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1

Functions in the Real World

1.1 Functions Are All Around Us

The notion of *function* is a fundamental idea in mathematics. Functions are the basis of most mathematical applications in nearly all areas of human endeavor. To see how functions can arise in unexpected places, look at the graph shown in Figure 1.1.

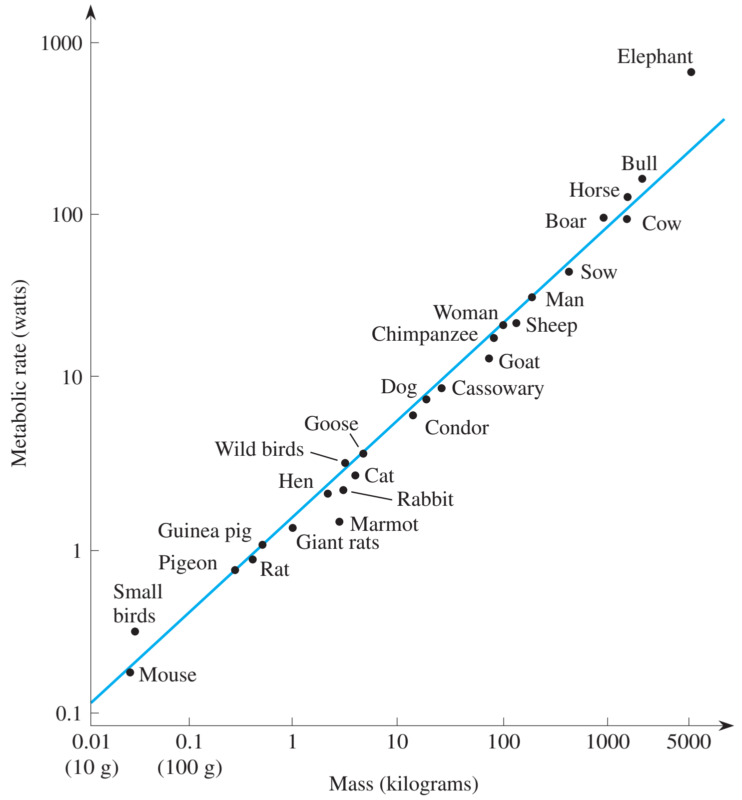


FIGURE 1.1

This graph appears in many introductory biology textbooks. It shows the results of a study comparing the masses of various mammals and birds with their metabolic rates. The biologist who conducted the study first plotted the *data*—the raw measurements on body mass (measured in kilograms) and metabolic rate (measured in watts)—on a graph and then drew a line that passes very close to most of the points. What does this graph show? It is clear from the pattern of the data points that there must be some relationship between the body mass and the metabolic rate of mammals and birds. If there were no relationship, the points would not fall into such a clear pattern. Thus we conclude that, in some way, the metabolic rate of an organism depends on the mass of that organism. Such a relationship is a *function,* and we say that metabolic rate R is a function of body mass W.

Informally, a function is a rule that associates a set of values of one quantity with a set of values of another quantity. Functions are usually represented in four different ways:

1. by formulas or equations,
2. by graphs,
3. by tables, and
4. in words.

For instance, a function might be expressed as a mathematical formula such as $A = s^2$, which gives the area A of a square in terms of its side s. The equation might be $D = 50t$, which gives the distance D you travel at a constant rate of 50 mph in terms of the time t that you drive. A function might be given as a graph, as in the relationship between metabolic rate R as a function of body mass W of various organisms illustrated in Figure 1.1. A function might be given as a table of data. For instance, you compute your income tax for the Internal Revenue Service by using a table—for each level of taxable income, there is a corresponding tax levied, as shown in Figure 1.2. The rule for a function might be expressed in words, as in

If line 37 (taxable income) is—		**And you are—**		
At least	But less than	Single	Married filing jointly	Married filing separately
			Your tax is—	
29,000				
29,000	**29,050**	5,092	4,354	5,592
29,050	**29,100**	5,106	4,361	5,606
29,100	**29,150**	5,120	4,369	5,620
29,150	**29,200**	5,134	4,376	5,634
29,200	**29,250**	5,148	4,384	5,648
29,250	**29,300**	5,162	4,391	5,662
29,300	**29,350**	5,176	4,399	5,676
29,350	**29,400**	5,190	4,406	5,690
29,400	**29,450**	5,204	4,414	5,704
29,450	**29,500**	5,218	4,421	5,718
29,500	**29,550**	5,232	4,429	5,732
29,550	**29,600**	5,246	4,436	5,746
29,600	**29,650**	5,260	4,444	5,760
29,650	**29,700**	5,274	4,451	5,774
29,700	**29,750**	5,288	4,459	5,788
29,750	**29,800**	5,302	4,466	5,802
29,800	**29,850**	5,316	4,474	5,816
29,850	**29,900**	5,330	4,481	5,830
29,900	**29,950**	5,344	4,489	5,844
29,950	**30,000**	5,358	4,496	5,858

FIGURE 1.2

"The cost of postage is 37 cents for the first ounce and 23 cents for each additional ounce."

Typically, when a functional relationship exists between two quantities, the values of one of the quantities depends on the values of the other quantity. That is:

> A **function** is a rule that assigns to each value of one quantity precisely one related value of another quantity.

But, if one value of a quantity leads to two or more values of the other quantity, the relationship between them is *not* a function. For instance, consider the relationship between the number of home runs that a batter has hit by the end of the baseball season and the number of runs he has batted in (RBIs). How many RBIs are associated with 10 home runs? Many different players hit 10 home runs say, but each likely had a different number of RBIs, so this relationship is not a function.

Representing Functions with Formulas and Equations

When you think of functions, the first thing you probably think of is a relationship between two quantities that is given by a formula, such as $A = \pi r^2$, which gives the area A of a circle in terms of its radius r. Similarly, the ideal gas law from chemistry, which says that $P = kT/V$, expresses the pressure P of a gas as a function of its temperature T, where V is the volume of the container that holds the gas and k is a constant. The conversion between Fahrenheit and Celsius temperature readings, $F = \frac{9}{5}C + 32$, expresses the functional relationship between the two temperature scales.

Frequently, when we observe that one quantity is a function of another, we would like to determine an appropriate formula that expresses this relationship. For example, throughout most of human history, people believed that objects fall at a constant speed. Then, in about 1590, Galileo realized that this belief might not necessarily be true. He also had the insight to realize that this conjecture could be tested experimentally. Galileo conducted his now-famous experiments of dropping objects from the top of the Tower of Pisa and found that they fell at ever-increasing rates and that the weight of the objects didn't affect how fast they fell. Galileo's study of the relationship between the distance that an object falls and the time it takes to fall was the key connection for Newton that enabled him to develop his theories of motion that transformed the physical sciences. Based on either Newton's laws of motion or the analysis of data from such an experiment, a formula for the height y of an object dropped from the top of the 180-foot-high Tower of Pisa is $y = 180 - 16t^2$, where t is the number of seconds since the object was dropped. We show where this formula comes from later in the book.

Representing Functions with Graphs

Many effective ways are used to display functions graphically in everyday life—in newspapers, magazines, and scientific, business, and government reports. The graph of a function is valuable because it displays accurate information about a quantity while simultaneously giving an overview of the behavior of that quantity. In particular, a graph can show any trends or patterns in the process being studied.

The graph shown in Figure 1.3 shows the increase in life expectancy in the United States in years since the beginning of the twentieth century. This graph is a function of time t because, for any given year, there is a single value for the life

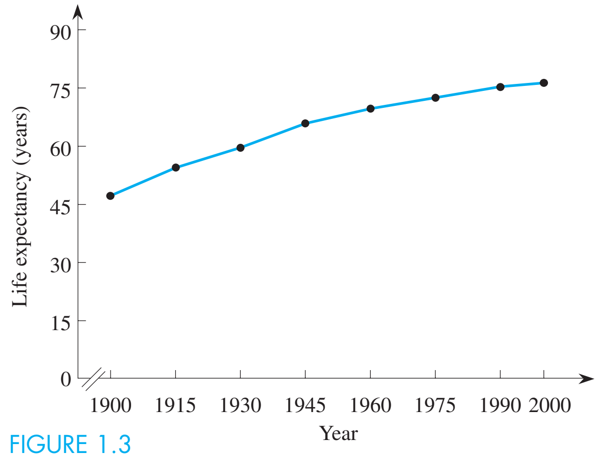


FIGURE 1.3

expectancy of a child born in that year. From the graph, for example, we can estimate that a child born in 1900 would have had, on average, a life expectancy of about 47 years, and that a child born in 1990 would have a life expectancy of about 75 years. The rise in life expectancy is a remarkable achievement due to advances in science and medicine and improvements in lifestyles. However, there are also some unfortunate aspects connected with living longer. Can you think of any?

From this graph, not only can you observe the rising trend, but you can also look ahead to predict life expectancies in the not-too-distant future. Note that life expectancy is not merely increasing, but it is actually increasing more slowly as time goes by.

Think About This

What is the significance of this growth pattern for life expectancy if it continues?

Functions are displayed graphically in many ways in newspapers and magazines. Keep an eye out for them in your daily activities.

Representing Functions with Tables

Consider the following table of values, which shows the acceleration of a Pontiac Trans Am. The table gives the time in seconds needed to reach different speeds.

Final speed, *v* (mph)	30	40	50	60	70
Time, *t* (sec)	3.00	4.29	5.52	7.38	9.81

Although you may not have thought of something such as this as being a function, the time t needed for a Trans Am to achieve a certain speed v is a function of the speed. There may not be an explicit formula for this time as a function of the final speed, but it nevertheless satisfies the definition of a function: For each final speed v, there is a unique time t needed for a Trans Am to accelerate to that speed.

We can plot these points and connect them with a series of straight line segments or even by a smooth curve, as shown in Figure 1.4. Note that the times depend on the speeds. Thus we plot the speeds on the horizontal axis and the times needed to achieve those speeds on the vertical axis. Also, you should realize that the values in the table represent only the actual measured points. Drawing a smooth curve through the points requires making assumptions about what happens be-

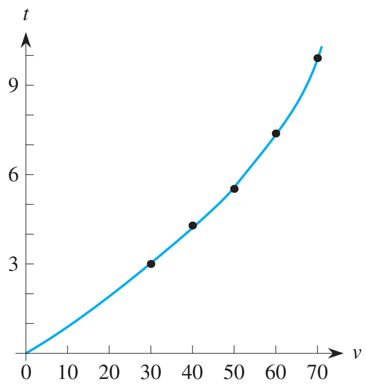


FIGURE 1.4

tween those points. The curve is just an artist's rendition of what the pattern could be; the actual pattern might have some minor variations. Note that we have now represented the same function by both a table and a graph.

Think About This

Estimate the time needed for a Trans Am to accelerate from 0 to 45 mph and from 0 to 75 mph. Which estimate do you think is more accurate? Why?

Now consider the following daily high temperatures in Phoenix during a severe heat wave in June 1990.

Date	19	20	21	22	23	24	25	26	27	28	29
Temperature (°F)	109	113	114	113	113	113	120	122	118	118	108

Note that a single high-temperature reading is associated with each day, so high temperature is a function of the day. This function makes sense only for the 11 days—June 19 through June 29—and its values consist of the high-temperature readings 108, 109, 113, 114, 118, 120, and 122.

However, the date is *not* a function of the high temperature because a given temperature (say, 113°) was reached on *more than one* date (in this case the 20th, the 22nd, the 23rd, and the 24th).

The function that associates the high temperature in Phoenix with the corresponding day of the month can be depicted graphically by plotting the individual points, as shown in Figure 1.5, so again we have represented the same function by both a table and a graph. The points in the figure can be joined by a series of line segments or by a smooth curve to give a sense of an overall trend or pattern, as shown in Figure 1.6. However, doing so requires some careful thought. When we connect the points, we are *not* indicating that this graph represents temperature as a function of time; we are just connecting the maximum temperatures recorded each day, and the curve shown gives absolutely no information about the temperature at any intermediate time. In fact, the actual graph of temperature versus time would typically show the type of oscillatory effect depicted in Figure 1.7.

Representing Functions with Words

Functions are expressed verbally in many different ways. The maximum load that a jet plane can lift is *related* to its wingspan. The number of different species that can live on an island *depends* on the size of the island. The population of the world *over*

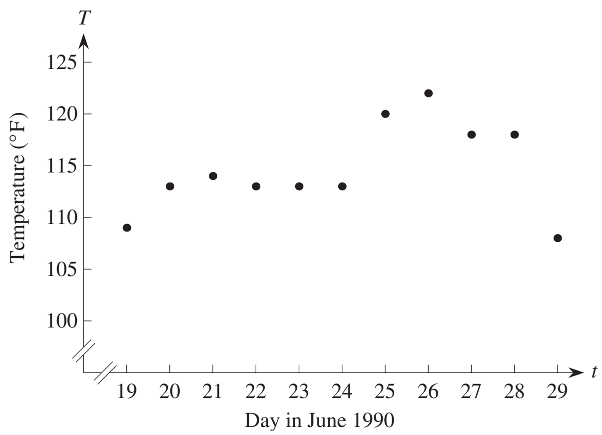


FIGURE 1.5

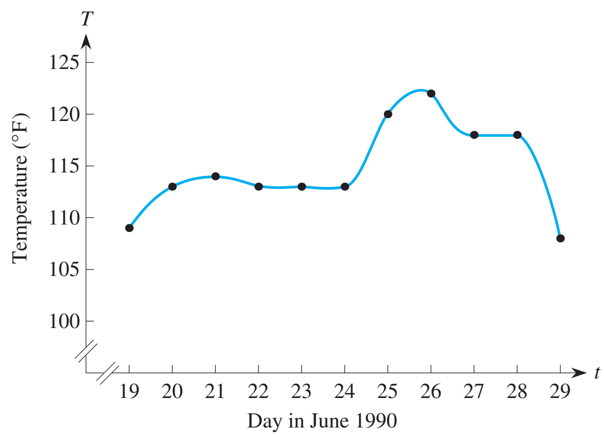


FIGURE 1.6

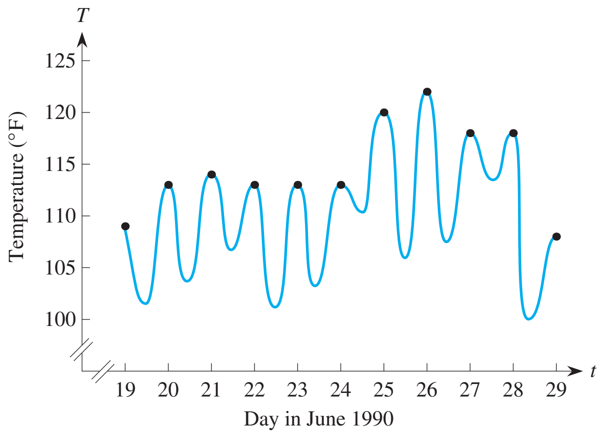


FIGURE 1.7

time is a *function of* time. If money is borrowed at simple interest (no compounding), the amount of interest earned is *a multiple of* the amount borrowed.

