Exceptional Coupling Constants for the Coulomb-Dirac Operator with Anomalous Magnetic Moment

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Dedicated to E.B. Davies on the occasion of his 60th birthday

Abstract. It was recently shown that the point spectrum of the separated Coulomb-Dirac operator $H_0(k)$ is the limit of the point spectrum of the Dirac operator with anomalous magnetic moment $H_a(k)$ as the anomaly parameter tends to 0; this spectral stability holds for all Coulomb coupling constants c for which $H_0(k)$ has a distinguished self-adjoint extension if the angular momentum quantum number k is negative, but for positive k there are certain exceptional values for c. Here we obtain an explicit formula for these exceptional values. In particular, it implies spectral stability for the threedimensional Coulomb-Dirac operator if |c| < 1, covering all physically relevant cases.

1. Introduction

By separation of variables in spherical polar coordinates, the Dirac operator of relativistic quantum mechanics with a Coulomb potential,

$$\mathcal{H}_0 := -i\alpha \cdot \nabla + \beta + \frac{c}{|x|},$$

where c < 0 and $\alpha_1, \alpha_2, \alpha_3$ and $\beta = \alpha_0$ are symmetric 4×4 matrices satisfying the anticommutation relations

$$\alpha_i \alpha_j + \alpha_j \alpha_i = \delta_{ij} \qquad (i, j \in \{0, 1, 2, 3\}),$$

is unitarily equivalent to a direct sum of one-dimensional Dirac operators on the halfline,

$$H_0(k) = -i\sigma_2 \frac{d}{dr} + \sigma_3 + \frac{k}{r}\sigma_1 + \frac{c}{r} \qquad (r \in (0,\infty)),$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices and $k \in \mathbb{Z} \setminus \{0\}$ is the angular momentum quantum number [13, Appendix to Section 1]. For an electron orbiting a nucleus of charge number $Z \in \mathbb{N}$, the Coulomb coupling constant is $c = -Z\alpha$, with the Sommerfeld fine structure constant $\alpha \approx 1/137$. $H_0(k)$ is essentially self-adjoint on its minimal domain if and only if $c^2 \leq k^2 - 1/4$ [13, Theorem 6.9]. For $c^2 \in (k^2 - 1/4, k^2)$, there still exists a distinguished self-adjoint extension, characterised by the requirement that the wave-functions in its domain have a finite potential (or kinetic) energy [13, Theorem 6.10].

Pauli suggested a modification of the Dirac operator which takes into account the anomalous magnetic moment of the electron (for the historical background cf. [7]). With suitably normalised constants, the operator with a potential V for an electron of magnetic moment $(1 + a)\mu_B$, where μ_B is the Bohr magneton, takes the form

$$\mathcal{H}_a = -i\alpha \cdot \nabla + \beta + V - ia\alpha \cdot \nabla V.$$

In the case of the Coulomb potential, the corresponding half-line operators after variable separation will be

$$H_a(k) = -i\sigma_2 \frac{d}{dr} + \sigma_3 + \left(\frac{k}{r} + \frac{a}{r^2}\right)\sigma_1 + \frac{c}{r} \qquad (r \in (0, \infty)).$$

The mathematical investigation of the properties of this operator was initiated by Behncke [2, 3, 4]. $H_a(k)$ is essentially self-adjoint on its minimal domain for all values c < 0 of the coupling constant (cf. [1, 5]). The essential spectrum of $H_a(k)$, as that of $H_0(k)$, is $(-\infty, -1] \cup [1, \infty)$.

Let us now focus on the case $c^2 < k^2$, so that a distinguished self-adjoint realisation of $H_0(k)$ exists. We always assume a < 0 in the following; the case of positive a can be reduced to this (with a sign change of k) by a suitable unitary transformation. Although only integer values of k are relevant for the three-dimensional operator, we admit general non-zero real values of k. As is well known, $H_0(k)$ has infinitely many eigenvalues in the spectral gap (-1, 1), which accumulate at 1. One would expect that the eigenvalues of $H_a(k)$ will be perturbations of those of $H_0(k)$, such that each eigenvalue of $H_0(k)$ will be the limit of exactly one eigenvalue branch of $H_a(k)$ in the limit $a \to 0$ (spectral stability, cf. [8, Chapter VIII §1.4]). This expectation is partly corroborated by the strong resolvent convergence of $H_a(k)$ to $H_0(k)$, at least for $c^2 \leq k^2 - 1/4$, which implies that the spectrum cannot suddenly expand in the limit. Nevertheless, due to the strong singularity of the a/r^2 term at the origin, this limit problem is highly nontrivial and does not subject itself to the standard perturbation techniques. Behncke [4, Theorem 2] was able to prove spectral stability for $c^2 < k^2 - (3/2)^2$ if k < 0, and for $c^2 < k - 5/2$ if k > 0.

