CHAPTER 8

QUADRATICS, POLYNOMIALS, AND BEYOND

OVERVIEW

In this chapter we learn how to transform familiar functions into new ones. We start by adding power functions to create quadratics, polynomials of degree 2, which can be used to describe motion, model traffic flow, or predict the spread of a wildfire. We continue adding power functions to create polynomial functions, and then take the ratio of two polynomial functions to create a rational function. Finally, we look at many ways of transforming or combining functions from all of the families of functions we have studied.

After reading this chapter you should be able to

- understand the behavior and construct graphs of quadratic and other polynomial functions
- determine the vertex and intercepts of a quadratic function
- convert quadratic functions from one form to another
- transform any function using stretches, compressions, shifts, and reflections
- combine any two functions using basic algebraic operations or composition
- identify when a function has an inverse

8.1 *An Introduction to Quadratic Functions*

The Simplest Quadratic

A power function of degree 2 (in the form $y = ax^2$) is also the simplest member of a family of functions called *quadratics.* We've already encountered quadratics such as $S = 6x^2$ (the surface area of a cube) and $S = \pi r^2$ (the area of a circle). Since $y = ax^2$ has an even integer power, its graph has the classic ∪ -shaped curve called a *parabola.* (See Figure 8.1.) Certain properties are suggested by the graph:

- The parabola is concave up if $a > 0$ (and concave down if $a < 0$).
- It has a minimum (or maximum) point called the *vertex.*
- It is symmetric across a vertical line called an *axis of symmetry* that runs through the vertex.

A parabola has another unique and useful property. There is an associated point off any parabola called the *focus* or *focal point*. It is located $\left|\frac{1}{4a}\right|$ units above (or below) the vertex and lies within the arms of the parabola.

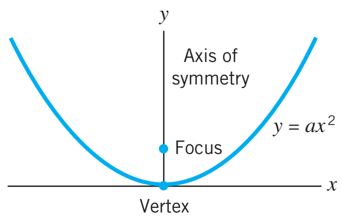


Figure 8.1 The parabolic graph of $y = ax^2$, where $a > 0$.

EXAMPLE 1 Determine whether the graph of $y = 2x^2$ is concave up or down, find its vertex and focal point, and then graph the function.

SOLUTION Since $2x^2 \geq 0$, the minimum value for $y = 2x^2$ is 0, which will occur when $x = 0$. So the function has a minimum and its vertex is at the origin (0, 0). Since $a > 0$, the graph is concave up. The coefficient, a, of x^2 is 2, so the focal point is $1/(4a) = 1/(4 \cdot 2) = 1/8$ or 0.125 units above the vertex. The coordinates of the focal point (or focus) are (0, 0.125). Figure 8.2 shows the graph of the function.

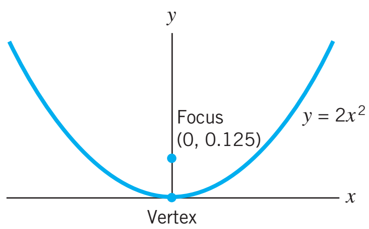


Figure 8.2 Graph of $y = 2x^2$ with vertex at (0, 0) and focal point at (0, 0.125).

The focal point merits its name. A three-dimensional parabolic bowl has vertical cross sections that are all the same parabola. If the bowl is built of a reflective material, the parabolic shape will concentrate parallel rays from the sun or a satellite TV channel at the focal point (see Figure 8.3). This property is used to focus the sun's rays to construct primitive cooking devices or to concentrate electromagnetic waves from satellites or distant stars.

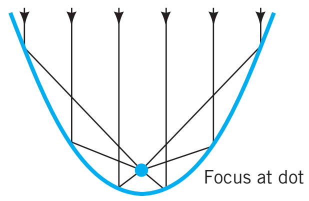


Figure 8.3 A parabola with parallel incoming rays concentrating at its focus.

The course software "Q10: Parabolic Reflector" gives an interactive demonstration of the reflective property of parabolas.

This process also works in reverse. For example, if you have a light bulb centered at the focus of a parabola, the light rays will bounce off a reflective parabolic surface as a set of parallel rays. This property is used to construct lamps, car headlights, or radio transmitters that send messages to robot rovers on Mars or broadcast our existence to the universe.

Designing parabolic devices

The easiest way to design a parabolic device is to place the vertex of the parabola at the origin (0, 0)—in other words, to construct a function of the form $y = ax^2$.

EXAMPLE 2 **Designing a cooking device**

In developing countries, such as Burkina Faso,[1] solar parabolic cooking devices are cheap ways to cook food and pasteurize water. The cooking devices can be made any size and can even be constructed of hardboard and aluminum foil. If a large parabolic cooking device is built that is 4′ wide and 1′ deep, where should the cooking pot be centered for maximum efficiency?

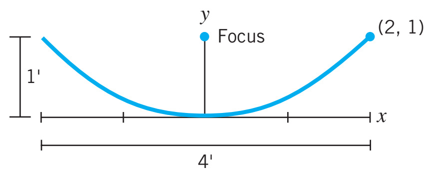


Figure 8.4 Parabolic cooking device.

SOLUTION The cooking pot should be placed at the focal point of the parabola, where the sun's rays will be concentrated. If we think of the bottom of the cooker as placed at (0, 0), we can construct a function of the form $y = ax^2$ and use it to find the focal point (see Figure 8.4). To find the value of a, we need the coordinates of a point on the parabolic cross section. Since the cooker is 4′ wide, if we move 2′ to the right of the cooker's base, and 1′ up, we will be at the point (2, 1), which lies on the rim of the cross section. Given the equation

	$y = ax^2$
let $x = 2$ and $y = 1$	$1 = a \cdot 2^2$
evaluate	$1 = a \cdot 4$
solve for a	$a = \dfrac{1}{4}$

[1]In 2004 the La Trame documentary film studio released *Bon Appétit, Monsieur Soleil,* a film about solar cooking in Burkina Faso. The studio has produced several versions in different languages, including English. The film covers the essential points of solar cooking, the acceptance of these devices by the population, the advantages achieved by using them (savings in money and time), and their contribution to fighting deforestation. The film is a good tool to inform people living in countries with energy problems about other ways to cook. It also serves to introduce this cooking method to decision makers in government agencies and NGOs.

So the formula for the parabolic oven cross section is $y = \left(\frac{1}{4}\right)x^2$. The focal point is at $\left|\frac{1}{4a}\right| = \left|\frac{1}{4 \cdot \left(\frac{1}{4}\right)}\right| = \frac{1}{1}$ or 1 foot above the vertex, the bottom of the cooker. Since the height of the cooker is also 1 foot, then the pot should be suspended 1 foot above the center, at the level of the rim, for maximum efficiency.

The General Quadratic

A quadratic function may include a linear term added to the ax^2 term. The *general quadratic* in *standard* or *a-b-c form* is

$$y = ax^2 + bx + c \qquad (a \neq 0)$$

where a, b, and c are constants. A quadratic is also called a *polynomial of degree 2*. For example, the quadratic function $y = 3x^2 + 7$ is a polynomial of degree 2 where $a = 3$, $b = 0$, and $c = 7$.

EXAMPLE 3 **A quadratic model for tuition revenue**

Because of state financial problems, many publicly funded colleges are faced with large budget cuts. To raise more revenue, one state college plans to raise both tuition and student body size over the next 10 years. Its goal is to increase the current tuition of \$6000 by \$600 a year and increase the student population of 1200 by 40 students a year.

a. Generate two equations, one to describe the projected increases in tuition and the other to describe increases in student body size over time.

b. Generate an equation to describe the projected total tuition revenue over time.

c. Compare the current tuition revenue to the projected revenue in 10 years.

SOLUTION

a. If we let S = student body size and N = number of years after the present, then $S = 1200 + 40N$.

If we let T = cost of tuition in thousands of dollars, then $T = 6 + 0.6N$.

b. The total tuition revenue, R, is the product of the number of students, S, times the cost of tuition, T. So

	$R = S \cdot T$
Substitute	$= (1200 + 40N)(6 + 0.6N)$
multiply out[2]	$= 7200 + 720N + 240N + 24N^2$
combine terms	$= 7200 + 960N + 24N^2$

So the total revenue R (in thousands of dollars) from tuition is given by the quadratic function

$$R = 7200 + 960N + 24N^2$$

c. At the present time $N = 0$, so the total tuition revenue R = \$7200 thousand or \$7.2 million. In 10 years, $N = 10$, so

$$\begin{aligned} R &= 7200 + (960 \cdot 10) + (24 \cdot 10^2) \\ &= 7200 + 9600 + 2400 \\ &= 19{,}200 \end{aligned}$$

So in 10 years the projected total tuition revenue is \$19,200 thousand or \$19.2 million.

[2]Recall that the product $(a + b)(c + d)$ is the sum of four terms: the product of the first (F) two terms, the outside (O) terms, the inside (I) terms, and the last (L) two terms of the factors.

$$(a + b)(c + d) = \underset{\text{F}}{ac} + \underset{\text{O}}{ad} + \underset{\text{I}}{bc} + \underset{\text{L}}{bd}$$

Properties of Quadratic Functions

The vertex and the axis of symmetry

The graph of the general quadratic, $y = ax^2 + bx + c$, is a parabola that is symmetric about its axis of symmetry. For any quadratic other than the simplest (of the form $y = ax^2$), the axis of symmetry is not the vertical axis. The vertex lies on the axis of symmetry and is a maximum (if the parabola is concave down) or a minimum (if the parabola is concave up) (see Figure 8.5). In Section 8.2 we'll learn how to find the coordinates of the vertex.

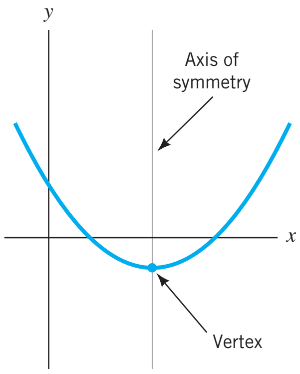


Figure 8.5 Each parabola has a vertex that lies on an axis of symmetry.

Vertical and horizontal intercepts

The general properties of vertical and horizontal intercepts are the same for all functions.

> **Intercepts of a Function**
>
> For any function $f(x)$:
>
> the vertical intercept is at $f(0)$
>
> the horizontal intercepts occur at values of x, called *zeros,* where $f(x) = 0$

For a quadratic function $f(x) = ax^2 + bx + c$, we have $f(0) = c$. So $f(x)$ has a vertical intercept at $(0, c)$. We usually shorten this to say that the vertical intercept is at c, since we know the other coordinate is 0.

Because of the ∪ shape of its graph, a quadratic function may have no, one, or two horizontal intercepts (see Figure 8.6). If $(r, 0)$ are the coordinates of a horizontal intercept, we abbreviate this to say that the horizontal intercept is at r. These values of r are also called *zeros* of the function since $f(r) = 0$. In Section 8.3 we'll learn how to find the horizontal intercepts.

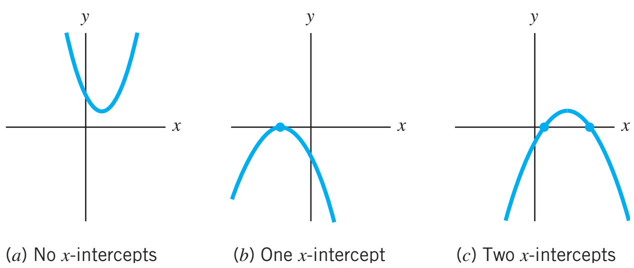


Figure 8.6 Graphs of quadratic functions showing the three possible cases for the number of horizontal or x-intercepts.

Since a parabola is symmetric about its axis of symmetry, each point on one arm of the parabola has a mirror image on the other arm. In particular, if there are two horizontal intercepts at r_1 and r_2, then $(r_1, 0)$ and $(r_2, 0)$ are mirror images across the parabola's axis of symmetry. The horizontal intercepts will lie at an equal distance, d, to the left and right of the axis of symmetry (see Figure 8.7). Equivalently, the axis of symmetry crosses the horizontal axis halfway between r_1 and r_2, at the one-dimensional point where x is $\frac{(r_1 + r_2)}{2}$. So the equation of the axis of symmetry is the two-dimensional line $x = \frac{(r_1 + r_2)}{2}$.

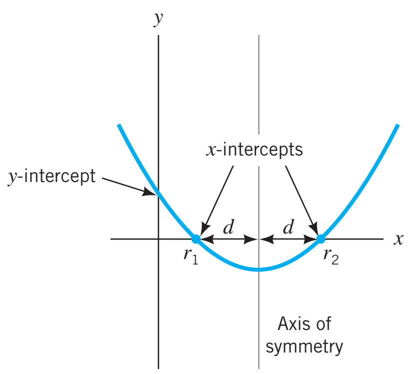


Figure 8.7 Each parabola has a vertical intercept and zero, one, or two horizontal intercepts.

SOMETHING TO THINK ABOUT

? *Does increasing $|a|$ bring the focus closer to or farther away from the vertex?*

The focal point

The focal point for the general parabola $y = ax^2 + bx + c$ is still $\left|\frac{1}{4a}\right|$ units above (or below) the vertex. The distance $\left|\frac{1}{4a}\right|$ from the vertex to the focal point is called the *focal length.*

The Quadratic Function and Its Graph

A *quadratic function* can be written in standard form as

$$f(x) = ax^2 + bx + c \qquad \text{(where } a \neq 0\text{)}$$

Its graph:

is called a *parabola*

is symmetric about its *axis of symmetry*

has a minimum (if the parabola is concave up) or a maximum (if the parabola is concave down) point called its *vertex*

has one vertical intercept, but may have zero, one, or two horizontal intercepts

has a *focal point* $\left|\frac{1}{4a}\right|$ units above the minimum (or below the maximum); the value $\left|\frac{1}{4a}\right|$ is called the *focal length.*

Estimating the Vertex and Horizontal Intercepts

Quadratics arise naturally in area and motion problems. The earliest problems we know of that led to quadratic equations are on Babylonian tablets dating from 1700 B.C. The writings suggest a problem similar to the following example.

EXAMPLE 4 **Maximizing area**

What is the maximum rectangular area you can enclose within a fixed perimeter of 24 meters? What are the dimensions of the rectangle with the maximum area?

SOLUTION If the rectangular region has length L and width W, then

$$\begin{array}{ll} & 2L + 2W = \text{perimeter} \\ \text{Substitute} & 2L + 2W = 24 \\ \text{divide by 2} & L + W = 12 \\ \text{subtract } L \text{ from both sides} & W = 12 - L \end{array}$$

$$\begin{array}{ll} \text{Since area, } A\text{, is given by} & A = L \cdot W \\ \text{substitute for } W & = L \cdot (12 - L) \\ \text{multiply through} & = 12L - L^2 \end{array}$$

So area, A, is a quadratic function of L. Table 8.1 shows values of L from 0 to 12 meters and corresponding values for W and A.

Figure 8.8 shows a graph of area, A, versus length, L. In the equation $A = 12L - L^2$ the coefficient of L^2 is negative (-1), so the parabola is concave down. From the table and the graph, it appears that the vertex of the parabola is at (6, 36); that is, at a length of 6 meters the area reaches a maximum of 36 square meters. Since $W = 12 - L$, when $L = 6$, $W = 6$. Hence the rectangle has maximum area when the length equals the width, or in other words when the rectangle is a square.

Length, L (m)	Width, W (m)	Area, A (m^2)
0	12	0
1	11	11
2	10	20
3	9	27
4	8	32
5	7	35
6	6	36
7	5	35
8	4	32
9	3	27
10	2	20
11	1	11
12	0	0

Table 8.1

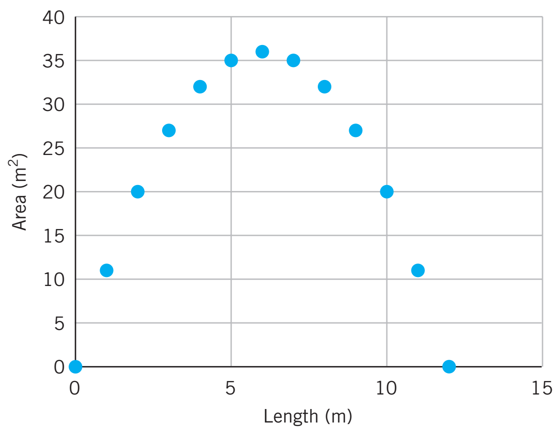


Figure 8.8 Graph of area vs. length for a rectangle.

EXAMPLE 5 **The trajectory of a projectile**

Figure 8.9 shows a plot of the height above the ground (in feet) of a projectile for the first 5 seconds of its trajectory. How could we:

a. Estimate the height from which the projectile was launched?

b. Estimate when the projectile will hit the ground?

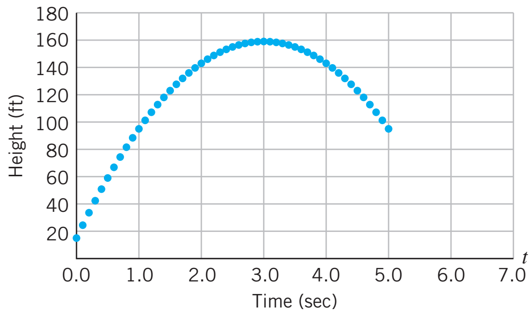


Figure 8.9 The height of a projectile.

SOLUTION

a. The initial height of the projectile corresponds to the vertical intercept of the parabola (where time $t = 0$). A quick look at Figure 8.9 gives us an estimate somewhere between 10 and 20 feet.

If we use technology to generate a best-fit quadratic, we get $H(t) = -16t^2 + 96t + 15$, where t = time (in seconds) and $H(t)$ = height (in feet). Since $H(0) = 15$, then 15 is the vertical intercept. In other words, when $t = 0$ seconds, the initial height is $H(0) = 15$ feet.

"The Mathematics of Motion" following this chapter offers an extended exploration into constructing mathematical models to describe the motion of freely falling bodies.

b. Figure 8.10 overlays on Figure 8.9 the graph of $H(t) = -16t^2 + 96t + 15$, our function model for the projectile's path. The projectile will hit the ground when the height above the ground $H(t) = 0$. This occurs at a horizontal intercept of the parabola. One horizontal intercept (not shown) would occur at a negative value of time, t, which would be meaningless here. From the graph we can estimate that the other intercept occurs at $t \approx 6.2$ seconds.

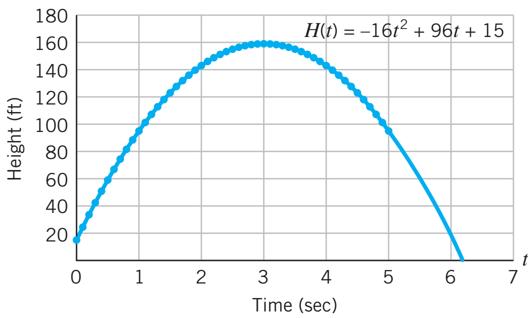


Figure 8.10 A graph of the function model for the projectile's path.

Algebra Aerobics 8.1

1. Find the coordinates of the vertex and the focal point for each quadratic function. Then specify whether each vertex is a maximum or minimum.

   a. $y = 3x^2$ **c.** $y = \frac{1}{24}x^2$

   b. $y = -6x^2$ **d.** $y = -\frac{1}{12}x^2$

2. A designer is planning an outdoor concert place for solo performers. The stage area is to have a parabolic wall 30′ wide by 10′ deep. If the performer stands at the focal point, the sound will be reflected out in parallel sound waves directly to the audience in front. See Figure 8.11.

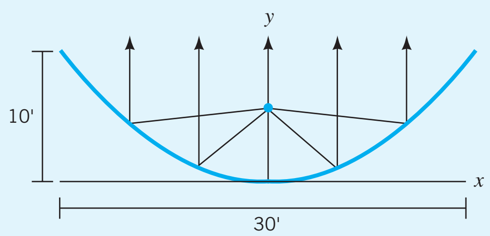


Figure 8.11 A parabolic design for an outdoor concert stage back wall.

a. Where would the focal point be?

b. What is the equation for the reflecting wall?

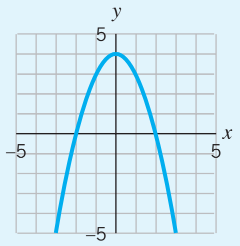
c. Do you see any problem with this idea for concert acoustics? (*Hint:* What if the audience is noisy?)

3. Evaluate each of the following quadratic functions at 2, -2, 0, and z.

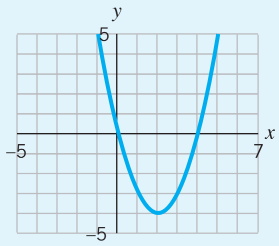
a. $g(x) = x^2$
b. $h(w) = -w^2$
c. $Q(t) = -t^2 - 3t + 1$
d. $m(s) = 5 + 2s - 3s^2$
e. $D(r) = -(r - 3)^2 + 4$
f. $k(x) = 5 - x^2$

4. For each of the graphs of the quadratic functions $f(x)$, $g(x)$, $h(x)$, and $j(x)$ given below, determine:

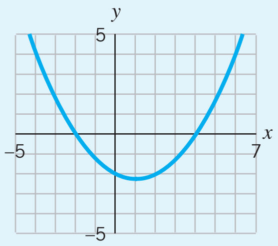
i. If the parabola is concave up or concave down
ii. If the parabola has a maximum or a minimum
iii. The equation of the axis of symmetry
iv. The coordinates of the vertex
v. Estimated coordinates of the horizontal and vertical intercepts



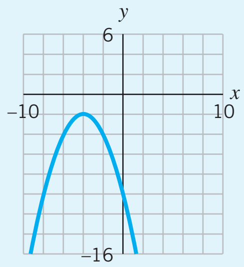
a. Graph of $f(x)$



c. Graph of $h(x)$



b. Graph of $g(x)$



d. Graph of $k(x)$

5. A manufacturer wants to make a camp stove using a parabolic reflector to concentrate sun rays at the focal point of the parabola, where the food to be cooked would be placed. The reflector must be no wider than 24″, and the depth is to be the same as the focal length, f, the distance from the focal point to the bottom. See Figure 8.12.

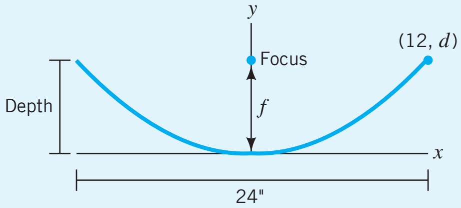


Figure 8.12 A parabolic design for a camp stove.

a. What focal length would the camp stove need to have? (*Hint:* The focal length is $\left|\frac{1}{4a}\right|$).

b. What formula is needed for manufacturing the stove reflector?

6. Put the following quadratic functions into standard *a-b-c* form:

$f(x) = (x - 3)(x + 5)$ $\quad g(x) = (2x + 5)(x + 1)$

$h(x) = 10(x - 3)(x - 5)$ $\quad j(x) = 2(x - 3)(x + 3)$

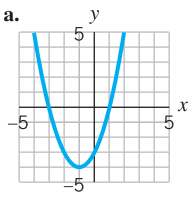
Exercises for Section 8.1

The "Extended Exploration: The Mathematics of Motion," which follows this chapter, contains many additional exercises using quadratics to describe freely falling and thrown bodies.

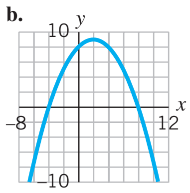
A graphing program is recommended or required for several exercises.

1. From the graph of each quadratic function, identify whether the parabola is concave up or down and hence whether the function has a maximum or minimum. Then estimate the vertex, the axis of symmetry, and any horizontal and vertical intercepts.

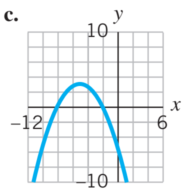
a.



b.



c.



2. Using the quadratic function $y = \frac{1}{8}x^2$:

a. Fill in the values in the table.

x	-12	-8	-4	0	4	8	12
y							

b. Plot the points by hand and sketch the graph of the function.

c. Determine the coordinates of the focal point and place the focal point on the graph.

d. What is the equation of the axis of symmetry?

e. What are the coordinates of the vertex? Is the vertex a maximum or minimum?

3. (Graphing program recommended.) On the same graph, plot the three functions.

$$y_1 = x^2 \qquad y_2 = \tfrac{1}{4}x^2 \qquad y_3 = \tfrac{1}{12}x^2$$

a. Calculate the focal point for each parabola.

b. Compare y_2 and y_3 with y_1.

c. Complete this sentence: As the value of $|a|$ gets smaller, the focal point gets ________ the vertex and the graph gets ________.

4. (Graphing program recommended.) On the same graph, plot the three functions.

$$y_1 = x^2 \qquad y_2 = 4x^2 \qquad y_3 = 8x^2$$

a. Calculate the focal point for each parabola.

b. Compare y_3 and y_2 with y_1.

c. Complete this sentence: As the value of $|a|$ gets larger, the focal point gets ________ to the vertex and the graph of the parabola gets ________ .

5. (Graphing program optional.) Given a point P on a parabola of the form $y = ax^2$, find the equation of the parabola. If available, use technology to verify your equations by plotting each parabola.

a. $P = (12, 6)$ **c.** $P = (4, 12)$

b. $P = (-12, -6)$ **d.** $P = (3, 144)$

6. Find the coordinates of the vertex for each quadratic function listed. Then specify whether each vertex is a maximum or minimum.

a. $y = 4x^2$ **c.** $P(n) = \left(\frac{1}{12}\right)n^2$

b. $f(x) = -8x^2$ **d.** $Q(t) = -\left(\frac{1}{24}\right)t^2$

7. Given the following focal points, write the equation of a parabola in the form $y = ax^2$ by finding a.

a. $(0, 4)$ **c.** $\left(0, \frac{1}{16}\right)$

b. $(0, -8)$ **d.** $\left(0, -\frac{1}{24}\right)$

8. Find the focal length (the distance from the focal point to the vertex) for each of the following.

a. $f(x) = x^2$ **b.** $f(x) = 3x^2$ **c.** $f(x) = \frac{1}{3}x^2$

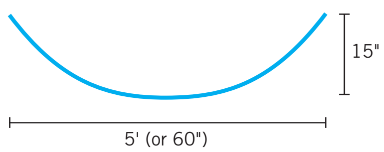
9. For each of the following functions, evaluate $f(2)$ and $f(-2)$.

a. $f(x) = x^2 - 5x - 2$

b. $f(x) = 3x^2 - x$

c. $f(x) = -x^2 + 4x - 2$

10. A designer proposes a parabolic satellite dish 5 feet in diameter and 15 inches deep.



a. What is the equation for the cross section of the parabolic dish?

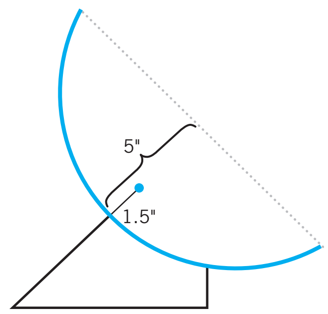
b. What is the focal length (the distance between the vertex and the focal point)?

c. What is the diameter of the dish at the focal point?

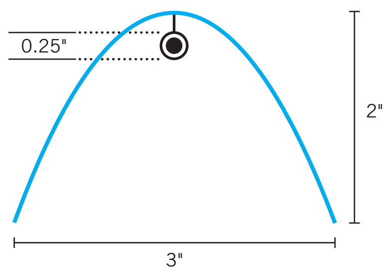
11. An electric heater is designed as a parabolic reflector that is 5″ deep. To prevent accidental burns, the centerline of the heating element, placed at the focus, must be set in 1.5″ from the base of the reflector.

a. What equation could you use to design the reflector? (*Hint:* Tip the parabolic reflector so it opens upward.)

b. How wide would the reflector be at its rim?



12. A parabolic reflector 3″ in diameter and 2″ deep is proposed for a spotlight.



a. What formula is needed for manufacturing the reflector?

b. Where would the focus be?

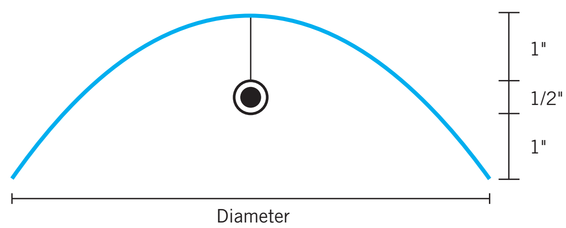
c. Will a $\frac{1}{4}$″-diameter light source fit if it is centered at the focus?

13. A slimline fluorescent bulb $\frac{1}{2}$″ in diameter needs 1″ clearance top and bottom in a parabolic reflecting shade.

a. What are the coordinates of the focus for this parabola?

b. What is the equation for the parabolic curve of the reflector?

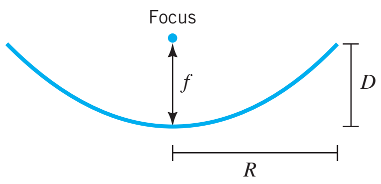
c. What is the diameter of the opening of the shade?



14. If we know the radius and depth of a parabolic reflector, we also know where the focus is.

a. Find a generic formula for the focal length f of a parabolic reflector expressed in terms of its radius R and depth D. The focal length $\left|\frac{1}{4a}\right|$ is the distance between the vertex and the focal point. Assume $a > 0$.

b. Under what conditions does $f = D$?



15. Construct several of your own equations of the form $y = ax^2$ and then describe in words how the focal length varies depending on how open or closed the parabolic curve is.

16. For each of the following quadratics with their respective vertices, calculate the distance from the vertex to the focal point. Then determine the coordinates of the focal point.

a. $f(x) = x^2 - 2x - 3$ with vertex at $(1, -4)$

b. $g(t) = 2t^2 - 16t + 24$ with vertex at $(4, -8)$

17. Put each of the following quadratics into standard form.

a. $f(x) = (x + 3)(x - 1)$

b. $P(t) = (t - 5)(t + 2)$

c. $H(z) = (2 + z)(1 - z)$

18. Put each of the following quadratics into standard form.

a. $g(x) = (2x - 1)(x + 3)$

b. $h(r) = (5r + 2)(2r + 5)$

c. $R(t) = (5 - 2t)(3 - 4t)$

19. Determine the dimensions for enclosing the maximum area of a rectangle if:

a. The perimeter is held constant at 200 meters.

b. The perimeter is held constant at P meters.

20. A gardener wants to grow carrots along the side of her house. To protect the carrots from wild rabbits, the plot must be enclosed by a wire fence. The gardener wants to use 16 feet of fence material left over from a previous project. Assuming that she constructs a rectangular plot, using the side of her house as one edge, estimate the area of the largest plot she can construct.

21. Which of the following are true statements for quadratic functions?

a. The vertex and focal point always lie on the axis of symmetry.

b. The graph of a parabola could have three horizontal intercepts.

c. The graph of a parabola does not necessarily have a vertical intercept.

d. If $f(2) = 0$, then f has a horizontal intercept at 2.

e. The focal point always lies above the vertex.

22. The management of a company is negotiating with a union over salary increases for the company's employees for the next 5 years. One plan under consideration gives each worker a bonus of \$1500 per year. The company currently employs 1025 workers and pays them an average salary of \$30,000 a year. It also plans to increase its workforce by 20 workers a year.

a. Construct a function $C(t)$ that models the projected cost of this plan (in dollars) as a function of time t (in years).

b. What will the annual cost be in 5 years?

23. (Graphing program required for part (c).) A landlady currently rents each of her 50 apartments for \$1250 per month. She estimates that for each \$100 increase in rent, two additional apartments will remain vacant.

a. Construct a function that represents the revenue $R(n)$ as a function of the number of rent increases, n. (*Hint:* Find the rent per unit after n increases and the number of units rented after n increases.)

b. After how many rent increases will all the apartments be empty? What is a reasonable domain for this function?

c. Using technology, plot the function. From the graph, estimate the maximum revenue. Then estimate the number of rent increases that would give you the maximum revenue.

24. (Graphing program required for part (c).)

a. In economics, revenue R is defined as the amount of money derived from the sale of a product and is equal to the number x of units sold times the selling price p of each unit. What is the equation for revenue?

b. If the selling price is given by the equation $p = -\frac{1}{10}x + 20$, express revenue R as a function of the number x of units sold.

c. Using technology, plot the function and estimate the number of units that need to be sold to achieve maximum revenue. Then estimate the maximum revenue.

8.2 *Finding the Vertex: Transformations of* $y = x^2$

In this section we show that every quadratic can be generated through transformations of the basic quadratic $y = x^2$. We start by investigating what happens to the equation of a function when we change its graph.

Stretching and Compressing Vertically

When we stretch or compress a graph we change its shape, creating a graph of a new function. To find the equation of this new function, we need to ask, "What is the relationship between the graphs of the new function and the old function?"

We can build on what we know from Chapter 7, where we learned that multiplying a power function by a constant stretches or compresses its graph. Specifically, for power functions in the form $y = kx^p$, the value of the coefficient k compresses or stretches the graph of $y = x^p$. A quadratic in the form $y = ax^2$ is a power function, so the graph of $y = ax^2$ is the graph of $y = x^2$

vertically stretched by a factor of a, if $a > 1$

vertically compressed by a factor of a, if $0 < a < 1$

The magnitude (or absolute value) of a tells us how much the graph of $y = x^2$ is stretched or compressed. As $|a|$ increases, a acts as a vertical stretch factor, which pulls harder and harder on the arms of the parabola anchored at the vertex, narrowing the graph. As $|a|$ decreases, a acts as a compression factor, which weighs down the graph, flattening the parabola's arms. So the value of a determines the shape of the parabola. See Figure 8.13.

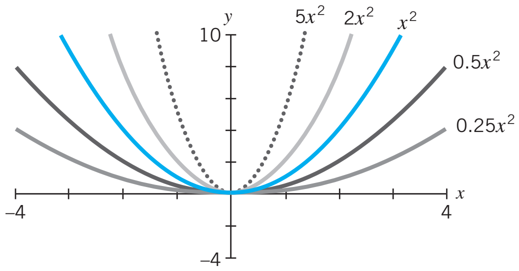


Figure 8.13 The graph of $y = x^2$ vertically compressed and stretched.

Reflections across the Horizontal Axis

If $a < 0$, the graph of $y = ax^2$ is the graph of $y = x^2$ stretched or compressed by a factor of $|a|$ and then reflected across the x-axis. In general, the graphs of $y = ax^2$ and $y = -ax^2$ are reflections of each other across the x-axis. See Figure 8.14.

The sign of a tells us whether the parabola opens up or down and if the vertex represents a maximum or a minimum value of the function. If a is positive, the parabola is concave up (opens upward) and the vertex is a minimum. If a is negative, the parabola is concave down (opens downward) and the vertex is a maximum.

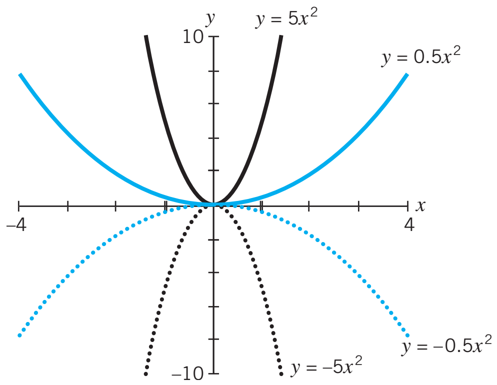


Figure 8.14 Graphs that are reflected across the x-axis.

EXAMPLE 1 Describe how the following pairs of functions are related to each other and to $y = x^2$. Sketch each pair and $y = x^2$ on the same grid.

a. $y = 3x^2$ and $y = -3x^2$ **b.** $y = 0.3x^2$ and $y = -0.3x^2$

SOLUTION **a.** The functions $y = 3x^2$ and $y = -3x^2$ are reflections of each other across the x-axis. For both functions, the absolute value of the coefficient a of x^2 is 3, so in each case $|a| > 1$. Hence both graphs are the graph of $y = x^2$ stretched by a factor of 3. See Graph *A* in Figure 8.15.

b. The functions $y = 0.3x^2$ and $y = -0.3x^2$ are also reflections of each other across the x-axis. For both functions $|a| = 0.3$, so in each case $0 < |a| < 1$. Hence both of their graphs are the graph of $y = x^2$ compressed by a factor of 0.3. See Graph *B* in Figure 8.15.

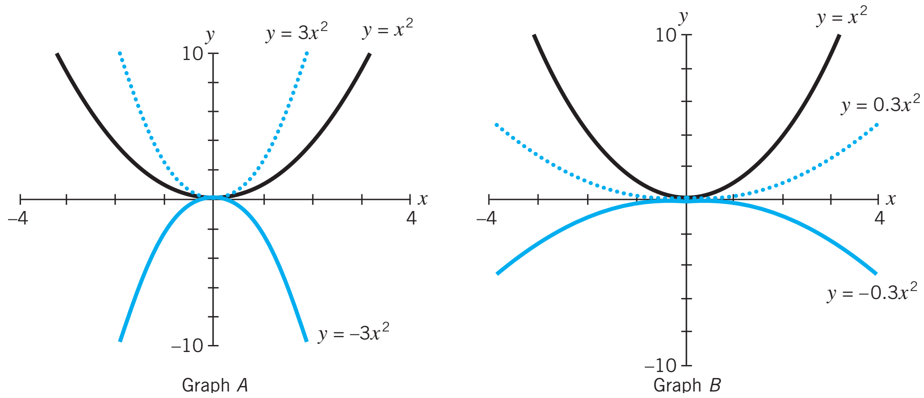


Figure 8.15 Stretching and compressing $y = x^2$.

Stretching, Compressing, and Reflecting the Graph of Any Function

If f is a function and a is a constant, then the graph of $a \cdot f(x)$ is the graph of $f(x)$

vertically stretched by a factor of a, if $a > 1$

vertically compressed by a factor of a, if $0 < a < 1$

vertically stretched or compressed by a factor of $|a|$ and reflected across x-axis, if $a < 0$

Shifting Vertically and Horizontally

What happens when we keep the shape of the graph of $y = ax^2$ but change its position on the grid, shifting the graph vertically or horizontally? Clearly we will get a graph of a new function where the coordinates of the vertex are no longer (0, 0). Let's see how to construct the equation of the new function from the original function.

Shifting a graph vertically

How does a function change when its graph is shifted up or down? Examine the graphs in Figure 8.16, Graphs *A* and *B*.

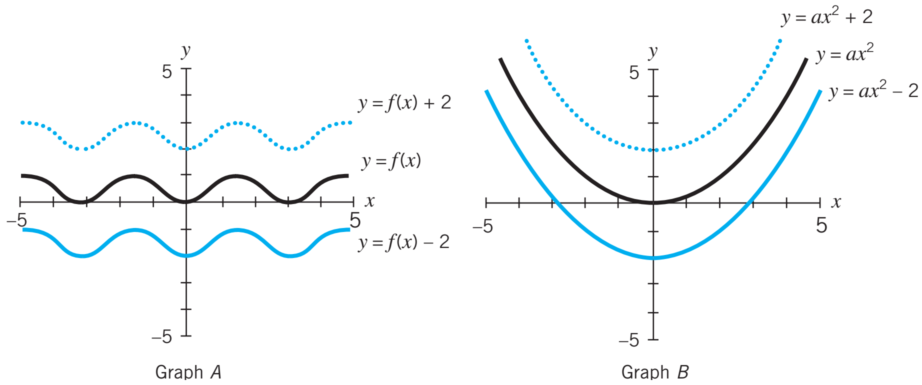


Figure 8.16 Graphs that are shifted vertically.

What happens to the output value, y, when the graph of $y = f(x)$ is shifted up two units? Every y value increases by two units. Translating this shift into equation form, our new function is $y = f(x) + 2$. (See Figure 8.16, Graph *A*.)

What happens when the graph of $y = f(x)$ is shifted down two units? Our new function is of the form $y = f(x) - 2$. In general, if the vertical shift is k units, then

$$y = f(x) + k$$

is the graph of $y = f(x)$ shifted up by k units if k is positive and shifted down by k units if k is negative.

Figure 8.16, Graph *B* shows the graph of $f(x) = ax^2$ raised or lowered by two units. If the vertical shift is k units, then the graph of

$$y = ax^2 + k$$

is the graph of $y = ax^2$ shifted up by k units if k is positive and shifted down by k units if k is negative. The vertex is at $(0, k)$.

Algebra Aerobics 8.2a

In Problems 1–5, without drawing the graphs, compare the graph of part (b) to the graph of part (a).

1. **a.** $r(x) = x^2 + 2$ **b.** $s(x) = 2x^2 + 2$
2. **a.** $h(t) = t^2 + 5$ **b.** $k(t) = -t^2 + 5$
3. **a.** $f(z) = -5z^2$ **b.** $g(z) = -0.5z^2$
4. **a.** $f(x) = x^2 + 3x + 2$ **b.** $g(x) = x^2 + 3x + 8$
5. **a.** $f(t) = -3t^2 + t - 5$ **b.** $g(t) = -3t^2 + t - 2$
6. Create a quadratic equation of the form $y = ax^2 + k$ with the given values for a and the vertex. Sketch by hand the graph of each equation.
   - **a.** $a = 3$ and the vertex is at $(0, 5)$.
   - **b.** $a = \frac{1}{3}$ and the vertex is at $(0, -2)$.
   - **c.** $a = -2$ and the vertex is at $(0, 4)$
7. Create new functions by performing the following transformations on $f(x) = x^2$.
   - **a.** $g(x)$ is $f(x)$ stretched by a factor of 3.
   - **b.** $h(x)$ is $f(x)$ stretched by a factor of 5 and reflected across the x-axis.
   - **c.** $j(x)$ is $f(x)$ compressed by a factor of 1/2.
   - **d.** $k(x)$ is $f(x)$ reflected across the x-axis.
8. Create a function in the form $y = ax^2 + k$ for each of the following transformations of $f(x) = x^2$:
   - **a.** Stretched by a factor of 5 and shifted down 2 units
   - **b.** Concave up and shifted 3 units up
   - **c.** Multiplied by a factor of 0.5, concave down, and shifted 4.7 units down
   - **d.** Opens up and is $f(x)$ shifted 71 units down

Shifting a graph horizontally

How does a function change when its graph is shifted to the right or left? Examine the graphs in Figure 8.17, Graphs *A* and *B*.

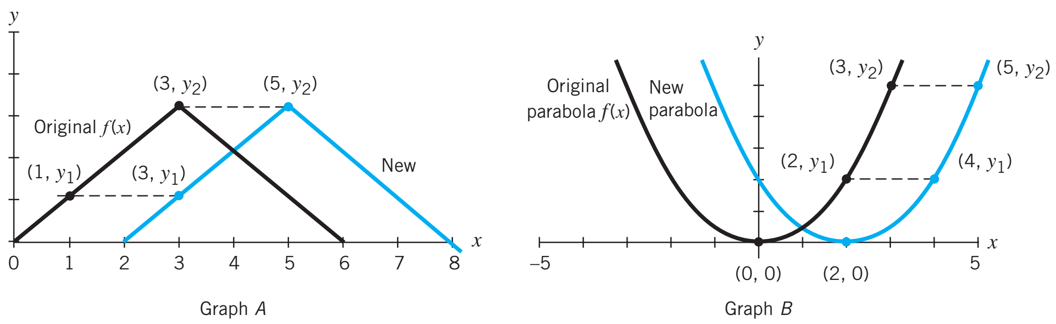


Figure 8.17 Graphs that are shifted horizontally to the right.

When we shift the graph of $y = f(x)$ two units to the right, how is the new function related to our original function? Recall that adding a constant to the output value of a function shifts its graph vertically. When we shift the graph of $f(x)$ horizontally, the input value, x, is shifted. The dotted lines in Figure 8.17 show that for any particular output value, y, the corresponding input value for the new function is $(x - 2)$. So the graph of $f(x)$ has been shifted to the *right* two units. Translating this shift into equation form, we get

$$y = f(x - 2)$$

At first this may seem counterintuitive. But remember, we are expressing the new function in terms of the original function. If you evaluate the new function at x, the same y-value of the original function $f(x)$ is now at $f(x - 2)$. For example, in Figure 8.17 if you evaluate each of the new functions in Graph A and Graph B at $x = 5$, you get an output of y_2. The same output for the original function occurs at $5 - 2 = 3$.

