Quaternionic Electron Theory: Geometry, Algebra and Dirac's Spinors

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The use of complexified quaternions and i-complex geometry in formulating the Dirac equation allows us to give *interesting* geometric interpretations hidden in the conventional matrix-based approach.

I. INTRODUCTION

Among the many alternative mathematical systems in which the Dirac equation can be written [1–7], we showed in a recent paper [8] how the matrix and vector algebras can be replaced by a single mathematical system, complexified quaternionic algebra, with which geometric interpretations can be carried out more efficiently. The power of our (complexified) quaternionic formalism becomes evident within the electron theory, and it derives from the fact that the elements of the complexified quaternionic algebra are subject to direct geometrical identifications. The theory presented in our previous paper [8] is algebraically isomorphic to the matrix Dirac theory; it can be provided with an equivalent physical interpretation as well. It differs from the Dirac approach in that all its algebraic ingredients have a geometrical significance determined by geometric properties of the complexified quaternionic algebra. In contrast to the complex matrix algebra of Dirac, the complexified quaternionic algebra has a clear geometrical significance. By introducing new imaginary units, we give a geometrical interpretation for the "complex" imaginary unit, i, which characterizes standard quantum theories.

By working with complexified quaternions and i-complex geometry [8,9], we also obtain the surprising conclusion that the Schrödinger theory contains additional informations. It is possible to affirm that spin and positron are already present in the complexified quaternionic Schrödinger theory. At first sight such a conclusion may seem preposterous. Schrödinger knew nothing of spin and positron when he framed his equation. "But it is no more preposterous than the incredible fact that Schrödinger wrote down his equation and solved the first problems of modern quantum theory without any mention of probability, thought it now appears that probability was already in the theory. The Born interpretation of $\psi^\dagger\psi$ as a probability density can be adopted without the slights modification of the Schrödinger work. Indeed, the Born interpretation is now an indispensable part of the theory" [11].

Let us give an *informal* discussion concerning the additional solutions in the complexified quaternionic Schrödinger equation. By working with complex numbers we have no possibility to accommodate the two freedom degrees characterizing the spin of the electron

$$\psi \in C(1,i) \qquad [\text{ electron with “frozen” spin }]\ . \tag{1}$$

By allowing real quaternions as underlying numerical field we double the complex freedom degrees, and consequently we have rotations in the plane

$$\left(\ C_{(1,i)}\ ,\ jC_{(1,i)}\ \right)\ .$$

In this case we can accommodate the spin up and down of the electron respectively on the complex and *pure* quaternionic axis. Thus, (real) quaternionic wave-functions naturally describe spin in the Schrödinger theory,

$$\psi \in \mathcal{H} \qquad [\text{ electron with spin up and down }]\ . \tag{2}$$

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Finally, by considering complexified quaternionic numbers we can introduce the dual space $\boldsymbol{\iota}\mathcal{H}$ and quadruple the complex degrees of freedom

$$(\mathcal{H} \ , \ \boldsymbol{\iota}\mathcal{H}) \ .$$

This means that the wave-function, Ψ, in $\mathcal{H}_c$, possesses the needed degrees of freedom to represent the spin and positron content

$$\Psi \in \mathcal{H}_c \qquad [\text{ electron/positron with spin up and down }] \ . \tag{3}$$

Obviously, the previous discussion represents only a "rough" approach. In section III, by analyzing the complexified quaternionic Dirac spinors, we shall give a more clear geometric interpretation. Positive and negative energy solutions are respectively characterized (in the rest frame) by real and pure complexified quaternionic spinors. For the most general solutions, we will be able to related them to rotation angles in the complexified quaternionic hyper-plane

$$(\mathcal{H}_c \ , \ \boldsymbol{\iota}\mathcal{H}_c) \ .$$

Finally, the *polar form* of complexified quaternionic spinors, given in section II, will allow to identify the spin operator by a *new* hypercomplex imaginary unit $\mathcal{I}$, individued by the components of the momentum operator $\vec{p}$ of our particle.

II. GEOMETRIC ALGEBRA OF COMPLEXIFIED QUATERNIONS

In this section we draw out the *polar form* of complexified quaternions, which will be very useful in discussing geometric interpretations of Dirac spinors.

