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A Brief Course in Spontaneous Symmetry Breaking
I. The Paleolitic Age[1]

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Abstract

The physical word is marked by the phenomenom of spontaneous broken symmetry (SBS) i.e. where the state of a system is assymmetric with respect to the symmetry principles that govern its dynamics. For material systems this is not surprising since more often than not energetic considerations dictate that the ground state or low lying excited states of many body system become ordered i.e. a collective variable, such as magnetization or the Fourier transform of the density of a solid, picks up expectation values which otherwise would vanish by virtue of the dynamical symmetry(isotropy or translational symmetry in the aforementioned examples). More surprising was the discovery of the role of SBS in describing the vacuum or low lyng excitations of a quantum field theory. First came spontaneously broken chiral symmetry which was then applied to soft pion physics. When combined with current algebra, this field dominated particle physics in the 60's. Then came the application of the notion of SBS to situations where the symmetry is locally implemented by gauge fields. In that case the concept of order becomes more subtle. This development lead the way to electroweak unification and it remains one of the principal tools of the theorist in the quest for physics beyond the standard model. This brief review is intended to span the history of SBS with emphasis on conceptual rather than quantitative content. It is a written version of lectures of R.Brout on the "Paleolithic Age" and on "Modern Times" by F.Englert, i.e. respectively without and with gauge fields.

[1]Invited talks presented at the 2001 Corfu Summer Institute on Elementary Particle Physics

I. The Early Ancestors (van der Waals and Weiss)[1]

At the beginning of the 20th century, van der Waals proposed the idea of a "molecular field" in order to explain the deviation of the equation of state of gases from ideality and from there to condensation. (What this has to do with SBS will emerge subsequently.) His idea was to consider that each molecule was surrounded by others which interact with it. Thus its energy is

$$V_{mol} = \left[\int d^3r' v(r - r') \right] \rho \,, \tag{1}$$

where $v(r)$ is the intermolecular potential, taken to be attractive, in van der Waals's eyes, for $r > r_0$; ρ is the mean density. The "molecular field approximation" (MFA) is to neglect the correlation of density at r' to the presence of a molecule at r. Though this neglect does some injustice to the situation, we have learnt over the years that, in the large, the essential physics is respected. One exception is the quantitative theory of critical phenomena, so beautifully executed by Wilson, Fisher and others. However, throughout this review we shall work in MFA since the main progress which has been made in analyzing the order encountered in a great variety situations has been in MFA. (Once more a notable exception is in 2 dimensional systems wherein topological considerations are often vital). In general as the dimensionality increases so does the reliability of MFA and for $d > 4$, it becomes reliable in all thermodynamic conditions (In this review we shall not touch upon lattice gauge theory where dimensionality plays a different role from the more conventional many body and field theoretic systems treated here. Thus confinement will not be included).

From Eq.(1), van der Waals deduced the existence of an internal pressure, p_{int} given by

$$p_{int} = -\frac{\partial V_{mol}}{\partial v} = \rho^2 \frac{\partial V_{mol}}{\partial \rho} = \rho^2 \widetilde{v}(o) \,, \tag{2}$$

where $\widetilde{v}(o)$ is the Fourier transform of $v(r)$ at $q = 0$. The total pressure is thus $p + p_{int}$; p is the external pressure. Under normal conditions ($p \cong 1atm$ and $T \cong$ room temperature) a typical liquid exhibits $p_{int} = 10^3 atm$, which gives one an idea of just how essential are the intermolecular forces in maintaining the cohesion of the liquid, as against a vapor where p_{int} more often than not is negligible away from critical conditions.

Whereas in an ideal gas has $p = kT/v$ (v being the volume per molecule), van der Waals proposed that in a general fluid one should replace v by the "free volume", that which is unoccupied by the molecule itself. Thus he set $p_{total} = (kT/v - b)$ where b is volume occupied by stuff within a single molecule. He thus set

$$p_{Tot} = p + \rho^2 \tilde{v}(o) = kT[\rho^{-1} - b]^{-1}\,, \tag{3}$$

the famous van der Waals equation. This equation of state has been qualitatively successful but fails quantitatively near the critical point, as is to be expected. In Fig.1 we sketch schematically a few isotherms

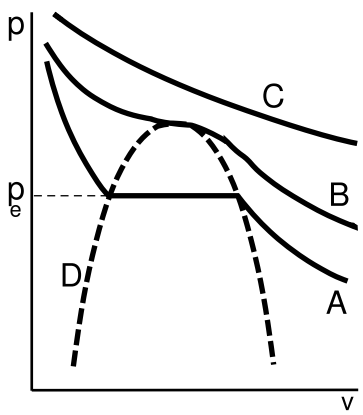

Fig. 1

Only vapor exists for all p once $T \geq T(c)$, but for $T < T(c)$, along, for example, isotherm A, the system is in the liquid (vapor) phase for $p > p(e)$ ($p < p(e)$) respectively. At $p = p(e)$, the system has a choice between liquid and vapor. They can coexist. It is this choice, seemingly arbitrary, which in this case is SBS. We shortly bring more clarity into this question, but for the moment we ask the reader to bear in mind the coexistence curve D, which on the left side marks the locus of where liquid isotherms begin as the pressure increases and on the right where vapor isotherms take over as the pressure decreases.

Whilst van der Waals was busy in Holland explaining why it rains or shines, Weiss in France was proposing a similar mechanism to explain ferromagnetism. Like van der Waals, he was toying with this new fangled idea of the atomic hypothesis. Each atom was endowed with a magnetic moment and ferromagnetism was the alignement of these elementary entities occasioned by the existence of an external magnetic field. The only trouble was that these fields were far too small to maintain the alignement at room temperature, owing to their thermal agitation. Weiss therefore proposed that there was an internal magnetic field proportional to the magnetization itself. Thus

$$\vec{H}_{Tot} = \vec{H}_{ext} + \vec{H}_{int} = \vec{H}_{ext} + \alpha\vec{M} \,, \tag{4}$$

where M is the magnetization per atom.

The idea was matched neatly with van der Waals internal pressure in the 20's when Heisenberg proposed his exchange mechanism wherein there was an energy due to spin-spin interactions brought about by interatomic coulombic forces (like those which were invoked to explain chemical binding or Hund's rule for atoms). Thus Heisenberg proposed an interaction

$$V_{spin-spin} = -v(r - r')\vec{S}(r).\vec{S}(r') \,, \tag{5}$$

$\vec{S}(r)$ being the spin of the atom located at r, hence proportional to its magnetic moment. At typical interatomic distances in a solid, $v(r)$ could be estimated to be something less than chemical bond energies, rather like $O(10^{-1}ev) = 10^3$ K. The v in Eq.(5) is not be confused with the v of Eq.(1). It is the spin-spin part. This advance lent more credence to Weiss's suggestion since when the idea was first proposed, the energy one could come by was in dipole-dipole magnetic interactions and these were too small by 3 orders of magnitude (Typically ferromagnetic transitions occur at $O(10^3K)$).

Thus Heisenberg proposed to furnish Weiss's hypothesis with a model which contained spin-spin interactions in the form

$$V = -\frac{1}{2}\sum_{i,j} v_{ij}\vec{S}_i.\vec{S}_j \,, \tag{6}$$

$\vec{S}_i$ being the spin on site i. This implies the existence of an internal field given at site i by

$$\vec{H}_i = \sum_j v_{ij}\vec{S}_j \,. \tag{7}$$

MFA is then the analog of van der Waals' approximation. One neglects the correlation of $\vec{S}_j$ to $\vec{S}_i$ and approximates

$$\langle\vec{H}_i\rangle = \vec{H}_{int} = \sum_j v_{ij}\langle\vec{S}_j\rangle = \sum_j v_{ij}\langle\vec{M}\rangle \,, \tag{8}$$

where we have used translational symmetry so that $\langle \vec{S}_j \rangle$ is site independent. We have set the elementary magneton of each atom equal to unity so that spin and magnetic moment mean the same thing. Then H has the dimensions of energy.

