# Exceptional Coupling Constants for the Coulomb-Dirac Operator with Anomalous Magnetic Moment 

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#### Abstract

It was recently shown that the point spectrum of the separated CoulombDirac operator $H_{0}(k)$ is the limit of the point spectrum of the Dirac operator with anomalous magnetic moment $H_{a}(k)$ as the anomaly parameter tends to 0 ; this spectral stability holds for all Coulomb coupling constants $c$ for which $H_{0}(k)$ has a distinguished self-adjoint extension if the angular momentum quantum number $k$ is negative, but for positive $k$ there are certain exceptional values for $c$. Here we obtain an explicit formula for these exceptional values. In particular, it implies spectral stability for the threedimensional Coulomb-Dirac operator if $|c|<1$, covering all physically relevant cases.


## 1. Introduction

By separation of variables in spherical polar coordinates, the Dirac operator of relativistic quantum mechanics with a Coulomb potential,

$$
\mathcal{H}_{0}:=-i \alpha \cdot \nabla+\beta+\frac{c}{|x|},
$$

where $c<0$ and $\alpha_{1}, \alpha_{2}, \alpha_{3}$ and $\beta=\alpha_{0}$ are symmetric $4 \times 4$ matrices satisfying the anticommutation relations

$$
\alpha_{i} \alpha_{j}+\alpha_{j} \alpha_{i}=\delta_{i j} \quad(i, j \in\{0,1,2,3\}),
$$

is unitarily equivalent to a direct sum of one-dimensional Dirac operators on the halfline,

$$
H_{0}(k)=-i \sigma_{2} \frac{d}{d r}+\sigma_{3}+\frac{k}{r} \sigma_{1}+\frac{c}{r} \quad(r \in(0, \infty)),
$$

where $\sigma_{1}=\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right), \sigma_{2}=\left(\begin{array}{cc}0 & -i \\ i & 0\end{array}\right)$ and $\sigma_{3}=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$ are the Pauli matrices and $k \in \mathbb{Z} \backslash\{0\}$ is the angular momentum quantum number [13, Appendix to Section $1]$. For an electron orbiting a nucleus of charge number $Z \in \mathbb{N}$, the Coulomb coupling constant is $c=-Z \alpha$, with the Sommerfeld fine structure constant $\alpha \approx 1 / 137$.
$H_{0}(k)$ is essentially self-adjoint on its minimal domain if and only if $c^{2} \leq k^{2}-1 / 4$ [13, Theorem 6.9]. For $c^{2} \in\left(k^{2}-1 / 4, k^{2}\right)$, there still exists a distinguished self-adjoint extension, characterised by the requirement that the wave-functions in its domain have a finite potential (or kinetic) energy [13, Theorem 6.10].

Pauli suggested a modification of the Dirac operator which takes into account the anomalous magnetic moment of the electron (for the historical background cf. [7]). With suitably normalised constants, the operator with a potential $V$ for an electron of magnetic moment $(1+a) \mu_{B}$, where $\mu_{B}$ is the Bohr magneton, takes the form

$$
\mathcal{H}_{a}=-i \alpha \cdot \nabla+\beta+V-i a \alpha \cdot \nabla V
$$

In the case of the Coulomb potential, the corresponding half-line operators after variable separation will be

$$
H_{a}(k)=-i \sigma_{2} \frac{d}{d r}+\sigma_{3}+\left(\frac{k}{r}+\frac{a}{r^{2}}\right) \sigma_{1}+\frac{c}{r} \quad(r \in(0, \infty))
$$

The mathematical investigation of the properties of this operator was initiated by Behncke $[2,3,4] . H_{a}(k)$ is essentially self-adjoint on its minimal domain for all values $c<0$ of the coupling constant (cf. [1, 5]). The essential spectrum of $H_{a}(k)$, as that of $H_{0}(k)$, is $(-\infty,-1] \cup[1, \infty)$.