Why Study Functions?

Any situation involving two quantities usually raises several questions:

1. Is there a functional relationship between the two quantities?
2. If there is a relationship, can we find a formula for it?
3. Can we construct a table or graph relating the two quantities, especially if we can't find a formula?
4. If we can find a formula, or if we have a graph of the relationship, or if we have a table of values relating the two quantities, how do we use it? That is, how can knowledge of the function aid in understanding the relationship between the two quantities or allow us to make predictions or informed decisions about one of the variables based on the other?

Figure 1.1 clearly suggests a relationship between metabolic rates R in mammals and birds and their body mass W. This same relationship can then be used to predict the metabolic rate of other species—say, lions or Kodiak bears—based on knowledge of their mass. This relationship could even be used to predict the metabolic rate of an extinct pterodactyl from estimates of its body mass made from its skeletal remains.

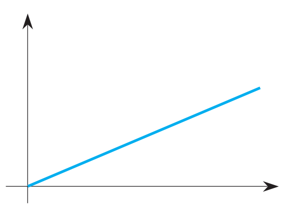
However, it could not be used to predict the metabolic rate for a crocodile because a crocodile is neither a mammal nor a bird; the relationship observed in Figure 1.1 for mammals and birds may not apply to reptiles.

Think About This

Based on the graph shown in Figure 1.1, would you use the relationship to predict the metabolic rate for extinct mammoths, which were slightly larger than today's elephant?

Problems

1. Which of the following relationships are functions and which are not? Explain your reasoning. For those that are functions, identify which of the two quantities depends on the other. Again, explain your reasoning.
   - **a.** The number of miles driven in a car versus the number of gallons of gas used.
   - **b.** The price of a diamond versus the number of carats.
   - **c.** The major league baseball player who has a certain number of home runs at the end of the season.
   - **d.** The student who has a specific score on the SAT in a particular year.
   - **e.** The amount of rain that falls on any particular day of the year in Seattle.
   - **f.** The day of the year on which given amounts of snow, in inches, fall in Buffalo.
2. Match each of the following functions with a corresponding graph. Explain your reasoning.
   - **a.** The population of a country as a function of time.
   - **b.** The path of a thrown football as a function of time.
   - **c.** The distance driven at a constant speed as a function of time.
   - **d.** The daily high temperature in a city as a function of time over several years.
   - **e.** The number of cases of a disease as a function of time.
   - **f.** The percentage of families owning VCRs as a function of time.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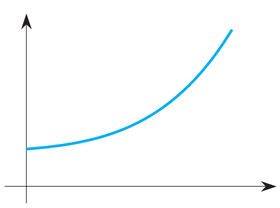
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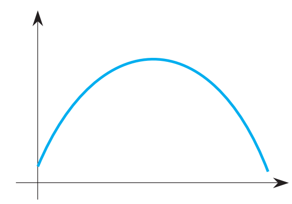
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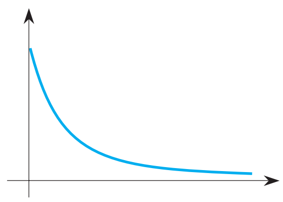
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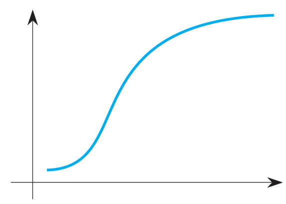
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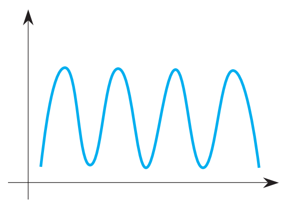
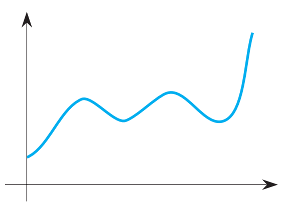

(vi)

3. The following graphs show the noise level of a crowd of college students watching their school's basketball team playing at home in the championship finals for the league title. Match the three graphs with the corresponding scenarios (reactions) and then draw a graph for the remaining scenario.
   - **a.** Our team started slowly but eventually began to pull away.
   - **b.** It was a disaster from start to finish.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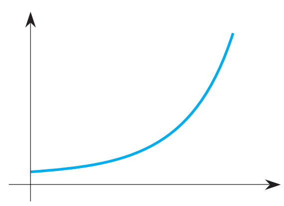
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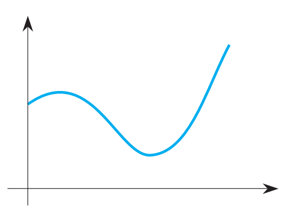

(iii)

c. The score kept seesawing back and forth, but we finally won on a three-point shot at the buzzer.
d. Our team started well, then the opposition took the lead, but we finally won.

4. Consider the scenario: "You left home to run to the local gym. You started at a constant rate of speed but sped up when you realized how energetic you felt. About halfway there, you began to tire, so you started slowing down." Sketch a graph of your distance from home as a function of time.

5. Sketch a graph of your distance from home as a function of time for each situation.
   a. You drove steadily across town, speeding up as traffic diminished until the road turned into a highway.
   b. You drove steadily toward town but slowed down as the traffic increased. Eventually you inched forward around a car that had broken down before you could resume normal speed.
   c. You drove steadily but realized you had left something behind, so you returned home and then drove all the way to school without any further trouble.
   d. You drove steadily across town but then had a flat tire; after changing it, you drove much faster so that you wouldn't be too late for class.

6. For each of the scenarios in Problem 5, sketch a graph of the *total distance* you've traveled as a function of time.

7. Consider again the graph in Figure 1.3. Write a paragraph or two interpreting what the increase in life expectancy over the past century means. For example, you might consider it in terms of your own expected life span compared to those of your children and grandchildren. Alternatively, you might consider the effects on the overall distribution of people of different ages in the population at large, or you might discuss the question of whether there is a natural limit to how long the human life span can be extended in the future. Compare the values for life expectancies in the United States in Figure 1.3 with the values for life expectancies of other nations given in Appendix G.

8. Which table of values represents a function and which doesn't? Explain your reasoning.

a.

x	0	3	6	1	5	2	4
y	8	6	2	2	4	5	3

b.

x	0	2	3	4	1	3	5
y	8	4	7	2	6	10	9

9. The Dow-Jones average of 30 industrial stocks is probably the most closely watched measure of stock market performance. Below are the Dow values at the beginning of each year from 1980 to 2000.

   Write a short paragraph describing the behavior of the stock market over this period of time. When did it rise? When did it fall? Which years would have been the best times to buy stocks? Which would have been the worst times to do so?

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	
Dow	839	964	875	1047	1259	1212	1547	1896	1939	2169	
Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Dow	2753	2634	3169	3301	3758	3834	5177	6447	7965	9184	11358

Source: *Wall Street Journal.*

1.2 Describing the Behavior of Functions

Functions are used to represent quantities in the real world. Because most of these quantities change over time or depend on some other quantity, we need some terminology to describe the *behavior of the function*—that is, how the function changes. We can describe the behavior of a function in two different ways.

Increasing and Decreasing Functions

The first and most immediate aspect of behavior is whether the function is increasing or decreasing. The graph of a function is *increasing* if, as you look from left to right, the vertical values get larger; that is, the graph rises. We also describe this behavior as *growth.* Similarly, the graph of a function is *decreasing* if, as you look from left to right, the vertical values get smaller; that is, the graph falls. We also describe this behavior as *decay.* Figure 1.8 illustrates these characteristics.

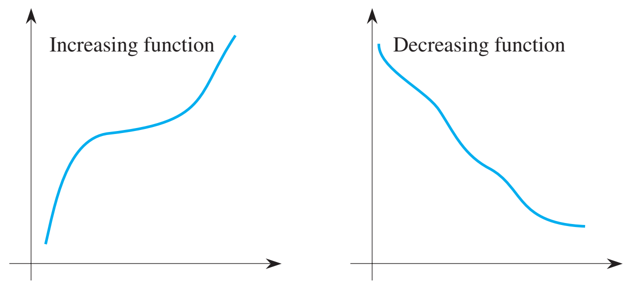


FIGURE 1.8

For instance, the world's population is growing. Therefore the function that expresses the population over time is an increasing function. Also, the heavier a car is, the lower its gas mileage will be. Therefore the function that relates gas mileage to the weight of a car is a decreasing function.

Of course, not every quantity merely increases or decreases. Often, a quantity will rise some of the time and fall some of the time, such as the height of a bouncing ball, the value of the Dow-Jones average, or the high temperature recorded in a particular location each day of the year. Thus a function whose graph looks like the one shown in Figure 1.9 increases for some values of the variable and decreases for others. Here the function rises (increases) to a maximum or largest value compared to nearby points, then falls (decreases) to a minimum value compared to nearby points, and then rises again. We call any point where the behavior of the function changes from increasing to decreasing or from decreasing to increasing a **turning point** of the function. Turning points occur at points where a function reaches a *local maximum* (the value where the function is larger than any nearby value) or a *local minimum* (the value where the function is smaller than any nearby value).

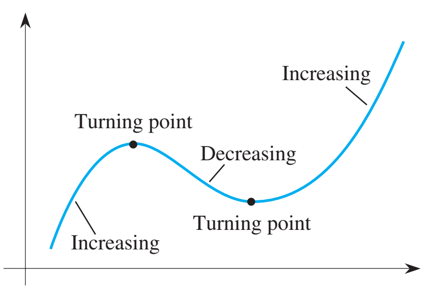


FIGURE 1.9

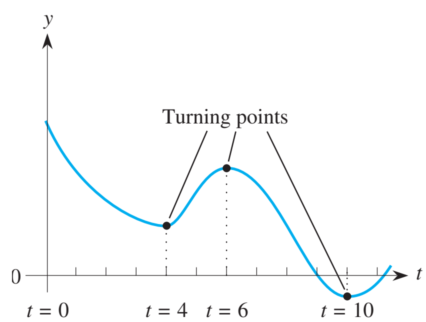


FIGURE 1.10

Note that a function increases or decreases over an interval of values on the horizontal axis; it has a turning point at a particular point corresponding to a single value along the horizontal axis. Figure 1.10 shows a function of t decreasing from $t = 0$ to $t = 4$, increasing from $t = 4$ to $t = 6$, decreasing from $t = 6$ to $t = 10$, and then increasing after $t = 10$. This function has turning points at $t = 4$, $t = 6$, and $t = 10$.

Think About This

Sketch the graph of a different function having the same behavior.

We describe a function that is always increasing or always decreasing (it has no turning points) as being *strictly increasing* or *strictly decreasing*.

Concavity: How A Function Bends

There is a second aspect to a function's behavior. Figure 1.11 shows two increasing functions. How do they differ? In Figure 1.11(a), the function isn't merely increasing; it is actually increasing faster and faster as time goes by. Think of the curve as bending upward. For instance, population growth typically follows this type of growth pattern. The function shown in Figure 1.11(b) is also increasing, but it is increasing more and more slowly as time goes by. Think of the curve as bending downward. For instance, the increasing human lifespan previously depicted in Figure 1.3 grows in this way.

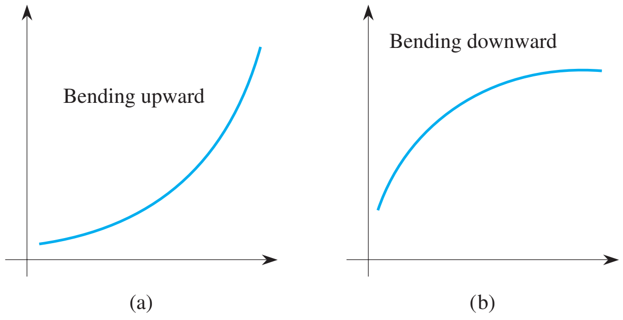


FIGURE 1.11

Now look at the two decreasing functions in Figure 1.12. The function shown in Figure 1.12(a) decreases very rapidly at first and then more slowly as time passes—it is decreasing at a decreasing rate. For instance, if a pollutant is released into a lake, the level of pollution in the lake will decrease ever more slowly as time goes by. Like the function shown in Figure 1.11(a), this curve is also bending upward. The graph in Figure 1.12(b) also decreases, but it is decreasing slowly at first and then more and more rapidly—it is decreasing at an increasing rate. For instance, if an object is tossed off the roof of a tall building, its height above ground will decrease in this manner as it speeds up in its descent because of the effects of gravity. Note that this curve is bending downward, as is the curve shown in Figure 1.11(b).