The surprising asymmetry with respect to the sign of k is not an artefact of Behncke's method, but inherent in the problem, as observed in the recent treatment [7], where spectral stability was proved, based on an asymptotic analysis of the (non-linear) Prüfer and Riccati equations equivalent to the Dirac system

$$(H_a(k) - \lambda)u = 0, \tag{1}$$

for all $c \in (k, 0)$ if k < 0, and for all $c \in (-k, 0) \setminus \{c_0, c_1, \ldots\}$ if k > 0. Here $c_j \in (-k, 0)$, with $c_j > c_{j+1}$, are certain exceptional values of the coupling constant at which a transition in the behaviour of the eigenfunctions of $H_a(k)$ occurs: for $c \in (c_n, c_{n-1})$, the eigenfunctions of $H_a(k)$ show *n* additional rapid oscillations very close to the origin, compared to the corresponding eigenfunctions of $H_0(k)$. It remains a fairly subtle open question whether or not spectral stability holds if *c* is equal to one of the exceptional values.

The present note is devoted to a study of the number and positions of the exceptional values of c for a given k > 0.

2. Exceptional values

The exceptional values for c appear in a stability analysis of the differential equation [7, Eq.(1)]

$$\varrho\vartheta'(\varrho) = c + (k - \frac{1}{\varrho})\sin 2\vartheta \qquad (\varrho > 0).$$
(2)

This equation arises from the Prüfer transformation (cf. Appendix A) of the Dirac system (1) after omitting the lowest-order term at 0, $\sigma_3 - \lambda$, and rescaling $r = |a|\varrho$ to absorb the parameter a.

In the limit $\rho \to \infty$, the right-hand side of (2) — which is π -periodic in ϑ — has asymptotic zeros $\vartheta_{\pm}(c, k)$ satisfying $0 < \vartheta_{-}(c, k) < \pi/4 < \vartheta_{+}(c, k) < \pi/2$,

$$\sin 2\vartheta_{\pm}(c,k) = -\frac{c}{k}, \qquad \tan \vartheta_{\pm}(c,k) = \frac{k \pm \sqrt{k^2 - c^2}}{-c}.$$

An asymptotic study of the direction field of (2) for $\rho \to 0$ and $\rho \to \infty$ yields the following result, which shows that this differential equation has (up to addition of a constant multiple of π) a unique unstable solution at 0 and at ∞ (see [7, Lemma 2.1, 2.2]).

Lemma 1. Let k > 0 and $c \in (-k, 0)$. Then the following statements hold. a) There is a unique solution $\vartheta_0(\cdot, c)$ of (2) such that $\lim_{\varrho \to 0} \vartheta_0(\varrho, c) = \pi$.

All other solutions either differ from ϑ_0 by a constant integer multiple of π , or else satisfy $\lim_{\varrho \to 0} \vartheta(\varrho, c) = \frac{\pi}{2} \mod \pi$.

For fixed $\hat{\varrho} > 0$, $\vartheta_0(\hat{\varrho}, \cdot)$ is continuous non-decreasing.

b) There is a unique solution ϑ_∞(·, c) of (2) such that lim ϑ_∞(ρ, c) = ϑ_−(c, k). All other solutions either differ from ϑ_∞ by a constant integer multiple of π, or else satisfy lim ϑ(ρ, c) = ϑ₊(c, k) mod π. For fixed ρ̂ > 0, ϑ_∞(ρ̂, ·) is continuous and strictly decreasing.

The solution $\vartheta_0(\cdot, c)$ asymptotically corresponds to the Prüfer angle of the $L^2(0, \infty)$ solution of (1).

