Complex Analysis 2002-2003

1 Complex Functions

In this section we will define what we mean by a complex function. We will then generalise the definitions of the exponential, sine and cosine functions using complex power series. To deal with complex power series we define the notions of convergent and absolutely convergent, and see how to use the ratio test from real analysis to determine convergence and radius of convergence for these complex series.

We start by defining domains in the complex plane. This requires the prelimary definition.

Definition 1.1
The ε**-neighbourhood** of a complex number z is the set of complex numbers $\{w \in \mathbb{C} : |z - w| < \varepsilon\}$ where ε is positive number.

Thus the ε-neigbourhood of a point z is just the set of points lying within the circle of radius ε centred at z. Note that it doesn't contain the circle.

Definition 1.2
A **domain** is a non-empty subset D of $\mathbb{C}$ such that for every point in D there exists a ε-neighbourhood contained in D.

Examples 1.3
The following are domains.

(i) $D = \mathbb{C}$. (Take $c \in \mathbb{C}$. Then, any $\varepsilon > 0$ will do for an ε-neighbourhood of c.)

(ii) $D = \mathbb{C}\backslash\{0\}$. (Take $c \in D$ and let $\varepsilon = \frac{1}{2}(|c|)$. This gives a ε-neighbourhood of c in D.)

(iii) $D = \{z : |z - a| < R\}$ for some $R > 0$. (Take $c \in \mathbb{C}$ and let $\varepsilon = \frac{1}{2}(R - |c - a|)$. This gives a ε-neighbourhood of c in D.)

Example 1.4
The set of real numbers $\mathbb{R}$ is not a domain. Consider any real number, then any ε-neighbourhood must contain some complex numbers, i.e. the ε-neighbourhood does not lie in the real numbers.

We can now define the basic object of study.

Definition 1.5
Let D be a domain in $\mathbb{C}$. A **complex function**, denoted $f : D \to \mathbb{C}$, is a map which assigns to each z in D an element of $\mathbb{C}$, this value is denoted $f(z)$.

Common Error 1.6
Note that f is the *function* and $f(z)$ is the *value* of the function at z. It is wrong to say $f(z)$ is a function, but sometimes people do.

Examples 1.7

(i) Let $f(z) = z^2$ for all $z \in \mathbb{C}$.

(ii) Let $f(z) = |z|$ for all $z \in \mathbb{C}$. Note that here we have a complex function for which every value is real.

(iii) Let $f(z) = 3z^4 - (5 - 2i)z^2 + z - 7$ for all $z \in \mathbb{C}$. All complex polynomials give complex functions.

(iv) Let $f(z) = 1/z$ for all $z \in \mathbb{C}\backslash\{0\}$. This function cannot be extended to all of $\mathbb{C}$.

Remark 1.8
Functions such as $\sin x$ for x real are not complex functions since the real line in $\mathbb{C}$ is not a domain. Later we see how to extend the concept of the sine so that it is complex function on the whole of the complex plane.

Obviously, if f and g are complex functions, then $f + g$, $f - g$, and fg are functions given by $(f + g)(z) = f(z) + g(z)$, $(f - g)(z) = f(z) - g(z)$, and $(fg)(z) = f(z)g(z)$, respectively. We can also define $(f/g)(z) = f(z)/g(z)$ provided that $g(z) \neq 0$ on D. Thus we can build up lots of new functions by these elementary operations.

The aim of complex analysis

We wish to study complex functions. Can we define differentiation? Can we integrate? Which theorems from Real Analysis can be extended to complex analysis? For example, is there a version of the mean value theorem? Complex analysis is essentially the attempt to answer these questions. The theory will be built upon real analysis but in many ways it is easier than real analysis. For example if a complex function is differentiable (defined later), then its derivative is also differentiable. This is not true for real functions. (Do you know an example of a differentiable real function with non-differentiable derivative?)

Real and imaginary parts of functions

We will often use z to denote a complex number and we will have $z = x + iy$ where x and y are both real. The value $f(z)$ is a complex number and so has a real and imaginary part. We often use u to denote the real part and v to denote the imaginary part. Note that u and v are functions of z.

We often write $f(x + iy) = u(x, y) + iv(x, y)$. Note that u is a function of *two* real variables, x and y. I.e. $u : \mathbb{R}^2 \to \mathbb{R}$. Similarly for v.

Examples 1.9

(i) Let $f(z) = z^2$. Then, $f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy$. So, $u(x, y) = x^2 - y^2$ and $v(x, y) = xy$.

(ii) Let $f(z) = |z|$. Then, $f(x + iy) = \sqrt{x^2 + y^2}$. So, $u(x, y) = \sqrt{x^2 + y^2}$ and $v(x, y) = 0$.

Exercises 1.10

Find u and v for the following:

(i) $f(z) = 1/z$ for $z \in \mathbb{C}\backslash\{0\}$.

(ii) $f(z) = z^3$.

Visualising complex functions

In Real Analysis we could draw the graph of a function. We have an axis for the variable and an axis for the value, and so we can draw the graph of the function on a piece of paper.

For complex functions we have a complex variable (that's two real variables) and the value (another two real variables), so if we want to draw a graph we will need $2 + 2 = 4$ *real* variables, i.e. we will have to work in 4-dimensional space. Now obviously this is a bit tricky because we are used to 3 space dimensions and find visualising 4 dimensional space very hard.

Thus, it is very difficult to visualise complex functions. However, there are some methods available:

(i) We can draw two complex planes, one for the domain and one for the range.

(ii) The two-variable functions u and v can be visualised separately. The graph of a function of two variables is a surface in three space.

$$u(x, y) = \cos x + \sin y \text{ and } v(x, y) = x^2 - y^2$$

(iii) Make one of the variables time and view the graph as something that evolves over time. This is not very helpful.

Defining e^z, $\cos z$ and $\sin z$

First we will try and define some elementary complex functions to play with. How shall we define functions such as e^z, $\cos z$ and $\sin z$? We require that their definition should coincide with the real version when z is a real number, and we would like them to have properties similar to the real versions of the functions, e.g. $\sin^2 z + \cos^2 z = 1$ would be nice. However, sine and cosine are defined using trigonometry and so are hard to generalise: for example, what does it mean for a triangle to have an hypotenuse of length $2 + 3i$? The exponential is defined using differential calculus and we have not yet defined differentiation of complex functions.

However, we know from Real Analysis that the functions can be described using a power series, e.g.,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Thus, for $z \in \mathbb{C}$, we shall *define* the exponential, sine and cosine of z as follows:

$$\begin{aligned} e^z &:= \sum_{n=0}^{\infty} \frac{z^n}{n!}, \\ \sin z &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \\ \cos z &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}. \end{aligned}$$

Thus,

$$e^{3+2i} = \sum_{n=0}^{\infty} \frac{(3+2i)^n}{n!} = 1 + (3+2i) + \frac{(3+2i)^2}{2!} + \frac{(3+2i)^3}{3!} + \dots$$

These definitions obviously satisfy the requirement that they coincide with the definitions we know and love for real z, but how can we be sure that the series converges? I.e. when we put in a z, such as $3+2i$, into the definition, does a complex number comes out?

To answer this we will have to study complex series and as the theory of *real series* was built on the theory of *real sequences* we had better start with complex sequences.

Complex Sequences

The definition of convergence of a complex sequence is the same as that for convergence of a real sequence.

Definition 1.11
A complex sequence $\langle c_n \rangle$ **converges** to $c \in \mathbb{C}$, if given any $\varepsilon > 0$, then there exists N such that $|c_n - c| < \varepsilon$ for all $n \geq N$.

We write $c_n \to c$ or $\lim_{n\to\infty} c_n = c$.

Example 1.12
The sequence $c_n = \left(\frac{4-3i}{7}\right)^n$ converges to zero.

Consider

$$|c_n - 0| = |c_n| = \left|\left(\frac{4-3i}{7}\right)^n\right| = \left|\frac{4-3i}{7}\right|^n = \left(\sqrt{\frac{25}{49}}\right)^n = \left(\frac{5}{7}\right)^n.$$

So

$$\begin{aligned} |c_n - 0| < \varepsilon \quad &\iff \quad (5/7)^n < \varepsilon \\ &\iff \quad n \log(5/7) < \log \varepsilon \\ &\iff \quad n > \frac{\log \varepsilon}{\log(5/7)}. \end{aligned}$$

So, given any ε we can choose N to be any natural number greater than $\log \varepsilon / \log(5/7)$. Thus the sequence converges to zero.

Remark 1.13
Notice that $a_n = |c_n - c|$ is a *real* sequence, and that $c_n \to c$ if and only if the real sequence $|c_n - c| \to 0$. Hence, we are saying something about a complex sequence using real analysis.

Paradigm 1.14
The remark above gives a good example of the paradigm[1] we will be using. We can apply results from real analysis to produce results in complex analysis. In this case we take the modulus, but we can also take real and imaginary parts.

This is a key observation. Note it well!

Let's apply the paradigm. The next proposition shows that a sequence converges if and only its real and imaginary parts do.

Proposition 1.15
Let $c_n = a_n + ib_n$ where a_n and b_n are real sequences, and $c = a + ib$. Then

$$c_n \to c \iff a_n \to a \text{ and } b_n \to b.$$

Proof. [$\Rightarrow$] If $c_n \to c$, then $|c_n - c| \to 0$. But

$$0 \leq |a_n - a| = |Re(c_n) - Re(c)| = |Re(c_n - c)| \leq |c_n - c|.$$

So by the squeeze rule $|a_n - a| \to 0$, i.e. $a_n \to a$. Similarly, $b_n \to b$.

[$\Leftarrow$] Suppose $a_n \to a$ and $b_n \to b$, then $|a_n - a| \to 0$, and $|b_n - b| \to 0$. We have

$$0 \leq |c_n - c| = |(a_n - a) + i(b_n - b)| \leq |a_n - a| + |b_n - b|.$$

The last inequality follows from the triangle inequality applied to $z = a_n - a$ and $w = i(b_n - b)$. Because $|a_n - a| \to 0$ and $|b_n - b| \to 0$ we deduce $|c_n - c| \to 0$, i.e. $c_n \to c$. $\square$

HTTLAM 1.16
Try not to use the definition of convergence to prove that a sequence converges.

Example 1.17

$$\frac{n^2 + in^3}{n^3 + 1} = \frac{n^2}{n^3 + 1} + i\frac{n^3}{n^3 + 1} \to 0 + i.1 = i.$$

Exercises 1.18
(i) Which of the following sequences converge(s)?

$$\frac{(n+1)^5}{n^5 i} \qquad \text{and} \qquad \left(\frac{5 - 12i}{6}\right)^n.$$

(ii) Show that the limit of a complex sequence is unique.

[1] Paradigm: a conceptual model underlying the theories and practice of a scientific subject. (Oxford English Dictionary).

Complex Series

Now that we have defined convergence of complex sequences we can define convergence of complex *series*.

Definition 1.19
A complex series $\sum_{k=0}^{\infty} w_k$ *converges* if and only if the sequence $\langle s_n \rangle$ formed by its partial sums $s_n = \sum_{k=0}^{n} w_k$ converges.

That is, the following sequences converges

$$\begin{aligned} s_0 &= w_0 \\ s_1 &= w_0 + w_1 \\ s_2 &= w_0 + w_1 + w_2 \\ s_3 &= w_0 + w_1 + w_2 + w_3 \\ &\vdots \end{aligned}$$

Let's apply the paradigm and give a result on complex series using real series.

Proposition 1.20
Let $w_k = x_k + iy_k$ where x_k and y_k are real for all k. Then,

$$\sum_{k=0}^{\infty} w_k \text{ converges } \iff \sum_{k=0}^{\infty} x_k \text{ and } \sum_{k=0}^{\infty} y_k \text{ converge.}$$

In this case

$$\sum_{k=0}^{\infty} w_k = \sum_{k=0}^{\infty} x_k + i \sum_{k=0}^{\infty} y_k.$$

Proof. Let $a_n = \sum_{k=0}^{n} x_k$, $b_n = \sum_{k=0}^{n} y_k$, and $s_n = \sum_{k=0}^{n} w_k$, and apply Proposition 1.15. The second part of the statement comes from equating real and imaginary parts. $\square$

Example 1.21
The series $\sum_{n=0}^{\infty} \frac{(-1)^n i}{n!}$ converges. Let $x_k = 0$ and $y_k = \frac{(-1)^k}{k!}$. Then $\sum x_k = 0$, obviously, and $\sum \frac{(-1)^k}{k!} = e^{-1}$.

Thus $\sum_{n=0}^{\infty} \frac{(-1)^n i}{n!}$ converges to i/e.

In real analysis we have some great ways to tell if a series is convergent, for example, the ratio test and the integral test. Can we use the real analysis tests in complex analysis? The next theorem says we can, but first let us make a definition.

Definition 1.22
We say $\sum_{k=0}^{\infty} w_k$ is **absolutely convergent** if the real series $\sum_{k=0}^{\infty} |w_k|$ converges.

This definition is really the same as in Real Analysis, it has merely been extended to complex numbers in a natural way.

Now for a very important theorem which says that if a series is absolutely convergent, then it is convergent.

Theorem 1.23
If $\sum_{k=0}^{\infty} |w_k|$ converges, then $\sum_{k=0}^{\infty} w_k$ converges.

This is a fantastic tool. Remember it. The assumption says something about a *real* series (we know lots about these!) and gives a conclusion about a *complex* series. Thus, we can apply the ratio test or comparison test to the real series and say something about the complex series. Great!

Proof. Let $w_k = x_k + iy_k$, with x_k and y_k real. Then $\sum_{k=0}^{\infty} |w_k|$ convergent implies that $\sum_{k=0}^{\infty} |x_k|$ is convergent (because $0 \leq |x_k| = |Re(w_k)| \leq |w_k|$ and we can apply the comparison test). So the real series $\sum_{k=0}^{\infty} x_k$ converges absolutely and we know from Real Analysis I that this implies that $\sum_{k=0}^{\infty} x_k$ converges. Similarly, the series $\sum_{k=0}^{\infty} y_k$ converges.

Then, $\sum_{k=0}^{\infty} w_k = \sum_{k=0}^{\infty} x_k + i\sum_{k=0}^{\infty} y_k$, by Proposition 1.20. $\square$

HTTLAM 1.24
When asked to show a series converges, show it absolutely converges.

Remark 1.25
Note that the converse to Theorem 1.23 is *not* true. We already know this from Real Analysis. For example, $\sum_{k=0}^{\infty} \frac{(-1)^k}{k}$ converges but $\sum_{k=0}^{\infty} \left|\frac{(-1)^k}{k}\right| = \sum_{k=0}^{\infty} \frac{1}{k}$ diverges.

We now prove an infinite version of the triangle inequality.

Lemma 1.26
Suppose that $\sum_{k=0}^{\infty} w_k$ converges absolutely. Then

$$\left|\sum_{k=0}^{\infty} w_k\right| \leq \sum_{k=0}^{\infty} |w_k|.$$

Proof. For $n \geq 1$,

$$\begin{aligned}
\left|\sum_{k=0}^{\infty} w_k\right| &= \left|\sum_{k=0}^{\infty} w_k - \sum_{k=0}^{n} w_k + \sum_{k=0}^{n} w_k\right| \\
&\leq \left|\sum_{k=0}^{\infty} w_k - \sum_{k=0}^{n} w_k\right| + \left|\sum_{k=0}^{n} w_k\right| \\
&\leq \left|\sum_{k=0}^{\infty} w_k - \sum_{k=0}^{n} w_k\right| + \sum_{k=0}^{n} |w_k|.
\end{aligned}$$

As $n \to \infty$ then obviously, $|\sum_{k=0}^{\infty} w_k - \sum_{k=0}^{n} w_k| \to 0$, hence the result. $\square$

Definition 1.27
A **complex power series** is a sum of the form $\sum_{k=0}^{m} c_k z^k$, where $c_k \in \mathbb{C}$ and m is possibly infinite.

Such a power series is a function of z. Much of the theory of differentiable complex functions is concerned with power series, because as we shall see later, any differentiable complex function can be represented as a power series.

Radius of Convergence

Just as with real power series we can have complex power series that do not converge on the whole of the complex plane.

Example 1.28
Consider the series $\sum_0^{\infty} z^n$, where $z \in \mathbb{C}$. We know for $z = 1$ this series does not converge because then we have $\sum_0^{\infty} 1^n = \sum_0^{\infty} 1 = 1 + 1 + 1 + \ldots$.

We also know it converges for $z = 0$, because $\sum_0^{\infty} 0^n = \sum_0^{\infty} 0 = 0 + 0 + 0 + \cdots = 0$. Hopefully, you remember from Real Analysis I that for *real* z the power series converges only for $-1 < z < 1$.

So, for which complex values of z does it converge? Let us use the ratio test. Let $a_n = |z^n|$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|z^{n+1}|}{|z^n|} = |z|.$$

As $n \to \infty$ we have $|z| \to |z|$, because there is no dependence on n. So by the ratio test the series $\sum a_n$ converges if $|z| < 1$, diverges if $|z| > 1$ and for $|z| = 1$ we don't know what will happen. So $\sum z^n$ converges absolutely, and hence converges, for $|z| < 1$.

That the set of complex numbers for which the series converges is given by something of the form $|z| < R$ for some R is a general phenomenon, as the next theorem shows.

Theorem 1.29
Let $\sum_0^{\infty} a_n z^n$ be some complex power series. Then, there exists R, with $0 \leq R \leq \infty$, such that

$$\sum_0^{\infty} a_n z^n \begin{cases} \text{converges } \underline{\text{absolutely}} \text{ for } |z| < R, \\ \text{diverges for } |z| > R. \end{cases}$$

Proof. The proof is similar to that for real power series used in Real Analysis I. Stewart and Tall also have a good proof, see p56-57. □

HTTLAM 1.30
Given a power series, immediately ask 'What is its radius of convergence?'

Exercise 1.31
Show that $\sum_0^\infty z^n/n$ has radius of convergence 1.

In the last exercise note that for $z = -1$ the series converges, but for $z = 1$ the series diverges, (both these fact should be well known from Real Analysis). This tells us that for $|z| = 1$ we can get some values of z for which the series converges and some for which the series diverges.

Sine, cosine, and exponential are defined for all complex numbers

Let us now return to showing that the sine, cosine and exponential functions are defined on the whole of $\mathbb{C}$.

Example 1.32
(I'll do this example in great detail. The next example will be more like the solution I would expect from you.)

The series $e^z = \sum_{n=0}^\infty \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$.

For any $z \in \mathbb{C}$ let $a_n = \left|\frac{z^n}{n!}\right|$. We want $\sum_{n=0}^\infty a_n$ to converge, so we use the ratio test on this real series. We have

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\left(\left|\frac{z^{n+1}}{(n+1)!}\right| \Big/ \left|\frac{z^n}{n!}\right|\right)\right| \\ &= \left|\frac{z^{n+1}}{z^n}\frac{n!}{(n+1)!}\right| \\ &= \frac{|z|}{n+1} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

The last part is true because for fixed z the real number $|z|$ is of course a finite constant.

So by the ratio test $\sum_{n=0}^\infty a_n = \sum_{n=0}^\infty \left|\frac{z^n}{n!}\right|$ converges. Thus by Theorem 1.23 the series $\sum_{n=0}^\infty \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$.

The following is an example with some of the small detail missing. This is how I would expect the solution to be given if I had set this as an exercise.

Example 1.33
The series $\sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$ converges for all $z \in \mathbb{C}$.

Let $a_n = \left|(-1)^n \frac{z^{2n+1}}{(2n+1)!}\right|$. Then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{z^{2(n+1)+1}}{(2(n+1)+1)!}\right| \Big/ \left|\frac{z^{2n+1}}{(2n+1)!}\right| \\
&= \left|\frac{z^{2n+3}}{(2n+3)!}\Big/\frac{z^{2n+1}}{(2n+1)!}\right| \\
&= \left|\frac{z^2}{(2n+3)(2n+2)}\right| \\
&= \frac{|z|^2}{(2n+3)(2n+2)} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

So by the ratio test the complex series converges absolutely, and hence converges.

Exercise 1.34
Prove that $\cos z$ converges for all z.

Properties of the exponential

We have defined the exponential function and shown that is defined on all of $\mathbb{C}$, let's now look at its properties. Most of these you may already from Numbers and Proofs, but the proofs may not have been rigorous.

Theorem 1.35
(i) $e^{\bar{z}} = \overline{e^z}$, for all $z \in \mathbb{C}$.

(ii) $e^{iz} = \cos z + i \sin z$, for all $z \in \mathbb{C}$.

(iii) $e^{z+w} = e^z e^w$, for all $z, w \in \mathbb{C}$.

(iv) $e^z \neq 0$, for all $z \in \mathbb{C}$.

(v) $e^{-z} = 1/e^z$, for all $z \in \mathbb{C}$.

(vi) $e^{nz} = (e^z)^n$, for all $z \in \mathbb{C}$ and $n \in \mathbb{Z}$.

(vii) $|e^z| = e^{Re(z)}$, for all $z \in \mathbb{C}$.

(viii) $|e^{iy}| = 1$, for all $y \in \mathbb{R}$.

Proof. (i) We have

$$e^{\bar{z}} = \sum_{n=0}^{\infty} \frac{(\bar{z})^n}{n!} = \sum_{n=0}^{\infty} \frac{\overline{(z^n)}}{n!} = \overline{\sum_{n=0}^{\infty} \frac{z^n}{n!}} = \overline{e^z}.$$

(ii) Exercise. (Just put iz into the power series and separate the real and imaginary parts.)

(iii) This will be delayed until we deal with differentiability.

(iv) Note that e^z and e^{-z} both exist. We have

$$\begin{aligned} e^z e^{-z} &= e^{z-z} \text{ by (iii),} \\ &= e^0 \\ &= 1, \text{ by calculation.} \end{aligned}$$

Thus, e^z cannot be zero.

(v) This is obvious from the proof of (iv).

(vi) Follows from repeated application of (iii).

