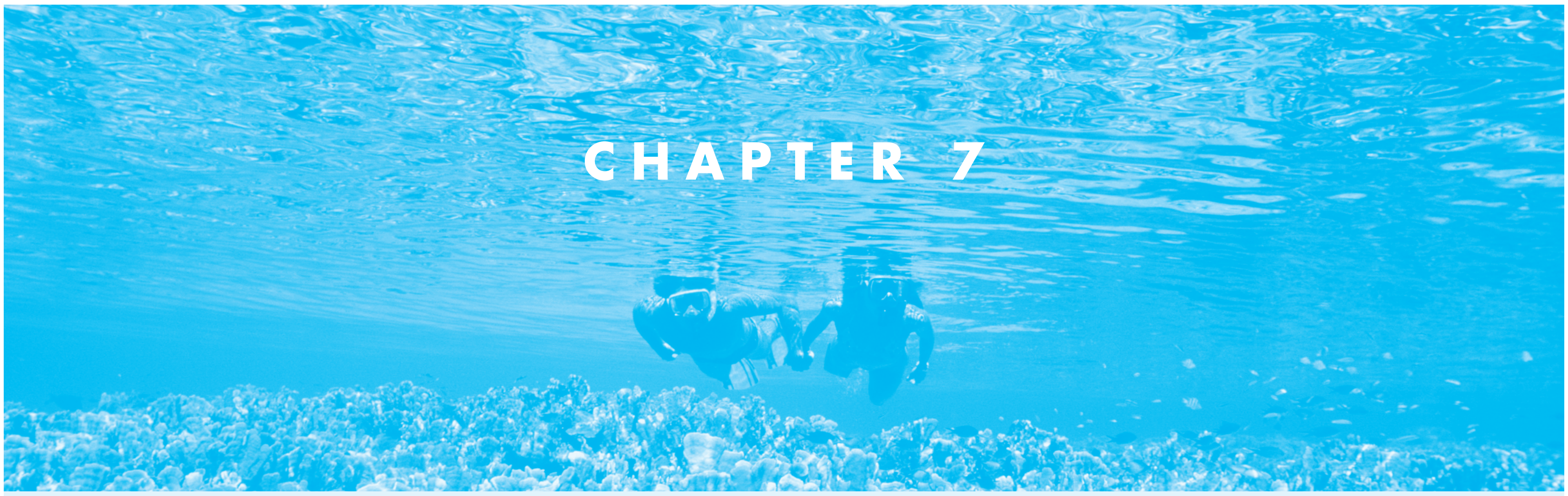


CHAPTER 7

POWER FUNCTIONS

OVERVIEW

Power functions help us answer questions such as "Why is the most important rule of scuba diving 'never hold your breath'?" and "Why do small animals have faster heartbeats and higher metabolic rates than large ones?"

After reading this chapter you should be able to

- recognize the properties of power functions
- construct and interpret graphs of power functions
- understand direct and inverse proportionality
- use logarithmic scales to determine whether a linear, power, or exponential function is the best function model for a set of data
- develop a sense about the relationship between size and shape

7.1 The Tension between Surface Area and Volume

Why do small animals have faster heartbeats and higher metabolic rates than large ones? Why are the shapes of the bodies and organs of large animals often quite different from those of small ones? To find answers to these questions, we examine how the relationship between surface area and volume changes as objects increase in size.

Scaling Up a Cube

Let's look at what happens to the surface area and volume of a simple geometric figure, the cube, as we increase its size. In Figure 7.1 we have drawn a series of cubes where the lengths of the edges are 1, 2, 3, and 4 units.[1]

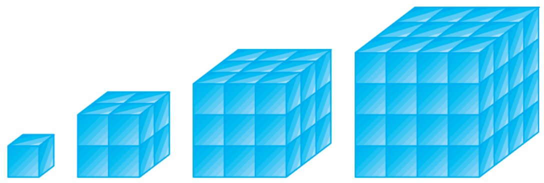


Figure 7.1 Four cubes for which the lengths of the edges are 1, 2, 3, and 4 units, respectively.

Surface area of a cube

If we were painting a cube, the surface area would tell us how much area we would have to cover. Each cube has six identical faces, so

$$\text{surface area} = 6 \cdot (\text{area of one face})$$

If the edge length of one face of the cube is x, then the surface area of that face is x^2. So the total surface area $S(x)$ of the cube is

$$S(x) = 6x^2$$

$S(x)$ is called a *power function* of degree 2.

The second column of Table 7.1 lists the surface areas of cubes for various edge lengths. Figure 7.2 shows a graph of the function $S(x)$.

Ratio of Surface Area to Volume of Cube with Length of Edge x

Edge Length x	Surface Area $S(x) = 6x^2$	Volume $V(x) = x^3$	$\frac{\text{Surface Area}}{\text{Volume}}$ $\frac{S(x)}{V(x)} = \frac{6x^2}{x^3} = \frac{6}{x} = R(x)$
1	6	1	6.00
2	24	8	3.00
3	54	27	2.00
4	96	64	1.50
6	216	216	1.00
8	384	512	0.75
10	600	1000	0.60

Table 7.1

[1]For this discussion it doesn't matter which unit we use, but if you prefer, you may think of "unit" as being "centimeter" or "foot."

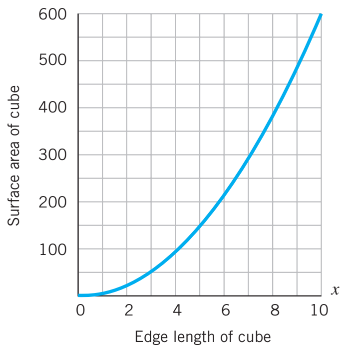


Figure 7.2 Graph of the surface area of a cube, $S(x) = 6x^2$.

Since the surface area of a cube with edge length x can be represented as

$$\text{surface area} = \text{constant} \cdot x^2$$

we say that the surface area is *directly proportional* to the *square* (or second power) of the length of its edge. In Table 7.1, observe what happens to the surface area when we double the length of an edge. If we double the length from 1 to 2 units, the surface area becomes four times larger, increasing from 6 to 24 square units. If we double the length from 2 to 4 units, the surface area is again four times larger, increasing this time from 24 to 96 square units. In general, if we double the length of the edge from x to $2x$, the surface area will increase by a factor of 2^2, or 4:

$$\begin{aligned}
&\text{Surface area of a cube with edge length } x\text{:} & S(x) &= 6x^2 \\
&\text{Surface area of a cube with edge length } 2x\text{:} & S(2x) &= 6(2x)^2 \\
&\quad \text{If we apply rules of exponents} & &= 6 \cdot 2^2 \cdot x^2 \\
&\quad \text{simplify and rearrange terms} & &= 4(6x^2) \\
&\quad \text{substitute } S(x) \text{ for } 6x^2\text{, we get} & S(2x) &= 4 \cdot S(x)
\end{aligned}$$

Volume of a cube

If the edge length of one face of the cube is x, then the volume $V(x)$ of the cube is given by

$$V(x) = x^3$$

$V(x)$ is an example of a power function of degree 3.

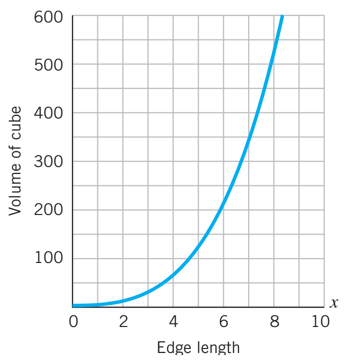


Figure 7.3 Graph of the volume of a cube, $V(x) = x^3$.

The third column of Table 7.1 lists values for $V(x)$, and Figure 7.3 is the graph of $V(x)$. Since

$$\text{volume} = \text{constant} \cdot x^3 \qquad \text{(the constant in this case is 1)}$$

we say that the volume is *directly proportional* to the *cube* (or the third power) of the length of its edge.

What happens to the volume when we double the length? In Table 7.1, if we double the length from 1 to 2 units, the volume increases by a factor of 2^3 or 8, increasing from 1 to 8 cubic units. If we double the length from 2 to 4 units, the volume again becomes eight times larger, increasing from 8 to 64 cubic units. In general, if we double the edge length from x to $2x$, the volume will increase by a factor of 8:

$$\begin{aligned}
&\text{Volume of a cube with edge length } x\text{:} & V(x) &= x^3 \\
&\text{Volume of a cube with edge length } 2x\text{:} & V(2x) &= (2x)^3 \\
&\quad\text{If we apply rules of exponents} & &= 2^3 \cdot x^3 \\
&\quad\text{simplify} & &= 8x^3 \\
&\quad\text{substitute } V(x) \text{ for } x^3\text{, we get} & V(2x) &= 8 \cdot V(x)
\end{aligned}$$

Ratio of the surface area to the volume

The ratio of (surface area)/volume generates a new function $R(x)$, where

$$R(x) = \frac{S(x)}{V(x)} = \frac{6x^2}{x^3} = \frac{6}{x} = 6x^{-1}$$

$R(x)$ is called a power function of degree –1. As we increase the value of x, the edge length of the cube, the value of $R(x)$ decreases, as we can see in Table 7.1 and Figure 7.4

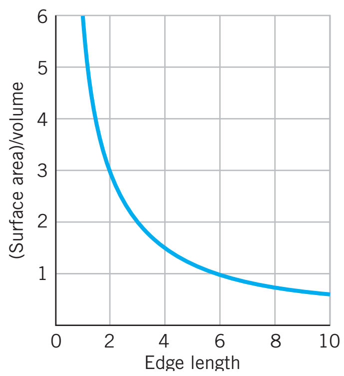


Figure 7.4 Graph of $R(x) =$ (surface area)/volume.

As the size of the cube increases, the volume increases faster than the surface area. For example, if we double the edge length, the surface area increases by a factor of 4, but the volume increases by a factor of 8. So the ratio

$$\frac{\text{surface area}}{\text{volume}}$$

decreases as the edge length increases.

Size and Shape

What we learned about the cube is true for any three-dimensional object, no matter what the shape. In general,

For any shape, as an object becomes larger while keeping the same shape, the ratio of its surface area to its volume decreases.

In Exploration 7.1 you can study further the effects of scaling up an object.

Thus a larger object has relatively less surface area than a smaller one. This fact allows us to understand some basic principles of biology and to answer the questions we asked at the beginning of this section.

Biological functions such as respiration and digestion depend upon surface area but must service the body's entire volume.[2] The biologist J. B. S. Haldane wrote that "comparative anatomy is largely the story of the struggle to increase surface in proportion to volume." This is why the shapes of the bodies and organs of large animals are often quite different from those of small ones. Many large species have adapted by developing complex organs with convoluted exteriors, thus greatly increasing the organs' surface areas. Human lungs, for instance, are heavily convoluted to increase the amount of surface area, thereby increasing the rate of exchange of gases. Stephen Jay Gould wrote that "the villi of our small intestine increase the surface area available for absorption of food."

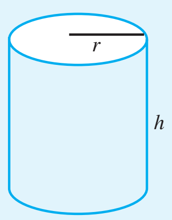
Stephen Jay Gould's essay "Size and Shape" in *Ever Since Darwin: Reflections in Natural History* offers an interesting perspective on the relationship between the size and shape of objects.

Body temperature also depends upon the ratio of surface area to volume. Animals generate the heat needed for their volume by metabolic activity and lose heat through their skin surface. Small animals have more surface area in proportion to their volume than do large animals. Since heat is exchanged through the skin, small animals lose heat proportionately faster than large animals and have to work harder to stay warm. Hence their heartbeats and metabolic rates are faster. As a result, smaller animals burn more energy per unit mass than larger animals.

Algebra Aerobics 7.1

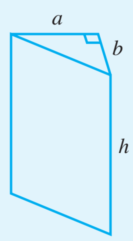
The inside back cover of the text contains geometric formulas.

1. The function $S(r) = 4\pi r^2$ gives the surface area of a sphere with radius r.
   - **a.** Compare $S(r)$, $S(2r)$, and $S(3r)$.
   - **b.** What happens to the surface area when the radius is doubled? When the radius is tripled?
2. The function $V(r) = \frac{4}{3}\pi r^3$ gives the volume of a sphere with radius r.
   - **a.** Compare $V(r)$, $V(2r)$, and $V(3r)$.
   - **b.** What happens to the volume when the radius is doubled? When the radius is tripled?
3. As the radius of a sphere increases, which grows faster, the surface area or the volume? What happens to the ratio $S(r)/V(r)$ = (surface area)/volume as the radius increases?
4. The two-dimensional analog to the ratio of (surface area)/volume for a sphere is the ratio of circumference/area for a circle.
   - **a.** The circumference $C(r)$ of a circle with radius r is $2\pi r$. Compare $C(r)$, $C(2r)$, and $C(4r)$. What happens to the circumference when the radius is doubled? Quadrupled (multiplied by 4)?
   - **b.** The area $A(r)$ of a circle with radius r is πr^2. Compare $A(r)$, $A(2r)$, and $A(4r)$. What happens to the area when the radius is doubled? Quadrupled (multiplied by 4)?
   - **c.** Which grows faster, the circumference or the area? As a result, what happens to the ratio $C(r)/A(r)$ = circumference/area as the radius increases?
5. Compare a sphere with radius r and a cube with edge length r.
   - **a.** For what value(s) of r, if any, is the volume of the cube equal to the volume of the sphere?
   - **b.** For what value(s) of r, if any, is the volume of the cube greater than the volume of the sphere? Less than?
6. Two different cylinders have the same height of 25 ft, but the base of one has a radius of 5 feet and the base of the other has a radius of 10 feet. Is the volume of one cylinder double the volume of the other? Justify your answer.
7. A cylindrical silo has a radius of 12 feet. If a certain amount of feed fills the silo to a depth of 5 feet, will twice the amount of feed fill the silo to a depth of 10 feet? Justify your answer.
8. The volume V of some regular figures can be defined as the area B of the base times the height h, or $V = Bh$.
   - **a.** Use this definition to find the formulas for the volumes in Figures 7.5 and 7.6.
   - **b.** In each case, if the height is doubled, by what factor is the volume increased?
   - **c.** If all dimensions are doubled, by what factor is the volume increased?



Cylinder

Figure 7.5



Triangular prism

Figure 7.6

[2]For those who want to investigate how species have adapted and evolved over time, see D.W. Thompson, *On Growth and Form* (New York: Dover, 1992), and T. McMahon and J. Bonner, *On Size and Life* (New York: Scientific American Books, 1983).

Exercises for Section 7.1

A calculator that can evaluate powers is required throughout. Formulas for geometric figures are on the inside back cover of the text.

1. Solve the following formulas for the indicated variable.

a. $V = lwh$, solve for l **c.** $P = 2l + 2w$, solve for w

b. $A = \dfrac{hb}{2}$, solve for b **d.** $S = 2x^2 + 4xh$, solve for h

2. In parts (a)–(c), evaluate the functions at R and $3R$.

a. $C(r) = 2\pi r$, the circumference of a circle with radius r

b. $A(r) = \pi r^2$, the area of a circle with radius r

c. $V(r) = \frac{4}{3}\pi r^3$, the volume of a sphere with radius r

d. Describe what happens to $C(r)$, $A(r)$, and $V(r)$ when the radius triples from R to $3R$.

3. Using the formulas in Exercise 2, simplify the following ratios and identify which ones are increasing as r increases. Describe each result in geometric terms.

a. $\dfrac{A(r)}{C(r)}$ **b.** $\dfrac{V(r)}{A(r)}$ **c.** $\dfrac{V(r)}{C(r)}$

4. Consider the formulas $V_1 = \frac{4}{3}\pi r^3$ (volume of a sphere) and $V_2 = \pi r^2 h$ (volume of a cylinder). Simplify the following ratios. Which one(s) increase as r increases (assume h is held constant)? Describe your result in geometric terms.

a. $\dfrac{V_2}{V_1}$ **b.** $\dfrac{V_1}{V_2}$

5. A tiny sphere has a radius of 0.000 000 000 1 $= 10^{-10}$ meter, which is roughly equivalent to the radius of a protein molecule. Answer the following questions. Express your answers in scientific notation.

a. Find its surface area, S, in square meters, where $S = 4\pi r^2$.

b. Find its volume, V, in cubic meters, where $V = \frac{4}{3}\pi r^3$.

c. Find the ratio of the surface area to the volume.

d. As r increases, does the ratio in part (c) increase or decrease?

6. The radius of Earth is about 6400 kilometers. Assume that Earth is spherical. Express your answers to the questions below in scientific notation.

a. Find the surface area of Earth in square meters.

b. Find the volume of Earth in cubic meters.

c. Find the ratio of the surface area to the volume.

7. If the radius of a sphere is x meters, what happens to the surface area and to the volume of a sphere when you:

a. Quadruple the radius?

b. Multiply the radius by n?

c. Divide the radius by 3?

d. Divide the radius by n?

8. Assume a box has a square base and the length of a side of the base is equal to twice the height of the box.

a. If the height is 4 inches, what are the dimensions of the base?

b. Write functions for the surface area and the volume that are dependent on the height, h.

c. If the volume has increased by a factor of 27, what has happened to the height?

d. As the height increases, what will happen to the ratio of (surface area)/volume?

9. A box has volume V = length · width · height.

a. Find the volume of a cereal box with dimensions of length = 19.5 cm, width = 5 cm, and height = 27 cm. (Be sure to specify the unit.)

b. If the length and width are doubled, by what factor is the volume increased?

c. What are two ways you could increase the volume by a factor of 4 and keep the height the same?

10. A box with a lid has a square base with each base side x inches and a height of h inches.

a. Write the formula for the volume of the box.

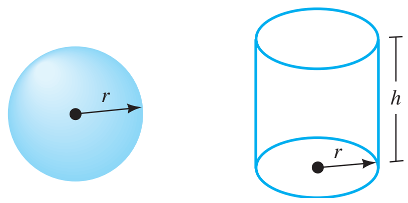
b. Write the formula for the surface area. (*Hint:* The surface area of the box is the sum of the areas of each side of the box.)

c. If the side of the base is tripled and the height remains constant, by what factor is the volume increased?

d. If the side of the base triples (from x to $3x$) with h held constant, what is the change in the surface area?

11. Consider a cylinder with volume $V = \pi r^2 h$. What happens to its volume when you double its height, h? When you double its radius, r?

12. Consider two solid figures, a sphere and cylinder, where each has radius r. The volume of a sphere is $V_s = \frac{4}{3}\pi r^3$ and the volume of a cylinder is $V_c = \pi r^2 h$.



a. If $h = 1$, when is the volume of the sphere greater than the volume of the cylinder?

b. What value of h would make the volumes the same?

c. Complete this statement: "The volume of the cylinder with radius r is greater than the volume of a sphere of radius r, when $h >$ ____."

13. The volume, V, of a cylindrical can is $V = \pi r^2 h$ and the total surface area, S, of the can is

$$S = \text{area of curved surface} + 2 \cdot (\text{area of base})$$
$$= 2\pi rh + 2\pi r^2$$

where r is the radius of the base and h is the height.

a. Assume the height is three times the radius. Write the volume and the surface area as functions of the radius.

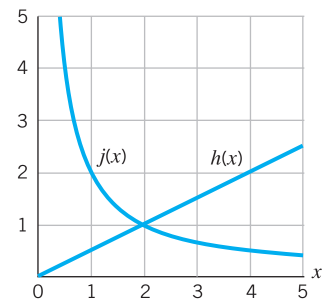
   b. As r increases, which grows faster, the volume or the surface area? Explain.

14. (*Hint:* See Exploration 7.1.) To celebrate Groundhog Day, a gourmet candy company makes a solid, quarter-pound chocolate groundhog that is 2 inches tall.

   a. If the company wants to introduce a solid chocolate groundhog in the same shape but twice as tall (4 inches high), how much chocolate is required?

   b. How tall would a solid chocolate groundhog be if made with twice the original quarter-pound amount (for a total of 8 ounces)?

15. A circle has a radius of x units. Which graph could represent the ratio of area to circumference? Which could represent the ratio of circumference to area? Explain your answers.



16. A cylinder (with no top or bottom) has a radius of x feet and a height of 3 feet.

   a. Construct a function for the volume and another for the surface area of the cylinder.

   b. Construct the ratio $\frac{\text{volume}}{\text{surface area}}$. What would happen to the ratio as x increases?

7.2 Direct Proportionality: Power Functions with Positive Powers

In the last section we encountered three functions of the edge length x of a cube:

Surface area	$S(x) = 6x^2$
Volume	$V(x) = x^3$
Ratio of $\dfrac{\text{surface area}}{\text{volume}}$	$R(x) = \dfrac{6}{x}$ or $6x^{-1}$

All three are called *power functions* since they are of the form

$$\text{dependent variable} = \text{constant} \cdot (\text{independent variable})^{\text{power}}$$

or

$$\text{output} = \text{constant} \cdot \text{input}^{\text{power}}$$

> **Power Functions**
>
> A power function $y = f(x)$ can be represented by an equation of the form
>
> $$y = kx^p$$
>
> where k and p are constants.

For the functions: $S(x) = 6x^2$, we have $k = 6$ and $p = 2$.
$V(x) = x^3$, we have $k = 1$ and $p = 3$.
$R(x) = 6x^{-1}$, we have $k = 6$ and $p = -1$.

EXAMPLE 1 Decide whether or not each of the following functions is a power function of the form $y = kx^p$, and if it is, identify the values of k and p.

a. $y = 3x^4$ **b.** $y = 3x^4 + 1$ **c.** $y = 2x^{-4}$ **d.** $y = -6\sqrt{x}$ **e.** $y = 4 \cdot 2^x$

SOLUTION

a. $y = 3x^4$ is a power function where $k = 3$ and $p = 4$.

b. $y = 3x^4 + 1$ is not a power function since it cannot be written in the form $y = kx^p$.

c. $y = 2x^{-4}$ is a power function where $k = 2$ and $p = -4$.

d. $y = -6\sqrt{x} = -6x^{1/2}$ is a power function where $k = -6$ and $p = \frac{1}{2}$.

e. $y = 4 \cdot 2^x$ is not a power function, since the variable x is in the exponent. It's an exponential function.

Direct Proportionality

In Chapter 2, for linear functions, we said that y is *directly proportional to x* if y equals a constant times x. For example, if $y = 4x$, then y is directly proportional to x. We can extend the same concept to any power function with positive exponents. If $y = kx^p$ and p is positive, we say that y is *directly proportional to x^p*.

In Section 7.1, we saw that the surface area of a cube is directly proportional to the square of its edge length and that the volume is directly proportional to the cube of its edge length. The symbol $\propto$ is used to indicate direct proportionality. For example, if $y = 5x^3$, then y is directly proportional to x^3, which we write as $y \propto x^3$.

Direct Proportionality

If

$$y = kx^p \qquad (k \neq 0 \text{ and } p > 0)$$

we say that y is *directly proportional to x^p*. We write this as

$$y \propto x^p$$

The coefficient k is called the *constant of proportionality.*

EXAMPLE 2 Write formulas to represent the following relationships.

a. The circumference, C, of a circle is directly proportional to its radius, r.

b. The area, A, of a circle is directly proportional to its radius, r, squared.

c. The volume, V, of a liquid flowing through a tube is directly proportional to the fourth power of the radius, r, of the tube.

SOLUTION **a.** $C = kr$, where $k = 2\pi$. **c.** $V = kr^4$ for some constant k.

b. $A = kr^2$, where $k = \pi$.

Properties of Direct Proportionality

Consider a general power function $f(x) = kx^p$ where $p > 0$. If we double the input from x to $2x$, we have

evaluating f at $2x$	$f(2x) = k(2x)^p$
using rules of exponents	$= k(2^p x^p)$
and rearranging terms	$= 2^p(kx^p)$
substituting $f(x)$ for kx^p	$= 2^p f(x)$

Doubling the input multiplies the output by 2^p. We saw in Section 7.1 that if we double the input for the function $S(x) = 6x^2$ (the surface area of a cube), the output is multiplied by 2^2 or 4.

If we multiply the input by m, changing the input from x to mx, then

evaluating f at mx	$f(mx) = k(mx)^p$
using rules of exponents	$= k(m^p x^p)$
and rearranging terms	$= m^p(kx^p)$
substituting $f(x)$ for kx^p	$= m^p f(x)$

SOMETHING TO THINK ABOUT

Weight is directly proportional to length cubed, but the ability to support the weight, as measured by the cross-sectional area of bones, is proportional to length squared. Why does this mean that Godzilla or King Kong could exist only in the movies?

So multiplying the input by m multiplies the output by m^p. For example, if we triple the input for $S(x) = 6x^2$, the output is multiplied by 3^2 or 9. Note that the value for k (the constant of proportionality) is irrelevant in these calculations.

In general,

> If y is directly proportional to x^p (where $p > 0$), then $y = kx^p$ for some nonzero constant k.
>
> Multiplying the input by m multiplies the output by m^p.
>
> For example, tripling the input multiplies the output by 3^p.

EXAMPLE 3 Given each function, what happens if the input is doubled? Increased by a factor of 10? Cut in half?

a. $y = 2x^4$ **b.** $h(z) = -2x^5$

SOLUTION

a. If the input is doubled, the output is multiplied by 2^4 or 16. If the input is multiplied by 10, the output is multiplied by 10^4 or 10,000. If the input is cut in half, the output would be multiplied by $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

b. If the input is doubled, the output is multiplied by 2^5 or 32. If the input is multiplied by 10, the output is multiplied by 10^5 or 100,000. If the input is cut in half, the output would be multiplied by $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.

EXAMPLE 4

a. What is the difference among $2f(x)$, $f(2x)$, and $f(x) + 2$?

b. If $f(x) = x^2$, evaluate the three expressions in part (a) when $x = 4$.

SOLUTION

a. $2f(x)$ means to multiply the value of $f(x)$ by 2.
$f(2x)$ means to use $2x$ as the input for the function f.
$f(x) + 2$ means to evaluate $f(x)$ and then add 2.

b. If $f(x) = x^2$ and $x = 4$, then

$$2f(4) = 2 \cdot 4^2 = 2 \cdot 16 = 32$$
$$f(2 \cdot 4) = f(8) = 8^2 = 64, \text{ or equivalently,}$$
$$= (2 \cdot 4)^2 = 2^2 \cdot 4^2 = 4 \cdot 16 = 64$$
$$f(4) + 2 = 4^2 + 2 = 16 + 2 = 18$$

EXAMPLE 5 **Pedaling into the wind**

When you pedal a bicycle, it's much easier to pedal with the wind than into the wind. That's because as the wind blows against an object, it exerts a force upon it. This force is directly proportional to the wind velocity squared.

a. Construct an equation to describe the relationship between the wind force and the wind velocity.

b. If the wind velocity doubles, by how much does the wind force go up?

SOLUTION

a. $F(v) = kv^2$ for some constant k, where v is the wind velocity and $F(v)$ is the wind force.

b. If the velocity doubles, then the wind force is multiplied by 2^2 or 4. So pedaling into a 20-mph wind requires four times as much effort as pedaling into a 10-mph wind. (See Figure 7.7.)

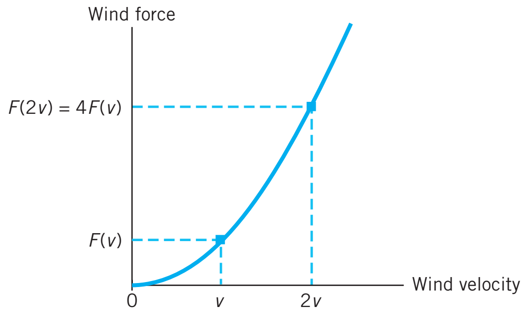


Figure 7.7 Doubling the wind velocity quadruples the wind force.

EXAMPLE 6 **Earth's core**

The radius of the core of Earth is slightly over half the radius of Earth as a whole, yet the core is only about 16% of the total volume of Earth. How is this possible?

SOLUTION We can think of both Earth and its core as approximately spherical in shape (see Figure 7.8).

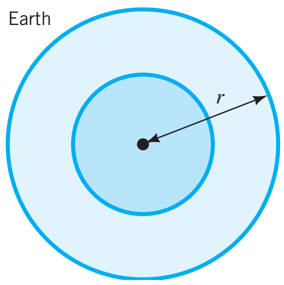


Figure 7.8 A sketch of a spherical Earth with radius r and its core.

If we let r = radius of Earth, then Earth's volume is

$$V_{\text{Earth}} = \frac{4}{3}\pi r^3$$

If the radius of the core were exactly half that of Earth $\left(\frac{r}{2}\right)$, then the core's volume would be

$$
\begin{aligned}
V_{\text{core}} &= \tfrac{4}{3}\pi\left(\tfrac{r}{2}\right)^3 \\
\text{use rules of exponents} \qquad &= \tfrac{4}{3}\pi\left(\tfrac{r^3}{2^3}\right) \\
\text{multiply and regroup terms} \qquad &= \left(\tfrac{1}{2^3}\right)\left(\tfrac{4}{3}\pi r^3\right) \\
\text{substitute } \tfrac{1}{8} \text{ for } \left(\tfrac{1}{2^3}\right) \text{ and } V_{\text{Earth}} \text{ for } \tfrac{4}{3}\pi r^3 \qquad &= \tfrac{1}{8}V_{\text{Earth}} \\
&= 12.5\% \text{ of the volume of Earth}
\end{aligned}
$$

Since the radius of the core is slightly more than half the radius of Earth, 16% of Earth's volume is a reasonable estimate for the volume of the core.

EXAMPLE 7 In Chapter 4 we encountered the following formula used by police. It estimates the speed, S, at which a car must have been traveling given the distance, d, the car skidded on a dry tar road after the brakes were applied:

$$S = \sqrt{30d} \approx 5.48d^{1/2}$$

Speed, S, is in miles per hour and distance, d, is in feet.

a. Use the language of proportionality to describe the relationship between S and d.

b. If the skid marks were 50 feet long, approximately how fast was the driver going when the brakes were applied?

c. What happens to S if d doubles? Quadruples?

d. Is d directly proportional to S?