What if we shift $y = f(x)$ to the *left* two units? We need to think about the effect of the shift and the direction of the shift. If you evaluate the new function at x, the comparable point on the graph of $f(x)$ is $f(x + 2)$. We can think of this shift as replacing x with $x - (-2)$ or $x + 2$, so our new function is

$$y = f(x + 2)$$

Figure 8.18 shows the graph of $y = 3x^2$ after being shifted to the right and then to the left 2 units. Note that the value of the x-coordinate of the vertex changed by ± 2 units. If the horizontal shift is h units, then

$$y = a(x - h)^2$$

is the graph of $y = ax^2$ shifted right by h if h is positive, and shifted left by h if h is negative. The vertex is now at $(h, 0)$. Note that if $h = 2$, the expression $(x - h)$ becomes $x - 2$, and if $h = -2$, the expression $(x - h)$ becomes $x - (-2) = x + 2$.

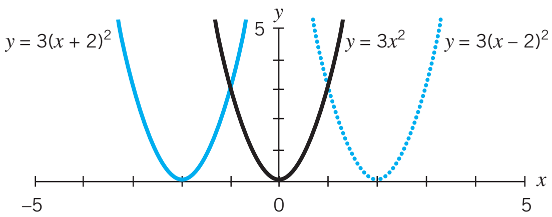


Figure 8.18 Graph of $y = 3x^2$ shifted horizontally to the left and to the right two units.

EXAMPLE 2

a. Identify the vertex for each of the following functions and indicate whether it represents a maximum or minimum value:

$$g(x) = 5(x - 3)^2 \qquad j(x) = 5x^2 + 3$$
$$h(x) = -5(x + 3)^2 \qquad k(x) = -5x^2 - 3$$

b. Describe how the graphs in part (a) are related to the graph of $f(x) = 5x^2$. Specify the order of any transformations.

SOLUTION

a. Vertex for: g is at $(3, 0)$; j is at $(0, 3)$; h is at $(-3, 0)$; k is at $(0, -3)$.

Vertices for g and j represent minimum values since in both cases the coefficient for x^2 is positive ($a = 5$).

Vertices for h and k represent maximum values since in both cases the coefficient for x^2 is negative ($a = -5$).

b. The graph of $f(x) = 5x^2$

shifted horizontally to the right three units is the graph of $g(x)$

shifted vertically up three units is the graph of $j(x)$

shifted horizontally to the left three units and then reflected across the x-axis is the graph of $h(x)$

reflected across the x-axis and then shifted vertically down three units is the graph of $k(x)$

We can generalize to any function $f(x)$.

Horizontal and Vertical Shifts

The graph of $f(x) + k$ is the graph of $f(x)$ shifted vertically $|k|$ units.

If k is positive, the shift is up; if k is negative, the shift is down.

The graph of $f(x - h)$ is the graph of $f(x)$ shifted horizontally $|h|$ units.

If h is positive, the shift is to the right; if h is negative, the shift is to the left.

Using Transformations to Get the Vertex Form

We can use the previous transformations to generate a quadratic function in what is called the *vertex form*. We start by stretching or compressing the graph of $y = x^2$ by a factor of a. So our new function will be

$$y = ax^2$$

Next we shift the graph of the function $y = ax^2$ horizontally h units and vertically k units, to get

$$y = a(x - h)^2 + k$$

"Q3: a, h, k Sliders," "Q8: $y = ax^2$ vs. $y = a(x - h)^2 + k$," and "Q9: Finding 3 Points: a-h-k Form" in *Quadratic Functions* can help you visualize quadratic functions in the a-h-k form.

The vertex is now at (h, k). This quadratic is in the *vertex* or *a-h-k form*. Its axis of symmetry is the vertical line at $x = h$. See Figure 8.19.

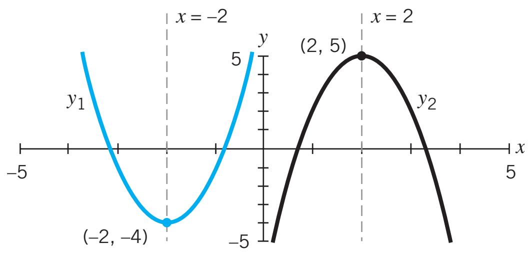


Figure 8.19 Graphs of $y_1 = 3(x + 2)^2 - 4$ and $y_2 = -3(x - 2)^2 + 5$.

The Vertex Form of a Quadratic Function

The *vertex* or *a-h-k form* of the quadratic function is

$$f(x) = a(x - h)^2 + k$$

where the vertex is at (h, k).

EXAMPLE 3 **Transforming graphs**

Given $f(x) = x^2$ and $g(x) = -2(x + 3)^2 + 5$, show how to transform the graph of $f(x)$ into the graph of $g(x)$.

SOLUTION One way to transform the graph of $f(x)$ into the graph of $g(x)$ is shown in Figure 8.20.

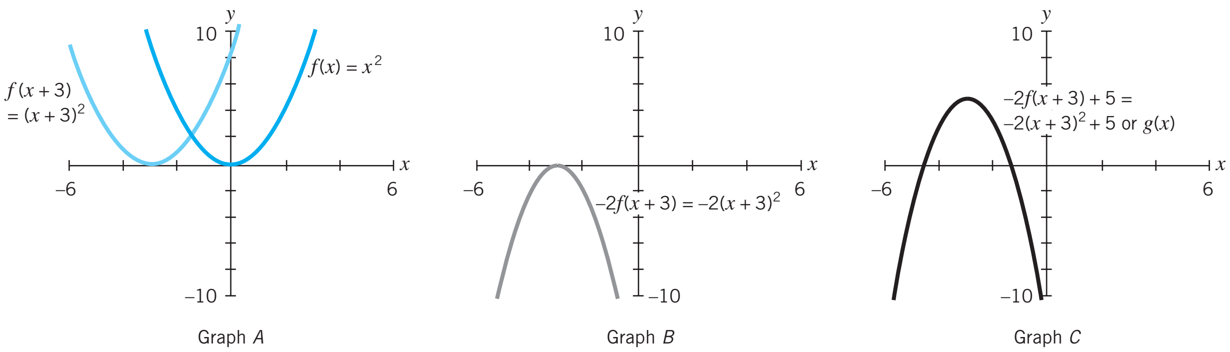


Figure 8.20 The graph of $f(x) = x^2$ in A is shifted horizontally to the left three units, then in B stretched vertically by a factor of 2 and reflected across the x-axis, and finally in C shifted up vertically five units to generate $g(x) = -2(x + 3)^2 + 5$.

EXAMPLE 4 **Finding the function from its graph**

Figure 8.21 shows the graph of $f(x) = 2x^2$ transformed into three new parabolas. Assume that each of the three new graphs retains the shape of $f(x)$.

a. Estimate the coordinates of the vertex for each new parabola.

b. Use your estimates from part (a) to write equations for each parabola in Figure 8.21.

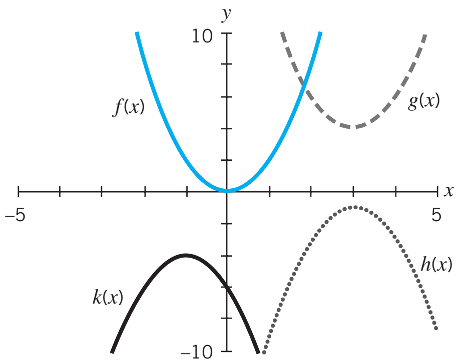


Figure 8.21 Three transformations of $f(x) = 2x^2$

SOLUTION

a. Vertex for: $g(x)$ is at $(3, 4)$; $h(x)$ is at $(3, -1)$; $k(x)$ is at $(-1, -4)$

b. $g(x) = 2(x - 3)^2 + 4$; $h(x) = -2(x - 3)^2 - 1$; $k(x) = -2(x + 1)^2 - 4$

EXAMPLE 5 **Identifying the vertex**

For the following functions, identify the coordinates of the vertex and specify whether the vertex represents a maximum or minimum.

a. $y = -5x^2$ **b.** $y = 2(x - 3)^2 - 15$ **c.** $y = -3(x + 5)^2 + 10$

SOLUTION

a. The vertex is at (0, 0) and represents a maximum.

b. The vertex is at (3, −15) and represents a minimum.

c. The vertex is at (−5, 10) and represents a maximum.

Algebra Aerobics 8.2b

For Problems 1 to 3, without graphing, compare the positions of the vertices of parts (b) and (c) to that of part (a).

1. a. $y = x^2$ **b.** $y = (x + 3)^2$ **c.** $y = (x - 2)^2$

2. a. $f(x) = 0.5x^2$ **b.** $f(x) = 0.5(x - 1)^2$ **c.** $f(x) = 0.5(x + 4)^2$

3. a. $r = -2t^2$ **b.** $r = -2(t + 1.2)^2$ **c.** $r = -2(t - 0.9)^2$

For Problems 4 and 5, without graphing, compare the graphs of parts (b), (c), and (d) to that of part (a).

4. a. $y = x^2$ **b.** $y = (x - 2)^2$ **c.** $y = (x - 2)^2 + 4$ **d.** $y = (x - 2)^2 - 3$

5. a. $y = -x^2$ **b.** $y = -(x + 3)^2$ **c.** $y = -(x + 3)^2 - 1$ **d.** $y = -(x + 3)^2 + 4$

6. Create new functions by performing the following transformations on $f(x) = x^2$. Give the coordinates of the vertex for each new parabola.

a. $g(x)$ is $f(x)$ shifted right 2 units, stretched by a factor of 3, and then shifted down by 1 unit.

b. $h(x)$ is $f(x)$ shifted left 3 units, stretched by a factor of 2, then reflected across the x-axis, and finally shifted up by 5 units.

c. $j(x)$ is $f(x)$ shifted left 4 units, compressed by a factor of 5, and then shifted down by 3.5 units.

d. $k(x)$ is $f(x)$ shifted right 1 unit, reflected across the x-axis, and then shifted up by 4 units.

7. Give the coordinates of the vertex for each of the following functions and indicate whether the vertex represents a maximum or minimum.

a. $y = 2(x - 3)^2 - 4$ **b.** $y = -3(x + 1)^2 + 5$ **c.** $y = -0.5(x - 4)^2$ **d.** $y = \frac{2}{3}x^2 - 7$

Finding the Vertex from the Standard Form

What if our quadratic function is in the standard *a-b-c* form and we want to find the vertex of its graph? We can use a formula to find the coordinates of the vertex or, as we show in the next section, we can convert the quadratic from the standard to vertex form, in which the vertex is easy to identify.

Using a formula to find the vertex

The following formula can be used to find the coordinates of the vertex of a parabola when the function is in standard form.

See the reading "Why the Formula for the Vertex and the Quadratic Formula Work" for a derivation of this vertex formula.

Formula for Finding the Vertex from the Standard Form

The vertex of a quadratic function in the form

$$f(x) = ax^2 + bx + c$$

has coordinates $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

EXAMPLE 6 **Finding the vertex**

Find the vertex and sketch the graph of $f(x) = x^2 - 10x + 100$.

SOLUTION The function f is in standard form, $f(x) = ax^2 + bx + c$. So $a = 1$, $b = -10$, and $c = 100$. Since a is positive, the graph is concave up, so the vertex represents a minimum.

Using the formula for the horizontal coordinate of the vertex, we have

$$x = -\frac{b}{2a} = -\frac{(-10)}{2(1)} = \frac{10}{2} = 5$$

To find the vertical coordinate of the vertex, we need to find the value of $f(x)$ when $x = 5$.

Given	$f(x) = x^2 - 10x + 100$
let $x = 5$	$f(5) = 5^2 - 10(5) + 100$
simplify	$= 75$

The coordinates of the vertex are (5, 75) and the y-intercept is at $f(0) = 100$. Figure 8.22 shows a sketch of the graph.

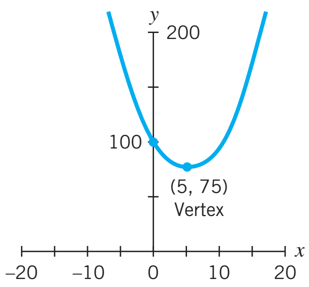


Figure 8.22 Graph of $f(x) = x^2 - 10x + 100$.

EXAMPLE 7 **Measuring traffic flow**

Urban planners and highway designers are interested in maximizing the number of cars that pass along a section of roadway in a certain amount of time. Observations indicate that the primary variable controlling traffic flow (and hence a good choice for the independent variable) is the density of cars on the roadway: the closer each driver is to the car ahead, the more slowly he or she drives. The following quadratic relationship between traffic flow rate and density of cars was derived from observing traffic patterns in the Lincoln Tunnel, which connects New York and New Jersey:

$$t = -0.21d^2 + 34.66d$$

where t = traffic flow rate (cars/hour) and d = density of cars (cars/mile).

a. Find the coordinates of the vertex of the parabola.

b. What does the vertex represent in terms of the traffic flow rate?

c. Graph the function and estimate the value of the horizontal intercepts.

d. What do the horizontal intercepts mean in terms of the traffic flow and density of cars?

SOLUTION **a.** Our equation $t = -0.21d^2 + 34.66d$ is in standard form $(t = ad^2 + bd + c)$, so we can use the formula for finding the coordinates of the vertex. Since $a = -0.21$, $b = 34.66$, and $c = 0$, the horizontal coordinate of the vertex is

$$-\frac{b}{2a} = \frac{-34.66}{2(-0.21)}$$

simplify	≈ 82.52
round to the nearest integer	≈ 83

We can find the corresponding value of t by substituting $d = 83$ into our equation:

Given	$t = -0.21d^2 + 34.66d$
substitute for d	$t \approx -0.21(83)^2 + 34.66(83)$
simplify	$= 1430.09$
round to the nearest integer	≈ 1430

So the coordinates of the vertex are approximately (83, 1430).

b. A vertex at (83, 1430) means that when the density is 83 cars/mile, the traffic flow rate reaches a maximum of 1430 cars/hour. When the density is above or below 83 cars/mile, the traffic flow rate is less than 1430 cars/hour.

c. Figure 8.23 shows the graph of $t = -0.21d^2 + 34.66d$. The horizontal intercepts are the values of d when $t = 0$. From the graph we estimate that when $t = 0$, then d is either 0 or about 165.

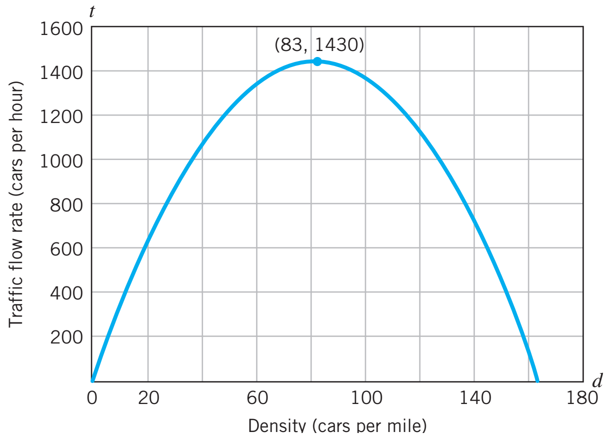


Figure 8.23 The quadratic relationship between traffic flow rate and density.
Source: Adapted from G. B. Whitman, *Linear and Nonlinear Waves* (New York: John Wiley, 1974), p. 68.

d. When the traffic flow rate is equal to zero, traffic is at a standstill. This happens when the density is either zero (when there are no cars) or approximately 165 cars per mile.

Converting between Standard and Vertex Forms

Every quadratic can be written in either standard or vertex form. In this section we examine how to convert from one form to the other.

Converting from a-h-k to a-b-c form

Every function written in the vertex or a-h-k form can be rewritten as a function in the standard or a-b-c form if we multiply out and group terms with the same power of x.

EXAMPLE 8 **Converting from vertex to standard form**

Rewrite the quadratic $f(x) = 3(x + 7)^2 - 9$ in the a-b-c form.

SOLUTION The function $f(x)$ is in the a-h-k form where $a = 3$, $h = -7$, and $k = -9$.

Given	$f(x) = 3(x + 7)^2 - 9$
write out the factors	$= 3(x + 7)(x + 7) - 9$
multiply the factors	$= 3(x^2 + 14x + 49) - 9$
distribute the 3	$= 3x^2 + 42x + 147 - 9$
group the constant terms	$= 3x^2 + 42x + 138$

This function is now in the a-b-c format with $a = 3$, $b = 42$, and $c = 138$.

For a graphic illustration of the shift from *a-b-c* to *a-h-k* form, see "Q7: From *a-b-c* to *a-h-k* Form" in *Quadratic Functions.*

Converting from a-b-c to a-h-k form

Here we examine two strategies that can be used to convert from the standard or *a-b-c* form to the vertex or *a-h-k* form.

Strategy 1: "Completing the Square."

We can convert the function $f(x) = x^2 + 14x + 9$ into *a-h-k* form using a method called *completing the square*.

When a function is in *a-h-k* form, the term $(x - h)^2$ is a perfect square; that is, $(x - h)^2$ is the product of the expression $x - h$ times itself. We can examine separately the expression $x^2 + 14x$ and ask what constant term we would need to add to it in order to make it a perfect square.

A perfect square is in the form

$$(x + m)^2 = x^2 + 2mx + m^2$$

for some number m. Notice that the coefficient of x is two times the number m in our expression for a perfect square. So to turn $x^2 + 14x$ into a perfect square $(x + m)^2$, we need to find m. Since the coefficient of x corresponds to $2m$, then $2m = 14 \Rightarrow m = 7 \Rightarrow m^2 = 49$. So if we add 49 to $x^2 + 14x$ we have a perfect square.

$$(x + 7)^2 = x^2 + 14x + 49$$

To translate $f(x) = x^2 + 14x + 9$ into the *a-k-h* form, we add 49 to make a perfect square, and then subtract 49 to preserve equality. So we have

	$f(x) = x^2 + 14x + 9$
add and subtract 49	$= x^2 + 14x + (49 - 49) + 9$
regroup terms	$= (x^2 + 14x + 49) - 49 + 9$
factor and simplify	$= (x + 7)^2 - 40$

Now $f(x)$ is in *a-h-k* form. The vertex is at $(-7, -40)$.

EXAMPLE 9 **Converting from standard to vertex form**

Convert the function $g(t) = -2t^2 + 12t - 23$ to *a-h-k* form.

SOLUTION This function is more difficult to convert by completing the square, since a is not 1. We first need to factor out -2 *from the t terms only,* getting

$$g(t) = -2(t^2 - 6t) - 23 \tag{1}$$

It is the expression $t^2 - 6t$ for which we must complete the square. Since $\left(\frac{1}{2} \cdot 6\right)^2 = 9$, then adding 9 to $t^2 - 6t$ gives us

$$t^2 - 6t + 9 = (t - 3)^2$$

We must add the constant term 9 *inside the parentheses* in Equation (1) in order to make $t^2 - 6t$ a perfect square. Since everything inside the parentheses is multiplied by -2, we have essentially subtracted 18 from our original function, so we need to add 18 *outside the parentheses* to preserve equality. We have

Given	$g(t) = -2t^2 + 12t - 23$
factor out -2 from t terms	$= -2(t^2 - 6t) - 23$
add 9 inside parentheses and 18 outside parentheses	$= -2(t^2 - 6t + 9) + 18 - 23$
factor and simplify	$= -2(t - 3)^2 - 5$

We now have $g(t)$ in *a-h-k* form. The vertex is at $(3, -5)$.

Strategy 2: Using the Formula for the Vertex.

We could also convert $g(x) = 3x^2 - 12x + 5$ to the *a-h-k* form by using the formula for the coordinates of the vertex. Since the coefficient a is the same in both the *a-b-c* and the *a-h-k* forms, we have $a = 3$. The coordinates of the vertex of a quadratic $f(x)$ in the *a-b-c* form are given by $\left(-\frac{b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

For $g(x)$ we have $a = 3$, $b = -12$, and $c = 5$. So

$$-\frac{b}{2a} = -\frac{(-12)}{2 \cdot 3} = \frac{12}{6} = 2$$

and

$$\begin{aligned} g(2) &= (3 \cdot 2^2) - (12 \cdot 2) + 5 \\ &= 12 - 24 + 5 \\ &= -7 \end{aligned}$$

The vertex is at $(2, -7)$. In the *a-h-k* form the vertex is at (h, k), so $h = 2$ and $k = -7$. Substituting for a, h, and k in $g(x) = a(x - h)^2 + k$, we get

$$g(x) = 3(x - 2)^2 - 7$$

and the transformation is complete.

EXAMPLE 10 Mystery parabola

Find the equation of the parabola in Figure 8.24. Write it in *a-h-k* and in *a-b-c* form.

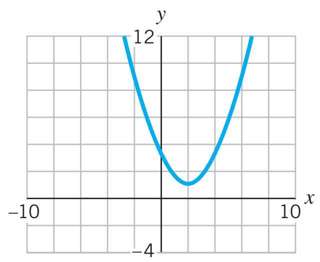


Figure 8.24 A mystery parabola.

SOLUTION The vertex of the graph appears to be at (2, 1) and the graph is concave up. We can substitute the coordinates of the vertex in the *a-h-k* form of the quadratic equation to get

$$y = a(x - 2)^2 + 1 \tag{1}$$

How can we find the value for a? If we can identify values for any other point (x, y) that lies on the parabola, we can substitute these values into Equation (1) and solve it for a. The y-intercept, estimated at (0, 3), is a convenient point to pick. Setting $x = 0$ and $y = 3$, we get

$$\begin{aligned} 3 &= a(0 - 2)^2 + 1 \\ 3 &= 4a + 1 \\ 2 &= 4a \\ \text{so} \qquad a &= 0.5 \end{aligned}$$

The equation in the *a-h-k* form is

$$y = 0.5(x - 2)^2 + 1$$

If we wanted it in the equivalent *a-b-c* form, we could square, multiply, and collect like terms to get

$$y = 0.5x^2 - 2x + 3$$

The Vertex of a Quadratic Function

A quadratic function in vertex form

$$f(x) = a(x - h)^2 + k \qquad \text{has a vertex at } (h, k)$$

A quadratic function in standard form

$$f(x) = ax^2 + bx + c \qquad \text{has a vertex at } \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

Whether in standard or vertex form, the value of the coefficient a determines if the parabola is relatively narrow or flat.

If $a > 0$, the parabola is concave up and has a minimum at the vertex.

If $a < 0$, the parabola is concave down and has a maximum at the vertex.

SOMETHING TO THINK ABOUT

Using "Q1: a, b, c Sliders" in Quadratic Functions *in the course software, can you describe the effect on the parabola of changing the value for b while you hold a and c fixed?*

EXAMPLE 11 Match the following functions to the graphs in Figure 8.25. What allows you to match them easily?

a. $f(x) = x^2 - 8x + 18$

b. $g(x) = 2x^2 - 16x + 34$

c. $h(x) = -0.5x^2 + 4x - 6$

d. $j(x) = -x^2 + 8x - 14$

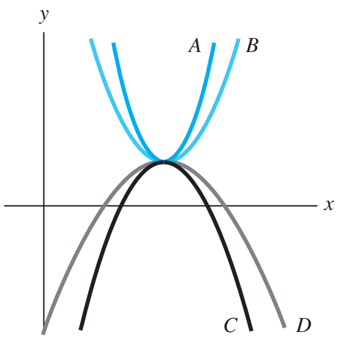


Figure 8.25 Graphs of four quadratic functions.

SOLUTION All the graphs share the same vertex. Graphs A and B are concave up, so they could match with $f(x)$ or $g(x)$, which both have a positive value of a (the coefficient of x^2). Graphs C and D are concave down, so they could match with $h(x)$ or $j(x)$, which both have a negative value of a.

Graph A is narrower than B, so $g(x)$ matches with graph A and $f(x)$ matches with graph B. Graph C is narrower than D, so $j(x)$ matches with graph C and $h(x)$ matches with graph D. Reasoning: The larger the absolute value of a, the narrower the graph.

Algebra Aerobics 8.2c

Use of a graphing program is optional for Problems 3 and 13.

1. Find the vertex of the graph of each of the following quadratic functions:

   a. $f(x) = 2x^2 - 4$

   b. $g(z) = -z^2 + 6$

   c. $w = 4t^2 + 1$

2. Find the vertex of the graph of each of the following functions and then sketch the graphs on the same grid:

   a. $y = x^2 + 3$

   b. $y = -x^2 + 3$

3. In parts (a) to (d) determine the vertex and whether the graph is concave up or down. Then predict the number of x-intercepts. Graph the function to confirm your answer. Use technology if available.

   a. $f(x) = -3x^2$

   b. $f(x) = -2x^2 - 5$

   c. $f(x) = x^2 + 4x - 7$

   d. $f(x) = 4 - x - 2x^2$

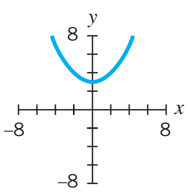
4. Without drawing the graph, describe whether the graph of each of the following functions has a maximum or minimum at the vertex and is narrower or broader than $y = x^2$.
   a. $y = 2x^2 - 5$
   b. $y = 0.5x^2 + 2x - 10$
   c. $y = 3 + x - 4x^2$
   d. $y = -0.2x^2 + 11x + 8$
5. For each of the following functions, use the formula $h = -\frac{b}{2a}$ to find the horizontal coordinate of the vertex, and then find its vertical coordinate. Draw a rough sketch of each function.
   a. $y = x^2 + 3x + 2$
   b. $f(x) = 2x^2 - 4x + 5$
   c. $g(t) = -t^2 - 4t - 7$
6. Find the coordinates of the vertex and the vertical intercept, and then sketch the function by hand.
   a. $y = 0.1(x + 5)^2 - 11$
   b. $y = -2(x - 1)^2 + 4$
7. Rewrite each expression in the form $(x + m)^2 - m^2$.
   a. $x^2 + 6x$ **c.** $x^2 - 30x$
   b. $x^2 - 10x$ **d.** $x^2 + x$
8. Convert the following functions to vertex form by completing the square. Identify the stretch factor and the vertex.
   a. $f(x) = x^2 + 2x - 1$
   b. $j(z) = 4z^2 - 8z - 6$
   c. $h(x) = -3x^2 - 12x$
   d. $h(t) = -16t^2 + 96t$
   e. $h(t) = -4.9t^2 - 98t + 200$
9. Express each of the following functions in standard form $y = ax^2 + bx + c$. Identify the stretch or compression factor and the vertex.
   a. $y = 2\left(x - \frac{1}{2}\right)^2 + 5$
   b. $y = -\frac{1}{3}(x + 2)^2 + 4$
   c. $y = 10(x - 5)^2 + 12$
   d. $y = 0.1(x + 0.2)^2 + 3.8$
10. Express each of the following functions in vertex form.
   a. $y = x^2 + 6x + 7$ **b.** $y = 2x^2 + 4x - 11$
11. Convert the following quadratic functions to vertex form. Identify the coordinates of the vertex.
   a. $y = x^2 + 8x + 11$ **b.** $y = 3x^2 + 4x - 2$
12. The daily profit, f (in dollars), of a hot pretzel stand is a function of the price per pretzel, p (in dollars), given by $f(p) = -1875p^2 + 4500p - 2400$.
   a. Find the coordinates of the vertex of the parabola.
   b. Give the maximum profit and the price per pretzel that gives that profit.
13. Find the equations in vertex form of the parabolas that satisfy the following conditions. Then check your solutions by using a graphing program, if available.
   a. The vertex is at $(-1, 4)$ and the parabola passes through the point $(0, 2)$.
   b. The vertex is at $(1, -3)$ and one of the parabola's two horizontal intercepts is at $(-2, 0)$.

Exercises for Section 8.2

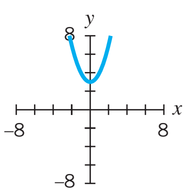
A graphing program is required or recommended for several exercises. Technology is needed in Exercises 30, 36, 38, and 40 to generate a best-fit quadratic.

1. Which of the following quadratics have parabolic graphs that are concave up? Concave down? Explain your reasoning.
   a. $y = 20 + x - x^2$ **c.** $y = -3x + 2 + x^2$
   b. $y = 0.5t^2 - 2$ **d.** $y = 3(5 - x)(x + 2)$
2. Match each function with its graph. Explain your reasoning for each choice.

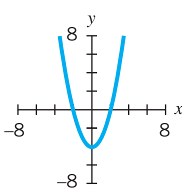
   $f(x) = x^2 + 3$ $g(x) = x^2 - 4$ $h(x) = 0.25x^2 + 3$



Graph *A*



Graph *B*



Graph *C*

3. On the same graph, sketch by hand the plots of the following functions and label each with its equation.
   $y = 2x^2$ $y = -2x^2 + 3$
   $y = -2x^2$ $y = -2x^2 - 3$
4. In each case sketch by hand a quadratic function $y = f(x)$ with the indicated characteristics.
   a. Does not cross the x-axis and has a negative vertical intercept
   b. Has a vertex at (h, k) where $h < 0$ and $k > 0$ and a positive y-intercept
   c. Crosses the x-axis at $x = 2$ and $x = -4$ and is concave up
   d. Has the same shape as $y = x^2 - 3x$ but is raised up four units
   e. Has the same y-intercept as $y = x^2 - 3x - 2$ but is concave down and narrower
   f. Has the same shape as $y = \frac{x^2}{2}$, but is shifted to the left by three units

5. Identify the stretch/compression factor and the vertex for each of the following.

a. $y_1 = 0.3(x - 1)^2 + 8$ **c.** $y_3 = 0.01(x + 20)^2$
b. $y_2 = 30x^2 - 11$ **d.** $y_4 = -6x^2 + 12x$

6. For each of the following functions, identify the vertex and specify whether it represents a maximum or minimum, and then sketch its graph by hand.

a. $y = (x - 2)^2$ **c.** $y = -2(x + 1)^2 + 5$
b. $y = \frac{1}{2}(x - 2)^2 + 3$ **d.** $y = -0.4(x - 3)^2 - 1$

7. For each of the following quadratic functions, find the vertex (h, k) and determine if it represents the maximum or minimum of the function.

a. $f(x) = -2(x - 3)^2 + 5$ **c.** $f(x) = -5(x + 4)^2 - 7$
b. $f(x) = 1.6(x + 1)^2 + 8$ **d.** $f(x) = 8(x - 2)^2 - 6$

8. For each quadratic function identify the vertex and specify whether it represents a maximum or minimum. Evaluate the function at 0 and then sketch a graph of the function by hand.

a. $f(t) = 0.25(t - 2)^2 + 1$ **c.** $h(x) = -(x + 3)^2 + 4$
b. $g(x) = 3 - (x - 5)^2$ **d.** $k(x) = \frac{(x + 5)^2}{3} - \frac{1}{3}$

9. (Graphing program optional.) Create a quadratic function in the vertex form $y = a(x - h)^2 + k$, given the specified values for a and the vertex (h, k). Then rewrite the function in the standard form $y = ax^2 + bx + c$. If available, use technology to check that the graphs of the two forms are the same.

a. $a = 1, (h, k) = (2, -4)$ **c.** $a = -2, (h, k) = (-3, 1)$
b. $a = -1, (h, k) = (4, 3)$ **d.** $a = \frac{1}{2}, (h, k) = (-4, 6)$

10. Transform the function $f(x) = x^2$ into a new function $g(x)$ by compressing $f(x)$ by a factor of $\frac{1}{4}$, then shifting the result horizontally left three units, and finally shifting it down by six units. Find the equation of $g(x)$ and sketch it by hand.

11. Transform the function $f(x) = 3x^2$ into a new function $h(x)$ by shifting $f(x)$ horizontally to the right four units, reflecting the result across the x-axis, and then shifting it up by five units.

a. What is the equation for $h(x)$?
b. What is the vertex of $h(x)$?
c. What is the vertical intercept of $h(x)$?

12. For the following quadratic functions in vertex form, $f(x) = a(x - h)^2 + k$, determine the values for a, h, and k. Then compare each to $f(x) = x^2$, and identify which constants represent a stretch/compression factor, or a shift in a particular direction.

a. $p(x) = 5(x - 4)^2 - 2$
b. $g(x) = \frac{1}{3}(x + 5)^2 + 4$
c. $h(x) = -0.25\left(x - \frac{1}{2}\right)^2 + 6$
d. $k(x) = -3(x + 4)^2 - 3$

13. Using the strategy of "completing the square," fill in the missing numbers that would make the statement true.

a. $x^2 + 6x +$ _______ $= (x +$ _____$)^2$
b. $x^2 + 8x +$ _______ $= (x +$ _____$)^2$
c. $x^2 - 4x +$ _______ $= (x -$ _____$)^2$
d. $x^2 - 3x +$ _______ $= (x -$ _____$)^2$
e. $2(x^2 + 2x +$ _______$) = 2(x +$ _____$)^2$
f. $-3(x^2 + x +$ _______$) = -3(x +$ _____$)^2$

14. (Graphing program optional.) For each quadratic function use the method of "completing the square" to convert to the *a-h-k* form, and then identify the vertex. If available, use technology to confirm that the two forms are the same.

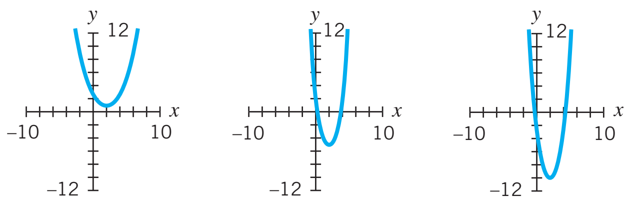
a. $y = x^2 + 8x + 15$ **d.** $r(s) = -5s^2 + 20s - 10$
b. $f(x) = x^2 - 4x - 5$ **e.** $z = 2m^2 + 6m - 5$
c. $p(t) = t^2 - 3t + 2$

15. (Graphing program optional.) For each quadratic function convert to *a-h-k* form by using $h = -\frac{b}{2a}$ and then find k. If available, use technology to graph the two forms of the function to confirm that they are the same.

a. $y = x^2 + 6x + 13$ **d.** $p(r) = -3r^2 + 18r - 9$
b. $f(x) = x^2 - 5x - 5$ **e.** $m(z) = 2z^2 + 8z - 5$
c. $g(x) = x^2 - 3x + 6$

16. Match each of the following graphs with one of following equations. Explain your reasoning. (*Hint:* Find the vertex.) Note that one function does not have a match.

$f(x) = 2x^2 - 8x - 2$ $h(x) = 0.5x^2 - 2x + 3$
$g(x) = 2x^2 - 8x + 3$ $i(x) = 0.5x^2 - 2x + 8$



Graph *A* Graph *B* Graph *C*

17. Without drawing the graph, list the following parabolas in order, from the narrowest to the broadest. Verify your results with technology.

a. $y = x^2 + 20$ **d.** $y = 4x^2$
b. $y = 0.5x^2 - 1$ **e.** $y = 0.1x^2 + 2$
c. $y = \frac{1}{3}x^2 + x + 1$ **f.** $y = -2x^2 - 5x + 4$

18. Convert the following functions from the *a-b-c* or standard form to the *a-h-k* or vertex form.

a. $f(x) = x^2 + 6x + 5$ **d.** $y = 2x^2 + 3x - 5$
b. $g(x) = x^2 - 3x + 7$ **e.** $h(x) = 3x^2 + 6x + 5$
c. $y = 3x^2 - 12x + 12$ **f.** $y = -x^2 + 5x - 2$

19. (Graphing program required.) Given $f(x) = -x^2 + 8x - 15$:

a. Estimate by graphing: the x-intercepts, the y-intercept, and the vertex.
b. Calculate the coordinates of the vertex.

20. Write each of the following quadratic equations in function form (i.e., solve for y in terms of x). Find the vertex and the y-intercept using any method. Finally, using these points, draw a rough sketch of the quadratic function.

a. $y + 12 = x(x + 1)$ **d.** $y - 8x = x^2 + 15$
b. $2x^2 + 6x + 14.4 - 2y = 0$ **e.** $y + 1 = (x - 2)(x + 5)$
c. $y + x^2 - 5x = -6.25$ **f.** $y + 2x(x - 6) = 20$

21. a. Find the equation of the parabola with a vertex of (2, 4) that passes through the point (1, 7).

b. Construct two different quadratic functions both with a vertex at (2, −3) such that the graph of one function is concave up and the graph of the other function is concave down.

c. Find two different equations of a parabola that passes through the points (−2, 5) and (4, 5) and that opens downward. (*Hint:* Find the axis of symmetry.)

22. For each part construct a function that satisfies the given conditions.

a. Has a constant rate of increase of $15,000/year

b. Is a quadratic that opens upward and has a vertex at (1, −4)

c. Is a quadratic that opens downward and the vertex is on the x-axis

d. Is a quadratic with a minimum at the point (10, 50) and a stretch factor of 3

e. Is a quadratic with a vertical intercept of (0, 3) that is also the vertex

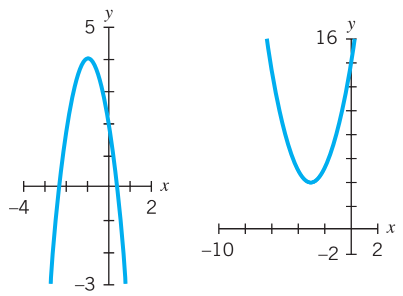
23. If a parabola is the graph of the equation $y = a(x - 4)^2 - 5$:

a. What are the coordinates of the vertex? Will the vertex change if a changes?

b. What is the value of stretch factor a if the y-intercept is (0, 3)?

c. What is the value of stretch factor a if the graph goes through the point (1, −23)?

24. Construct an equation for each of the accompanying parabolas.



Graph *A* Graph *B*

25. Determine the equation of the parabola whose vertex is at (2, 3) and that passes through the point (4, −1). Show your work, including a sketch of the parabola.

26. Students noticed that the path of water from a water fountain seemed to form a parabolic arc. They set a flat surface at the level of the water spout and measured the maximum height of the water from the flat surface as 8 inches and the distance from the spout to where the water hit the flat surface as 10 inches. Construct a function model for the stream of water.

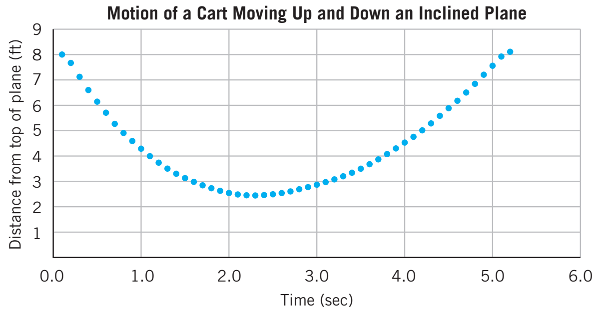
27. Marketing research by a company has shown that the profit, $P(x)$ (in thousands of dollars), made by the company is related to the amount spent on advertising, x (in thousands of dollars), by the equation $P(x) = 230 + 20x - 0.5x^2$. What expenditure (in thousands of dollars) for advertising gives the maximum profit? What is the maximum profit?

28. Tom has a taste for adventure. He decides that he wants to bungee-jump off the Missouri River bridge. At any time t (in seconds from the moment he jumps) his height $h(t)$ (in feet above the water level) is given by the function $h(t) = 20.5t^2 - 123t + 190.5$. How close to the water will Tom get?

29. A manager has determined that the revenue $R(x)$ (in millions of dollars) made on the sale of supercomputers is given by $R(x) = 48x - 3x^2$, where x represents the number of supercomputers sold. How many supercomputers must be sold to maximize revenue? According to this model, what is the maximum revenue (in millions of dollars)?

30. (Technology required to generate a best-fit quadratic.) The accompanying graph of the data file INCLINE shows the motion of a cart, initially at the bottom of an inclined plane, after it was given a push toward a motion detector positioned at the top of the plane. The distance (in feet) of the cart from the top of the plane is plotted vs. time (in seconds). The motion can be modeled with a quadratic function.

DATA



a. Estimate the coordinates of the vertex. Describe what is happening to the cart at the vertex.

b. Use technology to generate a best-fit quadratic to the data.

c. Calculate the coordinates of the vertex.

31. (Graphing program required.) A baseball hit straight up in the air is at a height

$$h = 4 + 50t - 16t^2$$

feet above ground level at time t seconds after being hit. (This formula is valid for $t \geq 0$ until the ball hits the ground.)

a. What is the value of h when $t = 0$? What does this value represent in this context?

b. Construct a table of values for $t = 0, 1, 2, 3, 4$. Roughly when does the ball hit the ground? How can you tell?

c. Graph the function. Does the graph confirm your estimate in part (b)?

d. Explain why negative values for h make no sense in this situation.

e. Estimate the maximum height that the baseball reaches. When does it reach that height?

32. (Graphing program optional.) The following function represents the relationship between time t (in seconds) and height h (in feet) for objects thrown upward on *Pluto*. For an initial velocity of 20 ft/sec and an initial height above the ground of 25 feet, we get

$$h = -t^2 + 20t + 25$$

a. Find the coordinates of the point where the graph intersects the h-axis.

b. Find the coordinates of the vertex of the parabola.

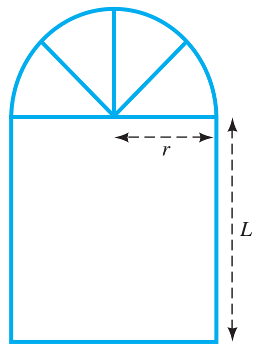
c. Sketch the graph. Label the axes.

d. Interpret the vertex in terms of time and height.

e. For what values of t does the mathematical model make sense?

33. In ancient times, after a bloody defeat that made her flee her city, the queen of Carthage, Dido, found refuge on the shores of Northern Africa. Sympathetic to her plight, the local inhabitants offered to build her a new Carthage along the shores of the Mediterranean Sea. However, her city had to be rectangular in shape, and its perimeter (excluding the coastal side) could be no larger than the length of a ball of string that she could make using fine strips from only one cow hide. Queen Dido made the thinnest string possible, whose length was one mile. Dido used the string to create three non-coastal sides enclosing a rectangular piece of land (assuming the coastline was straight). She made the width exactly half the length. This way, she claimed, she would have the maximum possible area the ball of string would allow her to enclose. Was Dido right?

34. A Norman window has the shape of a rectangle surmounted by a semicircle of diameter equal to the width of the rectangle (see the accompanying figure). If the perimeter of the window is 20 feet (including the semicircle), what dimensions will admit the most light (maximize the area)? (*Hint:* Express L in terms of r. Recall that the circumference of a circle $= 2\pi r$, and the area of a circle $= \pi r^2$, where r is the radius of the circle.)



35. A pilot has crashed in the Sahara Desert. She still has her maps and knows her position, but her radio is destroyed. Her only hope for rescue is to hike out to a highway that passes near her position. She needs to determine the closest point on the highway and how far away it is.

a. The highway is a straight line passing through a point 15 miles due north of her and another point 20 miles due east. Draw a sketch of the situation on graph paper, placing the pilot at the origin and labeling the two points on the highway.

b. Construct an equation that represents the highway (using x for miles east and y for miles north).

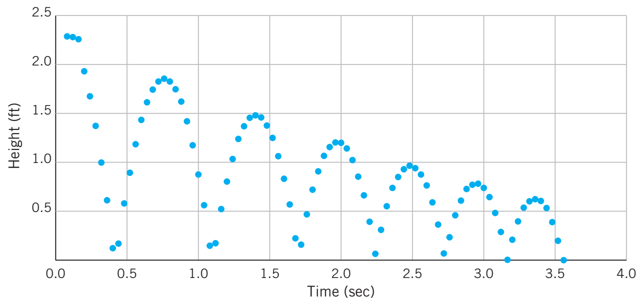
c. Now use the Pythagorean Theorem to describe the square of the distance, d, of the pilot to any point (x, y) on the highway.

d. Substitute the expression for y from part (b) into the equation from part (c) in order to write d^2 as a quadratic in x.

e. If we minimize d^2, we minimize the distance d. So let $D = d^2$ and write D as a quadratic function in x. Now find the minimum value for D.

f. What are the coordinates of the closest point on the highway, and what is the distance, d, to that point?