(vii) We have

$$\begin{aligned} |e^z|^2 &= e^z \overline{e^z} \text{ by definition,} \\ &= e^z e^{\overline{z}} \text{ by (i),} \\ &= e^{z+\overline{z}} \text{ by (iii),} \\ &= e^{2Re(z)} \\ &= \left(e^{Re(z)}\right)^2 \text{ by (vi).} \end{aligned}$$

As both $|e^z|$ and $e^{Re(z)}$ are real and positive we deduce that (vii) is true.

(viii) From (vii) we get $|e^{iy}| = e^{Re(iy)} = e^0 = 1$. □

Corollary 1.36

(i) $e^{2\pi i} = 1$.

(ii) (De Moivre's Theorem) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $\theta \in \mathbb{R}$.

The proofs are left as simple exercises. Part (i) is one of the **best theorems in mathematics**. It relates so many different important numbers, e, $\sqrt{-1}$, π, and of course 1 and 2, in a simple expression.

Warning! 1.37
We have **not** shown that $e^{zw} = (e^z)^w$ for all $z, w \in \mathbb{C}$. This is because we have not yet *defined* a^b for all complex a and b. Consider $z = 2\pi i$ and $w = i$. Then $(e^z)^w = (e^{2\pi i})^i = 1^i$. What could 1^i be? [2]

Exercise 1.38
Prove that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{and} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

[2] The astute reader may say 'define it to be $e^{-2\pi}$.'

Another property of the complex exponential is that it is periodic.

Theorem 1.39
For any complex numbers z and w we have

$$e^z = e^w \iff z - w = 2\pi i n \text{ for some } n \in \mathbb{Z}.$$

Proof. [$\Rightarrow$] Let $z - w = x + iy$ where x and y are real. Then,

$$\begin{aligned}
e^z = e^w \quad &\iff \quad e^z/e^w = 1 \\
&\iff \quad e^{z-w} = 1 \\
&\iff \quad e^{x+iy} = 1 \qquad (*) \\
&\implies \quad |e^{x+iy}| = 1, \text{ (the implication does not reverse!)} \\
&\iff \quad |e^x e^{iy}| = 1 \\
&\iff \quad |e^x||e^{iy}| = 1 \\
&\iff \quad |e^x| = 1 \\
&\iff \quad e^x = 1, \text{ since the exponential of a real number is positive,} \\
&\iff \quad x = 0.
\end{aligned}$$

By $(*)$ we know that $e^{x+iy} = 1$, so $e^{iy} = 1$ as $x = 0$. Then,

$$\begin{aligned}
e^{iy} = 1 \quad &\iff \quad \cos y + i \sin y = 1 \\
&\iff \quad \cos y = 1 \text{ and } \sin y = 0 \\
&\iff \quad y = 2\pi n \text{ for some } n \in \mathbb{Z}.
\end{aligned}$$

So $z - w = x + iy = 0 + i.2\pi n = 2\pi i n$.

[$\Leftarrow$] Suppose that $z - w = 2\pi i n$ for some $n \in \mathbb{Z}$. Then, $e^{z-w} = e^{2\pi i n} = (e^{2\pi i})^n = 1^n = 1$. But from the working in the earlier part of the proof we know this is equivalent to $e^z = e^w$. $\square$

This theorem has serious repercussions for defining the inverse of e^z, i.e. defining the log function.

Definition of the complex logarithm

We all know that the real exponential function has an inverse function called $\log_e$ or just $\ln$. Is there an inverse for the complex exponential?

Well, to define the real log of a number x we want some *unique* number y such that $e^y = x$, that is the crux of the definition of inverse. So let's suppose we have a complex number w, then we want some z such that $e^z = w$. Let's investigate this.

Proposition 1.40
For complex numbers z and $w \neq 0$ we have

$$e^z = w \iff z = \ln|w| + i(\arg(w) + 2k\pi), \text{ for some } k \in \mathbb{Z}.$$

Proof. [$\Rightarrow$] Write $z = x+iy$, so we get $e^{x+iy} = e^x(\cos y + i \sin y) = w$. Now let us take the modulus of both sides:

$$\begin{aligned}
|e^{x+iy}| &= |w| \\
|e^x||e^{iy}| &= |w| \\
|e^x| &= |w| \\
e^x &= |w| \\
\log_e e^x &= \log_e |w| \text{ (using the real log function)} \\
x &= \ln |w|.
\end{aligned}$$

Now suppose that $w = r(\cos\theta + i \sin\theta)$ for some real r and θ, i.e. $r = |w|$ and $\theta = \arg(w)$. Then $e^z = w$ implies that $r = \ln |w|$ and $y = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. So z has the form in the statement.

[$\Leftarrow$] If $z = \ln |w| + i(\arg(w) + 2\pi k)$ for some $k \in \mathbb{Z}$ then

$$\begin{aligned}
e^z &= e^{\ln|w| + i(\arg(w)+2\pi k)} = e^{\ln|w|} e^{i\arg(w)} e^{2\pi i k} \\
&= |w| e^{i \arg(w)} \left(e^{2\pi i}\right)^k = |w| e^{i\arg(w)} = w.
\end{aligned}$$

$\square$

Example 1.41
Solve $e^z = 1 + i\sqrt{3}$.

Solution: Let $w = 1 + i\sqrt{3}$. The modulus of w is $|w| = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2$. By drawing a picture (or through careful use of calculator) we can see that $\arg(w) = \dfrac{\pi}{3} + 2n\pi$, $n \in \mathbb{Z}$. So

$$z = \ln 2 + i\left(\frac{\pi}{3} + 2n\pi\right), \ n \in \mathbb{Z}.$$

HTTLAM 1.42
Notice how well working out the modulus and argument serves us. Conclusion: calculate modulus and argument.

Common Error 1.43
Don't forget the $2k\pi$ with the argument.

Exercise 1.44
Solve $e^{2iz} = i$. (It's not $i(\pi/2 + 2k\pi)$.)

So does the theorem allow us to define the log of a complex number? Yes, if we define the log to be the complex number with $-\pi < \arg(w) \leq \pi$. (The point is that if we have a w then the proposition gives us lots of zs to choose from. If z is such that $e^z = w$, then $z + 2\pi i$ will work just as well ($e^{z+2\pi i} = e^z e^{2\pi i} = e^z.1 = e^z = w$). Thus, there is some ambiguity and we make a choice.) However, if we are trying to solve an equation and take the log of both sides using this definition, then we may be losing solutions. So in fact the best definition is to make the function *multi-valued.* This is something we will not go into in great depth just now.

Another worked example

Example 1.45
Solve the equation $\sin z = 2$.

Solution: We can rewrite this as $\dfrac{e^{iz} - e^{-iz}}{2i}$. Let $w = e^{iz}$. Then the equation becomes $\dfrac{1}{2i}\left(w - \dfrac{1}{w}\right) = 2$. So,

$$\begin{aligned}
w^2 - 1 &= 4iw \\
w^2 - 4iw - 1 &= 0 \\
w &= \frac{4i \pm \sqrt{(4i)^2 + 4}}{2} \\
&= \frac{4i \pm \sqrt{-12}}{2} \\
&= 2i \pm \sqrt{-3} \\
&= (2 \pm \sqrt{3})i.
\end{aligned}$$

Now, $e^{iz} = w = (2 \pm \sqrt{3})i$, so

$$\begin{aligned}
iz &= \ln|w| + i\arg(w) \\
&= \ln|(2 \pm \sqrt{3})i| + i\left(\frac{\pi}{2} + 2n\pi\right) \\
&= \ln|2 \pm \sqrt{3}| + i\left(\frac{\pi}{2} + 2n\pi\right) \\
z &= \frac{1}{i}\left(\ln|2 \pm \sqrt{3}| + i\left(\frac{\pi}{2} + 2n\pi\right)\right) \\
&= -i\left(\ln|2 \pm \sqrt{3}| + i\left(\frac{\pi}{2} + 2n\pi\right)\right) \\
&= \left(\frac{\pi}{2} + 2n\pi\right) - i\ln|2 \pm \sqrt{3}| \\
&= \left(\frac{\pi}{2} + 2n\pi\right) - i\ln(2 \pm \sqrt{3}).
\end{aligned}$$

(The last equality is true because $\ln(2+\sqrt{3}) > 0$ and $\ln(2-\sqrt{3}) > 0$.)

HTTLAM 1.46
Note that in the above example we replaced e^{iz} with another complex number w, because we could then get a polynomial equation.

Exercise 1.47
Show that

$$\begin{aligned}
\sin z = 0 \quad &\iff \quad z = k\pi,\ k \in \mathbb{Z}, \\
\cos z = 0 \quad &\iff \quad z = \frac{1}{2}(2k+1)\pi,\ k \in \mathbb{Z}.
\end{aligned}$$

These results will be used later.

Summary

- Paradigm: Complex analysis is developed by reducing to real analysis, often through taking the modulus.
- We define exponential, sine and cosine by power series.
- If $\sum_{n=0}^{\infty} |w_k|$ converges, then $\sum_{n=0}^{\infty} w_k$ converges.
- Apply the ratio test, comparison test, etc, to the modulus of terms of a complex series to determine convergence.
- For power series use the ratio test to find radius of convergence.

2 Complex Riemann Integration

In a later section we define contour integration, that is integration over a complex variable. This notion is fundamental in complex analysis. But let us first generalise integration and differentiation to complex-valued functions of a *real* variable.

A complex-valued function of a real variable is a map $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{R}$. E.g. If $f(t) = (2+3i)t^3$, $t \in \mathbb{R}$, then $f(1) = 2+3i \in \mathbb{C}$.

Such a function is different to a complex function. A complex-valued function of a real variable takes a real number and produces a complex number. A complex function takes a complex number from a domain and produces a complex number.

Differentiation of complex valued real functions

Suppose that $f : \mathbb{R} \to \mathbb{C}$ is given by $f(t) = (1+3i)t^2$. If we define $f'(t)$ using the standard definition:

$$f'(t) = \lim_{\delta \to 0} \frac{f(t+\delta) - f(t)}{\delta}, \ (\delta \in \mathbb{R}),$$

then we get $f'(t) = 2(1+3i)t$. That is, in this case, the rule $f'(t) = nct^{n-1}$ for $f(t) = ct^n$, holds even though c is complex.

Basically, all similar rules work in this way, any constants can be real or complex. So, for instance,

$$\begin{aligned} \frac{d}{dt} e^{ct} &= ce^{ct}, \\ \frac{d}{dt} \sin(ct) &= c\cos(ct), \\ \frac{d}{dt} \cos(ct) &= -c\sin(ct). \end{aligned}$$

Exercise 2.1
Let $\phi(x) = 3x^3 + 2ix - i + \tan((4+2i)x)$. Then,

$$\phi'(x) =$$

Remark 2.2
This is not the same as differentiation with respect to a complex variable.[3] That will come later.

Complex Riemann Integrals

Now we shall *integrate* complex-valued functions with respect to one real variable. We shall do this with a bit more care than differentiation.

[3] They do behave in much the same way though.

Definition 2.3
Let $g : [a, b] \to \mathbb{C}$ be given $g(t) = u(t) + iv(t)$, We say that g is **complex Riemann integrable** (abbreviated $\mathbb{C}$-RI) if both u and v are RI as real functions, and we define $\int_a^b g(t)\, dt$ by

$$\int_a^b g = \int_a^b g(t)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt.$$

Example 2.4

$$\begin{aligned}
\int_a^b e^{3it}\, dt &= \int_a^b \cos 3t\, dt + i \int_a^b \sin 3t\, dt \\
&= \frac{1}{3}\left[\sin 3b - \sin 3a\right] + \frac{i}{3}\left[-\cos 3b + \cos 3a\right] \\
&= \frac{1}{3i}\left(-e^{3ib} + e^{3ia}\right) \\
&= \frac{i}{3}\left(e^{3ib} - e^{3ia}\right).
\end{aligned}$$

Many properties of $\mathbb{C}$-RI can be derived from the corresponding properties for $\mathbb{R}$-RI by considering the real and imaginary parts. For example, if u and v are continuously differentiable, then since $g' = u' + iv'$ we get a version of the Fundamental Theorem of Calculus:

$$\int_a^b g' = \int_a^b u' + i \int_a^b v' = [u(b) - u(a)] + i[v(b) - v(a)] = g(b) - g(a).$$

Other standard methods, such as substitution also work.

Obviously, separating functions into real and imaginary parts can get a bit tedious. Fortunately, just as for differentiation above, we can use standard integrals, replacing real constants by complex ones. It is not difficult to prove the following, where a and C are complex constants.

Example 2.5

$$\begin{aligned}
\int at^n\, dt &= a\frac{t^{n+1}}{n+1} + C, \\
\int e^{at}\, dt &= \frac{e^{at}}{a} + C, \\
\int \sin at\, dt &= -\frac{1}{a}\cos(at) + C, \\
\int \cos at\, dt &= \frac{1}{a}\sin(at) + C.
\end{aligned}$$

So a lot of the time we can use standard integrals to calculate complex-valued integrals with respect to a real variable t.

Exercise 2.6
Calculate $\int_0^2 t^2 - it^3 - \cos(2t)\,dt$.

Triangle inequality for $\mathbb{C}$-RI

We need the following result later. Its format should be familiar from real analysis, the only difference here is that the functions can be complex-valued.

Lemma 2.7
If $g : [a, b] \to \mathbb{C}$ is $\mathbb{C}$-RI and $|g|$ is $\mathbb{R}$-RI, then

$$\left| \int_a^b g(t)\,dt \right| \leq \int_a^b |g(t)|\,dt.$$

Proof. If $LHS = 0$, then the statement is trivial. Hence, assume $LHS \neq 0$. Let $\alpha = |\int_a^b g| / \int_a^b g$. (Hence $|\alpha| = 1$).
So,

$$\begin{aligned}
\left| \int_a^b g(t)\,dt \right| &= \alpha \int_a^b g(t)\,dt \\
&= \int_a^b Re\left(\alpha g(t)\right)\,dt + i \int_a^b Im(\alpha g(t))\,dt \\
&= \int_a^b Re\left(\alpha g(t)\right)\,dt, \text{ because LHS is real,} \\
&\leq \int_a^b |\alpha g(t)|\,dt, \text{ as } Re(z) \leq |z|, \\
&= \int_a^b |\alpha|\,|g(t)|\,dt \\
&= \int_a^b |g(t)|\,dt, \text{ as } |\alpha| = 1.
\end{aligned}$$

$\square$

Summary

- We can integrate and differentiate complex-valued functions of real variables in the same way as real-valued functions of real variables.

3 Contours

In the next section define integration along a contour in the complex plane.[4] This is a fairly abstract process, the *meaning* of which usually takes a little time to understand. Fortunately, it is easy to *do* as it has similar properties to Riemann integration of one real variable, and you have years of experience of that.

In case you think that it is too abstract and not relevant to real problems, then consider the integral

$$\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin n\theta)\, d\theta.$$

This is a seriously nasty integral! Imagine trying to solve it via the methods we know. Using contour integration we shall show that it is very simple to calculate.

First though we will define contours.

Contours

Definition 3.1
A **contour** (also called a **path**) is a continuous map $\gamma : [a, b] \to \mathbb{C}$ which is piecewise smooth, i.e. there exist $a = a_0 < a_1 < a_2 < \cdots < a_n = b$ such that

(i) $\gamma|[a_{j-1}, a_j]$ is differentiable, for all j,

(ii) γ' is continuous on $[a_{j-1}, a_j]$, for all j.

(The left and right derivatives of γ at a_j may differ.)

We say γ is **closed** if $\gamma(a) = \gamma(b)$.

Warning! 3.2
A contour is not a complex function. It is a complex-valued function of a real variable. Its image is usually some curve in the plane.

Examples 3.3

(i) Straight line from α to β: This is $\gamma : [0, 1] \to \mathbb{C}$ given by $\gamma(t) = \alpha + t(\beta - \alpha)$.

(ii) Circle of radius r based at the origin: $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = re^{it}$.

(iii) Circle of radius r based at z_0: $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = z_0 + re^{it}$.

[4] Those of you who have done MATH2360 or MATH2420 will see that this is just a line integral.

(iv) Circular arc of radius r based at w: $\gamma(t) = w + re^{it}$, $\theta_1 \leq t \leq \theta_2$. (So $0 \leq t \leq 2\pi$ gives the circle above).

(v) Let $\alpha : [-1, \pi/2] \to \mathbb{C}$ be given by

$$\alpha(t) = \begin{cases} t+1, & \text{for } -1 \leq t \leq 0, \\ e^{it} & \text{for } 0 \leq t \leq \pi/2. \end{cases}$$

Draw the image:

We draw an arrow to show the direction we go in.

(vi) Let $\gamma : [0, 4\pi] \to \mathbb{C}$ be given by $\gamma(t) = e^{it}$. Then the image of γ is the unit circle centred at zero. The contour goes round the circle twice. This subtlety will be important later.

HTTLAM 3.4
Given a contour, try to draw its image.

Common Error 3.5
There is often confusion between a contour and its image. A contour is not a set of points in the complex plane, it is a map.

Consider the contours (ii) and (vi) above, taking $r = 1$ in (ii). They have the same image, the unit circle. However, the contours are different, one maps from $[0, 2\pi]$, the other $[0, 4\pi]$.

We do use the notation $z \in \gamma$ later, by which we mean $z \in \gamma([a, b])$. Strictly speaking, writing $z \in \gamma$ is incorrect because γ is not a set.

Since contours are complex-valued functions of a real variables, we can differentiate them, etc, with ease.

Summary

- A contour is a continuous map $\gamma : [a, b] \to \mathbb{C}$ which is piecewise smooth. It is a complex-valued function of a real variable.
- Straight line from α to β: $\gamma : [0, 1] \to \mathbb{C}$ given by $\gamma(t) = \alpha + t(\beta - \alpha)$.
- Circle of radius r based at z_0: $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = z_0 + re^{it}$.

4 Contour Integration

We now come to probably the most important definition in complex analysis: contour integral. It is central to the module. If you don't understand this section, then the rest of the course will be a complete mystery to you.

Definition 4.1
Let $f : D \to \mathbb{C}$ be a continuous complex function and $\gamma : [a, b] \to \mathbb{C}$ be a contour. Then, the **integral of** f **along** γ is

$$\int_\gamma f = \int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

Note that $f(\gamma(t))$ and $\gamma'(t)$ are complex-valued functions of a real variable, and hence so is their product. Thus we can integrate this product.

Example 4.2
Let $\gamma(t) = t + it^2$ for $0 \le t \le 2$ and $f(z) = z$. Then, $\gamma'(t) = 1 + 2it$, and

$$\begin{aligned}
\int_\gamma f &= \int_0^2 (t + it^2)(1 + 2it)\, dt \\
&= \int_0^2 t + 2it^2 + it^2 + 2i^2t^3\, dt \\
&= \int_0^2 t + 3it^2 - 2t^3\, dt \\
&= \left[\frac{1}{2}t^2 + \frac{3it^3}{3} - \frac{2t^4}{4}\right]_0^2 \\
&= \left[\frac{1}{2}t^2 + it^3 - \frac{t^4}{2}\right]_0^2 \\
&= \frac{1}{2}2^2 + i2^3 - \frac{2^4}{2} \\
&= -6 + 8i.
\end{aligned}$$

Example 4.3
Let $\gamma(t) = 2 + it^2$ for $0 \le t \le 1$ and $f(z) = z^2$. Then,

$$\begin{aligned}
\int_\gamma z^2\, dz &= \int_0^1 (2+it^2)^2(2it)\, dt \\
&= \int_0^1 \frac{d}{dt}\left(\frac{(2+it^2)^3}{3}\right) dt \\
&= \left[\frac{(2+it^2)^3}{3}\right]_0^1 \\
&= \frac{(2+i.1^2)^3}{3} - \frac{(2+i.0^2)^3}{3} \\
&= \frac{1}{3}\left((2+i)^3 - 8\right) \\
&= -2 + \frac{11}{3}i.
\end{aligned}$$

This is just the sort of example you need to be able to do with ease.

Exercise 4.4
Draw the contours and calculate the integrals of the functions along the contours.

(i) $f_1(z) = Re(z)$ and $\gamma_1(t) = t$, $0 \le t \le 1$.

(ii) $f_2(z) = Re(z)$ and $\gamma_2(t) = t + it$, $0 \le t \le 1$.

(iii) $f_3(z) = Re(z)$ and $\gamma_3(t) = 1 - t + i(1-t)$, $0 \le t \le 1$.

(iv) $f_4(z) = 1/z$ and $\gamma_4(t) = 2e^{-it}$, $0 \le t \le \pi$.

(v) $f_5(z) = z^2$ and $\gamma_5(t) = e^{it}$, $0 \le t \le \pi/2$.

Can you justify the results in (ii) and (iii)? Can you make any conjectures, say, involving $f(z) = z^n$ in (v)?

Remark 4.5
Note that in the definition of contour integral we only require f to be continuous. The resulting integrand $f(\gamma(t))\gamma'(t)$ is $\mathbb{C}$-RI because it is continuous except possibly at finitely many points where $\gamma'(t)$ is discontinuous. In practice we subdivide $[a, b]$ into pieces $[a_{j-1}, a_j]$ and calculate

$$\int_a^b f(\gamma(t))\gamma'(t)\, dt = \sum_{j=1}^n \int_{a_{j-1}}^{a_j} f(\gamma(t))\gamma'(t)\, dt.$$

Example 4.6
Let γ be as in Example 3.3(v). Find $\int_\gamma z^2\,dz$.

$$\begin{aligned}
\int_\gamma z^2\,dz &= \int_{-1}^{\pi/2} \gamma(t)^2\gamma'(t)\,dt \\
&= \int_{-1}^{0} (t+1)^2.1\,dt + \int_0^{\pi/2} \left(e^{it}\right)^2 ie^{it}\,dt \\
&= \int_{-1}^{0} (t+1)^2\,dt + \int_0^{\pi/2} ie^{3it}\,dt \\
&= \left[\frac{1}{3}(t+1)^3\right]_{-1}^{0} + \left[\frac{i}{3i}e^{3it}\right]_0^{\pi/2} \\
&= \left[\frac{1}{3} - 0\right] + \frac{i}{3i}\left[-i-1\right] \\
&= -\frac{i}{3}.
\end{aligned}$$

Remarks 4.7

(i) Suppose $f : D \to \mathbb{C}$ is a complex function such that $f(x)$ is real for x real, for example, $\sin x$. If we take $\gamma : [a,b] \to \mathbb{C}$ given by $\gamma(t) = t$ for $a \le t \le b$, then

$$\int_\gamma f(z)\,dz = \int_a^b f(t)\gamma'(t)\,dt = \int_a^b f(t)\,dt.$$

Thus, by taking a contour along the real line, we can see that contour integration includes the theory of real integration as a special case.

