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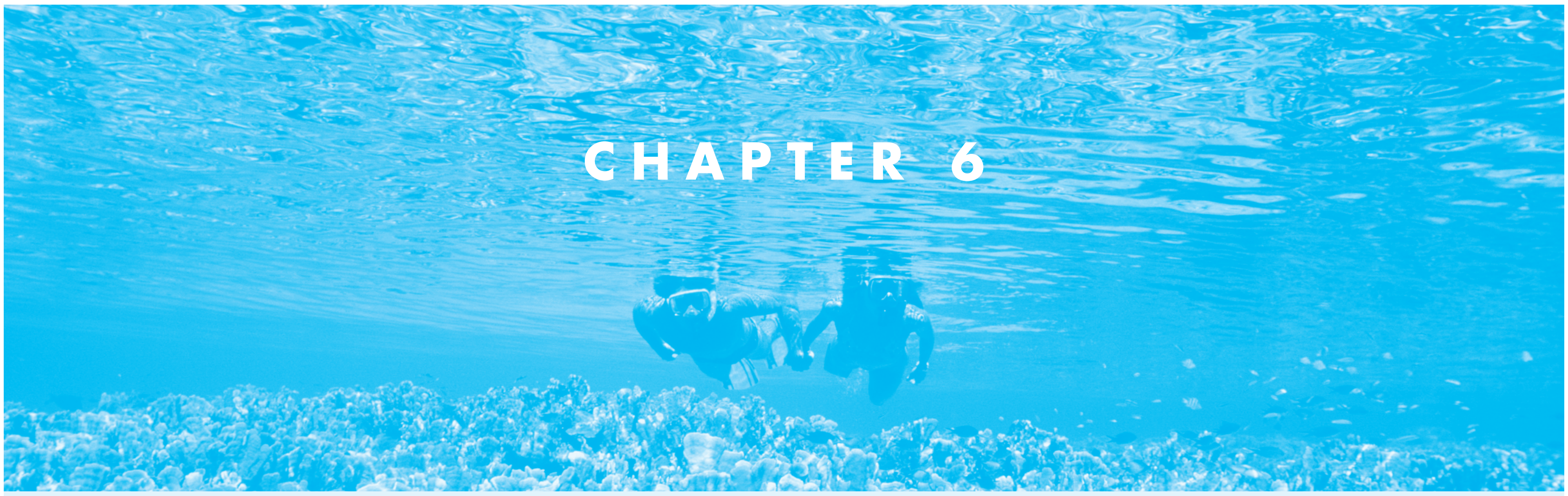

CHAPTER 6

LOGARITHMIC LINKS: LOGARITHMIC AND EXPONENTIAL FUNCTIONS

OVERVIEW

If we know a specific output for an exponential function, how can we find the associated input? To answer this question, we can use logarithmic functions, the close relatives of exponential functions. We return to the *E. coli* model to calculate the doubling time.

After reading this chapter you should be able to

- use logarithms to solve exponential equations
- apply the rules for common and natural (base *e*) logarithms
- create an exponential model for continuous compounding
- understand the properties of logarithmic functions
- describe the relationship between logarithmic and exponential functions
- find the equation of an exponential function on a semi-log plot

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6.1 *Using Logarithms to Solve Exponential Equations*

Estimating Solutions to Exponential Equations

So far, we have found a function's output from particular values of the input. For example, in Chapter 5, we modeled the growth of *E. coli* bacteria with the function

$$N = 100 \cdot 2^t$$

where N = number of *E. coli* bacteria and t = time (in 20-minute periods). If the input t is 5, then the corresponding output N is $100 \cdot 2^5 = 100 \cdot 32 = 3200$ bacteria.

Now we do the reverse, that is, find a function's input when we know the output. Starting with a value for N, the output, we find a corresponding value for t, the input. For example, at what time t will the value for N, the number of *E. coli,* be 1000? This turns out to be a harder question to answer than it might first appear. First we will estimate a solution from a table and a graph, and then we will learn how to find an exact solution using logarithms.

From a data table and graph

In Table 6.1, when $t = 3$, $N = 800$. When $t = 4$, $N = 1600$. Since N is steadily increasing, if we know $N = 1000$, then the value of t is somewhere between 3 and 4.

Values for $N = 100 \cdot 2^t$

t	N
0	100
1	200
2	400
3	800
4	1,600
5	3,200
6	6,400
7	12,800
8	25,600
9	51,200
10	102,400

Table 6.1

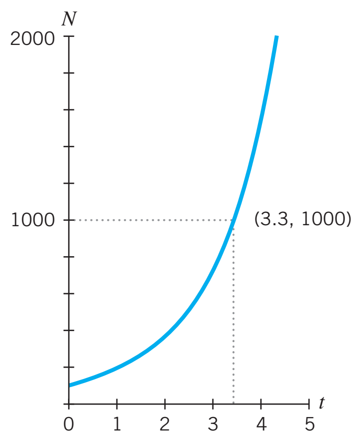

Figure 6.1 Estimating a value for t when $N = 1000$ on a graph of $N = 100 \cdot 2^t$.

We can also estimate the value for t when $N = 1000$ by looking at a graph of the function $N = 100 \cdot 2^t$ (Figure 6.1). By locating the position on the vertical axis where $N = 1000$, we can move over horizontally to find the corresponding point on the function graph. By moving from this point vertically down to the t-axis, we can estimate the t value for this point. The value for t appears to be approximately 3.3, so after about 3.3 time periods (or 66 minutes), the number of bacteria is 1000.

From an equation

An alternative strategy is to set $N = 1000$ and solve the equation for the corresponding value for t.

Start with the equation	$N = 100 \cdot 2^t$
Set $N = 1000$	$1000 = 100 \cdot 2^t$
Divide both sides by 100	$10 = 2^t$

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Then we are left with the problem of finding a solution to the equation

$$10 = 2^t$$

We can estimate the value for t that satisfies the equation by bracketing 2^t between consecutive powers of 2. Since $2^3 = 8$ and $2^4 = 16$, then $2^3 < 10 < 2^4$. So $2^3 < 2^t < 2^4$, and therefore t is between 3 and 4, which agrees with our previous estimates. Using this strategy, we can find an approximate solution to the equation $1000 = 100 \cdot 2^t$. Strategies for finding exact solutions to such equations require the use of logarithms.

Algebra Aerobics 6.1a

1. Given the equation $M = 250 \cdot 3^t$, find values for M when:

a. $t = 0$ **b.** $t = 1$ **c.** $t = 2$ **d.** $t = 3$

2. Use Table 6.1 to determine between which two consecutive integers the value of t lies when N is:

a. 2000 **b.** 50,000

3. Use the graph of $y = 3^x$ in Figure 6.2 to estimate the value of x in each of the following equations.

a. $3^x = 7$ **b.** $3^x = 0.5$

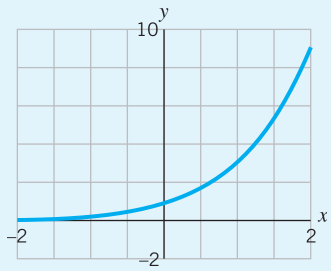

Figure 6.2 Graph of $y = 3^x$.

4. Use Table 6.2 to determine between which two consecutive integers the value of x in each of the following equations lies.

a. $5^x = 73$ **b.** $5^x = 0.36$

x	5^x
−2	0.04
−1	0.2
0	1
1	5
2	25
3	125
4	625

Table 6.2

5. For each of the following, find two consecutive integers for the exponents a and b that would make the statement true.

a. $2^a < 13 < 2^b$ **c.** $5^a < 0.24 < 5^b$

b. $3^a < 99 < 3^b$ **d.** $10^a < 1500 < 10^b$

Rules for Logarithms

In Chapter 4 we defined logarithms. Recall that:

> The *logarithm base 10 of x* is the exponent of 10 needed to produce x.
>
> $\log_{10} x = c$ means that $10^c = x$ where $x > 0$
>
> Logarithms base 10 are called *common logarithms* and $\log_{10} x$ is written as $\log x$.

So,

$10^5 = 100{,}000$ is equivalent to saying that $\log 100{,}000 = 5$

$10^{-3} = 0.001$ is equivalent to saying that $\log 0.001 = -3$

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Using a calculator we can solve equations such as

	$10^x = 80$
Rewrite it in equivalent form	$\log 80 = x$
use a calculator to compute the log	$1.903 \approx x$

But to solve equations such as $2^t = 10$ that involve exponential expressions where the base is not 10, we need to know more about logarithms. As the following expressions suggest, the rules of logarithms follow directly from the definition of logarithms and from the rules of exponents.

Rules for Exponents

If a is any positive real number and p and q are any real numbers, then

1. $a^p \cdot a^q = a^{p+q}$ **3.** $(a^p)^q = a^{p \cdot q}$

2. $a^p/a^q = a^{p-q}$ **4.** $a^0 = 1$

Corresponding Rules for Logarithms

If A and B are positive real numbers and p is any real number, then

1. $\log (A \cdot B) = \log A + \log B$ **3.** $\log (A^p) = p \log A$

2. $\log (A/B) = \log A - \log B$ **4.** $\log 1 = 0$ (since $10^0 = 1$)

SOMETHING TO THINK ABOUT

Try expressing in words all the other rules for logarithms in terms of exponents.

Finding the common logarithm of a number means finding its exponent when the number is written as a power of 10. So when you see "logarithm," think "exponent," and the rules of logarithms make sense.

As we learned in Chapter 4, the log of 0 or a negative number is not defined. But when we take the log of a number, we can get 0 or a negative value. For example, $\log 1 = 0$ and $\log 0.1 = -1$.

We will list a rationale for each of the rules of logarithms and prove Rule 1. We leave the other proofs as exercises.

Rule 1 $\log (A \cdot B) = \log A + \log B$

Rationale

Rule 1 of exponents says that when we multiply two terms with the same base, we keep the base and *add* the exponents; that is, $a^p \cdot a^q = a^{p+q}$.

Rule 1 of logs says that if we rewrite A and B each as a power of 10, then the exponent of $A \cdot B$ is the sum of the exponents of A and B.

Proof

If we let $\log A = x$, then	$10^x = A$
and if $\log B = y$, then	$10^y = B$
We have two equal products	$A \cdot B = 10^x \cdot 10^y$
by laws of exponents	$A \cdot B = 10^{x+y}$
Taking the log of each side	$\log(A \cdot B) = \log (10^{x+y})$
by definition of log	$\log(A \cdot B) = x + y$
Substituting $\log A$ for x and $\log B$ for y	$\log (A \cdot B) = \log A + \log B$
we arrive at our desired result.	

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Rule 2 $\log (A/B) = \log A - \log B$

Rationale

Rule 2 of exponents says that when we divide terms with the same base, we keep the base and *subtract* the exponents, that is, $a^p/a^q = a^{p-q}$. Rule 2 of logs says that if we write A and B each as a power of 10, then the exponent of A/B equals the exponent of A *minus* the exponent of B.

EXAMPLE 1 **Using the rules for logs**

Indicate whether or not the statement is true using the rules of exponents and the definition of logarithms.

a. $\log (10^2 \cdot 10^3) \stackrel{?}{=} \log 10^2 + \log 10^3$

b. $\log (10^5 \cdot 10^{-7}) \stackrel{?}{=} \log 10^5 + \log 10^{-7}$

c. $\log (10^2/10^3) \stackrel{?}{=} \log 10^2 - \log 10^3$

d. $\log (10^3/10^{-3}) \stackrel{?}{=} \log 10^3/\log 10^{-3}$

SOLUTION To see if an expression is a true statement, we must verify that the two sides of the expression are equal.

a. $\log (10^2 \cdot 10^3) \stackrel{?}{=} \log 10^2 + \log 10^3$

Rule 1 of exponents $\quad \log 10^5 \stackrel{?}{=} \log 10^2 + \log 10^3$

definition of log $\quad 5 = 2 + 3$

$5 = 5$

Since the values on the two sides of the equation are equal, our original equation is a true statement.

b. $\log (10^5 \cdot 10^{-7}) \stackrel{?}{=} \log 10^5 + \log 10^{-7}$

Rule 1 of exponents $\quad \log 10^{-2} \stackrel{?}{=} \log 10^5 + \log 10^{-7}$

definition of log $\quad -2 = 5 + (-7)$

$-2 = -2$

Since the values on the two sides of the equation are equal, our original equation is a true statement.

c. $\log (10^2/10^3) \stackrel{?}{=} \log 10^2 - \log 10^3$

Rule 2 of exponents $\quad \log (10^{-1}) \stackrel{?}{=} \log 10^2 - \log 10^3$

definition of log $\quad -1 = 2 - 3$

$-1 = -1$

Since the values on the two sides of the equation are equal, our original equation is a true statement.

d. $\log (10^3/10^{-3}) \stackrel{?}{=} \log 10^3/\log 10^{-3}$

Rule 2 of exponents $\quad \log 10^{3-(-3)} \stackrel{?}{=} \log 10^3/\log 10^{-3}$

combine terms in exponent $\quad \log 10^6 \stackrel{?}{=} \log 10^3/\log 10^{-3}$

definition of log $\quad 6 \stackrel{?}{=} 3/(-3)$

$6 \neq -1$

Since the values on the two sides of the equation are not equal, our original equation is not a true statement.

SOMETHING TO THINK ABOUT

Why does $\log \sqrt{AB}$ lie halfway between $\log A$ and $\log B$?

Rule 3 $$\log A^p = p \log A$$

Rationale

Since

$$\log A^2 = \log (A \cdot A) = \log A + \log A = 2 \log A$$
$$\log A^3 = \log (A \cdot A \cdot A) = \log A + \log A + \log A = 3 \log A$$
$$\log A^4 = \log (A \cdot A \cdot A \cdot A) = 4 \log A$$

it seems reasonable to expect that in general

$$\log A^p = p \log A$$

Rule 4 $$\log 1 = 0$$

Rationale

Since $10^0 = 1$ by definition, the equivalent statement using logarithms is $\log 1 = 0$.

EXAMPLE 2 **Simplifying expressions with logs**

Simplify each expression, and if possible evaluate with a calculator.

$$\log 10^3 = 3 \log 10 = 3 \cdot 1 = 3$$
$$\log 2^3 = 3 \log 2 \approx 3 \cdot 0.301 \approx 0.903$$
$$\log \sqrt{3} = \log 3^{1/2} = \tfrac{1}{2} \log 3 \approx \tfrac{1}{2} \cdot 0.477 \approx 0.239$$
$$\log x^{-1} = (-1) \cdot \log x = -\log x$$
$$\log 0.01^a = a \log 0.01 = a \cdot (-2) = -2a$$
$$(\log 5) \cdot (\log 1) = (\log 5) \cdot 0 = 0$$

EXAMPLE 3 **Expanding expressions with logs**

Use the rules of logarithms to write the following expression as the sum or difference of several logs.

$$\log \left(\frac{x(y-1)^2}{\sqrt{z}} \right)$$

SOLUTION

By Rule 2 $$\log \left(\frac{x(y-1)^2}{\sqrt{z}} \right) = \log x(y-1)^2 - \log \sqrt{z}$$

by Rule 1 and exponent notation $$= \log x + \log (y-1)^2 - \log z^{1/2}$$

by Rule 3 $$= \log x + 2 \log (y-1) - \tfrac{1}{2} \log z$$

We call this process *expanding the expression*.

EXAMPLE 4 **Contracting expressions with logs**

Use the rules of logarithms to write the following expression as a single logarithm.

$$2 \log x - \log(x-1)$$

SOLUTION

By Rule 3 $$2 \log x - \log (x-1) = \log x^2 - \log(x-1)$$

by Rule 2 $$= \log \left(\frac{x^2}{x-1} \right)$$

We call this process *contracting the expression*.

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Common Error

Probably the most common error in using logarithms stems from confusion over the division property.

$$\log A - \log B = \log\left(\frac{A}{B}\right) \quad \text{but} \quad \log A - \log B \neq \frac{\log A}{\log B}$$

For example,

$$\log 100 - \log 10 = \log\left(\frac{100}{10}\right) = \log 10 = 1$$

$$\text{but} \quad \log 100 - \log 10 \neq \frac{\log 100}{\log 10} = 2$$

EXAMPLE 5 **Solving equations that contain logs**

Solve for x in the equation $\log x + \log x^2 = 15$

SOLUTION

Given	$\log x + \log x^2 = 15$
Rule 1 for logs	$\log x^3 = 15$
Rule 3 for logs	$3 \log x = 15$
divide by 3	$\log x = 5$
rewrite using definition of logs	$x = 10^5$ or 100,000

Algebra Aerobics 6.1b

1. Using only the rules of exponents and the definition of logarithm, verify that:
   a. $\log (10^5/10^7) = \log 10^5 - \log 10^7$
   b. $\log [10^5 \cdot (10^7)^3] = \log 10^5 + 3 \log 10^7$
2. Determine the rule(s) of logarithms that were used in each statement.
   a. $\log 3 = \log 15 - \log 5$
   b. $\log 1024 = 10 \log 2$
   c. $\log \sqrt{31} = \frac{1}{2} \log 31$
   d. $\log 30 = \log 2 + \log 3 + \log 5$
   e. $\log 81 - \log 27 = \log 3$
3. Determine if each of the following is true or false. For the true statements tell which rule of logarithms was used.
   a. $\log (x + y) \stackrel{?}{=} \log x + \log y$
   b. $\log (x - y) \stackrel{?}{=} \log x - \log y$
   c. $7 \log x \stackrel{?}{=} \log x^7$
   d. $\log 10^{1.6} \stackrel{?}{=} 1.6$
   e. $\dfrac{\log 7}{\log 3} \stackrel{?}{=} \log 7 - \log 3$
   f. $\dfrac{\log 7}{\log 3} \stackrel{?}{=} \log (7 - 3)$
4. Expand, using the properties of logarithms:
   a. $\log\sqrt{\dfrac{2x - 1}{x + 1}}$
   b. $\log \dfrac{xy}{z}$
   c. $\log \dfrac{x\sqrt{x + 1}}{(x - 1)^2}$
   d. $\log \dfrac{x^2(y - 1)}{y^3 z}$
5. Contract, expressing your answer as a single logarithm: $\frac{1}{3} [\log x - \log (x + 1)]$.
6. Use rules of logarithms to combine into a single logarithm (if necessary), then solve for x.
   a. $\log x = 3$
   b. $\log x + \log 5 = 2$
   c. $\log x + \log 5 = \log 2$
   d. $\log x - \log 2 = 1$
   e. $\log x - \log (x - 1) = \log 2$
   f. $\log (2x + 1) - \log (x + 5) = 0$
7. Show that $\log 10^3 - \log 10^2 \neq \dfrac{\log 10^3}{\log 10^2}$

Solving Exponential Equations

Answering our original question: Solving $1000 = 100 \cdot 2^t$

Remember the question that started this chapter? We wanted to find out how many time periods it would take 100 *E. coli* bacteria to become 1000. To find an exact solution, we need to solve the equation $1000 = 100 \cdot 2^t$ or, dividing by 100, the equivalent equation, $10 = 2^t$. We now have the necessary tools.

Given	$10 = 2^t$
take the logarithm of each side	$\log 10 = \log 2^t$
use Rule 3 of logs	$\log 10 = t \log 2$
divide both sides by log 2	$\dfrac{\log 10}{\log 2} = t$
use a calculator	$\dfrac{1}{0.3010} \approx t$
divide	$3.32 \approx t$

which is consistent with our previous estimates of a value for t between 3 and 4, approximately equal to 3.3.

In our original model the time period represents 20 minutes, so 3.32 time periods represents $3.32 \cdot (20 \text{ minutes}) = 66.4$ minutes. So the bacteria would increase from the initial number of 100 to 1000 in a little over 66 minutes.

EXAMPLE 6 **Doubling your money**

As we saw in Chapter 5, the equation $P = 250(1.05)^n$ gives the value of \$250 invested at 5% interest (compounded annually) for n years. How many years does it take for the initial \$250 investment to double to \$500?

SOLUTION

a. *Estimating the answer:* If $R = 5\%$ per year, then the rule of 70 (discussed in Section 5.6) estimates the doubling time as

$$70/R = 70/5 = 14 \text{ years}$$

b. *Calculating a more precise answer:* We can set $P = 500$ and solve the equation.

	$500 = 250(1.05)^n$
Divide both sides by 250	$2 = (1.05)^n$
take the log of both sides	$\log 2 = \log (1.05)^n$
use Rule 3 of logs	$\log 2 = n \log 1.05$
divide by log 1.05	$\dfrac{\log 2}{\log 1.05} = n$
evaluate with a calculator	$\dfrac{0.3010}{0.0212} \approx n$
divide and switch sides	$n \approx 14.2$ years

So the estimate of 14 years using the rule of 70 was pretty close.

EXAMPLE 7 Solve the following equation for x in two ways.

$$8^x = 2^{x+1}$$

SOLUTION *Method 1:* Make the bases the same.

	$8^x = 2^{x+1}$
Make the base the same	$2^{3x} = 2^{x+1}$
set exponents equal	$3x = x + 1$
combine terms	$2x = 1$
solve for x	$x = \dfrac{1}{2}$

Method 2: Use the rules for logarithms.

	$8^x = 2^{x+1}$
Take logs of both sides	$\log 8^x = \log 2^{x+1}$
Rule 3 of logs	$x \log 8 = (x + 1) \log 2$
distributive law	$x \log 8 = x \log 2 + \log 2$
subtract $x \log 2$ from both sides	$x \log 8 - x \log 2 = \log 2$
factor out x	$x(\log 8 - \log 2) = \log 2$
Rule 2 of logs	$x \log 4 = \log 2$
divide by log 4	$x = \dfrac{\log 2}{\log 4} = \dfrac{\log 2}{\log 2^2} = \dfrac{\log 2}{2 \log 2} = \dfrac{1}{2}$

We can double check by both evaluating 8^x at $x = \frac{1}{2}$ $\qquad 8^{1/2} \approx 2.8284$

or evaluating 2^{x+1} at $x = \frac{1}{2}$ $\qquad 2^{(1/2)+1} = 2^{3/2} = (2^3)^{1/2} = 8^{1/2} \approx 2.8284$

EXAMPLE 8 **Time to decay to a specified amount**

In Chapter 5 we used the function $f(t) = 100(0.976)^t$ to measure the remaining amount of radioactive material as 100 milligrams (mg) of strontium-90 decayed over time t (in years). How many years would it take for there to be only 10 mg of strontium-90 left?