SOLUTION **a.** The speed, S, is directly proportional to $d^{1/2}$ and the constant of proportionality is approximately 5.48.

b. If the length of the skid marks, d, is 50 feet, then the estimated speed of the car $S \approx 5.48 \cdot (50)^{1/2} \approx 5.48(7.07) \approx 39$ mph.

c. If the skid marks double in length from d to $2d$, then the estimate for the speed of the car increases from $\sqrt{30d}$ to $\sqrt{30 \cdot (2d)} = \sqrt{2} \cdot \sqrt{30d}$. So the estimated speed goes up by a factor of $\sqrt{2} \approx 1.414$ or, equivalently, by about 41.4%.

If the skid marks quadruple in length from d to $4d$, then the estimated speed of the car goes from $\sqrt{30d}$ to $\sqrt{30 \cdot (4d)} = \sqrt{4} \cdot \sqrt{30d} = 2\sqrt{30d}$. So the speed estimate goes up by a factor of 2, or by 100%.

For example, if the skid marks doubled from 50 to 100 feet, the speed estimate would go up from 39 mph to $1.414 \cdot 39 \approx 55$ mph. If the skid marks quadrupled from 50 to 200 feet, then the speed estimate would double from about 39 to almost 78 mph.

d. To determine if d is directly proportional to S, we need to solve our original equation for d.

Given	$S = \sqrt{30d}$
square both sides of the equation	$S^2 = 30d$
divide both sides by 30	$\frac{S^2}{30} = d$
or	$d = \frac{S^2}{30}$

So d is directly proportional to S^2 but not to S.

Direct Proportionality with More Than One Variable

When a quantity depends directly on more than one other quantity, we no longer have a simple power function. For example, the volume, V, of a cylindrical can depends on both the radius, r, of the base and the height, h. The equation describing this relationship is

$$V = \text{area of base} \cdot \text{height}$$
$$V = \pi r^2 h$$

We say V is directly proportional to both r^2 and h.

EXAMPLE 8 **Deflection**

A person stands at the center of a fir plank (with a cross section of 2″ by 12″) that is anchored at both ends (see Figure 7.9). Architects estimate the downward deflection of the plank using the formula

$$D = (6.6 \cdot 10^{-6}) \cdot P \cdot L^3$$

where D is the deflection in inches, P is the weight of the person in pounds, and L is the length of the plank in feet.[3]

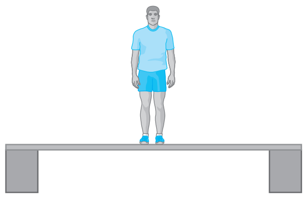


Figure 7.9 A person standing at the center of a plank.

a. Describe the relationship among D, P, and L using the language of proportionality.

b. What is the downward deflection of the plank if the person weighs 200 lb and the plank is 6′ long?

c. What happens to D if we increase P by 50%? If we increase L by 50%?

SOLUTION

a. D is directly proportional to P and to L^3.

b. If $P = 200$ lb and $L = 6'$, then

$$D = (6.6 \cdot 10^{-6}) \cdot 200 \cdot 6^3$$

rewrite 200 and rearrange terms $\quad = (6.6 \cdot 6^3 \cdot 2) \cdot 10^2 \cdot 10^{-6}$

evaluate $\quad = 2851.2 \cdot 10^{-4}$

≈ 0.3 inches

So the plank is about a third of an inch lower in the middle, under the 200-lb person.

c. Increasing the person's weight by 50% means multiplying the weight by 1.5. Since the deflection is directly proportional to the person's weight, that means that the deflection is multiplied by 1.5 or, equivalently, increased by 50%. So if we replace a 200-lb person (who causes a deflection of 0.3 inches) with someone 300 lb (or 50% heavier), the deflection would become $(1.5) \cdot (0.3) = 0.45$, or almost half an inch.

Increasing the plank's length by 50% means multiplying the length by 1.5. Since the deflection is directly proportional to the cube of the plank length, the deflection is multiplied by $(1.5)^3 \approx 3.4$ or, equivalently, increased by 240%. So for a 200-lb person, if the plank length of 6′ is increased by 50% [to $(1.5) \cdot 6 = 9'$], the projected deflection would be $(3.4) \cdot (0.3$ inches$) = 1.02$ inches, which is at the edge of reliability for this model.

[3]The deflection formula for the plank is quite accurate for deflections up to about an inch. After that it starts to produce unrealistic D values very quickly since L is cubed.

Algebra Aerobics 7.2

1. In each case, indicate whether or not the function is a power function. If it is, identify the independent and dependent variables, the constant of proportionality, and the power.
   a. $A = \pi r^2$ **c.** $z = w^5 + 10$ **e.** $y = 3x^5$
   b. $y = z^5$ **d.** $y = 5^x$
2. Identify which (if any) of the following equations represent direct proportionality:
   a. $y = 5.3x^2$ **c.** $y = 5.3 \cdot 2^x$
   b. $y = 5.3x^2 + 10$
3. Given $g(x) = 5x^3$
   a. Calculate $g(2)$ and compare this value with $g(4)$.
   b. Calculate $g(5)$ and compare this value with $g(10)$.
   c. What happens to the value of $g(x)$ if x doubles in value?
   d. What happens to $g(x)$ if x is divided by 2?
4. Given $h(x) = 0.5x^2$
   a. Calculate $h(2)$ and compare this value with $h(6)$.
   b. Calculate $h(5)$ and compare this value with $h(15)$.
   c. What happens to the value of $h(x)$ if x triples in value?
   d. What happens to $h(x)$ if x is divided by 3?
5. Express each of the following relationships with an equation.
   a. The volume of a sphere is directly proportional to the cube of its radius.
   b. The volume of a prism is directly proportional to its length, width, and height.
   c. The electrostatic force, f, is directly proportional to the particle's charge, c.
6. Express in your own words the relationship between y and x in the following functions.
   a. $y = 3x^5$ **b.** $y = 2.5x^3$ **c.** $y = \dfrac{x^5}{4}$
7. Express each of the relationships in Problem 6 in terms of direct proportionality.
8. **a.** Solve $P = aR^2$ for R.
   b. Solve $V = \left(\frac{1}{3}\right)\pi r^2 h$ for h.
9. If $f(x) = 0.1x^3$, evaluate and describe the difference between the following functions.
   a. $f(2x)$ and $2f(x)$
   b. $f(3x)$ and $3f(x)$
   c. Evaluate each function in part (a) for $x = 5$.

Exercises for Section 7.2

A calculator that can evaluate powers is useful here. A graphing program is recommended for Exercises 15 and 16.

1. Evaluate the following functions when $x = 2$ and $x = -2$:
   a. $f(x) = 5x^2$ **c.** $h(x) = -5x^2$
   b. $g(x) = 5x^3$ **d.** $k(x) = -5x^3$
2. Identify which of the following are power functions. For each power function, identify the value of k, the constant of proportionality, and the value of p, the power.
   a. $y = -3x^2$ **c.** $y = x^2 + 3$
   b. $y = 3x^{10}$ **d.** $y = 3^x$
3. Each of the following represents direct proportionality with powers of both x and y. Identify the constant of proportionality, k, and then evaluate each function for $x = 4$ and $y = 3$.
   a. $P = \frac{1}{2}x^3y^2$ **c.** $T = 4\pi x^2y^3$
   b. $M = 14x^{1/2}y$ **d.** $N = (18xy)^{1/3}$
4. Find the constant of proportionality, k, for the given conditions.
   a. $y = kx^3$, and $y = 64$ when $x = 2$.
   b. $y = kx^{3/2}$, and $y = 96$ when $x = 16$.
   c. $A = kr^2$, and $A = 4\pi$ when $r = 2$.
   d. $v = kt^2$, and $v = -256$ when $t = 4$.
5. Write the general formula to describe each variation and then solve the variation problem.
   a. y is directly proportional to x, and $y = 2$ when $x = 10$. Find y when x is 20.
   b. p is directly proportional to the square root of s, and $p = 12$ when $s = 4$. Find p when $s = 16$.
   c. A is directly proportional to the cube of r, and $A = 8\pi$ when $r = \sqrt[3]{6}$. Find A when $r = 2 \cdot \sqrt[3]{6}$.
   d. P is directly proportional to the square of m, and $P = 32$ when $m = 8$. If m is halved, what happens to P?
6. The data in the table satisfy the equation $y = kx^n$, where n is a positive integer. Find k and n.

x	2	3	4	5
y	1	2.25	4	6.25

7. Assume Y is directly proportional to X^3.
   a. Express this relationship as a function where Y is the dependent variable.
   b. If $Y = 10$ when $X = 2$, then find the value of the constant of proportionality in part (a).
   c. If X is increased by a factor of 5, what happens to the value of Y?

d. If X is divided by 2, what happens to the value of Y?

e. Rewrite your equation from part (a), solving for X. Is X directly proportional to Y?

8. The distance, d, a ball travels down an inclined plane is directly proportional to the square of the total time, t, of the motion.

a. Express this relationship as a function where d is the dependent variable.

b. If a ball starting at rest travels a total of 4 feet in 0.5 second, find the value of the constant of proportionality in part (a).

c. Complete the equation and solve for t. Is t directly proportional to d?

9. a. Assume L is directly proportional to x^5. What is the effect of doubling x?

b. Assume M is directly proportional to x^p, where p is a positive integer. What is the effect of doubling x?

10. In "Love That Dirty Water" (*Chicago Reader,* April 5, 1996), Scott Berinato interviewed Ernie Vanier, captain of the towboat *Debris Control.* The Captain said, "We've found a lot of bowling balls. You wouldn't think they'd float, but they do." When will a bowling ball float in water? The bowling rule book specifies that a regulation ball must have a circumference of exactly 27 inches. Recall that the circumference of a circle with radius r is $2\pi r$. The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

a. What is a regulation bowling ball's radius in inches?

b. What is the volume of a regulation bowling ball in cubic inches? (Retain at least two decimal places in your answer.)

c. What is the weight in pounds of a volume of water equivalent in size to a regulation bowling ball? (Water weighs 0.03612 lb/in^3).

d. A bowling ball will float when its weight is less than or equal to the weight of an equivalent volume of water. What is the heaviest weight of a regulation bowling ball that will float in water?

e. Typical men's bowling balls are 15 or 16 pounds. Women commonly use 12-pound bowling balls. What will happen to the men's and to the women's bowling balls when dropped into the water? Will they sink or float?

11. When a variable is directly proportional to the product of two or more variables, we say that the variable is *jointly* proportional to those variables. Express each of the following as an equation.

a. x is jointly proportional to y and the square of z.

b. V is jointly proportional to l, w, and h.

c. w is jointly proportional to the square of x and the cube root of y.

d. The volume of a cylinder is jointly proportional to its height and the square of its radius.

12. Write a general formula to describe each variation. Use the information given to find the constant of proportionality.

a. Q is directly proportional to both the cube root of t and the square of d, and $Q = 18$ when $t = 8$ and $d = 3$.

b. A is directly proportional to both h and the square of the radius, r, and $A = 100\pi$ when $r = 5$ and $h = 2$.

c. V is directly proportional to B and h, and $V = 192$ when $B = 48$ and $h = 4$.

d. T is directly proportional to both the square root of p and the square of u, and $T = 18$ when $p = 4$ and $u = 6$.

13. The cost, C, in dollars, of insulating a wall is directly proportional to the area, A, of the wall (measured in ft^2) and the thickness, t, of the insulation (measured in inches).

a. Write a cost equation for insulating a wall.

b. If the insulation costs are \$12 when the area is 50 ft^2 and the thickness is 4 inches, find the constant of proportionality.

c. A storage room is 15 ft by 20 ft with 8-ft-high ceilings. Assuming no windows and one uninsulated door that is 3 feet by 7 feet, what is the total area of the insulated walls for this room?

d. What is the cost of insulating this room with 4-inch insulation? With 6-inch insulation?

14. Suppose you are traveling in your car at speed S and you suddenly brake hard, leaving skid marks on the road. A "rule of thumb" for the distance, D, that the car will skid is given by

$$D = \frac{S^2}{30f}$$

where D = distance the car skids (in feet), S = speed of the car (in miles per hour), and f is a number called the coefficient of friction that depends on the road surface and its condition. For a dry tar road, $f \approx 1.0$. For a wet tar road, $f \approx 0.5$. (We saw a variation of this problem in Example 7, Section 7.2.)

a. What is the equation giving distance skidded as a function of speed for a dry tar road? For a wet tar road?

b. Generate a small table of values for both functions in part (a), including speeds between 0 and 100 miles per hour.

c. Plot both functions on the same grid.

d. Why do you think the coefficient of friction is less for a wet road than for a dry road? What effect does this have on the graph in part (c)?

e. In the accompanying table, estimate the speed given the distances skidded on dry and on wet tar roads. Describe the method you used to find these numbers.

Distance Skidded	Estimated Speed (mph)	
(ft)	Dry Tar	Wet Tar
25		
50		
100		
200		
300		

f. If one car is going twice as fast as another when they both jam on the brakes, how much farther will the faster car skid? Explain. Does your answer depend on whether the road is dry or wet?

15. (Graphing program recommended.) Assume a person weighing P pounds is standing at the center of a 4″ × 12″ fir plank that spans a distance of L feet. The downward deflection $D_{\text{deflection}}$ (in inches) of the plank can be described by

$$D_{\text{deflection}} = (5.25 \cdot 10^{-7}) \cdot P \cdot L^3$$

a. Graph the deflection formula, $D_{\text{deflection}}$, assuming $P = 200$ pounds. Put values of L (from 0 to 25 feet) on the horizontal axis and deflection (in inches) on the vertical axis.

b. A rule used by architects for estimating acceptable deflection, D_{safe}, in inches, of a beam L feet long bent downward as a result of carrying a load is

$$D_{\text{safe}} = 0.05L$$

Add to your graph from part (a) a plot of D_{safe}.

c. Is it safe for a 200-pound person to sit in the middle of a $4'' \times 12''$ fir plank that spans 20 feet? What maximum span is safe for a 200-pound person on a $4'' \times 12''$ plank?

16. (Graphing program recommended.) In Exercise 15 we looked at how much a single load (a person's weight) could bend a plank downward by being placed at its midpoint. Now we look at what a continuous load, such as a solid row of books spread evenly along a shelf, can do. A long row of paperback fiction weighs about 10 pounds for each foot of the row. Typical hardbound books weigh about 20 lb/ft, and oversize hardbounds such as atlases, encyclopedias, and dictionaries weigh around 36 lb/ft. The following function is used to model the deflection D (in inches) of a $1'' \times 12''$ common pine board spanning a length of L feet carrying a continuous row of books:

$$D = (4.87 \cdot 10^{-4}) \cdot W \cdot L^4$$

where W is the weight per foot of the type of books along the shelf. This deflection model is quite good for deflections up to 1 inch; beyond that the fourth power causes the deflection value to increase very rapidly into unrealistic numbers.

a. How much deflection does the formula predict for a shelf span of 30 inches with oversize books? Would you recommend a stronger, thicker shelf?

b. Plot $D_{\text{hardbound}}$, $D_{\text{paperback}}$, and D_{oversize} on the same graph. Put L on the horizontal axis with values up to 4 feet.

c. For each kind of book identify what length, L, will cause a deflection of 0.5 inch in a $1'' \times 12''$ pine shelf.

17. Construct formulas to represent the following relationships.

a. The distance, d, traveled by a falling object is directly proportional to the square of the time, t, traveled.

b. The energy, E, released is directly proportional to the mass, m, of the object and the speed of light, c, squared.

c. The area, A, of a triangle is directly proportional to its base, b, and height, h.

d. The reaction rate, R, is directly proportional to the concentration of oxygen, $[O_2]$, and the square of the concentration of nitric oxide, [NO].

e. When you drop a small sphere into a dense fluid such as oil, it eventually acquires a constant velocity, v, that is directly proportional to the square of its radius, r.

18. a. An insulation blanket for a cylindrical hot water heater is sold in a roll 48 in $\times$ 75 in $\times$ 2 in. Assuming a hot water heater 48 inches high, for what diameter water heater is this insulation blanket made? (Round to the nearest $\frac{1}{2}$ inch.)

b. What is the volume of the hot water heater in part (a)? (*Note:* Ignore the thickness of the heater walls.)

c. If 1 gallon $= 231$ in^3, what is the maximum number of gallons of water that a cylinder the size of this water heater could hold?

19. The volume, V, of a cylinder, with radius r and height h, is given by the formula $V = \pi r^2 h$. Describe what happens to V under the following conditions.

a. The *radius* is doubled; the *radius* is tripled.

b. The *height* is doubled; the *height* is tripled.

c. The *radius* r is multiplied by n, where n is a positive integer.

d. The *height* is multiplied by n, where n is a positive integer.

20. If you were designing cylinders, what are two different ways you could quadruple the volume? (See Exercise 19.)

21. Given $h(x) = 0.5x^2$:

a. Calculate $h(2)$ and compare this value with $h(6)$.

b. Calculate $h(5)$ and compare this value with $h(15)$.

c. What happens to the value of $h(x)$ if x triples in value?

d. What happens to the value of $h(x)$ if x is divided by 3?

22. a. Let $f(x) = 2x^2$. Describe the difference between $f(3x)$ and $3f(x)$.

b. Let $f(x) = 5x^3$. Describe the difference between $f(4x)$ and $4f(x)$.

c. In general, given $f(x) = kx^p$, describe the difference between $f(nx)$ and $nf(x)$, where p and n are positive integers.

7.3 Visualizing Positive Integer Powers

What do the graphs of power functions look like when the input values are infinitely large or assume negative values? Let's examine two basic power functions, $f(x) = x^2$ and $g(x) = x^3$.

The Graphs of $f(x) = x^2$ and $g(x) = x^3$

What happens when $x \geq 0$?

For $f(x) = x^2$ and $g(x) = x^3$, as $x \to +\infty$, both x^2 and $x^3 \to +\infty$. If $x = 0$, then x^2 and x^3 are both equal to 0, and if $x = 1$, both x^2 and x^3 equal 1. So $f(x)$ and $g(x)$ intersect at (0, 0) and (1, 1).

We can see from the graph in Figure 7.10 that when $0 < x < 1$, then $x^3 < x^2$, but when $x > 1$, then $x^3 > x^2$. So the graph of $g(x) = x^3$ eventually dominates the graph of $f(x) = x^2$.

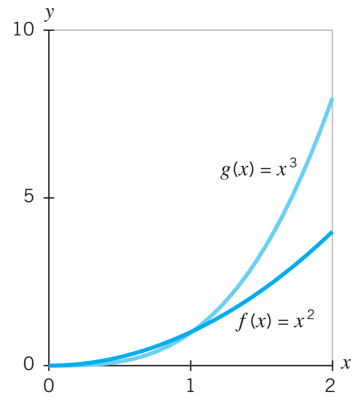


Figure 7.10 Comparing the graphs of $f(x) = x^2$ and $g(x) = x^3$ when $x \geq 0$.

What happens when x < 0?

When x is negative, the functions $f(x) = x^2$ and $g(x) = x^3$ exhibit quite different behaviors (see Table 7.2 and Figure 7.11). When x is negative, x^2 is positive but x^3 is negative. As $x \to -\infty$, $x^2 \to +\infty$ but $x^3 \to -\infty$. So the domain for both functions is $(-\infty, +\infty)$, but the range for $f(x) = x^2$ is $[0, +\infty)$. The range for $g(x) = x^3$ is $(-\infty, +\infty)$.

x	$f(x) = x^2$	$g(x) = x^3$
-4	16	-64
-3	9	-27
-2	4	-8
-1	1	-1
0	0	0
1	1	1
2	4	8
3	9	27
4	16	64

Table 7.2

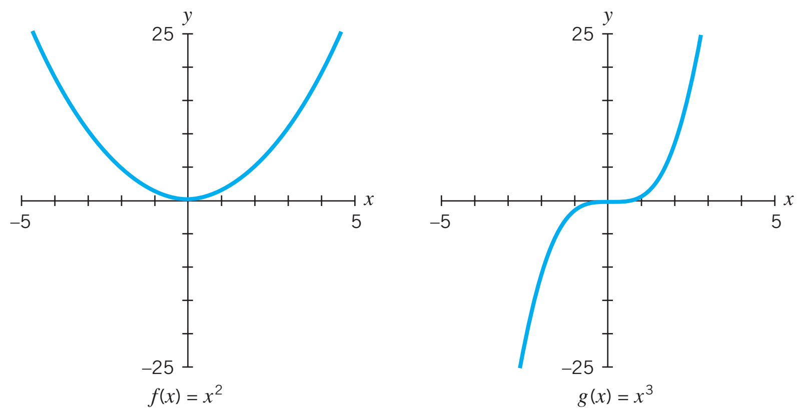


Figure 7.11 The graphs of $f(x) = x^2$ and $g(x) = x^3$.

Exploration 7.2 can help reinforce your understanding of power functions.

Odd vs. Even Powers

As we saw with the graphs of $f(x) = x^2$ and $g(x) = x^3$, the exponent of the power function affects the shape of the graph. The graphs of power functions with positive even powers are ∪-shaped and the graphs of positive odd powers greater than 1 are -shaped. Why does it matter if the exponent is even or odd?

If n is a positive integer and if x is positive, then x^n is positive. But if x is negative, we have to consider whether n is even or odd. If n is even, then x^n is positive, since

$$(\text{negative number})^{\text{even power}} = \text{positive number}$$

If n is odd, then x^n is negative, since

$$(\text{negative number})^{\text{odd power}} = \text{negative number}$$

So whether the exponent of a power function is odd or even will affect the shape of the graph.

If we graph the simplest power functions, $y = x$, $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$, $y = x^6$, and so on, we see quickly that the graphs fall into two groups: the odd powers and the even powers (Figure 7.12).

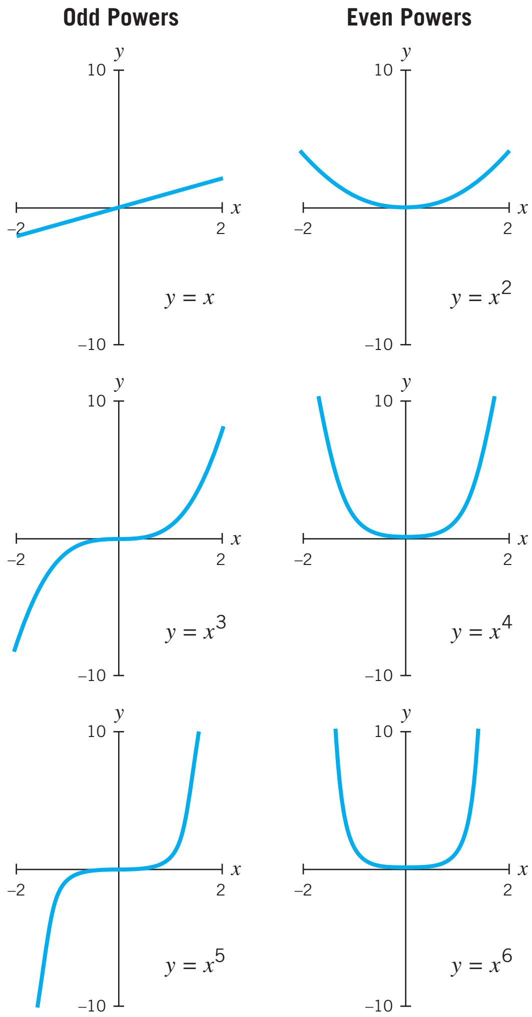


Figure 7.12 Graphs of odd and even power functions.

All the power functions graphed in Figure 7.12 have a positive coefficient k $(=1)$. So the graphs of the even powers in Figure 7.12 are all concave up. Graphs of the odd powers (except for the power 1) change concavity at $x = 0$. When $x < 0$, the graphs are concave down. When $x > 0$, the graphs are concave up.

Symmetry of the graphs

For the odd positive powers, the graphs are rotationally symmetric about the origin; that is, if you hold the graph fixed at the origin and then rotate it 180°, you end up with the same graph. As can be seen in Figure 7.13(a), the net result is the same as a double reflection, first across the y-axis and then across the x-axis.

The graphs of even powers are symmetric across the y-axis. You can think of the y-axis as a dividing line: the "left" side of the graph is a mirror image, or reflection, of the "right" side, as shown in Figure 7.13(b).

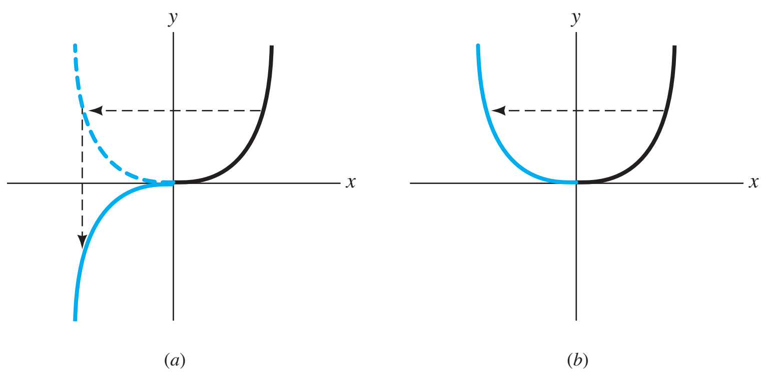


Figure 7.13 (a) Power functions with positive odd powers are symmetric about the origin. (b) Power functions with positive even powers are symmetric across the y-axis.

The Effect of the Coefficient *k*

The program "P1: k & p Sliders" in *Power Functions* can help you visualize the graphs of $y = kx^p$ for different values of k and p.

What is the effect of different values for the coefficient k on graphs of power functions in the form

$$y = kx^p \quad \text{where } p \text{ is a positive integer and } k \neq 0?$$

What happens when k > 0?

We know from our work with power functions of degree 1 (that is, linear functions of the form $y = kx$) that k affects the steepness of the line. For all power functions, where $k > 1$, the larger the value for k, the more vertical the graph of $y = kx^p$ becomes compared with $y = x^p$. As k increases, the steepness of the graph increases. We say the graph is *stretched vertically*.

When $0 < k < 1$, the graph of $y = kx^p$ is flatter than the graph of $y = x^p$ and lies closer to the x-axis. We say the graph is *compressed vertically*. The graphs in Figure 7.14 illustrate this effect for power functions of degrees 3 and 4.

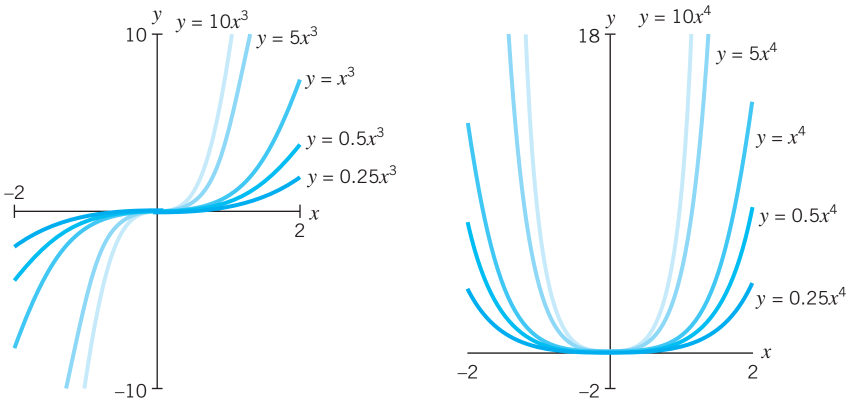


Figure 7.14 When $k > 0$, the larger the value of k, the more closely the power function $y = kx^p$ "hugs" the y-axis for both odd and even powers.

What happens when k < 0?

We know from our work with linear functions that the graphs of $y = kx$ and $y = -kx$ are mirror images across the x-axis. Similarly, the graphs of $y = kx^p$ and $y = -kx^p$ are mirror images of each other across the x-axis. For example, $y = -7x^3$ is the mirror image of $y = 7x^3$. Figure 7.15 shows various pairs of power functions of the type $y = kx^p$ and $y = -kx^p$.

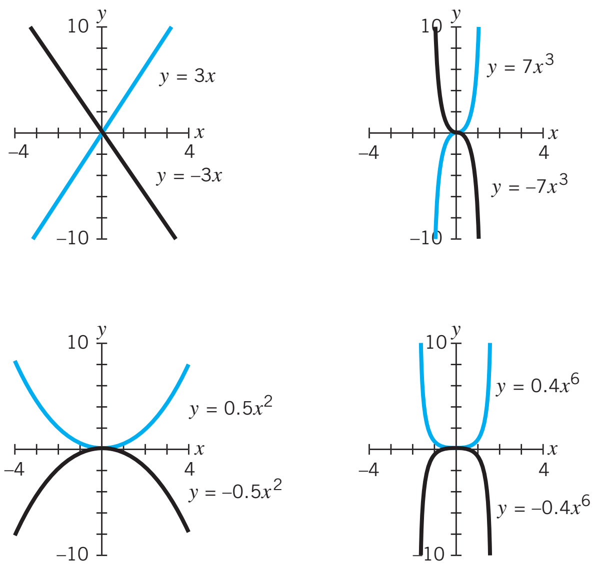


Figure 7.15 In each case, the graphs of $y = kx^p$ and $y = -kx^p$ (shown in blue and black, respectively) are mirror images of each other across the x-axis.

In sum: the value of k can stretch, compress, or reflect the graph

The graph of a power function in the form $y = kx^p$ is the graph of $y = x^p$

- vertically stretched by a factor of k, if $k > 1$
- vertically compressed by a factor of k, if $0 < k < 1$
- vertically stretched or compressed by a factor of $|k|$ and reflected across the x-axis, if $k < 0$

Graphs of Power Functions with Positive Integer Powers

For functions of the form $y = kx^p$ (where $p > 0$ and $k \neq 0$)

The graph goes through the origin.