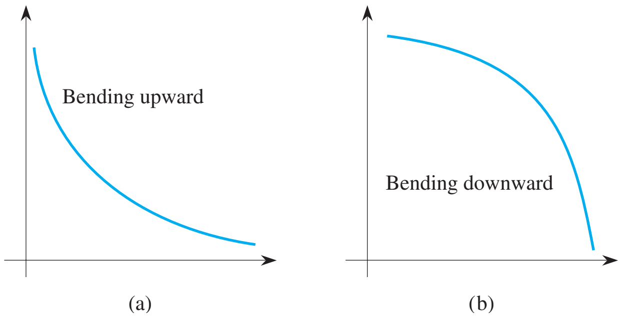


FIGURE 1.12

We use the term **concavity** to describe the way a function *bends.* Curves that bend upward, such as those shown in Figures 1.11(a) and 1.12(a), are **concave up.**

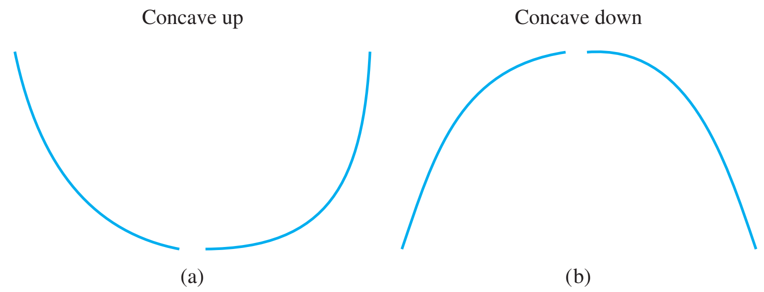


FIGURE 1.13

Note that one curve is increasing and that the other is decreasing, so concavity is a completely different concept from increasing/decreasing. Similarly, curves that bend downward, such as those shown in Figures 1.11(b) and 1.12(b), are **concave down.** Again, note that one is increasing and the other is decreasing. Figure 1.13(a) illustrates the two types of concave up behavior, and Figure 1.13(b) illustrates the two types of concave down behavior.

Think About This

Imagine a ball bouncing up and down across the floor in front of you. Is the path of the ball concave up or concave down?

Just as a function can be increasing over one interval and decreasing over another, a function can be concave up over one interval and concave down over another. For instance, think of the behavior of the Dow-Jones average. This function is increasing during some time intervals and is decreasing during other time intervals, as shown in Figure 1.14. It is also concave up over some time intervals and is concave down over other time intervals.

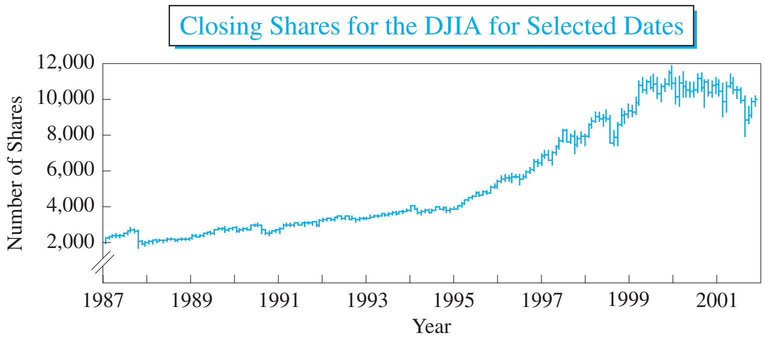


FIGURE 1.14

A point on a graph where the concavity changes from concave up to concave down or vice versa is called a **point of inflection** or an **inflection point.** In Figure 1.15, we show two curves, one having a point of inflection where the curve changes from concave up to concave down and the other where the curve changes from concave down to concave up. Observe that neither point of inflection occurs at the turning points where the curve reaches a local maximum or a local minimum, so turning points are not the same as inflection points.

Note that the function on the left in Figure 1.15 grows faster and faster to the left of the inflection point and then grows slower and slower to the right of the inflection point. As a result, the function is growing most rapidly *at* the inflection point.

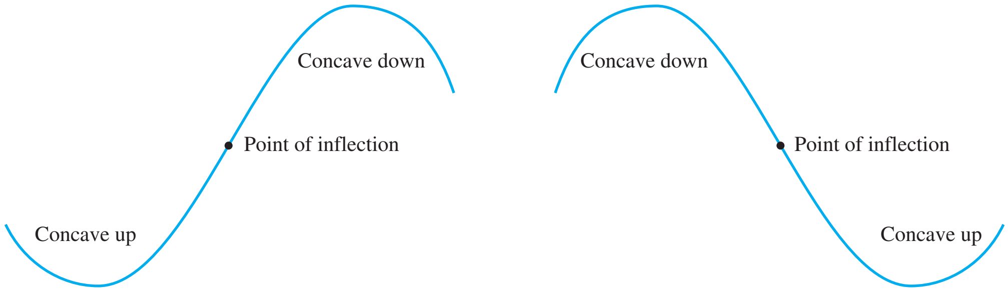


FIGURE 1.15

Think About This What happens at the inflection point of the function on the right in Figure 1.15?

We summarize the preceding information about functions as follows:

A function of x is *increasing* if the values of the function increase as x increases.

A function of x is *decreasing* if the values of the function decrease as x increases.

The graph of an *increasing* function *rises* from left to right.

The graph of a *decreasing* function *falls* from left to right.

The points where a function changes from increasing to decreasing or from decreasing to increasing are the *turning points.*

The graph of a function is *concave up* if it bends upward.

The graph of a function is *concave down* if it bends downward.

The points where the concavity changes from concave up to concave down or from concave down to concave up are the *points of inflection* or *inflection points.*

In addition, the rate of change and concavity of the graph of a function are related in the following ways.

If the graph of a function is increasing and concave up, it is increasing at an increasing rate.

If the graph of a function is increasing and concave down, it is increasing at a decreasing rate.

If the graph of a function is decreasing and concave up, it is decreasing at a decreasing rate.

If the graph of a function is decreasing and concave down, it is decreasing at an increasing rate.

A function grows fastest or decays fastest *at* a point of inflection.

EXAMPLE 1

Identify all intervals where the function *f* shown in Figure 1.16 is

a. increasing;
b. decreasing;
c. concave up;
d. concave down.

Then indicate all points where the function has a

e. turning point;
f. local maximum;
g. local minimum;
h. point of inflection.

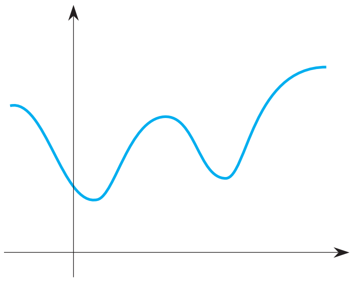


FIGURE 1.16

Solution For a–d, the task is to find the *intervals* of *x*-values where the different types of behavior occur. We begin by redrawing the graph and introducing all the points x_1, $x_2, \ldots, x_9$ where the behavior of the function changes, as shown in Figure 1.17.

a. The function is increasing for values of x between x_3 and x_5 and again between x_7 and x_9, as shown in Figure 1.17(a).
b. The function is decreasing between x_1 and x_3 and again between x_5 and x_7, as shown in Figure 1.17(a).
c. The curve is concave up between x_2 and x_4 and again between x_6 and x_8, as shown in Figure 1.17(b).
d. The function is concave down between x_1 and x_2, between x_4 and x_6, and again from x_8 to x_9, as shown in Figure 1.17(b).

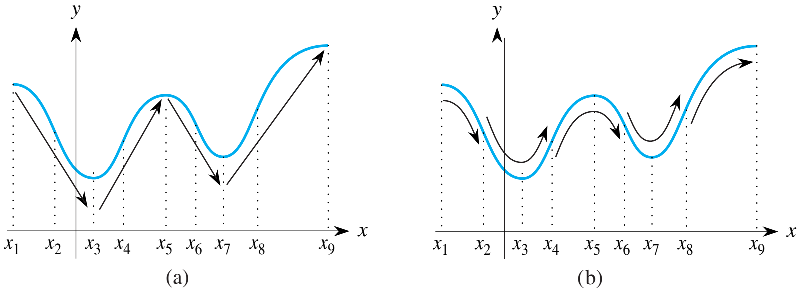


FIGURE 1.17

Next, we look for particular points on the curve.

e. The turning points for this function are at $x = x_3$, at $x = x_5$, and at $x = x_7$.
f. The function has a local maximum at $x = x_1$ (when compared to other nearby points); the function also has a local maximum at $x = x_5$ and again at $x = x_9$.
g. Similarly, the function reaches a local minimum at $x = x_3$ (when compared to other nearby points) and again at $x = x_7$.
h. The points of inflection occur where the concavity changes, which happens at $x = x_2$, at $x = x_4$, at $x = x_6$, and at $x = x_8$.

EXAMPLE 2

x	1	2	3	4
y	6	11	15	18

x	1	2	3	4
y	6	11	17	24

Two functions are defined in the accompanying tables of values. Describe the behavior of each function.

Solution Both functions are obviously increasing as x increases. Note that the first function grows first by 5 (from 6 to 11), then by 4 (from 11 to 15), and then by 3 (from 15 to 18), so it is growing at a decreasing rate. Therefore it is concave down. The second function, however, grows by larger and larger amounts—first by 5 (from 6 to 11), then by 6 (from 11 to 17), and then by 7 (from 17 to 24)—so the function is increasing at an increasing rate and thus is concave up. Plot the points to verify both behaviors.

Periodic Behavior

Another behavior pattern for functions is extremely common in real life. Many natural processes have the property of being *periodic*—that is, the pattern repeats over and over. We see this in the height of tides that rise and fall in the same pattern roughly every 12 hours in most coastal locations. It also occurs in the pattern of temperature readings in any location from one year to the next. Spotting a periodic function from its graph is easy: The identical pattern appears repeatedly. For instance, consider the following data based on historical records giving the average number of tornados reported in the United States, per month, in a typical year.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Tornados	16	24	60	111	191	179	96	66	41	26	31	22

Source: National Oceanic and Atmospheric Administration.

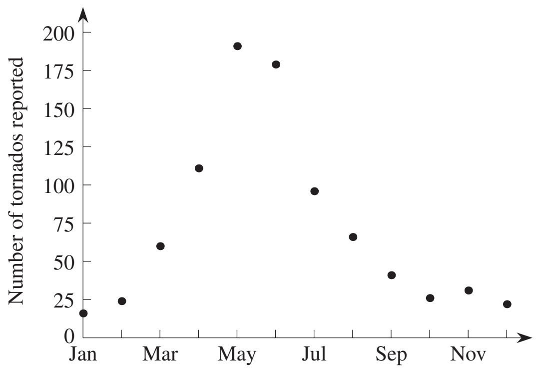


FIGURE 1.18

Figure 1.18 shows a graph of these points. Note, either from the table or the graph, how the values increase from a minimum level of tornado sightings in January to a maximum number in May and then decrease toward the minimum as the year ends. Because these values are based on historical averages, this cycle will likely repeat yearly with little change from one year to the next. It is therefore a roughly periodic phenomenon. Figure 1.19 shows a smooth curve that captures the longer term behavior of this roughly periodic function.

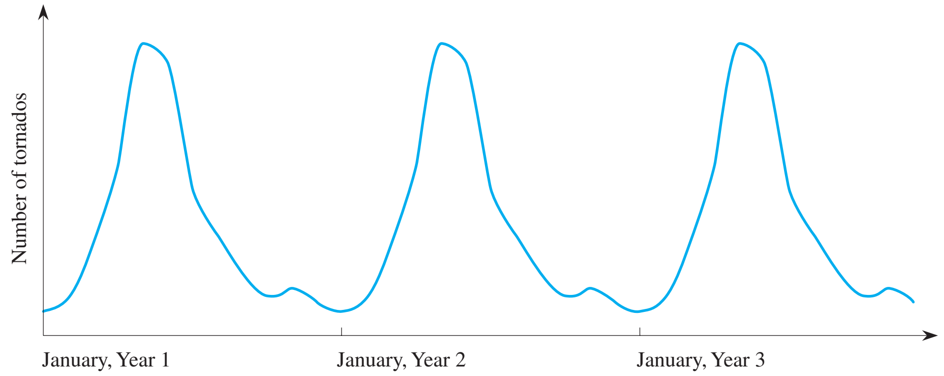
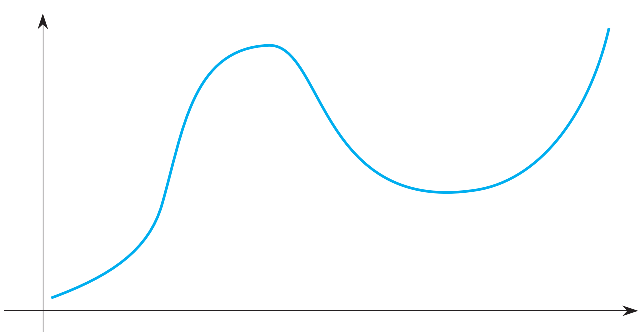


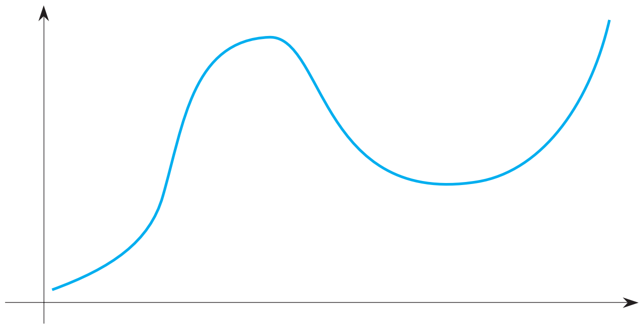
FIGURE 1.19

Problems

1. Which of the functions are strictly increasing, strictly decreasing, or neither?
   - **a.** The cost of first-class postage on January first of each year.
   - **b.** The time of sunrise associated with each day of the year.
   - **c.** The high temperature associated with each day of the year.
   - **d.** The closing price of one share of IBM stock for each trading day on the stock exchange.
   - **e.** The area of an equilateral triangle in terms of its base *b*.
   - **f.** The height of a bungee jumper *t* seconds after leaping off a bridge.
   - **g.** The height of liquid in a 55-gallon tank *h* hours after a leak develops.
   - **h.** The daily cost of heating a home as a function of the day's average temperature.
   - **i.** The world record times for running the 100 meter dash.
2. Consider the function shown in the accompanying graph. Use two different colored pens or pencils. With one, trace all parts of the curve where the function is increasing. With the other, trace all parts of the curve where the function is decreasing. Then mark all turning points on the curve.