SOLUTION Set $f(t) = 10$ and solve the equation.

	$10 = 100(0.976)^t$
Divide both sides by 100	$0.1 = (0.976)^t$
take the log of both sides	$\log 0.1 = \log (0.976)^t$
use Rule 3 of logs	$\log 0.1 = t \log 0.976$
divide by log 0.9755	$\dfrac{\log 0.1}{\log 0.976} = t$
evaluate logs	$\dfrac{-1}{-0.0106} \approx t$
divide and switch sides	$t \approx 94$ years

So it takes almost a century for 100 mg of strontium-90 to decay to 10 mg.

Algebra Aerobics 6.1c

These problems require a calculator that can evaluate logs.

1. Solve the following equations for t.

a. $60 = 10 \cdot 2^t$ **c.** $80(0.95)^t = 10$

b. $500(1.06)^t = 2000$

2. Using the model $N = 100 \cdot 2^t$ for bacteria growth, where t is measured in 20-minute time periods, how long will it take for the bacteria count:

a. To reach 7000? **b.** To reach 12,000?

3. First use the rule of 70 to estimate how long it would take \$1000 invested at 6% compounded annually to double to \$2000. Then use logs to find a more precise answer.

4. Use the function in Example 8 to determine how long it will take for 100 milligrams of strontium-90 to decay to 1 milligram.

5. Solve each equation for t (in years). Which equation(s) asks you to find the time necessary for the initial amount to double? For the initial amount to drop to half?

a. $30 = 60(0.95)^t$ **c.** $500 = 200(1.045)^t$

b. $16 = 8(1.85)^t$

6. Find the half-life of a substance that decays according to the following models.

a. $A = 120(0.983)^t$ (t in days)

b. $A = 0.5(0.92)^t$ (t in hours)

c. $A = A_0(0.89)^t$ (t in years)

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Exercises for Section 6.1

Many of these problems (and those in later sections) require a calculator that can evaluate powers and logs. Some require a graphing program.

1. Use the accompanying table to estimate the number of years it would take \$100 to become \$300 at the following interest rates compounded annually.

a. 3% **b.** 7%

Compound Interest over 40 Years

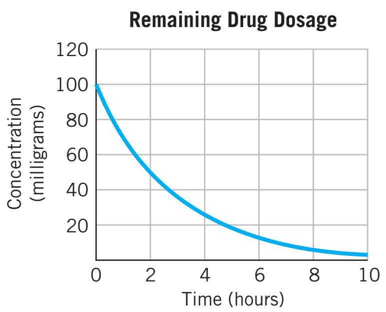
Number of Years	Value of \$100 at 3% (\$)	Value of \$100 at 7% (\$)
0	100	100
10	134	197
20	181	387
30	243	761
40	326	1497

2. a. Generate a table of values to estimate the half-life of a substance that decays according to the function $y = 100(0.8)^x$, where x is the number of time periods, each time period is 12 hours, and y is in grams.

b. How long will it be before there is less than 1 gram of the substance remaining?

3. The accompanying graph shows the concentration of a drug in the human body as the initial amount of 100 mg dissipates over time. Estimate when the concentration becomes:

a. 60 mg **b.** 40 mg **c.** 20 mg



4. (Requires a graphing program.) Assume throughout that x represents time in seconds.

a. Plot the graph of $y = 6(1.3)^x$ for $0 \le x \le 4$. Estimate the doubling time from the graph.

b. Now plot $y = 100(1.3)^x$ and estimate the doubling time from the graph.

c. Compare your answers to parts (a) and (b). What does this tell you?

5. Without a calculator, determine x if we know that $\log x$ equals:

a. -3 **c.** 0 **e.** -1

b. 6 **d.** 1

6. Given that $\log 5 \approx 0.699$, without using a calculator determine the value of:

a. $\log 25$ **b.** $\log \frac{1}{25}$ **c.** $\log 10^{25}$ **d.** $\log 0.0025$

Check your answers with a scientific calculator.

7. Expand each logarithm using only the numbers 2, 3, log 2, and log 3.

a. $\log 9$ **b.** $\log 18$ **c.** $\log 54$

8. Identify the rules of logarithms that were used to expand each expression.

a. $\log 14 = \log 2 + \log 7$

b. $\log 14 = \log 28 - \log 2$

c. $\log 36 = 2 \log 6$

d. $\log 9z^3 = 2 \log 3 + 3 \log z$

e. $\log 3x^4 = \log 3 + 4 \log x$

f. $\log\left(\dfrac{16}{3x}\right) = 4 \log 2 - (\log 3 + \log x)$

9. Use the rules of logarithms to find the value of x. Verify your answer with a calculator.

a. $\log x = \log 2 + \log 6$

b. $\log x = \log 24 - \log 2$

c. $\log x^2 = 2 \log 12$

d. $\log x = 4 \log 2 - 3 \log 2$

10. Use the rules of logarithms to show that the following are equivalent.

a. $\log 144 = 2 \log 3 + 4 \log 2$

b. $7 \log 3 + 5 \log 3 = 12 \log 3$

c. $2(\log 4 - \log 3) = \log\left(\dfrac{16}{9}\right)$

d. $-4 \log 3 + \log 3 = \log\left(\dfrac{1}{27}\right)$

11. Prove Rule 2 of logarithms:

$$\log(A/B) = \log A - \log B \quad (A, B > 0)$$

12. Expand, using the rules of logarithms.

a. $\log\left(x^2y^3\sqrt{z-1}\right)$ **c.** $\log\left(t^2 \cdot \sqrt[4]{tp^3}\right)$

b. $\log \dfrac{A}{\sqrt[3]{BC}}$

13. Contract, using the rules of logarithms, and express your answer as a single logarithm.

a. $3 \log K - 2 \log(K + 3)$

b. $-\log m + 5 \log(3 + n)$

c. $4 \log T + \frac{1}{2} \log T$

d. $\frac{1}{3}(\log x + 2 \log y) - 3(\log x + 2 \log y)$

14. For each of the following equations either prove that it is correct (by using the rules of logarithms and exponents) or else show that it is not correct (by finding numerical values for the variables that make the values of the two sides of the equation different).

a. $\log\left(\frac{x}{y}\right) = \frac{\log x}{\log y}$

b. $\log x - \log y = \log\left(\frac{x}{y}\right)$

c. $\log(2x) = 2 \log x$

d. $2 \log x = \log(x^2)$

e. $\log\left(\frac{x+1}{x+3}\right) = \log(x+1) - \log(x+3)$

f. $\log\left(x\sqrt{x^2+1}\right) = \log x + \frac{1}{2}(x^2+1)$

g. $\log(x^2+1) = 2 \log x + \log 1$

15. Prove Rule 3 of common logarithms: $\log A^p = p \log A$ (where $A > 0$).

16. Solve for t.

a. $10^t = 4$

b. $3(2^t) = 21$

c. $1 + 5^t = 3$

d. $10^{-t} = 5$

e. $5^t = 7^{t+1}$

f. $6 \cdot 2^t = 3^{t-1}$

17. Solve for x.

a. $\log x = 3$

b. $\log(x+1) = 3$

c. $3 \log x = 5$

d. $\log(x+1) - \log x = 1$

e. $\log x - \log(x+1) = 1$

18. Solve for x.

a. $2^x = 7$

b. $\left(\sqrt{3}\right)^{x+1} = 9^{2x-1}$

c. $12(1.5)^{x+1} = 13$

d. $\log(x+3) + \log 5 = 2$

e. $\log(x-1) = 2$

19. In Exercise 1, we estimated the number of years it would take \$100 to become \$300 at each of the interest rates listed below, compounded annually. Now calculate the number of years by constructing and solving the appropriate equations.

a. 3% **b.** 7%

20. Solve for the indicated variables.

a. $825 = 275 \cdot 3^T$

b. $45{,}000 = 15{,}000 \cdot 1.04^t$

c. $12 \cdot 10^{10} = (6 \cdot 10^8) \cdot 5^x$

d. $100 = 25(1.18)^t$

e. $32{,}000 = 8000(2.718)^t$

f. $8 \cdot 10^5 = 4 \cdot 10^5(2.5)^x$

21. (Requires a graphing program.) Let $f(x) = 500(1.03)^x$ and $g(x) = 4500$.

a. Using technology, graph the two functions on the same screen.

b. Estimate the point of intersection.

c. Solve the equation $4500 = 500(1.03)^x$ using logarithms.

d. Compare your answers.

22. (Requires a graphing program.) Let $f(x) = 100(0.8)^x$ and $g(x) = 10$.

a. Using technology, graph the two functions on the same screen.

b. Estimate the point of intersection.

c. Solve the equation $10 = 100(0.8)^x$ using logarithms.

d. Compare your answers.

23. Find the doubling time or half-life for each of the following functions (where x is in years).

a. $f(x) = 100 \cdot 4^x$

b. $g(x) = 100 \cdot \left(\frac{1}{4}\right)^x$

c. $h(x) = A(4)^x$ (*Hint:* Set $h(x) = 2A$)

d. $j(x) = A \cdot \left(\frac{1}{4}\right)^x$ (*Hint:* Set $j(x) = \frac{1}{2}A$)

24. The yearly per capita consumption of whole milk in the United States reached a peak of 40 gallons in 1945, at the end of World War II. It has been steadily decreasing at a rate of about 2.8% per year.

a. Construct an exponential model $M(t)$ for per capita whole milk consumption (in gallons) where t = years since 1945.

b. Use your model to estimate the year in which per capita whole milk consumption dropped to 7 gallons per person. How does this compare with the actual consumption of 7 gallons per person in 2005?

c. What might have caused this decline?

25. Wikipedia is a popular, free online encyclopedia (at *en.wikipedia.org*) that anyone can edit. (So articles should be taken with "a grain of salt.") One Wikipedia article claims that the number of articles posted on Wikipedia has been growing exponentially since October 23, 2002. At that date there were approximately 90 thousand articles posted, and the growth rate was about 0.2% *per day*.

a. Create an exponential model for the growth in Wikipedia articles.

b. What is the doubling time? Interpret your answer.

26. If the amount of drug remaining in the body after t hours is given by $f(t) = 100\left(\frac{1}{2}\right)^{t/2}$ (graphed in Exercise 3), then calculate:

a. The number of hours it would take for the initial 100 mg to become:

i. 60 mg **ii.** 40 mg **iii.** 20 mg

b. The half-life of the drug.

27. In Chapter 5 we saw that the function $N = N_0 \cdot 1.5^t$ described the actual number N of *E. coli* bacteria in an experiment after t time periods (of 20 minutes each) starting with an initial bacteria count of N_0.

a. What is the doubling time?

b. How long would it take for there to be ten times the original number of bacteria?

28. (Requires a graphing program.) A woman starts a training program for a marathon. She starts in the first week by doing 10-mile runs. Each week she increases her run length by 20% of the distance for the previous week.

a. Write a formula for her run distance, D, as a function of week, W.

b. Use technology to graph your function, and then use the graph to estimate the week in which she will reach a marathon length of approximately 26 miles.

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c. Now use your formula to calculate the week in which she will start running 26 miles.

29. The half-life of bismuth-214 is about 20 minutes.

a. Construct a function to model the decay of bismuth-214 over time. Be sure to specify your variables and their units.

b. For any given sample of bismuth-214, how much is left after 1 hour?

c. How long will it take to reduce the sample to 25% of its original size?

d. How long will it take to reduce the sample to 10% of its original size?

30. The atmospheric pressure at sea level is approximately 14.7 lb/in.^{2}, and the pressure is reduced by half for each 3.6 miles above sea level.

a. Construct a model that describes the atmospheric pressure as a function of miles above sea level.

b. At how many miles above sea level will the atmospheric pressure have dropped to half, i.e., to 7.35 lb/in.^{2}?

31. A department store has a discount basement where the policy is to reduce the selling price, S, of an item by 10% of its current price each week. If the item has not sold after the tenth reduction, the store gives the item to charity.

a. For a \$300 suite, construct a function for the selling price, S, as a function of week, W.

b. After how many weeks might the suite first be sold for less than \$150? What is the selling price at which the suite might be given to charity?

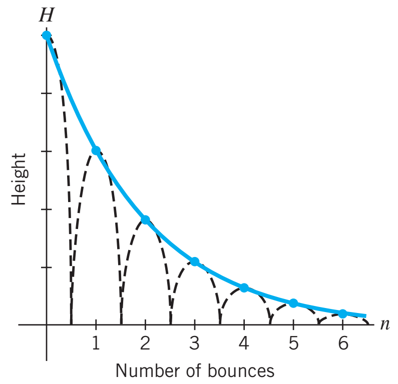
32. If you drop a rubber ball on a hard, level surface, it will usually bounce repeatedly. (See the accompanying graph at the top of the next column.) Each time it bounces, it rebounds to a height that is a percentage of the previous height. This percentage is called the *rebound height*.

a. Assume you drop the ball from a height of 5 feet and that the rebound height is 60%. Construct a table of values that shows the rebound height for the first four bounces.

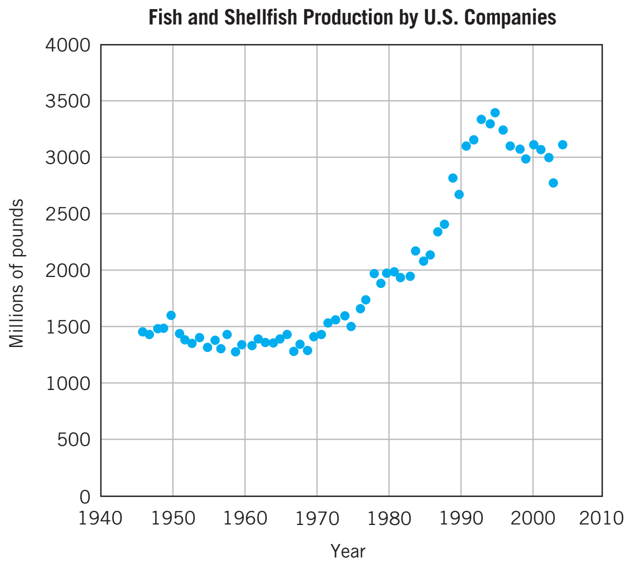
b. Construct a function to model the ball's rebound height, H, on the nth bounce.

c. How many bounces would it take for the ball's rebound height to be 1 foot or less?

d. Construct a general function that would model a ball's rebound height H on the nth bounce, where H_0 is the initial height of the ball and r is the ball's rebound height.



33. The accompanying graph shows fish and shellfish production (in million of pounds) by U.S. companies.



Source: U.S. Department of Agriculture, Economic Research Service, Data set on food availability, *www.ers.usda.gov.*

The growth appears to be exponential between 1970 and 1995. The exponential function $F(t)$ models that growth, where $F(t) = 1355(1.036)^t$ where $F(t)$ represents millions of pounds of fish and shellfish produced by U.S. companies since 1970 (where $0 \leq t \leq 25$).

a. Suppose we used the model to predict when fish production would reach 4000 million pounds. What year would that be? Using the graph, estimate the actual production in that year.

b. What might have caused the decline in U.S. fish production after 1995? Do you think that America's appetite for fish has waned?

6.2 *Base e and Continuous Compounding*

What is *e*?

Any positive number can be used as the base for an exponential or logarithmic expression. However, there is a "natural" base called e that is used in scientific applications. This number was named after Euler, a Swiss mathematician, and its value

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is approximately 2.71828. (You can use 2.72 as an estimate for e in most calculations. A scientific calculator has an e^x key for more accurate computations.) The number e is irrational; it cannot be written as the quotient of two integers or as a repeating decimal. Like π, the number e is a fundamental mathematical constant.

> The irrational number e is a fundamental mathematical constant whose value is approximately 2.71828.

We first learn why e is important, and then how to write any exponential function using e.

Continuous Compounding

The number e arises naturally in cases of continuous growth at a specified rate. For example, suppose we invest \$100 (called the principal) in a bank account that pays interest of 6% per year. To compute the amount of money we have at the end of 1 year, we must also know how often the interest is credited (added) to our account, that is, how often it is *compounded.*

What is compound interest?

Compounding Annually. If we invest \$100 in a bank account that pays 6% interest per year, then at the end of one year we would have \$100(1.06) = \$106. When the interest is applied to your account once a year, we say that the interest is *compounded* annually.

Compounding Twice a Year. Now suppose that the interest is compounded twice a year. This means that instead of applying the annual rate of 6% once, it is divided by 2 and applied twice, at the end of each 6-month period. At the end of the first 6 months we earn 6%/2 or 3% interest, so our balance is \$100(1.03) = \$103. At the end of the second 6 months we earn 3% interest on our new balance of \$103. So, after 1 year we have

$$\begin{aligned}\text{\$100 at 6\% interest compounded twice a year} &= \$100 \cdot (1.03) \cdot (1.03)\\ &= \$100 \cdot (1.03)^2\\ &= \$106.09\end{aligned}$$

We earn 9 cents more when interest is credited twice per year than when it is credited once per year. The difference is a result of the interest earned during the second half-year on the \$3 in interest credited at the end of the first half-year. In other words, we're starting to earn interest on interest. To earn the same amount with only annual compounding, we would need an interest rate of 6.09%.

When 6% interest is compounded twice a year, then

6% is the *nominal interest rate* (in name only)
or the annual percentage rate (APR)

and

6.09% is the *effective interest rate*
or the annual percentage yield (APY)

The effective interest rate is how much you actually earn (or pay) on an account. Banks and credit card companies are required by law to list both the nominal (the APR) and effective (the APY) interest rates.

Compounding Four Times a Year. Next, suppose that interest is compounded quarterly, or four times per year. In each quarter, we receive one-quarter of 6%, or 1.5% interest. Each quarter, our investment is multiplied by $1 + 0.015 = 1.015$ and, after the first quarter, we earn interest on the interest we have already received. At the end of 1 year we have received interest four times, so our initial \$100 investment has become

$$\$100 \cdot (1.015)^4 \approx \$106.14$$

In this case, the effective interest rate (or annual percentage yield) is about 6.14%.

Compounding n Times a Year. We may imagine dividing the year into smaller and smaller time intervals and computing the interest earned at the end of 1 year. The effective interest rate will be slightly more each time (Table 6.3).

Investing \$100 for One Year at a Nominal Interest Rate of 6%

Number of Times Interest Computed During the Year	Value of \$100 at End of One Year (\$)	Effective Annual Interest Rate (%)
1	$100(1 + 0.06) = 100(1.06) = 106.00$	6.00
2	$100(1 + 0.06/2)^2 = 100(1.03)^2 = 100(1.0609) = 106.09$	6.09
4	$100(1 + 0.06/4)^4 = 100(1.015)^4 \approx 100(1.0614) \approx 106.14$	6.14
6	$100(1 + 0.06/6)^6 = 100(1.010)^6 \approx 100(1.0615) \approx 106.15$	6.15
12	$100(1 + 0.06/12)^{12} = 100(1.005)^{12} \approx 100(1.0617) \approx 106.17$	6.17
24	$100(1 + 0.06/24)^{24} = 100(1.0025)^{24} \approx 100(1.0618) \approx 106.18$	6.18
.		
.		
.		
n	$100(1 + 0.06/n)^n$	

Table 6.3

In general, if we calculate the interest on \$100 n times a year when the nominal interest rate is 6%, we get

$$\$100\left(1 + \frac{0.06}{n}\right)^n$$

At the End of t Years. What if we invest \$100 for t years at a nominal interest rate of 6% compounded n times a year? The annual growth factor is $(1 + 0.06/n)^n$; that is, every year the \$100 is multiplied by $(1 + 0.06/n)^n$. After t years the \$100 is multiplied by $(1 + 0.06/n)^n$ a total of t times, or equivalently, multiplied by $[(1 + 0.06/n)^n]^t = (1 + 0.06/n)^{nt}$. So \$100 will be worth

$$\$100(1 + 0.06/n)^{nt}$$

We can generalize our results:

> **Compounding n Times a Year for t Years**
>
> The value P of P_0 dollars (called the principal) invested at a nominal interest rate r (expressed in decimal form) compounded n times a year for t years is
>
> $$P = P_0\left(1 + \frac{r}{n}\right)^{nt}$$

Continuous compounding using e

Imagine increasing the number of periods, n, without limits, so that interest is computed every week, every day, every hour, every second, and so on. The surprising result is that the term by which \$100 gets multiplied, namely,

$$\left(1 + \frac{0.06}{n}\right)^n$$

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does not get arbitrarily large. Examine Table 6.4.

Value of $(1 + 0.06/n)^n$ as n Increases

Compounding Period	n (Number of Compoundings per Year)	Approximate Value of $(1 + 0.06/n)^n$
Once a day	365	1.0618313
Once an hour	$365 \cdot 24 =$ 8,760	1.0618363
Once a minute	$365 \cdot 24 \cdot 60 =$ 525,600	1.0618365
Once a second	$365 \cdot 24 \cdot 60 \cdot 60 =$ 31,536,000	1.0618365

Table 6.4

Where does the irrational constant e fit in? As n, the number of compounding periods per year, increases, the value of $(1 + 0.06/n)^n$ approaches $1.0618365 \approx e^{0.06}$. You can confirm this on your calculator. As n gets arbitrarily large, we can think of the compounding occurring at each instant. We call this *continuous compounding*.

If we invest \$100 at 6% continuously compounded, at the end of 1 year we will have

$$\begin{aligned}\$100 \cdot e^{0.06} &\approx \$100 \cdot 1.0618365\\ &= \$106.18365\end{aligned}$$

When 6% annual interest is compounded continuously, then

6% is the *nominal interest rate* (in name only) or the annual percentage rate (APR)

and 6.18365% is the *effective interest rate* or the annual percentage yield (APY), which tells you how much interest you will earn after one year.