Let us now focus on the case $c^{2}<k^{2}$, so that a distinguished self-adjoint realisation of $H_{0}(k)$ exists. We always assume $a<0$ in the following; the case of positive $a$ can be reduced to this (with a sign change of $k$ ) by a suitable unitary transformation. Although only integer values of $k$ are relevant for the three-dimensional operator, we admit general non-zero real values of $k$. As is well known, $H_{0}(k)$ has infinitely many eigenvalues in the spectral gap $(-1,1)$, which accumulate at 1 . One would expect that the eigenvalues of $H_{a}(k)$ will be perturbations of those of $H_{0}(k)$, such that each eigenvalue of $H_{0}(k)$ will be the limit of exactly one eigenvalue branch of $H_{a}(k)$ in the limit $a \rightarrow 0$ (spectral stability, cf. [8, Chapter VIII §1.4]). This expectation is partly corroborated by the strong resolvent convergence of $H_{a}(k)$ to $H_{0}(k)$, at least for $c^{2} \leq k^{2}-1 / 4$, which implies that the spectrum cannot suddenly expand in the limit. Nevertheless, due to the strong singularity of the $a / r^{2}$ term at the origin, this limit problem is highly nontrivial and does not subject itself to the standard perturbation techniques. Behncke [4, Theorem 2] was able to prove spectral stability for $c^{2}<k^{2}-(3 / 2)^{2}$ if $k<0$, and for $c^{2}<k-5 / 2$ if $k>0$.

The surprising asymmetry with respect to the sign of $k$ is not an artefact of Behncke's method, but inherent in the problem, as observed in the recent treatment [7], where spectral stability was proved, based on an asymptotic analysis of the (non-linear) Prüfer and Riccati equations equivalent to the Dirac system

$$
\begin{equation*}
\left(H_{a}(k)-\lambda\right) u=0 \tag{1}
\end{equation*}
$$

for all $c \in(k, 0)$ if $k<0$, and for all $c \in(-k, 0) \backslash\left\{c_{0}, c_{1}, \ldots\right\}$ if $k>0$. Here $c_{j} \in(-k, 0)$, with $c_{j}>c_{j+1}$, are certain exceptional values of the coupling constant at which a
transition in the behaviour of the eigenfunctions of $H_{a}(k)$ occurs: for $c \in\left(c_{n}, c_{n-1}\right)$, the eigenfunctions of $H_{a}(k)$ show $n$ additional rapid oscillations very close to the origin, compared to the corresponding eigenfunctions of $H_{0}(k)$. It remains a fairly subtle open question whether or not spectral stability holds if $c$ is equal to one of the exceptional values.

The present note is devoted to a study of the number and positions of the exceptional values of $c$ for a given $k>0$.

## 2. Exceptional values

The exceptional values for $c$ appear in a stability analysis of the differential equation [7, Eq.(1)]

$$
\begin{equation*}
\varrho \vartheta^{\prime}(\varrho)=c+\left(k-\frac{1}{\varrho}\right) \sin 2 \vartheta \quad(\varrho>0) . \tag{2}
\end{equation*}
$$

This equation arises from the Prüfer transformation (cf. Appendix A) of the Dirac system (1) after omitting the lowest-order term at $0, \sigma_{3}-\lambda$, and rescaling $r=|a| \varrho$ to absorb the parameter $a$.

In the limit $\varrho \rightarrow \infty$, the right-hand side of (2) — which is $\pi$-periodic in $\vartheta$ - has asymptotic zeros $\vartheta_{ \pm}(c, k)$ satisfying $0<\vartheta_{-}(c, k)<\pi / 4<\vartheta_{+}(c, k)<\pi / 2$,

$$
\sin 2 \vartheta_{ \pm}(c, k)=-\frac{c}{k}, \quad \tan \vartheta_{ \pm}(c, k)=\frac{k \pm \sqrt{k^{2}-c^{2}}}{-c}
$$

An asymptotic study of the direction field of (2) for $\varrho \rightarrow 0$ and $\varrho \rightarrow \infty$ yields the following result, which shows that this differential equation has (up to addition of a constant multiple of $\pi$ ) a unique unstable solution at 0 and at $\infty$ (see [7, Lemma 2.1, 2.2]).

Lemma 1. Let $k>0$ and $c \in(-k, 0)$. Then the following statements hold.
a) There is a unique solution $\vartheta_{0}(\cdot, c)$ of $(2)$ such that $\lim _{\varrho \rightarrow 0} \vartheta_{0}(\varrho, c)=\pi$.