Let us start with the standard discussion about plane geometry and complex algebra. We may relate the geometry to the complex algebra by representing complex numbers in a plane

$$x + iy = re^{i\theta} \ . \tag{4}$$

In fact, we know that a rotation of α-angle around the z axis, can be represented by $e^{i\alpha}$,

$$e^{i\alpha} (x + iy) = re^{i(\alpha+\theta)} \ .$$

Feynman called Eq. (4), the unification of algebra and geometry [10]. We like talking of "partial" unification. Our aim in this section is to show how it is possible generalize the connection between geometry and algebra by using a noncommutative numerical field, or more generally hypercomplex numbers.

Similarly to rotations in a plane, a rotation about an axis passing through the origin and parallel to a given unitary vector $\hat{u} \equiv (u_x, u_y, u_z)$ by an angle α can be obtained taking the real quaternionic transformation

$$\exp\left(\frac{iu_x + ju_j + ku_z}{2}\alpha\right) (ix + jy + kz) \exp\left(-\frac{iu_x + ju_j + ku_z}{2}\alpha\right) \ . \tag{5}$$

The vector part $(q^{\dagger} = -q)$ of a generic quaternion

$$q = r_0 + \vec{h} \cdot \vec{r} \ ,$$

with the real axis $q = r_0$ specifies a unique "complex" plane with imaginary axis given by the unit

$$i\,\frac{r_1}{r} + j\,\frac{r_2}{r} + k\,\frac{r_3}{r} \ , \qquad r = \sqrt{r_1^2 + r_2^2 + r_3^2} \ .$$

Our quaternionic number will be represented in this plane by

$$r_0 + Ir = \rho e^{I\alpha} \ , \tag{6}$$

where

$$I = \frac{\vec{h} \cdot \vec{r}}{r} \ , \qquad \vec{h} \equiv (i, j, k) \ .$$

Let us analyze what happens for complexified quaternions. What is the *polar form* of complexified quaternions? How can we individuate a "complex" phase? It is possible to define an hyper-plane?

A generic complexified quaternion is expressed by

$$q_c = c_0 + ic_1 + jc_2 + kc_3 \; , \qquad c_{0,1,2,3} \in \mathcal{C}(1, \boldsymbol{\iota}) \; . \tag{7}$$

We define

$$c = \sqrt{c_1^2 + c_2^2 + c_3^2} \; ,$$

and consequently we have

$$q_c = c_0 + \mathcal{I}c \; , \tag{8}$$

where

$$\mathcal{I} = i\,\frac{c_1}{c} + j\,\frac{c_2}{c} + k\,\frac{c_3}{c} = \frac{\vec{h}\cdot\vec{c}}{c} \; , \qquad \mathcal{I}^2 = -1 \; , \qquad \mathcal{I}^\star \mathcal{I} = 1 \; .$$

The vector part ($q_c^\star = -q_c$) of a generic complexified quaternion q_c with the $\boldsymbol{\iota}$-complex axis $q_c = c_0$ specifies a unique *hypercomplex* plane with "imaginary" axis given by the unit $\mathcal{I}$.

By taking the $\star$-involution, we find

$$q_c^\star q_c = c_0^2 + c^2 \in \mathcal{C}(1, \boldsymbol{\iota}) \; .$$

Thus, we can write

$$q_c^\star q_c = \rho^2 e^{2\boldsymbol{\iota}\alpha} \qquad \rho, \alpha \in \mathcal{R} \; ,$$

and consequently to identify

$$c_0 = \rho e^{\boldsymbol{\iota}\alpha} \cos z \; , \qquad c = \rho e^{\boldsymbol{\iota}\alpha} \sin z \qquad z \in \mathcal{C}(1, \boldsymbol{\iota}) \; .$$

Then, the *polar form* of a complexified quaternion reads

$$q_c = \rho e^{\boldsymbol{\iota}\alpha} e^{\mathcal{I}z} \; , \tag{9}$$

where

$$\rho e^{\boldsymbol{\iota}\alpha} = \sqrt{c_0^2 + c_1^2 + c_2^2 + c_3^2} \; ,$$

and

$$\cos z = \frac{c_0}{\sqrt{c_0^2 + c_1^2 + c_2^2 + c_3^2}} \; , \qquad \sin z = \frac{\sqrt{c_1^2 + c_2^2 + c_3^2}}{\sqrt{c_0^2 + c_1^2 + c_2^2 + c_3^2}} \; .$$

