



Pantograph

# 9

## Mechanisms with Lower Pairs

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### 9.1. Introduction

We have already discussed, that when the two elements of a pair have a surface contact and a relative motion takes place, the surface of one element slides over the surface of the other, the pair formed is known as **lower pair**. In this chapter we shall discuss such mechanisms with lower pairs.

### 9.2. Pantograph

A pantograph is an instrument used to reproduce to an enlarged or a reduced scale and as exactly as possible the path described by a given point.

It consists of a jointed parallelogram  $ABCD$  as shown in Fig. 9.1.

It is made up of bars connected by turning pairs. The bars  $BA$  and  $BC$  are extended to  $O$  and  $E$  respectively, such that

$$OA/OB = AD/BE$$

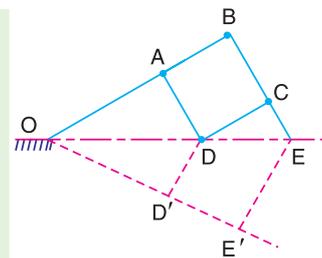


Fig. 9.1. Pantograph.

Thus, for all relative positions of the bars, the triangles  $OAD$  and  $OBE$  are similar and the points  $O, D$  and  $E$  are in one straight line. It may be proved that point  $E$  traces out the same path as described by point  $D$ .

From similar triangles  $OAD$  and  $OBE$ , we find that

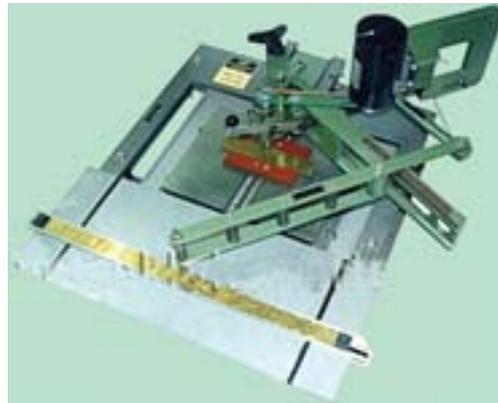
$$OD/OE = AD/BE$$

Let point  $O$  be fixed and the points  $D$  and  $E$  move to some new positions  $D'$  and  $E'$ . Then

$$OD/OE = OD'/OE'$$

A little consideration will show that the straight line  $DD'$  is parallel to the straight line  $EE'$ . Hence, if  $O$  is fixed to the frame of a machine by means of a turning pair and  $D$  is attached to a point in the machine which has rectilinear motion relative to the frame, then  $E$  will also trace out a straight line path. Similarly, if  $E$  is constrained to move in a straight line, then  $D$  will trace out a straight line parallel to the former.

A pantograph is mostly used for the reproduction of plane areas and figures such as maps, plans etc., on enlarged or reduced scales. It is, sometimes, used as an indicator rig in order to reproduce to a small scale the displacement of the crosshead and therefore of the piston of a reciprocating steam engine. It is also used to guide cutting tools. A modified form of pantograph is used to collect power at the top of an electric locomotive.



Pantograph.

### 9.3. Straight Line Mechanisms

One of the most common forms of the constraint mechanisms is that it permits only relative motion of an oscillatory nature along a straight line. The mechanisms used for this purpose are called **straight line mechanisms**. These mechanisms are of the following two types:

1. in which only turning pairs are used, and
2. in which one sliding pair is used.

These two types of mechanisms may produce exact straight line motion or approximate straight line motion, as discussed in the following articles.

### 9.4. Exact Straight Line Motion Mechanisms Made up of Turning Pairs

The principle adopted for a mathematically correct or exact straight line motion is described in Fig.9.2. Let  $O$  be a point on the circumference of a circle of diameter  $OP$ . Let  $OA$  be any chord and  $B$  is a point on  $OA$  produced, such that

$$OA \times OB = \text{constant}$$

Then the locus of a point  $B$  will be a straight line perpendicular to the diameter  $OP$ . This may be proved as follows:

Draw  $BQ$  perpendicular to  $OP$  produced. Join  $AP$ . The triangles  $OAP$  and  $OBQ$  are similar.

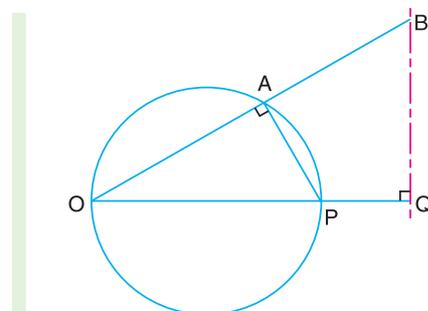


Fig. 9.2. Exact straight line motion mechanism.

$$\begin{aligned} \therefore \quad & \frac{OA}{OP} = \frac{OQ}{OB} \\ \text{or} \quad & OP \times OQ = OA \times OB \\ \text{or} \quad & OQ = \frac{OA \times OB}{OP} \end{aligned}$$

But  $OP$  is constant as it is the diameter of a circle, therefore, if  $OA \times OB$  is constant, then  $OQ$  will be constant. Hence the point  $B$  moves along the straight path  $BQ$  which is perpendicular to  $OP$ .

Following are the two well known types of exact straight line motion mechanisms made up of turning pairs.

**1. Peaucellier mechanism.** It consists of a fixed link  $OO_1$  and the other straight links  $O_1A, OC, OD, AD, DB, BC$  and  $CA$  are connected by turning pairs at their intersections, as shown in Fig. 9.3. The pin at  $A$  is constrained to move along the circumference of a circle with the fixed diameter  $OP$ , by means of the link  $O_1A$ . In Fig. 9.3,

$$AC = CB = BD = DA ; OC = OD ; \text{ and } OO_1 = O_1A$$

It may be proved that the product  $OA \times OB$  remains constant, when the link  $O_1A$  rotates. Join  $CD$  to bisect  $AB$  at  $R$ . Now from right angled triangles  $ORC$  and  $BRC$ , we have

$$\begin{aligned} OC^2 &= OR^2 + RC^2 && \dots(i) \\ \text{and} \quad BC^2 &= RB^2 + RC^2 && \dots(ii) \end{aligned}$$

Subtracting equation (ii) from (i), we have

$$\begin{aligned} OC^2 - BC^2 &= OR^2 - RB^2 \\ &= (OR + RB)(OR - RB) \\ &= OB \times OA \end{aligned}$$

Since  $OC$  and  $BC$  are of constant length, therefore the product  $OB \times OA$  remains constant. Hence the point  $B$  traces a straight path perpendicular to the diameter  $OP$ .

**2. Hart's mechanism.** This mechanism requires only six links as compared with the eight links required by the Peaucellier mechanism. It consists of a fixed link  $OO_1$  and other straight links  $O_1A, FC, CD, DE$  and  $EF$  are connected by turning pairs at their points of intersection, as shown in Fig. 9.4. The links  $FC$  and  $DE$  are equal in length and the lengths of the links  $CD$  and  $EF$  are also equal. The points  $O, A$  and  $B$  divide the links  $FC, CD$  and  $EF$  in the same ratio. A little consideration will show that  $BOCE$  is a trapezium and  $OA$  and  $OB$  are respectively parallel to  $FD$  and  $CE$ .

Hence  $OAB$  is a straight line. It may be proved now that the product  $OA \times OB$  is constant.

\* In  $\Delta FCE$ ,  $O$  and  $B$  divide  $FC$  and  $EF$  in the same ratio, i.e.

$$CO/CF = EB/EF$$

$\therefore OB$  is parallel to  $CE$ . Similarly, in triangle  $FCD$ ,  $OA$  is parallel to  $FD$ .



A modified form of pantograph is used to collect electricity at the top of electric trains and buses.

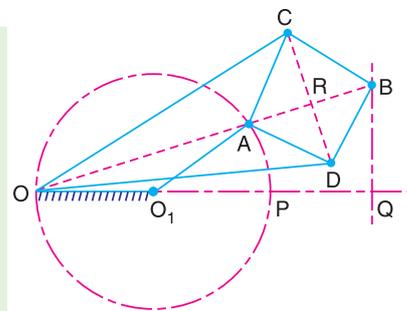


Fig. 9.3. Peaucellier mechanism.

From similar triangles  $CFE$  and  $OFB$ ,

$$\frac{CE}{FC} = \frac{OB}{OF} \quad \text{or} \quad OB = \frac{CE \times OF}{FC} \quad \dots(i)$$

and from similar triangles  $FCD$  and  $OCA$

$$\frac{FD}{FC} = \frac{OA}{OC} \quad \text{or} \quad OA = \frac{FD \times OC}{FC} \quad \dots(ii)$$

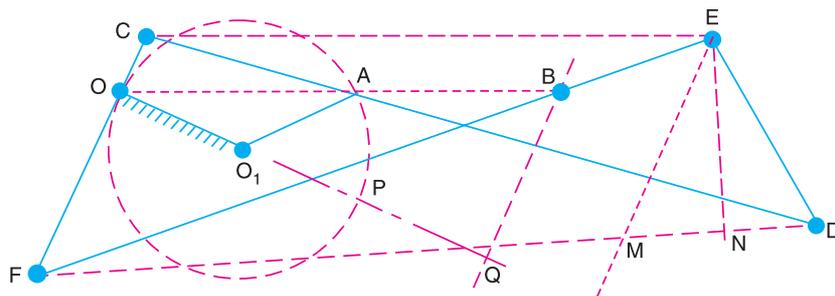


Fig. 9.4. Hart's mechanism.

Multiplying equations (i) and (ii), we have

$$OA \times OB = \frac{FD \times OC}{FC} \times \frac{CE \times OF}{FC} = FD \times CE \times \frac{OC \times OF}{FC^2}$$

Since the lengths of  $OC$ ,  $OF$  and  $FC$  are fixed, therefore

$$OA \times OB = FD \times CE \times \text{constant} \quad \dots(iii)$$

$$\dots \left( \text{substituting } \frac{OC \times OF}{FC^2} = \text{constant} \right)$$

Now from point  $E$ , draw  $EM$  parallel to  $CF$  and  $EN$  perpendicular to  $FD$ . Therefore

$$FD \times CE = FD \times FM \quad \dots(\because CE = FM)$$

$$= (FN + ND)(FN - MN) = FN^2 - ND^2 \quad \dots(\because MN = ND)$$

$$= (FE^2 - NE^2) - (ED^2 - NE^2)$$

...(From right angled triangles  $FEN$  and  $EDN$ )

$$= FE^2 - ED^2 = \text{constant} \quad \dots(iv)$$

...(because Length  $FE$  and  $ED$  are fixed)

From equations (iii) and (iv),

$$OA \times OB = \text{constant}$$

It therefore follows that if the mechanism is pivoted about  $O$  as a fixed point and the point  $A$  is constrained to move on a circle with centre  $O_1$ , then the point  $B$  will trace a straight line perpendicular to the diameter  $OP$  produced.