All other solutions either differ from $\vartheta_{0}$ by a constant integer multiple of $\pi$, or else satisfy $\lim _{\varrho \rightarrow 0} \vartheta(\varrho, c)=\frac{\pi}{2} \bmod \pi$.
For fixed $\hat{\varrho}>0, \vartheta_{0}(\hat{\varrho}, \cdot)$ is continuous non-decreasing.
b) There is a unique solution $\vartheta_{\infty}(\cdot, c)$ of $(2)$ such that $\lim _{\varrho \rightarrow \infty} \vartheta_{\infty}(\varrho, c)=\vartheta_{-}(c, k)$.

All other solutions either differ from $\vartheta_{\infty}$ by a constant integer multiple of $\pi$, or else satisfy $\lim _{\varrho \rightarrow \infty} \vartheta(\varrho, c)=\vartheta_{+}(c, k) \bmod \pi$.
For fixed $\widehat{\varrho}>0, \vartheta_{\infty}(\hat{\varrho}, \cdot)$ is continuous and strictly decreasing.

The solution $\vartheta_{0}(\cdot, c)$ asymptotically corresponds to the Prüfer angle of the $L^{2}(0, \infty)$ solution of (1).

The exceptional values are defined as those values of $c$ for which $\vartheta_{0}(\cdot, c)$ and $\vartheta_{\infty}(\cdot, c)$ match up $\bmod \pi$, so that the unstable solution of $(2)$ at 0 is unstable at
infinity as well. More precisely, in view of the monotonicity properties of $\vartheta_{0}$ and $\vartheta_{\infty}$ with respect to $c$, we have

$$
\lim _{\varrho \rightarrow \infty} \vartheta_{0}\left(\varrho, c_{m}\right)=\vartheta_{-}\left(c_{m}, k\right)-m \pi \quad(m=0,1,2, \ldots)
$$

By the transformation $\varrho=e^{t}, \vartheta(\varrho)=\varphi(\log \varrho)$, the equation (2) is equivalent to

$$
\begin{equation*}
\varphi^{\prime}(t)=c+\left(k-e^{-t}\right) \sin 2 \varphi(t) \quad(t \in \mathbb{R}) \tag{3}
\end{equation*}
$$

the differential equation for the Prüfer angle $\varphi$ (cf. Appendix A) of a $\mathbb{R}^{2}$-valued solution $u$ of the Dirac system

$$
\begin{equation*}
\left(-i \sigma_{2} \frac{d}{d t}+\left(k-e^{-t}\right) \sigma_{1}+c\right) u(t)=0 \tag{4}
\end{equation*}
$$

which can be rewritten using the definition of the Pauli matrices as

$$
\begin{aligned}
u_{1}^{\prime} & =\left(e^{-t}-k\right) u_{1}-c u_{2}, \\
u_{2}^{\prime} & =c u_{1}+\left(k-e^{-t}\right) u_{2} .
\end{aligned}
$$

In (4), $c$ takes on the role of a spectral parameter, while the coefficient of $\sigma_{1}$ can be interpreted as a constant mass term with an exponentially decaying perturbation. In the sense of the analogue of Kneser's Theorem for Dirac systems [9], this perturbation is subcritical, and indeed the method developed in [9] can be used to show that, for each $k>0$, there are only finitely many exceptional values for $c$. Moreover, an asymptotic analysis of (4) along the lines of [10; see also 12] reveals that the number of exceptional values is asymptotic to $k$ in the limit $k \rightarrow \infty$ with fixed ratio $c / k$.

This was verified in a computational investigation of (4) based on a piecewiseconstant approximation of the exponential function, following the approach of [11]. The numerical findings suggested that the number of exceptional values in $(-k, 0)$ is always equal to the unique integer in $[k-1, k)$; more precisely, whenever $k$ reaches an integer value, an additional exceptional value, initially equal to $-k$, appears and moves into the interval $(-k, 0)$ as $k$ increases.

This regularity raised the suspicion that (4) has an underlying solvable structure, and indeed, on closer scrutiny it turns out that this equation can be analysed by means of a variant of the factorisation method, resulting not only in a proof of the above conjecture, but even in an explicit formula for the exceptional values, which makes any asymptotic and computational analysis of (4) superfluous.

Theorem 1. For given $k>0$, the exceptional values for $H_{a}(k)$ are given by

$$
c_{n-1}(k)=-\sqrt{2 k n-n^{2}} \quad(n \in \mathbb{N}, n<k)
$$

Consequently there are exactly $N$ exceptional values in $(-k, 0)$ if $k \in(N, N+1]$, $N \in \mathbb{N}_{0}$.