At the End of t Years. If the interest is compounded continuously, the annual growth factor is $e^{0.06}$; that is, every year the \$100 is multiplied by $e^{0.06}$. After t years the \$100 is multiplied by $e^{0.06}$ a total of t times, or equivalently multiplied by $(e^{0.06})^t = e^{0.06t}$. So \$100 will be worth

$$\$100e^{0.06t}$$

Hence, if we invest \$100 over t years at a nominal interest rate of 6%, we will have

$\$100(1 + 0.06/n)^{nt}$	if the interest is compounded n times a year
$\$100e^{0.06t}$	if the interest is compounded continuously

Just as $(1 + 0.06/n)^n$ approaches $e^{0.06}$ as n gets very large, $(1 + r/n)^n$ approaches e^r. So if P_0 dollars are invested at an annual interest rate r (in decimal form) compounded continuously, then after t years we have

$$P_0 \cdot (e^r)^t = P_0 \cdot e^{rt}$$

Compounding Continuously for t Years

The value P of P_0 dollars invested at a nominal interest rate r (expressed in decimal form) *compounded continuously* for t years is

$$P = P_0 \cdot e^{rt}$$

EXAMPLE 1 If you have \$250 to invest, and you are quoted a nominal interest rate of 4%, construct the equations that will tell you how much money you will have if the interest is compounded once a year, quarterly, once a month, or continuously. In each case calculate the value after 10 years.

SOLUTION See Table 6.5.

Investing $250 at a Nominal Interest Rate of 4% for Different Compoundings Intervals

Number of Compoundings per Year	$ Value after t Years	Approximate $ Value When $t = 10$ Years
1	$250 \cdot (1 + 0.04)^t = 250 \cdot (1.04)^t$	370.06
4	$250 \cdot (1 + 0.04/4)^{4t} = 250 \cdot (1.01)^{4t}$	372.22
12	$250 \cdot (1 + 0.04/12)^{12t} \approx 250 \cdot (1.00333)^{12t}$	372.71
Continuous	$250 \cdot e^{0.04t}$	372.96

Table 6.5

EXAMPLE 2 **Continuously compounding debt**

Suppose you have a debt on which the nominal annual interest rate (APR) is 7% compounded continuously. What is the effective interest rate (APY)?

SOLUTION The nominal interest rate (APR) of 7% is compounded continuously, so the equation $P = P_0e^{0.07t}$ describes the amount P that an initial debt P_0 becomes after t years. Using a calculator, we find that $e^{0.07} \approx 1.073$. The equation could be rewritten as $P = P_0(1.073)^t$. So the effective interest rate (APY) on your debt is about 7.3% per year.

EXAMPLE 3 You have a choice between two bank accounts. One is a passbook account in which you receive simple interest of 5% per year, compounded once per year. The other is a 1-year certificate of deposit (CD), which pays interest at the rate of 4.9% per year, compounded continuously. Which account is the better deal?

SOLUTION Since the interest on the passbook account is compounded once a year, the nominal and effective interest rates are both 5%. The equation $P = P_0(1.05)^t$ can be used to describe the amount P that the initial investment P_0 is worth after t years.

The 1-year certificate of deposit has a nominal interest rate of 4.9%. Since this rate is compounded continuously, the equation $P = P_0e^{0.049t}$ describes the amount P that the initial investment P_0 is worth after t years. Since $e^{0.049} \approx 1.0502$, the equation can also be written as $P = P_0(1.0502)^t$, and the effective interest rate is 5.02%. So the CD is a better deal.

Exponential Functions Base *e*

The notation of continuous compounding is useful in scientific as well as financial contexts. We can convert any exponential function in the form $f(t) = Ca^t$ into a continuous growth (or decay) function using a power of e as the base. Since $a > 0$, we can always find a value for k such that

$$a = e^k$$

So we can rewrite the function f as

$$\begin{aligned} f(t) &= C(e^k)^t \\ &= Ce^{kt} \end{aligned}$$

In general applications, we call k the *instantaneous* or *continuous growth* (or *decay*) *rate*. The value of k may be given as either a decimal or a percent.

If $k > 0$, then the function represents exponential growth. Why is this true? If an exponential function represents growth, then the growth factor $a > 1$. If we rewrite a as e^k and 1 as e^0, then $e^k > e^0$, so $k > 0$. For example, the equation $P(t) = 100\, e^{0.06t}$ could describe the growth of 100 cells with a continuous growth rate of 0.06 or 6% per time period t.

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If $k < 0$, then the function represents exponential decay. For exponential decay, the decay factor a is such that $0 < a < 1$. Rewriting a as e^k and 1 as e^0, we have $0 < e^k < e^0$. We know the value of e^k is always > 0 since e is a positive number. But if $e^k < e^0$, then $k < 0$. For example, the function $Q(t) = 50e^{-0.03t}$ could describe the decay of 50 cells with a continuous decay rate of 0.03 or 3% per time period t.

Continuous Growth and Decay

If $Q(t) = Ce^{kt}$, then k is called the *instantaneous* or *continuous growth* (or *decay*) *rate.*

For exponential growth, k is positive.

For exponential decay, k is negative.

EXAMPLE 4 Continuous growth or decay rates

Identify the continuous growth (or decay) rate for each of the following functions and graph each function using technology.

$$f(t) = 100 \cdot e^{0.055t}$$
$$g(t) = 100 \cdot e^{0.02t}$$
$$h(t) = 100 \cdot e^{-0.055t}$$
$$j(t) = 100 \cdot e^{-0.02t}$$

SOLUTION The function f has a continuous growth rate of 0.055 or 5.5% and g has a continuous growth rate of 0.02 or 2%.

The function h has a continuous decay rate of 0.055 or 5.5% and j has a continuous decay rate of 0.02 or 2%.

The graphs of these four functions are shown in Figure 6.3.

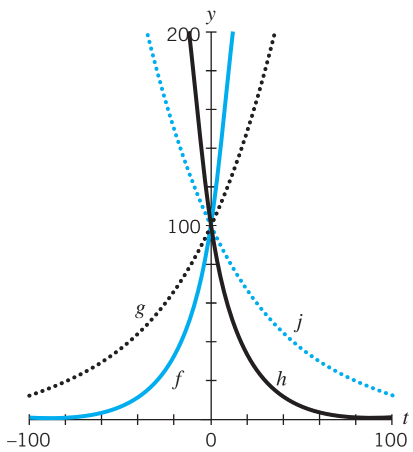

Figure 6.3 Graphs of four exponential functions.

Converting e^k into a

Using the rules of exponents, we can rewrite e^{kt} as $(e^k)^t$. When we know the value of k, we can calculate the value of e^k. For example,

	$P = 100 \cdot e^{0.06t}$
Rule 3 of exponents	$= 100 \cdot (e^{0.06})^t$
use a calculator to evaluate $e^{0.06}$	$\approx 100 \cdot 1.0618^t$

The two functions

$$P = 100 \cdot e^{0.06t} \qquad \text{and} \qquad P = 100 \cdot 1.0618^t$$

are equivalent. The first function (base e) suggests growth that occurs *continuously* throughout a time period, so we call 0.06 or 6% the *continuous* growth rate per time period t. The other function suggests growth that happens all at once at the end of each time period, so 0.0618 or 6.18% is just called the growth rate per time period t.

To do the reverse, that is, convert any base a into e^k, we need to know about logarithms base e, which we'll meet in the next section.

EXAMPLE 5

For each of the following functions, identify the *continuous* growth (or decay) rate per year and the growth (or decay) rate based on the growth (or decay) factor. Assume t is measured in years.

a. $f(t) = 240 \cdot e^{0.127t}$ **b.** $g(t) = 5700 \cdot e^{-0.425t}$

SOLUTION

a. The function $f(t)$ has a *continuous* growth rate of 0.127 or 12.7% per year. To find the growth factor, we need to convert $f(t)$ into the form $f(t) = Ca^t$. To do this we need to evaluate $e^{0.127}$. Using a calculator, we find that $e^{0.127} \approx 1.135$. So $f(t) = 240 \cdot e^{0.127t}$ can be rewritten as

$$f(t) = 240 \cdot 1.135^t$$

Since the growth factor $a = 1.135$, the growth rate is 0.135 or 13.5% per year.

b. The function $g(t)$ has a *continuous* decay rate of 0.425 or 42.5% per year. To find the decay factor, we need to convert $g(t)$ into the form $g(t) = Ca^t$. To do this, we need to evaluate $e^{-0.425}$. Using a calculator, we find that $e^{-0.425} \approx 0.654$. So $g(t) = 5700 \cdot e^{-0.425t}$ can be rewritten as

$$g(t) = 5700 \cdot 0.654^t$$

So the decay factor $a = 0.654$ and the decay rate is $1 - 0.654 = 0.346$ or 34.6% per year.

EXAMPLE 6

The cost of bottled water

In 1976 approximately 0.28 billion gallons of bottled water were sold in the United States, according to Beverage Marketing Corp., a New York research and consulting firm. Between 1976 and 2004 the bottled water industry in the United States had a continuous growth rate of about 11.5% a year.

a. Construct a model that represents the continuous growth of bottled water sales between 1976 and 2004.

b. Beverage Marketing later reported that in 2005 sales of bottled water were nearly 7.5 billion gallons. If we extrapolate, what would the model predict for sales in 2005? How does this compare with the actual sales?

SOLUTION

a. To represent continuous growth, we construct an exponential function using base e and a continuous growth rate of 11.5% a year. If we have an initial value of 0.28 billion gallons and t = number of years since 1976, then

$$f(t) = 0.28 \cdot e^{0.115t} \qquad (0 \le t \le 28)$$

models the continuous growth of bottled water sales in the United States between 1976 and 2004.

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b. If we extrapolate our model to predict sales for the year 2005, then $t = 2005 - 1976 = 29$ years and

$$\begin{aligned} f(29) &= 0.28 \cdot e^{(0.115)(29)} \\ &\approx 0.28 \cdot 28.1 \\ &\approx 7.86 \text{ billion gallons} \end{aligned}$$

Our model's prediction is somewhat over the actual sales of nearly 7.5 billion gallons in 2005.

You can check the current sales of bottled water at *www.beveragemarketing.com.*

Algebra Aerobics 6.2

Most of these problems require a calculator that can evaluate powers of e and, for Problem 4, evaluate logs.

1. Find the amount accumulated after 1 year on an investment of \$1000 at 8.5% compounded:

a. Annually **b.** Quarterly **c.** Continuously

2. Find the effective interest rate for each given nominal interest rate that is compounded continuously.

a. 4% **b.** 12.5% **c.** 18%

3. Assume that each of the following describes the value of an investment, A, over t years. Identify the principal, nominal rate, effective rate, and number of interest periods per year.

a. $A = 6000 \cdot 1.05^t$ **d.** $A = 50{,}000 \cdot 1.025^{2t}$

b. $A = 10{,}000 \cdot 1.02^{4t}$ **e.** $A = 125e^{0.076t}$

c. $A = 500 \cdot 1.01^{12t}$

4. Fill in the missing values, translating from e^k to a (the growth or decay factor).

a. $5e^{0.03t} = 5(\_\_\_)^t$

b. $3500e^{0.25t} = 3500(\_\_\_)^t$

c. $660e^{1.75t} = 660(\_\_\_)^t$

d. $55{,}000e^{-0.07t} = 55{,}000(\_\_\_)^t$

e. $125{,}000e^{-0.28t} = 125{,}000(\_\_\_)^t$

5. The value for e is often defined as the number that $(1 + 1/n)^n$ approaches as n gets arbitrarily large. Use your calculator to complete Table 6.6 at the bottom of the page. Use your exponent key (x^y or y^x) to evaluate the last column. Is your value consistent with the approximate value for e of 2.71828 given in the text?

6. If a principal of \$10,000 is invested at the rate of 12% compounded quarterly, the amount accumulated at the end of t years is given by the formula

$$A = 10{,}000 \cdot \left(1 + \frac{0.12}{4}\right)^{4t} = 10{,}000(1.03)^{4t}$$

The graph of this function is given in Figure 6.4. Use the graph to estimate for parts (a)–(c) the amount, A, accumulated after:

Figure 6.4 Graph of $A = 10{,}000(1.03)^{4t}$.

a. 1 year **b.** 5 years **c.** 10 years

d. Use the graph to estimate the number of years it will take to double the original investment.

e. Use the equation to calculate the amount A after the years specified in parts (a)–(c) and the doubling time for A.

n	$1/n$	$1 + (1/n)$	$[1 + (1/n)]^n$
1	1	$1 + 1 = 2$	$2^1 = 2$
100	0.01	$1 + 0.01 = 1.01$	$(1.01)^{100} \approx 2.7048138$
1,000			
1,000,000			
1,000,000,000			

Table 6.6

7. At birth, Maria's parents set aside $8000 in an account designated to help pay for her college education. How much will Maria's account be worth by her 18th birthday if the interest rate was:

a. 8% compounded quarterly?

b. 8% compounded continuously?

c. 8.4% compounded annually?

8. Suppose Maria (from the previous problem) earns a full scholarship and is able to save all the money in her college account. How much would be in her account on her 30th birthday for each of the rates above?

9. For each of the following functions, identify the continuous growth rate and then determine the effective growth rate. Assume t is measured in years.

a. $A(t) = Pe^{0.6t}$ **b.** $N(t) = N_0e^{2.3t}$

10. For each of the following functions, identify the continuous decay rate and then determine the effective decay rate. Assume t is measured in years.

a. $Q(t) = Q_0e^{-0.055t}$ **b.** $P(t) = P_0e^{-0.15t}$

Exercises for Section 6.2

Many of these exercises require a calculator that can evaluate powers (including powers of base e) and logs. One (Exercise 14) requires a graphing program.

1. Use a calculator to find the value of each expression to four decimal places.

a. $\left(1 + \dfrac{0.035}{2}\right)^2$ **d.** $e^{0.035}$

b. $\left(1 + \dfrac{0.035}{4}\right)^4$ **e.** How are these values related?

c. $\left(1 + \dfrac{0.035}{12}\right)^{12}$

2. Assume $y = Pe^{rx}$ represents P dollars invested at an annual interest rate r (in decimal form) compounded continuously for x years. Then for each of the following, calculate the nominal and effective rates in percentage form.

	Nominal Rate (APR)	Effective Rate (APY) (*Hint:* Evaluate at $e^k = a$)
a. $y = Pe^{0.025x}$		
b. $y = Pe^{0.039x}$		
c. $y = Pe^{0.062x}$		

3. Assume $10,000 is invested at a nominal interest rate of 8.5%. Write the equations that give the value of the money after n years and determine the effective interest rate if the interest is compounded:

a. Annually **c.** Quarterly

b. Semiannually **d.** Continuously

4. Assume you invest $2000 at 3.5% compounded continuously.

a. Construct an equation that describes the value of your investment at year t.

b. How much will $2000 be worth after 1 year? 5 years? 10 years?

5. Construct functions for parts (a) and (b) and compare them in parts (c) and (d).

a. $25,000 is invested at 5.75% compounded quarterly.

b. $25,000 is invested at 5.75% compounded continuously.

c. What is the amount in each account at the end of 5 years?

d. Explain, using the concept of effective rates, why one amount is larger than the other.

6. Fill in the following chart assuming that the principal is $10,000 in each case.

Nominal Interest Rate (APR)	Compounding Period	Expression for the Value of Your Account after t Years	Effective Interest Rate (APY)
5.25%	Monthly		
		$10{,}000(1 + 0.045/4)^{4t}$	
	Semiannually		8.16%
3.25%	Daily		
		$10{,}000(1.02)^t$	

7. Insert the symbol $>, <$, or $\approx$ to make the statement true.

a. $e^{0.045}$ ___ 1.046 **d.** $e^{-0.10}$ ___ 0.90

b. 1.068 ___ $e^{0.068}$ **e.** 0.8607 ___ $e^{-0.15}$

c. 1.269 ___ $e^{0.238}$

8. Assume that $5000 was put in each of two accounts. Account A gives 4% interest compounded semiannually. Account B gives 4% compounded continuously.

a. What are the total amounts in each of the accounts after 10 years?

b. Show that account B gives 0.04% annually more interest than account A.

9. The half-life of uranium-238 is about 5 billion years. Assume you start with 10 grams of U-238 that decays continuously.

a. Construct an equation to describe the amount of U-238 remaining after x billion years.

b. How long would it take for 10 grams of U-238 to become 5 grams?

10. You want to invest money for your newborn child so that she will have $50,000 for college on her 18th birthday. Determine

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how much you should invest if the best annual rate that you can get on a secure investment is:

a. 6.5% compounded annually

b. 9% compounded quarterly

c. 7.9% compounded continuously

11. Determine the doubling time for money invested at the rate of 12% compounded:

a. Annually **b.** Quarterly

12. a. Phosphorus-32 is used to mark cells in biological experiments. If phosphorus-32 has a continuous daily decay rate of 0.0485 or 4.85%, what is its half-life? (*Hint:* Rewrite the function as $y = \text{C}a^x$ and set $y = 0.5C$.)

b. Phosphorus-32 can be quite dangerous to work with if the experimenter fails to use the proper shields, since its high-energy radiation extends out to 610 cm or about 20 feet. Because disposal of radioactive wastes is increasingly difficult and expensive, laboratories often store the waste until it is within acceptable radioactive levels for disposal with non-radioactive trash. For instance, the rule of thumb for the laboratories of a large East Coast university and medical center is that any waste containing radioactive material with a half-life under 65 days must be stored for 10 half-lives before disposal with the non-radioactive trash.

i. For how many days would phosphorus-32 have to be stored?

ii. What percentage of the original phosphorus-32 would be left at that time?

13. A city of population 1.5 million is expected to experience a 15% decrease in population every 10 years.

a. What is the 10-year decay factor? What is the *yearly* decay factor? The *yearly* decay rate?

b. Use part (a) to create an exponential population model $g(t)$ that gives the population (in millions) after t years.

c. Create an exponential population model $h(t)$ that gives the population (in millions) after t years, assuming a 1.625% continuous yearly decrease.

d. Compare the populations predicted by the two functions after 20 years. What can you conclude?

14. (Requires a graphing program.) Using technology, graph the functions $f(x) = 15{,}000e^{0.085x}$ and $g(x) = 100{,}000$ on the same grid.

a. Estimate the point of intersection. (*Hint:* Let x go from 0 to 60.)

b. If $f(x)$ represents the amount of money accumulated by investing at a continuously compounded rate (where x is in years), explain what the point of intersection represents.

15. Rewrite each continuous growth function in its equivalent form $f(t) = Ca^t$. In each case identify the continuous growth rate, and the effective growth rate. (Assume that t is in years.)

a. $P(t) = 500e^{0.02t}$ **c.** $Q(t) = 45e^{0.06t}$

b. $N(t) = 3000e^{1.5t}$ **d.** $G(t) = 750e^{0.035t}$

16. Rewrite each continuous decay function in its equivalent form $f(t) = Ca^t$. In each case identify the continuous decay rate and the effective decay rate. (Assume that t is in years.)

a. $P(t) = 600e^{-0.02t}$ **c.** $Q(t) = 7145e^{-0.06t}$

b. $N(t) = 30{,}000e^{-0.5t}$ **d.** $G(t) = 750e^{-0.035t}$

17. Find the nominal interest rate (APR) if a bank advertises that the effective interest rate (APY) on an account compounded continuously is:

a. 3.43% on a checking account

b. 4.6% on a savings account

18. An investment pays 6% compounded four times a year.

a. What is the annual growth factor?

b. What is the annual growth rate?

c. Develop a formula to represent the total value of the investment after each compounding period.

d. If you invest $2000 for a child's college fund, how much will it total after 15 years?

e. For how many years would you have to invest to increase the total to $5000?

6.3 *The Natural Logarithm*

The *common logarithm* uses 10 as a base. The *natural logarithm* uses e as a base and is written $\ln x$ rather than $\log_e x$. Scientific calculators have a key that computes $\ln x$.

The Natural Logarithm

The *logarithm base* e of x is the exponent of e needed to produce x. Logarithms base e are called *natural logarithms* and are written as $\ln x$.

$$\ln x = c \qquad \text{means that} \qquad e^c = x \quad (x > 0)$$

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The properties for natural logarithms (base e) are similar to the properties for common logarithms (base 10). Like the common logarithm, $\ln A$ is not defined when $A \leq 0$.

If A and B are positive real numbers and p is any real number, then the following rules hold.

Rules of Common Logarithms

1. $\log(A \cdot B) = \log A + \log B$	**3.** $\log A^p = p \log A$
2. $\log(A/B) = \log A - \log B$	**4.** $\log 1 = 0$ (since $10^0 = 1$)

Rules of Natural Logarithms

1. $\ln(A \cdot B) = \ln A + \ln B$	**3.** $\ln A^p = p \ln A$
2. $\ln(A/B) = \ln A - \ln B$	**4.** $\ln 1 = 0$ (since $e^0 = 1$)

In the following examples we show how to use natural logarithms to manipulate expressions and solve exponential equations.

EXAMPLE 1 **Effective vs. nominal rates**

If the effective annual interest rate on an account is 5.21%, estimate the nominal annual interest rate that is compounded continuously.

SOLUTION A 5.21% effective interest rate is 0.0521 in decimal form. So the equation $P = P_0(1.0521)^x$ represents the value P of P_0 dollars after x years. To find the nominal interest rate that is continuously compounded, we must convert $(1.0521)^x$ into the form e^{rx} or $(e^r)^x$. So e^r must equal 1.0521. We need to solve for r in the equation.

	$1.0521 = e^r$
Take ln of both sides	$\ln 1.0521 = \ln e^r$
use a calculator	$0.0508 \approx \ln e^r$
definition of ln	$0.0508 \approx r$

So $1.0521 \approx e^{0.0508}$. A nominal interest rate of 0.0508 or 5.08% compounded continuously is equivalent to an effective interest rate of 0.0521 or 5.21%.

EXAMPLE 2 Expand, using the laws of logarithms, the expression: $\ln\sqrt{\dfrac{x+3}{x-2}}$.

