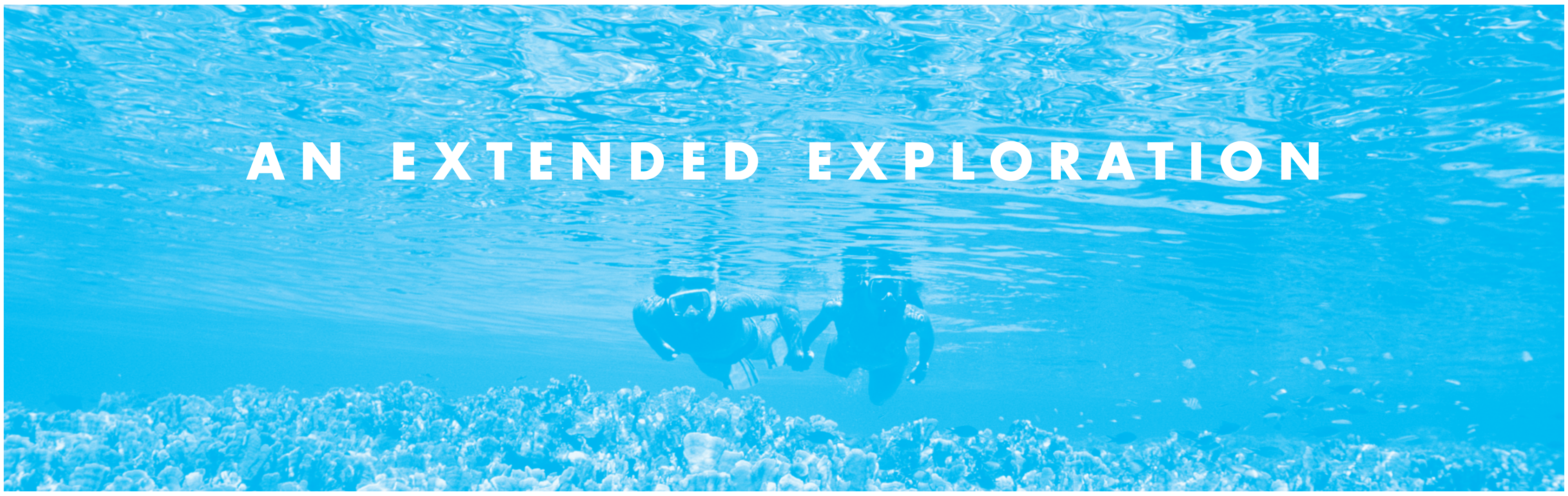


THE MATHEMATICS OF MOTION

OVERVIEW

In this extended exploration we use the laboratory methods of modern physicists to collect and analyze data about freely falling bodies and then examine the questions asked by Galileo about bodies in motion.

After conducting this exploration, you should be able to

- understand the importance of the scientific method
- describe the relationship between distance and time for freely falling bodies
- derive equations describing the velocity and acceleration of a freely falling body

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 560

The Scientific Method

Today we take for granted that scientists study physical phenomena in laboratories using sophisticated equipment. But in the early 1600s, when Galileo did his experiments on motion, the concept of laboratory experiments was unknown. In his attempts to understand nature, Galileo asked questions that could be tested directly in experiments. His use of observation and direct experimentation and his discovery that aspects of nature were subject to quantitative laws were of decisive importance, not only in science but in the broad history of human ideas.

Ancient Greeks and medieval thinkers believed that basic truths existed within the human mind and that these truths could be uncovered through reasoning, not empirical experimentation. Their scientific method has been described as a "qualitative study of nature." Greek and medieval scientists were interested in *why* objects fall. They believed that a heavier object fell faster than a lighter one because "it has weight and it falls to the Earth because it, like every object, seeks its natural place, and the natural place of heavy bodies is the center of the Earth. The natural place of a light body, such as fire, is in the heavens, hence fire rises."[1]

Galileo changed the question from *why* things fall to *how* things fall. This question suggested other questions that could be tested directly by experiment: "By alternating questions and experiments, Galileo was able to identify details in motion no one had previously noticed or tried to observe."[2] His quantitative descriptions of objects in motion led not only to new ways of thinking about motion, but also to new ways of thinking about science. His process of careful observation and testing began the critical transformation of science from a qualitative to a quantitative study of nature.[3] Galileo's decision to search for quantitative descriptions "was the most profound and the most fruitful thought that anyone has had about scientific methodology."[4] This approach became known as the scientific method.

The Free Fall Experiment

Instructions for conducting the free fall experiment are in the last section.

In this extended exploration, you will conduct a modern version of Galileo's free fall experiment. This classic experiment records the distance that a freely falling object falls during each fraction of a second. The experiment can be performed either with a graphing calculator connected to a motion sensor or in a physics laboratory with an apparatus that drops a heavy weight and records its position on a tape.

In this experiment, Galileo sought to answer the following questions:

> How can we describe mathematically the distance an object falls over time? Do freely falling objects fall at a constant speed? If the speed of freely falling objects is not constant, is it increasing at a constant rate?

The software "Q11: Freely Falling Objects" in *Quadratic Functions* provides a simulation of the free fall experiment.

You can try to find answers to these questions by collecting and analyzing your own data or by using the data provided as both Excel and graph link files. Instructions for using technology to collect and analyze data are provided in the last section. The following discussion will help you analyze your results and provide answers to Galileo's questions.

[1]M. Kline, *Mathematics for the Nonmathematician* (New York: Dover, 1967), p. 287.
[2]E. Cavicchi, "Watching Galileo's Learning," in the Anthology of Readings on the course website.
[3]Galileo's scientific work was revolutionary in terms not only of science but also of the politics of the time; his work was condemned by the ruling authorities, and he was arrested.
[4]M. Kline, op. cit., p. 288.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 561

Interpreting Data from a Free Fall Experiment

See the Excel or graph link file FREEFALL, which contains the data in Table 1.

The sketch of a tape given in Figure 1 gives data collected by a group of students from a falling-object experiment. Each dot represents how far the object fell in each succeeding 1/60 of a second.

Since the first few dots are too close together to get accurate measurements, we start measurements at the sixth dot, which we call dot_0. At this point, the object is already in motion. This dot is considered to be the starting point, and the time, t, at dot_0 is set at 0 seconds. The next dot represents the position of the object 1/60 of a second later. Time increases by 1/60 of a second for each successive dot. In addition to assigning a time to each point, we also measure the total distance fallen, d (in cm), from the point designated dot_0. For every dot we have two values: the time, t, and the distance fallen, d. At dot_0, we have $t = 0$ and $d = 0$.

