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# 14

# Gyroscopic Couple and Precessional Motion

## Features



## 14.1. Introduction

We have already discussed that,

**1.** When a body moves along a curved path with a uniform linear velocity, a force in the direction of centripetal acceleration (known as centripetal force) has to be applied externally over the body, so that it moves along the required curved path. This external force applied is known as ***active force.***

**2.** When a body, itself, is moving with uniform linear velocity along a circular path, it is subjected to the centrifugal force* radially outwards. This centrifugal force is called ***reactive force.*** The action of the reactive or centrifugal force is to tilt or move the body along radially outward direction.

**Note:** Whenever the effect of any force or couple over a moving or rotating body is to be considered, it should be with respect to the reactive force or couple and not with respect to active force or couple.

* Centrifugal force is equal in magnitude to centripetal force but opposite in direction.



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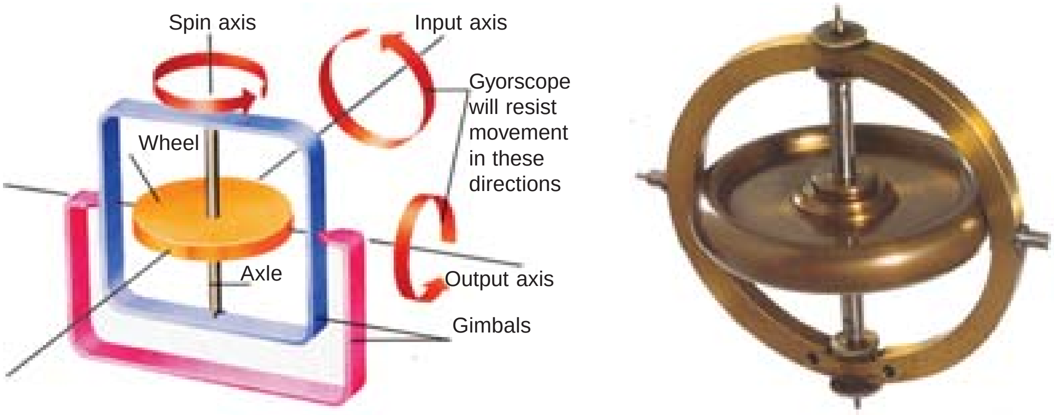



Gyroscopic inertia prevents a spinning top from falling sideways.

## 14.2. Precessional Angular Motion

We have already discussed that the angular acceleration is the rate of change of angular velocity with respect to time. It is a vector quantity and may be represented by drawing a vector diagram with the help of right hand screw rule (see chapter 2, Art. 2.13).

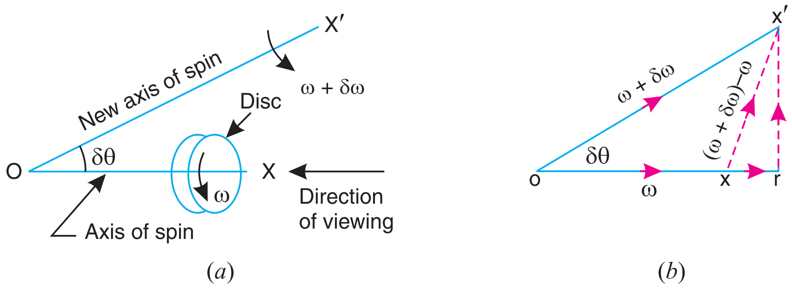



**Fig. 14.1.** Precessional angular motion.

Consider a disc, as shown in Fig. 14.1 (*a*), revolving or spinning about the axis *OX* (known as **axis of spin**) in anticlockwise when seen from the front, with an angular velocity ω in a plane at right angles to the paper.

After a short interval of time δ*t*, let the disc be spinning about the new axis of spin *OX′* (at an angle δθ) with an angular velocity (ω + δω). Using the right hand screw rule, initial angular velocity of the disc (ω) is represented by vector *ox*; and the final angular velocity of the disc (ω + δω) is represented by vector *ox′* as shown in Fig. 14.1 (*b*). The vector *xx′* represents the change of angular velocity in time δ*t* *i.e.* the angular acceleration of the disc. This may be resolved into two components, one parallel to *ox* and the other perpendicular to *ox*.

Component of angular acceleration in the direction of *ox*,

$$\alpha_t = \frac{xr}{\delta t} = \frac{or - ox}{\delta t} = \frac{ox' \cos\delta\theta - ox}{\delta t}$$

$$= \frac{(\omega + \delta\omega)\ \cos\delta\theta - \omega}{\delta t} = \frac{\omega\cos\delta\theta + \delta\omega\ \cos\delta\theta - \omega}{\delta t}$$

Since δθ is very small, therefore substituting cos δθ = 1, we have

$$\alpha_t = \frac{\omega + \delta\omega - \omega}{\delta t} = \frac{\delta\omega}{\delta t}$$

In the limit, when $\delta t \to 0$,

$$\alpha_t = \underset{\delta t \to 0}{\text{Lt}} \left(\frac{\delta\omega}{\delta t}\right) = \frac{d\omega}{dt}$$

Component of angular acceleration in the direction perpendicular to *ox*,

$$\alpha_c = \frac{rx'}{\delta t} = \frac{ox' \sin\delta\theta}{\delta t} = \frac{(\omega + \delta\omega)\sin\delta\theta}{\delta t} = \frac{\omega\sin\delta\theta + \delta\omega.\sin\delta\theta}{\delta t}$$

Since $\delta\theta$ in very small, therefore substituting $\sin\delta\theta = \delta\theta$, we have

$$\alpha_c = \frac{\omega.\delta\theta + \delta\omega.\delta\theta}{\delta t} = \frac{\omega.\delta\theta}{\delta t}$$

...(Neglecting $\delta\omega.\delta\theta$, being very small)

In the limit when $\delta t \to 0$,

$$\alpha_c = \underset{\delta t \to 0}{\text{Lt}} \frac{\omega.\delta\theta}{\delta t} = \omega \times \frac{d\theta}{dt} = \omega.\omega_P \qquad \ldots\left(\text{Substituting } \frac{d\theta}{dt} = \omega_P\right)$$

∴ Total angular acceleration of the disc

$$= \text{vector } xx' = \text{vector sum of } \alpha_t \text{ and } \alpha_c$$

$$= \frac{d\omega}{dt} + \omega \times \frac{d\theta}{dt} = \frac{d\omega}{dt} + \omega.\omega_P$$

where $d\theta/dt$ is the angular velocity of the axis of spin about a certain axis, which is perpendicular to the plane in which the axis of spin is going to rotate. This angular velocity of the axis of spin (*i.e.* $d\theta/dt$) is known as ***angular velocity of precession*** and is denoted by $\omega_P$. The axis, about which the axis of spin is to turn, is known as ***axis of precession.*** The angular motion of the axis of spin about the axis of precession is known as ***precessional angular motion.***

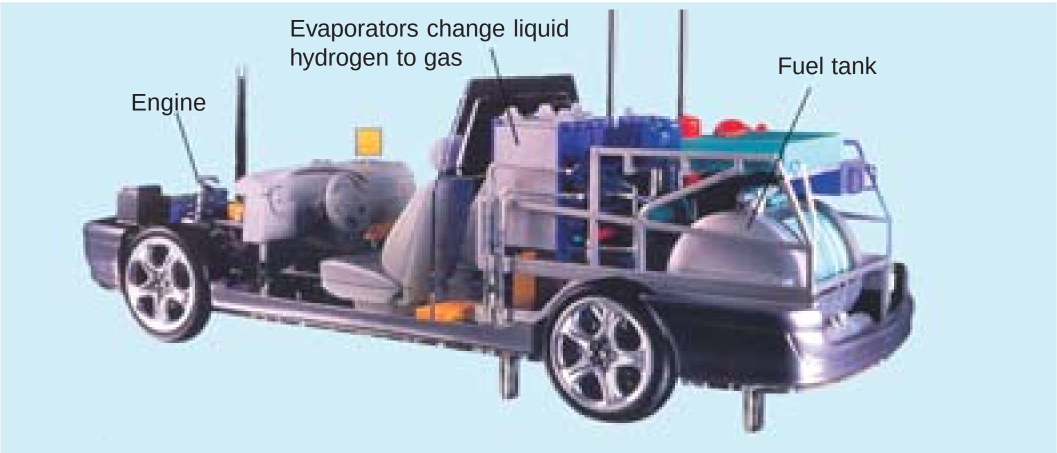
**Notes:1.** The axis of precession is perpendicular to the plane in which the axis of spin is going to rotate.

**2.** If the angular velocity of the disc remains constant at all positions of the axis of spin, then $d\theta/dt$ is zero; and thus $\alpha_c$ is zero.

**3.** If the angular velocity of the disc changes the direction, but remains constant in magnitude, then angular acceleration of the disc is given by

$$\alpha_c = \omega.d\theta/dt = \omega.\omega_P$$

The angular acceleration $\alpha_c$ is known as ***gyroscopic acceleration.***




This experimental car burns hydrogen fuel in an ordinary piston engine. Its exhaust gases cause no pollution, because they contain only water vapour.

Note : This picture is given as additional information and is not a direct example of the current chapter.

## 14.3. Gyroscopic Couple

Consider a disc spinning with an angular velocity ω rad/s about the axis of spin *OX*, in anticlockwise direction when seen from the front, as shown in Fig. 14.2 (*a*). Since the plane in which the disc is rotating is parallel to the plane *YOZ*, therefore it is called ***plane of spinning.*** The plane *XOZ* is a horizontal plane and the axis of spin rotates in a plane parallel to the horizontal plane about an axis *O Y*. In other words, the axis of spin is said to be rotating or processing about an axis *O Y*. In other words, the axis of spin is said to be rotating or processing about an axis *OY* (which is perpendicular to both the axes *OX* and *OZ*) at an angular velocity $\omega_P$ rap/s. This horizontal plane *XOZ* is called ***plane of precession*** and *O Y* is the ***axis of precession.***

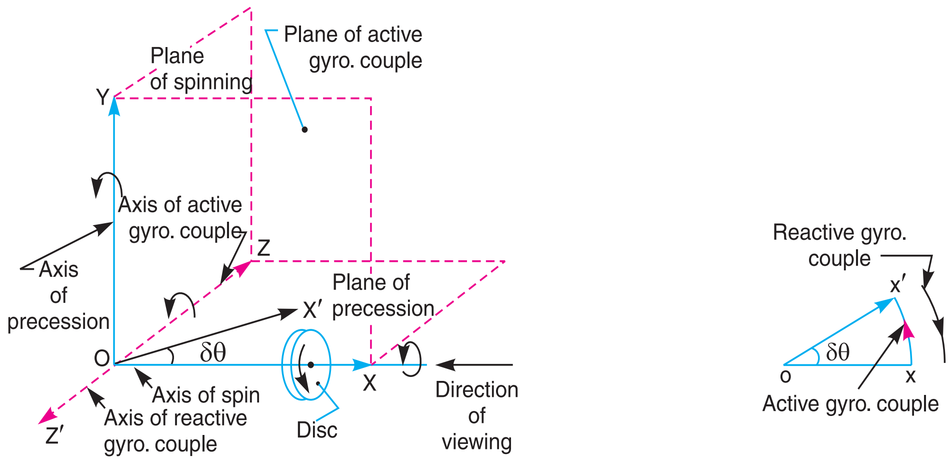
Let $I$ = Mass moment of inertia of the disc about *OX*, and

$\omega$ = Angular velocity of the disc.

∴ Angular momentum of the disc

$= I.\omega$

Since the angular momentum is a vector quantity, therefore it may be represented by the vector $\vec{ox}$, as shown in Fig. 14.2 (*b*). The axis of spin *OX* is also rotating anticlockwise when seen from the top about the axis *O Y*. Let the axis *O X* is turned in the plane *XOZ* through a small angle δθ radians to the position *OX′*, in time δ*t* seconds. Assuming the angular velocity ω to be constant, the angular momentum will now be represented by vector *ox*′.




(*a*) (*b*)

**Fig. 14.2.** Gyroscopic couple.

∴ Change in angular momentum

$$= \vec{ox'} - \vec{ox} = \vec{xx'} = \vec{ox}.\delta\theta \quad \text{...(in the direction of } \vec{xx'})$$

$$= I.\ \omega.\delta\theta$$

and rate of change of angular momentum

$$= I.\omega \times \frac{\delta\theta}{dt}$$

Since the rate of change of angular momentum will result by the application of a couple to the disc, therefore the couple applied to the disc causing precession,

$$C = \underset{\delta t \to 0}{\text{Lt}}\ I.\omega \times \frac{\delta\theta}{\delta t} = I.\omega \times \frac{d\theta}{dt} = I.\omega.\omega_P \quad \text{...}\left(\because \frac{d\theta}{dt} = \omega_P\right)$$

where $\omega_P$ = Angular velocity of precession of the axis of spin or the speed of rotation of the axis of spin about the axis of precession *O Y*.

In S.I. units, the units of *C* is N-m when *I* is in kg-m$^2$.

It may be noted that

**1.** The couple $I.\omega.\omega_p$, in the direction of the vector $xx'$ (representing the change in angular momentum) is the ***active gyroscopic couple,*** which has to be applied over the disc when the axis of spin is made to rotate with angular velocity $\omega_P$ about the axis of precession. The vector $xx'$ lies in the plane *XOZ* or the horizontal plane. In case of a very small displacement $\delta\theta$, the vector $xx'$ will be perpendicular to the vertical plane *X O Y*. Therefore the couple causing this change in the angular momentum will lie in the plane *X O Y*. The vector $xx'$, as shown in Fig. 14.2 (*b*), represents an anticlockwise couple in the plane *X O Y*. Therefore, the plane *XOY* is called the ***plane of active gyroscopic couple*** and the axis *OZ* perpendicular to the plane *X O Y*, about which the couple acts, is called the axis of active gyroscopic couple.