If the power p is even, the graph is ∪-shaped and symmetric across the y-axis.

If the power p is odd, and > 1, the graph is shaped roughly like and is symmetric about the origin.

The coefficient k compresses or stretches the graph of $y = x^p$.

The graphs of $y = kx^p$ and $y = -kx^p$ are mirror images of each other across the x-axis.

EXAMPLE 1 Match each set of functions with the appropriate graph in Figure 7.16.

a. $i(x) = -x^2$
$j(x) = -3x^2$
$k(x) = -0.5x^2$

b. $l(x) = x^3$
$m(x) = 3x^3$
$n(x) = 0.5x^3$

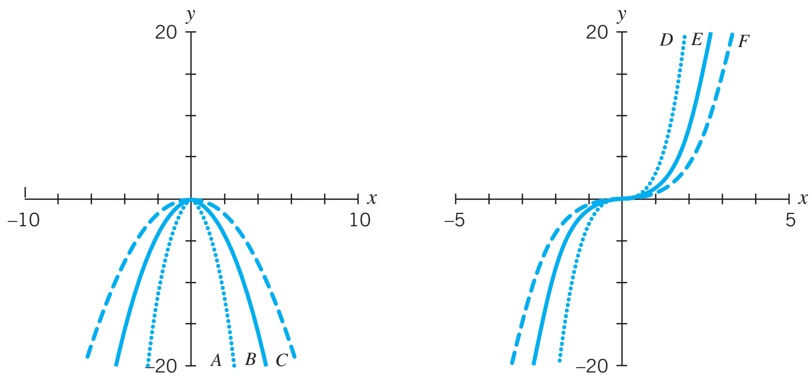


Figure 7.16 Two families of power functions.

SOLUTION **a.** A: $j(x) = -3x^2$
B: $i(x) = -x^2$
C: $k(x) = -0.5x^2$

b. D: $m(x) = 3x^3$
E: $l(x) = x^3$
F: $n(x) = 0.5x^3$

Algebra Aerobics 7.3

A graphing program is useful for Problems 5, 6, 9, and 10.

1. If $f(x) = 4x^3$, evaluate the following.

a. $f(2)$ **b.** $f(-2)$ **c.** $f(s)$ **d.** $f(3s)$

2. If $g(t) = -4t^3$, evaluate the following.

a. $g(2)$ **b.** $g(-2)$ **c.** $g\left(\frac{1}{2}t\right)$ **d.** $g(5t)$

3. If $f(x) = 3x^2$, evaluate:

a. $f(4)$ **c.** $f(s)$ **e.** $f(2s)$ **g.** $f\left(\frac{s}{2}\right)$

b. $f(-4)$ **d.** $2f(s)$ **f.** $f(3s)$ **h.** $f\left(\frac{s}{4}\right)$

4. a. For each of the following functions, generate a small table of values, including positive and negative values for x, and sketch the functions on the same grid:

$$f(x) = 4x^2 \quad \text{and} \quad g(x) = 4x^3$$

b. What happens to $f(x)$ and to $g(x)$ as $x \to +\infty$?

c. What happens to $f(x)$ and to $g(x)$ as $x \to -\infty$?

d. Specify the domain and range of each function.

e. Where do the graphs of these functions intersect?

f. For what values of x is $g(x) > f(x)$?

5. Graph both functions $f(x) = 2x^2$ and $g(x) = 2x^3$ over the interval $[-2, 2]$ and then determine values for x where:

a. $f(x) = g(x)$ **b.** $f(x) > g(x)$ **c.** $f(x) < g(x)$

6. Plot each pair of functions on the same grid.

a. $y_1 = -2x^2$ and $y_2 = -2x^3$

b. $y_1 = -0.1x^3$ and $y_2 = -0.1x^4$

7. Without graphing the functions, describe the differences in the graphs of each pair of functions.

a. $y_1 = 0.2x^3$ and $y_2 = 0.3x^3$

b. $y_1 = -0.15x^4$ and $y_2 = -0.2x^4$

8. Determine whether p in each power function of the form $y = ax^p$ in Figures 7.17 and 7.18 is even or odd and whether $a > 0$ or $a < 0$. Then find the value of a.

a.

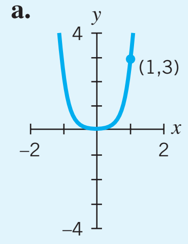


Figure 7.17

b.

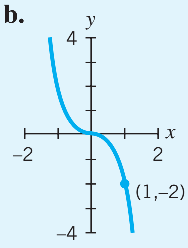


Figure 7.18

9. Draw a quick sketch by hand of the power functions.

a. $y = x^9$ **b.** $y = x^{10}$

If possible, check your graphs using technology.

10. Draw a rough sketch of:

a. $y = x$, $y = 4x$, and $y = -4x$ (all on the same grid)

b. $y = x^4$, $y = 0.5x^4$, and $y = -0.5x^4$ (all on the same grid)

If possible, check your graphs using technology.

Exercises for Section 7.3

A graphing program is useful for many of the exercises in this section.

1. Sketch by hand and compare the graphs of the following:

$y_1 = x^2$ $y_2 = -x^2$ $y_3 = 2x^2$

$y_4 = -2x^2$ $y_5 = \frac{1}{2}x^2$ $y_6 = -\frac{1}{2}x^2$

2. (Graphing program option.) For each part sketch by hand the three graphs on the same grid and clearly label each function. Describe how the three graphs are alike and not alike.

a. $y = x^1$ $y = x^3$ $y = x^5$

b. $y = x^2$ $y = x^4$ $y = x^6$

c. $y = x^3$ $y = 2x^3$ $y = -2x^3$

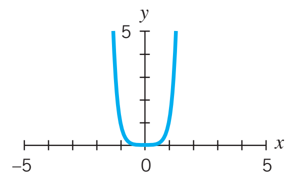
d. $y = x^2$ $y = 4x^2$ $y = -4x^2$

If possible, check your results using a graphing program.

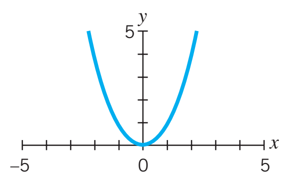
3. Match the function with its graph. Explain why you have chosen each graph.

a. $f(x) = x^2$ **c.** $h(x) = 2x^2$

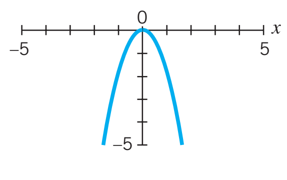
b. $g(x) = x^6$ **d.** $j(x) = -2x^2$



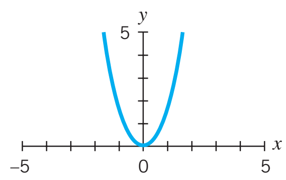
Graph *A*



Graph *C*



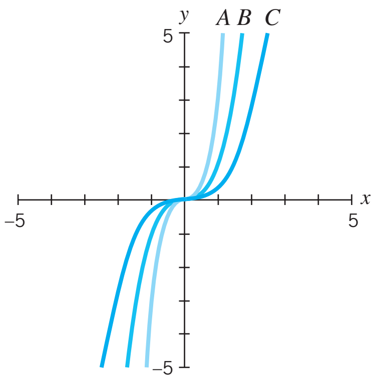
Graph *B*



Graph *D*

4. Match the function with the appropriate graph. Explain why you have chosen each graph.

a. $f(x) = x^3$ **b.** $g(x) = \frac{1}{3}x^3$ **c.** $h(x) = 3x^3$



5. Begin with the function $f(x) = x^4$.

a. Create a new function $g(x)$ by vertically stretching $f(x)$ by a factor of 6.

b. Create a new function $h(x)$ by vertically compressing $f(x)$ by a factor of $\frac{1}{2}$.

c. Create a new function $j(x)$ by first vertically stretching $f(x)$ by a factor of 2 and then reflecting it across the x-axis.

6. (Graphing program optional.) Plot the functions $f(x) = x^2$, $g(x) = 3x^2$, and $h(x) = \frac{1}{4}x^2$ on the same grid. Insert the symbol $>$ or $<$ to make the relation true.

a. For $x > 0$, $g(x)$ ___ $f(x)$ ___ $h(x)$

b. For $x < 0$, $g(x)$ ___ $f(x)$ ___ $h(x)$

7. (Graphing program recommended.) Plot the functions $f(x) = x^2$ and $g(x) = x^5$ on the same grid. Insert the symbol $>$, $<$, or $=$ to make the relation true.

a. For $x = 0$, $f(x)$ ___ $g(x)$

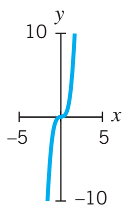
b. For $0 < x < 1$, $f(x)$ ___ $g(x)$

c. For $x = 1$, $f(x)$ ___ $g(x)$

d. For $x > 1$, $f(x)$ ___ $g(x)$

e. For $x < 0$, $f(x)$ ___ $g(x)$

8. Consider the accompanying graph of $f(x) = kx^n$, where n is a positive integer.



a. Is n even or odd?

b. Is $k > 0$ or is $k < 0$?

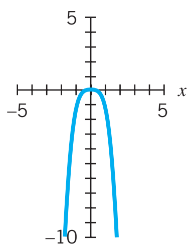
c. Does $f(-2) = -f(2)$?

d. Does $f(-x) = -f(x)$?

e. As $x \to +\infty$, $f(x) \to$ ___

f. As $x \to -\infty$, $f(x) \to$ ___

9. Consider the accompanying graph of $f(x) = kx^n$, where n is a positive integer.



a. Is n even or odd?

b. Is $k > 0$ or is $k < 0$?

c. Does $f(-2) = f(2)$?

d. Is $f(-x) = f(x)$?

e. As $x \to +\infty$, $f(x) \to$ ___

f. As $x \to -\infty$, $f(x) \to$ ___

10. Consider the accompanying graphs of four functions of the form $y = kx^n$. Which (if any) individual functions:

a. Are symmetric across the x-axis?

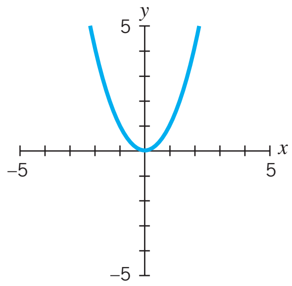
b. Are symmetric across the y-axis?

c. Are symmetric about the origin?

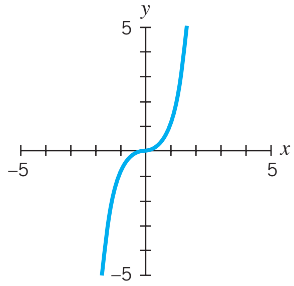
d. Have a positive even power for n?

e. Have a positive odd power for n?

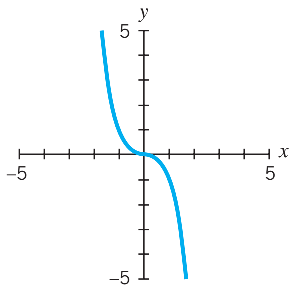
f. Have $k < 0$?



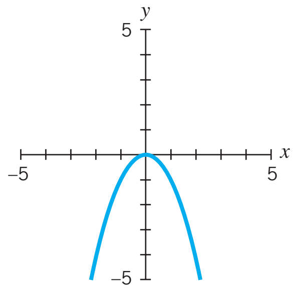
Graph *A*



Graph *C*



Graph *B*



Graph *D*

11. Sketch by hand the graph of each function:

$$f(x) = x^3, \quad g(x) = -x^3, \quad h(x) = \tfrac{1}{2}x^3, \quad j(x) = -2x^3$$

a. Identify the k value, the constant of proportionality, for each function.

b. Which graph is a reflection of $f(x)$ across the x-axis?

c. Which graph is both a stretch and a reflection of $f(x)$ across the x-axis?

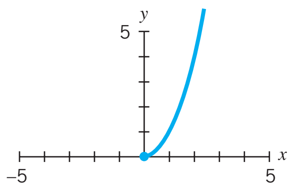
d. Which graph is a compression of $f(x)$?

12. a. Complete the partial graph in the accompanying figure in three different ways by:

i. Reflecting the graph across the y-axis to create Graph *A*.

ii. Reflecting the graph across the x-axis to create Graph *B*.

iii. Reflecting the graph first across the y-axis and then across the x-axis to create Graph *C*.



b. Which of the finished graphs is symmetric across the y-axis?

c. Which of the finished graphs is symmetric across the x-axis?

d. Which of the finished graphs is symmetric about the origin?

13. (Graphing program optional.) Given $f(x) = 4x^2$, construct a function that is a reflection of $f(x)$ across the horizontal axis. Graph the functions and confirm your answer.

14. (Graphing program optional.) Are $f(x) = x^3$ and $g(x) = -x^3$ reflections of each other across the vertical axis? Graph the functions and confirm your answer.

15. (Graphing program optional.) Evaluate each of the following functions at 0, 0.5, and 1. Then, on the same grid, graph each over the interval [0, 1]. Compare the graphs.

a. $y_1 = x$ $\quad y_2 = x^{1/2}$ $\quad y_3 = x^{1/3}$ $\quad y_4 = x^{1/4}$

b. $y_5 = x^2$ $\quad y_6 = x^3$ $\quad y_7 = x^4$

16. (Graphing program required). Now graph the functions in Exercise 15 over the interval [1, 4] and compare the graphs.

7.4 Comparing Power and Exponential Functions

Which Eventually Grows Faster, a Power Function or an Exponential Function?

Although power and exponential functions may appear to be similar in construction, in each function type the independent variable or input assumes a very different role. For power functions the input, x, is the *base*, which is raised to a fixed power. Power functions have the form

$$\text{output} = k \cdot (\text{input})^{\text{power}}$$
$$y = kx^p$$

If k and p are both positive, the power function describes growth.

For exponential functions the input, x, is the *exponent* applied to a fixed base. Exponential functions have the form

$$\text{output} = C \cdot a^{\text{input}}$$
$$y = Ca^x$$

If $C > 0$ and $a > 1$, the exponential function describes growth.

Consider the functions $y = x^3$, a power function; $y = 3^x$, an exponential function; and $y = 3x$, a linear function. Table 7.3 compares the role of the independent variable, x, in the three functions.

x	Linear Function $y = 3x$ (x is multiplied by 3)	Power Function $y = x^3$ (x is the *base* raised to the third power)	Exponential Function $y = 3^x$ (x is the *exponent* for base 3)
0	$3 \cdot 0 = 0$	$0 \cdot 0 \cdot 0 = 0$	$3^0 = 1$
1	$3 \cdot 1 = 3$	$1 \cdot 1 \cdot 1 = 1$	$3^1 = 3$
2	$3 \cdot 2 = 6$	$2 \cdot 2 \cdot 2 = 8$	$3^2 = 9$
3	$3 \cdot 3 = 9$	$3 \cdot 3 \cdot 3 = 27$	$3^3 = 27$
4	$3 \cdot 4 = 12$	$4 \cdot 4 \cdot 4 = 64$	$3^4 = 81$
5	$3 \cdot 5 = 15$	$5 \cdot 5 \cdot 5 = 125$	$3^5 = 243$

Table 7.3

Visualizing the difference

Table 7.3 and Figure 7.19 show that the power function $y = x^3$ and the exponential function $y = 3^x$ both grow very quickly relative to the linear function $y = 3x$.

Yet there is a vast difference between the growth of an exponential function and the growth of a power function. In Figure 7.20 we zoom out on the graphs. Notice that now the scale on the x-axis goes from 0 to 10 (rather than just 0 to 4 as in Figure 7.19) and the y-axis extends to 8000, which is still not large enough to show the value of 3^x once x is slightly greater than 8. The linear function $y = 3x$ is not shown, since at this scale it would appear to lie flat on the x-axis.

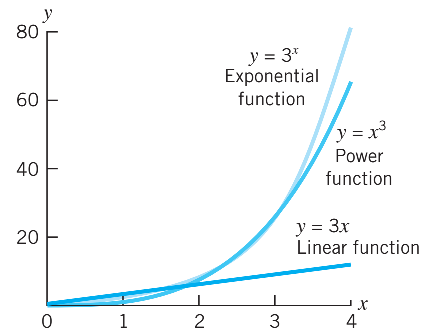


Figure 7.19 A comparison of $y = 3x$, $y = x^3$, and $y = 3^x$.

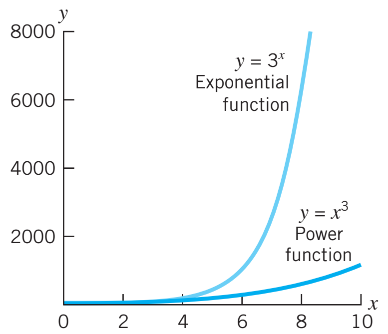


Figure 7.20 "Zooming out" on Figure 7.19.

The exponential function $y = 3^x$ clearly dominates the power function $y = x^3$. The exponential function continues to grow so rapidly that its graph appears almost vertical relative to the graph of the power function.

What if we had picked a larger exponent for the power function? Would the exponential function still overtake the power function? The answer is yes. Let's compare, for instance, the graphs of $y = 3^x$ and $y = x^{10}$. If we could zoom in on the graph in Figure 7.21, we would see that for a while the graph of $y = 3^x$ lies below the graph of $y = x^{10}$. For example, when $x = 2$, then $3^2 < 2^{10}$. But eventually $3^x > x^{10}$. Figure 7.21 shows that somewhere after $x = 30$, the values for 3^x become substantially larger than the values for x^{10}.

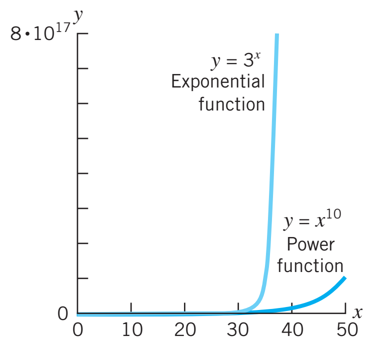


Figure 7.21 Graph of $y = x^{10}$ and $y = 3^x$.

> Any exponential growth function will eventually dominate any power function.

EXAMPLE 1 **Comparing power and exponential functions**

a. Construct a table using values of $x \geq 0$ for each of the following functions:

$$y = x^2 \quad \text{and} \quad y = 2^x$$

Then plot the functions on the same grid.

b. If $x > 0$, for what value(s) of x is $x^2 = 2^x$?

c. Does one function eventually dominate? If so, which one and after what value of x?

SOLUTION **a.** Table 7.4 and Figure 7.22 compare values for $y = x^2$ and $y = 2^x$.

x	$y = x^2$	$y = 2^x$
0	0	1
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32
6	36	64

Table 7.4

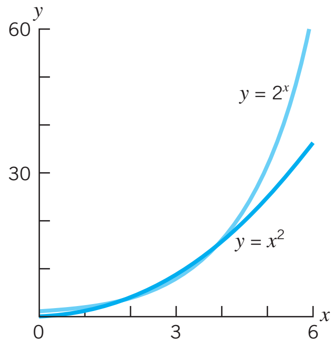


Figure 7.22 Comparison of $y = x^2$ and $y = 2^x$.

b. For positive values of x, if $x = 2$ or $x = 4$, then $x^2 = 2^x$.

c. In Table 7.4 and Figure 7.22, we can see that if $x > 4$, then $2^x > x^2$, so the function $y = 2^x$ will dominate the function $y = x^2$ for $x > 4$.

Algebra Aerobics 7.4

1. **a.** For each of the following functions construct a table using integer values of x between 0 and 5. Then sketch the functions on the same grid.

$$y = 4^x \quad \text{and} \quad y = x^3$$

   b. Does one function eventually dominate? If so, which one and after approximately what value of x?

2. By inspection, determine if each of the graphs in Figure 7.23 is more likely to be the graph of a power function or of an exponential function.

3. Which function eventually dominates?

   a. $y = x^{10}$ or $y = 2^x$

   b. $y = (1.000\,005)^x$ or $y = x^{1{,}000{,}000}$

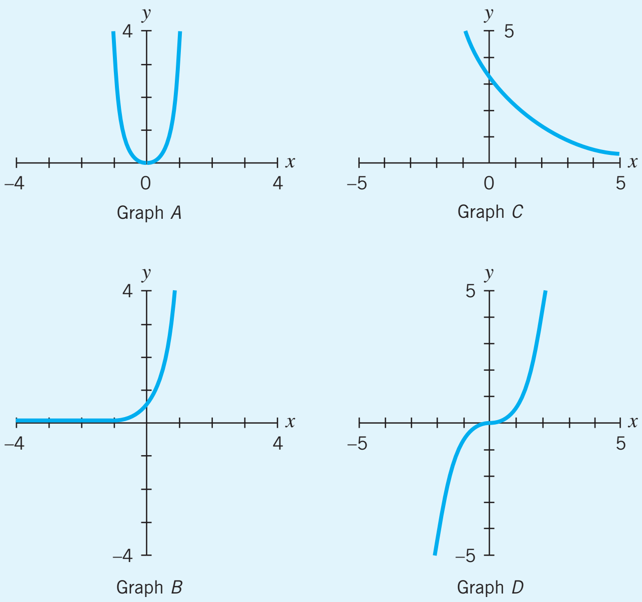


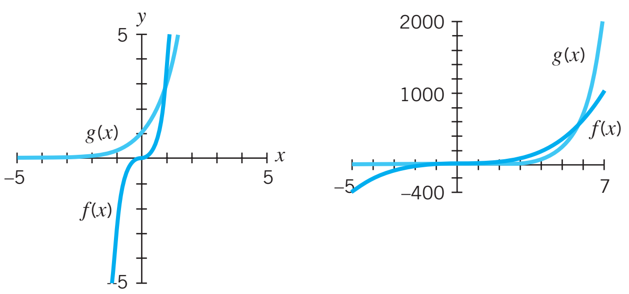
Figure 7.23 Graphs of power and exponential functions.

Exercises for Section 7.4

A graphing program is useful for several of the exercises.

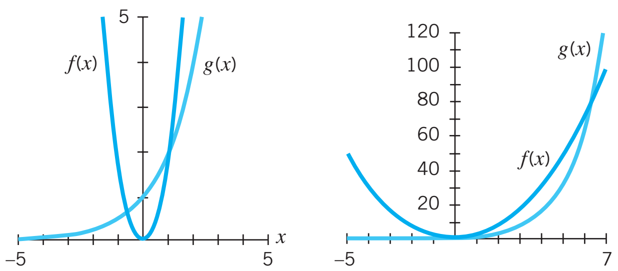
1. (Graphing program recommended.) Graph $y = x^4$ and $y = 4^x$ on the same grid.
   a. For positive values of x, where do your graphs intersect? Do they intersect more than once?
   b. For positive values of x, describe what happens to the right and left of any intersection points. You may need to change the scales on the axes or change the windows on a graphing calculator in order to see what is happening.
   c. Which eventually dominates, $y = x^4$ or $y = 4^x$?

2. Examine the following two different versions of the graphs of both $f(x) = 3x^3$ and $g(x) = 3^x$. Notice that each version uses different scales on both axes. Make sure to check each version when answering the following questions.



   a. When $x < 0$, how many times do $f(x)$ and $g(x)$ intersect?
   b. When $x > 0$, how many times do $f(x)$ and $g(x)$ intersect?
   c. Estimate the points of intersection.
   d. As $x \to +\infty$, does $f(x) = 3x^3$ or $g(x) = 3^x$ dominate?
   e. Which function, $10 \cdot f(x)$ or $g(x)$, will dominate as $x \to +\infty$?

3. The following figures show two different versions of the graphs of $f(x) = 2x^2$ and $g(x) = 2^x$. Notice that each version uses different scales on the axes. Make sure to check each version when answering the following questions.



   Supply the appropriate inequality symbol.
   a. As $x \to -\infty$, $f(x)$ ___ $g(x)$.
   b. As $x \to +\infty$, $f(x)$ ___ $g(x)$.
   c. When x lies in the interval $(-\infty, -1)$, then $f(x)$ _____ $g(x)$.
   d. When x lies in the interval $(-0.5, 1)$ then $f(x)$ _____ $g(x)$.
   e. When x lies in the interval $(1, 6)$, then $f(x)$ _____ $g(x)$.
   f. When x lies in the interval $(7, +\infty)$, then $f(x)$ _____ $g(x)$.

4. a. Which eventually dominates, $y = (1.001)^x$ or $y = x^{1000}$?
   b. As the independent variable approaches $+\infty$, which function eventually approaches zero faster, an exponential decay function or a power function with negative integer exponent?

5. (Graphing program optional.) Use a table or graphing utility to determine where $2^x > x^2$ and where $3^x > x^3$ for nonnegative values of x.

6. Match the following data tables with the appropriate function.
   a. $f(x) = 2/x$ b. $f(x) = x^2$ c. $f(x) = 2^x$

x	y
1	2
2	1
3	2/3
4	1/2

Table 1

x	y
1	2
2	4
3	8
4	16

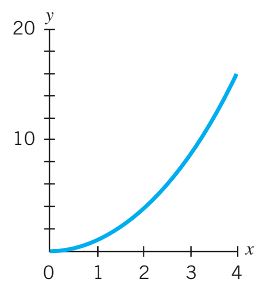
Table 2

x	y
1	1
2	4
3	9
4	16

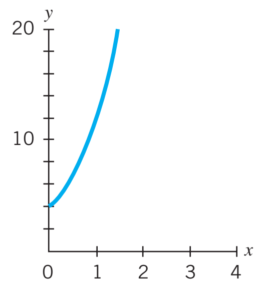
Table 3

7. (Graphing program recommended.) If x is positive, for what values of x is $3 \cdot 2^x < 3 \cdot x^2$? For what values of x is $3 \cdot 2^x > 3 \cdot x^2$?

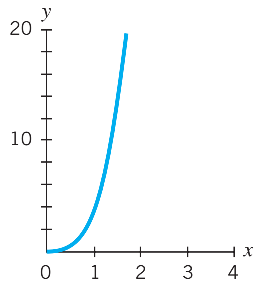
8. For $x > 0$, match each function with its graph.
   a. $y = x^2$ c. $y = 4x^3$
   b. $y = 2^x$ d. $y = 4(3^x)$



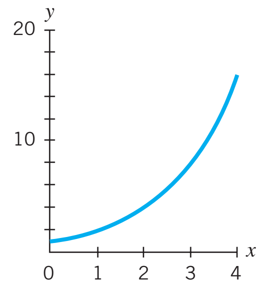
Graph A



Graph C



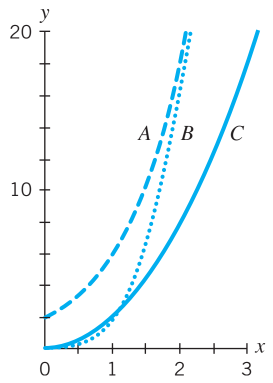
Graph B



Graph D

9. Match the function with its graph.

a. $h(x) = 2x^2$ **b.** $i(x) = 2(3)^x$ **c.** $j(x) = 2x^3$



10. Given the functions $f(x) = x^4$ and $g(x) = (4)^x$:

a. Find $f(3x)$ and $3f(x)$. Summarize the difference between these functions and $f(x)$.

b. Find $g(3x)$ and $3g(x)$. Summarize the difference between these functions and $g(x)$.

11. Consider a power function $f(x) = x^n$ where n is a positive integer, and an exponential function $g(x) = b^x$ where $b > 1$. Describe how $f(ax)$ and $g(ax)$ differ from $f(x)$ and $g(x)$ (see Exercise 10).

12. Given the points (1, 2) and (6, 72):

a. Find a linear function of the form $y = ax + b$ that goes through the two points.

b. Find an exponential function of the form $y = ab^x$ that goes through the two points.

c. Find a power function of the form $y = ax^p$ that goes through the two points.

13. Find power, exponential, and linear equations that go through the two points (1, 0.5) and (4, 32).

14. Think of the graphs of $f(x) = ax^3$ and $g(x) = a(3)^x$, where $a > 0$, and then decide whether each of the following statements is true or false.

a. $f(x)$ and $g(x)$ have the same vertical intercept.

b. $f(x)$ intersects $g(x)$ only once.

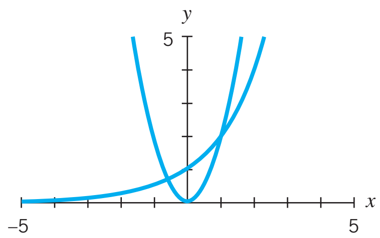
c. As $x \to +\infty$, $g(x) > f(x)$.

d. As $x \to -\infty$, both $f(x)$ and $g(x)$ approach 0.

15. If you know that one of the graphs in the accompanying figure is a power function of the form $y = ax^p$ and the other is an exponential function of the form $y = Cb^x$, then determine if each of the following is true or false.

a. $C \geq 1$ **b.** p is an even integer. **c.** $0 < b < 1$

d. The graphs of the functions will only intersect twice. (*Hint:* Think of the long-term behavior of these functions.)



7.5 *Inverse Proportionality: Power Functions with Negative Integer Powers*

Recall that the general form of an equation for a power function is

$$\text{output} = \text{constant} \cdot (\text{input})^{\text{power}}$$

In Sections 7.1 through 7.4 we focused on functions where the power was a positive integer. We now consider power functions where the power is a negative integer.

Using the rules for negative exponents, we can rewrite power functions in the form

$$y = kx^{\text{negative power}}$$

where k is a constant, as

$$y = \frac{k}{x^{\text{positive power}}}$$

For example, $y = 3x^{-2}$ can be rewritten $y = \frac{3}{x^2}$. In this form (with a positive power) it is easier to make calculations and to see what happens to y as x increases or decreases in value.