3. Consider the function shown in the accompanying graph. Use two different colored pens or pencils. With one, trace all parts of the curve where the function is concave up. With the other, trace all parts of the curve where the function is concave down. Then mark all points of inflection on the curve.



4. Sketch the graph of a single smooth curve that is first increasing and concave up, then increasing and concave down, and finally decreasing and concave down. Mark all turning points and points of inflection on your curve.
5. Sketch the graph of a single smooth curve that is first decreasing and concave up, then increasing and concave up, and finally increasing and concave down. Mark all turning points and points of inflection on your curve.
6. Sketch a possible graph of the temperature in your hometown over an entire week as a function of time. On the graph indicate all the turning points. Where

is the temperature function increasing? Where is it decreasing? Where is the temperature function concave down? Where is it concave up?

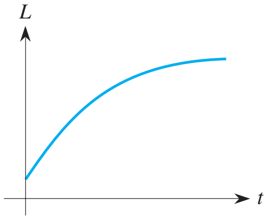
7. Each year the world's annual consumption of water rises, as does the amount of increase in water consumption. Sketch a graph of the annual world consumption of water as a function of time.

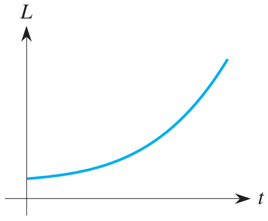
8. A human fetus grows rapidly at first and then grows with decreasing rapidity. Draw a graph showing the size of a fetus as a function of time.

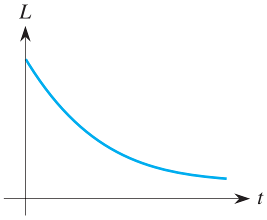
9. Sales of microwave ovens grew slowly when they were first introduced and then increased dramatically as more people appreciated their usefulness. Eventually, sales began to slow as most households already owned one. Sketch the graph of microwave oven sales as a function of time. Indicate the location of the point of inflection.

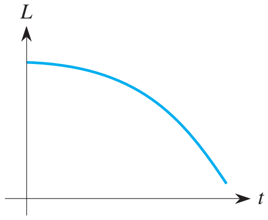
10. Sales of VCRs grew slowly at first and then increased tremendously as people came to accept them widely. Eventually new sales began to level off as market saturation neared. Sketch a possible graph of the percentage of U.S. homes owning a VCR as a function of time, paying careful attention to the behavior of the function. Indicate any turning points and points of inflection.

11. The Environmental Protection Agency (EPA) monitors the levels of industrial pollutants in many lakes and rivers. The following graphs show the level of pollutants L in four different lakes as a function of time t. For each, write a short paragraph either from the point of view of the EPA bringing charges against a company for polluting or from the point of view of a company defending itself against such charges.

a.


b.


c.


d.


12. Each part of the table of values below defines a function. Determine the concavity of each function.

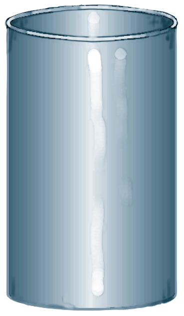
x	y	x	y	x	y
1	36	10	160	3	84
2	31	15	172	7	74
3	27	20	189	11	61
4	24	25	209	15	45
5	22	30	243	19	22

13. The Apollo-12 mission involved a flight to the moon (250,000 miles from Earth), five circular orbits about the moon, and a return to Earth.

a. Assume (incorrectly) that the spacecraft traveled at a constant speed between the Earth and the moon. Sketch a rough graph of the distance from Earth as a function of time.

b. Assume (correctly) that the spacecraft's speed diminished the farther it got from Earth's gravity until it neared the moon and then increased due to the moon's gravitational force. The behavior of the spacecraft's speed reversed on the return trip. Sketch a rough graph of the spacecraft's distance from the Earth as a function of time. (Think concavity!)

14. Water is being poured, at a constant rate, into vases having the shapes shown. Sketch a graph showing

a.


b.

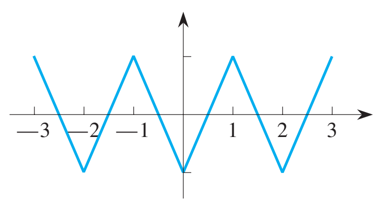

the level of the water as a function of time, paying careful attention to concavity. What is the significance of each of the points of inflection, if any?

c.

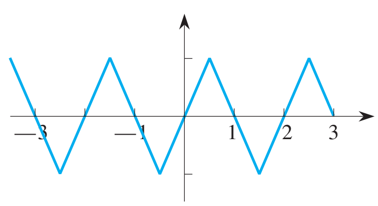
d.

15. Decide which functions in a–j are periodic. (Assume that the graphs continue indefinitely to the left and right in the same pattern.)

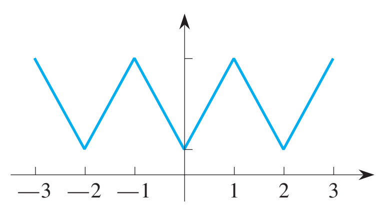
a.



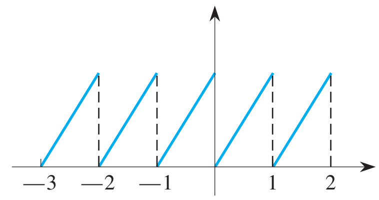
b.



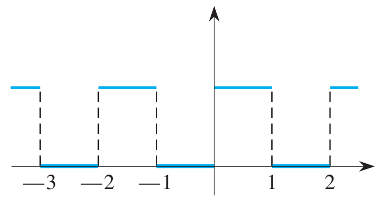
c.



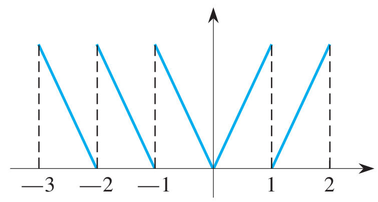
d.



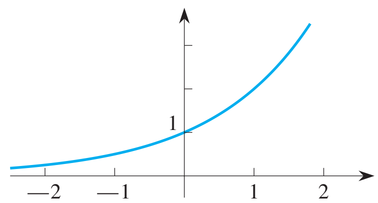
e.



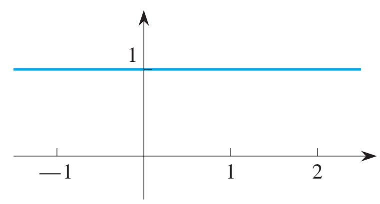
f.



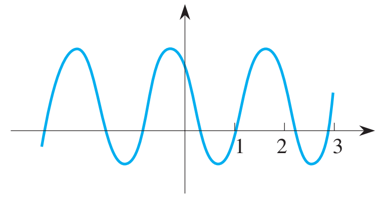
g.



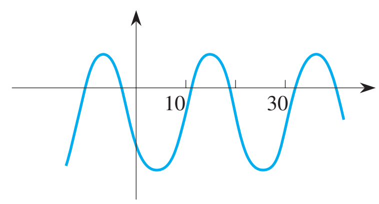
h.



i.



j.

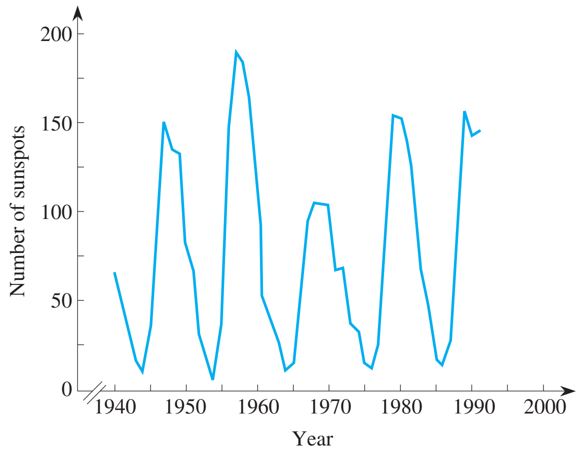


16. Janis trims her fingernails every Saturday morning. Sketch the graph of the length of her nails as a function of time. Is this process periodic?

17. Craig is a perfectly normal individual with a pulse rate of 60 beats per minute and a blood pressure of 120 over 80. Thus his heart is beating 60 times/minute and his blood pressure is oscillating between a low (diastolic) reading of 80 and a high (systolic) reading of 120. Sketch the graph of his blood pressure as a function of time. Be sure to indicate appropriate scales on each axis.

18. **a.** The thermostat in Sylvia's home in Baltimore is set at 66°F during the winter. Whenever the temperature drops to 66° (roughly every half-hour), the furnace comes on and stays on until the temperature reaches 70°. Sketch the graph of the temperature in her house as a function of time. Be sure to indicate appropriate scales on each axis.
    b. Gary, who lives in upstate New York, also has his thermostat set to come on at 66°F. How will a sketch of the temperature in his house differ from the one you drew in part (a) for Sylvia's house?
    c. Jodi, who lives in central Florida, likewise has her thermostat set to come on at 66°. How will a sketch of the temperature in her house differ from the other two?

19. Astronomers have been observing sunspots on the face of the sun for centuries. These dark spots on the sun, which are accompanied by the release of bursts of electromagnetic radiation that disrupt radio and TV signals, occur in periodic cycles. The accompanying figure is a graph of the number of sunspots observed each year.
    a. Estimate the period of the sunspot cycle.
    b. Estimate when the next two peaks will occur in the cycle.
    c. Suppose that you were required to come up with a reasonable estimate for the maximum number of sunspots that will occur during the next peak in the cycle. How might you create such an estimate based on the information given in the figure?



1.3 Representing Functions Symbolically

A function is a rule that associates one and only one value of a quantity (say, y) with each value of another quantity (say, x). The quantities x and y are variables. We use a single letter, such as f, g, or h, as the name of a function. The particular formula for the function, if it is known, is usually written as $y = f(x)$. It is read as "y equals f of x" or possibly "y is a function of x." For instance, the function f that takes any real number x and squares it can be written as

$$y = f(x) = x^2.$$

Some particular values of this function are

$$f(3) = 3^2 = 9; \qquad f(\tfrac{1}{3}) = (\tfrac{1}{3})^2 = \tfrac{1}{9};$$
$$f(-5) = (-5)^2 = 25; \qquad f(0.02) = (0.02)^2 = 0.0004;$$
$$f(-0.01) = (-0.01)^2 = 0.0001; \qquad f(\pi) = \pi^2 \approx 9.8696.$$

In each case we replaced the variable x in $f(x) = x^2$ with the indicated value, $x = 3$, $x = \frac{1}{3}$, and so on, and then evaluated the expression 3^2, $(\frac{1}{3})^2$, and so on.

Similarly, the function g that takes the square root of any nonnegative real number x can be written as

$$y = g(x) = \sqrt{x}.$$

For instance, some values of this function g are

$$g(16) = \sqrt{16} = 4;$$
$$g(0.04) = \sqrt{0.04} = 0.2;$$
$$g(\tfrac{1}{4}) = \sqrt{\tfrac{1}{4}} = \tfrac{1}{2}.$$

But $g(-25) = \sqrt{-25}$ does not make sense because it isn't possible to take the square root of a negative number in the real number system. That is, the function g is *not defined* for $x = -25$.

The function h that gives the reciprocal of any nonzero number x can be written as

$$y = h(x) = \tfrac{1}{x}.$$

Some values of this function h are

$$h(5) = \tfrac{1}{5} = 0.2;$$
$$h(200) = \tfrac{1}{200} = 0.005;$$
$$h(-0.125) = -\tfrac{1}{0.125} = -8.$$

But $h(0)$ does not make sense because division by 0 is not possible. That is, the function h is *not defined* at $x = 0$.

To work with functions requires some terminology. In the form $y = f(x)$, we call x the **independent variable** because it can take on any appropriate value. We call y the **dependent variable** because its value depends on the choice of x.

You can use letters other than f, g, or h to represent functions; other common choices are F, G, or f_1, f_2, f_3, and so on. You can use letters other than x to represent the independent variable; other common choices are t (for time), θ (for an angle), and r (for radius). Similarly, you can use letters other than y to represent the dependent variable; for instance, you can use A for area, D for distance, P for population, and C for cost.

The area A of a circle is a function of its radius r—for each radius r, there is one and only one area A. We write this function as $A = f(r) = \pi r^2$. Here r is the independent variable, A is the dependent variable because the area depends on the choice of r, and f is the function. The distance D that a car moves in t hours at a steady speed of 50 miles per hour (mph) is given by $D = g(t) = 50t$. Here t is the independent variable, D is the dependent variable, and g is the function.

Suppose that you toss a ball straight up with an initial velocity of 64 feet per second. The function

$$y = f(t) = 64t - 16t^2$$

gives the height in feet of the ball above ground level after t seconds, until the instant that the ball hits the ground. Picture what happens. As the ball rises, it slows due to the effect of gravity. Eventually it reaches a maximum height and then begins to fall back to the ground. As the ball falls, its speed increases, again because of gravity.

Now let's see how the function f gives the height of the ball above ground at any time t. After half a second, when $t = \tfrac{1}{2}$, the ball is 28 feet above the ground because

$$y = f(\tfrac{1}{2}) = 64(\tfrac{1}{2}) - 16(\tfrac{1}{2})^2 = 32 - 4 = 28 \text{ feet.}$$

After 1 second, it is at a height of

$$y = f(1) = 64(1) - 16(1)^2 = 48 \text{ feet.}$$

After 2 seconds, it is at a height of

$$y = f(2) = 64(2) - 16(2)^2 = 64 \text{ feet},$$

which happens to be the maximum height the ball reaches. After 3 seconds, the height is

$$y = f(3) = 64(3) - 16(3)^2 = 48 \text{ feet},$$

and the ball is on its way down. After 4 seconds, the ball is back at ground level because

$$f(4) = 64(4) - 16(4)^2 = 0.$$

Domain and Range of a Function

In each of the preceding functions, there were some natural limitations on the possible values for both the independent variable and the dependent variable. The ball is released at time $t = 0$ and returns to the ground at $t = 4$ seconds. It therefore makes no sense in this problem to think about what happens before time $t = 0$ or after time $t = 4$. Thus the permissible values for t are between 0 and 4 seconds. Furthermore, the ball rises to its maximum height of 64 feet and then falls back to the ground. Therefore the only meaningful values for the height of the ball are between $y = 0$ and $y = 64$ feet. (Of course, it is more realistic to think of throwing a ball upward from about 4 or 5 feet above the ground rather from ground level—we used ground level here just to simplify the mathematics.)