The exceptional values are defined as those values of c for which $\vartheta_0(\cdot, c)$ and $\vartheta_{\infty}(\cdot, c)$ match up mod π , so that the unstable solution of (2) at 0 is unstable at

infinity as well. More precisely, in view of the monotonicity properties of ϑ_0 and ϑ_{∞} with respect to c, we have

$$\lim_{\varrho \to \infty} \vartheta_0(\varrho, c_m) = \vartheta_-(c_m, k) - m\pi \qquad (m = 0, 1, 2, \ldots).$$

By the transformation $\rho = e^t$, $\vartheta(\rho) = \varphi(\log \rho)$, the equation (2) is equivalent to

$$\varphi'(t) = c + (k - e^{-t})\sin 2\varphi(t) \qquad (t \in \mathbb{R}), \tag{3}$$

the differential equation for the Prüfer angle φ (cf. Appendix A) of a \mathbb{R}^2 -valued solution u of the Dirac system

$$(-i\sigma_2 \frac{d}{dt} + (k - e^{-t})\sigma_1 + c) u(t) = 0,$$
(4)

which can be rewritten using the definition of the Pauli matrices as

$$u'_{1} = (e^{-t} - k)u_{1} - cu_{2},$$

$$u'_{2} = cu_{1} + (k - e^{-t})u_{2}.$$

In (4), c takes on the role of a spectral parameter, while the coefficient of σ_1 can be interpreted as a constant mass term with an exponentially decaying perturbation. In the sense of the analogue of Kneser's Theorem for Dirac systems [9], this perturbation is subcritical, and indeed the method developed in [9] can be used to show that, for each k > 0, there are only finitely many exceptional values for c. Moreover, an asymptotic analysis of (4) along the lines of [10; see also 12] reveals that the number of exceptional values is asymptotic to k in the limit $k \to \infty$ with fixed ratio c/k.

This was verified in a computational investigation of (4) based on a piecewiseconstant approximation of the exponential function, following the approach of [11]. The numerical findings suggested that the number of exceptional values in (-k, 0) is always equal to the unique integer in [k - 1, k); more precisely, whenever k reaches an integer value, an additional exceptional value, initially equal to -k, appears and moves into the interval (-k, 0) as k increases.

This regularity raised the suspicion that (4) has an underlying solvable structure, and indeed, on closer scrutiny it turns out that this equation can be analysed by means of a variant of the factorisation method, resulting not only in a proof of the above conjecture, but even in an explicit formula for the exceptional values, which makes any asymptotic and computational analysis of (4) superfluous.

Theorem 1. For given k > 0, the exceptional values for $H_a(k)$ are given by

$$c_{n-1}(k) = -\sqrt{2kn - n^2}$$
 $(n \in \mathbb{N}, n < k).$

Consequently there are exactly N exceptional values in (-k, 0) if $k \in (N, N + 1]$, $N \in \mathbb{N}_0$.

In view of Theorems 1.1 and 1.2 of [7], this implies in particular that the eigenvalues of $H_a(k)$ converge to those of $H_0(k)$ as $a \to 0$ if $c^2 < 2k - 1$, confirming Behncke's linear bound (although with a different constant). However, his conjecture [4, p. 2558] that spectral stability may hold for all $c^2 < k^2 - 5/2$ turns out to be very questionable.

Since the three-dimensional Coulomb-Dirac Hamiltonian \mathcal{H}_a is the direct sum of $H_a(k)$ for $k \in \mathbb{Z} \setminus \{0\}$, we can thus infer spectral stability for this operator in all cases in which \mathcal{H}_0 has a distinguished self-adjoint realisation.

Corollary 1. Let \mathcal{H}_0 be the Dirac operator and \mathcal{H}_a the Dirac operator with anomalous magnetic moment with a Coulomb potential with coupling constant $c \in (-1, 0)$. Then the eigenvalues of \mathcal{H}_0 are the limits of the eigenvalues of \mathcal{H}_a as $a \to 0$.

3. Proof of Theorem 1

The following proof of Theorem 1 will be based on squaring the Dirac-type operator in (4), which is equivalent to deriving second-order differential equations for u_1 and u_2 . In a general situation, this is usually not a good idea except to obtain a quick and rough heuristic result, since it involves derivatives of the coefficients and consequently requires unnecessarily strong regularity. In the present instance, however, we are dealing with a specific Dirac system with analytic coefficients, and as it turns out, the resulting second-order equation system decouples and can be solved by the factorisation method in the relevant cases (the second-order equation (5) is closely related to the radial Schrödinger equation with a Morse potential [6, Section 5.2]). Here the factorisation is incidentally provided by the original Dirac system; see the discussion in Appendix B. The key observation is contained in the following result.

