

Bark Frequency Transform Using an Arbitrary Order Allpass Filter

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Abstract—We propose an arbitrary order stable allpass filter structure for frequency transformation from Hertz to Bark scale. According to the proposed filter structure, the first order allpass filter is causal, but the second and higher order allpass filters are non-causal. We find that the accuracy of the transformation significantly improves when a second or higher order allpass filter is designed compared to a first order allpass filter. We also find that the RMS error of the transformation monotonically decreases by increasing the order of the allpass filter.

Index Terms—Allpass filter, bark scale.

I. INTRODUCTION

ANALYSIS and modeling of signal spectra over the Bark frequency scale is widely adopted in speech and audio signal processing. The Bark frequency scale closely resembles the frequency analysis scale in the human ear. It ranges from 1 to 24 Barks, corresponding to the first 24 critical bands of hearing [1]. The published Bark band edge frequencies (in Hertz) are 0, 100, 200, 300, 400, 510, 630, 770, 920, 1080, 1270, 1480, 1720, 2000, 2320, 2700, 3150, 3700, 4400, 5300, 6400, 7700, 9500, 12 000, 15 500. Let us denote them by b_k , $0 \leq k \leq 24$.

For discrete-time signal processing with sampling frequency F_s (say $F_s = 2b_L$, $1 \leq L \leq 24$), the Bark band edges correspond to points $\omega_l = (b_l)/(b_L)\pi$, $0 \leq l \leq L$ on the unit circle placed non-uniformly from 0 to π . Thus a proper frequency transformation is required to convert the linear frequency scale to the Bark scale. The use of a first-order allpass filter $D_\lambda(z)$ in place of a unit delay z^{-1} is a common approach [2], [3] to map uniformly spaced points on the unit circle to nonuniformly spaced points on the unit circle.

$$D_\lambda(z) = \frac{-\lambda + z^{-1}}{1 - \lambda z^{-1}} \quad (1)$$

where λ is the free parameter of the allpass filter. When $\lambda = 0$, $D_0(z) = z^{-1}$, a unit delay. By varying λ ($-1 < \lambda < 1$), one gets various types of warping on the unit circle [3]. Smith *et al.* [4] have shown that by properly selecting λ , such a first-order allpass transformation provides a good match to the Bark frequency scale. For $F_s = 31$ kHz, the Bark transformation and the best first-order allpass transformation obtained by Smith *et al.* [4] are shown in Fig. 1.

Manuscript received February 03, 2010; revised March 11, 2010. Date of publication March 22, 2010; date of current version April 21, 2010. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Jen-Tzung Chien.

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Digital Object Identifier 10.1109/LSP.2010.2046192

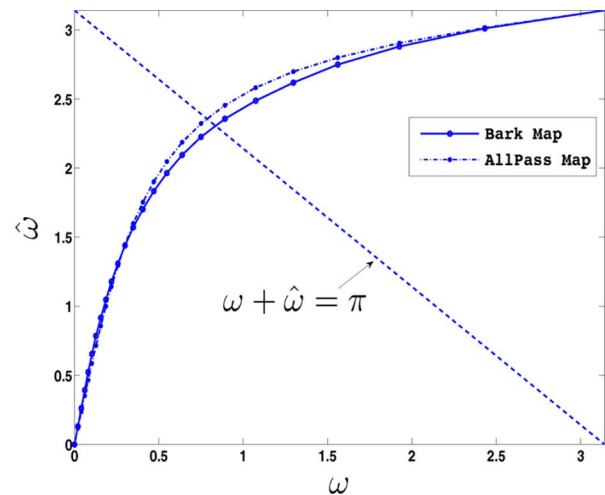


Fig. 1. Bark and allpass frequency warping at a sampling rate of 31 kHz. The parameter λ for the allpass warping is 0.707806 as reported by Smith *et al.* [4]. The allpass warping function is symmetric w.r.t. the line $\omega + \hat{\omega} = \pi$.