**Note:** This mechanism has a great practical disadvantage that even when the path of  $B$  is short, a large amount of space is taken up by the mechanism.

### 9.5. Exact Straight Line Motion Consisting of One Sliding Pair-Scott Russell's Mechanism

It consists of a fixed member and moving member  $P$  of a sliding pair as shown in Fig. 9.5.

The straight link  $PAQ$  is connected by turning pairs to the link  $OA$  and the link  $P$ . The link  $OA$  rotates about  $O$ . A little consideration will show that the mechanism  $OAP$  is same as that of the reciprocating engine mechanism in which  $OA$  is the crank and  $PA$  is the connecting rod. In this mechanism, the straight line motion is not generated but it is merely copied.

In Fig. 9.5,  $A$  is the middle point of  $PQ$  and  $OA = AP = AQ$ . The instantaneous centre for the link  $PAQ$  lies at  $I$  in  $OA$  produced and is such that  $IP$  is perpendicular to  $OP$ . Join  $IQ$ . Then  $Q$  moves along the perpendicular to  $IQ$ . Since  $OPIQ$  is a rectangle and  $IQ$  is perpendicular to  $OQ$ , therefore  $Q$  moves along the vertical line  $OQ'$  for all positions of  $QP$ . Hence  $Q$  traces the straight line  $OQ'$ . If  $OA$  makes one complete revolution, then  $P$  will oscillate along the line  $OP$  through a distance  $2 OA$  on each side of  $O$  and  $Q$  will oscillate along  $OQ'$  through the same distance  $2 OA$  above and below  $O$ . Thus, the locus of  $Q$  is a copy of the locus of  $P$ .

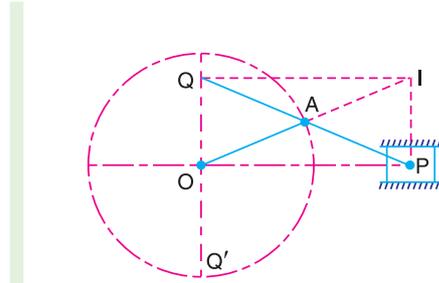


Fig. 9.5. Scott Russell's mechanism.

**Note:** Since the friction and wear of a sliding pair is much more than those of turning pair, therefore this mechanism is not of much practical value.

### 9.6. Approximate Straight Line Motion Mechanisms

The approximate straight line motion mechanisms are the modifications of the four-bar chain mechanisms. Following mechanisms to give approximate straight line motion, are important from the subject point of view :

**1. Watt's mechanism.** It is a crossed four bar chain mechanism and was used by Watt for his early steam engines to guide the piston rod in a cylinder to have an approximate straight line motion.

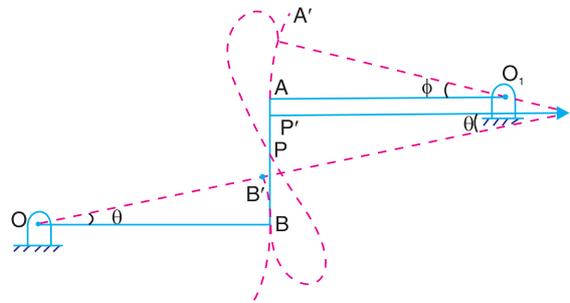


Fig. 9.6. Watt's mechanism.

In Fig. 9.6,  $OBAO_1$  is a crossed four bar chain in which  $O$  and  $O_1$  are fixed. In the mean position of the mechanism, links  $OB$  and  $O_1A$  are parallel and the coupling rod  $AB$  is perpendicular to  $O_1A$  and  $OB$ . The tracing point  $P$  traces out an approximate straight line over certain positions of its movement, if  $PB/PA = O_1A/OB$ . This may be proved as follows :

A little consideration will show that in the initial mean position of the mechanism, the instantaneous centre of the link  $BA$  lies at infinity. Therefore the motion of the point  $P$  is along the vertical line  $BA$ . Let  $OB'A'O_1$  be the new position of the mechanism after the links  $OB$  and  $O_1A$  are displaced through an angle  $\theta$  and  $\phi$  respectively. The instantaneous centre now lies at  $I$ . Since the angles  $\theta$  and  $\phi$  are very small, therefore

$$\text{arc } B B' = \text{arc } A A' \quad \text{or} \quad OB \times \theta = O_1A \times \phi \quad \dots(i)$$

$$\begin{aligned} \therefore \quad OB / O_1A &= \phi / \theta \\ \text{Also} \quad A'P' &= IP' \times \phi, \text{ and } B'P' = IP' \times \theta \\ \therefore \quad A'P' / B'P' &= \phi / \theta \end{aligned} \quad \dots(ii)$$

From equations (i) and (ii),

$$\frac{OB}{O_1A} = \frac{A'P'}{B'P'} = \frac{AP}{BP} \quad \text{or} \quad \frac{O_1A}{OB} = \frac{PB}{PA}$$

Thus, the point  $P$  divides the link  $AB$  into two parts whose lengths are inversely proportional to the lengths of the adjacent links.

**2. Modified Scott-Russel mechanism.** This mechanism, as shown in Fig. 9.7, is similar to Scott-Russel mechanism (discussed in Art. 9.5), but in this case  $AP$  is not equal to  $AQ$  and the points  $P$  and  $Q$  are constrained to move in the horizontal and vertical directions. A little consideration will show that it forms an elliptical trammel, so that any point  $A$  on  $PQ$  traces an ellipse with semi-major axis  $AQ$  and semi-minor axis  $AP$ .

If the point  $A$  moves in a circle, then for point  $Q$  to move along an approximate straight line, the length  $OA$  must be equal  $(AP)^2 / AQ$ . This is limited to only small displacement of  $P$ .

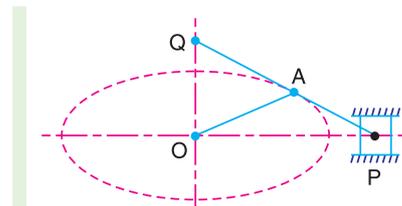


Fig. 9.7. Modified Scott-Russel mechanism.

**3. Grasshopper mechanism.** This mechanism is a modification of modified Scott-Russel's mechanism with the difference that the point  $P$  does not slide along a straight line, but moves in a circular arc with centre  $O$ .

It is a four bar mechanism and all the pairs are turning pairs as shown in Fig. 9.8. In this mechanism, the centres  $O$  and  $O_1$  are fixed. The link  $OA$  oscillates about  $O$  through an angle  $AOA_1$  which causes the pin  $P$  to move along a circular arc with  $O_1$  as centre and  $O_1P$  as radius. For small angular displacements of  $OP$  on each side of the horizontal, the point  $Q$  on the extension of the link  $PA$  traces out an approximately a straight path  $QQ'$ , if the lengths are such that  $OA = (AP)^2 / AQ$ .

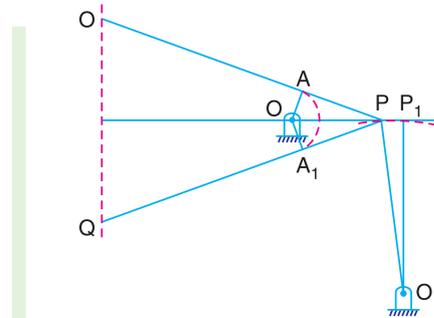


Fig. 9.8. Grasshopper mechanism.

**Note:** The Grasshopper mechanism was used in early days as an engine mechanism which gave long stroke with a very short crank.

**4. Tchebicheff's mechanism.** It is a four bar mechanism in which the crossed links  $OA$  and  $O_1B$  are of equal length, as shown in Fig. 9.9. The point  $P$ , which is the mid-point of  $AB$  traces out an approximately straight line parallel to  $OO_1$ . The proportions of the links are, usually, such that point  $P$  is exactly above  $O$  or  $O_1$  in the extreme positions of the mechanism *i.e.* when  $BA$  lies along  $OA$  or when  $BA$  lies along  $BO_1$ . It may be noted that the point  $P$  will lie on a straight line parallel to  $OO_1$ , in the two extreme positions and in the mid position, if the lengths of the links are in proportions  $AB : OO_1 : OA = 1 : 2 : 2.5$ .

**5. Roberts mechanism.** It is also a four bar chain mechanism, which, in its mean position, has the form of a trapezium. The links  $OA$  and  $O_1B$  are of equal length and  $OO_1$  is fixed. A bar  $PQ$  is rigidly attached to the link  $AB$  at its middle point  $P$ .

A little consideration will show that if the mechanism is displaced as shown by the dotted lines in Fig. 9.10, the point  $Q$  will trace out an approximately straight line.

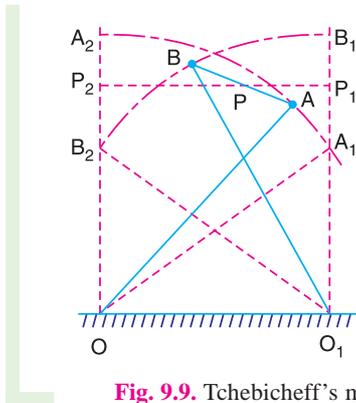


Fig. 9.9. Tehebicheff's mechanism.