The time and distance measurements from the tape are recorded in Table 1 and plotted on the graph in Figure 2. Time, t, is the independent variable, and distance, d, is the dependent variable. The graph gives a representation of the data collected on distance fallen over time, not a picture of the physical motion of the object. The graph of the data looks more like a curve than a straight line, so we expect the average rates of change between different pairs of points to be different. We know how to calculate the average rate of change between two points and that it represents the slope of a line segment connecting the two points:

$$\text{average rate of change} = \frac{\text{change in distance}}{\text{change in time}} = \text{slope of line segment}$$

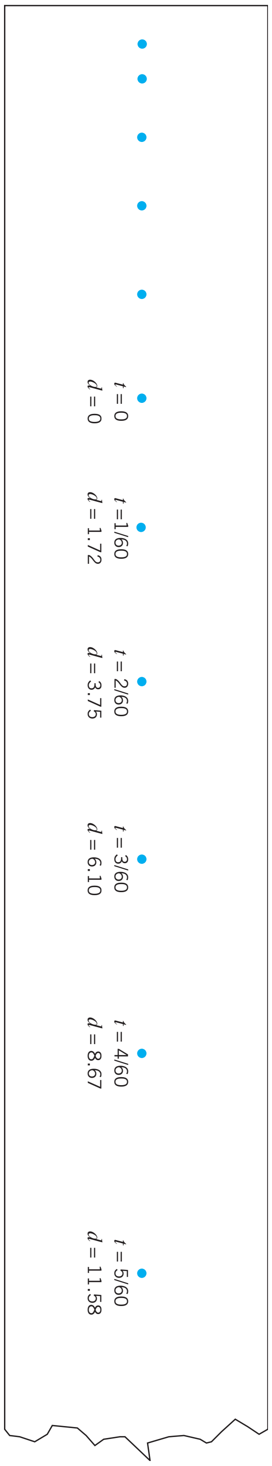


Figure 1 Tape from a free fall experiment.

Time (sec)	Total Distance Fallen (cm)
0.0000	0.00
0.0167	1.72
0.0333	3.75
0.0500	6.10
0.0667	8.67
0.0833	11.58
0.1000	14.71
0.1167	18.10
0.1333	21.77
0.1500	25.71
0.1667	29.90
0.1833	34.45
0.2000	39.22
0.2167	44.22
0.2333	49.58
0.2500	55.15
0.2667	60.99
0.2833	67.11
0.3000	73.48
0.3167	80.10
0.3333	87.05
0.3500	94.23

Table 1

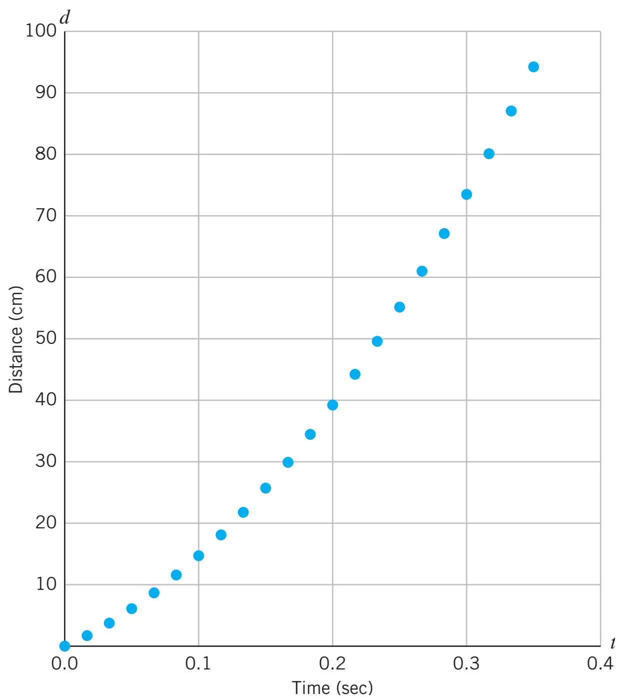


Figure 2 Free fall: distance versus time.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 562

Table 2 and Figure 3 show the increase in the average rate of change over time for three different pairs of points. The time interval nearest the start of the fall shows a relatively small change in the distance per time step and therefore a relatively gentle slope of 188 cm/sec. The time interval farthest from the start of the fall shows a greater change of distance per time step and a much steeper slope of 367 cm/sec.

t	d	Average Rate of Change
0.0500 0.1333	6.10 21.77	$\dfrac{21.77 - 6.10}{0.1333 - 0.0500} \approx 188$ cm/sec
0.0833 0.2333	11.58 49.58	$\dfrac{49.58 - 11.58}{0.2333 - 0.0833} \approx 253$ cm/sec
0.2167 0.3333	44.22 87.05	$\dfrac{87.05 - 44.22}{0.3333 - 0.2167} \approx 367$ cm/sec

Table 2

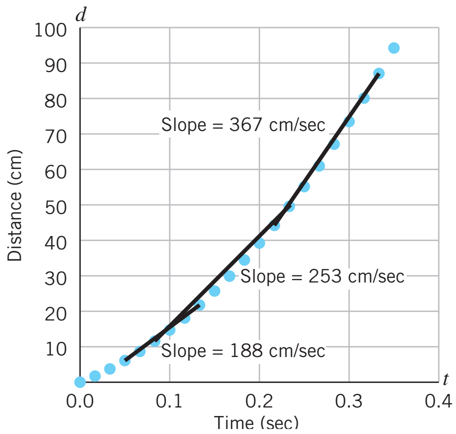


Figure 3 Slopes (or average velocities) between three pairs of end points.

In this experiment the average rate of change has an additional important meaning. For objects in motion, the change in distance divided by the change in time is also called the *average velocity* for that time period. For example, in the calculations in Table 2, the average rate of change of 188 cm/sec represents the average velocity of the falling object between 0.0500 and 0.1333 second.

$$\textit{average velocity} = \frac{\text{change in distance}}{\text{change in time}}$$

Important Questions

Do objects fall at a constant speed?[5] The rate-of-change calculations and the graph in Figure 3 indicate that the average rate of change of position with respect to time—the

[5]In everyday usage, "speed" and "velocity" are used interchangeably. In physics, "velocity" gives the direction of motion by the sign of the number—positive for forward, negative for backward. "Speed" means the absolute value, or magnitude, of the velocity. So speed is never negative, whereas velocity can be positive or negative.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 563

velocity—of the falling object is not constant. Moreover, the average velocity appears to be increasing over time. In other words, as the object falls, it is moving faster and faster. Our calculations agree with Galileo's observations. He was the first person to show that the velocity of a freely falling object is not constant.

This finding prompted Galileo to ask more questions. One of these questions was: If the velocity of freely falling bodies is *not constant,* is it increasing at a *constant rate?* Galileo discovered that the velocity of freely falling objects does increase at a constant rate. If the rate of change of velocity with respect to time is constant, then the graph of velocity versus time is a straight line. The slope of that line is constant and equals the rate of change of velocity with respect to time. A theory of gravity has been built around Galileo's discovery of a constant rate of change for the velocity of a freely falling body. This constant of nature, the gravitational constant of Earth, is denoted by g and is approximately 980 cm/sec^2.

Deriving an Equation Relating Distance and Time

Galileo wanted to describe mathematically the distance an object falls over time. Using mathematical and technological tools not available in Galileo's time, we can describe the distance fallen over time in the free fall experiment using a "best-fit" function for our data. Galileo had to describe his finding in words. Galileo described the free fall motion first by direct measurement and then abstractly with a time-squared rule. "This discovery was revolutionary, the first evidence that motion on Earth was subject to mathematical laws."[6]

Using Galileo's finding that distance is related to time by a time-squared rule, we use technology to find the following best-fit quadratic function for the free fall data in Table 1:

$$d = 487.8t^2 + 98.73t - 0.0528$$

If you are interested in learning more about how Galileo made his discoveries, read Elizabeth Cavicchi's "Watching Galileo's Learning."

