## Stability Analysis of Slopes

## 14-1 INTRODUCTION

Whenever a mass of soil has an inclined surface, the potential always exists for part of the soil mass to slide from a higher location to a lower one. Sliding will occur if shear stresses developed in the soil exceed the corresponding shear strength of the soil. This phenomenon is of importance in the case of highway cuts and fills, embankments, earthen dams, and so on.

This principle-that sliding will occur if shear stresses developed in the soil exceed the corresponding shear strength the soil possesses-is simple in theory; however, certain practical considerations make precise stability analyses of slopes difficult in practice. In the first place, sliding may occur along any of a number of possible surfaces. In the second place, a given soil's shear strength generally varies throughout time, as soil moisture and other factors change. Obviously, stability analysis should be based on the smallest shear strength a soil will ever have in the future. This is difficult, if not impossible, to ascertain. It is, therefore, normal in practice to use appropriate safety factors when one is analyzing slope stability.

There are several techniques available for stability analysis. Section 14-2 covers the analysis of a soil mass resting on an inclined layer of impermeable soil. Section 14-3 discusses slopes in homogeneous cohesionless soils. Section 14-4 gives two methods of analyzing stability for homogeneous soils that have cohesion. The first is known as the Culmann method. It is applicable to only vertical, or nearly vertical, slopes. The second might be called the stability number method. Section 14-5 presents the method of slices.

## 14-2 ANALYSIS OF A MASS RESTING ON AN INCLINED LAYER OF IMPERMEABLE SOIL

One situation for which slope stability analysis is fairly simple is that of a soil mass resting on an inclined layer of impermeable soil (see Figure 14-1). There exists a tendency for the upper mass to slide downward along its plane of contact with the lower layer of impermeable soil.

The force tending to cause sliding is the component of the upper mass's weight along the plane of contact. By referring to Figure 14-2 and considering a unit width of slope (i.e., perpendicular to wedge $a b c$ ), one can compute the upper mass's weight ( $W$ ) (i.e., weight of wedge $a b c$ ) by using the following equation:

$$
\begin{equation*}
W=\frac{L h \gamma}{2} \tag{14-1}
\end{equation*}
$$

where $\gamma$ is the unit weight of the upper mass. Hence, the force tending to cause sliding $\left(F_{s}\right)$ is given by the following equation:

$$
\begin{equation*}
F_{s}=W \sin \alpha \tag{14-2}
\end{equation*}
$$



FIGURE 14-1 Sketch showing soil mass resting on an inclined layer of impermeable soil.


FIGURE 14-2 Sketch showing forces acting on inclined layer of impermeable soil.

Forces that resist sliding result from cohesion and friction. In quantitative terms, the cohesion (i.e., adhesion) component is the product of the soil's cohesion (c) times the length of the plane of contact ( $L$ in Figure 14-2). The friction component is obtained by multiplying the coefficient of friction between the two strata $(\tan \phi)$ by the component of the upper mass's weight that is perpendicular to the plane of contact $(W \cos \alpha)$. Hence, the resistance (to sliding) force, $R_{s^{\prime}}$, is given by the following equation:

$$
\begin{equation*}
R_{s}=c L+W \cos \alpha \tan \phi \tag{14-3}
\end{equation*}
$$

where $\phi$ is the angle of friction between the upper mass and the lower layer of impermeable soil.

The factor of safety (F.S.) against sliding is determined by dividing the resistance (to sliding) force, $R_{s}$ [Eq. (14-3)], by the sliding force, $F_{s}$ [Eq. (14-2)]. Hence,

$$
\begin{equation*}
\text { F.S. }=\frac{c L+W \cos \alpha \tan \phi}{W \sin \alpha} \tag{14-4}
\end{equation*}
$$

Figure 14-3 gives the formulation required to evaluate $L$ and $h$, which are needed in applying Eqs. (14-1) and (14-4). Table 14-1 gives the significance of factors of


FIGURE 14-3 Sketch showing formulation required to evaluate $L$ and $h$.
TABLE 14-1
Safety Factor Design Significance

| Safety Factor | Significance |
| :--- | :--- |
| Less than 1.0 | Unsafe |
| 1.0 to 1.2 | Questionable safety |
| 1.3 to 1.4 | Satisfactory for cuts, fills; questionable for dams |
| 1.5 to 1.75 | Safe for dams |

safety against sliding for design. Example 14-1 illustrates the computation of the factor of safety for stability analysis of a soil mass resting on an inclined layer of impermeable soil.

## EXAMPLE 14-1

## Given

1. Figure $14-4$ shows a $15-\mathrm{ft}$ cut through two soil strata. The lower is a highly impermeable cohesive soil.
2. Shearing strength data between the two strata are as follows:

$$
\begin{aligned}
\text { Cohesion } & =150 \mathrm{lb} / \mathrm{ft}^{2} \\
\text { Angle of friction } & =25^{\circ} \\
\text { Unit weight of upper layer } & =105 \mathrm{lb} / \mathrm{ft}^{3}
\end{aligned}
$$

3. Neglect the effects of soil water between the two strata.

## Required

Factor of safety against sliding.

## Solution

From Figure 14-3,

$$
\begin{aligned}
& L=\frac{D}{\sin \alpha} \\
& L=\frac{10 \mathrm{ft}}{\sin 30^{\circ}}=20.0 \mathrm{ft}
\end{aligned}
$$



FIGURE 14-4

Again, from Figure 14-3,

$$
\begin{aligned}
& h=\left(\frac{D}{\sin \beta}\right) \sin (\beta-\alpha) \\
& h=\left(\frac{10 \mathrm{ft}}{\sin 45^{\circ}}\right) \sin \left(45^{\circ}-30^{\circ}\right)=3.66 \mathrm{ft}
\end{aligned}
$$

From Eq. (14-1),

$$
\begin{align*}
& W=\frac{L h \gamma}{2}  \tag{14-1}\\
& W=\frac{(20.0 \mathrm{ft})(3.66 \mathrm{ft})\left(105 \mathrm{lb} / \mathrm{ft}^{3}\right)}{2}=3843 \mathrm{lb} / \mathrm{ft}
\end{align*}
$$