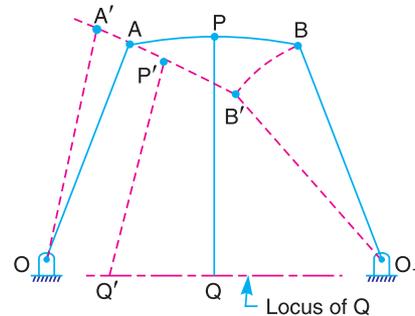
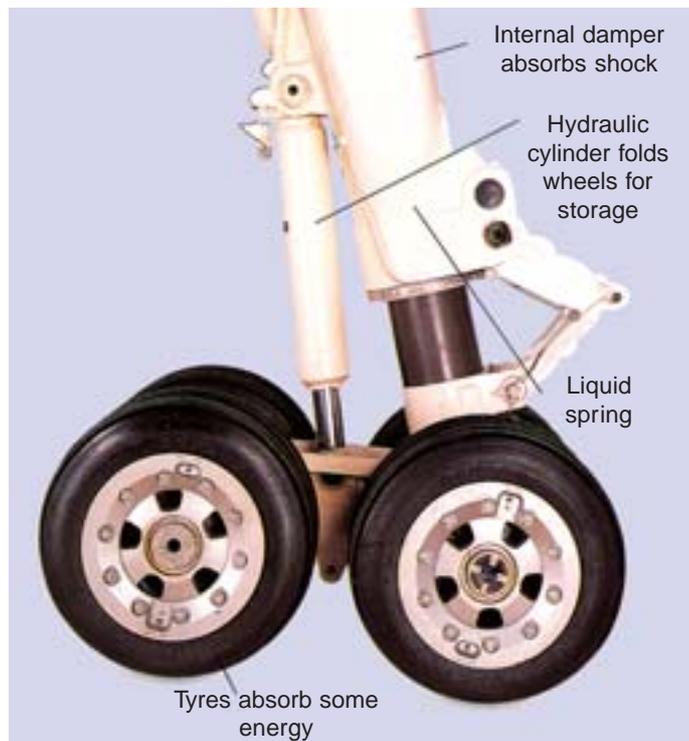


Fig. 9.10. Roberts mechanism

### 9.7. Straight Line Motions for Engine Indicators

The application of straight line motions is mostly found in the engine indicators. In these instruments, the cylinder of the indicator is in direct communication with the steam or gas inside the cylinder of an engine. The indicator piston rises and falls in response to pressure variation within the engine cylinder. The piston is resisted by a spring so that its displacement is a direct measure of the steam or gas pressure acting upon it. The displacement is communicated to the pencil which traces the variation of pressure in the cylinder (also known as indicator diagram) on a sheet of paper wrapped on the indicator drum which oscillates with angular motion about its axis, according to the motion of the engine piston. The variation in pressure is recorded to an enlarged scale. Following are the various engine indicators which work on the straight line motion mechanism.



Airplane's Landing Gear.

Note : This picture is given as additional information and is not a direct example of the current chapter.

**1. Simplex indicator.** It closely resembles to the pantograph copying mechanism, as shown in Fig. 9.11. It consists of a fixed pivot  $O$  attached to the body of the indicator. The links  $AB, BC, CD$

and  $DA$  form a parallelogram and are pin jointed. The link  $BC$  is extended to point  $P$  such that  $O$ ,  $D$  and  $P$  lie in one straight line. The point  $D$  is attached to the piston rod of the indicator and moves along the line of stroke of the piston (*i.e.* in the vertical direction). A little consideration will show that the displacement of  $D$  is reproduced on an enlarged scale, on the paper wrapped on the indicator drum, by the pencil fixed at point  $P$  which describes the path similar to that of  $D$ . In other words, when the piston moves vertically by a distance  $DD_1$ , the path traced by  $P$  is also a vertical straight line  $PP_1$ , as shown in Fig. 9.11.

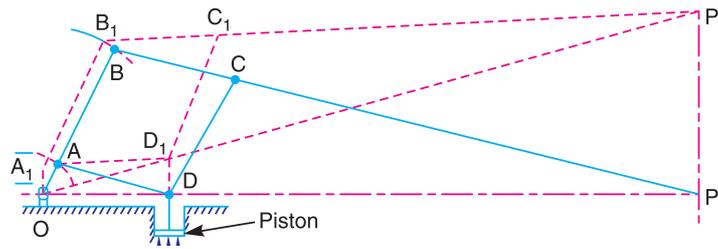


Fig. 9.11. Simplex indicator.

The magnification may be obtained by the following relation :

$$\frac{OP}{OD} = \frac{OB}{OA} = \frac{BP}{BC} = \frac{PP_1}{DD_1}$$

From the practical point of view, the following are the serious objections to this mechanism:

- (a) Since the accuracy of straight line motion of  $P$  depends upon the accuracy of motion of  $D$ , therefore any deviation of  $D$  from a straight path involves a proportionate deviation of  $P$  from a straight path.
- (b) Since the mechanism has five pin joints at  $O$ ,  $A$ ,  $B$ ,  $C$  and  $D$ , therefore slackness due to wear in any one of pin joints destroys the accuracy of the motion of  $P$ .

**2. Cross-by indicator.** It is a modified form of the pantograph copying mechanism, as shown in Fig. 9.12.

In order to obtain a vertical straight line for  $P$ , it must satisfy the following two conditions:

1. The point  $P$  must lie on the line joining the points  $O$  and  $A$ , and
2. The velocity ratio between points  $P$  and  $A$  must be a constant.

This can be proved by the instantaneous centre method as discussed below :

The instantaneous centre  $I_1$  of the link  $AC$  is obtained by drawing a horizontal line from  $A$  to meet the line  $ED$  produced at  $I_1$ . Similarly, the instantaneous centre  $I_2$  of the link  $BP$  is obtained by drawing a horizontal line from  $P$  to meet the line  $BO$  at  $I_2$ . We see from Fig. 9.12, that the points  $I_1$  and  $I_2$  lie on the fixed pivot  $O$ . Let  $v_A$ ,  $v_B$ ,  $v_C$  and  $v_P$  be the velocities of the points  $A$ ,  $B$ ,  $C$  and  $P$  respectively.

We know that 
$$\frac{v_C}{v_A} = \frac{I_1 C}{I_1 A} = \frac{I_2 C}{I_2 A} \quad \dots(i)$$

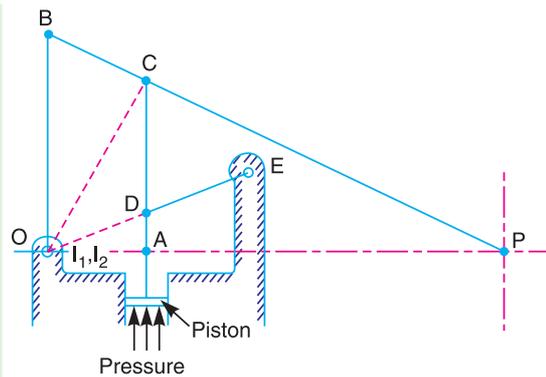


Fig. 9.12. Cross-by indicator.

and 
$$\frac{v_P}{v_C} = \frac{I_2 P}{I_2 C} \quad \dots(ii)$$

Multiplying equations (i) and (ii), we get

$$\frac{v_C}{v_A} \times \frac{v_P}{v_C} = \frac{I_2 C}{I_2 A} \times \frac{I_2 P}{I_2 C} \quad \text{or} \quad \frac{v_P}{v_A} = \frac{I_2 P}{I_2 A} = \frac{OP}{OA} \quad \dots(iii)$$

...( $\because$   $O$  and  $I_2$  are same points.)

Since  $AC$  is parallel to  $OB$ , therefore triangles  $PAC$  and  $POB$  are similar.

$$\therefore \frac{OP}{OA} = \frac{BP}{BC} \quad \dots(iv)$$

From equations (iii) and (iv),

$$\frac{v_P}{v_A} = \frac{OP}{OA} = \frac{BP}{BC} = \text{constant} \quad \dots(\because \text{Lengths } BP \text{ and } BC \text{ are constant.})$$

**3. Thompson indicator.** It consists of the links  $OB$ ,  $BD$ ,  $DE$  and  $EO$ . The tracing point  $P$  lies on the link  $BD$  produced. A little consideration will show that it constitutes a straight line motion of the Grasshopper type as discussed in Art.9.6. The link  $BD$  gets the motion from the piston rod of the indicator at  $C$  which is connected by the link  $AC$  at  $A$  to the end of the indicator piston rod. The condition of velocity ratio to be constant between  $P$  and  $A$  may be proved by the instantaneous centre method, as discussed below :

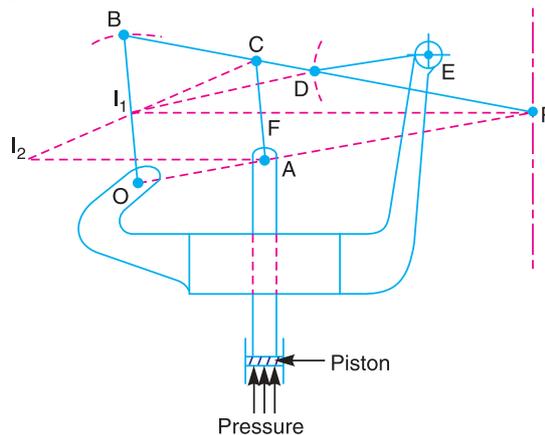


Fig. 9.13. Thompson indicator.

Draw the instantaneous centres  $I_1$  and  $I_2$  of the links  $BD$  and  $AC$  respectively. The line  $I_1 P$  cuts the links  $AC$  at  $F$ . Let  $v_A$ ,  $v_C$  and  $v_P$  be the velocities of the points  $A$ ,  $C$  and  $P$  respectively.

$$\therefore \frac{v_C}{v_A} = \frac{I_2 C}{I_2 A} \quad \dots(i)$$

From similar triangles  $I_1 CF$  and  $I_2 CA$

$$\frac{I_2 C}{I_2 A} = \frac{I_1 C}{I_1 F} \quad \text{or} \quad \frac{v_C}{v_A} = \frac{I_2 C}{I_2 A} = \frac{I_1 C}{I_1 F} \quad \dots(ii)$$

...[From equation (i)]

Also 
$$\frac{v_P}{v_C} = \frac{I_1 P}{I_1 C} \quad \dots(iii)$$

Multiplying equations (ii) and (iii), we get

$$\frac{v_C}{v_A} \times \frac{v_P}{v_C} = \frac{I_1 C}{I_1 F} \times \frac{I_1 P}{I_1 C} \quad \text{or} \quad \frac{v_P}{v_A} = \frac{I_1 P}{I_1 F} \quad \dots(iv)$$



Also 
$$\frac{v_P}{v_D} = \frac{I_1 P}{I_1 D} \quad \dots(v)$$

Multiplying equations (iii), (iv) and (v), we get

$$\frac{v_C}{v_A} \times \frac{v_D}{v_C} \times \frac{v_P}{v_D} = \frac{I_1 D}{I_1 H} \times \frac{ED}{EC} \times \frac{I_1 P}{I_1 D} \quad \text{or} \quad \frac{v_P}{v_A} = \frac{I_1 P}{I_1 H} \times \frac{ED}{EC} \quad \dots(vi)$$

From similar triangles  $I_1BP$  and  $PDH$ ,

$$\frac{I_1 P}{I_1 H} = \frac{PB}{BD}$$

$\therefore \frac{v_P}{v_A} = \frac{PB}{BD} \times \frac{ED}{EC} = \text{constant} \quad \dots[\text{From equation (vi)}]$

$\dots[\because \text{Lengths } PB, BD, ED \text{ and } EC \text{ are constant.}]$

### 9.8. Steering Gear Mechanism

The steering gear mechanism is used for changing the direction of two or more of the wheel axles with reference to the chassis, so as to move the automobile in any desired path. Usually the two back wheels have a common axis, which is fixed in direction with reference to the chassis and the steering is done by means of the front wheels.