Figure 4 shows a plot of the data and the function. If your curve-fitting program does not provide a measure of closeness of fit, such as the correlation coefficient for regression lines, you may have to rely on a visual judgment. Rounding the coefficients to the nearest unit, we obtain the equation

$$\begin{aligned} d &= 488t^2 + 99t - 0 \\ &= 488t^2 + 99t \end{aligned} \tag{1}$$

We now have a mathematical model for our free fall data.

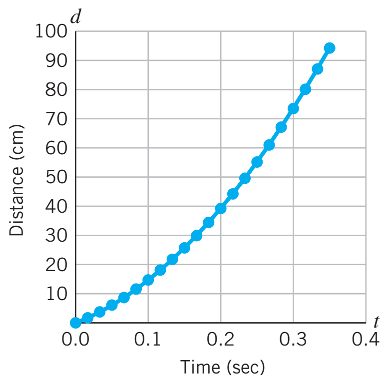


Figure 4 Best-fit function for distance versus time.

[6]E. Cavicchi, "Watching Galileo's Learning," in the Anthology of Readings on the course website.

What are the units for each term of the equation? Since d is in centimeters, each term on the right-hand side of Equation (1) must also be in centimeters. Since t is in seconds, the coefficient, 488, of t^2 must be in centimeters per second squared:

$$\frac{\text{cm}}{\cancel{\text{sec}^2}} \cdot \frac{\cancel{\text{sec}^2}}{1} = \text{cm}$$

The coefficient, 99, of t must be in centimeters per second, and the constant term, 0, in centimeters.

If we ran the experiment again, how would the results compare? In one class, four small groups did the free fall experiment, plotted the data, and found a corresponding best-fit second-degree polynomial. The functions are listed below, along with Equation (1). In each case we have rounded the coefficients to the nearest unit. All of the constant terms rounded to 0.

$$d = 488t^2 + 99t \tag{1}$$
$$d = 486t^2 + 72t \tag{2}$$
$$d = 484t^2 + 173t \tag{3}$$
$$d = 486t^2 + 73t \tag{4}$$
$$d = 495t^2 + 97t \tag{5}$$

Examine the coefficients of each of the terms in these equations. All the functions have similar coefficients for the t^2 term, very different coefficients for the t term, and zero for the constant term. Why is this the case? Using concepts from physics, we can describe what each of the coefficients represents.

The coefficients of the t^2 term found in Equations (1) to (5) are all close to one-half of 980 cm/sec^2, or half of g, Earth's gravitational constant. The data from this simple experiment give very good estimates for $\frac{1}{2}g$.

The coefficient of the t term represents the initial velocity, v_0, of the object when $t = 0$. In Equation (1), $v_0 = 99$ cm/sec. Recall that we didn't start to take measurements until the sixth dot, the dot we called dot_0. So at dot_0, where we set $t = 0$, the object was already in motion with a velocity of approximately 99 cm/sec. The initial velocities, or v_0 values, in Equations (2) to (5) range from 72 to 173 cm/sec. Each v_0 represents approximately how fast the object was moving when $t = 0$, the point chosen to begin recording data in each of the various experiments.

The constant term rounded to zero in each of Equations (1) to (5). On the tape where we set $t = 0$, we set $d = 0$. So we expect that in all our best-fit equations the constant terms, which represent the distance at time zero, are approximately zero. If we substitute zero for t in Equations (1) to (5), the value for d is indeed zero. If we looked at additional experimental results, we might encounter some variation in the constant term, but all should have values of approximately zero.

Galileo's discoveries are the basis for the following equations relating distance and time:

> The general equation of motion of freely falling bodies that relates distance fallen, d, to time, t, is
>
> $$d = \frac{1}{2}gt^2 + v_0 t$$
>
> where v_0 is the initial velocity and g is the acceleration due to gravity on Earth.

For example, in our original model, $d = 488t^2 + 99t$, the coefficient 488 approximates $\frac{1}{2}g$ (in centimeters per second squared) and 99 approximates the initial velocity (in centimeters per second).

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 565

Returning to Galileo's Question

If the velocity for freely falling bodies is not constant, is it increasing at a constant rate? Galileo discovered that the rate of change of the velocity of a freely falling object is constant. In this section we confirm his finding with data from the free fall experiment.

Velocity: Change in Distance over Time

If the rate of change of velocity is constant, then the graph of velocity vs. time should be a straight line. Previously we calculated the average rates of change of distance with respect to time (or average velocities) for three arbitrarily chosen pairs of points. Now, in Table 3 we calculate the average rates of change for all the pairs of adjacent points in our free fall data. The results are in column 4. Since each computed velocity is the average over an interval, for increased precision we associate each velocity with the midpoint time of the interval instead of one of the end points. In Figure 5, we plot velocity from the fourth column against the midpoint times from the third column. The graph is strikingly linear.

See "C3: Average Velocity and Distance" in *Rates of Change.*

Time, t (sec)	Distance Fallen, d (cm)	Midpoint Time, t (sec)	Velocity, v (cm/sec)
0.0000	0.00		
		0.0083	103.2
0.0167	1.72		
		0.0250	121.8
0.0333	3.75		
		0.0417	141.0
0.0500	6.10		
		0.0583	154.2
0.0667	8.67		
		0.0750	174.6
0.0833	11.58		
		0.0917	187.8
0.1000	14.71		
		0.1083	203.4
0.1167	18.10		
		0.1250	220.2
0.1333	21.77		
		0.1417	236.4
0.1500	25.71		
		0.1583	251.4
0.1667	29.90		
		0.1750	273.0
0.1833	34.45		
		0.1917	286.2
0.2000	39.22		
		0.2083	300.0
0.2167	44.22		
		0.2250	321.6
0.2333	49.58		
		0.2417	334.2
0.2500	55.15		
		0.2583	350.4
0.2667	60.99		
		0.2750	367.2
0.2833	67.11		
		0.2917	382.2
0.3000	73.48		
		0.3083	397.2
0.3167	80.10		
		0.3250	417.0
0.3333	87.05		
		0.3417	430.8
0.3500	94.23		

Table 3

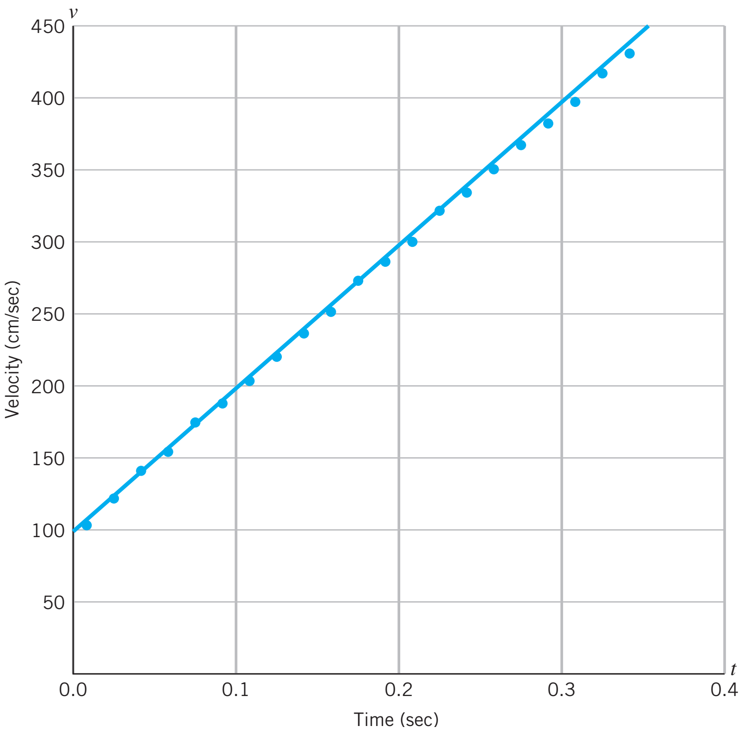


Figure 5 Best-fit linear function for average velocity versus time.