(ii) From a purely formal viewpoint, we can justify the definition of contour integral by saying that we are replacing z by $\gamma(t)$ so we need to replace dz by $\gamma'(t)dt$, (which can be thought of as $(dz/dt)dt$).

Fundamental Example

Take the function defined by $f(z) = (z - w)^n$ where $n \in \mathbb{Z}$, (so for $n < 0$ the map is not defined at w). Let γ be a circle with centre w and radius $r > 0$, i.e. $\gamma(t) = w + re^{it}$, $0 \leq t \leq 2\pi$. Then,

$$\begin{aligned}
\int_\gamma (z-w)^n \, dz &= \int_0^{2\pi} (\gamma(t) - w)^n \, \gamma'(t) \, dt \\
&= \int_0^{2\pi} \left(w + re^{it} - w\right)^n . ire^{it} \, dt \\
&= \int_0^{2\pi} r^n e^{int} . ire^{it} \, dt \\
&= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt \\
&= \begin{cases} \dfrac{r^{n+1}}{n+1} \left[e^{i(n+1)t}\right]_0^{2\pi} & = 0, \quad \text{if } n \neq -1, \\ i \int_0^{2\pi} 1.dt & = 2\pi i, \quad \text{if } n = -1. \end{cases}
\end{aligned}$$

Thus $\displaystyle\int_\gamma \frac{1}{z-w} \, dz = 2\pi i$ and $\displaystyle\int_\gamma (z-w)^n \, dz = 0$ for $n \neq -1$.

Note this well, this innocuous looking calculation will be used to devastating effect later!

Summary

- The integral of f along γ is

$$\int_\gamma f = \int_\gamma f(z) \, dz = \int_a^b f(\gamma(t))\gamma'(t) \, dt.$$

- $\displaystyle\int_\gamma \frac{1}{z-w} \, dz = 2\pi i, \qquad \gamma(t) = w + re^{it}, \; 0 \leq t \leq 2\pi.$

- $\displaystyle\int_\gamma (z-w)^n \, dz = 0$ for $n \neq -1, \qquad \gamma(t) = w + re^{it}, \; 0 \leq t \leq 2\pi.$

5 Properties of Contour Integration

There are a number of well known properties of ordinary integration:

$$\begin{aligned}
\int_a^b \lambda f(x) + \mu g(x)\, dx &= \lambda \int_a^b f(x)\, dx + \mu \int_a^b g(x)\, dx,\ \ \lambda, \mu \in \mathbb{R}, \\
\int_a^b f(x)\, dx &= \int_a^c f(x)\, dx + \int_c^b f(x)\, dx, \quad a < c < b, \\
\int_a^b f(x)\, dx &= -\int_b^a f(x)\, dx, \\
\int_a^b f(x)\, dx &= \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\phi(y))\phi'(y)\, dy \text{ (change of variables)}.
\end{aligned}$$

All of these have analogues in contour integration. We shall now describe them.

In the following the functions will be continuous on some domain D and the contours will be maps into D.

Linearity.

If f, g are continuous on D, γ a contour in D, $\lambda, \mu \in \mathbb{C}$, then

$$\int_\gamma \lambda f + \mu g = \lambda \int_\gamma f + \mu \int_\gamma g.$$

So this has the same format as in integration over a real variable.

The proof follows directly from the linearity property of Riemann integration. The details are as follows:

$$\begin{aligned}
\int_\gamma \lambda f + \mu g &= \int_a^b (\lambda f + \mu g)(\gamma(t))\gamma'(t)\, dt \\
&= \int_a^b (\lambda f(\gamma(t)) + \mu g(\gamma(t)))\, \gamma'(t)\, dt \\
&= \lambda \int_a^b f(\gamma(t))\gamma'(t)\, dt + \mu \int_a^b g(\gamma(t))\gamma'(t)\, dt \\
&= \lambda \int_\gamma f + \mu \int_\gamma g.
\end{aligned}$$

Integration over joins

What if produce a contour by doing one after another?

Definition 5.1
If $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [b,c] \to \mathbb{C}$ are two contours such that $\gamma_1(b) = \gamma_2(b)$, then their **join** (or **sum**) is the contour $\gamma_1 + \gamma_2 : [a,c] \to \mathbb{C}$ given by

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & a \leq t \leq b, \\ \gamma_2(t) & b \leq t \leq c. \end{cases}$$

The join is continuous by the glue rule.

Example 5.2
Consider Example 3.3(v). The contour $\alpha : [-1, \pi/2] \to \mathbb{C}$ is given by

$$\alpha(t) = \begin{cases} t+1, & \text{for } -1 \leq t \leq 0, \\ e^{it} & \text{for } 0 \leq t \leq \pi/2. \end{cases}$$

This can be produced from $\gamma_1(t) = t + 1$ for $-1 \leq t \leq 0$ and $\gamma_2(t) = e^{it}$ for $0 \leq t \leq \pi/2$.

Proposition 5.3
For any continuous f

$$\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

Again, this follows from applying the definition and using a $\mathbb{C}$-RI property: $\int_a^b = \int_a^c + \int_c^b$.

Example 5.4
Find $\displaystyle\int_\gamma \frac{1}{(z-w)}\,dz$, where γ is the boundary of the square with corners $w \pm l \pm il$, starting at $w + l - li$ and going anticlockwise.

So $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. By Proposition 5.3 we have

$$\int_\gamma \frac{dz}{z-w} = \sum_{i=1}^{4} \int_{\gamma_i} \frac{dz}{z-w}.$$

To compute $\int_{\gamma_1}$ let $\gamma_1(t) = w + l + it$, $-l \le t \le l$. We have,

$$\begin{aligned}
\int_{\gamma_1} \frac{dz}{z-w} &= \int_{-l}^{l} \frac{\gamma_t'(t)}{\gamma_1(t) - w}\, dt \\
&= \int_{-l}^{l} \frac{i}{l+it}\, dt \\
&= \int_{-l}^{l} \frac{l-it}{l^2+t^2}\, i\, dt \\
&= \int_{-l}^{l} \frac{t}{l^2+t^2}\, dt + i \int_{-l}^{l} \frac{l}{l^2+t^2}\, dt \\
&= 0 + i \left[\tan^{-1} \frac{t}{l}\right]_{-l}^{l} \\
&= i\frac{\pi}{2}.
\end{aligned}$$

Similarly $\int_{\gamma_2} = \int_{\gamma_3} = \int_{\gamma_4} = \dfrac{\pi}{2} i$ (check!). So,

$$\int_{\gamma} \frac{dz}{z-w} = 2\pi i.$$

Notice that this coincides with the value of the integral when γ is a circle, see the Fundamental Example.

Reverse contour

Definition 5.5
If $\gamma : [a,b] \to \mathbb{C}$ is a contour, then its **reverse** is the contour $(-\gamma)(t) = \gamma(a+b-t)$, for $a \le t \le b$.

The point is that we do γ backwards. Instead of starting at $\gamma(a)$ we end there, etc.:

$$(-\gamma)(a) = \gamma(a+b-a) = \gamma(b),$$
$$(-\gamma)(b) = \gamma(a+b-b) = \gamma(a).$$

Proposition 5.6
For any continuous f, we have

$$\int_{-\gamma} f = -\int_{\gamma} f.$$

Proof. Again we apply definitions and use RI properties, but in this case we also need a simple change variables: $s = a+b-t$, so $dt = -ds$. Note that $(-\gamma)'(t) = (-1)\gamma(a+b-t)$. We have,

$$\begin{aligned}
\int_{-\gamma} f &= \int_a^b f((-\gamma)(t))(-\gamma)'(t)\, dt \\
&= \int_a^b f(\gamma(a+b-t))(-1)\gamma'(a+b-t)\, dt \\
&= -\int_b^a f(\gamma(s))\gamma'(s)\,(-ds) \\
&= \int_b^a f(\gamma(s))\gamma'(s)\, ds \\
&= -\int_a^b f(\gamma(s))\gamma'(s)\, ds \\
&= -\int_\gamma f.
\end{aligned}$$

□

Example 5.7
See Exercise 4.4(ii) and (iii). Then $\gamma_3 = -\gamma_2$ as follows. We have $a = 0$ and $b = 1$, so

$$(-\gamma_2)(t) = \gamma_2(0+1-t) = \gamma_2(1-t) = (1-t) + i(1-t) = \gamma_3(t).$$

This explains why $\int_{\gamma_3} f_3(z)\, dz = -\int_{\gamma_2} f_2(z)\, dz$.

Reparametrisation

Now we shall do the analogue of a change of variables.

Definition 5.8
Let $\gamma : [a,b] \to \mathbb{C}$ be a contour and let $\phi : [c,d] \to [a,b]$ be a function such that

(i) ϕ is continuously differentiable ,

(ii) $\phi'(s) > 0$ for all $s \in [c,d]$,

(iii) $\phi(c) = a$, and $\phi(d) = b$.

Then the contour $\widetilde{\gamma} : [c,d] \to \mathbb{C}$ defined by $\widetilde{\gamma}(s) = \gamma(\phi(s))$ is called a **reparametrisation** of γ.

Example 5.9
Let $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$. Let $\phi(s) = 2\pi s$ for $0 \le s \le 1$. Then $\widetilde{\gamma}(s) = e^{2\pi i s}$ for $0 \le s \le 1$.

We think of reparametrised contours as equivalent since we have the following.

Proposition 5.10
If $\widetilde{\gamma}$ is a reparametrisation of γ, then

$$\int_{\widetilde{\gamma}} f = \int_{\gamma} f,$$

for all continuous f.

Proof. We have,

$$\begin{aligned}\int_{\widetilde{\gamma}} f &= \int_c^d f(\widetilde{\gamma}(s))\,\widetilde{\gamma}'(s)\,ds \\ &= \int_c^d f(\gamma(\phi(s))\,\gamma'(\phi(s))\,\phi'(s)\,ds \\ &= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt, \text{ using } t=\phi(s), \\ &= \int_{\gamma} f.\end{aligned}$$

□

Thus, it does not matter whether we evaluate the integral $\int_\gamma 1/(z-w)\,dz$ using $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$ or $\gamma(t) = e^{2\pi it}$ for $0 \le t \le 1$.

Remarks 5.11

(i) If $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ are two contours such that $\gamma_1(b) = \gamma_2(c)$, then we can reparametrise γ_2 as

$$\widetilde{\gamma}_2 : [b, d+b-c] \to \mathbb{C} \text{ given by } \widetilde{\gamma}(s) = \gamma_2(c-b+s).$$

Then $\gamma_1(b) = \widetilde{\gamma}_2(b)$ so we can form the join $\gamma_1 + \widetilde{\gamma}_2$. Often we abuse notation and write this simply as $\gamma_1 + \gamma_2$.

(ii) If γ is a **simple** (i.e. it doesn't cross itself) closed contour with no orientation specified, then it is traversed anticlockwise.

(iii) If $\widetilde{\gamma}$ is a reparametrisation of γ, then it has the same image as γ. The converse is not true.

Summary

- If f, g are continuous on D, γ a contour in D, $\lambda, \mu \in \mathbb{C}$, then
$$\int_\gamma \lambda f + \mu g = \lambda \int_\gamma f + \mu \int_\gamma g.$$
- If f is continuous on D, γ a contour in D, then
$$\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$
- If $\gamma : [a, b] \to \mathbb{C}$ is a contour, then its reverse is the contour $(-\gamma)(t) = \gamma(a + b - t)$, for $a \leq t \leq b$.
- For f continuous, we have
$$\int_{-\gamma} f = -\int_\gamma f.$$
- If $\widetilde{\gamma}$ is a reparametrisation of γ, then
$$\int_{\widetilde{\gamma}} f = \int_\gamma f,$$

6 The Estimation Lemma

Recall for a real continuous function that

$$\left| \int_a^b f(x)\, dx \right| \leq \sup_{x \in [a,b]} \{|f(x)|\} \times (b-a).$$

We would like a complex version of this, that is, is there some bound on $|\int_\gamma f(z)\, dz|$? It is obvious that the first part in the product above can be generalised, but what does $b-a$ correspond to? Those of you familiar with measure theory will know it is the length of the interval from a to b, and so the generalisation to complex analysis is the length of the contour γ.

Definition 6.1
Let $\gamma : [a,b] \to \mathbb{C}$ be a contour. The **length of** γ, denoted $L(\gamma)$, is defined to be

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

This really does measure the length of the curve. Intuitively speaking, $\gamma'(t)$ is the velocity of a contour, so $|\gamma'(t)|$ is the speed, and the integral of speed over time gives the length of a path.

Example 6.2
Let $\gamma(t) = \alpha + t(\beta - \alpha)$ with $0 \leq t \leq 1$ be the straight line contour from α to β.

We have

$$\begin{aligned} L(\gamma) &= \int_0^1 |\gamma'(t)|\, dt = \int_0^1 |\beta - \alpha|\, dt = |\beta - \alpha| \int_0^1 dt = |\beta - \alpha| [t]_0^1 \\ &= |\beta - \alpha|. \end{aligned}$$

So, the length of the contour from α to β is $|\beta - \alpha|$, which is reassuring.

Exercise 6.3
Show that the length of the contour given by traversing once round the the circle of radius r based at the origin is 2π.

The next theorem is similar to one from real analysis:

Lemma 6.4 (Estimation Lemma)
Let $f : D \to \mathbb{C}$ be a continuous complex function and $\gamma : [a,b] \to D$ be a contour. Suppose that $|f(z)| \leq M$ for all $z \in \gamma$. Then

$$\left| \int_\gamma f(z)\, dz \right| \leq M\, L(\gamma).$$

Proof. We have,

$$\begin{aligned}
\left|\int_\gamma f(z)\,dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \\
&\leq \int_a^b |f(\gamma(t))\gamma'(t)|\;dt \text{ by Lemma 2.7} \\
&\leq \int_a^b M|\gamma'(t)|dt \\
&= ML(\gamma).
\end{aligned}$$

□

Example 6.5
Show that

$$\left|\int_\gamma \frac{e^z}{z+1}\,dz\right| \leq \frac{\pi R}{R-1}$$

where γ describes the semi-circle from iR to $-iR$ in the left half-plane $\{z : Re(z) \leq 0\}$ and $R > 1$.

Solution: If $z = x + iy$ lies on the image of γ, then $x \leq 0$ and so $|e^z| = e^{Re(z)} = e^x \leq e^0 = 1$. Also, when z lies on γ, $|z| = R$ so $R = |z+1-1| \leq |z+1| + 1$. Thus $|z+1| \geq R-1$. Hence,

$$\left|\frac{e^z}{z+1}\right| \leq \frac{1}{R-1},\; z \in \gamma.$$

We know that $L(\gamma) = \pi R$ as γ is semi-circle of radius R, so by the Estimation Lemma (6.4)

$$\left|\int_\gamma \frac{e^z}{z+1}\,dz\right| \leq \frac{1}{R-1}L(\gamma) = \frac{\pi R}{R-1}.$$

Remarks 6.6

(i) The constant M always exists: On the image of γ the function $|f(z)|$ will always be bounded because the map $t \mapsto |f(\gamma(t))|$ is a continuous real function on $[a, b]$, and hence is bounded. So $M = \sup_{z\in\gamma}\{|f(z)|\}$ will do as a bound. Anything bigger than this is also useful.

(ii) We can show some integral is zero by finding an M that tends to zero or some contour, which can be made smaller, so that $L(\gamma) \to 0$.

Termwise integration of series

The lemma will be useful in a number of contexts. To begin with, we use it to prove that we can integrate certain series term-by-term. (We know that an infinite series can be differentiated term-by-term.)

Corollary 6.7 (Term-by-term integration of series)
Let γ be a contour in a domain D. Let $f : D \to \mathbb{C}$ and $f_k : D \to \mathbb{C}$ be continuous complex functions, $k \in \mathbb{N}$. Suppose that

(i) $\sum_{k=0}^{\infty} f_k(z)$ converges to $f(z)$ for all $z \in \gamma$;

(ii) there exist real constants M_k such that $|f_k(z)| \leq M_k$ for all $z \in \gamma$;

(iii) $\sum_{k=0}^{\infty} M_k$ converges.

Then,

$$\sum_{k=0}^{\infty} \int_{\gamma} f_k(z)\,dz = \int_{\gamma} \sum_{k=0}^{\infty} f_k(z)\,dz = \int_{\gamma} f(z)\,dz.$$

Proof. The second equality is obvious. We show the first. Let $M = \sum_{k=0}^{\infty} M_k$. We have

$$\begin{aligned}
\left| \int_{\gamma} f(z)\,dz - \sum_{k=0}^{n} \int_{\gamma} f_k(z)\,dz \right| &= \left| \int_{\gamma} \left(f(z) - \sum_{k=0}^{n} f_k(z) \right) dz \right| \\
&\leq \sup_{z\in\gamma} \left\{ \left| f(z) - \sum_{k=0}^{n} f_k(z) \right| \right\} \times L(\gamma) \\
&= \sup_{z\in\gamma} \left\{ \left| \sum_{k=n+1}^{\infty} f_k(z) \right| \right\} \times L(\gamma) \\
&\leq \sup_{z\in\gamma} \left\{ \sum_{k=n+1}^{\infty} |f_k(z)| \right\} \times L(\gamma) \\
&\leq \left(\sum_{k=n+1}^{\infty} M_k \right) \times L(\gamma) \\
&= \left(M - \sum_{k=0}^{n} M_k \right) \times L(\gamma) \\
&\to 0 \times L(\gamma), \text{ as } n \to \infty, \\
&= 0.
\end{aligned}$$

Thus, the first equality is true. □

Remark 6.8
I could have defined uniform convergence and so on for complex series in order to state the theorem. Rather than waste time doing so I just used a version of the Weierstrass M-test in the assumptions above. Hopefully, you remember from Real Analysis II that this implies uniform convergence.

Example 6.9
For any contour γ the integral $\int_\gamma e^z\,dz$ can be calculated by term-by-term integration of the series for e^z.

Let $f_k(z) = \dfrac{z^k}{k!}$, then $e^z = \displaystyle\sum_{k=0}^{\infty} f_k(z)$, so condition (i) is fulfilled. We have

$$|f_k(z)| = \left|\frac{z^k}{k!}\right| = \frac{|z|^k}{k!}.$$

But $|z| = |\gamma(t)|$ and $\gamma(t)$ is a continuous map from $[a,b]$ to $\mathbb{C}$ so its image must lie within an origin-centred circle of radius R, for some large enough R. Thus $|z| \le R$ for all $z \in \gamma$.

Hence, let $M_k = \dfrac{R^k}{k!}$, then $|f_k(z)| \le M_k$ for all $z \in \gamma$. Therefore, condition (ii) holds.

Also,

$$\sum_0^\infty M_k = \sum_0^\infty \frac{R^k}{k!} = e^R,$$

so $\sum_0^\infty M_k$ converges. Condition (iii) holds.

Thus,

$$\int_\gamma e^z\,dz = \int_\gamma \sum_{k=0}^\infty \frac{z^k}{k!} = \sum_{k=0}^\infty \int_\gamma \frac{z^k}{k!}\,dz.$$

This of course begs the question, 'how do we find $\displaystyle\int_\gamma \frac{z^k}{k!}\,dz$'? Obviously, we can apply the definition of integration, but there are better ways, such as the Fundamental Theorem of Calculus, as we shall see in the next two sections.

Summary

- The length of γ, denoted $L(\gamma)$, is defined to be

$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt.$$

- Suppose that $|f(z)| \le M$ for all $z \in \gamma$. Then

$$\left|\int_\gamma f(z)\,dz\right| \le M\,L(\gamma).$$

- We can integrate term-by-term complex series that satisfy a Weierstrass M-test type condition.

7 Complex Differentiation

If we were inventing the theory of differentiation of complex functions for the the first time, then we might be tempted to define complex differentiable to mean that the real and imaginary parts of the function are differentiable with respect to x and y. This would give us a theory, but not a great one; it would be the same as theory of differentiable maps from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Instead we develop a theory that really uses the complex numbers. This gives a much richer theory. It looks a lot better, but more importantly, it allows us to solve some hard problems concerning real functions in a simple manner.

Continuity

Let's first give a definition of continuity.

Definition 7.1
Suppose that $f : D \to \mathbb{C}$ is a complex function on a domain D. We say f is **continuous at** c if $\lim_{z\to c} f(z) = f(c)$

This is not remarkably different to real analysis.

Remark 7.2
This is equivalent to either of the following.

(i) For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(c)| < \varepsilon$, whenever $|z - c| < \delta$.

(ii) For any sequence $\langle z_n \rangle$, we have $z_n \to c$ implies that $f(z_n) \to f(c)$.

The proofs of these facts are the same as in real analysis.

Definition 7.3
We say f is **continuous on** D if f is continuous at c for all $c \in D$.

We need some examples and the next result helps supply some.

Proposition 7.4
Suppose $f(x + iy) = u(x, y) + iv(x, y)$. Then, f is continuous if and only if u and v are continuous.

Proof. Use Proposition 1.15. □

Thus, $f(z) = z^n$ is continuous. Sums and products of continuous functions are continuous, and so on.

Differentiation

Now we come to the crucial definition, that of differentiation. It doesn't look different to the real situation, but the consequences are far more profound.