$z = e^{i\omega}$ and $\hat{z} = e^{i\hat{\omega}}$ in Fig. 1 correspond to original frequency and warped frequency scales respectively. It is easy to show [3] that,

$$\begin{aligned} \hat{z}^{-1}|_{\hat{z}=e^{i\hat{\omega}}} &= D_\lambda(z)|_{z=e^{i\omega}} = \frac{-\lambda + z^{-1}}{1 - \lambda z^{-1}}|_{z=e^{i\omega}} \\ \Rightarrow \tan\left(\frac{\hat{\omega}}{2}\right) &= \frac{1 + \lambda}{1 - \lambda} \tan\left(\frac{\omega}{2}\right). \end{aligned} \quad (2)$$

Thus the relation between ω and $\hat{\omega}$ in a first-order allpass map is linear in their tangent (illustrated in Fig. 1 for $F_s = 31$ kHz and $\lambda = 0.707806$). It is important to note that the relation between ω and $\hat{\omega}$ using a first-order allpass transformation is symmetric with respect to the line $\omega + \hat{\omega} = \pi$. This can be easily shown by replacing ω and $\hat{\omega}$ by $\pi - \hat{\omega}$ and $\pi - \omega$, respectively, in (2) and observing that the equation remains identical. On the other hand, the Bark frequency transformation does not have such symmetry property.

Here we propose an arbitrary order allpass filter structure; the first order allpass filter is a special case of the proposed filter structure. We find that by increasing the order of the allpass map, we can approximate the asymmetric Bark transformation better compared to a first-order allpass transformation. Higher order allpass transformations have been used to convert low-pass or highpass prototype filters into multiple bandpass/band-stop filters [5]. However, to the best of our knowledge, there has not been any previous work to design high-order allpass transformation to approximate the Bark scale. There are two issues in using high-order allpass transformation [4]: 1) an allpass transformation of order $R (> 1)$ maps the unit circle to

R traversals of the unit circle, hence, the mapping does not remain one-to-one anymore, 2) optimizing R free parameters ($\lambda_i, 1 \leq i \leq R$) becomes a nonlinear optimization problem in general and may suffer from local minima issues, which may not guarantee the stability of the allpass filter. Our proposed allpass filter overcomes the first problem at the cost of its non-causality. To overcome the second problem we use an appropriate initialization strategy for solving the optimization problem such that the designed allpass transformation is stable. Time-domain implementation of our proposed allpass filter (of order R) requires a look-ahead of $R - 1$ samples. Thus in practice, one needs to trade-off between the accuracy of the Bark scale transformation desired and the amount of delay permitted in a specific application.

II. PROPOSED HIGHER-ORDER ALLPASS TRANSFORMATION

We propose an allpass transformation $A_R(z)$ of order R between the original frequency ω and the warped frequency $\hat{\omega}$ as follows:

$$A_R(z) = z^{R-1} \prod_{r=1}^R \frac{-\lambda_r + z^{-1}}{1 - \lambda_r z^{-1}} \quad (3)$$

where $\lambda_r, 1 \leq r \leq R$ are free parameters. Note that if $\lambda_r = 0, 2 \leq r \leq R$, then $A_R(z)$ is a first-order allpass filter. Let $A_R(z)|_{z=e^{i\omega}} = e^{-ia(\omega)}$. Using (2), it is easy to show that

$$a(\omega) = \sum_{r=1}^R \phi_r(\omega) - (R-1)\omega$$

where,

$$\phi_r(\omega) = 2 \arctan \left(\frac{1 + \lambda_r}{1 - \lambda_r} \tan \left(\frac{\omega}{2} \right) \right). \quad (4)$$

Under such an allpass transformation, the relation between ω and $\hat{\omega}$ is $\hat{\omega} = a(\omega)$. Note that $a(\omega = 0) = 0$ and $a(\omega = \pi) = \pi$. It can be easily shown that the ‘‘winding number’’ of the mapping is only one, which is the difference between number of poles and zeros inside the unit circle. Also due to the factor z^{R-1} in $A_R(z)$, the time domain implementation of $A_R(z)$ requires a look-ahead of $R-1$ samples to compute the output at the current sample index. This means that $A_R(z)$ is non-causal.