Above picture shows an aircraft propeller.
These rotors play role in gyroscopic couple.

**2.** When the axis of spin itself moves with angular velocity $\omega_P$, the disc is subjected to ***reactive couple*** whose magnitude is same (*i.e.* $I.\ \omega.\omega_P$) but opposite in direction to that of active couple. This reactive couple to which the disc is subjected when the axis of spin rotates about the axis of precession is known as ***reactive gyroscopic couple.*** The axis of the reactive gyroscopic couple is represented by $OZ'$ in Fig. 14.2 (*a*).

**3.** The gyroscopic couple is usually applied through the bearings which support the shaft. The bearings will resist equal and opposite couple.

**4.** The gyroscopic principle is used in an instrument or toy known as ***gyroscope.*** The gyroscopes are installed in ships in order to minimize the rolling and pitching effects of waves. They are also used in aeroplanes, monorail cars, gyrocompasses etc.

**Example 14.1.** *A uniform disc of diameter 300 mm and of mass 5 kg is mounted on one end of an arm of length 600 mm. The other end of the arm is free to rotate in a universal bearing. If the disc rotates about the arm with a speed of 300 r.p.m. clockwise, looking from the front, with what speed will it precess about the vertical axis*?

**Solution.** Given: $d$ = 300 mm or $r$ = 150 mm = 0.15 m ; $m$ = 5 kg ; $l$ = 600 mm = 0.6 m ; $N$ = 300 r.p.m. or $\omega = 2\pi \times 300/60 = 31.42$ rad/s

We know that the mass moment of inertia of the disc, about an axis through its centre of gravity and perpendicular to the plane of disc,

$$I = m.r^2/2 = 5(0.15)^2/2 = 0.056 \text{ kg-m}^2$$

and couple due to mass of disc,

$$C = m.g.l = 5 \times 9.81 \times 0.6 = 29.43 \text{ N-m}$$

Let $\omega_P$ = Speed of precession.

We know that couple (*C*),

$$29.43 = I.\omega.\omega_P = 0.056 \times 31.42 \times \omega_P = 1.76\ \omega_P$$

$$\therefore \quad \omega_P = 29.43/1.76 = 16.7 \text{ rad/s}$$ **Ans.**

**Example 14.2.** *A uniform disc of 150 mm diameter has a mass of 5 kg. It is mounted centrally in bearings which maintain its axle in a horizontal plane. The disc spins about it axle with a constant speed of 1000 r.p.m. while the axle precesses uniformly about the vertical at 60 r.p.m. The directions of rotation are as shown in Fig. 14.3. If the distance between the bearings is 100 mm, find the resultant reaction at each bearing due to the mass and gyroscopic effects.*

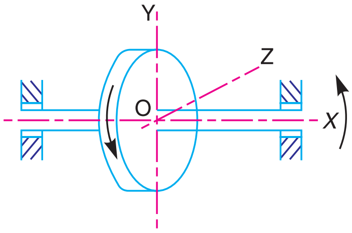



**Fig. 14.3**

**Solution.** Given: $d$ = 150 mm or $r$ = 75 mm = 0.075 m ; $m$ = 5 kg ; $N$ = 1000 r.p.m. or $\omega = 2\pi \times 1000/60$ = 104.7 rad/s (anticlockwise); $N_P$ = 60 r.p.m. or $\omega_P = 2\pi \times 60/60$ = 6.284 rad/s (anticlockwise); $x$ = 100 mm = 0.1 m

We know that mass moment of inertia of the disc, about an axis through its centre of gravity and perpendicular to the plane of disc,

$$I = m.r^2/2 = 5\,(0.075)^2/2 = 0.014 \text{ kg m}^2$$

∴ Gyroscopic couple acting on the disc,

$$C = I.\ \omega.\ \omega_P = 0.014 \times 104.7 \times 6.284 = 9.2 \text{ N-m}$$

The direction of the reactive gyroscopic couple is shown in Fig.14.4 (*b*). Let *F* be the force at each bearing due to the gyroscopic couple.

$$\therefore \quad F = C/x = 9.2/0.1 = 92 \text{ N}$$

The force *F* will act in opposite directions at the bearings as shown in Fig. 14.4 (*a*). Now let $R_A$ and $R_B$ be the reaction at the bearing *A* and *B* respectively due to the weight of the disc. Since the disc is mounted centrally in bearings, therefore,

$$R_A = R_B = 5/2 = 2.5 \text{ kg} = 2.5 \times 9.81 = 24.5 \text{ N}$$

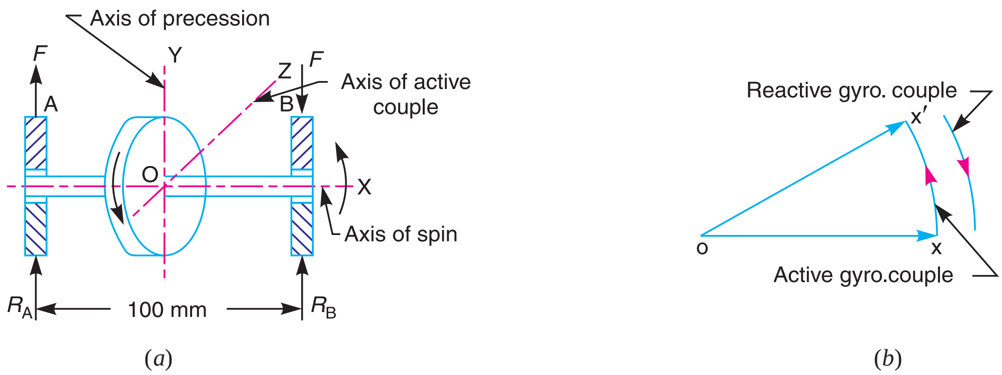



**Fig. 14.4**

***Resultant reaction at each bearing***

Let $R_{A1}$ and $R_{B1}$ = Resultant reaction at the bearings *A* and *B* respectively.

Since the reactive gyroscopic couple acts in clockwise direction when seen from the front, therefore its effect is to increase the reaction on the left hand side bearing (*i.e. A*) and to decrease the reaction on the right hand side bearing (*i.e. B*).

$$\therefore \quad R_{A1} = F + R_A = 92 + 24.5 = 116.5 \text{ N (upwards)} \quad \textbf{Ans.}$$

and $$R_{B1} = F - R_B = 92 - 24.5 = 67.5 \text{ N (downwards)} \quad \textbf{Ans.}$$

## 14.4. Effect of the Gyroscopic Couple on an Aeroplane

The top and front view of an aeroplane are shown in Fig 14.5 (*a*). Let engine or propeller rotates in the clockwise direction when seen from the rear or tail end and the aeroplane takes a turn to the left.

Let $\omega$ = Angular velocity of the engine in rad/s,

$m$ = Mass of the engine and the propeller in kg,

$k$ = Its radius of gyration in metres,

$I$ = Mass moment of inertia of the engine and the propeller in kg-m$^2$ $= m.k^2$,

$v$ = Linear velocity of the aeroplane in m/s,

$R$ = Radius of curvature in metres, and

$\omega_P$ = Angular velocity of precession $= \dfrac{v}{R}$ rad/s

$\therefore$ Gyroscopic couple acting on the aeroplane,

$$C = I.\omega.\omega_P$$

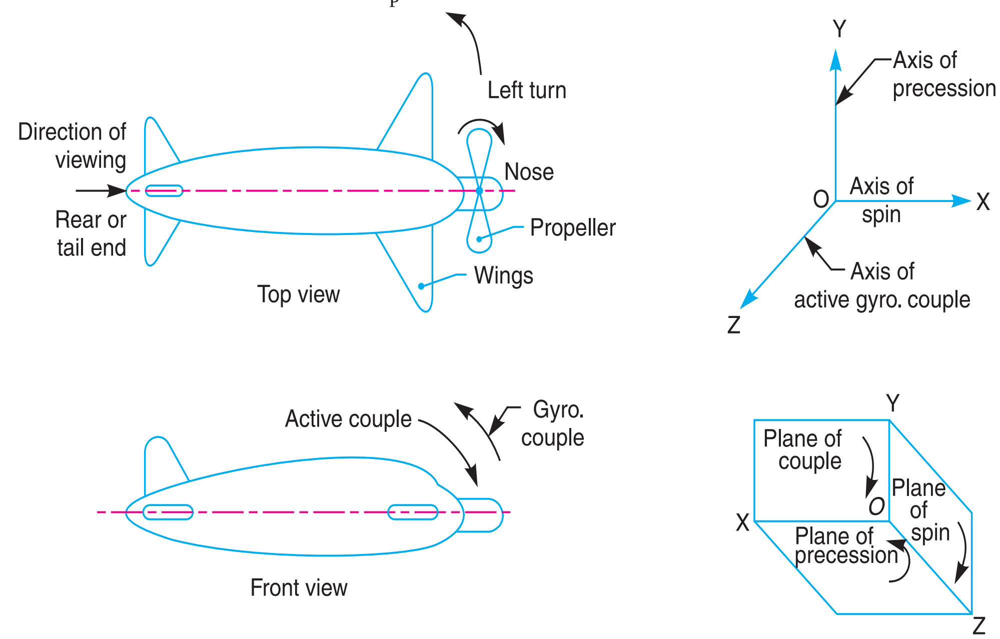
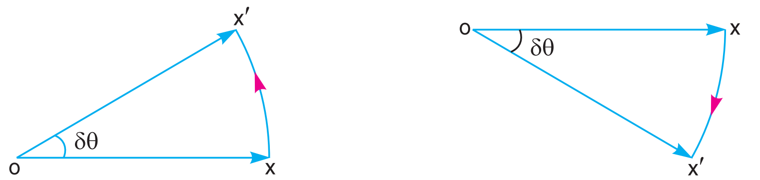



**Fig. 14.5.** Aeroplane taking a left turn.

Before taking the left turn, the angular momentum vector is represented by *ox*. When it takes left turn, the active gyroscopic couple will change the direction of the angular momentum vector from *ox* to *ox*′ as shown in Fig. 14.6 (*a*). The vector *xx*′, in the limit, represents the change of angular momentum or the active gyroscopic couple and is perpendicular to *ox*. Thus the plane of active gyroscopic couple *XOY* will be perpendicular to *xx*′ , *i.e.* vertical in this case, as shown in Fig 14.5 (*b*). By applying right hand screw rule to vector *xx*′, we find that the direction of active gyroscopic couple is clockwise as shown in the front view of Fig. 14.5 (*a*). In other words, for left hand turning, the active gyroscopic couple on the aeroplane in the axis *OZ* will be clockwise as shown in Fig. 14.5 (*b*).The reactive gyroscopic couple (equal in magnitude of active gyroscopic couple) will act in the opposite direction (*i.e.* in the anticlockwise direction) and the effect of this couple is, therefore, to **raise the nose** and **dip the tail** of the aeroplane.




(*a*) Aeroplane taking left turn. (*b*) Aeroplane taking right turn.

**Fig. 14.6.** Effect of gyroscopic couple on an aeroplane.

**Notes : 1.** When the aeroplane takes a **right turn** under similar conditions as discussed above, the effect of the reactive gyroscopic couple will be to **dip the nose** and **raise the tail** of the aeroplane.

**2.** When the engine or propeller rotates in **anticlockwise direction** when viewed from the rear or tail end and the aeroplane takes a **left turn,** then the effect of reactive gyroscopic couple will be to **dip the nose** and **raise the tail** of the aeroplane.

**3.** When the aeroplane takes a **right turn** under similar conditions as mentioned in note 2 above, the effect of reactive gyroscopic couple will be to **raise the nose** and **dip the tail** of the aeroplane.

**4.** When the engine or propeller rotates in **clockwise direction** when viewed from the front and the aeroplane takes a left turn, then the effect of reactive gyroscopic couple will be to **raise the tail** and **dip the nose** of the aeroplane.

**5.** When the aeroplane takes a **right turn** under similar conditions as mentioned in note 4-above, the effect of reactive gyroscopic couple will be to **raise the nose** and **dip the tail** of the aeroplane.