From Eq. (14-4),

$$
\begin{align*}
\text { F.S. } & =\frac{c L+W \cos \alpha \tan \phi}{W \sin \alpha}  \tag{14-4}\\
\text { F.S. } & =\frac{\left(150 \mathrm{lb} / \mathrm{ft}^{2}\right)(20.0 \mathrm{ft})+(3843 \mathrm{lb} / \mathrm{ft})\left(\cos 30^{\circ}\right)\left(\tan 25^{\circ}\right)}{(3843 \mathrm{lb} / \mathrm{ft})\left(\sin 30^{\circ}\right)} \\
& =2.37>1.5 \quad \therefore \text { O.K. }
\end{align*}
$$

## 14-3 SLOPES IN HOMOGENEOUS COHESIONLESS SOILS ( $c=0, \Phi>0$ )

When the slope angle $(\beta)$ of a sand slope exceeds the sand's angle of internal friction $(\phi)$, the sand slope tends to fail by sliding in a downhill direction parallel to the slope. This phenomenon can be inferred by visualizing individual grains of sand being blocks resting on an inclined plane at the slope angle. If the slope angle is increased, the sand grains will begin to slide down the slope when the slope angle exceeds the sand's $\phi$ angle. Accordingly, the greatest slope for a free-standing cohesionless soil is at an angle approximately equal to the soil's $\phi$ angle.

The slope angle at which a loose sand fails may be estimated by its angle of repose, the angle formed (with the horizontal) by sand as it forms a pile below a funnel through which it passes. A sand's angle of repose is roughly equal to its angle of internal friction in a loose condition, and sand at or near ground surface is ordinarily in a loose condition and therefore near its maximum value of $\phi$.

The factor of safety for slopes in homogeneous cohesionless soils is given by the following equation:

$$
\begin{equation*}
\text { F.S. }=\frac{\tan \phi}{\tan \beta} \tag{14-5}
\end{equation*}
$$

Clearly, when slope angle $\beta$ equals angle of internal friction $\phi$, the factor of safety is 1 . For slopes with $\beta$ less than $\phi$, the factor of safety is greater than 1 .

## 14-4 SLOPES IN HOMOGENEOUS SOILS POSSESSING COHESION ( $c>0, \Phi=0$, and $c>0, \phi>0$ )

In this section, two methods are presented for analyzing slope stability in homogeneous soils possessing cohesion. One is known as the Culmann method; the other might be called the stability number method.

## Culmann Method

In the Culmann method, the assumption is made that failure (sliding) will occur along a plane that passes through the toe of the fill. Such a plane is indicated in Figure 14-5. As with the analysis of a mass resting on an inclined layer of impermeable soil (Section 14-2), the force tending to cause sliding is given by Eq. (14-2):

$$
\begin{equation*}
F_{s}=W \sin \alpha \tag{14-2}
\end{equation*}
$$

Also similarly, resistance to sliding results from cohesion and friction and is given by Eq. (14-3):

$$
\begin{equation*}
R_{s}=c_{d} L+W \cos \alpha \tan \phi_{d} \tag{14-3}
\end{equation*}
$$

where $c_{d}$ is the developed cohesion ( $c /$ F.S. ${ }_{c}$ ), $\tan \phi_{d}$ is the developed coefficient of friction $\left(\tan \phi /\right.$ F.S. $\left._{\phi}\right)$ ), and the other terms are as defined in Figure 14-5. (F.S. ${ }_{c}$ and F.S. ${ }_{\phi}$ denote factors of safety for cohesion and angle of internal friction, respectively.) As in Section 14-2, the weight of soil in the upper triangle $a b c(W)$ can be computed by using Eq. (14-1):

$$
\begin{equation*}
W=\frac{L h \gamma}{2} \tag{14-1}
\end{equation*}
$$



FIGURE 14-5 Sketch showing assumed failure plane in the Culmann method.
but $h$, the height of the triangle, can be evaluated as follows:

$$
\begin{equation*}
h=\left(\frac{H}{\sin \beta}\right) \sin (\beta-\alpha) \tag{14-6}
\end{equation*}
$$

Substituting Eq. (14-6) into Eq. (14-1) gives the following:

$$
\begin{equation*}
W=\left(\frac{1}{2}\right) L\left(\frac{H}{\sin \beta}\right) \sin (\beta-\alpha)(\gamma) \tag{14-7}
\end{equation*}
$$

Equating Eqs. (14-2) and (14-3), substituting $W$ from Eq. (14-7) into the new equation, and then solving for $c_{d}$ gives the following:

$$
\begin{equation*}
c_{d}=\left(\frac{\gamma H}{2 \sin \beta}\right)\left[\frac{\sin (\beta-\alpha) \sin \left(\alpha-\phi_{d}\right)}{\cos \phi_{d}}\right] \tag{14-8}
\end{equation*}
$$

The critical angle for $\alpha$ (i.e., $\alpha_{c}$ ) can be determined by equating the first derivative of $c_{d}$ with respect to $\alpha$ to zero [i.e., $d\left(c_{d}\right) / d(\alpha)=0$ ] and solving for $\alpha$. The result of this operation is as follows:

$$
\begin{equation*}
\alpha_{c}=\frac{\beta+\phi_{d}}{2} \tag{14-9}
\end{equation*}
$$

Substituting $\alpha_{c}$ from Eq. (14-9) into Eq. (14-8) for $\alpha$ gives the following:

$$
\begin{equation*}
c_{d}=\frac{\gamma H\left[1-\cos \left(\beta-\phi_{d}\right)\right]}{4 \sin \beta \cos \phi_{d}} \tag{14-10}
\end{equation*}
$$

Solving for $H$ gives the following:

$$
\begin{equation*}
H=\frac{4 c_{d} \sin \beta \cos \phi_{d}}{\gamma\left[1-\cos \left(\beta-\phi_{d}\right)\right]} \tag{14-11}
\end{equation*}
$$

where $H=$ safe depth of cut
$c_{d}=$ developed cohesion
$\beta=$ angle from horizontal to cut surface (Figure 14-5)
$\phi_{d}=$ developed angle of internal friction of the soil
$\gamma=$ unit weight of the soil
In using Eq. (14-11) to compute the safe depth of a cut, one must determine developed cohesion $\left(c_{d}\right)$ and the developed angle of internal friction ( $\phi_{d}$ ) by dividing cohesion and the tangent of the angle of internal friction by their respective safety factors.