Thus outfitted,Weiss's molecular field becomes (with $\tilde{v}(q)$ = Fourier transform of v_{ij})

$$\vec{H}_{mol} = \vec{H} + \tilde{v}(o)\vec{M} \,. \tag{9}$$

From statistical mechanics one may then calculate $\langle \vec{M} \rangle$ self consistently

$$\langle \vec{M} \rangle = \frac{tr \exp\left[\beta \vec{H}_{mol} \cdot \vec{S}\right] \vec{S}}{tr \exp\left[\beta \vec{H}_{mol} \cdot \vec{S}\right]} \quad ; \quad \beta = (1/kT)\,. \tag{10}$$

II. Broken Discrete Symmetry

In the next few paragraphs we shall develop the idea using the Ising model (proposed by Heisenberg to his student as a thesis project). We shall see that this model is the prototype of a broken discrete symmetry, as opposed to a continuous symmetry wherein S is a vector.

One treats S as a 2-valued function, taking on values ± 1. Then

$$\begin{aligned}\langle M \rangle = \frac{\exp \beta H_{mol} - \exp -\beta H_{mol}}{\exp \beta H_{mol} + \exp -\beta H_{mol}} &= \tanh \beta H_{mol} \\ &= \tanh \beta \left[H + \tilde{v}(o)\langle M \rangle\right] \end{aligned} \tag{11}$$

a self consistent equation for $\langle M \rangle$. In Eq.(11) H designates the external field. To see the consequences of Eq.(11) set $H = 0$. It is then seen that in addition to $\langle M \rangle = 0$ two additional solutions arise of equal and opposite values when $\beta \tilde{v}(o) > 1$ since the slope of $\tanh x$ at $x = 0$ exceeds unity when $x > 1$. These are the solutions which encode spontaneous magnetization below the critical temperature ($kT < kT_c$ where $kT_c = \tilde{v}(o)$). We shall shortly see that these are stable solutions whereas $\langle M \rangle = 0$ is unstable for $T < T_c$. This is SBS.

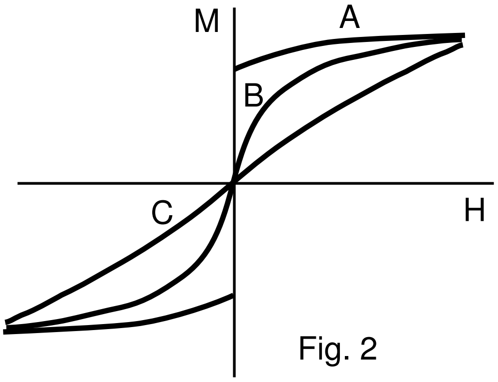

Fig. 2

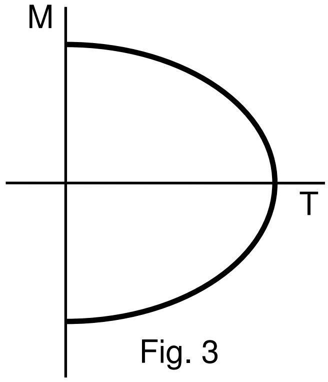

Fig. 3

In Fig.2, curve A is the isotherm for $T < T_c$, B for $T = T_c$ and C for $T > T_c$.

The values of $\langle M \rangle$ at $H = 0$ are sketched in Fig.3.

Contact with the van der Waals theory is made as follows. Define

$$\epsilon_i = \frac{1}{2}(1 + \mu_i) . \tag{12}$$

Thus $\epsilon = 0$ means absence of a particle on a site and $\epsilon = 1$ means its presence. Then $\langle M \rangle$ is mean density and H is like pressure. Turning Fig.2 on its side and tinkering with some thermodynamic identities converts Fig.2 into Fig.1 . Fig.3 looked at on its side is the coexistence curve of Fig.1 . In this case SBS is the choice of whether sites are occupied by particles or holes. It has recently been proven by Fisher that the analogy between liquid-vapor condensation in the case of continuum space (as opposed to the lattice) and ferromagnetism runs very deeply, even to the most minute details of their critical behaviour.

We shall continue with discrete SBS as exemplified by the Ising model by deriving these results from an important construction called the effective potential, a method developed by Bragg and Williams in the early 30's (in another connection, but this is irrelevant).

The partition function is

$$Z = tr \exp \left[\frac{\beta}{2} \sum v_{ij} \mu_i \mu_j + \beta \sum \mu_i H_i \right] . \tag{13}$$

For the nonce take H_i independent of i. The trace is over all 2^N spin states.

Carry it out piecemeal M by M where M now means the total spin

$$M = \sum \mu_i = N_{up} - N_{down} \,, \qquad (14)$$

so that

$$Z = \sum_M e^{\beta MH} tr_M \exp\left[\frac{\beta}{2}\sum v_{ij}\mu_i\mu_j\right] \,, \qquad (15)$$

where tr_M means summing over the $\binom{N}{(N+M)/2}$ states characterized by M.

Though one can carry through the construction in all rigor we shall only develop the theory here in MFA in order to bring out the essential ideas. It is the essence of MFA to neglect inter-spin correlations. Thus

$$\begin{aligned} \frac{1}{2}\sum_{i,j} v_{ij}\langle\mu_i\mu_j\rangle_{MFA} &= \frac{1}{2}\sum_{i,j} v_{ij}\langle\mu_i\rangle\langle\mu_j\rangle & (16) \\ &= \frac{1}{2}\sum_{i,j} v_{ij}\langle\mu\rangle_M^2 \,, & (17) \end{aligned}$$

where we used translational symmetry to set $\langle\mu_i\rangle$ independent of site i. The symbol $\langle\mu\rangle_M$ means the average of a spin in the subensemble characterized by M i.e.

$$\langle\mu\rangle_M = \frac{1}{N}\sum_i \langle\mu_i\rangle_M = \frac{M}{N} = m \,. \qquad (18)$$

Thus in MFA one has

$$\begin{aligned} Energy &= -\frac{1}{2}\sum_{i,j} v_{ij}\mu_i\mu_j - H\sum \mu_i \\ &= -\frac{1}{2}N\tilde{v}(0)m^2 - NHm \,. & (19) \end{aligned}$$

Thus

$$Z_M = \Omega(M)e^{-\beta E(M)} \,, \qquad (20)$$

where

$$\Omega(M) = \binom{N}{(N+M)/2} \,, \qquad (21)$$

whence

$$\begin{aligned} \ln Z_M &= \ln\Omega(M) - \beta E(M) \\ \ln\Omega(M) &= N\ln 2 - \frac{N}{2}\left[(1+m)\ln(1+m) + (1-m)\ln(1-m)\right] \,, & (22) \end{aligned}$$

where we have used Stirling's approximation and Eq (18). $\ln Z_M$ has a sharp maximum at $N = Mm^*$ where

$$m^* = \tanh\left[\beta\widetilde{v}(o)m^* + H\right] , \tag{23}$$

the relative width of which is $O(1/\sqrt{N})$ so that in the thermodynamic limit (i.e. $\lim_{N\to\infty} \ln Z$) one has

$$\ln Z = \ln(Nm^*) . \tag{24}$$

Since $\ln Z = -\beta[\textit{Helmholtz free energy}]$, we identify $\ln\Omega(Nm^*)$ with the entropy (because the energy has already been identified in Eq (19)). Over the years we have come to call $(-1/N)\ln Z$ the effective potential and this has become the standard way to approach SBS in field theory (since $Z(M)$ is the functional integral over configurations of $\exp(-S)$ where S is the eucledeanized action; in our case the functional integral is the discrete sum over 2^N configurations).

One gets a first glimpse into the field formulation by looking at V_{eff} for small m

$$\begin{aligned} V_{eff} = -\lim(\frac{1}{N})\ln Z_N &= -\frac{1}{2}\beta\widetilde{v}(o)m^2 - \beta mH + \frac{1}{2}m^2 + \frac{1}{12}m^4 + \ldots \\ &= \frac{1}{2}(1-\beta\widetilde{v}(o))m^2 + \frac{1}{12}m^4 - \beta mH , \end{aligned} \tag{25}$$

m is to be considered a field taking on a continuum of values in the $N \to \infty$ limit and from now on we shall use the symbols m and φ (for field) interchangeably. In Eq.(25) the irrelevant constant $N\ln 2$ has been dropped.