III. DETERMINATION OF ALLPASS TRANSFORMATION PARAMETERS

Let us consider $F_s = 2bL, 1 \leq L \leq 24$. We want to minimize the error in frequency transformation in the discrete frequency grid corresponding to the Bark band-edges. Formally, we want to find $\lambda_r, 1 \leq r \leq R$, such that the following error function is minimized

$$J(\{\lambda_r, 1 \leq r \leq R\}) = \sum_{k=0}^L (\hat{\omega}_k - a(\omega_k))^2 \quad (5)$$

where $\omega_k = (\pi)/(F_s/2)b_k$ and $\hat{\omega}_k = (\pi)/(L)k, 0 \leq k \leq L$ are the normalized Bark band edge frequencies and the corresponding Bark frequencies, respectively. Note that $0 = \omega_0 < \omega_1 < \dots < \omega_L = \pi$ and $0 = \hat{\omega}_0 < \hat{\omega}_1 < \dots < \hat{\omega}_L = \pi$.

Let $\alpha_r = (1 + \lambda_r)/(1 - \lambda_r)$. Since α_r and λ_r have an one-to-one mapping, we can rewrite the optimization problem, using (4), as follows:

$$\begin{aligned} \{\alpha_r^*\} &= \arg \min_{\alpha_r, 1 \leq r \leq R} J(\{\alpha_r, 1 \leq r \leq R\}) \\ &= \arg \min_{\alpha_r, 1 \leq r \leq R} \sum_{k=0}^L \left(\hat{\omega}_k + (R-1)\omega_k \right. \\ &\quad \left. - \sum_{r=1}^R \left\{ 2 \arctan \left(\alpha_r \tan \left(\frac{\omega_k}{2} \right) \right) \right\} \right)^2. \end{aligned} \quad (6)$$

Once $\{\alpha_r^*\}$ are obtained, λ_r^* can be obtained by $\lambda_r^* = (\alpha_r^* - 1)/(\alpha_r^* + 1)$.

Note that the first and last terms in J amount to zero and hence do not contribute to J ; these terms correspond to $\hat{\omega}_k = \omega_k = 0$ and $\hat{\omega}_k = \omega_k = \pi$. Note also that J is a continuous function of $\alpha_r, \forall r$. The differentiation of J w.r.t. $\alpha_n, 1 \leq n \leq R$ is as follows:

$$\begin{aligned} \frac{\partial J}{\partial \alpha_n} &= \sum_{k=0}^L \left(\hat{\omega}_k + (R-1)\omega_k \right. \\ &\quad \left. - \sum_{r=1}^R \left\{ 2 \arctan \left(\alpha_r \tan \left(\frac{\omega_k}{2} \right) \right) \right\} \right) \\ &\quad \times \frac{-2 \tan \left(\frac{\omega_k}{2} \right)}{1 + \alpha_n^2 \tan^2 \left(\frac{\omega_k}{2} \right)}. \end{aligned} \quad (7)$$

Solving for $\alpha_r, 1 \leq r \leq R$ where $(\partial J)/(\partial \alpha_n) = 0$ requires the solution of a nonlinear equation. We also have to ensure that the allpass filter is stable because unstable allpass filters can not be implemented in time-domain; unstable allpass filters also fail to preserve stability when mapping a stable digital filter to a warped frequency scale. The allpass filter $A_R(z)$ will be stable if and only if $|\lambda_r^*| < 1$. This can only happen if we constrain $\alpha_r^* > 0$.

Let

$$\begin{aligned} \eta_{\min} &= \min_k \left\{ \frac{\tan \left(\frac{\hat{\omega}_k}{2} \right)}{\tan \left(\frac{\omega_k}{2} \right)}, 1 \leq k \leq L-1 \right\} \\ &\quad \text{and} \\ \eta_{\max} &= \max_k \left\{ \frac{\tan \left(\frac{\hat{\omega}_k}{2} \right)}{\tan \left(\frac{\omega_k}{2} \right)}, 1 \leq k \leq L-1 \right\}. \end{aligned}$$

If $0 \leq \alpha_m < \eta_{\min}$ and $0 \leq \alpha_r < 1, \forall r \neq m$, it is easy to show that

$$\begin{aligned} \hat{\omega}_k - 2 \arctan \left(\alpha_m \tan \left(\frac{\omega_k}{2} \right) \right) &> 0, \forall k \\ &\quad \text{and} \\ \omega_k - 2 \arctan \left(\alpha_m \tan \left(\frac{\omega_k}{2} \right) \right) &> 0, \forall k, \forall r \neq m. \end{aligned} \quad (8)$$