The Culmann method gives reasonably accurate results if the slope is vertical or nearly vertical (i.e., angle $\beta$ is equal to, or nearly equal to, $90^{\circ}$ ) (Taylor, 1948). Examples 14-2 and 14-3 illustrate the Culmann method.

## EXAMPLE 14-2

## Given

1. A vertical cut is to be made through a soil mass.
2. The soil to be cut has the following properties:

$$
\begin{aligned}
\text { Unit weight }(\gamma) & =105 \mathrm{lb} / \mathrm{ft}^{3} \\
\text { Cohesion }(c) & =500 \mathrm{lb} / \mathrm{ft}^{2} \\
\text { Angle of internal friction }(\phi) & =21^{\circ}
\end{aligned}
$$

## Required

Safe depth of cut in this soil, by the Culmann method, using a factor of safety of 2.

## Solution

From Eq. (14-11),

$$
\begin{equation*}
H=\frac{4 c_{d} \sin \beta \cos \phi_{d}}{\gamma\left[1-\cos \left(\beta-\phi_{d}\right)\right]} \tag{14-11}
\end{equation*}
$$

Here,

$$
\begin{aligned}
c_{d}= & \frac{c}{\text { F.S. }_{c}}=\frac{500 \mathrm{lb} / \mathrm{ft}^{2}}{2}=250 \mathrm{lb} / \mathrm{ft}^{2} \\
& \left(\text { F.S. }{ }_{c} \text { is the factor of safety with respect to cohesion }\right) \\
\tan \phi_{d}= & \frac{\tan \phi}{\text { F.S. }_{\phi}}=\frac{\tan 21^{\circ}}{2}=0.192 \\
& \quad\left(\text { F.S. }{ }_{\phi} \text { is the factor of safety with respect to } \tan \phi\right) \\
\phi_{d}= & \arctan 0.192=10.87^{\circ} \\
\beta= & 90^{\circ} \quad(\text { vertical cut }) \\
H= & \frac{(4)\left(250 \mathrm{lb} / \mathrm{ft}^{2}\right) \sin 90^{\circ} \cos 10.87^{\circ}}{\left(105 \mathrm{lb} / \mathrm{ft}^{3}\right)\left[1-\cos \left(90^{\circ}-10.87^{\circ}\right)\right]}=11.5 \mathrm{ft}
\end{aligned}
$$

## EXAMPLE 14-3

## Given

A $1.8-\mathrm{m}$-deep vertical-wall trench is to be dug in soil without shoring. The soil's unit weight, angle of internal friction, and cohesion are $19.0 \mathrm{kN} / \mathrm{m}^{3}, 28^{\circ}$, and $20.2 \mathrm{kN} / \mathrm{m}^{2}$, respectively.

## Required

Factor of safety of this trench, using the Culmann method.

## Solution

From Eq. (14-11),

$$
\begin{equation*}
H=\frac{4 c_{d} \sin \beta \cos \phi_{d}}{\gamma\left[1-\cos \left(\beta-\phi_{d}\right)\right]} \tag{14-11}
\end{equation*}
$$

$\operatorname{Try}$ F.S. ${ }_{\phi}=1.0$

$$
\tan \phi_{d}=\frac{\tan \phi}{\text { F.S. }_{\phi}}=\frac{\tan 28^{\circ}}{1.0}=\tan 28^{\circ}
$$

Therefore, $\phi_{d}=28^{\circ}$

$$
\beta=90^{\circ} \quad \text { (for a vertical wall) }
$$

Substituting into Eq. (14-11) yields the following:

$$
\begin{aligned}
1.8 \mathrm{~m} & =\frac{(4)\left(c_{d}\right) \sin 90^{\circ} \cos 28^{\circ}}{\left(19.0 \mathrm{kN} / \mathrm{m}^{3}\right)\left[1-\cos \left(90^{\circ}-28^{\circ}\right)\right]} \\
c_{d} & =5.14 \mathrm{kN} / \mathrm{m}^{2} \\
\text { F.S. }_{c} & =\frac{c}{c_{d}}=\frac{20.2 \mathrm{kN} / \mathrm{m}^{2}}{5.14 \mathrm{kN} / \mathrm{m}^{2}}=3.93
\end{aligned}
$$

Because [F.S $\left.{ }_{c}=3.93\right] \neq\left[\right.$ F.S $\left._{\phi}=1.0\right]$, another trial factor of safety must be attempted.

Try F.S. ${ }_{\phi}=2.0$

$$
\tan \phi_{d}=\frac{\tan \phi}{\text { F.S. }_{\phi}}=\frac{\tan 28^{\circ}}{2.0}=0.2659
$$

Therefore, $\phi_{d}=14.89^{\circ}$

$$
\begin{aligned}
1.8 \mathrm{~m} & =\frac{(4)\left(c_{d}\right) \sin 90^{\circ} \cos 14.89^{\circ}}{\left(19.0 \mathrm{kN} / \mathrm{m}^{3}\right)\left[1-\cos \left(90^{\circ}-14.89^{\circ}\right)\right]} \\
c_{d} & =6.57 \mathrm{kN} / \mathrm{m}^{2} \\
\text { F.S. }_{c} & =\frac{c}{c_{d}}=\frac{20.2 \mathrm{kN} / \mathrm{m}^{2}}{6.57 \mathrm{kN} / \mathrm{m}^{2}}=3.07
\end{aligned}
$$

Because $\left[\right.$ F.S. $\left.{ }_{c}=3.07\right] \neq\left[\right.$ F.S. $\left._{\phi}=2.0\right]$, another trial factor of safety must be attempted.

Try F.S. ${ }_{\phi}=3.0$

$$
\tan \phi_{d}=\frac{\tan \phi}{\text { F.S. }_{\phi}}=\frac{\tan 28^{\circ}}{3.0}=0.1772
$$

Therefore, $\phi_{d}=10.05^{\circ}$

$$
\begin{aligned}
1.8 \mathrm{~m} & =\frac{(4)\left(c_{d}\right) \sin 90^{\circ} \cos 10.05^{\circ}}{\left(19.0 \mathrm{kN} / \mathrm{m}^{3}\right)\left[1-\cos \left(90^{\circ}-10.05^{\circ}\right)\right]} \\
c_{d} & =7.17 \mathrm{kN} / \mathrm{m}^{2} \\
\text { F.S }_{c} & =\frac{c}{c_{d}}=\frac{20.2 \mathrm{kN} / \mathrm{m}^{2}}{7.17 \mathrm{kN} / \mathrm{m}^{2}}=2.82
\end{aligned}
$$

Because [F.S. $\left.{ }_{c}=2.82\right] \neq\left[\right.$ F.S $\left._{{ }_{\phi}}=3.0\right]$, the correct factor of safety has not yet been found. Rather than continue this trial-and-error procedure, the values of F.S. ${ }_{c}$ and F.S. ${ }_{\phi}$ are plotted in Figure 14-6, from which the applicable factor of safety of about 2.84 can be read.