Similarly, the function $y = g(x) = \sqrt{x}$ makes sense only if the independent variable x is not negative. The possible corresponding values for y must be positive or zero. The function $y = h(x) = 1/x$ makes sense only if x is not zero. The possible corresponding y-values of this function can be any number other than 0 because there is no value of x such that $y = 1/x = 0$. Finally, for the function $y = f(x) = x^2$, there is no limitation on the possible values of x, but there certainly is a limitation on the corresponding values for $y = x^2$ because they can never be negative.

For any function f, the set of *all possible values for the independent variable* is called the **domain** of f; the set of *all possible values for the dependent variable* is called the **range** of f.

Typically, the domain and range consist of intervals of values for the independent variable and the dependent variable, respectively. For instance, with the function representing the height of the ball, the domain consists of the interval from $t = 0$ to $t = 4$ and the range consists of the interval from $y = 0$ to $y = 64$. We can also use inequalities to write these intervals expressing the domain as $0 \leq t \leq 4$ and the range as $0 \leq y \leq 64$.

Because each of these intervals contains endpoints ($t = 0$ and $t = 4$ for the domain and $y = 0$ and $y = 64$ for the range), they are called *closed intervals*. We can also write these intervals using *interval notation,* so that the domain is the closed interval $[0, 4]$ and the range is the closed interval $[0, 64]$. We use square brackets to indicate that the endpoint value is included in the interval.

If one or both endpoint values is *not* included in an interval, we use parentheses instead of square brackets. For instance, if an interval is $3 < x < 6$, where both endpoints are not included, we write it in interval notation as the *open interval* $(3, 6)$. (Caution: Don't misinterpret this notation as the coordinates of the point with $x = 3$ and $y = 6$. The symbols are identical, but the meaning should be clear from the context.)

An interval can also contain one endpoint, but not the other, as in $-2 < x \le 8$. We write this in interval notation as $(-2, 8]$, where the square bracket on the right indicates that $x = 8$ is included and the left parenthesis indicates that $x = -2$ is not included. For instance, the domain of the function $y = g(x) = \sqrt{x}$ is the interval $[0, \infty)$, which indicates that $x \ge 0$; it is closed on the left, and extends toward ∞, but never reaches ∞, and so is open on the right.

EXAMPLE 1

Find the values of the function $y = g(x) = \sqrt{x}$ at $x = 0, \frac{1}{4}, 1, 2, \pi$, and 4. What is the domain and range of this function?

Solution For each of the given values of x in the domain of this function, the corresponding y-values in its range are

$$y = g(0) = \sqrt{0} = 0;$$
$$y = g(\tfrac{1}{4}) = \sqrt{\tfrac{1}{4}} = \tfrac{1}{2};$$
$$y = g(1) = \sqrt{1} = 1;$$
$$y = g(2) = \sqrt{2} \approx 1.41421\ldots;$$
$$y = g(\pi) = \sqrt{\pi} \approx \sqrt{3.14159\ldots} \approx 1.77245;$$
$$y = g(4) = \sqrt{4} = 2.$$

Note that you can take the square root of any positive value of x or of 0, so the domain of the function g consists of all nonnegative numbers. Similarly, the square root of any such number is positive or 0, so the range of g also consists of all nonnegative numbers.

We can use inequality notation to write $x \ge 0$ for the domain and $y \ge 0$ for the range. Alternatively, using interval notation, we have $[0, \infty)$ for the domain and $[0, \infty)$ for the range.

EXAMPLE 2

Discuss the range of the function

$$y = F(x) = x + \frac{1}{x}$$

when the domain for F is restricted to the set of all positive numbers.

Solution We start by looking at the graph of the function F, as shown in Figure 1.20. Note that the function is decreasing rapidly to the right of $x = 0$. It has a turning point at

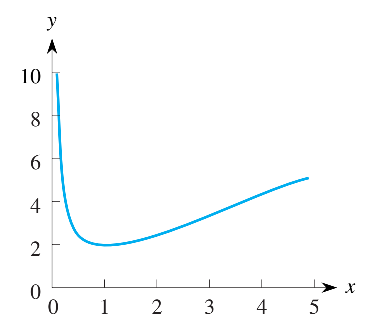


FIGURE 1.20

about $x = 1$, and then it increases slowly thereafter. Try different values for x with your calculator to verify that this result is indeed the case numerically. Also, extend the graph farther to the right with your function grapher to see that this pattern continues indefinitely. It turns out that the smallest possible value for y, which is $y = 2$ exactly, corresponds to $x = 1$ at the turning point. For any other value of x, the value for y is larger. Therefore the range of F is all values $y \geq 2$.

You can visualize a function f as an operation that *transforms* each value x from its domain into the corresponding value y in its range. Figure 1.21 illustrates how each point x in the domain is transformed into a single point in the range. Thus x_1 is transformed into y_1. We also can say that x_1 is carried into y_1 or that x_1 is *mapped* into y_1. Similarly, x_2 is carried into y_2 and x_3 is mapped into y_3. Note that x_4 and x_5 are both transformed into y_4, which is perfectly legitimate for a function. Each x-value must be mapped into a single y-value, although it is certainly possible for several different x's to be mapped into the same y. Think about the function $y = f(x) = x^2$, where both $x = 2$ and $x = -2$ are transformed into $y = 4$.

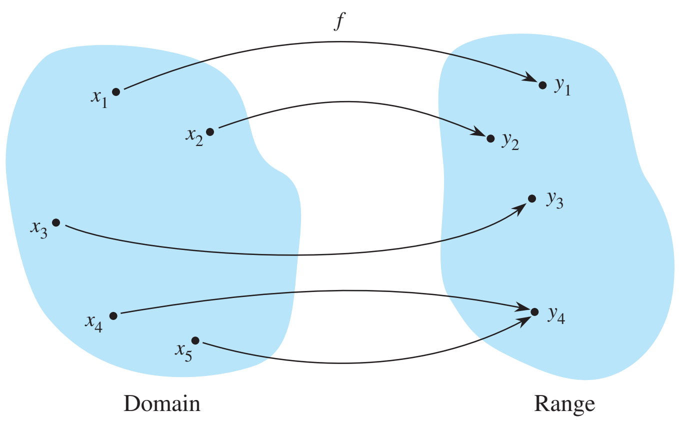


FIGURE 1.21

We now summarize the preceding ideas in a formal definition of a function.

Definition of a Function

A **function** f is a rule that assigns to each permissible value of the independent variable x one and only one value of the dependent variable y.

The **domain** of f is the set of all possible values for the independent variable.

The **range** of f is the set of all possible values for the dependent variable.

EXAMPLE 3

Discuss the domain and range for the function relating acceleration time t to final speed v for a Trans Am, based on the following set of data.

Final speed, *v* (mph)	30	40	50	60	70
Time, *t* (sec)	3.00	4.29	5.52	7.38	9.81

Solution The independent variable is the final speed v, so the domain of this function consists of all possible speeds. We therefore might conclude that the domain would be 0 to 70 mph; however, if we want to use the function to predict the time needed to reach a higher speed, we would need a somewhat larger domain—say, 0 to 100 mph. It probably isn't reasonable to think of speeds any faster than that. The dependent variable is the time t needed for a Trans Am to accelerate to a given speed, v. If we use only speeds between 0 and 70 mph, the associated range would be 0 to 9.81 seconds. If we use the extended domain of v of 0 to 100 mph, however, the associated range might be more like 0 to 20 seconds. It takes about 2.5 seconds to accelerate from 60 to 70 mph. The pattern suggests that it will take even more time to accelerate from 80 to 90 mph and still more time to go from 90 to 100 mph. Thus an estimate of 20 seconds to accelerate from 0 to 100 mph is reasonable.

Consider the relationship between people and their telephone numbers. Is this relationship a function? If there is even one person who has two different telephone numbers, the relationship does not satisfy the definition and so *is not* a function. But a person's height *is* a function of the person—each individual has one and only one height at any particular time.

Often a verbal description of a function includes the idea of *proportionality* from elementary algebra. Recall that y is proportional to x means that $y = k \cdot x$, for some constant of proportionality k. For instance, the area of a circle is proportional to the square of the radius because $A = \pi r^2$ and π is a constant (it is the constant of proportionality). Similarly, y is inversely proportional to x if $y = k \cdot 1/x = k/x$, where k is a constant of proportionality.

Throughout this book, unless some restriction is indicated, we assume that all functions discussed are defined (either mathematically or practically) on the largest possible domain that makes sense.

Problems

1. Which of the relationships are functions and which are not? For those that are not functions, explain why. For those that are functions, identify the independent and dependent variables and give a reasonable domain.
   - **a.** The cost of first-class postage on January first of each year since 1900.
   - **b.** The weight of letters you can mail with $n = 1, 2, 3, \ldots$ postage stamps.
   - **c.** The time of sunrise associated with each day of the year.
   - **d.** The time of high tide associated with each day of the year.
   - **e.** The high temperature associated with each day of the year.
   - **f.** The closing price of one share of IBM stock each trading day on the stock exchange.
   - **g.** The area of a rectangle whose base is b.
   - **h.** The area of an equilateral triangle whose base is b.
   - **i.** The height of a bungee jumper t seconds after leaping off a bridge.
   - **j.** The time it takes the bungee jumper to reach a height H above the ground.
   - **k.** The number of baseball players who have n home runs in a full season.
   - **l.** The height of liquid in a 55-gallon tank h hours after a leak develops.
   - **m.** The daily cost to a family of heating their home versus the average temperature that day.
2. The balance B, in thousands of dollars, in a CD account at a bank is a function of time t, in years, since you opened the account, so $B = f(t)$.
   - **a.** What does $f(4) = 2$ tell you? What are appropriate units?
   - **b.** Is f an increasing or decreasing function of t?
   - **c.** Discuss the concavity of f.

3. The height H in inches of a child is a function of the child's age a, so $H = f(a)$.
   - **a.** What does $f(10) = 50$ tell you? What are appropriate units?
   - **b.** Is f an increasing or decreasing function of a?
   - **c.** Discuss the concavity of f.
4. The surface area S of a sphere of radius r is 4π times the square of the radius. Write a formula for S as a function of r.
5. The pressure P of a gas in a container of fixed size is proportional to the temperature T of the gas. Write a formula for the pressure as a function of temperature.
6. The pressure P of a gas held at a constant temperature in a container is inversely proportional to the volume V of the container. Write a formula for the pressure as a function of volume.
7. The force of gravity F between two objects is inversely proportional to the square of the distance d between the objects. Write a formula for F as a function of d.
8. When a cup of hot coffee is left to cool on the table where the air temperature is 70°F, the change ΔT in the temperature T of the coffee is proportional to the difference between the temperature of the coffee and the room temperature. Write a formula for ΔT as a function of T.
9. Kim has a peanut butter sandwich on white bread each day. The number of calories C in the sandwich, as a function of the number of grams P of peanut butter, is $C = f(P) = 150 + 6P$.
   - **a.** What is $f(1)$? What does it mean?
   - **b.** What is $f(10)$? $f(15.5)$? $f(20)$? $f(30)$?
   - **c.** How many calories come from the bread alone?
   - **d.** Explain why using $P = -1$ makes no sense.
   - **e.** What is a reasonable domain and range for this function?
10. Suppose that Jim wants his peanut butter sandwich on rye bread instead of white bread. Rye bread contains 85 calories per slice. What would be the corresponding formula for the number of calories in Jim's sandwich?
11. The number of calories in a peanut butter and jelly sandwich on white bread is $C = 150 + 6P + 2.7J$, where P and J are the number of grams of peanut butter and jelly, respectively.
   - **a.** How many calories are in a sandwich with 24 g of peanut butter and 20 g of jelly?
   - **b.** Suppose that Adam is on a diet and wants to limit his calorie intake from a peanut butter and jelly sandwich to a maximum of 300 calories. Find two reasonable combinations of amounts of peanut butter and jelly that produce a sandwich with exactly 300 calories.
   - **c.** Which is more caloric, a gram of peanut butter or a gram of jelly? Explain how you know.
12. A car rental company charges a fixed daily rate for a midsize car plus a charge for each mile more than 100 miles that the car is driven per day. A formula for the cost of a rental car driven more than 100 miles is $c = f(m) = 35 + 0.25(m - 100)$, where m is the number of miles that the car is driven.
   - **a.** Find $f(100)$. What does it mean?
   - **b.** Find $f(150)$, $f(200)$, and $f(500)$.
   - **c.** What is a reasonable domain and range for this function?
13. Suppose that you throw a ball upward, with an initial velocity of 60 ft/sec, from the roof of a 120-ft-high building.
   - **a.** Sketch a possible graph of the height of the ball as a function of time, as you visualize it.
   - **b.** Suppose that the height of the ball as a function of time is given by
     $$H(t) = 120 + 60t - 16t^2.$$
     Find the height of the ball when $t = 1$; when $t = 4$.
   - **c.** Find $H(2)$ and $H(3)$. What do they represent?
   - **d.** Use your function grapher to estimate how long it takes for the ball to reach its maximum height. What is the maximum height?
   - **e.** How long does it take until the ball first hits the street below?
   - **f.** What are the domain and range for this function?
14. For the function $f(t) = t^2 - 5$, find the values corresponding to $t = 2, 4, 6, 10$.
15. For the function[1]
    $$F(x) = \frac{1}{x^2 - 4},$$
    find $F(0)$, $F(1)$, $F(3)$, $F(4)$, $F(5)$. Why did we skip $x = 2$? Are there any other values of x that should be skipped? What is the domain of this function?