SOLUTION

Rewrite using exponents	$\ln\sqrt{\dfrac{x+3}{x-2}} = \ln\left(\dfrac{x+3}{x-2}\right)^{1/2}$
Rule 3 of ln	$= \dfrac{1}{2}\ln\left(\dfrac{x+3}{x-2}\right)$
Rule 2 of ln	$= \dfrac{1}{2}[\ln(x+3) - \ln(x-2)]$

EXAMPLE 3 Contract, expressing the answer as a single logarithm: $\frac{1}{3}\ln(x - 1) + \frac{1}{3}\ln(x + 1)$

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SOLUTION

Distributive property	$\frac{1}{3}\ln(x-1)+\frac{1}{3}\ln(x+1)=\frac{1}{3}[\ln(x-1)+\ln(x+1)]$
Rule 1 of ln	$=\frac{1}{3}\ln[(x-1)(x+1)]$
Rule 3 of ln	$=\ln[(x-1)(x+1)]^{1/3}$
multiply binomials	$=\ln(x^2-1)^{1/3}$ or $\ln\sqrt[3]{x^2-1}$

EXAMPLE 4 Solve the equation $10 = e^t$ for t.

SOLUTION

Given	$10 = e^t$
take ln of both sides	$\ln 10 = \ln e^t$
definition of ln	$\ln 10 = t$
evaluate and switch sides	$t \approx 2.303$

Algebra Aerobics 6.3

Problems 4, 5, 6, and 8 require a calculator that can evaluate natural logs.

1. Evaluate without a calculator:
   a. $\ln e^2$
   b. $\ln 1$
   c. $\ln \frac{1}{e}$
   d. $\ln \frac{1}{e^2}$
   e. $\ln \sqrt{e}$
2. Expand the following:
   a. $\ln\sqrt{xy}$
   b. $\ln\left(\frac{3x^2}{y^3}\right)$
   c. $\ln\left((x+y)^2(x-y)\right)$
   d. $\ln\frac{\sqrt{x+2}}{x(x-1)}$
3. Contract, expressing your answer as a single logarithm:
   a. $\ln x + \ln(x-1)$
   b. $\ln(x+1) - \ln x$
   c. $2\ln x - 3\ln y$
   d. $\frac{1}{2}\ln(x+y)$
   e. $\ln x - 2\ln(2x-1)$
4. Find the nominal rate on an investment compounded continuously if the effective rate is 6.4%.
5. Determine how long it takes for \$10,000 to grow to \$50,000 at 7.8% compounded continuously.
6. Solve the following equations for x.
   a. $e^{x+1} = 10$ b. $e^{x-2} = 0.5$
7. Determine which of the following are true statements. If the statement is false, rewrite the right-hand side so that the statement becomes true.
   a. $\ln 81 \stackrel{?}{=} 4\ln 3$
   b. $\ln 7 \stackrel{?}{=} \frac{\ln 14}{\ln 2}$
   c. $\ln 35 \stackrel{?}{=} \ln 5 + \ln 7$
   d. $2\ln 10 \stackrel{?}{=} \ln 20$
   e. $\ln\sqrt{e} \stackrel{?}{=} \frac{1}{2}$
   f. $5\ln 2 \stackrel{?}{=} \ln 25$
8. Use the rules of logarithms to contract each expression into a single logarithm (if necessary), then solve for x.
   a. $\ln 2 + \ln 6 = x$
   b. $\ln 2 + \ln x = 2.48$
   c. $\ln(x+1) = 0.9$
   d. $\ln 5 - \ln x = -0.06$

Exercises for Section 6.3

Some of these exercises require a calculator that can evaluate powers and logs.

1. Determine the rule(s) of logarithms that were used to expand each expression.
   a. $\ln 15 = \ln 3 + \ln 5$
   b. $\ln 15 = \ln 30 - \ln 2$
   c. $\ln 49 = 2\ln 7$
   d. $\ln 25z^3 = 2\ln 5 + 3\ln z$
   e. $\ln 5x^4 = \ln 5 + 4\ln x$
   f. $\ln\left(\frac{125}{3x}\right) = 3\ln 5 - (\ln 3 + \ln x)$
2. Expand each logarithm using only the numbers 2, 5, ln 2, and ln 5.
   a. $\ln 25$ b. $\ln 250$ c. $\ln 625$
3. Write an equivalent expression using exponents.
   a. $n = \log 35$
   b. $\ln 75 = x$
   c. $\ln x = \frac{3}{4}$
   d. $\ln\left(\frac{N}{N_0}\right) = -kt$
4. Write an equivalent equation in logarithmic form.
   a. $N = 10^{-t/c}$
   b. $I = I_0 \cdot e^{-k/x}$
   c. $e^{3x} = 27$
   d. $\frac{1}{2} = e^{-kt}$

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5. Use rules of logarithms to find the value of x. Verify your answer with a calculator.

a. $\ln x = \ln 2 + \ln 5$
b. $\ln x = \ln 24 - \ln 2$
c. $\ln x^2 = 2 \ln 11$
d. $\ln x = 3 \ln 2 + 2 \ln 6$
e. $\ln x = 6 \ln 2 - 2 \ln 3$
f. $\ln x = 4 \ln 2 - 3 \ln 2$

6. Use rules of logarithms to contract to a single logarithm. Use a calculator to verify your answer.

a. $2 \ln 3 + 4 \ln 2$ **c.** $2(\ln 4 - \ln 3)$
b. $3 \ln 7 - 5 \ln 3$ **d.** $-4 \ln 3 + \ln 3$

7. Use rules of logarithms to expand.

a. $\ln\left(\sqrt{4xy}\right)$ **b.** $\ln\left(\frac{\sqrt[3]{2x}}{4}\right)$ **c.** $\ln\left(3 \cdot \sqrt[4]{x^3}\right)$

8. Use rules of logarithms to contract to a single logarithm.

a. $\frac{1}{2} \ln x - 5 \ln y$ **c.** $\ln 2 + \frac{1}{3} \ln x - 4 \ln y$
b. $\frac{2}{5} \ln x + \frac{4}{5} \ln y$

9. Contract, expressing your answer as a single logarithm.

a. $\frac{1}{4} \ln (x + 1) + \frac{1}{4} \ln (x - 3)$
b. $3 \ln R - \frac{1}{2} \ln P$
c. $\ln N - 2 \ln N_0$

10. Expand:

a. $\ln \sqrt[3]{\frac{x^2 - 1}{x + 2}}$ **b.** $\ln\left(\frac{x}{y\sqrt{2}}\right)^2$ **c.** $\ln\left(\frac{K^2 L}{M + 1}\right)$

11. Solve for the indicated variable, by changing to logarithmic form. Round your answer to three decimal places.

a. $e^r = 1.0253$ **b.** $3 = e^{0.5t}$ **c.** $\frac{1}{2} = e^{3x}$

12. Solve for x by changing to exponential form. Round your answer to three decimal places.

a. $\ln 3x = 1$ **b.** $3 \ln x = 5$ **c.** $\ln 3 + \ln x = 1.5$

13. Solve for x.

a. $e^x = 10$ **c.** $2 + 4^x = 7$ **e.** $\ln(x + 1) = 3$
b. $10^x = 3$ **d.** $\ln x = 5$ **f.** $\ln x - \ln(x + 1) = 4$

14. Solve for t.

a. $5^{t+1} = 6^t$ **d.** $\ln\left(\frac{t}{t - 2}\right) = 1$
b. $e^{t^2} = 4$ **e.** $\ln t - \ln(t - 2) = 1$
c. $5 \cdot 2^{-t} = 4$

15. Prove that $\ln(A \cdot B) = \ln A + \ln B$, where A and B are positive real numbers.

16. For each of the following equations, either prove that it is correct (by using the rules of logarithms and exponents) or else show that it is not correct (by finding numerical values for x that make the values on the two sides of the equation different).

a. $e^{x + \ln x} = x \cdot e^x$
b. $\ln\left(\frac{(x + 1)^2}{x}\right) = 2 \ln(x + 1) - \ln x$
c. $\ln\left(\frac{x}{x + 1}\right) = \frac{\ln x}{\ln(x + 1)}$
d. $\ln(x + x^2) = \ln x + \ln x^2$
e. $\ln(x + x^2) = \ln x + \ln(x + 1)$

17. How long would it take \$15,000 to grow to \$100,000 if invested at 8.5% compounded continuously?

18. The effective annual interest rate on an account compounded continuously is 4.45%. Estimate the nominal interest rate.

19. The effective annual interest rate on an account compounded continuously is 3.38%. Estimate the nominal interest rate.

20. A town of 10,000 grew to 15,000 in 5 years. Assuming exponential growth:

a. What is the annual growth rate?
b. What was its annual continuous growth rate?

21. Solve the following for r.

a. $1.025 = e^r$ **b.** $\frac{1}{2} = e^r$ **c.** $1.08 = e^r$

22. Solve the following for r.

a. $0.9 = e^r$ **b.** $2 = e^r$ **c.** $0.75 = e^r$

23. Without using a calculator, identify which functions represent growth and which decay. Then, using a calculator, find the corresponding instantaneous growth or decay rate.

a. $f(t) = 50e^{(\ln 1.45)t}$ **d.** $j(p) = 2000e^{(\ln 0.3)p}$
b. $g(x) = 125e^{(\ln 3.15)x}$ **e.** $k(v) = 600e^{(\ln 1.75)v}$
c. $h(m) = 1500e^{(\ln 0.83)m}$ **f.** $l(x) = e^{(\ln 0.75)x}$

24. Use a calculator to determine the value of each expression. Then rewrite each expression in the form $e^{\ln a}$.

a. $e^{0.083}$ **b.** $e^{-0.025}$ **c.** $e^{-0.35}$ **d.** $e^{0.83}$

6.4 *Logarithmic Functions*

Exploration 6.1 and course software "E8: Logarithmic Sliders" in *Exponential & Log Functions* will help you understand the properties of logarithmic functions.

Up until now we have been dealing with logarithms of specific numbers, such as log 2 or ln 10. But since for any $x > 0$ there is a unique corresponding value of $\log x$ or $\ln x$, we can define two logarithmic functions:

$$y = \log x \qquad \text{and} \qquad y = \ln x \qquad \text{where } x > 0$$

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The Graphs of Logarithmic Functions

What will the graphs look like? We know something about the graphs since

Properties of logarithms

If $x > 1$,	$\log x$ and $\ln x$ are both positive
If $x = 1$,	$\log 1 = 0$ and $\ln 1 = 0$
If $0 < x < 1$,	$\log x$ and $\ln x$ are both negative
If $x \leq 0$,	neither logarithm is defined

Table 6.7 and Figure 6.5 show some data points and the graphs of $y = \log x$ and $y = \ln x$.

Evaluating log x and ln x

x	$y = \log x$	$y = \ln x$
0.001	−3.000	−6.908
0.01	−2.000	−4.605
0.1	−1.000	−2.303
1	0.000	0.000
2	0.301	0.693
3	0.477	1.099
4	0.602	1.386
5	0.699	1.609
6	0.778	1.792
7	0.845	1.946
8	0.903	2.079
9	0.954	2.197
10	1.000	2.303

Table 6.7

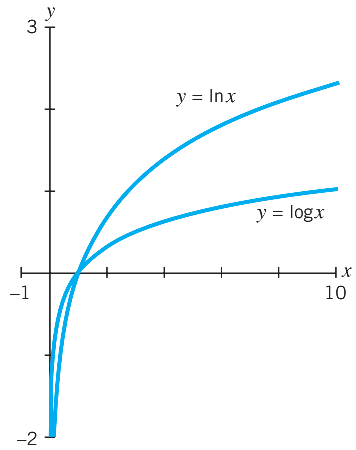

Figure 6.5 Graphs of $y = \log x$ and $y = \ln x$.

The graphs of common and natural logarithms share a distinctive shape. They are both defined only when $x > 0$ and they are both concave down. Both graphs increase throughout with no maximum or minimum value, although they grow more slowly when $x > 1$.

Vertical asymptotes

The graphs of both $y = \log x$ and $y = \ln x$ are vertically asymptotic to the y-axis. This means that as x approaches 0 (through positive values), the graphs come closer and closer to the vertical y-axis but never touch it. For both functions, as $x \to 0$, the values for y get increasingly negative, plunging down near the y-axis toward $-\infty$.

The Graphs of Logarithmic Functions

The graphs of both $y = \log x$ and $y = \ln x$

- lie to the right of the y-axis since they are defined only for $x > 0$
- share a horizontal intercept of (1, 0)
- are concave down
- increase throughout with no maximum or minimum
- are asymptotic to the y-axis

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EXAMPLE 1 Match each function with the appropriate graph in Figure 6.6.

a. $y = \log(3x)$ **b.** $y = 3 + \log x$ **c.** $y = \log(x^3)$ **d.** $y = 3\log x$

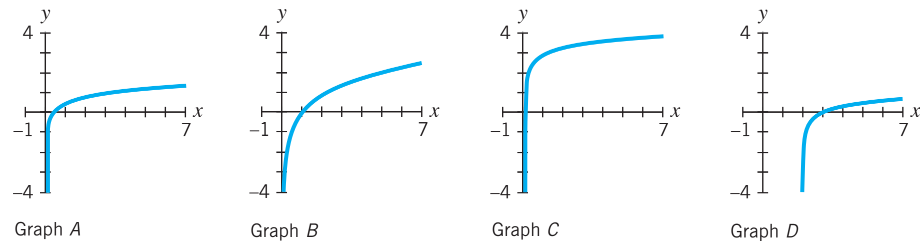

Figure 6.6 Four graphs involving logarithms.

SOLUTION **a.** Graph *A*. When $x = 1/3$, then $y = \log(3 \cdot 1/3) = \log 1 = 0$, so the horizontal intercept is $(1/3, 0)$.

b. Graph *C*. When $x = 1$, then $y = 3 + \log 1 = 3 + 0 = 3$, so the graph passes through $(1, 3)$.

c. and **d.** Graph *B*. Since $\log(x^3) = 3\log x$, the graphs for functions (c) and (d) are the same. When $x = 1$, then $y = 3\log 1 = 0$, so the horizontal intercept is $(1, 0)$.

There is no match for Graph *D*.

EXAMPLE 2 Graph $y = \ln(x - 2)$. Describe its relationship to $y = \ln x$.

SOLUTION Figure 6.7 shows the two graphs.

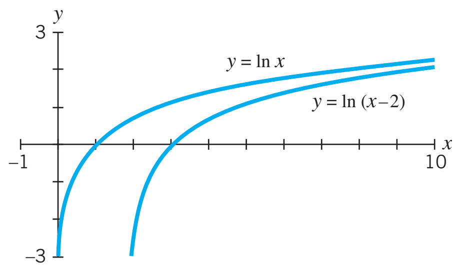

Figure 6.7 Graphs of $y = \ln x$ and $y = \ln(x-2)$.

The function $y = \ln(x - 2)$ tells us to subtract 2 from x and then apply the function ln. So the graph of $y = \ln(x - 2)$ is the graph of $y = \ln x$ shifted two units to the right. For $y = \ln x$, the horizontal intercept is at 1 since $\ln(1) = 0$. For $y = \ln(x - 2)$, the horizontal intercept is at 3, since $\ln(3 - 2) = \ln 1 = 0$.

The Relationship between Logarithmic and Exponential Functions

Logarithmic vs. exponential growth

Logarithmic and exponential growth are both unbounded. They both increase forever, never reaching a maximum value. But exponential growth is rapid—not only increasing, but doing so at a rate that is speeding up (accelerating). Logarithmic growth is slow—increasing, but at rate that is slowing down (decelerating). For example, if we let $t = 1$, 3, and 5, then 10^t is respectively 10, 1000, and 100,000, but $\log(t)$ is respectively 0, 0.477, and 0.699.

Logarithmic and exponential functions are inverses of each other

What does it mean for two functions to be inverses of each other? It means that what one function does, the other undoes. Log and exponential functions are inverses of each other. For example,

$$\log(10^x) = x \qquad \text{and} \qquad 10^{\log x} = x$$

Why are these true?

Rationale for $\log(10^x) = x$

Finding the logarithm of a number base 10 involves finding the exponent of 10 needed to produce the number. Since 10^x is already written as 10 to the power x, then $\log 10^x = x$.

Rationale for $10^{\log x} = x$

By definition, $\log x$ is the number such that when 10 is raised to that power the result is x.

Proof Let $y = \log x$

rewrite using definition of logarithm $10^y = x$

substitute $\log x$ for y $10^{\log x} = x$

The two functions

$$y = 10^x \qquad \text{and} \qquad y = \log x$$

are *inverses* of each other. Similarly, for natural logarithms

$$\ln(e^x) = x \qquad \text{and} \qquad e^{\ln x} = x$$

So the functions $y = e^x$ and $y = \ln x$

are also inverses of each other.

More Rules for Common and Natural Logarithms

5. $\log(10^x) = x$ and $\ln(e^x) = x$

6. $10^{\log x} = x$ and $e^{\ln x} = x$

The graphs of two inverse functions are mirror images across the diagonal line y = x

The graphs of two inverse functions such as $y = 10^x$ and $y = \log x$ are mirror images across the line $y = x$. If you imagine folding the graph along the dotted line $y = x$, the two curves would lie right on top of each other. See Figure 6.8 on the following page.

Why are these graphs mirror images? Choose any point (a, b) on the graph of $y = \log x$. To reach that point you would need to move a units horizontally (on the x-axis) and b units vertically (on the y-axis). Now imagine folding at the dotted line $y = x$. What are the coordinates of the point's mirror image? You would need to move a units vertically (on the y-axis) and b units horizontally (on the x-axis). The mirror-image coordinates would be (b, a). So the points (a, b) and (b, a) are mirror images across the line $y = x$. Table 6.8 lists some pairs of mirror-image points that lie on the graphs of $y = 10^x$ and $y = \log x$.

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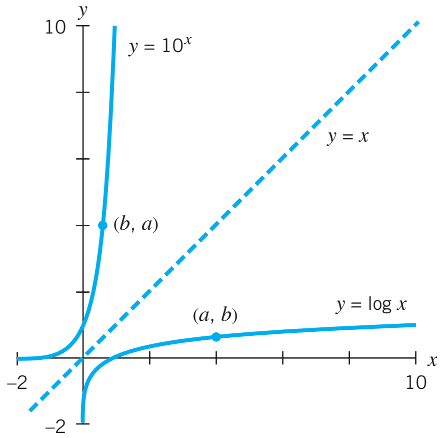

Figure 6.8 The graphs of $y = \log x$ and $y = 10^x$ are mirror images across the dotted line $y = x$.

Mirror-Image Points on $y = 10^x$ and $y = \log x$

$y = 10^x$	$y = \log x$
(0, 1)	(1, 0)
(1, 10)	(10, 1)
(2, 100)	(100, 2)
(−1, 0.1)	(0.1, −1)
(−2, 0.01)	(0.01, −2)

Table 6.8

We can use similar arguments to show that the graphs of $y = e^x$ and $y = \ln x$ are mirror images as well.

Inverse Functions and Their Graphs

The functions in each pair

$$y = 10^x \quad \text{and} \quad y = \log x$$
$$y = e^x \quad \text{and} \quad y = \ln x$$

are *inverses* of each other. What one does, the other undoes.

The graphs in each pair are mirror images across the line $y = x$.

EXAMPLE 3 Graph $y = \ln x$ and $y = e^x$. Identify three pairs of points on the function graphs that are mirror images across the line $y = x$.

SOLUTION Figure 6.9 shows the graphs of $y = \ln x$ and $y = e^x$, and Table 6.9 contains some pairs of points on $y = e^x$ and $y = \ln x$.

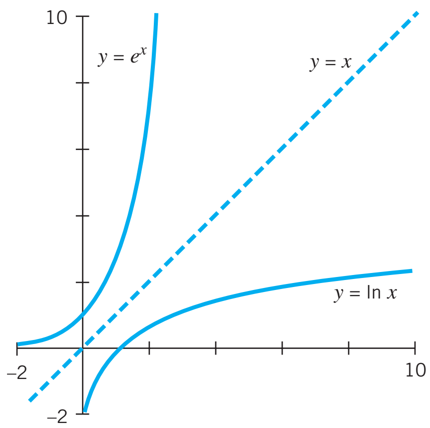

Figure 6.9 Graphs of $y = \ln x$ and $y = e^x$ are mirror images across the dotted line $y = x$.

Mirror-Image Points on $y = e^x$ and $y = \ln x$

$y = e^x$	$y = \ln x$
(0, 1)	(1, 0)
(1, e)	(e, 1)
(−2, e^{-2})	(e^{-2}, −2)

Table 6.9

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Applications of Logarithmic Functions

Just as with logarithmic scales, logarithmic functions are used in dealing with quantities that vary widely in size. We'll examine two such functions used to measure acidity and noise levels.

Measuring acidity: The pH scale

Chemists use a logarithmic scale called pH to measure acidity. pH values are defined by the function

$$\text{pH} = -\log[\text{H}^+]$$

where $[\text{H}^+]$ designates the concentration of hydrogen ions. Chemists use the symbol H^+ for hydrogen ions (hydrogen atoms stripped of their one electron), and the brackets [ ] mean "concentration of." Ion concentration $[\text{H}^+]$ is measured in moles per liter, *M*, where one mole equals $6.022 \cdot 10^{23}$ or Avogadro's number of ions. A pH value is the negative of the logarithm of the number of moles per liter of hydrogen ions.

Table 6.10 and Figure 6.10 show a set of values and a graph for pH. The graph is the standard logarithmic graph flipped over the horizontal axis because of the negative sign in front of the log.

Calculating pH Values

$[\text{H}^+]$ (moles per liter)	pH
10^{-15}	15.000
10^{-10}	10.000
1	0.000
5	−0.699
10	−1.000

Table 6.10

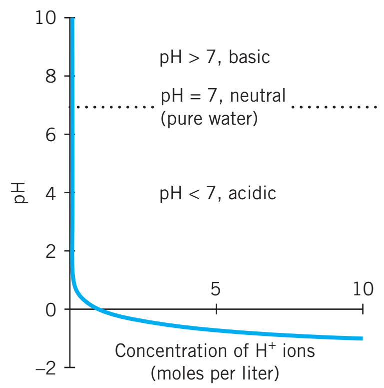

Figure 6.10 Graph of the pH function.