In view of Theorems 1.1 and 1.2 of [7], this implies in particular that the eigenvalues of $H_{a}(k)$ converge to those of $H_{0}(k)$ as $a \rightarrow 0$ if $c^{2}<2 k-1$, confirming Behncke's linear bound (although with a different constant). However, his conjecture [4, p. 2558] that spectral stability may hold for all $c^{2}<k^{2}-5 / 2$ turns out to be very questionable.

Since the three-dimensional Coulomb-Dirac Hamiltonian $\mathcal{H}_{a}$ is the direct sum of $H_{a}(k)$ for $k \in \mathbb{Z} \backslash\{0\}$, we can thus infer spectral stability for this operator in all cases in which $\mathcal{H}_{0}$ has a distinguished self-adjoint realisation.

Corollary 1. Let $\mathcal{H}_{0}$ be the Dirac operator and $\mathcal{H}_{a}$ the Dirac operator with anomalous magnetic moment with a Coulomb potential with coupling constant $c \in(-1,0)$. Then the eigenvalues of $\mathcal{H}_{0}$ are the limits of the eigenvalues of $\mathcal{H}_{a}$ as $a \rightarrow 0$.

## 3. Proof of Theorem 1

The following proof of Theorem 1 will be based on squaring the Dirac-type operator in (4), which is equivalent to deriving second-order differential equations for $u_{1}$ and $u_{2}$. In a general situation, this is usually not a good idea except to obtain a quick and rough heuristic result, since it involves derivatives of the coefficients and consequently requires unnecessarily strong regularity. In the present instance, however, we are dealing with a specific Dirac system with analytic coefficients, and as it turns out, the resulting second-order equation system decouples and can be solved by the factorisation method in the relevant cases (the second-order equation (5) is closely related to the radial Schrödinger equation with a Morse potential [6, Section 5.2]). Here the factorisation is incidentally provided by the original Dirac system; see the discussion in Appendix B. The key observation is contained in the following result.

Theorem 2. Let $\kappa \geq 0$. Let $v_{1}(t, \kappa):=e^{-e^{-t}-\kappa t}$, and define $v_{j}$ recursively for $j \in \mathbb{N}+1$ by

$$
v_{j+1}(t, \kappa):=\left(\kappa+j-e^{-t}\right) v_{j}(t)-v_{j}^{\prime}(t) \quad(t \in \mathbb{R})
$$

Then $v_{j}(\cdot, \kappa)$ is a nontrivial solution of

$$
\begin{equation*}
v^{\prime \prime}=\left(e^{-2 t}-(2 \kappa+2 j-1) e^{-t}+\kappa^{2}\right) v \tag{5}
\end{equation*}
$$

and it has the asymptotic properties
$\lim _{t \rightarrow \infty} v_{j}(t, \kappa)=0, \quad \lim _{t \rightarrow \infty} \frac{v_{j}^{\prime}(t, \kappa)}{v_{j}(t, \kappa)}=-\kappa, \quad$ and $\quad \lim _{t \rightarrow-\infty} \frac{v_{j}^{\prime}(t, \kappa)}{v_{j}(t, \kappa)} e^{t}=1 \quad(j \in \mathbb{N})$.

The proof can be done by induction with respect to $j$. The fact that $v_{j}$ is a solution of (5) is readily verified by differentiating twice. The asymptotic properties are obvious for

$$
\frac{v_{1}^{\prime}(t, \kappa)}{v_{1}(t, \kappa)}=e^{-t}-\kappa .
$$

Now assume that $j \in \mathbb{N}$ is such that the assertion is true. If $v_{j+1}$ were trivial, this would imply that (up to multiplication by a constant) $v_{j}(t, \kappa)=e^{(\kappa+j) t+e^{-t}} \rightarrow \infty$ $(t \rightarrow \infty)$, contradicting the first limit property of $v_{j}$. The asymptotic properties of $v_{j+1}$ are easily checked using the identity

$$
\frac{v_{j+1}^{\prime}(t, \kappa)}{v_{j+1}(t, \kappa)}=\frac{\left.-e^{-2 t}+2(\kappa+j) e^{-t}-\kappa^{2}\right)+\left(\kappa+j-e^{-t}\right) \frac{v_{j}^{\prime}}{v_{j}}(t, \kappa)}{\kappa+j-e^{-t}-\frac{v_{j}^{\prime}}{v_{j}}(t, \kappa)}
$$