In conclusion we can characterize a complexified quaternionic number by *two* "imaginary" units

$$\boldsymbol{\iota} \qquad \text{and} \qquad \mathcal{I} \; ,$$

respectively related to the following "hypercomplex" planes

$$\Lambda^{(E)} \equiv \left(\rho e^{\mathcal{I}z} \cos\alpha \, , \, \rho e^{\mathcal{I}z} \sin\alpha\right) \qquad \text{and} \qquad \Lambda^{(s)} \equiv \left(\rho e^{\boldsymbol{\iota}\alpha} \cos z \, , \, \rho e^{\boldsymbol{\iota}\alpha} \sin z\right) \; .$$

The first plane will identify the mixing of positive and negative energy solutions, whereas the second one will represent the spin plane.

The real quaternionic theory essentially distinguishes two types of objects, the scalars and the vectors. Nevertheless, we know that the vectors split into two disjoint sets, polar and axial vectors (often called pseudo-vectors) and, in three-dimensions we have also scalars and pseudo-scalars. These distinctions are well illustrated by complexified quaternions [12]. Following the convention used by Hestenes [13], the terms in complexified quaternions

$$\alpha_0 + \boldsymbol{\iota}\beta_0 + i(\alpha_1 + \boldsymbol{\iota}\beta_1) + j(\alpha_2 + \boldsymbol{\iota}\beta_2) + k(\alpha_3 + \boldsymbol{\iota}\beta_3) \qquad \alpha_{0,1,2,3}, \, \beta_{0,1,2,3} \in \mathcal{R} \; ,$$

naturally separate into four groups

$$\begin{array}{rcl} \alpha & \rightarrow & \text{scalars} \;, \\ \boldsymbol{\iota}\beta & \rightarrow & \text{pseudo-scalars} \;, \\ \boldsymbol{\iota}\vec{h}\cdot\vec{\beta} & \rightarrow & \text{vectors} \;, \\ \vec{h}\cdot\vec{\alpha} & \rightarrow & \text{bivectors} \;. \end{array}$$

In the complexified quaternionic electron theory we can define the dual of a complexified quaternion, q_c, to be $\boldsymbol{\iota} q_c$. The dual operation turns a scalar into a pseudo-scalar and a vector into a bivector (and vice-versa). We showed [8] that parity operation, in the Dirac theory, is related to the •-involution. Thus, in the present formalism obtaining the dual is simply achieved by multiplying by $\boldsymbol{\iota}$.

III. QUATERNIONIC DIRAC'S SPINORS

The possibility to express the Dirac spinor as a simple complexified quaternions instead of vector column (as in real quaternionic and complex formulations) represents the main difference between complexified quaternionic and real quaternionic or complex version of the Dirac equation.

"The representation of a Dirac spinor by a generic (invertible) element of the Pauli algebra has given rise to new insights in Dirac's theory of the electron: the polar form of Dirac-Hestenes spinor presents the Dirac spinor as the product of a *complex* number by a Lorentz transformation, where the argument of the complex number has been identified with the Yvone-Takabayashi angle and its module with the charge density of the mixture of positrons and electrons. Nevertheless the proof of the equivalence between the original Dirac equation and the Dirac-Hestenes equation is usually made raising an explicit relation between the components of Dirac spinor and the Dirac-Hestenes spinor, that require a lot of calculations to be done" - *Zeni* [14].

Our complexified quaternionic approach to Dirac theory reproduce quickly the standard results, avoids a lot of calculations and gives the desired *geometrical interpretations* which characterize the Dirac-Hestenes theory [13]. The plane wave solutions of the complexified quaternionic Dirac equation

$$\left(\partial_t + \boldsymbol{\iota}\vec{h}\cdot\vec{\partial}\right)\Psi(x)\, i = m\Psi^{\bullet}(x) \;, \tag{10}$$

are

$$\sqrt{\frac{|E|+m}{2}} \times \begin{array}{|lll|} \hline 1+\frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\;, & \left(1+\frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right) j \;, & E>0 \\ \hline \left(1-\frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right)\boldsymbol{\iota}\;, & \left(1-\frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right)\boldsymbol{\iota} j \;, & E<0 \\ \hline \end{array} \times e^{-ipx} \;.$$

The wave-function does not have a direct physical significance, and a crucial part of the Dirac theory is to relate Ψ to observable quantities. The *polar* decomposition of Ψ

$$\rho\, \exp(\boldsymbol{\iota}\alpha)\, \exp(\mathcal{I}z) \tag{11}$$

greatly facilitates this task and, in addition, makes the geometric content of the theory explicit. The quantities

$$\rho,\; \alpha,\; \mathcal{I},\; z$$

have distinctive geometrical and physical interpretations which are independent of any matrix representation. So, it is best to use them instead of the α's and β's which appear in the standard Dirac equation.