Theorem 2. Let $\kappa \ge 0$. Let $v_1(t, \kappa) := e^{-e^{-t}-\kappa t}$, and define v_j recursively for $j \in \mathbb{N} + 1$ by

$$v_{j+1}(t,\kappa) := (\kappa + j - e^{-t})v_j(t) - v'_j(t) \qquad (t \in \mathbb{R}).$$

Then $v_i(\cdot, \kappa)$ is a nontrivial solution of

$$v'' = (e^{-2t} - (2\kappa + 2j - 1)e^{-t} + \kappa^2) v,$$
(5)

and it has the asymptotic properties

$$\lim_{t \to \infty} v_j(t,\kappa) = 0, \qquad \lim_{t \to \infty} \frac{v'_j(t,\kappa)}{v_j(t,\kappa)} = -\kappa, \quad and \quad \lim_{t \to -\infty} \frac{v'_j(t,\kappa)}{v_j(t,\kappa)} e^t = 1 \qquad (j \in \mathbb{N}).$$

The proof can be done by induction with respect to j. The fact that v_j is a solution of (5) is readily verified by differentiating twice. The asymptotic properties are obvious for

$$\frac{v_1'(t,\kappa)}{v_1(t,\kappa)} = e^{-t} - \kappa.$$

Now assume that $j \in \mathbb{N}$ is such that the assertion is true. If v_{j+1} were trivial, this would imply that (up to multiplication by a constant) $v_j(t, \kappa) = e^{(\kappa+j)t+e^{-t}} \to \infty$ $(t \to \infty)$, contradicting the first limit property of v_j . The asymptotic properties of v_{j+1} are easily checked using the identity

$$\frac{v'_{j+1}(t,\kappa)}{v_{j+1}(t,\kappa)} = \frac{-e^{-2t} + 2(\kappa+j)e^{-t} - \kappa^2) + (\kappa+j-e^{-t})\frac{v'_j}{v_j}(t,\kappa)}{\kappa+j-e^{-t} - \frac{v'_j}{v_j}(t,\kappa)}$$

Corollary 2. Let $\kappa \ge 0$, $j \in \mathbb{N}$ and v_j be as in Theorem 1. Then

$$\lim_{t \to \infty} \frac{v_j(t)}{v_{j+1}(t)} = \frac{1}{2\kappa + j}, \qquad \lim_{t \to -\infty} \frac{v_j(t)}{v_{j+1}(t)} = 0.$$

This is a direct consequence of Theorem 1 in view of

$$\frac{v_j(t,\kappa)}{v_{j+1}(t,\kappa)} = \frac{1}{\kappa + j - e^{-t} - \frac{v'_j}{v_j}(t,\kappa)}$$

Furthermore, the following conclusion can be verified by a straightforward calculation.

Corollary 3. Let v_j be defined as in Theorem 1, and let $j \in \mathbb{N}$, $k \ge j$. Then

$$u(t) = \begin{pmatrix} \frac{1}{\sqrt{2kj-j^2}} v_{j+1}(t,k-j) \\ v_j(t,k-j) \end{pmatrix} \qquad (t \in \mathbb{R})$$

is a nontrivial solution of the Dirac system (4) with $c = -\sqrt{2kj - j^2}$. The Prüfer angle φ of u satisfies

$$\lim_{t \to \infty} \tan \varphi(t) = \frac{j}{\sqrt{2kj - j^2}}, \qquad \lim_{t \to -\infty} \tan \varphi(t) = 0.$$

Since, with c as in Corollary 3,

$$\tan \vartheta_{-}(c) = \frac{k - \sqrt{k^2 - c^2}}{-c} = \frac{k - \sqrt{k^2 - 2kj + j^2}}{\sqrt{2kj - j^2}} = \frac{j}{\sqrt{2kj - j^2}},$$

where we have used $j \leq k$, the Prüfer angle φ in Corollary 3 corresponds via $\vartheta(\varrho) = \varphi(\log \varrho)$ to a solution of (3) which coincides (up to addition of integer multiples of π) with both $\vartheta_0(\cdot, c)$ and $\vartheta_{\infty}(\cdot, c)$; hence such values of c are exceptional values.