In Section 7.1 we constructed a function $R(x) = \frac{6}{x} = 6x^{-1}$ from the ratio of surface area to volume of a cube with edge length x.We described $R(x) = 6x^{-1}$ as a power function of degree -1. Repeating the table and graph from Section 7.1 here as Table 7.5 and Figure 7.24 reminds us that as x increases, $R(x)$ decreases. So the shape of this power function with a negative exponent is quite different from the shape of those with a positive exponent.

Edge Length x	$\frac{\text{Surface Area}}{\text{Volume}}$ $R(x) = \frac{6}{x}$
1	6.00
2	3.00
3	2.00
4	1.50
6	1.00
8	0.75
10	0.60

Table 7.5

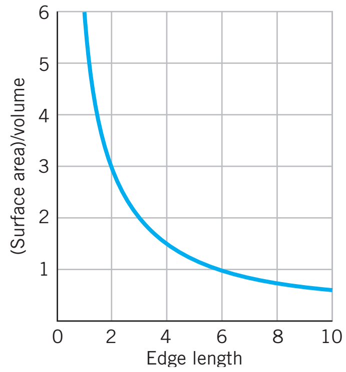


Figure 7.24 The graph of $R(x) = \frac{6}{x}$ shows that as the edge length, x, increases, the ratio of (surface area)/volume decreases.

Inverse Proportionality

The headline for a March 8, 2004, article in *The Economic Times* read "Grace is inversely proportional to the crisis an individual faces." The author was arguing that the greater the crisis, the less gracious an individual is likely to be. Mathematics has a formal definition for the same concept.

For power functions in the form

$$y = \frac{\text{constant}}{x^p}$$

where p is positive, we say that y is *inversely proportional to* x^p. For example, if $y = \frac{8}{x^3}$ we say that y is *inversely proportional to* x^3. If y is inversely proportional to x^p, then as x increases, y decreases.

Direct and Inverse Proportionality

Let p be a positive number and $k \neq 0$.

If $y = kx^p$, then y is *directly proportional to* x^p.

If $y = \dfrac{k}{x^p} = kx^{-p}$, then y is *inversely proportional to* x^p.

In both cases k is called the *constant of proportionality*.

EXAMPLE 1 **Examples of direct and inverse proportionality**

Write formulas to represent the following relationships.

a. Boyle's Law says that the volume, V, of a fixed quantity of gas is inversely proportional to the pressure, P, applied to it.

b. The force, F, keeping an electron in orbit is inversely proportional to the square of the distance, d, between the electron and the nucleus.

c. The acceleration, a, of an object is directly proportional to the force, F, applied upon the object and inversely proportional to the object's mass, m.

SOLUTION

a. $V = \frac{k}{P}$ for some constant k

b. $F = \frac{k}{d^2}$ for some constant k

c. $a = \frac{kF}{m}$ for some constant k

EXAMPLE 2

In each formula, identify which variables are directly or inversely proportional to each other, and specify the constant of proportionality.

a. $a = \frac{v}{t}$, where a = acceleration, v = velocity, and t = time.

b. $F = \frac{GM_1M_2}{d^2}$, Newton's Law of Universal Gravitation, where F is the force of gravity, G is a gravitational constant, M_1 and M_2 are the masses of two bodies, and d is the distance between the two bodies.

SOLUTION

a. Acceleration, a, is directly proportional to velocity, v, and inversely proportional to time, t. The constant of proportionality is 1.

b. The force of gravity, F, is directly proportional to the masses of the two bodies, M_1 and M_2, and inversely proportional to d^2, the square of the distance between them. The constant of proportionality is G, a gravitational constant.

EXAMPLE 3

You've just found a great house to rent for $2,400 a month. You would need to share the rent with several friends.

a. Construct an equation for the function $R(n)$ that shows the rental cost per person for n people. What kind of function is this?

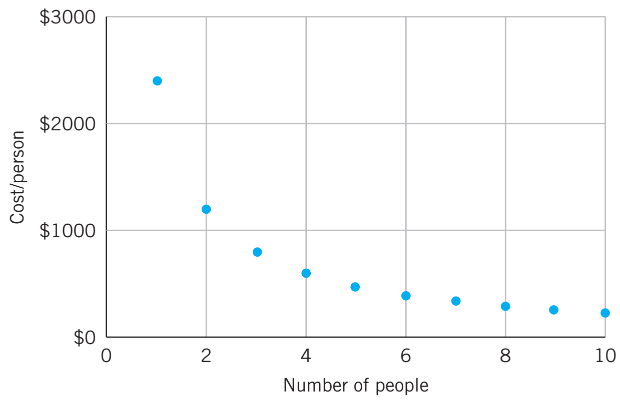
b. Graph the function for a reasonable domain.

c. Evaluate $R(4)$ and $R(6)$, and describe what they mean in this context.

SOLUTION

a. $R(n) = 2400/n$ is a power function where $R(n)$ is inversely proportional to n.

b.



c. $R(4) = 600$, which means that the cost per person would be $600 if there are four renters. $R(6) = 400$, which means that the cost per person would be $400 if there are six renters.

Properties of Inverse Proportionality

If y is *inversely proportional* to x^p, then $y = \frac{k}{x^p}$ (where $p > 0$) for some nonzero constant k. For an inversely proportional relationship, when we multiply the input by m, the output is multiplied by $\frac{1}{m^p}$ (where $p > 0$). Similar to our argument for direct proportionality, if $f(x) = \frac{k}{x^p}$, then

evaluate f at mx	$f(mx) = \dfrac{k}{(mx)^p}$
use rules of exponents	$= \dfrac{k}{(m^p x^p)}$
factor out $\dfrac{1}{m^p}$	$= \left(\dfrac{1}{m^p}\right) \cdot \left(\dfrac{k}{x^p}\right)$
substitute $f(x)$ for $\left(\dfrac{k}{x^p}\right)$	$= \left(\dfrac{1}{m^p}\right) \cdot f(x)$

Note that k, the constant of proportionality, does not play a role in these calculations.

For example, if $f(x) = \frac{3}{x^2}$, then doubling the input would multiply the output by $\frac{1}{2^2}$ or $\frac{1}{4}$. So if we double the input, the output would be reduced to one-fourth or 25% of the original amount. These doubling calculations do not depend on 3, the constant of proportionality.

> If y is inversely proportional to x^p, then $y = \frac{k}{x^p}$, where $p > 0$.
>
> Multiplying the input by m multiplies the output by $\frac{1}{m^p}$.
>
> For example, doubling the input multiplies the output by $\frac{1}{2^p}$.

EXAMPLE 4 For each of the following functions, what happens if the input is doubled? Multiplied by 10? Cut in half?

a. $y = \frac{1}{x^4}$ **b.** $y = 2x^{-3}$

SOLUTION

a. If the input is doubled, the output is multiplied by $\frac{1}{2^4} = \frac{1}{16}$ or 0.0625. If the input is multiplied by 10, the output is multiplied by $\frac{1}{10^4} = \frac{1}{10{,}000} = 0.0001$. If the input is cut in half, the output is multiplied by $\frac{1}{(0.5)^4} = \frac{1}{0.0625} = 16$.

b. The function $y = 2x^{-3}$ can be rewritten as $y = \frac{2}{x^3}$. In this form it's easier to see that if the input is doubled, the output is multiplied by $\frac{1}{2^3} = \frac{1}{8} = 0.125$. If the input is multiplied by 10, the output is multiplied by $\frac{1}{10^3} = \frac{1}{1000} = 0.001$. If the input is cut in half, the output is multiplied by $\frac{1}{(0.5)^3} = \frac{1}{0.125} = 8$.

EXAMPLE 5 **The cardinal rule of scuba diving**

The most important rule in scuba diving is "Never, ever, hold your breath." Why?

SOLUTION

Think of your lungs as balloons filled with air. As a balloon descends underwater, the surrounding water applies pressure, compressing the balloon. As the balloon ascends, the pressure is lessened and the balloon expands until it attains its original size when it reaches the surface. The volume V of the balloon is inversely proportional to the pressure P; that is, as the pressure increases, the volume decreases. The relationship is given by a special case of Boyle's Law for Gases,

$$V = \frac{1}{P} \quad \text{or equivalently} \quad V = P^{-1}$$

where V is the volume in cubic feet and P the pressure measured in atmospheres (atm). (One atm is 15 lb/in^2, the atmospheric pressure at Earth's surface.) Each additional 33 feet of water depth increases the pressure by 1 atm. Table 7.6 and Figure 7.25 describe the relationship between pressure and volume.

Pressure vs. Volume

Depth (ft)	Pressure (atm)	Volume (ft^3)
0	1	1
33	2	1/2
66	3	1/3
99	4	1/4

Table 7.6

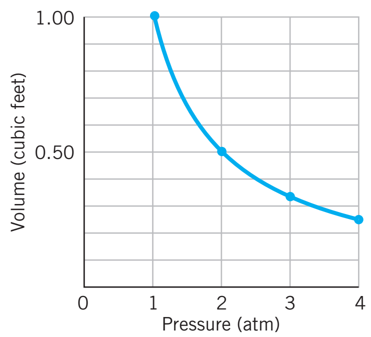


Figure 7.25 Graph of volume vs. pressure for a balloon descending underwater.

We can see in Table 7.6 that if we start with 1 cubic foot of air in the balloon at the surface and descend 33 feet, the surrounding pressure has doubled, from 1 to 2 atm, and the volume of air is cut in half. If the balloon descends to 99 feet, doubling the pressure again from 2 to 4 atm, the volume of air is cut in half again—leaving only one-fourth of the original volume. Why does this matter to divers?

Suppose you are swimming in a pool, take a lung full of air at the surface, and then dive down to the bottom. As you descend, the build-up of pressure will decrease the volume of air in your lungs. When you ascend back to the surface, the volume of air in your lungs will expand back to its original size, and everything is fine.

But when you are scuba diving, you are constantly breathing air that has been pressurized at the surrounding water pressure. If you are scuba diving 33 feet below the surface of the water, the surrounding water pressure is at 2 atm, twice that at the surface. What will happen then if you fill your lungs from your tank, hold your breath, and ascend to the surface? When you reach the surface, the pressure will drop in half, from 2 atm down to 1 atm, so the volume of air in your lungs will double, rupturing your lungs! Hence the first rule of scuba diving is "Never, ever, hold your breath."

EXAMPLE 6 **Designing a soda can**

A designer is asked to redesign a 12-ounce soda can. The volume must remain constant at 22 cubic inches (just enough to hold the 12 ounces and a little air) and the shape must remain cylindrical. What are her options?

SOLUTION If r = radius of the can (in inches) and h = height (in inches), then

$$\begin{aligned} \text{volume of can} &= \text{area of base} \cdot \text{height of can} \\ &= \pi r^2 h \end{aligned}$$

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Since the volume of the can must be 22 cubic inches,

$$22 = \pi r^2 h$$

Solving for h gives

$$h = \frac{22}{\pi r^2}$$

So h is inversely proportional to r^2. If we substitute an approximation of 3.14 for π, we get

$$h \approx \frac{22}{3.14r^2} \approx \frac{7}{r^2}$$

Figure 7.26 shows a graph of the relationship between the height of the can and its radius.

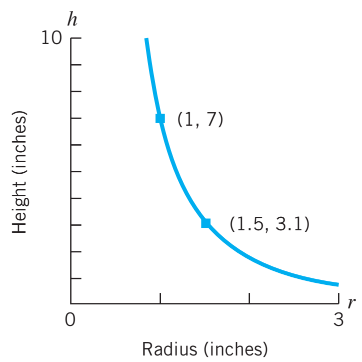


Figure 7.26 The relationship between height, h, and radius, r, of a can holding 22 cubic inches.

The designer can pick any point on the curve to determine the potential dimensions of the can. For example, if $r = 1$, then $h = 7$. So the point (1, 7) on the curve represents a radius of 1″ (hence a diameter of 2″) and a height of 7″. If $r = 1.5$, then $h = \frac{7}{(1.5)^2} = \frac{7}{2.25} \approx 3.1$. The point (1.5, 3.1) on the curve represents a radius of 1.5″ (diameter of 3″) and a height of 3.1″. The two points are labeled on the graph (Figure 7.26), and the corresponding can sizes are drawn in Figure 7.27.

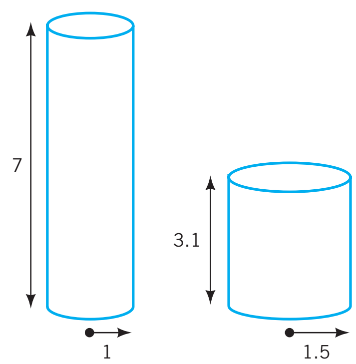


Figure 7.27 Two possible sizes for the soda can.

Inverse Square Laws

When the output is inversely proportional to the square of the input, the functional relationship is called an *inverse square law*. In the preceding example, where we held the volume of the soda can fixed at 22, the resulting function $h = 7/r^2$ is an inverse square law. Inverse square laws are quite common in the sciences.

EXAMPLE 7 **Seeing the light**

The intensity of light is inversely proportional to the square of the distance between the light source and the viewer.

a. Describe light intensity as a function of the distance from the light source.

b. What happens to the intensity of the light if the distance doubles? Triples?

SOLUTION

a. If d is the distance from the light source, then $I(d)$, the intensity of the light, is inversely proportional to d, so we can write $I(d) = \frac{k}{d^2}$ for some constant k.

b. If the distance doubles, then the light intensity output is multiplied by $\frac{1}{2^2} = \frac{1}{4}$. So the intensity drops to one-fourth of the original intensity.

If the distance triples, then the light intensity is multiplied by $\frac{1}{3^2} = \frac{1}{9}$. So the intensity drops to one-ninth of the original intensity.

For example, if you are reading a book that is 3 feet away from a lamp, and you move the book to 6 feet away (doubling the distance between the book and the light), the light will be one-fourth as intense. If you move the book from 3 to 9 feet away (tripling the distance), the light will be only one-ninth as intense. The reverse is also true; for example, if the book is 6 feet away and the light seems too dim for reading, by cutting the distance in half (to 3 feet), the illumination will be four times as intense.

EXAMPLE 8 **Gravitational force between objects**

The gravitational force between you and Earth is inversely proportional to the square of the distance between you and the center of Earth.

a. Express this relationship as a power function.

b. What happens to the gravitational force as the distance between you and the center of Earth increases by a factor of 10?

c. Why do astronauts appear to be weightless in space?

SOLUTION

a. The power function

$$F(d) = \frac{k}{d^2} = kd^{-2}$$

describes the gravitational force $F(d)$ between you and Earth in terms of a constant k times d^{-2}, where d is the distance between you and the center of Earth.

b. If the distance between you and Earth's center increases by a factor of 10, then the gravitational force is multiplied by $\frac{1}{10^2} = \frac{1}{100}$; that is, multiplying the distance by 10 decreases the gravitational force to one-hundredth of its original size.

c. Suppose an astronaut starts at the surface of Earth (roughly 4000 miles from Earth's center) and travels to 40,000 miles above Earth's center. She will have increased her distance from Earth's center by a factor or 10. The pull of Earth's gravity there would be one-hundredth that on Earth's surface. So she would appear to be weightless.

Why many inverse square laws work

Inverse square laws in physics often depend on a power source and simple geometry. Imagine a single point as a source of power, emitting perhaps heat, sound, or light. We can think of the power radiating out from the point as passing through an infinite number of concentric spheres. The farther away you are from the point source, the lower the intensity of the power, since it is spread out over the surface area of a sphere that increases in size as you move away from the point source. Therefore, the intensity, I, of the power you receive at any point on the sphere will be a function of the distance you are from the point source. The distance can be thought of as the radius, r, of a sphere with the point source at its center:

$$\text{intensity} = \frac{\text{power from a source}}{\text{surface area of sphere}}$$

$$I = \frac{\text{power}}{4\pi r^2}$$

$$\text{factor out } \frac{1}{r^2} \qquad = \frac{\left(\frac{\text{power}}{4\pi}\right)}{r^2}$$

If the power from the source is constant, we can simplify the expression by substituting a constant $k = \frac{\text{power}}{4\pi}$ and rewrite our equation as

$$I = \frac{k}{r^2}$$

The intensity you receive, I, is inversely proportional to the square of your distance from the source if the power from the point source is constant. If you double the distance, the intensity you receive is one-fourth of the original intensity. If you triple the distance, the intensity you receive is one-ninth of the original intensity. In particular, if $r = 1$, then $I = k$; if r doubles to 2, then $I = \frac{k}{4} = 0.25k$. The graph of $I = \frac{k}{r^2}$ is sketched in Figure 7.28.

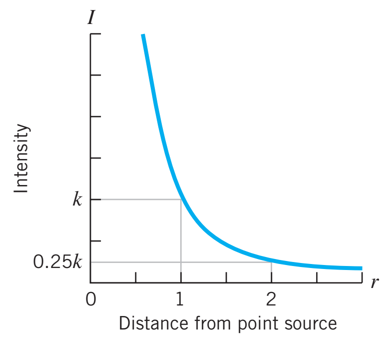


Figure 7.28 Graph of $I = \frac{k}{r^2}$, the relationship between intensity and distance from a point source.

EXAMPLE 9 **Direct and inverse proportionality in the same equation: The relative effects of the moon and the sun on tides**

The force that creates ocean tides on the surface of Earth varies inversely with the cube of the distance from Earth to any other large body in space and varies directly with the mass of the other body.

a. Construct an equation that describes this relationship.

b. The sun has a mass $2.7 \cdot 10^7$ times larger than the moon's, but the sun is about 390 times farther away from Earth than the moon. Would you expect the sun or the moon to have a greater effect on Earth's tides?

SOLUTION **a.** Using T for the tide-generating force, d for the distance between Earth and another body, and m for the mass of the other body, we have

$$T = \frac{km}{d^3} \quad \text{or} \quad T = kmd^{-3}$$

for some constant of proportionality k.

b. If M_{moon} is the mass of the moon and M_{sun} is the mass of the sun, then

$$M_{\text{sun}} = 2.7 \cdot 10^7 \cdot M_{\text{moon}}$$

If d_{moon} is the distance from the moon to Earth and d_{sun} is the distance from the sun to Earth, then

$$d_{\text{sun}} = 390 d_{\text{moon}}$$

If T_{sun} is the tidal force of the sun, then using the formula from part (a), we have

$$T_{\text{sun}} = \frac{k \cdot M_{\text{sun}}}{(d_{\text{sun}})^3}$$

substitute for M_{sun} and d_{sun}
$$= \frac{k \cdot (2.7 \cdot 10^7 \cdot M_{\text{moon}})}{(390 d_{\text{moon}})^3}$$

regroup terms
$$= \frac{2.7 \cdot 10^7}{390^3} \cdot \frac{k \cdot M_{\text{moon}}}{(d_{\text{moon}})^3}$$

substitute in T_{moon} and simplify
$$T_{\text{sun}} \approx \frac{1}{2} \cdot T_{\text{moon}}$$

Thus, despite the fact that the sun is more massive than the moon, because it is much farther away from Earth than the moon, its effect on Earth's tides is about one-half that of the moon.

Algebra Aerobics 7.5

1. Describe each proportionality with a sentence. (Assume k is a constant.)

   a. $x = kyz$ **b.** $y = \frac{k}{x^2}$ **c.** $D = \frac{k\sqrt{y}}{z^3}$

2. Which of the following represent quantities that vary inversely? (Assume k is a constant.)

   a. $y = \sqrt{x}$ **c.** $y = kx^{-3}$ **e.** $y = 8 \cdot 2^{-x}$

   b. $xy = k$ **d.** $V = \frac{1}{3}\pi r^2 h$

3. You inflate a balloon with 1 ft^3 of compressed air from your scuba tank while diving 99 feet underwater. The pressure at 99 ft^3 is equal to 4 atm. When you have ascended to the surface, by how much will the volume have increased (assuming the balloon doesn't burst)?

4. **a.** Rewrite each of the following expressions using positive exponents.

   i. $15x^{-3}$ **iii.** $3.6x^{-1}$

   ii. $-10x^{-4}$ **iv.** $2x^{-2}y^{-3}$

   b. Rewrite each of the following expressions using negative exponents.

   i. $\frac{1.5}{x^2}$ **iii.** $-\frac{2}{3x^2}$

   ii. $-\frac{6}{x^3}$ **iv.** $\frac{6}{x^3y^4z}$

5. The time in seconds, t, needed to fill a tank with water is inversely proportional to the square of the diameter, d, of the pipe delivering the water. Write an equation describing this relationship.

6. A light is 4 feet above the book you are reading. The light seems too dim, so you move the light 2 feet closer to the book. What is the change in light intensity?

7. If $g(x) = \frac{3}{x^4}$, what happens to $g(x)$ when:

   a. x doubles? **b.** x is divided by 2?

8. If $h(x) = -\frac{2}{x^3}$, what happens to $h(x)$ if you:

   a. Triple x? **b.** Divide x by 3?

9. Write an equation of variation where:

a. x is directly proportional to both y and the square of z.

b. a is directly proportional to both b and the cube of c and inversely proportional to d.

c. a is directly proportional to both the square of b and the square root of c and inversely proportional to both d and e.

10. Construct an equation of variation and find the constant of proportionality for each of the following situations. (*Hint:* Substitute values for x, y, and z to find the constant of proportionality, k. Then solve for the indicated value.)

a. x is inversely proportional to the square root of y. When $x = 3$, $y = 4$. Find x when $y = 400$.

b. x is directly proportional to both y and z. When $x = 3$ and $y = 4$, $z = 0.5$. Find y when $x = 6$ and $z = 2$.

c. x is directly proportional to the square of y and inversely proportional to the cube root of z. When $x = 6$, $y = 6$ and $z = 8$. Find x when $y = 9$ and $z = 0.027$.

Exercises for Section 7.5

Exercises 21 and 22 require technology that can generate a best-fit power function.

1. For $y = \dfrac{4}{x^3}$ answer the following:

a. As x changes from 1 to 4, y changes from ____ to ____.

b. When $x > 0$, as x increases, y ____.

c. As x changes from 2 to $\frac{1}{2}$, y changes from ____ to ____.

d. When $x > 0$, as x decreases, y ____ .

e. We say that y is ________ proportional to ____, and ____ is the constant of proportionality.

2. For $f(x) = kx^{-1}$ and $g(x) = kx^{-2}$, find:

a. $f(2x)$, $f(3x)$, and $f\left(\frac{x}{2}\right)$

b. $g(2x)$, $g(3x)$, and $g\left(\frac{x}{2}\right)$

3. For $h(x) = kx^{-3}$ and $j(x) = -kx^{-4}$, find:

a. $h(2x)$, $h(-3x)$, and $h\left(\frac{1}{3}x\right)$

b. $j(2x)$, $j(-3x)$, and $j\left(\frac{1}{3}x\right)$

4. Find the value for k, the constant of proportionality, if:

a. $y = \frac{k}{x}$ and $y = 3$ when $x = 2$.

b. $y = \frac{k}{x^2}$ and $y = \frac{1}{4}$ when $x = 8$.

c. $y = \frac{k}{x^2}$ and $y = \frac{1}{16}$ when $x = 2$.

d. $y = \frac{k}{\sqrt{x}}$ and $y = 1$ when $x = 9$.

5. Create an equation that meets the given specifications and then solve for the indicated variable.

a. If P is inversely proportional to f, and $P = 0.16$ when $f = 0.1$, then what is the value for P when $f = 10$?

b. If Q is inversely proportional to the square of r, and $Q = 6$ when $r = 3$, then what is the value for Q when $r = 9$?

c. If S is inversely proportional to w and p, and $S = 8$ when $w = 4$ and $p = \frac{1}{2}$, what is the value for S when $w = 8$ and $p = 1$?

d. W is inversely proportional to the square root of u and $W = \frac{1}{3}$ when $u = 4$. Find W when $u = 16$.

6. Assume y is inversely proportional to the cube of x.

a. If x doubles, what happens to y?

b. If x triples, what happens to y?

c. If x is halved, what happens to y?

d. If x is reduced to one-third of its value, what happens to y?

7. a. B is inversely proportional to x^4. What is the effect on B of doubling x?

b. Z is inversely proportional to x^p, where p is a positive integer. What is the effect on Z of doubling x?

8. The time, t, required to empty a tank is inversely proportional to r, the rate of pumping. If a pump can empty the tank in 30 minutes at a pumping rate of 50 gallons per minute, how long will it take to empty the tank if the pumping rate is doubled?

9. The intensity of light from a point source is inversely proportional to the square of the distance from the light source. If the intensity is 4 watts per square meter at a distance of 6 m from the source, find the intensity at a distance of 8 m from the source. Find the intensity at a distance of 100 m from the source.

10. In Exercise 9, it you wanted to quadruple the intensity, what would need to be done to the distance?

11. A light fixture is mounted flush on a 10-foot-high ceiling over a 3-foot-high counter. How much will the illumination (the light intensity) increase if the light fixture is lowered to 4 feet above the counter?

12. The frequency, F (the number of oscillations per unit of time), of an object of mass m attached to a spring is inversely proportional to the square root of m.

a. Write an equation describing the relationship.

b. If a mass of 0.25 kg attached to a spring makes three oscillations per second, find the constant of proportionality.

c. Find the number of oscillations per second made by a mass of 0.01 kg that is attached to the spring discussed in part (b).

13. Boyle's Law says that if the temperature is held constant, then the volume, V, of a fixed quantity of gas is inversely proportional to the pressure, P. That is, $V = \frac{k}{P}$ for some constant k. What happens to the volume if:

a. The pressure triples?

b. The pressure is multiplied by n?

c. The pressure is halved?

d. The pressure is divided by n?

14. The pressure of the atmosphere around us is relatively constant at 15 lb/in^2 at sea level, or 1 atmosphere of pressure (1 atm). In other words, the column of air above 1 square inch of Earth's surface is exerting 15 pounds of force on that square inch of Earth. Water is considerably more dense. As we saw in Section 7.5, pressure increases at a rate of 1 atm for each additional 33 feet of water. The accompanying table shows a few corresponding values for water depth and pressure.

Water Depth (ft)	Pressure (atm)
0	1
33	2
66	3
99	4

a. What type of relationship does the table describe?

b. Construct an equation that describes pressure, P, as a function of depth, D.

c. In Section 7.5 we looked at a special case of Boyle's Law for the behavior of gases, $P = \frac{1}{V}$ (where V is in cubic feet, P is in atms).

i. Use Boyle's Law and the equation you found in part (b) to construct an equation for volume, V, as a function of depth, D.

ii. When $D = 0$ feet, what is V?

iii. When $D = 66$ feet, what is V?

iv. If a snorkeler takes a lung full of air at the surface, dives down to 10 feet, and returns to the surface, describe what happens to the volume of air in her lungs.

v. A large, flexible balloon is filled with a cubic foot of compressed air from a scuba tank at 132 feet below water level, sealed tight, and allowed to ascend to the surface. Use your equation to predict the change in the volume of its air.

15. a. Construct an equation to represent a relationship where x is directly proportional to y and inversely proportional to z.

b. Assume that $x = 4$ when $y = 16$ and $z = 32$. Find k, the constant of proportionality.

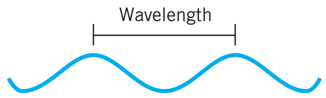
c. Using your equation from part (b), find x when $y = 25$ and $z = 5$.

16. a. Construct an equation to represent a relationship where w is directly proportional to both y and z and inversely proportional to the square of x.

b. Assume that $w = 10$ when $y = 12$, $z = 15$, and $x = 6$. Find k, the constant of proportionality.

c. Using your equation from part (b), find x when $w = 2$, $y = 5$, and $z = 6$.

17. Waves on the open ocean travel with a velocity that is directly proportional to the square root of their wavelength, the distance from one wave crest to the next. [See D. W. Thompson, *On Growth and Form* (New York: Dover Publications, 1992).]



a. If you are in a fixed spot in the ocean, the time interval between successive waves equals $\frac{\text{wave length}}{\text{wave velocity}}$. Show that this time interval is directly proportional to the square root of the wavelength.

b. On one day waves crash on the beach every 3 seconds. On the next day, the waves crash every 6 seconds. On the open ocean, how much farther apart do you expect the wave crests to be on the second day than on the first?

18. When installing Christmas lights on the outside of your house, you read the warning "Do not string more than four sets of lights together." This is because the electrical resistance, R, of wire varies directly with the length of the wire, l, and inversely with the square of the diameter of the wire, d.

a. Construct an equation for electrical wire resistance.

b. If you double the wire diameter, what happens to the resistance?

c. If you increase the length by 25% (say, going from four to five strings of lights), what happens to the resistance?

19. The rate of vibration of a string under constant tension is inversely proportional to the length of the string.

a. Write an equation for the vibration rate of a string, v, as a function of its length, l.

b. If a 48-inch string vibrates 256 times per second, then how long is a string that vibrates 512 times per second?

c. In general, it can be said that if the length of the string increases, the vibration rate will _______.

d. If you want the vibration rate of a string to increase, then you must _________ the length of the string.

e. Playing a stringed instrument, such as a guitar, dulcimer, banjo, or fiddle, requires placing your finger on a fret, effectively shortening the string. Doubling the vibration produces a note pitched one octave higher, and halving the vibration produces a note pitched one octave lower. If the number of vibrations decreased from 440 to 220 vibrations per second, what happened to the length of the string to cause the change in vibration?

20. The weight of a body is inversely proportional to the square of the distance from the body to the center of Earth. Assuming an Earth radius of 4000 miles, a man who weighs 200 pounds on Earth's surface would weigh how much 20 miles above Earth?

21. (Requires technology to generate a best-fit power function.) Oil is forced into a closed tube containing air. The height, H, of the air column is inversely proportional to the pressure, P.

a. Construct a general equation to describe the relationship between H and P.

b. Using technology, construct a best-fit power function for the accompanying data collected on height and pressure.