From Eq.(25) we have,

$$V_{eff} = \frac{1}{2}\mu^2\varphi^2 + \lambda\varphi^4 - \varphi H , \tag{26}$$

where $\mu^2 = (1 - \beta\widetilde{v}(o)) = (1 - (T_c/T))$. It is seen that μ^2 changes sign at $T = T_c$, becoming negative for $T < T_c$. V_{eff} is sketched in Fig.4

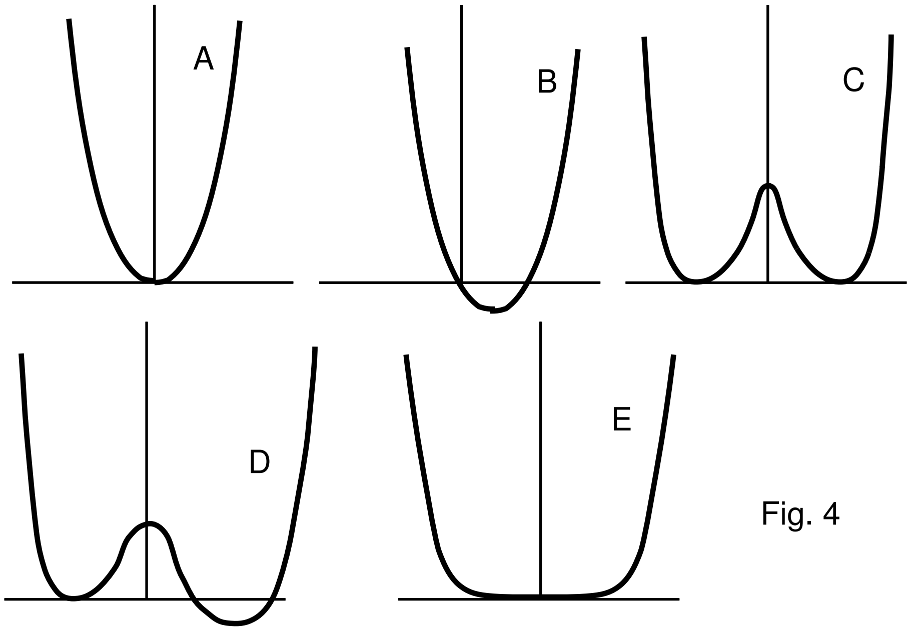

Fig. 4

A: $T > T_c\,; H = 0$ B: $T > T_c\,; H > 0$ C: $T < T_c\,; H = 0$ D: $T < T_c\,; H > 0$ E: $T = T_c\,; H = 0$

These pictures essentially contain the whole story and nothing more.

For $2 < d \leq 4$, small modifications exist in the critical region defined by $\left|1 - \frac{T_c}{T}\right| = O$ (a few percent) and H/kT a few percent. Otherwise MFA has quantitative significance, and always qualitative significance. For example for $d = 3$ one has $\mu^2 \sim |T - T_c|^{0.62}$ in the critical region at $H = 0$ rather than $\mu^2 \sim |T - T_c|^{0.5}$. The latter estimate becomes valid once $\left|1 - \frac{T_c}{T}\right| > 10\%$. The next section will afford further insight into the whys and wherefores of these facts.

III. Correlation Function (Green's Function)

Consider $H = 0$ and $T = T_c + \epsilon$ where ϵ is small and positive. As $\epsilon \to 0$, long range order comes into being i.e. if one fixes the orientation of a single spin out of N, then all N get oriented in the same direction. It then must be expected that for $\epsilon > 0$, there must be a precursor of this phenomenom i.e. long range order should be heralded in by correlations among spins over increasingly longer range as $\epsilon \to 0$. The theory of this was worked out in the first decades of the 20th century by Ornstein-Zernike, by Smolochowski and by Einstein, the interest being the tremendous enhancement of light scattering by critical fluctuations (i.e. long range correlations) giving rise to critical opalescence. In our case the cross section is proportional to $\langle |\mu_q|^2 \rangle$ where

$$\mu_{\vec{q}} = \sum_i \mu_i \exp\left[i\vec{q} \cdot \vec{R}_i\right] , \tag{27}$$

hence to the Fourier transform $\langle \mu_i \mu_j \rangle$, therefore at small q sensitive to long range correlations.

A first shot at the problem is contained in a rather obvious generalization of MFA. Turn on an external field which varies from site to site

$$\begin{aligned} Hamiltonian = \mathcal{H} &= -\frac{1}{2}\sum v_{ij}\mu_i\mu_j - \sum \mu_i H_i \\ Z &= tre^{-\beta\mathcal{H}} . \end{aligned} \tag{28}$$

Then

$$\langle \mu_i \rangle = \frac{\partial \ln Z}{\partial \beta H_i} , \tag{29}$$

$$\langle \mu_i \mu_j \rangle - \langle \mu_i \rangle \langle \mu_j \rangle = \frac{\partial \langle \mu_i \rangle}{\partial \beta H_j} = \frac{\partial^2 \ln Z}{\partial \beta H_i \partial \beta H_j} . \tag{30}$$

For $T > T_c$ and in $\lim_{H_i \to 0}$, all i, Eq.(30) allows to compute the (connected) correlation function (Green's function in field theory). We shall approximate this by use of MFA in this extended local sense.

The field on μ_i is $\sum_j v_{ij}\mu_j + H_i$. Thus for small H_i the average of μ_i for $T > T_c$ is

$$\langle \mu_i \rangle = \tanh \beta(\sum v_{ij} \langle \mu_j \rangle + H_i) \to \beta \left[\sum v_{ij} \langle \mu_j \rangle + H_i\right] . \tag{31}$$

The essential approximation that has been made is that $\langle \mu_j \rangle$ is calculated with a probability distribution that is independent of the orientation of μ_i . This is not exact since the distributions differ according to $\mu_i = +1$ or $\mu_i = -1$. This neglect of correlation in the present context is then a sort of local MFA. Taking a derivative of Eq.(31) with respect of to H_k and using Eq (30) gives

$$\langle \mu_i \mu_k \rangle = \delta_{ik} + \beta \sum v_{ij} \langle \mu_j \mu_k \rangle . \tag{32}$$

This is like an integral equation for $G_{ij}(\equiv \langle\mu_i\mu_j\rangle)$. It may be solved by Fourier transform. Denoting by $G(q)$ and $\tilde{v}(q)G(q)$ the Fourier transforms of G_{ij} and v_{ij}, one has

$$G(q) = 1 + \beta\tilde{v}(q)G(q)\,, \tag{33}$$

$$G(q) = \frac{1}{1-\beta\tilde{v}(q)}\,. \tag{34}$$

An interaction which is ferromagnetic over its whole range has $v(\vec{R}_i - \vec{R}_j) > 0$ for all distances $\left|\vec{R}_i - \vec{R}_j\right|$. Therefore $\tilde{v}(q)$ is maximal at $q = 0$ and has the form

$$\tilde{v}(q) = \tilde{v}(0) - \alpha q^2\,, \tag{35}$$

valid small q ($qa \ll 1$ where a = lattice distance).

Thus

$$G(q) = \frac{1}{(1-\beta\tilde{v}(0)) + \alpha q^2} \simeq \frac{1}{\mu^2 + \alpha q^2}\,, \tag{36}$$

and we see that the curvature of the effective potential at its minimum (for $T > T_c$ and $H = 0$) is equal to the $(mass)^2$ in $G(q)$. In this way one sees that for small values of $[(T-T_c)/T_c]$ with $(T-T_c) > 0$ and for small values of (H/kT_c) the spin system is governed by an effective action density equal to

$$\frac{1}{2}(\nabla\varphi)^2 + (\mu^2/2)\varphi^2 + \lambda\varphi^4 + \varphi H\,, \tag{37}$$

with $\mu^2 \sim (T-T_c)$ and $\lambda > 0$. We have dropped irrelevant factors of $O(1)$ which may be absorbed into the definition of φ, μ and λ. The important point is that $\mu^2 \to 0$ as $T \to T_c$ and one is confronted with an infra red problem at $T = T_c$, $H = 0$. This gives rise to the theory of critical phenomena which results in a dynamical theory of renormalization. In particular mass renormalization shifts μ^2 to $\sim (T-T_c)^{0.62}$ for $d = 3$. This is of little interest to us in this review which is an exploration of the physical mechanism behind SBS. Nevertheless it is interesting to understand how it is that there is a threshold value of d ($d > 4$) for which these renormalization effects become insignificant.