We now conclude the proof of Theorem 1 by showing, based on the continuity and monotonicity properties of ϑ_0 and ϑ_∞ , that all exceptional values are obtained in this way. Corollary 3 specifies the asymptotic behaviour of the relevant angle functions mod π ; a study of the zeros of v_j reveals their global behaviour as shown in Lemma 2 below. Theorem 1 then follows in view of

$$\vartheta(\varrho, -\sqrt{2kj - j^2}) = \varphi_j(\log \varrho, k - j) \qquad (\varrho > 0, k > 0, j \in \mathbb{N}, j < k),$$

where (with $\arctan 0 = 0$)

$$\varphi_j(\cdot,\kappa) := \arctan(\frac{v_j(\cdot,\kappa)}{v_{j+1}(\cdot,\kappa)}\sqrt{2\kappa j + j^2}) + \pi.$$

Lemma 2. Let $\kappa \ge 0$. Then $v_j(\cdot, \kappa)$ has exactly j - 1 zeros. The associated angle function φ_j satisfies

$$\lim_{t \to -\infty} \varphi_j(t, \kappa) = \pi$$

and

$$\lim_{t \to \infty} \varphi_j(t, \kappa) = \vartheta_-(-\sqrt{2\kappa j + j^2}, \kappa + j) - (j - 1)\pi \qquad (j \in \mathbb{N}).$$

Proof. A look at the direction field of (3) with $k = \kappa + j$ shows that $\varphi_j(\cdot, \kappa)$, being the unstable solution at both $\pm \infty$, is monotone decreasing. $v_1(\cdot, \kappa)$ has no zeros, so $\varphi_1(\cdot, \kappa)$ has no zeros mod π , and the assertion follows for j = 1.

To conclude the proof by induction, assume now that $j \in \mathbb{N}$ is such that the assertion is true. Then $\varphi_j(\cdot, \kappa) - \pi/2$ has exactly j zeros mod π , so $v_{j+1}(\cdot, \kappa)$ has exactly j zeros. Since the $v_{j+1}(\cdot, \kappa)$ is a non-trivial solution of a second-order equation, $v'_{j+1}(\cdot, \kappa)$ does not vanish at zeros of $v_{j+1}(\cdot, \kappa)$. Consequently, $v_{j+1}(\cdot, \kappa)$ and $v_{j+2}(\cdot, \kappa)$ have no common zeros. Hence $\varphi_{j+1}(\cdot, \kappa)$ has exactly j zeros mod π , and therefore must converge to $\vartheta_{-}(-\sqrt{2\kappa j + j^2}, \kappa + j) - (j - 1)\pi$ as $t \to \infty$.

Appendix

A. The Prüfer Transformation

Consider a general Dirac system

$$(-i\sigma_2 \frac{d}{dx} + m(x)\sigma_3 + l(x)\sigma_1 + q(x))u(x) = 0 \qquad (x \in I)$$
(6)

on an interval $I \subset \mathbb{R}$ with locally integrable, real-valued coefficients m, l and q (we have absorbed the spectral parameter in the latter). If u is a solution of (6), then so is its (component-wise) complex conjugate, so it is sufficient to study \mathbb{R}^2 -valued solutions. Since (6) is linear, a non-trivial solution will never take the value $\begin{pmatrix} 0\\0 \end{pmatrix}$, so u(x) traces out an absolutely continuous curve in the punctured plane as x varies.

Introducing polar coordinates in this plane by writing

$$u(x) = |u(x)| \left(\begin{array}{c} \cos \vartheta(x) \\ \sin \vartheta(x) \end{array} \right),$$

where the absolutely continuous function ϑ (the *Prüfer angle* of u) is determined uniquely up to addition of an integer multiple of 2π — or indeed π , since -u is again a solution of (6) —, a straightforward calculation yields the differential equation for ϑ (*Prüfer equation*)

$$\vartheta' = m\cos 2\vartheta + l\sin 2\vartheta + q.$$

This non-linear equation is equivalent to (6), because one can recover u from ϑ by noticing that (choosing $x_0 \in I$)

$$|u(x)| = |u(x_0)| \exp \int_{x_0}^x (m \sin 2\vartheta - l \cos 2\vartheta) \qquad (x \in I).$$

B. Dirac systems and the factorisation method

Known to students of quantum mechanics as a trick to treat the Schrödinger equation for the harmonic oscillator, the factorisation method is in fact a tool of much wider scope for studying second-order differential equations (see the extensive discussion in [6]). It is based on the close link between the eigenvalue equations

$$A^*Au = \lambda u$$
 and $AA^*v = \lambda v$,

where A is typically a first-order differential operator: a solution u of the first equation gives rise to a solution v = Au of the second.