Height, H (in)	Pressure, P (lb/in^2)
13.3	6.7
10.7	8.1
9.2	10.2
7.1	12.9
6.3	14.4
5.1	17.6
4.1	21.4
3.5	25.1
3.0	29.4

22. (Requires technology to generate a best-fit power function.) Recall that Boyle's Law states that for a fixed mass of gas at a constant temperature, the volume, V, of the gas is inversely proportional to the pressure, P, exerted on the gas; that is, $V = \frac{k}{P}$ for some constant k. Use technology to determine a best-fit function model for the accompanying measurements collected on the volume of air as the pressure on it was increased.

Pressure, P (atm)	Volume of Air, V (cm^3)
1.0098	20
1.1610	18
1.3776	16
1.6350	14
1.9660	12
2.3828	10
2.9834	8
3.9396	6
5.0428	4
6.2687	2

7.6 Visualizing Negative Integer Power Functions

The Graphs of $f(x) = x^{-1}$ and $g(x) = x^{-2}$

Let's take a close look at two power functions with negative exponents, $f(x) = x^{-1}$ and $g(x) = x^{-2}$. Two important questions are "What happens to these functions when x is close to 0?" and "What happens as x approaches $+\infty$ or $-\infty$?"

What happens when x = 0?

If we rewrite $f(x) = x^{-1}$ as $f(x) = \frac{1}{x}$ and $g(x) = x^{-2}$ as $g(x) = \frac{1}{x^2}$, it is clear that both functions are undefined when $x = 0$. So neither domain includes 0. The graph of each function is split into two pieces: where $x > 0$ and $x < 0$.

What happens when $x \to +\infty$?

Table 7.7 and Figure 7.29 show that when x is positive and increasing, both $f(x)$ and $g(x)$ are positive and decreasing. As $x \to +\infty$, both $\frac{1}{x}$ and $\frac{1}{x^2}$ grow smaller and smaller, approaching, but never reaching, zero. This means that the range does not include 0. Graphically, as x gets larger and larger, both curves get closer and closer to the x-axis but never touch it. So both graphs are asymptotic to the x-axis.

Values for $f(x)$ and $g(x)$ when $x > 0$

x	$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{x^2}$
0	Undefined	Undefined
1/100	100	10,000
1/4	4	16
1/3	3	9
1/2	2	4
1	1	1
2	1/2	1/4
3	1/3	1/9
4	1/4	1/16
100	1/100	1/10,000

Table 7.7

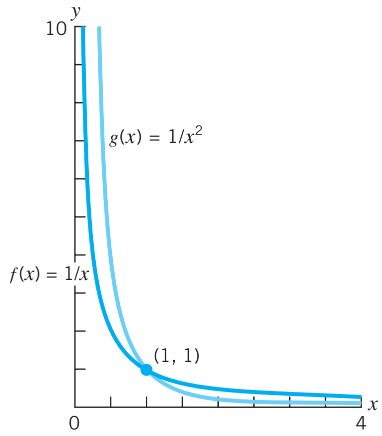


Figure 7.29 Graphs of $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ when $x > 0$.

What happens when x approaches 0?

We can also see in Table 7.7 and Figure 7.29 that when x is positive and grows smaller and smaller, approaching zero, the values for both $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ get larger and larger. Let's examine some of the underlying calculations in Table 7.7 for fractional values of x between 0 and 1.

$$f\left(\frac{1}{4}\right) = \frac{1}{1/4} = 1 \div \frac{1}{4} = 1 \cdot \frac{4}{1} = 4$$

$$f\left(\frac{1}{100}\right) = \frac{1}{1/100} = 1 \div \frac{1}{100} = 1 \cdot \frac{100}{1} = 100$$

$$g\left(\frac{1}{4}\right) = \frac{1}{(1/4)^2} = 1 \div \frac{1}{16} = 1 \cdot \frac{16}{1} = 16$$

$$g\left(\frac{1}{100}\right) = \frac{1}{(1/100)^2} = 1 \div \frac{1}{10{,}000} = 1 \cdot \frac{10{,}000}{1} = 10{,}000$$

If x is positive and approaches zero, then $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ approach positive infinity. Graphically, as x gets closer and closer to 0, both curves get closer and closer to the y-axis but never touch it. So both graphs are asymptotic to the y-axis.

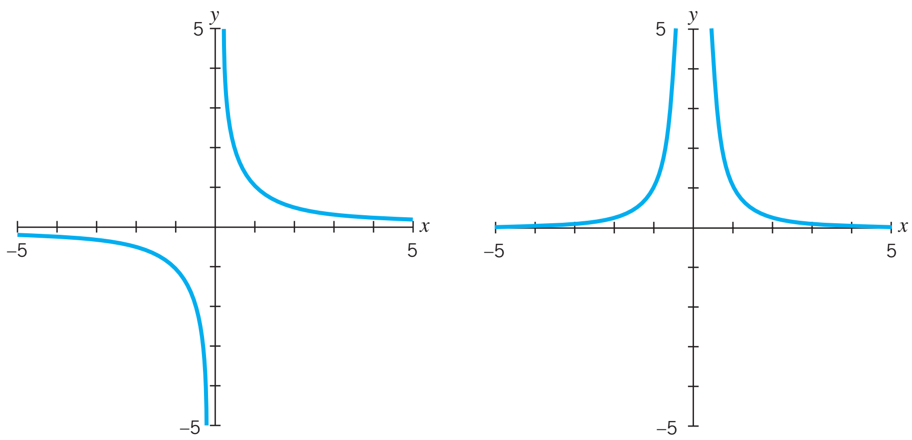
What happens when x < 0?

When x is negative, the graphs of $f(x)$ and $g(x)$ have very different shapes. If $x < 0$, then $1/x$ is negative but $1/x^2$ is positive (see Table 7.8 and Figure 7.30). As $x \to 0$ through negative values, $1/x \to -\infty$ and $1/x^2 \to +\infty$. So both $f(x)$ and $g(x)$ are again asymptotic to the y-axis, but in different directions.

Values for $f(x)$ and $g(x)$ when $x < 0$

x	$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{x^2}$
−100	−1/100	1/10,000
−4	−1/4	1/16
−3	−1/3	1/9
−2	−1/2	1/4
−1	−1	1
−1/2	−2	4
−1/3	−3	9
−1/4	−4	16
−1/100	−100	10,000

Table 7.8



Graph of $f(x) = 1/x$ Graph of $g(x) = 1/x^2$

Figure 7.30 The graphs of $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$.

As $x \to -\infty$, both $1/x$ and $1/x^2 \to 0$. So both functions are again asymptotic to the x-axis, but on different sides of the x-axis.

The graphs of $f(x)$ and $g(x)$, for both positive and negative values of x, are in Figure 7.30. Each graph consists of two pieces, split by the y-axis. When $x > 0$, the graphs are somewhat similar, both positive and concave up. However, when $x < 0$, the graphs lie on opposite sides of the x-axis. One graph is negative and concave down, the other positive and concave up. The graphs of $f(x)$ and $g(x)$ are asymptotic to both the x- and y-axes.

Odd vs. Even Powers

The graphs of power functions with negative powers are all similar in that they are composed of two non-intersecting curves that are each asymptotic to both the x- and y-axes. However, like the positive power functions, the graphs of negative power functions fall into two categories, even and odd powers, that are distinctly different as shown in Figure 7.31.

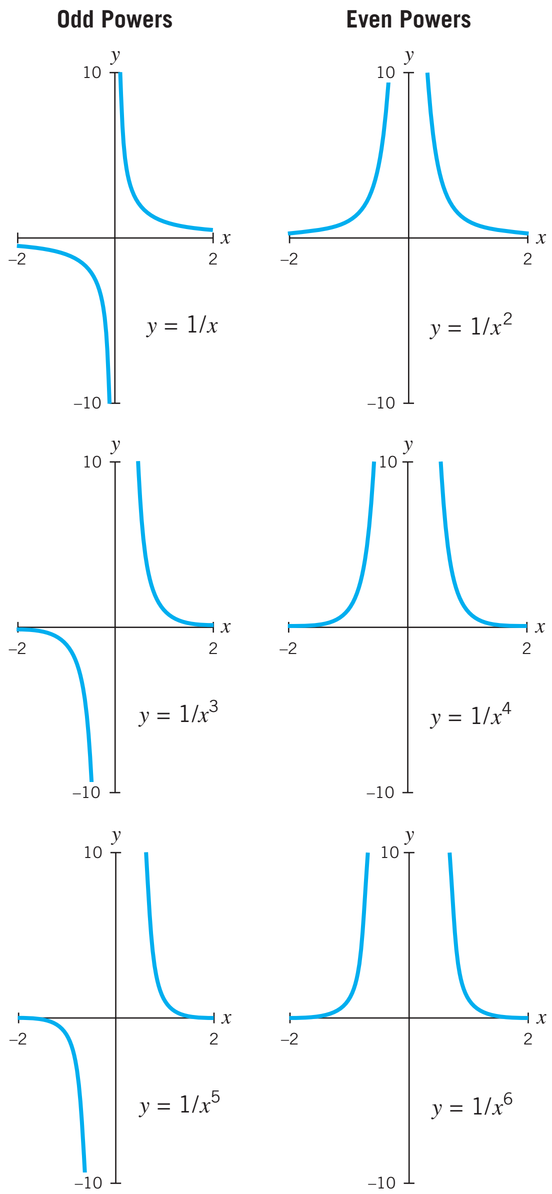


Figure 7.31 Graphs of power functions with even and odd powers.

All of the graphs in Figure 7.31 are split into two pieces and have a positive coefficient $k\,(= 1)$. So both pieces of the even-power graphs are concave up. However, the two pieces of the odd-power graphs differ in concavity. Here when $x < 0$, the graphs are concave down. When $x > 0$, the graphs are concave up.

Asymptotes

Figure 7.31 confirms that all odd and even negative power functions are asymptotic to both axes. (Note that positive power functions—odd or even—have no asymptotes.) All negative power functions "blow up" at the point 0; that is, the function is not defined at 0, but as x gets closer and closer to 0, the values of $y = k/x^p$ (where $p > 0$) approach either $+\infty$ or $-\infty$. We call 0 a *singularity*. This singularity forces the negative power functions to "explode" at $x = 0$, causing them to be asymptotic to the y-axis.

Similarly, as x approaches either $+\infty$ or $-\infty$ (at either end of the x-axis), the values of y approach, but never reach, 0. This means that the functions are asymptotic to the x-axis.

Symmetry of the graphs

All of the odd negative powers exhibit rotational symmetry about the origin. If you hold the graph fixed at the origin and rotate it 180º, you end up with the same image. The net result is equivalent to a double reflection, first about the y-axis and then about the x-axis.

All of the even negative power functions are symmetric about the y-axis. If you think of the y-axis as the dividing line, the "left" side of the graph is a mirror image, or reflection, of the "right" side.

The Effect of the Coefficient *k*

In Exploration 7.3 you can study more about the effect of k on the graphs of negative power functions.

We saw in Section 7.3 that the value of the coefficient k compresses or stretches the graphs of power functions with positive integer powers. We also saw that $y = kx^p$ and $y = -kx^p$ are mirror images of each other across the x-axis. These properties hold true for all power functions.

The graph of power functions in the form $y = \frac{k}{x^p}$ (where $p > 0$ and $k \neq 0$) is the graph of $y = \frac{1}{x^p}$

- vertically stretched by a factor of k, if $k > 1$
- vertically compressed by a factor of k, if $0 < k < 1$
- vertically stretched or compressed by a factor $|k|$, and reflected across the x-axis, if $k < 0$

EXAMPLE 1

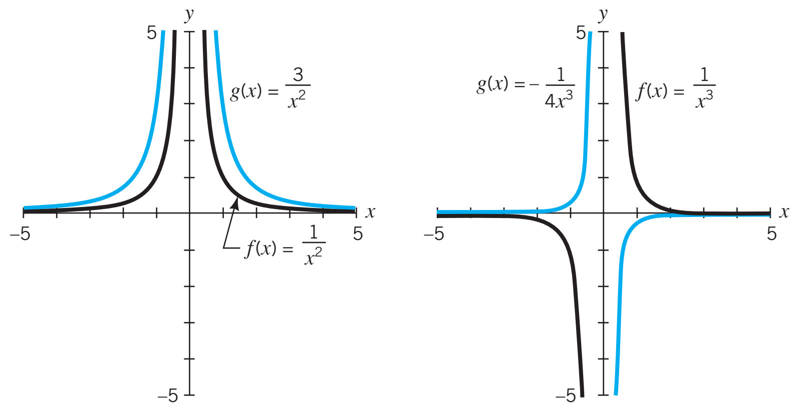
a. Describe how the graph of $g(x) = \frac{3}{x^2}$ is related to the graph of $f(x) = \frac{1}{x^2}$. Support your answer by graphing the functions on the same grid using technology.

b. Describe how the graph of $g(x) = -\frac{1}{4x^3}$ is related to the graph of $f(x) = \frac{1}{x^3}$. Support your answer by graphing the functions on the same grid using technology.

SOLUTION

a. $g(x) = \frac{3}{x^2} = 3\left(\frac{1}{x^2}\right)$. Since the coefficient 3 is greater than 1, the graph of $g(x)$ is the graph of $f(x) = \frac{1}{x^2}$ vertically stretched by a factor of 3. See Graph A in Figure 7.32.

b. $g(x) = -\frac{1}{4x^3} = -\frac{1}{4}\left(\frac{1}{x^3}\right)$. Since the coefficient $-\frac{1}{4}$ is negative and $0 < \left|-\frac{1}{4}\right| < 1$, $g(x)$ is the graph of $f(x) = \frac{1}{x^3}$, compressed by a factor of $\frac{1}{4}$ and then reflected across the x-axis. See Graph B in Figure 7.32.



Graph *A* Graph *B*

Figure 7.32 Graph A: The black graph is $f(x) = \frac{1}{x^2}$ and the blue graph is $g(x) = \frac{3}{x^2}$. Graph B: The black graph is $f(x) = \frac{1}{x^3}$ and the blue graph is $g(x) = -\frac{1}{4x^3}$.

Graphs of Power Functions with Negative Integer Powers

The graph of $y = \frac{k}{x^p}$ (where $p > 0$ and $k \neq 0$):

never goes through the origin, since the function is not defined when $x = 0$,

is composed of two non-intersecting curves that are each asymptotic to both the x- and y-axes.

If the power p is even, the graph is symmetric across the y-axis.
If the power p is odd, the graph is symmetric about the origin.

The value of the coefficient k compresses or stretches the graph of $y = \frac{1}{x^p}$.

The graphs of $y = \frac{k}{x^p}$ and $y = -\frac{k}{x^p}$ are mirror images of each other across the x-axis.

EXAMPLE 2 For $f(x) = x^{-2}$ and $g(x) = x^{-3}$, construct the following functions and their graphs.

a. $f(-x)$ and $-f(x)$

b. $3f(x)$ and its reflection across the x-axis

c. $g(-x)$ and $-g(x)$

d. $4g(x)$ and its reflection across the x-axis

SOLUTION **a.** We can rewrite $f(x) = x^{-2}$ as $f(x) = \frac{1}{x^2}$. Then

$$f(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} \quad \text{and} \quad -f(x) = -\frac{1}{x^2}$$

See Figure 7.33 Graph A.

b. $3f(x) = 3\left(\frac{1}{x^2}\right) = \frac{3}{x^2}$

The function $y = -\frac{3}{x^2}$ is the reflection of $3f(x)$ across the x-axis. See Figure 7.33 Graph B.

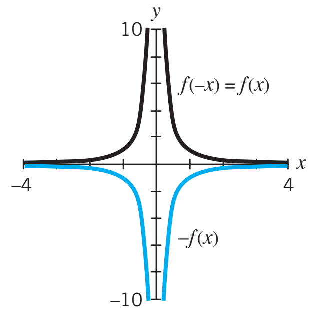
c. We can rewrite $g(x) = x^{-3}$ as $g(x) = \frac{1}{x^3}$. Then

$$g(-x) = \frac{1}{(-x)^3} = -\frac{1}{x^3} \quad \text{and} \quad -g(x) = -\frac{1}{x^3}$$

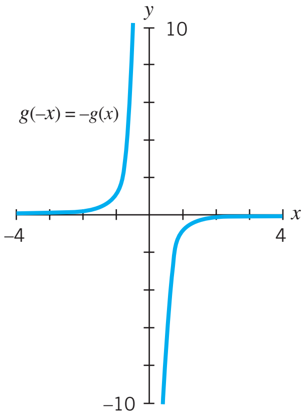
So $g(-x)$ and $-g(x)$ are equivalent functions. See Figure 7.33 Graph C.

d. $4g(x) = 4\left(\frac{1}{x^3}\right) = \frac{4}{x^3}$

The function $y = -\frac{4}{x^3}$ is the reflection of $4g(x)$ across the x-axis. See Figure 7.33 Graph D.

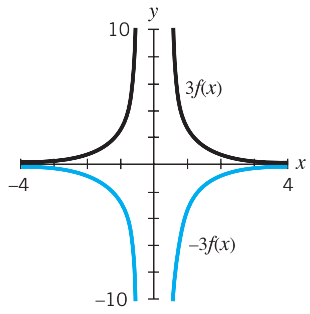


Graph *A*

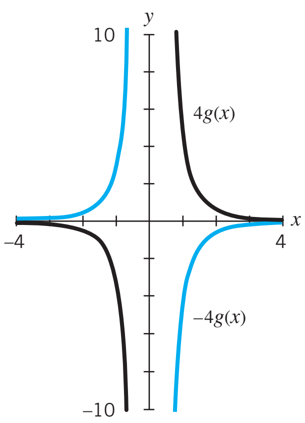


Graph *C*

Graph of two equivalent functions $g(-x)$ and $-g(x)$.



Graph *B*



Graph *D*

Figure 7.33 Four graphs of pairs of functions.

EXAMPLE 3

a. Graph the power function $f(x) = \frac{2}{x^3}$.

b. Evaluate $f(0.5)$ and $f(-2)$.

c. Use part (b) and the fact that $f(x)$ is a power function to predict the values for $f(10)$ and $f(-0.5)$.

d. Now evaluate $f(10)$ and $f(-0.5)$ to confirm your predictions in part (c).

SOLUTION

a. See Figure 7.34 on next page.

b. $f(0.5) = \frac{2}{(0.5)^3} = \frac{2}{(0.125)} = 16$ and $f(-2) = \frac{2}{(-2)^3} = \frac{2}{(-8)} = -0.25$. So the points $(0.5, 16)$ and $(-2, -0.25)$ lie on the graph.

c. Changing the input from 0.5 to 10 is equivalent to multiplying the original input by 20. So the output will be multiplied by $\frac{1}{20^3} = \frac{1}{8000}$. The value of $f(10)$ should be $\left(\frac{1}{8000}\right) \cdot f(0.5) = \left(\frac{1}{8000}\right) \cdot 16 = 0.002$.

Changing the input from 0.5 to -0.5 is equivalent to multiplying the input of 0.5 by -1. The output will be multiplied by $\frac{1}{(-1)^3} = -1$. So $f(-0.5) = -f(0.5) = -16$.

d. Evaluating the functions directly, we have $f(10) = \frac{2}{10^3} = \frac{2}{1000} = 0.002$ and $f(-0.5) = \frac{2}{(-0.5)^3} = \frac{2}{(-0.125)} = -16$, confirming our predictions in part (c).

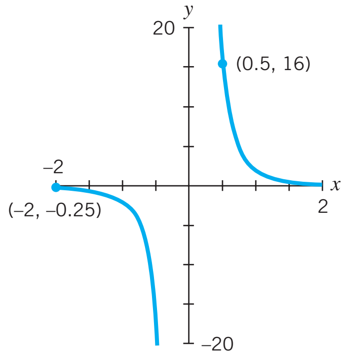


Figure 7.34 The graph of $f(x) = \frac{2}{x^3}$.

Algebra Aerobics 7.6

A graphing program is useful for Problems 3 and 4.

1. Rewrite each of the following expressions with positive exponents and then evaluate each expression for $x = -2$.
   a. x^{-2} **c.** $4x^{-3}$ **e.** $2^{-1}x^{-4}$ **g.** $-2x^{-4}$
   b. x^{-3} **d.** $-4x^{-3}$ **f.** $-2x^{-3}$ **h.** $2x^{-4}$
2. **a.** Select the most likely function type for each graph in Figure 7.35.

$$y = ax \qquad y = \frac{a}{x} \qquad y = ax^2$$
$$y = \frac{a}{x^2} \qquad y = ax^3 \qquad y = \frac{a}{x^3}$$

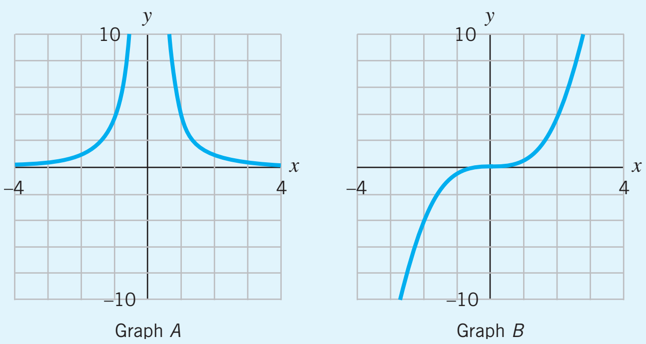


Figure 7.35 Two unidentified graphs.

   b. After you have decided on the function type, determine the value of a for each graph.

3. Sketch the graph for each equation:
   a. $y = x^{-10}$ **b.** $y = x^{-11}$
4. On the same grid, graph the following equations.
   a. $y = x^{-2}$ and $y = x^{-3}$. Where do these graphs intersect? How are the graphs similar and how are they different?
   b. $y = 4x^{-2}$ and $y = 4x^{-3}$. Where do these graphs intersect? How are the graphs similar and how are they different?
5. Match the appropriate function with each of the graphs in Figure 7.36

$$f(x) = \frac{5}{x^2} \qquad g(x) = \frac{1}{x^2} \qquad h(x) = \frac{-5}{x^2}$$

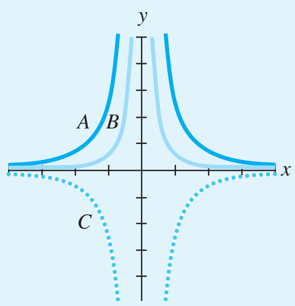


Figure 7.36 Three unidentified graphs.

Exercises for Section 7.6

For many of the exercises in this section the use of a graphing program is required or recommended.

1. A cube of edge length x has a surface area $S(x) = 6x^2$ and a volume $V(x) = x^3$. We constructed the function $R(x) = \frac{S(x)}{V(x)} = \frac{6x^2}{x^3} = \frac{6}{x}$. Consider R(x) as an abstract function. What is the domain? Construct a small table of values, including negative values of x, and plot the graph. Describe what happens to R(x) when x is positive and $x \to 0$. What happens to R(x) when x is negative and $x \to 0$?

2. a. (Graphing program recommended.) Make a table of values and sketch a graph for each of the following functions. Be sure to include negative and positive values for x, as well as values for x that lie close to zero.

$$y_1 = \frac{1}{x} \qquad y_2 = \frac{1}{x^2} \qquad y_3 = \frac{1}{x^3} \qquad y_4 = \frac{1}{x^4}$$

b. Describe the domain and range of each function.

c. Describe the behavior of each function as x approaches positive infinity and as x approaches negative infinity.

d. Describe the behavior of each function when x is near 0.

3. a. (Graphing program recommended.) Generate a table of values and a graph for each of the following functions:

$$g(x) = 5x \qquad h(x) = \frac{x}{5} \qquad t(x) = \frac{1}{x} \qquad f(x) = \frac{5}{x}$$

b. Describe the ways in which the graphs in part (a) are alike and the ways in which they are not alike.

4. (Graphing program optional.) In each part, sketch the three graphs on the same grid and label each function. Describe how the three graphs are similar and how they are different.

a. $y_1 = x^{-1}$ $\quad y_2 = x^{-3}$ $\quad y_3 = x^{-5}$

b. $y_1 = x^{0}$ $\quad y_2 = x^{-2}$ $\quad y_3 = x^{-4}$

c. $y_1 = 2x^{-1}$ $\quad y_2 = 4x^{-1}$ $\quad y_3 = -2x^{-1}$

5. (Graphing program required.) Using graphing technology, on the same grid graph $y = x^2$ and $y = x^{-2}$.

a. Over what interval does each function increase? Decrease?

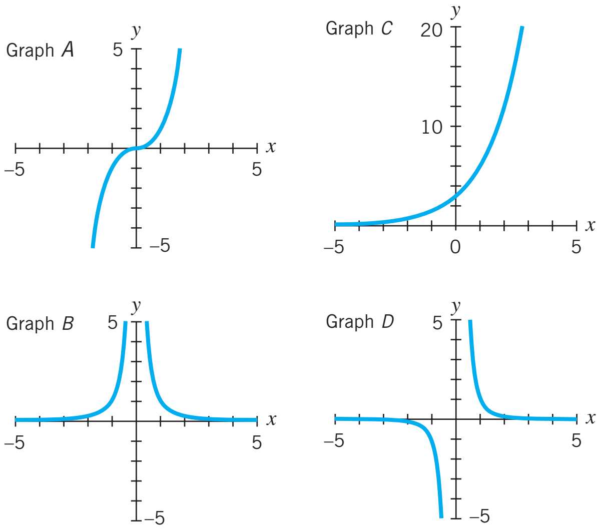
b. Where do the graphs intersect?

c. What happens to each function as x approaches positive infinity? Negative infinity?

6. Match each of the following functions with its graph:

a. $y = 3(2^x)$ $\qquad$ **c.** $y = x^{-3}$

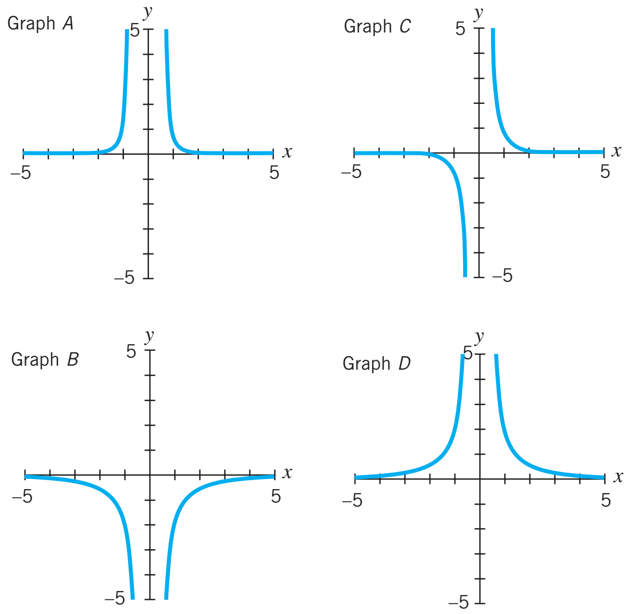
b. $y = x^3$ $\qquad$ **d.** $y = x^{-2}$



7. Match each function with its graph. Explain your choices.

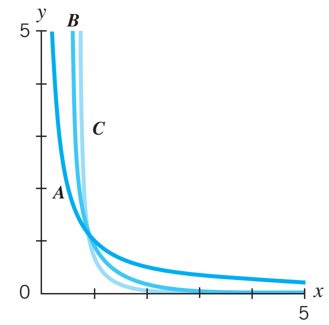
a. $f(x) = x^{-4}$ $\qquad$ **c.** $h(x) = \dfrac{2}{x^2}$

b. $g(x) = x^{-3}$ $\qquad$ **d.** $j(x) = -\dfrac{2}{x^2}$



8. Match each function (where $x > 0$) with the appropriate graph. Explain your choices.

a. $f(x) = x^{-3}$ $\qquad$ **b.** $g(x) = \dfrac{1}{x^5}$ $\qquad$ **c.** $h(x) = \dfrac{1}{x}$



9. Begin with the function $f(x) = x^{-3}$. Then:

a. Create a new function $g(x)$ by vertically stretching $f(x)$ by a factor of 4.

b. Create a new function $h(x)$ by vertically compressing $f(x)$ by a factor of $\frac{1}{2}$.

c. Create a new function $j(x)$ by first vertically stretching $f(x)$ by a factor of 3 and then by reflecting the result across the x-axis.

10. (Graphing program optional.) Graph the functions $f(x) = \frac{1}{x^2}$, $g(x) = \frac{4}{x^2}$, and $h(x) = \frac{1}{4x^2}$.

For parts (a)–(c), insert the symbol $>$ or $<$ to make the relation true.

a. For $x > 0$, $f(x)$ ________ $g(x)$

b. For $x > 0$, $g(x)$ ________ $h(x)$

c. For $x > 0$, $h(x)$ ________ $f(x)$

d. Describe how the graphs are related to each other when $x < 0$. Generalize your findings to any value for the coefficient $k > 0$.

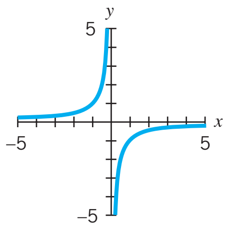
11. (Graphing program required.) Graph the functions $f(x) = \frac{1}{x^5}$ and $g(x) = \frac{1}{x^4}$ and then insert the symbol $>$ or $<$ to make the relation true.

a. For $x > 1$, $f(x)$ ________ $g(x)$

b. For $0 < x < 1$, $f(x)$ ________ $g(x)$

c. For $x < 0$, $f(x)$ ________ $g(x)$

12. Consider the accompanying graph of $f(x) = k \cdot \frac{1}{x^n}$, where n is a positive integer.



a. Is n even or odd?

b. Is $k > 0$ or is $k < 0$?