FIGURE 14-6

## Stability Number Method

The stability number method is also based on the premise that resistance of a soil mass to sliding results from cohesion and internal friction of the soil along the failure surface. Unlike the Culmann method, in this method the failure surface is assumed to be a circular arc (see Figure 14-7). A parameter called the stability number is introduced, which groups factors affecting the stability of soil slopes. The stability number $\left(N_{s}\right)$ is defined as follows:

$$
\begin{equation*}
N_{s}=\frac{\gamma H}{c} \tag{14-12}
\end{equation*}
$$

where $\quad \gamma=$ unit weight of soil
$H=$ height of cut (Figure 14-7)
$c=$ cohesion of soil
For the embankment illustrated in Figure 14-7, three types of failure surfaces are possible. These are shown in Figure 14-8. For the toe circle (Figure 14-8a), the failure surface passes through the toe. In the case of the slope circle (Figure 14-8b), the failure surface intersects the slope above the toe. For the midpoint circle (Figure $14-8 \mathrm{c}$ ), the center of the failure surface is on a vertical line passing through the midpoint of the slope.

Both the type of failure surface and the stability number can be determined for a specific case based on given values of $\phi$ (angle of internal friction) and $\beta$ (slope angle, Figure 14-7). If the value of $\phi$ is zero, or nearly zero, Figure 14-9 may be used to determine both the type of failure surface and the stability number. One enters along the abscissa at the value of $\beta$ and moves upward to the line that indicates the appropriate value of $n_{d}$. ( $n_{d}$ is a depth factor related to the distance to the underlying layer of stiff material or bedrock and is determined from the relationship indicated in Figure 14-8a.) The type of line for $n_{d}$ indicates the type of failure surface, and the value of stability number is determined by moving leftward and reading from the ordinate. Observation of Figure 14-9 indicates that if $\beta$ is greater than $53^{\circ}$, the


FIGURE 14-7 Sketch showing assumed failure surface as a circular arc.


FIGURE 14-8 Types of failure surfaces: (a) toe circle; (b) slope circle; (c) midpoint circle.
failure surface is always a toe circle, and if $n_{d}$ is greater than 4 , the failure surface is always a midpoint circle ( $\mathrm{Wu}, 1976$ ).

If the value of $\phi$ is greater than $3^{\circ}$, the failure surface is always a toe circle ( Wu , 1976). Figure 14-10 may be used to determine the stability number for different values of $\phi$. One enters along the abscissa at the value of $\beta$, moves upward to the line that indicates the $\phi$ angle, and then leftward to the ordinate where the stability number is read.

The factor of safety for highly cohesive soils (that have $\phi=0$ ) can be obtained from Figure 14-9. This is illustrated in Example 14-5. For soils possessing cohesion and having $\phi>0$, the procedure is more complicated. One procedure is to estimate F.S. ${ }_{\phi}$ and determine $\phi_{\text {required }}$. Using this value and slope angle $\beta$, one can find the stability number from Figure $14-10$. With this stability number, $c_{\text {required }}$ can be computed from Eq. $(14-12)$. F.S. ${ }_{c}$ is the quotient of $c_{\text {given }}$ divided by $c_{\text {required. }}$. If F.S. ${ }_{\phi}$ equals F.S..$_{c^{\prime}}$ the overall factor of safety is equal to F.S..$_{\phi}$ (or F.S. ${ }_{c}$ ). If F.S. ${ }_{\phi}$ and F.S. . are not equal, additional values of F.S. ${ }_{\phi}$ can be estimated and the preceding procedure repeated to determine corresponding values of F.S. ${ }_{c}$ until the factor of safety is found where F.S. ${ }_{\phi}$ equals F.S. ${ }_{c}$. If the correct factor of safety has not been found after several such trials, it may be expedient to plot corresponding values of F.S. ${ }_{\phi}$ and F.S. ${ }_{c}$ on a graph, from which the overall factor of safety (i.e., where F.S. ${ }_{\phi}$ equals F.S. ${ }_{c}$ ) can be read. This procedure is illustrated in Example 14-4.


FIGURE 14-9 Stability numbers and types of slope failures for $\phi=0$.
Source: D. W. Taylor, "Stability of Earth Slopes." J. Boston Soc. Civil Eng., 24 (1937), and K. Terzaghi and R. B. Peck, Soil Mechanics in Engineering Practice, 2nd ed. Copyright $\odot^{\odot} 1967$ by John Wiley \& Sons, Inc. Reprinted by permission of John Wiley \& Sons, Inc.

EXAMPLE 14-4

## Given

The slope and data shown in Figure 14-11.

## Required

Factor of safety against failure, by the stability number method.

## Solution

Because the given angle of internal friction $(\phi)$ of $10^{\circ}$ is greater than $3^{\circ}$, the failure surface will be a toe circle.
Try F.S $._{\phi}=1$

$$
\begin{aligned}
\tan \phi_{\text {required }} & =\frac{\tan \phi_{\text {given }}}{\text { F.S. }_{\phi_{\phi}}}=\frac{\tan 10^{\circ}}{1} \\
\phi_{\text {required }} & =10^{\circ}
\end{aligned}
$$



FIGURE 14-10 Stability numbers for soils having cohesion and friction.
Source: D. W. Taylor, "Stability of Earth Slopes," J. Boston Soc. Civil Eng., 24 (1937), and K. Terzaghi and R. B. Peck, Soil Mechanics in Engineering Practice, 2nd ed. Copyright ${ }^{\circledR} 1967$ by John Wiley \& Sons, Inc. Reprinted by permission of John Wiley \& Sons, Inc.