Generating a best-fit linear function and rounding to the nearest unit, we obtain the equation

$$\text{average velocity} = 977t + 98$$

where the average velocity, v, is in centimeters per second, and time t, is in seconds. The graph of this function appears in Figure 5. The slope of the line is constant and equals the rate of change of velocity with respect to time. So although the velocity is not constant, its rate of change with respect to time *is* constant.

The coefficient of t, 977, is the slope of the line and in physical terms represents g, the acceleration due to gravity. The conventional value for g is 980 cm/sec^2. So the velocity of the freely falling object increases by about 980 cm/sec during each second of free fall.

With this equation we can estimate the velocity at any given time t. When $t = 0$, then $v = 98$ cm/sec. This means that the object was already moving at about 98 cm/sec when we set $t = 0$. In our experiment, the velocity when $t = 0$ depends on where we choose to start measuring our dots. If we had chosen a dot closer to the beginning of the free fall, we would have had an initial velocity lower than 98 cm/sec. If we had chosen a dot farther away from the start, we would have had an initial velocity higher than 98 cm/sec. Note that 98 cm/sec closely matches the value of 99 cm/sec in our best-fit quadratic function (Equation 1).

> The general equation that relates, v, the velocity of a freely falling body, to t, time, is
>
> $$v = gt + v_0$$
>
> where v_0 = initial velocity (velocity at time $t = 0$) and g is the acceleration due to gravity.

Acceleration: Change in Velocity over Time

Acceleration means a change in velocity or speed. If you push the accelerator pedal in a car down just a bit, the speed of the car increases slowly. If you floor the pedal, the speed increases rapidly. The rate of change of velocity with respect to time is called *acceleration*. Calculating the average rate of change of velocity with respect to time gives an estimate of acceleration. For example, if a car is traveling at 20 mph and 1 hour later the car has accelerated to 60 mph, then

$$\frac{\text{change in velocity}}{\text{change in time}} = \frac{(60 - 20)\text{ mph}}{1\text{ hr}} = (40\text{ mph})/\text{hr} = 40\text{ mi/hr}^2$$

In 1 hour, the velocity of the car changed from 20 to 60 mph, so its average acceleration was 40 mph/hr, or 40 mi/hr^2.

> $$\textit{average acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

Table 4 uses the average velocity data and midpoint time from Table 3 to calculate average accelerations. Figure 6 shows the plot of average acceleration in centimeters per second squared (the third column) versus time in seconds (the first column).

The data lie along a roughly horizontal line. The average acceleration values vary between a low of 756 cm/sec^2 and a high of 1296 cm/sec^2 with a mean of 982.8. Rounding off, we have

$$\text{acceleration} \approx 980\text{ cm/sec}^2$$

This expression confirms that for each additional second of free fall, the velocity of the falling object increases by approximately 980 cm/sec. The longer it falls, the faster it goes. We have verified a characteristic feature of gravity near the surface of Earth: It causes objects to fall at a velocity that increases every second by about 980 cm/sec. We say that the acceleration due to gravity near Earth's surface is 980 cm/sec^2.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 567

Midpoint Time, t (sec)	Average Velocity, v (cm/sec)	Average Acceleration (cm/sec^2)
0.0083	103.2	n.a.
0.0250	121.8	1116
0.0417	141.0	1152
0.0583	154.2	792
0.0750	174.6	1224
0.0917	187.8	792
0.1083	203.4	936
0.1250	220.2	1008
0.1417	236.4	972
0.1583	251.4	900
0.1750	273.0	1296
0.1917	286.2	792
0.2083	300.0	828
0.2250	321.6	1296
0.2417	334.2	756
0.2583	350.4	972
0.2750	367.2	1008
0.2917	382.2	900
0.3083	397.2	900
0.3250	417.0	1188
0.3417	430.8	828

Table 4

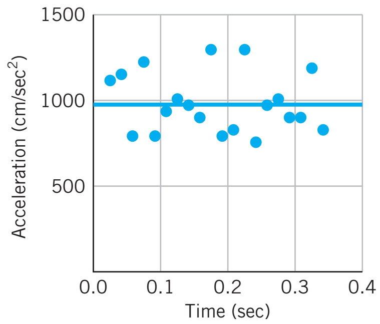


Figure 6 Average acceleration for free fall data.

In order to express g in feet per second squared, we need to convert 980 centimeters to feet. We start with the fact that 1 ft = 30.48 cm. So the conversion factor for centimeters to feet is (1 ft)/(30.48 cm) = 1. If we multiply 980 cm by (1 ft)/(30.48 cm) to convert centimeters to feet, we get

$$980 \text{ cm} = (980 \cancel{\text{cm}})\left(\frac{1 \text{ ft}}{30.48 \cancel{\text{cm}}}\right) \approx 32.15 \text{ ft} \approx 32 \text{ ft}$$

So a value of 980 cm/sec^2 for g is equivalent to approximately 32 ft/sec^2.

The numerical value used for the constant g depends on the units being used for the distance, d, and the time, t. The exact value of g also depends on where it is measured.[7]

The conventional values for g, the acceleration due to gravity near the surface of Earth, are

$$g = 32 \text{ ft/sec}^2$$

or equivalently $$g = 980 \text{ cm/sec}^2 = 9.8 \text{ m/sec}^2$$

[7] Because Earth is rotating, is not a perfect sphere, and is not uniformly dense, there are variations in g according to the latitude and elevation. The following are a few examples of local values for g.

Location	North Latitude (deg)	Elevation (m)	g (cm/sec^2)
Panama Canal	9	0	978.243
Jamaica	18	0	978.591
Denver, CO	40	1638	979.609
Pittsburgh, PA	40.5	235	980.118
Cambridge, MA	42	0	980.398
Greenland	70	0	982.534

Source: H. D. Young, *University Physics,* Vol. I, 8th ed. (Reading, MA: Addison-Wesley, 1992), p. 336.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 568

Deriving an Equation for the Height of an Object in Free Fall

Assume we have the following motion equation relating distance fallen, d (in centimeters), and time, t (in seconds):

$$d = 490t^2 + 45t$$

Also assume that when $t = 0$, the height, h, of the object was 110 cm above the ground. Until now, we have considered the distance from the point the object was dropped, a value that *increases* as the object falls. How can we describe a different distance, the *height above ground* of an object, as a function of time, a value that *decreases* as the object falls?

At time zero, the distance fallen is zero and the height above the ground is 110 centimeters. After 0.05 second, the object has fallen about 3.5 centimeters, so its height would be $110 - 3.5 = 106.5$ cm. For an arbitrary distance d, we have $h = 110 - d$. Table 5 gives associated values for time, t, distance fallen, d, and height above ground, h. The graphs in Figure 7 show distance versus time and height versus time.