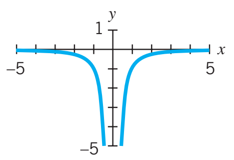
c. Is $f(-1) > 0$ or is $f(-1) < 0$?

d. Does $f(-x) = -f(x)$?

e. As $x \to +\infty$, $f(x) \to$ ___

f. As $x \to -\infty$, $f(x) \to$ ___

13. Consider the accompanying graph of $f(x) = k \cdot \frac{1}{x^n}$, where n is a positive integer.



a. Is n even or odd?

b. Is $k > 0$ or is $k < 0$?

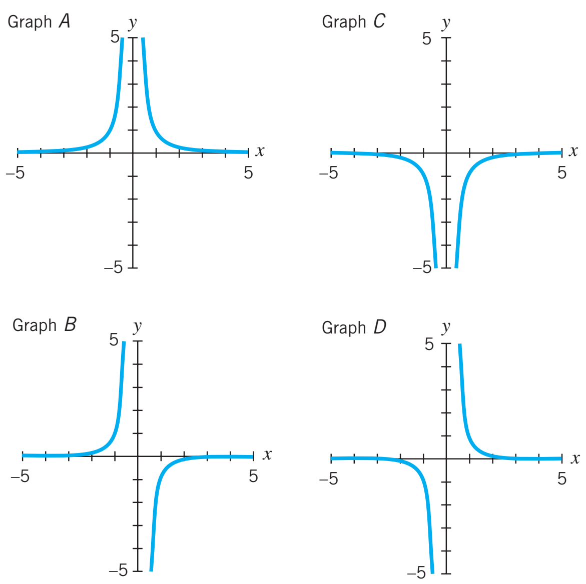
c. Is $f(-1) > 0$ or is $f(-1) < 0$?

d. Does $f(-x) = f(x)$?

e. As $x \to +\infty$, $f(x) \to$ ___

f. As $x \to -\infty$, $f(x) \to$ ___

14. Consider the following graphs of four functions of the form $y = \frac{k}{x^n}$, where $n > 0$.



Which functions:

a. Are symmetric across the y-axis?

b. Are symmetric about the origin?

c. Have an even power n?

d. Have an odd power n?

e. Have $k < 0$?

15. (Graphing program optional.) Sketch the graph of the function $f(x) = x^{-3}$. Then consider $f(x)$ in relation to the following functions:

$$g(x) = -x^{-3}, \quad h(x) = \frac{1}{2x^3} \quad \text{and} \quad j(x) = -\frac{2}{x^3}$$

a. Identify k, the constant of proportionality, for each function.

b. Which function has a graph that is a reflection of $f(x)$ across the x-axis?

c. Which function has a graph that is both a stretch and a reflection of $f(x)$ across the x-axis?

d. Which function has a graph that is a compression of $f(x)$?

16. **a.** Complete the partial graph in the accompanying figure three different ways by adding:

i. The reflection of the graph across the y-axis to create Graph A.

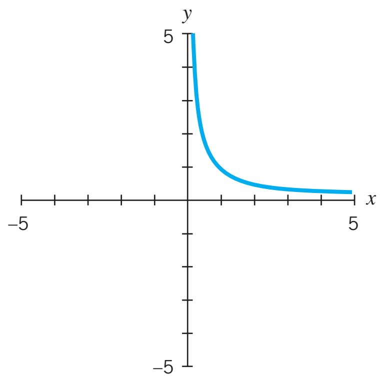
ii. The reflection of the graph across the x-axis to create Graph B.

iii. The result of reflecting the graph first across the y-axis and then across the x-axis to create Graph C.

b. Which of the finished graphs is symmetric across the y-axis?

c. Which of the finished graphs is symmetric about the origin?

d. Which of the finished graphs is symmetric across the x-axis?



17. (Graphing program required.) Given $f(x) = 4x^{-2}$, construct a function $g(x)$ that is a reflection of $f(x)$ across the horizontal axis. Graph the function and confirm your answer.

18. (Graphing program required.) Are $f(x) = x^{-3}$ and $g(x) = -x^{-3}$ reflections of each other across the vertical axis? Graph the functions and confirm your answer.

19. If $f(x) = \frac{1}{x^4}$ and $g(x) = \frac{1}{x^5}$, construct the following functions.

a. $f(-x)$ and $-f(x)$ **c.** $g(-x)$ and $-g(x)$

b. $2f(x)$ and $f(2x)$ **d.** $2g(x)$ and $g(2x)$

e. The function whose graph is the reflection of $f(x)$ across the x-axis

f. The function whose graph is the reflection of $g(x)$ across the x-axis

20. (Graphing program required.) Begin with a function $f(x) = \frac{1}{x^3}$. In each part describe how the graph of $g(x)$ is related to the graph of $f(x)$. Support your answer by graphing each function with technology.

a. $g(x) = \frac{2}{x^3}$ **b.** $g(x) = -\frac{1}{x^3}$ **c.** $g(x) = -\frac{1}{2}x^{-3}$

21. (Graphing program required.) Determine which of the graphs of the following pairs of functions intersect. If the graphs intersect, find the point or points of intersection.

a. $y = 2x$ $y = 4x^2$ **d.** $y = x^{-1}$ $y = x^{-2}$

b. $y = 4x^2$ $y = 4x^3$ **e.** $y = 4x^{-2}$ $y = 4x^{-3}$

c. $y = x^{-2}$ $y = 4x^2$

7.7 Using Logarithmic Scales to Find the Best Functional Model

Looking for Lines

Throughout this course, we have examined several different families of functions: linear, exponential, logarithmic, and power. In this section we address the question "How can we determine a reasonable functional model for a given set of data?" The simple answer is "Look for a way to plot the data as a straight line." We look for straight-line representations not because the world is intrinsically linear, but because straight lines are easy to recognize and manipulate.

In Figure 7.37 we plot the functions (for $x > 0$) of

$y = 3 + 2x$ linear function

$y = 3 \cdot 2^x$ exponential function

$y = 3x^2$ power function

first with a linear scale on both axes (a standard plot), then with a linear scale on the x-axis and a logarithmic scale on the y-axis (a semi-log plot), and finally with a logarithmic scale on both axes (a *log-log* plot).

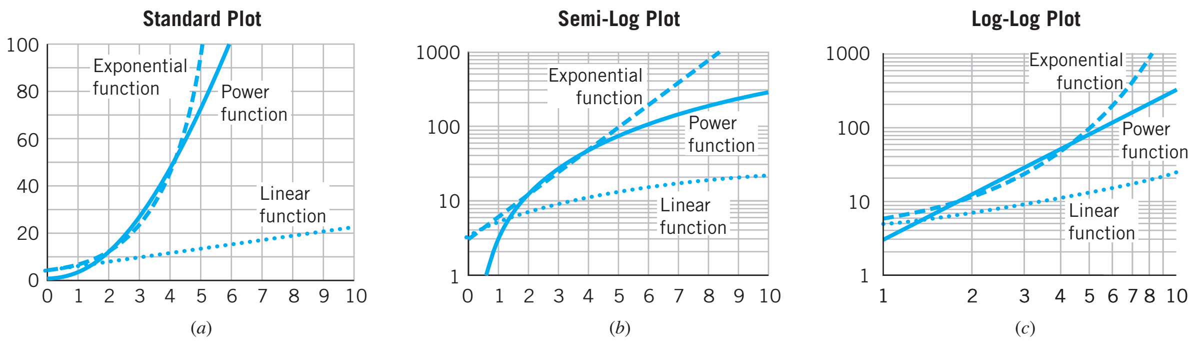


Figure 7.37 (*a*) *Standard plot.* The graph of a linear function appears as a straight line. (*b*) *Semi-log plot.* The graph of an exponential function appears as a straight line. (*c*) *Log-log plot.* The graph of a power function appears as a straight line.

The course software "E11: Semi-Log Plots of $y = Ca^x$" in *Exponential & Log Functions* and "P2: Log-Log Plots of Power Functions" in *Power Functions* can help you visualize the ideas in this section.

On the standard plot in Figure 7.37(*a*), only the linear function appears as a straight line; the power and exponential functions curve steeply upward. On the semi-log plot in Figure 7.37(*b*), the exponential function now is a straight line, and the power and linear functions curve downward. As we saw in Chapter 5, exponential growth will always appear as a straight line on a *semi-log* plot, where the independent variable (input) is plotted on a linear scale and the dependent variable (output) on a logarithmic scale. On the log-log plot in Figure 7.37(*c*), only the power function appears as a straight line.

Why Is a Log-Log Plot of a Power Function a Straight Line?

As we did with exponential functions and semi-log plots in Chapter 6, we can use the properties of logarithms to understand why power functions appear as straight lines on a log-log plot. Let's analyze our power function:

$$
\begin{array}{lrcll}
 & y &=& 3x^2 \quad (x > 0) & (1)\\
\text{Take the logarithm of both sides} & \log y &=& \log(3x^2) & \\
\text{use Rule 1 of logs} & &=& \log 3 + \log x^2 & \\
\text{use Rule 3 of logs} & &=& \log 3 + 2\log x & \\
\text{evaluating log 3, we get} & \log y &\approx& 0.48 + 2\log x & (2)
\end{array}
$$

If we let $X = \log x$ and $Y = \log y$, we can rewrite the equation as

$$Y \approx 0.48 + 2X \qquad (3)$$

So Y (or $\log y$) is a linear function of X (or $\log x$). Equations (1), (2), and (3) are all equivalent. The graph of Equation (3) on a log-log plot, with $\log x$ values on the horizontal and $\log y$ values on the vertical axis, is a straight line (see Figure 7.38). The slope of the line is 2, the power of x in Equation (1). The vertical intercept is 0.48 or log 3, the logarithm of the constant of proportionality in Equation (1).

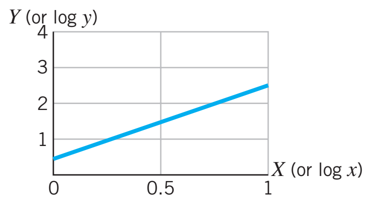


Figure 7.38 The graph of $Y \approx 0.48 + 2X$.

> **SOMETHING TO THINK ABOUT**
>
> *On what type of plot would a logarithmic function appear as a straight line?*

The slope, 2, described by Equation (3) means that each time X (or $\log x$) increases by one unit, Y (or $\log y$) increases by two units. Remember that a log scale is multiplicative. So, adding 1 to $\log x$ is equivalent to multiplying x by 10. Adding 2 to $\log y$ is equivalent to multiplying y by $10 \cdot 10$ or 10^2. We can confirm this by examining Equation (1), where we also see that each time x increases by a factor of 10, y increases by a factor of 10^2.

Translating Power Functions into Equivalent Logarithmic Functions

In general, we can translate a power function in the form

$$y = kx^p \qquad \text{where } k \text{ and } x > 0$$

into an equivalent linear function

$$
\begin{array}{rcll}
\log y &=& \log k + p\log x & \text{or}\\
Y &=& \log k + pX & \text{where } X = \log x \text{ and } Y = \log y
\end{array}
$$

The graph of the power function will appear as a straight line on a log-log plot with $\log x$ on the horizontal and $\log y$ on the vertical axis. The slope of the line is the power p of x and the vertical intercept is $\log k$, where k is the constant of proportionality.

Finding the Equation of a Power Function on a Log-Log Plot

The graph of a power function $y = kx^p$, where k and $x > 0$, appears as a straight line on a log-log plot and the equation of the line is

$$Y = \log k + pX \qquad \text{where } X = \log x \text{ and } Y = \log y$$

The slope of the line is p, where p is the power of x for $y = kx^p$.

The vertical intercept is $\log k$, where k is the constant of proportionality for $y = kx^p$.

Analyzing Weight and Height Data

In 1938, Katherine Simmons and T. Wingate Todd measured the average weight and height of children in Ohio between the ages of 3 months and 13 years. Table 7.9 shows their data.

Using a standard plot

Let's examine the relationship between height and weight. Figure 7.39 shows height versus weight for boys and for girls using a standard linear scale on both axes.

DATA See Excel or graph link file CHILDSTA.

Measuring Children

Age (yr)	Weight (kg)		Height (cm)	
	Boys	Girls	Boys	Girls
$\frac{1}{4}$	6.5	5.9	61.3	59.3
1	10.8	9.9	76.1	74.2
2	13.2	12.5	87.4	86.2
3	15.2	14.7	96.2	95.5
4	17.4	16.8	103.9	103.2
5	19.6	19.2	110.9	110.3
6	22.0	22.0	117.2	117.4
7	24.8	24.5	123.9	123.2
8	28.2	27.9	130.1	129.3
9	31.5	32.1	136.0	135.7
10	35.6	35.2	141.4	140.8
11	39.2	39.5	146.5	147.8
12	42.0	46.6	151.1	155.3
13	46.6	52.0	156.7	159.9

Table 7.9
Source: Data adapted from D. W. Thompson, *On Growth and Form* (New York: Dover Paperback, 1992), p. 105.

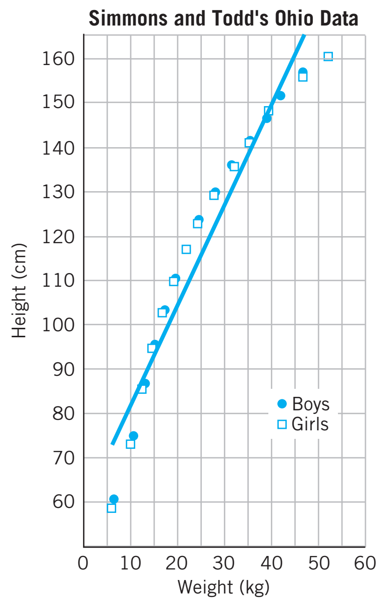


Figure 7.39 Standard linear plot of height vs. weight data for boys and girls.

The filled circles are the data for boys, the open squares the data for girls. Table 7.9 also gives their respective ages, and in general, baby girls are smaller than baby boys and teenage girls are larger than teenage boys. But in the graph in Figure 7.39, we see that at the same weight the heights of boys and girls are roughly the same.

The line in Figure 7.39 shows the linear model that best approximates the combined data. The line does not describe the data very well, and it doesn't seem reasonable that height is a linear function of weight for growing children. Other models may describe the data better.

Using a semi-log plot

SOMETHING TO THINK ABOUT

Why is there no zero on an axis with a logarithmic scale?

Figure 7.40 shows the same data, this time using a linear scale for weight and a logarithmic scale for height. Figure 7.40 also shows a best-fit exponential function, which appears as a straight line on the semi-log plot. This model is not a good fit to the data, and it doesn't seem reasonable to suppose that height is an exponential function of weight.

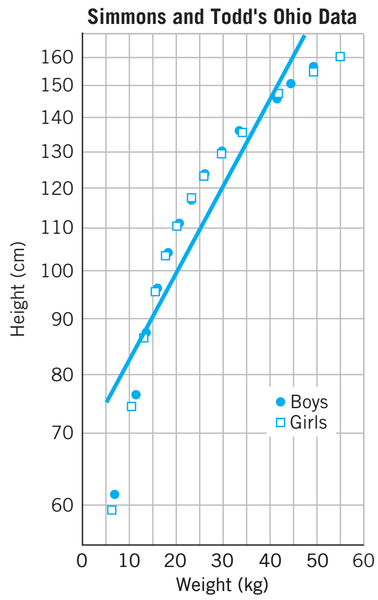


Figure 7.40 Semi-log plot of height vs. weight data for boys and girls.

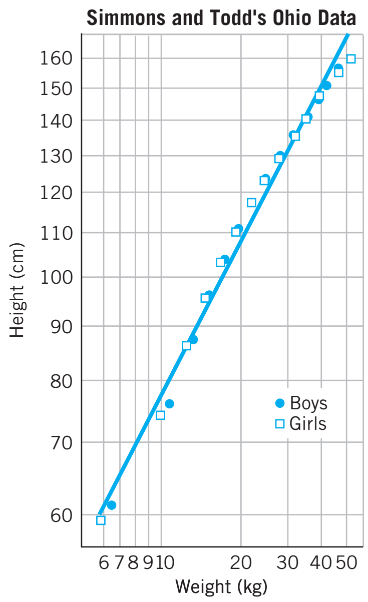


Figure 7.41 Log-log plot of height vs. weight data.

Using a log-log plot

The same data are plotted again on a log-log plot (a plot with logarithmic scales on both axes) in Figure 7.41. In this case, the line is a reasonably good approximation to the data and certainly the best fit of the three models considered. Since we know that a straight line on a log-log plot represents a power function, we can find the equation for the best-fit straight line on the log-log plot and transform the equation into a power function.

Finding the equation of a power function on a log-log plot

To construct a linear function for the best-fit line on the log-log plot in Figure 7.41, we can regraph the line using $H(=\log h)$ and $W(=\log w)$ scales on the axes, where h = height (cm) and w = weight (kg) (see Figure 7.42).

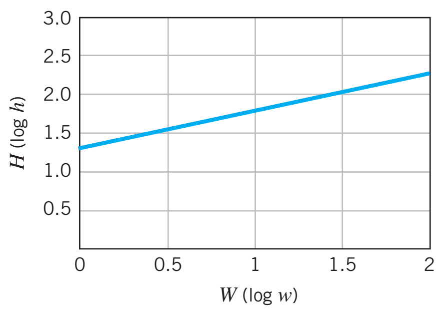


Figure 7.42 Best-fit line for height vs. weight data graphed on a log-log plot.

The equation will then be of the form

$$H = b + mW \tag{1}$$

where $H = \log h$ and $W = \log w$. To find values for the slope m and the vertical intercept b, we can estimate two points on the line, say (0.5, 1.5) and (1.5, 2.0). So

$$\text{slope} = \frac{2.0 - 1.5}{1.5 - 0.5} = 0.5$$

From the graph, the vertical intercept appears to be approximately 1.3. So the linear equation would be

$$H = 1.3 + 0.5W \tag{2}$$

where $H = \log h$ and $W = \log w$. We can transform this equation into an equivalent power function in the form

$$h = kw^p \tag{3}$$

where p is the slope of the best-fit line on the log-log plot and $\log k$ is the vertical intercept (see box on p. 431). The slope of the best-fit line is 0.5, so $p = 0.5$. The vertical intercept is approximately 1.3, so

	$\log k = 1.3$
definition of log	$k = 10^{1.3}$
evaluate $10^{1.3}$	$k \approx 20$

Substituting $p = 0.5$ and $k = 20$ in Equation (3), we get

$$h = 20w^{0.5} \qquad (\text{or } 20w^{(1/2)}) \tag{4}$$

We now have a power function that models the relationship between height, h, and weight, w.

Interpreting our results

Is it plausible to argue that height is a power law function of weight for growing children? Equation (4) says that

$$h \propto w^{1/2} \qquad \text{or equivalently} \qquad w \propto h^2$$

where the symbol $\propto$ means "is directly proportional to."

Let's think about whether this is a reasonable exponent. Suppose children grew *self-similarly;* that is, they kept the same shape as they grew from the age of 3 months to 13 years. Then, as we have discussed in Section 7.1, their volume, and hence their weight, would be proportional to the cube of their height:

$$w = kh^3 \qquad (\text{self-similar growth})$$

But of course children do not grow self-similarly; they become proportionately more slender as they grow from babies to young adults (see Figure 7.43). Their weight therefore grows less rapidly than would be predicted by self-similar growth, and we expect an exponent less than 3 for weight as a function of height.

Figure 7.43 The change in human body shape with increasing age.
Source: Medawar, P. B. 1945. "Size and Shape in Age" in *Essays on Growth and Form Presented to D'Arcy Thompson* (W. E. LeGros Clark and P. B. Medawar, eds.), pp. 157–187. Oxford: Clarendon Press.

On the other hand, suppose that children's bodies grew no wider as their height increased. Since weight is proportional to height times cross-sectional area, with constant cross-sectional area we would expect weight to increase linearly with height. So $w \propto h^1$. But certainly, children do become wider as they grow taller, so we expect the exponent of h to be greater than 1. Therefore, the experimentally determined exponent of approximately 2 seems reasonable.

Other considerations

There is a fourth possibility for plotting our height-weight data: We could use a linear scale for the height axis and a logarithmic scale for the weight axis. On such a plot, a straight line will represent a model in which weight is an exponential function of height.

If we generate this plot, the data fall along what is very close to a straight line. Based only on the data, we might argue that an exponential function is a reasonable model. However, it is very difficult to construct a plausible physical explanation of such a model, and it would not help us understand how children grow. It may be, of course, that this relationship reveals some unknown physical or biological law. Perhaps you, the reader, will be the one to find some previously unsuspected explanation.

Allometry: The Effect of Scale

Allometry is the study of how the relationships among different physical attributes of an object change with scale. In biology, allometric studies naturally focus on relationships in living organisms. Biologists look for general "laws" that can describe the relationship, for instance, between height and weight (as in this section) or between surface area and volume (as in Section 7.1) as the organism size increases. The relationships may occur within one species or across species. These laws are characteristically power functions. Sometimes we can predict the exponent by simple reasoning about physical properties and verify our prediction by examining the data; other times we can measure the exponent from data but cannot give a simple explanation for the observed value.

Surface area vs. body mass

In Section 7.1 we made the argument that larger animals have relatively less surface area than smaller ones. Let's see if we can describe this relationship as a power law and then look at some real data to see if our conclusion was reasonable.

Making Predictions. Since all animals have roughly the same mass per unit volume, their mass should be directly proportional to their volume. We will substitute mass for volume in our discussion since we can determine mass easily by weighing an animal. Since volume is measured in cubic units of some length, if animals of different sizes have roughly the same shape, then we would expect mass, M, to be proportional to the cube of the length, L, of the animal,

$$M \propto L^3 \tag{1}$$

that is, $M = k_1 \cdot L^3$ for some constant k_1. Also, we would expect surface area, S, to be proportional to the square of the length:

$$S \propto L^2 \tag{2}$$

That is, $S = k_2 \cdot L^2$ for some constant k_2.

By taking the cube root and square root, respectively, in relationships (1) and (2), we may rewrite them in the form

$$M^{1/3} \propto L \quad \text{and} \quad S^{1/2} \propto L$$

Since $M^{1/3} \propto L$ and $L \propto S^{1/2}$, we can now eliminate L and combine these two statements into the prediction that

$$S^{1/2} \propto M^{1/3}$$

By squaring both sides, we have the equivalent relationship

$$S \propto M^{2/3} \tag{3}$$

It is useful to eliminate length as a measure of the size of an animal, since it is a little more ambiguous than mass. Should a tail, for instance, be included in the length measurement?

Testing Our Predictions. This prediction, that surface area is directly proportional to the two-thirds power of body mass, can be tested experimentally. Figure 7.44 shows data collected for a wide range of mammals, from mice, with a body mass on the order of 1 gram, to elephants, whose body mass is more than a million grams (1 metric ton). Here surface area, S, measured in square centimeters, is plotted on the vertical axis and body mass, M, in grams, is on the horizontal. The scales on both axes are logarithmic.

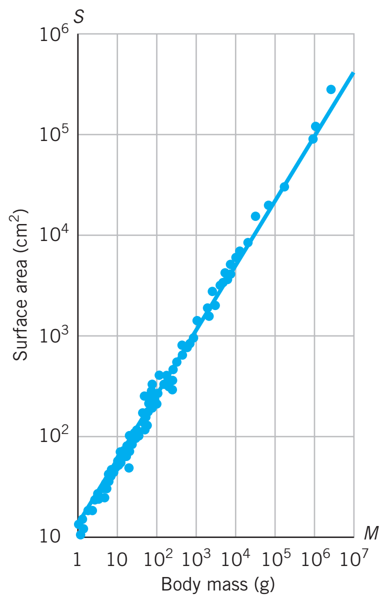


Figure 7.44 A log-log plot of body surface vs. body mass for a wide range of mammals.
Source: McMahon, T. A. 1973. "Size and Shape in Biology," *Science,* 179: 1201–1204.

The data lie pretty much in a straight line. Since this is a log-log plot, that implies that S is directly proportional to M^p for some power p, where p is the slope of the best-fit line. To estimate the slope we can convert the scales on the axes in Figure 7.44 to log(surface area) and log(body mass); that is, we use only the exponents of 10 on both the vertical and the horizontal axes. Then the best-fit line passes through (3, 3) and (6, 5), so the slope is $\frac{5-3}{6-3} = \frac{2}{3}$. This implies that

$$S \propto M^{2/3}$$

confirming our simple prediction of a two-thirds power law. This means that every time the horizontal coordinate increases by a factor of 10^3, the vertical coordinate increases by a factor of 10^2. We have approximate verification of the fact that mammals of different body masses have roughly the same proportion of surface area to the two-thirds power of body mass. This power function is just another way of describing our finding in Section 7.1 on the relationship between surface area and volume.

Metabolic rate vs. body mass

Another example of scaling among animals of widely different sizes is metabolic rate as a function of body mass. This relationship is very important in understanding the mechanisms of energy production in biology. Figure 7.45 shows a log-log plot of metabolic heat production, H, in kilocalories per day, versus body mass, M, in kilograms, for a range of land mammals.

The data fall in a straight line, so we would expect H to be directly proportional to a power of M. The slope of this line, and hence the exponent of M, is $\frac{3}{4} = 0.75$. As body mass increases by four factors of 10, say from 10^{-1} to 10^3 kg, metabolic rate increases by three factors of 10, from 10 to 10^4 kcal/day. Thus $H \propto M^{3/4}$, where M is mass in kilograms and H is metabolic heat production in kilocalories (thousands of calories) per day.

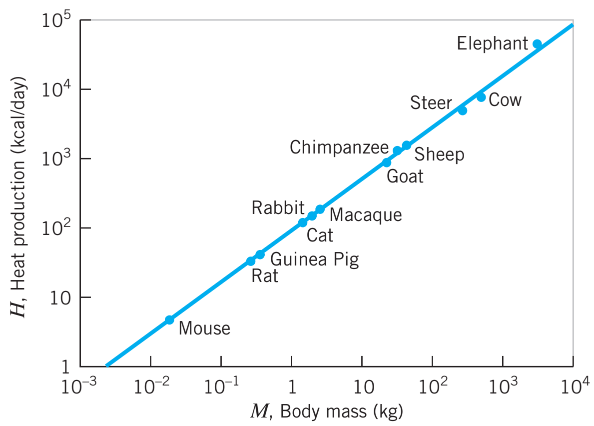


Figure 7.45 A log-log plot of heat production vs. body mass.
Source: Kleiber, M. 1951. "Body Size and Metabolism," *Hilgardia,* 6: 315–353.

Kleiber's Law. This scaling relationship is called "Kleiber's Law," after the American veterinary scientist who first observed it in 1932. It has been verified by many series of subsequent measurements, though its cause is not fully understood.

Animals have evolved biological modifications partially to avoid the consequences of scaling laws. For example, our argument for the two-thirds power law of surface area vs. body mass was based on the assumption that the shapes of animals stay roughly the same as their size increases. But elephants have relatively thicker legs than gazelles in order to provide the extra strength needed to support their much larger weight. Such changes in shape are very important biologically but are too subtle to be seen in the overall trend of the data we have shown. Despite these adaptations, inexorable scaling laws give an upper limit to the size of land animals, since eventually the animal's weight would become too heavy to be supported by its body. That's why large animals such as whales must live in the ocean and why giant creatures in science fiction movies couldn't exist in real life.

Allometric Laws. Allometric laws provide an overview of the effects of scale, valid over several orders of magnitude. They help us compare important traits of elephants and mice, or of children and adults, without getting lost in the details. They help us to understand the limitations imposed by living in three dimensions.

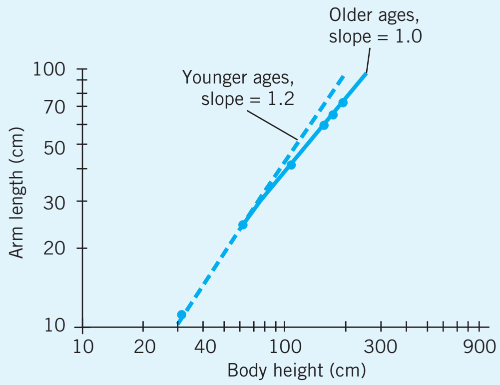
Algebra Aerobics 7.7

A calculator that can evaluate powers and logs is required for Problems 2, 3, and 7.

1. Rewrite each of the following as an equivalent equation by taking the log of both sides.
   - **a.** $y = 3 \cdot 2^x$ **c.** $y = 12 \cdot 10^x$
   - **b.** $y = 4x^3$ **d.** $y = 0.15x^{-2}$
2. Rewrite each of the following so that y is a power or exponential function of x.
   - **a.** $\log y = 0.067 + 1.63 \log x$
   - **b.** $\log y = 2.135 + 1.954x$
   - **c.** $\log y = -1.963 + 0.865x$
   - **d.** $\log y = 0.247 - 0.871 \log x$
3. By inspection, determine whether y is an exponential or a power function of x. Then rewrite the expression so that y is a (power or exponential) function of x.
   - **a.** $\log y = \log 2 + 3 \log x$
   - **b.** $\log y = 2 + x \log 3$
   - **c.** $\log y = 0.031 + 1.25x$
   - **d.** $\log y = 2.457 - 0.732 \log x$
   - **e.** $\log y = -0.289 - 0.983x$
   - **f.** $\log y = -1.47 + 0.654 \log x$
4. **a.** Identify the type of function.
   - **i.** $y = 4x^3$ **ii.** $y = 3x + 4$ **iii.** $y = 4 \cdot 3^x$

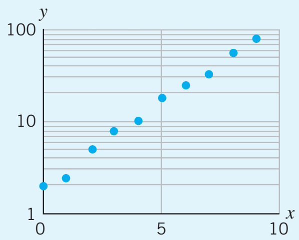
   b. Which of the functions in part (a) would have a straight-line graph on a standard linear plot? On a semi-log plot? On a log-log plot? If possible, use technology to check your answers.

   c. For each straight-line graph in part (b), use the original equation in part (a) to predict the slope of the line.
5. Interpret the slopes of 1.2 and 1.0 on the accompanying graph in terms of arm length and body height. [*Note:* Slopes refer to the slopes of the lines, where the units of the axes are translated to log(body height) and log(arm length).]