In automobiles, the front wheels are placed over the front axles, which are pivoted at the points  $A$  and  $B$ , as shown in Fig. 9.15. These points are fixed to the chassis. The back wheels are placed over the back axle, at the two ends of the differential tube. When the vehicle takes a turn, the front wheels along with the respective axles turn about the respective pivoted points. The back wheels remain straight and do not turn. Therefore, the steering is done by means of front wheels only.

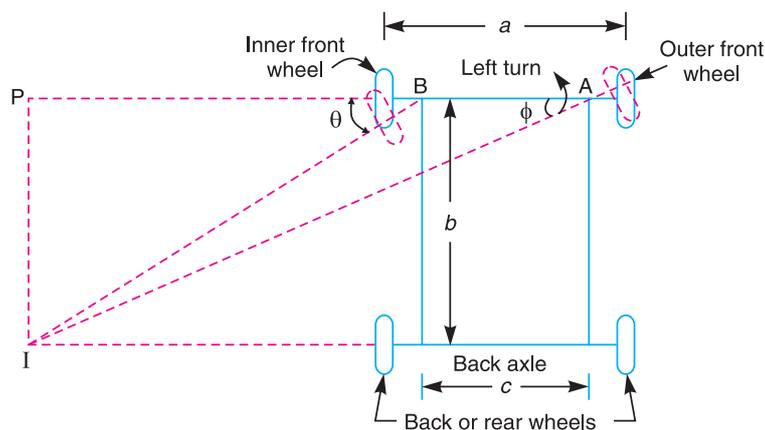


Fig. 9.15. Steering gear mechanism.

In order to avoid skidding (*i.e.* slipping of the wheels sideways), the two front wheels must turn about the same instantaneous centre  $I$  which lies on the axis of the back wheels. If the instantaneous centre of the two front wheels do not coincide with the instantaneous centre of the back wheels, the skidding on the front or back wheels will definitely take place, which will cause more wear and tear of the tyres.



Thus, the condition for correct steering is that all the four wheels must turn about the same instantaneous centre. The axis of the inner wheel makes a larger turning angle  $\theta$  than the angle  $\phi$  subtended by the axis of outer wheel.

Let  $a$  = Wheel track,  
 $b$  = Wheel base, and  
 $c$  = Distance between the pivots  $A$  and  $B$  of the front axle.

Now from triangle  $IBP$ ,

$$\cot \theta = \frac{BP}{IP}$$

and from triangle  $IAP$ ,

$$\cot \phi = \frac{AP}{IP} = \frac{AB + BP}{IP} = \frac{AB}{IP} + \frac{BP}{IP} = \frac{c}{b} + \cot \theta \quad \dots(\because IP = b)$$

$$\therefore \cot \phi - \cot \theta = c / b$$

This is the fundamental equation for correct steering. If this condition is satisfied, there will be no skidding of the wheels, when the vehicle takes a turn.

### 9.9. Davis Steering Gear

The Davis steering gear is shown in Fig. 9.16. It is an exact steering gear mechanism. The slotted links  $AM$  and  $BH$  are attached to the front wheel axle, which turn on pivots  $A$  and  $B$  respectively. The rod  $CD$  is constrained to move in the direction of its length, by the sliding members at  $P$  and  $Q$ . These constraints are connected to the slotted link  $AM$  and  $BH$  by a sliding and a turning pair at each end. The steering is affected by moving  $CD$  to the right or left of its normal position.  $C'D'$  shows the position of  $CD$  for turning to the left.

Let  $a$  = Vertical distance between  $AB$  and  $CD$ ,  
 $b$  = Wheel base,  
 $d$  = Horizontal distance between  $AC$  and  $BD$ ,  
 $c$  = Distance between the pivots  $A$  and  $B$  of the front axle.  
 $x$  = Distance moved by  $AC$  to  $AC' = CC' = DD'$ , and  
 $\alpha$  = Angle of inclination of the links  $AC$  and  $BD$ , to the vertical.

From triangle  $AA'C'$ ,

$$\tan (\alpha + \phi) = \frac{A'C'}{AA'} = \frac{d + x}{a} \quad \dots(i)$$

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From triangle  $AA'C$ ,

$$\tan \alpha = \frac{A'C}{AA'} = \frac{d}{a} \quad \dots(ii)$$

From triangle  $BB'D'$ ,

$$\tan (\alpha - \theta) = \frac{B'D'}{BB'} = \frac{d - x}{a} \quad \dots(iii)$$

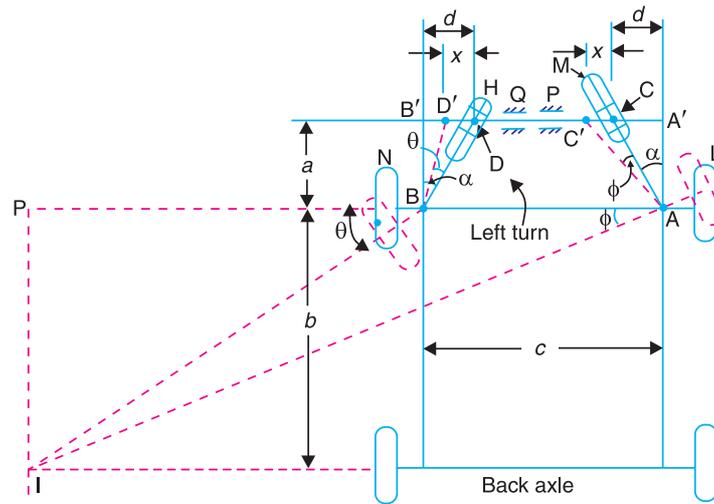


Fig. 9.16. Davis steering gear.

We know that  $\tan (\alpha + \phi) = \frac{\tan \alpha + \tan \phi}{1 - \tan \alpha \cdot \tan \phi}$

or  $\frac{d+x}{a} = \frac{d/a + \tan \phi}{1 - d/a \times \tan \phi} = \frac{d + a \tan \phi}{a - d \tan \phi}$

...[From equations (i) and (ii)]

$$(d+x)(a-d \tan \phi) = a(d+a \tan \phi)$$

$$a \cdot d - d^2 \tan \phi + a \cdot x - d \cdot x \tan \phi = a \cdot d + a^2 \tan \phi$$

$$\tan \phi (a^2 + d^2 + d \cdot x) = ax \quad \text{or} \quad \tan \phi = \frac{a \cdot x}{a^2 + d^2 + d \cdot x} \quad \dots(iv)$$

Similarly, from  $\tan (\alpha - \theta) = \frac{d-x}{a}$ , we get

$$\tan \theta = \frac{ax}{a^2 + d^2 - d \cdot x} \quad \dots(v)$$

We know that for correct steering,

$$\cot \phi - \cot \theta = \frac{c}{b} \quad \text{or} \quad \frac{1}{\tan \phi} - \frac{1}{\tan \theta} = \frac{c}{b}$$

$$\frac{a^2 + d^2 + d \cdot x}{a \cdot x} - \frac{a^2 + d^2 - d \cdot x}{a \cdot x} = \frac{c}{b} \quad \dots[\text{From equations (iv) and (v)}]$$

or  $\frac{2d \cdot x}{a \cdot x} = \frac{c}{b}$  or  $\frac{2d}{a} = \frac{c}{b}$

$\therefore 2 \tan \alpha = \frac{c}{b}$  or  $\tan \alpha = \frac{c}{2b}$  ...( $\because d/a = \tan \alpha$ )

**Note:** Though the gear is theoretically correct, but due to the presence of more sliding members, the wear will be increased which produces slackness between the sliding surfaces, thus eliminating the original accuracy. Hence Davis steering gear is not in common use.

**Example 9.1.** In a Davis steering gear, the distance between the pivots of the front axle is 1.2 metres and the wheel base is 2.7 metres. Find the inclination of the track arm to the longitudinal axis of the car, when it is moving along a straight path.

**Solution.** Given :  $c = 1.2 \text{ m}$  ;  $b = 2.7 \text{ m}$

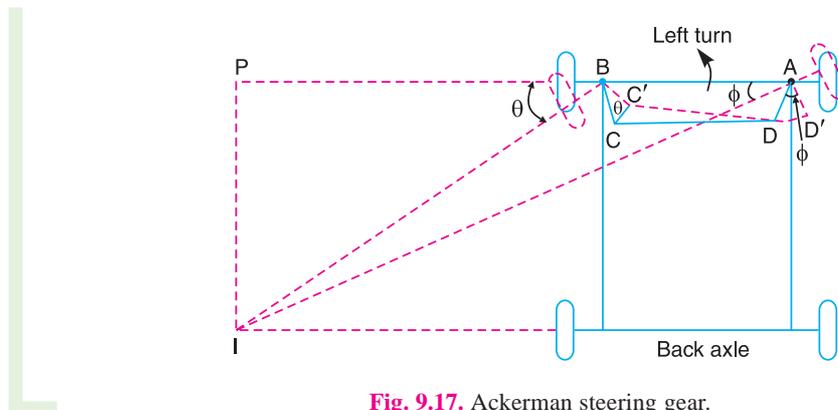
Let  $\alpha =$  Inclination of the track arm to the longitudinal axis.

We know that  $\tan \alpha = \frac{c}{2b} = \frac{1.2}{2 \times 2.7} = 0.222$  or  $\alpha = 12.5^\circ$  **Ans.**

### 9.10. Ackerman Steering Gear

The Ackerman steering gear mechanism is much simpler than Davis gear. The difference between the Ackerman and Davis steering gears are :

1. The whole mechanism of the Ackerman steering gear is on back of the front wheels; whereas in Davis steering gear, it is in front of the wheels.
2. The Ackerman steering gear consists of turning pairs, whereas Davis steering gear consists of sliding members.



**Fig. 9.17.** Ackerman steering gear.

In Ackerman steering gear, the mechanism  $ABCD$  is a four bar crank chain, as shown in Fig. 9.17. The shorter links  $BC$  and  $AD$  are of equal length and are connected by hinge joints with front wheel axles. The longer links  $AB$  and  $CD$  are of unequal length. The following are the only three positions for correct steering.