**Example 14.3.** *An aeroplane makes a complete half circle of 50 metres radius, towards left, when flying at 200 km per hr. The rotary engine and the propeller of the plane has a mass of 400 kg and a radius of gyration of 0.3 m. The engine rotates at 2400 r.p.m. clockwise when viewed from the rear. Find the gyroscopic couple on the aircraft and state its effect on it.*

**Solution.** Given : $R$ = 50 m ; $v$ = 200 km/hr = 55.6 m/s ; $m$ = 400 kg ; $k$ = 0.3 m ; $N$ = 2400 r.p.m. or $\omega = 2\pi \times 2400/60 = 251$ rad/s

We know that mass moment of inertia of the engine and the propeller,

$$I = m.k^2 = 400(0.3)^2 = 36 \text{ kg-m}^2$$

and angular velocity of precession,

$$\omega_p = v/R = 55.6/50 = 1.11 \text{ rad/s}$$

We know that gyroscopic couple acting on the aircraft,

$$C = I.\ \omega.\ \omega_p = 36 \times 251.4 \times 1.11 = 100\ 46 \text{ N-m}$$

$$= 10.046 \text{ kN-m}$$ **Ans.**

We have discussed in Art. 14.4 that when the aeroplane turns towards left, the effect of the gyroscopic couple is to lift the nose upwards and tail downwards. **Ans.**

## 14.5. Terms Used in a Naval Ship

The top and front views of a naval ship are shown in Fig 14.7. The fore end of the ship is called ***bow*** and the rear end is known as ***stern*** or ***aft***. The left hand and right hand sides of the ship, when viewed from the stern are called ***port*** and ***star-board*** respectively. We shall now discuss the effect of gyroscopic couple on the naval ship in the following three cases:

**1.** Steering, **2.** Pitching, and **3.** Rolling.

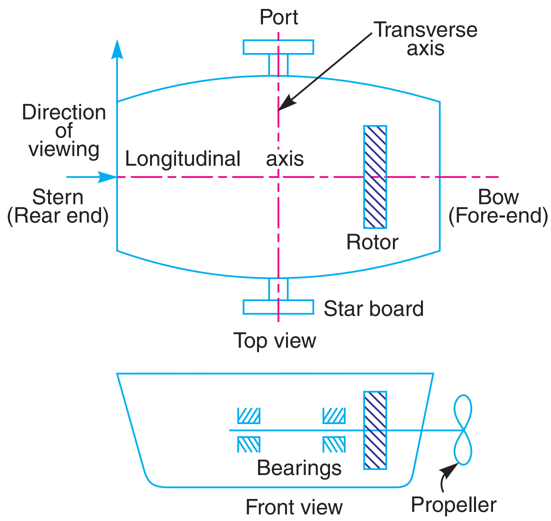



**Fig. 14.7.** Terms used in a naval ship.

## 14.6. Effect of Gyroscopic Couple on a Naval Ship during Steering

Steering is the turning of a complete ship in a curve towards left or right, while it moves forward. Consider the ship taking a left turn, and rotor rotates in the clockwise direction when viewed from the stern, as shown in Fig. 14.8. The effect of gyroscopic couple on a naval ship during steering taking left or right turn may be obtained in the similar way as for an aeroplane as discussed in Art.14.4.

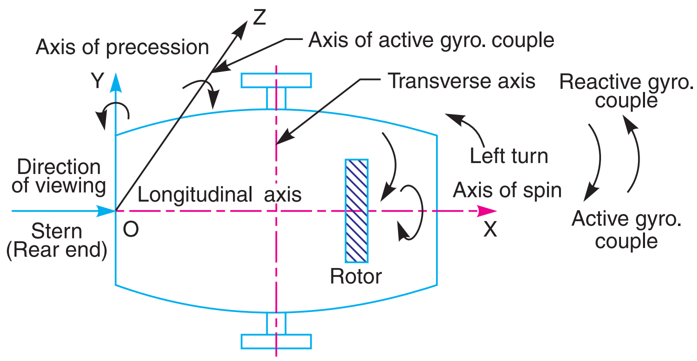



**Fig. 14.8.** Naval ship taking a left turn.

When the rotor of the ship rotates in the clockwise direction when viewed from the stern, it will have its angular momentum vector in the direction *ox* as shown in Fig. 14.9 (*a*). As the ship steers to the left, the active gyroscopic couple will change the angular momentum vector from *ox* to *ox*′. The vector *xx*′ now represents the active gyroscopic couple and is perpendicular to *ox*. Thus the plane of active gyroscopic couple is perpendicular to *xx*′ and its direction in the axis *OZ* for left hand turn is clockwise as shown in Fig. 14.8. The reactive gyroscopic couple of the same magnitude will act in the opposite direction (*i.e.* in anticlockwise direction). The ***effect of this reactive gyroscopic couple is to raise the bow and lower the stern.***


(*a*) Streeing to the left (*b*) Streeing to the right

**Fig. 14.9.** Effect of gyroscopic couple on a naval ship during steering.

**Notes: 1.** When the ship steers to the right under similar conditions as discussed above, the effect of the reactive gyroscopic couple, as shown in Fig. 14.9 (*b*), will be to **raise the stern** and **lower the bow.**

**2.** When the rotor rates in the anticlockwise direction, when viewed from the stern and the ship is steering to the left, then the effect of reactive gyroscopic couple will be to **lower the bow** and **raise the stern.**

**3.** When the ship is steering to the right under similar conditions as discussed in note 2 above, then the effect of reactive gyroscopic couple will be to **raise the bow** and **lower the stern.**

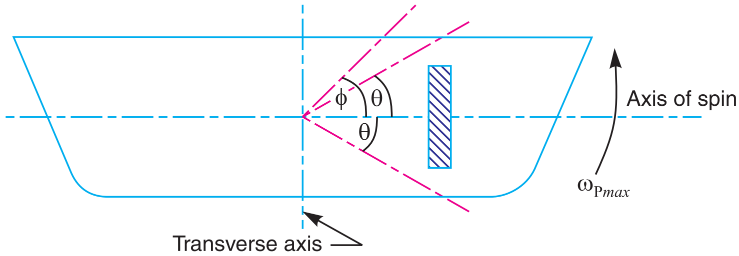
**4.** When the rotor rotates in the clockwise direction when viewed from the bow or fore end and the ship is steering to the left, then the effect of reactive gyroscopic couple will be to **raise the stern** and **lower the bow.**

**5.** When the ship is steering to the right under similar conditions as discussed in note 4 above, then the effect of reactive gyroscopic couple will be to **raise the bow** and **lower the stern.**

**6.** The effect of the reactive gyroscopic couple on a boat propelled by a turbine taking left or right turn is similar as discussed above.

## 14.7. Effect of Gyroscopic Couple on a Naval Ship during Pitching

Pitching is the movement of a complete ship up and down in a vertical plane about transverse axis, as shown in Fig. 14.10 (*a*). In this case, the transverse axis is the axis of precession. The pitching of the ship is assumed to take place with simple harmonic motion *i.e.* the motion of the axis of spin about transverse axis is simple harmonic.




(*a*) Pitching of a naval ship


(*b*) Pitching upward (*c*) Pitching downward

**Fig. 14.10.** Effect of gyroscopic couple on a naval ship during pitching.

Gryroscopic couple plays its role during ship's turning and pitching.

∴ Angular displacement of the axis of spin from mean position after time $t$ seconds,

$$\theta = \phi \sin \omega_1 . t$$

where $\phi$ = Amplitude of swing *i.e.* maximum angle turned from the mean position in radians, and

$\omega_1$ = Angular velocity of S.H.M.

$$= \frac{2\pi}{\text{Time period of S.H.M. in seconds}} = \frac{2\pi}{t_p} \text{ rad/s}$$

Angular velocity of precession,

$$\omega_P = \frac{d\theta}{dt} = \frac{d}{dt}(\phi \sin \omega_1 . t) = \phi\, \omega_1 \cos \omega_1 \, t$$

The angular velocity of precession will be maximum, if $\cos \omega_1 . t = 1$.

∴ Maximum angular velocity of precession,

$$\omega_{Pmax} = \phi . \omega_1 = \phi \times 2\pi / t_p \qquad \text{...(Substituting } \cos \omega_1 . t = 1)$$

Let $I$ = Moment of inertia of the rotor in kg-m$^2$, and

$\omega$ = Angular velocity of the rotor in rad/s.

∴ Mamimum gyroscopic couple,

$$C_{max} = I . \omega . \omega_{Pmax}$$

When the pitching is upward, the effect of the reactive gyroscopic couple, as shown in Fig. 14.10 (*b*), will try to move the ship toward star-board. On the other hand, if the pitching is downward, the effect of the reactive gyroscopic couple, as shown in Fig. 14.10 (*c*), is to turn the ship towards port side.

**Notes : 1.** The effect of the gyroscopic couple is always given on specific position of the axis of spin *i.e.* whether it is pitching downwards or upwards.

**2.** The pitching of a ship produces forces on the bearings which act horizontally and perpendicular to the motion of the ship.

**3.** The maximum gyroscopic couple tends to shear the holding-down bolts.

**4.** The angular acceleration during pitching,

$$\alpha = \frac{d^2\theta}{dt^2} = -\phi(\omega_1)^2 \sin \omega_1 t \qquad \text{...}\left(\text{Differentiating } \frac{d\theta}{dt} \text{ with respect to } t\right)$$

The angular acceleration is maximum, if $\sin \omega_1 t = 1$.

∴ Maximum angular acceleration during pitching,

$$\alpha_{max} = (\omega_1)^2$$

## 14.8. Effect of Gyroscopic Couple on a Naval Ship during Rolling

We know that, for the effect of gyroscopic couple to occur, the axis of precession should always be perpendicular to the axis of spin. If, however, the axis of precession becomes parallel to the axis of spin, there will be no effect of the gyroscopic couple acting on the body of the ship.

In case of rolling of a ship, the axis of precession (*i.e.* longitudinal axis) is always parallel to the axis of spin for all positions. Hence, there is no effect of the gyroscopic couple acting on the body of a ship.

**Example 14.4.** *The turbine rotor of a ship has a mass of 8 tonnes and a radius of gyration 0.6 m. It rotates at 1800 r.p.m. clockwise, when looking from the stern. Determine the gyroscopic couple, if the ship travels at 100 km/hr and steer to the left in a curve of 75 m radius.*

**Solution.** Given: $m$ = 8 t = 8000 kg ; $k$ = 0.6 m ; $N$ = 1800 r.p.m. or $\omega = 2\pi \times 1800/60 = 188.5$ rad/s ; $v$ = 100 km/h = 27.8 m/s ; $R$ = 75 m

We know that mass moment of inertia of the rotor,

$$I = m.k^2 = 8000\,(0.6)^2 = 2880 \text{ kg-m}^2$$

and angular velocity of precession,

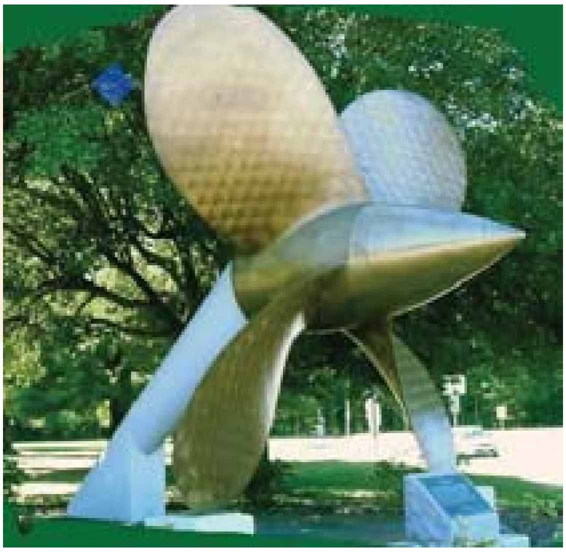
$$\omega_P = v / R = 27.8 / 75 = 0.37 \text{ rad/s}$$

We know that gyroscopic couple,

$$C = I.\omega.\omega_P = 2880 \times 188.5 \times 0.37 = 200\,866 \text{ N-m}$$

$$= 200.866 \text{ kN-m}$$ **Ans.**

We have discussed in Art. 14.6, that when the rotor rotates in clockwise direction when looking from the stern and the ship steers to the left, the effect of the reactive gyroscopic couple is to raise the bow and lower the stern.



Ship's propeller shown as a separate part. A ship's propeller is located at backside (stern) of the ship below the water surface.

**Example 14.5.** *The heavy turbine rotor of a sea vessel rotates at 1500 r.p.m. clockwise looking from the stern, its mass being 750 kg. The vessel pitches with an angular velocity of 1 rad/s. Determine the gyroscopic couple transmitted to the hull when bow is rising, if the radius of gyration for the rotor is 250 mm. Also show in what direction the couple acts on the hull?*

**Solution.** Given: $N$ = 1500 r.p.m. or $\omega = 2\pi \times 1500/60 = 157.1$ rad/s; $m$ = 750 kg; $\omega_P$ = 1 rad/s; $k$ = 250 mm = 0.25 m

We know that mass moment of inertia of the rotor,

$$I = m.k^2 = 750\,(0.25)^2 = 46.875 \text{ kg-m}^2$$

∴ Gyroscopic couple transmitted to the hull (*i.e.* body of the sea vessel),

$$C = I.\omega.\omega_P = 46.875 \times 157.1 \times 1 = 7364 \text{ N-m} = 7.364 \text{ kN-m}$$

We have discussed in Art. 14.7, that when the bow is rising *i.e.* when the pitching is upward, the reactive gyroscopic couple acts in the clockwise direction which moves the sea vessel towards star-board.