36. (Requires technology to find a best-fit quadratic.) The accompanying figure is a plot of the data in the file BOUNCE, which shows the height of a bouncing racquetball (in feet) over time (in seconds). The path of the ball between each pair of bounces can be modeled using a quadratic function.



a. Select from the file BOUNCE the subset of the data that represents the motion of the ball between the first and second bounces (that is, between the first and second times the ball hits the floor). Use technology to generate a best-fit quadratic function for this subset.

b. From the graph, estimate the maximum height the ball reaches between the first and second bounces.

c. Use the best-fit function to calculate the maximum height the ball reaches between the first and second bounces.

37. (Graphing program required.) At low speeds an automobile engine is not at its peak efficiency; efficiency initially rises with speed and then declines at higher speeds. When efficiency is at its maximum, the consumption rate of gas (measured in gallons per hour) is at a minimum. The gas consumption rate of a particular car can be modeled by the following equation, where G is the gas consumption rate in gallons per hour and M is speed in miles per hour:

$$G = 0.0002M^2 - 0.013M + 1.07$$

a. Construct a graph of gas consumption rate versus speed. Estimate the minimum gas consumption rate from your graph and the speed at which it occurs.

b. Using the equation for G, calculate the speed at which the gas consumption rate is at its minimum. What is the minimum gas consumption rate?

c. If you travel for 2 hours at peak efficiency, how much gas will you use and how far will you go?

d. If you travel at 60 mph, what is your gas consumption rate? How long does it take to go the same distance that you calculated in part (c)? (Recall that travel distance = speed × time traveled.) How much gas is required for the trip?

e. Compare the answers for parts (c) and (d), which tell you how much gas is used for the same-length trip at two different speeds. Is gas actually saved for the trip by traveling at the speed that gives the minimum gas consumption rate?

f. Using the function G, generate data for gas consumption rate measured in gallons per mile by completing the following table. Plot gallons per mile (on the vertical axis) vs. miles per hour (on the horizontal axis). At what speed is gallons per mile at a minimum?

	Measures of the Rate of Gas Consumption	
Speed of Car (mph)	(gal/hr)	(gal/hr)/mph = gal/mile
0		
10		
20		
30		
40		
50		
60		
70		
80		

g. Add a fourth column to the data table. This time compute miles/gal = mph/(gal/hr). Plot miles per gallon vs. miles per hour. At what speed is miles per gallon at a maximum? This is the inverse of the preceding question; we are normally used to maximizing miles per gallon instead of minimizing gallons per mile. Does your answer make sense in terms of what you found for parts (b) and (f)?

38. (Requires technology to create best-fit functions.) The following data show the average growth of the human embryo prior to birth.

Embryo Age (weeks)	Weight (g)	Length (cm)
8	3	2.5
12	36	9
20	330	25
28	1000	35
36	2400	45
40	3200	50

Source: Reprinted with permission from Kimber et al., *Textbook of Anatomy and Physiology* (Upper Saddle River, NJ: Prentice Hall, 1955), "Embryo Age, Weight and Height," p. 785.

a. Plot weight (on vertical axis) vs. age (on horizontal axis). Then use technology to find the best-fit quadratic model to approximate the data.

b. According to your model, what would an average 32-week-old embryo weigh?

c. Comment on the domain for which your formula is reliable.

d. Plot length versus age; then use technology to construct an appropriate mathematical model for the length as a function of embryo age from 20 to 40 weeks.

e. Using your model, compute the age at which an embryo would be 42.5 centimeters long.

39. A shot-put athlete releases the shot at a speed of 14 meters per second, at an angle of 45 degrees to the horizontal (ground level). The height y (in meters above the ground) of the shot is given by the function

$$y = 2 + x - \frac{1}{20}x^2$$

where x is the horizontal distance the shot has traveled (in meters).

a. What was the height of the shot at the moment of release?

b. How high is the shot after it has traveled 4 meters horizontally from the release point? 16 meters?

c. Find the highest point reached by the shot in its flight.

d. Draw a sketch of the height of the shot and indicate how far the shot is from the athlete when it lands.

40. (Technology required to create a best-fit quadratic.) When people meet for the first time, it is customary for all people in the group to shake hands. Below is a table that shows the number of handshakes that occur depending on the size of the group.

Group size	2	3	4	5	6
Total number of handshakes	1	3	6	10	15

a. Draw a scatterplot of the data in the table (with group size on the horizontal axis) and find a quadratic model of best fit.

b. Use the model to find the total number of handshakes for a group of seven people, and for a group of ten people.

c. Factor the model of best fit. Describe in words how you could find the number of handshakes knowing the group size.

8.3 *Finding the Horizontal Intercepts*

We often want to find the exact values for horizontal intercepts when we use a function to model a real-world situation. We know that a quadratic function may have two, one, or no horizontal intercepts. To find the horizontal intercepts (if any) of a function $f(x)$, we need to find the *zeros* of the function, the value(s) of x such that $f(x) = 0$. If $f(x) = ax^2 + bx + c$, then setting $f(x) = 0$, we have $0 = ax^2 + bx + c$. If, as in the following example, we can factor $ax^2 + bx + c$ as a product of two linear factors (each with an x term), solving for the intercepts is easy. Later on in this chapter we'll use the quadratic formula to find the horizontal intercepts when the function is not easily factored.

Using Factoring to Find the Horizontal Intercepts

EXAMPLE 1 **A quadratic model for a battleship gun range**

Iowa class battleships have large naval guns that have a muzzle velocity of about 2000 feet per second. If the gun is set to maximize range, then the relationship between the height of the projectile fired and the distance it travels is $h(d) = d - 8 \cdot 10^{-6} d^2$, where d = horizontal distance in feet from the battleship, and $h(d)$ = height in feet of the projectile. What is the maximum range of the battleship gunfire?

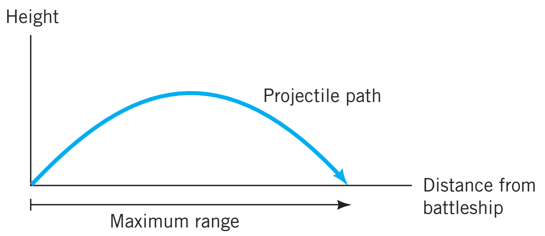


Figure 8.26 Graph of height versus distance for a projectile fired from a battleship.

SOLUTION Assuming that the battleship and the target are at the same level, to find the maximum range we need to find where the projectile will hit the ground. At that point $h(d) = 0$, so the point represents a horizontal intercept on the graph (see Figure 8.26). To find the horizontal intercepts we set $h(d) = 0$ and solve for d.

Given the function	$h(d) = d - 8 \cdot 10^{-6} d^2$
if we set $h(d) = 0$	$0 = d - 8 \cdot 10^{-6} d^2$
and factor out d, we get	$0 = d \cdot (1 - 8 \cdot 10^{-6} d)$

So when $h(d) = 0$, then either

$d = 0$ (at the gun) *or* $1 - 8 \cdot 10^{-6} d = 0$ (when the projectile hits the ground)

To solve the second expression for d	$1 - 8 \cdot 10^{-6} d = 0$
subtract 1 from each side	$-8 \cdot 10^{-6} d = -1$
divide each side by $-8 \cdot 10^{-6}$	$d = \dfrac{-1}{(-8 \cdot 10^{-6})}$
and simplify	$= 0.125 \cdot 10^{6}$
to get	$= 125{,}000$ feet

So a projectile fired from this gun is able to hit a target that is 125,000 feet, or almost 24 miles, from the battleship.

> **Finding the Horizontal Intercepts of a Function**
>
> Given a function $f(x)$:
>
> To find the x-intercepts, set $f(x) = 0$ and solve for x.
>
> Every x-intercept is a *zero* of the function.

One of the properties of real numbers we used in the previous example was the "zero product rule"—the notion that if the product of two terms is 0, then at least one term must be 0.

> **Zero Product Rule**
>
> For any two numbers r and s, if the product $rs = 0$, then r or s or both must equal 0.

We'll use this rule repeatedly in our search for the horizontal intercepts of a function.

Factoring Quadratics

In the battleship example, we wrote $h(d) = d - 8{\cdot}10^{-6}d^2$, which is in standard form, as $h(d) = d(1 - 8{\cdot}10^{-6}d)$, which is in *factored form*. If a quadratic is in factored form, it's easy to find the horizontal intercepts. We set the product equal to 0 and use the zero product rule.

Factoring review

To convert $ax^2 + bx + c$ to factored form requires thinking, practice, and a few hints. It is often a trial-and-error process. We usually restrict ourselves to finding factors with integer coefficients.

First, look for common factors in all of the terms. For example, $10x^2 + 2x$ can be factored as $2x(5x + 1)$.

Second, look for two linear factors. This is easiest to do when the coefficient of x^2 is 1. For example, to factor $x^2 + 7x + 12$, we want to rewrite it as

$$(x + m)(x + n)$$

for some m and n. Note that the coefficients of both x's in the factors equal 1, since x times x is equal to the x^2 in the original expression. If we multiply out the two factors, we get $x^2 + (m + n)x + m \cdot n$. So we need $m + n = 7$ and $m \cdot n = 12$. Since 12 is positive, m and n must have the same sign, either both positive or both negative. But since $m + n$ $(=7)$ is positive, m and n must both be positive. So we consider pairs of positive integers whose product is 12, namely, 1 and 12, 2 and 6, or 3 and 4. We can then narrow our list of factors of 12 to those whose sum equals 7, the coefficient of the x term. Only the factors 3 and 4 fit this criterion. We can factor our polynomial as:

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

We can check that these factors work by multiplying them out.

Third, look for the special case of the difference of two squares. In this case the middle terms cancel out when multiplying:

$$\begin{aligned} x^2 - 25 &= (x - 5)(x + 5) \\ &= x^2 - 5x + 5x - 25 \\ &= x^2 - 25 \end{aligned}$$

In general, $$x^2 - n^2 = (x - n)(x + n)$$

Guidelines for Factoring $ax^2 + bx + c$

First factor out any common terms.

If $a = 1$,

Find the factors of c that add to give b.

$$x^2 + bx + c = (x + m)(x + n), \qquad \text{where } c = mn \text{ and } b = m + n$$

If the quadratic is the difference of two squares (so $b = 0$ and $c = n^2$ for some number n), factor it into the product of a sum and a difference.

$$x^2 - n^2 = (x - n)(x + n)$$

It's a good idea to double-check your answer by multiplying out the factored terms.

EXAMPLE 2 **Finding factors**

Put each of the following functions into factored form and then identify any horizontal intercepts.

a. $f(x) = 300x^2 + 195x$ **c.** $g(z) = -3z^2 + 12z - 12$ **e.** $Q(w) = 2w^2 - 3w - 5$

b. $h(t) = t^2 - 5t + 6$ **d.** $H(v) = -2v^2 + 18$

SOLUTION

a. In factored form $f(x) = 15x(20x + 13)$. (You can double-check this by multiplying the factors to return to $300x^2 + 195x$.) To find the horizontal intercepts, we need to find values for x such that $f(x) = 0$. If we set $15x(20x + 13) = 0$, then according to the zero product rule, either

$$\begin{aligned} 15x &= 0 \quad \text{or} \quad & 20x + 13 &= 0 \\ x &= 0 & 20x &= -13 \\ & & x &= \frac{-13}{20} \quad \text{or} \quad -0.65 \end{aligned}$$

So there are two horizontal intercepts, at $x = 0$ and $x = -0.65$

b. In factored form $h(t) = (t - 3)(t - 2)$. (Again we can double-check our answer by multiplying the two factors to get $t^2 - 3t - 2t + 6 = t^2 - 5t + 6$.) To find the horizontal intercepts we need to find values for t such that $h(t) = 0$. If we set $(t - 3)(t - 2) = 0$, then either

$$\begin{aligned} t - 3 &= 0 \quad \text{or} \quad & t - 2 &= 0 \\ t &= 3 & t &= 2 \end{aligned}$$

So there are two horizontal intercepts, at $t = 3$ and $t = 2$.

c. In factored form
$$\begin{aligned} g(z) &= -3(z^2 - 4z + 4) \\ &= -3(z - 2)(z - 2) \\ &= -3(z - 2)^2 \end{aligned}$$

To find the horizontal intercepts, we need to find values for z such that $g(z) = 0$. When $-3(z - 2)^2 = 0$, then $z - 2 = 0$, so $z = 2$. Hence $g(z)$ has a single horizontal intercept at $z = 2$.

d. In factored form
$$\begin{aligned} H(v) &= -2(v^2 - 9) \\ &= -2(v + 3)(v - 3) \end{aligned}$$

When $(v + 3)(v - 3) = 0$, then either

$$\begin{aligned} v + 3 &= 0 \quad \text{or} \quad & v - 3 &= 0 \\ v &= -3 & v &= 3 \end{aligned}$$

So $H(v)$ has two horizontal intercepts, at -3 and 3.

e. In factored form $Q(w) = (2w - 5)(w + 1)$. When $(2w - 5)(w + 1) = 0$, then either

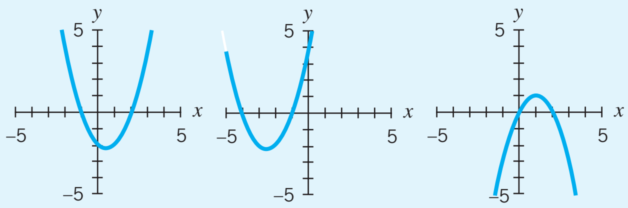
$$\begin{aligned} 2w - 5 &= 0 \quad \text{or} \quad & w + 1 &= 0 \\ 2w &= 5 & w &= -1 \\ w &= \tfrac{5}{2} \end{aligned}$$

So $Q(w)$ has two horizontal intercepts, at $\frac{5}{2}$ (or 2.5) and at -1.

Algebra Aerobics 8.3a

1. Put the function into factored form with integer coefficients and then identify any horizontal intercepts.
   a. $y = -16t^2 + 50\,t$
   b. $y = t^2 - 25$
   c. $h(z) = z^2 - 3z - 4$
   d. $g(x) = 4x^2 - 9$
   e. $y = 15 - 8x + x^2$
   f. $v(x) = x^2 + 2x + 1$
   g. $p(q) = q^2 - 6q + 9$
2. When possible, put the function into factored form with integer coefficients and then identify any horizontal intercepts.
   a. $f(x) = 5 - x - 4x^2$
   b. $h(t) = 64 - 9t^2$
   c. $y = 10 - 13t - 3t^2$
   d. $z = 4w^2 - 20w + 25$
   e. $y = 2x^2 - 3x - 5$
   f. $Q(t) = 6t^2 + 11t - 10$
3. Identify which of the following quadratic functions can be factored into the product of a sum and a difference, $y = (a + b)(a - b)$, which can be factored into the square of the sum or difference, $y = (a \pm b)^2$, and which can be factored into neither.
   a. $y = x^2 - 9$
   b. $y = x^2 + 4x + 4$
   c. $y = x^2 + 5x + 25$
   d. $y = 9x^2 - 25$
   e. $y = x^2 - 8x + 16$
   f. $y = 16 - 25x^2$
   g. $y = 4 + 16x^2$
4. Find any horizontal and vertical intercepts for the following functions and explain the meaning of each within the context of the problem.
   a. An object is thrown vertically into the air at time $t = 0$ seconds. Its height, $h(t)$, in feet, is given by $h(t) = -16t^2 + 64t$.
   b. The monthly profit $P(q)$ (in dollars) is a function of q, the number of items sold. The relationship is described by $P(q) = -q^2 + 60q - 800$.
5. Match the factored form of the quadratic function with its graph.

$$y_1 = -x(x - 2)$$
$$y_2 = (x - 2)(x + 1)$$
$$y_3 = (x + 4)(x + 1)$$



Graph *A* Graph *B* Graph *C*

6. Given $f(x) = x^2 + x - 30$:
   a. Factor $f(x)$.
   b. Find the horizontal intercepts.
   c. Sketch the graph of $f(x)$.
   d. Describe the relationship between the factored form and the horizontal intercepts in this problem.

Using the Quadratic Formula to Find the Horizontal Intercepts

Most quadratic functions are not easily factored. When this is the case, we still set $f(x) = 0$ and solve for x. But now we use the quadratic formula to find the solutions or *roots* to the equation $0 = ax^2 + bx + c$.

The Quadratic Formula

For any quadratic equation of the form $0 = ax^2 + bx + c$ (where $a \neq 0$), the solutions, or *roots,* of the equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The term under the radical sign, $b^2 - 4ac$, is called the *discriminant*.

See the reading "Why the Formula for the Vertex and the Quadratic Formula Work" for a derivation of this famous formula.

The symbol $\pm$ lets us use one formula to write the two roots as

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note on Terminology. The language of roots and zeros can be confusing. The numbers 3 and -3 are called the *roots* or *solutions* of the *equation* $x^2 - 9 = 0$ and the *zeros* of the *function* $f(x) = x^2 - 9$. The zeros of the function $f(x)$ are the roots of the equation $f(x) = 0$.

The discriminant

One shortcut for predicting the number of horizontal intercepts is to use the discriminant, the term $b^2 - 4ac$.

Using the Discriminant

A quadratic function $f(x) = ax^2 + bx + c$ has a discriminant of $b^2 - 4ac$.

If the discriminant > 0, there are two distinct real roots and hence two x-intercepts, at

$$x_1 = \frac{-b + \sqrt{\text{discriminant}}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{\text{discriminant}}}{2a}$$

If the discriminant $= 0$, there is only one distinct real root and hence only one x-intercept, at

$$x = \frac{-b}{2a}$$

If the discriminant < 0, then $\sqrt{\text{discriminant}}$ is not a real number; so there are no x-intercepts and the zeros of the function are not real numbers.

EXAMPLE 3 Identify whether each of the graphs in Figure 8.27 has a discriminant > 0, < 0, or equal to 0.

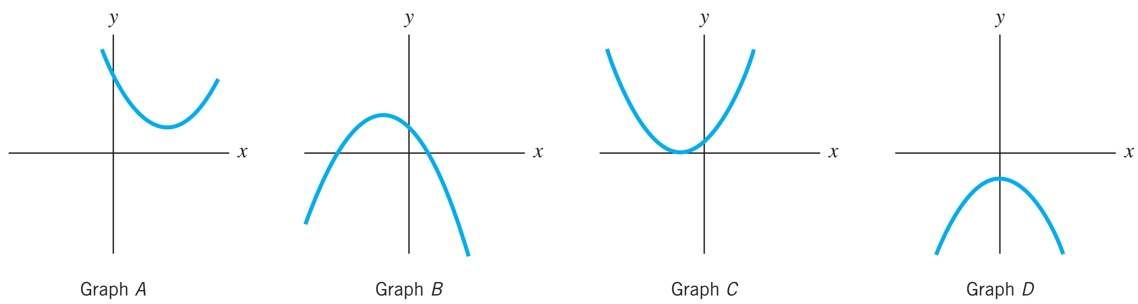


Figure 8.27 Graphs of four parabolas.

SOLUTION Graph B has two horizontal intercepts, so the discriminant is > 0. Graph C has only one horizontal intercept, so the discriminant $= 0$. Graphs A and D have no horizontal intercepts, so the discriminant is < 0.

EXAMPLE 4 For each of the following functions, use the discriminant to predict the number of horizontal intercepts. If there are any, use the quadratic formula to find them. Then using technology, graph the function to confirm your predictions.

a. $f(z) = z^2 + 3z + 2.25$

b. $h(t) = 34 + 32t - 16t^2$

c. $g(x) = -x^2 - 6x - 10$

SOLUTION

a. Setting $f(z) = 0$, we have $0 = z^2 + 3z + 2.25$. Here $a = 1$, $b = 3$, and $c = 2.25$, so the discriminant is $b^2 - 4ac = 3^2 - (4 \cdot 1 \cdot 2.25) = 9 - 9 = 0$. Since $\sqrt{0} = 0$, the quadratic formula says that there is only one root, at $-b/(2a) = -3/(2 \cdot 1) = -1.5$. So $f(z)$ has one real zero, and hence one z-intercept at $(-1.5, 0)$, which must also be the vertex of the parabola. (See Figure 8.28.)

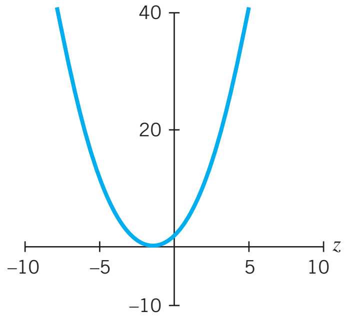


Figure 8.28 Graph of $f(z) = z^2 + 3z + 2.25$, with one horizontal intercept, at the vertex.

b. Setting $h(t) = 0$, we have $0 = 34 + 32t - 16t^2$. If we rearrange the terms as $0 = -16t^2 + 32t + 34$, it's easier to see that we should set $a = -16$, $b = 32$, and $c = 34$. The discriminant is $b^2 - 4ac = (32)^2 - (4 \cdot (-16) \cdot 34) = 1024 + 2176 = 3200$. So there are two real roots:

$$\text{one at} \quad \frac{-32 + \sqrt{3200}}{2(-16)} \approx \frac{-32 + 56.6}{-32} = \frac{24.6}{-32} \approx -0.77$$

$$\text{the other at} \quad \frac{-32 - \sqrt{3200}}{2(-16)} \approx \frac{-32 - 56.6}{-32} = \frac{-88.6}{-32} \approx 2.77$$

Therefore, the parabola for $h(t)$ has two real zeros, and hence two t-intercepts, at approximately $(-0.77, 0)$ and $(2.77, 0)$. (See Figure 8.29.)

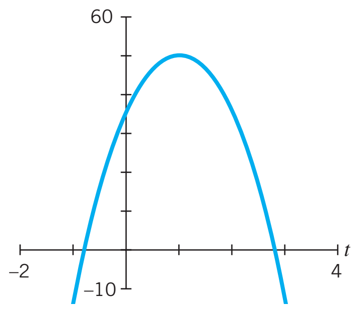


Figure 8.29 Graph of $h(t) = 34 + 32t - 16t^2$, with two horizontal intercepts.

c. Setting $g(x) = 0$, we have $0 = -x^2 - 6x - 10$. Here $a = -1$, $b = -6$, and $c = -10$, so the discriminant is $b^2 - 4ac = (-6)^2 - 4(-1)(-10) = 36 - 40 = -4$. The discriminant is negative, so taking its square root presents a problem. There is no real number r such that $r^2 = -4$. Therefore, the roots at $\frac{6 \pm \sqrt{-4}}{-2}$ are not real. Since there are no real zeros for the function, there are no horizontal intercepts. (See Figure 8.30.)

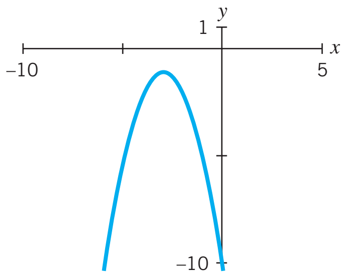


Figure 8.30 Graph of $g(x) = -x^2 - 6x - 10$, with no horizontal intercepts.

Imaginary and complex numbers

Mathematicians were uncomfortable with the notion that certain quadratic equations did not have solutions, so they literally invented a number system in which such equations would be solvable. In the process, they created new numbers, called *imaginary numbers*. The imaginary number i is defined as a number such that

$$i^2 = -1$$

or equivalently $$i = \sqrt{-1}$$

A number such as $\sqrt{-4}$ is also an imaginary number. We can write $\sqrt{-4}$ as

$$\sqrt{(4)(-1)} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1} = 2i$$

When a number is called imaginary, it sounds as if it does not exist. But imaginary numbers are just as legitimate as real numbers. Imaginary numbers are used to extend the real number system to a larger system called the complex numbers.

> **Complex Numbers**
>
> A complex number is defined as any number that can be written in the form
>
> $$z = a + bi$$
>
> where a and b are real numbers and $i = \sqrt{-1}$.
>
> The real part of z is the number a, and the imaginary part is the number b.

Note that the real numbers are a subset of the complex numbers, since any real number a can be written as $a + 0 \cdot i$.

EXAMPLE 5 Write each expression as a complex number of the form $a + bi$.

a. $-2 + 7i$ **b.** $4 + \sqrt{-9}$ **c.** $13 - \sqrt{36}$ **d.** $\sqrt{-25}$ **e.** $5 + 3i^2$

SOLUTION

a. $-2 + 7i$ is complex and already in $a + bi$ form.

b. $4 + \sqrt{-9} = 4 + \sqrt{9}\sqrt{-1} = 4 + 3i$

c. $13 - \sqrt{36} = 13 - 6 = 7 = 7 + 0 \cdot i$ (a real number)

d. $\sqrt{-25} = \sqrt{25}\sqrt{-1} = 5i = 0 + 5i$ (an imaginary number)

e. Since $i^2 = -1$, then $5 + 3i^2 = 5 + 3(-1) = 5 - 3 = 2 = 2 + 0 \cdot i$ (a real number)

The Factored Form

We started this section by finding the horizontal intercepts from the factored form. We now show that any quadratic function can be put into factored form, whether the zeros are real or complex. The Factor Theorem relates the zeros of a function to the factors of a function.

> **The Factor Theorem**
>
> Given a function $f(x)$, if $f(r) = 0$, then r is a *zero* of the function and $(x - r)$ is a factor of $f(x)$.

Using the Factor Theorem, if r_1 and r_2 are zeros of the function $f(x)$, then both $(x - r_1)$ and $(x - r_2)$ are factors of $f(x)$. So we can write a quadratic function $f(x) = ax^2 + bx + c$ in factored form as $f(x) = a(x - r_1)(x - r_2)$. Note that if you multiply out the factored form, the coefficient of x^2 is a, as it is in the standard form. The factored form is useful when we want to emphasize the zeros of a function.

> **The Factored Form**
>
> The quadratic function $f(x) = ax^2 + bx + c$ can be written in *factored form* as
>
> $$f(x) = a(x - r_1)(x - r_2)$$
>
> where r_1 and r_2 are the *zeros* of $f(x)$.
>
> If r_1 and r_2 are real numbers, then $f(x)$ has x-intercepts at r_1 and r_2.

EXAMPLE 6 Constructing a quadratic function with complex zeros

a. Construct a quadratic function that is concave up and has zeros at $5 + i$ and $5 - i$. Put it into standard form.

b. What do we know about the graph of this function? Are there any other functions with the same characteristics?

SOLUTION

a. The function $f(x) = (x - (5 + i))\,(x - (5 - i))$ has zeros at $5 + i$ and $5 - i$. If we multiply it out (using the FOIL technique), we get

$$(x - (5 + i))(x - (5 - i)) = x^2 - (5 - i)x - (5 + i)x + (5 + i)(5 - i)$$

use the distributive law and FOIL again	$= x^2 - 5x + ix - 5x - ix + (25 - 5i + 5i - i^2)$
simplify	$= x^2 - 10x + (25 - i^2)$
and substitute -1 for i^2	$= x^2 - 10x + 26$

So we can rewrite $f(x)$ in standard form as $f(x) = x^2 - 10x + 26$. (You can double-check this by using the quadratic formula.) Since the coefficient of x^2 is 1, and hence positive, the graph is concave up.

b. Since the zeros are not real, the graph of the function has no x-intercepts.

Any function of the form $af(x) = a(x^2 - 10x + 26)$, where $a > 0$, will be concave up and have zeros at $5 + i$ and $5 - i$. Since there are an infinite number of values of a, there are an infinite number of functions with these characteristics.

EXAMPLE 7 A parabola has horizontal intercepts at $d = -1$ and $d = 2$, and passes through the point $(d, h) = (1.5, 1.25)$.

a. Find the equation for the parabola.

b. Graph the parabola using a dotted line. Now restrict the domain from $d = 0$ to the horizontal intercept that is positive and then color in the corresponding section of the parabola with a solid line. This section of the parabola, drawn with a solid line, describes the path of a water jet located at the center of a circular fountain, where d = distance (in feet) from the base of the fountain, and h = height (in feet) of the water.

c. At what height is the nozzle of the water jet?

d. What is the greatest height the stream of water reaches?

SOLUTION **a.** If the parabola has horizontal intercepts at -1 and 2, then the equation for the parabola is in the form $h = a(d - (-1))(d - 2)$ or $a(d + 1)(d - 2)$. The point (1.5, 1.25) lies on the parabola, so it must satisfy the equation. Hence

given that	$h = a(d + 1)(d - 2)$
if we substitute 1.5 for d and 1.25 for h	$1.25 = a\,(1.5 + 1)(1.5 - 2)$
simplify	$1.25 = a(2.5)(-0.5)$
multiply	$1.25 = a(-1.25)$
and divide by -1.25, we have	$-1 = a$

So the equation of the parabola is $h = -(d + 1)(d - 2)$.

b. See Figure 8.31.

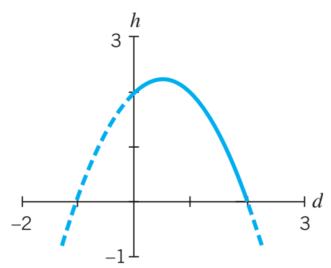


Figure 8.31 Graph of $h = -(d + 1)(d - 2)$.

c. When $d = 0$, $h = -(0 + 1)(0 - 2) = 2$. So the water nozzle is at 2 feet.

d. There are several strategies for finding the vertex. You could calculate it using the formula in Section 8.2. Or recall that the vertex lies on the axis of symmetry, halfway between the two horizontal intercepts (-1 and 2) at $d = \frac{(-1 + 2)}{2} = \frac{1}{2}$ or 0.5. Substituting 0.5 for d in the equation in part (a), we get

$$\begin{aligned} h &= -(0.5 + 1)\,(0.5 - 2) \\ &= -(1.5)(-1.5) \\ &= 2.25 \end{aligned}$$

The coordinates of the vertex are (0.5, 2.25). So the maximum height of the water (at the vertex of the parabola) is 2.25 feet (or 27″), which occurs at 0.5 feet (or 6″) from the base of the fountain.

The Horizontal Intercepts of a Quadratic Function

The graph of a quadratic function may have zero, one, or two horizontal intercepts, which can be found by factoring or using the quadratic formula.

The discriminant in the quadratic formula can be used to predict the number of horizontal intercepts.

Any quadratic function in standard form $f(x) = ax^2 + bx + c$ can be written in *factored form* as

$$f(x) = a(x - r_1)(x - r_2) \quad \text{where } r_1 \text{ and } r_2 \text{ are the zeros of } f(x)$$

If the zeros r_1 and r_2 are real numbers, they are the horizontal intercept(s) of $f(x)$.

Algebra Aerobics 8.3b

1. Find any real numbers that satisfy the following equations.

a. $4x + 7 = 0$
b. $4x^2 - 7 = 0$
c. $4x^2 - 7x = 0$
d. $2(x + 3) = x^2$
e. $(2x - 11)^2 = 0$
f. $(x + 1)^2 = 81$
g. $x = x^2 - 5$

2. Evaluate the discriminant $b^2 - 4ac$ for each of the following quadratic functions of the form $f(x) = ax^2 + bx + c$. Use the discriminant to determine the nature of the zeros of the function and the number (if any) of horizontal intercepts.

a. $f(x) = 2x^2 + 3x - 1$
b. $f(x) = x^2 + 7x + 2$
c. $f(x) = 4x^2 + 4x + 1$
d. $f(x) = 2x^2 + x + 5$

3. Find and interpret the horizontal and vertical intercepts for the following height equations.

a. $h = -4.9t^2 + 50t + 80$ (h is in meters and t is in seconds)
b. $h = 150 - 80t - 490t^2$ (h is in centimeters and t is in seconds)
c. $h = -16t^2 + 64t + 3$ (h is in feet and t is in seconds)
d. $h = 64t - 16t^2$ (h is in feet and t is in seconds)

4. Evaluate the discriminant and then predict the number of x-intercepts for each function. Use the quadratic formula to find all the zeros of each function and then identify the coordinates of any x-intercept(s).

a. $y = 4 - x - 5x^2$
b. $y = 4x^2 - 28x + 49$
c. $y = 2x^2 + 5x + 4$
d. $y = 2x^2 - 3x - 1$
e. $y = 2 - 3x^2$

5. From the descriptions given in parts (a) and (b), determine the coordinates of the vertex, find the equation of the parabola, and then sketch the parabola.

a. A parabola with horizontal intercepts at $x = -2$ and $x = 4$ that passes through the point (3, 2).
b. A parabola with horizontal intercepts at $x = 2$ and $x = 8$, and a vertical intercept at $y = 10$.

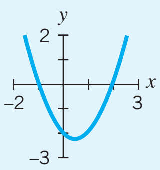
6. Identify the number of x-intercepts for the following functions without converting into standard form.

a. $y = 3(x - 1)^2 + 5$
b. $y = -2(x + 4)^2 - 1$
c. $y = -5(x + 3)^2$
d. $y = 3(x - 1)^2 - 2$

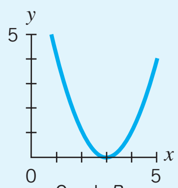
7. Write an equation for a quadratic function, in factored form, with the specified zeros.

a. 2 and -3 **b.** 0 and -5 **c.** 8

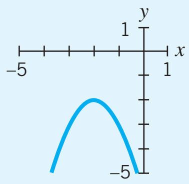
8. For each quadratic function in the accompanying graphs, specify the number of real zeros and whether the corresponding discriminant would be positive, negative, or zero.



Graph *A*



Graph *B*



Graph *C*

Exercises for Section 8.3

Several exercises either require or recommend the use of a graphing program.

1. Solve the following quadratic equations by factoring.

a. $x^2 - 9 = 0$
b. $x^2 - 4x = 0$
c. $3x^2 = 25x$
d. $x^2 + x = 20$
e. $4x^2 + 9 = 12x$
f. $3x^2 = 13x + 10$
g. $(x + 1)(x + 3) = -1$
h. $x(x + 2) = 3x(x - 1) - 3$

2. Find the x-intercepts for each of the following functions. Will the vertex lie above, below, or on the x-axis? Find the vertex and sketch the graph, labeling the x-intercepts.

a. $y = (x + 2)(x + 1)$
b. $y = 3(1 - 2x)(x + 3)$
c. $y = -4(x + 3)^2$
d. $y = \frac{1}{2}(x)(x - 5)$
e. $q(x) = 2(x - 3)(x + 2)$
f. $f(x) = -2(5 - x)(3 - 2x)$

3. Factor the quadratic expression and then sketch the graph of the function, labeling the axes and horizontal intercepts.

a. $y = x^2 + 6x + 8$ **d.** $w = t^2 - 25$
b. $z = 3x^2 - 6x - 9$ **e.** $r = 4s^2 - 100$
c. $f(x) = x^2 - 3x - 10$ **f.** $g(x) = 3x^2 - x - 4$

4. a. Construct a quadratic function with zeros at $x = 1$ and $x = 2$.
b. Is there more than one possible quadratic function for part (a)? Why or why not?

5. (Graphing program required.) Using a graphing program, estimate the real solutions to the following equations. (*Hint:* Think of the equations as resulting from setting $f(x) = 0$.) Verify by factoring, if possible.

a. $x^2 - 5x + 6 = 0$ **d.** $-3x^2 - 12x + 15 = 0$
b. $3x^2 - 2x + 5 = 0$ **e.** $0.05x^2 + 1.1x = 0$
c. $3x^2 - 12x + 12 = 0$ **f.** $-2x^2 - x + 3 = 0$

6. Write each function in factored form, if possible, using integer coefficients.

a. $f(x) = x^2 + 2x - 15$ **d.** $k(p) = p^2 + 5p + 7$
b. $g(x) = x^2 - 6x + 9$ **e.** $l(s) = 5s^2 - 37s - 24$
c. $h(a) = a^2 + 6a - 16$ **f.** $m(t) = 5t^2 + t + 1$

7. Solve the following equations using the quadratic formula. (*Hint:* Rewrite each equation so that one side of the equation is zero.)

a. $6t^2 - 7t = 5$ **e.** $6s^2 - 10 = -17s$
b. $3x(3x - 4) = -4$ **f.** $2t^2 = 3t + 9$
c. $(z + 1)(3z - 2) = 2z + 7$ **g.** $5 = (4x + 1)(x - 3)$
d. $(x + 2)(x + 4) = 1$ **h.** $(2x - 3)^2 = 7$

8. Solve using the quadratic formula.

a. $x^2 - 3x = 12$ **e.** $\dfrac{1}{x - 2} = \dfrac{x + 1}{x - 1}$
b. $3x^2 = 4x + 2$ **f.** $\dfrac{x^2}{3} + \dfrac{x}{2} - \dfrac{1}{6} = 0$
c. $3(x^2 + 1) = x + 2$ **g.** $\dfrac{1}{x^2} - \dfrac{3}{x} = \dfrac{1}{6}$
d. $(3x - 1)(x + 2) = 4$

9. Calculate the coordinates of the x- and y-intercepts for the following quadratics.

a. $y = 3x^2 + 2x - 1$ **c.** $y = (5 - 2x)(3 + 5x)$
b. $y = 3(x - 2)^2 - 1$ **d.** $f(x) = x^2 - 5$

10. Use the discriminant to predict the number of horizontal intercepts for each function. Then use the quadratic formula to find all the zeros. Identify the coordinates of any horizontal or vertical intercepts.

a. $y = 2x^2 + 3x - 5$
b. $f(x) = -16 + 8x - x^2$
c. $f(x) = x^2 + 2x + 2$
d. $y = 2(x - 1)^2 + 1$
e. $g(z) = 5 - 3z - z^2$
f. $f(t) = (t + 2)(t - 4) + 9$

11. In each part (a) to (e), graph a parabola with the given characteristics. Then write an equation of the form $y = ax^2 + bx + c$ for that parabola.

a. $a > 0,\ b^2 - 4ac > 0, c > 0$
b. $a > 0,\ b^2 - 4ac > 0, c < 0$
c. $a > 0,\ b^2 - 4ac < 0, b \neq 0$
d. $a < 0,\ b^2 - 4ac = 0, c \neq 0$
e. $b \neq 0,\ c < 0, b^2 - 4ac > 0$

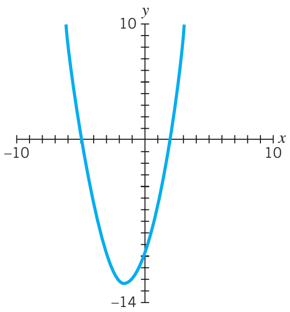
12. For each part, draw a rough sketch of a graph of a function of the type $f(x) = ax^2 + bx + c$

a. Where $a > 0$, $c > 0$, and the function has no real zeros.
b. Where $a < 0$, $c > 0$, and the function has two real zeros.
c. Where $a > 0$ and the function has one real zero.

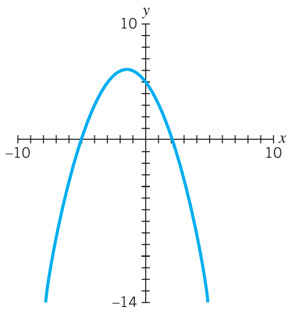
13. a. Construct a quadratic function $Q(t)$ with exactly one zero at $t = -1$ and a vertical intercept at -4.
b. Is there more than one possible quadratic function for part (a)? Why or why not?
c. Determine the axis of symmetry. Describe the vertex.

14. a. Construct a quadratic function $P(s)$ that goes through the point $(5, -22)$ and has two real zeros, one at $s = -6$ and the other at $s = 4$.
b. What is the axis of symmetry?
c. What are the coordinates of the vertex?
d. What is the vertical intercept?

15. Construct a quadratic function for each of the given graphs. Write the function in both factored form and standard form.



Graph *A*



Graph *B*

16. Complete the following table, and then summarize your findings.

$i^1 = \sqrt{-1} = i$ $\quad i^5 = ?$
$i^2 = i \cdot i = -1$ $\quad i^6 = ?$
$i^3 = i \cdot i^2 = ?$ $\quad i^7 = ?$
$i^4 = i^2 \cdot i^2 = ?$ $\quad i^8 = ?$

17. Complex number expressions can be simplified by combining the real parts and then the imaginary parts. Add (or subtract) the following complex numbers and then simplify.

a. $(4 + 3i) + (-5 + 7i)$ **c.** $(7i - 3) + (2 - 4i)$
b. $(-2 + 3i) - (-3 + 3i)$ **d.** $(7i - 3) - (2 - 4i)$

18. Complex number expressions can be multiplied using the distributive property or the FOIL technique. Multiply and simplify the following. (*Note:* $i^2 = -1$.)

a. $(3 + 2i)(-2 + 3i)$
b. $(4 - 2i)(3 + i)$
c. $(2 + i)(2 - i)$
d. $(5 - 3i)(5 + 3i)$
e. $(3 - i)^2$
f. $(4 + 5i)^2$

19. A quadratic function has two complex roots, $r_1 = 1 + i$ and $r_2 = 1 - i$. Use the Factor Theorem to find the equation of this quadratic, assuming $a = 1$, and then put it into standard form.

20. The factored form of a quadratic function is $y = -2(x - (3 + i))(x - (3 - i))$. Answer the following.

a. Will the graph open up or down? Explain.
b. What are the zeros of the quadratic function?
c. Does the graph cross the x-axis? Explain.
d. Write the quadratic in standard form. (*Hint:* Multiply out; see Exercise 18.)
e. Verify your answer in part (b) by using the quadratic formula and your answer for part (d).

21. Use the quadratic formula to find the zeros of the function $f(x) = x^2 - 4x + 13$ and then write the function in factored form. Without graphing this function, how can you tell if it intersects the x-axis?

22. Let (h, k) be the coordinates of the vertex of a parabola. Then h is equal to the average of the two real zeros of the function (if they exist). For parts (a) and (b) use this to find h, and then construct an equation in vertex form, $y = a(x - h)^2 + k$.

a. A parabola with x-intercepts of 4 and 8, and a y-intercept of 32
b. A parabola with x-intercepts of -3 and 1, and a y-intercept of -1
c. Can you find the equation of a parabola knowing only its x-intercepts? Explain.

23. Let (h, k) be the coordinates of the vertex of a parabola. Then h equals the average of the two real zeros of the function (if they exist). For each of the following use this to find h, and then put the equations into the vertex form, $y = a(x - h)^2 + k$.

a. A parabola with equation $y = x^2 + 2x - 8$
b. A parabola with equation $y = -x^2 - 3x + 4$

24. (Graphing program optional)

a. Write each of the following functions in both the a-b-c and the a-h-k forms. Is one form easier than the other for finding the vertex? The x- and y-intercepts?

$y_1 = 2x^2 - 3x - 20$ $\qquad$ $y_3 = 3x^2 + 6x + 3$
$y_2 = -2(x - 1)^2 - 3$ $\qquad$ $y_4 = -(2x + 4)(x - 3)$

b. Find the vertex and x- and y-intercepts and construct a graph by hand for each function in part (a). If you have access to a graphing program, check your work.

25. Find the equation of the graph of a parabola that has the following properties:

- The x-intercepts of the graph are at (2, 0) and (3, 0), *and*
- The parabola is the graph of $y = x^2$ vertically stretched by a factor of 4.

Explain your reasoning. Sketch the parabola.

26. (Graphing program required for part (b)). We dealt previously with systems of lines and ways to determine the coordinates of points where lines intersect. Once you know the quadratic formula, it's possible to determine where a line and a parabola, or two parabolas, intersect. As with two straight lines, at the point where the graphs of two functions intersect (*if* they intersect), the functions share the same x value and the same y value.

a. Find the intersection of the parabola $y = 2x^2 - 3x + 5.1$ and the line $y = -4.3x + 10$.
b. Plot both functions, labeling any intersection point(s).