Ordinarily, Dirac spinors are said to be representations of the Lorentz group because they transform in a certain way under Lorentz transformation. In contrast, we say that Ψ represents a Lorentz transformation, the spinor "$e^{\mathcal{I}z}$" may be regarded as a representation of a Lorentz transformation. Complexified quaternionic spinors, Ψ, consist of Lorentz rotations, dilatation and duality transformation.

By looking at the Dirac plane wave solutions in the rest frame of the particle ($\vec{p}=\vec{0}$), we immediately find the following structures for our spinors

$$\Psi_{E=+m} \in \mathcal{H} \quad \rightarrow \quad \alpha = 0 \; ,$$
$$\Psi_{E=-m} \in \boldsymbol{\iota}\mathcal{H} \quad \rightarrow \quad \alpha = \frac{\pi}{2} \; .$$

Thus, the hyper-plane $(\mathcal{H}, \boldsymbol{\iota}\mathcal{H})$, gives the mixing of positive/negative energy solutions in the rest frame of the particle. This was anticipated in our introduction. The situations appears more complicated for the general case in which the particle is in motion. In such a case the positive energy solutions are not identified by simple (real) quaternions and so it is not immediate to extract $\alpha = 0$. Nevertheless, by using the $\star$-conjugation operation ($\vec{h} \rightarrow -\vec{h}$, $\boldsymbol{\iota} \rightarrow \boldsymbol{\iota}$) we find

$$\Psi^\star \Psi = \rho^2 \exp(2\boldsymbol{\iota}\alpha) \; ,$$

and consequently the Eqs.

$$\Psi^\star_{E>0}\Psi_{E>0} = \left(1 - \frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right)\left(1 + \frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right) = -j\left(1 - \frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right)\left(1 + \frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right)j = \frac{2m}{|E|+m} \; ,$$
$$\Psi^\star_{E<0}\Psi_{E<0} = \boldsymbol{\iota}^2\left(1 + \frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right)\left(1 - \frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right) = -\boldsymbol{\iota}^2 j\left(1 - \frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right)\left(1 + \frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right)j = -\frac{2m}{|E|+m} \; ,$$

imply that positive energy solutions are characterized by the α-angle value 0, whereas negative energy solutions by the α-angle value $\frac{\pi}{2}$. This means that we can think to an "complex" hyper-plane

$$\left(\mathcal{H}_c^{\alpha=0} \, , \, \boldsymbol{\iota}\mathcal{H}_c^{\alpha=0}\right) \;\; \equiv \;\; \left(\mathcal{H}_c^{\alpha=0} \, , \, \mathcal{H}_c^{\alpha=\frac{\pi}{2}}\right) \; ,$$

where the positive energy solutions are mapped in the "real" axis, whereas the negative energy solutions are obtained by rotations of $\frac{\pi}{2}$ angles.

Let us now give the explicit *polar* form for our complexified quaternionic Dirac spinors. Consider

$$\sqrt{\frac{|E|+m}{2}} \;\times\; \left(1 + \frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right) \; .$$

It is immediate to recognize

$$c_0 = \sqrt{\frac{|E|+m}{2}} \; , \qquad \vec{c} = \boldsymbol{\iota}\frac{\vec{p}}{\sqrt{2(|E|+m)}} \; ,$$

thus

$$\rho e^{\boldsymbol{\iota}\alpha} = \sqrt{\frac{|E|+m}{2} - \frac{\vec{p}^{\,2}}{2(|E|+m)}} = \sqrt{m} \; ,$$

and

$$\cos z = \sqrt{\frac{|E|+m}{2m}} \; , \qquad \sin z = \boldsymbol{\iota}\frac{|\vec{p}|}{\sqrt{2m(|E|+m)}} \; .$$