**Example 14.6.** *The turbine rotor of a ship has a mass of 3500 kg. It has a radius of gyration of 0.45 m and a speed of 3000 r.p.m. clockwise when looking from stern. Determine the gyroscopic couple and its effect upon the ship:*

*1. when the ship is steering to the left on a curve of 100 m radius at a speed of 36 km/h.*

*2. when the ship is pitching in a simple harmonic motion, the bow falling with its maximum velocity. The period of pitching is 40 seconds and the total angular displacement between the two extreme positions of pitching is 12 degrees.*

**Solution.** Given : $m$ = 3500 kg ; $k$ = 0.45 m; $N$ = 3000 r.p.m. or $\omega = 2\pi \times 3000/60 = 314.2$ rad/s

### 1. *When the ship is steering to the left*

Given: $R$ =100 m ; $v$ = km/h = 10 m/s

We know that mass moment of inertia of the rotor,

$$I = m.k^2 = 3500\,(0.45)^2 = 708.75 \text{ kg-m}^2$$

and angular velocity of precession,

$$\omega_p = v/R = 10/100 = 0.1 \text{ rad/s}$$

∴ Gyroscopic couple,

$$C = I.\omega.\omega_p = 708.75 \times 314.2 \times 0.1 = 22\,270 \text{ N-m}$$

$$= 22.27 \text{ kN-m } \textbf{Ans.}$$

We have discussed in Art. 14.6, that when the rotor rotates clockwise when looking from the stern and the ship takes a left turn, the effect of the reactive gyroscopic couple is to raise the bow and lower the stern. **Ans.**

### 2. *When the ship is pitching with the bow falling*

Given: $t_p$ = 40 s

Since the total angular displacement between the two extreme positions of pitching is 12° (*i.e.* $2\phi = 12°$), therefore amplitude of swing,

$$\phi = 12 / 2 = 6° = 6 \times \pi/180 = 0.105 \text{ rad}$$

and angular velocity of the simple harmonic motion,

$$\omega_1 = 2\pi / t_p = 2\pi / 40 = 0.157 \text{ rad/s}$$

We know that maximum angular velocity of precession,

$$\omega_p = \phi.\omega_1 = 0.105 \times 0.157 = 0.0165 \text{ rad/s}$$

∴ Gyroscopic couple,

$$C = I.\omega.\omega_p = 708.75 \times 314.2 \times 0.0165 = 3675 \text{ N-m}$$

$$= 3.675 \text{ kN-m } \textbf{Ans.}$$

We have discussed in Art. 14.7, that when the bow is falling (*i.e.* when the pitching is downward), the effect of the reactive gyroscopic couple is to move the ship towards port side. **Ans.**

**Example 14.7.** *The mass of the turbine rotor of a ship is 20 tonnes and has a radius of gyration of 0.60 m. Its speed is 2000 r.p.m. The ship pitches 6° above and 6° below the horizontal position. A complete oscillation takes 30 seconds and the motion is simple harmonic. Determine the following:*

***1.*** *Maximum gyroscopic couple,* ***2.*** *Maximum angular acceleration of the ship during pitching, and* ***3.*** *The direction in which the bow will tend to turn when rising, if the rotation of the rotor is clockwise when looking from the left.*

**Solution.** Given : $m$ = 20 t = 20 000 kg ; $k$ = 0.6 m ; $N$ = 2000 r.p.m. or $\omega = 2\pi \times 2000/60$ = 209.5 rad/s; $\phi = 6° = 6 \times \pi/180 = 0.105$ rad ; $t_p$ = 30 s

**1. *Maximum gyroscopic couple***

We know that mass moment of inertia of the rotor,

$$I = m.k^2 = 20\,000\,(0.6)^2 = 7200 \text{ kg-m}^2$$

and angular velocity of the simple harmonic motion,

$$\omega_1 = 2\pi / t_p = 2\pi/30 = 0.21 \text{ rad/s}$$

∴ Maximum angular velocity of precession,

$$\omega_{Pmax} = \phi.\omega_1 = 0.105 \times 0.21 = 0.022 \text{ rad/s}$$

We know that maximum gyroscopic couple,

$$C_{max} = I.\omega.\omega_{Pmax} = 7200 \times 209.5 \times 0.022 = 33\,185 \text{ N-m}$$

$$= 33.185 \text{ kN-m}$$ **Ans.**

**2. *Maximum angular acceleration during pitching***

We know that maximum angular acceleration during pitching

$$= \phi(\omega_1)^2 = 0.105\,(0.21)^2 = 0.0046 \text{ rad/s}^2$$

**3. *Direction in which the bow will tend to turn when rising***

We have discussed in Art. 14.7, that when the rotation of the rotor is clockwise when looking from the left (*i.e.* rear end or stern) and when the bow is rising (*i.e.* pitching is upward), then the reactive gyroscopic couple acts in the clockwise direction which tends to turn the bow towards right (*i.e.* towards star-board). **Ans.**

**Example 14.8.** *A ship propelled by a turbine rotor which has a mass of 5 tonnes and a speed of 2100 r.p.m. The rotor has a radius of gyration of 0.5 m and rotates in a clockwise direction when viewed from the stern. Find the gyroscopic effects in the following conditions:*

***1.*** *The ship sails at a speed of 30 km/h and steers to the left in a curve having 60 m radius.*

***2.*** *The ship pitches 6 degree above and 6 degree below the horizontal position. The bow is descending with its maximum velocity. The motion due to pitching is simple harmonic and the periodic time is 20 seconds.*

***3.*** *The ship rolls and at a certain instant it has an angular velocity of 0.03 rad/s clockwise when viewed from stern.*

*Determine also the maximum angular acceleration during pitching. Explain how the direction of motion due to gyroscopic effect is determined in each case.*

**Solution.** Given : $m$ = 5 t = 5000 kg ; $N$ = 2100 r.p.m. or $\omega = 2\pi \times 2100/60$ = 220 rad/s ; $k$ = 0.5 m

**1. *When the ship steers to the left***

Given: $v$ = 30 km / h = 8.33 m / s ; $R$ = 60 m

We know that angular velocity of precession,

$$\omega_P = v/R = 8.33/60 = 0.14 \text{ rad/s}$$

and mass moment of inertia of the rotor,

$$I = m.k^2 = 5000(0.5)^2 = 1250 \text{ kg-m}^2$$

∴ Gyroscopic couple,

$$C = I.\omega.\omega_P = 1250 \times 220 \times 0.14 = 38\,500 \text{ N-m} = 38.5 \text{ kN-m}$$

We have discussed in Art. 14.6, that when the rotor in a clockwise direction when viewed from the stern and the ship steers to the left, the effect of reactive gyroscopic couple is to raise the bow and lower the stern. **Ans.**

**2. *When the ship pitches with the bow descending***

Given: $\phi = 6° = 6 \times \pi/180 = 0.105$ rad/s ; $t_p = 20$ s

We know that angular velocity of simple harmonic motion,

$$\omega_1 = 2\pi / t_p = 2\pi / 20 = 0.3142 \text{ rad/s}$$

and maximum angular velocity of precession,

$$\omega_{Pmax} = \phi.\omega_1 = 0.105 \times 0.3142 = 0.033 \text{ rad/s}$$

∴ Maximum gyroscopic couple,

$$C_{max} = I.\omega.\omega_{Pmax} = 1250 \times 220 \times 0.033 = 9075 \text{ N-m}$$

Since the ship is pitching with the bow descending, therefore the effect of this maximum gyroscopic couple is to turn the ship towards port side. **Ans.**

**3. *When the ship rolls***

Since the ship rolls at an angular velocity of 0.03 rad / s, therefore angular velocity of precession when the ship rolls,

$$\omega_P = 0.03 \text{ rad /s}$$

∴ Gyroscopic couple,

$$C = I.\omega.\omega_P = 1250 \times 220 \times 0.03 = 8250 \text{ N-m}$$

In case of rolling of a ship, the axis of precession is always parallel to the axis of spin for all positions, therefore there is no effect of gyroscopic couple. **Ans.**

***Maximum angular acceleration during pitching***

We know that maximum angular acceleration during pitching.

$$\alpha_{max} = \phi\,(\omega_1)^2 = 0.105\,(0.3142)^2 = 0.01 \text{ rad/s}^2 \textbf{ Ans.}$$

**Example 14.9.** *The turbine rotor of a ship has a mass of 2000 kg and rotates at a speed of 3000 r.p.m. clockwise when looking from a stern. The radius of gyration of the rotor is 0.5 m.*

*Determine the gyroscopic couple and its effects upon the ship when the ship is steering to the right in a curve of 100 m radius at a speed of 16.1 knots (1 knot = 1855 m/hr).*

*Calculate also the torque and its effects when the ship is pitching in simple harmonic motion, the bow falling with its maximum velocity. The period of pitching is 50 seconds and the total angular displacement between the two extreme positions of pitching is 12°. Find the maximum acceleration during pitching motion.*

**Solution.** Given : $m = 2000$ kg ; $N = 3000$ r.p.m. or $\omega = 2\pi \times 3000/60 = 314.2$ rad/s ; $k = 0.5$ m ; $R = 100$ m ; $v = 16.1$ knots $= 16.1 \times 1855 / 3600 = 8.3$ m/s

***Gyroscopic couple***

We know that mass moment of inertia of the rotor,

$$I = m.k^2 = 2000\,(0.5)^2 = 500 \text{ kg-m}^2$$

and angular velocity of precession,

$$\omega_P = v/R = 8.3/100 = 0.083 \text{ rad /s}$$

∴ Gyroscopic couple,

$$C = I.\omega.\omega_P = 500 \times 314.2 \times 0.083 = 13\ 040 \text{ N-m} = 13.04 \text{ kN-m}$$

We have discussed in Art. 14.6, that when the rotor rotates clockwise when looking from a stern and the ship steers to the right, the effect of the reactive gyroscopic couple is to raise the stern and lower the bow. **Ans.**

***Torque during pitching***

Given : $t_p$ =50 s ; 2 $\phi$ = 12° or $\phi$ = 6° × $\pi$/180 = 0.105 rad

We know that angular velocity of simple harmonic motion,

$$\omega_1 = 2\pi / t_p = 2\pi / 50 = 0.1257 \text{ rad/s}$$

and maximum angular velocity of precession,

$$\omega_{Pmax} = \phi.\omega_1 = 0.105 \times 0.1257 = 0.0132 \text{ rad/s}$$

∴ Torque or maximum gyroscopic couple during pitching,

$$C_{max} = I.\omega.\omega_{P\,max} = 500 \times 314.2 \times 0.0132 = 2074 \text{ N-m}$$ **Ans.**

We have discussed in Art. 14.7, that when the pitching is downwards, the effect of the reactive gyroscopic couple is to turn the ship towards port side.

***Maximum acceleration during pitching***

We know that maximum acceleration during pitching

$$\alpha_{max} = \phi\,(\omega_1)^2 = 0.105\,(0.1257)^2 = 0.00166 \text{ rad/s}^2$$ **Ans.**

## 14.9. Stability of a Four Wheel Drive Moving in a Curved Path

Consider the four wheels *A*, *B*, *C* and *D* of an automobile locomotive taking a turn towards left as shown in Fig. 14.11. The wheels *A* and *C* are inner wheels, whereas *B* and *D* are outer wheels. The centre of gravity (*C.G.*) of the vehicle lies vertically above the road surface.

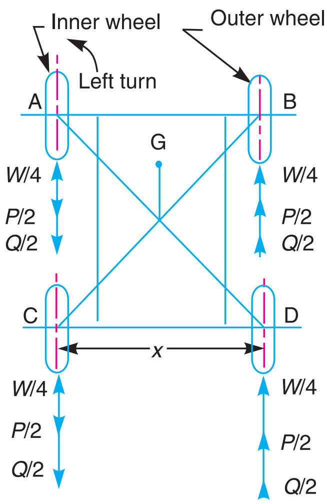



**Fig. 14.11.** Four wheel drive moving in a curved path.

Let $m$ = Mass of the vehicle in kg,

$W$ = Weight of the vehicle in newtons = $m.g$,

$r_W$ = Radius of the wheels in metres,

$R$ = Radius of curvature in metres ($R > r_W$),

$h$ = Distance of centre of gravity, vertically above the road surface in metres,

$x$ = Width of track in metres,

$I_W$ = Mass moment of inertia of one of the wheels in kg-m$^2$,

$\omega_W$ = Angular velocity of the wheels or velocity of spin in rad/s,

$I_E$ = Mass moment of inertia of the rotating parts of the engine in kg-m$^2$,

$\omega_E$ = Angular velocity of the rotating parts of the engine in rad/s,

$G$ = Gear ratio = $\omega_E / \omega_W$,

$v$ = Linear velocity of the vehicle in m/s = $\omega_W . r_W$


A little considereation will show, that the weight of the vehicle ($W$) will be equally distributed over the four wheels which will act downwards. The reaction between each wheel and the road surface of the same magnitude will act upwards. Therefore

Road reaction over each wheel
$= W/4 = m.g/4$ newtons

Let us now consider the effect of the gyroscopic couple and centrifugal couple on the vehicle.

### 1. *Effect of the gyroscopic couple*

Since the vehicle takes a turn towards left due to the precession and other rotating parts, therefore a gyroscopic couple will act.

We know that velocity of precession,

$$\omega_P = v/R$$

∴ Gyroscopic couple due to 4 wheels,

$$C_W = 4\, I_W.\omega_W.\omega_P$$

and gyroscopic couple due to the rotating parts of the engine,

$$C_E = I_E.\omega_E.\omega_P = I_E.G.\omega_W.\omega_P \qquad \ldots (\because G = \omega_E/\omega_W)$$

∴ Net gyroscopic couple,

$$C = C_W \pm C_E = 4\, I_W.\omega_W.\omega_P \pm I_E.G.\omega_W.\omega_P$$
$$= \omega_W.\omega_P\,(4\, I_W \pm G.I_E)$$

The ***positive*** sign is used when the wheels and rotating parts of the engine rotate in the same direction. If the rotating parts of the engine revolves in opposite direction, then ***negative*** sign is used.

Due to the gyroscopic couple, vertical reaction on the road surface will be produced. The reaction will be vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at the two outer or inner wheels be $P$ newtons. Then

$$P \times x = C \quad \text{or} \quad P = C/x$$

∴ Vertical reaction at each of the outer or inner wheels,

$$P/2 = C/2x$$

**Note:** We have discussed above that when rotating parts of the engine rotate in opposite directions, then –ve sign is used, *i.e.* net gyroscopic couple,

$$C = C_W - C_E$$

When $C_E > C_W$, then $C$ will be –ve. Thus the reaction will be vertically downwards on the outer wheels and vertically upwards on the inner wheels.