[1]Note that when you enter this expression in a calculator or most computer programs, you must key the expression in as 1/(x^2 − 4). Pay careful attention to when you need to use parentheses in any such expression.

16. For the function

$$g(x) = \frac{x^2 + 4}{x^2 - 9},$$

find $g(0)$, $g(1)$, $g(2)$, $g(4)$, $g(-1)$. Why did we skip $x = 3$? Are there any other values of x that should be skipped? What is the domain of this function?

17. For the function $g(s) = s + \sqrt{s}$, find the values corresponding to $s = 4, 16, 25, 100$. Are there any values for s that will make the function come out negative? What does that tell you about the range of g? What is its domain?

18. For the function $z = f(q) = q^3 + 5$, find the value of the dependent variable that corresponds to a value of the independent variable of 4. Find the value of the independent variable that corresponds to a value of the dependent variable of 6.

19. For the function $f(x) = x^3 - 8x^2 + 15x - 1$, find three different values of x between 1 and 8 for which $f(x) < 0$. Then find at least two noninteger values of x for which $f(x) < 0$.

20. A simple substitution code in which each letter is replaced by a different letter can be thought of as a function f whose domain is the letters of the alphabet $A, B, \ldots, Z$. Suppose that $f(A) = M$, $f(B) = D$, $f(C) = K$, $f(D) = V$, $f(E) = X$, $f(F) = B$, $f(G) = P$, $f(H) = T$, $f(I) = J$, $f(J) = S$, $f(K) = Z$, $f(L) = Q$, $f(M) = H$, $f(N) = O$, $f(O) = A$, $f(P) = L$, $f(Q) = W$, $f(R) = C$, $f(S) = F$, $f(T) = Y$, $f(U) = R$, $f(V) = G$, $f(W) = I$, $f(X) = U$, and $f(Y) = N$.
    a. What is $f(Z)$?
    b. What is the solution to the equation $f(x) = R$?

1.4 Connecting Geometric and Symbolic Representations

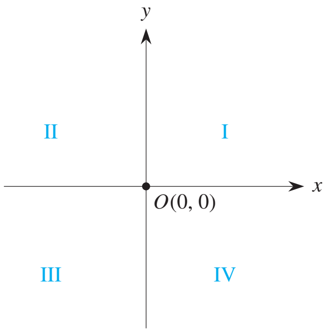


FIGURE 1.22

One of the most significant advances in mathematics is based on the idea of connecting the geometric and symbolic representations of functions. It allows you to think of functions from a visual rather than an exclusively symbolic perspective.

Begin by drawing two perpendicular *axes,* as shown in Figure 1.22, whose point of intersection O is the *origin.* The *horizontal axis* represents values of the *independent variable* (in this case, x); by convention, these values increase from left to right, as indicated by the arrow. The *vertical axis* represents values of the *dependent variable* (in this case, y); by convention these values increase upward, as indicated by the vertical arrow. The two axes divide the plane into four *quadrants:* the first quadrant (I), the second quadrant (II), the third quadrant (III), and the fourth quadrant (IV). Whenever appropriate, indicate the units used for each variable and label the axes accordingly.

This representation is called a *rectangular* or *Cartesian coordinate system,* and it is a way of associating points in the plane with ordered pairs of numbers. Every point P in the plane can be represented by an ordered pair of numbers, (x, y). Alternatively, every ordered pair, such as $(2, 5)$, $(27, 1)$, or $(23.84, 21.02)$, represents a point in the plane. We call (x, y) the **coordinates** of the point P. In mathematics, the letters x and y generally represent the independent and dependent variables, respectively. However, in any given context, you should use letters that suggest the quantities being studied.

Consider again the function

$$y = f(t) = 64t - 16t^2,$$

which represents the height, at any time t, of a ball tossed vertically upward with initial velocity of 64 feet per second. The formula for the function f gives the vertical height y of the ball at any instant t. When $t = 0$, $y = 0$ and the corresponding point $(0, 0)$ is the origin. Further, as we calculated before, $f(\frac{1}{2}) = 28$, $f(1) = 48$,

$f(2) = 64$, $f(3) = 48$, and $f(4) = 0$, which give rise to the points $(\frac{1}{2}, 28)$, $(1, 48)$, $(2, 64)$, $(3, 48)$, and $(4, 0)$ in the (t, y) coordinate system. These six points are plotted in Figure 1.23.

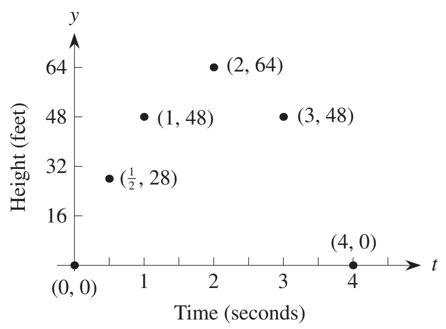


FIGURE 1.23

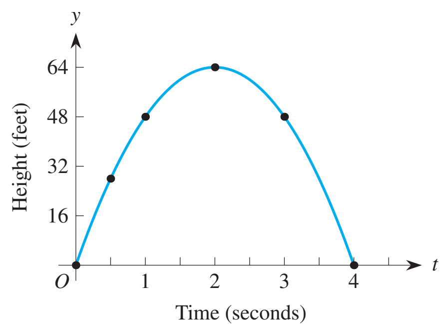


FIGURE 1.24

We can determine many other ordered pairs (t, y) satisfying the equation $y = 64t - 16t^2$. (Simply pick any other value for t between 0 and 4 and calculate the associated value of y by using the equation.) Each such ordered pair can be plotted as a point in the coordinate system. When all possible points are plotted, they form the curve shown in Figure 1.24. This curve is the *graph of the function f.* It consists of *all* points in the plane whose coordinates (t, y) satisfy the given equation. Thus we have a direct connection between the graph of a function and its algebraic equation. The graph of a function is therefore another representation of the same function. Note that the graph shown in Figure 1.24 represents the *height* of the ball at any time t; it doesn't show the *path* of the ball, which goes straight up and then down.

> The **graph** of a function $y = f(x)$ consists of *all* points (x, y) in the plane whose coordinates satisfy the equation of the function.

A table of values for a function is also useful when you're creating a hand-drawn graph of the function f from the formula $y = f(x)$. It provides a simple method of organizing the values of the independent variable x and the associated values of the dependent variable y that produce each point to be plotted. The number of points that you need to calculate for a table to draw a reasonable graph of a function depends on how complicated the behavior of the function is. For a line, all you need is two points because two points completely determine a line. We used six points to produce the graph of the height of the thrown ball shown in Figure 1.24. For comparison, a graphing calculator uses about 100 points to construct a curve.

When drawing the graph of a function, you should determine several key points. One point is where the graph crosses the vertical axis. You can easily find this point if you have a formula for the function: Just set the independent variable x equal to zero in the algebraic formula for the function and calculate the corresponding y-value. Although often desirable, finding the point(s) where the curve crosses the horizontal axis is usually more complicated. To find them, set the dependent variable y equal to zero and then solve the resulting equation. For the function representing the height of the ball, we can factor the expression for y and then set $y = 0$:

$$y = 64t - 16t^2 = 16t(4 - t) = 0.$$

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When you solve this equation for t, you get either $t = 0$ or $t = 4$. The time $t = 0$ is the instant when the ball is first released, so $y = 0$. At the instant when $t = 4$, the corresponding value for y, which represents the height of the ball, is also zero. That is, at time $t = 4$, the ball has come back to the ground. You can see the pattern for the values of this function (and thus the pattern for the height of the ball) in the following table.

Time *t* (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Height *y* (ft)	0	28	48	60	64	60	48	28	0

EXAMPLE 1

Determine the domain and range of the function shown in Figure 1.25.

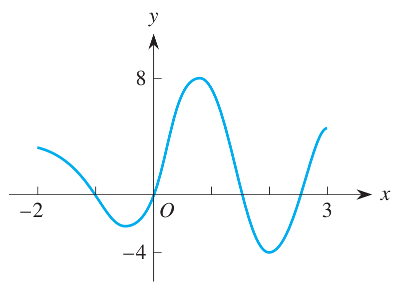


FIGURE 1.25

Solution Note that the axes shown are labeled x and y; x is the independent variable, and y is the dependent variable. Further, observe that the graph extends from $x = -2$ at the left to $x = 3$ at the right, so the domain of this function is from -2 to 3. We can write this domain in terms of inequalities as $-2 \le x \le 3$. Similarly, the graph extends vertically from a low of $y = -4$ to a high of $y = 8$, so the range is $-4 \le y \le 8$.

In many situations, we typically start with a set of data collected from some experiment or from measurements taken on some process. We then graph the data to get a feel for the behavior of the quantity. Often, we try to connect the points on the graph with a smooth curve to get a better indication of the behavior of the quantity. Finally, we would like to obtain an equation for a function that fits these data points because many questions can be answered far more easily and accurately when an equation is available. We illustrate this methodology in Examples 2–4.

EXAMPLE 2

The snow tree cricket, which lives in the Colorado Rockies, has been studied by field biologists who have gathered the following measurements on how the chirp rate depends on the air temperature.

Temperature *T* (°F)	50	55	60	65	70	75	80
Rate *R* (chirps/min)	40	60	80	100	120	140	160

Plot the points to determine the kind of trend in the data.

Solution Plotting these data points gives a visual dimension, as shown in Figure 1.26. Note that the chirp rate is growing at a constant rate as the temperature increases. Moreover, the corresponding points in the figure seem to fall into a straight line pattern, as indicated by the line drawn through them. In Chapter 2, we discuss how to find the equation of this line and how to predict the chirp rate R of the cricket based on the temperature T, or vice versa.

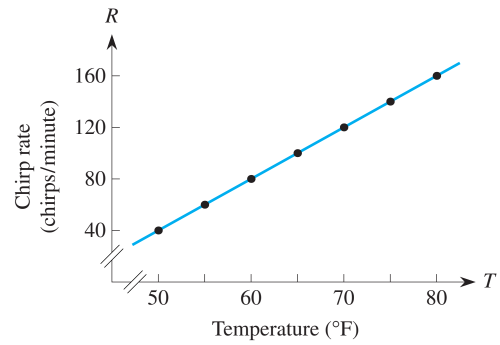


FIGURE 1.26

EXAMPLE 3

The following table of values gives the population, in millions, of the state of Florida since 1990.

a. Plot the data points and describe the behavior pattern.

b. If this trend continues, estimate the population in the year 2000.

Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Population	12.94	13.32	13.70	14.10	14.51	14.93	15.36	15.81	16.27	16.74	?

Solution

a. The graph of this set of data is shown in Figure 1.27. The growth pattern clearly is not a straight line pattern; rather, the population grows ever faster. The function is both increasing and concave up.

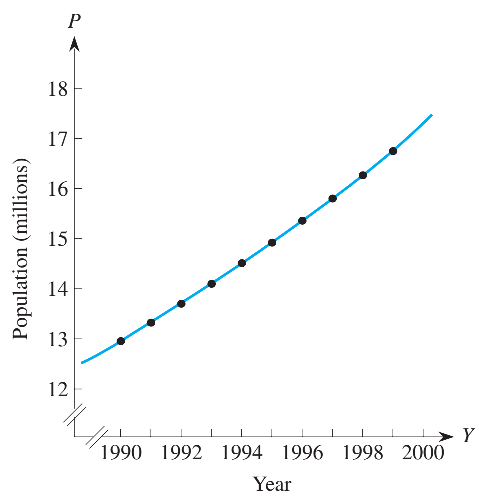


FIGURE 1.27

b. The increase from 1997 to 1998 was $16.27 - 15.81 = 0.46$ million, and the increase from 1998 to 1999 was $16.74 - 16.27 = 0.47$ million. As a result, we could estimate that the increase from 1999 to 2000 might be about 0.48 million, so our prediction for the year 2000 is about $16.74 + 0.48 = 17.22$ million people. We determine a formula for this function in Chapter 2 so that we can make such a prediction in a much simpler and more confident way.

EXAMPLE 4

The following table of values shows measurements, at different times, of the height of an object dropped from the top of the 1250-foot-high Empire State Building. Construct a graph of the height as a function of time and describe its behavior.

Time (sec)	0	1	2	3	4	5	6	7	8
Height (ft)	1250	1234	1186	1106	994	850	674	466	226

Solution The graph of the height of the object versus time in Figure 1.28 shows that the object is falling ever faster as time goes by. The function is decreasing and concave down. Again note that the graph represents the height of the object, not its path, which is straight down.

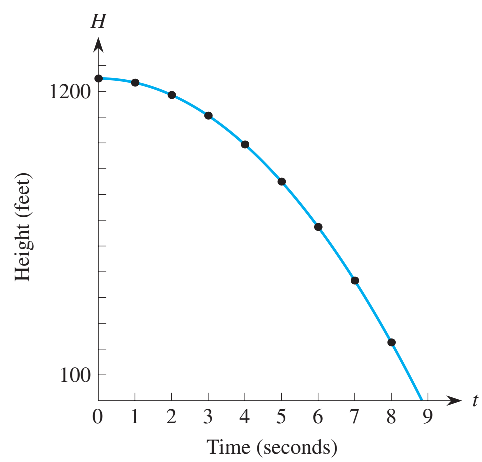


FIGURE 1.28

Although we could estimate from either the table or the graph how long it takes the object to hit the ground or to pass, say, the 30th floor, we could answer such questions more precisely if we knew the formula for the function.

In Examples 2–4, we simply connected the points to construct a smooth curve that seemed to fit the pattern. Doing so, however, can sometimes lead to serious errors. Suppose that we had some data on the turkey population of the United States taken on January 1 each year. It would likely show a growth trend similar to that in Example 3 on the population of Florida. However, a little thought will convince you that this population will change quite drastically about the middle of November each year. The smooth curve drawn using the January 1 turkey census data would therefore be a rather poor description of the actual population over all intermediate times.