Typically, pH values are between 0 and 14, indicating the level of acidity. Pure water has a pH of 7.0 and is considered *neutral*. A substance with a pH < 7 is called *acidic*. A substance with a pH > 7 is called *basic* or *alkaline*.[1] The lower the pH, the more acidic the substance. The higher the pH, the less acidic and the more alkaline the substance. Table 6.11 shows approximate pH values for some common items. Most foods have a pH between 3 and 7. Substances with a pH below 3 or above 12 can be dangerous to handle with bare hands.

Remember that pH is the *negative* of a logarithmic function. So the larger the hydrogen ion concentration, the smaller the pH. (See Figure 6.10.) Multiplying the hydrogen ion concentration by 10 decreases the pH by 1. So vinegar (with a pH of 3) has a hydrogen ion concentration 10 times greater than that of wine (pH 4) and 10,000 times greater than that of pure water (pH 7).

[1]Things that are alkaline tend to be slimy and sticky, like a bar of soap. If you wash your hands with soap and don't rinse, there will be an alkaline residue. Acids can be used to neutralize alkalinity. For example, shampoo often leaves a sticky alkaline residue. So we use acidic conditioners to neutralize the alkalinity.

pH of Various Substances

Acidic (more hydrogen ions than pure water)	pH	Neutral	pH	Basic or Alkaline (fewer hydrogen ions than pure water)	pH
Gastric juice	2	Pure water	7	Egg whites, sea water	8
Coca-Cola, vinegar	3			Soap, baking soda	9
Grapes, wine	4			Detergents, toothpaste, ammonia	10
Coffee, tomatoes	5			Household cleaner	12
Bread	5.5			Caustic oven cleaner	13
Beef, chicken	6				

Table 6.11

EXAMPLE 4 **Comparing hydrogen ion concentrations**

a. Compare the hydrogen ion concentration of Coca-Cola with the hydrogen ion concentrations of coffee and ammonia.

b. Calculate the hydrogen ion concentration of Coca-Cola.

SOLUTION **a.** Table 6.11 gives the pH of Coca-Cola as 3, coffee as 5 (so both are acidic), and ammonia as 10 (which is alkaline). Increasing the pH by 1 corresponds to decreasing the hydrogen ion concentration by a factor of 10. So Coca-Cola will have more ions than coffee, and even more hydrogen than ammonia. The hydrogen ion concentration of Coca-Cola is $10^{5-3} = 10^2 = 100$ times more than that of coffee, and $10^{10-3} = 10^7 = 10{,}000{,}000$ times more than that of ammonia!

b. To calculate the hydrogen ion concentration of Coca-Cola, we need to solve the equation

$$3 = -\log[H^+]$$

Multiply by -1 $$-3 = \log[H^+]$$

rewrite as powers of 10 $$10^{-3} = 10^{\log[H+]}$$

evaluate and use Rule 6 of logs $$0.001 = [H^+]$$

So the hydrogen ion concentration of Coca-Cola is 10^{-3} or 0.001 moles per liter. That means that each liter of Coca-Cola contains $10^{-3} \cdot (6.022 \cdot 10^{23}) = 6.022 \cdot 10^{23-3} = 6.022 \cdot 10^{20}$ hydrogen ions.

EXAMPLE 5 **Calculating the pH level**

Sulfuric acid has a hydrogen ion concentration $[H^+]$ of 0.109 moles per liter. Calculate its pH.

SOLUTION The pH of sulfuric acid equals $-\log 0.109 \approx 0.96$.

The rain in many parts of the world is becoming increasingly acidic. The burning of fossil fuels (such as coal and oil) by power plants and automobile emissions release gaseous impurities into the air. The impurities contain oxides of sulfur and nitrogen that combine with moisture in the air to form droplets of dilute sulfuric and nitric acids. An acid releases hydrogen ions in water. High concentrations of hydrogen ions damage plants and water resources (such as the lakes of New England and Sweden) and erode structures (such as the Parthenon in Athens) by removing oxygen molecules. Some experts feel that the acid rain dilemma may be one of the greatest environmental problems facing the world in the near future.

EXAMPLE 6 **The pH function is not linear**

It is critical that health care professionals understand the nonlinearity of the pH function. An arterial blood pH of 7.35 to 7.45 for a patient is quite normal, whereas a pH of 7.1 means that the patient is severely acidotic and near death.

a. Determine the hydrogen ion concentration, $[H^+]$, first for a patient with an arterial blood pH of 7.4, then for one with a blood pH of 7.1.

b. How many more hydrogen ions are there in blood with a pH of 7.1 than in blood with a pH of 7.4?

SOLUTION To find $[H^+]$ if the pH is 7.4, we must solve the equation

$$
\begin{array}{lrl}
 & 7.4 &= -\log[H^+] \\
\text{Multiply both sides by } -1 & -7.4 &= \log[H^+] \\
\text{then these powers of 10 are equal} & 10^{-7.4} &= 10^{\log[H^+]} \\
\text{use Rule 6 of logs} & 10^{-7.4} &= [H^+] \\
\text{evaluate and switch sides} & [H^+] &\approx 0.000\,000\,040 \\
 & &\approx 4.0 \cdot 10^{-8}\, M
\end{array}
$$

where M is in moles per liter.

Similarly, if the pH is 7.1, we can solve the equation

$$
\begin{array}{lrl}
 & 7.1 &= -\log[H^+] \\
\text{to get} & [H^+] &\approx 7.9 \cdot 10^{-8}\, M
\end{array}
$$

Comparing the two concentrations gives us

$$\frac{[H^+] \text{ in blood with a pH of 7.1}}{[H^+] \text{ in blood with a pH of 7.4}} \approx \frac{7.9 \cdot 10^{-8}\, M}{4.0 \cdot 10^{-8}\, M} = \frac{7.9}{4.0} \approx 2$$

So there are approximately twice as many hydrogen ions in blood with a pH of 7.1 than in blood with a pH of 7.4.

SOMETHING TO THINK ABOUT

A noise emission statute enforced by the Massachusetts Registry of Motor Vehicles requires that the noise level of a motorcycle not exceed 82–86 decibels. Jay McMahon, chairman of the Modified Motorcycle Association, says, "It's just another way for the Registry to stick it to motorcycle operators. No one complains when a Maserati or a tractor trailer drives by." Does the statute seem fair to you? Why?

Measuring noise: The decibel scale

The decibel scale was designed to reflect the human perception of sounds.[2] When it is very quiet, it is easy to notice a small increase in sound intensity. The same increase in intensity in a noisy environment would not be noticed; it would take a much bigger change to be detected by humans. The same is true for light. If a 50-watt light bulb is replaced with a 100-watt light bulb, it is easy to notice the difference in brightness. But if you replaced 500 watts with 550 watts, it would be very hard to distinguish the 50-watt difference. The decibel scale, like the pH scale for acidity or the Richter scale for earthquakes, is logarithmic; that is, it measures order-of-magnitude changes.

Noise levels are measured in units called *decibels,* abbreviated dB. The name is in honor of the inventor of the telephone, Alexander Graham Bell. If we designate I_0 as the intensity of a sound at the threshold of human hearing (10^{-16} watts/cm^2) and we let I represent the intensity of an arbitrary sound (measured in watts/cm^2), then the noise level N of that sound measured in decibels (dB) is defined to be

$$N = 10 \log\left(\frac{I}{I_0}\right)$$

[2] Two scientists, Weber and Fechner, studied the psychological response to intensity changes in stimuli. Their discovery, that the perceived change is proportional to the logarithm of the intensity change of the stimulus, is called the Weber-Fechner stimulus law.

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The unitless expression I/I_0 gives the *relative intensity* of a sound compared with the reference value of I_0. For example, if $I/I_0 = 100$, then the noise level, N, is equal to

$$N = 10\log(100) = 10\log(10^2) = 10(2) = 20 \text{ dB}$$

Table 6.12 shows relative intensities, the corresponding noise levels (in decibels), and how people perceive these noise levels. Note how much the relative intensity (the ratio I/I_0) of a sound source must increase for people to discern differences. Each time we *add* 10 units on the decibel scale, we *multiply* the relative intensity by 10, increasing it by one order of magnitude.

How Decibel Levels Are Perceived

Relative Intensity I/I_0	Decibels (dB)	Average Perception
1	0	Threshold of hearing
10	10	Soundproof room, very faint
100	20	Whisper, rustle of leaves
1,000	30	Quiet conversation, faint
10,000	40	Quiet home, private office
100,000	50	Average conversation, moderate
1,000,000	60	Noisy home, average office
10,000,000	70	Average radio, average factory, loud
100,000,000	80	Noisy office, average street noise
1,000,000,000	90	Loud truck, police whistle, very loud
10,000,000,000	100	Loud street noise, noisy factory
100,000,000,000	110	Elevated train, deafening
1,000,000,000,000	120	Thunder of artillery, nearby jackhammer
10,000,000,000,000	130	Threshold of pain, ears hurt

Table 6.12

EXAMPLE 7 **How loud is a rock band?**

What is the decibel level of a typical rock band playing with an intensity of 10^{-5} watts/cm^2? How much more intense is the sound of the band than an average conversation?

SOLUTION Given $I_0 = 10^{-16}$ watts/cm^2 and letting $I = 10^{-5}$ watts/cm^2 and N represent the decibel level,

$$\begin{array}{ll} \text{by definition} & N = 10\log\left(\dfrac{I}{I_0}\right) \\ \text{substitute for } I \text{ and } I_0 & \quad = 10\log(10^{-5}/10^{-16}) \\ \text{Rule 2 of exponents} & \quad = 10\log(10^{11}) \\ \text{Rule 5 of logs} & \quad = 10 \cdot 11 \\ & \quad = 110 \text{ decibels} \end{array}$$

So the noise level of a typical rock band is about 110 decibels.

According to Table 6.12, an average conversation measures about 50 decibels. So the noise level of the rock band is 60 decibels higher. Each increment of 10 decibels corresponds to a one-order-of-magnitude increase in intensity. So the sound of a rock

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band is about six orders of magnitude, or 10^6 (a million times), more intense than an average conversation.

EXAMPLE 8 **Perceiving sound**

What's wrong with the following statement? "A jet airplane landing at the local airport makes 120 decibels of noise. If we allow three jets to land at the same time, there will be 360 decibels of noise pollution."

SOLUTION There will certainly be three times as much sound intensity, but would we perceive it that way? According to Table 6.12, 120 decibels corresponds to a relative intensity of 10^{12}. Three times that relative intensity would equal $3 \cdot 10^{12}$. So the corresponding decibel level would be

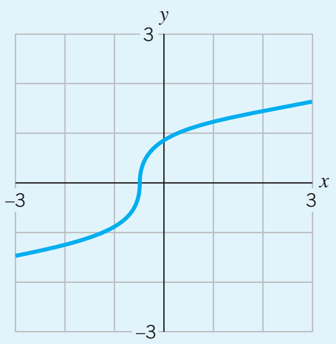
$$
\begin{aligned}
& & N &= 10 \log(3 \cdot 10^{12}) \\
&\text{Rule 1 of logs} & &= 10(\log 3 + \log(10^{12})) \\
&\text{use a calculator} & &\approx 10(0.477 + 12) \\
&\text{combine} & &\approx 10(12.477) \\
&\text{multiply and round off} & &\approx 125 \text{ decibels}
\end{aligned}
$$

So three jets landing will produce a decibel level of 125, not 360. We would perceive only a slight increase in the noise level.

Algebra Aerobics 6.4

A graphing program is recommended for Problem 3 and a scientific calculator for Problem 7.

1. How would the graphs of $y = \log x^2$ and $y = 2 \log x$ compare?
2 Draw a rough sketch of the graph $y = -\ln x$. Compare it with the graph of $y = \ln x$.
3. Compare the graphs of $f(x) = \log x$ and $g(x) = \ln x$.
   a. Where do they intersect?
   b. Where does each have an output value of 1? Of 2?
   c. Describe each graph for values of x such that $0 < x < 1$.
   d. Describe each graph for values of x where $x > 1$.
4. Given the accompanying graph of a function f, sketch the graph of its inverse.



5. Use the rules for logarithms to evaluate the following expressions.
   a. $\log 10^3$
   b. $\log 10^{-5}$
   c. $3 \log 10^{0.09}$
   d. $10^{\log 3.4}$
   e. $\ln e^5$
   f. $\ln e^{0.07}$
   g. $\ln e^{3.02} + \ln e^{-0.27}$
   h. $e^{\ln 0.9}$
6. A typical pH value for rain or snow in the northeastern United States is about 4. Is this basic or acidic? What is the corresponding hydrogen ion concentration? How does this compare with the hydrogen ion concentration of pure water?
7. What is the decibel level of a sound whose intensity is $1.5 \cdot 10^{-12}$ watts/cm^2?
8. If the intensity of a sound increases by a factor of 100, what is the increase in the decibel level? What if the intensity is increased by a factor of 10,000,000?

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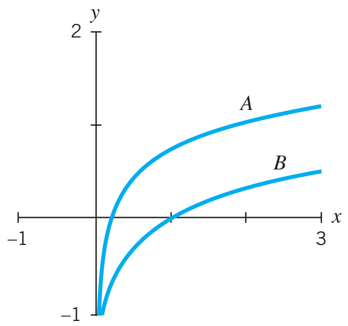
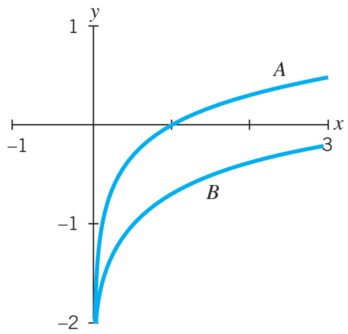
Exercises for Section 6.4

Some of these exercises require a calculator that can evaluate powers and logs. Exercise 6 requires a graphing program.

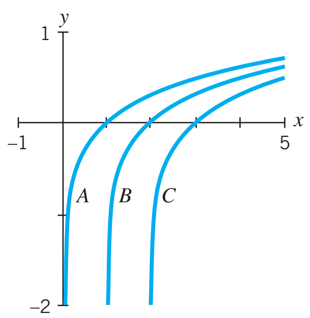
1. Use the rules of logarithms to explain how you can tell which graph is $y = \log x$ and which is $y = \log(5x)$.



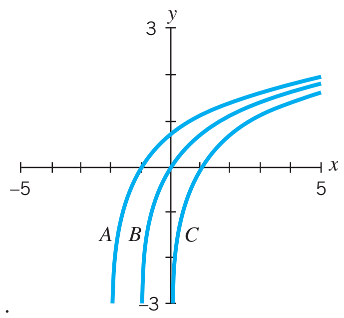
2. Use the rules of logarithms to explain how you can tell which graph is $y = \log x$ and which is $y = \log(x/5)$.



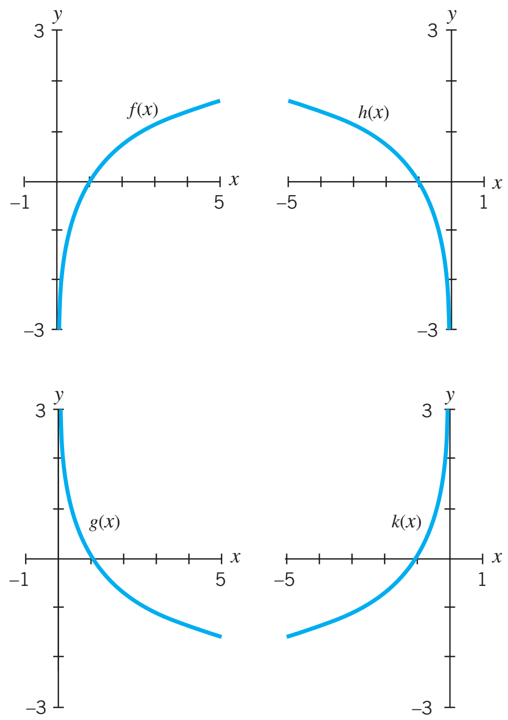
3. The functions $f(x) = \log x$, $g(x) = \log(x - 1)$, and $h(x) = \log(x - 2)$ are graphed below.
   a. Match each function with its graph.
   b. Find the value of x for each function that makes that function equal to zero.
   c. Identify the x-intercept for each function.
   d. Assuming $k > 0$, describe how the graph of $f(x)$ moves if you replace x by $(x - k)$.



4. The functions $f(x) = \ln x$, $g(x) = \ln(x + 1)$, and $h(x) = \ln(x + 2)$ are graphed below.
   a. Match each function with its graph.
   b. Find the value of x for each function that makes that function equal to zero.
   c. Identify the coordinates of the horizontal intercept for each function.
   d. Assuming $k > 0$, describe how the graph of $f(x)$ moves if you replace x by $(x + k)$.
   e. Determine the y-intercept, if possible, for each function.



5. Examine the following graphs of four functions.
   a. Which function graphs are mirror images of each other across the y-axis?
   b. Which function graphs are mirror images of each other across the x-axis?



6. (Requires a graphing program.) On the same grid graph $y_1 = \ln(x)$, $y_2 = -\ln(x)$, $y_3 = \ln(-x)$ and $y_4 = -\ln(-x)$.

a. Which pairs of function graphs are mirror images across the y-axis?

b. Which pairs of function graphs are mirror images across the x-axis?

c. What predictions would you make about the graphs of the functions $f(x) = a\ln x$ and $g(x) = -a\ln x$? Using technology, test your predictions for different values of a.

d. What predictions would you make about the graphs of the functions $f(x) = a\ln x$ and $g(x) = a\ln(-x)$? Using technology, test your predictions for different values of a.

7. Logarithms can be constructed using any positive number except 1 as a base:

$$\log_a x = y \text{ means that } a^y = x$$

a. Complete the accompanying table and sketch the graph of $y = \log_3 x$.

x	$y = \log_3 x$
$\frac{1}{9}$	
$\frac{1}{3}$	
1	
3	
9	
27	

b. Now make a small table and sketch the graph of $y = \log_4 x$. (*Hint:* To simplify computations, try using powers of 4 for values of x.)

8. The stellar magnitude M of a star is approximately $-2.5\log(B/B_0)$, where B is the brightness of the star and B_0 is a constant.

a. If you plotted B on the horizontal and M on the vertical axis, where would the graph cross the B axis?

b. Without calculating any other coordinates, draw a rough sketch of the graph of M. What is the domain?

c. As the brightness B increases, does the magnitude M increase or decrease? Is a sixth-magnitude star brighter or dimmer than a first-magnitude star?

d. If the brightness of a star is increased by a factor of 5, by how much does the magnitude increase or decrease?

9. [*Source:* H. D. Young, *University Physics,* Vol. 1 (Reading, MA: Addison-Wesley, 1992), p. 591] If you listen to a 120-decibel sound for about 10 minutes, your threshold of hearing will typically shift from 0 dB up to 28 dB for a while. If you are exposed to a 92-dB sound for 10 years, your threshold of hearing will be permanently shifted to 28 dB. What intensities correspond to 28 dB and 92 dB?

10. In all of the sound problems so far, we have not taken into account the distance between the sound source and the listener. Sound intensity is inversely proportional to the square of the distance from the sound source; that is, $I = k/r^2$, where I is intensity, r is the distance from the sound source, and k is a constant.

Suppose that you are sitting a distance R from the TV, where its sound intensity is I_1. Now you move to a seat twice as far from the TV, a distance $2R$ away, where the sound intensity is I_2.

a. What is the relationship between I_1 and I_2?

b. What is the relationship between the decibel levels associated with I_1 and I_2?

11. If there are a number of different sounds being produced simultaneously, the resulting intensity is the sum of the individual intensities. How many decibels louder is the sound of quintuplets crying than the sound of one baby crying?

12. An ulcer patient has been told to avoid acidic foods. If he drinks coffee, with a pH of 5.0, it bothers him, but he can tolerate both tap water, with a pH of 5.8, and milk, with a pH of 6.9.

a. Will a mixture of half coffee and half milk be at least as tolerable as tap water?

b. What pH will the half coffee–half milk mixture have?

c. In order to make 10 oz of a milk-coffee drink with a pH of 5.8, how many ounces of each are required?

13. Lemon juice has a pH of 2.1. If you make diet lemonade by mixing $\frac{1}{4}$ cup of lemon juice with 2 cups of tap water, with a pH of 5.8, will the resulting acidity be more or less than that of orange juice, with a pH of 3?

6.5 *Transforming Exponential Functions to Base e*

In Section 6.2 we saw that an exponential function can be written in two equivalent forms:

$$f(t) = Ca^t \qquad \text{or} \qquad f(t) = C(e^k)^t$$

where $e^k = a$. Given a value for k, we can evaluate e^k to find a. Now we can use logarithms to do the reverse; that is, given a value for a we find a value for k such that $a = e^k$.

Converting a to e^k

In general, if a is the growth (or decay) factor, then we can always find a value for k such that

$$e^k = a$$

by taking ln of both sides $\qquad \ln e^k = \ln a$

using Rule 5 for ln $\qquad k = \ln a$

So the function $f(t) = Ca^t$ can be rewritten as

$$\begin{aligned} f(t) &= C(e^k)^t \\ &= C(e^{\ln a})^t \end{aligned}$$

where $k = \ln a$.

Recall that k is called the *instantaneous* or *continuous* growth (or decay) rate.

For exponential growth, $a > 1$, so $e^k > 1$ and k is positive.

For exponential decay, $0 < a < 1$, so $0 < e^k < 1$ and k is negative.

Distinguishing Different Rates

For functions in the form $f(t) = C(e^k)^t$, we call the exponent k the *continuous* growth rate (if $k > 0$) or the *continuous* decay rate (if $k < 0$).

For functions in the form $f(t) = Ca^t$, we call the base a the growth (or decay) factor.

If $a > 1$, $\qquad$ the growth factor $= 1 +$ growth rate

If $0 < a < 1$, $\qquad$ the decay factor $= 1 -$ decay rate

EXAMPLE 1

a. Rewrite the function $f(t) = 250(1.3)^t$ using base e.

b. Identify the growth rate (based on the growth factor) and the *continuous* growth rate, both per time period t.