Time, t (sec)	Distance Fallen, d (cm) ($d = 490t^2 + 45t$)	Height above Ground, h (cm) ($h = 110 - d$)
0.00	0.0	110.0
0.05	3.5	106.5
0.10	9.4	100.6
0.15	17.8	92.2
0.20	28.6	81.4
0.25	41.9	68.1
0.30	57.6	52.4
0.35	75.8	34.2
0.40	96.4	13.6

Table 5

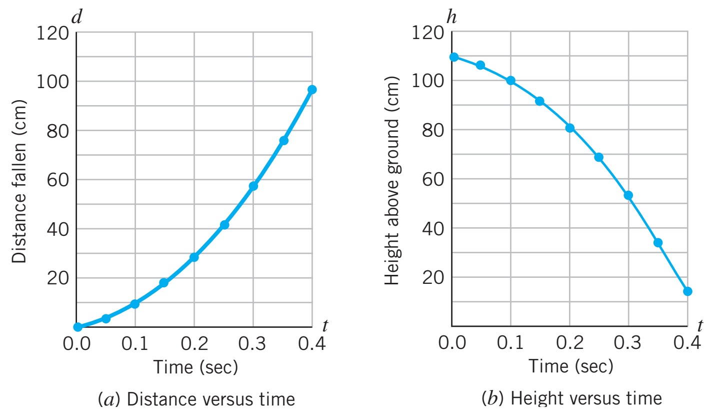


Figure 7 Representations of free fall data.

How can we convert the equation $d = 490t^2 + 45t$, relating distance fallen and time, to an equation relating height above ground and time? We know that the relationship between height and distance is $h = 110 - d$. We can substitute the expression for d into the height equation:

$$\begin{aligned} h &= 110 - d \\ &= 110 - (490t^2 + 45t) \\ &= 110 - 490t^2 - 45t \qquad (6) \end{aligned}$$

Switching the order of the terms, we could rewrite this equation as $h = -490t^2 - 45t + 110$. The constant term, here 110 cm, represents the initial height when $t = 0$. By placing the constant term first as in Equation (6), we emphasize 110 cm as the initial or starting value. Height equations often appear in the form $h = c + bt + at^2$ to emphasize the constant term c as the starting height. This is similar to writing linear equations in the form $y = b + mx$ to emphasize the constant term b as the base, or starting, value.

Note that in Equation (6) for height, the coefficients of both t and t^2 are negative. If we consider what happens to the height of an object in free fall, this makes sense. As time increases, the height decreases. (See Table 5 and Figure 7.) When we were measuring the increasing distance an object fell, we did not take into account the direction in which it was going (up or down). We cared only about the magnitudes (the absolute values) of distance and velocity, which were positive. But when we are measuring a decreasing height or distance, we have to worry about direction. In this case we define downward motion to be negative and upward motion to be positive. In the height equation $h = 110 - 45t - 490t^2$, the constant term, the initial height, is 110 cm. The change in height resulting from the initial velocity, $-45t$, is negative because the object was moving down when we started to measure it. The change in height caused by acceleration, $-490t^2$, is also negative because gravity pulls objects downward in what we are now considering as a negative direction, reducing the height of a falling object.

Once we have introduced the notion that downward motion is negative and upward motion is positive, we can also deal with situations in which the initial velocity is upward and the acceleration is downward. The velocity equation for this situation is

$$v = -gt + v_0$$

where v_0 could be either positive or negative, depending on whether the object is thrown upward or downward, and the sign for the g term is negative because gravity accelerates downward in the negative direction.

> If we treat upward motion as positive and downward motion as negative, then the acceleration due to gravity is negative. So the general equations of motion of freely falling bodies that relate height, h, and velocity, v, to time, t, are
>
> $$h = h_0 + v_0 t - \tfrac{1}{2}gt^2$$
> $$v = -gt + v_0$$
>
> where h_0 = initial height, g = acceleration due to gravity, and v_0 = initial velocity (which can be positive or negative).

Working with an Initial Upward Velocity

If we want to use the general equation to describe the height of a thrown object, we need to understand the meaning of each of the coefficients. Suppose a ball is thrown upward with an initial velocity of 97 cm/sec from a height of 87 cm above the ground. Describe the relationship between the height of the ball and time with an equation.

The initial height of the ball is 87 cm when $t = 0$, so the constant term is 87 cm. The coefficient of t, or the initial-velocity term, is $+97$ cm/sec since the initial motion is upward. The coefficient of t^2, the gravity term, is -490 cm/sec^2, since gravity causes objects to fall down.

Substituting these values into the equation for height, we get

$$h = 87 + 97t - 490t^2$$

Table 6 gives a series of values for heights corresponding to various times. Figure 8 plots height above ground (cm) vs. time (sec.).

t (sec)	h (cm)
0.00	87.00
0.05	90.63
0.10	91.80
0.15	90.53
0.20	86.80
0.25	80.63
0.30	72.00
0.35	60.93
0.40	47.40
0.45	31.43
0.50	13.00

Table 6

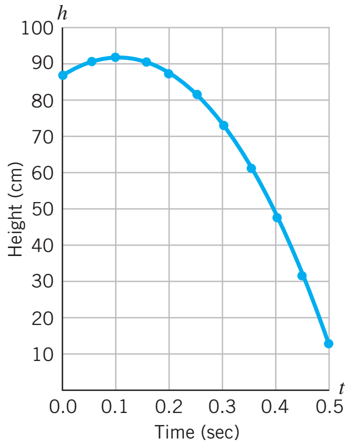


Figure 8 Height of a thrown ball.

The graph of the heights at each time in Figure 8 should not be confused with the trajectory of a thrown object. The actual motion we are talking about is purely vertical—straight up and straight down. The graph shows that the object travels up for a while before it starts to fall. This corresponds with what we all know from practical experience throwing balls. The upward (positive) velocity is decreased by the pull of gravity until the object stops moving upward and begins to fall. The downward (negative) velocity is then increased by the pull of gravity until the object strikes the ground.

Collecting and Analyzing Data from a Free Fall Experiment

Objective

- to describe mathematically how objects fall

Equipment/Materials

- graphing calculator with best-fit function capabilities or computer with spreadsheet and function graphing program
- notebook for recording measurements and results (sample Lab Book on course website)

DATA If using precollected data, see the Excel or graph link file FREEFALL.

Equipment needed for collecting data in physics laboratory:

a. Free fall apparatus
b. Meter sticks 2 meters long
c. Masking tape

Equipment needed for collecting data with CBL® (Calculator-Based Laboratory System®):

a. CBL® unit with AC-9201 power adapter
b. Vernier CBL® ultrasonic motion detector
c. Graphing calculator
d. Extension cord and some object to drop, such as a pillow or rubber ball

Preparation

If collecting data in a physics laboratory, schedule a time for doing the experiment and have the laboratory assistant available to set up the equipment and assist with the

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 571

experiment. If collecting data with a CBL® unit with graphing calculator, instructions for using a CBL® unit are in the Instructor's Manual.

Procedure

The following procedures can be used for collecting data in a physics laboratory.[8] If you are collecting data with a CBL®, collect the data and go to the Results section. If you are using the precollected data in the file FREEFALL, go directly to the Results section.