Nevertheless, the idea of connecting a series of points to form a curve is precisely how a computer or graphing calculator produces the graph of a function. We strongly urge you to become comfortable with using a graphing calculator or a computer program to investigate the graph of any desired function given by a formula. The visual dimension invariably provides a wealth of information about the behavior of the function, and we continually turn to graphical images throughout this book.

Connections Between Geometric, Numerical, and Symbolic Representations

We have shown that a function can be represented in a variety of ways—as a formula giving a symbolic representation, as a graph giving a geometric representation, as a table giving a numerical representation, or in words giving a verbal representation. The first three ways (formula, graph, or table) are the most useful, but each approach provides a very different perspective.

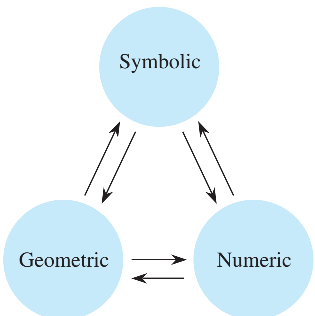


FIGURE 1.29

The problem is: Can you always move back and forth between these different representations? Figure 1.29 illustrates schematically the interrelationships between the three most useful representations—symbolic, geometric, and numeric—by arrows. Ideally, you should be able to start with any one of these representations for a function and shift to the other two representations. Some of these shifts are very simple. If you know a formula for a function, you can create a table of numerical values. Similarly, if you know a formula, you can create its graph at the push of a button, using a graphing calculator or computer graphics program or even by hand, as the graph of a function consists of all points (x, y) that satisfy the equation of the function. If you have the graph of a function, you can easily read off a set of points on the curve and so produce a table of values. If you have a table of values, you can easily plot the points and connect them with an appropriate, usually smooth, curve to generate a graphical representation.

Unfortunately, the two remaining shifts in perspective are considerably more complicated. If you start with a table of values, how do you produce a formula for the function? Similarly, if you start with the graph of a function, how do you construct a formula for it? Both shifts can be extremely difficult, but fortunately modern technology provides the tools by which you can create reasonably accurate formulas. That often is the key step in most real-life applications of mathematics.

Does Every Curve Represent a Function?

Let's consider one last question: Is every curve the graph of some function $y = f(x)$? Consider the five curves shown in Figure 1.30. Are they all the graphs of functions? That is, does each value of the independent variable x correspond to one and only one value of the dependent variable y? We can test a curve in the following way. Imagine a vertical line moving across the curve from left to right so that it passes through every possible value of x in the domain. If, for each x, the line crosses the curve at only one point, there is exactly one y-value for that x and so the curve represents a function. If the vertical line crosses the curve at more than one point for *any* value of x, the curve does not represent a function. This criterion, called the *vertical line test,* shows that the curves (a), (b), and (c) are all graphs of functions. However, when the vertical line test is applied to curves (d) and (e), the line crosses both curves at more than one y-value and thus neither are graphs of functions.

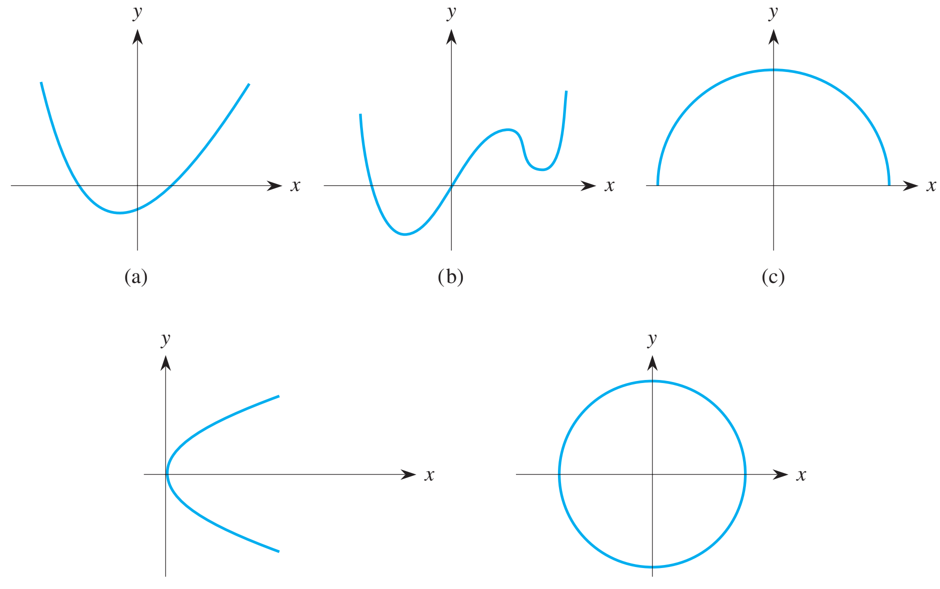
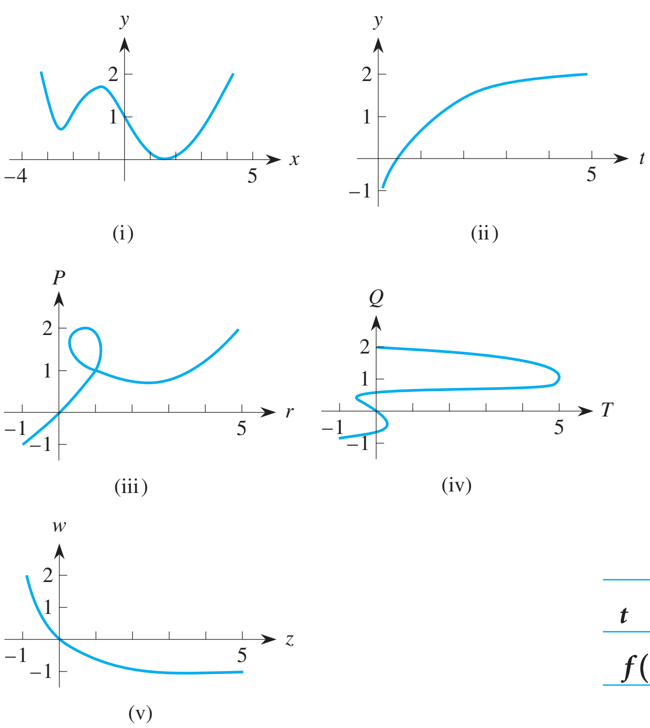


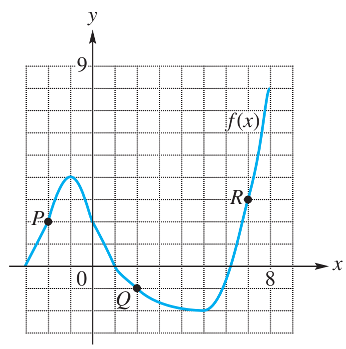
FIGURE 1.30

Problems

1. Which of the following graphs are functions? For each function, (**a**) give its domain and range, (**b**) identify where it is increasing or decreasing, and (**c**) identify where it is concave up or concave down.



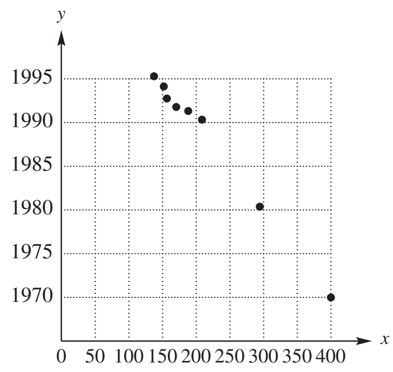
2. The following questions all relate to the accompanying graph of a function $y = f(x)$.



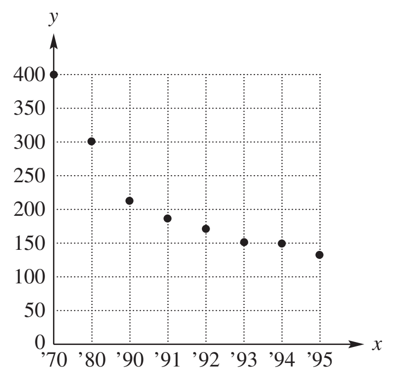
a. What are the coordinates of the points P, Q, and R?
b. Is the point $(-2, 5)$ on the curve?
c. Is the point $(5, -2)$ on the curve?
d. What is $f(5)$?
e. Find x if $f(x) = -1$.
f. Is $f(4) = -1$ true or false?
g. Find y when $x = 2$.
h. Find x when $y = 2$.
i. Solve $f(x) = 0$ for x.
j. Solve $f(0) = y$ for y.

t	1970	1980	1990	1991	1992	1993	1994	1995
$f(t)$	400	300	210	190	175	162	150	135

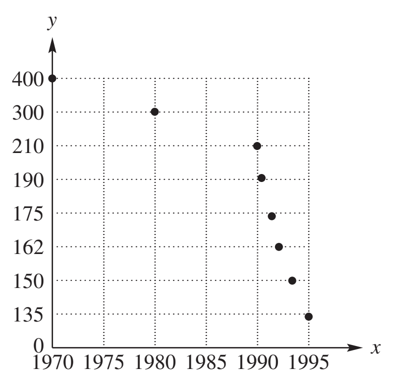
3. The accompanying graphs are based on the set of data above, but something is wrong with each graph. What was done incorrectly in each instance?



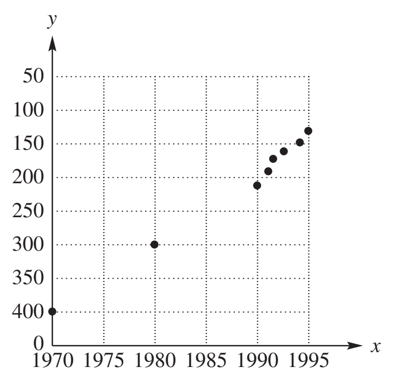
(a)



(b)



(c)



(d)

4. Refer to the function f relating the snow cricket's chirp rate to the air temperature in Example 2.
   a. What is $f(60)$?
   b. Solve $f(x) = 120$.
   c. In a complete sentence, tell what the equation $f(62) = 88$ means.
5. For the function $f(x) = x^2 - 3x + 2$, find the values of y corresponding to $x = -3, -2, -1, \ldots, 4, 5$. Plot the corresponding points and connect them with a smooth curve. Then use your function grapher to graph the function. How do the two graphs compare? Find the values of the function corresponding to $x = \frac{1}{2}, x = \frac{3}{2}, x = -\frac{5}{2}$ and indicate the location of the corresponding points on the curve you drew.
6. For the function $g(t) = 9 - t^2$, use an appropriate set of values for t and the corresponding y values to get a feel for the behavior of the curve when you draw and connect the points. How does your sketch compare to what you see when you use your function grapher?
7. Repeat Problem 6 for $h(s) = s^3 - 7s + 5$.
8. One of the functions f or g in the following table of values is concave up and the other is concave down. Which is which? Explain how you know.

x	5	10	15	20
$f(x)$	80	70	62	56
$g(x)$	80	70	58	43

9. A function $f(x)$ whose values are given in the following table is increasing and concave up. Give a possible value for $f(5)$.

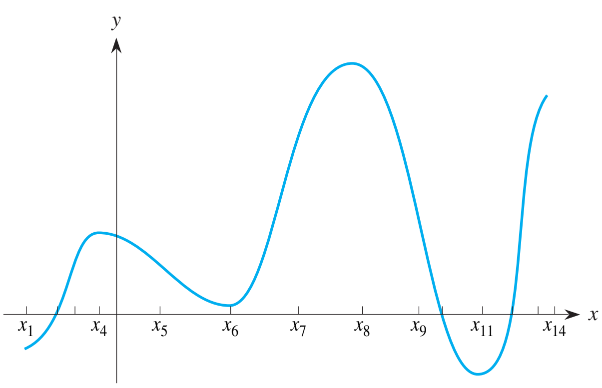
x	4	5	6
$f(x)$	10	??	20

10. A function $f(x)$ whose values are given in the following table is increasing and concave down. Give a possible value for $f(50)$.

x	30	40	50
$f(x)$	12	20	??

11. For the function shown in the accompanying figure, indicate

a. the intervals of x-values where the function is increasing.
b. the intervals where the function is decreasing.
c. all points x where the function has a turning point.
d. all points x where the function has a local maximum.
e. all points x where the function has a local minimum.
f. all points x where the function has points of inflection.
g. the intervals of x-values where the function is concave up.
h. the intervals where the function is concave down.
i. approximately where the function is increasing most rapidly.
j. approximately where the function is decreasing most rapidly.
k. the location of any *zeros* of the function (points where the curve crosses the x-axis).
l. For any of parts (a)–(k) that asks for intervals, write the interval both in terms of inequalities and interval notation.



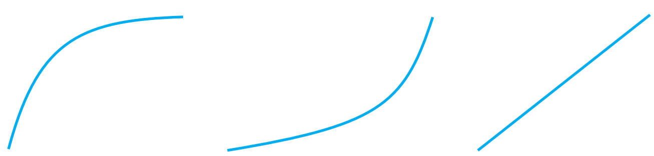
12. Consider the data below. Assume that these values represent a sample of values for a smooth, or continuous, function.

a. Over what intervals of x-values is the function increasing?
b. Over what intervals is the function decreasing?
c. Near what x-values is the function at a local maximum?
d. Near what x-values is the function at a local minimum?
e. Between what pair of successive x-values is the function increasing most rapidly?
f. Between what pair of successive x-values is the function decreasing most rapidly?
g. Over what intervals is the function concave up?
h. Over what intervals is the function concave down?
i. Near what x-values does the function have points of inflection?
j. Estimate the location of any *zeros* of the function.

x	−2.5	−2.0	−1.5	−1.0	−0.5	0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	62.3	28.4	6.8	4.3	11.9	33.2	14.7	2.3	−12.5	−38.8	−5.2	11.7

13. Functions f, g, and h in the following table are increasing functions of x, but each function increases according to a different behavior pattern. Which of the accompanying graphs best fits each function?

x	$f(x)$	$g(x)$	$h(x)$
1	11	30	5.4
2	12	40	5.8
3	14	49	6.2
4	17	57	6.6
5	21	64	7.0
6	26	70	7.4


(a) (b) (c)

14. Functions f, g, and h in the following table are decreasing functions of t, but each function decreases according to a different behavior pattern. Which of the accompanying graphs best fits each function?

t	$f(t)$	$g(t)$	$h(t)$
1	200	30	5.4
2	180	27.6	5.2
3	164	25.2	4.8
4	151	22.8	4.1
5	139	20.4	3.1
6	129	18.0	1.8

(a) (b) (c)

15. Sketch the graph of a function that passes through the point $(0, 1)$ and is

a. increasing and concave up for $x < 0$ and increasing and concave down for $x > 0$.
b. increasing and concave up for $x < 0$ and decreasing and concave up for $x > 0$.
c. decreasing and concave up for $x < 0$ and increasing and concave up for $x > 0$.
d. decreasing and concave up for $x < 0$ and decreasing and concave down for $x > 0$.