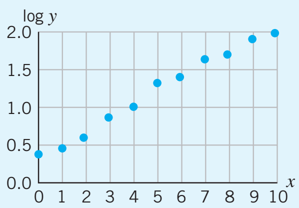


Source: T. A. McMahon and J. T. Bonner, *On Size and Life* (New York: Scientific American Books, 1983), p. 32.

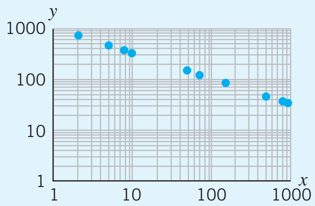
6. For each of the three graphs below, examine the scales on the axes. Then decide whether a linear, exponential, or power function would be the most appropriate model for the data.



Graph *A*



Graph *B*



Graph *C*

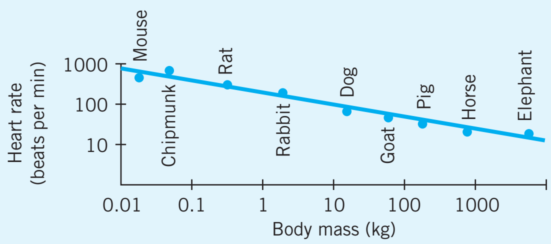
7. a. Using Figure 7.44 on page 435, estimate the surface area for a human being whose mass is 70 kg (or $7 \cdot 10^4$ g).

   b. Construct a power function for the plot in Figure 7.44 and use it to calculate the value for part (a).

   c. How do your answers in parts (a) and (b) compare?

   d. Translate your answers into units of pounds and square inches.

8. The accompanying graph shows the relationship between heart rate (in beats per minute) and body mass (in kilograms) for mammals ranging over many orders of magnitude in size.



Source: Robert E. Ricklefs, *Ecology* (San Francisco: W. H. Freeman, 1990), p. 66.

   a. Would a linear, power, or exponential function be the best model for the relationship between heart rate and body mass?

   b. The slope of the best-fit line shown on the graph is −0.23. Construct a function for heart rate in terms of body mass.

   c. Interpret the slope of −0.23 in terms of heart rate and body mass.

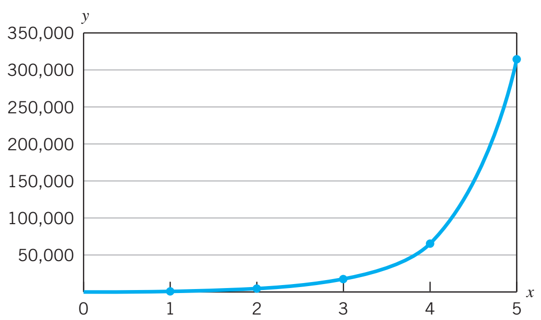
9. Using the graph in Figure 7.45 on page 436, what rate of heat production does this model predict for a 70-kg human being? Considering that each kilocalorie of heat production requires consumption of one food Calorie, does this seem about right?

Exercises for Section 7.7

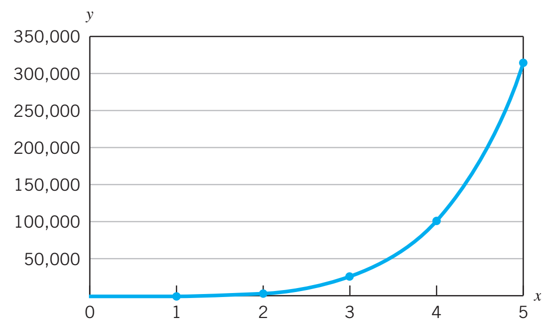
Several exercises require a graphing program that can convert axes from a linear to a logarithmic scale.

1. Match each function with its standard linear-scale graph and its logarithmic-scale graph. (*Hint:* Evaluate each function for $x = 4$.)

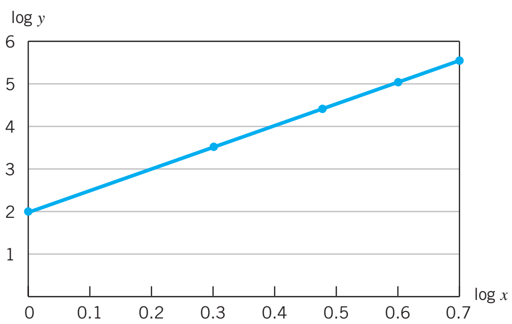
   a. $y = 100\,(5)^x$ b. $y = 100x^5$



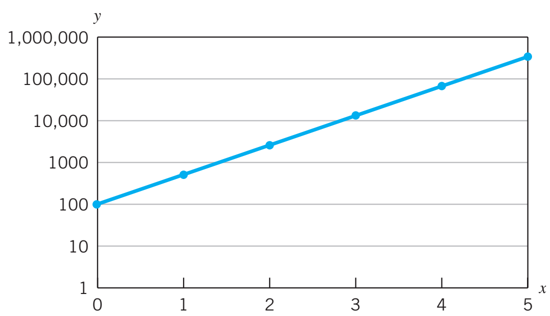
Graph *A*



Graph *C*



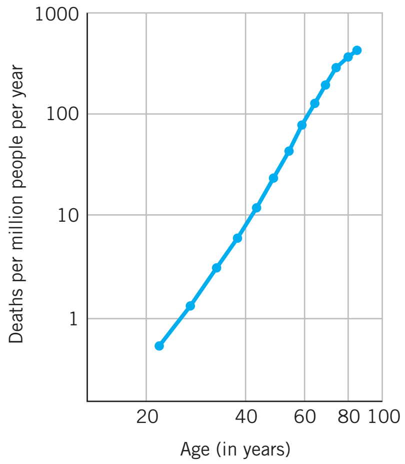
Graph *B*



Graph *D*

2. The accompanying graph shows data on U.S. annual death rate from cancer of the large intestine as a function of age. Would an exponential or power function be a better model for the data? Justify your answer.

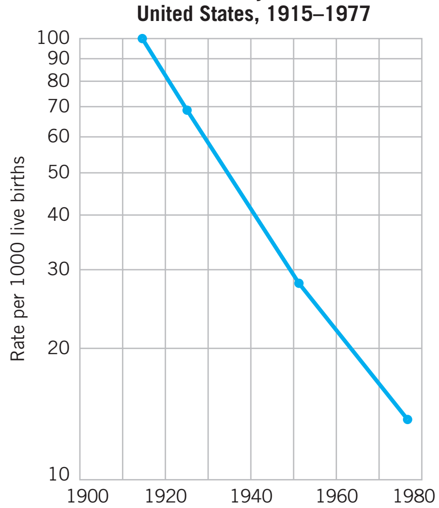
Annual U.S. Death Rate from Cancer of the Large Intestine in Relation to Age, 1968



Source: J. Cairns, *Cancer and Society* (San Francisco: W. H. Freeman and Company, 1978), p. 37.

3. Does the accompanying graph suggest that an exponential or a power function would be a better choice to model infant mortality rates in the United States from 1915 to 1977? Explain your answer.

Infant Mortality Rates in the United States, 1915–1977



Source: Healthy People: The Surgeon General's Report on Health Promotion and Disease Prevention, DHEW (PHS) Publication 79-55071, Department of Health, Education, and Welfare, Washington, DC, 1979.

4. Given the following three power functions in the form $y = kx^p$,

$$y_1 = x^3 \qquad y_2 = 5x^3 \qquad y_3 = 2x^4$$

a. Use the rules of logarithms to change each power function to the form: $\log y = \log k + p \log x$.

b. Substitute in each equation in part (a), $Y = \log y$ and $X = \log x$ and the value of $\log k$ to obtain a linear function in X and Y.

c. Compare your functions in parts (a) and (b) to the original functions. What does the value of $\log k$ represent in the linear equation? What does the value of the slope represent in the linear equation?

5. Using rules of logarithms, convert each equation to its power function equivalent in the form $y = kx^p$.

a. $\log y = \log 4 + 2 \log x$ **c.** $\log y = \log 1.25 + 4 \log x$

b. $\log y = \log 2 + 4 \log x$ **d.** $\log y = \log 0.5 + 3 \log x$

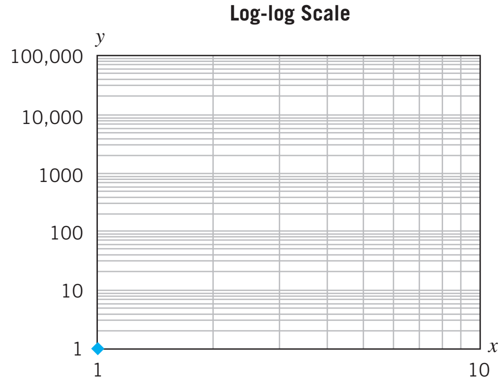
6. Given linear equations in the form $Y = B + mX$, where $Y = \log y$, $X = \log x$, and $B = \log k$, change each linear equation to its power function equivalent, in the form $y = kx^p$. (*Note:* Round values to the nearest hundredth.)

a. $Y = 0.34 + 4X$ **c.** $Y = 1.0 + 3X$

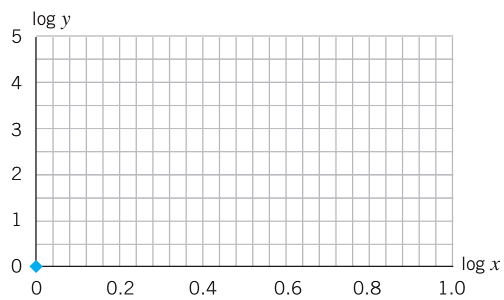
b. $Y = -0.60 + 4X$ **d.** $Y = 1.0 - 3X$

7. a. Create a table of values for the function $y = x^5$ for $x = 1, 2, 3, 4, 5, 6$. Then plot the values on the given log-log scale.

Log-log Scale



b. Add to the table you created in part (a) the values for $\log x$ and $\log y$. Then graph the results on the $\log y$ vs. $\log x$ scale provided below.

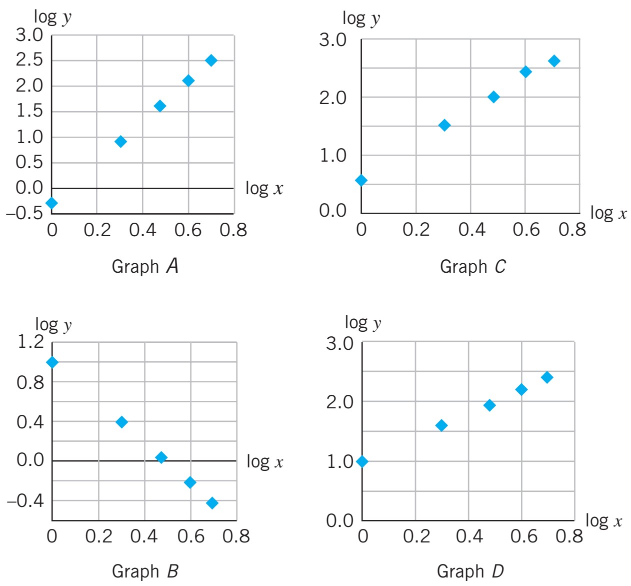


c. Describe the relationship between the plots in parts (a) and (b).

8. Match each power function with its log y vs. log x plot.

a. $y = 10x^2$ **c.** $y = \frac{1}{2}x^4$

b. $y = 4x^3$ **d.** $y = 10x^{-2}$



9. (Technology recommended for graphing and changing axis scales.) The following data give the typical masses for some birds and their eggs:

Species	Adult Bird Mass (g)	Egg Mass (g)
Ostrich	113,380.0	1,700.0
Goose	4,536.0	165.4
Duck	3,629.0	94.5
Pheasant	1,020.0	34.0
Pigeon	283.0	14.0
Hummingbird	3.6	0.6

Source: W. A. Calder III, *Size, Function and Life History* (Boston: Harvard University Press, 1984).

a. Create a fourth column in the table with the values for the ratio $\frac{\text{egg mass}}{\text{adult bird mass}}$. Is the ratio the same for all of the birds? (*Note:* The first ratio value should be 0.015.) Write a sentence that describes what you discover.

b. Graph egg mass (vertical axis) vs. adult bird mass (horizontal axis) three times using standard linear, semi-log, and log-log plots.

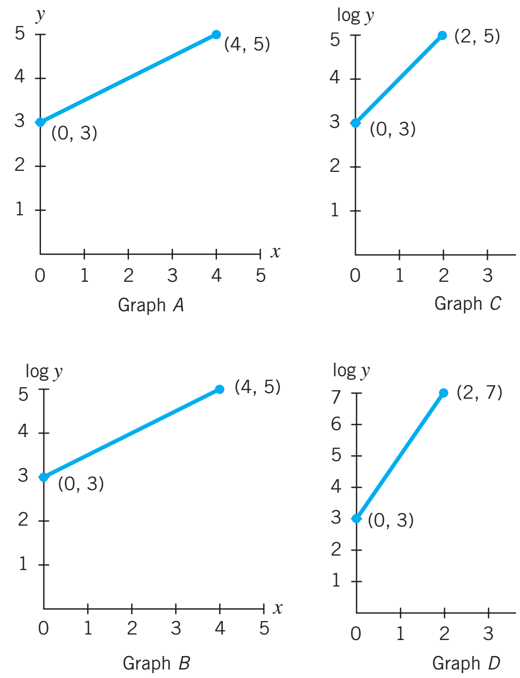
c. Examine the three graphs you made and determine which looks closest to linear. Find a linear equation in the form $Y = b + mX$ to model the line. (*Hint:* To find the slope of the line, you will need to use log y on the vertical axis, and possibly log x on the horizontal axis.)

d. Once you have a linear model, transform it to a form that gives egg mass as a function of adult bird mass.

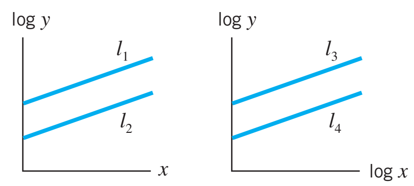
e. What egg mass does your formula predict for a 12.7-kilogram turkey?

f. A giant hummingbird (*Patagona gigas*) lays an egg of mass 2 grams. What size does your formula predict for the mass of the adult bird?

10. Find the equation of the line in each of the accompanying graphs. Rewrite each equation, expressing y in terms of x.



11. Use the accompanying graphs to answer the following questions.



a. Assume l_1 and l_2 are straight lines that are parallel. In each case, what type of equation would describe y in terms of x? How are the equations corresponding to l_1 and l_2 similar? How are they different?

b. Assume l_3 and l_4 are also parallel straight lines. For each case, what type of equation would describe y in terms of x? How are the equations corresponding to l_3 and l_4 similar? How are they different?

12. (Technology recommended for graphing and changing axis scales.)

a. Assume that $f(1) = 5$ and $f(3) = 45$. Then find an equation for f assuming f is:

i. A linear function

ii. An exponential function

iii. A power function

b. Verify that you get a straight line when you plot:
  i. Your linear function on a standard plot
  ii. Your exponential function on a semi-log plot
  iii. Your power function on a log-log plot

13. (Technology optional.) For each table, create a linear, exponential, or power function that best models the data. Support your answer with a graph.

a.

x	1	2	3	4	5
y	3	12	27	48	75

b.

x	1	2	3	4	5
y	8	16	32	64	128

c.

x	1	2	3	4	5
y	9	12	15	18	21

14. (Graphing program required.) Examine the following table.

x	y	$\log x$	$\log y$
1	5	0	0.698 97
2	80	0.301 03	1.903 09
3	405	0.477 12	2.607 46
4	1280	0.602 06	3.107 21
5	3125	0.698 97	3.494 85
6	6480	0.778 15	3.811 58

Treating x as the independent variable:

a. Create a scatter plot of y versus x.

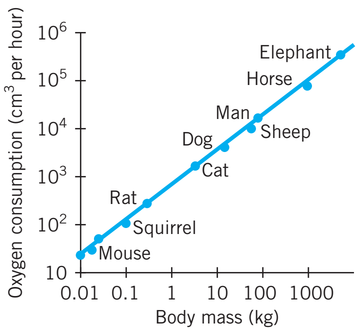
b. Create a scatter plot of $\log y$ versus x.

c. Create a scatter plot of $\log y$ versus $\log x$.

d. Does the table represent a linear, power, or exponential function? Explain your reasoning.

e. Create the equation that models the data.

15. The accompanying graph shows oxygen consumption versus body mass in mammals.



Source: R. E. Ricklefs, *Ecology* (San Francisco: W. H. Freeman, 1990), p. 67.

a. Would a power or an exponential function be the best model of the relationship between oxygen consumption and body mass?

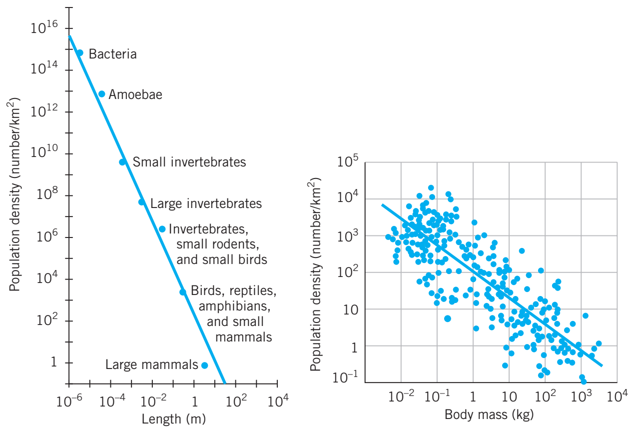
b. The slope of the best-fit line (for log(oxygen) vs. log(mass)) shown on the graph is approximately $\frac{3}{4}$. Construct the basic form of the functional model that you chose in part (a).

c. Interpret the slope in terms of oxygen consumption and body mass. In particular, by how much does oxygen consumption increase when body mass increases by a factor of 10? By a factor of 10^4?

16. a. The left-hand graph below shows the relationship between population density and length of an organism. The slope of the line for log(density) vs. log(length) is -2.25. Express the relationship between population density and length in terms of direct proportionality.

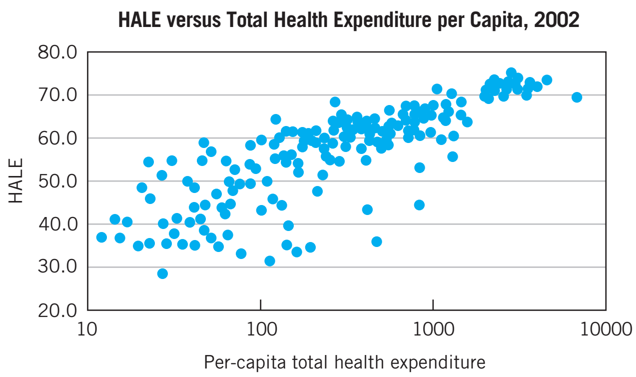
b. The right-hand graph shows the relationship between population density and body mass of mammals. The slope of the line for log(density) vs. log(mass) is -0.75. Express the relationship between population density and body mass in terms of direct proportionality.

c. Are your statements in parts (a) and (b) consistent with the fact that body mass is directly proportional to length^3? (*Hint:* Calculate the cube of the length to the -0.75 power.)



Source: T. A. McMahon and John T. Bonner, *On Size and Life* (New York: Scientific American Books, 1983), p. 228.

17. Public health researchers have developed a new measure of overall population health called "health-adjusted life expectancy" (HALE). HALE represents the number of expected years of life lived in full health. The following chart shows health-adjusted life expectancy versus the health care expenditure per capita for various countries. Note that here the log scale is on the *horizontal* axis.



a. Sketch a best-fit line through the data. Replace each label x on the horizontal axis with $\log(x) = X$. (So 10 becomes $\log 10 = 1$, 100 becomes $\log 100 = 2$, etc. You end up with the new labels 1, 2, 3, and 4 on the horizontal axis.)

b. Estimate two points on the line to construct an equation for your line for y in terms of X. (Note that the vertical axis, where $x = 0$, is not on the graph, so you can't use it to estimate the vertical intercept.) Then rewrite your equation as y in terms of $\log x$.

c. Why is it reasonable that HALE is a log function of per capita health expenditure?

d. What does your result suggest about functions that appear to be linear on a semi-log plot, where the logarithmic scale is on the horizontal axis?

CHAPTER SUMMARY

Power Functions

A power function has the form

$$y = kx^p$$

where k is called the *constant of proportionality*.

If $p > 0$ (and $k \neq 0$), then

y is *directly proportional to* x^p if $y = kx^p$

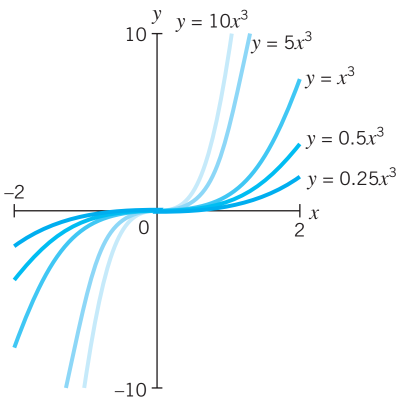
y is *inversely proportional to* x^p if $y = kx^{-p}\left(= \dfrac{k}{x^p}\right)$

Positive Integer Powers

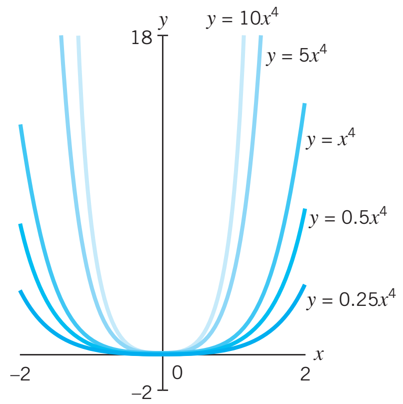
For power functions of the form $y = kx^p$ (where $p > 0$ and $k \neq 0$)

- The graph goes through the origin.
- If the power p is odd (and $p > 1$), the graph is shaped roughly like ~ and is symmetric about the origin.
- If the power p is even, the graph is ∪-shaped and symmetric across the y-axis.
- The value of the coefficient k compresses or stretches the graph of $y = x^p$.
- The graphs of $y = kx^p$ and $y = -kx^p$ are mirror images of each other across the x-axis.

Odd Positive Powers



Even Positive Powers

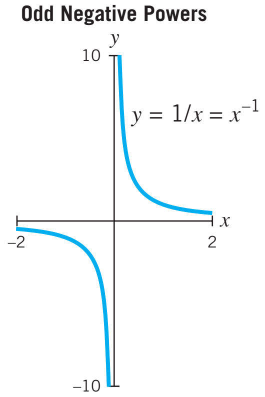


Negative Integer Powers

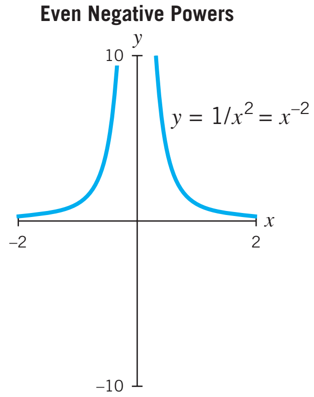
For power functions of the form $y = \frac{k}{x^p}$ or kx^{-p} (where $p > 0$ and $k \neq 0$)

- The graph never goes through the origin, since the function is not defined when $x = 0$.
- The graph is composed of two non-intersecting curves each asymptotic to both the x- and the y-axes.
- If the power p is even, the graph is symmetric across the y-axis.
- If the power p is odd, the graph is symmetric about the origin.
- The value of the coefficient k compresses or stretches the graph of $y = \frac{1}{x^p}$.
- The graphs of $y = \frac{k}{x^p}$ and $y = -\frac{k}{x^p}$ are mirror images of each other across the x-axis.

Odd Negative Powers



Even Negative Powers



Semi-Log and Log-Log Graphs

To find the best function to model a relationship between two variables, we look for a way to plot the data so they appear as a straight line:

- The graph of a linear function appears as a straight line on a standard plot that has linear scales on both axes.
- The graph of an exponential function appears as a straight line on a semi-log plot that has a linear scale on the horizontal axis and a logarithmic scale on the vertical axis. The slope is the logarithm of the growth (or decay) factor.
- The graph of a power function appears as a straight line on a log-log plot that has logarithmic scales on both axes. The slope is the exponent of the power function.

Finding the Equation of a Power Function on a Log-Log Plot

The graph of a power function $y = kx^p$ (where k and $x > 0$) appears as a straight line on a log-log plot. The equation of the line is

$$Y = \log k + pX \qquad \text{where } X = \log x \text{ and } Y = \log y$$

- The slope of the line is p, where p is the power of x for $y = kx^p$.
- The vertical intercept is $\log k$, where k is the constant of proportionality for $y = kx^p$.

CHECK YOUR UNDERSTANDING

I. Is each of the statements in Problems 1–21 true or false? Give an explanation for your answer.

1. The graph of $y = -3x^5$ is decreasing.

2. The graph of $y = 2x^{-1}$ increases when $x < 0$ and decreases when $x > 0$.

3. The graphs of $y = 3.5x^4$ and $y = 5.1x^2$ intersect only at the origin.

4. If $y = x^3$ and if x increases by 1, then y increases by a factor of 3.

5. If $y = 5x^{-1}$ and if x doubles, then y decreases by a factor of $\frac{1}{2}$.

6. The function $y = 3 \cdot 2^x$ is a power function.

7. The graphs of the functions $y = 3x^2$ and $y = 3x^{-2}$ are reflections of each other across the x-axis.

8. The graphs of the functions $y = 4x^{-1}$ and $y = -4x^{-1}$ are reflections of each other across the x-axis.

9. If $y = x^{-2}$, as x becomes very large, then y approaches zero.

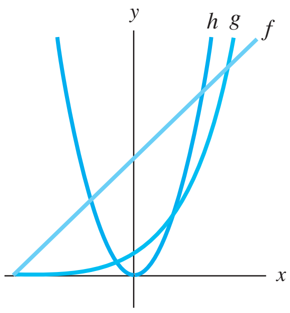
10. If $y = \frac{1}{x^2}$, as x approaches $+\infty$, then y approaches $+\infty$.

11. The function $y = x^{10}$ eventually grows faster than the function $y = 1.5^x$.

12. The functions $y = x^2$ and $y = x^{-1}$ intersect at the point (1, 1).

13. The graph of $y = \frac{2}{x^5}$ on a log-log plot is a straight line with slope of -5.

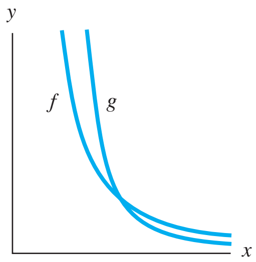
14. Of the three functions f, g, and h in the accompanying figure, only function h could be a power function.



15. If $F = km^{-3}$ and k is a nonzero constant, then F is directly proportional to m.

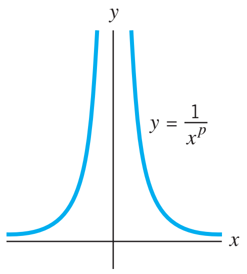
16. If $K(d) = 3d^{-2}$, then $K(-2) = \frac{-3}{4}$.

17. The graphs of two power functions $f(x) = \frac{1}{x^p}$ and $g(x) = \frac{1}{x^q}$ are shown in the accompanying figure. For these functions, $q < p$.



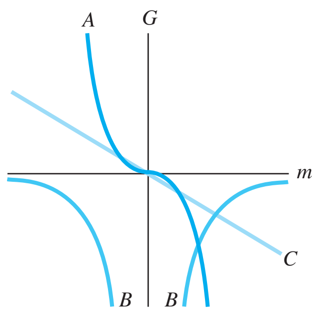
18. If a quantity M is inversely proportional to the cube of a quantity Q, then $M = \frac{k}{Q^3}$ for k, a nonzero constant.

19. In the power function $y = \frac{1}{x^p}$ graphed in the accompanying figure, p is an even positive integer.



20. If $f(q) = -q^5$, then $f(-2) = -32$.

21. The graphs of the power functions A, B, and C, all of the form $G = k \cdot m^p$, are shown in the accompanying figure. For each of these functions, $k < 0$.



II. In Problems 22–29, construct a function or functions with the specified properties.

22. A power function $y = f(x)$ whose graph is steeper than the graph of $y = 4x^6$ when $x > 1$.

23. A power function $y = g(x)$ whose graph is a reflection of the graph of $y = -3.2x^4$ across the horizontal axis.

24. A power function of even degree whose graph opens downward.

25. A function $w = h(m)$ whose graph will eventually dominate the function $w = 500m^{10}$.

26. A function that is not a power function.

27. A function whose graph on a log-log plot is a straight line.

28. A power function whose graph would be similar to the graph of $y = -\frac{3}{x}$ but would have a different power of x.

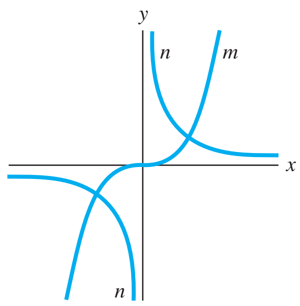
29. A power function $T(m)$ where doubling the value of m increases the value of $T(m)$ by a factor of 16.

III. Is each of the statements in Problems 30–40 true or false? If a statement is true, explain why. If a statement is false, give a counterexample.

30. All linear functions are power functions.

31. All power functions $y = kx^p$, where p is a positive integer, pass through the origin.

32. For the two power functions $m(x) = x^p$ and $n(x) = x^q$ in the accompanying figure, both p and q have negative values.



33. The graphs of all power functions $y = kx^p$, where p is a negative integer, lie in quadrants I (where $x \geq 0$ and $y \geq 0$) and III (where $x \leq 0$ and $y \leq 0$).