Corollary 2. Let $\kappa \geq 0, j \in \mathbb{N}$ and $v_{j}$ be as in Theorem 1. Then

$$
\lim _{t \rightarrow \infty} \frac{v_{j}(t)}{v_{j+1}(t)}=\frac{1}{2 \kappa+j}, \quad \lim _{t \rightarrow-\infty} \frac{v_{j}(t)}{v_{j+1}(t)}=0
$$

This is a direct consequence of Theorem 1 in view of

$$
\frac{v_{j}(t, \kappa)}{v_{j+1}(t, \kappa)}=\frac{1}{\kappa+j-e^{-t}-\frac{v_{j}^{\prime}}{v_{j}}(t, \kappa)}
$$

Furthermore, the following conclusion can be verified by a straightforward calculation.

Corollary 3. Let $v_{j}$ be defined as in Theorem 1, and let $j \in \mathbb{N}, k \geq j$. Then

$$
u(t)=\binom{\frac{1}{\sqrt{2 k j-j^{2}}} v_{j+1}(t, k-j)}{v_{j}(t, k-j)} \quad(t \in \mathbb{R})
$$

is a nontrivial solution of the Dirac system (4) with $c=-\sqrt{2 k j-j^{2}}$. The Prüfer angle $\varphi$ of $u$ satisfies

$$
\lim _{t \rightarrow \infty} \tan \varphi(t)=\frac{j}{\sqrt{2 k j-j^{2}}}, \quad \lim _{t \rightarrow-\infty} \tan \varphi(t)=0
$$

Since, with $c$ as in Corollary 3 ,

$$
\tan \vartheta_{-}(c)=\frac{k-\sqrt{k^{2}-c^{2}}}{-c}=\frac{k-\sqrt{k^{2}-2 k j+j^{2}}}{\sqrt{2 k j-j^{2}}}=\frac{j}{\sqrt{2 k j-j^{2}}}
$$

where we have used $j \leq k$, the Prüfer angle $\varphi$ in Corollary 3 corresponds via $\vartheta(\varrho)=$ $\varphi(\log \varrho)$ to a solution of (3) which coincides (up to addition of integer multiples of $\pi$ ) with both $\vartheta_{0}(\cdot, c)$ and $\vartheta_{\infty}(\cdot, c)$; hence such values of $c$ are exceptional values.

We now conclude the proof of Theorem 1 by showing, based on the continuity and monotonicity properties of $\vartheta_{0}$ and $\vartheta_{\infty}$, that all exceptional values are obtained in this way. Corollary 3 specifies the asymptotic behaviour of the relevant angle functions $\bmod \pi$; a study of the zeros of $v_{j}$ reveals their global behaviour as shown in Lemma 2 below. Theorem 1 then follows in view of

$$
\vartheta\left(\varrho,-\sqrt{2 k j-j^{2}}\right)=\varphi_{j}(\log \varrho, k-j) \quad(\varrho>0, k>0, j \in \mathbb{N}, j<k)
$$

where (with $\arctan 0=0$ )

$$
\varphi_{j}(\cdot, \kappa):=\arctan \left(\frac{v_{j}(\cdot, \kappa)}{v_{j+1}(\cdot, \kappa)} \sqrt{2 \kappa j+j^{2}}\right)+\pi
$$

Lemma 2. Let $\kappa \geq 0$. Then $v_{j}(\cdot, \kappa)$ has exactly $j-1$ zeros. The associated angle function $\varphi_{j}$ satisfies

$$
\lim _{t \rightarrow-\infty} \varphi_{j}(t, \kappa)=\pi
$$

and

$$
\lim _{t \rightarrow \infty} \varphi_{j}(t, \kappa)=\vartheta_{-}\left(-\sqrt{2 \kappa j+j^{2}}, \kappa+j\right)-(j-1) \pi \quad(j \in \mathbb{N})
$$

Proof. A look at the direction field of (3) with $k=\kappa+j$ shows that $\varphi_{j}(\cdot, \kappa)$, being the unstable solution at both $\pm \infty$, is monotone decreasing. $v_{1}(\cdot, \kappa)$ has no zeros, so $\varphi_{1}(\cdot, \kappa)$ has no zeros $\bmod \pi$, and the assertion follows for $j=1$.