Definition 7.5
Let $f : D \to \mathbb{C}$ be a complex function. Then, f is **complex differentiable at** c if

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}, \; h \in \mathbb{C}$$

exists. We write $f'(c)$ for this limit.

We say f is **complex differentiable on** D if it is complex differentiable at c for all $c \in D$.

The definition looks the same as in the real case, but the fact that h can go to zero from *any* direction in the complex plane makes a huge difference, it imposes considerable restrictions.

Remark 7.6
Complex differentiable functions are also called **holomoprhic** or **analytic**.

Example 7.7
The function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$ is differentiable on $\mathbb{C}$.

Let $c \in \mathbb{C}$. Then

$$\begin{aligned}
\lim_{h\to 0} \frac{(c+h)^2 - c^2}{h} &= \lim_{h\to 0} \frac{c^2 + 2ch + h^2 - c^2}{h} \\
&= \lim_{h\to 0} 2c + h \\
&= 2c.
\end{aligned}$$

That is, $f'(c) = 2c$, as expected.

More generally, we have that, if $f(z) = bz^n$ for some complex constant b, then $f'(z) = nbz^{n-1}$. The proof of this is the same as the real situation, at least symbolically. Of course, mathematicians do not want to do anything as clumsy or ugly as using first principles. We would use a theorem such as the following.

Proposition 7.8
Let f and g be complex functions on the domain D.

(i) If f and g are differentiable at $c \in D$, then so are

(a) $f+g$, and $(f+g)'(c) = f'(c) + g'(c)$;

(b) fg, and $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$;

(c) f/g, and $(f/g)'(c) = \dfrac{g(c)f'(c) - f(c)g'(c)}{g(c)^2}$ provided $g(c) \neq 0$.

(ii) (Chain Rule) Suppose $f : D \to \mathbb{C}$ and $g : E \to \mathbb{C}$ are complex functions with $f(D) \subseteq E$. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c))f'(c)$.

Proof. These are proved in the same way as the real case. □

Warning! 7.9
Unlike continuity, complex differentiablity isn't the same as being differentiable with respect to two real variables. There is a connection as we see shortly. In fact it is this difference that makes complex analysis so different to real analaysis.

Exercise 7.10
Show that if $f : D \to \mathbb{C}$ is differentiable at c, then f is continuous at c.

We don't yet know that we can differentiate series term-by-term and so can't immediately prove that e^z, $\cos z$ and $\sin z$ are differentiable, or that their derivatives are what we expect them to be. It is possible to work find their derivatives from first principles, (you can try this as an exercise if you are keen), but will delay a proof till later and just state the following.

Theorem 7.11
The elementary functions have the expected derivatives:

(i) $\dfrac{d}{dz}\, e^z = e^z$, for all $z \in \mathbb{C}$.

(ii) $\dfrac{d}{dz} \sin z = \cos z$ for all $z \in \mathbb{C}$.

(iii) $\dfrac{d}{dz} \cos z = -\sin z$ for all $z \in \mathbb{C}$.

Exercise 7.12
Where are the following funtions *not* differentiable?

$$z^2,\ \frac{1}{z},\ \frac{1}{z^2},\ \frac{\sin z}{z(z^2+1)},\ \tan z.$$

Summary

- The complex function f is continuous at c if $\lim_{z \to c} f(z) = f(c)$.
- The complex function f is complex differentiable at c if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \ h \in \mathbb{C}.$$

- Differentiablity $\implies$ continuity.
- The elementary functions have the expected derivatives.

8 Cauchy-Riemann Equations

If f is a differentiable complex function, then the real and imaginary parts are differentiable as real functions. (But not vice versa, see earlier warning.)

Theorem 8.1
Suppose $f(x+iy) = u(x,y) + iv(x,y)$, where $z = x+iy$, and that f is differentiable at $c = a+ib$. Then the partial derivatives u_x, u_y, v_x and v_y all exist at (a,b) and

$$f'(c) = u_x(a,b) + iv_x(a,b) = v_y(a,b) - iu_y(a,b).$$

Proof. We know $\dfrac{f(c+h)-f(c)}{h} \to f'(c)$ as $h \to 0$. Let $h = k+il$, we get

$$\frac{u(a+k,b+l) - u(a,b) + iv(a+k,b+l) - iv(a,b)}{k+il} \to f'(c) \text{ as } k+il \to 0.$$

So if we let $h \to 0$ through *real* values, i.e. $l = 0$, then

$$\frac{u(a+k,b) - u(a,b) + iv(a+k,b) - iv(a,b)}{k} \to f'(c) \text{ as } k \to 0.$$

Therefore, $\dfrac{u(a+k,b)-u(a,b)}{k} \to Re(f'(c))$ as $k \to 0$. So $u_x(a,b)$ exists and equals $Re(f'(c))$. Likewise, $v_x(a,b) = Im(f'(c))$.

The second equation follows from letting $h \to 0$ through imaginary values, i.e. $k = 0$. □

And now for a result that is fundamental in all complex analysis courses.

Corollary 8.2 (Cauchy-Riemann equations)
If $f(z) = u(x,y) + iv(x,y)$ is differentiable, then

$$u_x = v_y \text{ and } v_x = -u_y.$$

Proof. Equate real and imaginary parts in the theorem above. □

The two equations are called the **Cauchy-Riemann equations** after two of the founders of complex analysis.

Note that the corollary says that if f is differentiable, then the equations hold, but says nothing of the converse, which is not true anyway!

Examples 8.3

(i) Let $f(z) = z^2 = (x+iy)^2$. Then, $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We see that

$$u_x = v_y = 2x \text{ and } u_y = -v_x = -2y,$$

so the C-R equations hold.

(ii) Let $f(z) = |z|^2 = x^2 + y^2$. Then $u = x^2 + y^2$ and $v = 0$. Thus,

$$\begin{aligned} u_x &= 2x, \quad & v_y &= 0, \\ u_y &= 2y, \quad & v_x &= 0. \end{aligned}$$

The only place where the C-R equations are satisfied is $x = y = 0$.

So f is differentiable nowhere, except possibly at 0. Is it differentiable at 0? Well,

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|^2 - 0}{h} = \frac{h\bar{h}}{h} = \bar{h} \to 0 \text{ as } h \to 0,$$

so $f'(0) = 0$.

Exercise 8.4
Show that the function $f(z) = |z|$ is not differentiable anywhere in $\mathbb{C}$.

Remark 8.5
The preceding exercise shows that complex differentiability is imposing a stronger condition real differentiability, because for real points of $\mathbb{C}$ the function $f(x) = |x|$ *is* differentiable, provided $x \neq 0$. (You know this last fact well, I hope!).

The condition is stronger because we require $f'(c)$ to exist as $h \to 0$ from all directions, not just real ones.

Remark 8.6
The converse to Corollary 8.2 is false. For example, let

$$f(z) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then, the equations are satisfied, (check!), but f is not continuous, so can't be differentiable.

However, if u_x, u_y, v_x, v_y exist near c and are continuous at c, and satisfy the C-R equations, then f is differentiable at c. This fact won't be used later, but is useful to know.

Harmonic functions

Definition 8.7
Let $w(x, y)$ be a C^2-function of two real variables[5]. Then w is a **harmonic function** if it satisfies Laplace's equation:

$$w_{xx} + w_{yy} = 0.$$

Example 8.8
Let $w(x, y) = x^2 - y^2$. Then $w_{xx} = 2$ and $w_{yy} = -2$, so $w_{xx} + w_{yy} = 0$.

[5] A C^k-function is a function that is differentiable k times

Laplace's equation is important in potential theory and many other areas, and so, since they are solutions of it, harmonic functions are of particular interest. Complex analysis provides many examples of harmonic functions, as the next theorem shows.

Theorem 8.9
If $f : D \to \mathbb{C}$ is complex differentiable, and u and v are C^2-functions, then u and v are both harmonic:

$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0.$$

Proof. We use the C-R equations:

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{xy} - v_{yx} = 0.$$

Similarly for v. □

Conversely, given an harmonic function u there is *locally* (i.e. in some ε-neighbourhood) a harmonic function v such that $f(x + iy) = u(x, y) + i(v, y)$ is complex differentiable. Such a v is called a **harmonic conjugate** of u.

Example 8.10
Let $u(x, y) = 2xy$, then u is harmonic. We can construct v using the C-R equations (they're very useful, aren't they!) Because

$$2y = u_x = v_y \implies v = y^2 + g(x) \text{ where } g \text{ is a function of } x,$$

and

$$2x = u_y = -v_x \implies v = -x^2 + h(y) \text{ where } h \text{ is a function of } y,$$

we can deduce that $v = y^2 - x^2 + C$ where C is a constant.

Summary

- (Cauchy-Riemann equations) If $f(z) = u(x, y) + iv(x, y)$ is differentiable, then

$$u_x = v_y \text{ and } v_x = -u_y.$$

- The function $w : \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function if it satisfies Laplace's equation:

$$w_{xx} + w_{yy} = 0.$$

- For every harmonic u there is a harmonic function v (called a harmonic conjugate of u) such that $f(x + iy) = u(x, y) + i(v, y)$ is complex differentiable.

9 Fundamental Theorem of Calculus for Complex Functions

One of the best theorems in Real Calculus is the Fundamental Theorem of Calculus. We now see this in a contour integration setting.

Theorem 9.1 (Fundamental Theorem of Calculus)
Let $f : D \to \mathbb{C}$ be a continuous complex function and $\gamma : [a, b] \to D$ be a contour. Suppose there exists a complex differentiable $F : D \to \mathbb{C}$ such that $F' = f$. Then,

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. Let $a = a_0 < a_1 < \cdots < a_n = b$ be a dissection of $[a, b]$ such that $\gamma'|[a_{j-1}, a_j]$ is continuous for all j. Then,

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_\gamma F'(z)\,dz \\
&= \int_a^b F'(\gamma(t))\gamma'(t)\,dt \\
&= \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} F'(\gamma(t))\gamma'(t)\,dt \\
&= \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} (F \circ \gamma)'(t)\,dt \\
&= \sum_{j=1}^{n} [(F \circ \gamma)(t)]_{a_{j-1}}^{a_j} \text{ by the usual FTC for RI functions,} \\
&= F(\gamma(b)) - F(\gamma(a)).
\end{aligned}$$

□

Example 9.2
Consider Example 4.3. Then, $f(z) = z^2$, $\gamma(0) = 2$ and $\gamma(1) = 2+i$.

Obviously, $F(z) = \frac{1}{3}z^3$ is an antiderivative for f, i.e. $F'(z) = f(z)$. Then, by the FTC,

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \frac{1}{3}\left((\gamma(1))^3 - (\gamma(0))^3\right) \\
&= \frac{1}{3}\left(2^3 - (2+i)^3\right) \\
&= -2 + \frac{11}{3}i.
\end{aligned}$$

How common are antiderivatives for continuous complex functions? Do they always exist? For real continuous functions we know that the Riemann integral can be found and this will be an antiderivative. Unfortunately, the analogue is not true for continuous complex functions, not even differentiable ones. Consider this corollary of the FTC and the following example.

Corollary 9.3
With the assumptions of the above theorem suppose that γ is any closed contour. Then

$$\int_\gamma f(z)\,dz = 0.$$

Proof. The definition of a closed contour is that $\gamma(a) = \gamma(b)$. So,

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(b)) = 0.$$

□

Example 9.4
Let $f(z) = 1/z$. Then, f is differentiable on $D = \mathbb{C}\backslash\{0\}$. Let γ be the unit circle round the origin, traversed once anti-clockwise. Then, by the fundamental example we know

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{1}{z}\,dz = 2\pi i.$$

Therefore, if there existed an $F : D \to \mathbb{C}$ such that $F' = f$, then the corollary would be contradicted. Hence, antiderivatives do not always exist.

Property of exponential

We can now prove a result we would expect to be true by analogy with real analysis: if a function has zero derivative, then it is constant.

First we need a definition.

Definition 9.5
A domain D is called **connected** if for each pair $\alpha, \beta \in D$, there exists a contour $\gamma : [a, b] \to D$ such that $\gamma(a) = \alpha$ and $\gamma(b) = \beta$.

So a domain is connected if we can draw a curve from any point to any other.

Examples 9.6
(i) The domains $D = \mathbb{C}$ and $D = \mathbb{C}\backslash\{0\}$ are both connected.

(ii) The domain $D = \mathbb{C}\backslash\{z : z \text{ is real}\}$ is not connected. There is no way to construct a contour starting below and finishing above the real line, without crossing that line. But the real line is not in D.

Theorem 9.7
Suppose that D is is connected domain, and $f : D \to \mathbb{C}$ is analytic, such that $f'(z) = 0$ for all $z \in D$. Then, f is constant.

Proof. Take any α and β in D. Then, as D is connected, there exists a path $\gamma : [a, b] \to D$, such that $\gamma(a) = \alpha$ and $\gamma(b) = \beta$. By the FTC

$$f(\beta) - f(\alpha) = f(\gamma(b)) - f(\gamma(a)) = \int_\gamma f' = \int_\gamma 0 = 0.$$

Thus $f(\beta) = f(\alpha)$. Since these were general points of D, we deduce that f is constant. $\square$

This theorem allows us to prove the property of e^z in Theorem 1.35 we did not prove earlier.

Corollary 9.8
For all complex numbers z and w we have $e^{z+w} = e^z e^w$.

Proof. Let $f(z) = e^z e^{\alpha-z}$, where α is any complex number. By the product rule we get $f'(z) = -e^z e^{\alpha-z} + e^z e^{\alpha-z} = 0$. Thus, by the theorem, the function is constant. So, if we let $\alpha = w + z$, then we get

$$\begin{array}{rcl} f(w) & = & f(0) \\ e^w e^{(w+z)-w} & = & e^0 e^{w+z+0} \\ e^w e^z & = & e^{w+z}. \end{array}$$

$\square$

* Existence of antiderivatives

The next proposition gives some conditions equivalent to the existence of antiderivatives.

Proposition 9.9
Let $f : D \to \mathbb{C}$ be a continuous complex function on a connected domain D. The following statements are equivalent.

(i) There exists an $F : D \to \mathbb{C}$ such that $F' = f$.

(ii) $\int_\gamma f = 0$ for every closed contour γ in D.

(iii) $\int_\gamma f$ only depends on the end points of γ for any contour γ in D.

Proof. We shall prove (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii): This is just Corollary 9.3.

(ii)⇒(iii): Suppose that α and β are two points in D, then we want to prove that the integral of f over any two contours γ_1 and γ_2 both starting at α and finishing at β are equal.

Consider the contour $\gamma_1 - \gamma_2$. This is a closed contour because it starts and finishes at α. (Draw a picture!) Hence, (ii) implies that $\int_{\gamma_1-\gamma_2} f = 0$. We have

$$0 = \int_{\gamma_1-\gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f.$$

Hence (iii) is true.

(iii)⇒(i): We shall define a function F and show that it is differentiable. Fix $z_0 \in D$. Since D is connected, for any point $z \in D$, there exists a contour γ from z_0 to z. We define

$$F(z) = \int_\gamma f(w)\,dw.$$

[Note that this really is a function of z because γ has z as its endpoint.]

Now we show that $F'(z) = f(z)$. Let $z_1 \in D$ be any point. Then by definition F is differentiable at z_1 with derivative $f(z_1)$ if

$$\lim_{h\to 0} \frac{F(z_1+h) - F(z_1)}{h} = f(z_1).$$

So, as D is a domain, there exists an ε_1-neighbourhood at z_1 for some $\varepsilon > 0$. If $|h| < \varepsilon_1$, then there exists a contour $\Lambda : [0,1] \to D$ which gives the straight line from z_1 to $z_1 + h$, i.e. $\Lambda(t) = z_1 + ht$.

Using our definition of F, we get

$$F(z_1+h) = \int_{\gamma+\Lambda} f = \int_\gamma f + \int_\Lambda f.$$

Therefore,

$$\frac{F(z_1+h) - F(z_1)}{h} = \frac{1}{h}\left(\int_\gamma f + \int_\Lambda f - \int_\gamma f\right) = \frac{1}{h}\int_\Lambda f(w)\,dw.$$

We have,

$$\int_\Lambda f(z_1)\,dw = f(z_1)\int_\Lambda dw = f(z_1)\,[w]_{z_0}^{z_0+h} = f(z_1)h.$$

Hence,

$$\frac{F(z_1+h) - F(z_1)}{h} - f(z_1) = \int_\Lambda \frac{f(w) - f(z_1)}{h}\,dw.$$

We want the LHS to tend to zero as $h \to 0$, so we use the Estimation Lemma on the RHS. Since f is continuous, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|w - z_1| < \delta$ implies that $|f(w) - f(z_1)| < \varepsilon$. We can assume that $\delta < \varepsilon_1$. So, when $|z| < \delta$ and $w \in \Lambda$, we have

$$\left|\frac{f(w) - f(z_1)}{h}\right| < \frac{\varepsilon}{|h|}.$$

The length of Λ is $|h|$, so we deduce from the Estimation Lemma that

$$\left|\int_\Lambda \frac{f(w) - f(z_1)}{h}\, dw\right| \leq \frac{\varepsilon}{|h|}|h|.$$

Thus,

$$\left|\frac{F(z_1 + h) - F(z_1)}{h} - f(z_1)\right| \leq \varepsilon \text{ for all } |h| < \delta.$$

Since ε was arbitrary we deduce that the LHS is zero. This implies that $F'(z_1) = f(z_1)$. So, F is what we were seeking. □

Summary

- Fundamental Theorem of Calculus: Suppose $f : D \to \mathbb{C}$ is a continuous complex function and there exists F such that $F' = f$. Then,

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

- Not all functions have an antiderivative.

- A domain D is called connected if for each pair $\alpha, \beta \in D$, there exists a contour $\gamma : [a, b] \to D$ such that $\gamma(a) = \alpha$ and $\gamma(b) = \beta$.

10 Differentiability of Power Series

We have successfully defined functions such as $\exp$, $\sin$ and $\cos$ by power series. Now we would like to show that they are differentiable. We do this by investigating the differentiability of power series with positive radius of convergence. We will show later that if a function is differentiable at a point z_0, then near that point it can be given as a power series, so this investigation wil be useful for more than just the elementary functions.

We prove:

- A power series is differentiable in the obvious way: term-by-term.
- The derivative of a power series has the same radius of convergence.
- Power series are infinitely differentiable.
- The coefficients of the power series can be given in terms of the derivatives.
- The coefficients of the power series are unique.

Suppose that $\sum_{n=0}^{\infty} a_n z^n$ is a series. Then, the obvious candidate for the derivative is the one produced by term-by-term differentiation[6], i.e. $\sum_{n=0}^{\infty} n a_n z^{n-1}$. But, initially, we do do not even know that this sequence converges.

So, we begin by proving that a power series and its 'obvious derivative' have same radius of convergence, and then show that this obvious derivative really is the derivative of the series.

Lemma 10.1
The series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.

Proof. The following proof is taken from Priestley p20.

If $\sum_{n=0}^{\infty} n a_n z^{n-1}$ had radius of convergence $R \neq 0$, then it is not difficult to show that $\sum_{n=0}^{\infty} a_n z^n$ has the same radius of convergence.

Therefore, we prove the converse: If $\sum_{n=0}^{\infty} a_n z^n$ had radius of convergence $R \neq$, then so does $\sum_{n=0}^{\infty} n a_n z^{n-1}$.

Fix z with $0 < |z| < R$, and choose ρ such that $|z| < \rho < R$. Then

$$|n a_n z^{n-1}| = \frac{n}{|z|} \left(\frac{|z|}{\rho} \right)^n |a_n \rho^n|.$$

[6] Hopefully, studying maths has taught you that just because something is obvious doesn't mean it's true!

The series $\sum n(|z|/\rho)^n$ is easily shown to converge, by the ratio test. (Exercise!).

Just as in the real case, if the series $\sum b_n$ converges, then $b_n \to 0$ as $n \to \infty$ and so there exists a constant M such that $|b_n| \leq M$ for all n. Thus, there exists a constant M such that for all n

$$n\left(\frac{|z|}{\rho}\right) \leq M,$$

thus,

$$|na_n z^{n-1}| \leq \frac{M}{|z|}|a_n \rho^n|,$$

and so by the comparison test, $\sum na_n z^{n-1}$ is absolutely convergent. Hence, the result. □

Now, let's show that this really is the derivative of the series.

Theorem 10.2
Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$. Then, $f'(z) = \sum_{n=0}^{\infty} na_n z^{n-1}$, for all $|z| < R$.

Proof. Again we follow Priestley's proof. We know that $g(z) = \sum na_n z^{n-1}$ is well-defined for $|z| < R$. We shall show that $f'(z)$ exists and is equal to $g(z)$.

[If we are thinking like mathematicians, then we know a good way of doing this is to show that $|f'(z) - g(z)| = 0$. In other words we want

$$\left|\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} - g(z)\right| = 0.$$

Certainly, this is true if

$$\left|\frac{f(z+h) - f(z)}{h} - g(z)\right| \to 0 \text{ as } h \to 0.$$

This is our method of attack.]

For any z and h such that $|z| < R$ and $|z+h| < R$, we have

$$\begin{aligned}
\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| &= \left|\sum_{n=0}^{\infty} a_n \frac{(z+h)^n - z^n}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1}\right| \\
&= \left|\sum_{n=1}^{\infty} a_n \left(\frac{(z+h)^n - z^n}{h} - n z^{n-1}\right)\right| \\
&= |h| \left|\sum_{n=1}^{\infty} a_n \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^{k-2}\right| \quad \text{use binom thm} \\
&\leq |h| \sum_{n=1}^{\infty} \frac{1}{2} n(n-1)|a_n| \sum_{m=2}^{n-2} \binom{n-2}{m} |z|^{n-2-m} |h|^m \\
&\leq |z| \sum_{n=1}^{\infty} \frac{1}{2} n(n-1)|a_n| (|z|+|h|)^{n-2}.
\end{aligned}$$

Fix z and choose ρ with $|z| < \rho < R$. By Lemma 10.1 $\sum_{n=1}^{\infty} n(n-1)|a_n|\rho^{n-2}$ converges to K say. For $|h| < \rho - |z|$,

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| \leq \frac{1}{2} K |h|.$$

So as $h \to 0$ we get $\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| \to 0$ as required. □

The proofs are quite technical, probably the most technical in the course, and I wouldn't expect you to reproduce them in exams, but I would like you to understand them. These two results will certainly be very important to the course, so understand what they mean:

The derivative of a series can be found by term-by-term differentation and the resulting series has the same radius of convergence.