FIGURE 14-11

With $\phi_{\text {required }}=10^{\circ}$ and $\beta=45^{\circ}$, from Figure 14-10,

$$
\begin{align*}
N_{s} & =9.2 \\
N_{s} & =\frac{\gamma H}{c}  \tag{14-12}\\
\gamma & =120 \mathrm{lb} / \mathrm{ft}^{3} \\
H & =30 \mathrm{ft} \\
9.2 & =\frac{\left(120 \mathrm{lb} / \mathrm{ft}^{3}\right)(30 \mathrm{ft})}{c_{\text {required }}} \\
c_{\text {required }} & =391 \mathrm{lb} / \mathrm{ft}^{2} \\
\text { F.S }_{c_{c}} & =\frac{c_{\text {given }}}{c_{\text {required }}}=\frac{600 \mathrm{lb} / \mathrm{ft}^{2}}{391 \mathrm{lb} / \mathrm{ft}^{2}}=1.53
\end{align*}
$$

Because F.S..$_{\phi}$ and F.S. ${ }_{c}$ are not the same value, another value of F.S. ${ }_{\phi}$ must be tried.
Try F.S. ${ }_{\phi}=1.2$

$$
\begin{aligned}
\tan \phi_{\text {required }} & =\frac{\tan \phi_{\text {given }}}{\mathrm{F} . \mathrm{S}_{\phi}}=\frac{\tan 10^{\circ}}{1.2}=0.147 \\
\phi_{\text {required }} & =8.36^{\circ}
\end{aligned}
$$

With $\phi_{\text {required }}=8.36^{\circ}$ and $\beta=45^{\circ}$, from Figure 14-10,

$$
\begin{aligned}
N_{s} & =8.6 \\
c_{\text {required }} & =\frac{\left(120 \mathrm{lb} / \mathrm{ft}^{3}\right)(30 \mathrm{ft})}{8.6}=419 \mathrm{lb} / \mathrm{ft}^{2} \\
\text { F.S. }{ }_{c} & =\frac{c_{\text {given }}}{c_{\text {required }}}=\frac{600 \mathrm{lb} / \mathrm{ft}^{2}}{419 \mathrm{lb} / \mathrm{ft}^{2}}=1.43
\end{aligned}
$$

Again, F.S. ${ }_{\phi}$ and F.S. ${ }_{c}$ are not the same value; hence, another value of F.S. ${ }_{\phi}$ must be tried.
Try F.S. ${ }_{\phi}=1.5$

$$
\begin{aligned}
\tan \phi_{\text {required }} & =\frac{\tan \phi_{\text {given }}}{\text { F.S. }}=\frac{\tan 10^{\circ}}{1.5}=0.118 \\
\phi_{\text {required }} & =6.73^{\circ}
\end{aligned}
$$

With $\phi_{\text {required }}=6.73^{\circ}$ and $\beta=45^{\circ}$, from Figure 14-10,


FIGURE 14-12

$$
\begin{aligned}
N_{s} & =7.9 \\
c_{\text {required }} & =\frac{\left(120 \mathrm{lb} / \mathrm{ft}^{3}\right)(30 \mathrm{ft})}{7.9}=456 \mathrm{lb} / \mathrm{ft}^{2} \\
\text { F.S. }_{c} & =\frac{c_{\text {given }}}{c_{\text {required }}}=\frac{600 \mathrm{lb} / \mathrm{ft}^{2}}{456 \mathrm{lb} / \mathrm{ft}^{2}}=1.32
\end{aligned}
$$

Again, F.S. ${ }_{\phi}$ and F.S. ${ }_{c}$ are not the same value. Rather than continue a trial-and-error solution, plot the values computed. From Figure 14-12, the factor of safety of the slope against failure is observed to be 1.36 .

## EXAMPLE 14-5

## Given

1. A cut 25 ft deep is to be made in a stratum of highly cohesive soil (see Figure 14-13).
2. The slope angle $\beta$ is $30^{\circ}$.
3. Soil exploration indicated that bedrock is located 40 ft below the original ground surface.
4. The soil has a unit weight of $120 \mathrm{lb} / \mathrm{ft}^{3}$, and its cohesion and angle of internal friction are $650 \mathrm{lb} / \mathrm{ft}^{2}$ and $0^{\circ}$, respectively.

## Required

Factor of safety against slope failure.


FIGURE 14-13

## Solution

From Figure 14-8a,

$$
\begin{aligned}
n_{d} H & =40 \mathrm{ft} \\
H & =25 \mathrm{ft} \\
n_{d} & =\frac{40 \mathrm{ft}}{25 \mathrm{ft}}=1.60
\end{aligned}
$$

with $\beta=30^{\circ}$ and $n_{d}=1.60$, from Figure 14-9,

$$
\begin{gather*}
N_{s}=6.0 \\
N_{s}=\frac{\gamma H}{c_{\text {required }}}  \tag{14-12}\\
\gamma=120 \mathrm{lb} / \mathrm{ft}^{3} \\
H=25 \mathrm{ft} \\
6.0=\frac{\left(120 \mathrm{lb} / \mathrm{ft}^{3}\right)(25 \mathrm{ft})}{c_{\text {required }}} \\
c_{\text {required }}=500 \mathrm{lb} / \mathrm{ft}^{2} \\
\text { F.S. }=\frac{c_{\text {given }}}{c_{\text {required }}}=\frac{650 \mathrm{lb} / \mathrm{ft}^{2}}{500 \mathrm{lb} / \mathrm{ft}^{2}}=1.30
\end{gather*}
$$

## EXAMPLE 14-6

Given

1. A cut 30 ft deep is to be made in a deposit of highly cohesive soil that is 60 ft thick and underlain by rock (see Figure 14-14).


FIGURE 14-14
2. The properties of the soil to be cut are as follows:

$$
\begin{aligned}
c & =750 \mathrm{lb} / \mathrm{ft}^{2} \\
\phi & =0^{\circ} \\
\gamma & =120 \mathrm{lb} / \mathrm{ft}^{3}
\end{aligned}
$$

3. The factor of safety against slope failure must be 1.25 .

## Required

Estimate the slope angle $(\beta)$ at which the cut should be made.