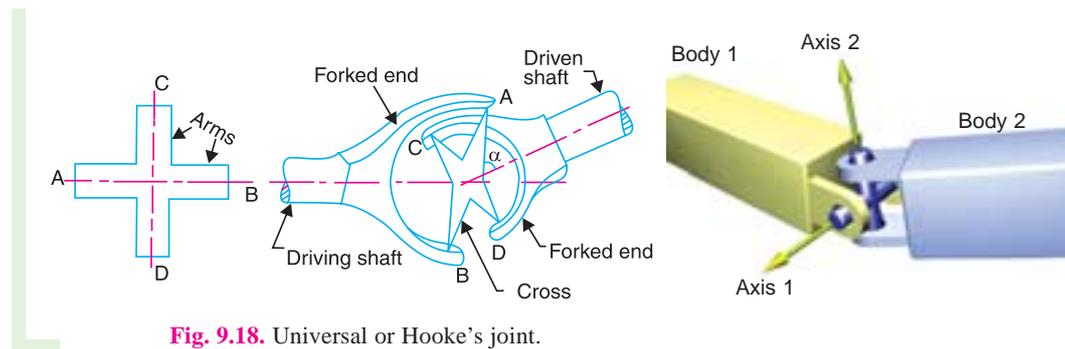
1. When the vehicle moves along a straight path, the longer links  $AB$  and  $CD$  are parallel and the shorter links  $BC$  and  $AD$  are equally inclined to the longitudinal axis of the vehicle, as shown by firm lines in Fig. 9.17.
2. When the vehicle is steering to the left, the position of the gear is shown by dotted lines in Fig. 9.17. In this position, the lines of the front wheel axle intersect on the back wheel axle at  $I$ , for correct steering.

3. When the vehicle is steering to the right, the similar position may be obtained.

In order to satisfy the fundamental equation for correct steering, as discussed in Art. 9.8, the links  $AD$  and  $DC$  are suitably proportioned. The value of  $\theta$  and  $\phi$  may be obtained either graphically or by calculations.

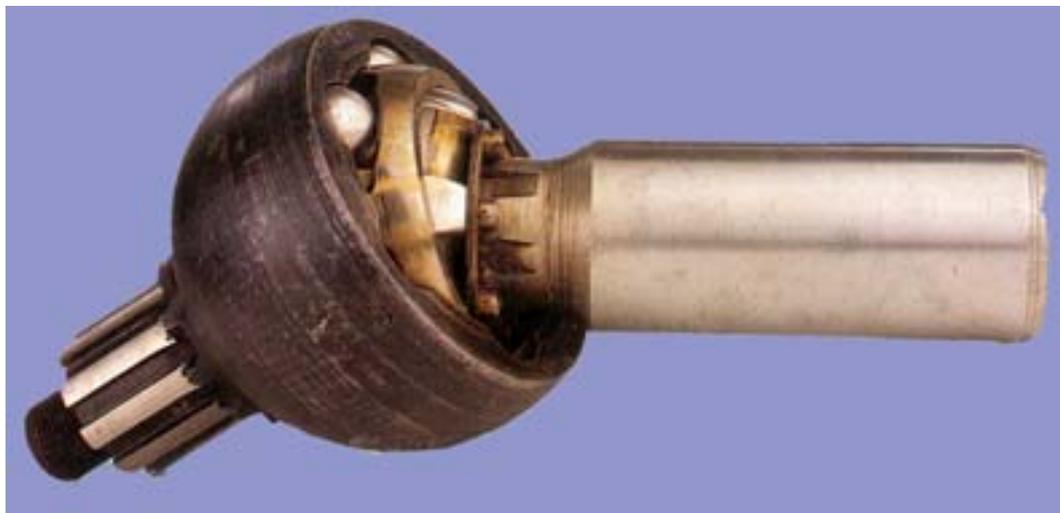
### 9.11. Universal or Hooke's Joint

A \*Hooke's joint is used to connect two shafts, which are intersecting at a small angle, as shown in Fig. 9.18. The end of each shaft is forked to U-type and each fork provides two bearings



**Fig. 9.18.** Universal or Hooke's joint.

for the arms of a cross. The arms of the cross are perpendicular to each other. The motion is transmitted from the driving shaft to driven shaft through a cross. The inclination of the two shafts may be constant, but in actual practice it varies, when the motion is transmitted. The main application of the Universal or Hooke's joint is found in the transmission from the \*\*gear box to the differential or back axle of the automobiles. It is also used for transmission of power to different spindles of multiple drilling machine. It is also used as a knee joint in milling machines.



Universal Joint.

\* This joint was first suggested by Da Vinci and was named after English physicist and mathematician Robert Hooke who first applied it to connect two offset misaligned shafts.

\*\* In case of automobiles, we use two Hooke's joints one at each end of the propeller shaft, connecting the gear box on one end and the differential on the other end.

9.12. Ratio of the Shafts Velocities

The top and front views connecting the two shafts by a universal joint are shown in Fig. 9.19. Let the initial position of the cross be such that both arms lie in the plane of the paper in front view, while the arm  $AB$  attached to the driving shaft lies in the plane containing the axes of the two shafts. Let the driving shaft rotates through an angle  $\theta$ , so that the arm  $AB$  moves in a circle to a new position  $A_1B_1$  as shown in front view. A little consideration will show that the arm  $CD$  will also move in a circle of the same size. This circle when projected in the plane of paper appears to be an ellipse. Therefore the arm  $CD$  takes new position  $C_1D_1$  on the ellipse, at an angle  $\theta$ . But the true angle must be on the circular path. To find the true angle, project the point  $C_1$  horizontally to intersect the circle at  $C_2$ . Therefore the angle  $COC_2$  (equal to  $\phi$ ) is the true angle turned by the driven shaft. Thus when the driving shaft turns through an angle  $\theta$ , the driven shaft turns through an angle  $\phi$ . It may be noted that it is not necessary that  $\phi$  may be greater than  $\theta$  or less than  $\theta$ . At a particular point, it may be equal to  $\theta$ .

In triangle  $OC_1M$ ,  $\angle OC_1M = \theta$

$$\therefore \tan \theta = \frac{OM}{MC_1} \quad \dots(i)$$

and in triangle  $OC_2N$ ,  $\angle OC_2N = \phi$

$$\therefore \tan \phi = \frac{ON}{NC_2} = \frac{ON}{MC_1}$$

Dividing equation (i) by (ii),

$$\frac{\tan \theta}{\tan \phi} = \frac{OM}{MC_1} \times \frac{MC_1}{ON} = \frac{OM}{ON}$$

But  $OM = ON_1 \cos \alpha = ON \cos \alpha$

...(where  $\alpha$  = Angle of inclination of the driving and driven shafts)

$$\therefore \frac{\tan \theta}{\tan \phi} = \frac{ON \cos \alpha}{ON} = \cos \alpha$$

or  $\tan \theta = \tan \phi \cdot \cos \alpha \quad \dots(iii)$

Let  $\omega =$  Angular velocity of the driving shaft  $= d\theta / dt$

$\omega_1 =$  Angular velocity of the driven shaft  $= d\phi / dt$

Differentiating both sides of equation (iii),

$$\sec^2 \theta \times d\theta / dt = \cos \alpha \cdot \sec^2 \phi \times d\phi / dt$$

$$\sec^2 \theta \times \omega = \cos \alpha \cdot \sec^2 \phi \times \omega_1$$

$$\therefore \frac{\omega_1}{\omega} = \frac{\sec^2 \theta}{\cos \alpha \cdot \sec^2 \phi} = \frac{1}{\cos^2 \theta \cdot \cos \alpha \cdot \sec^2 \phi} \quad \dots(iv)$$

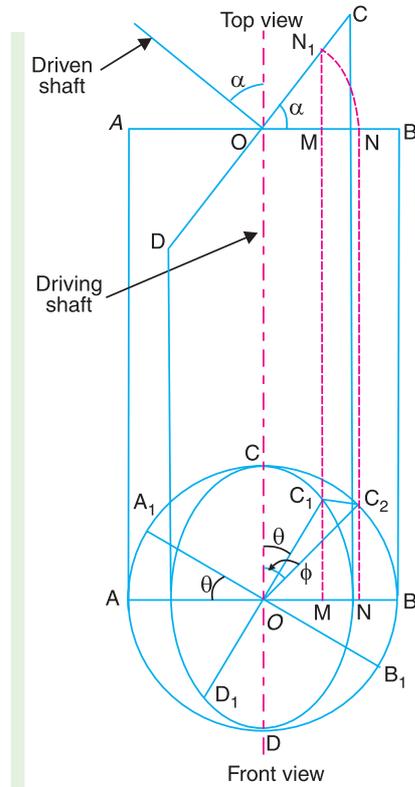


Fig. 9.19. Ratio of shafts velocities.

$$\dots(\because NC_2 = MC_1) \quad \dots(ii)$$

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We know that  $\sec^2 \phi = 1 + \tan^2 \phi = 1 + \frac{\tan^2 \theta}{\cos^2 \alpha}$  ...[From equation (iii)]

$$= 1 + \frac{\sin^2 \theta}{\cos^2 \theta \cdot \cos^2 \alpha} = \frac{\cos^2 \theta \cdot \cos^2 \alpha + \sin^2 \theta}{\cos^2 \theta \cdot \cos^2 \alpha}$$

$$= \frac{\cos^2 \theta (1 - \sin^2 \alpha) + \sin^2 \theta}{\cos^2 \theta \cdot \cos^2 \alpha} = \frac{\cos^2 \theta - \cos^2 \theta \cdot \sin^2 \alpha + \sin^2 \theta}{\cos^2 \theta \cdot \cos^2 \alpha}$$

$$= \frac{1 - \cos^2 \theta \cdot \sin^2 \alpha}{\cos^2 \theta \cdot \cos^2 \alpha} \quad \dots(\because \cos^2 \theta + \sin^2 \theta = 1)$$

Substituting this value of  $\sec^2 \phi$  in equation (iv), we have velocity ratio,

$$\frac{\omega_1}{\omega} = \frac{1}{\cos^2 \theta \cdot \cos \alpha} \times \frac{\cos^2 \theta \cdot \cos^2 \alpha}{1 - \cos^2 \theta \cdot \sin^2 \alpha} = \frac{\cos \alpha}{1 - \cos^2 \theta \cdot \sin^2 \alpha} \quad \dots(v)$$

**Note:** If

$N$  = Speed of the driving shaft in r.p.m., and

$N_1$  = Speed of the driven shaft in r.p.m.