Examples 10.3

(i) We can show that $\sum_0^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$. So, differentiating both sides gives $\sum_0^{\infty} n z^{n-1} = \frac{1}{(1-z)^2}$ for $|z| < 1$.

(ii) The series $S = \displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is absolutely convergent for $x \in \mathbb{R}$ but term-by-term differentiation gives $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n}$. This clearly diverges at $x = 0$. Thus, S is not a power series!

Exercise 10.4

Prove Theorem 7.11.

We have seen that, if f is defined on the disc of convergence, then so is f', and f' is a series. We can differentiate f' term-by-term to get f'' and so on. This suggests the following:

Corollary 10.5
If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then f is infinitely differentiable.

Proof. By induction using the theorem. $\square$

Note that this is to be expected as a similar statement is true for real power series. This result is one of the reasons that power series are so great. Later, we will see that any complex function differentiable at a point can be given by a power series.

Corollary 10.6
If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then $a_k = \dfrac{f^{(k)}(0)}{k!}$ for all k.

Proof. By induction we can prove

$$f^{(k)}(z) = \sum_{n=0}^{\infty} n(n-1)\dots(n-k+1)a_n z^{n-k}$$

and this of course holds for $|z| < R$.

Then,

$$\begin{aligned} f^{(k)}(z) &= (k(k-1)\dots(k-k+1)a_k) \\ &\quad + ((k+1)((k+1)-1)\dots((k+1)-k+1)a_{k+1}z) + \dots \end{aligned}$$

If we put $z = 0$ into this, then we get $f^{(k)}(0) = k!a_k$. $\square$

Again, you should already know this is true for real series.

Lemma 10.7 (Uniqueness Lemma)
Suppose for some $R > 0$, $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n$, for all $|z| < R$. Then, $a_n = b_n$ for all n.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n$. Then, by Corollary 10.6,

$$a_n = \frac{f^{(n)}(0)}{n!} \text{ and } b_n = \frac{f^{(n)}(0)}{n!}.$$

Hence, $a_n = b_n$. $\square$

Note that the Uniqueness Lemma is not trivial. A priori we don't know that we can equate coefficients for infinite series.

Remark 10.8
Just like in Real Analysis it is possible to prove that the product of two power series S_1 and S_2 is again a power series with radius of convergence at least the minimum of those for S_1 and S_2. Its coefficients can be given in terms of those of S_1 and S_2.

However, we shall delay proving this as later work will give a particularly simple proof of this fact. (See Theorem 15.6.)

Power series about points other than zero

So far all our power series have been centred at the origin. This is rather limiting. Just as in real analysis giving an expansion about a different point is very useful. Let's do that now.

Suppose we are interested in the point $z_0 \in \mathbb{C}$. If we let h be a complex variable, then in a neighbourhood of z_0 we can get $z_0 + h$ to give any point. For example, if h is zero we get z_0. If $|h| < R$ for some R, then we have a disc around z_0.

Suppose we have $f(z_0 + h) = \sum_0^\infty a_n h^n$, with the series convergent for $|h| < R$. I.e. a power series in h about 0, but which defines the function f *around* the point z_0.

Let $z = z_0 + h$, well z_0 is a constant and h is a single variable so z is a variable. By substitution, $f(z) = \sum_0^\infty a_n(z - z_0)^n$, which converges for $|h| = |z - z_0| < R$.

Thus we can deal with power series centred at a particular point z_0 and we will have a disc of convergence centred at z_0.

The main point is that we can prove results about power series centred at zero and just translate to another point. So for example, if we have the power series $\sum a_n(z - z_0)^n$, then we can let $h = z - z_0$ to get the series $\sum a_n h^n$ and we can apply ratio test, etc., to that. Then we translate back to say something about the series $\sum a_n(z - z_0)^n$.

For example, we can differentiate term-by-term:

Example 10.9
If $f(z) = \sum a_n(z - z_0)^n$ converges for $|z - z_0| < R$, then $f'(z) = \sum n a_n (z - z_0)^{n-1}$ converges for $|z - z_0| < R$.

Proof: Let $h = z - z_0$. Then $g(h) = \sum a_n h^n$ converges for all $|h| = |z - z_0| < R$ as $f(z) = g(z - z_0)$. We have

$$f'(z) = \frac{d}{dz} g(z-z_0) = g'(z-z_0)\frac{d}{dz}(z-z_0) = g'(z-z_0) = \sum n a_n (z-z_0)^n.$$

Summary

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $R > 0$.

- f can be differentiated term-by-term to get the derivative.
- f' has the same radius of convergence as f.
- f is infinitely differentiable.
- The coefficients a_n are unique and $a_n = \dfrac{f^{(n)}(0)}{n!}$ for all n.
- We can translate these results to series of the following form: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, for $|z - z_0| < R$.

11 Win a Million Dollars!

The Riemann Zeta function

We define the Riemann Zeta function as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

So, for example if $s = 1$, then we get $\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ which we know does not converge, so $\zeta(1)$ is not defined. If $s = 2$ we get $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which does converge, and so on for all real numbers s.

If we let s be complex, then what happens? We first we need to define n^s when n is a real number and s is a complex number, the only number we have done this for is e. We know that $n = e^{\log n}$ and so we could define n^s by

$$n^s = e^{s \log n}.$$

(Since obviously we want $n^s = (e^{\log n})^s = e^{s \log n}$.)

Riemann was not the first to study this type of function but did leave us an interesting hypothesis.

The Riemann Hypothesis

We now come to the million dollar question. Where do the roots of the function lie? I.e. the $s \in \mathbb{C}$ such that $\zeta(s) = 0$.

What Riemann found is that all roots (apart from some trivial real ones) seem to have $Re(s) = \frac{1}{2}$, and so he conjectured that *all* non-trivial roots have the property of being on this line. This is called the Riemann Hypothesis and it is regarded as being the most important unsolved problem in pure mathematics.

Anyone who proves that the hypothesis is true can claim a million dollars from the Clay Institute. You can find the details on the Web at http://www.claymath.org/ under the heading Millenium Prize Problems.

There are six other problems from various areas of mathematics for which a million dollar prize is offered. One of them (the Poincaré Conjecture) is explained in Homotopy and Surfaces in Year 3.

For futher reading see Prime Time by Erica Klarreich in New Scientist, Vol 168 issue 2264, 11/11/2000, p32, (Edward Boyle Library Floor 11), for the connection with number theory and the distribution of primes.

See also the classic text by E.C. Titchmarsh, The theory of the Riemann-Zeta function.

12 Winding numbers

Cauchy's theorem is one of the most remarkable theorems in mathematics. To state it we need the notions of winding number and interior point. Both these notions are intuitively simple. We start with the winding number.

Definition 12.1
Let γ be a closed contour and w a point not on γ. Then the **winding number** of γ about w, written $n(\gamma, w)$, is the net number of times that γ winds about w, with anticlockwise counted positively.

Example 12.2
The winding numbers for points in the regions enclosed by the contour are shown below.

Exercise 12.3
Find the winding numbers for points in the various regions.

We can calculate winding numbers mathematically.

Lemma 12.4 (Winding Number Lemma)
Let γ be a closed contour, $w \in \mathbb{C}\backslash\gamma$. Then

$$n(\gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}.$$

Proof. If we write $\gamma(t) = w + r(t)e^{i\theta(t)}$, then $r(t)$ and $\theta(t)$ are continuous, piecewise continuous differentiable functions on $[a, b]$. We have,

$$\begin{aligned}
\int_\gamma \frac{dz}{z-w} &= \int_a^b \frac{\gamma'(t)}{\gamma(t)-w}\,dt \\
&= \int_a^b \frac{r'(t)e^{i\theta(t)} + i\theta'(t)r(t)e^{i\theta(t)}}{r(t)e^{i\theta(t)}}\,dt \\
&= \int_a^b \frac{r'(t)}{r(t)}\,dt + i\int_a^b \theta'(t)\,dt \\
&= \left[\log r(t)\right]_a^b + i\left[\theta(t)\right]_a^b \\
&= \log r(a) - \log r(b) + i\left(\theta(a) - \theta(b)\right).
\end{aligned}$$

Now, γ is a closed contour and so $\gamma(a) = \gamma(b)$, i.e. $r(a)e^{i\theta(a)} = r(b)e^{i\theta(b)}$. Equating the moduli and arguments we get $r(a) = r(b)$ and $\theta(b) = \theta(a) + 2\pi k$ for some $k \in \mathbb{Z}$.

Putting these results into the integral above,

$$\int_\gamma \frac{dz}{z-w} = 2\pi i k.$$

Moreover, $\theta(b) - \theta(a)$ is the net increase in $\arg(\gamma(t) - w)$ as t runs from a to b, so the number of times that γ winds round w is $\dfrac{\theta(b)-\theta(a)}{2\pi}$, i.e. $k = n(\gamma, w)$. □

Exercises 12.5

(i) Let γ be a closed contour and $w \in \mathbb{C}$. Prove that $n(-\gamma, w) = -n(\gamma, w)$.

(ii) Let γ_1 and γ_2 be closed contours, so that the join can be taken. Prove that $n(\gamma_1 + \gamma_2, w) = n(\gamma_1, w) + n(\gamma_2, w)$

An easy method of calculation

Calculating winding numbers can appear to be complicated to calculate, you have to trace your finger round the curve bearing in mind how many revolutions have been made. And you have to do it for each region. Fortunately, there is an easier method. In the examples above note that when passing from one region to another via an edge (rather than via a crossing of two line) the winding number for points in the regions only changes by 1.

Now, consider the complicated contour image drawn below with a line passing through it.

Start at the left side of the line. Obviously the winding number there is 0. As we go from left to right on the line we will cross the contour. We apply the following rules:

- If we cross the contour so that it is travelling *up*, then the winding number *decreases* by 1.
- If we cross the contour so that it is travelling *down*, then the winding number *increases* by 1.

In the above diagram, the contour is travelling up when we first meet it, so the point in the region we pass into have winding number -1. At the next crossing the contour is going down, and so the winding number of points in the next region is 0. We can carry this out for all regions on the line. To get other regions we can use different lines.

Exercise 12.6
Find the winding numbers for the points in the diagram.

Justification of the method

We can justify the method by considering two points w and z that lie on opposite sides of a contour line that goes up. We can assume that the contour starts and finishes at a point near w and z. (If it didn't we can do some reparametrisation and join work.)

Now, take a loop C round w. Then, $n(\gamma + C, w) = n(\gamma, z)$, as the diagram shows. Thus,

$$\begin{aligned} n(\gamma + C, w) &= n(\gamma, z) \\ n(\gamma, w) + n(C, w) &= \\ n(\gamma, w) + 1 &= \\ n(\gamma, w) &= n(\gamma, z) - 1. \end{aligned}$$

This shows that if we pass from z to w, then the winding number decreases by 1. Similarly, one can show an increase by 1 for a contour locally heading down.

Warning! 12.7
We have been using the *image* of the contour to calculate the winding number of a point. Recall that the contour and its image are different. Effectively, we have assumed that the contour is traversed only once. If we go round the contour twice, then the numbers calculated by eye have to be doubled. More generally, if we go round k times, then we multiply the 'by eye' calculations by k.

Interior points

The method of counting the number of times a contour wraps round a point helps define the notion of an interior point.

Definition 12.8
Let γ be a closed contour. The **interior** of γ is the set of points $w \in \mathbb{C}\backslash\gamma$ for which $n(\gamma, w) \neq 0$.

We denote the set of interior points of γ by $\text{Int}(\gamma)$.

Example 12.9
The interior points of the contour in the next diagram are shaded.

Example 12.10
Let $\gamma(t) = w + Re^{2\pi i t}$, $0 \leq t \leq 1$. Then, $\text{Int}(\gamma) = \{z : |z - w| < R\}$.

Summary

- The winding number is the number of times a contour wraps round a point in an anticlockwise direction.
- Let γ be a closed contour, $w \in \mathbb{C}\backslash\gamma$. Then
$$n(\gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - w}.$$
- It is easy to calculate a winding number by eye.
- An interior point is any point with non-zero winding number.

13 Cauchy's (Fantastic) Theorem

We now come to the fundamental theorem in complex analysis. There is no analogue in real analysis. It has far reaching, deep consequences, and most of what we will prove from now on relies on this theorem.

Theorem 13.1 (Cauchy's Theorem)
Let $D \subseteq \mathbb{C}$ be a domain, and $f : D \to \mathbb{C}$ be a differentiable complex function. Let γ be a closed contour such that γ and its interior points lie in D.

Then, $\displaystyle\int_\gamma f = 0$.

Remarks 13.2

(i) This is truly a great theorem. It refers to *any* domain in $\mathbb{C}$, *any* analytic function on D, and *any* contour with all interior points in D. And it says that *any* integral arising from this is zero. Thus, weak assumptions lead to a strong conclusion.

(ii) It is important to note that the interior of γ lies within D. If w is in the interior of γ but not in D, consider $f(z) = 1/(z-w)$ which is analytic on $D = \mathbb{C}\backslash\{w\}$, but

$$\int_\gamma f = \int_\gamma \frac{dz}{z-w} = 2\pi i\, n(\gamma, w) \neq 0,$$

so Cauchy's theorem does not apply.

(iii) The proof of theorem is too complicated for the moment and we will do it later. Note that we can't just use the FTC since we don't know that f has an anti-derivative on D.

(iv) At first sight it may appear that the theorem will only tell us about the behaviour of differentiable functions. However, it has strong implications for non-differentiable functions as well.

Exercise 13.3
Using the contours in Exercises 2 Question 1, to which of the following integrals does Cauchy's theorem apply? (There is no need to evaluate them.)

$$\int_{\gamma_1} |z|^2\, dz, \qquad \int_{\gamma_1} \frac{z^2}{z-2}\, dz, \qquad \int_{\gamma_2} z\, dz, \qquad \int_{\gamma_2+\gamma_3} \sin\left(\frac{1}{z-1}\right)\, dz.$$

The calculation trick

Recall, the trick I performed where I calculated an integral over a path, *before* the path had even been defined. How was this done?

The trick follows from Cauchy's theorem and the fundamental example, (recall that the example says that $\int_\gamma \frac{1}{z}\,dz = 2\pi i$ where γ is a small circle round the origin).

I was trying to integrate $\int_\gamma \frac{a}{z}\,dz$, where a was a number chosen at random, and γ was a path chosen at random. However, you were forced into choosing γ so that its winding number was 1.

Now, we can take a small circle C going *clockwise* round the origin, and we can make it so small that we can assume it lies totally within the interior of γ. Now take another path from the end of γ to the start of the circle C. Call this contour β.

Consider the contour $\Gamma = \gamma + \beta + C - \beta$. This is a closed contour. Its interior does not include the disc encircled by C. So it doesn't contain the origin. But a/z is differentiable on $D = \mathbb{C}\backslash\{0\}$, and the interior of Γ is a subset of D. Thus, we can apply Cauchy's theorem:

$$\begin{aligned}
\int_\Gamma \frac{a}{z}\,dz &= 0 \\
\int_{\gamma+\beta+C-\beta} \frac{a}{z}\,dz &= 0 \\
\int_\gamma \frac{a}{z}\,dz + \int_\beta \frac{a}{z}\,dz \int_C \frac{a}{z}\,dz + \int_{-\beta} \frac{a}{z}\,dz &= 0 \\
\int_\gamma \frac{a}{z}\,dz + \int_\beta \frac{a}{z}\,dz \int_C \frac{a}{z}\,dz - \int_\beta \frac{a}{z}\,dz &= 0 \\
\int_\gamma \frac{a}{z}\,dz + \int_C \frac{a}{z}\,dz &= 0 \\
\int_\gamma \frac{a}{z}\,dz &= -\int_C \frac{a}{z}\,dz \\
&= a\int_{-C} \frac{1}{z}\,dz \\
&= a \times 2\pi i \\
\int_\gamma \frac{a}{z}\,dz &= 2a\pi i
\end{aligned}$$

Thus we get the answer expected. This method – of cutting out a disc where the function is not defined – will be used again later in more generality so make sure you understand this simpler example.

Summary

- Let $D \subseteq \mathbb{C}$ be a domain, and $f : D \to \mathbb{C}$ be a differentiable complex function. Let γ be a closed contour such that γ and its interior points lie in D.

  Then, $\displaystyle\int_\gamma f = 0$.

14 Strange Consequences of Cauchy's Theorem

We will prove a number of surprising theorems that can be deduced from Cauchy's theorem.

(i) Cauchy's Integral Formula: The value of a function at z_0 is determined by the values on a contour round z_0.

(ii) Liouville's Theorem: Any differential function bounded on the whole of $\mathbb{C}$ is contant.

(iii) The Maximum Modulus Principle: The modulus of a function on a domain achieves its maximum on the boundary of the domain.

(iv) Fudamental Theorem of Algebra: Every complex polynomial has a complex root.

Cauchy's Integral Formula

Theorem 14.1
Let $D \subseteq \mathbb{C}$ be a domain, and $f : D \to \mathbb{C}$ be differentiable. Let γ be a closed contour such that γ and its interior points lie in D.

If $w \in D \backslash \gamma$, then

$$\int_\gamma \frac{f(z)}{z-w}\, dz = 2\pi i\, n(\gamma, w) f(w).$$

Remarks 14.2

(i) If w is in the interior of γ, then

$$f(w) = \frac{1}{2\pi i\, n(\gamma, w)} \int_\gamma \frac{f(z)}{z-w}\, dz.$$

This says that the values of f inside γ are completely determined by those on γ! Remarkable! This contrasts with real analysis.

(ii) This behaviour is sometimes called 'action at a distance'.

(iii) The special case $f(z) = 1$ for all $z \in D$ gives the winding number lemma.

Proof (of Theorem 14.1). Let γ_r be the circular contour $\gamma_r = w + re^{it}$, $(0 \leq t \leq 2\pi)$, where $r > 0$ is sufficiently small so that γ_r is contained in the interior of γ.

[Step 1] Let β be a contour in $D\backslash\{w\}$ from the start point of γ to the start point of γ_r.

The contour

$$\widetilde{\gamma} = \gamma + \beta + \underbrace{(-\gamma_r) + \cdots + (-\gamma_r)}_{n(\gamma,w)\ \text{times}} + (-\beta),$$

where there are $n(\gamma, w)$ copies of $(-\gamma_r)$, has winding number zero about w, so by Cauchy's theorem applied to $\dfrac{f(z)}{z-w}$ on $D\backslash\{w\}$,

$$\begin{aligned}
\int_{\widetilde{\gamma}} \frac{f(z)}{z-w}\,dz &= 0 \\
\left(\int_\gamma + \int_\beta - \int_{\gamma_r} - \cdots - \int_{\gamma_r} + \int_{-\beta}\right) \frac{f(z)}{z-w}\,dz &= 0 \\
\int_\gamma \frac{f(z)}{z-w}\,dz - n(\gamma,w)\int_{\gamma_r} \frac{f(z)}{z-w}\,dz &= 0 \\
\int_\gamma \frac{f(z)}{z-w}\,dz &= n(\gamma,w)\int_{\gamma_r} \frac{f(z)}{z-w}\,dz.
\end{aligned}$$

[Step 2] We shall now show that

$$\lim_{r\to 0}\int_{\gamma_r} \frac{f(z)}{z-w}\,dz = 2\pi i f(w).$$

We have,

$$\begin{aligned}
\int_{\gamma_r} \frac{f(z)}{z-w}\,dz &= f(w)\int_{\gamma_r}\frac{dz}{z-w} + \int_{\gamma_r}\frac{f(z)-f(w)}{z-w}\,dz \\
&= 2\pi i f(w) + \int_{\gamma_r}\frac{f(z)-f(w)}{z-w}\,dz, \text{ by winding lemma.}
\end{aligned}$$

Therefore

$$\begin{aligned}
\left|\int_{\gamma_r} \frac{f(z)}{z-w}\,dz - 2\pi i f(w)\right| &= \left|\int_{\gamma_r}\frac{f(z)-f(w)}{z-w}\,dz\right| \\
&\le \sup_{z\in\gamma_r}\left|\frac{f(z)-f(w)}{z-w}\right| \times 2\pi r,
\end{aligned}$$

by the Estimation Lemma. Now $\dfrac{f(z)-f(w)}{z-w} \to f'(w)$ as $z \to w$. (Why does $f'(w)$ exist?) Hence RHS $\to |f'(w)|.0 = 0$ as $r \to 0$. Therefore,

$$\lim_{r\to 0}\int_{\gamma_r} \frac{f(z)}{z-w}\,dz = 2\pi i f(w).$$

Thus we can combine these steps to get

$$\begin{aligned}
\int_\gamma \frac{f(z)}{z-w}\,dz - 2\pi i n(\gamma,w) f(w) &= \left(n(\gamma,w)\int_{\gamma_r}\frac{f(z)}{z-w}\,dz\right) - 2\pi i n(\gamma,w) f(w) \\
&= n(\gamma,w)\left(\int_{\gamma_r}\frac{f(z)}{z-w}\,dz - 2\pi i f(w)\right).
\end{aligned}$$

The RHS tends to 0 as $r \to 0$, but the LHS is independent of r and so must be 0. $\square$

Examples 14.3

(i) Calculate $\displaystyle\int_\gamma \frac{\sin z}{z-\pi}\,dz$, where $\gamma(t) = 3e^{-it}$, $0 \le t \le 4\pi$.

We apply CIF with $f(z) = \sin z$ and $w = \pi$. The point w lies within the circle formed by γ, and $n(\gamma, w) = -2$, because γ winds round w clockwise twice. (Draw a picture!)