Let us denote the hyper-rectangle in the positive hyper-quadrant by

$\mathcal{S}_\alpha = \{\alpha_r : 0 \leq \alpha_m < \eta_{\min}, 0 \leq \alpha_r < 1, \forall r \neq m\}$. Thus, it is clear from (7) that

$$\frac{\partial J}{\partial \alpha_r} < 0, \forall r, \quad \text{if } \underline{\alpha} = [\alpha_1, \dots, \alpha_R]^T \in \mathcal{S}_\alpha. \quad (9)$$

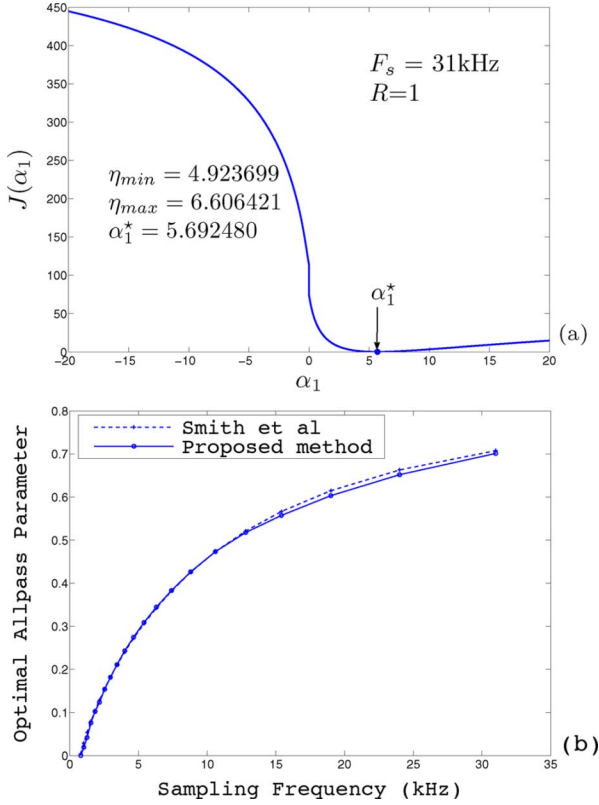


Fig. 2. (a) $J(\alpha_1)$ vs α_1 for $F_s = 31$ kHz. (b) Comparison between α_1^* and allpass parameter reported by Smith *et al.* [4] for different sampling frequencies.

This in turn implies that in \mathcal{S}_α , J decreases as α_r increases. On the other hand, when $\eta_{\max} \leq \alpha_r < \infty$, $(\partial J)/(\partial \alpha_r) > 0, \forall r$. This means J increases as α_r increases beyond $\eta_{\max}, \forall r$. Thus $(\partial J)/(\partial \alpha_r) = 0$ one or more times for $\eta_{\min} < \alpha_r < \eta_{\max}$.

Hence, there exist one or more minima of J in the positive hyper-quadrant of the R dimensional space ($\alpha_r > 0, \forall r$). We use the Nelder–Mead Simplex Method [6], [7] to find the minima of J . This is performed by using the widely available function `fminsearch` in the MATLAB optimization toolbox. The advantage of the Nelder–Mead Simplex Method is that it avoids the computation of the derivative of the objective function [7]. In the following subsections, we describe details about the optimization process. It should be noted that it is not guaranteed that the Nelder–Mead Simplex Method will provide the global optimum of J . But it, surprisingly, turns out that with appropriate initialization, the RMS error in the Bark scale approximation monotonically decreases for increasing allpass transformation order R , although the allpass parameter may not correspond to the global minimum of J for each choice of R .

A. $R = 1$ Case

$R = 1$ corresponds to a first-order allpass filter. We need to find only one free parameter α_1 such that $J(\alpha_1) = \sum_{k=0}^L (\hat{\omega}_k - 2 \arctan(\alpha_1 \tan(\omega_k)/(2)))^2$ is minimized. Fig. 2(a) plots $J(\alpha_1)$ as a function of α_1 for $L = 24$. η_{\min} and η_{\max} are indicated in the figure.

