sent an e-mail and you open up an attachment that contains SoBig, two things happen. First, SoBig installs a program that can be remotely activated in the future to send spam messages or shut down your computer. Then SoBig e-mails itself to everyone in your address book.
a. Construct a function that could model the spread of SoBig. Assume that everyone has in his or her address book ten people, none of whom have received SoBig.
b. Assume everyone in America has a computer. How many levels in a fractal tree would it take for SoBig to affect 300,000,000 Americans? (Hint: To find the total number of people who receive SoBig you need to add those at level 0 , level 1 , level, 2 , etc.)
c. Why is a worm like SoBig so dangerous?
25. [A calculator that can evaluate powers is recommended for part (c).] Lung cancer is one of the leading causes of death, especially for smokers. There are various forms of cancer that attack the lungs, including cancers that start in some other organ and metastasize to the lungs. Doubling times for lung cancers vary considerably but are likely to fall within the range of 2 to 8 months. Although individual cancers are unpredictable and may speed up or slow down, the following examples give an idea of the range of time possibilities.
a. Two people are found to have lung cancer tumors whose volumes are each estimated at 0.5 cubic centimeters. A former asbestos worker has a tumor with an expected doubling time of 8 months. A heavy smoker has a tumor with an expected doubling time of 2 months. Write two tumor growth exponential functions, $A(t)$ and $S(t)$, for the asbestos worker and smoker, respectively, where $t=$ number of 2-month time periods.
b. Graph $S(t)$ and $A(t)$ over 12 time periods on the same grid and compare the graphs.
c. Assuming both tumors grow in a spherical shape, what would be the diameter of each tumor after one year? (Note: The volume of a sphere $V=4 / 3 \pi r^{3}$.)
26. Which of the following graphs suggests a linear model for the original data? Which suggests an exponential model for the original data? Justify your answers.




[^2]27. (A calculator that can evaluate powers is recommended.) In February 2007 the U.S. Intergovernmental Panel on Climate Change reported that human activities have increased greenhouse gases in the atmosphere, resulting in global warming and other climate changes. Methane (a greenhouse gas) in the atmosphere has more than doubled since preindustrial times from about 750 parts per billion in 1850 to 1,750 in 2005 . The following graph shows the increase in methane gas since 1850 .


Source: The New York Times International, February 3, 2007.
a. Construct an exponential equation, $M(t)$ that could be used to model the increase in methane gas in the atmosphere, where $t=$ number of years since 1850 .
b. From the graph, estimate the approximate doubling time. Now calculate the approximate doubling time using the "rule of 70." How close are your answers?
c. Describe the exponential growth in methane gas since 1850 in two different ways.

## EXPLORATION 5.A

## Properties of Exponential Functions

## Objectives

- explore the effects of $a$ and $C$ on the graph of the exponential function in the form

$$
y=C a^{x} \quad \text { where } \quad a>0 \text { and } a \neq 0
$$

## Material/Equipment

- computer and software "E3: $y=C a^{x}$ Sliders" in Exponential \& Log Functions, or graphing calculator
- graph paper


## Procedure

We start by choosing values for $a$ and $C$ and graphing the resulting equations by hand. From these graphs we make predictions about the effects of $a$ and $C$ on the graphs of other equations. Take notes on your predictions and observations so you can share them with the class. Work in pairs and discuss your predictions with your partner.

## Making Predictions

1. Start with the simplest case, where $C=1$. The equation will now have the form

$$
y=a^{x}
$$

Make a data table and by hand sketch on the same grid the graphs for $y=2^{x}$ (here $a=2$ ) and $y=3^{x}$ (here $a=3$ ). Use both positive and negative values for $x$. Predict where the graphs of $y=2.7^{x}$ and $y=5^{x}$ would be located on your graph. Check your work and predictions with your partner.

| $x$ | $y=2^{x}$ | $\mathrm{y}=3^{x}$ |
| ---: | :--- | :--- |
| -2 |  |  |
| -1 |  |  |
| 0 |  |  |
| 1 |  |  |
| 2 |  |  |

How would you describe your graphs? Do they have a maximum or a minimum value? What happens to $y$ as $x$ increases? What happens to $y$ as $x$ decreases? Which graph shows $y$ changing faster compared with $x$ ?
2. Now create two functions in the form $y=a^{x}$ where $0<a<1$. Create a data table and graph your functions on the same grid. Make predictions for other functions where $C=1$ and $0<a<1$.
3. Now consider the case where $C$ has a value other than 1 for the general exponential function

$$
y=C a^{x}
$$

Create a table of values and sketch the graphs of $y=0.5\left(2^{x}\right)$ (in this case $C=0.5$ and $a=2$ ) and $y=3\left(2^{x}\right)$ (in this case $C=3$ and $a=2$ ). What do all these graphs have in common? What do you think will happen when $a=2$ and $C=10$ ? What do you think will happen to the graph if $a=2$ and $C=-3$ ? Check your predictions with your partner.

| $x$ | $y=0.5\left(2^{x}\right)$ | $y=3\left(2^{x}\right)$ | $y=-3\left(2^{x}\right)$ |
| ---: | :--- | :--- | :--- |
| -2 |  |  |  |
| -1 |  |  |  |
| 0 |  |  |  |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |

How would you describe your graphs? Do they have a maximum or a minimum value? What happens to $y$ as $x$ increases? What happens to $y$ as $x$ decreases? What is the $y$-intercept for each graph?

## Testing Your Predictions

Now test your predictions by using a program called "E3: $y=C a^{x}$ Sliders" in the Exponential \& Log Functions software package or by creating graphs using technology.

1. What effect does $a$ have?

Make predictions when $a>1$ and when $0<a<1$, based on the graphs you constructed by hand. Explore what happens when $C=1$ and you choose different values for $a$. Check to see whether your observations about the effect of $a$ hold true when $C \neq 1$.
How does changing $a$ change the graph? When does $y=a^{x}$ describe growth? When does it describe decay? When is it flat? Write a rule that describes what happens when you change the value for $a$. You only have to deal with cases where $a>0$.
2. What effect does $C$ have?

Make a prediction based on the graphs you constructed by hand. Now choose a value for $a$ and create a set of functions with different $C$ values. Graph these functions on the same grid.
How does changing $C$ change the graph? What does the value of $C$ tell you about the graphs of functions in the form $y=C a^{x}$ ? Describe your graphs when $C>0$ and when $C<0$. Use technology to test your generalizations.

## Exploration-Linked Homework

Write a 60 -second summary of your results, and present it to the class.


[^0]:    ${ }^{1}$ Most types of $E$. coli are beneficial to humans, aiding in digestion. A few types are lethal. You may have read about deaths resulting from people eating certain deadly strains of $E$. coli bacteria in undercooked hamburgers or tainted spinach. The explosive nature of exponential growth shows how a few dangerous bacteria can multiply rapidly to become a deadly quantity in a very short time.

[^1]:    ${ }^{5}$ In such cases sooner or later the government usually intervenes. In 1985 the Brazilian government imposed an anti-inflationary wage and price freeze. When the controls were dropped, inflation soared again, reaching a high in March 1990 of $80 \%$ per month! At this level, it begins to matter whether you buy groceries in the morning or wait until that night. On August 2, 1993, the government devalued the currency by defining a new monetary unit, the cruzeiro real, equal to 1000 of the old cruzeiros. Inflation still continued, and on July 1, 1994, yet another unit, the real, was defined equal to 2740 cruzeiros reales. By 1997, inflation had slowed considerably to about $0.1 \%$ per month.

[^2]:    Source: http://www.kurzweilai.net.