### 2. *Effect of the centrifugal couple*

Since the vehicle moves along a curved path, therefore centrifugal force will act outwardly at the centre of gravity of the vehicle. The effect of this centrifugal force is also to overturn the vehicle. We know that centrifugal force,

$$F_C = \frac{m \times v^2}{R}$$

∴ The couple tending to overturn the vehicle or overturning couple,

$$C_O = F_C \times h = \frac{m.v^2}{R} \times h$$

This overturning couple is balanced by vertical reactions, which are vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at the two outer or inner wheels be $Q$. Then

$$Q \times x = C_O \quad \text{or} \quad Q = \frac{C_O}{x} = \frac{m.v^2.h}{R.x}$$

∴ Vertical reaction at each of the outer or inner wheels,

$$\frac{Q}{2} = \frac{m.v^2.h}{2R.x}$$

∴ Total vertical reaction at each of the outer wheel,

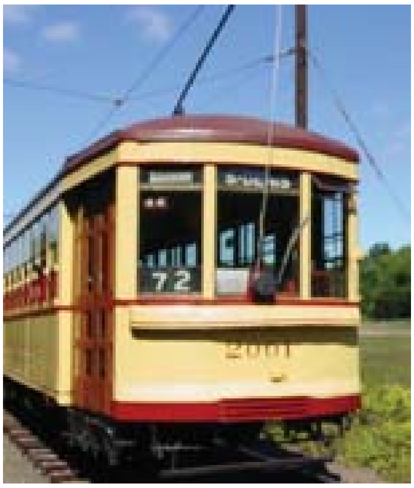
$$P_O = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2}$$

and total vertical reaction at each of the inner wheel,

$$P_I = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$

A little consideration will show that when the vehicle is running at high speeds, $P_I$ may be zero or even negative. This will cause the inner wheels to leave the ground thus tending to overturn the automobile. In order to have the contact between the inner wheels and the ground, the sum of $P/2$ and $Q/2$ must be less than $W/4$.

**Example 14.10.** *A four-wheeled trolley car of mass 2500 kg runs on rails, which are 1.5 m apart and travels around a curve of 30 m radius at 24 km / hr. The rails are at the same level. Each wheel of the trolley is 0.75 m in diameter and each of the two axles is driven by a motor running in a direction opposite to that of the wheels at a speed of five times the speed of rotation of the wheels. The moment of inertia of each axle with gear and wheels is 18 kg-m². Each motor with shaft and gear pinion has a moment of inertia of 12 kg-m². The centre of gravity of the car is 0.9 m above the rail level. Determine the vertical force exerted by each wheel on the rails taking into consideration the centrifugal and gyroscopic effects. State the centrifugal and gyroscopic effects on the trolley.*




**Solution.** Given : $m$ = 2500 kg ; $x$ = 1.5 m ; $R$ = 30 m ; $v$ = 24 km/h = 6.67 m/s ; $d_W$ = 0.75 m or $r_W$ = 0.375 m ; $G = \omega_E/\omega_W = 5$ ; $I_W$ = 18 kg-m$^2$ ; $I_E$ = 12 kg-m$^2$ ; $h$ = 0.9 m

The weight of the trolley ($W = m.g$) will be equally distributed over the four wheels, which will act downwards. The reaction between the wheels and the road surface of the same magnitude will act upwards.

∴ Road reaction over each wheel $= W/4 = m.g/4 = 2500 \times 9.81/4 = 6131.25$ N

We know that angular velocity of the wheels,

$$\omega_W = v/r_W = 6.67/0.375 = 17.8 \text{ rad/s}$$

and angular velocity of precession, $\omega_P = v/R = 6.67/30 = 0.22$ rad/s

∴ Gyroscopic couple due to one pair of wheels and axle,

$$C_W = 2\,I_W.\omega_W.\omega_P = 2 \times 18 \times 17.8 \times 0.22 = 141 \text{ N-m}$$

and gyroscopic couple due to the rotating parts of the motor and gears,

$$C_E = 2\,I_E.\omega_E.\omega_P = 2\,I_E.G.\omega_W.\omega_P \qquad \ldots (\because\ \omega_E = G.\,\omega_W)$$

$$= 2 \times 12 \times 5 \times 17.8 \times 0.22 = 470 \text{ N-m}$$

∴ Net gyroscopic couple, $C = C_W - C_E = 141 - 470 = -329$ N-m

... (–ve sign is used due to opposite direction of motor)

Due to this net gyroscopic couple, the vertical reaction on the rails will be produced. Since $C_E$ is greater than $C_W$, therefore the reaction will be vertically downwards on the outer wheels and vertically upwards on the inner wheels. Let the magnitude of this reaction at each of the outer or inner wheel be $P/2$ newton.

∴ $P/2 = C/2x = 329 / 2 \times 1.5 = 109.7$ N

We know that centrifugal force, $F_C = m.v^2/R = 2500\,(6.67)^2/30 = 3707$ N

∴ Overturning couple, $C_O = F_C \times h = 3707 \times 0.9 = 3336.3$ N-m

This overturning couple is balanced by the vertical reactions which are vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at each of the outer or inner wheels be $Q/2$ newton.

∴ $Q/2 = C_O / 2x = 3336.3 / 2 \times 1.5 = 1112.1$ N

We know that vertical force exerted on each outer wheel,

$$P_O = \frac{W}{4} - \frac{P}{2} + \frac{Q}{2} = 6131.25 - 109.7 + 1112.1 = 7142.65 \text{ N}$$ **Ans.**

and vertical force exerted on each inner wheel,

$$P_I = \frac{W}{4} + \frac{P}{2} - \frac{Q}{2} = 6131.25 + 109.7 - 1112.1 = 5128.85 \text{ N}$$ **Ans.**

**Example 14.11.** *A rear engine automobile is travelling along a track of 100 metres mean radius. Each of the four road wheels has a moment of inertia of 2.5 kg-m² and an effective diameter of 0.6 m. The rotating parts of the engine have a moment of inertia of 1.2 kg-m². The engine axis is parallel to the rear axle and the crankshaft rotates in the same sense as the road wheels. The ratio of engine speed to back axle speed is 3 : 1. The automobile has a mass of 1600 kg and has its centre of gravity 0.5 m above road level. The width of the track of the vehicle is 1.5 m.*

*Determine the limiting speed of the vehicle around the curve for all four wheels to maintain contact with the road surface. Assume that the road surface is not cambered and centre of gravity of the automobile lies centrally with respect to the four wheels.*

**Solution.** Given : $R = 100$ m ; $I_W = 2.5$ kg-m² ; $d_W = 0.6$ m or $r_W = 0.3$ m ; $I_E = 1.2$ kg-m²; $G = \omega_E/\omega_W = 3$ ; $m = 1600$ kg ; $h = 0.5$ m ; $x = 1.5$ m

The weight of the vehicle ($m.g$) will be equally distributed over the four wheels which will act downwards. The reaction between the wheel and the road surface of the same magnitude will act upwards.

∴ Road reaction over each wheel

$$= W/4 = m.g / 4 = 1600 \times 9.81/4 = 3924 \text{ N}$$

Let $v$ = Limiting speed of the vehicle in m/s.

We know that angular velocity of the wheels,

$$\omega_W = \frac{v}{r_W} = \frac{v}{0.3} = 3.33\, v \text{ rad/s}$$

and angular velocity of precession,

$$\omega_P = \frac{v}{R} = \frac{v}{100} = 0.01\, v \text{ rad/s}$$

∴ Gyroscopic couple due to 4 wheels,

$$C_W = 4\, I_W.\omega_W.\omega_P = 4 \times 2.5 \times \frac{v}{0.3} \times \frac{v}{100} = 0.33\, v^2 \text{ N-m}$$

and gyroscopic couple due to rotating parts of the engine,

$$C_E = I_E.\omega_E.\omega_P = I_E.G.\omega_W.\omega_P$$

$$= 1.2 \times 3 \times 3.33v \times 0.01v = 0.12\, v^2 \text{ N-m}$$

∴ Total gyroscopic couple,

$$C = C_W + C_E = 0.33\, v^2 + 0.12\, v^2 = 0.45\, v^2 \text{ N-m}$$

Due to this gyroscopic couple, the vertical reaction on the rails will be produced. The reaction will be vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at each of the outer or inner wheel be $P/2$ newtons.

∴ $P/2 = C/2x = 0.45v^2/2 \times 1.5 = 0.15\, v^2$ N

We know that centrifugal force,

$$F_C = m.v^2/R = 1600 \times v^2/100 = 16\, v^2 \text{ N}$$

∴ Overturning couple acting in the outward direction,

$$C_O = F_C \times h = 16\, v^2 \times 0.5 = 8\, v^2 \text{ N-m}$$

This overturning couple is balanced by vertical reactions which are vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at each of the outer or inner wheels be $Q/2$ newtons.

∴ $Q/2 = C_O/2x = 8\, v^2/2 \times 1.5 = 2.67\, v^2$ N

We know that total vertical reaction at each of the outer wheels,

$$P_O = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2} \quad ...(i)$$

and total vertical reaction at each of the inner wheels,

$$P_I = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2} = \frac{W}{4} - \left(\frac{P}{2} + \frac{Q}{2}\right) \quad ...(ii)$$

From equation (*i*), we see that there will always be contact between the outer wheels and the road surface because $W/4$, $P/2$ and $Q/2$ are vertically upwards. In order to have contact between the inner wheels and road surface, the reactions should also be vertically upwards, which is only possible if

$$\frac{P}{2} + \frac{Q}{2} \leq \frac{W}{4}$$

*i.e.* $0.15\, v^2 + 2.67\, v^2 \le 3924$ or $2.82\, v^2 \le 3924$

$\therefore$ $v^2 \le 3924/2.82 = 1391.5$

or $v \le 37.3$ m/s $= 37.3 \times 3600 / 1000 = 134.28$ km/h **Ans.**

**Example 14.12.** *A four wheeled motor car of mass 2000 kg has a wheel base 2.5 m, track width 1.5 m and height of centre of gravity 500 mm above the ground level and lies at 1 metre from the front axle. Each wheel has an effective diameter of 0.8 m and a moment of inertia of 0.8 kg-m². The drive shaft, engine flywheel and transmission are rotating at 4 times the speed of road wheel, in a clockwise direction when viewed from the front, and is equivalent to a mass of 75 kg having a radius of gyration of 100 mm. If the car is taking a right turn of 60 m radius at 60 km/h, find the load on each wheel.*

**Solution.** Given : $m = 2000$ kg : $b = 2.5$ m ; $x = 1.5$ m ; $h = 500$ mm $= 0.5$ m ; $L = 1$ m ; $d_W = 0.8$ m or $r_W = 0.4$ m ; $I_W = 0.8$ kg-m² ; $G = \omega_E / \omega_W = 4$ ; $m_E = 75$ kg ; $k_E = 100$ mm $= 0.1$ m ; $R = 60$ m ; $v = 60$ km/h $= 16.67$ m/s

Since the centre of gravity of the car lies at 1 m from the front axle and the weight of the car ($W = m.g$) lies at the centre of gravity, therefore weight on the front wheels and rear wheels will be different.

Let $W_1$ = Weight on the front wheels, and

$W_2$ = Weight on the rear wheels.

Taking moment about the front wheels,

$$W_2 \times 2.5 = W \times 1 = m.g \times 1 = 2000 \times 9.81 \times 1 = 19\,620$$

$$\therefore \quad W_2 = 19\,620 / 2.5 = 7848 \text{ N}$$

We know that weight of the car or on the four wheels,

$$W = W_1 + W_2 = m.g = 2000 \times 9.81 = 19\,620 \text{ N}$$

or $$W_1 = W - W_2 = 19\,620 - 7848 = 11\,772 \text{ N}$$

$\therefore$ Weight on each of the front wheels

$$= W_1 / 2 = 11\,772 / 2 = 5886 \text{ N}$$

and weight on each of the rear wheels

$$= W_2 /2 = 7874 / 2 = 3924 \text{ N}$$

Since the weight of the car over the four wheels will act downwards, therefore the reaction between each wheel and the road surface of the same magnitude will act upwards as shown in Fig. 14.12.

Let us now consider the effect of gyroscopic couple due to four wheels and rotating parts of the engine.