Hence,

$$\int_\gamma \frac{\sin z}{z-\pi}\,dz = 2\pi i \times (-2) \times \sin(\pi) = -4\pi i.$$

(ii) Let γ be a circle of radius 2 about 2, i.e. z such that $|z-2| = 2$.

Draw the contour and indicate where the integrand is not differentiable.

Then,

$$\begin{aligned}
\int_{|z-2|=2} \frac{e^z}{z^2-9}\,dz &= \int_{|z-2|=2} \frac{1}{6}\frac{e^z}{z-3} - \frac{1}{6}\frac{e^z}{z+3}\,dz \\
&= \frac{1}{6}\int_{|z-2|=2} \frac{e^z}{z-3}\,dz - \frac{1}{6}\int_{|z-2|=2} \frac{e^z}{z+3}\,dz \\
&= \frac{1}{6}2\pi i e^3 - 0, \text{ by 14.1}, \\
&= \frac{i\pi e^3}{3}.
\end{aligned}$$

Liouville's Theorem

The next theorem is also rather surprising.

Theorem 14.4
Suppose f is differentiable on the whole of $\mathbb{C}$, and is bounded, i.e. there exists M such that $|f(z)| \le M$ for all $z \in \mathbb{C}$. Then, f is constant.

Proof. [HTTLAM: We want to show the function is constant. This is true if $f(\alpha) = f(0)$ for an arbitrary α. So we need to show $|f(\alpha) - f(0)| = 0$ for every $\alpha \in \mathbb{C}$.]

Let $\alpha \in \mathbb{C}$. We let $R \geq 2|\alpha|$ and $\gamma(t) = e^{2\pi i t}$ for $0 \leq t \leq 1$ be a circle of radius R round the origin. Then,

$$\begin{aligned}
|f(\alpha) - f(0)| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-\alpha}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z}\,dz\right|, \text{ by CIF} \\
&= \left|\frac{1}{2\pi i}\right| \left|\int_\gamma f(z)\left(\frac{1}{z-\alpha} - \frac{1}{z}\right)dz\right| \\
&= \frac{1}{2\pi}\left|\int_\gamma \frac{f(z)\alpha}{z(z-\alpha)}\,dz\right| \\
&\leq \frac{1}{2\pi} M \sup_{z\in\gamma}\left\{\left|\frac{\alpha}{z(z-\alpha)}\right|\right\} L(\gamma) \text{ by} \\
&\leq \frac{1}{2\pi} M \frac{|\alpha|}{R(R-|\alpha|)}.2\pi R \\
&= \frac{M|\alpha|}{R-|\alpha|}.
\end{aligned}$$

But R was effectively arbitrary, so as $R \to \infty$, the LHS$\to 0$. As neither $f(\alpha)$ nor $f(0)$ depend on R we must have $f(\alpha) - f(0) = 0$. That is, f is constant. $\square$

Remark 14.5
Contrast this with real analysis. Being bounded does not imply constant, for example, consider $\sin$. This is differentiable on all $\mathbb{R}$ and $|\sin x| \leq 1$ for all $x \in \mathbb{R}$ but it $\sin$ is not constant.

Exercise 14.6
Show $\cos$ and $\sin$ are not bounded on $\mathbb{C}$.

Maximum Modulus Principle

Theorem 14.7
Let $f : D \to \mathbb{C}$ be a differentiable function and γ be a closed contour such that $\mathrm{Int}(\gamma) \subset D$.

If $|f(z)| \leq M$ for all $z \in \gamma$, then $|f(w)| \leq M$ for all $w \in \mathrm{Int}(\gamma)$.

Remark 14.8
The theorem says that the modulus of a function within the interior of a contour is never bigger than the modulus of the function on the contour. In other words the maximum modulus occurs on the boundary of a region.

Proof. This proof contains a nice trick. We shall apply CIF to $w \in \operatorname{Int}(\gamma)$ and $f(z)^k$, where k is a natural number:

$$f(w)^k = \frac{1}{2\pi i n(\gamma, w)} \int_\gamma \frac{f(z)^k}{z-w}\, dz.$$

Define the distance from w to γ by

$$\operatorname{dist}(\gamma, w) = \inf\{|z-w| \,:\, z \in \gamma\}.$$

Then, obviously $|z-w| \geq \operatorname{dist}(\gamma, w)$ for all $z \in \gamma$.

So,

$$|f(w)|^k \leq \frac{1}{2\pi\,|n(\gamma, w)|} \frac{M^k}{\operatorname{dist}(\gamma, w)} L(\gamma), \text{ by the Estimation Lemma,}$$

since

$$\frac{|f(w)|^k}{|z-w|} \leq \frac{M^k}{\operatorname{dist}(\gamma, w)} \text{ on } \gamma.$$

Therefore,

$$|f(w)| \leq \left(\frac{L(\gamma)}{2\pi\,|n(\gamma, w)|\,\operatorname{dist}(\gamma, w)}\right)^{1/k} M.$$

Now, we use the fact that $\lim_{k\to\infty} x^{1/k} = 1$ for all $x > 0$, (this is true because $x^{1/k} = \exp(\frac{1}{k}\ln x) \to \exp(0) = 1$). Thus, letting $k \to \infty$ gives $|f(w)| \leq M$. □

Fundamental Theorem of Algebra

We are now in a position to prove the Fundamental Theorem of Algebra, (which is in fact not really a theorem of algebra but of analysis!)

Theorem 14.9
Every polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \ldots a_1 z + a_0$, $n \geq 1$, has a root in $\mathbb{C}$.

Proof. Suppose not and derive a contradiction. Since $p(z) \neq 0$ for all $z \in \mathbb{C}$ the function f defined by $f(z) = 1/p(z)$ is differentiable on all of $\mathbb{C}$. Now, for $z \neq 0$,

$$\left|\frac{p(z)}{z^n}\right| = \left|1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \to 1 \text{ as } |z| \to \infty.$$

So there exists an R such that $|z| \geq R$ implies that $\left|\dfrac{p(z)}{z^n}\right| \geq \dfrac{1}{2}$.

This in turn means

$$|f(z)| = \left|\frac{1}{p(z)}\right| \leq \frac{2}{|z^n|} \leq \frac{2}{R^n} \text{ for } |z| > R.$$

By the Maximum Modulus Principle applied to f with γ being the standard circle of radius R round the origin we have $|z| < R \implies |f(z)| \leq \dfrac{2}{R^n}$.

So $|f(z)| \leq \dfrac{2}{R^n}$ on all of $\mathbb{C}$. By Liouville's theorem, f is constant, and hence so is p. This is a contradiction, so p has a root. $\square$

Summary

- Cauchy's theorem can be used to prove surprising theorems that have no analogues in real analysis.
- Cauchy's Integral Formula: The value of a function at z_0 is determined by the values on contours round z_0.
- Liouville's Theorem: Any differential function bounded on the whole of $\mathbb{C}$ is contant.
- The Maximum Modulus Principle: The modulus of a function on a domain achieves its maximum on the boundary of the domain.
- Fudamental Theorem of Algebra: Every complex polynomial has a complex root.

15 Taylor's Theorem

Taylor's theorem is a consequence of Cauchy's theorem but is so important we give it a separate section.

Theorem 15.1 (Taylor's theorem for complex functions) Suppose $f : D \to \mathbb{C}$ is a differentiable function. Let $z_0 \in D$ and $R = \operatorname{dist}(z_0, \mathbb{C}\backslash D)$ (which equals infinity if $D = \mathbb{C}$.) Then, there exists a Taylor expansion of f about z_0, i.e. there exist $a_n \in \mathbb{C}$ such that

$$f(z) = \sum_{0}^{\infty} a_n (z - z_0)^n, \text{ for all } |z - z_0| < R.$$

Furthermore,

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz,$$

where C_r is a small circle about z_0 and r is any number with $0 < r < R$.

Proof. Fix r with $0 < r < R$. (We will replace z with $z_0 + h$ so that we have a power series in h centred at 0). For each h with $|h| < r$, Cauchy's Integral Formula gives

$$f(z_0 + h) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - (z_0 + h)}\, dz.$$

Now, with $|h| < r$ and $|z - z_0| = r$,

$$\begin{aligned} \frac{f(z)}{z - (z_0 + h)} &= \frac{f(z)}{(z - z_0)\left(1 - \frac{h}{z - z_0}\right)} = \frac{f(z)}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{h}{z - z_0}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{f(z) h^n}{(z - z_0)^{n+1}}. \end{aligned}$$

Now we shall integrate term-by-term, (see Corollary 6.7), using $g(z) = \dfrac{f(z)}{z - z_0}$, and $g_k(z) = \dfrac{f(z)h^n}{(z - z_0)^{n+1}}$. Then, define M_k by

$$M_k = \sup_{|z - z_0| = r} |g_k(z)| = \sup_{|z - z_0| = r} \left| \frac{f(z)h^n}{(z - z_0)^{n+1}} \right| = \sup_{|z - z_0| = r} |f(z)| \frac{|h|^n}{r^{n+1}},$$

so, $\sum_{k=0}^{\infty} M_k < \infty$ as $|h| < r$. (Use the ratio test). Thus, by

Corollary 6.7,

$$
\begin{aligned}
\int_{C_r} \frac{f(z)}{z-(z_0+h)}\,dz &= \sum_{k=0}^{\infty} \int_{C_r} \frac{f(z)h^n}{(z-z_0)^{n+1}}\,dz \\
2\pi i f(z_0+h) &= \sum_{k=0}^{\infty} \left(\int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \right) h^n \\
f(z_0+h) &= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \right) h^n
\end{aligned}
$$

Therefore, if we substitute $z = z_0 + h$ and let $a_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$, we get the series

$$
f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n.
$$

That we also have $a_n = \dfrac{f^{(n)}(z_0)}{n!}$ follows from Corollary 10.6. □

The theorem says that there exists an R-neighbourhood of z_0 upon which f is a power series. That is, f is **analytic**. This explains why some authors use the word analytic to mean complex differentiable:

differentiable and analytic are equivalent in complex analysis.

Analytic and differentiable are not equivalent in real analysis as we see in a later example.

The expression for a_n is sometimes known as **Cauchy's formula for derivatives**.

HTTLAM 15.2

When solving problems concerning differentiable complex functions, whether theoretical or practical ones, use the fact that the function can be written locally as a power series.

Example 15.3

(i) Find the Taylor series of $f(z) = (z+i)^3$ at $z = 0$. We have,

$$
\begin{array}{rclcrcl}
f(z) &=& (z+i)^3 & \Longrightarrow & f(0) &=& (0+i)^3 = -i, \\
f'(z) &=& 3(z+i)^2 & \Longrightarrow & f'(0) &=& 3(0+i)^2 = -3, \\
f''(z) &=& 6(z+i) & \Longrightarrow & f''(0) &=& 6(0+i) = 6i, \\
f^{(3)}(z) &=& 6 & \Longrightarrow & f^{(3)}(0) &=& 6, \\
f^{(4)}(z) &=& 0 & \Longrightarrow & f^{(4)}(0) &=& 0.
\end{array}
$$

So,

$$
\begin{aligned}
f(z) &= f(0) + f'(0)(z-0) + \frac{f''(0)}{2!}(z-0)^2 + \frac{f^{(3)}(0)}{3!}(z-0)^3 \\
&= -i - 3z + 3iz^2 + z^3.
\end{aligned}
$$

Note that this expansion is valid for all $z \in \mathbb{C}$ as f is differentiable on all of $\mathbb{C}$, i.e. $R = \operatorname{dist}(0, \mathbb{C}\backslash\mathbb{C}) = \infty$.

(ii) Expand $f(z) = \dfrac{1}{1-z}$ about 0. It is easy to show that $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$, so $f^{(n)}(0) = n!$.

Thus,

$$\frac{1}{1-z} = f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z-0)^n = \sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \sum_{n=0}^{\infty} z^n.$$

This is valid for all $|z| < 1$ as f is differentiable on $\mathbb{C}\backslash\{1\}$ so $R = \operatorname{dist}(0, \mathbb{C}\backslash(\mathbb{C}\backslash\{1\})) = \operatorname{dist}(0, \{1\}) = 1$.

Note that this confirms a result we already know.

Now we come to a very surprising and useful theorem, which contrasts sharply with real differentiability.

Corollary 15.4 (Infinite Differentiability)
Suppose $f : D \to \mathbb{C}$ is differentiable on the domain D. Then, f is infinitely differentiable, i.e. it has derivatives of all orders.

Proof. Suppose z_0 is any point in D. From the theorem we know that, in some neighbourhood of z_0, f can be given as a power series, and so is infinitely differentiable by Corollary 10.5. □

This is really astounding. It doesn't happen for real functions as the next example illustrates.

Example 15.5
Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Then, f is differentiable with derivative

$$f(x) = \begin{cases} 2x & x > 0 \\ 0 & x \leq 0. \end{cases}$$

This derivative is not differentiable at 0, so f is not infinitely differentiable.

Some facts about power series

We now use Taylor's Theorem to prove some facts about series which might otherwise be difficult to demonstrate.

Theorem 15.6
Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $|z| < R_1$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ for all $|z| < R_2$. Then, $f(z)g(z) = (fg)(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $|z| < \min\{R_1, R_2\}$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Proof. Let $R = \min\{R_1, R_2\}$. Then, f and g are complex differentiable for all $|z| < R$. The product of two differentiable functions is differentiable by the product rule. So, fg is differentiable. By Theorem 15.1 it has a series expansion for $|z| < R$, and by Corollary 10.6, we get

$$(fg)(z) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(0)}{n!} z^n.$$

Now we can apply Leibnitz's rule

$$\begin{aligned} (fg)^{(n)}(0) &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(k)}(0) g^{(n-k)}(0) \\ &= \sum_{k=0}^{n} n! \frac{f^{(k)}(0)}{k!} \frac{g^{(n-k)}(0)}{(n-k)!} \\ &= n! \sum_{k=0}^{n} a_k b_{n-k}. \end{aligned}$$

Therefore,

$$(fg)(z) = \sum_{n=0}^{\infty} c_n z^n, \ (|z| < R), \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

□

Corollary 15.7
Suppose f and g are power series with positive radii of convergence. If $g(0) \neq 0$, then f/g has a power series expansion at 0 with positive radius of convergence.

Proof. Define $h(z) = 1/g(z)$. By continuity of g and the fact that $g(0) \neq 0$, there is a ε-neighbourhood of 0 upon which h is differentiable. Thus, h has a power series expansion, and the result follows from the theorem applied to fh. □

Remark 15.8
The precise value of the radius of convergence will depend on where g is zero.

* Morera's Theorem

There is a partial converse to Cauchy's theorem. It relies on the earlier optional material on the Fundamental Theorem of Calculus.

Theorem 15.9
Let $f : D \to \mathbb{C}$ be a continuous map on a connected domain D such that $\int_\gamma f = 0$ for all contours f. Then, f is differentiable.

Proof. We know from Theorem 9.9 that there exists $F : D \to \mathbb{C}$ such that $F' = f$. But by Corollary 15.4 the derivative of a differentiable function is itself differentiable. $\square$

Summary

- If a complex function is differentiable on a domain, then at every point there is power series expansion valid on some ε-neighbourhood.
- Differentiable complex functions are infinitely differentiable.
- If S_1 and S_2 are power series with radii of convergence R_1 and R_2, then S_1S_2 is a power series with radius of convergence $\min\{S_1, S_2\}$.
- If S is a power series at z_0 with positive radius of convergence, then $1/S$ has a power series expansion at z_0 provided $S(z_0) \neq 0$.
- There is a partial converse to Cauchy's Theorem.

16 Zeros of functions

We have met a number of functions that are not analytic (i.e. differentiable) at certain points. For example, $1/z$ is not analytic at 0, because it is undefined when z is zero. Therefore, it seems natural to investigate zeros of functions first.

Zeros

Definition 16.1
An analytic function $f : D \to \mathbb{C}$ has a **zero** at z_0 if $f(z_0) = 0$.

Examples 16.2

(i) The function $\sin$ has zeros at $k\pi$, for all $k \in \mathbb{Z}$.

(ii) By Theorem 1.35(iv) $f(z) = e^z$ has no zeros.

(iii) $f(z) = z^3(z^2 + 1)$ has a zero at $z = i$, one at $z = -i$, and one at $z = 0$.

Exercise 16.3
Find the zeros of:

$$\text{(i) } z^2 + 9, \quad \text{(ii) } e^{z^2+9}(z + 4i), \quad \text{(iii) } e^{z-2} - 1.$$

We now know that if $f : D \to \mathbb{C}$ is differentiable at z_0, it has a Taylor series expansion about z_0: $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all z with $|z - z_0| < R$ for some $R > 0$.

Definition 16.4
We say f has a **zero of order** m **at** z_0 if

$$a_0 = a_1 = \cdots = a_{m-1} = 0 \text{ but } a_m \neq 0.$$

Remark 16.5
Obviously, by Theorem 15.1, f has a zero of order m at z_0 if and only if $f^{(j)}(z_0) = 0$ for all $j < m$ and $f^{(m)}(z_0) \neq 0$.

In particular, if $f(z_0) = 0$ and $f'(z_0) \neq 0$, then f has a zero of order 1.

Examples 16.6

(i) The function $f(z) = z^2$ has a zero of order 2 at 0.

(ii) The function $f(z) = z(z+2i)^3$ has a zero of order 1 at 0 and one of order 3 at $-2i$.

(iii) More generally, suppose that f is a polynomial with a root of multiplicity m at z_0. Then, f has a zero of order m at z_0.

Exercise 16.7
Find the zeroes and their orders of the following:

(i) $(z-1)^3(z+1)$ (ii) $(z^2+1)^2$, (iii) ze^{z^2}, (iv) $(2z-3i)^4$, (v) z^3+1.

Exercise 16.8
Suppose that $f : D \to \mathbb{C}$ is an analytic function with a zero of order m at z_0. Then, there exists a differentiable g and an $R > 0$, such that $f(z) = (z - z_0)^m g(z)$, for all $|z - z_0| < R$, and $g(z_0) \neq 0$.

Summary

- An analytic function $f : D \to \mathbb{C}$ has a **zero** at z_0 if $f(z_0) = 0$.
- Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all z with $|z - z_0| < R$ for some R. Then f has a **zero of order** m **at** z_0 if

$$a_0 = a_1 = \cdots = a_{m-1} = 0 \text{ but } a_m \neq 0.$$

17 Poles

The notions of pole and residue are crucial to the application of complex analysis to real problems, such as real integrals, or Fourier and Laplace Transforms[7]. We define poles in this section and show how to locate them in the next. After that we define their residue.

A motivating example

Let us consider an example: Integrate $\displaystyle\int_{|z|=1} \frac{e^z}{z^2}\,dz$, where $|z|=1$ denotes the standard contour of a unit circle round the origin.

We can't use Cauchy's Integral Formula as the integrand is not analytic at 0 and the Fundamental Theorem of Calculus doesn't help as we can't see any obvious antiderivative.

Let's just integrate term-by-term[8] and see what happens.

$$\begin{aligned}
\int_{|z|=1} \frac{e^z}{z^2}\,dz &= \int_{|z|=1} \frac{1}{z^2}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\right)dz \\
&= \int_{|z|=1} \left(\frac{1}{z^2}+\frac{1}{z}+\frac{1}{2}+\frac{z}{6}+\dots\right)dz \\
&= \int_{|z|=1} \frac{1}{z^2}\,dz+\int_{|z|=1} \frac{1}{z}\,dz+\int_{|z|=1} \frac{1}{2}\,dz+\int_{|z|=1} \frac{z}{6}\,dz+\dots \\
&= 0+2\pi i+0+0+\dots.
\end{aligned}$$

Notice that **only the $1/z$ term mattered**. By the Fundamental Example, all the other terms were irrelevant. This is key to understanding the terms pole and residue we are going to define.

Laurent expansions

First we shall look at functions, like the integrand e^z/z^2 example above, that can be represented by a series with negative powers.

Recall for an *analytic function* we can represent it as

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ for } |z-z_0|<R.$$

Compare this with the following series.

[7]These transforms are used in Applied Mathematics. If you know what they are, then fine. If not, then don't worry about it.

[8]Obviously, we should check that we can do this!

Definition 17.1
Suppose that $f : D \to \mathbb{C}$ is complex function. Then we say f has a **Laurent expansion at** z_0 if there exist $a_n \in \mathbb{C}$ and $R > 0$, such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for all z with $0 < |z - z_0| < R$.

Note that the limits go from $-\infty$ to ∞, and that the expansion, does not necessarily equal $f(z_0)$ when $z = z_0$, (which may be undefinable anyway).

Example 17.2
(i) $f(z) = 1/z$ has an expansion at 0 with $a_n = 0$ for $n \neq -1$, $a_{-1} = 1$, and $R = \infty$.

(ii) The expansion of $f(z) = e^z/z^2$ at 0,

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{z}{6} + \dots$$

has $a_n = \dfrac{1}{(n+2)!}$ for $n \geq 0$, $a_{-1} = a_{-2} = 1$, and $a_j = 0$ for $j \leq -3$. The radius is $R = \infty$.

(iii) Any power series with a positive radius of convergence is a Laurent expansion. Hence, any function differentiable at z_0 has a Laurent expansion.

Exercise 17.3
Find a Laurent expansion for $f(z) = \dfrac{\sin z}{z^4}$ defined on $\mathbb{C} \backslash \{0\}$.

The definition of a pole

Definition 17.4
Suppose that $f : D \to \mathbb{C}$ is a complex function. We say that f has a **pole of order** N at $z_0 \in D$, if there exists a Laurent expansion at z_0 of the form

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n \quad (0 < |z - z_0| < R), \text{ with } a_{-N} \neq 0.$$

Poles are also called **singularities**.

If there is no $N \geq 0$ such that $a_n = 0$ for all $n < -N$, then we say f has an **essential singularity** at z_0.

Remark 17.5
The point is that for a pole of finite order there are only a *finite* number of negative exponents of z in the series.

Examples 17.6

(i) order$(1/z^5) = 5$ at 0.