Collecting the data

Since the falling times are too short to record with a stopwatch, we use a free fall apparatus. Every sixtieth of a second a spark jumps between the falling object or "bob" and the vertical metal pole supporting the tape. Each spark burns a small dot on the fixed tape, recording the bob's position. The procedure is to:

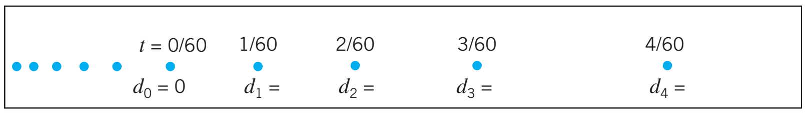
1. Position the bob at the top of the column in its holder.
2. Pull the tape down the column so that a fresh tape is ready to receive spark dots.
3. Be sure that the bob is motionless before you turn on the apparatus.
4. Turn on the spark switch and bob release switch as demonstrated by the laboratory assistant.
5. Tear off the length of tape recording the fall of the bob.

Obtaining and recording measurements from the tapes

The tape is a record of the distance fallen by the bob between each sixtieth-of-a-second spark dot. Each pair of students should measure and record the distance between the dots on the tape. Let d = the distance fallen in centimeters and t = time in seconds.

1. Fasten the tape to the table using masking tape.
2. Inspect the tape for missing dots. *Caution:* The sparking apparatus sometimes misses a spark. If this happens, take proper account of it in numbering the dots.
3. Position the 2-meter stick on its edge along the dots on the tape. Use masking tape to fasten the meter stick to the table, making sure that the spots line up in front of the bottom edge of the meter stick so you can read their positions off of the stick.
4. Beginning with the sixth visible dot, mark the time for each spot on the tape; that is, write $t = 0/60$ sec by the sixth dot, $t = 1/60$ by the next dot, $t = 2/60$ by the next dot, and so on, until you reach the end of the tape.

   Note: The first five dots are ignored in order to increase accuracy of measurements. One cannot be sure that the object is released exactly at the time of the spark, instead of between sparks, and the first few dots are too close together to get accurate measurements. When the body passes the sixth dot, it already has some velocity, which we call v_0, and this point is arbitrarily taken as the initial time, $t = 0$.



5. Measure the distances (accurate to a fraction of a millimeter) from the sixth dot to each of the other dots. Record each distance by the appropriate dot on the tape.

[8]These procedures are adapted from "Laboratory Notes for Experiment 2: The Kinematics of Free Fall," University of Massachusetts, Boston, Elementary Physics 181.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 572

6. Recheck your measurements.
7. Clean your work area.

Results

Use your notebook to keep a record of your data, observations, graphs, and analysis of the data.

a. Record the data obtained from your measurements on the tape or from using a CBL® unit. If you are entering your data into a function graphing program or a spreadsheet, you can use a printout of the data and staple it into your laboratory notework. Your data should include time, t, and distance fallen, d, as in the following table:

t (sec)	d (cm)
0/60	0
1/60	–
2/60	–
....	–
To last record	–

This table assumes regular time intervals of one-sixtieth of a second. Check your equipment to see whether it uses a different interval size.

b. Note at which dot on the tape you started to make your measurements.

Analysis of data

1. By hand:

a. Graph your data, using the vertical axis for distance fallen, d, in centimeters and the horizontal axis for time, t, in seconds. What does your graph suggest about the average rate of change of distance with respect to time?

b. Calculate the average rate of change for distance, d, with respect to time, t, for three pairs of points from your data table:

$$\text{average rate of change} = \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta d}{\Delta t}$$

Show your work. This average rate of change is called the *average velocity* of the falling object between these two points. Do your calculations support your answer in part (a)?

c. Jot down your observations from your graph and calculations in your notebook. Staple your graph into your notebook.

2. With graphing calculators or computers:

a. Use technology to graph your data for the free fall experiment. Plot time, t, on the horizontal axis and distance fallen, d, on the vertical axis.

b. Find a best-fit function for distance fallen versus time.

c. Use your spreadsheet or graphing calculator to calculate the average rate of change in distance over each of the small time intervals. This average rate of change is the average velocity over each of these time intervals.

d. Plot average velocity versus time, with time on the horizontal axis and average velocity on the vertical axis.

e. Jot down your observations from your graphs and calculations in your notebook. Be sure to specify the units for any numbers you recorded.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 573

Conclusions

Summarize your conclusions from the experiment:

- Describe what you found out from your graph of distance vs. time and your calculations for the average rate of change of distance with respect to time. Is the average rate of change of distance with respect to time the same for each small time interval?
- What does your graph of the average velocity vs. time tell you about the average velocity of the freely falling body? Is the average rate of change in velocity from one interval to the next roughly constant?
- In light of the readings and class discussion, interpret your graphs for distance and average velocity and interpret the coefficients in the equation you found for distance.

In his own version of this experiment, Galileo sought to answer the following questions:

How can we describe mathematically the distance an object falls over time?

Do freely falling objects fall at a constant speed?

If the velocity of a freely falling object is not constant, is it increasing at a constant rate?

Use your results to answer these questions.

EXERCISES

(A graphing program is optional for many exercises and required for Exercise 30.)

1. Complete the accompanying table. What happens to the average velocity of the object as it falls?

Time (sec)	Distance Fallen (cm)	Average Velocity (average rate of change for the previous 1/30 of a second)
0.0000	0.00	n.a.
0.0333	3.75	$\frac{3.75 - 0.00}{0.0333 - 0.0000} \approx 113$ cm/sec
0.0667	8.67	$\frac{8.67 - 3.75}{0.0667 - 0.0333} \approx 147$ cm/sec
0.1000	14.71	
0.1333	21.77	
0.1667	29.90	

2. The essay "Watching Galileo's Learning" examines the learning process that Galileo went through to come to some of the most remarkable conclusions in the history of science. Write a summary of one of Galileo's conclusions about motion. Include in your summary the process by which Galileo made this discovery and some aspect of your own learning or understanding of Galileo's discovery.

3. (Graphing program optional.) The equation $d = 490t^2 + 50t$ describes the relationship between distance fallen, d, in centimeters, and time, t, in seconds, for a particular freely falling object.

a. Interpret each of the coefficients and specify its units of measurement.

b. Generate a table for a few values of t between 0 and 0.3 second.

c. Graph distance versus time by hand. Check your graph using a computer or graphing calculator if available.

4. A freely falling body has an initial velocity of 125 cm/sec. Assume that $g = 980$ cm/sec^2.

a. Write an equation that relates d, distance fallen in centimeters, to t, time in seconds.

b. How far has the body fallen after 1 second? After 3 seconds?

c. If the initial velocity were 75 cm/sec, how would your equation in part (a) change?

5. (Graphing program optional.) The equation $d = 4.9t^2 + 1.7t$ describes the relationship between distance fallen, d, in meters, and time, t, in seconds, for a particular freely falling object.

a. Interpret each of the coefficients and specify its units of measurement.

b. Generate a table for a few values of t between 0 and 0.3 second.

c. Graph distance versus time by hand. Check your graph using a computer or graphing calculator if available.

d. Relate your answers to earlier results in this chapter.

6. In the equation of motion $d = \frac{1}{2}gt^2 + v_0t$, we specified that distance was measured in centimeters, velocity in centimeters per second, and time in seconds. Rewrite this as an equation that shows only units of measure. Verify that you get centimeters = centimeters.

7. The equation $d = \frac{1}{2}gt^2 + v_0t$ could also be written using distances measured in meters. Rewrite the equation showing only units of measure and verify that you get meters = meters.