We know angular velocity of wheels,

$$\omega_W = v/r_W = 16.67 / 0.4 = 41.675 \text{ rad /s}$$

and angular velocity of precession,

$$\omega_P = v/R = 16.67 / 60 = 0.278 \text{ rad /s}$$

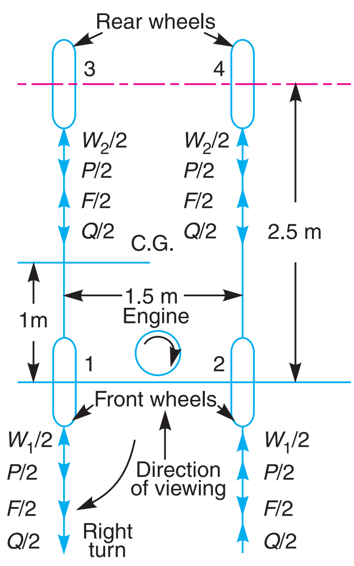



Fig. 14.12

∴ Gyroscopic couple due to four wheels,

$$C_W = 4\, I_W.\omega_W.\omega_P$$
$$= 4 \times 0.8 \times 41.675 \times 0.278 = 37.1 \text{ N-m}$$

This gyroscopic couple tends to lift the inner wheels and to press the outer wheels. In other words, the reaction will be vertically downward on the inner wheels (*i.e.* wheels 1 and 3) and vertically upward on the outer wheels (*i.e.* wheels 2 and 4) as shown in Fig. 14.12. Let *P*/2 newtons be the magnitude of this reaction at each of the inner or outer wheel.

$$\therefore \quad P/2 = C_W / 2x = 37.1 / 2 \times 1.5 = 12.37 \text{ N}$$

We know that mass moment of inertia of rotating parts of the engine,

$$I_E = m_E\,(k_E)^2 = 75\,(0.1)^2 = 0.75 \text{ kg-m}^2 \qquad ...(\because I = m.k^2)$$

∴ Gyroscopic couple due to rotating parts of the engine,

$$C_E = I_E.\omega_E.\omega_P = m_E\,(k_E)^2\,G.\,\omega_W.\omega_P$$
$$= 75\,(0.1)^2\,4 \times 41.675 \times 0.278 = 34.7 \text{ N-m}$$

This gyroscopic couple tends to lift the front wheels and to press the outer wheels. In other words, the reaction will be vertically downwards on the front wheels and vertically upwards on the rear wheels as shown in Fig. 14.12. Let *F*/2 newtons be the magnitude of this reaction on each of the front and rear wheels.

$$\therefore \quad F/2 = C_E / 2b = 34.7/2 \times 2.5 = 6.94 \text{ N}$$

Now let us consider the effect of centrifugal couple acting on the car. We know that centrifugal force,

$$F_C = m.v^2 / R = 2000\,(16.67)^2/\,60 = 9263 \text{ N}$$

∴ Centrifugal couple tending to overturn the car or over turning couple,

$$C_O = F_C \times h = 9263 \times 0.5 = 4631.5 \text{ N-m}$$

This overturning couple tends to reduce the pressure on the inner wheels and to increase on the outer wheels. In other words, the reactions are vertically downward on the inner wheels and vertically upwards on the outer wheels. Let *Q*/2 be the magnitude of this reaction on each of the inner and outer wheels.

$$\therefore \quad Q/2 = C_O / 2x = 4631.5 / 2 \times 1.5 = 1543.83 \text{ N}$$

From Fig. 14.12, we see that

Load on the front wheel 1

$$= \frac{W_1}{2} - \frac{P}{2} - \frac{F}{2} - \frac{Q}{2} = 5886 - 12.37 - 6.94 - 1543.83 = 4322.86 \text{ N}$$ **Ans.**

Load on the front wheel 2

$$= \frac{W_1}{2} + \frac{P}{2} - \frac{F}{2} + \frac{Q}{2} = 5886 + 12.37 - 6.94 + 1543.83 = 7435.26 \text{ N}$$ **Ans.**

Load on the rear wheel 3

$$= \frac{W_2}{2} - \frac{P}{2} + \frac{F}{2} - \frac{Q}{2} = 3924 - 12.37 + 6.94 - 1543.83 = 2374.74 \text{ N}$$ **Ans.**

Load on the rear wheel 4

$$= \frac{W_2}{2} + \frac{P}{2} + \frac{F}{2} + \frac{Q}{2} = 3924 + 12.37 + 6.94 + 1543.83 = 5487.14 \text{ N}$$ **Ans.**

**Example 14.13.** *A four-wheeled trolley car of total mass 2000 kg running on rails of 1.6 m gauge, rounds a curve of 30 m radius at 54 km/h. The track is banked at 8°. The wheels have an external diameter of 0.7 m and each pair with axle has a mass of 200 kg. The radius of gyration for each pair is 0.3 m. The height of centre of gravity of the car above the wheel base is 1 m. Determine, allowing for centrifugal force and gyroscopic couple actions, the pressure on each rail.*

**Solution.** Given : $m$ = 2000 kg ; $x$ = 1.6 m ; $R$ = 30 m ; $v$ = 54 km / h = 15 m / s ; θ = 8° ; $d_W$ = 0.7 m or $r_W$ = 0.35 m ; $m_1$ = 200 kg ; $k$ = 0.3 m ; $h$ = 1 m

First of all, let us find the reactions $R_A$ and $R_B$ at the wheels $A$ and $B$ respectively. The various forces acting on the trolley car are shown in Fig. 14.13.

Resolving the forces perpendicular to the track,

$$R_A + R_B = W\cos\theta + F_C\sin\theta = m.g\cos\theta + \frac{m.v^2}{R}\sin\theta$$

$$= 2000 \times 9.81\cos 8° + \frac{2000\,(15)^2}{30} \times \sin 8°$$

$$= 19\,620 \times 0.9903 + 15\,000 \times 0.1392 = 21\,518 \text{ N}$$

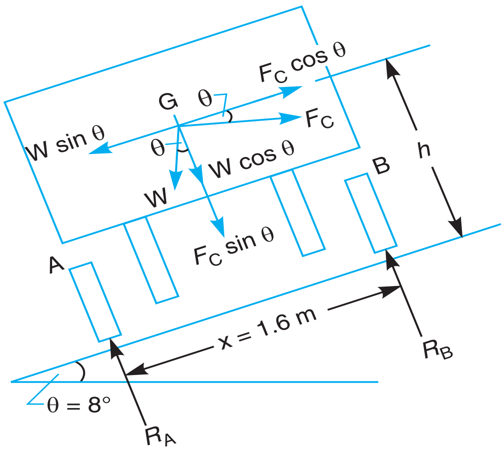



**Fig. 14.13**

Now taking moments about $B$,

$$R_A \times x = (W\cos\theta + F_C\sin\theta)\,\frac{x}{2} + W\sin\theta \times h - F_C\cos\theta \times h$$

$$\therefore \quad R_A = \left(m.g\cos\theta + \frac{m.v^2}{R}\sin\theta\right)\frac{1}{2} + \left(m.g\sin\theta - \frac{m.v^2}{R}\cos\theta\right)\frac{h}{x}$$

$$= \left(2000 \times 9.81\cos 8° + \frac{2000\,(15)^2}{30}\sin 8°\right)\frac{1}{2}$$

$$+ \left(2000 \times 9.81\sin 8° - \frac{2000\,(15)^2}{30}\cos 8°\right)\frac{1}{1.6}$$

$$= (19\,620 \times 0.9903 + 15\,000 \times 0.1392)\,\frac{1}{2}$$

$$+ (19\,620 \times 0.1392 - 15\,000 \times 0.9903)\,\frac{1}{1.6}$$

$$= (19\,430 + 2088)\frac{1}{2} + (2731 - 14\,855)\frac{1}{1.6}$$

$$= 10\,759 - 7577 = 3182 \text{ N}$$

$$\therefore \quad R_B = (R_A + R_B) - R_A = 21\,518 - 3182 = 18\,336 \text{ N}$$

We know that angular velocity of wheels,

$$\omega_W = \frac{v}{r_W} = \frac{15}{0.35} = 42.86 \text{ rad/s}$$

and angular velocity of precession,

$$\omega_P = \frac{v}{R} = \frac{15}{30} = 0.5 \text{ rad/s}$$

∴ Gyroscopic couple,

$$C = {}^{*}I\,\omega_W \cos\theta \times \omega_P = m_I.k^2.\omega_W \cos\theta.\omega_P \qquad \ldots(\because I = m_I.k^2)$$

$$= 200\,(0.3)^2\, 42.86 \cos 8° \times 0.5 = 382 \text{ N-m}$$

Due to this gyroscopic couple, the car will tend to overturn about the outer wheels. Let $P$ be the force at each pair of wheels or each rail due to the gyroscopic couple,

$$\therefore \quad P = C / x = 382 / 1.6 = 238.75 \text{ N}$$

We know that pressure (or total reaction) on the inner rail,

$$P_I = R_A - P = 3182 - 238.75 = 2943.25 \text{ N} \quad \textbf{Ans.}$$

and pressure on the outer rail,

$$P_O = R_B + P = 18\,336 + 238.75 = 18\,574.75 \text{ N} \quad \textbf{Ans.}$$

**Example 14.14.** *A pair of locomotive driving wheels with the axle, have a moment of inertia of 180 kg-m². The diameter of the wheel treads is 1.8 m and the distance between wheel centres is 1.5 m. When the locomotive is travelling on a level track at 95 km/h, defective ballasting causes one wheel to fall 6 mm and to rise again in a total time of 0.1 s. If the displacement of the wheel takes place with simple harmonic motion, find :* **1.** *The gyroscopic couple set up, and* **2.** *The reaction between the wheel and rail due to this couple.*

**Solution.** Given : $I = 180$ kg-m² ; $D = 1.8$ m or $R = 0.9$ m ; $x = 1.5$ m ; $v = 95$ km / h = 26.4 m /s

**1. *Gyroscopic couple set up***

We know that angular velocity of the locomotive,

$$\omega = v/R = 26.4/\,0.9 = 29.3 \text{ rad/s}$$

Since the defective ballasting causes one wheel to fall 6 mm and to rise again in a total time ($t$) of 0.1 s, therefore

$$\text{Amplitude, } A = \frac{1}{2}\text{ Fall} = \frac{1}{2}\text{ Rise} = \frac{1}{2} \times 6 = 3 \text{ mm}$$

and maximum velocity while falling,

$$v_{max} = \frac{2\pi}{t} \times A = \frac{2\pi}{0.1} \times 3 = 118.5 \text{ mm/s} = 0.1885 \text{ m/s}$$

∴ Maximum angular velocity of tilt of the axle or angular velocity of precession,

$$\omega_{P\,max} = \frac{v_{max}}{x} = \frac{0.1885}{1.5} = 0.126 \text{ rad/s}$$

---

* Angular momentum about axle = $I.\omega_W$

∴ Angular momentum about horizontal = $I.\omega_W \cos\theta$

We know that gyroscopic couple set up,

$$C = I.\omega.\omega_{P\,max} = 180 \times 29.3 \times 0.126 = 664.5 \text{ N-m } \textbf{Ans.}$$

The gyroscopic couple will act in a horizontal plane and this couple will tend to produce ***swerve*** *i.e.* it tends to turn the locomotive aside.

**2.** ***Reaction between the wheel and rail due to the gyroscopic couple***

We know that the reaction between the wheel and rail due to the gyroscopic couple is

$$P = C / x = 664.5 / 1.5 = 443 \text{ N } \textbf{Ans.}$$

## 14.10. Stability of a Two Wheel Vehicle Taking a Turn

Consider a two wheel vehicle (say a scooter or motor cycle) taking a right turn as shown in Fig. 14.14 (*a*).

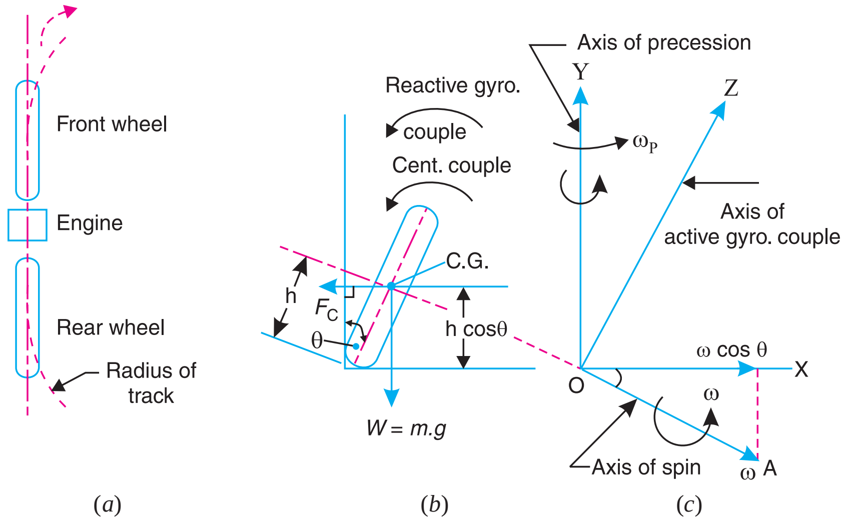



**Fig. 14.14.** Stability of a two wheel vehicle taking a turn.

Let $m$ = Mass of the vehicle and its rider in kg,

$W$ = Weight of the vehicle and its rider in newtons = $m.g$,

$h$ = Height of the centre of gravity of the vehicle and rider,

$r_W$ = Radius of the wheels,

$R$ = Radius of track or curvature,


Motorcycle taking a turn.

$I_W$ = Mass moment of inertia of each wheel,

$I_E$ = Mass moment of inertia of the rotating parts of the engine,

$\omega_W$ = Angular velocity of the wheels,

$\omega_E$ = Angular velocity of the engine,

$G$ = Gear ratio = $\omega_E / \omega_W$,

$v$ = Linear velocity of the vehicle = $\omega_W \times r_W$,

$\theta$ = Angle of heel. It is inclination of the vehicle to the vertical for equilibrium.

Let us now consider the effect of the gyroscopic couple and centrifugal couple on the vehicle, as discussed below.

### 1. *Effect of gyroscopic couple*

We know that $v = \omega_W \times r_W$ or $\omega_W = v / r_W$

and $$\omega_E = G.\omega_W = G \times \frac{v}{r_W}$$

$\therefore$ Total $$(I \times \omega) = 2\, I_W \times \omega_W \pm I_E \times \omega_E$$

$$= 2\, I_W \times \frac{v}{r_W} \pm I_E \times G \times \frac{v}{r_W} = \frac{v}{r_W}(2\, I_W \pm G.I_E)$$

and velocity of precession, $\omega_P = v / R$

A little consideration will show that when the wheels move over the curved path, the vehicle is always inclined at an angle $\theta$ with the vertical plane as shown in Fig. 14.14 (*b*). This angle is known as ***angle of heel.*** In other words, the axis of spin is inclined to the horizontal at an angle $\theta$, as shown in Fig. 14.14 (*c*). Thus the angular momentum vector $I\omega$ due to spin is represented by $OA$ inclined to $OX$ at an angle $\theta$. But the precession axis is vertical. Therefore the spin vector is resolved along $OX$.

$\therefore$ Gyroscopic couple,

$$C_1 = I.\omega \cos\theta \times \omega_P = \frac{v}{r_W}(2\, I_W \pm G.I_E)\cos\theta \times \frac{v}{R}$$

$$= \frac{v^2}{R.r_W}(2\, I_W \pm G.I_E)\cos\theta$$

**Notes : (*a*)** When the engine is rotating in the same direction as that of wheels, then the ***positive*** sign is used in the above expression and if the engine rotates in opposite direction, then ***negative*** sign is used.