From Eq.(32), one sees that $G(R_i - R_j)$ is built out of chains of the interactions v_{kl} (Fig.5).

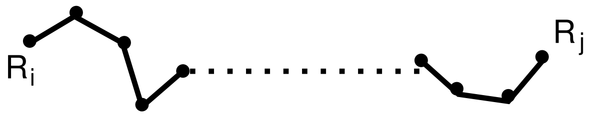

Fig. 5

The sum of all these random walks is G_{ij}. One can recast these considerations so as to take into account the fact that in the correct rule for each walk, a given intermediate spin is visited one and only one time. To this must be added walks which do have intersections. These have different weights. One can then convert the problem into a random walk without restriction giving rise to our $G(q)$ plus corrections due to intersections (there has been some oversimplification but without injustice to the essential physics). It is a fact, (and one can show it from the field theory itself) that for $d > 4$, the probability of self intersection becomes so small that it has no effect on the Green's functions. There is too much space around and that is why MFA becomes exact (except for irrelevant factors of O(1)) for $d > 4$.

Now let us see what happens $T < T_c$. Then $\langle \mu_i \rangle = m$ and the Green's function is

$$G_{ij} = \langle \mu_i \mu_j \rangle - \langle \mu_i \rangle \langle \mu_j \rangle = \langle \mu_i \mu_j \rangle - m^2 \,. \tag{38}$$

One can go through some formalism to establish the rules for how to construct the random walk in this case. Suffice it to say that, once again, one sums on random walks in MFA, but one must weight each vertex with the factor $(1 - m^2)$. To understand this physically, note that when one asks for the infinitesimal variation of $\langle \mu_i \rangle$ due to a variation of H_i , the field on μ_i , then

$$\frac{\partial \langle \mu_i \rangle}{\partial \beta H_i} = \frac{\partial}{\partial \beta H_i} \left[\tanh \beta H_i \right]_{H_i = H_i^0} = 1 - m^2(H_i^0) \,. \tag{39}$$

Here H_i^0 is the value of the field on μ_i before the variation, be it an external or internal field!

Since in the chain of interactions, an intermediate spin, say k , is submitted to a variation of the field H_k upon it, coming from the link which precedes it, (where one considers the chain originating at i and terminating at j). The response of μ_k to this variation is equal to $\beta v_{lk}(1 - m^2)$ which is taken to be small.

The net result is for $T < T_c$, $H = 0$

$$G(q) = \frac{1 - m^2}{1 - (1 - m^2)\beta\tilde{v}(q)} , \tag{40}$$

which for small q reads

$$G(q) = \frac{1}{\mu'^2 - \alpha q^2} , \tag{41}$$

where

$$\mu'^2 = \frac{1}{(1 - m^2)^{-1} - \beta\tilde{v}(0)} . \tag{42}$$

In this expression m is the spontaneous magnetization i.e.

$$m = \tanh\left[\beta\tilde{v}(0)m\right] . \tag{43}$$

From Eq.(43) and Eq.(42) it is very easy to show that $\mu'^2 > 0$ and in fact for small $(T_c - T)/T_c$ one has $\mu'^2 \sim (T_c - T)$.

Thus the wild infra-red fluctuations encountered as $T \to T_c + \epsilon$ with $\epsilon > 0$ become quenched through the existence of m for $T < T_c$. One easily shows that μ'^2 is the curvature of V_{eff} in Fig.4C at one of the minima.

One defines the susceptibility, χ , as $(\partial m/\partial H)|_{H=0}$ and in both cases ($(T - T_c)$ positive or negative) one has $\chi \sim |T - T_c|^{-1}$.

IV. Broken Continuous Symmetry

Rather than the spin taking on discrete values like $\mu_i = \pm 1$, one can now study a spin which is a vector. This can be done classically by placing a unit vector on each lattice site. Then the trace that is used to calculate Z is

$$h = \int \prod d\vec{S}_i \; \delta(\sum \vec{S}_i^2 - 1) , \tag{44}$$

where $\vec{S}_i$ is an n dimensional vector in some internal space. Or one can do a quantum calculation where $\vec{S}_i$ is an operator and the trace is the sum over the eigenvalues of $\vec{S}_i$ in some group representation. For example if $\vec{S}_i = \vec{\sigma}_i$ (=Pauli matrices) then the trace is once again over the values ± 1 for each spin. The Hamiltonian in either case is taken symmetric with respect to the transformation of the symmetry group which is represented by $\vec{S}_i$

$$\mathcal{H} = -\frac{1}{2}\sum v_{ij}(\vec{S}_i \cdot \vec{S}_j)\,, \tag{45}$$

and one represents an external breaking by fields H_i according to

$$\mathcal{H}_{ext} = -\sum \vec{S}_i \cdot \vec{H}_i\,. \tag{46}$$

For $T > T_c$ the physics of this case resembles strongly that of the Ising model. For example if $\vec{S}_i = \vec{\sigma}_i$ then the chain of interactions contributing to $G(q)$ contains chains like

$$tr\,[\ldots v_{kl}\vec{\sigma}_k.\vec{\sigma}_l\; v_{lm}\vec{\sigma}_l.\vec{\sigma}_m \ldots\,]\,. \tag{47}$$

Since $tr\vec{\sigma}_l\vec{\sigma}_l$ is $\neq 0$ only for equal spatial components, one sees that the trace is the same as that for the Ising model. Then the chain which contributes to $\left\langle \sigma_i^x \sigma_j^x \right\rangle$ is the same as for the Ising model. Moreover $\left\langle \sigma_i^x \sigma_j^y \right\rangle = 0$. Thus once again,in MFA one has

$$G(q) \sim \frac{1}{q^2+\mu^2}\,. \tag{48}$$

However for $T < T_c$, new physics emerges. The easiest road to Rome passes by the effective potential. From Eq.(26), one sees that the system duplicates in detail the theory of discrete SBS. Thus the quadratic part of the potential is proportional to $(\nabla\vec{\varphi})^2 + \mu^2\vec{\varphi}^2$ where $\mu^2 = T - T_c$. As before, there will be corrections due to self intersections of walks (and other dynamic effects which in fact are dependent on the representation $\vec{\varphi}$ of the symmetry group in question. But, whatever, the form of the quartic interaction is dictated by symmetry to be $\lambda\left[(\vec{\varphi}_i)^2\right]^2$). [This is going a little too fast since sometimes other polynomial forms are available, (like $d_{ijk}\varphi_i\varphi_j\varphi_k$ for SU_3 with φ_i in the regular representation), but this cursory review is not the place to enter into such niceties]. Thus one is led to

$$V_{eff} = \frac{1}{2}\left[\nabla\vec{\varphi}^2 + \mu^2(\vec{\varphi})^2\right] + \lambda(\vec{\varphi}^2)^2 + \vec{\varphi}\cdot\vec{H}\,. \tag{49}$$

Whereas μ^2 changes sign at $T = T_c$ (being proportional to $(1-\beta\tilde{v}(0))$ up to a scaling of $\tilde{v}(0)$), one has $\lambda > 0$. This is most easily seen in MFA where λ arise from the entropy factor as in the Ising model.

It then follows that all the pictures of Fig.4 remain applicable for use of SBS in the case of continuous symmetry provided they become multidimensional in "φ-space". For ease of representation let the symmetry be $U(1)$. Then $\vec{\varphi} = (\varphi_1, \varphi_2)$ and the pictures must be interpreted as planes cut through figures of revolution about a central axis in φ space.