## Solution

From Figure 14-8a,

$$
\begin{aligned}
n_{d} H & =60 \mathrm{ft} \\
H & =30 \mathrm{ft} \\
n_{d} & =\frac{60 \mathrm{ft}}{30 \mathrm{ft}}=2.0
\end{aligned}
$$

From Eq. (14-12),

$$
\begin{align*}
N_{s} & =\frac{\gamma H}{c_{\text {required }}}  \tag{14-12}\\
\gamma & =120 \mathrm{lb} / \mathrm{ft}^{3} \\
H & =30 \mathrm{ft} \\
c_{\text {required }} & =\frac{c_{\text {given }}}{\text { F.S. }}=\frac{750 \mathrm{lb} / \mathrm{ft}^{2}}{1.25}=600 \mathrm{lb} / \mathrm{ft}^{2} \\
N_{s} & =\frac{\left(120 \mathrm{lb} / \mathrm{ft}^{3}\right)(30 \mathrm{ft})}{600 \mathrm{lb} / \mathrm{ft}^{2}}=6.0
\end{align*}
$$

From Figure 14-9, with $N_{s}=6.0$ and $n_{d}=2.0$,

$$
\beta=23^{\circ}
$$

## EXAMPLE 14-7

Given
A cut 10 m deep is to be made in soil that has the following properties:

$$
\begin{aligned}
\gamma & =17.66 \mathrm{kN} / \mathrm{m}^{3} \\
c & =19.2 \mathrm{kN} / \mathrm{m}^{2} \\
\phi & =16^{\circ}
\end{aligned}
$$

## Required

Using a factor of safety of 1.25 , estimate the slope angle at which the cut should be made.

## Solution

$$
c_{d}=\frac{c}{\text { F.S. }_{c}}=\frac{19.2 \mathrm{kN} / \mathrm{m}^{2}}{1.25}=15.36 \mathrm{kN} / \mathrm{m}^{2}
$$

From Eq. (14-12),

$$
\begin{align*}
N_{s} & =\frac{\gamma H}{c_{d}}  \tag{14-12}\\
N_{s} & =\frac{\left(17.66 \mathrm{kN} / \mathrm{m}^{3}\right)(10 \mathrm{~m})}{15.36 \mathrm{kN} / \mathrm{m}^{2}}=11.5 \\
\tan \phi_{d} & =\frac{\tan \phi}{\text { F.S. }_{\phi}}=\frac{\tan 16^{\circ}}{1.25}=0.2294 \\
\phi_{d} & =12.9^{\circ}
\end{align*}
$$

From Figure 14-10, with $\phi_{d}=12.9^{\circ}$ and $N_{s}=11.5$,

$$
\beta=44^{\circ}
$$

## 14-5 METHOD OF SLICES

In Section 14-4, the assumption was made in the Culmann method that failure (sliding) would occur along a plane that passes through the toe of the slope. It is probably more likely, and observations suggest, that failure will occur along a curved surface (rather than a plane) within the soil. Like the stability number method, the method of slices, which was developed by Swedish engineers, performs slope stability analysis assuming failure occurs along a curved surface.

The first step in applying the method of slices is to draw to scale a cross section of the slope such as that shown in Figure 14-15. A trial curved surface along which sliding is assumed to take place is then drawn. This trial surface is normally approximately circular. Soil contained between the trial surface and the slope is then divided into a number of vertical slices of equal width. The weight of soil within each slice is


FIGURE 14-15 Sketch showing assumed curved failure surface for method of slices.
calculated by multiplying the slice's volume by the soil's unit weight. (This problem is, of course, three-dimensional; however, by assuming a unit thickness throughout the computations, the problem can be treated as two-dimensional.)

Figure 14-16 shows a sketch of a single slice. The weight of soil within the slice is a vertically downward force ( $W$ in Figure 14-16). This force can be resolved into two components-one normal to the base of the slice $\left(W_{n}\right)$ and one parallel to the base of the slice $\left(W_{p}\right)$. It is the parallel component that tends to cause sliding. Resistance to sliding is afforded by the soil's cohesion and internal friction. The cohesion force is equal to the product of the soil's cohesion times the length of the slice's curved base. The friction force is equal to the component of $W$ normal to the base $\left(W_{n}\right)$ multiplied by the friction coefficient $(\tan \phi$, where $\phi$ is the angle of internal friction).

FIGURE 14-16 Sketch showing forces on a single slice in method of slices.


Because $W_{p^{\prime}}$, the component tending to cause sliding of the slice, is equal to $W$ multiplied by $\sin \alpha$ (Figure 14-16), the total force tending to cause sliding of the entire soil mass is the summation of products of the weight of each slice times the respective value of $\sin \alpha$, or $\Sigma W \sin \alpha$. Because $W_{n}$ is equal to $W$ multiplied by $\cos \alpha$, the total friction force resisting sliding of the entire soil mass is the summation of products of the weight of each slice times the respective value of $\cos \alpha$ times $\tan \phi$, or $\Sigma W \cos \alpha \tan \phi$. The total cohesion force resisting sliding of the entire soil mass can be computed simply by multiplying the soil's cohesion by the (total) length of the trial curved surface, or $c L$. Based on the foregoing, the factor of safety can be computed by using the following equation:

$$
\begin{equation*}
\text { F.S. }=\frac{c L+\sum W \cos \alpha \tan \phi}{\sum W \sin \alpha} \tag{14-13}
\end{equation*}
$$

(As related subsequently in Example 14-8, the term $W \sin \alpha$ may be negative in certain situations.)

This method gives the factor of safety for the specific assumed failure surface. It is quite possible that the circular surface selected may not be the weakest, or the one along which sliding would occur. The location of the most critical or most dangerous failure circle must usually be determined by method of trial. It is essential, therefore, that several circular surfaces be analyzed until the designer is satisfied that the worst condition has been considered.

## EXAMPLE 14-8

Given

1. The stability of a slope is to be analyzed by the method of slices.
2. On a particular trial curved surface through the soil mass (see Figure 14-17), the shearing component (i.e., sliding force) and the normal component (i.e., normal to the base of each slice) of each slice's weight are tabulated as follows:

| Slice <br> Number | Shearing Component <br> $(W \sin \boldsymbol{\alpha})(\mathrm{lb} / \mathrm{ft})$ | Normal Component <br> $(W \cos \boldsymbol{\alpha})(\mathrm{lb} / \mathrm{ft})$ |
| :---: | :---: | :---: |
| 1 | $-63^{1}$ | 358 |
| 2 | $-51^{1}$ | 1450 |
| 3 | 86 | 2460 |
| 4 | 722 | 3060 |
| 5 | 1470 | 3300 |
| 6 | 1880 | 3130 |
| 7 | 2200 | 2270 |
| 8 | 950 | 91 |

${ }^{1}$ Because the trial surface curves upward near its lower end, the shearing components of the weights of slices 1 and 2 will act in a direction opposite to those along the remainder of the trial curve, resulting in a negative sign.