8. The equation $d = \frac{1}{2}gt^2 + v_0t$ could be written using distance measured in feet. Rewrite the equation showing only units of measure and verify that you get feet = feet.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 574

9. A freely falling object has an initial velocity of 50 cm/sec.

a. Write two motion equations, one relating distance and time and the other relating velocity and time.

b. How far has the object fallen and what is its velocity after 1 second? After 2.5 seconds? Be sure to identify units in your answers.

10. A freely falling object has an initial velocity of 20 ft/sec.

a. Write one equation relating distance fallen (in feet) and time (in seconds) and a second equation relating velocity (in feet per second) and time.

b. How many feet has the object fallen and what is its velocity after 0.5 second? After 2 seconds?

11. (Graphing program optional.) A freely falling object has an initial velocity of 12 ft/sec.

a. Construct an equation relating distance fallen and time.

b. Generate a table by hand for a few values of the distance fallen between 0 and 5 seconds.

c. Graph distance vs. time by hand. Check your graph using a computer or graphing calculator if available.

12. (Graphing program optional.) Use the information in Exercise 11 to do the following:

a. Construct an equation relating velocity and time.

b. Generate a table by hand for a few values of velocity between 0 and 5 seconds.

c. Graph velocity versus time by hand. If possible, check your graph using a computer or graphing calculator.

13. If the equation $d = 4.9t^2 + 11t$ represents the relationship between distance and time for a freely falling body, in what units is distance now being measured? How do you know?

14. The distance that a freely falling object with no initial velocity falls can be modeled by the quadratic function $d = 16t^2$, where t is measured in seconds and d in feet. There is a closely related function $v = 32t$ that gives the velocity, v, in feet per second at time t, for the same freely falling body.

a. Fill in the missing values in the following table:

Time, t (sec)	Distance, d (ft)	Velocity, v (ft/sec)
1		
1.5		
2		
		80
	144	

b. When $t = 3$, describe the associated values of d and v and what they tell you about the object at that time.

c. Sketch both functions, distance versus time and velocity versus time, on two different graphs. Label the points from part (b) on the curves.

d. You are standing on a bridge looking down at a river. How could you use a pebble to estimate how far you are above the water?

One screen in "Q11: Freely Falling Objects" in *Quadratic Functions* simulates this activity.

15. (This exercise requires a free fall data tape created using a spark timer.)

a. Make a graph from your tape: Cut the tape with scissors crosswise at each spark dot, so you have a set of strips of paper that are the actual lengths of the distances fallen by the object during each time interval. Arrange them evenly spaced in increasing order, with the bottom of each strip on a horizontal line. The end result should look like a series of steps. You could paste or tape them down on a big piece of paper or newspaper.

b. Use a straight edge to draw a line that passes through the center of the top of each strip. Is the line a good fit? Each separate strip represents the distance the object fell during a fixed time interval, so we can think of the strips as representing change in distance over time, or average velocity. Interpret the graph of the line you have constructed in terms of the free fall experiment.

16. In the Anthology Reading "Watching Galileo's Learning," Cavicchi notes that Galileo generated a sequence of odd integers from his study of falling bodies. Show that in general the odd integers can be constructed from the difference of the squares of successive integers, that is, that the terms $(n + 1)^2 - n^2$ (where $n = 0, 1, 2, 3, \ldots$) generate a sequence of all the positive odd integers.

17. The data from a free fall tape generate the following equation relating distance fallen in centimeters and time in seconds:

$$d = 485.7t^2 + 7.6t$$

a. Give a physical interpretation of each of the coefficients along with its appropriate units of measurement.

b. How far has the object fallen after 0.05 second? 0.10 second? 0.30 second?

18. What would the free fall equation $d = 490t^2 + 90t$ become if d were measured in feet instead of centimeters?

19. In the equation $d = 4.9t^2 + 500t$, time is measured in seconds and distance in meters. What does the number 500 represent?

20. In the height equation $h = 300 + 50t - 4.9t^2$, time is measured in seconds and height in meters.

a. What does the number 300 represent?

b. What does the number 50 represent? What does the fact that 50 is positive tell you?

21. (Graphing program optional.) The height of an object that was projected vertically from the ground with initial velocity of 200 m/sec is given by the equation $h = 200t - 4.9t^2$, where t is in seconds.

a. Find the height of the object after 0.1, 2, and 10 seconds.

b. Sketch a graph of height vs. time.

c. Use the graph to determine the maximum height of the projectile and the approximate number of seconds that the object traveled before hitting the ground.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 575

22. (Graphing program optional.) The height of an object that was shot downward from a 200-meter platform with an initial velocity of 50 m/sec is given by the equation $h = -4.9t^2 - 50t + 200$, where t is in seconds and h is in meters. Sketch the graph of height versus time. Use the graph to determine the approximate number of seconds that the object traveled before hitting the ground.

23. (Graphing program optional.) Let $h = 85 - 490t^2$ be a motion equation describing height, h, in centimeters and time, t, in seconds.

a. Interpret each of the coefficients and specify its units of measurement.

b. What is the initial velocity?

c. Generate a table for a few values of t between 0 and 0.3 second.

d. Graph height versus time by hand. Check your graph using a computer or graphing calculator if possible.

24. (Graphing program optional.) Let $h = 85 + 20t - 490t^2$ be a motion equation describing height, h, in centimeters and time, t, in seconds.

a. Interpret each of the coefficients and specify its units of measurement.

b. Generate a table for a few values of t between 0 and 0.3 second.

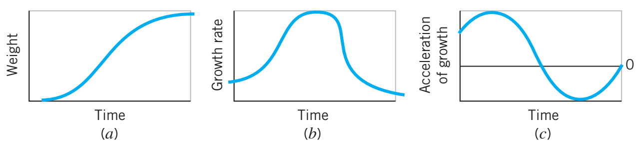
c. Graph height versus time by hand. Check your graph using a computer or graphing calculator if possible.

25. At $t = 0$, a ball is thrown upward at a velocity of 10 ft/sec from the top of a building 50 feet high. The ball's height is measured in feet above the ground.

a. Is the initial velocity positive or negative? Why?

b. Write the motion equation that describes height, h, at time, t.

26. The concepts of velocity and acceleration are useful in the study of human childhood development. The accompanying figure shows (a) a standard growth curve of weight over time, (b) the rate of change of weight over time (the *growth rate* or *velocity*), and (c) the rate of change of the growth rate over time (or *acceleration*). Describe in your own words what each of the graphs shows about a child's growth.



Source: Adapted from B. Bogin, "The Evolution of Human Childhood," *BioScience,* Vol. 40, p. 16.

27. The relationship between the velocity of a freely falling object and time is given by

$$v = -gt - 66$$

where g is the acceleration due to gravity and the units for velocity are centimeters per second.

a. What value for g should be used in the equation?

b. Generate a table of values for t and v, letting t range from 0 to 4 seconds.

c. Graph velocity vs. time by hand and interpret your graph.

d. What was the initial condition? Was the object dropped or thrown? Explain your reasoning.