27. (Graphing program recommended for part (b)).

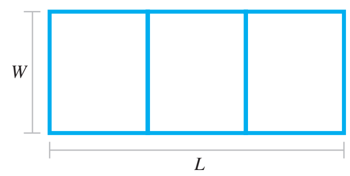
a. Find the intersection of the two parabolas $y = 7x^2 - 5x - 9$ and $y = -2x^2 + 4x + 9$.
b. Plot both functions, labeling any intersection points.

28. Market research suggests that if a particular item is priced at x dollars, then the weekly profit $P(x)$, in thousands of dollars, is given by the function

$$P(x) = -9 + \tfrac{11}{2}x - \tfrac{1}{2}x^2$$

a. What price range would yield a profit for this item?
b. Describe what happens to the profit as the price increases. Why is a quadratic function an appropriate model for profit as a function of price?
c. What price would yield a maximum profit?

29. A dairy farmer has 1500 feet of fencing. He wants to use all 1500 feet to construct a rectangle and two interior separators that together form three rectangular pens. See the accompanying figure.



a. If W is the width of the larger rectangle, express the length, L, of the larger rectangle in terms of W.
b. Express the total area, $A(W)$, of the three pens as a polynomial in terms of W.
c. What is the domain of the function $A(W)$?
d. What are the dimensions of the larger rectangle that give a maximum area? What is the maximum area?

8.4 The Average Rate of Change of a Quadratic Function

In previous chapters we argued that the average rate of change of a linear function is constant, and that the average rate of change of an exponential function is exponential. What about the average rate of change of a quadratic function?

EXAMPLE 1 **Finding the average rate of change of the simplest quadratic function**

Given $y = x^2$, calculate the average rate of change of y with respect to x at unit intervals from -3 to 3. Then calculate the average rate of change of the average rate of change. What do these data points suggest?

SOLUTION Column 3 in Table 8.2 shows the values for the average rate of change of y with respect to x. For these values the average rate of change is linear, since adding 1 to x increases the average rate of change by 2 over each interval. Column 4 computes the average rate of change of column 3 with respect to x. Since these values all are constant at 2, this confirms that column 3 (the average rate of change) is linear with respect to x. This suggests that when $y = x^2$ the average rate of change of y with respect to x is a linear function.

x	$y = x^2$	Average Rate of Change	Average Rate of Change of the Average Rate of Change
-3	9	n.a.	n.a.
-2	4	$\frac{4-9}{-2-(-3)} = -5$	n.a.
-1	1	$\frac{(1-4)}{1} = -3$	$\frac{-3-(-5)}{-1-(-2)} = 2$
0	0	$\frac{(0-1)}{1} = -1$	$\frac{(-1-(-3))}{1} = 2$
1	1	$\frac{(1-0)}{1} = 1$	$\frac{(1-(-1))}{1} = 2$
2	4	$\frac{(4-1)}{1} = 3$	$\frac{(3-1)}{1} = 2$
3	9	$\frac{(9-4)}{1} = 5$	$\frac{(5-3)}{1} = 2$

Table 8.2

EXAMPLE 2 **Finding the average rate of change of a quadratic function**

Given $y = 3x^2 - 8x - 23$, calculate the average rate of change of y with respect to x at unit intervals from -3 to 3. Then calculate the average rate of change of the average rate of change. What do the data suggest about the original function?

SOLUTION Column 3 in Table 8.3 shows the values for the average rate of change. Again the values suggest that the average rate of change is linear, since adding 1 to x increases the average rate of change by 6 over each interval. This is confirmed by column 4, which shows that the average rate of change of the third column with respect to x is constant at 6. This suggests that when $y = 3x^2 - 8x - 23$, the average rate of change of y with respect to x is a linear function.

x	$y = 3x^2 - 8x - 23$	Average Rate of Change	Average Rate of Change of the Average Rate of Change
−3	28	n.a.	n.a.
−2	5	−23	n.a.
−1	−12	−17	6
0	−23	−11	6
1	−28	−5	6
2	−27	1	6
3	−20	7	6

Table 8.3

We have just seen two numerical examples that suggest that the average rate of change of a quadratic function is a linear function. We now show algebraically that this is true for *every* quadratic function.

Suppose we have $y = f(x) = ax^2 + bx + c$. In the previous examples we fixed an interval size of 1 over which to calculate the average rate of change, since it is easy to make comparisons. Now we pick a constant interval size r and for each position x compute the average rate of change of f over the interval from x to $x + r$.

First we must compute $f(x + r)$:

$$\begin{aligned} f(x+r) &= a(x+r)^2 + b(x+r) + c \\ \text{apply exponent} \qquad &= a(x^2 + 2rx + r^2) + b(x+r) + c \\ \text{multiply through} \qquad &= ax^2 + 2arx + ar^2 + bx + br + c \\ \text{regroup terms} \qquad &= ax^2 + bx + c + (2ax + b)r + ar^2 \end{aligned}$$

Then the average rate of change of $f(x)$ between x and $x + r$ is

$$\begin{aligned} \frac{\text{change in } f(x)}{\text{change in } x} &= \frac{f(x+r) - f(x)}{(x+r) - x} \\ &= \frac{[ax^2 + bx + c + (2ax + b)r + ar^2] - (ax^2 + bx + c)}{r} \\ &= \frac{(2ax + b)r + ar^2}{r} \\ &= 2ax + (b + ar) \end{aligned}$$

The average rate of change between $(x, f(x))$ and $(x + r, f(x + r))$ is a linear function of x in its own right, where

$$\text{slope} = 2a \qquad y\text{-intercept} = b + ar$$

Note that the slope depends only on the value of a in the original equation. The y-intercept depends not only on a and b in the original equation, but also on r, the interval size over which we calculate the average rate of change. (See Figure 8.32.)

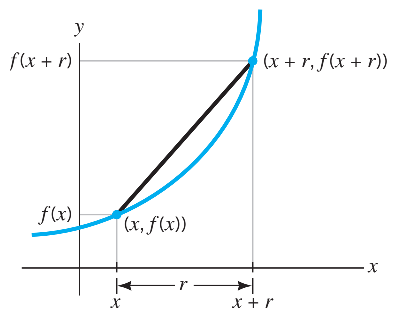


Figure 8.32 The slope of the line segment connecting two points on the parabola separated by a horizontal distance of r is $2ax + (b + ar)$.

If we took smaller and smaller values for r, then the term ar would get closer and closer to zero, and hence the y-intercept $b + ar$ would get closer and closer to b. For very small r's the average rate of change would get closer and closer to the linear expression $2ax + b$.

So given a quadratic function $f(x) = ax^2 + bx + c$, the average rate of change over small intervals around any x value can be approximated by the linear expression $2ax + b$. So we talk about the function $g(x) = 2ax + b$ as representing the average rate of change of $f(x)$ with respect to x. This is a central idea in calculus.

The Average Rate of Change of a Quadratic Function

Given a quadratic function $f(x) = ax^2 + bx + c$, we can think of the linear function $g(x) = 2ax + b$ as representing the average rate of change of $f(x)$ with respect to x.

EXAMPLE 3 Graph each of the following functions. Construct and graph the equation for the average rate of change in each case.

a. $f(x) = x^2$ **b.** $f(x) = 3x^2 - 8x - 23$

SOLUTION **a.** The average rate of change of the quadratic function $f(x) = x^2$ is the linear function $g(x) = 2x$ (see Figure 8.33).

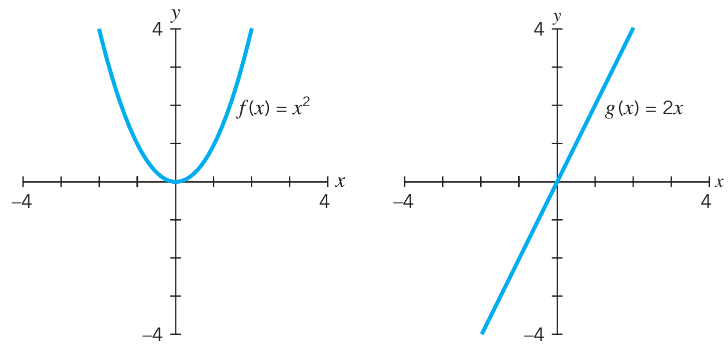


Figure 8.33 Graph of $f(x) = x^2$ and graph of its average rate of change.

b. The average rate of change of the quadratic function $f(x) = 3x^2 - 8x - 23$ is the linear function $g(x) = 6x - 8$ (see Figure 8.34).

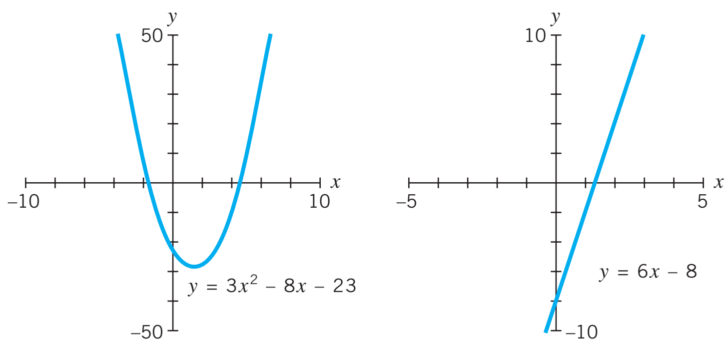


Figure 8.34 Graph of $y = 3x^2 - 8x - 23$ and graph of its average rate of change.

Algebra Aerobics 8.4

1. a. Complete the table below for the function $y = 5 - x^2$.

x	y	Average Rate of Change	Average Rate of Change of Average Rate of Change
−1	4	n.a.	n.a.
0	5	1	n.a.
1	4	−1	−2
2	1		
3			
4			

b. What do the third and fourth columns tell you?

2. Determine from each of the three following tables whether you would expect the original function $y = f(x)$ to be linear, exponential, or quadratic.

a.

x	y	Average Rate of Change	Average Rate of Change of Average Rate of Change
−1	2	n.a.	n.a.
0	1	−1	n.a.
1	6	5	6
2	17	11	6
3	34	17	6
4	57	23	6

b.

x	y	Average Rate of Change	Average Rate of Change of Average Rate of Change
−1	7	n.a.	n.a.
0	4	−3	n.a.
1	1	−3	0
2	−2	−3	0
3	−5	−3	0
4	−8	−3	0

c.

x	y	Average Rate of Change	Average Rate of Change of Average Rate of Change
−1	0.5	n.a.	n.a.
0	1	0.5	n.a.
1	2	1	0.5
2	4	2	1
3	8	4	2
4	16	8	4

3. Construct a function that represents the average rate of change for the following three quadratic functions.

a. $f(t) = t^2 + t$

b. $f(x) = 3x^2 + 5x$

c. $f(x) = 5x^2 + 2x + 7$

Exercises for Section 8.4

Exercise 6 requires a graphing program.

1. Complete these sentences.

a. For a quadratic function, the graph of its average rate of change represents a ________________ function.

b. When a quadratic function is increasing, its average rate of change is ____________, and when a quadratic function is decreasing, its average rate of change is ______________.

2. Complete the table for the function $y = 3 - x - x^2$.

x	y	Average Rate of Change	Average Rate of Change of Average Rate of Change
−3	−3	n.a.	n.a.
−2	1	$\frac{1-(-3)}{-2-(-3)} = \frac{4}{1} = 4$	n.a.
−1	3	$\frac{3-1}{-1-1(-2)} = \frac{2}{1} = 2$	$\frac{2-4}{-1-(-2)} = \frac{-2}{1} = -2$
0	3	$\frac{3-3}{0-(-1)} = \frac{0}{1} = 0$	$\frac{0-2}{0-(-1)} =$
1	1		
2			
3			

What type of function does the average rate of change represent? What is its slope (the rate of change of the average rate of change)?

3. a. What is the average rate of change of the linear function $y = 3x - 1$? Of $y = -2x + 5$? Of $y = ax + b$? What equation could represent the average rate of change of the general linear function $y = ax + b$? Describe its graph.

b. We have seen that the average rate of change of a quadratic function $y = ax^2 + bx + c$ can be represented by a linear function of the form $y = 2ax + b$ (where $a \neq 0$). Describe the linear graph. Could such a graph ever represent the average rate of change of a linear function?

c. What would you guess the equation representing the average rate of change of the cubic function $y = ax^3 + bx^2 + cx + d$ to be?

4. a. Complete the table for the function $f(x) = 3x^2 - 2x - 5$.

x	y	Average Rate of Change	Average Rate of Change of Average Rate of Change
-3	28	n.a.	n.a.
-2	11	$\frac{11 - 28}{(-2) - (-3)} = -17$	n.a.
-1	0	$\frac{0 - 11}{(-1) - (-2)} = -11$	$\frac{(-11) - (-17)}{(-1) - (-2)} = 6$
0	-5		
1			
2			
3			

b. What is the slope of the linear function that represents the average rate of change? How is this slope related to the average rate of change of the average rate of change?

c. What type of function represents the average rate of change of $f(x)$?

5. Complete the table for the function $Q = 2t^2 + t + 1$.

a. Plot $Q = 2t^2 + t + 1$. What type of function is this?

b. What does the third column tell you about the function that represents the average rate of change for Q?

t	Q	Average Rate of Change
-3		n.a.
-2		
-1		
0		
1		
2		
3		

6. a. (Graphing program required.) Plot the function $h(t) = 4 + 50t - 16t^2$ for the restricted domain $0 \le t \le 3$.

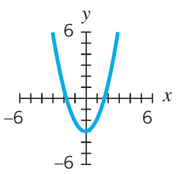
b. For what interval is this function increasing? Decreasing?

c. Estimate the maximum point.

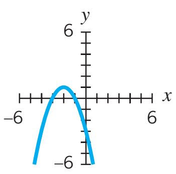
d. Construct and graph a function $g(t)$ that represents the average rate of change of $h(t)$.

e. What does $g(t)$ tell you about the function $h(t)$?

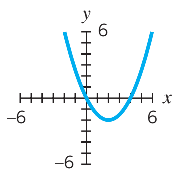
7. Match the quadratic function with the graph of its average rate of change.



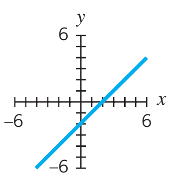
Graph *A*



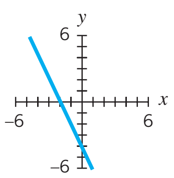
Graph *B*



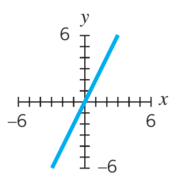
Graph *C*



Graph *D*



Graph *E*



Graph *F*

8. Having found the matched pairs of graphs in Exercise 7, explain the relationship between the horizontal intercept of the linear function and the vertex of the quadratic function.

9. Construct a function that represents the average rate of change for each given function.

a. $f(t) = 3t^2 + t$

b. $g(x) = -5x^2 + 0.4x + 3$

c. $h(z) = 2z + 7$

10. Construct a function that represents the average rate of change for each given function.

a. $r(t) = -3t^2 - 7$

b. $s(v) = \frac{1}{4}v^2 - \frac{3}{2}v + 5$

c. $w(x) = -5x + 0.25$

d. $m(p) = 4p - \frac{1}{2}$

11. Determine from each of the tables whether you would expect the original function $y = f(x)$ to be linear, exponential, or quadratic.

a.

x	y	Average Rate of Change	Average Rate of Change of Average Rate of Change
-1	-2	n.a.	n.a.
0	0	2	n.a.
1	8	8	6
2	22	14	6
3	42	20	6
4	68	26	6

b.

x	y	Average Rate of Change	Average Rate of Change of Average Rate of Change
-1	9	n.a.	n.a.
0	5	-4	n.a.
1	1	-4	0
2	-3	-4	0
3	-7	-4	0
4	-11	-4	0

c.

x	y	Average Rate of Change	Average Rate of Change of Average Rate of Change
-1	0.25	n.a.	n.a.
0	1	0.75	n.a.
1	4	3	2.25
2	16	12	9
3	64	48	36
4	256	192	144

8.5 *An Introduction to Polynomial Functions*

Defining a Polynomial Function

Linear, power, and quadratic functions are all part of a larger family of functions called *polynomials*. We can think of a polynomial as a sum of power functions with positive integer powers. For example, if we sum the power functions

$$y_1 = 2x^4, \quad y_2 = x^3, \quad y_3 = -2x, \quad \text{and} \quad y_4 = 11$$

to get $\quad y = y_1 + y_2 + y_3 + y_4$

we obtain $\quad y = 2x^4 + x^3 - 2x + 11$

This function is called a polynomial of degree 4, since the highest power of the independent variable (x) is 4.

Each of the separate power functions is called a *term* of the polynomial. In standard form we arrange the terms from the highest power down to the lowest.

Definition of a Polynomial Function

A *polynomial function of degree n* is of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$.

The constants $a_n, a_{n-1}, \ldots, a_0$ are called the *coefficients*.
$a_n x^n$ is called the *leading term*.
a_0 is called the *constant term*.

Polynomials of certain degrees have special names.

Polynomials of Degree	Are Called	Example
0	Constant	$y = 3$
1	Linear	$y = -4x - 8$
2	Quadratic	$y = 3x^2 + 5x - 10$
3	Cubic	$y = 5x^3 - 4x - 14$
4	Quartic	$y = -2x^4 - x^3 + 4x^2 + 4x - 14$
5	Quintic	$y = 8x^5 - 3x^4 + 4x^2 - 4$

Notice that the example for the cubic function, $y = 5x^3 - 4x - 14$, does not include an x^2 term. In this case the coefficient for the x^2 term is zero, since this function could be rewritten as $y = 5x^3 + 0x^2 - 4x - 14$.

Early algebraists believed that higher-degree polynomials were not relevant to the physical world and hence were useless. "Going beyond the cube just as if there were more than three dimensions . . . is against nature."[3] But as we shall see shortly, there are real applications for polynomials with degrees greater than 3.

[3]Stifel, as cited by M. Kline, *Mathematical Thought from Ancient to Modern Times* (Oxford: Oxford University Press, 1972).

EXAMPLE 1 **Identifying function types**

Specify the type(s) of each function (linear, exponential, power, quadratic, and/or polynomial). If it's polynomial, identify the degree and any special name.

a. $y = 3x + 5$ **c.** $Q(t) = 5 \cdot 2^t$ **e.** $g(z) = 2(z^2 - 1)(z^2 + 3)$

b. $f(x) = 4x^2$ **d.** $v = 1 - 13w - 5w^2 + 2w^5$

SOLUTION All of the functions are polynomials, except for (c).

a. This function is a polynomial of degree 1, so it's a linear function.

b. This function is a polynomial of degree 2, so it's quadratic. It's also a power function.

c. This is an exponential function.

d. This function is a polynomial of degree 5, called a quintic.

e. Multiplying out, we get $g(z) = 2z^4 + 4z^2 - 6$. So this function is a polynomial of degree 4, called a quartic.

EXAMPLE 2 **Curing poison ivy**

A patient with an acute poison ivy rash is given a 5-day "prednisone taper." The daily dosage of the drug prednisone is respectively 20 mg, 15 mg, 10 mg, 5 mg, and 5 mg. The daily decay rate r (in decimal form) is the rate at which the drug is naturally absorbed and removed from the body. The rate varies from patient to patient. The decay factor, $x = 1 - r$, gives the percentage (in decimal form) of the drug left in the patient's body after each day.

a. If one patient has a prednisone decay rate of 0.35 (or 35%) per day, what is the patient's decay factor? If another has a decay rate of 50%, what would this patient's decay factor be?

b. Construct functions to describe a patient's prednisone level as a function of x, the decay factor, for each of the 5 days of treatment.

c. Use technology to graph the prednisone level on day 5. What type of function is this, and what is its domain in this context?

d. If a patient's daily decay factor is 40%, use the graph to estimate the total dosage of prednisone remaining on day 5. Then use the equation to calculate a more exact answer.

e. Construct a function to describe the patient's prednisone level after 6 days, after 7 days, and after n days. How do the functions for which $n > 5$ differ from those with $n \leq 5$?

SOLUTION

a. A decay rate of 0.35 corresponds to a decay factor of 0.65. A decay rate of 0.50 (or 50%) corresponds to a decay factor of 0.50.

b. On day 1, the patient would receive 20 mg of prednisone. Then on day 2 the patient would have $20x$ remaining of the original 20 mg of prednisone, plus the additional 15 mg he took that day. On day 3 he'd have $20x^2$ left of his 20-mg dosage from day 1, plus $15x$ left from his 15-mg dosage from day 2, plus the 10 mg he took that day. And so on. If we let $\text{Day}_i(x)$ = total amount of prednisone in the body for day i, then

$$\text{Day}_1(x) = 20 \qquad \text{Day}_4(x) = 20x^3 + 15x^2 + 10x + 5$$
$$\text{Day}_2(x) = 20x + 15 \qquad \text{Day}_5(x) = 20x^4 + 15x^3 + 10x^2 + 5x + 5$$
$$\text{Day}_3(x) = 20x^2 + 15x + 10$$

c. The graph of $\text{Day}_5(x)$ is given in Figure 8.35. The function is a polynomial of degree 4. Since its domain is restricted to decimal values of decay rates, $0 < x < 1$.

d. Estimating from the graph, a patient with a decay factor of 0.40 has about 10 mg of prednisone left in the body on day 5. Evaluating with a calculator, the same patient on day 5 has

$$\text{Day}_5(0.40) = 20(0.40)^4 + 15(0.40)^3 + 10(0.40)^2 + 5(0.40) + 5 \approx 10.07 \text{ mg.}$$

e. When $n > 5$ there are no more added dosages. Thus $\text{Day}_6(x) = \text{Day}_5(x) \cdot x$, and in general $\text{Day}_n(x) = \text{Day}_{n-1}(x) \cdot x$, if $n > 5$. So for $n \geq 5$,

$$\text{Day}_n(x) = 20x^{n-1} + 15x^{n-2} + 10x^{n-3} + 5x^{n-4} + 5x^{n-5}.$$

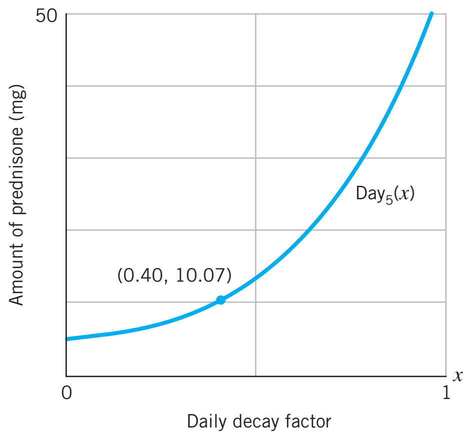


Figure 8.35 Amount of prednisone in the body on day 5 for various decay factors.

Algebra Aerobics 8.5a

1. For each of the following polynomials, specify the degree and evaluate each function at $x = -1$.
   - **a.** $f(x) = 4x^3 + 11x^5 - 11$
   - **b.** $y = 1 + 7x^4 - 5x^3$
   - **c.** $g(x) = -2x^4 - 20$
   - **d.** $z = 3x - 4 - 2x^2$
2. Find the degree of the polynomial function without multiplying out.
   - **a.** $f(x) = x^5(3x - 2)^3(5x^2 + 1)$
   - **b.** $g(x) = (x^2 - 1)(x^3 + x - 5)^4$
3. Given the polynomial function
   $f(t) = 0.5 - 2t^5 + 4t^3 - 6t^2 - t$:
   - **a.** What is the degree of the polynomial? What is the name for polynomials of this degree?
   - **b.** What is the leading term?
   - **c.** What is the constant term?
   - **d.** What is $f(0)$? $f(0.5)$? $f(-1)$?

Visualizing Polynomial Functions

What can we predict about the graph of a polynomial function from its equation? Examine the graphs of polynomials of different degrees in Figure 8.36. What can we observe from each of these pairs?

Turning points

The first thing we might notice is the number of turning points on each graph—the number of times the graph bends and changes direction. The quadratics bend once, the cubics seem to bend twice, the quartics three times, one quintic seems to bend four times, and the other quintic appears to bend twice. In general, a polynomial function of degree n will have at most $n - 1$ turning points.

Horizontal intercepts

Second, we might notice the number of times each graph crosses the horizontal axis. Each quadratic crosses at most two times, the cubics each cross at most three times, the quartics cross at most four times, and the quintics cross at most five times. In general, a polynomial function of degree n will cross the horizontal axis at most n times.

Global behavior

Finally, imagine zooming out on a graph to look at it on a global scale. All the polynomials in Figure 8.36 were graphed using small values for x (at most between -10 and 10) in order for us to see all the turning points and horizontal intercepts. What

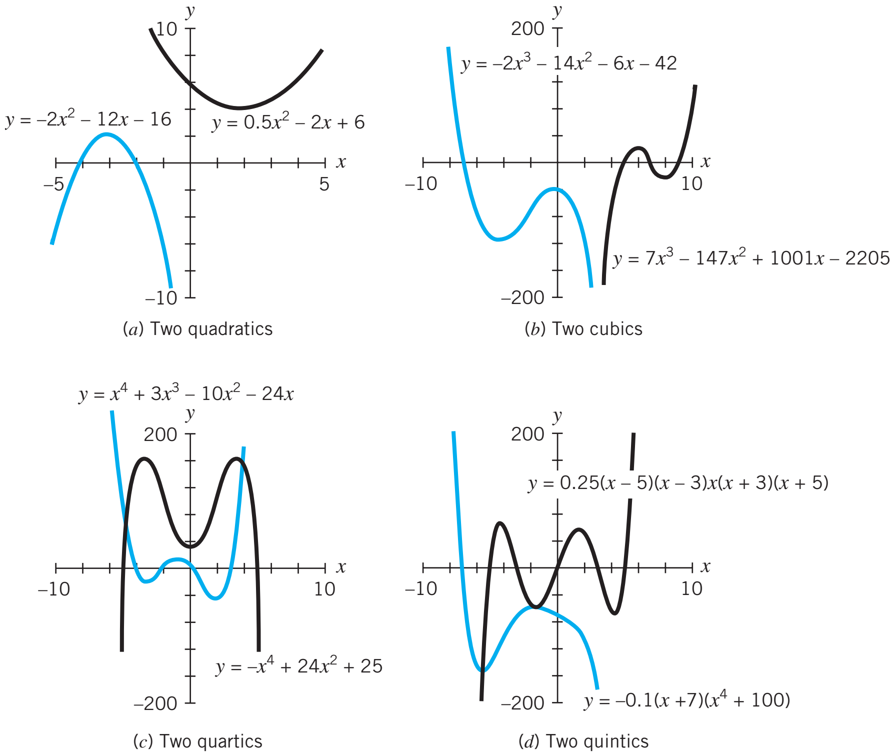


Figure 8.36 Graphs of pairs of polynomial functions (*a*) of degree 2 (quadratics), (*b*) of degree 3 (cubics), (*c*) of degree 4 (quartics), and (*d*) of degree 5 (quintics).

if we graphed the functions using a much wider range of x values, say between -1000 and 1000 or between $-1{,}000{,}000$ and 1,000,000?

Figure 8.37 shows two graphs of the same quintic, $y = 0.25(x - 5)(x - 3)x(x + 3)(x + 5)$, that is graphed in Figure 8.36(*d*). The scale for x on the left-hand graph is -10 to 10. The scale for x on the right-hand graph is -1000 to 1000. On the larger-scale graph on the right, the global behavior is clear, but the five horizontal intersection points and four turning points are no longer visible. The dominant features now are the arms of polynomials that extend infinitely upward on the right and downward on the left. Their direction is dictated by the leading term, which eventually dominates all the other terms. So at this large scale, the graph of our quintic function $y = 0.25(x - 5)(x - 3)x(x + 3)(x + 5)$ looks like the graph of $y = 0.25x^5$.

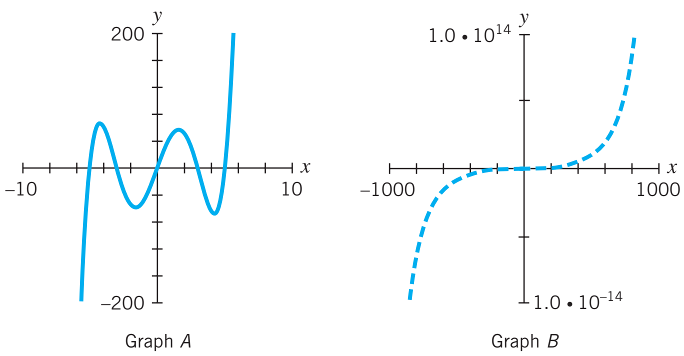


Figure 8.37 Two graphs of the same quintic using different scales.

For polynomial functions of odd degree, the two arms extend in opposite directions, one up and one down. For polynomial functions of even degree, both arms extend in the same direction, either both up or both down (see Figure 8.36). This is because given a polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$$

of degree n, for large values of x the values of the leading term $a_n x^n$ will dominate the values of the other terms of smaller degree. In other words, for large values of x the function behaves like the power function $g(x) = a_n x^n$. The leading term of a polynomial determines its global shape.

The Graph of a Polynomial Function

The graph of a polynomial function of degree n will

have at most $n - 1$ turning points

cross the horizontal axis at most n times

For large values of x, the graph of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ will resemble the graph of the power function $g(x) = a_n x^n$.

EXAMPLE 3 What are the number of visible turning points and the number of horizontal intercepts for each polynomial graphed in Figure 8.38? What is the minimal degree for each polynomial function?

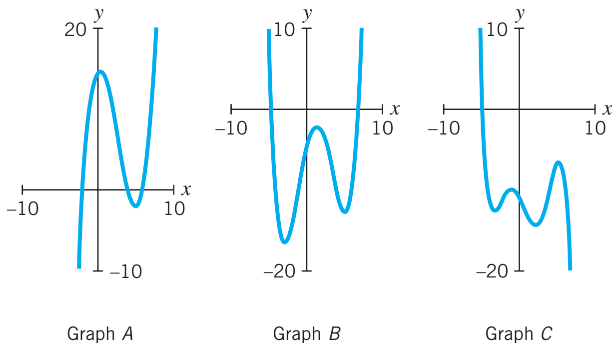


Figure 8.38 Graphs of three polynomial functions.

SOLUTION

a. Graph A has two turning points, three horizontal intercepts, and the global shape of an increasing odd power function. This is at least a third-degree polynomial with a positive leading term.

b. Graph B has three turning points but only two horizontal intercepts and has the global shape of an even power function. This is at least a fourth-degree polynomial with a positive leading term.

c. Graph C has four turning points but only one horizontal intercept. It has the global shape of a decreasing odd power function. This is at least a fifth-degree polynomial with a negative leading term.

Finding the Vertical Intercept

The polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has a vertical intercept at a_0, since $f(0) = a_0$. For example, the function $f(x) = x^4 - 16$ has a vertical intercept at -16 (see Figure 8.39).

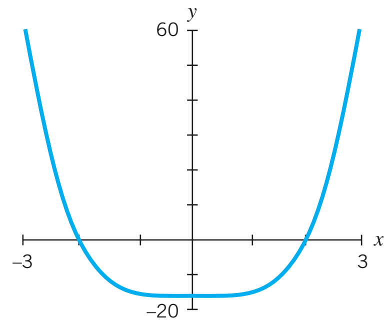


Figure 8.39 The graph of $f(x) = x^4 - 16$ has a vertical intercept at -16 and two horizontal intercepts at 2 and -2.

Finding the Horizontal Intercepts

As with any function, to find the zeros of a polynomial function $f(x)$, we set $f(x) = 0$ and solve for x. If a zero is a real number, it represents a horizontal intercept. For example, revisiting the relatively simple function $f(x) = x^4 - 16$, we can set $f(x) = 0$ and solve the corresponding equation for x to find the zeros.

Setting $f(x) = 0$ we have	$0 = x^4 - 16$
adding 16 to both sides	$16 = x^4$
taking the fourth root	$x = \pm 2$

So -2 and $+2$ are both real zeros, and hence x-intercepts, for the function $f(x) = x^4 - 16$ (see Figure 8.39).

But for a general polynomial function $f(x)$, solving the equation resulting from setting $f(x) = 0$ can be difficult. There is no simple analogue to the quadratic formula for polynomials of degree ≥ 3. The formulas for finding the zeros for third- and fourth-degree polynomials are extremely complicated. For polynomials of degree 5 or higher, there are no general algebraic formulas for finding the zeros. But there are algebraic approximation methods that allow us to compute the values for the real zeros (the horizontal intercepts) accurate to as many decimal places as we wish.

Estimating the horizontal intercepts

The complicated algebraic strategies for finding real zeros for a polynomial function of degree ≥ 3 are beyond the scope of this course. However, we can estimate the horizontal intercepts from a graph.

EXAMPLE 4 **Estimating**

Estimate the horizontal intercepts of the polynomial function in Figure 8.40. What is the minimum degree of the polynomial?

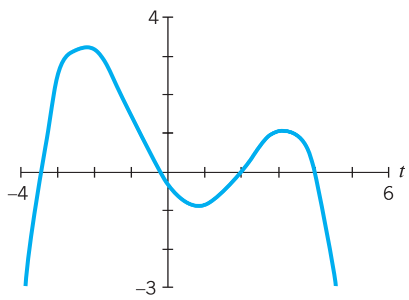


Figure 8.40 Graph of a polynomial function $Q(t)$.

SOLUTION The t-intercepts are at approximately -3.5, -0.25, 2, and 4. Since the polynomial has four horizontal intercepts and three turning points, it must be at least of degree 4.

EXAMPLE 5 Zooming in on a graph

Suppose you make four yearly deposits of \$2000 in a risky high-yield account, starting today. What annual interest must you earn if you want to have \$10,000 in the account after 4 years?[4]

SOLUTION Let's track the four separate \$2000 deposits made in years 1, 2, 3, and 4. If r is the annual interest rate (in decimal form), then $x = 1 + r$ is the annual multiplier or the growth factor.

The first \$2000 invested at the beginning of year 1 will be multiplied by x, the annual growth factor, four times, once for each year. So at the end of 4 years this \$2000 investment will be worth $2000x^4$ dollars.

The second \$2000 invested at the beginning of year 2 will only have 3 years in which to earn interest, years 2, 3, and 4. At the end of year 4, the second \$2000 investment will have been multiplied by the growth factor x only three times, making it worth $2000x^3$ dollars. Similarly, the \$2000 investments made at the beginning of years 3 and 4 will be worth $2000x^2$ and $2000x$ dollars, respectively, by the end of year 4. So at the end of the fourth year, assuming you do not make a final deposit, the total dollars in your account are

$$2000x^4 + 2000x^3 + 2000x^2 + 2000x$$

Since you want to have \$10,000 in your account after 4 years, our equation is

$$10{,}000 = 2000x^4 + 2000x^3 + 2000x^2 + 2000x \tag{1}$$

We need to solve Equation (1) for x to find the annual interest rate r (where $x = 1 + r$). If we subtract 10,000 from each side and then divide by 2000, we have

$$0 = x^4 + x^3 + x^2 + x - 5 \tag{2}$$

A solution to Equation (2) will be a value x for the function

$$f(x) = x^4 + x^3 + x^2 + x - 5$$

such that $f(x) = 0$. So x-intercepts for $f(x)$ correspond to real solutions to Equation (2).

We can estimate the x-intercepts of $f(x)$ by graphing the function (Figure 8.41) and then zooming in (Figure 8.42).

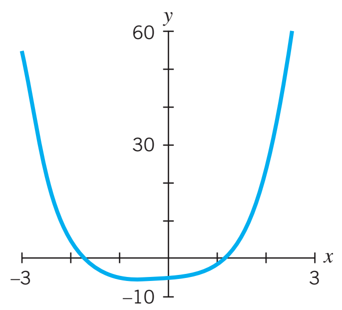


Figure 8.41 Graph of $f(x) = x^4 + x^3 + x^2 + x - 5$.

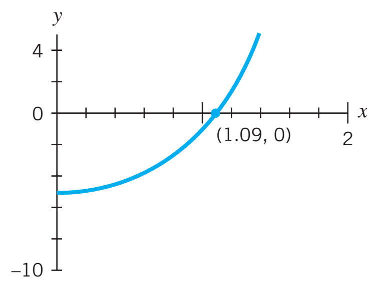


Figure 8.42 Zooming in on the x-intercept between 1 and 2.

[4]Example modeled after a problem in E. Connally, D. Hughes-Hallett, A. M. Gleason, et al., *Functions Modeling Change: A Preparation for Calculus,* 2nd ed. (New York: John Wiley & Sons, 1998), pp. 384–385.

Only positive solutions have meaning in this model, and there is only one positive x-intercept, at approximately (1.09, 0). We have $x = 1 + r$, where r is the annual interest rate. Since $x = 1.09$, then $r = x - 1 = 0.09$ in decimal form, or 9% written as a percentage. So you would need an annual interest rate of approximately 9% to end up with \$10,000 when investing \$2000 each year for 4 years.

Using factoring to find the horizontal intercepts

The Factor Theorem extends to polynomials in general.

The Factored Form of a Polynomial

If $r_1, r_2, \ldots, r_n$ are zeros of a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $f(x)$ can be written in *factored form* as

$$f(x) = a_n(x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n)$$

Any real zero is an x-intercept for $f(x)$.

So if you are given the zeros, it's easy to construct a function in factored form with those zeros. Conversely, if you have a function in factored form, it is easy to find the zeros.

EXAMPLE 6 **Finding horizontal intercepts**

a. What are the t-intercepts of the function $Q(t) = 1.5t(t - 7)(t - 5)(t + 3)$?

b. For large values of t (both positive and negative), the graph of $Q(t)$ resembles what power function? Would the left arm of the parabola eventually extend up or down? And the right arm?

SOLUTION

a. The factors of $Q(t)$ tell us it has real zeros and hence t-intercepts at 0, 7, 5, and -3.

b. For large values of t, the leading term will dominate. If we put $Q(t)$ into standard form, the leading term will be $1.5t^4$, so in the long run $Q(t)$ behaves like the power function $f(t) = 1.5t^4$. So both arms of the graph of $Q(t)$ will eventually extend upward.

EXAMPLE 7 **Constructing polynomial functions**

Create a polynomial function with x-intercepts at 0, -2, and 1.

SOLUTION

If $x = 0$ is an x-intercept, then x is a factor of the polynomial function.
If $x = -2$ is an x-intercept, then $(x - (-2))$, or $(x + 2)$, is also a factor.
If $x = 1$ is an x-intercept, then $(x - 1)$ is a factor as well.

So the function $f(x) = x(x + 2)(x - 1)$ has x-intercepts at 0, -2, and 1.

There are many other polynomial functions with the same horizontal intercepts, as we will see in the next example.

EXAMPLE 8 **Multiple functions with the same horizontal intercepts**

a. Describe two other polynomial functions of degree 3 that have x-intercepts at 0, -2, and 1.

b. Construct one that passes through the point $(2, -16)$.

SOLUTION **a.** The functions $g(x) = 3x(x + 2)(x - 1)$ and $h(x) = -5x(x + 2)(x - 1)$ both have x-intercepts at 0, -2, and 1 (see Figure 8.43). Note that both are multiples of the function $f(x) = x(x + 2)(x - 1)$; that is, $g(x) = 3f(x)$ and $h(x) = -5f(x)$. In general, any function of the form $af(x) = ax(x + 2)(x - 1)$ where $a \neq 0$ will have x-intercepts at 0, -2, and -1. So there are infinitely many possibilities. (See Figure 8.43.)

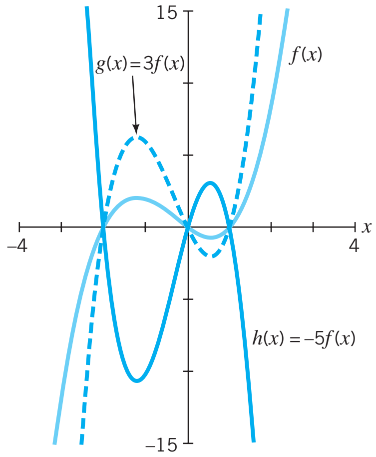


Figure 8.43 The graphs of three polynomial functions all with x-intercepts at 0, -2, and 1.

b. Given the general form of a polynomial with x-intercepts at 0, -2, and 1,

	$f(x) = ax(x + 2)(x - 1)$
If we substitute in $x = 2$ and $f(x) = -16$	$-16 = a(2)(2 + 2)(2 - 1)$
simplify	$-16 = a(8)$
and solve for a, we have	$a = -2$

So the polynomial function $f(x) = -2x(x + 2)(x - 1)$ has x-intercepts at 0, -2, and 1 and passes through the point $(2, -16)$ (see Figure 8.44).

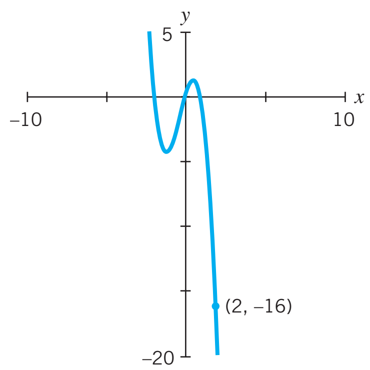


Figure 8.44 A polynomial with x-intercepts at 0, -2, and 1 that passes through $(2, -16)$.

SOMETHING TO THINK ABOUT

Are the only functions with x-intercepts at 0, −2, and 1 of the form $k \cdot f(x)$, where k is a constant and $f(x) = x(x + 2)(x - 1)$?

From Examples 7 and 8, we see that we can create a polynomial function with any finite set of real numbers as its zeros. Looking at it another way, we can create new polynomials by multiplying together existing polynomials.

A Summary of Polynomial Behavior

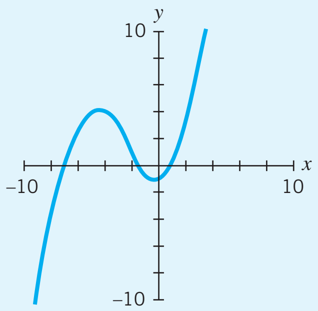
Given a polynomial of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$,

The graph has at most $n - 1$ turning points and n horizontal intercepts.
For large values of x, the graph will resemble the graph of $g(x) = a_n x^n$.
The *zeros* of $f(x)$ are the values of x such that $f(x) = 0$.
If $r_1, r_2, r_3, \ldots, r_n$ are all the zeros of a function $f(x)$, then $f(x)$ can be written in factored form as $f(x) = a_n(x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n)$.
Each real zero corresponds to an *x-intercept*.
The *vertical intercept* is at $f(0) = a_0$.

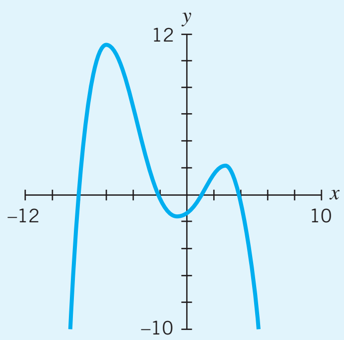
Algebra Aerobics 8.5b

A graphing program is required for Problems 1 and 2, and recommended for Problems 4 and 5.