34. The graphs of all power functions with integer powers are asymptotic to the horizontal axis.

35. The graphs of all power functions $y = kx^p$, where p is a positive even integer, are symmetric across the y-axis.

36. The ratio of (surface area)/volume is less for a larger sphere than for a smaller one.

37. The graphs of the functions $F(m) = a \cdot m^p$ and $G(m) = -a \cdot m^p$, where $a \neq 0$ and p is a positive integer, are reflections of each other across the vertical axis.

38. The graphs of power functions $y = kx^p$, where p is a negative integer, never pass through the origin.

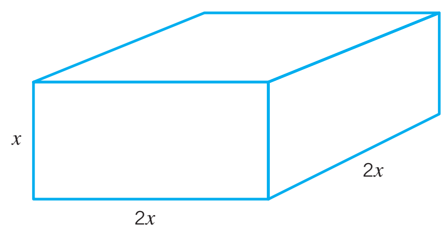
39. The graphs of power functions $f(x) = kx^p$, where p is a positive integer, on a log-log plot are straight lines with a positive slope.

40. The graphs of exponential functions $g(x) = a \cdot b^x$ on a log-log plot are also straight lines.

CHAPTER 7 REVIEW: PUTTING IT ALL TOGETHER

A standard calculator that can do multiplication and division is required throughout. Some problems are identified as needing a calculator that can calculate fractional powers (such as square roots) or a graphing program (that for one problem can change axis scales from linear to logarithmic).

1. Suppose you have a solid rectangular box, with dimensions x by $2x$ by $2x$.



a. What is its volume $V(x)$? If you double the value of x, from X to $2X$, what happens to the volume?

b. What is the surface area $S(x)$? If you double the value of x, from X to $2X$, what happens to the surface area?

c. Construct the ratio $R(x) = S(x)/V(x)$. If you double the value of x, from X to $2X$, what happens to $R(x)$? Describe your result in terms of surface area and volume.

d. As x increases, what happens to $R(x) = \text{(surface area)/volume}$?

2. Inside a bottle of Bayer aspirin is a small plastic cylinder (about 5/8" in height and 1/2" in diameter) whose tightly packed contents are used to absorb water to keep the aspirin dry.

a. What is the volume (in cubic inches) of the cylinder (and hence a good approximation of the volume of its contents)? [*Note:* The volume of a cylinder with radius r and height h is $\pi r^2 h$.

b. If you examine the contents of the cylinder under a microscope, you would see an incredibly convoluted surface. The total surface area is about the size of a football field (160' wide and 360' long, including the two end zones). What is the area in square feet? In square inches?

c. Interpret your results. Why would the manufacturers want such a large surface area in such a tiny volume?

3. The website for *La Riviera Gourmet* in Lexington, Massachusetts, states, "A 10" cake takes up 3 quarts of volume, whereas a 12" cake takes up 6 quarts of volume. That's why a 12" cake feeds twice as many and costs almost twice as much!" Does this seem reasonable?

4. Which of the following functions represent power functions? Which power functions represent direct and which inverse proportionality?

a. $Q = 4t$ **c.** $Q = 4 + t^2$ **e.** $Q = 4t^{-2}$

b. $Q = 4t^2$ **d.** $Q = 4 \cdot 2^t$

5. A bathtub holds about 15 gallons of water.

a. Construct a function to describe the water volume $V_F(t)$ in gallons at minute t if the tub is being filled at a rate of 3 gallons/minute. What would be a reasonable domain for the function? Graph $V_F(t)$ by hand. Does $V_F(t)$ represent direct proportionality?

b. Assuming the tub is full to the brim, construct a new function $V_D(t)$ to describe the water volume (in gallons) at minute t if the tub has a leak that is draining water at a rate of 0.5 gallons/minute. What would be a reasonable domain for $V_D(t)$? Graph $V_D(t)$ by hand. Does $V_D(t)$ represent inverse proportionality?

6. In some areas wind turbines offer a viable clean source of energy. The amount of power, W, (in watts) delivered by a wind turbine is a function of S, wind speed, and A, the area swept out by the wind turbine rotors. Wind engineers can estimate the wattage using the formula

$$W = 0.0052AS^3$$

where W is in watts, A is in square feet , and S is in miles per hour.

a. Is W directly proportional to A? To S? To S^3?

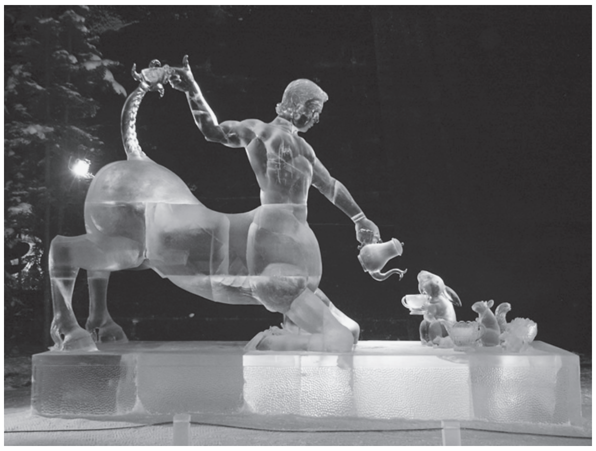
b. Calculate the circular area A (in square feet) swept out by wind turbine blades with a rotor radius of 10 ft. (Round off to the nearest integer.)

c. Using your answer in part (b), alter the general formula for W to represent the wattage W_{10} generated by a wind turbine with 10-ft blades. Using the new formula, how many watts would be generated by a wind speed of 30 mph? How many kilowatts is that? If a typical suburban house needs on average about 1.8 kilowatts of power at any point in the day, how many houses could this turbine support?

d. If the wind speed doubles, what happens to the wattage output? If the wind speed is cut in half, what happens to the wattage output?

e. Is it better to have ten wind turbines in an area that averages 20-mph winds or two turbines in an area that averages 40-mph winds?

7. Some northern cities celebrate New York's eve with First Night events, which include artists making large, elaborate ice sculptures. (See the accompanying photo from Fairbanks, Alaska.)



An artist planned to build a sculpture 4' high to be carved out of a 4'×2'×2' block of ice. But she decides it would look better proportionally scaled up to 6' high. Her friend says, "You'll have to order 50% more ice if you make it 6 ft instead of 4 ft."

a. Is the friend correct?

b. Give the dimensions of the ice block she should order for the 6' sculpture? What percent increase in the volume of ice would that be?

c. If the temperature gets above freezing and the 6' sculpture loses one-tenth of its volume through melting, how tall will it be then?

8. (Requires a calculator that can compute square roots.) In any stringed instrument, such as a violin, guitar, or ukulele, each note is determined by the frequency F of a vibrating string. F is directly proportional to the square root of the tension T and inversely proportional to the length L of the string. The higher the frequency, the higher the note.

a. Construct a general formula for F.

b. If the string length is cut in half (by holding down a string in the middle against the neck of the instrument), is the resulting note higher or lower? By how much?

c. If the tension is increased by 10%, will the resulting note be higher or lower? By how much?

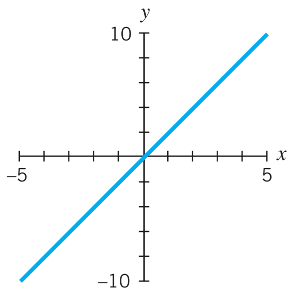
9. Construct a formula for each situation.

a. The time, t, taken for a journey is directly proportional to the distance traveled, d, and inversely proportional to the rate r at which you travel.

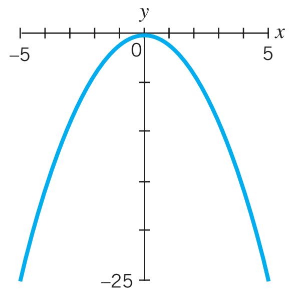
b. On a map drawn to scale, the distance D_1 between two points is directly proportional to the distance D_2 between the corresponding two points in real life.

c. The resistance R is directly proportional to the voltage, V, squared and inversely proportional to the wattage, W.

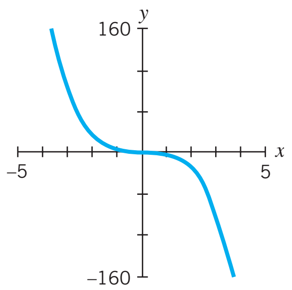
10. The following four graphs are power functions with positive exponents (and hence represent direct proportionality). Which graphs depict even and which depict odd power functions?



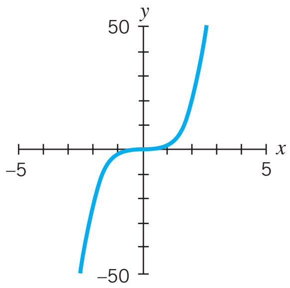
Graph *A*



Graph *C*



Graph *B*



Graph *D*

11. Describe any symmetries and asymptotes for each function graphed in the preceding problem.

12. Of the following three functions, as $x \to +\infty$, which would eventually dominate (have the largest values for y)?

$$y_1 = x^{1000}, \quad y_2 = 1.000001^x, \quad y_3 = 10^{100{,}000}x$$

13. (Graphing program optional.)

a. In Nashua, New Hampshire, you can rent an indoor skydiving facility for \$450 per half hour. What is the cost per person, $P(n)$, if there are n friends sharing the experience?

b. What is the cost per person for two people? For five people? What is the trade-off between number of people and cost per person?

c. Now consider your function $P(n)$ as an abstract function, where n can be any positive or negative real number. Graph this extended version of $P(n)$ and identify any asymptotes.

14. (Requires a calculator that can compute square roots.) Film studios, photographers, and the lighting industry usually measure light intensity in foot-candles—historically, the amount of light one candle generates on a surface one foot away. Light intensity I (in foot-candles) is directly proportional to candlepower P (which measures how much light is generated by a light source) and inversely proportional to the square of the distance, d, from the light source. (See Section 7.5, Example 7.) The relationship is the inverse square law

$$I = P/d^2$$

where I is measured in foot-candles, P in candlepower, and d in feet.

a. A student's angle-arm lamp can be positioned as close as 1 foot above the work surface and as far as 3 feet, measuring to the center of the bulb. If the 60-watt bulb has 70 candlepower, how many foot-candles illuminate the surface at the lowest and the highest positions?

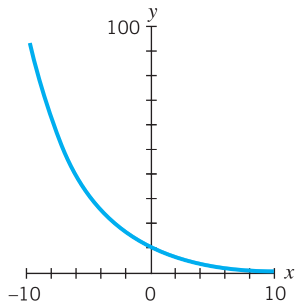
b. Where would you position the bulb to get 50 foot-candles of illumination, the standard level recommended for reading?

15. (Requires a calculator that can compute square roots.) Surgeons use a lamp that supplies 4000 foot-candles when suspended 3 feet above the operating surface. (See the preceding problem.)

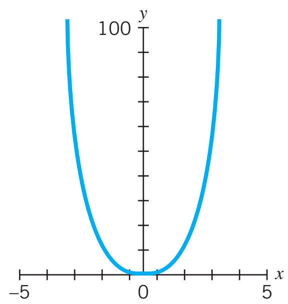
a. What candlepower would the source have to provide?

b. Lighting engineers assume that surgery requires somewhere in the range of 2000 to 5000 foot-candles. How high above the operating surface would you hang the lamp to give 2000 foot-candles?

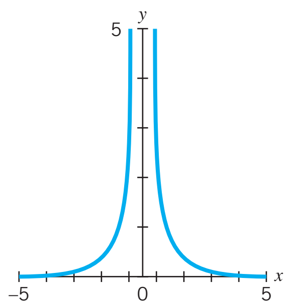
16. Which of the following graphs could depict power functions with a negative exponent (and hence inverse proportionality)?



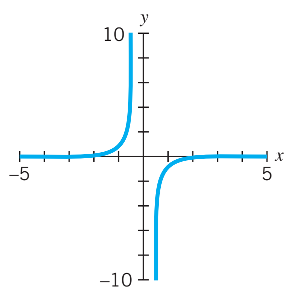
Graph *A*



Graph *C*



Graph *B*



Graph *D*

17. Describe any apparent symmetries and asymptotes for each of the function graphs in the preceding problem.

18. (Requires a calculator that can compute fractional powers.) Because wind speed increases with height, wind turbines are placed as high as possible, and certainly above the level of any existing structures or trees that may block the wind. In order to estimate typical wind speeds at various heights, wind engineers first measure the average wind speed S_{33} (in mph) at a height of 33 feet. Then they can estimate the wind speed S_h at height h using the formula

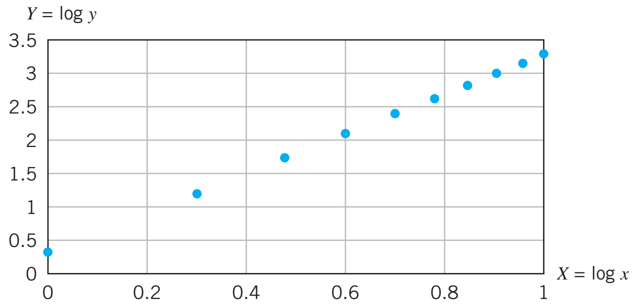
$$S_h = S_{33}(h/33)^{2/7}$$

where h is in feet, and S_h and S_{33} are both in miles per hour.

a. For an area likely to have hurricanes, wind engineers design for a wind speed of 90 mph at a height of 33 feet. Estimate the wind speed under hurricane conditions at the top of a building 120 feet tall.

b. Rearrange the formula to express h in terms of S_h and S_{33}. Using the hurricane conditions in part (a), at what height dose the formula predict 156-mph winds?

19. a. Given the following graph, what kind of function is Y with respect to X?



b. Sketch in a best-fit line and then generate its equation using Y in terms of X. [*Hint:* Use the points (0, 0.3) and (1, 3.3).]

c. Transform your equation in part (b) to one using y and x. What sort of function do you have now (for y in terms of x)?

20. (Requires a graphing program that can convert the scale on either axis from linear to logarithmic.) These data give the sound reduction in decibels through walls of different densities, measured in pounds per square foot of the wall. (*Note:* This assumes a fixed depth to the wall material.)

Sound Reduction through Walls

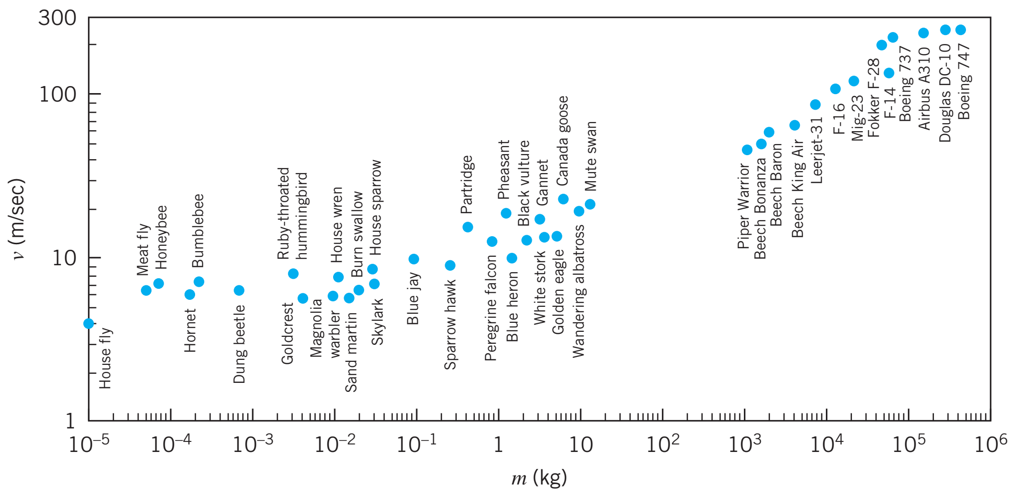
Wall Density (lb/sq. ft)	Sound Loss (Db)
1	22.7
5	32.0
10	37.0
20	41.0
40	45.0
60	48.0
100	51.0
400	60.0

a. Graph these data with wall density on the horizontal axis. Try linear, log-linear (with the log scale on the horizontal and then on the vertical axis), and log-log plots. In which format do the data look most linear?

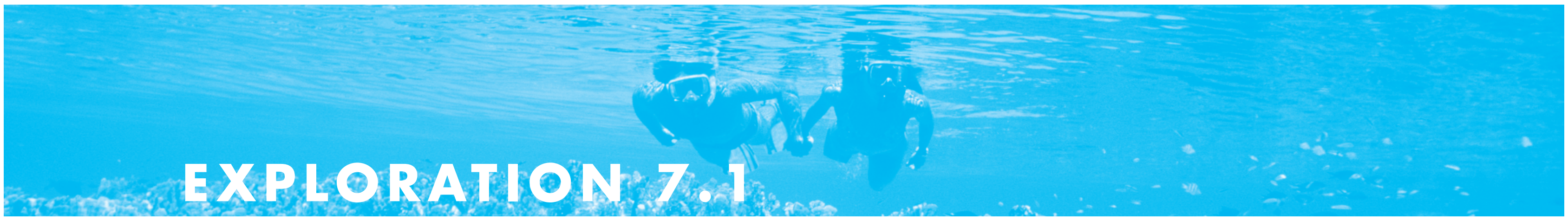
b. Use the format of the data from part (a) to construct a formula for sound loss as a function of wall density.

c. What does your result suggest about data that appear linear when graphed on a semi-log plot, where the log scale is on the horizontal axis?

21. According to Adrian Bejan's Constructal Theory, the optimal cruising speed v (in meters per second) of flying bodies (whether insects, birds, or airplanes) is directly proportional to the one-sixth power of body mass m (in kilograms). Use the accompanying graph below to confirm or refute this.



Source: Image from Wikipedia, derived from the work of Adrian Bejan, Professor of Mechanical Engineering at Duke University.



EXPLORATION 7.1

Scaling Objects

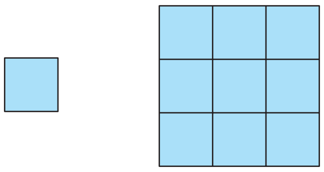
Objectives

- find and use general formulas for scaling different types of objects

Procedure

1. *Scaling factors.*

When a two- or three-dimensional object is enlarged or shrunk, each linear dimension is multiplied by a constant called the *scaling factor*. For the two squares below, the scaling factor, F, is 3. That means that any linear measurement (for example, the length of the side or of the diagonal) is three times larger in the bigger square.



Scaling up a square by a factor of 3.

a. What is the relationship between the areas of the two squares above? Show that for all squares

$$\text{area scaled} = (\text{original area}) \cdot F^2 \qquad \text{where } F \text{ is the scaling factor}$$

b. Given a scaling factor, F, find an equation to represent the surface area S_1 of a scaled-up cube, in terms of the surface area S_0 of the original cube.

c. Given a scaling factor, F, find an equation to represent the volume V_1 of a scaled-up cube, in terms of the volume V_0 of the original object.

2. *Representing 3-D objects.*

We can describe the volume, V, of any three-dimensional object as $V = kL^3$, where V depends on any length, L, that describes the size of the object. The coefficient, k, depends upon the shape of the object and the particular length L and the measurement units we choose. For a statue of a deer, for example, L could represent the deer's width or the length of an antler or the tail. If we let L_0 represent the overall height of the deer in Figure 1, then L_0^3 is the volume of a cube with edge length L_0 that contains the entire deer. The actual volume, V_0, of the deer is some fraction k of L_0^3; that is, $V_0 = kL_0^3$ for some constant k.

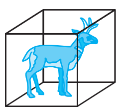


Figure 1

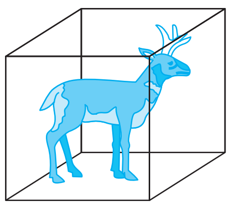


Figure 2

3. *Finding the scaling factor.*

If we scale up the deer to a height L_1 (see Figure 2), then the volume of the scaled-up deer equals kL_1^3. The scaled-up object inside the cube will still occupy the same fraction, k, of the volume of the scaled-up cube.

a. If an original object has volume V_0 and some measure of its length is L_0, then we know $V_0 = kL_0^3$. Rewrite the equation, solving for the coefficient, k, in terms of L_0 and V_0.

b. Write an equation for the volume, V_1, of a replica scaled to have length L_1. Substitute for k the expression you found in part (a). Simplify the expression such that V_1 is expressed as some term times V_0.

c. The ratio L_1/L_0 is the *scaling factor*. A scaling factor is the number by which each linear dimension of an original object is *multiplied* when the size is changed. Write an equation for the volume of a scaled object, V_1, in terms of V_0, the volume of the original object, and F, the scaling factor.

4. *Using the scaling factor*

a. Draw a circle with a 1-inch radius. If you want to draw a circle with an area four times as large, what is the change in the radius?

b. A photographer wants to blow up a 3-cm by 5-cm photograph. If he wants the final print to be double the area of the original photograph, what scaling factor should he use?

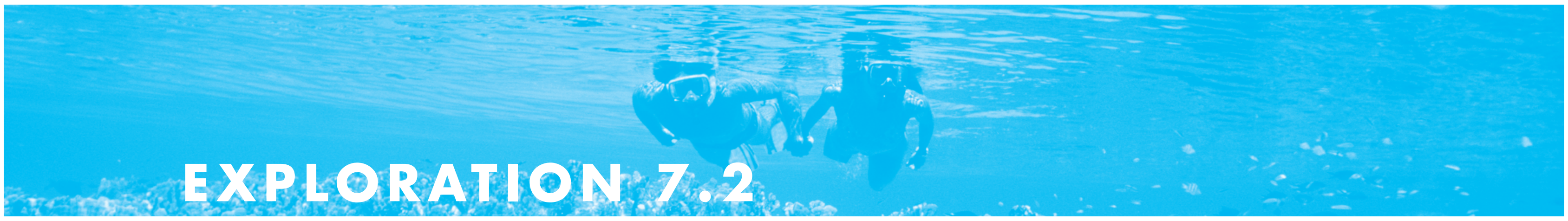
c. A sculptor is commissioned to make a bronze statue of George Washington sitting on a horse. To fit into its intended location, the final statue must be 15 feet long from the tip of the horse's nose to the end of its tail. In her studio, the sculptor experiments with smaller statues that are only 1 foot long. Suppose the final version of the small statue requires 0.15 cubic feet of molten bronze. When the sculptor is ready to plan construction of the larger statue, how can she figure out how much metal she needs for the full-size one?

d. One of the famous problems of Greek antiquity was the *duplication of the cube*. Our knowledge of the history of the problem comes down to us from Eratosthenes (circa 284 to 192 B.C.), who is famous for his estimate of the circumference of Earth. According to him, the citizens of Delos were suffering from a plague. They consulted the oracle, who told them that to rid themselves of the plague, they must construct an altar to a particular god. That altar must be the same shape as the existing altar but double the volume. What should the scaling factor be for the new altar? [Adapted from COMAP, *For All Practical Purposes* (New York: W. H. Freeman, 1988), p. 370.]

e. Pyramids have been built by cultures all over the world, but the largest and most famous ones were built in Egypt and Mexico.

i. The Great Pyramid of Khufu at Giza in Egypt has a square base 755 feet on a side. It originally rose about 481 feet high (the top 31 feet have been destroyed over time). Find the original volume of the Great Pyramid. [The volume V of a pyramid is given by $V = (B \cdot H)/3$, where B is the area of the base and H is its height.]

ii. The third largest pyramid at Giza, the Pyramid of Menkaure, occupies approximately one-quarter of the land area covered by the Great Pyramid. Menkaure's pyramid is the same shape as the original shape of the Great Pyramid, but it is scaled down. Find the volume of the Pyramid of Menkaure.



Predicting Properties of Power Functions

Objectives

- construct power functions with positive integer exponents
- find patterns in the graphs of power functions

Materials

- graphing calculator or function graphing program or "P1: k & p Sliders" in *Power Functions*
- graph paper

Procedure

Working in Pairs

1. **a.** Construct the equation of a power function whose graph is symmetric across the y-axis. Construct a second power function whose graph is symmetric about the origin.

   b. Construct the equations for two power functions whose graphs are mirror images across the x-axis. Construct two more whose graphs are mirror images across the y-axis.

2. Using the same power for each function, construct two different power functions such that:

   a. Both functions have even powers and the graph of one function "hugs" the y-axis more closely than that of the other function.

   b. Both functions have odd powers and the graph of one function "hugs" the y-axis more closely than that of the other function.

3. Using the same value for the coefficient, k, construct two different power functions such that:

   a. Both functions have even powers and the graph of one function "hugs" the y-axis more closely than that of the other function.

   b. Both functions have odd powers and the graph of one function "hugs" the y-axis more closely than that of the other function.

4. **a.** Choose a value for k that is greater than 1 and construct the following functions:

   i. $y = kx^0 = k$ $\quad y = kx^2$ $\quad y = kx^4$ $\qquad$ **ii.** $y = kx^1$ $\quad y = kx^3$ $\quad y = kx^5$

   b. Graph the functions in part (i) on the same grid. Choose a scale for your axes so you can examine what happens when $0 < x < 1$. Now regraph, choosing a scale for your axes so you can examine what happens when $x > 1$. Then repeat for part (a. ii).

   c. Describe your findings.

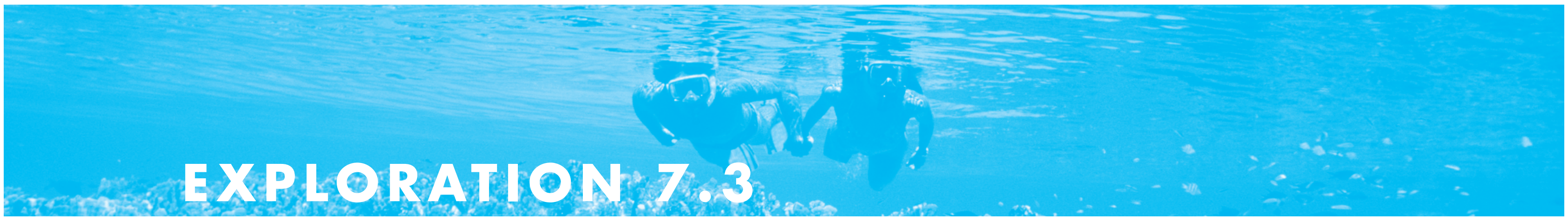
Class Discussion

Compare your findings. As a class, develop a 60-second summary describing the results.

Exploration-Linked Homework: A Challenge Problem

In Chapter 7 we primarily studied power functions with integer exponents. Try extending your analyses to power functions with positive fractional exponents. Start with fractions that in reduced form have 1 in the numerator. What would the graphs of $y = x^{1/2}$ and $y = x^{1/4}$ look like? What are the domains? The ranges? What would the graphs of $y = x^{1/3}$ or $y = x^{1/5}$ look like? What about their domains and ranges? Generalize your results for fractions of the form $\frac{1}{n}$, where n is a positive integer.

For an even harder challenge, consider fractional exponents of the form m/n.



EXPLORATION 7.3

Visualizing Power Functions with Negative Integer Powers

Objectives

- examine the effect of k on negative integer power functions

Materials

- graphing calculator or function graphing program
- graph paper

Related Software

- "P1: k & p Sliders" in *Power Functions*

Procedure

Class Demonstration: Constructing Negative Integer Power Functions

1. A power function has the form $y = kx^p$, where k and p are constants. Consider the following power functions with negative integer exponents where k is 4 and 6, respectively:

$$y = 4x^{-2} \quad \text{and} \quad y = 6x^{-2}$$

We can also write these as

$$y = \frac{4}{x^2} \quad \text{and} \quad y = \frac{6}{x^2}$$

What are the constraints on the domain and the range for each of these functions?

2. Construct a table of values for both functions using positive and negative values for x. How do different values for k lead to different values for y for these two functions? Sketch a graph for each function. Check your graphs with a function graphing program or graphing calculator.

a. Describe the overall behavior of these graphs. In each case, when is y increasing? When is y decreasing? How do the graphs behave for values of x near 0?

b. Describe how these graphs are similar and how they are different.

Working in Small Groups

In the following exploration you will predict the effect of the coefficient k on the graphs of power functions with negative integer exponents. In each part, compare your findings, and then write down your observations.

1. a. Choose a value for p in the function $y = \frac{k}{x^p}$, where p is a positive integer. Construct several functions where p has the same value but k assumes different *positive* values (as in the example above, where $k = 4$ and $k = 6$). What effect do you think k has on the graphs of these equations? Graph the functions. As you choose larger and larger values for k, what happens to the graphs? Try choosing values for k between 0 and 1. What happens to the graphs?

b. Using the same value of p as in part (a), construct several functions with *negative* values for the constant k. What effect do you think k has on the graphs of these equations? Graph the functions. Describe the effect of k on the graphs of your equations. Do you think your observations about k will hold for any value of p?

c. Choose a new value for p and repeat your experiment. Are your observations still valid? Compare your observations with those of your partners. Have you examined both odd and even negative integer powers?

In your own words, describe the effect of k on the graphs of functions of the form $y = \frac{k}{x^p}$, where p is a positive integer. What is the effect of the sign of the coefficient k?

2. a. Choose a value for k where $k > 1$ and rewrite each function in the form $y = \frac{k}{x^p}$.

i. $y = kx^0 = k$ $\quad y = kx^{-2}$ $\quad y = kx^{-4}$

ii. $y = kx^{-1}$ $\quad y = kx^{-3}$ $\quad y = kx^{-5}$

b. Graph the functions in part (i) on the same grid. Choose a scale for your graphs such that you can examine what happens when $0 < x < 1$. Now choose scales for your axes so you can examine what happens when $x > 1$.

c. Graph the functions in part (ii) on the same grid. Choose scales for your graphs such that you can examine what happens when $0 < x < 1$. Now choose scales for your axes so you can examine what happens when $x > 1$.

d. Describe your findings.

Class Discussion

Compare the findings of each of the small groups. As a class, develop a 60-second summary on the effect of k on power functions in the form $y = \frac{k}{x^p}$, where p is a positive integer.