(ii) order$(e^z/z^2) = 2$ at 0 as $f(z) = \dfrac{1}{z^2} + \dfrac{1}{z} + \dfrac{1}{2} + \dfrac{z}{6} + \dots$.

(iii) order $\left(\dfrac{i}{(z-4i)^3} + (z-4i)^2 - 2i(z-4i)^5\right) = 3$ at $4i$.

(iv) $f(z) = \exp(1/z)$ has an essential singularity at 0 because

$$f(z) = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!},$$

and this obviously has an infinite number of terms with negative exponent.

Remarks 17.7

(i) If such a Laurent series exists, then the coefficients are unique. To see this look at $(z-z_0)^N f(z)$, this is a power series and so has unique coefficients.

(ii) Suppose that $f(z)$ has a Laurent expansion with $N = 0$, i.e.

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \text{ for } 0 < |z-z_0| < R.$$

Then, we can define $f(z_0) = a_0$ and thus make f analytic at z_0.

If we can do this, we say f has a **removable singularity**.

Example 17.8

The function $f(z) = \dfrac{\sin z}{z}$ is defined on $\mathbb{C}\backslash\{0\}$, and we have

$$\begin{aligned} f(z) &= \frac{\sin z}{z} \\ &= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots. \end{aligned}$$

So define $f(0) = 1$ to make f analytic on all of $\mathbb{C}$. That is,

$$f(z) = \begin{cases} \dfrac{\sin z}{z}, & \text{for } z \neq 0, \\ 1, & \text{for } z = 0. \end{cases}$$

Exercise 17.9

Show that $(e^z - 1)/z$ has a removable singularity.

Summary

- An expansion of f of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

  for all z with $0 < |z - z_0| < R$, is called a Laurent expansion at z_0.

- We say that f has a pole of order N at $z_0 \in D$, if there exists a Laurent expansion at z_0 of the form

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n \ (0 < |z - z_0| < R), \text{ with } a_{-N} \neq 0.$$

18 How to find poles

It is hard to give non-trivial examples of poles without using some theory. You may have noticed that the order is somehow connected with the order of multiplicity of the polynomial in a denominator. The following puts that intuition into a precise mathematical statement, and does it for general functions not just polynomials.

Lemma 18.1
Suppose that $f(z) = \dfrac{p(z)}{q(z)}$, where

(i) p is analytic at z_0, and $p(z_0) \neq 0$,

(ii) q is analytic at z_0, and has a zero of order N at z_0.

Then, f has a pole of order N at z_0.

Proof. By assumption, we have $q(z) = (z - z_0)^N r(z)$ where $r(z)$ is analytic with $r(z_0) \neq 0$ (see Exercise 16.8). Then, $g(z) = \dfrac{p(z)}{r(z)}$ is analytic at z_0, so by Theorem 15.1 it has a power series expansion:

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \text{ valid for } |z - z_0| < R, \text{ some } R > 0.$$

Thus,

$$\begin{aligned} f(z) &= \frac{p(z)}{q(z)} \\ &= \frac{p(z)}{(z - z_0)^N r(z)} \\ &= \frac{g(z)}{(z - z_0)^N} \\ &= \frac{1}{(z - z_0)^N} \sum_{n=0}^{\infty} a_n (z - z_0)^n \\ &= \frac{a_0}{(z - z_0)^N} + \frac{a_1}{(z - z_0)^{N-1}} + \frac{a_2}{(z - z_0)^{N-2}} + \dots \end{aligned}$$

But,

$$a_0 = g(z_0) = \frac{p(z_0)}{r(z_0)} \neq 0,$$

because $p(z_0) \neq 0$ by assumption and $r(z_0) \neq 0$ by definition. So as $a_0 \neq 0$, we deduce that f has a pole of order N at z_0. $\square$

Common Error 18.2
The above lemma gives a method for finding a pole and its order. The definition of a pole is that given in Definition 17.4. It is not the multiplicity of the denominator in an expression. Do not confuse the definition of an object with a process by which we find the object.

Remark 18.3
Recall that a zero of order N of a polynomial is just a root of multiplicity N.

Example 18.4

(i) The function $\dfrac{\sin z}{(z-3)^2}$ has a pole of order 2 at $z=3$, since $\sin$ is analytic and $\sin 3 \neq 0$, and $(z-3)^2$ is analytic with a root of multiplicity 2 at 3.

(ii) The quotient $\dfrac{z(z-1)^4}{(z^2-2z+5)^2}$ has poles of order 2 at $z=1\pm 2i$. This is because we can factorize the polynomial z^2-2z+5 to get $(z-(1+2i))(z-(1-2i))$. The resulting roots are repeated as

$$(z^2-2z+5)^2 = [(z-(1+2i))]^2\,[(z-(1-2i))]^2 .$$

Exercises 18.5
Find the poles and their orders for

(a) $\dfrac{1+e^z}{z(z^2-1)}$, (b) $\dfrac{1+z}{z^3-2z^2}$, (c) $\dfrac{e^{z^2}-1}{z^5}$ (It ain't 5!)

Simple poles

Poles of order 1 are given a special name.

Definition 18.6
A pole of order 1 is called a **simple pole**.

Example 18.7

(i) $1/z$ has a simple pole at 0.

(ii) $\dfrac{\sin z}{z^2}$ has a simple pole at 0:

$$\begin{aligned}\frac{\sin z}{4z^2} &= \frac{z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dots}{4z^2}\\ &= \frac{1}{4z}-\frac{z}{3!4}+\frac{z^3}{5!4}-\dots.\end{aligned}$$

There is a useful way of locating simple poles.

Theorem 18.8
Suppose $f(z) = \dfrac{p(z)}{q(z)}$ where,

(i) p is analytic at z_0, and $p(z_0) \neq 0$,

(ii) q is analytic at z_0, $q(z_0) = 0$ and $q'(z_0) \neq 0$.

Then, f has a simple pole at z_0.

Proof. By Remark 16.5 q has a zero of order 1 at z_0. Thus, by Lemma 18.1, f has a pole of order 1 at z_0, i.e. the pole is simple. □

Examples 18.9

(i) The function $f(z) = \dfrac{\cos z}{z}$ has a simple pole at 0. We have $p(z) = \cos z$ and $q(z) = z$, so $p(0) = 1$, $q(0) = 0$, and $q'(0) = 1$.

(ii) The function $f(z) = \dfrac{z+3}{\sin z}$ has a simple pole at 0. We have $p(z) = z + 3$ and $q(z) = \sin z$, so $p(0) = 3$, $q(0) = 0$, and $q'(0) = \cos 0 = 1$.

Summary

- Suppose that $f(z) = \dfrac{p(z)}{q(z)}$, where

  (i) p is analytic at z_0, and $p(z_0) \neq 0$,

  (ii) q is analytic at z_0, and has a zero of order N at z_0.

  Then, f has a pole of order N at z_0.

- A **simple pole** is a pole of order 1.

19 Residues

We saw earlier that to integrate

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{z}{6} \dots$$

we could use the Laurent expansion about zero to integrate term-by-term. The only information that mattered was the coefficient of the z^{-1} term. Let us now give that term an official name.

Definition 19.1
Suppose that f has a pole at z_0. The **residue of** f **at** z_0 is the coefficient a_{-1} in the expansion $\sum_{n=-N}^{\infty} a_n(z-z_0)^n$. This is denoted $\operatorname{res}(f, z_0)$.

Examples 19.2

(i) $\operatorname{res}(e^z/z^2, 0) = 1$.

(ii) $\operatorname{res}(z^2, 0) = 0$. (The residue is always 0 at differentiable points.)

(iii) From earlier,

$$\frac{\sin z}{4z^2} = \frac{1}{4z} - \frac{z}{3!4} + \frac{z^3}{5!4} - \dots$$

so $\operatorname{res}\left(\frac{\sin z}{4z^2}, 0\right) = \frac{1}{4}$.

Remark 19.3
The residue at a point is unique because the coefficients of a Laurent expansion with a finite number of negative terms are unique.

The next theorem gives a hint of where we are going with residues: They allow us to quickly calculate integrals. This theorem is really just generalising the example of e^z/z^2 used at the start of the section on poles.

Theorem 19.4 (Cauchy's Residue Theorem for a Circle)
Suppose that the analytic function $f : D \to \mathbb{C}$ has a pole at p. Let $\gamma(p, r)$ denote the circle of radius r round p. Then, there exists $R > 0$ such that

$$\int_{\gamma(p,r)} f(z)\, dz = 2\pi i \operatorname{res}(f, p),$$

for all $0 < r < R$.

Proof. At p the function f has a Laurent expansion $f(z) = \sum_{n=-N}^{\infty} a_n(z-p)^n$, for $0 < |z-p| < R$, some $R > 0$. Therefore, let r be such that $0 < r < R$, and then

$$\begin{aligned}
\int_{\gamma(p,r)} f(z)\,dz &= \int_{\gamma(p,r)} \sum_{n=-N}^{\infty} a_n(z-p)^n\,dz \\
&= \sum_{n=-N}^{\infty} \int_{\gamma(p,r)} a_n(z-p)^n\,dz.
\end{aligned}$$

The latter equality follows from Theorem 6.7 (you can check this!). But, by the Fundamental Example,

$$\begin{aligned}
\int_{\gamma(p,r)} a_n(z-p)^n\,dz &= a_n \int_{\gamma(p,r)} (z-p)^n\,dz \\
&= \begin{cases} a_{-1} \times 2\pi i, & n = -1 \\ 0, & n \neq -1. \end{cases}
\end{aligned}$$

Thus, $\int_{\gamma(p,r)} f(z)\,dz = 2\pi i a_{-1} = 2\pi i \operatorname{res}(f,p)$. □

Remark 19.5
The theorem says that to calculate certain integrals all we have to do is calculate the residue of a function at a point.

Example 19.6
Let γ be the unit circle round the origin. We take $f(z) = e^z/z^2$ from earlier, then $\operatorname{res}(f,0) = 1$ has already been calculated. So $\int_\gamma e^z/z^2 = 2\pi i \operatorname{res}(f,0) = 2\pi i$, as we have already seen.

We shall generalise the theorem further, but in the next section we will see investigate some methods for calculating residues.

Summary

- The **residue of** f **at** z_0 is the coefficient a_{-1} in the expansion $\sum_{n=-N}^{\infty} a_n(z-z_0)^n$. This is denoted $\operatorname{res}(f,z_0)$.
- Suppose that the analytic function $f : D \to \mathbb{C}$ has a pole at p. Let $\gamma(p,r)$ denote the circle of radius r round p. Then, there exists $R > 0$ such that

  $$\int_{\gamma(p,r)} f(z)\,dz = 2\pi i \operatorname{res}(f,p),$$

  for all $0 < r < R$.

20 Bluffer's Guide to Calculating Residues

Residues are an important part of any course on Complex Analysis. They allow us to easily calculate otherwise complicated integrals – without doing any integration!

This section contains the main methods for calculating the residues of poles. You should know them and their proofs.

Method 1: Simple Poles

Suppose f has a *simple pole* (i.e. order 1) at w then

$$\operatorname{res}(f, w) = \lim_{z \to w} (z - w) f(z).$$

Proof. As f has a simple pole at w we have

$$f(z) = \sum_{n=-1}^{\infty} a_n (z - w)^n.$$

So,

$$\begin{aligned}
& \lim_{z \to w} (z - w) f(z) \\
= & \lim_{z \to w} (z - w) \sum_{n=-1}^{\infty} a_n (z - w)^n \\
= & \lim_{z \to w} (z - w) \left(\frac{a_{-1}}{z - w} + a_0 + a_1(z - w) + a_2(z - w)^2 + \dots \right) \\
= & \lim_{z \to w} a_{-1} + a_0(z - w) + a_1(z - w)^2 + a_2(z - w)^3 + \dots \\
= & a_{-1} \\
= & \operatorname{res}(f, w).
\end{aligned}$$

□

Examples 20.1

(i) The function $f(z) = \dfrac{\sin(z)}{z - 3}$ has a pole of order 1 at $z = 3$, so

$$\operatorname{res}(f, 3) = \lim_{z \to 3} (z-3) f(z) = \lim_{z \to 3} (z-3) \frac{\sin(z)}{z - 3} = \lim_{z \to 3} \sin(z) = \sin(3).$$

(ii) The function $f(z) = \dfrac{z}{1 - \cos z}$ has a pole of order 1 at $z = 0$, so

$$\begin{aligned}
\operatorname{res}(f, 0) &= \lim_{z \to 0} z \frac{z}{1 - \cos z} = \lim_{z \to 0} \frac{z^2}{2 \sin^2(z/2)} = \lim_{z \to 0} \frac{4\,(z/2)^2}{2 \sin^2(z/2)} \\
&= 2 \lim_{z \to 0} \left(\frac{z/2}{\sin(z/2)} \right)^2 = 2 \left(\lim_{z \to 0} \frac{z/2}{\sin(z/2)} \right)^2 = 2.
\end{aligned}$$

Method 2: Some quotients (Really good method!)

Suppose $f(z) = \dfrac{p(z)}{q(z)}$ with p and q analytic, $p(w) \neq 0$, $q(w) = 0$ and $q'(w) \neq 0$. Then,

$$\operatorname{res}(f, w) = \frac{p(w)}{q'(w)}.$$

Proof. By Theorem 18.8 f has a simple pole at w, so

$$\begin{aligned}
\operatorname{res}(f, w) &= \lim_{z \to w} (z - w) f(z), \text{ by Method 1,} \\
&= \lim_{z \to w} (z - w) \frac{p(z)}{q(z)} \\
&= \lim_{z \to w} p(z) \Big/ \left(\frac{q(z)}{z - w} \right) \\
&= \lim_{z \to w} p(z) \Big/ \left(\frac{q(z) - q(w)}{z - w} \right), \text{ as } q(w) = 0, \\
&= \frac{\lim_{z \to w} p(z)}{\lim_{z \to w} \dfrac{q(z) - q(w)}{z - w}} \\
&= \frac{p(w)}{q'(w)}, \text{ by definition of differentiation.}
\end{aligned}$$

□

Remark 20.2
The conditions imply that f has a simple pole, so the method will only work in this case.

Examples 20.3

(i) The function $f(z) = \dfrac{1}{1 - z^3}$ has a pole at $z = 1$. We have $p(z) = 1$ and $q(z) = 1 - z^3$ so $q'(1) = -3 \times 1^2 = -3$. Thus $\operatorname{res}(f, 1) = \dfrac{p(1)}{q'(1)} = \dfrac{1}{-3} = -\dfrac{1}{3}$.

(ii) The function $f(z) = \dfrac{2z^2}{1 + z^4}$ has a pole at $z = e^{i\pi/4}$ (and more besides, but we'll ignore them). Let $p(z) = 2z^2$ and $q(z) = 1 + z^4$. Then $p(e^{i\pi/4}) = 2e^{i\pi/2} \neq 0$; $q'(z) = 4z^3$ so $q'(e^{i\pi/4}) = 4e^{3i\pi/4} \neq 0$. So

$$\operatorname{res}\left(f, e^{i\pi/4}\right) = \frac{p(e^{i\pi/4})}{q'(e^{i\pi/4})} = \frac{2e^{i\pi/2}}{4e^{3i\pi/4}} = \frac{e^{-i\pi/4}}{2} = \frac{1}{4}(1 - i)\sqrt{2}.$$

Method 3: Poles of order N

We now generalise Method 1 (which is the case $N = 1$ in the following).

Suppose that f has a pole of order N at w. Then

$$\operatorname{res}(f, w) = \frac{1}{(N-1)!} \lim_{z \to w} \frac{d^{N-1}}{dz^{N-1}} \left\{ (z-w)^N f(z) \right\}.$$

Proof. By assumption $f(z) = \sum_{n=-N}^{\infty} a_n (z-w)^n$ for some $a_i \in \mathbb{C}$ and $0 < |z - w| < R$. Thus,

$$\begin{aligned}
f(z) &= \sum_{n=-N}^{\infty} a_n (z-w)^n \\
(z-w)^N f(z) &= (z-w)^N \left(\frac{a_{-N}}{(z-w)^N} + \cdots + \frac{a_{-1}}{z-w} + a_0 + a_1(z-w) + \ldots \right) \\
&= a_{-N} + a_{-N+1}(z-w) + \cdots + a_{-1}(z-w)^{N-1} \\
&\qquad + a_0 (z-w)^N + \ldots .
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{1}{(N-1)!} \lim_{z \to w} \frac{d^{N-1}}{dz^{N-1}} \left\{ (z-w)^N f(z) \right\} \\
= {} & \frac{1}{(N-1)!} \lim_{z \to w} \frac{d^{N-1}}{dz^{N-1}} \{ a_{-N} + a_{-N+1}(z-w) + \ldots \\
& \qquad + a_{-1}(z-w)^{N-1} + a_0 (z-w)^N + \ldots \} \\
= {} & \frac{1}{(N-1)!} \lim_{z \to w} \{ (N-1)!\, a_{-1} + N!\, a_0 (z-w) + \ldots \} \\
= {} & \frac{1}{(N-1)!} (N-1)!\, a_{-1} \\
= {} & a_{-1} \\
= {} & \operatorname{res}(f, w).
\end{aligned}$$

□

Examples 20.4

(i) Let $f(z) = \left(\dfrac{z+i}{z-i} \right)^2$. By Lemma 18.1 this has a pole of order 2 at $z = i$. Hence,

$$\begin{aligned}
\operatorname{res}(f, i) &= \frac{1}{(2-1)!} \lim_{z \to i} \frac{d^{2-1}}{dz^{2-1}} \left\{ (z-i)^2 \left(\frac{z+i}{z-i} \right)^2 \right\} \\
&= 1. \lim_{z \to i} \frac{d}{dz} (z+i)^2 \\
&= \lim_{z \to i} 2(z+i) \\
&= 2(i+i) \\
&= 4i.
\end{aligned}$$

(ii) The function $f(z) = \dfrac{e^{e^z}}{z^2}$ has a pole of order 2 at $z = 0$, (again by Lemma 18.1). So,

$$\operatorname{res}(f,0) = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}z^2\frac{e^{e^z}}{z^2} = \lim_{z\to 0}\frac{d}{dz}\,e^{e^z} = \lim_{z\to 0}e^z.e^{e^z} = e.$$

Method 4: Direct expansion

We can use this as a last resort. We expand functions as power series, etc., and then calculate coefficients.

Examples 20.5

(i) Let $f(z) = \dfrac{ze^z}{(z-1)^3}$. This has a pole of order 3 at $z = 1$. So, expand about $w = 1$: (i.e. let $z = w + h$ and take h as our variable),

$$\begin{aligned} f(1+h) &= \frac{(1+h)e^{1+h}}{(1+h-1)^3} \\ &= (1+h)\frac{e.e^h}{h^3} \\ &= \frac{e}{h^3}(1+h)\left(1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\dots\right) \\ &= \frac{e}{h^3}\left(1+2h+\frac{3}{2}h^2+\frac{2}{3}h^3+\dots\right) \\ &= \frac{e}{h^3}+\frac{2e}{h^2}+\frac{3e}{2h}+\frac{2e}{3}+\dots \end{aligned}$$

So $\operatorname{res}(f,1) = \dfrac{3e}{2}$.

(ii) $*$ Find $\operatorname{res}\left(\dfrac{\pi\cot\pi z}{z^2}, 0\right)$.

We expand about $w = 0$:

$$\begin{aligned} \frac{\pi\cot\pi(0+h)}{(0+h)^2} &= \frac{\pi\cot\pi h}{h^2} \\ &= \frac{\pi}{h^2}\frac{\cos\pi h}{\sin\pi h} \\ &= \frac{\pi}{h^2}\frac{(1-(\pi h)^2/2!+\dots)}{(\pi h-(\pi h)^3/3!+\dots)} \\ &= \frac{1}{h^3}\frac{(1-\pi^2h^2/2+\dots)}{(1-\pi^2h^2/6+\dots)}. \end{aligned}$$

Consider the parts in brackets, this quotient will be a power series and so we need to work out its coefficients.

We need only work out the coeffient for h^2 since division by the h^3 will give the residue coefficient.

We work with the bottom bracket first. Suppose that $(1-\pi^2h^2/6+\dots)^{-1} = b_0 + b_1 h + b_2 h^2 + \dots$. Then,

$$\begin{aligned}
\left(1-\pi^2h^2/6+\dots\right)\left(1-\pi^2h^2/6+\dots\right)^{-1} &= 1 \\
\left(1-\pi^2h^2/6+\dots\right)\left(b_0+b_1h+b_2h^2+\dots\right) &= 1 \\
b_0+b_1h+b_2h^2-\frac{\pi^2h^2}{6}\left(b_0+b_1h+b_2h^2\right)+\dots &= 1 \\
b_0+b_1h+\left(b_2-\frac{\pi^2}{6}b_0\right)h^2+\dots &= 1.
\end{aligned}$$

Equating coefficients gives $b_0=1$, $b_1=0$ and $b_2-\frac{\pi^2}{6}b_0=0$, i.e. $b_2=\frac{\pi^2}{6}$.

Therefore,

$$\begin{aligned}
\frac{\pi\cot\pi h}{h^2} &= \frac{1}{h^3}\left(1-\frac{\pi^2h^2}{2}+\dots\right)\left(1-\frac{\pi^2h^2}{6}+\dots\right)^{-1} \\
&= \frac{1}{h^3}\left(1-\frac{\pi^2h^2}{2}+\dots\right)\left(1+\frac{\pi^2h^2}{6}+\dots\right) \\
&= \frac{1}{h^3}\left(1-\frac{\pi^2}{3}h^2+\dots\right) \\
&= \frac{1}{h^3}-\frac{\pi^2}{3h}+\dots
\end{aligned}$$

Thus $\operatorname{res}(f,0) = -\frac{\pi^2}{3}$.

Obviously, this method can get a bit calculation heavy.