FIGURE 14-17
3. The length of the trial curved surface is 36 ft .
4. The $\phi$ angle of the soil is $5^{\circ}$, and the cohesion $(c)$ is $400 \mathrm{lb} / \mathrm{ft}^{2}$.

## Required

Factor of safety of the slope along this particular trial surface.

## Solution

From Eq. (14-13),

$$
\begin{aligned}
& \text { F.S. }=\frac{c L+\sum W \cos \alpha \tan \phi}{\sum W \sin \alpha} \\
& c=400 \mathrm{lb} / \mathrm{ft}^{2} \\
& L=36 \mathrm{ft} \\
& \Sigma W \cos \alpha=358 \mathrm{lb} / \mathrm{ft}+1450 \mathrm{lb} / \mathrm{ft}+2460 \mathrm{lb} / \mathrm{ft}+3060 \mathrm{lb} / \mathrm{ft} \\
& +3300 \mathrm{lb} / \mathrm{ft}+3130 \mathrm{lb} / \mathrm{ft}+2270 \mathrm{lb} / \mathrm{ft}+91 \mathrm{lb} / \mathrm{ft} \\
& =16,119 \mathrm{lb} / \mathrm{ft} \\
& \phi=5^{\circ} \\
& \Sigma W \sin \alpha=-63 \mathrm{lb} / \mathrm{ft}-51 \mathrm{lb} / \mathrm{ft}+86 \mathrm{lb} / \mathrm{ft}+722 \mathrm{lb} / \mathrm{ft} \\
& +1470 \mathrm{lb} / \mathrm{ft}+1880 \mathrm{lb} / \mathrm{ft}+2200 \mathrm{lb} / \mathrm{ft}+950 \mathrm{lb} / \mathrm{ft} \\
& =7194 \mathrm{lb} / \mathrm{ft} \\
& \text { F.S. }=\frac{\left(400 \mathrm{lb} / \mathrm{ft}^{2}\right)(36 \mathrm{ft})+(16,119 \mathrm{lb} / \mathrm{ft}) \tan 5^{\circ}}{7194 \mathrm{lb} / \mathrm{ft}}=2.20
\end{aligned}
$$

It should be emphasized that the computed factor of safety of 2.20 is for the given trial surface, which is not necessarily the weakest surface.

## Bishop's Simplified Method of Slices

In 1955, Bishop (Bishop, 1955) presented a more refined method of analysis. His method uses static equilibrium considerations rather than finding a factor of safety against sliding by computing the ratio of the total force resisting sliding (of the entire soil mass) to the total force tending to cause sliding, as is done in the ordinary method of slices.

To understand Bishop's method, consider the representative slice shown in Figure 14-18. Unlike the slice shown in Figure 14-16, the one in Figure 14-18 shows all forces acting on the slice [i.e., its weight $W$, shear forces $T$, normal forces $H$ (on its sides), and a set of forces on its base (shear force $S$ and normal force $N$ )]. Bishop found that little error would accrue if the side forces are assumed equal and opposite. Equilibrium of the entire sliding mass requires (Figure 14-18) that

$$
\begin{equation*}
R \Sigma W \sin \alpha=R \Sigma S \tag{14-14}
\end{equation*}
$$

The shear force on the base of a slice, $S$, is given by the following (Figure 14-18):

$$
\begin{equation*}
S=\frac{s l}{\text { F.S. }}=\frac{s b}{\text { F.S. } \cos \alpha} \tag{14-15}
\end{equation*}
$$

where $s$ is shear strength; $l, b$, and $\alpha$ are as shown in Figure 14-18; and F.S. is the factor of safety. Substituting Eq. (14-15) into Eq. (14-14) yields the following:

$$
\begin{equation*}
\frac{R}{\text { F.S. }} \sum \frac{s b}{\cos \alpha}=R \Sigma W \sin \alpha \tag{14-16}
\end{equation*}
$$

from which

$$
\begin{equation*}
\text { F.S. }=\frac{\sum(s b / \cos \alpha)}{\sum W \sin \alpha} \tag{14-17}
\end{equation*}
$$

Shear strength $s$ can be determined from Eq. (2-17):

$$
\begin{equation*}
s=c+\bar{\sigma} \tan \phi \tag{2-17}
\end{equation*}
$$

where $\mathrm{c}=$ cohesion
$\bar{\sigma}=$ effective intergranular normal pressure (normal stress across the surface of sliding, $l$ )
$\phi=$ angle of internal friction
$\bar{\sigma}$ can be evaluated by analyzing the vertical equilibrium of the slice shown in Figure 14-18:

$$
\begin{equation*}
W=S \sin \alpha+N \cos \alpha \tag{14-18}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{\sigma}=\frac{N}{l}+\frac{N \cos \alpha}{b}=\frac{W}{b}-\frac{S}{b} \sin \alpha \tag{14-19}
\end{equation*}
$$



FIGURE 14-18 Sketch showing (a) assumed curved failure surface and (b) forces on a single slice for Bishop's simplified method of slices.

Substituting the latter value of $\bar{\sigma}$ from Eq. (14-19) into Eq. (2-17) gives the following:

$$
\begin{equation*}
s=c+\left(\frac{W}{b}-\frac{S}{b} \sin \alpha\right) \tan \phi \tag{14-20}
\end{equation*}
$$

But, substituting the value of $S$ from Eq. (14-15) yields the following:

$$
\begin{equation*}
s=c+\left(\frac{W}{b}-\frac{s}{\text { F.S. }} \tan \alpha\right) \tan \phi \tag{14-21}
\end{equation*}
$$

Solving for $s$ from Eq. (14-21) gives

$$
\begin{equation*}
s=\frac{c+(W / b) \tan \phi}{1+(\tan \alpha \tan \phi) / \text { F.S. }} \tag{14-22}
\end{equation*}
$$

To simplify computations, let

$$
\begin{equation*}
m_{\alpha}=\left[1+\frac{\tan \alpha \tan \phi}{\text { F.S. }}\right] \cos \alpha \tag{14-23}
\end{equation*}
$$