The "generalized" imaginary unit $\mathcal{I}$ is identified by

$$\mathcal{I} = \frac{\vec{h}\cdot\vec{p}}{|\vec{p}|} \; .$$

We have now all the needed ingredients to write down the *polar form* for our complexified quaternionic Dirac spinor

$$\sqrt{\frac{|E|+m}{2}} \;\times\; \left(1 + \frac{\boldsymbol{\iota}\vec{h}\cdot\vec{p}}{|E|+m}\right) \quad \rightarrow \quad \sqrt{m}\, e^{\boldsymbol{\iota}\mathcal{I}\beta} \; ,$$

where

$$\cosh\beta = \sqrt{\frac{|E|+m}{2m}}\ .$$

We can recognize in the definition of the "generalized" imaginary unit $\mathcal{I}$, the helicity operator [8]. For $\vec{p}\equiv(p_x,0,0)$, it reduces to the complex imaginary unit i, characterizing the standard Quantum Mechanics. Thus, the i in the Dirac equation can be interpreted geometrically (bivector) and its appearance is strictly related to the particle spin content. The *polar form* for the remaining Dirac spinors is very simply

$$\begin{array}{ll}
\sqrt{m}\, e^{\boldsymbol{\iota} i\beta}\ , & E>0\ ,\ \ \text{spin}\ \uparrow\ ,\\
\sqrt{m}\, e^{\boldsymbol{\iota} i\beta}\, j\ , & E>0\ ,\ \ \text{spin}\ \downarrow\ ,\\
\sqrt{m}\, e^{\boldsymbol{\iota} i\beta}\, \boldsymbol{\iota}\ , & E<0\ ,\ \ \text{spin}\ \uparrow\ ,\\
\sqrt{m}\, e^{\boldsymbol{\iota} i\beta}\, \boldsymbol{\iota} j\ , & E<0\ ,\ \ \text{spin}\ \downarrow\ ,
\end{array}$$

All the imaginary units present in the complexified quaternionic Dirac theory have now a geometric interpretation

$$\begin{array}{lcl}
\boldsymbol{\iota}\ \ \text{(pseudo-scalar)} & \rightarrow & \text{positive/negative energy flip}\ ,\\
j\ \ \ \ \text{(bivector)} & \rightarrow & \text{up/down spin flip}\ ,\\
i\ \ \ \ \text{(bivector)} & \rightarrow & \text{generator of rotations}\ .
\end{array}$$

IV. FROM QUATERNIONS TO CLIFFORD ALGEBRA

In the last years, different formulations of the Dirac equation have been done by using noncommutative hypercomplex number. The first attempt, performed by using real quaternions [6], gave the possibility to reduce the dimension of the γ^μ-matrices ($4\rightarrow 2$). The doubling of solutions, due to the i-complex geometry, allow us to obtain the standard results notwithstanding the halved dimension of Dirac spinors. Nevertheless, we also find a different representation for operators (2×2 matrices) and spinors (two-dimensional column vectors). The possibility to perform a one-dimensional complexified quaternionic version of the Dirac equation by using $\boldsymbol{\iota}$-complex geometry [7], overcome this problem, by putting on the same level the operators and the spinors. Yet, it was not possible to give a clear geometric interpretation because of the complicated structure of spinors and CPT operations. The formulation of Dirac equation by complexified quaternion and i-complex geometry maintains the possibility to treat operators and spin at the same level, and more it permits interesting geometric interpretations.

The Dirac theory can be formulated by complex or non commutative numbers. The passage from

$$\text{complex}\quad\rightarrow\quad\text{real quaternions}\quad\rightarrow\quad\text{complexified quaternions}$$

is achieved by performing a set of translation rules [9]. We can obtain the same result by working only with the general properties of the Clifford algebra, in special the concept of even sub-algebra. In doing it, we recall the main step performed in the agreeable and clear paper of Zeni [14].

In the standard matrix form the Dirac equation for a free particle is usually presented as

$$i\gamma^\mu\partial_\mu\psi = m\psi\ , \tag{12}$$

where the γ^μ matrices satisfy the relations:

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu = 2g^{\mu\nu}\ , \qquad \mu,\nu=0,1,2,3$$

where

$$g^{\mu\nu}\equiv \text{diag}\,(+,-,-,-)$$

is the Minkowski metric. The Dirac matrices can be represented by the following 4×4 complex matrices [15]

$$\gamma^0=\begin{pmatrix}1 & 0\\ 0 & \text{-}1\end{pmatrix}\ ,\qquad \vec{\gamma}=\begin{pmatrix}0 & \text{-}\vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}\ .$$