**(*b*)** The gyroscopic couple will act over the vehicle outwards *i.e.* in the anticlockwise direction when seen from the front of the vehicle. The tendency of this couple is to overturn the vehicle in outward direction.

An aircraft of 1920's model.

**2. *Effect of centrifugal couple***

We know that centrifugal force,

$$F_C = \frac{m.v^2}{R}$$

This force acts horizontally through the centre of gravity (*C.G.*) along the outward direction.

∴ Centrifugal couple,

$$C_2 = F_C \times h \cos\theta = \left(\frac{m.v^2}{R}\right) h \cos\theta$$

Since the centrifugal couple has a tendency to overturn the vehicle, therefore

Total overturning couple,

$$C_O = \text{Gyroscopic couple} + \text{Centrifugal couple}$$

$$= \frac{v^2}{R.r_W}(2\,I_W + G.I_E)\cos\theta + \frac{m.v^2}{R} \times h\cos\theta$$

$$= \frac{v^2}{R}\left[\frac{2\,I_W + G.I_E}{r_W} + m.h\right]\cos\theta$$

We know that balancing couple = $m.g.h \sin\theta$

The balancing couple acts in clockwise direction when seen from the front of the vehicle. Therefore for stability, the overturning couple must be equal to the balancing couple, *i.e.*

$$\frac{v^2}{R}\left(\frac{2\,I_W + G.I_E}{r_W} + m.h\right)\cos\theta = m.g.h\sin\theta$$

From this expression, the value of the angle of heel (θ) may be determined, so that the vehicle does not skid.

**Example 14.15.** *Find the angle of inclination with respect to the vertical of a two wheeler negotiating a turn. Given : combined mass of the vehicle with its rider 250 kg ; moment of inertia of the engine flywheel 0.3 kg-m² ; moment of inertia of each road wheel 1 kg-m² ; speed of engine flywheel 5 times that of road wheels and in the same direction ; height of centre of gravity of rider with vehicle 0.6 m ; two wheeler speed 90 km/h ; wheel radius 300 mm ; radius of turn 50 m.*

**Solution.** Given : $m = 250$ kg ; $I_E = 0.3$ kg-m$^2$ ; $I_W = 1$ kg-m$^2$ ; $\omega_E = 5\,\omega_W$ or $G = \frac{\omega_E}{\omega_W} = 5$ ; $h = 0.6$ m ; $v = 90$ km/h = 25 m/s ; $r_W = 300$ mm = 0.3 m ; $R = 50$ m

Let θ = Angle of inclination with respect to the vertical of a two wheeler.

We know that gyroscopic couple,

$$C_1 = \frac{v^2}{R \times r_W}(2\,I_W + G.I_E)\cos\theta = \frac{(25)^2}{50 \times 0.3}(2 \times 1 + 5 \times 0.3)\cos\theta$$

$$= 146\cos\theta \text{ N-m}$$

and centrifugal couple,

$$C_2 = \frac{m.v^2}{R} \times h\cos\theta = \frac{250\,(25)^2}{50} \times 0.6\cos\theta = 1875\cos\theta \text{ N-m}$$

∴ Total overturning couple,

$$= C_1 + C_2 = 146\cos\theta + 1875\cos\theta = 2021\cos\theta \text{ N-m}$$

We know that balancing couple

$$= m.g.h\sin\theta = 250 \times 9.81 \times 0.6\sin\theta = 1471.5\sin\theta \text{ N-m}$$

Since the overturning couple must be equal to the balancing couple for equilibrium condition, therefore

$$2021 \cos \theta = 1471.5 \sin \theta$$

$$\therefore \quad \tan \theta = \sin \theta / \cos \theta = 2021 / 1471.5 = 1.3734 \text{ or } \theta = 53.94° \quad \textbf{Ans.}$$

**Example 14.16.** *A gyrowheel D of mass 0.5 kg, with a radius of gyration of 20 mm, is mounted in a pivoted frame C as shown in Fig. 14.15. The axis AB of the pivots passes through the centre of rotation O of the wheel, but the centre of gravity G of the frame C is 10 mm below O. The frame has a mass of 0.30 kg and the speed of rotation of the wheel is 3000 r.p.m. in the anticlockwise direction as shown.*

*The entire unit is mounted on a vehicle so that the axis AB is parallel to the direction of motion of the vehicle. If the vehicle travels at 15 m/s in a curve of 50 metres radius, find the inclination of the gyrowheel from the vertical, when*

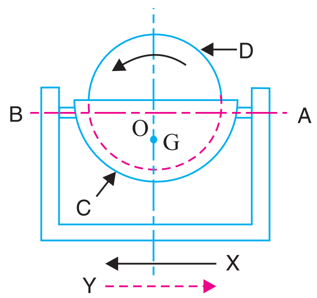



Fig. 14.15

*1. The vehicle moves in the direction of the arrow 'X' taking a left hand turn along the curve, and*

*2. The vehicle reverse at the same speed in the direction of arrow 'Y' along the same path.*

**Solution.** Given : $m_1 = 0.5$ kg ; $k = 20$ mm $= 0.02$ m ; $OG = h = 10$ mm $= 0.01$ m ; $m_2 = 0.3$ kg ; $N = 3000$ r.p.m. or $\omega = 2\pi \times 3000 / 60 = 314.2$ rad/s ; $v = 15$ m/s ; $R = 50$ m

We know that mass moment of inertia of the gyrowheel,

$$I = m_1.k^2 = 0.5\,(0.02)^2 = 0.0002 \text{ kg-m}^2$$

and angular velocity of precession,

$$\omega_P = v/R = 15 / 50 = 0.3 \text{ rad/s}$$

Let $\theta$ = Angle of inclination of gyrowheel from the vertical.

**1. *When the vehicle moves in the direction of arrow X taking a left turn along the curve***

We know that gyroscopic couple about $O$,

$$C_1 = I\,\omega.\omega_P \cos \theta = 0.0002 \times 314.2 \times 0.3 \cos \theta \text{ N-m}$$

$$= 0.019 \cos \theta \text{ N-m (anticlockwise)}$$

and centrifugal couple about $O$,

$$C_2 = \frac{m_2.v^2}{R} \times h \cos \theta = \frac{0.3\,(15)^2}{50} \times 0.01 \cos \theta \text{ N-m}$$

$$= 0.0135 \cos \theta \text{ N-m (anticlockwise)}$$

$\therefore$ Total overturning couple

$$= C_1 - C_2 = 0.019 \cos \theta - 0.0135 \cos \theta$$

... (– ve sign due to opposite direction)

$$= 0.0055 \cos \theta \text{ N-m (anticlockwise)}$$

We know that balancing couple due to weight ($W_2 = m_2.g$) of the frame about $O$,

$$= m_2.g.h \sin \theta = 0.3 \times 9.81 \times 0.01 \sin \theta \text{ N-m}$$

$$= 0.029 \sin \theta \text{ N-m (clockwise)}$$

Since the overturning couple must be equal to the balancing couple for equilibrium condition, therefore

$$0.0055 \cos \theta = 0.029 \sin \theta$$

or $$\tan \theta = \sin \theta / \cos \theta = 0.0055 / 0.029 = 0.1896$$

$\therefore$ $$\theta = 10.74° \textbf{ Ans.}$$

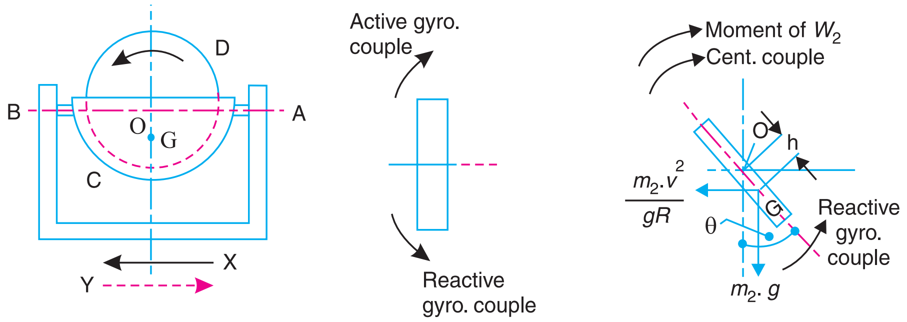



**Fig. 14.16**

### 2. *When the vehicle reverses at the same speed in the direction of arrow Y along the same path*

When the vehicle reverses at the same speed in the direction of arrow *Y*, then the gyroscopic and centrifugal couples ($C_1$ and $C_2$) will be in clockwise direction about *O* and the balancing couple due to weight ($W_2 = m_2.g$) of the frame about *O* will be in anticlockwise direction.

$\therefore$ Total overturning couple

$$= C_1 + C_2 = 0.019 \cos \theta + 0.0135 \cos \theta = 0.0325 \cos \theta \text{ N-m}$$

Equating the total overturning couple to the balancing couple, we have

$$0.0325 \cos \theta = 0.029 \sin \theta$$

or $$\tan \theta = \sin \theta / \cos \theta = 0.0325 / 0.029 = 1.1207$$

$\therefore$ $$\theta = 48.26° \textbf{ Ans.}$$

## 14.11. Effect of Gyroscopic Couple on a Disc Fixed Rigidly at a Certain Angle to a Rotating Shaft

Consider a disc fixed rigidly to a rotating shaft such that the polar axis of the disc makes an angle θ with the shaft axis, as shown in Fig. 14.17. Let the shaft rotates with an angular velocity ω rad/s in the clockwise direction when viewed from the front. A little consideration will show that the disc will also rotate about *OX* with the same angular velocity ω rad/s. Let *OP* be the polar axis and *OD* the diametral axis of the disc.

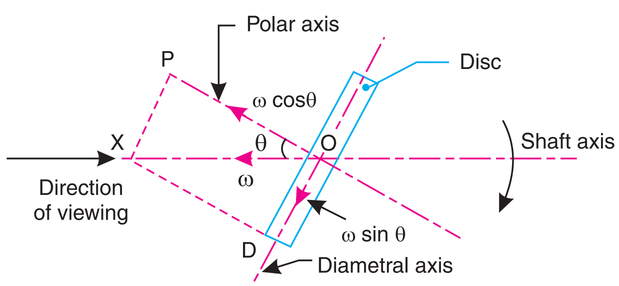



**Fig. 14.17.** Effect of gyroscopic couple on a disc fixed rigidly at a certain angle to a rotating shaft.

∴ Angular velocity of the disc about the polar axis *OP* or the angular velocity of spin

$= \omega \cos \theta$ ... (Component of ω in the direction of *OP*)

Since the shaft rotates, therefore the point *P* will move in a plane perpendicular to the plane of paper. In other words, precession is produced about *OD*.

∴ Angular velocity of the disc about the diametral axis *OD* or the angular velocity of precession

$= \omega \sin \theta$

If $I_P$ is the mass moment of inertia of the disc about the polar axis *OP*, then gyroscopic couple acting on the disc,

$$C_P = I_P.\omega \cos \theta.\omega \sin \theta = \frac{1}{2} \times I_P.\omega^2 \sin 2\theta$$

... ($\because$ 2 sin θ cos θ = sin 2θ)

The effect of this gyroscopic couple is to turn the disc in the anticlockwise when viewed from the top, about an axis through *O* in the plane of paper.

Now consider the movement of point *D* about the polar axis *OP*. In this case, *OD* is axis of spin and *OP* is the axis of precession.

∴ Angular velocity of disc about *OD* or angular velocity of spin

$= \omega \sin \theta$

and angular velocity of *D* about *OP* or angular velocity of precession

$= \omega \cos \theta$

If $I_D$ is the mass moment of inertia of the disc about the diametral axis *OD*, then gyroscopic couple acting on the disc,

$$C_D = I_D.\omega \sin \theta.\omega \cos \theta = \frac{1}{2} \times I_D.\omega^2 \sin 2\theta$$

The effect of this couple will be opposite to that of $C_P$.

∴ Resultant gyroscopic couple acting on the disc,

$$C = C_P - C_D = \frac{1}{2} \times \omega^2 \sin 2\theta \,(I_P - I_D)$$

This resultant gyroscopic couple will act in the anticlockwise direction as seen from the top. In other words, the shaft tends to turn in the plane of paper in anticlockwise direction as seen from the top, as a result the horizontal force is exerted on the shaft bearings.

**Notes: 1.** The mass moment of inertia of the disc about polar axis *OP*,

$$I_P = m.r^2/2$$

and mass moment of inertia of the disc about diametral axis *OD*,

$$I_D = m\left(\frac{l^2}{12} + \frac{r^2}{4}\right)$$

where $m$ = Mass of disc,

$r$ = Radius of disc, and

$l$ = Width of disc.