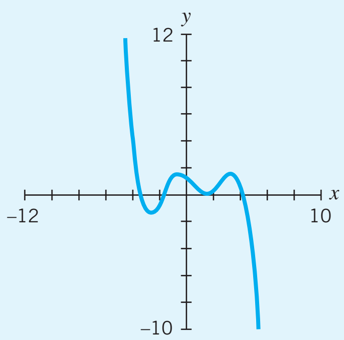
1. Use technology to graph the polynomial function $y = -2(x - 3)^2(x + 4)$.
   a. What is the degree of the function?
   b. How many turning points does it have?
   c. What happens to y as $x \to +\infty$? As $x \to -\infty$?
   d. The graph looks like what power function for very large positive or negative values of x?
   e. What are the horizontal intercepts?
   f. When $x = 0$, what is y? What is this point called?
2. Determine the lowest possible degree and the sign of the leading coefficient for the polynomial functions in the three graphs below. Justify your answers. (*Hint:* Use your knowledge of power functions.)
3. For each of the following functions, identify the vertical intercept. Then use a function graphing program (and its zoom feature) to estimate the number of horizontal intercepts and their approximate values.
   a. $y = 3x^3 - 2x^2 - 3$ b. $f(x) = x^2 + x + 3$
4. Identify the degree and the x-intercepts of each of the following polynomial functions. If technology is available, graph each function to verify your work.
   a. $y = 3x + 6$
   b. $f(x) = (x + 4)(x - 1)$
   c. $g(x) = (x + 5)(x - 3)(2x + 5)$
5. Construct three polynomial functions, all with x-intercepts at −3, 0, 5, and 7. Use technology, if available, to plot them.



Graph *A*



Graph *B*



Graph *C*

Exercises for Section 8.5

A calculator that can evaluate powers is required. Several exercises require a graphing program.

1. Identify which of the following are polynomial functions and, for those that are, specify the degree.

a. $y = 3x + 2$
b. $y = 2 - x^3$
c. $y = 5t^2 - 3\sqrt{t^5}$
d. $y = 3^x - 2$
e. $y = 3x^5 - 4x^3 - 6x^2 - 12$
f. $y = 6(x)(x - 5)(2x + 7)$

2. Identify the degree of any of the following functions that are polynomials, and for those that are not polynomials, explain why.

a. $y = 3x^2 - \dfrac{2}{x^3} + 1$
b. $y = 2x^5 + \dfrac{x^4}{3} - 3x$
c. $y = x^{5/3} + x^2$
d. $y = 2t^3 + 5t^4 + \sqrt{2}$

3. Evaluate the following expressions for $x = 2$ and $x = -2$.

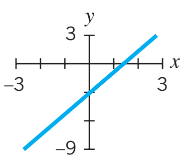
a. x^{-3} **b.** $4x^{-3}$ **c.** $-4x^{-3}$ **d.** $-4x^3$

4. Evaluate the following polynomials for $x = 2$ and $x = -2$, and specify the degree of each polynomial.

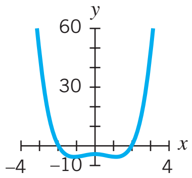
a. $y = 3x^2 - 4x + 10$
b. $y = x^3 - 5x^2 + x - 6$
c. $y = -2x^4 - x^2 + 3$

5. Match each of the following functions with its graph.

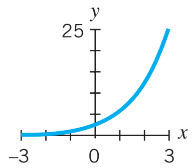
a. $y = 2x - 3$ **b.** $y = 3(2^x)$ **c.** $y = (x^2 + 1)(x^2 - 4)$



Graph *A*



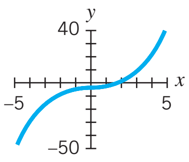
Graph *B*



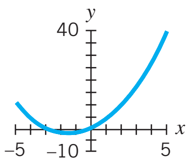
Graph *C*

6. Match each of the following functions with its graph.

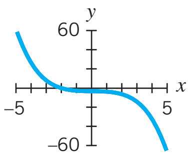
a. $f(x) = x^2 + 3x + 1$
b. $f(x) = \frac{1}{3}x^3 + x - 3$
c. $f(x) = -\frac{1}{2}x^3 + x - 3$



Graph *A*



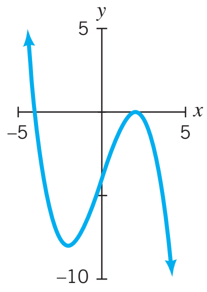
Graph *B*



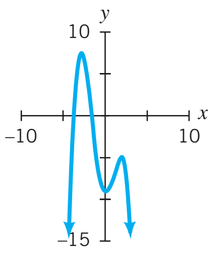
Graph *C*

7. For each of the graphs of polynomial functions, at the top of the next column, determine (assuming the arms extend indefinitely in the indicated direction):

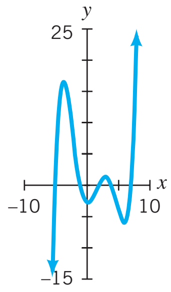
i. The number of turning points
ii. The number of x-intercepts
iii. The sign of the leading term
iv. The minimum degree of the polynomial



Graph *A*



Graph *B*



Graph *C*

8. Describe how $g(x)$ and $h(x)$ relate to $f(x)$.

$$f(x) = x^5 - 3x^2 + 4$$
$$g(x) = -x^5 + 3x^2 - 4$$
$$h(x) = x^5 - 3x^2 - 2$$

9. Divide the following functions into groups having the same global shape for large values of x. Explain your groupings.

a. $y = (x^2 - 3)(x + 9)$
b. $y = -x^4 + 3$
c. $y = x^5 - 3x^4 - 11x^3 + 3x^2 + 10x$
d. $y = x(3 - x)(x + 1)^2$
e. $y = 7x^3 - 3x^2 - 20x + 5$
f. $y = -(x^2 + 1)(x^2 - 4)$

10. Estimate the maximum number of turning points for each of the polynomial functions. If available, use technology to graph the function to verify the actual number.

a. $y = x^4 - 2x^2 - 5$
b. $y = 4t^6 + t^2$
c. $y = x^3 - 3x^2 + 4$
d. $y = 5 + x$

11. Describe the behavior of each polynomial function for large values (positive or negative) of the independent variable and estimate the maximum number of turning points. If available, use technology to verify the actual number.

a. $y = -2x^4 + 4x + 3$
b. $y = (t^2 + 1)(t^2 - 1)$
c. $y = x^3 + x + 1$
d. $y = x^5 - 3x^4 - 11x^3 + 3x^2 + 10x$

12. (Graphing program required.) Estimate the maximum number of horizontal intercepts for each of the polynomial functions. Then graph the function using technology to find the actual number. (See Exercise 10.)

a. $y = x^4 - 2x^2 - 5$
b. $y = 4t^6 + t^2$
c. $y = x^3 - 3x^2 + 4$
d. $y = 5 + x$

13. (Graphing program required.) Estimate the maximum number of horizontal intercepts for each of the polynomial functions. Then, using technology, graph the functions to find their approximate values.

a. $y = -2x^2 + 4x + 3$
b. $y = (t^2 + 1)(t^2 - 1)$
c. $y = x^3 + x + 1$
d. $y = x^5 - 3x^4 - 11x^3 + 3x^2 + 10x$

14. **a.** (Graphing program required.) Use a function graphing program to estimate the x-intercepts for each of the following. Make a table showing the degree of the polynomial and the number of x-intercepts. What can you conclude?

$y = 2x + 1$ $\qquad$ $y = x^3 - 5x^2 + 3x + 5$

$y = x^2 - 3x - 4$ $\qquad$ $y = 0.5x^4 + x^3 - 6x^2 + x + 3$

b. Repeat part (a) for the following functions. How do your results compare with those for part (a)? Are there any modifications you need to make to your conclusions in part (a)?

$y = 3x + 5$ $\qquad$ $y = x^3 - 2x^2 - 4x + 8$

$y = x^2 + 2x + 3$ $\qquad$ $y = (x - 2)^2(x + 1)^2$

15. (Graphing program required.) Use a function graphing program (and its zoom feature) to estimate the number of x-intercepts and their approximate values for:

a. $y = 3x^3 - 2x^2 - 3$ $\qquad$ **b.** $f(x) = x^2 + 5x + 3$

16. (Graphing program required.) Identify the x-intercepts of the following functions; then graph the functions to check your work.

a. $y = 3x + 6$

b. $y = (x + 4)(x - 1)$

c. $y = (x + 5)(x - 3)(2x + 5)$

17. **a.** If the degree of a polynomial is odd, then at least one of its zeros must be real. Explain why this is true.

b. Sketch a polynomial function that has no real zeros and whose degree is:

i. 2 $\qquad$ **ii.** 4

c. Sketch a polynomial function of degree 3 that has exactly:

i. One real zero $\qquad$ **ii.** Three real zeros

d. Sketch a polynomial function of degree 4 that has exactly two real zeros.

18. In each part, construct a polynomial function with the indicated characteristics.

a. Crosses the x-axis at least three times

b. Crosses the x-axis at -1, 3, and 10

c. Has a y-intercept of 4 and degree of 3

d. Has a y-intercept of -4 and degree of 5

19. Which of the following statements are true about the graph of the polynomial function

$$f(x) = x^3 + bx^2 + cx + d$$

a. It intersects the vertical axis at one and only one point.

b. It intersects the x-axis in at most three points.

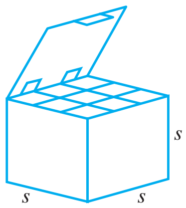
c. It intersects the x-axis at least once.

d. The vertical intercept is positive.

e. For large (positive or negative) values of x, the graph looks like $y = x^3$.

f. The origin is a point on the graph.

20. (Graphing program required.) A manufacturer sells children's wooden blocks packed tightly in a cubic tin box with a hinged lid. The blocks cost 3 cents a cubic inch to make. The box and lid material cost 1 cent per square inch. (Assume the sides of the box are so thin that their thickness can be ignored.) It costs 2 cents per linear inch to assemble the box seams. The hinges and clasp on the lid cost \$2.50, and the label costs 50 cents. (See the accompanying figure.)



A cubic box with edge length s filled with blocks.

a. If the edge length of the box is s inches, develop a formula for estimating the cost $C(s)$ of making a box that's filled with blocks.

b. Graph the function $C(s)$ for a domain of 0 to 20. What section of the graph corresponds to what the manufacturer actually produces—boxes between 4 and 16 inches in edge length?

c. What is the cost of this product if the cube's edge length is 8 inches?

d. Using the graph of $C(s)$, estimate the edge length of the cube when the total cost is \$100.

21. Polynomial expressions of the form $an^3 + bn^2 + cn + d$ can be used to express positive integers, such as 4573, using different powers of 10:

$$4573 = 4 \cdot 1000 + 5 \cdot 100 + 7 \cdot 10 + 3$$

or

$$4573 = 4 \cdot 10^3 + 5 \cdot 10^2 + 7 \cdot 10^1 + 3 \cdot 10^0$$

or if $n = 10$,

$$\begin{aligned} 4573 &= 4 \cdot n^3 + 5 \cdot n^2 + 7 \cdot n^1 + 3 \cdot n^0 \\ &= 4n^3 + 5n^2 + 7n + 3 \end{aligned}$$

Notice that in order to represent any positive number, the coefficient multiplying each power of 10 must be an integer between 0 and 9.

a. Express 8701 as a polynomial in n assuming $n = 10$.

b. Express 239 as a polynomial in n assuming $n = 10$.

Computers use a similar polynomial system, called binary numbers, to represent numbers as sums of powers of 2. The number 2 is used because each minuscule switch in a computer can have one of two states, on or off; the symbol 0 signifies off, and the symbol 1 signifies on. Each binary number is built up from a row of switch positions each set at 0 or 1 as multipliers for different powers of 2. For instance, in the binary number system 13 is represented as 1 1 0 1, which stands for

$$1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 =$$
$$1 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 1 \cdot 1 =$$
$$13$$

c. What number does the binary notation 1 1 0 0 1 represent?

d. Find a way to write 35 as the sum of powers of 2; then give the binary notation.

22. A typical retirement scheme for state employees is based on three things: age at retirement, highest salary attained, and total years on the job.

Annual retirement allowance =

$$\begin{pmatrix}\text{total years}\\ \text{worked}\end{pmatrix} \cdot \begin{pmatrix}\text{retirement}\\ \text{age factor}\end{pmatrix} \cdot \begin{pmatrix}\text{\% of highest}\\ \text{salary}\end{pmatrix}$$

where the maximum percentage is 80%. The highest salary is typically at retirement.

We define:

total years worked = retirement age − starting age

retirement age factor = 0.001 · (retirement age − 40)

salary at retirement = starting salary + all annual raises

a. For an employee who started at age 30 in 1973 with a salary of \$12,000 and who worked steadily, receiving a \$2000 raise every year, find a formula to express retirement allowance, R, as a function of employee retirement age, A.

b. Graph R versus A.

c. Construct a function S that shows 80% of the employee's salary at age A and add its graph to the graph of R.

From the graph estimate the age at which the employee annual retirement allowance reaches the limit of 80% of the highest salary.

d. If the rule changes so that instead of highest salary, you use the average of the three highest years of salary, how would your formula for R as a function of A change?

8.6 *New Functions from Old*

We've seen how the simple quadratic $y = x^2$ can be transformed (through stretching, compressing, and shifting) to generate all quadratic functions. We've also added power functions to generate polynomial functions. In the next three sections we'll look at many ways of transforming any function or combining any two or more functions to create new functions. Our focus, however, will be on combining members from the families of functions we have studied: linear, exponential, logarithmic, power, quadratic, and polynomial.

Transforming a Function

Stretching, compressing, and shifting

Recall from our discussion in Section 8.2 that to stretch or compress the graph of a function, we multiply the function by a constant. To shift the horizontal or vertical position of its graph, we add (or subtract) a constant to either the input or the output of the function.

Stretching, Compressing, or Shifting the Graph of $f(x)$

To stretch or compress:	Multiply the output of $f(x)$ *by* a constant to get $af(x)$. If $\lvert a\rvert > 1$, the graph of $f(x)$ is vertically stretched by a factor of a. If $0 < \lvert a\rvert < 1$, the graph of $f(x)$ is vertically compressed by a factor of a.
To shift vertically:	Add a constant to the output of $f(x)$ to get $f(x) + k$. If k is positive, the graph of $f(x)$ is shifted up. If k is negative, the graph of $f(x)$ is shifted down.
To shift horizontally:	Subtract a constant from the input of $f(x)$ to get $f(x - h)$. If h is positive, the graph of $f(x)$ is shifted to the right. If h is negative, the graph of $f(x)$ is shifted to the left.

Reflections across the horizontal axis

When we studied power functions, we learned that the graphs of $y = kx^p$ and $y = -kx^p$ are reflections of each other across the x-axis. What happens to the equation of any function when we reflect its graph across the horizontal axis?

Figure 8.45 shows two functions, f and g, that are reflections of each other across the x-axis. We can see from the graph that when a point is reflected vertically across the x-axis, the x value stays fixed while the y value changes sign. This means that for each input, the corresponding outputs for f and g are opposites; that is, they have the same magnitude but opposite signs, and are on opposite sides of the x-axis. So $g(x) = -f(x)$.

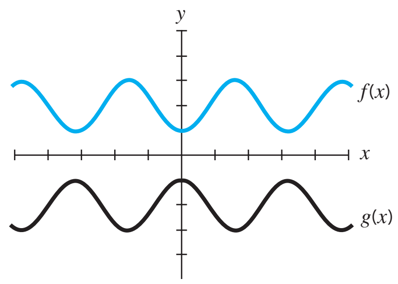


Figure 8.45 The graph of $g(x) = -f(x)$ is a reflection of the graph of $f(x)$ across the horizontal axis.

Reflections across the vertical axis

What if we reflected a function's graph across the vertical axis? What happens to the equation of the original function?

Figure 8.46 shows two functions, f and g, that are reflections of each other across the y-axis. We can see from the graph that when a point is reflected horizontally across the y-axis, the y value stays fixed while the x value changes sign. This means that when f and g have the same output, then the corresponding inputs, or x values, are opposites; that is, they have the same magnitude but opposite signs, and are on opposite sides of the y-axis. So $g(x) = f(-x)$.

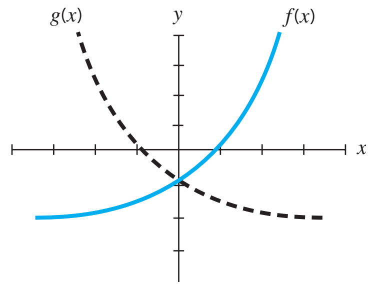


Figure 8.46 The graph of $g(x) = f(-x)$ is a reflection of the graph of $f(x)$ across the vertical axis.

Reflections across both the horizontal and vertical axes

The net result of reflecting the graph of $f(x)$ across both the x- and y-axes is equivalent to rotating the graph 180° about the origin, as shown in Figure 8.47. This is a double reflection, so we need to multiply both the input and the output by -1 to get $-f(-x)$.

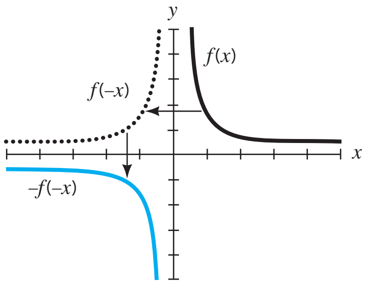


Figure 8.47 The graph of $-f(-x)$ is the graph of $f(x)$ reflected across the y-axis and then across the x-axis.

Reflecting the graph of $f(x)$

To reflect across the x-axis:	Multiply the output of $f(x)$ by -1 to get $-f(x)$.
To reflect across the y-axis:	Multiply the input of $f(x)$ by -1 to get $f(-x)$.
To reflect across both the x- and y-axes:	Multiply both the input and output of $f(x)$ by -1 to get $-f(-x)$.

? SOMETHING TO THINK ABOUT

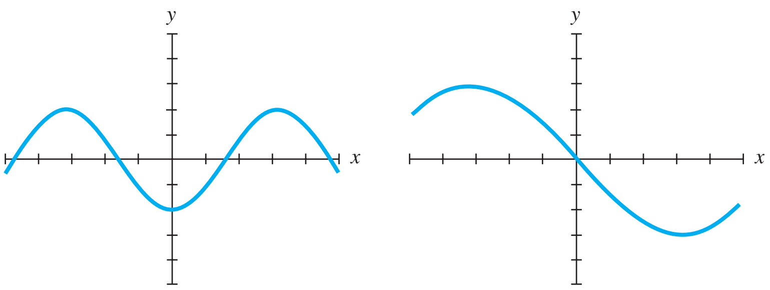
Can a function be symmetric across the x-axis? Why or why not?

Symmetry

If we reflect the graph of $f(x)$ across the y-axis and get the graph of $f(x)$ again, then $f(x)$ is *symmetric across the y-axis*. Symmetry across the y-axis means that the part of the graph to the left of the y-axis is the mirror image of the part to the right of the y-axis. More formally, it means $f(x) = f(-x)$ for all values of x in the domain of f.

If after we rotate the graph of $f(x)$ 180° about the origin we get the graph of $f(x)$, then $f(x)$ is *symmetric about the origin*. We can get the same net result by reflecting the graph of $f(x)$ across both the x- and y-axes. So $f(x) = -f(-x)$ for all values of x in the domain of f.

In Chapter 7 we learned that power functions with even integer powers are symmetric across the y-axis and those with odd integer powers are symmetric about the origin. Figure 8.48 shows the graphs of other functions that have symmetry.



(*a*) Symmetry across the y-axis (*b*) Symmetry about the origin

Figure 8.48 Examples of symmetric graphs.

Symmetry across the y-Axis and about the Origin

If for all values of x in the domain of f:

$f(x) = f(-x)$	then the graph of f is symmetric across the y-axis
$f(x) = -f(-x)$	then the graph of f is symmetric about the origin

EXAMPLE 1 **Reflections and symmetry**

a. Describe the functions $g(x)$ and $h(x)$ shown in Figure 8.49 in terms of transformations of $f(x)$.

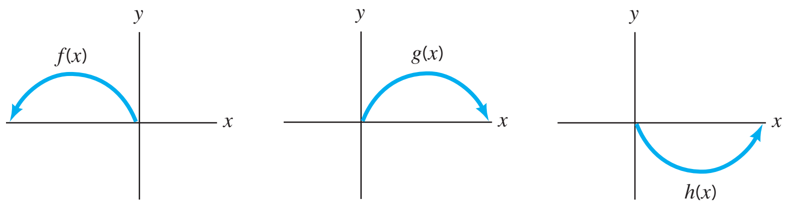


Figure 8.49 Two transformations of $f(x)$.

b. When two objects A and B collide, the force F_1 exerted on object A is equal in magnitude and opposite in direction to the force F_2 exerted on object B, as shown in Figure 8.50(a). The graph in Figure 8.50(b) shows F_1 and F_2 as functions of time. Describe F_1 in terms of F_2.

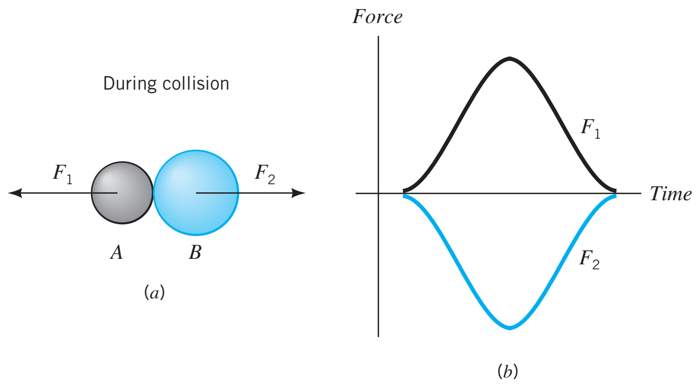


Figure 8.50 Objects colliding.

c. Identify any symmetries in the graphs in Figure 8.51.

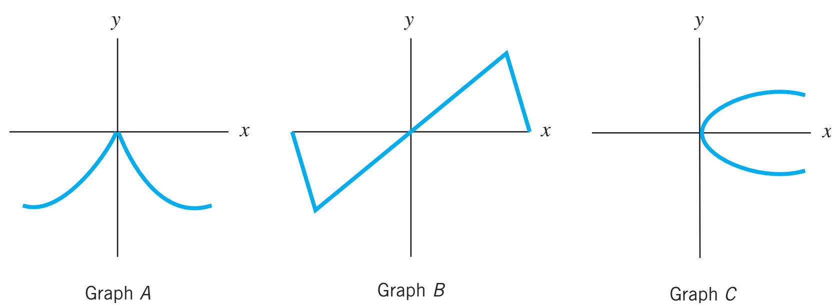


Figure 8.51 Different symmetries.

SOLUTION

a. The graph of g is a reflection of the graph of f across the y-axis, so $g(x) = f(-x)$. The graph of h is a refection of the graph of f across both the x- and y-axes, so $h(x) = -f(-x)$.

b. $F_1(t) = -F_2(t)$ since their graphs are reflections of each other across the time axis.

c. A is the graph of a function that is symmetric across the y-axis. B is the graph of a function that is symmetric about the origin. C is a graph that is symmetric across the x-axis, but it is not a function since there are two values of y for each $x > 0$.

EXAMPLE 2 Shifting and stretching

a. The time series in Figure 8.52 show two related measures of income inequality in the United States. Zero percent represents perfect equality. Construct an equation that approximates Z_{D+R} in terms of Z_D.

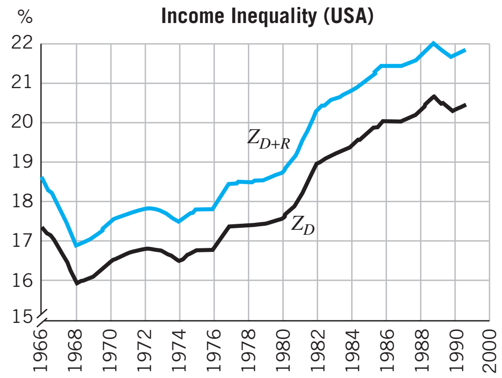


Figure 8.52 Measures of income inequality in the United States.
Source: G. Kluge, *Wealth and People: Inequality Measures* (2002), *www.poorcity.richcity.org/entkiss.htm.*

b. Use the properties of logs and function transformations to predict how the graphs of the following functions will differ from each other. Confirm your predictions by graphing all of the functions on the same grid.

$$y = \log x \qquad y = \log 10x \qquad y = \log 100x \qquad y = 10 \log x$$

SOLUTION

a. A rough approximation is $Z_{D+R} = Z_D + 1.3$.

b. The graph of $y = \log 10x$ is the graph of $y = \log x$ shifted vertically up one unit, since

Given	$y = \log 10x$
Rule 1 of logs	$= \log x + \log 10$
Evaluate log 10	$= \log x + 1$

Similarly, the graph of $y = \log 100x$ is the graph of $y = \log x$ shifted vertically up two units, since

$$\begin{aligned} y &= \log 100x \\ &= \log x + \log 100 \\ &= \log x + \log(10^2) \\ &= \log x + 2 \end{aligned}$$

The graph of $y = 10 \log x$ is the graph of $y = \log x$ stretched by a factor of 10. Figure 8.53 shows the graphs of the four functions on the same grid.

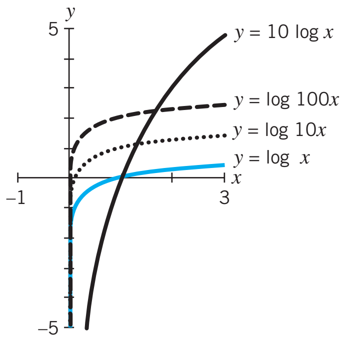


Figure 8.53 Four log functions.

EXAMPLE 3 **Multiple transformations**

a. Without using technology, sketch graphs of each of the following functions:

$$f(x) = x^3 \qquad g(x) = 2f(x+1) \qquad h(x) = f(-x) - 2$$

b. Construct a new function for each the following transformations of $g(t) = 100(1.03)^t$:

$$3g(t-2) \quad \text{and} \quad 0.5g(-t) + 3$$

SOLUTION

a. See Figure 8.54.

b. Given $g(t) = 100(1.03)^t$

$$\begin{aligned} 3g(t-2) &= (3)(100)(1.03)^{(t-2)} \\ &= 300(1.03)^{(t-2)} \end{aligned}$$

and

$$\begin{aligned} 0.5g(-t) + 3 &= (0.5)(100)(1.03)^{-t} + 3 \\ &= 50(1.03)^{-t} + 3 \\ &= 50\left(\tfrac{1}{1.03}\right)^t + 3 \\ &\approx 50(0.97)^t + 3 \end{aligned}$$

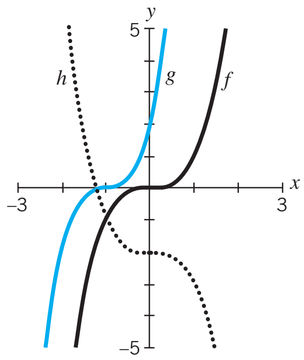


Figure 8.54 Two transformations of $f(x)$.

EXAMPLE 4 **Financial considerations: Horizontal shift of an exponential function**

Twins each receive \$10,000 from their grandmother on their shared 21st birthday. One invests her money right away in high-grade corporate bonds that return an annual 6% in interest, which the sister keeps reinvesting. The other twin can't decide what to do, so she puts her money in a non–interest-bearing checking account. At her 33rd birthday, after getting married and having kids, she decides she needs to save for the future. She now invests \$10,000 in the equivalent bonds, receiving the same interest rate as her sister.

a. Construct a function $S_1(t)$ that represents how much the first sister's investment is worth t years after her 21st birthday.

b. Use $S_1(t)$ to construct a function $S_2(t)$ for the second sister, who waits until she is 33 to invest her money.

c. Graph $S_1(t)$ and $S_2(t)$ on the same grid. What are the differences and similarities between the graphs in practical terms?

d. Use your graphs to estimate how long it will take for each investment to be worth \$40,000.

e. Use your graphs to estimate the difference in the worth of the sisters' investments when they are both 65 years old.

SOLUTION

a. $S_1(t) = 10{,}000(1.06)^t$ for $t \geq 0$

b. $S_2(t) = S_1(t - 12)$

$= 10{,}000(1.06)^{(t-12)}$ for $t \geq 12$

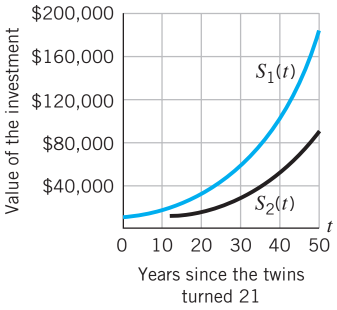


Figure 8.55 Twin investments.

c. The graphs are drawn in Figure 8.55. The graph of S_2 is the graph of S_1 shifted 12 units ($33 - 21 = 12$) to the right. So at age 33, the second sister has \$10,000 in her account, the same amount she had when she was 21 years old. Each successive year the second sister is 12 years behind her twin sister in terms of how much money her investment is worth.

d. It will take about 24 years for each investment to be worth about \$40,000. So the first twin would be about 45 and the second twin about 57, or 12 years older than her sister. Remember that $S_2(t) = S_1(t - 12)$, so $S_2(57) = S_1(57 - 12) = S_1(45)$.

e. The difference between the investments when the twins are 65 years old (44 years after their 21st birthday) is about \$65,000, since when $t = 44$ years, $S_1(44) \approx \$130{,}000$ and $S_2(44) \approx \$65{,}000$. The first twin has almost twice as much as the second.

EXAMPLE 5 Transformations of a power function

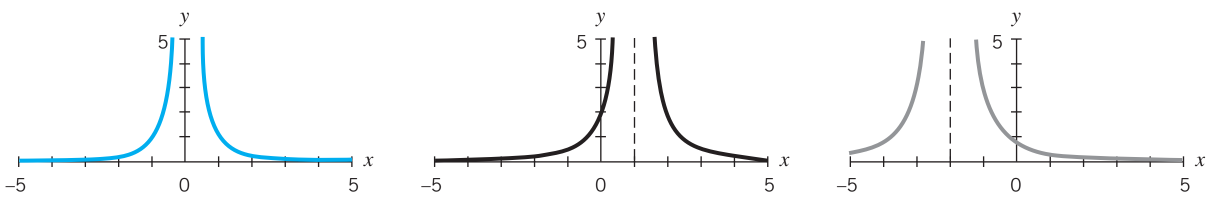
a. Construct an equation and sketch a graph of the following transformations of $f(x) = 1/x^2$:

$$2f(x - 1) \quad \text{and} \quad 3f(x + 2).$$

b. In what ways will the graph of $g(x) = 0.5f(x - 2)$ differ from the graph of $f(x)$? How will it be similar? Check your predictions by graphing $f(x)$ and $g(x)$ on the same grid.

SOLUTION

a. The graphs are drawn in Figure 8.56, with dotted vertical lines indicating vertical asymptotes.



Graph of $f(x) = \dfrac{1}{x^2}$ Graph of $2f(x-1) = \dfrac{2}{(x-1)^2}$ Graph of $3f(x+2) = \dfrac{3}{(x+2)^2}$

Figure 8.56 Graph of $f(x) = 1/x^2$ and two transformations of $f(x)$.

b. As shown in Figure 8.57, the graph of $g(x)$ has the same overall shape (containing two parts) as the graph of $f(x)$, but $g(x)$ is $f(x)$ compressed by a factor of 0.5 and shifted horizontally two units to the right. The y-axis is the vertical asymptote for $f(x)$, but the line $x = 2$ is the vertical asymptote (dotted line) for $g(x)$.

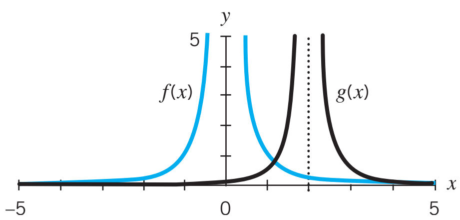


Figure 8.57 Graphs of $f(x) = \frac{1}{x^2}$ and $g(x) = 0.5f(x - 2)$.

EXAMPLE 6 **Cooling coffee: Vertical shift of an exponential function**

A cup of hot coffee (at 200° Fahrenheit) is left to stand in a 70° Fahrenheit room. The difference in temperature between the coffee and the room can be modeled by an exponential function where the temperature decay rate is 10% per minute.

a. Construct $D(t)$, the temperature difference, as a function of time t in minutes that the coffee has been left standing.

b. Using $D(t)$, construct a new function $C(t)$ that describes the coffee's temperature as it gradually approaches the temperature of the room.

c. What are the differences and similarities between the graphs of $D(t)$ and $C(t)$? Check your answer by graphing $D(t)$ and $C(t)$ on the same grid.

d. Use the graph of $C(t)$ to estimate how long it will take for the coffee to cool to 150°F, a comfortable temperature for drinking.

SOLUTION

a. $D(t) = 130(0.9)^t$ since the starting temperature difference is $200° - 70° = 130°$, and the decay factor is $1 - 0.1 = 0.9$.

b. $C(t) = 70 + 130(0.9)^t$

c. The graph of $C(t)$ is the graph of $D(t)$ shifted vertically up 70 units (see Figure 8.58).

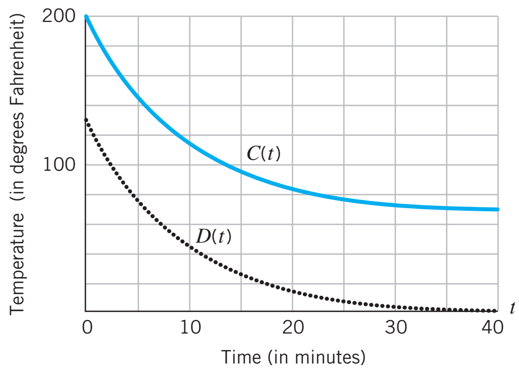


Figure 8.58

d. It will take approximately 5 minutes for the coffee to cool to 150°F.

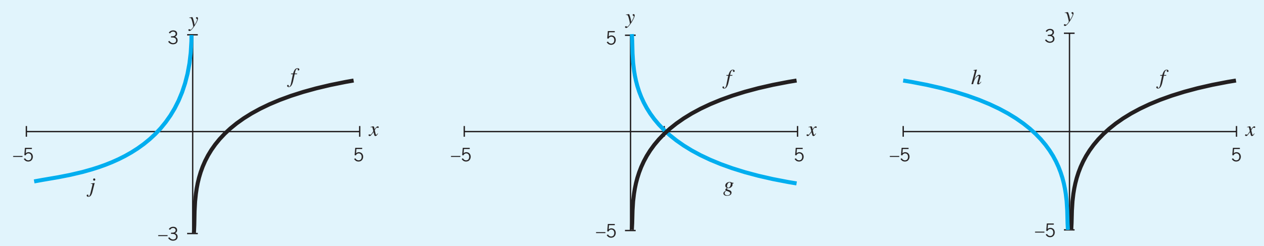
Algebra Aerobics 8.6

Graphing program optional in Problems 6 and 7.

1. Graphs A, B, and C show $f(x)$ and a transformation of $f(x)$. Identify the transformation as: $-f(x)$, $f(-x)$, or $-f(-x)$.

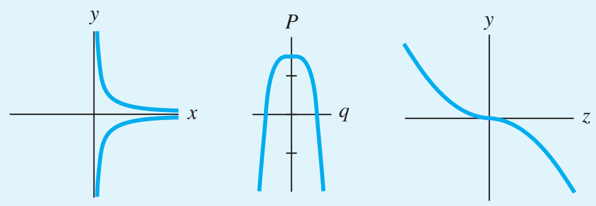
2. Apply the transformations specified in parts (a)–(e) to $f(x) = 2x - 3$ and $f(x) = 1.5^x$.

   a. $f(x + 2)$ **c.** $-f(x)$ **e.** $-f(-x)$
   b. $\frac{1}{2} f(x)$ **d.** $f(-x)$



Graph *A* Graph *B* Graph *C*

3. Identify which of the following graphs are symmetric across the vertical axis. Which are symmetric across the horizontal axis? Which are symmetric about the origin?



Graph *A* Graph *B* Graph *C*

4. Construct a new function from $h(t) = e^t$, where the graph of $h(t)$ is:

a. First shifted two units right

b. Then reflected across the t-axis

c. Then shifted down by one unit

5. Construct a new function from $Q(t) = 2 \cdot 1.06^t$ where the graph of $Q(t)$ is:

a. First shifted two units to the left

b. Then shifted down by one unit

c. Then reflected across the t-axis

(Graphing program optional.) Describe in each part of Problems 6 and 7 how $g(x)$ is related to $f(x)$. If you have a graphing tool available, check your answer by looking at the graphs.

6. a. $f(x) = 5 - x^2$ and $g(x) = x^2 - 5$

b. $f(x) = 3 \cdot 2^x$ and $g(x) = 3 \cdot 2^{-x}$

7. a. $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{1}{3x-6}$

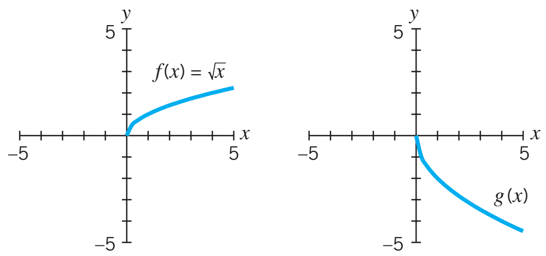
b. $f(x) = \ln x$ and $g(x) = \ln 3 + \ln x = \ln 3x$

Exercises for Section 8.6

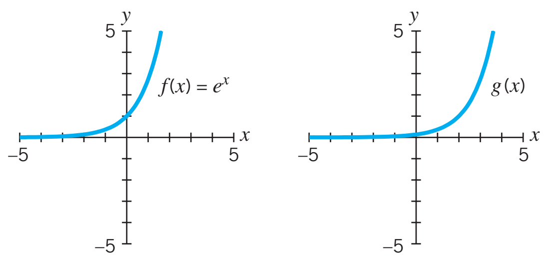
Graphing program optional for Exercises 7, 16, and 18, and required for Exercises 21 and 22.

1. In each part of the problem the graph of $f(x)$ to the left has been transformed into the graph of $g(x)$ to the right. First describe whether the graph of $f(x)$ was stretched/compressed, reflected, and/or shifted vertically/horizontally to form $g(x)$. Then write the equation for $g(x)$ in terms of $f(x)$.

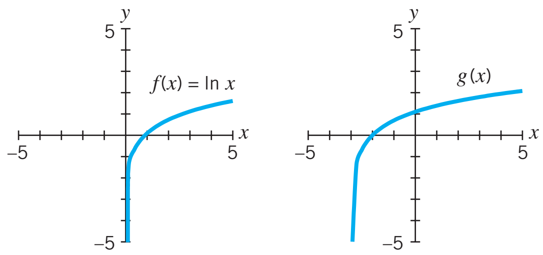
a.



b.



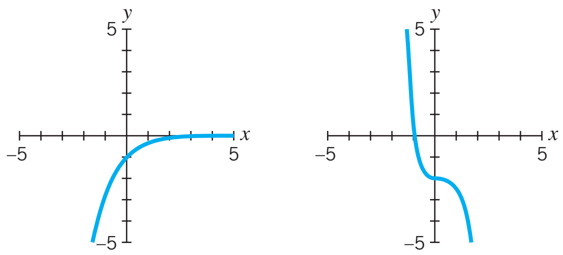
c.



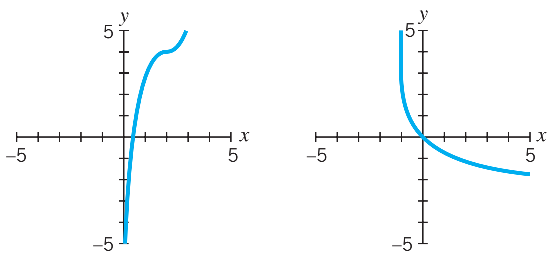
2. Match each of the following functions with its graph. Identify the parent (original) function $p(x)$ and the transformation(s) that took place.

a. $f(x) = (x - 2)^3 + 4$ **c.** $h(x) = -\ln(x + 1)$

b. $g(x) = -x^3 - 2$ **d.** $k(x) = -e^{-x}$



Graph *A* Graph *C*



Graph *B* Graph *D*

3. Let $f(x) = x^3$.

a. Write the equation for the new function $g(x)$ that results from each of the following transformations of $f(x)$. Explain in words the effect of the transformations.

i. $f(-x)$ **iii.** $f(x + 2)$

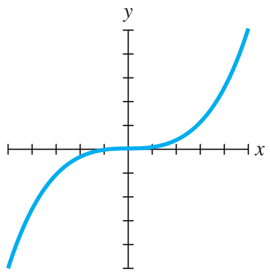
ii. $-2f(x) - 1$ **iv.** $-f(-x)$

b. Sketch by hand the graph of $f(x)$ and each function in part (a).

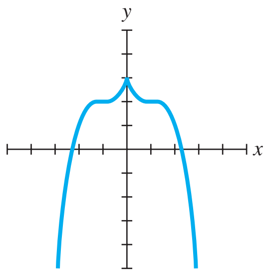
4. Explain in words the effect of the following transformations on the graph of $g(t)$.

a. $-5g(t)$
b. $\frac{1}{5}g(t-3)+1$
c. $-g(-t)-4$
d. $4g(-t)-2$

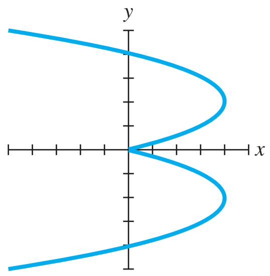
5. Decide if each graph (although not necessarily a function) is symmetric across the x-axis, across the y-axis, and/or about the origin.



Graph *A*



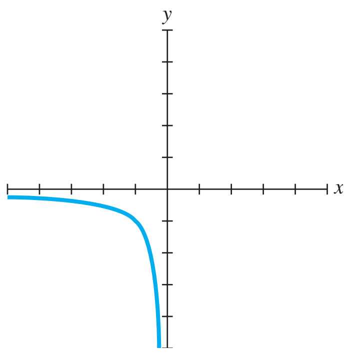
Graph *C*



Graph *B*

6. Complete the partial graph shown in three different ways to create a graph that is:

a. Symmetric across the x-axis
b. Symmetric across the y-axis
c. Symmetric about the origin



7. (Graphing program optional.) A function is said to be *even* if $f(-x)=f(x)$ and *odd* if $f(-x)=-f(x)$ for all x in f's domain. Use these definitions to:

a. Show that the even power functions are even.
b. Show that the odd power functions are odd.
c. Show whether each of the following functions is even, odd, or neither.

i. $f(x)=x^4+x^2$
ii. $u(x)=x^5+x^3$
iii. $h(x)=x^4+x^3$
iv. $g(x)=10.3^x$

d. For each function that you have identified as even or odd, what do you predict about the symmetry of its graph? If possible, check your predictions with a function graphing program.

8. Use the function $f(x)$ to create a new function $g(x)$ where the graph of $g(x)$ is:

a. The graph of $f(x)$ shifted three units to the left, then multiplied by 5, and finally shifted down by four units.
b. The graph of $f(x)$ shifted three units to the right, then shifted up by four units, and finally multiplied by five.

9. For each function, construct a new function whose graph is the graph of the original function shifted left by two units, then multiplied by $\frac{1}{3}$, and then shifted down by five units.

a. $f(x)=60\left(\frac{1}{2}\right)^x$
b. $g(x)=12x^3$
c. $y=\log x$

10. If $f(x)=\dfrac{1}{x}$, evaluate:

a. $f(x-3)+5$
b. $-3f(x+2)+\frac{1}{2}f(x-1)$
c. $f(t^2+2)$
d. $f(x+a)-f(a)$

11. If $f(x)=\dfrac{1}{x^2}$, determine:

a. $f(2-x)-1$
b. $\frac{1}{2}f(-t)$
c. $f(\sqrt{s}-3)$
d. $f(x+h)-f(x)$

12. If $f(x)=\frac{1}{x}$,

a. Describe the transformations of $f(x)$ used to create the new functions $g(x)$, $h(x)$, and $k(x)$.

$$g(x)=\frac{1}{3x}\qquad h(x)=-\frac{2}{x-3}\qquad k(x)=\frac{1}{2-x}+4$$

b. Determine the domain of each function in part (a).
c. Determine the equation of the vertical asymptote for each function in part (a).

13. Let $k(s)=\frac{1}{s}$. Construct a new function $j(s)$ that is the end result of the transformations of the graph of $k(s)$ described in the following steps. Show your work for each transformation.

a. First shift $k(s)$ to the right by two units.
b. Then compress your result by a factor of 1/3.
c. Reflect across the s-axis.
d. Finally, shift it up four units.