The dimple in Fig.4C is the unstable solution $\vec{\varphi} = 0$ for $T < T_c$. The more stable minima lie along a circle in the φ_1, φ_2 plane $\vec{\varphi}^2 = \varphi_1^2 + \varphi_2^2 = m^2$ where $\partial V/\partial(\vec{\varphi}^2) = 0$. In this case SBS is the choice of which vector $\vec{\varphi}$ is taken non vanishing. The reason for the words "more stable" rather than "stable" is clear. Suppose $\vec{H} \neq 0$ in some direction then the minimum becomes stable and $\vec{\varphi} \parallel \vec{H}$. Now let $H \to 0$. The solution then tends to that value of $\vec{\varphi}$ which is on the abovementioned circle without changing its direction as one takes the limit. But it is unstable with respect to directional changes upon applying an infinitesimal adjunction of $\vec{H}'$ in a direction different from the original $\vec{H}$ (which had been sent to zero). This will cause $\vec{\varphi}$ to swivel along the circle so as to lie in the direction of $\vec{H}'$, albeit this latter is infinitesimal but not zero.

This new element of SBS of continuous symmetry is essential to the physics of all kinds of situations and as will be seen in the gauge section, plays a vital role in the Brout-Englert-Higgs (BEH) mechanism[1]. It is an effect which was first consciously put on display by Felix Bloch in the mid 1930's in his spin wave theory of ferromagnetism. In the next sections we shall review his theory, as well as applications to superconductivity and superfluidity. This will be followed by Nambu's development of the theory of spontaneously broken chiral symetry and soft pion physics. The expression of these ideas in terms of a relativistic field theory, often called the Goldstone theorem, will be presented in the context of the BEH mechanism in the section "Modern Times" by François Englert since it is herein that this aspect of the theory has been particularly successful.

We briefly summarize the important result of SBS of continuous symmetry which has been deduced up to this point. In terms of the effective potential for $V_{eff}(\vec{\varphi})$, for $T > T_c$ the situation is the same as for discrete symmetry. There is a unique minimum at $\vec{\varphi}^2 = 0$, whose curvature is the inverse suscep-

[1] References to the gauge theory and relevant material are given in "Modern Times".

tibility $\chi = \partial m/\partial H = (\mu^2)^{-1} \sim (T - Tc)^{-1}$. For $T < T_c$, the point $\vec{\varphi}^2 = 0$ is a local maximum since at this point the curvature $(= T - T_c)$ is then negative. There is then an "orbit" of minima which for the case of broken $U(1)$ symmetry is a circle (for the general case see Modern Times for a description of this orbit in the space of a representation of a general group). From our discussion it is seen that the susceptibility becomes a tensor in "$\vec{\varphi}$" space. One defines a longitudinal susceptibility corresponding to the response of $\langle\vec{\varphi}\rangle$ with respect to $\vec{H}$ parallel to an priori fixed vector $\vec{\varphi}_0$ which is on the orbit of minima at $H = 0$. One thinks of this as a "stretching" mode of response of the magnetization. The transverse susceptibility is the response to $\vec{H}$ orthogonal to $\langle\vec{\varphi}\rangle_0$ and according to our discussion of instability this is infinity. One defines a $(mass)^2$ tensor which is χ^{-1} whereupon $\mu^2_{longitudinal} \sim (T_c - T)$ and $\mu^2_{transverse} = 0$. This vanishing of the mass in the transverse direction is in fact the terminal point of a continuous spectrum of excitations, the modes being sorted out according to Fourier transform. The expression of this in relativistic field theory (Goldstone's Theorem) is covered in "Modern Times". The application to feromagnetism is very instructive in this regard. This will be the subject of the next section.

V. Spin Wave Theory

The existence of zero mass modes as collective excitations (i.e. bosons in quantum field theory) is neatly revealed in spin wave theory. We here follow a procedure, due to Bloch, by studying the single quantum excitations from the the ground state (vacuum). For simplicity we work with $\vec{S}_i = \vec{\sigma}_i$ (the Pauli matrices). SBS is the choice of orientation of vacuum, the only group symmetric specification for which is "all spins parallel" i.e. all in the same spin state. We choose $\mid 0 >$ to the "all spins up" i.e. $\sigma_i^z \mid 0 >= \mid 0 >$.

Excitations then are generated by creating down spins, the most low lying being l.c.'s of $\sigma_i^- \mid 0 >$. These l.c.'s are determined from

$$[H, \varphi_\omega] \mid 0 >= i\omega\varphi_\omega \mid 0 > , \tag{50}$$

where φ_ω is an "eigen operator", i.e an l.c. of $\sigma_i^- \mid 0 >$ which satisfies $[H, \varphi_\omega] = i\omega\varphi_\omega$. Using the algebra of Pauli matrices, along with $[\vec{\sigma}_i, \vec{\sigma}_j] = 0$ for $i \neq j$, we get with $\mathcal{H} = -\frac{1}{2}\sum v_{ij}\vec{\sigma}_i \cdot \vec{\sigma}_j$,

$$\left[\mathcal{H}, \sigma_i^-\right] = -\sum v_{ij}(\left[\sigma_i^z, \sigma_i^-\right]\sigma_j^z + \left[\sigma_i^+, \sigma_i^-\right]\sigma_j^-) = \sum_j v_{ij}(\sigma_j^z\sigma_i^- - \sigma_i^z\sigma_j^-) \,. \tag{51}$$

Operating on vacuum, we get

$$\left[\mathcal{H},\sigma_i^-\right] \mid 0 >= \sum_j v_{ij}(\sigma_i^- - \sigma_j^-)\,. \tag{52}$$

By translational symmetry Eq.(52) is diagonalized by Fourier transform. Defining $\sigma_q^- = (1/\sqrt{N})\sum \sigma_i^- e^{i\vec{q}\cdot\vec{R}_i}$ one has

$$\left[\mathcal{H},\sigma_q^-\right] \mid 0 >= i\omega_q \sigma_q^- \mid 0 >= \left[\tilde{v}(0) - \tilde{v}(q)\right]\sigma_q^- \mid 0 >\,, \tag{53}$$

where

$$\omega_q = \tilde{v}(0) - \tilde{v}(q) \sim q^2 \quad , \quad \text{small } q\,. \tag{54}$$

The generalization to the case of an external field is equally interesting. Clearly our vacuum $\mid 0 >$, corresponds to $\vec{H}$ in the z direction. So adding to $\mathcal{H}$ a term $-H\sum \sigma_i^z$, going through the same steps then leads to

$$\omega_q = \tilde{v}(0) - \tilde{v}(q) + H \sim q^2 + H\,. \tag{55}$$

Thus H induces a $(mass)^2$ in the zero mode which is linear in the external breaking. This is especially important when applying these ideas to SBχS and soft pion physics. It is to be noted that the excitation operator, σ_q^-, reduces to the global rotation operator at $q=0$ i.e. $\left[H,\sigma_{q=0}^-\right] = 0$ (at $H=0$) in virtue of symmetry whence $\omega(0) = 0$, and we see that $\mu^2_{transverse} = 0$ is indeed the statement that the excitation energy of a continuous spectrum vanishes at $q=0$ in virtue of symmetry.

We also can now see why continuous SBS cannot apply in its naïve form to $d=2$. The number of spin waves, at temperature β^{-1}, is $\left[e^{\beta\omega_q} - 1\right]^{-1}$ for $kT \ll \tilde{v}(0)$ (for higher T they interact and the ideal gas of excitations is no a longer valid approximation). Then the total number of spin waves at low T is

$$\sim \int d^d q \frac{1}{e^{\beta\omega_q} - 1} \sim \beta^{-1} \int \frac{d^d q}{q^2}\,, \tag{56}$$

which diverges in the infra-red at $d=2$ i.e. $\mid 0 >$ is unstable for $H=0$. New methods are therefore required in continuous SBS. But for SBS in the discrete case, the naive notions are OK, albeit suffering severe quantitative modifications.

Some conceptual issues arise which we will now address. Their resolution is of pedagogical interest especially when compared with the corresponding situation in the gauge theory.

We shall first display the classical concept of broken symmetry given by the familiar picture of an arrow which points in the "direction of the vacuum state" picked by the broken symmetry. For example, in the above paragraphs this arrow points in the z-direction of group space. For simplicity, we continue with the example of broken $SU(2)$ symmetry represented by a Pauli matrix sitting on each lattice site, wherein the Hamiltonian is a group scalar as in Eq.(45). The generalization of these considerations to any group in any representation is straightforward.