Here we remark on how this method can serve to solve eigenvalue equations for certain one-dimensional Dirac operators, which in turn represent a factorisation of the Sturm-Liouville operators arising as their formal squares.

Starting from a Dirac system (6), or equivalently

$$u'_1 = -lu_1 + (m - q)u_2, u'_2 = (m + q)u_1 + lu_2,$$

with absolutely continuous coefficients l, m, q, we find that u_1 and u_2 satisfy the second-order equations

$$u_1'' = (m^2 - q^2 + l^2 - l')u_1 + (m' - q')u_2,$$

$$u_2'' = (m^2 - q^2 + l^2 + l')u_2 + (m' + q')u_1.$$

This ordinary differential equation system decouples if m and q are constant. If we assume this in the following and set c := m - q, d := m + q and a := cd, we are dealing with the Dirac system

$$u_1' = -lu_1 + cu_2$$

 $u_2' = du_1 + lu_2$

and corresponding second-order equations

$$u_1'' = (a + l^2 - l')u_1, (7)$$

$$u_2'' = (a + l^2 + l')u_2. aga{8}$$

These two Sturm-Liouville equations differ only in the sign of l'. Hence we can construct a chain of interlocking equation systems, along with special solutions, in the following way.

Theorem 3. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers with $a_1 = 0$, and let $(l_n)_{n \in \mathbb{N}}$ be a sequence of real-valued absolutely continuous functions on I such that

$$l_{n+1}^{2} + l_{n+1}' + a_{n+1} = l_{n}^{2} - l_{n}' + a_{n} \qquad (n \in \mathbb{N}).$$

Let $v_1(x) := \exp \int^x l_1$ ($x \in I$), and define recursively for $n \in \mathbb{N} + 1$

$$v_n := l_n v_{n-1} - v'_{n-1}.$$

Then v_n is a solution of (7) with $a = a_n$, $l = l_n$, and of (8) with $a = a_{n+1}$, $l = l_{n+1}$. Furthermore, for $n \in \mathbb{N} + 1$ and arbitrary $d_n \in \mathbb{R} \setminus \{0\}$,

$$u := \left(\begin{array}{c} -\frac{1}{d_n} v_n \\ v_{n-1} \end{array} \right)$$

is a solution of the Dirac system

$$u'_{1} = -l_{n}u_{1} + (a_{n}/d_{n})u_{2}$$

$$u'_{2} = d_{n}u_{1} + l_{n}u_{2}.$$
(9)

Proof. Because of $a_1 = 0$, the first equation of the Dirac system (9) with n = 1 decouples from the second and can easily be solved to obtain the formula for v_1 . As the first component of a solution of this Dirac system, v_1 satisfies equation (7) with $l = l_1$, $a = a_1$. In view of the assumptions on the coefficients l_n , this equation is the same as (8) with l = 2, $a = a_2$, the Sturm-Liouville equation satisfied by the second component of a solution of the Dirac system for n = 2. The corresponding first component of this solution can easily be calculated from the second equation of the Dirac system. It will

depend on d_2 , but note that this dependence can be eliminated by normalisation as far as obtaining a solution v_2 of the Sturm-Liouville equation (7) with $l = l_2$, $a = a_2$ is concerned.

This process is then iterated to find the solutions for $n \in \{3, 4, \ldots\}$.

Examples.

1. The classical application of the factorisation method to the one-dimensional Schrödinger equation for a quantum-mechanical harmonic oscillator,

$$-v''(x) + x^2v(x) = \lambda v(x),$$

is captured in the above scheme if we choose $l_n(x) := x$ $(n \in \mathbb{N})$. The hypothesis of Theorem 3 is then satisfied with $a_n := -2(n - 1)$ $(n \in \mathbb{N})$, and we obtain the well-known solutions

$$v_n(x) = (-\frac{d}{dx} + x)^{n-1} e^{-\frac{x^2}{2}}$$

with eigenvalues $\lambda_n := 2n - 1 \ (n \in \mathbb{N}).$

2. The case of equation (5) is treated by choosing $l_n(x) := \kappa + (n-1) - e^{-x}$ $(n \in \mathbb{N})$; we immediately find $v_1(x) = e^{-e^{-x} + \kappa x}$ and the recursion formula of Theorem 2.

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