**2.** If the disc is thin, *l* may be neglected. In such a case

$$I_D = m.r^2/4$$

$$\therefore \quad C = \frac{1}{2} \times \omega^2 \sin 2\theta \left(\frac{m.r^2}{2} - \frac{m.r^2}{4}\right) = \frac{m}{8} \times \omega^2.r^2 \sin 2\theta$$

**Example 14.17.** *A shaft carries a uniform thin disc of 0.6 m diameter and mass 30 kg. The disc is out of truth and makes an angle of 1° with a plane at right angles to the axis of the shaft. Find the gyroscopic couple acting on the bearing when the shaft rotates at 1200 r.p.m.*

**Solution.** Given : $d$ = 0.6 m or $r$ = 0.3 m , $m$ = 30 kg ; $\theta$ = 1° ; $N$ = 1200 r.p.m. or $\omega = 2\pi \times 1200/60 = 125.7$ rad /s

We know that gyroscopic couple acting on the bearings,

$$C = \frac{m}{8} \times \omega^2 . r^2 \sin 2\theta = \frac{30}{8} (125.7)^2 \; (0.3)^2 \; \sin 2° = 186 \text{ N-m} \quad \textbf{Ans.}$$

## EXERCISES

**1.** A flywheel of mass 10 kg and radius of gyration 200 mm is spinning about its axis, which is horizontal and is suspended at a point distant 150 mm from the plane of rotation of the flywheel. Determine the angular velocity of precession of the flywheel. The spin speed of flywheel is 900 r.p.m. **[Ans. 0.39 rad/s]**

**2.** A horizontal axle $AB$, 1 m long, is pivoted at the mid point $C$. It carries a weight of 20 N at $A$ and a wheel weighing 50 N at $B$. The wheel is made to spin at a speed of 600 r.p.m in a clockwise direction looking from its front. Assuming that the weight of the flywheel is uniformly distributed around the rim whose mean diameter is 0.6 m, calculate the angular velocity of precession of the system around the vertical axis through $C$. **[Ans. 0.52 rad/s]**

**3.** An aeroplane runs at 600 km / h. The rotor of the engine weighs 4000 N with radius of gyration of 1 metre. The speed of rotor is 3000 r.p.m. in anticlockwise direction when seen from rear side of the aeroplane.

If the plane takes a loop upwards in a curve of 100 metres radius, find : 1. gyroscopic couple developed; and 2. effect of reaction gyroscopic couple developed on the body of aeroplane. **[Ans. 213.5 kN-m]**

**4.** An aeroplane makes a complete half circle of 50 metres radius, towards left, when flying at 200 km per hour. The rotary engine and the propeller of the plane has a mass of 400 kg with a radius of gyration of 300 mm. The engine runs at 2400 r.p.m. clockwise, when viewed from the rear. Find the gyroscopic couple on the aircraft and state its effect on it. What will be the effect, if the aeroplane turns to its right instead of to the left ? **[Ans. 10 kN-m]**

**5.** Each paddle wheel of a steamer have a mass of 1600 kg and a radius of gyration of 1.2 m. The steamer turns to port in a circle of 160 m radius at 24 km / h, the speed of the paddles being 90 r.p.m. Find the magnitude and effect of the gyroscopic couple acting on the steamer. **[Ans. 905.6 N-m]**

**6.** The rotor of the turbine of a yacht makes 1200 r.p.m. clockwise when viewed from stern. The rotor has a mass of 750 kg and its radius of gyration is 250 mm. Find the maximum gyroscopic couple transmitted to the hull (body of the yacht) when yacht pitches with maximum angular velocity of 1 rad /s. What is the effect of this couple ? **[Ans. 5892 N-m]**

**7.** The rotor of a turbine installed in a boat with its axis along the longitudinal axis of the boat makes 1500 r.p.m. clockwise when viewed from the stern. The rotor has a mass of 750 kg and a radius of gyration of 300 mm. If at an instant, the boat pitches in the longitudinal vertical plane so that the bow rises from the horizontal plane with an angular velocity of 1 rad /s, determine the torque acting on the boat and the direction in which it tends to turn the boat at the instant. **[Ans. 10.6 kN-m]**

**8.** The mass of a turbine rotor of a ship is 8 tonnes and has a radius of gyration 0.6 m. It rotates at 1800 r.p.m. clockwise when looking from the stern. Determine the gyroscopic effects in the following cases:

1. If the ship travelling at 100 km / h strees to the left in a curve of 75 m radius, 2. If the ship is pitching and the bow is descending with maximum velocity. The pitching is simple harmonic, the periodic time being 20 seconds and the total angular movement between the extreme positions is 10°, and 3. If the ship is rolling and at a certain instant has an angular velocity of 0.03 rad/s clockwise when looking from stern.

In each case, explain clearly how you determine the direction in which the ship tends to move as a result of the gyroscopic action. **[Ans. 201 kN-m ; 14.87 kN-m ; 16.3 kN-m]**

**9.** The turbine rotor of a ship has a mass of 20 tonnes and a radius of gyration of 0.75 m. Its speed is 2000 r.p.m. The ship pitches 6° above and below the horizontal position. One complete oscillation takes 18 seconds and the motion is simple harmonic. Calculate :

1. the maximum couple tending to shear the holding down bolts of the turbine, 2. the maximum angular acceleration of the ship during pitching, and 3. the direction in which the bow will tend to turn while rising, if the rotation of the rotor is clockwise when looking from rear.

**[Ans. 86.26 kN-m ; 0.0128 rad /s$^2$, towards star-board]**

**10.** A motor car takes a bend of 30 m radius at a speed of 60 km / hr. Determine the magnitudes of gyroscopic and centrifugal couples acting on the vehicle and state the effect that each of these has on the road reactions to the road wheels. Assume that :

Each road wheel has a moment of inertia of 3 kg-m$^2$ and an effective road radius of 0.4 m.

The rotating parts of the engine and transmission are equivalent to a flywheel of mass 75 kg with a radius of gyration of 100 mm. The engine turns in a clockwise direction when viewed from the front.

The back-axle ratio is 4 : 1, the drive through the gear box being direct. The gyroscopic effects of the half shafts at the back axle are to be ignored.

The car has a mass of 1200 kg and its centre of gravity is 0.6 m above the road wheel.

The turn is in a right hand direction.

If the turn has been in a left hand direction, all other details being unaltered, which answers, if any, need modification. **[Ans. 347.5 N-m : 6670 N-m]**

**11.** A rail car has a total mass of 4 tonnes. There are two axles, each of which together with its wheels and gearing has a total moment of inertia of 30 kg-m$^2$. The centre distance between the two wheels on an axle is 1.5 metres and each wheel is of 375 mm radius. Each axle is driven by a motor, the speed ratio between the two being 1 : 3. Each motor with its gear has a moment of inertia of 15 kg-m$^2$ and runs in a direction opposite to that of its axle. The centre of gravity of the car is 1.05 m above the rails.

Determine the limiting speed for this car, when it rounding a curve of 240 metres radius such that no wheel leaves the rail. Consider the centrifugal and gyroscopic effects completely. Assume that no cant is provided for outer rail. **[Ans. 144 km / h]**

**12.** A racing car weighs 20 kN. It has a wheel base of 2 m, track width 1 m and height of C.G. 300 mm above the ground level and lies midway between the front and rear axle. The engine flywheel rotates at 3000 r.p.m. clockwise when viewed from the front. The moment of inertia of the flywheel is 4 kg-m$^2$ and moment of inertia of each wheel is 3 kg-m$^2$. Find the reactions between the wheels and the ground when the car takes a curve of 15 m radius towards right at 30 km / h, taking into consideration the gyroscopic and the centrifugal effects. Each wheel radius is 400 mm.

**[Ans. Front inner wheel = 3341.7 N ; Front outer wheel = 6309.5 N ;**
**Rear inner wheel = 3690.5 N ; Rear outer wheel = 6658.3 N]**

**13.** A four wheel trolley car of total mass 2000 kg running on rails of 1 m gauge, rounds a curve of 25 m radius at 40 km / h. The track is banked at 10°. The wheels have an external diameter of 0.6 m and each pair of an axle has a mass of 200 kg. The radius of gyration for each pair is 250 mm. The height of C.G. of the car above the wheel base is 0.95 m. Allowing for centrifugal force and gyroscopic couple action, determine the pressure on each rail. **[Ans. 4328 N ; 16 704 N]**

**14.** A 2.2 tonne racing car has a wheel base of 2.4 m and a track of 1.4 m from the rear axle. The equivalent mass of engine parts is 140 kg with radius of gyration of 150 mm. The back axle ratio is 5. The engine shaft and flywheel rotate clockwise when viewed from the front. Each wheel has a diameter of 0.8 m and a moment of inertia of 0.7 kg-m$^2$. Determine the load distribution on the wheels when the car is rounding a curve of 100 m radius at a speed of 72 km / h to the left.

**15.** A disc has a mass of 30 kg and a radius of gyration about its axis of symmetry 125 mm while its radius of gyration about a diameter of the disc at right angles to the axis of symmetry is 75 mm. The disc is pressed on to the shaft but due to incorrect boring, the angle between the axis of symmetry and the actual axis of rotation is 0.25°, though both these axes pass through the centre of gravity of the disc. Assuming that the shaft is rigid and is carried between bearings 200 mm apart, determine the bearing forces due to the misalignment at a speed of 5000 r.p.m. **[Ans. 1810 N]**

16. A wheel of a locomotive, travelling on a level track at 90 km / h, falls in a spot hole 10 mm deep and rises again in a total time of 0.8 seconds. The displacement of the wheel takes place with simple harmonic motion. The wheel has a diameter of 3 m and the distance between the wheel centres is 1.75 m. The wheel pair with axle has a moment of inertia of 500 kg-m$^2$. Determine the magnitude and the effect of gyrocouple produced in this case. **[Ans. 186.6 N-m]**

17. Each road wheel of a motor cycle has a mass moment of inertia of 1.5 kg-m$^2$. The rotating parts of the engine of the motor cycle have a mass moment of inertia of 0.25 kg-m$^2$. The speed of the engine is 5 times the speed of the wheels and is in the same sense. The mass of the motor cycle with its rider is 250 kg and its centre of gravity is 0.6 m above the ground level.

    Find the angle of heel if the cycle is travelling at 50 km / h and is taking a turn of 30 m radius. The wheel diameter is 0.6 m. **[Ans. 35.7°]**

18. A racing motor cyclist travels at 140 km/h round a curve of 120 m radius measured horizontally. The cycle and rider have mass of 150 kg and their centre of gravity lies at 0.7 m above the ground level when the motor cycle is vertical. Each wheel is 0.6 m in diameter and has moment of inertia about its axis of rotation 1.5 kg-m$^2$. The engine has rotating parts whose moment of inertia about their axis of rotation is 0.25 kg-m$^2$ and it rotates at five times the wheel speed in the same direction. Find : 1. the correct angle of banking of the track so that there is no tendency to side slip, and 2. the correct angle of inclination of the cycle and rider to the vertical. **[Ans. 52.12°; 55.57°]**

    [**Hint.** In calculating the angle of banking of the track, neglect the effect of gyroscopic couple]

## DO YOU KNOW ?

1. Write a short note on gyroscope.
2. What do you understand by gyroscopic couple ? Derive a formula for its magnitude.
3. Explain the application of gyroscopic principles to aircrafts.
4. Describe the gyroscopic effect on sea going vessels.
5. Explain the effect of the gyroscopic couple on the reaction of the four wheels of a vehicle negotiating a curve.
6. Discuss the effect of the gyroscopic couple on a two wheeled vehicle when taking a turn.
7. What will be the effect of the gyroscopic couple on a disc fixed at a certain angle to a rotating shaft ?

## OBJECTIVE TYPE QUESTIONS

1. A disc is spinning with an angular velocity ω rad/s about the axis of spin. The couple applied to the disc causing precession will be

   (*a*) $\frac{1}{2} I.\omega^2$ (*b*) $I.\omega^2$ (*c*) $\frac{1}{2} I.\omega.\omega_P$ (*d*) $I.\omega.\omega_P$

   where $I$ = Mass moment of inertia of the disc, and

   $\omega_P$ = Angular velocity of precession of the axis of spin.

2. A disc spinning on its axis at 20 rad/s will undergo precession when a torque 100 N-m is applied about an axis normal to it at an angular speed, if mass moment of inertia of the disc is the 1 kg-m$^2$

   (*a*) 2 rad/s (*b*) 5 rad/s (*c*) 10 rad/s (*d*) 20 rad/s

3. The engine of an aeroplane rotates in clockwise direction when seen from the tail end and the aeroplane takes a turn to the left. The effect of the gyroscopic couple on the aeroplane will be

   (*a*) to raise the nose and dip the tail (*b*) to dip the nose and raise the tail

   (*c*) to raise the nose and tail (*d*) to dip the nose and tail

**4.** The air screw of an aeroplane is rotating clockwise when looking from the front. If it makes a left turn, the gyroscopic effect will

(*a*) tend to depress the nose and raise the tail

(*b*) tend to raise the nose and depress the tail

(*c*) tilt the aeroplane

(*d*) none of the above

**5.** The rotor of a ship rotates in clockwise direction when viewed from the stern and the ship takes a left turn. The effect of the gyroscopic couple acting on it will be

(*a*) to raise the bow and stern

(*b*) to lower the bow and stern

(*c*) to raise the bow and lower the stern

(*d*) to lower the bow and raise the stern

**6.** When the pitching of a ship is upward, the effect of gyroscopic couple acting on it will be

(*a*) to move the ship towards port side

(*b*) to move the ship towards star-board

(*c*) to raise the bow and lower the stern

(*d*) to raise the stern and lower the bow

**7.** In an automobile, if the vehicle makes a left turn, the gyroscopic torque

(*a*) increases the forces on the outer wheels

(*b*) decreases the forces on the outer wheels

(*c*) does not affect the forces on the outer wheels

(*d*) none of the above

**8.** A motor car moving at a certain speed takes a left turn in a curved path. If the engine rotates in the same direction as that of wheels, then due to the centrifugal forces

(*a*) the reaction on the inner wheels increases and on the outer wheels decreases

(*b*) the reaction on the outer wheels increases and on the inner wheels decreases

(*c*) the reaction on the front wheels increases and on the rear wheels decreases

(*d*) the reaction on the rear wheels increases and on the front wheels decreases

## ANSWERS

| | | | |
|---|---|---|---|
| **1.** (*d*) | **2.** (*b*) | **3.** (*a*) | **4.** (*b*) |
| **5.** (*c*) | **6.** (*b*) | **7.** (*a*) | **8.** (*b*) |

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