Let $|0\rangle$ be the vacuum state: $S_z|0\rangle = N/2|0\rangle$ where

$$\vec{S} = \sum_i \frac{\vec{\sigma}_i}{2} \, . \tag{57}$$

Since the $S_\alpha(\alpha = x, y, z)$ represent group generators (i.e. $[S_\alpha, S_\beta] = i\epsilon_{\alpha\beta\gamma}S_\gamma$), one may construct a rotated vacuum from them. For example, a rotation about the x-axis of $|0\rangle$ gives the rotated vacuum $|\theta\rangle$ where

$$|\theta\rangle = e^{iS_x\theta}|0\rangle \, . \tag{58}$$

The states $|\theta\rangle$ and $|0\rangle$ are degenerate since $[\mathcal{H}, S_x] = 0$, $\mathcal{H}$ being scalar and $\vec{S}$ being a group vector.

Since S_x is a group generator, it follows that

$$\langle\theta|S_x|\theta\rangle = 0 \; ; \; \langle\theta|S_y|\theta\rangle = -\sin\theta \; ; \; \langle\theta|S_z|\theta\rangle = \cos\theta \, . \tag{59}$$

In this way, the classical notion of "arrow" is given by the expectation value of the operator $\vec{S}$ in the different rotated vacua.

We shall now prove that, for θ fixed, in the limit $N \to \infty$, $\langle\theta|0\rangle = 0$. Moreover, we shall show that the Hilbert space of excitations built upon different vacua are mutually exclusive as well (in the limit $N \to \infty$).

$$\langle 0|\theta\rangle \;\; = \;\; \langle 0|e^{iS_x\theta}|0\rangle = \langle 0|\prod_{i=1}^{N} e^{i(\sigma_i^x/2)\theta}|0\rangle$$

$$
\begin{aligned}
&= \prod_{i=1}^{N} \langle 0| \cos(\theta/2) + i(\sigma_i^x/2) \sin(\theta/2) |0\rangle \\
&= [\cos(\theta/2)]^N \quad \longrightarrow_{N\to\infty} \; 0\,. \qquad (60)
\end{aligned}
$$

If instead of the overlap of $\langle 0|$ with $|\theta\rangle$ we took excited states of $\langle 0|$, say containing n spin wawes, the overlap would then be $\sim [\cos(\theta/2)]^{N-n}$. So even if n is a finite fraction of N, the result vanishes in the limit. With more effort one can prove that the excited states built on $|\theta\rangle$ are orthogonal to excited states of $|0\rangle$. This remains true until one reaches some threshhold number of excitations proportional to N at which point one approaches critical conditions wherein these naive considerations break down.

For finite N one can always construct $N+1$ orthogonal "vacuum" states as one does in the conventional method of quantizing angular momenta. These are the states $(S^-)^p|0\rangle$; $p = 0, 1..., N$. States corresponding to a rotation θ, φ from $|0\rangle$ are obtainable as a linear combination of these. For finite N such states are not, in general, orthogonal. But they become approximately so when their angular difference exceeds $O(1/\sqrt{N})$. In this way one recovers their mutual orthogonality as $N \to \infty$ for any angular difference.

VI. Superfluidity and Superconductivity

We briefly indicate how SBS applies to these two interesting phenomena. A free boson gas of N particles condenses at a temperature for which the thermal Compton wave length $(mkT)^{-1/2}$ is O(interparticle distance). For $T < T_c$, a finite fraction of N occupies the state $k = 0$, and at $T = 0$ all N have zero momenta. For the interacting case, at $T = 0$ there is only a finite fraction which condenses i.e.

$$\left\langle a_0^+ a_0 \right\rangle = N_0 = \alpha N \quad , \quad \alpha < 1\,. \qquad (61)$$

This macroscopic occupation of the $k = 0$ state can be transcribed into a SBS as follows. The commutator $\left[a_0^+, a_0\right] = 1$ is negligible with respect to N_0 i.e. $N_0 \cong N_0 + 1$ in good approximation. Then one can treat a_0 as a c-number. But a_0 has a phase. The choice of this phase is SBS. Bogoljubov [2] built a system of excitations in analogy to spins waves, by building them from a vacuum with a fixed complex c-number value of a_0. They are linear combinations of the form $\Psi_q^+ = \alpha_q a_q^+ + \beta_q a_{-q}$ (note $a_{-q} \mid 0 > \neq 0$ because $\mid 0 >$ contains virtual occupation of states with $q \neq 0$, in virtue of the interatomic interactions). The Bogoljubov coefficients α_q, β_q (with $(|\alpha_q|^2 - |\beta_q|^2) = 1$) are proportional to a_0 and a_0^* respectively. The point to be made here is that as $q \to 0$, the operator Ψ_q^+ becomes a rotational generator in the "gauge plane" i.e. it generates infinitesimal changes of the phase of a_0.

Superfluidity is then a spectacular example of SBS where the symmetry is $U(1)$. The all important phase plays vital physical role since if one lets it vary from point to point, its gradient is the velocity of superfluid.

Superconductivity is an equally fascinating case of spontaneously broken $U(1)$ symmetry. Bound states (Cooper pairs) are s states in spin singlets, so causing correlations $\langle n_{k\uparrow} n_{-k\downarrow}\rangle - \langle n_{k\uparrow}\rangle \langle n_{-k\downarrow}\rangle$ which are $O(1)$ rather than the usual free gas value $O(1/N)$. In terms of a pseudo spin algebra which is isomorphic to Pauli spin matrices given by

$$\begin{aligned} b_k = a_{k\uparrow} a_{-k\downarrow} &\sim \sigma_k^- \\ b_k^+ = a_{-k\uparrow}^+ a_{k\downarrow}^+ &\sim \sigma_k^+ \\ 1 - n_{k\uparrow} - n_{-k\downarrow} &\sim \sigma_k^z , \end{aligned} \tag{62}$$

one invents a set of order parameters which are $\langle b_k \rangle$. Since b_k has a phase, one breaks $U(1)$ and since the hamiltonian is invariant under this $U(1)$ symmetry, the interactions being $v(k, k')b_k^+ b_{k'}$, one has SBS. Note that an otherwise $SU(2)$ symmetry is broken externally since the kinetic energy in the Hamiltonian is equal to $\sum \varepsilon(k)(n_{k\uparrow} + n_{-k\downarrow})$ hence up to a constant $= \sum \varepsilon(k)\sigma_k^z$.

A typical "vacuum" configuration may be depicted as follows as one spans the Fermi surface in k-space

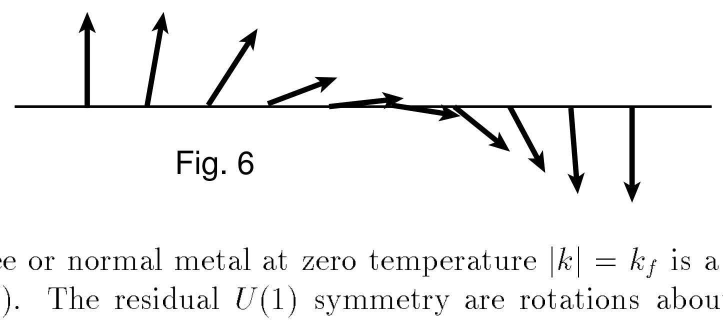

Fig. 6

(in the free or normal metal at zero temperature $|k| = k_f$ is a point of discontinuity). The residual $U(1)$ symmetry are rotations about the isospin Z axis which in the above picture is obtained by rotation around the horizontal axis. In this example the "molecular" field on the k^{th} subsystem is

$\varepsilon(k)\langle\sigma_k^z\rangle + \sum v(k,k')\langle\sigma_{k'}^x\rangle$ if the $\langle b_k\rangle$'s are chosen real. The zero mass is then the aforementioned rotation which for this ground state are excitations which are linear combinations of σ_k^y. There are also fermionic excitations which are massive. Their mass corresponds to the energy necessary to break up a Cooper pair. It is given by $\left[(\varepsilon_k - \varepsilon_F)^2 + H_k^2\right]^{1/2}$ where H_k is the "transverse" molecular field on $\vec{\sigma}_k$ given by $v(k,k')\left[\langle b_{k'}\rangle + \left\langle b_{k'}^+\right\rangle\right]$.