SOLUTION

a. The function $f(t) = 250(1.3)^t$ can be rewritten using base e, by

substituting $e^{\ln 1.3}$ for 1.3 $\qquad f(t) = 250(e^{\ln 1.3})^t$

evaluating ln 1.3 $\qquad \approx 250(e^{0.262})^t$

using Rule 3 of exponents $\qquad = 250e^{0.262t}$

b. Since $f(t) = 250(1.3)^t$, the growth rate per time period t is 0.3 or 30%. Since we also have $f(t) = 250e^{0.262t}$, the *continuous* growth rate per time period t is 0.262 or 26.2%.

EXAMPLE 2

a. Rewrite the function $g(t) = 340(0.94)^t$ using base e. Assume t is in years.

b. Identify the decay factor, the decay rate, and the continuous decay rate.

SOLUTION

a. Since the function $g(t) = 340(0.94)^t$ represents exponential decay, the value for k will be negative. The function g can be rewritten using base e, by

substituting $e^{\ln 0.94}$ for 0.94 $\qquad g(t) = 340(e^{\ln 0.94})^t$

evaluating ln 0.94 $\qquad \approx 340(e^{-0.062})^t$

using Rule 3 of exponents $\qquad = 340e^{-0.062t}$

b. The decay factor is 0.94. So the decay rate is $1 - 0.94 = 0.06$ or 6% per year. The *continuous* decay rate is 0.062 or 6.2% per year.

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See "E4: $y = Ce^{rx}$ Sliders" and "E5: Comparing $y = Ca^x$ to $y = Ce^{rx}$" in *Exponential & Log Functions.*

Writing an exponential function in the form $y = Ca^t$ or $y = Ce^{kt}$ (where $k = \ln a$) is a matter of emphasis, since the graphs and functional values are identical. When we use $y = Ca^t$, we may think of the growth taking place at discrete points in time, whereas the form $y = Ce^{kt}$ emphasizes the notion of continuous growth. For example, $y = 100(1.05)^t$ could be interpreted as giving the value of \$100 invested for t years at 5% compounded annually, whereas its equivalent form

$$y = 100(e^{\ln 1.05})^t \approx 100e^{0.049t}$$

suggests that the money is invested at 4.9% compounded continuously.

Exponential Functions Using Base *e*

The exponential function $y = Ca^t$ (where $a > 0$ and $a \neq 1$) can be rewritten as

$$y = C(e^k)^t \qquad \text{where } k = \ln a$$

We call k the *instantaneous* or *continuous growth* (or *decay*) *rate*.

Exponential growth:	$a > 1$	and	k is positive
Exponential decay:	$0 < a < 1$	and	k is negative

EXAMPLE 3 **Converting to a continuous growth rate**

Consider the bacterial growth we described with the equation $N = 100 \cdot 2^t$. The bacteria don't all double at the same time, precisely at the beginning of each time period t. A continuous growth pattern is much more likely. Rewrite this equation to reflect a continuous growth rate.

SOLUTION The equation $N = 100 \cdot 2^t$ can be rewritten to reflect a continuous growth rate by converting the base of 2 to its equivalent form using base e. We know the growth factor a can be rewritten as

$$\begin{array}{ll} & a = e^{\ln a} \\ \text{Substitute 2 for } a & 2 = e^{\ln 2} \\ \text{evaluate } \ln 2 & 2 \approx e^{0.693} \end{array}$$

So the equation that reflects a continuous growth rate is

$$N = 100 \cdot e^{0.693t}$$

EXAMPLE 4 **Converting from base *a* to base *e***

In Chapter 5 we saw that the function $f(t) = 100(0.976)^t$ measures the amount of radioactive strontium-90 remaining as 100 milligrams (mg) decay over time t (in years). Rewrite the function using base e.

SOLUTION To rewrite the function $f(t) = 100(0.976)^t$ using base e, we need to convert the base 0.976 to the form e^k, where $k = \ln 0.976$. Using a calculator to evaluate $\ln 0.976$, we get -0.0243. Substituting into our original function, we get

$$\begin{aligned} f(t) &= 100(0.976)^t \\ &\approx 100(e^{-0.0243})^t \\ &= 100e^{-0.0243t} \end{aligned}$$

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Since the original base 0.976 is less than 1, the function represents decay. So, as we would expect, when the function is rewritten using base e, the value of k (in this case -0.0243) is negative.

EXAMPLE 5 Use a continuous compounding model to describe the growth of Medicare expenditures.

SOLUTION In Chapter 5 we found a best-fit function for Medicare expenditures to be

$$C(n) = 10.6 \cdot (1.11)^n$$

where n = years since 1970 and $C(n)$ = Medicare expenditures in billions of dollars. This implies a growth rate of 11% compounded annually. To describe the same growth in terms of continuous compounding, we need to rewrite the growth factor 1.11 as e^k, where $k = \ln 1.11 \approx 0.104$. Substituting into the original function gives us

$$C(n) = 10.6e^{0.104n}$$

So Medicare expenditures are continuously compounding at about 10.4% per year.

EXAMPLE 6 **Proving the rule of 70**

We now have the tools to prove the rule of 70 introduced in Chapter 5. Recall that the rule said if a quantity is growing (or decaying) at R% per time period, then the time it takes the quantity to double (or halve) is approximately $70/R$ time periods. For example, if a quantity increases by a rate, R, of 7% each month, the doubling time is about $70/7 = 10$ months.

Proof

Let f be an exponential function of the form

$$f(t) = Ce^{rt}$$

where C is the initial quantity, r is the continuous growth rate, and t represents time. The rate r is in decimal form, and R is the equivalent amount expressed as a percentage. Since $f(t)$ represents exponential growth, $r > 0$. The doubling time for an exponential function is constant, so we need only calculate the time for any given quantity to double. In particular, we can determine the time it takes for the initial amount C (at time $t = 0$) to become twice as large; that is, we can calculate the value for t such that

	$f(t) = 2C$
Set	$Ce^{rt} = 2C$
divide by C	$e^{rt} = 2$
take ln of both sides	$\ln e^{rt} = \ln 2$
evaluate and use ln Rule 5	$rt \approx 0.693 \qquad (1)$

$R = 100r$, so $r = R/100$. If we round 0.693 up to 0.70 and

substitute in Equation (1)	$(R/100) \cdot t \approx 0.70$
multiply both sides by 100	$R \cdot t \approx 70$
divide by R we get	$t \approx 70/R$

So the time t it takes for the initial amount to double is approximately $70/R$, which is what the rule of 70 claims. We leave the similar proof about half-lives, where $r < 0$ represents decay, to the exercises.

Algebra Aerobics 6.5

A calculator that can evaluate logs and powers is required for Problems 2–4.

1. Identify each of the following exponential functions as representing growth or decay. (*Hint:* For parts (d)–(f) use rules for logarithms.)
   a. $M = Ne^{-0.029t}$
   b. $K = 100(0.87)^r$
   c. $Q = 375\,e^{0.055t}$
   d. $P = 250\,e^{(\ln 1.056)t}$
   e. $A = 20\,e^{(\ln 0.834)t}$
   f. $y = (1.2 \cdot 10^5)(e^{\ln 0.752})^t$
2. Rewrite each of the following as a continuous growth or decay model using base e.
   a. $y = 1000(1.062)^t$
   b. $y = 50(0.985)^t$
3. Determine the nominal and effective rates for each of the following. Assume t is measured in years. (*Hint:* To find the effective rate, convert from e^k to a.)
   a. $P = 25{,}000\,e^{(\ln 1.056)t}$
   b. $P = 10e^{(\ln 1.034)t}$
   c. $y = 2000\,e^{(\ln 1.083)t}$
   d. $y = (1.2 \cdot 10^5)(e^{\ln 1.295})^t$
4. Find the growth factor or decay factor for each of the following. (*Hint:* Convert from e^k to a.)
   a. $P = 50{,}000e^{0.08t}$
   b. $y = 30e^{-0.125t}$

Exercises for Section 6.5

Many of these exercises require a calculator that can evaluate powers and logs, some require a graphing program, and Exercise 18 requires technology that can generate a best-fit exponential function.

1. Rewrite each of the following functions using base e.
   a. $N = 10(1.045)^t$
   b. $Q = (5 \cdot 10^{-7}) \cdot (0.072)^A$
   c. $P = 500(2.10)^x$

2. Using technology, graph each of the following. Find, as appropriate, the doubling time or half-life.
   a. $A = 50\,e^{0.025t}$
   b. $A = 100\,e^{-0.046t}$
   c. $P = (3.2 \cdot 10^6)(e^{-0.15})^t$

3. Identify each of the following functions as representing growth or decay:
   a. $Q = Ne^{-0.029t}$
   b. $h(r) = 100(0.87)^r$
   c. $f(t) = 375e^{0.055t}$

4. Identify each function as representing growth or decay. Then determine the annual growth or decay factor, assuming t is in years.
   a. $A = A_0(1.0025)^{20t}$
   b. $A = A_0(1.0006)^{t/360}$
   c. $A = A_0(0.992)^{t/2}$
   d. $A = A_0\,e^{-0.063t}$
   e. $A = A_0\,e^{0.015t}$

5. For each of the following, find the doubling time, then rewrite each function in the form $P = P_0e^{rt}$. Assume t is measured in years.
   a. $P = P_0 2^{t/5}$
   b. $P = P_0 2^{t/25}$
   c. $P = P_0 2^{2t}$

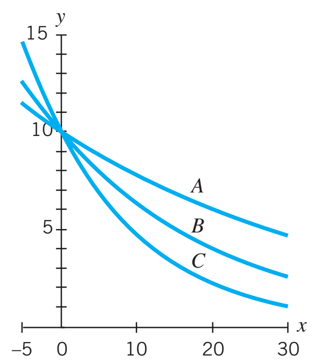
6. For each of the following, find the half-life, then rewrite each function in the form $P = P_0e^{rt}$. Assume t is measured in years.
   a. $P = P_0\left(\frac{1}{2}\right)^{t/10}$
   b. $P = P_0\left(\frac{1}{2}\right)^{t/215}$
   c. $P = P_0\left(\frac{1}{2}\right)^{4t}$

7. The barometric pressure, p, in millimeters of mercury, at height h, in kilometers above sea level, is given by the equation $p = 760e^{-0.128h}$. At what height is the barometric pressure 200 mm?

8. After t days, the amount of thorium-234 in a sample is $A(t) = 35e^{-0.029t}$ micrograms.
   a. How much was there initially?
   b. How much is there after a week?
   c. When is there just 1 microgram left?
   d. What is the half-life of thorium-234?

9. Assume $f(t) = Ce^{rt}$ is an exponential decay function (so $r < 0$). Prove the rule of 70 for halving times; that is, if a quantity is decreasing at $R\%$ per time period t, then the number of time periods it takes for the quantity to halve is approximately $70/R$. (*Hint:* $R = 100r$.)

10. Match the function with its graph.
   a. $f(x) = 10e^{-0.075x}$
   b. $g(x) = 10e^{-0.045x}$
   c. $h(x) = 10e^{-0.025x}$



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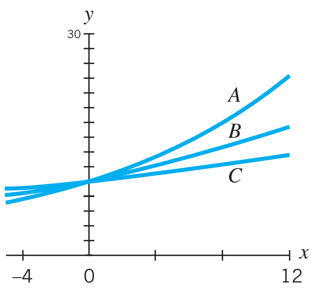
11. Match the function with its graph.

a. $f(x) = 10e^{0.025x}$

b. $g(x) = 10e^{0.045x}$

c. $h(x) = 10e^{0.075x}$



12. The functions $y = 50e^{0.04t}$ and $y = 50(2)^{t/n}$ are two different ways to write the same function.

a. What does the value 0.04 represent?

b. Set the functions equal to each other and use rules of natural logarithms to solve for n.

c. What does the value of n represent?

13. The functions $y = 2500e^{-0.02t}$ and $y = 2500\left(\frac{1}{2}\right)^{t/n}$ are two different ways to write the same function.

a. What does the value -0.02 represent?

b. Set the functions equal to each other and use rules of natural logarithms to solve for n.

c. What does the value of n represent?

14. (Requires a graphing program.) Radioactive lead-210 decays according to the exponential formula $Q = Q_0e^{-0.0311t}$, where Q_0 is the initial quantity in milligrams and t is in years. What is the half-life of lead-210? Verify your answer by graphing using technology.

15. Radioactive thorium-230 decays according to the formula $Q = Q_0\left(\frac{1}{2}\right)^{t/8000}$, where Q_0 is the initial quantity in milligrams and t is in years.

a. What is the half-life of thorium-230?

b. What is the annual decay rate?

c. Translate the equation into the form $Q = Q_0e^{rt}$. What does r represent?

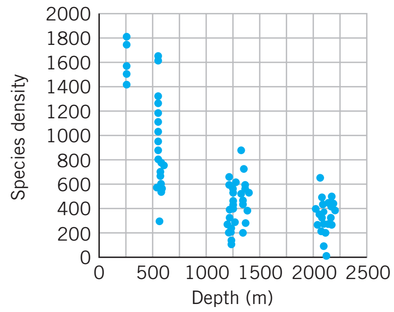
16. In 1859, the Victorian landowner Thomas Austin imported 12 wild rabbits into Australia and let them loose to breed. Since they had no natural enemies, the population increased very rapidly. By 1949 there were approximately 600 million rabbits.

a. Find an exponential function using a continuous growth rate to model this situation.

b. If the growth had gone unchecked, what would have been the rabbit population in 2000?

c. *Internet search:* Find out what was done to curb the population of rabbits in Australia, and find the current rabbit population.

17. Biologists believe that, in the deep sea, species density decreases exponentially with the depth. The accompanying graph shows data collected in the North Atlantic. Sketch an exponential decay function through the data. Then identify two points on your curve, and generate two equivalent equations that model the data, one in the form $y = Ca^t$ and the other in the form $y = Ce^{kt}$.



Source: Data collected by Ron Etter, Biology Department. University of Massachusetts–Boston.

18. (Requires technology to find a best-fit exponential.) According to another version of Moore's Law, the computing power built into chips doubles every 18 months (see Section 5.7). The accompanying table shows the computing power of some Intel chips (measured in calculations per second) between 1993 and 2005.

a. Graph the data points (if possible, on a semi-log plot). Explain why an exponential function would be an appropriate model.

Year	Chip Type	Chip Computing Power (millions of calculations per second)
1993	Pentium ®	66
1997	Pentium II	525
1999	Pentium III	1,700
2000	Pentium 4	3,400
2005	Dual-core Itanium 2	27,079

Source: Intel Corporation.

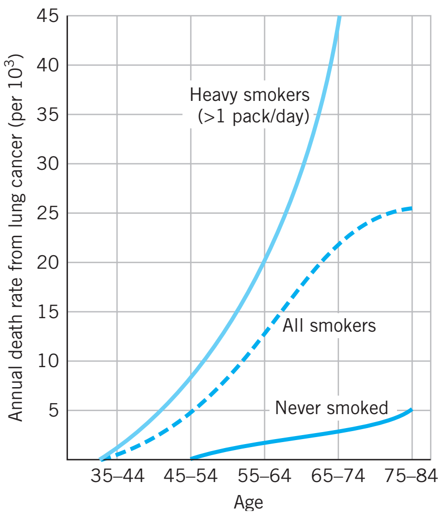
b. Construct an exponential function to model the data in the following two different ways. In each case let t = number of years since 1993.

i. Using the doubling time given by Moore's Law, construct an exponential function, $P_1(t)$, for chip computing power.

ii. Use technology to find a best-fit exponential function, $P_2(t)$, to the data in the table. Does this model come close to verifying Moore's Law ($P_1(t)$)?

19. The number of neutrons in a nuclear reactor can be predicted from the equation $n = n_0e^{(\ln 2)t/T}$, where n = number of neutrons at time t (in seconds), n_0 = the number of neutrons at time $t = 0$, and T = the reactor period, the doubling time of the neutrons (in seconds). When $t = 2$ seconds, $n = 11$, and when $t = 22$ seconds, $n = 30$. Find the initial number of neutrons, n_0, and the reactor period, T, both rounded to the nearest whole number.

20. According to Rubin and Farber's *Pathology*, "death from cancer of the lung, more than 85% of which is attributed to cigarette smoking, is today the single most common cancer death in both men and women in the United States." The accompanying graph shows the annual death rate (per thousand) from lung cancer for smokers and nonsmokers.



Source: E. Rubin and J. L. Farber, *Pathology,* 3rd ed. (Philadelphia: Lippincott-Raven, 1998), p. 312. Copyright © 1998 by Lippincott-Raven. Reprinted by permission.

a. The death rate for nonsmokers is roughly a linear function of age. After replacing each range of ages with a reasonable middle value (e.g., you could use 60 to approximate 55 to 64), estimate the coordinates of two points on the graph of nonsmokers and construct a linear model. Interpret your results.

b. By contrast, those who smoke more than one pack per day show an exponential rise in the annual death rate from lung cancer. Estimate the coordinates for two points on the graph for heavy smokers, and use the points to construct an exponential model (assume a continuous growth rate). Interpret your results.

21. If an object is put in an environment at a fixed temperature, A (the "ambient temperature"), then its temperature, T, at time t is modeled by Newton's Law of Cooling:

$$T = A + Ce^{-kt}$$

where k is a positive constant. Note that T is a function of t and that as $t \to +\infty$, then $e^{-kt} \to 0$, so the temperature T gets closer and closer to the ambient temperature, A.

a. Assume that a hot cup of tea (at 160ºF) is left to cool in a 75ºF room. If it takes 10 minutes for it to reach 100ºF, determine the constants A, C, and k in the equation for Newton's Law of Cooling. What is Newton's Law of Cooling in this situation?

b. Sketch the graph of your function.

c. What is the temperature of the tea after 20 minutes?

22. Newton's Law of Cooling (see Exercise 21) also works for objects being heated. At time $t = 0$, a potato at 70ºF (room temperature) is put in an oven at 375ºF. Thirty minutes later, the potato is at 220º.

a. Determine the constants A, C, and k in Newton's Law. Write down Newton's Law for this case.

b. When is the potato at 370ºF?

c. When is the potato at 374ºF?

d. According to your model, when (if ever) is the potato at 375ºF?

e. Sketch a graph of your function in part (a).

6.6 *Using Semi-Log Plots to Construct Exponential Models for Data*

In Chapter 5 we learned that an exponential function appears as a straight line on a semi-log plot (where the logarithmic scale is on the vertical axis). To decide whether an exponential function is an appropriate model for a data set, we can plot the data on a semi-log plot and see if it appears to be linear. This is one of the easiest and most reliable ways to recognize exponential growth in a data set. And, as we learned in Chapters 4 and 5, a logarithmic scale has the added advantage of being able to display clearly a wide range of values.

Why Do Semi-Log Plots of Exponential Functions Produce Straight Lines?

Consider the exponential function

$$y = 3 \cdot 2^x \qquad (1)$$

If we take the log of both sides $\qquad \log y = \log (3 \cdot 2^x)$

use Rule 1 of logs $\qquad = \log 3 + \log (2^x)$

use Rule 3 of logs	$= \log 3 + x \log 2$
evaluate logs and rearrange, we have	$\log y \approx 0.48 + 0.30x$
If we set $Y = \log y$, we have	$Y = 0.48 + 0.30x$ (2)

So Y (or $\log y$) is a linear function of x. Equations (1) and (2) are equivalent to each other. The graph of Equation (2) on a semi-log plot, with Y (or $\log y$) values on the vertical axis, is a straight line (see Figure 6.11). The slope is 0.30 or log 2, the logarithm of the growth factor 2 of Equation (1). The vertical intercept is 0.48 or log 3, the logarithm of the y-intercept 3 of Equation (1).

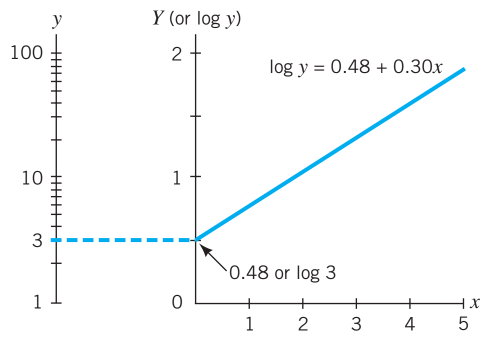


Figure 6.11 The graph of $Y = 0.48 + 0.30x$, showing the relationship between two equivalent logarithmic scales on the vertical axis.

Figure 6.11 shows two equivalent variations of logarithmic scales on the vertical axis. One plots the value of y on a logarithmic scale (using powers of 10), and the other plots the value of $\log y$ (using the exponents of the powers of 10). Spreadsheets and some graphing calculators have the ability to instantly switch axis scales between standard linear and the logarithmic scale using powers of 10. But one can in effect do the same thing by plotting $\log y$ instead of y. Notice that on the vertical $\log y$ scale, the units are now evenly spaced. This allows us to use the standard strategies for finding the slope and vertical intercept of a straight line.

In general, we can translate an exponential function in the form

$$y = Ca^x \quad \text{(where } C \text{ and } a > 0 \text{ and } a \neq 1)$$

into an equivalent linear function

$$\log y = \log C + (\log a)x \qquad \text{or}$$
$$Y = \log C + (\log a)x \qquad \text{where } Y = \log y$$

Finding the Equation of an Exponential Function on a Semi-Log Plot

The graph of an exponential function $y = Ca^x$ (where C and $a > 0$ and $a \neq 1$) appears as a straight line on a semi-log plot. The line's equation is

$$Y = \log C + (\log a)x \qquad \text{where } Y = \log y$$

The slope of the line is $\log a$, where a is the growth factor for $y = Ca^x$.
The vertical intercept is $\log C$, where C is the vertical intercept of $y = Ca^x$.

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Growth in the Dow Jones

The Dow Jones Industrial Average is based on the stock prices of thirty companies and is commonly used to measure the health of the stock market. Figure 6.12 is a semi-log plot of the Dow between the boom years of 1982 and 2000.