14. If $p(t)=\dfrac{1}{t^2}$, construct an expression for:

a. $p(t+1)$
b. $3p(t+2)-1$
c. $-2(p(t-3)+5)$

15. Apply the transformations specified in parts (a)–(e) to $f(x)=\ln x$ and $f(x)=\dfrac{1}{x^3}$.

a. $f(x+2)$
b. $\frac{1}{2}f(x)$
c. $-f(x)$
d. $f(-x)$
e. $-f(-x)$

16. (Graphing program optional.)

a. Starting with the function $f(x) = e^x$, create a new function $g(x)$ by performing the following transformations. At each step show the transformation in terms of $f(x)$ and e^x,

i. First shift the graph of $f(x)$ to the left by three units,

ii. Then compress your result by a factor of 1/4,

iii. Next reflect it across the x-axis,

iv. And finally shift it up by five units to create $g(x)$.

b. Graph $f(x)$ and $g(x)$ on the same grid.

17. Given the function $g(t)$, identify the simplest function $f(t)$ (linear, power, exponential, or logarithmic) from which $g(t)$ could have been constructed. Describe the transformations that changed $f(t)$ to $g(t)$.

a. $g(t) = \dfrac{t-1}{2}$

b. $g(t) = 3\left(\dfrac{1}{2}\right)^{t+4}$

c. $g(t) = \dfrac{-7}{t-5} - 2$

18. (Graphing program optional.)

a. On separate grids sketch the graphs of $f(x) = \sqrt{-x+2}$ and $g(x) = -\sqrt{x+2}$.

b. Using interval notation, describe the domains of $f(x)$ and $g(x)$.

c. Using interval notation, describe the ranges of $f(x)$ and $g(x)$.

d. What is the simplest function $h(x)$ from which both $f(x)$ and $g(x)$ could be created?

e. Describe the transformations of $h(x)$ to obtain $f(x)$. Of $h(x)$ to obtain $g(x)$.

f. Does the graph of $f(x)$, $g(x)$, or $h(x)$ have any symmetries (across the x- or y-axis, or about the origin)?

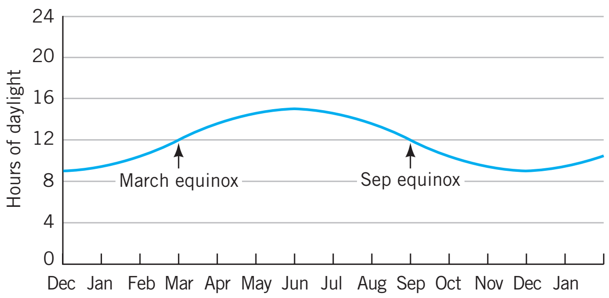
19. a. Given $f(x) = \ln x$, describe the transformations that created $g(x) = 3f(x+2) - 4$. Find $g(x)$.

b. Use your knowledge of properties of logarithms to find any vertical and horizontal intercepts for the function $g(x)$.

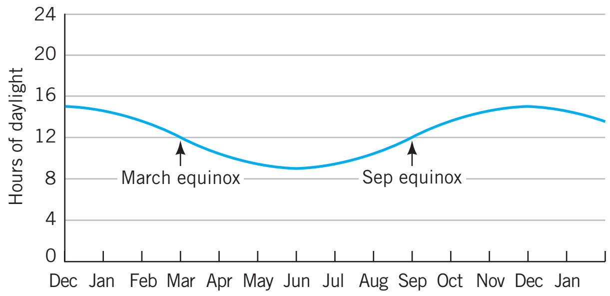
20. The following two graphs show the hours of daylight during the year for two different locations. One is for a latitude of 40 degrees above the equator (in the Northern Hemisphere), and the other for a latitude of 40 degrees below the equator (in the Southern Hemisphere).

a. Which graph is associated with which hemisphere?

b. Using the language of function transformation, describe Graph B in terms of Graph A.



Graph *A*



Graph *B*

Source: www.vcaa.vic.edu.au/prep10/csf/support/sampleunits/SolarHouseDaylight.xls.

21. (Graphing program required.) If an object is put in an environment with a fixed temperature A (the "ambient temperature"), then the object's temperature, T, at time t is modeled by Newton's Law of Cooling: $T = A + Ce^{-kt}$, where k is a positive constant. (Note that T is a function of t and as $t \to +\infty$, then $e^{-kt} \to 0$, so the temperature T of the object gets closer and closer to the ambient temperature, A.) A corpse is discovered in a motel room at midnight. The corpse's temperature is 80° and the room temperature is 60°. Two hours later the temperature of the corpse had dropped to 75°. (Problem adapted from one in the public domain site S.O.S. Math.)

a. Using Newton's Law of Cooling, construct an equation to model the temperature T of the corpse over time, t, in hours since the corpse was found.

b. Then determine the time of death. (Assume the normal body temperature is 98.6°.)

c. Graph the function from $t = -5$ to $t = 5$, and identify when the person was alive, and the coordinates where the temperature of the corpse was 98.6°, 80°, and 75°.

22. (Graphing program required.) Newton's Law of Cooling (see Exercise 21) also works for objects being heated. Suppose you place a frozen pizza (at 32°) into a preheated oven set at 350°. Thirty minutes later the pizza is at 320°—ready to eat.

a. Determine the constants A, C, and k in Newton's Law.

b. Sketch a graph of your function.

c. From your graph, estimate when the pizza will be at 200° and then calculate the time.

d. According to your model, if you kept the pizza in the oven indefinitely, would the pizza ever reach 350°? What would be a reasonable domain for the function as a model for cooking pizza?

8.7 Combining Two Functions

We can combine functions the way we combine numbers; that is, we can add, subtract, multiply, or divide two functions.

The Algebra of Functions

If $f(x)$ and $g(x)$ are two functions with the same domain, we can define new functions by:

Adding to get $f(x) + g(x) = (f + g)(x)$

Subtracting to get $f(x) - g(x) = (f - g)(x)$

Multiplying to get $f(x) \cdot g(x) = (f \cdot g)(x)$

Dividing to get $\frac{f(x)}{g(x)} = \left(\frac{f}{g}\right)(x)$ (where $g(x) \neq 0$)

For example, if $f(x) = x^3$ and $g(x) = 5(2)^x$, then

$$(f + g)(x) = f(x) + g(x) = x^3 + 5(2)^x$$
$$(f - g)(x) = f(x) - g(x) = x^3 - 5(2)^x$$
$$(f \cdot g)(x) = f(x) \cdot g(x) = x^3 \cdot 5(2)^x$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^3}{5(2)^x}$$

We've already seen several examples of combining functions. We initially described a polynomial function as a sum of power functions. Once a polynomial is put into factored form, we can think of it as a product of linear functions.

EXAMPLE 1 **Adding two functions: Skid distances**

In Example 7 of Section 7.2, we estimated the skid distance (in feet) of a car after the brakes were applied as $S^2/30$, where S (in mph) is the initial speed of the car. But the total braking distance has an additional component, the distance due to the driver's reaction time before hitting the brakes.

a. It takes the average driver about 0.75 second to react before putting on the brakes. If the car is traveling at S mph, how many feet has the car traveled during the driver's reaction time? Construct the equation for the reaction distance $d_r(S)$ in feet.

b. Construct an equation for the total braking distance as a function of car speed S.

c. If you suddenly see that there is a massive traffic accident 100 feet ahead, can you stop in time if you are traveling at 20 mph? At 40 mph? At 60 mph?

SOLUTION **a.** If you are traveling at S mph, the distance traveled during 0.75 second is

$$(0.75 \text{ sec}) \cdot \left(S \frac{\text{miles}}{\text{hour}}\right) = (0.75 \cancel{\text{sec}}) \cdot \left(S \frac{5280 \text{ feet}}{3600 \cancel{\text{sec}}}\right) = 1.1S \text{ feet}$$

So $d_r(S) = 1.1S$.

b. The total braking distance, $d_t(S)$, is the sum of two functions: the reaction distance $d_r(S)$ and the skid distance $d_s(S)$.

$$d_t(S) = d_r(S) + d_s(S)$$

substituting, we get $$d_t(S) = 1.1S + \frac{S^2}{30}$$

where S is in mph and $d_t(S)$ is in feet.

c. If the car is traveling at 20 mph, then the total braking distance is

$$d_t(20) = 1.1(20) + \frac{(20)^2}{30} = 22 + \frac{400}{30} \approx 22 + 13 = 35 \text{ feet}$$

So at 20 mph, you could stop in 35 feet, well before the accident.

At 40 mph, your total braking distance is

$$d_t(40) = 1.1(40) + \frac{(40)^2}{30} = 44 + \frac{1600}{30} \approx 44 + 53 = 97 \text{ feet}$$

So at 40 mph, you would be cutting it very close, stopping only about $100 - 97 = 3$ feet from the accident (too close really, since all the functions are estimates).

At 60 mph, your total braking distance is

$$d_t(60) = 1.1(60) + \frac{(60)^2}{30} = 66 + \frac{3600}{30} = 66 + 120 = 186 \text{ feet}$$

So you will not be able to stop in time.

EXAMPLE 2 **Subtracting one function from another: Making a profit**

You are the Chair of the Board of the "Friends" of a historical house in Philadelphia. You are trying to increase membership by sending out a mass mailing. The up-front fixed costs are \$1000 for designing a logo and printing 5000 brochures (the minimum for a discount rate). The mailing costs are \$0.51 per person, which includes stuffing the envelope, adhering address labels, and paying postage. Each person who joins would pay a \$35 annual membership fee; however, the usual response rate for such mass mailings is about 3%. How large should your mailing be for you to recover the costs of the mailing through the membership fees?

SOLUTION If n = the number of brochures mailed, then $C(n)$, the cost of the mailing, is

$$\begin{aligned} C(n) &= \text{fixed costs} + (\text{cost per mailing}) \cdot (\text{number of brochures}) \\ &= \$1000 + \$0.51n \end{aligned}$$

The total revenue $R(n)$ is

$$\begin{aligned} R(n) &= (\text{number of responses}) \cdot (\text{membership fee}) \\ &= (3\% \text{ of } n) \cdot (\$35) \\ &= (0.03n) \cdot (\$35) \\ &= \$1.05n \end{aligned}$$

Your profit $P(n)$ is the difference between the revenue and cost functions.

$$\begin{aligned} \text{Profit} &= \text{Revenue} - \text{Costs} \\ P(n) &= R(n) - C(n) \\ &= \$1.05n - (1000 + \$0.51n) \\ &= \$0.54n - 1000 \end{aligned}$$

$P(n)$ is a linear function of n, the number of mailings, with a vertical intercept at $-\$1000$ and a positive slope of \$0.54 per mailing (see Figure 8.59).

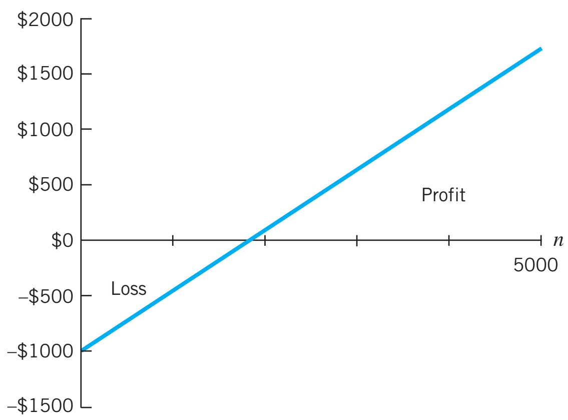


Figure 8.59 Graph of Profit = Revenue − Costs with breakeven point at $n = 1850$.

What is the breakeven point, where revenue equals costs? This happens when the profit (or loss) is 0.

Setting	$P(n) = 0$
substitute for $P(n)$	$\$0.54n - 1000 = 0$
add 1000 to each side	$\$0.54n = 1000$
divide and round to the nearest ten	$n \approx 1850$

The model suggests that you should mail out letters to at least 1850 people to cover your costs. If you mail to more than 1850 people, the model predicts you will have a profit. Mailing to fewer people, you would take a loss, so financially the mass mailing would not be worth doing.

EXAMPLE 3 **The quotient of two functions: The federal debt per person**

Since 1940 both the gross federal debt and the U.S. population have been growing exponentially. Best-fit curves describe the debt (in millions of dollars) as $D(t) = 78{,}800(1.073)^t$ and the U.S. population size (in millions) as $P(t) = 141(1.012)^t$, where for both functions t = years since 1940.

a. Construct a function for the estimated gross federal debt per person over time.

b. What was the estimated federal debt per person in 1990? In 2007? The projected federal debt per person in 2015?

c. Is the model realistic?

SOLUTION **a.** $F(t)$ = federal debt per person over time, which equals

$$\frac{\text{federal debt each year}}{\text{U.S. population each year}} = \frac{D(t)}{P(t)} = \frac{78{,}800(1.073)^t}{141(1.012)^t}$$

b. In 1990 we have $t = 50$. So

$$F(50) = \frac{D(50)}{P(50)} = \frac{78{,}800(1.073)^{50}}{141(1.012)^{50}} \approx \frac{\$2{,}670{,}000 \text{ million}}{256 \text{ million people}} \approx \$10{,}430 \text{ per person}$$

This means that if we had decided to pay off the gross federal debt in 1990, we would have had to spend the equivalent of over \$10,000 per person.

In 2007 we have $t = 67$. So

$$F(67) = \frac{D(67)}{P(67)} = \frac{78{,}800(1.073)^{67}}{141(1.012)^{67}} \approx \frac{\$88{,}445{,}200 \text{ million}}{314 \text{ million people}} \approx \$28{,}210 \text{ per person}$$

So only 17 years later, the federal debt had almost tripled to over \$28,000 per person.

If the same rates of growth continue, then in 2015 (when $t = 75$), we will have

$$F(75) = \frac{D(75)}{P(75)} = \frac{78{,}800(1.073)^{75}}{141(1.012)^{75}} \approx \frac{\$15{,}542{,}000 \text{ million}}{345 \text{ million people}} \approx \$45{,}050 \text{ per person}$$

or over $45,000 of gross federal debt for every man, woman, and child in the United States.

c. There are several U.S. "national debt clocks" (such *www.brilligcom/debt_clock*) that continuously track the gross federal debt and the debt per person. In May 2007 one clock estimated the gross federal debt per person at $29,790, which is slightly over our model's value of $28,000. So the model is reasonably accurate.

Check at least one such site and compare its figure for the current debt per person with what our model predicts for the current year.

Rational Functions: The Quotient of Two Polynomials

The quotient of two polynomial functions is called a *rational* function (from the word "ratio"). We've seen simple examples of rational functions before in Chapter 7 in the form of power functions with negative integer powers. For example, $R(x) = 6x^{-1} = \frac{6}{x}$, the ratio of surface area to volume of a cube. A more complex example would be the function

$$S(x) = \frac{x^3 - x + 5}{2x^2 - 7}$$

Rational Functions

A *rational function* $R(x)$ is of the form

$$R(x) = \frac{p(x)}{q(x)} \quad (q(x) \neq 0)$$

where $p(x)$ and $q(x)$ are both polynomials.

EXAMPLE 4 **Cost per unit**

The start-up costs for a small pizza company are $100,000, and it costs $3 to produce each additional pizza. (Economists call $3 the *marginal cost.*)

a. Construct a function $C(x)$ for the total cost of producing x pizzas.

b. Then create a function $P(x)$ for the total cost per pizza.

c. As more pizzas are produced, what happens to the cost per pizza?

SOLUTION

a. $C(x) = 100{,}000 + 3x$

b. $P(x) = \dfrac{100{,}000 + 3x}{x}$

c. The cost of producing one pizza is $100,003! But as more pizzas are produced, the total production cost per pizza goes down—allowing the producer to sell pizzas at a cheaper price.

EXAMPLE 5 **A rational function in disguise**

a. Show that $T(x) = \dfrac{1}{x - 2} + 3$ is a rational function.

b. Graph the function. What is the domain of $T(x)$?

SOLUTION **a.** Given $\quad T(x) = \dfrac{1}{x-2} + 3$

multiply 3 by $\dfrac{x-2}{x-2}(=1)$ $\quad = \dfrac{1}{x-2} + \dfrac{3(x-2)}{(x-2)}$

use the distributive law $\quad = \dfrac{1}{(x-2)} + \dfrac{3x-6}{(x-2)}$

combine terms with common denominators $\quad = \dfrac{3x-5}{x-2}$

$T(x)$ is now rewritten as the quotient of two polynomials.

b. The graph is shown in Figure 8.60. The domain is all real numbers except 2, since the denominator cannot equal zero.

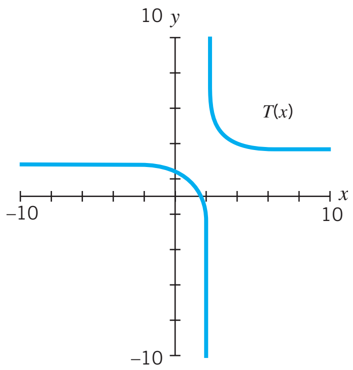


Figure 8.60 The graph of $T(x) = \dfrac{3x-5}{x-2}$.

Visualizing Rational Functions

Horizontal intercepts and vertical asymptotes

The graphs of rational functions can be quite complex. However, if $f(x)$ and $g(x)$ have no common factors, we do know two things about the graph $R(x) = \frac{f(x)}{g(x)}$.

- First, the horizontal intercepts of $f(x)$ are the horizontal intercepts of $R(x)$, since when $f(x) = 0$, $R(x) = 0$.
- Second, when $g(x) = 0$, then $R(x) = \frac{f(x)}{g(x)}$ is not defined since the denominator is 0. The function "blows up" at that value of x, which is called a *singularity*. The vertical line through that value of x is a *vertical asymptote* for the graph; that is, as the values of x approach the vertical asymptote, the values of the function approach $\pm\infty$.

EXAMPLE 6 **Finding horizontal intercepts and vertical asymptotes**

Given the rational function $f(x) = \frac{3x^2 - 13x - 10}{x^2 - 6x - 7}$:

a. Put the numerator and denominator into factored form.

b. What are the horizontal intercepts of $f(x)$?

c. What is the domain of the function? What are its vertical asymptotes?

d. Use technology to graph the function and confirm your answers in parts (b) and (c).

SOLUTION

a. Factoring, we get $f(x) = \frac{3x^2 - 13x - 10}{x^2 - 6x - 7} = \frac{(3x + 2)(x - 5)}{(x + 1)(x - 7)}$.

b. The horizontal intercepts occur when $f(x) = 0$ or, equivalently, when the numerator $(3x + 2)(x - 5) = 0$. This means either

$$3x + 2 = 0 \quad \text{or} \quad x - 5 = 0$$
$$3x = -2 \qquad\qquad x = 5$$
$$x = -\tfrac{2}{3}$$

So $f(x)$ has two horizontal intercepts, at $(-\frac{2}{3}, 0)$ and $(5, 0)$.

c. $f(x)$ is not defined when the denominator $(x + 1)(x - 7)$ is 0. This happens when either

$$x + 1 = 0 \quad \text{or} \quad x - 7 = 0$$
$$x = -1 \qquad\qquad x = 7$$

So the domain is all real numbers except the singularities at $x = -1$ and $x = 7$. The graph of $f(x)$ "blows up" at its singularities, creating two vertical asymptotes at the vertical lines $x = -1$ and $x = 7$.

d. Figure 8.61 shows the graph of this rational function, which confirms our answers in parts (b) and (c).

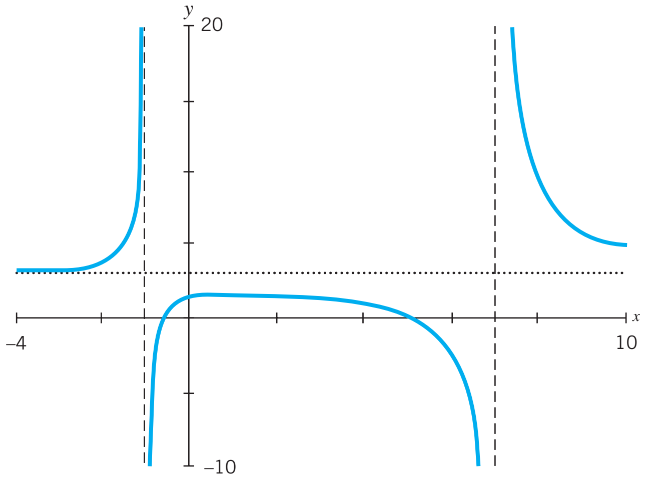


Figure 8.61 Graph of the rational function $f(x) = \frac{(3x + 2)(x - 5)}{(x + 1)(x - 7)}$.

The end behavior of rational functions: Additional asymptotes

What happens to the rational function $R(x) = \frac{p(x)}{q(x)}$ as $x \to \pm\infty$? This is called its end behavior. As $x \to \pm\infty$, the polynomial expressions $p(x)$ and $q(x)$ become dominated by their leading term—the one with the highest degree. For example, if we look at the rational function in Example 6, where $f(x) = \frac{3x^2 - 13x - 10}{x^2 - 6x - 7}$, then as $x \to \pm\infty$, $f(x) \approx \frac{3x^2}{x^2} = 3$. So as $x \to \pm\infty$, $f(x) \to 3$ (but never reaches 3). So $f(x)$ is asymptotic to the horizontal line $y = 3$. Examine Figure 8.61 to verify that this conclusion seems reasonable.

EXAMPLE 7 **End behavior**

Describe the behavior of the following rational functions as $x \to \pm\infty$. Then graph each function to confirm your descriptions.

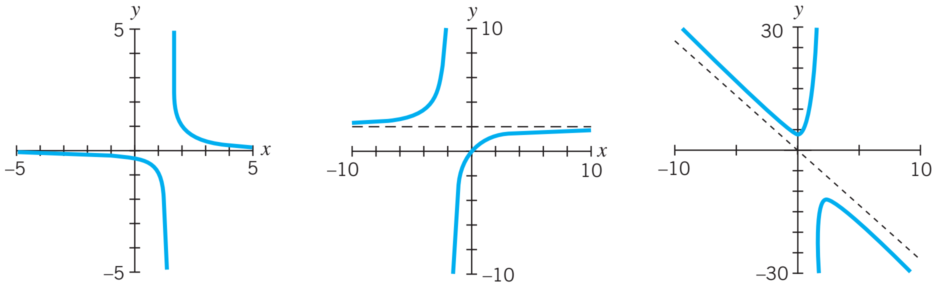
a. $f(x) = \dfrac{1}{2x - 3}$ **b.** $g(x) = \dfrac{6x + 1}{3x + 5}$ **c.** $h(x) = \dfrac{3x^2 + 2x - 4}{x - 1}$

SOLUTION As $x \to \pm\infty$, a rational function's behavior can be approximated by the ratio of the leading terms of the numerator and denominator.

a. As $x \to \pm\infty$, $f(x) = \dfrac{1}{2x - 3} \approx \dfrac{1}{2x}$. So as $x \to \pm\infty$, $f(x) \to 0$ (but never reaches 0). This means that $f(x)$ is asymptotic to the x-axis. (See Figure 8.62(a).)

b. As $x \to \pm\infty$, $g(x) = \dfrac{6x + 1}{3x + 5} \approx \dfrac{6x}{3x} = 2$. So as $x \to \pm\infty$, $g(x) \to 2$ (but never reaches 2). This means that $g(x)$ is asymptotic to the horizontal line $y = 2$. (See Figure 8.62(b).)

c. As $x \to \pm\infty$, $h(x) = \dfrac{-3x^2 + 2x - 4}{x - 1} \approx \dfrac{-3x^2}{x} = -3x$. So as $x \to \pm\infty$, $h(x) \to -3x$. This implies that $h(x)$ gets closer to the line $y = -3x$, but never touches it. So $h(x)$ is asymptotic to the line $y = -3x$. (See Figure 8.62(c).)



(a) Graph of $f(x) = \dfrac{1}{2x-3}$ (b) Graph of $g(x) = \dfrac{6x + 1}{3x + 5}$ (c) Graph of $h(x) = \dfrac{3x^2 + 2x - 4}{x - 1}$

Figure 8.62 Graphs of three rational functions.

When the end behavior is a horizontal line, we say the rational function has a *horizontal asymptote*. When the end behavior is a polynomial, we say it has an *oblique asymptote*.

Graphs of Rational Functions

For any rational function $R(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomials with no common factors and $q(x) \neq 0$, then:

The horizontal intercepts of R are the zeros of the numerator p; that is, the values of x where $p(x) = 0$.

The vertical asymptotes of R occur at the zeros of the denominator q; that is, the values of x where $q(x) = 0$.

The end behavior of R can be approximated by the ratio of the leading terms of p and q and is either a *horizontal* or an *oblique asymptote*.

Algebra Aerobics 8.7

Graphing program optional for Problems 5 and 6, and required for Problems 9 and 10.

1. Let $f(x) = x^3$, $g(x) = 2x - 1$, and $h(x) = \frac{1}{x}$. Evaluate each of the following.
   - **a.** $f(2)$
   - **b.** $g(2)$
   - **c.** $h(2)$
   - **d.** $(h \cdot g)(2)$
   - **e.** $(f + g)(2)$
   - **f.** $\left(\frac{h}{g}\right)(2)$
2. Let $Q(t) = 5 + 2t$ and $P(t) = 3t^3$. Evaluate each of the following.
   - **a.** $Q(1) + P(1)$
   - **b.** $Q(2) - P(2)$
   - **c.** $P(-1) \cdot Q(-1)$
   - **d.** $Q(3)/P(3)$
3. Let $f(t) = 3 - 2t$ and $h(t) = t^2 - 1$. Find general expressions for each of the following.
   - **a.** $f(t) - h(t)$
   - **b.** $f(t) + h(t)$
   - **c.** $f(t) \cdot h(t)$
   - **d.** $h(t)/f(t)$
4. Given $f(x) = x^2 + 2x - 3$ and $g(x) = \frac{1}{x - 1}$, find:
   - **a.** $f(x + 1)$
   - **b.** $f(x) + 1$
   - **c.** $g(x + 1)$
   - **d.** $g(x) + 1$
5. (Graphing program optional.) Identify any horizontal intercepts and any vertical asymptotes for the following functions. If possible, check your answers by graphing the functions with technology.
   - **a.** $f(x) = \dfrac{(x - 1)(x + 5)}{(x + 3)}$
   - **b.** $g(x) = \dfrac{(3x + 2)}{(x + 1)(x - 3)}$
6. (Graphing program optional.) Let $h(r) = r^2 - 4r - 12$ and $k(r) = r^2 - 4r + 3$. Construct the functions $f(r)$ and $g(r)$ and identify their horizontal intercepts and vertical asymptotes, if any exist. If possible, check your answers by using a graphing program.
   - **a.** $f(r) = \dfrac{h(r)}{k(r)}$
   - **b.** $g(r) = \dfrac{k(r)}{h(r)}$
7. Using the function $f(x) = \dfrac{1}{x^2}$,
   - **a.** Create a new function $g(x) = -f(x + 3) + 1$.
   - **b.** Show that $g(x)$ is a rational function of the form $g(x) = \dfrac{p(x)}{q(x)}$.
   - **c.** Identify the horizontal intercepts and vertical asymptotes of $g(x)$, if they exist.
8. If $f(x) = \frac{p(x)}{q(x)}$ is a rational function, what would you be looking for if you:
   - **a.** Set $f(x) = 0$?
   - **b.** Evaluated $f(0)$?
   - **c.** Set $p(x) = 0$?
   - **d.** Set $q(x) = 0$?

(Graphing program required.) For each function in Problems 9 and 10:

- **i.** Identify any horizontal and vertical intercepts.
- **ii.** Find any vertical asymptotes.
- **iii.** Describe the end behavior of the graph (and any other asymptotes).
- **iv.** Sketch the graph, labeling the intercepts.

9. $f(x) = \dfrac{2x + 6}{x - 3}$
10. $g(x) = \dfrac{x^2 + 2x - 3}{3x - 1}$

Exercises for Section 8.7

Some exercises recommend or require a graphing program.

1. Given $f(t) = 3t^2 + 4t - 5$ and $g(t) = 6t + 1$, find:
   - **a.** $f(t) + g(t)$
   - **b.** $g(t) - f(t)$
   - **c.** $f(t) \cdot g(t)$
   - **d.** $\dfrac{f(t)}{g(t)}$
2. Given $f(m) = \dfrac{3}{m - 4}$ and $g(m) = \dfrac{-3m}{2m - 5}$, find:
   - **a.** $f(m) + g(m)$
   - **b.** $(f - g)(m)$
   - **c.** $(f \cdot g)(m)$
   - **d.** $\dfrac{g(m)}{f(m)}$
   - **e.** $\left(\dfrac{f}{g}\right)(m)$
3. Let $f(x) = 3x^5 + x$ and $g(x) = x^2 - 1$.
   - **a.** Construct the following functions.

     $j(x) = f(x) + g(x)$, $k(x) = f(x) - g(x)$, $l(x) = f(x) \cdot g(x)$
   - **b.** Evaluate $j(2)$, $k(3)$, and $l(-1)$.
4. If $h(x) = f(x) \cdot g(x) = -x^2 - 3x + 4$, what are possible equations for $f(x)$ and $g(x)$?
5. You own a theater company and you have an upcoming event.
   - **a.** You decide to charge \$25 per ticket. Construct a basic ticket revenue function $R(n)$ (in dollars), where n is the number of tickets sold.
   - **b.** You need to pay \$500 to keep the box office open for ticket sales. Modify $R(n)$ to reflect this.
   - **c.** You decide to give 30 free tickets to the patrons of your company. Modify your function in part (b) to reflect this.
6. Many colleges around the country are finding they need to buy more computers every year, not only to replace broken or outmoded computers, but also because of the increasing use of computers in classrooms, labs, and studios. A college administrator is preparing a 5-year budget plan. She anticipates that her college, which now has 120 computers, will have to increase that amount by 40 per year for the next 5 years. She currently pays \$1000 per computer, but she expects the costs will go up by 3% per year because of inflation.

a. Construct a function $N(t)$ for the number of computers each year as a function of time t (in years since the present).

b. Construct a function $C(t)$ for the individual cost of a computer purchased in year t.

c. Construct a function that will describe the total cost of the computers each year.

d. For year 5, how much money should the budget allow for computers?

7. A worker gets \$20/hour for a normal work week of 40 hours and time-and-a-half for overtime. Assuming he works at least 40 hours a week, construct a function describing his weekly paycheck as a function of the number of hours worked.

8. (Graphing program required.) Using the accompanying table, evaluate the following expressions in parts (a)–(d).

x	0	1	2	3	4	5
$f(x)$	-3	-1	5	15	29	47

x	0	1	2	3	4	5
$g(x)$	-3	-5	-11	-21	-35	-53

a. $(f+g)(2)$ **b.** $(g-f)(0)$ **c.** $(f \cdot g)(3)$ **d.** $\left(\frac{g}{f}\right)(1)$

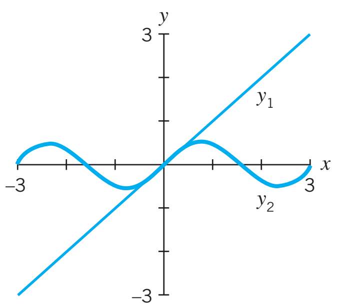
9. Use the table in Exercise 8 to create a new table for the functions

a. $h(x) = (f+g)(x)$ **c.** $k(x) = (f \cdot g)(x)$

b. $j(x) = (g-f)(x)$

10. One method of graphing functions is called "addition of ordinates." For example, to graph $y = x + \frac{1}{x}$ using this method, we would first graph $y_1 = x$. On the same coordinate plane, we would then graph $y_2 = \frac{1}{x}$. Then we would estimate the y-coordinates (called ordinates) for several selected x-coordinates by adding geometrically on the graph itself the values of y_1 and y_2 rather than by substituting numerically. This technique is often used in graphing the sum or difference of two different types of functions by hand, without the use of a calculator.

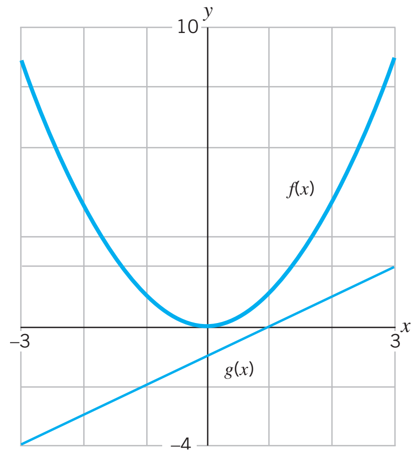
a. Use this technique to sketch the graph of the sum of the two functions graphed in the accompanying figure.



b. Use this technique to sketch the graph of $y = -x^2 + x^3$ for $-2 \le x \le 2$. Then use a graphing tool (if available) and compare.

c. Use this technique to graph $y = 2^x - x^2$ for $-2 \le x \le 5$.

11. Using the accompanying graph of $f(x)$ and $g(x)$, find estimates for the missing values in the following table.



x	-3	-2	-1	0	1	2	3
$f(x)$							
$g(x)$							
$f(x) + g(x)$							
$f(x) - g(x)$							
$f(x) \cdot g(x)$							
$g(x)/f(x)$							

12. From the graph and your results in Exercise 11, find the equations for:

a. $f(x)$ **c.** $(f+g)(x)$ **e.** $(f \cdot g)(x)$

b. $g(x)$ **d.** $(f-g)(x)$ **f.** $\left(\frac{g}{f}\right)(x)$

13. The Richland Banquet Hall charges \$500 to rent its facility and \$40 per person for dinner. The hall holds a minimum of 25 people and a maximum of 100. A sorority decides to hold its formal there, splitting all the costs among the attendees. Let n be the number of people attending the formal.

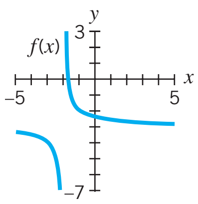
a. Create a function $C(n)$ for the total cost of renting the hall and serving dinner.

b. Create a function $P(n)$ for the cost per person for the event.

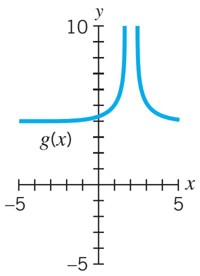
c. What is $P(25)$? $P(100)$? What do these numbers represent?

14. If $f(x) = \frac{1}{x^2}$, find $-f(x) + 2f(x-3)$; then find a common denominator and combine into one rational expression.

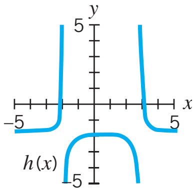
15. For each rational function graphed below, estimate the equation for any vertical or horizontal asymptote(s).



Graph *A*



Graph *B*



Graph *C*

For each of the functions in Exercises 16–18, identify any horizontal intercepts and vertical asymptotes. Then, if possible, use technology to graph each function and verify your results.

16. $f(x) = \dfrac{3x - 13}{x - 4}$

17. $g(x) = -\dfrac{2}{(x + 3)^2}$

18. $h(x) = \dfrac{1}{(x + 3)(x - 1)} - 2$

19. (Graphing program required.)

a. What is the domain of the rational function $S(x) = \dfrac{x}{x^2 + 1}$?

b. Does the function have any horizontal intercepts? Any vertical asymptotes?

c. What is its end behavior?

d. Graph the function. The graph is one of a set of curves called "serpentine" by Isaac Newton. Why would that name be appropriate?

20. (Graphing program required.) Let $g(x) = \dfrac{18}{(x - 1)^2} - 2$.

a. We can think of $g(x)$ as being created from transformations of the function $f(x) = \frac{1}{x^2}$. Describe the transformations and then write $g(x)$ as a function of $f(x)$.

b. Show that $g(x)$ is a rational function of the form $\frac{p(x)}{q(x)}$. (*Hint:* Find a common denominator.)

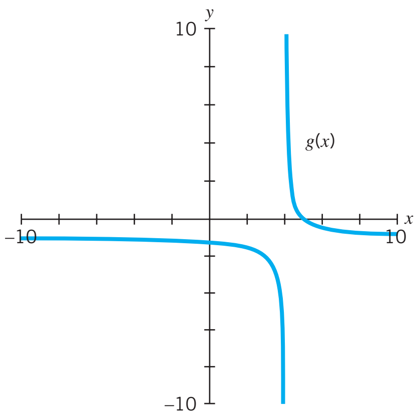
c. Identify any horizontal intercepts and any vertical asymptotes.

d. What is its end behavior?

e. Use graphing technology to confirm your answers and estimate the horizontal asymptote.

21. (Graphing program required.) Construct a rational function $f(x)$ that has horizontal intercepts at $(-3, 0)$ and $(4, 0)$ and vertical asymptotes at the lines $x = 1$ and $x = -5$. Use technology to sketch the graph of $f(x)$.

22. The function $f(x) = 1/x$ was transformed into the function $g(x)$ plotted on the accompanying graph. Construct $g(x)$ in terms of $f(x)$ and then write $g(x)$ in rational function form $g(x) = \frac{p(x)}{q(x)}$.

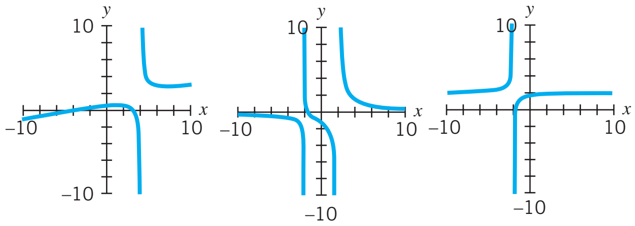


23. Without using technology, match each function with its graph.

a. $f(x) = -\dfrac{2}{3(x + 2)} + 2$

b. $g(x) = \dfrac{3x + 5}{x^2 - 4}$

c. $h(x) = \dfrac{x^2 - 9}{5x - 20}$

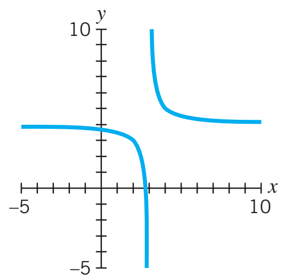


Graph *A* Graph *B* Graph *C*

24. (Graphing program required for part (c).) The rational function $g(x) = \frac{4x - 11}{x - 3}$ can be decomposed into a sum by using the following method:

write as sum of a fraction and a constant	$\dfrac{4x - 11}{x - 3} = \dfrac{A}{x - 3} + B$
find the common denominator	$\dfrac{4x - 11}{x - 3} = \dfrac{A}{x - 3} + \dfrac{B(x - 3)}{x - 3}$
multiply and simplify	$\dfrac{4x - 11}{x - 3} = \dfrac{A + Bx - 3B}{x - 3}$
Set numerators equal	$4x - 11 = Bx + A - 3B$
Set x values equal	$4x = Bx$
So	$B = 4$
Set the constants equal substitute 4 for B	$-11 = A - 3B$ $-11 = A - 3(4)$
So	$A = 1$
Therefore	$g(x) = \dfrac{4x - 11}{x - 3} = \dfrac{1}{x - 3} + 4$

which is the graph of $f(x) = \frac{1}{x}$ shifted to the right by three units, then shifted up by four units. See the accompanying graph.



a. Use the preceding method to decompose $g(x) = \frac{5x + 22}{x - 3}$.

b. Describe the transformation of the function $f(x) = \frac{1}{x}$ into $g(x) = \frac{5x + 22}{x - 3}$.

c. Using technology, plot the graphs of $f(x)$ and $g(x)$ to verify that the transformation described in part (b) is correct.

8.8 Composition and Inverse Functions

Composing Two Functions

Sometimes it is useful to use the output from one function as the input for another function. As an example, let's look at some simple "parent" functions. Assuming one's mother is a unique person (ignoring complexities such as adoption or cloning), we can define a "mother" function $M(p)$ as the mother of p. Similarly, we can define a "father" function $F(p)$ as the (unique) father of p. Then what would the expression $F(M(p))$ mean? We read the expression from the inside out. Starting with a person p, we apply the mother function to p and then apply the father function to the output. For example, if you are the person p, then

$$F(M(\text{you})) = F(\text{your mother}) = \text{the father of your mother}$$

In other words, this is your grandfather on your mother's side.

When we apply M and then F as above, we call it the *composition* of F and M, a new function denoted by $F \circ M$. We define

$$(F \circ M)(p) = F(M(p))$$

Warning: Be careful not to confuse the product of two functions $f \cdot g$ and the composition $f \circ g$. The product $(f \cdot g)(x) = f(x) \cdot g(x)$ means to evaluate f and g both at x and then multiply the results. The composition $(f \circ g)(x) = f(g(x))$ means to evaluate g at x, and then evaluate f at $g(x)$.

EXAMPLE 1 **Genealogy**

Using the parent functions, who is:

a. $(M \circ F)(\text{you})$? **b.** $(M \circ M)(\text{you})$? **c.** Your grandfather on your father's side?

SOLUTION

a. $(M \circ F)(\text{you}) = M(F(\text{you})) = M(\text{your father}) =$ the mother of your father, or your grandmother on your father's side.

b. $(M \circ M)(\text{you}) = M(M(\text{you})) = M(\text{your mother}) =$ the mother of your mother, or your grandmother on your mother's side.

c. Your grandfather on your father's side is the father of your father, that is, $F(F(\text{you}))$.

> **The Composition of Two Functions**
>
> If $f(x)$ and $g(x)$ are two functions, then the function $f \circ g$, called the *composition of f and g*, is defined by
>
> $$(f \circ g)(x) = f(g(x))$$

EXAMPLE 2 **The order of composition matters**

Let $f(x) = x^2$ and $g(x) = \frac{1}{x+1}$.

a. Evaluate $(f \circ g)(3)$.

b. Determine a general expression for $(f \circ g)(x)$.

c. Does $(f \circ g)(x) = (g \circ f)(x)$? Explain your answer.

SOLUTION

a. We have:

by definition	$(f \circ g)(3)$	$= f(g(3))$
evaluate $g(x)$ when $x = 3$		$= f\left(\frac{1}{3+1}\right)$
simplify		$= f\left(\frac{1}{4}\right)$
evaluate $f(x)$ when $x = \frac{1}{4}$		$= \left(\frac{1}{4}\right)^2$
or		$= \frac{1}{16}$

b. By definition $(f \circ g)(x) = f(g(x))$

substitute for $g(x)$ $= f\left(\frac{1}{x+1}\right)$

evaluate f $= \left(\frac{1}{x+1}\right)^2$

c. By definition $(g \circ f)(x) = g(f(x))$

substitute for $f(x)$ $= g(x^2)$

evaluate g $= \frac{1}{x^2+1}$

Since $\left(\frac{1}{x+1}\right)^2 \neq \frac{1}{x^2+1}$ (try evaluating both sides when $x = 1$), then $(f \circ g)(x) \neq (g \circ f)(x)$. So the order of composition does matter.

EXAMPLE 3 Toxic plumes

For two weeks during April 2004 the Pentagon, in its efforts to fight terrorism, conducted a study that involved the release of simulated airborne toxins at a fixed starting point. Scientists then measured the dispersion path and spread rate of the pseudo-toxic plume, using sensors placed in concentric circles from the point of release. The circles had radii ranging from a few feet to several thousand feet.

a. Assume that wind speed is not only the major, but the only dispersion factor. If the wind speed is 10 mph, or about 0.17 miles per minute (from any direction), construct an equation that calculates the circular area in square miles (within which the plume would lie) as a function of time in minutes.

b. What circular area would contain the "toxic" plume after 10 minutes? After 30 minutes?

c. If the wind speed doubles, will the area double?