To conclude the proof by induction, assume now that $j \in \mathbb{N}$ is such that the assertion is true. Then $\varphi_{j}(\cdot, \kappa)-\pi / 2$ has exactly $j$ zeros $\bmod \pi$, so $v_{j+1}(\cdot, \kappa)$ has exactly $j$ zeros. Since the $v_{j+1}(\cdot, \kappa)$ is a non-trivial solution of a second-order equation, $v_{j+1}^{\prime}(\cdot, \kappa)$ does not vanish at zeros of $v_{j+1}(\cdot, \kappa)$. Consequently, $v_{j+1}(\cdot, \kappa)$ and $v_{j+2}(\cdot, \kappa)$ have no common zeros. Hence $\varphi_{j+1}(\cdot, \kappa)$ has exactly $j$ zeros $\bmod \pi$, and therefore must converge to $\vartheta_{-}\left(-\sqrt{2 \kappa j+j^{2}, \kappa+j}\right)-(j-1) \pi$ as $t \rightarrow \infty$.

## Appendix

## A. The Prüfer Transformation

Consider a general Dirac system

$$
\begin{equation*}
\left(-i \sigma_{2} \frac{d}{d x}+m(x) \sigma_{3}+l(x) \sigma_{1}+q(x)\right) u(x)=0 \quad(x \in I) \tag{6}
\end{equation*}
$$

on an interval $I \subset \mathbb{R}$ with locally integrable, real-valued coefficients $m, l$ and $q$ (we have absorbed the spectral parameter in the latter). If $u$ is a solution of (6), then so is its (component-wise) complex conjugate, so it is sufficient to study $\mathbb{R}^{2}$-valued solutions.

Since (6) is linear, a non-trivial solution will never take the value $\binom{0}{0}$, so $u(x)$ traces out an absolutely continuous curve in the punctured plane as $x$ varies.

Introducing polar coordinates in this plane by writing

$$
u(x)=|u(x)|\binom{\cos \vartheta(x)}{\sin \vartheta(x)},
$$

where the absolutely continuous function $\vartheta$ (the Prüfer angle of $u$ ) is determined uniquely up to addition of an integer multiple of $2 \pi$ - or indeed $\pi$, since $-u$ is again a solution of (6) -, a straightforward calculation yields the differential equation for $\vartheta$ (Prüfer equation)

$$
\vartheta^{\prime}=m \cos 2 \vartheta+l \sin 2 \vartheta+q .
$$

This non-linear equation is equivalent to (6), because one can recover $u$ from $\vartheta$ by noticing that (choosing $x_{0} \in I$ )

$$
|u(x)|=\left|u\left(x_{0}\right)\right| \exp \int_{x_{0}}^{x}(m \sin 2 \vartheta-l \cos 2 \vartheta) \quad(x \in I) .
$$

## B. Dirac systems and the factorisation method

Known to students of quantum mechanics as a trick to treat the Schrödinger equation for the harmonic oscillator, the factorisation method is in fact a tool of much wider scope for studying second-order differential equations (see the extensive discussion in [6]). It is based on the close link between the eigenvalue equations

$$
A^{*} A u=\lambda u \quad \text { and } \quad A A^{*} v=\lambda v
$$

where $A$ is typically a first-order differential operator: a solution $u$ of the first equation gives rise to a solution $v=A u$ of the second.

Here we remark on how this method can serve to solve eigenvalue equations for certain one-dimensional Dirac operators, which in turn represent a factorisation of the Sturm-Liouville operators arising as their formal squares.

Starting from a Dirac system (6), or equivalently

$$
\begin{aligned}
& u_{1}^{\prime}=-l u_{1}+(m-q) u_{2}, \\
& u_{2}^{\prime}=(m+q) u_{1}+l u_{2},
\end{aligned}
$$

with absolutely continuous coefficients $l, m, q$, we find that $u_{1}$ and $u_{2}$ satisfy the second-order equations

$$
\begin{aligned}
& u_{1}^{\prime \prime}=\left(m^{2}-q^{2}+l^{2}-l^{\prime}\right) u_{1}+\left(m^{\prime}-q^{\prime}\right) u_{2} \\
& u_{2}^{\prime \prime}=\left(m^{2}-q^{2}+l^{2}+l^{\prime}\right) u_{2}+\left(m^{\prime}+q^{\prime}\right) u_{1}
\end{aligned}
$$