28. A certain baseball is at height $h = 4 + 64t - 16t^2$ feet at time t in seconds. Compute the average velocity over each of the following time intervals and indicate for which intervals the baseball is rising and for which it is falling. In which interval was the average velocity the greatest?

a. $t = 0$ to $t = 0.5$

b. $t = 0$ to $t = 0.1$

c. $t = 0$ to $t = 1$

d. $t = 1$ to $t = 2$

e. $t = 2$ to $t = 3$

f. $t = 1$ to $t = 3$

g. $t = 4$ to $t = 4.01$

29. At $t = 0$, an object is in free fall 150 cm above the ground, falling at a rate of 25 cm/sec. Its height, h, is measured in centimeters above the ground.

a. Is its velocity positive or negative? Why?

b. Construct an equation that describes its height, h, at time t.

c. What is the average velocity from $t = 0$ to $t = \frac{1}{2}$? How does it compare with the initial velocity?

30. (Graphing program required.) *The force of acceleration on other planets.* We have seen that the function $d = \frac{1}{2}gt^2 + v_0 t$ (where g is the acceleration due to Earth's gravity and v_0 is the object's initial velocity) is a mathematical model for the relationship between time and distance fallen by freely falling bodies near Earth's surface. This relationship also holds for freely falling bodies near the surfaces of other planets. We just replace g, the acceleration of Earth's gravitational field, with the acceleration for the planet under consideration. The following table gives the acceleration due to gravity for planets in our solar system:

Acceleration Due to Gravity

	m/sec^2	ft/sec^2
Mercury	3.7	12.1
Venus	8.9	29.1
Earth	9.8	32.1
Mars	3.7	12.1
Jupiter	24.8	81.3
Saturn	10.4	34.1
Uranus	8.5	27.9
Neptune	11.6	38.1

Note: Pluto is no longer considered a planet.
Source: The Astronomical Almanac, U.S. Naval Observatory, 1981.

a. Choose units of measurement (meters or feet) and three of the planets (other than Earth). For each of these planets, find an equation for the relationship between the distance an object falls and time. Construct a table as shown at the top of the next page. Assume for the moment that the initial velocity of the freely falling object is 0.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 576

Name of Planet	Function Relating Distance and Time (sec)	Units for Distance

b. Using a graphing program, plot the three functions, with time on the horizontal axis and distance on the vertical axis. What domain makes sense for your models? Why?

c. On which of your planets will an object fall the farthest in a given time? On which will it fall the least distance in a given time?

d. Examine the graphs and think about the *similarities* that they share. Describe their general shape. What happens to d as the value for t increases?

e. Think about the *differences* among the three curves. What effect does the coefficient of the t^2 term have on the shape of the graph; that is, when the coefficient gets larger (or smaller), how is the shape of the curve affected? Which graph shows d increasing the fastest compared with t?

31. Suppose an object is moving with constant acceleration, a, and its motion is initially observed at a moment when its velocity is v_0. We set time, t, equal to 0, at this point when velocity equals v_0. Then its velocity t seconds after the initial observation is $V(t) = at + v_0$. (Note that the product of acceleration and time is velocity.) Now suppose we want to find its average velocity between time 0 and time t. The average velocity can be measured in two ways. First, we can find the average of the initial and final velocities by calculating a numerical average or mean; that is, we add the two velocities and divide by 2. So, between time 0 and time t,

$$\text{average velocity} = \frac{v_0 + V(t)}{2} \qquad (1)$$

We can also find the average velocity by dividing the change in distance by the change in time. Thus, between time 0 and time t,

$$\text{average velocity} = \frac{\Delta\text{distance}}{\Delta\text{time}} = \frac{d-0}{t-0} = \frac{d}{t} \qquad (2)$$

If we substitute the expression for average velocity (from time 0 to time t) given by Equation (1) into Equation (2), we get

$$\frac{d}{t} = \frac{v_0 + V(t)}{2} \qquad (3)$$

We know that $V(t) = at + v_0$. Substitute this expression for $V(t)$ in Equation (3) and solve for d. Interpret your results.

32. In 1974 in Anaheim, California, Nolan Ryan threw a baseball at just over 100 mph. If he had thrown the ball straight upward at this speed, it would have risen to a height of over 335 feet and taken just over 9 seconds to fall back to Earth. Choose another planet and see what would have happened if he had been able to throw a baseball straight up at 100 mph on that planet. In your computations, use the table for the acceleration due to gravity on other planets from Exercise 30.

33. An object that is moving horizontally along the ground is observed to have (an initial) velocity of 60 cm/sec and to be accelerating at a constant rate of 10 cm/sec^2.

a. Determine its velocity after 5 seconds, after 60 seconds, and after t seconds.

b. Find the average velocity for the object between 0 and 5 seconds.

34. (Requires results from Exercise 33.) Find the distance traveled by the object described in the previous exercise after 5 seconds by using two different methods.

a. Use the formula distance = rate · time. For the rate, use the average velocity found in Exercise 33(b). For time, use 5 seconds.

b. Write an equation of motion $d = \frac{1}{2}at^2 + v_0 t$ using $a = 10$ cm/sec^2 and $v_0 = 60$ cm/sec and evaluate when $t = 5$. Does your answer agree with part (a)?

35. An object is observed to have an initial velocity of 200 m/sec and to be accelerating at 60 m/sec^2.

a. Write an equation for its velocity after t seconds.

b. Write an equation for the distance traveled after t seconds.

36. You may have noticed that when a basketball player or dancer jumps straight up in the air, in the middle of a blurred impression of vertical movement, the jumper appears to "hang" for an instant at the top of the jump.

a. If a player jumps 3 feet straight up, generate equations that describe his height above ground and velocity during his jump. What initial upward velocity must the player have to achieve a 3-foot-high jump?

b. How long does the total jump take from takeoff to landing? What is the player's downward velocity at landing?

c. How much vertical distance is traveled in the first third of the total time that the jump takes? In the middle third? In the last third?

d. Now explain in words why it is that the jumper appears suspended in space at the top of the jump.

37. Old Faithful, the most famous geyser at Yellowstone National Park, regularly shoots up a jet of water 120 feet high.

a. At what speed must the stream of water be traveling out of the ground to go that high?

b. How long does it take to reach its maximum height?

38. A vehicle trip is composed of the following parts:

i. Accelerate from 0 to 30 mph in 1 minute.

ii. Travel at 30 mph for 12 minutes.

iii. Accelerate from 30 to 50 mph in $\frac{1}{2}$ minute.

iv. Travel at 50 mph for 6 minutes.

v. Decelerate from 50 to 0 mph in $\frac{1}{2}$ minute.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 577

a. Sketch a graph of speed versus time for the trip.

b. What are the average velocities for parts (i), (iii), and (v) of the trip?

c. How much distance is covered in each part of the trip, and what is the total trip distance?

39. In general, for straight motion of a vehicle with constant acceleration, a, the velocity, v, at any time, t, is the original velocity, v_0, plus acceleration multiplied by time: $v = v_0 + at$. The distance traveled in time t is $d = v_0 t + \frac{1}{2}at^2$.

a. A criminal going at speed v_c passes a police car and immediately accelerates with constant acceleration a_c. If the police car has constant acceleration $a_p > a_c$, starting from 0 mph, how long will it take to pass the criminal? Give t in terms of v_c, a_p, and a_c.

b. At what time are the police and the criminal traveling at the same speed? If they are traveling at the same speed, does it mean the police have caught up with the criminal? Explain.

Kime07_Math on Mot_559-578.qxd 10/11/07 9:10 AM Page 578