Then the equation (v) may also be written as

$$\frac{N_1}{N} = \frac{\cos \alpha}{1 - \cos^2 \theta \cdot \sin^2 \alpha}.$$

### 9.13. Maximum and Minimum Speeds of Driven Shaft

We have discussed in the previous article that velocity ratio,

$$\frac{\omega_1}{\omega} = \frac{\cos \alpha}{1 - \cos^2 \theta \cdot \sin^2 \alpha} \quad \text{or} \quad \omega_1 = \frac{\omega \cdot \cos \alpha}{1 - \cos^2 \theta \cdot \sin^2 \alpha} \quad \dots(i)$$

The value of  $\omega_1$  will be maximum for a given value of  $\alpha$ , if the denominator of equation (i) is minimum. This will happen, when

$$\cos^2 \theta = 1, \quad \text{i.e. when } \theta = 0^\circ, 180^\circ, 360^\circ \text{ etc.}$$

$\therefore$  Maximum speed of the driven shaft,

$$\omega_{1(max)} = \frac{\omega \cos \alpha}{1 - \sin^2 \alpha} = \frac{\omega \cos \alpha}{\cos^2 \alpha} = \frac{\omega}{\cos \alpha} \quad \dots(ii)$$

or

$$N_{1(max)} = \frac{N}{\cos \alpha} \quad \dots(\text{where } N \text{ and } N_1 \text{ are in r.p.m.})$$

Similarly, the value of  $\omega_1$  is minimum, if the denominator of equation (i) is maximum. This will happen, when  $(\cos^2 \theta \cdot \sin^2 \alpha)$  is maximum, or

$$\cos^2 \theta = 0, \quad \text{i.e. when } \theta = 90^\circ, 270^\circ \text{ etc.}$$

$\therefore$  Minimum speed of the driven shaft,

$$\omega_{1(min)} = \omega \cos \alpha$$

or

$$N_{1(min)} = N \cos \alpha \quad \dots(\text{where } N \text{ and } N_1 \text{ are in r.p.m.})$$

Fig. 9.20, shows the polar diagram depicting the salient features of the driven shaft speed.

From above, we see that

1. For one complete revolution of the driven shaft, there are two points *i.e.* at  $0^\circ$  and  $180^\circ$  as shown by points 1 and 2 in Fig. 9.20, where the speed of the driven shaft is maximum and there are two points *i.e.* at  $90^\circ$  and  $270^\circ$  as shown by point 3 and 4 where the speed of the driven shaft is minimum.

2. Since there are two maximum and two minimum speeds of the driven shaft, therefore there are four points when the speeds of the driven and driver shaft are same. This is shown by points, 5,6,7 and 8 in Fig. 9.20 (See Art 9.14).

3. Since the angular velocity of the driving shaft is usually constant, therefore it is represented by a circle of radius  $\omega$ . The driven shaft has a variation in angular velocity, the maximum value being  $\omega/\cos \alpha$  and minimum value is  $\omega \cos \alpha$ . Thus it is represented by an ellipse of semi-major axis  $\omega/\cos \alpha$  and semi-minor axis  $\omega \cos \alpha$ , as shown in Fig. 9.20.

**Note:** Due to the variation in speed of the driven shaft, there will be some vibrations in it, the frequency of which may be decreased by having a heavy mass (a sort of flywheel) on the driven shaft. This heavy mass of flywheel does not perform the actual function of flywheel.

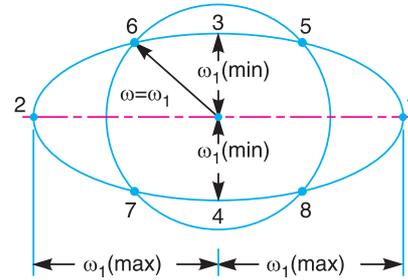


Fig. 9.20. Polar diagram-salient features of driven shaft speed.

### 9.14. Condition for Equal Speeds of the Driving and Driven Shafts

We have already discussed that the ratio of the speeds of the driven and driving shafts is

$$\frac{\omega_1}{\omega} = \frac{\cos \alpha}{1 - \cos^2 \theta \cdot \sin^2 \alpha} \quad \text{or} \quad \omega = \frac{\omega_1 (1 - \cos^2 \theta \cdot \sin^2 \alpha)}{\cos \alpha}$$

For equal speeds,  $\omega = \omega_1$ , therefore

$$\cos \alpha = 1 - \cos^2 \theta \cdot \sin^2 \alpha \quad \text{or} \quad \cos^2 \theta \cdot \sin^2 \alpha = 1 - \cos \alpha$$

and 
$$\cos^2 \theta = \frac{1 - \cos \alpha}{\sin^2 \alpha} \quad \dots(i)$$

We know that 
$$\sin^2 \theta = 1 - \cos^2 \theta = 1 - \frac{1 - \cos \alpha}{\sin^2 \alpha} = 1 - \frac{1 - \cos \alpha}{1 - \cos^2 \alpha}$$

$$= 1 - \frac{1 - \cos \alpha}{(1 + \cos \alpha)(1 - \cos \alpha)} = 1 - \frac{1}{1 + \cos \alpha} = \frac{\cos \alpha}{1 + \cos \alpha} \quad \dots(ii)$$

Dividing equation (ii) by equation (i),

$$\frac{\sin^2 \theta}{\cos^2 \theta} = \frac{\cos \alpha}{1 + \cos \alpha} \times \frac{\sin^2 \alpha}{1 - \cos \alpha}$$

or 
$$\tan^2 \theta = \frac{\cos \alpha \sin^2 \alpha}{1 - \cos^2 \alpha} = \frac{\cos \alpha \cdot \sin^2 \alpha}{\sin^2 \alpha} = \cos \alpha$$

$$\therefore \tan \theta = \pm \sqrt{\cos \alpha}$$

There are two values of  $\theta$  corresponding to positive sign and two values corresponding to negative sign. Hence, there are four values of  $\theta$ , at which the speeds of the driving and driven shafts are same. This is shown by point 5, 6, 7 and 8 in Fig. 9.20.

### 9.15. Angular Acceleration of the Driven Shaft

We know that 
$$\omega_1 = \frac{\omega \cos \alpha}{1 - \cos^2 \theta \cdot \sin^2 \alpha} = \omega \cdot \cos \alpha (1 - \cos^2 \theta \cdot \sin^2 \alpha)^{-1}$$

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Differentiating the above expression, we have the angular acceleration of the driven shaft,

$$\begin{aligned}\frac{d\omega_1}{dt} &= \omega \cos \alpha \left[ -1(1 - \cos^2 \theta \sin^2 \alpha)^{-2} \times (2 \cos \theta \sin \theta \sin^2 \alpha) \right] \frac{d\theta}{dt} \\ &= \frac{-\omega^2 \cos \alpha \times \sin 2\theta \cdot \sin^2 \alpha}{(1 - \cos^2 \theta \sin^2 \alpha)^2} \quad \dots(i)\end{aligned}$$

...(  $2 \cos \theta \sin \theta = \sin 2\theta$ , and  $d\theta/dt = \omega$  )

The negative sign does not show that there is always retardation. The angular acceleration may be positive or negative depending upon the value of  $\sin 2\theta$ . It means that during one complete revolution of the driven shaft, there is an angular acceleration corresponding to increase in speed of  $\omega_1$  and retardation due to decrease in speed of  $\omega_1$ .

For angular acceleration to be maximum, differentiate  $d\omega_1/dt$  with respect to  $\theta$  and equate to zero. The result is \* approximated as

$$\cos 2\theta = \frac{\sin^2 \alpha (2 - \cos^2 2\theta)}{2 - \sin^2 \alpha}$$

**Note:** If the value of  $\alpha$  is less than  $30^\circ$ , then  $\cos 2\theta$  may approximately be written as

$$\cos 2\theta = \frac{2 \sin^2 \alpha}{2 - \sin^2 \alpha}$$

### 9.16. Maximum Fluctuation of Speed

We know that the maximum speed of the driven shaft,

$$\omega_{1(max)} = \omega / \cos \alpha$$

and minimum speed of the driven shaft,

$$\omega_{1(min)} = \omega \cos \alpha$$

∴ Maximum fluctuation of speed of the driven shaft,

$$\begin{aligned}q &= \omega_{1(max)} - \omega_{1(min)} = \frac{\omega}{\cos \alpha} - \omega \cos \alpha \\ &= \omega \left( \frac{1}{\cos \alpha} - \cos \alpha \right) = \omega \left( \frac{1 - \cos^2 \alpha}{\cos \alpha} \right) = \frac{\omega \sin^2 \alpha}{\cos \alpha} \\ &= \omega \tan \alpha \cdot \sin \alpha\end{aligned}$$

Since  $\alpha$  is a small angle, therefore substituting  $\cos \alpha = 1$ , and  $\sin \alpha = \alpha$  radians.

∴ Maximum fluctuation of speed

$$= \omega \cdot \alpha^2$$

Hence, *the maximum fluctuation of speed of the driven shaft approximately varies as the square of the angle between the two shafts.*

**Note:** If the speed of the driving shaft is given in r.p.m. (i.e.  $N$  r.p.m.), then in the above relations  $\omega$  may be replaced by  $N$ .

### 9.17. Double Hooke's Joint

We have seen in the previous articles, that the velocity of the driven shaft is not constant, but varies from maximum to minimum values. In order to have a constant velocity ratio of the driving and driven shafts, an intermediate shaft with a Hooke's joint at each end as shown in Fig. 9.21, is used. This type of joint is known as *double Hooke's joint*.

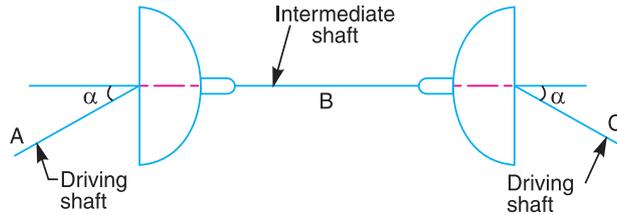
\* Since the differentiation of  $d\omega_1/dt$  is very cumbersome, therefore only the result is given.

Let the driving, intermediate and driven shafts, in the same time, rotate through angles  $\theta$ ,  $\phi$  and  $\gamma$  from the position as discussed previously in Art. 9.12.

Now for shafts A and B,  $\tan \theta = \tan \phi \cdot \cos \alpha$  ...**(i)**

and for shafts B and C,  $\tan \gamma = \tan \phi \cdot \cos \alpha$  ...**(ii)**

From equations **(i)** and **(ii)**, we see that  $\theta = \gamma$  or  $\omega_A = \omega_C$ .



**Fig. 9.21.** Double Hooke's joint.

This shows that the speed of the driving and driven shaft is constant. In other words, this joint gives a velocity ratio equal to unity, if

1. The axes of the driving and driven shafts are in the same plane, and
2. The driving and driven shafts make equal angles with the intermediate shaft.

**Example. 9.2.** Two shafts with an included angle of  $160^\circ$  are connected by a Hooke's joint. The driving shaft runs at a uniform speed of 1500 r.p.m. The driven shaft carries a flywheel of mass 12 kg and 100 mm radius of gyration. Find the maximum angular acceleration of the driven shaft and the maximum torque required.