Substituting this value of $m_{\alpha}$ into Eq. (14-22) yields the following:

$$
\begin{equation*}
s=\left[\frac{c+(W / b) \tan \phi}{m_{\alpha}}\right] \cos \alpha \tag{14-24}
\end{equation*}
$$

Then substitute Eq. (14-24) into Eq. (14-17):

$$
\begin{equation*}
\text { F.S. }=\frac{\sum \frac{c b+W \tan \phi}{m_{\alpha}}}{\sum W \sin \alpha} \tag{14-25}
\end{equation*}
$$

Equation (14-25) can be used to find the factor of safety for the given (trial) failure surface. This is complicated somewhat by the fact that the value of $m_{\alpha}$ to be substituted into Eq. $(14-25)$ to calculate the factor of safety must be determined from Eq. (14-23), which requires the value of the factor of safety (on the right side of the equation). Hence, Eq. (14-25) must be solved by trial and error-that is, assume a value for the factor of safety, substitute it into Eq. (14-23) to solve for $m_{\alpha^{\prime}}$ and substitute that value of $m_{\alpha}$ into Eq. $(14-25)$ to compute the factor of safety. If the computed value for the factor of safety is the same (or nearly the same) as the assumed value, then that value is the correct one. If not, another value must be assumed and the procedure repeated until the correct value for the factor of safety is found. Figure 14-19 may be used in lieu of Eq. (14-23) to evaluate $m_{\alpha}$.

As noted with the ordinary method of slices, it should be emphasized that Bishop's simplified method of slices also gives the factor of safety for the specific assumed failure surface. It is quite possible that the circular surface selected may not be the weakest, or the one along which sliding would occur. It is essential, therefore, that several circular surfaces be studied until the designer is satisfied that the worst condition has been analyzed.


FIGURE 14-19 Values of $m_{\alpha}$ for Bishop equation.
Source: A. W. Bishop, "The Use of Slip Circle in the Stability Analysis of Earth Slopes," Geotechnique, 5(1) (1955). Reprinted by permission.

Strictly speaking, both methods of slices apply only if the entire trial circle lies above the water table and no excess pore pressures are present. If these conditions are not met, additional analysis is required.

Generally, the Bishop method gives slightly higher factors of safety than those calculated from the ordinary slice method-hence, the latter is somewhat more conservative. The Bishop method provides too-high factors of safety if the negative alpha angle (see $-\alpha$ in Figure 14-19) approaches $30^{\circ}$. For the same situation, the ordinary method of slices tends to provide too-low values.

## 14-6 PROBLEMS

14-1. Figure 14-20 shows a $20-\mathrm{ft}$ cut through two soil strata. The lower is a highly impermeable cohesive clay. Shear strength data between the two strata are as follows:

$$
\begin{aligned}
c & =220 \mathrm{lb} / \mathrm{ft}^{2} \\
\phi & =12^{\circ}
\end{aligned}
$$

The unit weight of the upper layer is $110 \mathrm{lb} / \mathrm{ft}^{3}$. Determine if a slide is likely by computing the factor of safety against sliding. Neglect the effects of soil water.
$14-2$. A vertical cut is to be made in a deposit of homogeneous soil. The soil mass to be cut has the following properties: The soil's unit weight is $120 \mathrm{lb} / \mathrm{ft}^{3}$, cohesion is $350 \mathrm{lb} / \mathrm{ft}^{2}$, and the angle of internal friction is $10^{\circ}$. It has been


## FIGURE 14-20



FIGURE 14-21
specified that the factor of safety against sliding must be 1.50. Using the Culmann method, determine the safe depth of the cut.
$14-3$. A $1.5-\mathrm{m}$-deep vertical-wall trench is to be cut in a soil whose unit weight, angle of internal friction, and cohesion are $17.36 \mathrm{kN} / \mathrm{m}^{3}, 25^{\circ}$, and 20.6 $\mathrm{kN} / \mathrm{m}^{2}$, respectively. Determine the factor of safety of this trench by the Culmann method.
14-4. Determine the factor of safety against slope failure by means of the stability number method for the slope shown in Figure 14-21.
$14-5$. A cut 20 ft deep is to be made in a stratum of highly cohesive soil that is 80 ft thick and underlain by bedrock. The slope of the cut is 2:1 (i.e., 2 horizontal to 1 vertical). The clay's unit weight is $110 \mathrm{lb} / \mathrm{ft}^{3}$, and its $c$ and $\phi$ values are 500 $\mathrm{lb} / \mathrm{ft}^{2}$ and $0^{\circ}$, respectively. Determine the factor of safety against slope failure.
$\mathbf{1 4 - 6}$. A cut 25 ft deep is to be made in a deposit of cohesive soil with $c=700$ $\mathrm{lb} / \mathrm{ft}^{2}, \phi=0^{\circ}$, and $\gamma=115 \mathrm{lb} / \mathrm{ft}^{3}$. The soil is 30 ft thick and underlain by rock. The factor of safety of the slope against failure must be 1.50. At what slope angle should the cut be made?
14-7. A slope 8 m high is to be made in a soil whose unit weight, angle of internal friction, and cohesion are $16.7 \mathrm{kN} / \mathrm{m}^{3}, 10^{\circ}$, and $17.0 \mathrm{kN} / \mathrm{m}^{2}$, respectively. Using an overall factor of safety of 1.25 , estimate the slope angle that should be used.
$14-8$. The stability of a slope is to be analyzed by the method of slices. On a particular trial curved surface through the soil mass, the shearing and normal components of each slice's weight are tabulated as shown below. The length of the trial curved surface is 40 ft . The cohesion $c$ and $\phi$ angle of the soil are $225 \mathrm{lb} / \mathrm{ft}^{2}$ and $15^{\circ}$, respectively. Determine the factor of safety along this trial surface.

| Slice <br> Number | Shearing Component <br> $(W \sin \boldsymbol{\alpha})(\mathrm{lb} / \mathrm{ft})$ | Normal Component <br> $(W \cos \boldsymbol{\alpha})(\mathrm{lb} / \mathrm{ft})$ |
| :---: | :---: | :---: |
| 1 | -38 | 306 |
| 2 | -74 | 1410 |
| 3 | 124 | 2380 |
| 4 | 429 | 3050 |
| 5 | 934 | 3480 |
| 6 | 1570 | 3540 |
| 7 | 2000 | 3210 |
| 8 | 2040 | 2190 |
| 9 | 766 | 600 |