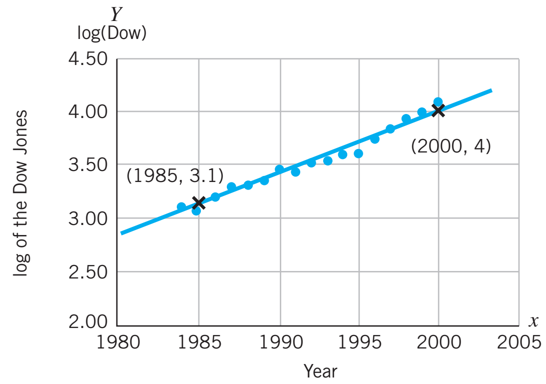

Figure 6.12 Graph of log(Dow) with two points on the best-fit line (not from the original data).
Source: Dow Jones website, *www.djindexes.com.*

The data points lie approximately on a straight line, so an exponential model is appropriate.

EXAMPLE 1 **Constructing an exponential function from a line on a semi-log plot**

Estimate the annual percent growth rate of the Dow Jones between 1982 and 2000.

SOLUTION We first construct a linear function for the best-fit line on the semi-log plot, then translate it into an exponential growth function to find the percent growth rate.

Estimating the coordinates of two points (1985, 3.1) and (2000, 4.0) on the best-fit line, *not* from the data (see Figure 6.12), the slope or average rate of change between them is

$$\text{slope} = \frac{\text{change in } Y}{\text{change in } x} = \frac{4.0 - 3.1}{2000 - 1985} = \frac{0.9}{15} = 0.06$$

If we let x = number of years since 1980, the slope remains the same. Reading off the graph, the vertical intercept is approximately 2.8. So the equation for the best-fit line is approximately

$$Y = 2.8 + 0.06x \qquad \text{where } Y = \log(\text{Dow}) \tag{1}$$

We can translate Equation (1) into an exponential function if we

substitute log(Dow) for Y	$\log(\text{Dow}) = 2.8 + 0.06x$
rewrite as a power of 10	$10^{\log(\text{Dow})} = 10^{(2.8+0.06x)}$
use Rule 6 of logs and Rule 1 of exponents	$\text{Dow} = 10^{2.8} \cdot 10^{0.06x}$
Rule 3 of exponents	$\text{Dow} = 10^{2.8} \cdot (10^{0.06})^x$
evaluate using calculator	$\text{Dow} \approx 631 \cdot 1.15^x$ (2)

The annual growth factor is approximately 1.15, and the annual growth rate in decimal form is 0.15. So between 1982 and 2000 the Dow Jones grew by about 15% each year.

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This strategy is equivalent to using the formula in the previous box, translating log(Dow) = 2.8 + 0.06x into the form Dow = Ca^x, by solving 2.8 = log C, and 0.06 = log a.

Algebra Aerobics 6.6

Many of the problems require a calculator that can evaluate logs and exponents.

1. a. Which of the following three functions would have a straight-line graph on a standard linear plot? On a semi-log plot?

$$y = 3x + 4, \quad y = 4 \cdot 3^x, \quad \log y = (\log 3) \cdot x + \log 4$$

b. For each straight-line graph in part (a), what is the slope of the line. The vertical intercept? (*Hint:* Substitute $Y = \log y$ and $X = \log x$ as needed.)

2. Change each exponential equation to a logarithmic equation.

a. $y = 5(3)^x$ **c.** $y = 10{,}000(0.9)^x$

b. $y = 1000(5)^x$ **d.** $y = (5 \cdot 10^6)(1.06)^x$

3. Change each logarithmic equation to an exponential equation.

a. $\log y = \log 7 + (\log 2) \cdot x$

b. $\log y = \log 20 + (\log 0.25) \cdot x$

c. $\log y = 6 + (\log 3) \cdot x$

d. $\log y = 6 + \log 5 + (\log 3) \cdot x$

4. a. Below are linear equations of log y (or Y) in x. Identify the slope and vertical intercept for each.

i. $\log y = \log 2 + (\log 5) \cdot x$

ii. $\log y = (\log 0.75) \cdot x + \log 6$

iii. $\log y = 0.4 + x \log 4$

iv. $\log y = 3 + \log 2 + (\log 1.05) \cdot x$

b. Use the slope and intercept to create an exponential function for each equation.

5. Solve each equation using the definition of log.

a. $0.301 = \log a$ **c.** $\log a = -0.125$

b. $\log C = 2.72$ **d.** $5 = \log C$

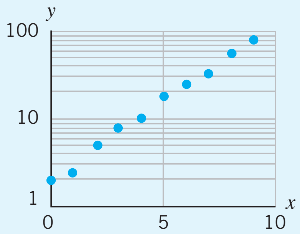
6. Change each function to exponential form, assuming that $Y = \log y$.

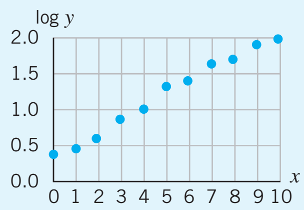
a. $Y = 0.301 + 0.477x$

b. $Y = 3 + 0.602x$

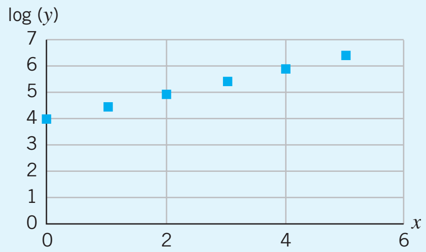
c. $Y = 1.398 - 0.046x$

7. For each of the two accompanying graphs examine the scales on the axes. Then decide whether a linear or an exponential function would be the most appropriate model for the data.





8. Generate an exponential function that could describe the data in the accompanying graph.



Exercises for Section 6.6

Some exercises require a calculator that can evaluate powers and logs. Exercises 10 and 13 require a graphing program.

1. Match each exponential function in parts (a)–(d) with its logarithmic form in parts (e)–(h).

a. $y = 10{,}000(2)^x$ **e.** $\log y = 6.477 - 0.097x$

b. $y = 1000(1.4)^x$ **f.** $\log y = 4 + 0.301x$

c. $y = (3 \cdot 10^6)(0.8)^x$ **g.** $\log y = 3 - 0.347x$

d. $y = 1000(0.45)^x$ **h.** $\log y = 3 + 0.146x$

2. Form the exponential function from its logarithmic equivalent for each of the following.

a. $\log y = \log 1400 + (\log 1.06)x$

b. $\log y = \log(25{,}000) + (\log 0.87)x$

c. $\log y = 2 + (\log 2.5)x$

d. $\log y = 4.25 + (\log 0.63)x$

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3. Change each exponential function to its logarithmic equivalent. Round values to the nearest thousandth.

a. $y = 30{,}000(2)^x$ **c.** $y = (4.5 \cdot 10^6)(0.7)^x$

b. $y = 4500(1.4)^x$ **d.** $y = 6000(0.57)^x$

4. Write the exponential equivalent for each function. Assume $Y = \log y$.

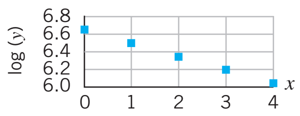
a. $Y = 2.342 + 0.123x$ **c.** $Y = 4.74 - 0.108x$

b. $Y = 3.322 + 0.544x$ **d.** $Y = 0.7 - 0.004x$

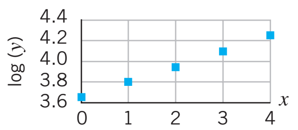
5. Match each exponential function with its semi-log plot.

a. $y = 30{,}000(2)^x$ **c.** $y = (4.5 \cdot 10^6)(0.7)^x$

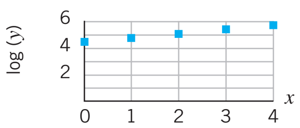
b. $y = 4500(1.4)^x$ **d.** $y = 6000(0.57)^x$



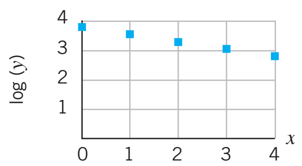
Graph *A*



Graph *C*



Graph *B*



Graph *D*

6. a. From the data in the following table, create a linear equation of Y in terms of x.

x	$\log y$ (or Y)
0	5.00000
1	5.60206
2	6.20412
3	6.80618
4	7.40824

b. Find the equivalent exponential function of y in terms of x.

7. Determine which data sets (if any) describe y as an exponential function of x, then construct the exponential function. (*Hint:* Find the average rate of change of Y with respect to x.)

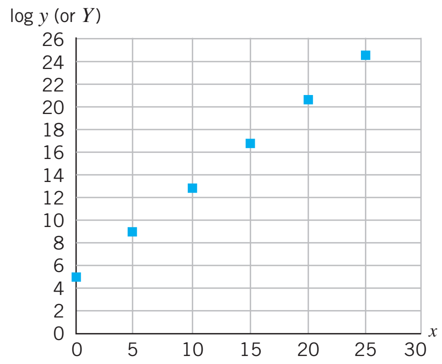
a.

x	$\log y$ (or Y)
0	2.30103
10	4.30103
20	4.90309
30	5.25527
40	5.50515

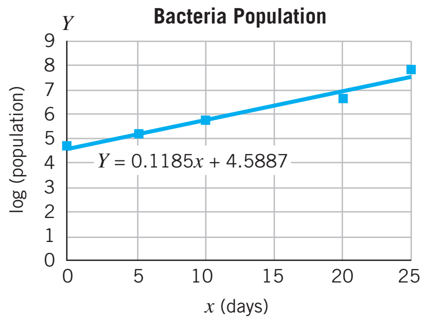
b.

x	$\log y$ (or Y)
0	4.77815
10	3.52876
20	2.27938
30	1.02999
40	−0.21945

8. Construct two functions that are equivalent descriptions of the data in the accompanying graph. Describe the change in $\log y$ (or Y) with respect to x.



9. The accompanying graph shows the growth of a bacteria population over a 25-day period.



a. Do the bacteria appear to be growing exponentially? Explain.

b. Translate the best-fit line shown on the graph into an exponential function and determine the daily growth rate.

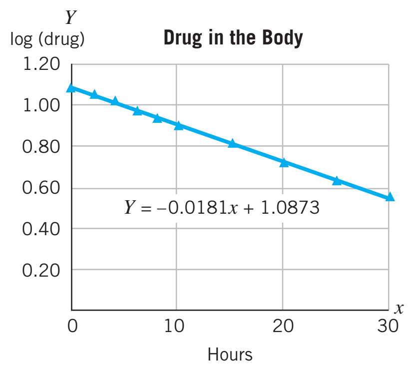
10. (Requires a graphing program.) The current population of a city is 1.5 million. Over the next 40 years, the population is expected to decrease by 12% each decade.

a. Create a function that models the population decline.

b. Create a table of values at 10-year intervals for the next 40 years.

c. Graph the population of the city on a semi-log plot.

11. An experiment by a pharmaceutical company tracked the amount of a certain drug (measured in micrograms/liter) in the body over a 30-hour period. The accompanying graph shows the results.



a. What was the initial amount of the drug?

b. Is the amount of drug in the body decaying exponentially? If yes, state the decay rate.

c. If the amount in the body is decaying exponentially, construct an exponential model.

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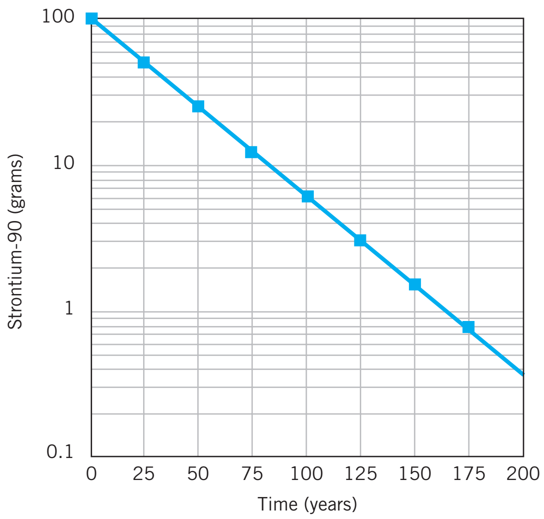
12. The accompanying graph shows the decay of strontium-90.



a. Is the graph a semi-log plot?

b. What is the initial amount of strontium-90 for this graph?

c. According to the graph, what is the half-life of strontium-90?

d. What would be the yearly decay rate for strontium-90?

e. Find the exponential model for the decay of strontium-90.

13. (Requires a graphing program.) The data in the following table are from the 2006 *Statistical Abstract* and show the rise of health care costs in the United States since 1960.

Year	U.S. Health Care Expenditures (billions of dollars)
1960	28
1970	75
1980	225
1985	442
1990	717
1995	1020
2000	1359
2005	2016

a. It is clear that the United States is spending more on health care as time goes on. Does this mean that we as individuals are paying more? How would you find out?

b. Is it fair to say that the expenses shown are growing exponentially? Use graphing techniques to find out; measure time in years since 1960.

c. Around 1985 the high cost of health care began to become an increasingly political issue, and insurance companies began to introduce "managed care" in an attempt to cut costs. Find a mathematical model that assumes a continuous growth for health costs from 1960 to 2005. What does your model predict for health costs in 2005? How closely does this reflect the actual cost in 2005?

CHAPTER SUMMARY

Logarithms

The *logarithm base 10 of x* is the *exponent* of 10 needed to produce x.

$$\log x = c \qquad \text{means that} \qquad 10^c = x$$

Logarithms base 10 are called *common logarithms*.

The *logarithm base e of x* is the *exponent* of e needed to produce x. The number e is an irrational natural constant ≈ 2.71828. Logarithms base e are called *natural logarithms* and are written as $\ln x$:

$$\ln x = c \qquad \text{means that} \qquad e^c = x$$

If $x > 1$,	$\log x$ and $\ln x$ are both positive
If $x = 1$,	$\log 1 = 0$ and $\ln 1 = 0$
If $0 < x < 1$,	$\log x$ and $\ln x$ are both negative
If $x \le 0$	neither logarithm is defined
and	$\log 10 = 1$ and $\ln e = 1$

The rules for logarithms follow directly from the definition of logarithms and from the rules for exponents.

Rules for Logarithms

If A and B are positive real numbers and p is any real number, then:

For Common Logarithms

1. $\log(A \cdot B) = \log A + \log B$
2. $\log(A/B) = \log A - \log B$
3. $\log A^p = p \log A$
4. $\log 1 = 0 \quad (\text{since } 10^0 = 1)$
5. $\log 10^x = x$
6. $10^{\log x} = x \quad (x > 0)$

For Natural Logarithms

1. $\ln(A \cdot B) = \ln A + \ln B$
2. $\ln(A/B) = \ln A - \ln B$
3. $\ln A^p = p \ln A$
4. $\ln 1 = 0 \quad (\text{since } e^0 = 1)$
5. $\ln e^x = x$
6. $e^{\ln x} = x \quad (x > 0)$

Solving Exponential Functions Using Logarithms

Logarithms can be used to solve equations such as $10 = 2^t$ by taking the log of each side to get $\log 10 = \log (2^t) \Rightarrow 1 = t \cdot \log 2 \Rightarrow t \approx 3.32$.

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Compounding

The value of P_0 dollars invested at a nominal interest rate r (in decimal form) compounded n times a year for t years is

$$P = P_0\left(1 + \frac{r}{n}\right)^{nt}$$

The value for r is called the *annual percentage rate* (APR). The actual interest rate per year is called the *effective interest rate* or the *annual percentage yield* (APY).

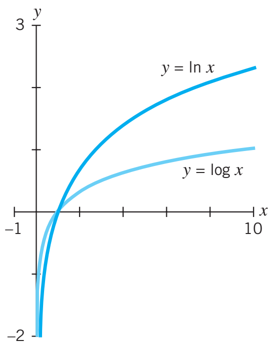
Compounding Continuously

The number e is used to describe *continuous compounding*. For example, the value of P_0 dollars invested at a nominal interest rate r (expressed in decimal form) compounded continuously for t years is

$$P = P_0e^{rt}$$

Logarithmic Functions

We can define two functions: $y = \log x$ and $y = \ln x$ (where $x > 0$).



Graphs of $y = \log x$ and $y = \ln x$

The graphs of these functions have similar shapes, and both

- lie to the right of the y-axis
- have a horizontal intercept of (1, 0)
- are concave down
- increase throughout with no maximum or minimum
- are asymptotic to the y-axis

Two functions are *inverses* of each other if what one function "does" the other "undoes." The graphs of two inverse functions are mirror images across the line $y = x$. The logarithmic and exponential functions are inverse functions.

Continuous Growth or Decay

Any exponential function $f(t) = Ca^t$ can be rewritten as

$$f(t) = Ce^{kt} \qquad \text{where } k = \ln a$$

We call k the *instantaneous* or *continuous growth* (or *decay*) *rate*.

Exponential growth:	$a > 1$	and	k is positive
Exponential decay:	$0 < a < 1$	and	k is negative

Using Semi-Log Plots

The graph of an exponential function $y = Ca^x$ appears as a straight line on a semi-log plot. The line's equation is

$$Y = \log C + (\log a)x \qquad \text{where } Y = \log y$$

The slope of the line is $\log a$, where a is the growth (or decay) factor for $y = Ca^x$.

The vertical intercept is $\log C$, where C is the vertical intercept of $y = Ca^x$.

CHECK YOUR UNDERSTANDING

I. Is each of the statements in Problems 1–30 true or false? If false, give an explanation for your answer.

1. For the function $y = \log x$, if x is increased by a factor of 10, then y increases by 1.
2. If $\log x = c$, then x is always positive but c can be positive, negative, or zero.
3. $\ln(1.08/2) = \ln(1.08)/\ln(2)$
4. Because the function $y = \log x$ is always increasing, it has no vertical asymptotes.
5. Both $\log 1$ and $\ln 1$ equal 0.
6. $\ln a = b$ means $b^e = a$.
7. $2 < e < 3$
8. $\log(10^2) = (\log 10)^2$
9. $\log(10^3 \cdot 10^5) = 15$
10. $\log\left(\frac{10^7}{10^2}\right) = \frac{\log(10^7)}{\log(10^2)}$
11. $\log\left(\frac{\sqrt{AB}}{(C + 2)^3}\right) = \frac{1}{2}(\log A + \log B) - 3\log(C + 2)$
12. $5\ln A + 2\ln B = \ln(A^5 + B^2)$
13. If $(2.3)^x = 64$, then $x = \frac{\log 64}{\log 2.3}$.
14. If $\log 20 = t\log 1.065$, then $t = \log\left(\frac{20}{1.065}\right)$.
15. The amount of dollars D compounded continuously for 1 year at a nominal interest rate of 8% yields $D \cdot e^{0.08}$.
16. $\ln t$ exists for any value of t.
17. If $f(t) = 100 \cdot a^t$ and $g(t) = 100 \cdot b^t$ are exponential decay functions and $a < b$, then the half-life of f is longer than the half-life of g.

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18. The graphs of the functions $y = \ln x$ and $y = \log x$ increase indefinitely.

19. The graphs of the functions $y = \ln x$ and $y = \log x$ have both vertical and horizontal asymptotes.

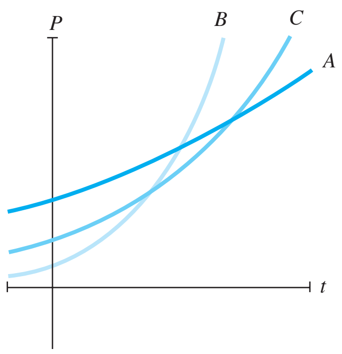
20. The graph of the function $y = \ln x$ lies above the graph of the function $y = \log x$ for all $x > 0$.

21. The graph of the function $y = \log x$ is always positive.

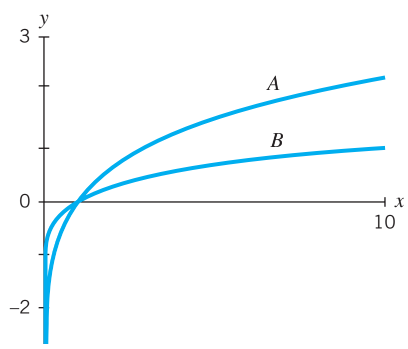
22. The graph of the function $y = \ln x$ is always decreasing.

23. The amount of \$100 invested at 8% compounded continously has a doubling time of about 8.7 years. (*Hint:* $e^{0.08} \approx 1.083$)

24. Of the functions of the form $P = P_0e^{rt}$ graphed in the accompanying figure, graph B has the smallest growth rate, r.



25. Of the two functions $f(x) = \ln x$ and $g(x) = \log x$ graphed in the accompanying figure, graph A is the graph of $f(x) = \ln x$.

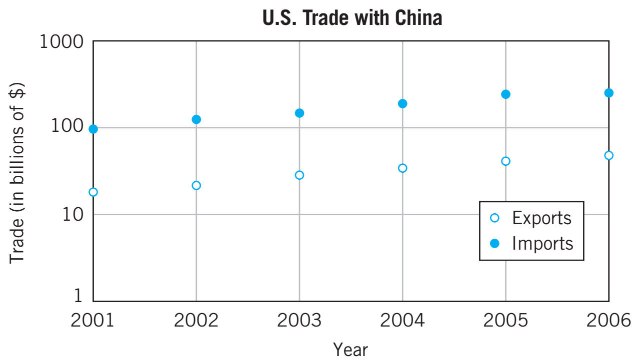


Problems 26–30 refer to the following table and semi-log plot.

U.S. Trade with China (in billions of dollars)

Year	Exports	Imports
2001	19.2	102.3
2002	22.1	125.2
2003	28.4	152.4
2004	34.7	196.7
2005	41.9	243.5
2006	50.0	263.6

Source: U.S. Bureau of the Census, Foreign Trade Statistics, *www.census.gov/foreigntrade/balance.*



26. Between 2001 and 2006 America's trade deficit (= exports − imports) with China remained roughly constant.

27. We can think of both U.S. exports to China and imports from China (between 2001 and 2006) as functions of the year. The plots of both data sets appear to be approximately linear on a semi-log plot, so both functions are roughly exponential.

28. If $\log(\text{imports}) = 2.015 + 0.086t$ (where t = years since 2001) is the equation of the best-fit line for imports on a semi-log plot, then the annual growth rate for imports is about 8.6%.