SOLUTION

a. The area $A(r)$ (in square miles) of the "toxic" circle is a function of its radius r (in miles),

$$A(r) = \pi r^2$$

In this case the radius r is not fixed but is a function $R(t)$ of time t (in minutes). So

$$r = R(t)$$

The wind is traveling at about 0.17 mile per minute. So if we start at the release point (when $t = 0$), then the radius $R(t)$ of the "toxic" circle is

$$R(t) = 0.17t$$

where t is in minutes and $R(t)$ is in miles. Substituting $R(t)$ for r in the equation for $A(r)$ gives us

$$A(R(t)) = A(0.17t) = \pi(0.17t)^2 \approx 0.09t^2$$

which represents the circular area (in square miles) at minute t that encloses the "toxic" plume.

b. Assuming the wind continues to blow at 10 mph (or 0.17 mile per minute), then after 10 minutes the circular area would be $A(R(10)) \approx 0.09(10)^2 = 9$ square miles. After 30 minutes, the circular area would be $A(R(30)) \approx 0.09(30)^2 = 81$ square miles, which is larger than all of Washington, D.C.

c. If the wind doubles from 10 to 20 mph, the radius at any point in time will be twice as large. Our new radius function will now be $R_{\text{new}}(t) = 2 \cdot 0.17t = 0.34t$. But the area would be four times as large, since now

$$A(R_{\text{new}}(t)) = A(0.34t) = \pi(0.34t)^2 \approx 0.36t^2 = 4 \cdot (0.09t^2) = 4 \cdot A(R(t))$$

or four times the area when the wind was at 10 mph. This makes sense if you think of the composite function $(A \circ R)$ as a direct proportionality with an exponent of 2.

Composing More Than Two Functions

We can think of the transformations of a single function (see Section 8.6) as the result of the composition of two or more functions. For example, the transformation of $f(x) = \frac{1}{x^2}$ to $2f(x-1) = \frac{2}{(x-1)^2}$ can be thought of as the composition of three functions:

$$f(x) = \frac{1}{x^2} \quad g(x) = 2x \quad \text{and} \quad h(x) = x - 1$$

Since

$$(g \circ f \circ h)(x) = g(f(h(x))) = g(f(x-1)) = g\left(\frac{1}{(x-1)^2}\right) = \frac{2}{(x-1)^2}$$

It is easy to construct the three functions if you just think of the steps taken to transform $\frac{1}{x^2}$ into $\frac{2}{(x-1)^2}$. Starting with x, first subtract 1 (applying h), then square and place the result into the denominator (applying f), and finally multiply by 2 (applying g).

EXAMPLE 4 Given $f(x) = \frac{1}{x^3}$, rewrite $\left(\frac{3}{4}\right)f(x+5) = \frac{3}{4(x+5)^3}$ as the composition of three functions.

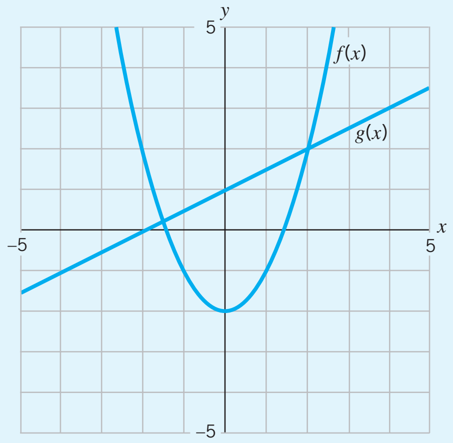
SOLUTION Letting $f(x) = \frac{1}{x^3}$, $g(x) = \left(\frac{3}{4}\right)x$, and $h(x) = x + 5$, the composition

$$(g \circ f \circ h)(x) = g(f(h(x))) = g(f(x+5)) = g\left(\frac{1}{(x+5)^3}\right) = \frac{3}{4(x+5)^3}$$

Algebra Aerobics 8.8a

1. Given $f(x) = 2x + 3$ and $g(x) = x^2 - 4$, find:
   - **a.** $f(g(2))$
   - **b.** $g(f(2))$
   - **c.** $f(g(3))$
   - **d.** $f(f(3))$
   - **e.** $(f \circ g)(x)$
   - **f.** $(g \circ f)(x)$
2. Given $P(t) = \frac{1}{t}$ and $Q(t) = 3t - 5$, find:
   - **a.** $(P \circ Q)(2)$
   - **b.** $(Q \circ P)(2)$
   - **c.** $(Q \circ Q)(3)$
   - **d.** $(P \circ Q)(t)$
   - **e.** $(Q \circ P)(t)$
3. Given $F(x) = \frac{2}{x-1}$ and $G(x) = 3x - 5$, find:
   - **a.** $(F \circ G)(x)$
   - **b.** $(G \circ F)(x)$
   - **c.** Are the composite functions in parts (a) and (b) equal?

Problems 4, 5, and 6 refer to the accompanying graph of $f(x)$ and $g(x)$.



4. Using the graphs of $f(x)$ and $g(x)$, determine the values of:
   - **a.** $f(-2)$
   - **b.** $g(-2)$
   - **c.** $f(0)$
   - **d.** $g(0)$
   - **e.** $(g \circ f)(-2)$
   - **f.** $(f \circ g)(-2)$
   - **g.** $(g \circ f)(0)$
   - **h.** $(f \circ g)(0)$
5. Using your knowledge of quadratic and linear functions, create algebraic expressions for the following:
   a. $f(x)$ **b.** $g(x)$ **c.** $(g \circ f)(x)$ **d.** $(f \circ g)(x)$
6. Using your functions from Problem 5, evaluate $(g \circ f)(-2)$ and $(f \circ g)(-2)$. Do they agree with your answers in Problem 4, parts (e) and (f)?
7. From the table, determine the values of the following compositions in parts (a) and (b).

x	0	1	2	3	4	5
$f(x)$	3	-1	1	3	5	7
$g(x)$	5	3	2	1	0	-1
$h(x)$	2	2	3	4	5	6

   a. $(h \circ f \circ g)(4)$ **b.** $(f \circ h \circ g)(1)$
8. Use the given functions f, g, and h to evaluate the compositions.

$$f(x) = \frac{1}{x-5}, \qquad g(x) = \sqrt{x+1}, \qquad h(x) = 3$$

   a. $(h \circ f \circ g)(3)$
   b. $(f \circ g \circ h)(100)$

Inverse Functions: Returning the Original Value

Sometimes when we compose two functions, the input is equal to the output. For example, if $f(x) = \log x$ and $g(x) = 10^x$, then

$$(f \circ g)(x) = f(g(x)) = f(10^x) = \log(10^x) = x$$

and

$$(g \circ f)(x) = g(f(x)) = g(\log x) = 10^{\log x} = x$$

What one function "does," the other "undoes." We say that f and g are *inverse functions* of each other. We denote this as $f^{-1} = g$ and $g^{-1} = f$. The notation f^{-1} is read as "f inverse" or "the inverse of f." A function that has an inverse is called *invertible.*

Inverse Functions

The functions $f(x)$ and $g(x)$ are *inverse functions* of each other if

$$(f \circ g)(x) = x \qquad \text{and} \qquad (g \circ f)(x) = x$$

We write $f^{-1} = g$ and $g^{-1} = f$.

Warning: The "-1" in the notation f^{-1} can be confusing. It is not an exponent, so f^{-1} does not mean $1/f$. The symbol "-1" simply indicates that f^{-1} is the inverse function of f.

EXAMPLE 5 Show that $f(x) = \ln x$ and $g(x) = e^x$ are inverses of each other.

SOLUTION

Since $\quad (f \circ g)(x) = f(g(x)) = f(e^x) = \ln(e^x) = x$

and $\quad (g \circ f)(x) = g(f(x)) = g(\ln x) = e^{\ln x} = x \quad (\text{where } x > 0)$

the functions are inverses of each other. So $f^{-1} = g$ (and equivalently $g^{-1} = f$).

EXAMPLE 6 Show that each pair of functions contains a function and its inverse.

a. $f(x) = 2x - 3$ and $g(x) = \frac{x+3}{2}$

b. $h(x) = x^3 + 1$ and $j(x) = (x - 1)^{1/3}$

SOLUTION

a. $(f \circ g)(x) = f(g(x)) = f((x+3)/2) = 2\left(\frac{x+3}{2}\right) - 3 = (x+3) - 3 = x$

$(g \circ f)(x) = g(f(x)) = g(2x - 3) = \frac{(2x-3)+3}{2} = x$

b. $(h \circ j)(x) = h(j(x)) = h(x-1)^{1/3} = [(x-1)^{1/3}]^3 + 1 = (x-1) + 1 = x$

$(j \circ h)(x) = j(h(x)) = j(x^3 + 1) = [(x^3 + 1) - 1]^{1/3} = (x^3)^{1/3} = x$

Changing perspectives

To get a better understanding of the relationship between a function and its inverse, let's return to the sales tax example from Chapter 1. There $T = 0.06P$ was the function that calculated the 6% sales tax (T) on the price (P). Here P is treated as the input and T the output; that is, given the price we can compute the tax. If we call the function rule F, we have

$$F(P) = T = 0.06P$$

But we also solved for P in terms of T, to get $P = T/0.06$. Now we have changed our perspective. We are thinking of T as the input and P the output; that is, given the tax paid, we can determine the original price. This function is the inverse of F (i.e., F^{-1}).

Table 8.4 and Figure 8.63 are the table and graph for the function

$$F(P) = T = 0.06P, \text{ where } P \text{ is the input and } T \text{ is the output}$$

Input, P	Output, T
0	0.00
2	0.12
4	0.24
6	0.36
8	0.48
10	0.60

Table 8.4 Table for $T = 0.06P$.

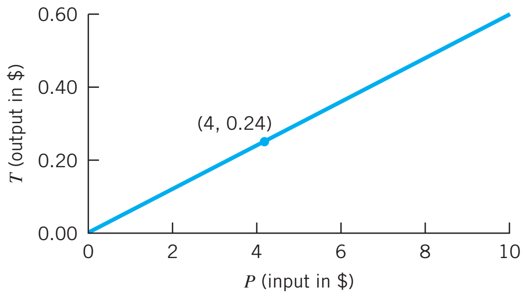


Figure 8.63 Graph of $T = 0.06P$.

Table 8.5 and Figure 8.64 are the table and graph of the inverse function:

$$F^{-1}(T) = P = \frac{T}{0.06}, \qquad \text{where } T \text{ is the input and } P \text{ is the output}$$

Input, T	Output, P
0.00	0
0.12	2
0.24	4
0.36	6
0.48	8
0.60	10

Table 8.5 Table for $P = \frac{T}{0.06}$.

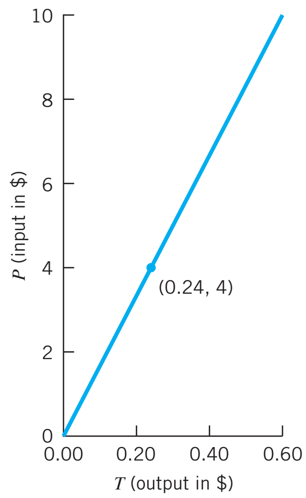


Figure 8.64 Graph of $P = \frac{T}{0.06}$.

The function F and its inverse F^{-1} represent exactly the same information, but from different viewpoints. Comparing Tables 8.4 and 8.5, we see that the columns have been swapped, indicating the reversal of the roles for the input (or independent variable) and the output (the dependent variable). Comparing the graphs (Figures 8.63 and 8.64), we see that the axes have been swapped. So, for example, while the point (4, 0.24) lies on the first graph (Figure 8.63), the point (0.24, 4) lies on the second graph (Figure 8.64). We could write this more formally as

$$F(P) = T \qquad \text{means that} \qquad F^{-1}(T) = P$$

so

$$F(4) = 0.24 \qquad \text{means that} \qquad F^{-1}(0.24) = 4$$

EXAMPLE 7 **Evaluating a function and its inverse**

Given $F(P) = 0.06P$ and $F^{-1}(T) = \frac{T}{0.06}$, evaluate and interpret $F(6)$ and $F^{-1}(6)$.

SOLUTION $F(6) = 0.06 \cdot 6 = \$0.36$, so if the price is \$6, the sales tax is \$0.36.

$F^{-1}(6) = 6/0.06 = \$100$, so if the sales tax is \$6, the price is \$100.

EXAMPLE 8 **Finding the inverse from a table**

If the following table represents a function mapping certain individuals to their cell phone numbers, construct the table of the inverse function.

Individual	Cell Phone Number
Janet Davidson	810 547-1832
John Harbison	919 287-3557
Elaine Woo	202 555-6911
Jon Stewart	212 376-1234

SOLUTION

Cell Phone Number	Individual
810 547-1832	Janet Davidson
919 287-3557	John Harbison
202 555-6911	Elaine Woo
212 376-1234	Jon Stewart

Does every function have an inverse? (Answer: No)

Not every function is invertible.

EXAMPLE 9 Does the function $y = x^2$ have an inverse?

SOLUTION If we try to solve $y = x^2$ for x in terms of y, we get the equation $x = \pm\sqrt{y}$. But for any positive value of y, there will be two values for x, namely, $+\sqrt{y}$ and $-\sqrt{y}$. For instance, if the input is $y = 100$, then the output could be either $x = -10$ or $x = 10$. So this new equation is not a function, and hence the original function, $y = x^2$, has no inverse.

EXAMPLE 10 Does the mother function, $M(p)$, from the previous section have an inverse?

SOLUTION Recall that $M(p)$ is the mother of a person p. Each person has only one mother (according to our definition), but each mother could have several children. For example, if both Fred and Jenny have the same mother, Sarah, then $M(\text{Fred}) = \text{Sarah} = M(\text{Jenny})$. (See Figure 8.65.)

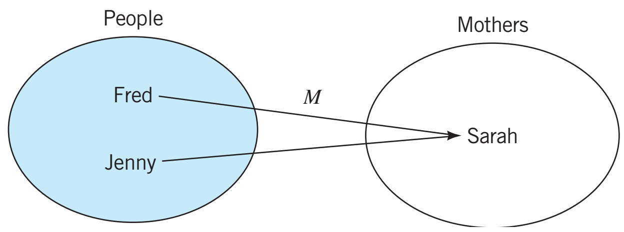


Figure 8.65 The mother function, M, showing that Fred and Jenny have the same mother, Sarah.

There cannot be an inverse function for M, since if the mother, Sarah, were now the input, her output would be both Fred and Jenny.

Functions that have an inverse must be "one-to-one"

In the last two examples (9 and 10), the original function had two different input values associated with the same output value, making it impossible to construct an inverse function. In order to be a function, each input for an inverse function must have a single output. But the inputs for an inverse function are the outputs of the original function. So for a function f to have an inverse function f^{-1}, then f must have only one input for each output value. Technically, that means if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. We call such functions *one-to-one*, which can abbreviated as *1-1*.

A good example of a 1-1 function is Social Security numbers. Each American is assigned a unique Social Security number. (See Figure 8.66.) We can think of this process as a function S that assigns to each person p a unique Social Security number $n = S(p)$. We can define an inverse function $S^{-1}(n) = p$ that maps any existing Social Security number back to its unique owner, p. (See Figure 8.67.)

S: The Social Security function is 1-1

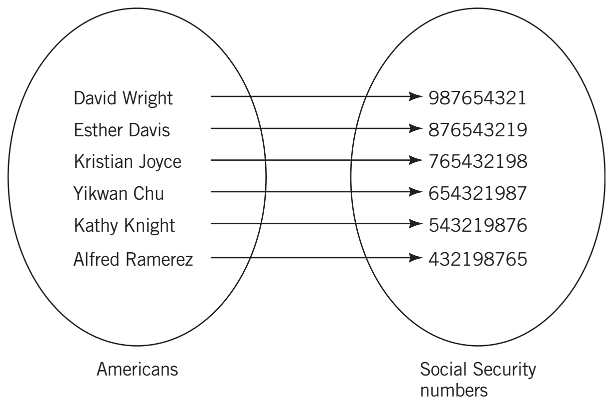


Figure 8.66 Mapping people to their Social Security numbers.

S^{-1}: The inverse of the Social Security function

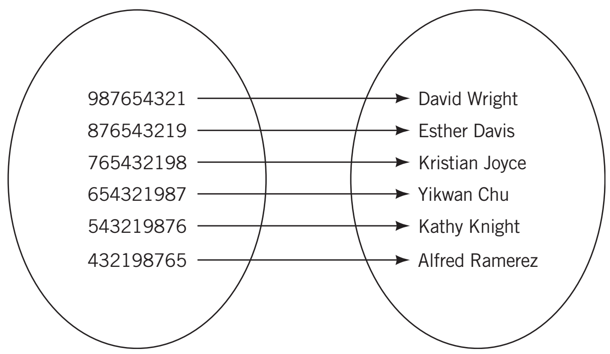


Figure 8.67 Mapping Social Security numbers to people.

The sales tax function is also 1-1 and has an inverse. But we've just seen that the function $y = x^2$ and the mother function, M, are not 1-1 and hence don't have inverses.

> A function $f(x)$ is called *one-to-one* (abbreviated as *1-1*) if no two distinct input values are mapped to the same output value; that is, whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
>
> Any one-to-one function f has an inverse function f^{-1} (restricted to an appropriate domain).

Finding the equation for an inverse function

If an invertible function is defined by a simple equation, it can be easy to find its inverse.

EXAMPLE 11 The circumference C of a sphere is a function of the radius r, $C = 2\pi r$. But in real-world situations, it is often easier to measure the circumference than the radius of a sphere.

a. Find the inverse function by solving for r in terms of C.

b. If the circumference of Earth is roughly 25,000 miles, estimate Earth's radius.

SOLUTION

a. C is a 1-1 function of r, so solving for r gives us the inverse function $r = \frac{C}{2\pi}$.

b. Letting $C = 25{,}000$ miles, we have $r = \frac{25{,}000}{2\pi} = \frac{12{,}500}{\pi} \approx 4000$ miles.

It is a little more complicated when we use function notation. To find the inverse of the 1-1 function $f(x) = 1 + \log x$, we need to name the output variable, say y, to get $y = 1 + \log x$. We can then solve this equation for x in terms of y.

Given	$y = 1 + \log x$	(where $x > 0$)
subtract 1 from each side	$y - 1 = \log x$	
write as a power of 10	$10^{y-1} = 10^{\log x}$	
simplify and switch sides	$x = 10^{y-1}$	

So we could write $f^{-1}(y) = 10^{y-1}$ or, more conventionally, since the variables are abstract (and unitless) we could use x as the input to get

$$f^{-1}(x) = 10^{x-1}$$

EXAMPLE 12

a. What is the domain of the function $f(x) = \frac{1}{x+1}$? Is f one-to-one?

b. If so, find its inverse f^{-1} and specify its domain.

SOLUTION

a. The domain of f is all real numbers except -1. The function $f(x)$ is 1-1 since if $f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1 + 1} = \frac{1}{x_2 + 1} \Rightarrow x_2 + 1 = x_1 + 1 \Rightarrow x_2 = x_1$. So no two distinct input values can be mapped to the same output value.

b. Thus f has an inverse function that we can determine by naming the output value $y = f(x)$ and solving the following formula for x in terms of y.

Given	$y = \dfrac{1}{x+1}$	(where $x \neq -1$)
cross-multiply	$x + 1 = \dfrac{1}{y}$	
subtract 1 from each side	$x = \dfrac{1}{y} - 1$	

So the inverse function is $f^{-1}(y) = \left(\frac{1}{y}\right) - 1$, where the domain is all real numbers except 0. Since this is an abstract function, we can use any variable name for the input. So following the standard convention of using x as the input, we could rewrite f^{-1} as

$$f^{-1}(x) = \frac{1}{x} - 1 \quad (\text{where } x \neq 0)$$

> **? SOMETHING TO THINK ABOUT**
>
> *Why must the graph of a function pass the horizontal line test in order to have an inverse?*

How to tell if graph represents a 1-1 function: The horizontal line test

It is easy to tell from its graph whether a function is 1-1 and hence has an inverse (on an appropriate domain). The graph must pass the *horizontal line test*, which means that no horizontal line can cross the graph twice. (Why?)

Compare Figures 8.68 and 8.69. Figure 8.68 shows the graph (from Section 8.3) of the height $h(d)$ of a projectile fired from a battleship gun as a function of d, its distance from the ship.

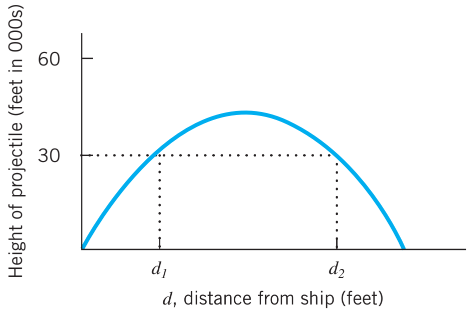


Figure 8.68 Graph of height vs. distance for a projectile fired from a battleship.

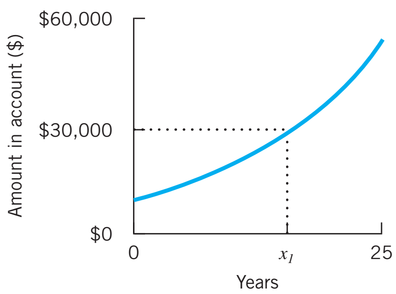


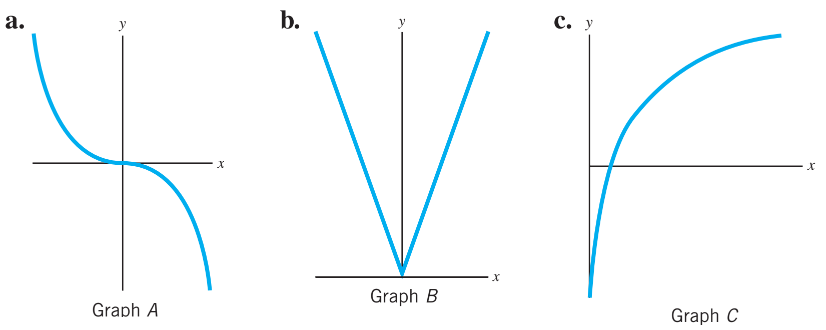
Figure 8.69 Amount in an account over time.

Choose any output height value, say 30 (thousand feet), that is below the maximum height. Then there will be two corresponding inputs, that is, two different distances from the battleship, d_1 and d_2, for which the projectile will be at 30 (thousand feet). So $d_1 \neq d_2$, but $h(d_1) = h(d_2) = 30{,}000$. (See Figure 8.68.) So the horizontal line at $h(d) = 30$ (thousand feet) crosses the graph twice. Hence the function h is not 1-1 and cannot have an inverse.

Figure 8.69 shows the amount $A(x)$ at year x, in an account that started with \$10,000 and grew by 7% a year. If we choose any output value (which must be $\geq$\$10,000), there will be associated with it a unique input value. For example, if we choose an output of \$30,000, we can see from Figure 8.69 that it will be associated with one and only one input value, labeled x_1. So the function A is 1-1, and there is an inverse function A^{-1} whose domain is restricted to values $\geq$\$10,000.

> Recall that a *function* must pass the *vertical line test*; that is, no vertical line crosses its graph more than once.
>
> A *one-to-one function* must pass the *horizontal line test*; that is, no horizontal line crosses its graph more than once.

EXAMPLE 13 Which of the functions in the accompanying graphs have inverses?



SOLUTION The functions in Graphs A and C would each have an inverse, since both function graphs pass the horizontal line test. The function in Graph B would not have an inverse, since we could construct a horizontal line (many, in fact) that would cross the graph twice.

EXAMPLE 14 Visualizing their graphs, identify which functions from the following function families have inverses.

a. Linear functions (of the form $y = b + mx$)

b. Exponential functions (of the form $y = Ca^x$)

c. Logarithmic functions (of the form $y = a\log(bx)$ or $y = a\ln(bx)$)

d. Power functions (of the form $y = kx^a$)

SOLUTION **a.** Every linear function, except those representing horizontal lines (where $m = 0$), has an inverse.

b, c. Every exponential and logarithmic function has an inverse.

d. Odd integer power functions have inverses, but even integer power functions (such as $y = ax^2$ or $y = ax^{-2}$) do not.

A Final Example

The following example combines many of the ideas covered in the course—exponential, log, and power functions as well as compositions and inverses. So it's a lengthier example than most, but it involves a reasonable model for an important issue—breast cancer.

EXAMPLE 15 **Modeling breast cancer tumor growth**

a. Depending on the aggressiveness of a breast tumor, its volume could double in weeks or months. On average, the volume of a breast tumor doubles every 100 days.[5] What is the daily growth factor? The yearly growth factor?

b. Describe the tumor volume as a function of time. Use an initial volume of 0.06 cubic cm (the minimum tumor volume detectable by a mammogram) and measure time in years after the tumor reached that volume.[6]

c. Breast tumors are roughly spherical. So their size is usually reported in terms of the diameter, which is easier to measure than the volume. Construct a function that describes the tumor diameter as a function of its volume.

d. Now compose the functions in parts (b) and (c) to create a new function that gives the tumor diameter as a function of time. What is the minimum tumor diameter a mammogram can detect?

e. Using technology, graph the function in part (d). What do the points where $t < 0$ represent?

f. Estimate and then calculate how many years it would take the tumor to reach a diameter of 2 cm, the smallest size detectable by touch. This is also the maximum size for a stage I breast cancer and is often used as the decision point for recommending a lumpectomy vs. a mastectomy.

[5]Breast cancer data are from the afterword by Dr. Susan M. Love, MD, of Joyce Wadler's *My Breast: One Woman's Cancer Story* (Reading, MA: Addison-Wesley, 1992).
[6]*Note:* The tumor may have been growing for 10 years before being detectable by a mammogram.

SOLUTION

a. *Growth factors*. Since the tumor volume has a fixed doubling time, its growth is exponential. Since the tumor volume doubles every 100 days, if a is the *daily* growth factor, then

$$2 = a^{100}$$

$$\text{taking the } \tfrac{1}{100}\text{th root} \qquad 2^{1/100} = a$$

The *yearly* growth factor is the daily growth factor, $2^{1/100}$, applied 365 times, or

$$(2^{1/100})^{365} = 2^{365/100} = 2^{3.65} \approx 12.6$$

So the yearly growth factor is 12.6. (Recall this means a growth rate of 11.6 (in decimal form) or 1,160% per year!)

SOMETHING TO THINK ABOUT

What is the yearly percentage increase in tumor volume?

b. *Tumor volume as a function of time*. Assuming an initial tumor volume of 0.06 cubic cm and a yearly growth factor of 12.6, the tumor volume $V(t)$ is a function of time t given by

$$V(t) = 0.06(12.6)^t \tag{1}$$

where $V(t)$ is measured in cubic centimeters and t in number of years since the tumor reached 0.06 cubic cm in volume.

c. *Tumor diameter as a function of volume.* The volume v of a sphere is a function of the radius r, where $v = \left(\frac{4}{3}\right)\pi r^3$. Since we are interested in the diameter d (twice the radius), we can substitute $r = \frac{d}{2}$ into the equation to get

$$v = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = \frac{4\pi d^3}{3 \cdot 8} = \frac{\pi d^3}{6} \tag{2}$$

We now have the volume as a function of the diameter.

We need the inverse of that function, one that describes the diameter as a function of the volume. We can solve Equation (2) for d in terms of v:

$$\text{Given} \qquad v = \frac{\pi d^3}{6}$$

$$\text{solving for } d \qquad \frac{6v}{\pi} = d^3$$

$$\text{taking the cube root} \qquad d = \sqrt[3]{\frac{6v}{\pi}} = \sqrt[3]{\frac{6}{\pi}}\sqrt[3]{v} \approx 1.2v^{1/3}$$

So using function notation, setting $d = D(v)$, we have a function that describes the tumor diameter $D(v)$ as a function of its volume v:

$$D(v) = 1.2v^{1/3} \tag{3}$$

where v is in cubic centimeters and $D(v)$ is in centimeters.

d. *Tumor diameter as a function of time.* Since the volume v can be also be written as a function of time as $V(t)$, then the tumor diameter as a function of time is given by the composition of Equations (1) and (3):

$$\begin{aligned}(D \circ V)(t) = D(V(t)) &= D(0.06 \cdot 12.6^t)\\ &= 1.2(0.06 \cdot 12.6^t)^{1/3}\\ &= 1.2 \cdot (0.06)^{1/3} \cdot (12.6^{1/3})^t\\ &\approx 0.5 \cdot 2.3^t\end{aligned}$$

where $(D \circ V)(t)$ is the diameter of the tumor at year t. When $t = 0$, then $(D \circ V)(t) \approx 0.5$ cm. So the minimum tumor diameter detectable by mammography is about 0.5 cm, less than a quarter of an inch.

e. *Visualizing tumor diameter over time.* Figure 8.70 shows the growth of a breast tumor over time.

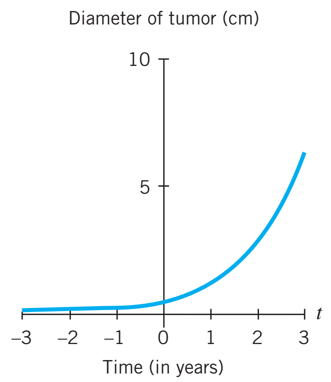


Figure 8.70 Graph of breast tumor diameter over time, where $t = 0$ corresponds to the minimum tumor diameter detectable by a mammogram.

When $t < 0$, then $(D \circ V)(t)$ gives the size of the tumor before it was detectable by mammogram. For example, when $t = -5$ (5 years before the tumor had a mammogram-detectable size), the tumor diameter was

$$(D \circ V)(-5) = 0.5 \cdot 2.3^{-5} \approx 0.008 \text{ cm (about three-thousandths of an inch)}$$

f. *Estimating and calculating tumor diameter growth.* Estimating from the graph in Figure 8.70, it would take about 2 years for the diameter to grow from 0.5 cm (when $t = 0$) to 2 cm. To get a more exact number, we can substitute 2 cm for $(D \circ V)(t)$ to get the equation

$$\begin{aligned}
& & 2 &= 0.5 \cdot 2.3^t \\
&\text{divide by } 0.5 & \frac{2}{0.5} &= 2.3^t \\
&\text{simplify} & 4 &= 2.3^t \\
&\text{take the log of both sides} & \log(4) &= \log(2.3^t) \\
&\text{use rules for logs} & \log(4) &= t\log(2.3) \\
&\text{solve for } t & t &= \frac{\log(4)}{\log(2.3)} \\
&\text{use a calculator} & t &\approx 1\tfrac{2}{3} \text{ years}
\end{aligned}$$

So it could take almost 2 years for a tumor detectable by a mammogram to grow to a size detectable by touch.

Algebra Aerobics 8.8b

Graphing program required in Problem 10.

1. Let $g(t) = 5 - 2t$ and $h(t) = \frac{5-t}{2}$.

a. Complete the following tables.

t	$g(t)$
0	
1	
2	
3	

t	$h(t)$
-1	
1	
3	
5	

b. Find $(g \circ h)(3)$ and $(h \circ g)(3)$.

c. Is $(g \circ h)(t) = (h \circ g)(t)$ for all t?

d. What is the relationship between g and h?

In Problems 2 and 3, verify that f and g are inverse functions by showing that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

2. $f(x) = 2x + 1$ and $g(x) = \frac{x-1}{2}$

3. $f(x) = \sqrt[3]{x+1}$ and $g(x) = x^3 - 1$

4. Verify that $f(x) = \frac{1}{x-1}$ and $f^{-1}(x) = \frac{1+x}{x}$ are inverse functions of one another.

Find the inverse (if it exists) for each of the functions in Problems 5, 6, and 7.

5. $f(x) = \frac{3}{x} + 5$

6. $g(x) = (x-2)^{2/3}$

7. $h(x) = 5x^3 - 4$

8. State the inverse action for each of the actions described below.

a. Saying "yes"

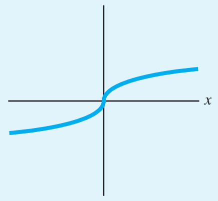
b. Going to class and then taking the bus home

c. Unlocking the door, opening the door, entering the room, and turning on the light

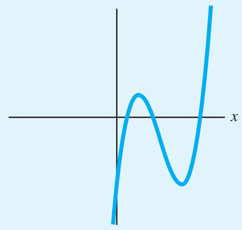
d. Subtracting 3 from x and multiplying the result by 5

e. Multiplying z by -3 and adding 2

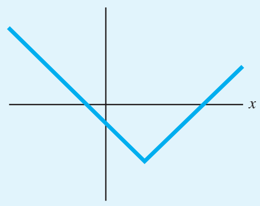
9. Determine from the accompanying function graphs which functions are 1-1.



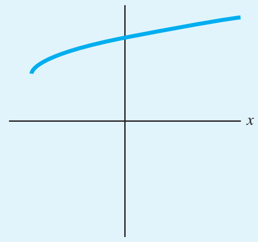
Graph *A*



Graph *C*



Graph *B*



Graph *D*

10. (Graphing program required.)

a. Sketch a graph of the function $f(x) = (x + 2)^2$. Does f have an inverse?

b. If not, restrict the domain of f so that f^{-1} does exist.

c. Given the restricted domain of f, find the equation for f^{-1} and then graph $f^{-1}(x)$.

11. An oil spill is spreading in a roughly circular shape. The radius, r, is growing by 10 feet per hour. The area $A(r)$ (in square feet) of the spill is a function of the radius r (in feet), given by $A(r) = \pi r^2$.

a. Construct a function $R(t)$ that represents the radius r as a function of time t (in hours since the oil spill).

b. If the oil spill has been spreading for 2 hours, what is the area of the spill?

c. How could you compose the functions A and R to give the area in terms of time t?

Exercises for Section 8.8

1. From the accompanying table, find:

a. $f(g(1))$ **c.** $f(g(0))$ **e.** $f(f(2))$

b. $g(f(1))$ **d.** $g(f(0))$

x	$f(x)$	$g(x)$
0	2	1
1	1	0
2	3	3
3	0	2

2. From the accompanying table, find:

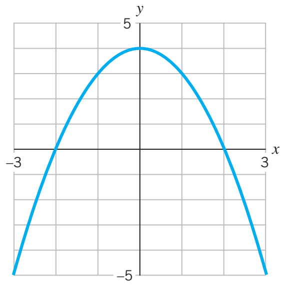
a. $f(g(1))$ **c.** $f(g(0))$ **e.** $f(f(2))$

b. $g(f(1))$ **d.** $g(f(0))$

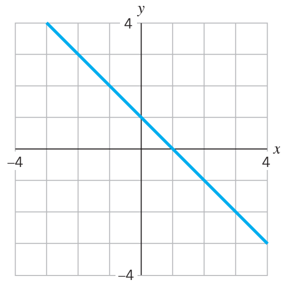
x	$f(x)$	$g(x)$
-1	-2	1
0	1	2
1	2	-1
2	0	-1

3. Using the accompanying graphs, find:

a. $g(f(2))$ **b.** $f(g(-1))$ **c.** $g(f(0))$ **d.** $g(f(1))$



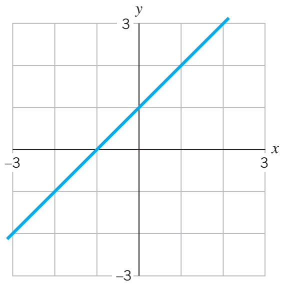
Graph of $f(x)$



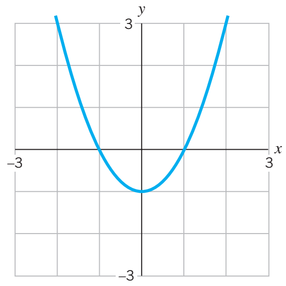
Graph of $g(x)$

4. Using the accompanying graphs, find:

a. $g(f(-2))$ **b.** $f(g(1))$ **c.** $g(f(0))$ **d.** $g(f(1))$



Graph of $f(x)$



Graph of $g(x)$

5. Given $F(x) = 2x + 1$ and $G(x) = \dfrac{x-1}{x+2}$, find:

a. $F(G(1))$ **d.** $F(F(0))$
b. $G(F(-2))$ **e.** $(F \circ G)(x)$
c. $F(G(2))$ **f.** $(G \circ F)(x)$

6. Given $f(x) = 3x - 2$ and $g(x) = (x + 1)^2$, find:

a. $f(g(1))$ **d.** $f(f(2))$
b. $g(f(1))$ **e.** $(f \circ g)(x)$
c. $f(g(2))$ **f.** $(g \circ f)(x)$

7. The winds are calm, allowing a forest fire to spread in a circular fashion at 5 feet per minute.

a. Construct a function $A(r)$ for the circular area burned, where r is the radius. Identify the units for the input and the output of $A(r)$.

b. Construct a function for the radius $r = R(t)$ for the increase in the fire radius as a function of time t. What are the units now for the input and the output for $R(t)$?

c. Construct a composite function that gives the burnt area as a function of time. What are the units now for the input and the output?

d. How much forest area is burned after 10 minutes? One hour?

8. The exchange rate a bank gave for Canadian dollars on March 2, 2007, was 1.18 Canadian dollars for 1 U.S. dollar. The bank also charges a constant fee of 3 U.S. dollars per transaction.

a. Construct a function F that converts U.S. dollars, d, to Canadian dollars.

b. Construct a function G that converts Canadian dollars, c, to U.S. dollars.

c. What would the function $F \circ G$ do? Would its input be U.S. or Canadian dollars (i.e., d or c)? Construct a formula for $F \circ G$.

d. What would the function $G \circ F$ do? Would its input be U.S. or Canadian dollars (i.e., d or c)? Construct a formula for $G \circ F$.

9. A stone is dropped into a pond, causing a circular ripple that is expanding at a rate of 13 ft/sec. Describe the area of the circle as a function of time.

10. The wind chill temperature is the apparent temperature caused by the extra cooling from the wind. A rule of thumb for estimating the wind chill temperature for an actual temperature t that is above 0°Fahrenheit is $W(t) = t - 1.5S_0$, where S_0 is any given wind speed in miles per hour.

a. If the wind speed is 25 mph and the actual temperature is 10°F, what is the wind chill temperature?

We know how to convert Celsius to Fahrenheit; that is, we can write $t = F(x)$, where $F(x) = 32 + \frac{9}{5}x$, with x the number of degrees Celsius and $F(x)$ the equivalent in degrees Fahrenheit.

b. Construct a function that will give the wind chill temperature as a function of degrees Celsius.

c. If the wind speed is 40 mph and the actual temperature is -10°C, what is the wind chill temperature?

11. Salt is applied to roads to decrease the temperature at which icing occurs. Assume that with no salt, icing occurs at 32°F, and that each unit increase in the density of salt applied decreases the icing temperature by 5°F.

a. Construct a formula for icing temperature, T, as a function of salt density, s.

Trucks spread salt on the road, but they do not necessarily spread it uniformly across the road surface. If the edges of the road get half as much salt as the middle, we can describe salt density $S(x)$ as a function of the distance, x, from the center of the road by $S(x) = [1 - \frac{1}{2}\left(\frac{x}{k}\right)^2]S_d$, where k is the distance from the centerline to the road edges and S_d is the salt density applied in the middle of the road.

b. What will the expression for $S(x)$ be if the road is 40 feet wide?

c. What will the value for x be at the middle of the 40-foot-wide road? At the edge of the road? Verify that at the middle of the road the value of the salt density $S(x)$ is S_d and that at the edge the value of $S(x)$ is $\frac{1}{2}S_d$.

d. Construct a function that describes the icing temperature, T, as a function of x, the distance from the center of the 40-foot-wide road.

e. What is the icing temperature at the middle of the 40-foot-wide road? At the edge?

12. Using the given functions f, g, and h where

$$f(x) = x + 1 \qquad g(x) = e^x \qquad h(x) = x - 2$$

a. Create the function $k(x) = (f \circ g \circ h)(x)$.

b. Describe the transformation from x to $k(x)$.

13. Using the given functions J, K, and L, where

$$J(x) = x^3 \qquad K(x) = \log(x) \qquad L(x) = \frac{1}{x}$$

a. Create the function $M(x) = (L \circ J \circ K)(x)$.

b. Describe the transformation from x to $M(x)$.

In Exercises 14 and 15, rewrite $j(x)$ as the composition of three functions, f, g, and h.

14. $j(x) = \dfrac{2}{(x-1)^3}$

15. $j(x) = 4e^{x-1}$

In Exercises 16–22, show that the two functions are inverses of each other.

16. $f(x) = 3x + 2$ and $g(x) = \dfrac{x-2}{3}$

17. $f(x) = \sqrt{x-1}$ (where $x > 1$) and $g(x) = x^2 + 1$ (where $x > 0$)

18. $f(x) = 2x - 1$ and $g(x) = \dfrac{x+1}{2}$

19. $f(x) = \sqrt[3]{4x+5}$ and $g(x) = \dfrac{x^3-5}{4}$

20. $f(x) = 10^{x/2}$ and $g(x) = \log(x^2)$

21. $F(t) = e^{3t}$ and $G(t) = \ln(t^{1/3})$

22. $H(r) = \frac{1}{2}\ln r$ and $J(r) = e^{2r}$

In Exercises 23 and 24, create a table of values for the inverse of the function $f(x)$.

23.

x	$f(x)$
−2	5
−1	1
0	2
1	4

24.

x	$f(x)$
0	5
1	3
−2	2
4	−7

25. Cryptology (the creation and deciphering of codes) is based on 1-1 functions. After you code a message using a 1-1 function, the decoder needs the inverse function in order to retrieve the original message. The following table matches each letter of the alphabet with its coded numerical form.

A	B	C	D	E	F	G	H	I	J	K	L	M
26	25	24	23	22	21	20	19	18	17	16	15	14

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
13	12	11	10	9	8	7	6	5	4	3	2	1

a. Does this code represent a 1-1 function? Is there an inverse function? If so, what is its domain?

b. Decode the message "14 26 7 19 9 6 15 22 8."

26. On March 2, 2007, the conversion rate from U.S. dollars to euros was 0.749; that is, on that day you could change \$1 for 0.749 euros, the currency of the European Union.

a. Was a U.S. dollar worth more or less than 1 euro?

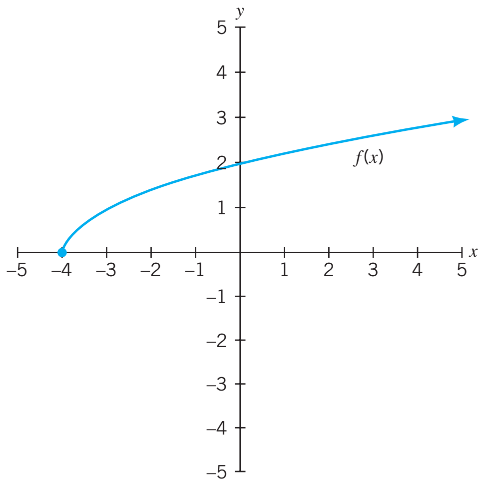
b. Using the March 2 exchange rate, construct a function $C_1(d)$ that converts d dollars to euros. What is $C_1(1)$? $C_1(25)$?

c. Now construct a second function $C_2(r)$ that converts r euros back to dollars. What is $C_2(1)$? $C_2(100)$?

d. Show that C_1 and C_2 are inverses of each other.

e. Reread the beginning of Exercise 8, which describes a conversion process between Canadian and U.S. dollars. In that process the two formulas are *not* inverses of each other. Why not?

27. Given the accompanying graph of $f(x)$, answer the following.



a. Does $f(x)$ have an inverse? Please explain.

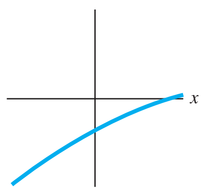
b. What is the domain of $f(x)$? Estimate the range of $f(x)$?

c. From the graph, determine $f(-4)$, $f(0)$, and $f(5)$.

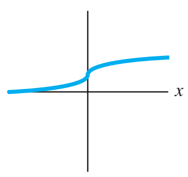
d. Determine $f^{-1}(0)$, $f^{-1}(2)$, and $f^{-1}(3)$.