**Solution.** Given :  $\alpha = 180^\circ - 160^\circ = 20^\circ$ ;  $N = 1500$  r.p.m.;  $m = 12$  kg ;  $k = 100$  mm = 0.1 m

We know that angular speed of the driving shaft,

$$\omega = 2 \pi \times 1500 / 60 = 157 \text{ rad/s}$$

and mass moment of inertia of the driven shaft,

$$I = m.k^2 = 12 (0.1)^2 = 0.12 \text{ kg} \cdot \text{m}^2$$

**Maximum angular acceleration of the driven shaft**

Let  $d\omega_1 / dt$  = Maximum angular acceleration of the driven shaft, and

$\theta$  = Angle through which the driving shaft turns.

We know that, for maximum angular acceleration of the driven shaft,

$$\cos 2\theta = \frac{2 \sin^2 \alpha}{2 - \sin^2 \alpha} = \frac{2 \sin^2 20^\circ}{2 - \sin^2 20^\circ} = 0.124$$

$\therefore 2\theta = 82.9^\circ$  or  $\theta = 41.45^\circ$

and

$$\begin{aligned} \frac{d\omega_1}{dt} &= \frac{\omega^2 \cos \alpha \cdot \sin 2\theta \cdot \sin^2 \alpha}{(1 - \cos^2 \theta \cdot \sin^2 \alpha)^2} \\ &= \frac{(157)^2 \cos 20^\circ \times \sin 82.9^\circ \times \sin^2 20^\circ}{(1 - \cos^2 41.45^\circ \times \sin^2 20^\circ)^2} = 3090 \text{ rad/s}^2 \text{ Ans.} \end{aligned}$$

**Maximum torque required**

We know that maximum torque required

$$= I \times d\omega_1 / dt = 0.12 \times 3090 = 371 \text{ N-m Ans.}$$

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**Example. 9.3.** The angle between the axes of two shafts connected by Hooke's joint is  $18^\circ$ . Determine the angle turned through by the driving shaft when the velocity ratio is maximum and unity.

**Solution.** Given :  $\alpha = 98^\circ$

Let  $\theta$  = Angle turned through by the driving shaft.

**When the velocity ratio is maximum**

We know that velocity ratio,

$$\frac{\omega_1}{\omega} = \frac{\cos \alpha}{1 - \cos^2 \theta \cdot \sin^2 \alpha}$$

The velocity ratio will be maximum when  $\cos^2 \theta$  is minimum, i.e. when

$$\cos^2 \theta = 1 \quad \text{or} \quad \text{when } \theta = 0^\circ \quad \text{or} \quad 180^\circ \quad \text{Ans.}$$

**When the velocity ratio is unity**

The velocity ratio ( $\omega / \omega_1$ ) will be unity, when

$$1 - \cos^2 \theta \cdot \sin^2 \alpha = \cos \alpha \quad \text{or} \quad \cos^2 \theta = \frac{1 - \cos \alpha}{\sin^2 \alpha}$$

$$\begin{aligned} \therefore \cos \theta &= \pm \sqrt{\frac{1 - \cos \alpha}{\sin^2 \alpha}} = \pm \sqrt{\frac{1 - \cos \alpha}{1 - \cos^2 \alpha}} = \pm \sqrt{\frac{1}{1 + \cos \alpha}} \\ &= \pm \sqrt{\frac{1}{1 + \cos 18^\circ}} = \pm \sqrt{\frac{1}{1 + 0.9510}} = \pm 0.7159 \\ \therefore \theta &= 44.3^\circ \quad \text{or} \quad 135.7^\circ \quad \text{Ans.} \end{aligned}$$

**Example. 9.4.** Two shafts are connected by a Hooke's joint. The driving shaft revolves uniformly at 500 r.p.m. If the total permissible variation in speed of the driven shaft is not to exceed  $\pm 6\%$  of the mean speed, find the greatest permissible angle between the centre lines of the shafts.

**Solution.** Given :  $N = 500$  r.p.m. or  $\omega = 2\pi \times 500 / 60 = 52.4$  rad/s

Let  $\alpha$  = Greatest permissible angle between the centre lines of the shafts.

Since the variation in speed of the driven shaft is  $\pm 6\%$  of the mean speed (i.e. speed of the driving speed), therefore total fluctuation of speed of the driven shaft,

$$q = 12\% \text{ of mean speed } (\omega) = 0.12 \omega$$

We know that maximum or total fluctuation of speed of the driven shaft ( $q$ ),

$$0.12 \omega = \omega \left( \frac{1 - \cos^2 \alpha}{\cos \alpha} \right) \quad \text{or} \quad \cos^2 \alpha + 0.12 \cos \alpha - 1 = 0$$

and 
$$\cos \alpha = \frac{-0.12 \pm \sqrt{(0.12)^2 + 4}}{2} = \frac{-0.12 \pm 2.0036}{2} = 0.9418$$

...(Taking + sign)

$$\alpha = 19.64^\circ \quad \text{Ans.}$$

**Example. 9.5.** Two shafts are connected by a universal joint. The driving shaft rotates at a uniform speed of 1200 r.p.m. Determine the greatest permissible angle between the shaft axes so that the total fluctuation of speed does not exceed 100 r.p.m. Also calculate the maximum and minimum speeds of the driven shaft.

**Solution.** Given :  $N = 1200$  r.p.m.;  $q = 100$  r.p.m.

**Greatest permissible angle between the shaft axes**

Let  $\alpha =$  Greatest permissible angle between the shaft axes.

We know that total fluctuation of speed ( $q$ ),

$$100 = N \left( \frac{1 - \cos^2 \alpha}{\cos \alpha} \right) = 1200 \left( \frac{1 - \cos^2 \alpha}{\cos \alpha} \right)$$

$$\therefore \frac{1 - \cos^2 \alpha}{\cos \alpha} = \frac{100}{1200} = 0.083$$

$$\cos^2 \alpha + 0.083 \cos \alpha - 1 = 0$$

$$\text{and} \quad \cos \alpha = \frac{-0.083 \pm \sqrt{(0.083)^2 + 4}}{2} = 0.9593 \quad \dots(\text{Taking } + \text{ sign})$$

$$\therefore \alpha = 16.4^\circ \quad \text{Ans.}$$

**Maximum and minimum speed of the driven shaft**

We know that maximum speed of the driven shaft,

$$N_{1(max)} = N / \cos \alpha = 1200 / 0.9593 = 1251 \text{ r.p.m.} \quad \text{Ans.}$$

and minimum speed of the driven shaft,

$$N_{1(min)} = N \cos \alpha = 1200 \times 0.9593 = 1151 \text{ r.p.m.} \quad \text{Ans.}$$

**Example 9.6.** The driving shaft of a Hooke's joint runs at a uniform speed of 240 r.p.m. and the angle  $\alpha$  between the shafts is  $20^\circ$ . The driven shaft with attached masses has a mass of 55 kg at a radius of gyration of 150 mm.

1. If a steady torque of 200 N-m resists rotation of the driven shaft, find the torque required at the driving shaft, when  $\theta = 45^\circ$ .

2. At what value of ' $\alpha$ ' will the total fluctuation of speed of the driven shaft be limited to 24 r.p.m. ?

**Solution.** Given :  $N = 240$  r.p.m or  $\omega = 2\pi \times 240/60 = 25.14$  rad/s ;  $\alpha = 20^\circ$  ;  $m = 55$  kg ;  $k = 150$  mm = 0.15 m ;  $T_1 = 200$  N-m ;  $\theta = 45^\circ$  ;  $q = 24$  r.p.m.

**1. Torque required at the driving shaft**

Let  $T' =$  Torque required at the driving shaft.

We know that mass moment inertia of the driven shaft,

$$I = m.k^2 = 55 (0.15)^2 = 1.24 \text{ kg-m}^2$$

and angular acceleration of the driven shaft,

$$\begin{aligned} \frac{d\omega_1}{dt} &= \frac{-\omega^2 \cos \alpha \cdot \sin 2\theta \cdot \sin^2 \alpha}{(1 - \cos^2 \theta \sin^2 \alpha)^2} = \frac{-(25.14)^2 \cos 20^\circ \times \sin 90^\circ \times \sin^2 20^\circ}{(1 - \cos^2 45^\circ \sin^2 20^\circ)^2} \\ &= -78.4 \text{ rad / s}^2 \end{aligned}$$

$\therefore$  Torque required to accelerate the driven shaft,

$$T_2 = I \times \frac{d\omega_1}{dt} = 1.24 \times -78.4 = -97.2 \text{ N-m}$$

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and total torque required on the driven shaft,

$$T = T_1 + T_2 = 200 - 97.2 = 102.8 \text{ N-m}$$

Since the torques on the driving and driven shafts are inversely proportional to their angular speeds, therefore

$$T' \cdot \omega = T \cdot \omega_1$$

or 
$$T' = \frac{T \cdot \omega_1}{\omega} = \frac{T \cos \alpha}{1 - \cos^2 \theta \cdot \sin^2 \alpha} \quad \dots \left( \because \frac{\omega_1}{\omega} = \frac{\cos \alpha}{1 - \cos^2 \theta \cdot \sin \alpha} \right)$$

$$= \frac{102.8 \cos 20^\circ}{1 - \cos^2 45^\circ \sin^2 20^\circ} = 102.6 \text{ N-m Ans.}$$

### 2. Value of $\alpha$ for the total fluctuation of speed to be 24 r.p.m.

We know that the total fluctuation of speed of the driven shaft ( $q$ ),

$$24 = N \left( \frac{1 - \cos^2 \alpha}{\cos \alpha} \right) = 240 \left( \frac{1 - \cos^2 \alpha}{\cos \alpha} \right)$$

or 
$$\frac{1 - \cos^2 \alpha}{\cos \alpha} = \frac{24}{240} = 0.1$$

$$\cos^2 \alpha + 0.1 \cos \alpha - 1 = 0$$

$$\cos \alpha = \frac{-0.1 \pm \sqrt{(0.1)^2 + 4}}{2} = 0.95 \quad \dots (\text{Taking + sign})$$

$\therefore \alpha = 18.2^\circ$  **Ans.**

**Example 9.7.** A double universal joint is used to connect two shafts in the same plane. The intermediate shaft is inclined at an angle of  $20^\circ$  to the driving shaft as well as the driven shaft. Find the maximum and minimum speed of the intermediate shaft and the driven shaft if the driving shaft has a constant speed of 500 r.p.m.