Summary

- Simple pole: $\operatorname{res}(f,w)=\lim_{z\to w}(z-w)f(z)$.
- Some quotients: Suppose $p(w)\neq 0$, $q(z)=0$ and $q'(w)\neq 0$. Then,
$$\operatorname{res}\left(\frac{p}{q},w\right)=\frac{p(w)}{q'(w)}.$$
- Pole of order N:
$$\operatorname{res}(f,w)=\frac{1}{(N-1)!}\lim_{z\to w}\frac{d^{N-1}}{dz^{N-1}}\left\{(z-w)^N f(z)\right\}.$$
- Calculate expansions directly, equate coefficients, etc.

21 Cauchy's Residue Formula

The next theorem shows that we can calculate certain complex integrals by merely calculating the residues and winding numbers at poles. Thus we have a very simple five-finger-exercise method for calculating integrals. Hooray!

Theorem 21.1 (Cauchy's Residue Formula)
Let D be domain and γ a closed contour such that $\operatorname{Int}(\gamma) \subseteq D$. Let $\{p_1, \ldots, p_m\} \in D \backslash \gamma$ and $f : D \to \mathbb{C}$ be analytic with poles at $p_1, \ldots, p_m$. Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{m} n(\gamma, p_i) \operatorname{res}(f, p_j).$$

Proof. We can assume that all the poles are in the interior of γ. Now, consider the following sketch.

Let C_j be a circle of radius $\varepsilon > 0$ at p_j, taken clockwise. Then, there exists $\varepsilon > 0$ such that $C_j \subseteq \operatorname{Int}(\gamma)$ and $C_j \cap C_l = \emptyset$ for all j and l with $j \neq l$ and so that C_j lies within the disc upon which the Laurent expansion for f at p_j holds.

There exist contours β_j from some point on the image of γ to the start of C_j. We can define γ_j as the part of γ from the start of β_j to β_{j+1}. So $\gamma = \gamma_1 + \cdots + \gamma_m$.

Next, we define a contour $\widetilde{\gamma}$ by

$$\begin{aligned} \widetilde{\gamma} &= \beta_1 + n(\gamma, p_1)C_1 - \beta_1 + \gamma_1 \\ &\quad + \beta_2 + n(\gamma, p_2)C_2 - \beta_2 + \gamma_2 + \ldots \\ &\quad + \beta_m + n(\gamma, p_m)C_m - \beta_m + \gamma_m. \end{aligned}$$

This is a path such that f is analytic on $\operatorname{Int}(\widetilde{\gamma})$, (just like in the calculation trick I did.) So by Cauchy's Theorem we have $\int_{\widetilde{\gamma}} f = 0$. But

$$\begin{aligned} \int_{\widetilde{\gamma}} f &= n(\gamma, p_1) \int_{C_1} f + n(\gamma, p_2) \int_{C_2} f + \cdots + n(\gamma, p_m) \int_{C_m} f + \int_{\gamma_1 + \ldots \gamma_m} f \\ 0 &= \left(\sum_{j=1}^{m} n(\gamma, p_j) \int_{C_j} f \right) + \int_{\gamma_1 + \ldots \gamma_m} f. \end{aligned}$$

Note that the integrals over all the β_j cancel. So, as $\gamma = \gamma_1 +$

$\cdots + \gamma_m$ we get,

$$\begin{aligned}
0 &= \left(\sum_{j=1}^{m} n(\gamma, p_j) \int_{C_j} f\right) + \int_\gamma f \\
-\sum_{j=1}^{m} n(\gamma, p_j) \int_{C_j} f &= \int_\gamma f \\
\sum_{j=1}^{m} n(\gamma, p_j) \int_{-C_j} f &= \int_\gamma f.
\end{aligned}$$

But, by Theorem 19.4, $\int_{-C_j} f = 2\pi i \operatorname{res}(f, p_j)$. Thus the proof is complete. $\square$

There we have it, one of the best theorems in mathematics!

Examples 21.2
Evaluate the integral $\displaystyle\int_\gamma \frac{1}{z^2 + (i-1)z - i}\, dz$, where γ is a square with sides of length 4 centred at the origin, oriented anti-clockwise.

Solution: Let $f(z) = \dfrac{1}{z^2 + (i-1)z - i} = \dfrac{1}{(z+i)(z-1)}$. Therefore, f has simple poles at $z = 1$ and $z = -i$. Let $q(z) = z^2 + (i-1)z - i$, then $q'(z) = 2z + i - 1$. We have $q'(1) = 2 + i - 1 = 1 + i$, and $q'(-i) = -2i + i - 1 = -1 - i$. Thus,

$$\operatorname{res}(f, 1) = \frac{1}{1+i} = \frac{1-i}{2} \text{ and } \operatorname{res}(f, -i) = \frac{1}{-1-i} = \frac{-1+i}{2}.$$

We have $n(\gamma, 1) = n(\gamma, -i) = 1$, so by Cauchy's Residue Theorem

$$\begin{aligned}
\int_\gamma \frac{1}{z^2 + (i-1)z - i}\, dz &= 2\pi i \left(\operatorname{res}(f, 1) + \operatorname{res}(f, -i)\right) \\
&= 2\pi i \left(\frac{1-i}{2} + \frac{-1+i}{2}\right) \\
&= 0.
\end{aligned}$$

Summary

- Let D be domain and γ a closed contour such that $\operatorname{Int}(\gamma) \subseteq D$. Let $\{p_1, \ldots, p_m\} \in D \backslash \gamma$ and $f : D \to \mathbb{C}$ be analytic with poles at $p_1, \ldots, p_m$. Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{m} n(\gamma, p_i) \operatorname{res}(f, p_j).$$

22 Evaluation of Definite Real Integrals

We shall now see that complex analysis allows us to solve, in a simple manner, problems involving real functions that are not easy to solve with real methods.

The basic idea

We want to calculate $\int_{-\infty}^{\infty} f(x)\,dx$ where f is a real function. The basic method is actually quite simple, but it has a number of parts, so may initially appear complicated. First, I'll outline the ideas behind it.

Let us note that we can calculate the real integral as a contour integral. Let $\Gamma_1(t) = t$, then $\Gamma'(t) = 1$ for $-R \leq t \leq R$. So,

$$\int_{\Gamma_1} f(z)dz = \int_{-R}^{R} f(t).1\,dt = \int_{-R}^{R} f(t)\,dt.$$

If we let $R \to \infty$, then we usually obtain the real integral $\int_{-\infty}^{\infty} f(x)\,dx$ (but see Remark 22.3 later).

This doesn't really get us much further forward. However, if we take a contour Γ_R from R to $-R$ via a path in the complex plane and join it to Γ_1, then we get a closed contour. Thus, we can use Cauchy's Residue Theorem to calculate the integral. Consider the following diagram in the complex plane, where Γ_R is the semi-circle in the upper half-plane.

If we integrate round the contour $\Gamma = \Gamma_1 + \Gamma_R$, then we get

$$\int_{\Gamma} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_R} f(z)\,dz.$$

Now, letting $R \to \infty$, we get

$$\begin{aligned}
\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz &= \lim_{R\to\infty} \int_{\Gamma_1} f(z)\,dz + \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz \\
&= \lim_{R\to\infty} \int_{-R}^{R} f(t)\,dt + \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz \\
&= \int_{-\infty}^{\infty} f(x)\,dx + \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz.
\end{aligned}$$

Now, the LHS is a contour integral for a closed contour and so by Cauchy's Residue Formula we can probably work it out using some method. The RHS contains the integral we want. Great! It also contains another term. Not so great! However, if that term is zero, then we have a way of calculating a real integral from residues!

Can we show that the integral $\int_{\Gamma_R} f(z)\,dz \to 0$ as $R \to \infty$? The following lemma gives us a supply of examples with this property.

Lemma 22.1 (Jordan's Lemma)
Suppose p and q are polynomials with $\deg(p) \leq \deg(q) - 2$, $\Gamma_R(t) = Re^{it}$, $0 \leq t \leq \pi$ is a semi-circular contour of radius R, and $a \geq 0$.

Then,

$$\int_{\Gamma_R} \frac{p(z)}{q(z)} e^{iaz}\,dz \to 0 \text{ as } R \to \infty.$$

Proof. As usual, we apply the Estimation Lemma. We have,

$$\left|e^{iaz}\right| = e^{Re(iaz)} = e^{Re(ia(x+iy))} = e^{Re(-ay+iax)} = e^{-ay} \leq 1,$$

the latter inequality holds because $y \geq 0$ for $z \in \gamma$, and because $a \geq 0$.

By the Fundamental Theorem of Algebra, the polynomial p can be written as $p(z) = c\prod_{j=1}^{\deg p}(z - \alpha_j)$, where c is some constant and α_j is a root of p. Similarly, $q(z) = d\prod_{j=1}^{\deg q}(z - \beta_j)$, for some d and β_j.

We then have,

$$|p(z)| = |c|\prod_{j=1}^{\deg p}|z - \alpha_j| \leq |c|\prod_{j=1}^{\deg p}(|z| + |\alpha_j|) \leq |c|\prod_{j=1}^{\deg p}(R + |\alpha_j|),$$

when $z \in \Gamma_R$, (i.e. $|z| = R$). Note that the RHS is a polynomial in R of degree $\deg p$.

Similarly,

$$|q(z)| = |d|\prod_{j=1}^{\deg q}(|z - \beta_j|) \geq |d|\prod_{j=1}^{\deg q}(R - |\beta_j|),$$

for $z \in \Gamma$ and $R \geq \max\{|\beta_j|\}$.

Thus, for large R, by the Estimation Lemma,

$$\left|\int_{\Gamma_R} \frac{p(z)}{q(z)} e^{iaz}\,dz\right| \leq \frac{|c|\prod_{j=1}^{\deg p}(R + |\alpha_j|)}{|d|\prod_{j=1}^{\deg q}(R - |\beta_j|)} \times \pi R.$$

So, if $\deg p \leq \deg q - 2$, then the RHS$\to 0$ as $R \to \infty$. Hence, the RHS is equal to 0. $\square$

Remark 22.2
We include an e^{iaz} term as this will be useful in determining integrals involving sines and cosines.

Remark 22.3
The integral $\int_{-\infty}^{\infty} f(x)\,dx$ is actually defined to be

$$\lim_{\substack{b\to\infty \\ a\to-\infty}} \int_a^b f(x)\,dx$$

and in fact, *may be different to*

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx.$$

Definition 22.4
The integral $\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$ is called the **principal part** of the integral, and is denoted $pv \int_{-\infty}^{\infty} f(x)\,dx$.

The principal part of an integral may exist even if $\int_{-\infty}^{\infty} f(x)\,dx$ does not. E.g. $f(x) = x$. However, if both integrals exist, then they are equal. In all our examples we may as well assume that both exist, but note that in practice we should check that this is the case.

Real integrals of the form $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$

For real integrals of the form $\displaystyle\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$, where p and q are polynomials with $q(x) \neq 0$ for all *real* x, the method is the following.

(i) Define $\Gamma = \Gamma_1 + \Gamma_R$, where

$$\Gamma_1(t) = t \text{ for } -R \leq t \leq R,$$

and

$$\Gamma_R(t) = Re^{it} \text{ for } 0 \leq t \leq \pi,$$

taking R large enough so that Γ contains all the poles of $\dfrac{p(z)}{q(z)}$ that lie in the upper half-plane.

(ii) Calculate $\displaystyle\int_\Gamma \frac{p(z)}{q(z)}\,dz$ using Cauchy's Residue Theorem.

(iii) Show that $\displaystyle\int_{\Gamma_R} \frac{p(z)}{q(z)}\,dz \to 0$ as $R \to \infty$, using Jordan's Lemma.

(iv) Conclude that $\int_{-\infty}^{\infty} f(x)\,dx$ and $\lim_{R\to\infty} \int_\Gamma f(z)\,dz$ are equal.

It is probably best if we do an example.

Example 22.5
Find the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$.

(i) Take Γ as the join of two contours above.

(ii) Let $f(z) = \dfrac{1}{(z^2+1)^2}$. Then $f(z) = \dfrac{1}{(z^2+1)^2} = \dfrac{1}{(z-i)^2(z+i)^2}$.
So f has poles of order 2 at $z = i$ and $z = -i$. Only the $z = i$ pole lies in the interior of Γ (when $R > 1$).

By Method 3 we get

$$\begin{aligned}
\operatorname{res}(f, i) &= \frac{1}{(2-1)!}\frac{d}{dz}\left((z-i)^2 \frac{1}{(z-i)^2(z+i)^2}\right)\bigg|_{z=i} \\
&= \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)\bigg|_{z=i} \\
&= \left(-2\frac{1}{(z+i)^3}\right)\bigg|_{z=i} \\
&= -2\frac{1}{(i+i)^3} \\
&= \frac{1}{4i} \\
&= -\frac{i}{4}.
\end{aligned}$$

(iii) By Jordan's Lemma (as $\deg q = \deg(z^2+1)^2 = 4$, $\deg p = \deg 1 = 0$, and $a = 0$)

$$\int_{\Gamma_R} \frac{1}{(z^2+1)^2}\, dz \to 0 \text{ as } R \to \infty.$$

(iv) Thus, as

$$\begin{aligned}
\int_{\Gamma} f(z)\, dz &= \int_{\Gamma_1} f(z)\, dz + \int_{\Gamma_R} f(z)\, dz \\
2\pi i \operatorname{res}(f, i) &= \int_{-R}^{R} \frac{1}{(x^2+1)^2}\, dx + \int_{\Gamma_R} f(z)\, dz \\
2\pi i\left(-\frac{i}{4}\right) &= \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx + 0 \text{ as } R \to \infty \\
\frac{\pi}{2} &= \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx
\end{aligned}$$

Can you calculate this integral using the standard methods of real analysis?

Real integrals of the form $\int_{-\infty}^{\infty} \left\{ \begin{array}{c} \sin x \\ \cos x \end{array} \right\} \frac{p(x)}{q(x)}\, dx$

We can apply a similar method to integrals of the form

$$\int_{-\infty}^{\infty} \left\{ \begin{array}{c} \sin x \\ \cos x \end{array} \right\} \frac{p(x)}{q(x)}\, dx$$

by making a good choice of complex function to integrate.

Let us try an example that really shows the power of complex analysis.

Example 22.6
Calculate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 2x + 2}\, dx$.

The key here is to chose $f(z) = \dfrac{e^{iz}}{z^2 - 2z + 2}$. (Remember that $e^{iz} = \cos z + i \sin z$ so we will get the $\cos x$, but we will also get a $\sin x$ term along the real axis. Surprisingly, this turns out to be a bonus, not a problem - just watch!)

(i) Let Γ be as in the previous example, so $\Gamma = \Gamma_1 + \Gamma_R$.

(ii) We want to evaluate

$$\int_{\Gamma} f(z)\, dz$$

first, so we use Cauchy's Residue Theorem.

We know that e^{iz} has no zeros and no poles, so all the poles arise from zeros of the denominator: $z^2 - 2z + 2$.

$$z^2 - 2z + 2 = 0 \iff z = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

Let $R > \sqrt{2}$, so $1+i$ is inside the contour, $1-i$ is outside it. The diagram shows the location of poles and the contour Γ_R.

The pole is simple so

$$\begin{aligned} \int_{\Gamma} \frac{e^{iz}}{z^2 - 2z + 2}\, dz &= 2\pi i \operatorname{res}\left(\frac{e^{iz}}{z^2 - 2z + 2}, 1+i \right) \\ &= 2\pi i \frac{e^{i(1+i)}}{2(1+i) - 2} \\ &= 2\pi i \frac{e^{-1+i}}{2i} \\ &= \pi e^{-1} e^{i} \\ &= \frac{\pi}{e}(\cos 1 + i \sin 1). \end{aligned}$$

(iii) By Jordan's Lemma

$$\int_{\Gamma_R} \frac{e^{iz}}{z^2-2z+2}\,dz \to 0 \text{ as } R \to \infty.$$

(iv) Now we shall calculate the integrals along the two parts of the contour. Along Γ_1 we get

$$\begin{aligned}
\int_{\Gamma_1} \frac{e^{iz}}{z^2-2z+2}\,dz &= \int_{-R}^{R} \frac{e^{ix}}{x^2-2x+2}\,dx \\
&= \int_{-R}^{R} \frac{\cos x}{x^2-2x+2}\,dx + i\int_{-R}^{R} \frac{\sin x}{x^2-2x+2}\,dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz &= \lim_{R\to\infty} \int_{\Gamma_1} f(z)\,dz + \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz \\
&= \lim_{R\to\infty} \left\{ \int_{-R}^{R} \frac{\cos x}{x^2-2x+2}\,dx + i\int_{-R}^{R} \frac{\sin x}{x^2-2x+2}\,dx \right\} + 0 \\
\frac{\pi}{e}(\cos 1 + i\sin 1) &= \int_{-\infty}^{\infty} \frac{\cos x}{x^2-2x+2}\,dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x^2-2x+2}\,dx.
\end{aligned}$$

Equating real parts we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2-2x+2}\,dx = \frac{\pi}{e}\cos 1.$$

Remark 22.7
Note that using the imaginary parts we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2-2x+2}\,dx = \frac{\pi}{e}\sin 1.$$

Thus, we have gained more information than we were looking for. What a method!

Summary

(i) Define $\Gamma = \Gamma_1 + \Gamma_R$ by

$$\Gamma_1(t) = t \text{ for } -R \le t \le R, \text{ and } \Gamma_R(t) = Re^{it} \text{ for } 0 \le t \le \pi.$$

Take R large enough to include the poles of f that lie in the upper half-plane.

(ii) Calculate $\int_\Gamma f(z)\,dz$ using Cauchy's Residue Theorem.

(iii) Show that $\int_{\Gamma_R} f(z)\,dz \to 0$ as $R \to \infty$, using Jordan's Lemma.

(iv) Relate $\int_{-\infty}^{\infty} f(x)\,dx$ to $\int_\Gamma f(z)\,dz$.

23 Integrals with sines and cosines

Let us consider the nasty integral

$$\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin n\theta)\, d\theta.$$

from Section 3.

If we consider contour integration, then is there some function and contour such that $\int_\gamma f$ gives the above? Let $\gamma(t) = e^{i\theta}$, for $0 \le \theta \le 2\pi$ and $f(z) = e^z$. Then, with our knowledge of poles and residues we see that

$$\int_\gamma \frac{e^z}{z^{n+1}}\, dz = 2\pi i \ \mathrm{res}(e^z/z^{n+1}, 0) = \frac{2\pi i}{n!}$$

So

$$\begin{aligned}
\int_\gamma \frac{e^z}{z^{n+1}}\, dz &= \int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{i\theta(n+1)}} i e^{i\theta}\, d\theta \\
\frac{2\pi i}{n!} &= i \int_0^{2\pi} e^{e^{i\theta}} e^{-i\theta n}\, d\theta \\
\frac{2\pi}{n!} &= \int_0^{2\pi} e^{e^{i\theta} - in\theta}\, d\theta \\
&= \int_0^{2\pi} e^{\cos\theta + i\sin\theta - in\theta}\, d\theta \\
&= \int_0^{2\pi} e^{\cos\theta} e^{i(\sin\theta - n\theta)}\, d\theta \\
&= \int_0^{2\pi} e^{\cos\theta} \left(\cos(\sin\theta - n\theta) + i\sin((\sin\theta - n\theta)\right)\, d\theta.
\end{aligned}$$

By equating real and imaginary parts we see that

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}$$

and because $\cos$ is an even function we deduce that

$$\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}$$

Rather spectacular, wouldn't you agree?

Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

Theorem 23.1
Let $\gamma(t) = e^{it}$, for $0 \le t \le 2\pi$. Then

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = -i \int_\gamma f\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) z^{-1}\, dz.$$

Proof. Let $z = e^{i\theta} = \gamma(\theta)$. Then $\cos\theta = \dfrac{z+z^{-1}}{2}$ and $\sin\theta = \dfrac{z-z^{-1}}{2i}$. We also have $\gamma'(\theta) = ie^{i\theta}$.

Thus,

$$\begin{aligned} -i\int_\gamma f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) z^{-1}\,dz &= -i\int_0^{2\pi} f(\cos\theta,\sin\theta)e^{-i\theta}ie^{i\theta}\,d\theta \\ &= \int_0^{2\pi} f(\cos\theta,\sin\theta)\,d\theta. \end{aligned}$$

$\square$

Remark 23.2

We can change the limits of integration by letting $\gamma(t) = e^{it}$ where $a \le t \le a+2\pi$ and $a \in \mathbb{R}$. So we get integrals of the form $\int_a^{a+2\pi} f(\cos\theta,\sin\theta)\,d\theta$.

A common example is $a = -\pi$, so we get $\int_{-\pi}^{\pi}$.

Example 23.3

Evaluate the real integral $\displaystyle\int_0^{2\pi} \frac{1}{13+12\cos t}\,dt$.

Solution: Applying the theorem we get, for $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$,

$$\begin{aligned} \int_0^{2\pi} \frac{1}{13+12\cos t}\,dt &= -i\int_\gamma \frac{1}{z\left(13+6\left(z+1/z\right)\right)}\,dz \\ &= -i\int_\gamma \frac{1}{6z^2+13z+6}\,dz. \end{aligned}$$

Let $g(z) = \dfrac{1}{6z^2+13z+6}$. This has poles at $-\frac{3}{2}$ and $-\frac{2}{3}$. Of these, only $-\frac{2}{3}$ lies within the unit circle given by γ. Hence, we calculate the residue for this pole:

$$\operatorname{res}\left(g, -\frac{2}{3}\right) = \left.\frac{1}{12z+13}\right|_{z=-2/3} = \frac{1}{12\left(-\frac{2}{3}+13\right)+13} = \frac{1}{5}.$$

Therefore, by Cauchy's Residue Theorem,

$$\int_0^{2\pi} \frac{1}{13+12\cos t}\,dt = -i\times 2\pi i\times\frac{1}{5} = \frac{2\pi}{5}.$$

Summary

- Let $\gamma(t) = e^{it}$, for $0 \le t \le 2\pi$. Then

$$\int_0^{2\pi} f(\cos\theta,\sin\theta)\,d\theta = -i\int_\gamma f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) z^{-1}\,dz.$$