28. Determine which of the accompanying graphs show functions that are one-to-one.

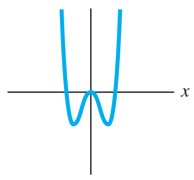
a. Graph of $f(x)$



c. Graph of $h(x)$



b. Graph of $g(x)$



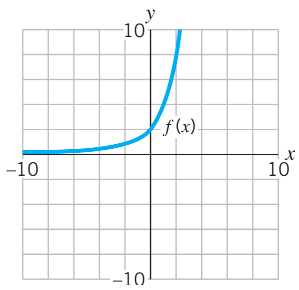
In Exercises 29–32, for each function Q find Q^{-1}, if it exists. For those functions with inverses, find $Q(3)$ and $Q^{-1}(3)$.

29. $Q(x) = \dfrac{2}{3}x - 5$

30. $Q(x) = 5e^{0.03x}$

31. $Q(x) = \dfrac{x + 3}{x}$

32. Use the graph of $f(x)$ to evaluate each expression.



a. $f(2)$ **b.** $f^{-1}(2)$ **c.** $f^{-1}(4)$ **d.** $(f \circ f^{-1})(8)$

33. The following tables represent a function f that converts cups to quarts and a function g that converts quarts to gallons (all measurements are for fluids).

a. Fill in the missing values in the chart. (*Hint:* One quart contains 4 cups, and one gallon contains 4 quarts.)

x (cups)	4	8	16	32
$f(x)$ (quarts)				

x (quarts)	2	4	8	16
$g(x)$ (gallons)				

b. Now evaluate each of the following and identify the units of the results.

i. $(g \circ f)(8)$ **iii.** $(f^{-1} \circ g^{-1})(1)$

ii. $g^{-1}(2)$ **iv.** $(f^{-1} \circ g^{-1})(2)$

c. Explain the significance of $(f^{-1} \circ g^{-1})(x)$ in terms of cups, quarts, and gallons.

34. Let $f(x) = mx + b$.

a. Does $f(x)$ always have an inverse? Explain.

b. If $f(x)$ has an inverse, find $f^{-1}(x)$.

c. Using the formula for $f^{-1}(x)$, explain in words how, given any linear equation (under certain constraints), you can find the inverse function knowing the slope m and y-intercept b.

35. If you do an Internet search on formulas for "ideal body weight" (IBW), one that comes up frequently was created by Dr. B. J. Devine. His formula states

IBW for men (in kilograms) =

50 + (2.3 kg per inch over 5 feet)

IBW for women (in kilograms) =

45.5 + (2.3 kg per inch over 5 feet)

a. Write the functions for IBW (in kg) for men and women, $W_{\text{men}}(h)$ and $W_{\text{women}}(h)$, where h is a person's height in inches. Give a reasonable domain for each.

b. Evaluate $W_{\text{men}}(70)$ and $W_{\text{women}}(66)$. Describe your results in terms of height and weight.

c. Evaluate $W^{-1}{}_{\text{men}}(77.6)$. What does this tell you?

d. Given that 1 lb = 0.4356 kg, alter the functions to create $W_{\text{newmen}}(h)$ and $W_{\text{newwomen}}(h)$ so that the weight is given in pounds rather than kilograms.

e. Use your functions in part (d) to find $\text{W}^{-1}{}_{\text{newwomen}}(125)$. What does this tell you?

[*Note:* More information can be found in the article by M. P. Pari and F. P. Paloucek, "The origin of the 'ideal' body weight equations," Annals of Pharmacology 34 (9), 2000: 10 :1066–69.]

36. The formula for the volume of a cone is $V = \left(\frac{1}{3}\right)\pi r^2 h$. Assume you are holding a 6-inch-high sugar cone for ice cream.

a. Construct a function $V(r)$ for the volume as a function of r. Why don't you need the variable h in this case? Find $V(1.5)$ and explain what have you found (using appropriate units).

b. Evaluate $V^{-1}(25)$. Describe your results. What are the units attached to the number 25?

c. When dealing with abstract functions where $f(x) = y$, we have sometimes used the convention of using x (rather than y) as the input to the inverse function $f^{-1}(x)$. Explain why it does not make sense to interchange V and r here to find the inverse function.

37. In Chapter 6 we learned that a logarithm can be constructed using any positive number (except 1) as a base: $\log_a x = y$ means that $a^y = x$. Show that $F(x) = a^x$ and $G(x) = \log_a x$ are inverse functions. The software "E10: Inverse Functions $y = a^x$ and $y = \log_a x$" in *Exponential and Log Functions* can help you visualize the relationship between the two functions.

38. The Texas Cancer Center website, *www.texascancercenter.com,* notes that the 5-year survival rate for stage I breast cancer (when the tumor diameter is ≤2 cm) is about 85%. The 5-year survival rate for stage II breast cancer (when the tumor diameter is ≤5 cm and the cancer has not spread to the lymph nodes) is about 65%.

a. Using the equations in Example 15 (the final one in the text) referring to a tumor that doubles in volume every 100 days, how long would it take such a tumor to grow from 0.5 cm in diameter to 5 cm? (Recall that 0.5 cm in diameter corresponds to an initial volume of 0.06 cubic cm, the minimum tumor size detectable by a mammogram.)

b. If the tumor were more aggressive, doubling in volume every 50 days, what would the yearly growth factor be? Use this to construct a new function to reflect the *volume* growth of this more aggressive tumor over time (again using 0.06 cubic cm as the initial volume).

c. Using your function from part (b), construct another function to represent the *diameter* growth over time.

d. How many years would it take the aggressive tumor to grow from 0.5 cm to 5 cm in diameter?

CHAPTER SUMMARY

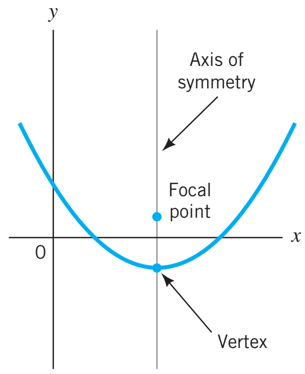
Quadratic Functions and Their Graphs

A *quadratic function* can be written in *standard form* as

$$f(x) = ax^2 + bx + c \quad (\text{where } a \neq 0)$$

Its graph

- has a distinctive ∪-shape called a *parabola*
- is symmetric across its *axis of symmetry*
- has a minimum or a maximum point called its *vertex*
- is concave up if $a > 0$ and concave down if $a < 0$
- becomes narrower as $|a|$ increases
- has a *focal point* $\left|\frac{1}{4a}\right|$ units above (or below) the vertex on the axis of symmetry



Finding the Vertex

Any quadratic function $f(x) = ax^2 + bx + c$ can also be written in *vertex* or *a-h-k* form as

$$f(x) = a(x - h)^2 + k$$

where the vertex is at $(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. The vertex "anchors" the graph, and the coefficient a determines the shape of the parabola.

Finding the Horizontal Intercepts

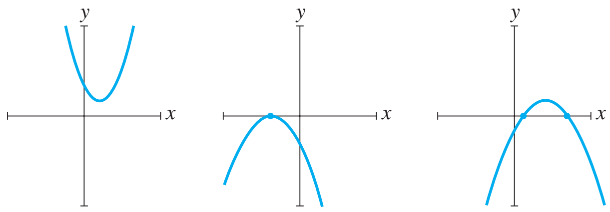
Every quadratic function $f(x)$ has two, one, or no horizontal intercepts x. To find the x-intercepts, we set $f(x) = 0$ and solve for x. The solutions are called the *zeros* of the function.

The Factor Theorem says that any quadratic function $f(x) = ax^2 + bx + c$ can be written in *factored form* as

$$f(x) = a(x - r_1)(x - r_2)$$

where r_1 and r_2 are the zeros of $f(x)$.

If the zeros, r_1 and r_2, are real numbers, they are the horizontal intercept(s) of the quadratic function $f(x)$.



(*a*) No x-intercepts
No real zeros

(*b*) One x-intercept
One real zero

(*c*) Two x-intercepts
Two real zeros

The Quadratic Formula

Setting the quadratic function $f(x) = ax^2 + bx + c$ equal to 0 and solving for x using the quadratic formula, gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The term $b^2 - 4ac$ is called the *discriminant* and can be used to predict the number of horizontal intercepts (or real zeros) of $f(x)$.

If the discriminant > 0, there are two distinct real roots and hence two x-intercepts.

If the discriminant $= 0$, there is only one distinct real root and hence only one x-intercept.

If the discriminant < 0, then the $\sqrt{b^2 - 4ac}$ is not a real number and hence there are no x-intercepts. These zeros are *complex numbers* of the form $a + bi$, where a and b are real numbers $(b \neq 0)$ and $i = \sqrt{-1}$.

The Average Rate of Change of a Quadratic Function

Given a quadratic function $f(x) = ax^2 + bx + c$, the average rate of change between two points on the parabola approaches $2ax + b$ over very small intervals. We can think of the linear function $g(x) = 2ax + b$ as representing the average rate of change of $f(x)$ with respect to x.

Polynomial Functions and Their Graphs

A *polynomial function of degree n* is of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$.

The graph of a polynomial function of degree n will

- Have at most $n - 1$ turning points
- Cross the horizontal axis at most n times

For large values for x, the graph of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ will resemble the graph of the power function $g(x) = a_n x^n$.

If $r_1, r_2, \ldots, r_n$ are zeros of a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $f(x)$ can be written in *factored form* as

$$f(x) = a_n(x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n)$$

Any real zero is an x-intercept for $f(x)$.

Creating New Functions from Old

Ways to Transform the Graph of f(x)

To stretch or compress:	Multiply the output of $f(x)$ by a constant to get $af(x)$.
To shift vertically:	Add a constant to the output of $f(x)$ to get $f(x) + k$.
To shift horizontally:	Subtract a constant from the input of $f(x)$ to get $f(x - h)$.
To reflect across the x-axis:	Multiply the output of $f(x)$ by -1 to get $-f(x)$.
To reflect across the y-axis:	Multiply the input of $f(x)$ by -1 to get $f(-x)$.

Symmetry

If $f(x) = f(-x)$, then f is symmetric across the y-axis.

If $f(x) = -f(-x)$, then f is symmetric about the origin.

The Algebra of Functions

If $f(x)$ and $g(x)$ are two functions with the same domain, we can define new functions by

Adding to get $f(x) + g(x) = (f + g)(x)$

Subtracting to get $f(x) - g(x) = (f - g)(x)$

Multiplying to get $f(x) \cdot g(x) = (f \cdot g)(x)$

Dividing to get $\dfrac{f(x)}{g(x)} = \left(\dfrac{f}{g}\right)(x)$ (where $g(x) \neq 0$)

A *rational function* $R(x)$ is the quotient of two polynomial functions. If $R(x) = \frac{p(x)}{q(x)}$ (where $p(x)$ and $q(x)$ have no common terms and $q(x) \neq 0$) then

Set $p(x) = 0$ to find horizontal intercepts.

Set $q(x) = 0$ to find vertical asymptotes.

The *end behavior* can be approximated by the ratio of the leading terms of p and q, and is either a *horizontal* or an *oblique* asymptote.

Composition and Inverses

If $f(x)$ and $g(x)$ are two functions, then the function $f \circ g$, called the *composition of f and g*, is defined by

$$(f \circ g)(x) = f(g(x))$$

If both $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, then the functions f and g are *inverses* of each other, written as $f^{-1} = g$ and $g^{-1} = f$.

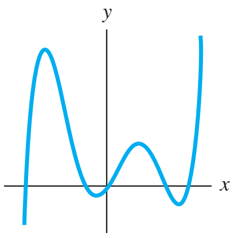
A function $f(x)$ is called *one-to-one* (or 1-1) if no two distinct input values are mapped to the same output value. A 1-1 function must pass the horizontal line test.

A function that has an inverse must be one-to-one.

CHECK YOUR UNDERSTANDING

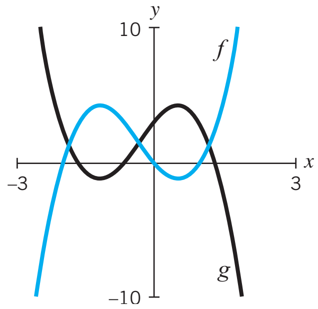
I. Is each of the statements in Problems 1–21 true or false? Give an explanation for your answer.

1. If $f(t) = 2(t - 1)^2$, then $f(0) = -2$.
2. If the vertical axis is the axis of symmetry for a quadratic function $g(x)$, then $g(-2) = g(2)$.
3. The polynomial function in the accompanying figure has a minimum degree of 5.



4. The function $y = 3(x - 2)^2 + 5$ has a focal point at $\left(2\frac{1}{12}, 5\right)$.
5. The graph of the quadratic function $y = 2x^2 - 3x + 1$ is steeper than the graph of $y = 3x^2 - 3x + 1$.
6. The graph of $y = 5 - x^2$ is concave down.
7. The graph of $y = x^2 + 2x + 3$ is three units higher than the graph of $y = x^2 + 2x$.
8. The graph of $y = (x + 4)^2$ lies four units to the right of the graph of $y = x^2$.
9. A quadratic function that passes through the points (1, 5) and (7, 5) will have an axis of symmetry at the vertical line $x = 4$.
10. If $f(x) = x^2 - 3x - 4$, then $f(4) = 0$.
11. The function $f(x) = (x + 2)(x + 5)$ has zeros at 2 and 5.

12. In the accompanying figure it appears that $g(x) = -2f(x)$.



13. The quadratic function $f(x) = 2x^2 - 3x - 1$ has a discriminant with a value of 1.

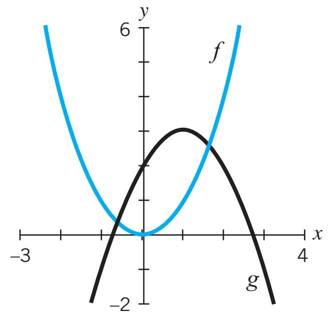
14. The function $f(x) = 3(x - 1)^2 + 2$ has an axis of symmetry at $x = 1$.

15. The function $f(x) = -2(x + 3)^2 - 1$ has a vertex at $(3, -1)$.

16. There is only one quadratic function $f(x)$ with x-intercepts at 3 and 0.

17. The function $h(t) = t^2 - 3t + 2$ has a zero at $t = 2$ because $h(0) = 2$.

18. In the accompanying figure it appears that $g(x) = -f(x - 1) + 3$.



19. Assume the height of a ball thrown vertically upward is modeled by the function $h(t) = -4.9t^2 + 38t + 55$, (where t is time in seconds, and $h(t)$ is the height in meters). Then the ball will hit the ground after approximately 9 seconds.

20. If revenue R (in dollars) from an item sold at price p (in dollars) is modeled by the function $R = p(100 - 5p)$, the revenue will be at a maximum when the price is \$10.

21. The functions $m(s) = \frac{1}{2s + 3}$ and $n(s) = \frac{1 - 3s}{2s}$ are inverses of each other.

II. For Problems 22–37 give an example of a function or functions with the specified properties. Express your answer using equations.

22. A polynomial function that does not intersect the horizontal axis.

23. A quadratic function with vertex at the point (0, 0) and with focal point $(0, -1)$.

24. A polynomial function with horizontal intercepts at -1, 3, and 4.

25. Two more polynomial functions with horizontal intercepts at -1, 3, and 4.

26. A quadratic function concave down with a vertex at (1, 3).

27. A quadratic function concave up with its axis of symmetry at the line $x = 3$.

28. A quadratic function concave down with vertical intercept at 2 and zeros at -2 and 2.

29. A quadratic function whose graph will be exactly the same shape as the graph of the function $r = s^2 - s$ but five units higher.

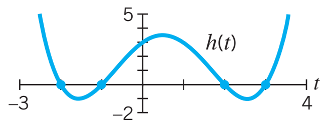
30. A quadratic function $G(x)$ whose graph will be exactly the same shape as the function $F(x) = x^2 + 2x$ but two units to the left.

31. A quadratic function whose graph will be the reflection across the t-axis of the graph of $h(t) = (t - 2)^2$.

32. Two distinct quadratic functions that intersect at the point (1, 1).

33. A quadratic function with one zero at $x = -4$.

34. A polynomial function $h(t)$ that could describe the function in the accompanying figure.



35. Two functions f and g such that $f(x) - g(x) = x^3 + 2x^2 - 5x + 2$.

36. Two functions Q and H such that $(Q \circ H)(t) = \sqrt{3t + 1}$.

37. A rational function with a horizontal intercept at (2, 0) and two vertical asymptotes at $x = 0$ and $x = -3$.

III. Is each of the statements in Problems 38–58 true or false? If a statement is true, explain how you know. If a statement is false, give a counterexample.

38. All polynomial functions are power functions.

39. All linear functions are polynomial functions.

40. All power functions are polynomial functions.

41. All quadratic functions are polynomial functions.

42. The quadratic function with vertex at the origin, (0, 0) and focal point at (0, 1) will be narrower than the quadratic function with the same vertex but with focal point at (0, 4).

43. A polynomial function of degree 4 will always have three turning points.

44. A polynomial function of degree 4 will cross the horizontal axis exactly four times.

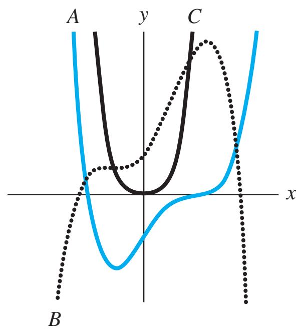
45. A polynomial function of odd degree must cross the horizontal axis at least one time.

46. Quadratic functions $f(x) = ax^2 + bx + c$ always have two distinct zeros because the equation $ax^2 + bx + c = 0$ always has two roots, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

47. If the discriminant is 0, then the associated quadratic function has no horizontal intercept.

48. The leading term determines the global shape of the graph of a polynomial function.

49. The three polynomial functions in the accompanying figure are all of even degree.



50. Quadratic functions that open upward have a minimum value at the vertex.

51. $(f \circ g)(x) = (g \circ f)(x)$ for any functions f and g.

52. The functions $f(x) = \ln x$ and $g(x) = e^x$ are inverses of each other.

53. $f^{-1}(x) = \frac{1}{f(x)}$ for any function f.

54. If $f(x) = \frac{1}{x}$ and $g(x) = 2x^2 + 1$, then $(f \circ g)(x) = \frac{1}{(2x^2 + 1)}$ and $f(x) \cdot g(x) = \frac{(2x^2 + 1)}{x}$.

55. If $f(x) = f(-x)$, the graph of f is symmetric across the x-axis.

56. If $f(x) = -f(-x)$, the graph of f is symmetric about the origin.

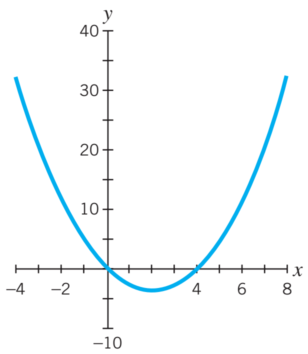
57. A function that passes the vertical line test has an inverse.

58. Every function is 1-1.

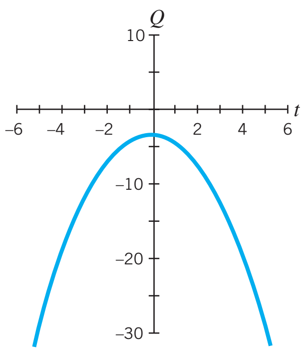
CHAPTER 8 REVIEW: PUTTING IT ALL TOGETHER

Some problems are identified as needing a calculator that calculates roots, a graphing program, or technology to generate a best-fit polynomial.

1. For each of the accompanying parabolas, identify the graph as concave up or down, and then estimate the minimum (or maximum) point, the axis of symmetry, and any horizontal intercepts.



Graph A



Graph B

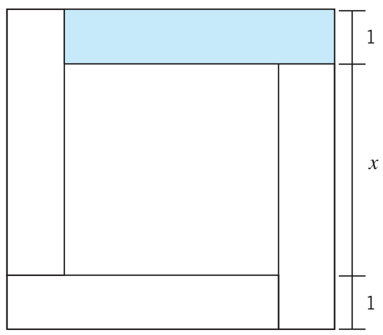
2. An electric heater is being designed as a parabolic reflector 6″ deep. To prevent accidental burns, the center of the heating element is placed at the focus, which is set 1.5″ from the vertex of the reflector.

a. What equation describes the shape of the reflector?

b. How wide will the reflector be?

c. Sketch an image of the parabolic reflector with the vertex at its origin. Put a circle at the focal point, and label the depth and width of the reflector.

3. (Requires graphing program for parts (e) and (f).) A wood craftsman has created a design for a parquet floor. The pattern for an individual tile is shown in the accompanying image. The square center (x inches wide) of the tile is made from white oak hardwood and is surrounded by 1-inch strips of maple hardwood.



a. What is the area of the interior white oak square (in terms of x)? The area of each of the maple 1-inch strips?

b. White oak costs \$2.39 per square foot; maple costs \$4.49 per square foot. Calculate the cost per square inch for white oak and for maple.

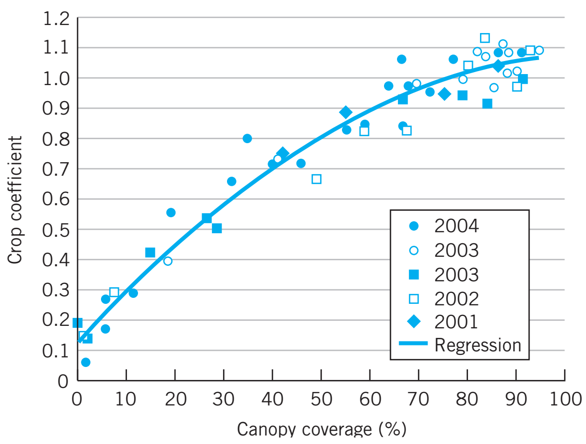
c. What is the cost for the white oak in one tile of the parquet? What is the cost of the four maple strips in one tile?

d. The approximate labor cost to make each tile is \$5.00. Create a cost function $C(x)$ (in dollars) for making one parquet tile. What type of function is this?

e. Use technology to graph the function $C(x)$, where x is the width of the inner white oak square. Use a domain of $0 \le x \le 15$ inches.

f. From your graph, estimate the size of a parquet tile if the total cost (including labor) is to be \$7.00 or less per tile.

4. California produces nearly 95% of the processing tomatoes grown in the United States. Therefore, managing irrigation water for tomatoes is a major issue. Agricultural researchers have been able to quantify the relationship between C, the canopy coverage, and K_C, the crop coefficient.[7] Canopy cover is the percentage of the total plot covered by shade produced by the leaves of the plants. The crop coefficient is a measure of the water needed by the plants. (Technically, it is the ratio of the amount of water plants need divided by the amount used for an equivalent area of well-watered grass.) The following graph shows the relationship between the crop canopy and the crop coefficient.



a. As the plants grow, will the canopy cover increase or decrease? Why?

b. The graph suggests that as the plant grows, the crop coefficient increases—but at a decreasing rate. Why would that be true?

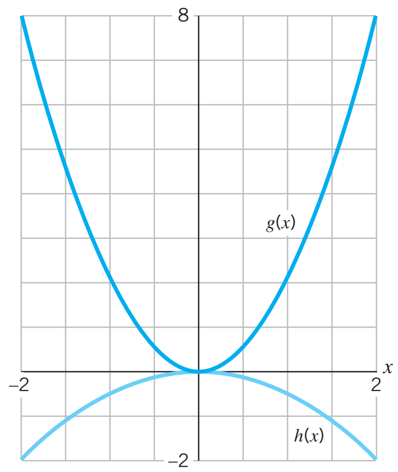
c. The best-fit function to the data is a quadratic:

$$K_C = 0.126 + (0.0172)C - (0.0000776)C^2$$
$$\text{where } 0 \le C \le 100\%$$

i. The initial growth stage has 10% canopy coverage. Estimate the corresponding crop coefficient from the graph, then calculate it using the quadratic model. What does this number mean in terms of well-watered lawn grass?

ii. In the crop development growth stage, the canopy is between 10% and 75%. Estimate the crop coefficient when the canopy is at 75% and then calculate it. What does this number mean?

5. Construct a function for each parabola $g(x)$ and $h(x)$in the accompanying graph.



6. (Graphing program optional.) On Earth, the distance d_{Earth} a freely falling object has traveled is a function of time t. It can be modeled by the equation $d_{\text{Earth}} = 16t^2$, where t is in seconds and d_{Earth} is in feet. On Mars, the comparable equation is $d_{\text{Mars}} = 6.1t^2$.

a. Will an object fall faster on Earth or on Mars?

b. Plot the two functions on the same graph for $0 \le t \le 2.5$ seconds.

c. Calculate the number of seconds on Earth it would take a freely falling object to fall 100 feet. Does your graph confirm your answer?

d. How far would a freely falling object fall on Mars during the same time (as in part (c))?

e. On Jupiter, the equation for the distance a freely falling object has traveled (in feet) is given by $d_{\text{Jupiter}} = 40.65t^2$ (where t is in seconds). Will an object fall faster or more slowly on Jupiter than on Earth? How would its graph compare with those of Earth and Mars?

f. If you double the time, what happens to the distance an object has fallen on Earth? On Mars? On Jupiter?

7. a. Identify the coordinates of the vertex for each the following quadratic functions.

$F(x) = x^2$, $G(x) = x^2 + 5$, $H(x) = (x + 2)^2$, and $J(x) = -(x - 1)^2 - 5$

b. Without using technology, draw a rough sketch on the same grid of all the functions for $-4 \le x \le 4$.

c. Describe how the graph of F was transformed into the graphs of G, H, and J, respectively.

[7] B. R. Hanson and D. M. May, "New crop coefficients developed for high-yield processing tomatoes," *California Agriculture* 60(2), April–June 2006.

8. Find any horizontal intercepts for the following functions.

a. $y = (x - 3)(2x + 1)$ **c.** $Q(t) = 2t^2 + t - 1$

b. $G(z) = 2z^2 - z + 3$

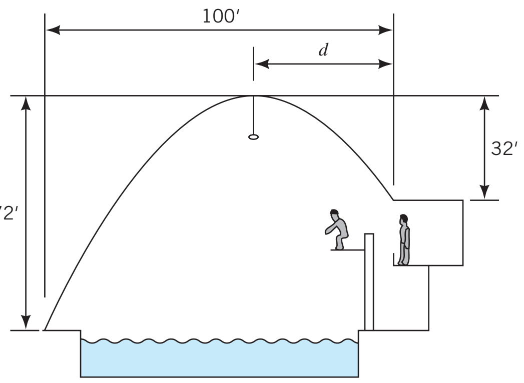
9. a. Construct a quadratic function $Q(t)$ that is concave up and has horizontal intercepts at $t = 4$ and $t = -2$. Write it in both factored and standard form. Find its vertex.

b. Construct a second function $M(t)$ that has the same horizontal intercepts as $Q(t)$ but is steeper. Write $M(t)$ in both factored and standard form. Do the two functions have the same vertex?

c. Add a term to $Q(t)$ to create a function $P(t)$ that has no horizontal intercepts.

10. Explain why you could (or couldn't) construct a parabola through any three points.

Problems 11, 12, and 13 refer to the accompanying diagram of the cross section of a swimming pool with a reflective parabolic roof.



11. Find an equation for the cross section of the parabolic roof of the swimming pool in the diagram. (*Hint:* Place the origin of your coordinate system at the vertex and identify two other points on the parabola in terms of d.)

12. (Requires results of Problem 11.) The pool designer wants to mount a light source at the parabolic focus so that it sheds light evenly on the water surface below. How many feet down from the vertex must that be?

13. A diver jumps up off the high board, which is 25 feet above the surface of the water. Her height, $H(t)$, in feet above the water at t seconds, can be modeled by the function $H(t) = 25 + 12t - 16t^2$.

a. What will be the highest point above the water of her dive?

b. When will she hit the water?

14. In the United States a "heat wave" is a period of three or more consecutive days at or above 90°F. A heat wave is often accompanied by high humidity, making the air feel even hotter. The following formula combines an air temperature of 90°F with relative humidity, H, to give the apparent temperature, A, the perceived level of heat:

$$A = 86.61 - 0.132H + 0.0059H^2$$

This formula uses relative humidity as a percentage (e.g., 70% relative humidity appears in the formula as H = 70). Remember, this formula applies only for an air temperature of 90°F.

a. If the relative humidity is 0%, what is the apparent temperature for an air temperature of 90°F? Does the apparent temperature feel lower or higher than the air temperature of 90°F?

b. On a 90°F day, if the relative humidity is 60%, what is the apparent temperature? How much hotter do you feel?

c. An apparent temperature of 105°F or above is considered dangerous, especially for children and elders. At what relative humidity on a 90°F day is an apparent temperature of 105°F reached?

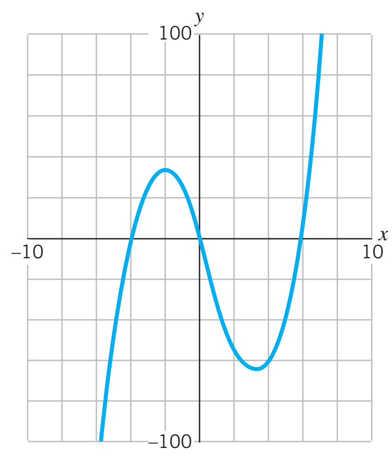
15. a. Complete the following table for the function $y = x^2 - 4x$.

x	y	Average Rate of Change	Average Rate of Change of Average Rate of Change
−1	5	n.a.	n.a.
0	0	−5	n.a.
1	−3	−3	$[-3 - (-5)]/(1 - 0) = 2$
2	−4		
3			
4			
5			

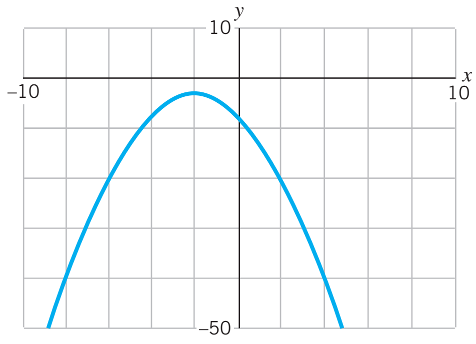
b. If you plotted the points with coordinates of the form (x, average rate of change) and connected adjacent points, what type of function would you get?

c. Does the fourth column verify your result in part (b)? Why or why not?

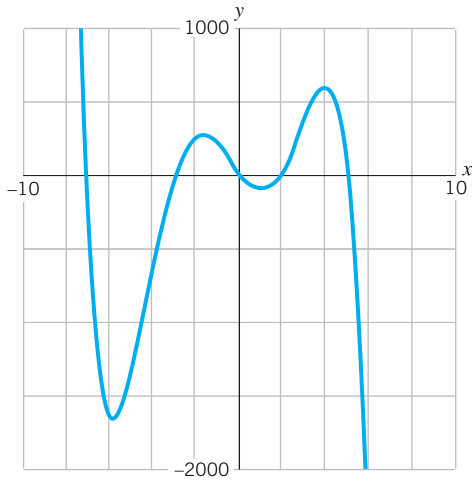
16. Using the three accompanying graphs of polynomial functions, determine whether the degree of the polynomial is odd or even, identify its minimum possible degree, and estimate any visible horizontal intercepts of the function.



Graph *A*



Graph *B*



Graph *C*

17. (Graphing technology that can generate a best-fit polynomial is required.) Commercial beekeepers rent out their bees to farmers. The table below gives the average price paid by California almond farmers for each hive of bees (according to Lance Sundberg, who owns and operates the honey business Sunshine Apiary in Columbus, Montana).

	2000	2001	2002	2003	2004	2005	2006
$/hive	$51.00	$51.50	$52.00	$52.50	$53.00	$82.50	$128.50

a. Plot the data. What sort of function do you think might best fit the data? Use technology to find that best-fit function $P(t)$, where t = years since 2000.

b. Use your model to predict the average price per hive a beekeeper would get in 2010.

c. On average, there are 2.5 hives per acre of almonds. Predict the amount a beekeeper might get per acre in 2010.

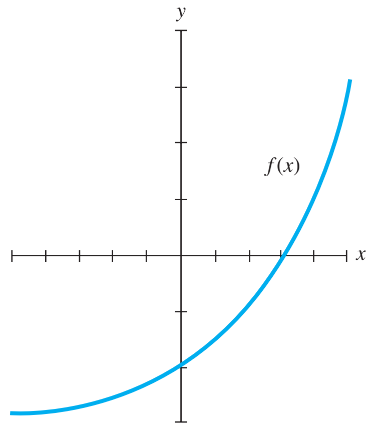
d. Unfortunately, in 2007, for some unknown reason, bees throughout the country have not been returning to their hives after being released. Do you think that the price per hive will go up or down? Why?

18 a. Generate two different polynomials, $M(z)$ and $N(z)$, that have horizontal intercepts at $z = -2$, 0, and 3.

b. Generate a third polynomial, $P(z)$, with the same horizontal intercepts but a higher degree.

19. On the same grid, hand-draw rough sketches of the three functions $f(x) = e^x$, $g(x) = 4e^x$, and $h(x) = -0.5e^x$ for $0 \leq x \leq 5$. Describe the relationships among the graphs.

20. a. Given the following graph of the function $f(x)$, sketch:



i. $g(x) = -f(x)$ **ii.** $h(x) = f(-x)$ **iii.** $j(x) = |f(x)|$

b. Describe each function in relation to $f(x)$.

21. (Graphing program required.)

a. If $g(x) = \frac{2}{3(x-4)} - 1$ and $f(x) = \frac{1}{x}$, describe the transformation of the graph of $f(x)$ into the graph of $g(x)$.

b. Rewrite $g(x)$ as a ratio of two polynomials.

c. What is the domain of $g(x)$? Sketch $g(x)$.

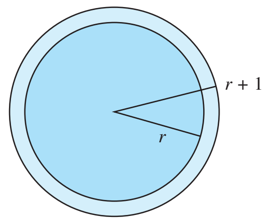
d. What are its horizontal and vertical intercepts (if any)?

e. Does $g(x)$ have a vertical asymptote?

f. What is its end behavior?

22. Global warming melts glaciers and polar ice, so scientists predict that the sea level will rise, flooding coastal areas.

a. Assuming Earth is a sphere with radius r and that roughly three-quarters of Earth's surface is ocean (with or without melt water), develop a formula to estimate the volume of melt water necessary to raise the sea level 1 foot. (*Note:* The volume of a sphere is $\frac{4}{3}\pi r^3$.)



b. Given the Earth's radius is currently about 3959 miles and 1 cubic foot = 7.481 gallons, how many gallons of melt water does your estimate predict? (*Recall:* 1 mile = 5280 feet.)

23. Retirement fund counselors often recommend a mixed portfolio of investments, including some higher-risk investments, which offer higher interest rates, and some more secure investments, with lower interest rates. A woman wants to put half of her $10,000 savings in a safe 4% fund, and the other half in a riskier

10% fund. She expects to retire in 30 years but would like to know how much she can expect to get if she retires earlier.

a. Create three functions where t is the number of years since the start of the investments and $S(t)$ is the amount of money in the 4% account, $R(t)$ is the amount in the 10% account, and $T(t)$ is the total amount invested in both accounts.

b. On one graph show how the 4% fund, the 10% fund, and the combined fund total accumulate over 30 years.

c. In the worst-case scenario, if she loses all the money in the 10% fund, how much will she be left with in 30 years?

d. You might think that if she is getting 4% on \$5000 plus 10% on another \$5000, this is the same as getting 14% on \$5000. Is it? If not, why not? You can explain your answer using a table and/or a graph.

24. Given the functions $f(x) = x^2$, $g(x) = x + 1$, and $h(x) = -3x$, evaluate each of the following compositions.

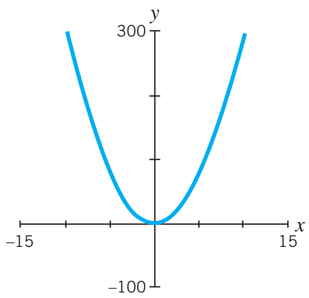
a. $(f \circ g)(x)$ and $(g \circ f)(x)$

b. $(h \circ g)(x)$ and $(g \circ h)(x)$

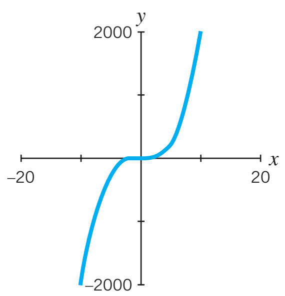
c. $(f \circ g \circ h)(x)$ and $(h \circ g \circ f)(x)$

25. Does the function $f(x) = (x - 2)^3 + 1$ have an inverse? If not, explain why. If so, what is it?

26. Which of these functions has an inverse? If there is one, what is it?



a. Graph of $y = 3x^2 - 1$



b. Graph of $y = 2x^3$

27. When lightning strikes, you seem to see it right away, but the associated thunder often comes a few seconds later. One rule of thumb is that each second of delay represents 1000 feet; that is, if you hear the thunder 3 seconds after the lightning strike, the strike was about 3000 feet away from you.

a. Light travels about 186,000 miles per second, so the light created from a lightning strike a few thousand feet away is seen virtually simultaneously with the strike. However, sound travels much more slowly, at about 761 mph at sea level.

i. Convert 761 mph into feet/second.

ii. Now construct a function that gives the distance $D(t)$ (in feet) that the thunder has traveled from the strike site in t seconds.

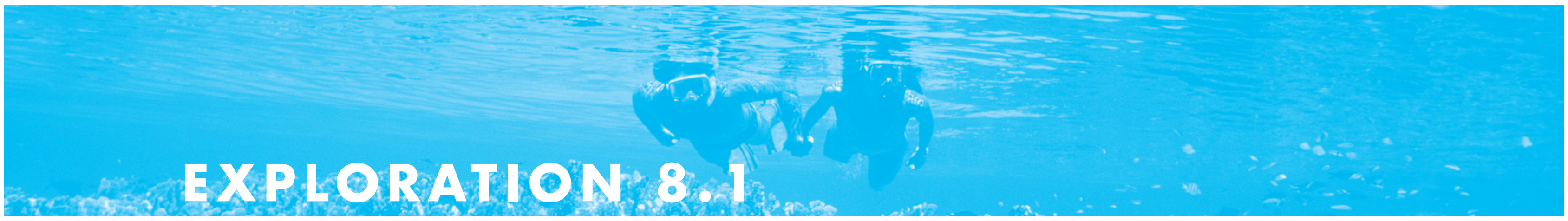
iii. Does the rule of thumb seem reasonable?

b. The sound travels in all directions, creating expanding "sound circles" that radiate out from the lightning strike.

i. Create a function $A(r)$ (in square feet) that gives the area of a sound circle with radius r (in feet).

ii. We can think of the radius r of the sound circle as $D(t)$, the distance thunder has traveled (in any direction) in t seconds. Substituting r for $D(t)$, construct a composite function $A(D(t))$ to describe the circular area at time t within which the thunder can be detected. What is the circular area (in square feet) within which thunder can be heard 4 seconds after the lightning strike? What is the area in square miles?

iii. When the time doubles, what happens to the distance the thunder has traveled? What happens to the area within which it can be heard?



EXPLORATION 8.1

How Fast Are You? Using a Ruler to Make a Reaction Timer[1]

Objective

- learn about the properties of freely falling bodies and your own reaction time

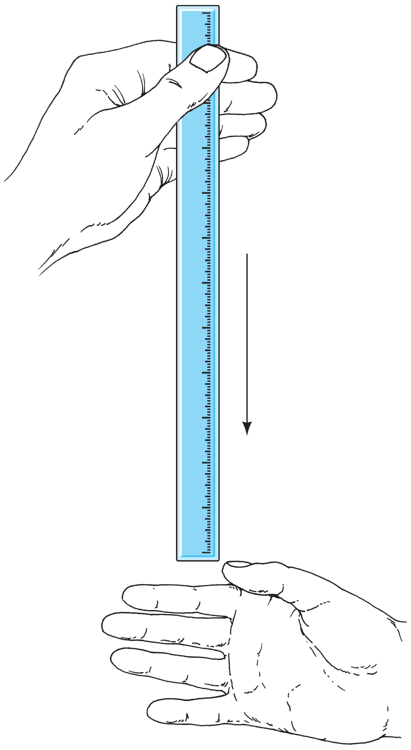
Materials/Equipment

- several 12″ rulers
- narrow strips of paper and tape
- calculators

Procedure

General Description

Work in groups of two or three. Each group has a 12″ ruler and will attach a 12″ paper strip to the ruler, adding some marks (specified below). One student drops the ruler between the thumb and forefinger of a second student. The second student tries to catch the ruler as quickly as possible (see image). The reaction time of the second student can be measured by how far the ruler falls before it is caught.



Mathematical Background

Near the surface of Earth, and neglecting air resistance, gravity causes dropped objects to fall approximately according to the formula

$$d = 16t^2$$

[1]This exploration was developed by Karl Schaffer, Mathematics Department, De Anza College, Cupertino, CA.

where d = distance fallen in feet and t = time of fall in seconds.

1. Thus an object in free fall for 1 second will fall 16 feet, and an object in free fall for 0.5 second will fall ________ feet. (Calculate.)

2. Can you give an intuitive explanation for why the object falling for 0.5 second does not fall half as far as the object that fell for 1 second?

3. What assumptions must we make about the shape of the dropped object for it to fall according to this formula? (*Hint:* Will a sheet of paper fall 16 feet in 1 second?)

4. Do you think a heavy object will fall faster or slower or at the same rate as a light object? Explain your reasoning.

5. For each of the following times, use the formula to calculate how far a dropped object will fall.

Time, t	Distance in feet, d	Distance in inches
0.05 second	________	________
0.10 second	________	________
0.15 second	________	________
0.20 second	________	________
0.25 second	________	________

6. Use tape to attach a strip of paper along the length of the ruler. Think of the ruler as measuring distance fallen. For each distance (in inches) in your previous table, put a mark on the paper that indicates the time corresponding to that distance. So you'll have unevenly spaced marks for the times 0.05 sec, 0.10 sec, up to 0.25 sec. Now the ruler is a reaction timer.

7. One member of your group holds the ruler just above the outstretched thumb and first finger of a second person. The "dropper" suddenly drops the ruler and the "catcher" tries to catch it. Use the time marks on the strip of paper to get an estimate for the reaction time of the catcher. Use the actual number of inches at the point where the catcher caught the ruler to calculate the reaction time. Record the reaction time, and average several tries. If you like, measure the reaction time for someone else in your group.

Person	Distance in inches	Distance in feet, d	Reaction time, t
________	________	________	________
________	________	________	________
________	________	________	________
________	________	________	________
________	________	________	________
________	________	________	________
________	________	________	________

What was your average reaction time? How did yours compare with that of others in your group?

Further Investigations

1. A popular party trick has one person drop a dollar bill between the fingers of a second person. Usually the bill will fall through the second person's fingers without being caught. How long is a dollar bill, and what must the second person's reaction time be for the bill to be caught? Does the use of money speed up the reaction times you measured with the ruler? Does the bill fall with only negligible air resistance?

2. How do medications or drugs affect our reaction times? Test the reaction times of someone who is taking cold or flu medication or aspirin, or has just drunk a cup of coffee or glass of alcohol (outside of class, of course). Is his or her reaction time impaired? Is there a correlation between the amount of alcohol consumed and reaction time that might enable you to use your ruler reaction timer as a portable tester to determine whether someone who has consumed alcohol should not drive?

3. Do reaction times measured in this activity improve with practice? Why or why not? (Try it!) Do you think the catcher learns to detect subtle indications of the dropper that she or he is about to drop the ruler? How might these biases be removed from this experiment?

4. Jugglers, athletes, and dancers need to understand, either intuitively or objectively, their own reaction times. In what other occupations is reaction time important? Using the Internet, can you find the reaction times necessary in any of these areas?

5. The *Guinness Book of World Records* lists the fastest times for drawing and firing a gun. Look this up. Are these times consistent with your results?

6. What else might you investigate about reaction times and your ability to measure them?