It is clear that $J(\alpha_1)$ is monotonically decreasing over $0 \leq \alpha_1 \leq \eta_{\min}$ and monotonically increasing for $\eta_{\max} < \alpha_1 <$

∞ . We initialize the optimization with $\alpha_1^0 = 1$ which corresponds to $\lambda_1^0 = 0$, i.e., no warping. Following Nelder–Mead Simplex optimization, the convergent solution is found to be $\alpha_1^* = 5.69248$, which corresponds to $\lambda_1^* = 0.701157$, which is very close to what Smith *et al.* reported (0.707806) in [4]. In fact, $J(\lambda_1^* = 0.701157) = 0.076081 < J(\lambda_1 = 0.707806) = 0.083455$ although Smith *et al.* did not directly minimize $J(\alpha_1)$. Fig. 2(b) compares λ_1^* for different sampling frequencies with those reported in [4]. It is clear from Fig. 2 that the best determined allpass parameters are very close to those obtained by Smith *et al.* They differ more at higher sampling frequencies.

B. $R > 1$ Case

$R > 1$ corresponds to higher-order allpass transformation. Thus the solution of the optimization problem (6) lies in R -dimensional space. We need to ensure that the final solution should be in the positive hyper-quadrant of the R dimensional space ($\alpha_r > 0, \forall r$) for stability of the optimized allpass filter $A_R(z)$. For this purpose we solve the optimization problem recursively. This means that we use the available optimal solution of order $R - 1$ to initialize the parameters for the case of order R with new dimension of the parameter vector initialized to 1 corresponding to no warping for the new allpass transformation factor¹. We find that by following such initialization strategy, the optimal solution using Nelder–Mead Simplex Method always turns out to be positive ($\alpha_R > 0$), which ensures stability of the overall allpass transformation $A_R(z)$. In addition, we observe that RMS error of the allpass transformation to the Bark scale monotonically decreases with increasing allpass order R for all sampling frequencies corresponding to the twice the Bark band edge frequencies. This is illustrated in Fig. 3(a). Let E_R denotes the squared error for using allpass transformation of order R . Fig. 3(a) plots $\delta E_R = E_R - E_{R+1}$ vs R for various sampling frequencies. Each curve in the figure corresponds to one sampling frequency. Logarithm of δE_R is plotted to illustrate the small reduction in δE_R with increasing R .

From Fig. 3(a), it is clear that $E_R > E_{R+1}, R = 1, \dots, 49$ for all choices of sampling frequency. The same trend is observed even for higher values of R . Note that the δE_R is maximum always for $R = 1$; this means the squared error reduces maximally when the order of the allpass transformation is changed from 1 to 2. For $R \geq 2$, the optimal allpass transformation does not alter significantly. This is clear from small δE_R values for higher R . This also becomes clear by noting the allpass parameters obtained as a result of optimization for high values of R (in Table I). With higher R , most of the λ_r turns out to be close to zero. Thus effectively the allpass transformation does not change significantly for $R > 3$.

Fig. 3(b) illustrates the RMS error of the allpass transformation across various F_s . For comparison, the RMS error is plotted for the best allpass transformation proposed in [4]. It is clear that our proposed method has lower RMS error at high sampling frequencies compared to that of [4]. It is also clear that the RMS error decreases as R increases from 1 to 2. But choosing an allpass transformation of order 3, 4, or 5 does not significantly reduce the RMS error compared to that for $R = 2$.

¹The Matlab implementation is available for download at <http://www-scf.usc.edu/~prasantg/BarkAllPass.tar>.