A left ideal in a matrix algebra is a linear subspace of the matrix space which is invariant under left multiplication. The set of all matrices which have all elements null except those in the first column is an example of an ideal. The main point in the ideal approach is to put on the same level the operators and the spinors, representing all of them by 4×4 complex matrices. This can be done if we use the following matrix representation for the Dirac spinor:

$$\Psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \quad \text{instead of} \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} .$$

The formulation of the Dirac equation using ψ or Ψ are completely equivalent. The ideals of an algebra are generated by idempotents, i.e. elements of the algebra whose squares are equal to themselves. For example, the following matrix is the generator of the ideal to which Ψ belongs:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

It is clear that $U^2 = U$ and also that for every $A \in C(4)$, AU is an element of the ideal defined above (only the elements in the first column are non null), so that every Ψ can be replaced by the product AU. Thus, the Dirac equation now reads:

$$\gamma^\mu \partial_\mu AUi = mAU . \tag{13}$$

The Clifford algebra $Cl_{4,1}$ is generated by the vectors ξ_x, $x \in [0,4]$ of $Cl_{4,1}$ that satisfy the following relation:

$$\xi_x \xi_y + \xi_y \xi_x = 2g_{xy} ,$$

where

$$g^{xy} \equiv \text{diag}\,(-,+,+,+,+) .$$

It is well know that the algebra $Cl_{4,1}$ is isomorphic to $C(4)$, so the Dirac equation has a natural representation in the Clifford algebra $Cl_{4,1}$ in the sense that all elements present in the Dirac equation belong to $Cl_{4,1}$. For example, the Dirac matrices γ^μ are the representatives in $C(4)$ of products of vectors ξ. We write the representative and the matrices with the same symbol since this does not cause any confusion,

$$\gamma^\mu \equiv \xi^\mu \xi^4 \qquad \mu = 0,1,2,3$$

and the imaginary unit of $C(4)$ is the volume element of $Cl_{4,1}$ given below:

$$i \equiv \xi^0 \xi^1 \xi^2 \xi^3 \xi^4 .$$

We can see from the standard representation of Dirac matrices that the idempotent U can be written as follows:

$$U = \frac{1}{4}\left(1+\gamma^0\right)\left(1-i\gamma^1\gamma^2\right) .$$

It is easy to verify the following identities, which will be useful afterwards:

$$\gamma^0 U = U\gamma^0 = U ,$$
$$iU = Ui = \gamma^2\gamma^1 U = U\gamma^2\gamma^1 .$$

The last equation shows to us that the imaginary unit, when multiplied by the idempotent, can be replaced by the product $\gamma^2\gamma^1$. The Dirac equation can be represented in the Clifford algebra $Cl_{4,1}$ through the following expression:

$$\boxed{\gamma^\mu \partial_\mu AU\gamma^2\gamma^1 = mAU\gamma^0}$$

It is well known that the Clifford algebras are Z_2 graded, i.e., we can divide the linear space of the algebra in two subspaces of even and odd grades, which are respectively composed of even and odd products of the generators. The even and odd subspaces of $Cl_{4,1}$ are spanned by the following sets:

$$Cl^+_{4,1} = \{1, \xi_x\xi_y, i\xi_x\} \qquad \text{and} \qquad Cl^-_{4,1} = \{\xi_x, i\xi_x\xi_y, i\} .$$

Every element of the odd subspace $Cl^-_{4,1}$ can be written as the product of an element of the even sub-algebra by an element of the odd subspace. In particular we have

$$Cl^-_{4,1} = Cl^+_{4,1} i\gamma^2\gamma^1 \ .$$

Now consider a generic element of the Dirac algebra, $A \in Cl_{4,1}$. According to previous equation

$$A = A^+ + A^- = A^+ + B^+ i\gamma^2\gamma^1 \ ,$$

where A^+ and B^+ belong to the even sub-algebra $Cl^+_{4,1} \sim Cl_{1,3}$, while A^- belongs to the odd subspace $Cl^-_{4,1}$. Taking the product of A by the idempotent U we get:

$$AU = (A^+ + B^+)U = \phi U \ .$$

The final step to reduce the Dirac equation to the space-time algebra (STA) $Cl_{1,3}$ is to take apart the idempotent which is the only element that does not belong to the even sub-algebra of $Cl_{4,1}$. We note that the idempotent U is the product of two idempotents

$$u = \frac{1}{2}\left(1 + \gamma^0\right) \qquad \text{and} \qquad v = \frac{1}{2}\left(1 + i\gamma^1\gamma^2\right) \ .$$