In the above model the interaction $v(k,k')$ is a small attractive force that issues from exchange of phonons (lattice waves) among the electrons. In addition there is a much stronger force due to Coulomb interaction. Whereas Bardeen Cooper Shrieffer [3] worked only with the former, Anderson [4] and Nambu [5] analysed the effects of the latter. The fermionic mass is essentially unaffected, but the collective mode is completely modified so as to become the massy plasmon. It is a "longitudinal" photon. There are also massy "transverse" photons. These give rise to the Meissner effect and the flux tubes of type II superconductors. The transverse and longitudinal masses are unequal since their origins differ dynamically. The plasmon uses the total electron density whereas the transverse photons refers to the condensate (i.e. the $\langle b_k\rangle$).

These effects were the precursor of the BEH mechanism which is studied in "Modern Times". Then because there is longitudinal and transverse isotropy in the quantum relativistic quantum field vacuum, there is only one mass.

VII. Spontanously Broken Chiral Symmetry(SBχS)

One of the first exercises for students in field theory is the calculation of the electron's self mass Δm, in QED with the result to $O(e^2)$

$$\Delta m \sim e^2 m_0 \ln(\Lambda/m_0)\,, \tag{63}$$

where m_0 is the bare mass, Λ the cut-off. The important point is that $\Delta m = 0$, if $m_0 = 0$. It is this circumstance that reduces the divergence of Δm from the naïve expectation that is linear in Λ to logarithmic. One says that the mass is "protected" by chiral symmetry. Chiral symmetry for a single fermion field is invariance of the action under

$$\Psi \to e^{i\alpha\gamma_5}\Psi\,. \tag{64}$$

Whereas under normal (global) gauge transformations the L and R components transform the same way (where $L, R = [(1 \pm \gamma_5)/2]\,\Psi$) they transform with opposite signs under the chiral gauge transformation.

Since $\overline{\Psi} = \Psi^+\gamma_0$ one has under Eq.(64) $\overline{\Psi} \rightarrow \overline{\Psi} e^{i\alpha\gamma_5}$. In consequence

$$\begin{aligned} \overline{\Psi}\Psi &\rightarrow (\cos 2\alpha)\,\overline{\Psi}\Psi + (\sin 2\alpha)\,\overline{\Psi} i\gamma_5\Psi \\ \overline{\Psi} i\gamma_5\Psi &\rightarrow (-\sin 2\alpha)\overline{\Psi} i\gamma_5\Psi + (\cos 2\alpha)\overline{\Psi}\Psi\,. \end{aligned} \tag{65}$$

Here γ_5 is hermitian with $(\gamma_5)^2 = 1$ and $\{\gamma_5, \gamma_\mu\} = 0$. Thus under Eq.(65) the couple $\left(\overline{\Psi}\Psi, \overline{\Psi}(i\gamma_5)\Psi\right)$ transforms as a vector under chiral transformations i.e. it rotates in the "chiral gauge plane" with angle (2α).

Whereas the electromagnetic interaction, as well the kinetic term in the action are chiral invariants (since $\{\gamma_5, \gamma_\mu\} = 0$) thereby securing the invariance of $\overline{\Psi}\gamma_\mu\Psi$, the mass term $(m_0\overline{\Psi}\Psi)$ is not, due to Eq.(65). One consequence is that every term in pertubation theory gives $m = 0$ if $m_0 = 0$. This is easily checked by making the count of the number of γ matrices appearing in vertices and fermion propagators. It is odd and the trace of such a term vanishes. A mass term appearing in the self energy is calculated by taking the trace. We shortly give a more synthetic demonstration of this fact from the chiral Ward identity.

Inspired by the BCS theory of superconductivity, wherein a mass gap was derived non perturbatively (through Cooper bound state formation), Nambu [6] showed that the same could arise in quantum field theory, the price being the existence of a dynamically generated pseudoscalar meson, which he then identified with the pion. During this same period, Gell-Mann and Levy [7] proposed a chiral invariant action which contained scalar and pseudoscalar fields coupled to the fermion (Yukawa coupling). SBχS was first generated through the bosonic action (effective potential method) wherein the scalar picked up an expectation value and the pseudoscalar had zero mass in consequence of SBS kinematics. At low momentum scales the two methods give equivalent physical results, whereas at large momenta the composite character of the effective boson fields in Nambu's methods could give a considerable modification of the dynamics, so as to augment the width of the massy scalar (i.e. the scalar which corresponds to the stretching mode or longitudinal susceptibility in the magnetic case).

It is the Gell-Mann Lévy phenomenological approach which until the present time has prevailed in standard model research in the implementation of the BEH mechanism. Research beyond the standard model is so tenuous that all avenues must be considered open. One also must bear in mind that the original dynamical mechanism of SBχS of Nambu Jona-Lasinio is now supplanted by the QCD confinement mechanism. In this case the zitterbewegung of quarks at the end of electric flux tubes (the model for mesons) provides for the "constituent" quark mass. The chiral symmetry of QCD then implies the existence of pions. These have zero mass if the "current" quark mass is zero and have a $(mass)^2$ proportional to the latter when it is not zero. An exception is the ninth pseudoscalar of the eightfold way which has mass due to an anomaly. The origin of quark and lepton masses in terms of some ultimate chiral, super or GUTS symmetry remains elusive, the Yukawa coupling in standard model reseach being most likely the phenomenological expression of a deeper theory.

In this review we shall adhere to Nambu's original approach since it carries a pedagogical message of both power and elegance. We first review the simple non perturbative approach of Nambu Jona-Lasinio [8], not that it need be of direct applicability, but rather because it sets the scene for more general considerations.

Consider a chiral invariant four point interaction (such as $\lambda\left[\overline{\Psi}\gamma_\mu\Psi\right]^2$) or its Fierz equivalent $\lambda\left[(\overline{\Psi}\Psi)^2-(\overline{\Psi}\gamma_5\Psi)\right]^2$. In lowest order the fermion self energy, Σ, is given by the graph (Fig.7)

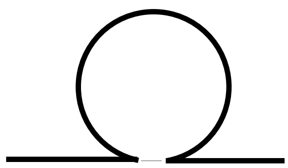

Fig. 7

where one may imagine some non locality over a distance Λ^{-1} at the vertex (say due to the exchange of a very heavy meson) supplies a U-V cut-off. Then

$$\Sigma(p)\sim g\int^\Lambda \Gamma\frac{d^4k}{\gamma^\mu(k+p)_\mu}\Gamma\,, \tag{66}$$

where irrelevant factors of $O(1)$ are dropped and Γ are the relevant γ matrices (e.g. take γ_μ for definiteness). Then $tr\Sigma=0$ and $\Delta m=0$.

Let us now make this self consistent by iterating in the fermion propagator so that

$$\Sigma(p) \sim g \int^{\Lambda} d^4k \, \Gamma \frac{1}{\gamma^{\mu}(k+p)_{\mu} - \Sigma(k+p)} \Gamma \, . \tag{67}$$

This corresponds to an infinite sum of graphs often called rainbow graphs i.e a rainbow is built on every fermion line ad infinitum. (It is amusing that one can build the Weiss molecular field of ferromagnetism using exactly the same set of graphs in a field theory which is equivalent to the original spin problem).

Let $\Sigma = A(p)\gamma^{\mu}p_{\mu} + M(p)$ and take the trace to give

$$M(p) \sim g \int^{\Lambda} d^4k \, \frac{M(p+k)}{(p+k)^2 - M^2(p+k)} \, , \tag{68}$$

where we have not included the effect due to the form factor A. The ensuing integral equation is difficult to solve but the ideas are brought out setting $M(p) = M=$ constant so as to give an eigenvalue equation for M. Taking into account factors for i, one gets a solution by making a Wick rotation provided the force is attractive ($g < 0$). This is the equivalent to the gap equation in superconductors.