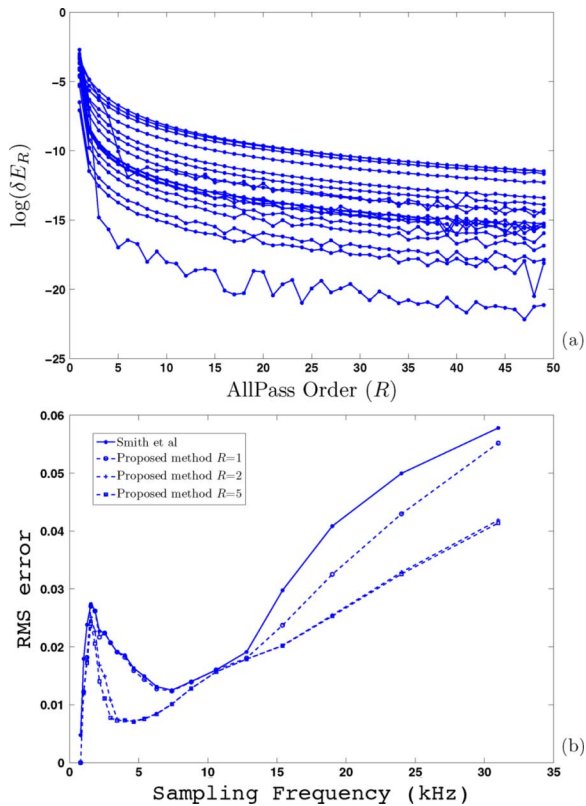


Fig. 3. (a) Logarithmic change in the RMS error ($\log(\delta E_R)$) between allpass transformation and the Bark scale for increasing allpass order R up to 50 ($\log(\delta E_R)$ at $R = i$ is the logarithmic difference in RMS error when R increases from i to $i + 1$). Each plot corresponds to a specific sampling frequency. The following sampling frequencies (in Hertz) are used—1540, 1840, 2160, 2540, 2960, 3440, 4000, 4640, 5400, 6300, 7400, 8800, 10600, 12 800, 15 400, 19 000, 24 000, 31 000. E_R decreases as R increases. (b) Comparison of the proposed allpass maps for $R = 1, 2, 5$ and the allpass transformation proposed by Smith *et al.* [4] in terms of RMS error versus F_s .

TABLE I
ALLPASS TRANSFORMATION PARAMETERS WITH
INCREASING R FOR $F_s = 31$ kHz

R	$\{\lambda_r, 1 \leq r \leq R\}$
1	0.70115
2	0.71390, -0.04820
3	0.71436, -0.02462, -0.02461
4	0.71452, -0.01653, -0.01653, -0.01653
5	0.71460, -0.01244, -0.01246, -0.01244, -0.01243

Finally, we analyze the relative bandwidth mapping error of the allpass transformation as defined in [4] for $F_s = 31$ kHz. Fig. 4(a) compares allpass maps of different orders with the Bark map. All of these allpass maps appear to have good match with the Bark frequency scale. However, it appears that with high order R , the relative bandwidth error increases at high frequency [Fig. 4(b)]. While the maximum relative bandwidth error is 20% for a first-order allpass transformation, the relative bandwidth error increases to 44% for the second and fifth order allpass transformation. This increase in relative bandwidth error

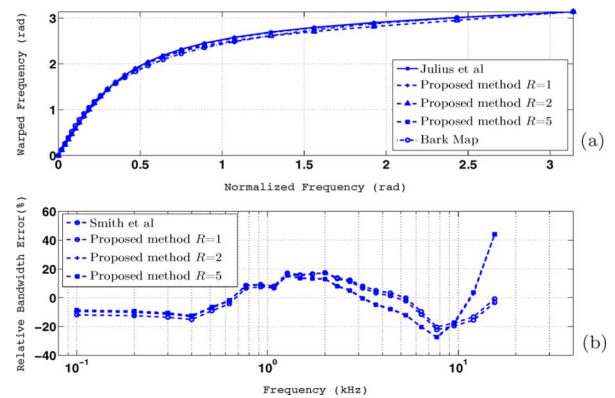


Fig. 4. (a) Visual comparison of the Bark map with the allpass maps proposed by Smith *et al.* [4] and obtained by the proposed method. (b) Relative bandwidth errors for different allpass maps.

happens only at the highest frequency. Such increase may occur since we are not explicitly minimizing relative bandwidth error in our optimization.

IV. CONCLUSION

We proposed a higher-order allpass transformation to approximate the Bark frequency scale of human auditory filters. The optimization problem to determine the best allpass filter parameters used for this purpose turns out to be nonlinear. With our proposed strategy for initialization, we found that we can design a high-order allpass transformation which has lower RMS error of approximation compared to that of a first-order allpass transformation. We also found that a second order ($R = 2$) allpass filter is sufficient to provide a good enough approximation to the Bark scale.

V. ACKNOWLEDGMENT

The insightful comments and advice of Prof. J. O. Smith, III are gratefully acknowledged. The authors also thank Dr. A. Patel for providing inputs to some derivations in the paper.

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