This ordinary differential equation system decouples if $m$ and $q$ are constant. If we assume this in the following and set $c:=m-q, d:=m+q$ and $a:=c d$, we are dealing with the Dirac system

$$
\begin{aligned}
& u_{1}^{\prime}=-l u_{1}+c u_{2} \\
& u_{2}^{\prime}=d u_{1}+l u_{2}
\end{aligned}
$$

and corresponding second-order equations

$$
\begin{align*}
u_{1}^{\prime \prime} & =\left(a+l^{2}-l^{\prime}\right) u_{1}  \tag{7}\\
u_{2}^{\prime \prime} & =\left(a+l^{2}+l^{\prime}\right) u_{2} \tag{8}
\end{align*}
$$

These two Sturm-Liouville equations differ only in the sign of $l^{\prime}$. Hence we can construct a chain of interlocking equation systems, along with special solutions, in the following way.

Theorem 3. Let $\left(a_{n}\right)_{n \in \mathbb{N}}$ be a sequence of real numbers with $a_{1}=0$, and let $\left(l_{n}\right)_{n \in \mathbb{N}}$ be a sequence of real-valued absolutely continuous functions on $I$ such that

$$
l_{n+1}^{2}+l_{n+1}^{\prime}+a_{n+1}=l_{n}^{2}-l_{n}^{\prime}+a_{n} \quad(n \in \mathbb{N})
$$

Let $v_{1}(x):=\exp \int^{x} l_{1}(x \in I)$, and define recursively for $n \in \mathbb{N}+1$

$$
v_{n}:=l_{n} v_{n-1}-v_{n-1}^{\prime} .
$$

Then $v_{n}$ is a solution of (7) with $a=a_{n}, l=l_{n}$, and of (8) with $a=a_{n+1}, l=l_{n+1}$. Furthermore, for $n \in \mathbb{N}+1$ and arbitrary $d_{n} \in \mathbb{R} \backslash\{0\}$,

$$
u:=\binom{-\frac{1}{d_{n}} v_{n}}{v_{n-1}}
$$

is a solution of the Dirac system

$$
\begin{align*}
u_{1}^{\prime} & =-l_{n} u_{1}+\left(a_{n} / d_{n}\right) u_{2}  \tag{9}\\
u_{2}^{\prime} & =d_{n} u_{1}+l_{n} u_{2} .
\end{align*}
$$

Proof. Because of $a_{1}=0$, the first equation of the Dirac system (9) with $n=1$ decouples from the second and can easily be solved to obtain the formula for $v_{1}$. As the first component of a solution of this Dirac system, $v_{1}$ satisfies equation (7) with $l=l_{1}$, $a=a_{1}$. In view of the assumptions on the coefficients $l_{n}$, this equation is the same as (8) with $l=2, a=a_{2}$, the Sturm-Liouville equation satisfied by the second component of a solution of the Dirac system for $n=2$. The corresponding first component of this solution can easily be calculated from the second equation of the Dirac system. It will
depend on $d_{2}$, but note that this dependence can be eliminated by normalisation as far as obtaining a solution $v_{2}$ of the Sturm-Liouville equation (7) with $l=l_{2}, a=a_{2}$ is concerned.

This process is then iterated to find the solutions for $n \in\{3,4, \ldots\}$.

## Examples.

1. The classical application of the factorisation method to the one-dimensional Schrödinger equation for a quantum-mechanical harmonic oscillator,

$$
-v^{\prime \prime}(x)+x^{2} v(x)=\lambda v(x)
$$

is captured in the above scheme if we choose $l_{n}(x):=x(n \in \mathbb{N})$. The hypothesis of Theorem 3 is then satisfied with $a_{n}:=-2(n-1)(n \in \mathbb{N})$, and we obtain the well-known solutions

$$
v_{n}(x)=\left(-\frac{d}{d x}+x\right)^{n-1} e^{-\frac{x^{2}}{2}}
$$

with eigenvalues $\lambda_{n}:=2 n-1(n \in \mathbb{N})$.
2. The case of equation (5) is treated by choosing $l_{n}(x):=\kappa+(n-1)-e^{-x}(n \in \mathbb{N})$; we immediately find $v_{1}(x)=e^{-e^{-x}+\kappa x}$ and the recursion formula of Theorem 2.

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