Of particular interest in particle theory is the accompanying pseudoscalar. Note the SBχS; one could have taken M as a linear combination of $M_1 + iM_2\gamma_5$ with $M^2 = M_1^2 + M_2^2$. Choosing $M_2 = 0$ is a choice of direction in the chiral gauge plane, along the axis $\overline{\Psi}\Psi$. Then $\overline{\Psi}\gamma_5\Psi$ should propagate with zero mass. It does as seen from its propagator $(\sim 1/1 - g\Pi)$ (Fig 8)

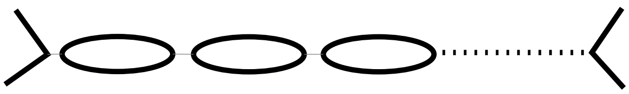

Fig. 8

$$\Pi(p) = tr \int d^4k \, \Gamma \frac{1}{\gamma^{\mu}(p+k)_{\mu} - M} \Gamma \frac{1}{\gamma^{\mu}k_{\mu} - M} \, . \tag{69}$$

At $q = 0$ one has

$$\Pi(0) \simeq \int \frac{d^4k}{k^2 - M^2} \, , \tag{70}$$

and from the eigenvalue condition Eq.(68) one checks $1 - g\Pi(0) = 0$. The diligent reader may check all the kinematic factors of $O(1)$. Thus the propagator of $\overline{\Psi}\gamma_5\Psi$ at $q = 0$ has a pole and one may check (for example by dispersion relations) that it corresponds to a pole at $q^2 = 0$ in the more general case when $q_\mu \neq 0$. (Here $q^2 \equiv q_0^2 - \underline{q}^2$)

This result is general, powerful and independent of approximations that have been made. That is the true powerful accomplishement of Nambu which we now present.

The chiral Ward identity, established in the same way as the usual vector Ward identity through use of the symmetry of the action under chiral transformations, reads

$$\lim_{q_\mu \to 0} q_\mu \Gamma_{\mu 5} = \gamma_5 \Sigma(p+q) + \Sigma(p)\gamma_5 \,. \tag{71}$$

$\Gamma_{\mu 5}$ is the vertex function formed from the chiral sources of momentum q_μ which scatters a fermion from p_μ to $p_\mu + q_\mu$. As $q_\mu \to 0$, one sees that the form factor $A(p)$ drops out of Eq.(71) (since $\{\gamma_5, \gamma_\mu\} = 0$). Whence

$$\lim_{q_\mu \to 0} q_\mu \Gamma_{\mu 5} = 2M(p)\gamma_5 \quad , \quad \Gamma_{\mu 5} \to \frac{2M(p)\gamma_5 q_\mu}{q^2} \,. \tag{72}$$

This pole at $q^2 = 0$ is the signal of a pseudoscalar meson which couples to the fermion field through the mass of the fermion.

Nambu recognized that in this way he had discovered the key to the success of the celebrated Goldberger-Treiman relation, one of the gems of particle physics in the 1950's-1960's, to which we now turn so closing out this review of the Paleolithic Age.

The original derivation by Goldberger and Treiman was based on a dispersion relation argument, involving two assumptions: an unsubtracted dispersion relation for one of the form factors occurring in the matrix element for β decay (see below) and pion dominance of the same. The quantitative success was remarkable, but there was little understanding of how to justify the assumptions. This was supplied by Nambu as follows. Whereas the Ward identity involves fields, one can also work directly with matrix elements of currents among physical states. In particular the matrix element of the axial current between nucleons is observed as the Gamow-Teller transition in β decay. Its most general form can be shown to be

$$\langle N\,|j_{\mu 5}|\,N'\rangle = F_A(q^2)\overline{u}_N(p+q)\gamma_\mu\gamma_5 u_{N'}(p) + F_p(q^2)\overline{u}_N(p+q)q_\mu\gamma_5 u_N(p)\,, \quad (73)$$

where q_μ is the 4-momentum tranfer carried by $j_{\mu 5}$.

Let us suppose that $\partial_\mu j_{\mu 5} = 0$ (i.e. chiral invariance). Then taking the divergence of Eq.(73) gives

$$0 = \left[(m_N + m_{N'})F_A(q^2) + q^2 F_p(q^2)\right]\left[\overline{u}_N(p+q)\gamma_5 u_N(p)\right]\,, \quad (74)$$

where we have used $\{\gamma_5, \gamma_\mu\} = 0$ and the Dirac equation $(\gamma^\mu p_\mu - M)\Psi = 0$. As $q^2 \to 0$, one finds

$$F_p(q^2) \to_{q^2\to 0} (m_N + m_{N'})\frac{1}{q^2}F_A(0)\,, \quad (75)$$

i.e. $F(p)$ has a pole which like $\Gamma_{\mu 5}$ (Eq.(72)) has residue proportional to the fermion mass, here the nucleon. Eqs. (74) and (75) have the interpretation given by the graphical structure for F_p (Fig.9)

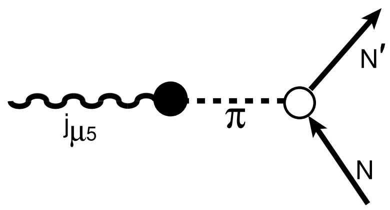

Fig. 9

The dark circle in the drawing is $\langle 0\,|j_{\mu 5}|\,\pi\rangle = -if_\pi q_\mu$. The dotted line is the pion propagator ($= 1/q^2$) and the light circle is $g_{\pi NN'}\overline{u}_{N'}\gamma_5 u_N$, where $g_{\pi NN'}$ is the pion nucleon coupling constant. The residue condition implied by Eq.(75) is

$$f_\pi g_{\pi NN'} = (m_N + m_{N'})F_A\,. \quad (76)$$

All quantities are measured; f_π is the pion decay constant into leptons, F_A the Gamow-Teller β decay constant, $g_{\pi NN'}$ is found from πN scattering. Agreement is found to $O(2\%)$. The deviation is attributed to the fact that

$\partial_\mu j_{\mu 5}$ does not quite vanish and this is reflected by the fact that $m_\pi^2 \neq 0$. Rather

$$\langle 0 \left| \partial_\mu j_{\mu 5} \right| \pi \rangle = q^2 f_\pi = m_\pi^2 f_\pi \, , \tag{77}$$

since the pion is "on shell". Nambu attributed $m_\pi^2 \neq 0$, (but small on the scale of hadron physics) to a small external breaking of chiral invariance induced by a bare mass, m_0. He then showed that $m_\pi^2 \sim m_0 M$ where $M=$ hadron scale ($m_0 \sim 10Mev, M \sim 1Gev, m_\pi \sim 130Mev$). Note the analogy to Eq.(55).

Now return to Eqs.(73), (74) and (75). The l.h.s. of Eq.(74) now contains the non vanishing value $\langle N \left| \partial_\mu j_{\mu 5} \right| N' \rangle$. Nevertheless the residue relation Eq.(76) should not change significantly. The $(momentum\ transfer)^2$ in $g_{\pi N N'}$ and in F_A are now shifted by $O(m_\pi^2)$. Therefore Eq.(76) ought to hold good at the 1% level. The corrections will be encoded in the contribution of high mass states not included in the pion dominance estimate of F_p. Assuming this true, one replaces Eq.(75) by

$$F_p(q) \simeq (m_N + m_{N'}) \frac{1}{q^2 - m_\pi^2} F_A(0) \, , \tag{78}$$

then yields Eq (76). This is the famous principle of PCAC wherein pion matrix elements are related to matrix elements of the axial current. See reference [9] for the phenomenology development of soft pion physics. When united with the Gell-Mann current algebra it becomes a very powerful tool which interrelates all kinds of hadronic phenomena, thus becoming one of the dominant elements of particle physics throughout the 1960's and early 70's. The success of the whole development bit by bit led to the QCD quark model and confinement which are now considered the theoretical bases of hadron physics as well as hadron-lepton interactions.

In the preceedings paragraphs, we have seen the important role that SBS has played in particle physics when the symmetry that has been broken is global (the chiral group). "Modern Times" is devoted to the other important facet of this development, to wit: the BEH mechanism wherein one adds to the previous consideration the complication of gauge symmetry. This chapter lead to the electroweak unification.

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