**Solution.** Given  $\alpha = 20^\circ$ ;  $N_A = 500$  r.p.m.

#### Maximum and minimum speed of the intermediate shaft

Let  $A$ ,  $B$  and  $C$  are the driving shaft, intermediate shaft and driven shaft respectively. We know that for the driving shaft ( $A$ ) and intermediate shaft ( $B$ ),

Maximum speed of the intermediate shaft,

$$N_{B(max)} = \frac{N_A}{\cos \alpha} = \frac{500}{\cos 20^\circ} = 532.1 \text{ r.p.m Ans.}$$

and minimum speed of the intermediate shaft,

$$N_{B(min)} = N_A \cos \alpha = 500 \times \cos 20^\circ = 469.85 \text{ r.p.m. Ans.}$$

#### Maximum and minimum speed of the driven shaft

We know that for the intermediate shaft ( $B$ ) and driven shaft ( $C$ ),

#### Maximum speed of the driven shaft,

$$N_{C(max)} = \frac{N_{B(max)}}{\cos \alpha} = \frac{N_A}{\cos^2 \alpha} = \frac{500}{\cos^2 20^\circ} = 566.25 \text{ r.p.m. Ans.}$$

and minimum speed of the driven shaft,

$$N_{C(min)} = N_{B(min)} \times \cos \alpha = N_A \cdot \cos^2 \alpha$$

$$= 500 \times \cos^2 20^\circ = 441.5 \text{ r.p.m. } \text{Ans.}$$

### EXERCISES

1. Fig. 9.22 shows the link  $GAB$  which oscillates on a fixed centre at  $A$  and the link  $FD$  on a fixed centre at  $F$ . The link  $AB$  is equal to  $AC$  and  $DB$ ,  $BE$ ,  $EC$  and  $CD$  are equal in length.

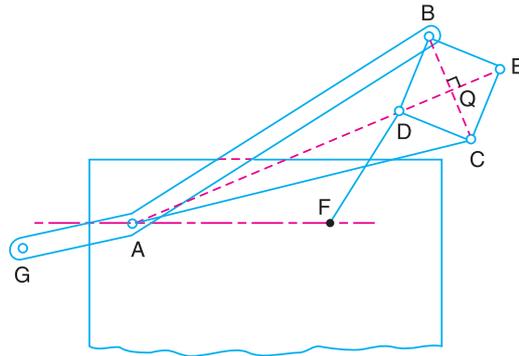
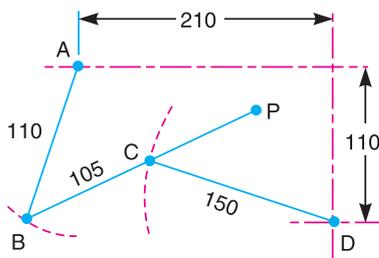


Fig. 9.22

- (a) Find the length of  $AF$  and the position of centre  $F$  so that the point  $E$  may move in a straight line.  
 (b) If the point  $E$  is required to move in a circle passing through centre  $A$ , what will be the path of point  $D$ ? [Ans.  $AF = FD$ ]

(Hint. The mechanism is similar to Peaucellier's mechanism)

2. Fig. 9.23 shows a part of the mechanism of a circuit breaker.  $A$  and  $D$  are fixed centres and the lengths of the links are :  $AB = 110$  mm,  $BC = 105$  mm, and  $CD = 150$  mm.



All dimensions in mm.

Fig. 9.23

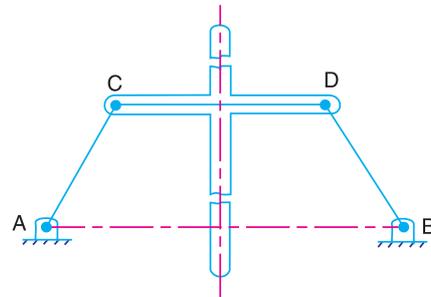


Fig. 9.24

Find the position of a point  $P$  on  $BC$  produced that will trace out an approximately straight vertical path 250 mm long.

3. The mechanism, as shown in Fig. 9.24, is a four bar kinematic chain of which the centres  $A$  and  $B$  are fixed. The lengths are :  $AB = 600$  mm,  $AC = BD = CD = 300$  mm. Find the point  $G$  on the centre line of the cross arm of which the locus is an approximately straight line even for considerable displacements from the position shown in the figure. [Ans. 400 mm.]

(Hint : It is a Robert's approximate straight line mechanism. Produce  $AC$  and  $BD$  to intersect at point  $E$ . Draw a vertical line from  $E$  to cut the centre line of cross arm at  $G$ . The distance of  $G$  from  $CD$  is the required distance).

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4. The distance between the fixed centres  $O$  and  $O_1$  of a Watt's straight line motion, as shown in Fig. 9.6, is 250 mm. The lengths of the three moving links  $OB$ ,  $BA$  and  $AO_1$  are 150 mm, 75 mm and 100 mm respectively. Find the position of a point  $P$  on  $BA$  which gives the best straight line motion.
5. A Watt's parallel motion has two bars  $OA$  and  $O'B$  pivoted at  $O$  and  $O'$  respectively and joined by the link  $AB$  in the form of a crossed four bar mechanism. When the mechanism is in its mean position, the bars  $OA$  and  $O'B$  are perpendicular to the link  $AB$ . If  $OA = 75$  mm,  $O'B = 25$  mm and  $AB = 100$  mm, find the position of the tracing point  $P$  and also find how far  $P$  is from the straight line given by the mean position of  $AB$ , when
1.  $OA$  and  $OB$  are in one straight line, and
  2.  $O'B$  and  $AB$  are in one straight line.
- [Ans. 37.5 mm, 6.5 mm, 12 mm]**
6. Design a pantograph for an indicator to obtain the indicator diagram of an engine. The distance from the tracing point of the indicator is 100 mm. The indicator diagram should represent four times the gas pressure inside the cylinder of an engine.
7. In a Davis steering gear, the distance between the pivots of the front axle is 1 metre and the wheel base is 2.5 metres. Find the inclination of the track arm to the longitudinal axis of the car, when it is moving along a straight path. **[Ans. 11.17°]**
8. A Hooke's joint connects two shafts whose axes intersect at  $150^\circ$ . The driving shaft rotates uniformly at 120 r.p.m. The driven shaft operates against a steady torque of 150 N-m and carries a flywheel whose mass is 45 kg and radius of gyration 150 mm. Find the maximum torque which will be exerted by the driving shaft. **[Ans. 187 N-m]**
- (Hint :** The maximum torque exerted by the driving shaft is the sum of steady torque and the maximum accelerating torque of the driven shaft).
9. Two shafts are connected by a Hooke's joint. The driving shaft revolves uniformly at 500 r.p.m. If the total permissible variation in speed of a driven shaft is not to exceed 6% of the mean speed, find the greatest permissible angle between the centre lines of the shafts. Also determine the maximum and minimum speed of the driven shaft. **[Ans. 19.6° ; 530 r.p.m. ; 470 r.p.m.]**
10. Two inclined shafts are connected by means of a universal joint. The speed of the driving shaft is 1000 r.p.m. If the total fluctuation of speed of the driven shaft is not to exceed 12.5% of this, what is the maximum possible inclination between the two shafts?
- With this angle, what will be the maximum acceleration to which the driven shaft is subjected and when this will occur ? **[Ans. 20.4° ; 1570 rad/s<sup>2</sup> ; 41.28°]**

### DO YOU KNOW ?

1. Sketch a pantograph, explain its working and show that it can be used to reproduce to an enlarged scale a given figure.
2. A circle has  $OR$  as its diameter and a point  $Q$  lies on its circumference. Another point  $P$  lies on the line  $OQ$  produced. If  $OQ$  turns about  $O$  as centre and the product  $OQ \times OP$  remains constant, show that the point  $P$  moves along a straight line perpendicular to the diameter  $OR$ .
3. What are straight line mechanisms ? Describe one type of exact straight line motion mechanism with the help of a sketch.
4. Describe the Watt's parallel mechanism for straight line motion and derive the condition under which the straight line is traced.
5. Sketch an intermittent motion mechanism and explain its practical applications.
6. Give a neat sketch of the straight line motion 'Hart mechanism.' Prove that it produces an exact straight line motion.
7. (a) Sketch and describe the Peaucellier straight line mechanism indicating clearly the conditions under which the point  $P$  on the corners of the rhombus of the mechanism, generates a straight line.  
(b) Prove geometrically that the above mechanism is capable of producing straight line.

8. Draw the sketch of a mechanism in which a point traces an exact straight line. The mechanism must be made of only revolute pairs. Prove that the point traces an exact straight line motion.  
(Hint. Peaucellier straight line mechanism)
9. Sketch the Dobbie-McInnes indicator mechanism and show that the displacement of the pencil which traces the indicator diagram is proportional to the displacement of the indicator piston.
10. What is the condition for correct steering ? Sketch and show the two main types of steering gears and discuss their relative advantages.
11. Explain why two Hooke's joints are used to transmit motion from the engine to the differential of an automobile.
12. Derive an expression for the ratio of shafts velocities for Hooke's joint and draw the polar diagram depicting the salient features of driven shaft speed.

### OBJECTIVE TYPE QUESTIONS

1. In a pantograph, all the pairs are
 

(a) turning pairs	(b) sliding pairs
(c) spherical pairs	(d) self-closed pairs
2. Which of the following mechanism is made up of turning pairs ?
 

(a) Scott Russel's mechanism	(b) Peaucellier's mechanism
(c) Hart's mechanism	(d) none of these
3. Which of the following mechanism is used to enlarge or reduce the size of a drawing ?
 

(a) Grasshopper mechanism	(b) Watt mechanism
(c) Pantograph	(d) none of these
4. The Ackerman steering gear mechanism is preferred to the Davis steering gear mechanism, because
 

(a) whole of the mechanism in the Ackerman steering gear is on the back of the front wheels.
(b) the Ackerman steering gear consists of turning pairs
(c) the Ackerman steering gear is most economical
(d) both (a) and (b)
5. The driving and driven shafts connected by a Hooke's joint will have equal speeds, if
 

(a) $\cos \theta = \sin \alpha$	(b) $\sin \theta = \pm \sqrt{\tan \alpha}$
(c) $\tan \theta = \pm \sqrt{\cos \alpha}$	(d) $\cot \theta = \cos \alpha$

where  $\theta$  = Angle through which the driving shaft turns, and  
 $\alpha$  = Angle of inclination of the driving and driven shafts.

### ANSWERS

- |        |             |        |        |        |
|--------|-------------|--------|--------|--------|
| 1. (a) | 2. (b), (c) | 3. (c) | 4. (d) | 5. (c) |
|--------|-------------|--------|--------|--------|