The Dirac equation is rewritten as follows:

$$\left(\gamma^\mu \partial_\mu \phi \gamma^2 \gamma^1 - m\phi\gamma_0\right) U = 0 \ ,$$

where the term in brackets is clearly an element of the STA, since γ^μ and ϕ belong to $Cl_{1,3}$. So the above equation looks like:

$$Cl^+_{4,1} U = Cl^+_{4,1} uv = Cl^+_{4,1} v \ .$$

The idempotent v is irrelevant to the above equation

$$Dv = 0 \qquad \to \qquad D = 0 \quad \text{and} \quad Di\gamma^1\gamma^2 = 0 \ ,$$

and because $i\gamma^1\gamma^2$ is an invertible element, we have simply $D = 0$. In conclusion, the Dirac equation can then be written using only elements of the STA:

$$\boxed{\gamma^\mu \partial_\mu \phi \gamma^2 \gamma^1 u = m\phi\gamma_0 u}$$

Before going on we observe that the idempotent u is irrelevant in the above equation. All mathematical and physical informations contained in $\phi = \phi(x) \in Cl_{1,3}$, which will be shown to be a sum of inhomogeneous even multivectors of $Cl_{1,3}$, and which looks as a superfield [16]. To reduce the Dirac equation to the Pauli algebra, $Cl_{3,0}$, we use the fact that the Pauli algebra is the even sub-algebra of STA. The first step is analogous to the previous case

$$\phi = \phi^+ + \phi^- = \phi^+ + \varphi^+\gamma^0 \ ,$$

and consequently we have

$$\phi u = (\phi^+ + \varphi^+)u = \eta u \ ,$$

where $u = \frac{1}{2}(1 + \gamma^0)$ is the idempotent present in the Dirac equation performed by STA and η is an element of the Pauli algebra. Multiplying the Dirac equation by γ^0 on the left and using η instead of ϕ, the Dirac equation becomes:

$$\left(\gamma^0\gamma^\mu \partial_\mu \eta \gamma^2 \gamma^1 - m\gamma^0\eta\gamma_0\right) u = 0 \ .$$

The term in brackets belongs to the Pauli algebra, $Cl_{3,0} \sim Cl^+_{1,3}$, since η and the products $\gamma^0\gamma^\mu$ and $\gamma^2\gamma^1$ belong to $Cl_{3,0}$. For the final step we have only to consider that

$$Du = 0 \qquad \to \qquad D = 0 \quad \text{and} \quad D\gamma^0 = 0 \ ,$$

and because γ^0 is an invertible element, we have simply

$$\boxed{\left(\partial_0 + \vec{\sigma} \cdot \vec{\partial}\right) \eta i \sigma_3 = m\eta^{\bullet}}$$

This equation contains only element of the Pauli algebra. By recalling the isomorphism between the Pauli algebra and complexified quaternions

$$\begin{aligned} \vec{\sigma} \quad &\leftrightarrow \quad \boldsymbol{\iota}\vec{h}\ , \\ i\vec{\sigma} \quad &\leftrightarrow \quad \vec{h}\ , \end{aligned}$$

we re-obtain the complexified quaternionic version of the Dirac equation. We only note that to have the desired geometric interpretations is important to adopt i-complex geometry in defining scalar products.

V. CONCLUSIONS

The main goal of this paper is to get geometric interpretations in the formulation of Dirac theory by complexified quaternions and i-complex geometry. The possibility to write down a one-dimensional version of Dirac equation simplify the solution of this equation. In the complexified quaternionic algebra we work in eight-dimensional space over the real numbers, while in the standard (complex) formulation we have to do with a 32-dimensional space on the reals. Moreover, all the elements in complexified quaternionic numbers have a geometric interpretation and this allow a clear interpretation for the complexified quaternionic imaginary units

$$\boldsymbol{\iota}\ , \quad \mathrm{i}\ , \quad \mathrm{j}\ ,$$

which appears in the spinor structure. We also recall that the Dirac spinors assume in the complexified quaternionic *polar* representation a very simple form. For these reasons, we think that the use of the complexified quaternionic algebra (together i-complex geometry) represents the "natural" mathematical language to write down the Dirac equation and discuss electron theory.

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