29. If we describe the growth in U.S. imports with the equation U.S. imports $= 103.59(1.22)^t$, then the annual growth rate for imports is about 22%.

30. If equations in Problems 28 and 29 are correct, they should be equivalent.

II. In Problems 31–35, give an example of a function or functions with the specified properties. Express your answers using equations, and specify the independent and dependent variables.

31. A function whose graph is identical to the graph of the function $y = \log\sqrt{\dfrac{x}{3}}$.

32. A function (that doesn't use e) whose graph is identical to the graph of the function $y = 50.3e^{0.06t}$. (*Hint:* $e^{0.06} \approx 1.062$)

33. A function (that uses e) whose graph is identical to the graph of the function $y = 100(0.974)^t$. (*Hint:* $0.974 \approx e^{-0.0263}$.)

34. An exponential model describing the number of farms in the United States over time if there were 3.3 million farms in 1966 and 2.1 million farms in 2006. (*Hint:* $\left(\frac{2.1}{3.3}\right)^{(1/40)} \approx 0.989$)

35. An exponential function $y = f(x)$ whose graph on a semi-log plot $(x, \log y)$ is the function $\log y = 2 + 0.08x$. (*Hint:* $10^{0.08} \approx 1.20$)

III. Are the statements in Problems 36–40 true or false? If a statement is true, explain how you know. If a statement is false, give a counterexample.

36. The number e used in the natural logarithm is a variable.

37. Compounding at 3% quarterly is the same as compounding at 12% annually.

38. Assuming your investment is growing, the effective interest rate is always greater than or equal to the nominal interest rate.

39. The graphs of the functions $y = Ca^x$ and $y = Ce^{rx}$, where $r = \ln a$, are identical.

40. The interest rate r for continuous compounding in $y = Ce^{rx}$ is called the nominal interest rate or the instantaneous interest rate, depending on the application.

CHAPTER 6 REVIEW: PUTTING IT ALL TOGETHER

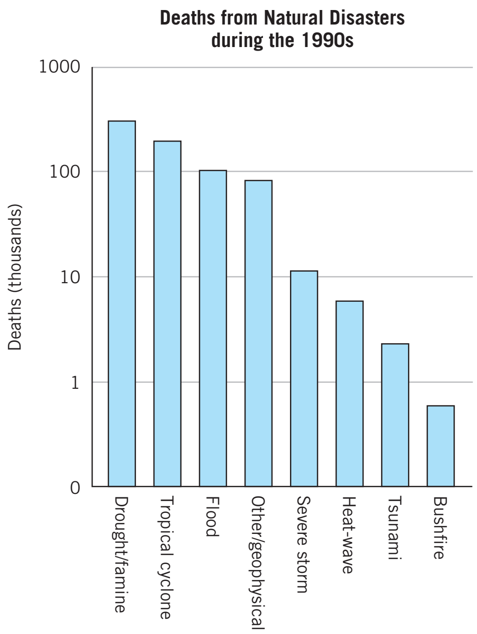
You will need a scientific calculator or its equivalent to evaluate logs and powers of e.

1. Solve each equation for the time t (in months) when the quantity $Q = 100$. Then interpret each result.

a. $Q = 50 \cdot 1.16^t$ **b.** $Q = 200 \cdot 0.92^t$

2. a. According to the following chart, approximately how many deaths from natural disasters in the 1990s were due to floods? To severe storms? To bushfires?

b. Why was this bar chart constructed with deaths displayed on a logarithmic scale?



Source: Australian government, Department of Meteorology.

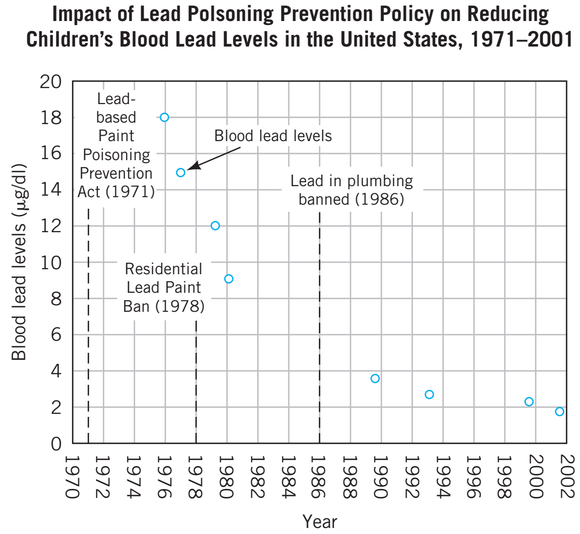
3. Which of the following expressions are equivalent?

a. $\log(xy^2)$ **d.** $2 \log x + \log y$

b. $2 \log(xy)$ **e.** $\log x + 2 \log y$

c. $\log(xy)^2$ **f.** $2 \log x + 2 \log y$

4. The Lead-based Paint Poisoning Prevention Act (1971) was passed to reduce the toxic blood levels of lead in young children who might eat pieces of peeling lead-based paint. The following chart shows the subsequent drop in the average blood lead level of children, along with additional, related legislative acts.



Source: Blood lead levels: National Health and Nutrition Examination Survey. National Center for Health Statistics, Centers for Disease Control and Prevention.

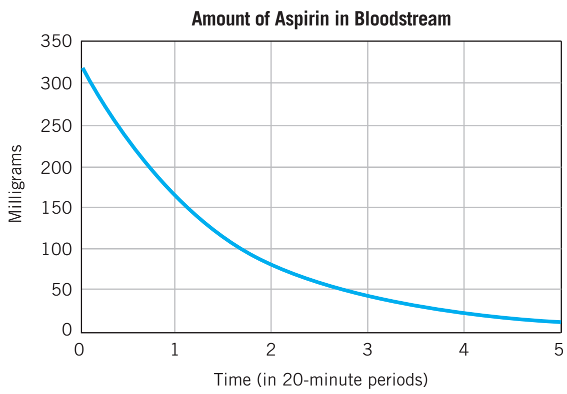
The best-fit exponential decay function for the data points on this graph is $L(t) = 18(0.88)^t$, where $L(t)$ is the average blood lead level (in micrograms per deciliter) of children and t = years since 1976 (the first year data are available).

a. What is the annual decay rate?

b. Estimate from the graph when the initial blood lead level (in 1976) would be reduced by 50%.

c. According to your model, when would the initial blood lead level be reduced by 50%?

5. The equation $A(t) = 325 \cdot (0.5)^t$ and accompanying graph describe the amount of aspirin in your bloodstream at time t (measured in 20-minute periods) after you have ingested a standard aspirin of 325 milligrams (mg).



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a. Using the equation, how much aspirin will remain after one time period? After two time periods?

b. Use the graph to estimate the number of time periods it will take for the aspirin level to reach 100 mg. Now calculate the time using your equation and compare your results.

c. Many doctors suggest that their adult patients take a daily dose of "baby aspirin" (81 mg) for long-term heart protection. Construct a new function $B(t)$ to describe the amount of aspirin in your bloodstream after taking an 81-mg aspirin. (Assume the decay rate is the same as for a 325-mg aspirin.) How do the functions $A(t)$ and $B(t)$ differ, and how are they alike?

6. The Japanese government is concerned about its decreasing population. The Japanese people joke about when Japan will disappear entirely. According to a UN report, the Japanese population is believed to have peaked at 127.5 million in 2005 and, if the current rate continues, will contract to about 105 million people in 2050.

a. What is the 45-year decay factor? The annual decay factor?

b. Use the annual decay factor to construct an exponential decay function to model the Japanese population $J(x)$, where x = years since 2005.

c. Using the rule of 70, estimate the half-life of the population. According to your model, what is the half-life?

7. India and China have the opposite problem from that of Japan (see Problem 6): a huge population that is growing. In 2005 India had an estimated population of 1.08 billion on a land area of 1.2 million square miles, and China had a population of 1.30 billion on 3.7 million square miles.

a. In 2005 did India or China have the larger population density (people per square mile)?

b. China has been trying to slow population growth with its one child-per-family policy. As a result, in 2005 its annual population growth rate (0.6%) was considerably lower than that of India (1.6%). Assuming that each population continues to grow at the 2005 rate, constuct functions $C(x)$ and $I(x)$ for the population growth in China and India, respectively, letting x = years since 2005.

c. Plot each function in part (b) on the same graph, for 30 years after 2005.

d. Looking at the graph, is there a year when India's population is projected to overtake China's? If so, use your models to predict the year and population level.

8. Polonium-210 is a toxic radioactive substance named after Poland by the Curies, who discovered it. Polonium-210 poisoning is the suspected cause of death of former Soviet spy Alexander Litvinenko, in London, November 2006.

a. Given that polonium has a half-life of approximately 138 days, construct a function to model the amount of polonium, $P(T)$, as a function of the original amount A and T, the number of half-life periods.

b. Most of the world's polonium-210 comes from Russia, which produces about 100 grams per year, which is sold commercially to the United States. How many time periods, T, would it take for 100 grams to decay until there is only 1 gram left? Translate this into days, and then into years.

c. The Soviet Union collapsed in 1991. What do you know about any pre-1991 Soviet-era stores of polonium-210?

9. At the birth of their granddaughter, the grandparents create a college fund in her name, investing \$10,000 at 7% per year.

a. Construct an equation to model the growth in the account.

b. How much will be in the account in 18 years?

c. If the annual inflation stays at 3%, how much will something that cost \$10,000 today cost in 18 years? Calculate the difference between your answer in part (b) and your answer in part (c).

d. Is this equivalent to the return you would have if you invested \$10,000 at 4% for 18 years? If not, what does investing \$10,000 at 4% represent?

10. Which of the following expressions are equivalent?

a. $\ln(\sqrt{x}/y)$ **d.** $\left(\frac{1}{2}\right)\ln x/\ln y$

b. $\left(\frac{1}{2}\right)\ln(x/y)$ **e.** $\left(\frac{1}{2}\right)\ln x - \left(\frac{1}{2}\right)\ln y$

c. $\left(\frac{1}{2}\right)\ln x - \ln y$ **f.** $\ln(\sqrt{x})/\ln y$

11. Solve each equation for t.

a. $10^t = 2.3$ **c.** $60 = 30\,e^{0.03t}$

b. $2\log t + \log 4 = 2$ **d.** $\ln(2t-5) - \ln(t-1) = 0$

12. Simplify these expressions without using a calculator.

a. $\ln(e^2)$ **c.** $10^{\log(t-2)}$

b. $e^{\ln(3)}$ **d.** $\ln(x^2 - x) - \ln x$

13. Convert each of the following expressions to a power of e.

a. 1.5 **b.** 0.7 **c.** 1

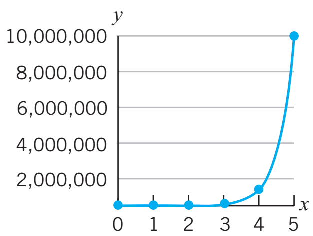
14. For each of the following functions identify the growth (or decay) factor, the growth (or decay) rate, and the continuous growth (or decay) rate.

a. $Q = 75(1.02)^t$ **b.** $P(x) = 50e^{-0.3t}$

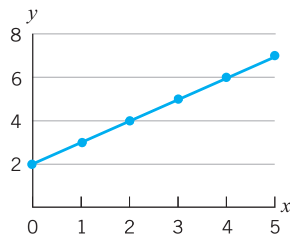
15. Match each function with one (or more) of the following graphs.

a. $y = 2 + x$ **c.** $y = 100(10)^x$

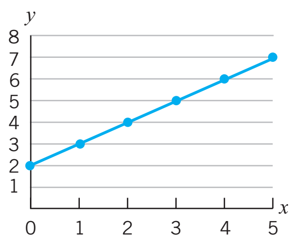
b. $y = 2 + \log(x)$ **d.** $\log(y) = 2 + x$



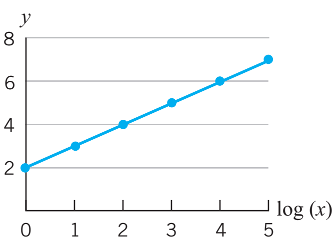
Graph *A*



Graph *C*



Graph *B*



Graph *D*

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16. a. In November 2006 the Bank of America offered a 12-month certificate of deposit (CD) at a nominal rate (or annual percentage rate, APR) of 3.6%. What is the effective interest rate (or annual percentage yield, APY) if the bank compounds:

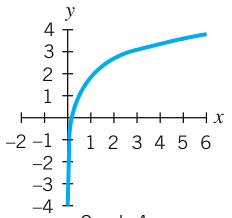
i. Annually? **ii.** Monthly? **iii.** Continuously?

b. The Bank lists the APY as 3.66%. Which of the above compounding schedules comes closest to this?

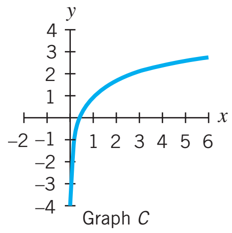
17. For each function, identify the corresponding graph.

a. $y_1 = 2 \ln x$ **c.** $y_3 = \ln(x + 2)$

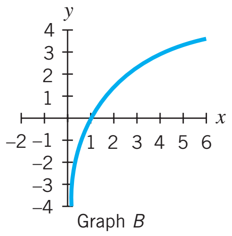
b. $y_2 = 2 + \ln x$ **d.** $y_4 = \ln(x^2)$



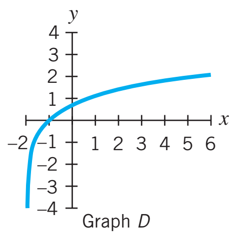
Graph *A*



Graph *C*

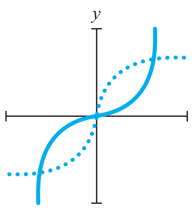


Graph *B*

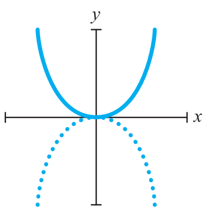


Graph *D*

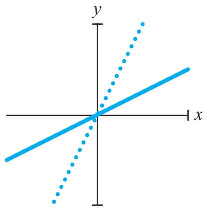
18. Which pair(s) of graphs show a function and its inverse?



Pair *A*



Pair *B*



Pair *C*

19. If the decibel level moves from 30 (a quiet conversation) to 80 (average street noise), by how many orders of magnitude has the intensity increased? Generate your answer in two ways:

a. Using the table in the text

b. Using the definition of decibels

20. Beer has a pH of 4.5 and household lye a pH of 13.5.

a. Which has the higher hydrogen ion concentration, and by how many orders of magnitude?

b. What is the difference between the two pH values? How does this relate to your answer in part (a)?

c. Why is it easier to use the pH number instead of $[H^+]$?

21. In Chapter 4 we encountered the Richter scale, which measures the amplitude of an earthquake. The Richter number R is defined as

$$R = \log(A/A_0)$$

where A is the amplitude of the shockwave caused by the earthquake and A_0 is the reference amplitude, the smallest earthquake amplitude that could be measured by a seismograph at the time this definition was adopted, in 1935.

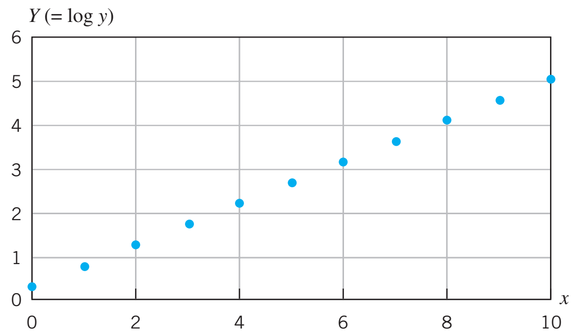
On October. 17, 1989, a magnitude 6.9 earthquake shook the San Francisco area. Robert Page from the U.S. Geological Survey said, "It was a wakeup call to prepare for the potentially even more devastating shocks that are inevitable in the future." Find the ratio A/A_0 for this earthquake. How many orders of magnitude larger was this earthquake's amplitude compared with the base-level amplitude, A_0?

22. The energy magnitude, M, radiated by an earthquake measures the potential damage to man-made structures. It can be described by the formula

$$M = \left(\tfrac{2}{3}\right) \log E - 2.9$$

where the seismic energy, E, is expressed in joules. Show that for every increase in M of one unit, the associated seismic energy E is increased by about a factor of 32.

23. a. Given the following graph, generate a linear equation for $Y(= \log y)$ in terms of x.



b. Now substitute $\log y$ for Y and solve your equation for y.

c. What type of function did you find in part (b)? What does this suggest about functions that appear linear on a semi-log graph (with the log scale on the vertical axis)?

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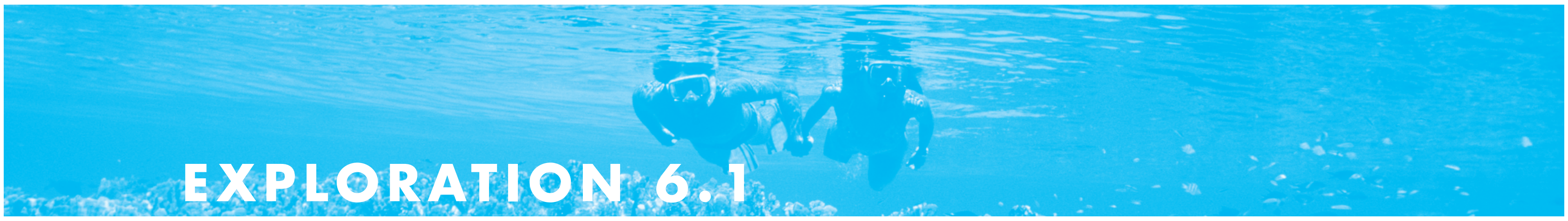

EXPLORATION 6.1

Properties of Logarithmic Functions

Objective

- explore the effects of a and c on the graphs of $y = c\log(ax)$ and $y = c\ln(ax)$

Materials/Equipment

- graphing calculator or computer with "E8: Logarithmic Sliders" in *Exponential & Log Functions* in course software or a function graphing program
- graph paper

Procedure

Making Predictions

1. *The effect of a on the equation* $y = \log(ax)$ when $a > 0$ *and* $x > 0$.

a. Why do we need to restrict a and x to positive values?

b. Using the properties of logarithms, write the expression $\log(ax)$ as the sum of two logs. Discuss with a partner what effect you expect a to have on the graph. Now using $\log 2 \approx 0.301$ and $\log 3 \approx 0.477$, complete the accompanying data table and, by hand, sketch the three graphs on the same grid. Do your results confirm your predictions? What do you expect to happen to the graph of $y = \log(ax)$ if larger and larger positive values are substituted for a?

Evaluating $y = \log(ax)$ when $a = 1, 2,$ and 3

x	$y = \log x$	$y = \log(2x)$	$y = \log(3x)$
0.001	−3.000		
0.010	−2.000		
0.100	−1.000		
1.000	0.000		
5.000	0.699		
10.000	1.000		

c. Discuss with your partner how you think the graphs of $y = \log x$ and $y = \log(ax)$ will compare if $0 < a < 1$. Complete the following small data table and sketch the graphs of $y = \log x$ and $y = \log(x/10)$ on the same grid. Do your predictions and your graph agree? Predict what would happen to the graph of $y = \log(ax)$ if smaller and smaller positive values were substituted for a.

Evaluating $y = \log(ax)$ when $a = 1$ and $\frac{1}{10}$

x	$y = \log x$	$y = \log(x/10)$
0.001	−3.000	
0.010	−2.000	
0.100	−1.000	
1.000	0.000	
5.000	0.699	
10.000	1.000	

d. How do you think your findings for $y = \log(ax)$ relate to the function $y = \ln(ax)$?

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2. *The effect of c on the equation* $y = c \log x$

a. When $c > 0$: Talk over with your partner your predictions for what will happen as c increases. Fill in the following table and on the same grid draw a quick sketch of the three functions. Were your predictions correct? What do you expect to happen to the graph of $y = c \log x$ as you substitute larger and larger positive values for c?

Evaluating $y = c \log x$ when $c = 1, 2,$ and 3

x	$y = \log x$	$y = 2 \log x$	$y = 3 \log x$
0.001	−3.000		
0.010	−2.000		
0.100	−1.000		
1.000	0.000		
5.000	0.699		
10.000	1.000		

b. When $c < 0$: Why can c be negative when a has to remain positive? How do you think the graphs of $y = \log x$ and $y = c \log x$ will compare if $c < 0$? What happens if c remains negative but $|c|$ gets larger and larger (for example, $c = -10, -150, -5000$, etc.)?

c. How do your findings on $y = c \log x$ relate to the function $y = c \ln x$?

3. *Generalizing your results*. Talk over with your partner the effect of varying both a and c on the general functions $y = c \log(ax)$ and $y = c \ln(ax)$. Try predicting the shapes of the graphs of such functions as $y = 3 \log(2x)$ and $y = -2 \log(x/10)$. Have each partner construct a small table, and graph the results of one such function. Compare your findings.

Testing Your Predictions

Test your predictions by either using "E8: Logarithmic Sliders" in *Exponential & Log Functions* or creating your own graphs with a graphing calculator or a function graphing program. Try changing the value for a to create different functions of the form $y = \log(ax)$ or $y = \ln(ax)$. Be sure to try values of $a > 1$ and values such that $0 < a < 1$. Then try changing the value for c in functions of the form $y = c \log x$ and $y = c \ln x$. Let c assume both positive and negative values.

Summarizing Your Results

Write a 60-second summary describing the effect of varying a and c in functions of the form $y = c \log(ax)$ and $y = c \ln(ax)$.

Exploration-Linked Homework

1. Use your knowledge of logarithmic functions to predict the shapes of the graphs of:

a. *The decibel scale:* given by the function $N = 10 \log \frac{I}{I_0}$, where I is the intensity of a sound and I_0 is the intensity of sound at the threshold of human hearing. (See Section 6.4, pages 359–361.)

b. *Stellar magnitude:* approximated by the function $M = -2.5 \log \frac{B}{B_0}$, where B is the brightness of a star and B_0 is a constant. (See Section 6.4, Exercise 8, p. 363.)

2. Explore the effect of changing the base, that is, the effect of changing b in $y = c \log_b(ax)$. (See Section 6.4, Exercise 7, p. 363.)

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