16. Sketch the graph of a single smooth curve that satisfies all the following conditions.

a. $f(0) = 4$ **b.** $f(5) = -2$
c. f has a turning point at $x = 3$.
d. f is decreasing from 3 to 5.
e. f is increasing for $x > 5$.
f. f is concave down from 0 to 4.
g. f has a point of inflection at $x = 7$.

17. Consider the function $f(x) = x^2 - 4$ for $x \geq 3$.

a. Is f increasing or decreasing?
b. Is f concave up or concave down?

18. Consider the function $f(x) = x^3 + 7$ for $x \leq -3$.

a. Is f increasing or decreasing?
b. Is f concave up or concave down?

19. For the function $f(x) = \sqrt{x}$, find the values of y corresponding to $x = 0, 1, 2, \ldots, 6$. Plot the corresponding points and connect them with a smooth curve. Then use your function grapher to graph the function. How do the two graphs compare? Find the values of the function corresponding to $x = \frac{1}{2}$, $x = \frac{3}{2}$, $x = \frac{5}{2}$ and indicate the location of the corresponding points on the curve you drew. What is the domain?

20. For the function

$$f(x) = \frac{x}{x + 1}$$

calculate $f(0), f(1), f(2), \ldots, f(8)$. Plot the corresponding points and connect them with a smooth curve. Then use your function grapher to graph the function. How do the two graphs compare? Find the values of the function corresponding to $x = \frac{1}{2}$, $x = \frac{3}{2}$, $x = \frac{5}{2}$ and indicate the location of the corresponding points on the curve you drew. What is the domain?

1.5 Mathematical Models

A *model* is an image or representation of an object or process. A diagram of the human circulatory system is a model of the veins and arteries in the human body; the picture can be used to help us understand how blood circulates throughout the body. Similarly, an architect's sketch of a proposed shopping center is a model of the actual center; a wooden model built to scale is a still more realistic representation for that shopping center.

> A **model** is a representation that highlights the most important characteristics of an object or process.

Models can be found everywhere: the tide tables used by fishermen; a computer scientist's flowchart for a new program; plastic replicas of jet fighters; and many, many more. Because our focus here is on mathematics, the models we present are *mathematical models.* A mathematical model is a representation of a process expressed by a formula, an equation, a graph, a sequence of numbers, or a table of values. Once we have developed any such mathematical representation, we can use it to examine the behavior of the actual process.

For example, the equation for the motion of the ball thrown vertically upward from ground level with an initial velocity of 64 ft/sec,

$$y = f(t) = 64t - 16t^2,$$

is a mathematical model that describes one important aspect of the motion of that ball—its height at any time t. There may be other aspects of the motion that may not be captured in this mathematical model (such as releasing it from some initial height above the ground or the effects of air resistance). These other factors may likely require development of a more sophisticated model, but often a simple model gives a reasonably accurate first approximation.

How can we use mathematics to describe the real world via a mathematical model? We begin by looking at some process in the real world, such as the motion of a ball thrown upward, the growth of a population, or a person's reaction to a drug. Typically, the process of trying to explain what is happening requires some simplifying assumptions. For instance, in modeling the motion of a ball, we assumed that the only force acting on the ball is the force of gravity and ignored the negligible effects of air resistance. (Of course, if the ball were replaced with a balloon, a feather, or a piece of paper, this assumption would be invalid.)

After making reasonable assumptions, we then express the process in a mathematical form, which leads to a mathematical representation of the process in terms of a formula, an equation, a graph, or a table. This mathematical model then has to be interpreted. Does it truly seem to reflect what happens in the real world? Does the behavior of the function mirror the behavior of the process? If so, we can use the mathematical model to describe of the process under study and as the basis for predictions about the process. If the model doesn't adequately reflect the actual process, we may have done something wrong—overlooked some important aspect of the situation or ignored some critical factor; made some erroneous assumptions; or made an error in our work. We illustrate this interplay between mathematics and the real world via mathematical modeling schematically in Figure 1.31.

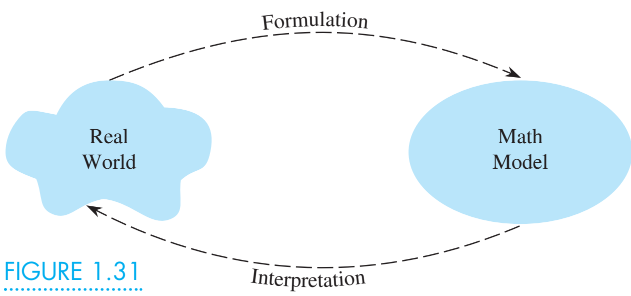


FIGURE 1.31

The concept of function is closely connected to the idea of a mathematical model, and most mathematical models are expressed as functions. Let's look at an example of this process. Researchers have studied the relationship between the level of animal fat in women's diets and the death rate from breast cancer in different countries. Some of their data are shown in the following table, which gives the average daily intake of animal fats in grams per day and the age-adjusted death rate from breast cancer per 100,000 women. We begin by looking at these data, first as a set of numerical values in the table and then visually on a graph, as shown in

Figure 1.32. There is obviously a relationship between the death rate and the daily fat intake. Clearly the death rate D increases as the daily fat intake F increases, so D is an increasing function of F. Moreover, the points fall into a straight line pattern.

Country	**Japan**	**Spain**	**Austria**	**U.S.**	**U.K.**
Daily fat intake(grams)	20	40	90	100	120
Death rate per 100,000	3	7	17	19	23

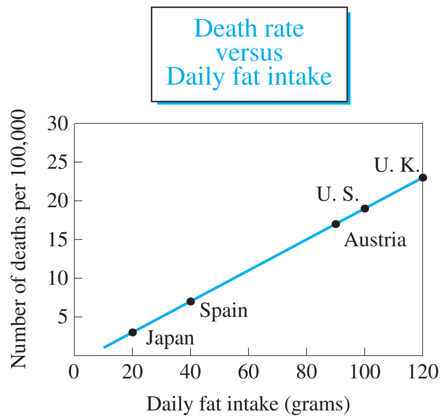


FIGURE 1.32

It turns out that the equation for this line is

$$D = f(F) = 0.2F - 1,$$

and we use this equation as our mathematical model in the following example.

EXAMPLE

The average daily animal fat intake in Mexico is $F = 23$ grams and the average daily animal fat intake in Denmark is $F = 135$ grams. Predict the death rate from breast cancer per hundred thousand women in Mexico and Denmark.

Solution We use the mathematical model to predict that the death rate from breast cancer in Mexico for the average daily fat intake $F = 23$ will be

$$D = 0.2(23) - 1 = 3.6 \text{ per } 100{,}00 \text{ Mexican women.}$$

Similarly, for the average daily fat intake $F = 135$ grams in Denmark, the equation predicts a death rate of

$$D = 0.2(135) - 1 = 26 \text{ per } 100{,}000 \text{ Danish women.}$$

The type of prediction for the death rate for breast cancer in Mexico is called *interpolation* because we are predicting the value of a quantity using a measurement *within* the set of data. The type of prediction for the death rate in Denmark is called *extrapolation* because we are predicting the value of a quantity *beyond* the set of data.

In Section 2.2, we show how to find an equation such as the one relating the death rate to the daily fat intake. Once we have such an equation as a mathematical model, we can base some informed judgments on it. This model is based on the *average* daily intake in each country, which can vary tremendously among individuals. Even so, there is obviously a link between consumption of animal fat and the incidence of breast cancer. Thus the mathematical relationship indicates that women should drastically reduce their daily animal fat intake to reduce their chance of breast cancer. Furthermore, knowing that such a link exists, researchers have since been conducting follow-up studies to determine why the link exists. They have also found links to other items in the diet as well as in the environment. Thus, the incidence of breast cancer depends not just on a single variable but on a number of different variables, so it is a function of several independent variables. Many situations that you encounter in real life are examples of functions of more than one variable. Although the study of such functions is somewhat beyond the scope of this book, we introduce and briefly discuss functions of several variables in Section 3.8.

Parameters and Mathematical Models

Consider again the formula for the height y at any time t of an object dropped from the top of the 180-foot-tall Tower of Pisa: $y = 180 - 16t^2$. In comparison, the comparable formula for the height of an object dropped from the top of the 555-foot-high Washington Monument at any time t is

$$y = 555 - 16t^2,$$

and the height of an object dropped from the top of the 1821-foot-high CN tower in Toronto at any time t is

$$y = 1821 - 16t^2.$$

Each of these functions has the same structure mathematically; what differs among them is the leading number that represents the initial height from which the object is dropped. Based on these specific functions, we can hypothesize a general formula for the height y at any time t of an object dropped from any initial height y_0:

$$y = y_0 - 16t^2.$$

In this formula the height y clearly is a function of time t— they are the dependent and independent variables, respectively. The quantity y_0 can also take on different values, but it isn't a variable in the same sense that t and y are. Although y_0 can take on different values, in any particular situation it has just one value—in this case the initial height of the object. That value doesn't change even though the variables t and y change during the event. A quantity such as y_0 is called a **parameter**. Note that each value of y_0 yields a different function, although each function has the same form.

Now suppose that you're driving steadily at a rate of 40 mph; the relationship between the distance D you travel and the time you drive is $D = 40t$. If you drive at a steady 50 mph, the relationship is $D = 50t$, and if you drive at a steady 65 mph, the relationship is $D = 65t$. Obviously, distance is a function of time. The independent variable is time t, the dependent variable is distance D, and we write the function as $D = r \cdot t$. This relationship holds for any choice of speed r and gives a slightly different function, but one having the identical form, for each possible value of r. The quantity r is a parameter in the formula for this distance function.

How Accurate is a Mathematical Model?

Recall that a mathematical model is only a mathematical description of a process, not the process itself. So, every model carries with it some degree of inaccuracy. For instance, we used the formula $y = 64t - 16t^2$ as a mathematical model for the height at any time t of an object thrown vertically upward with an initial velocity of 64 ft/sec. With this model, we can find how long it takes for the object to come back to the ground. That occurs when $y = 0$, so we solve the equation

$$y = 64t - 16t^2 = 0,$$

We factor out the common factor of $16t$ to get

$$16t(4 - t) = 0$$

so that the object is at ground level when $t = 0$ (when the object is initially released) or $t = 4$ (when it has come back to the ground).

However, according to the laws of physics, the coefficient of t^2 is actually one half of the Earth's gravitational constant. Its value is not precisely -16, but rather more like -16.1, so the solution $t = 4$ is not quite accurate. Instead, we really should say that t is *about* 4 or that it is *approximately equal* to 4, which we write symbolically as $t \approx 4$. We can improve on this *estimate* by using the more accurate value of -16.1 for the coefficient of t^2 and then solving the equation

$$64t - 16.1t^2 = 0.$$

The only common factor now is t, so that

$$t(64 - 16.1t) = 0,$$

giving either $t = 0$ or $t = 64/16.1 \approx 3.975$, which is correct to three decimal places. How many decimal places are reasonable for this answer? We could use more decimal places when we divide out the fraction—say $t \approx 3.97516$ or even $t \approx 3.97515528$—but when we are measuring time in seconds, both results are unrealistic levels of accuracy and should be avoided. Even using the three decimal places in $t \approx 3.975$ may be too many, both from a practical point of view—think about timing in Olympic events where time is usually measured to the hundredth of a second—and from a mathematical point of view—we used only one decimal place in the coefficient, -16.1. In any context, you should determine a reasonable number of decimal places for your final answer, both practically and in terms of the number of digits used.

In fact, rarely in applied situations do you get an "exact" answer such as $x = 5$ or $x = \sqrt{8}$. Even when you do get an exact answer involving a radical or a fraction, you should usually convert it to a decimal, which automatically introduces another level of inaccuracy. Thus $\sqrt{8} \approx 2.828$ or $\sqrt{8} \approx 2.82843$ or $\sqrt{8} \approx 2.82842712$. But $\sqrt{8}$ is an irrational number and its decimal equivalent is an infinite, nonrepeating decimal. Just because your calculator displays 10 or 12 decimal places does not necessarily mean that the result is exactly that number.

Problems

1. An uncooked chicken (temperature of 70° F) is placed in a hot oven at a temperature of 350° to cook. The chicken is removed when its internal temperature reaches 180°. Sketch a possible graph for the temperature T of the chicken as a function of time t. What would be appropriate values for the domain and range of this function? Describe the behavior (increasing/decreasing, concavity) for the graph.
2. A warm can of soda (80° F) is placed in a refrigerator at a temperature of 36° and left there to cool. Sketch a graph of the temperature T of the soda as a function of time t. Identify appropriate intervals for the domain and range of this temperature function. Describe the behavior (increasing/decreasing, concavity) of this function.
3. An Olympic diver dives off the 10 meter platform, enters the water cleanly, and rises slowly to the surface. Sketch a possible graph for the height of the diver above water level as a function of time. What might be appropriate values for the domain and range of this function? (Estimate how long it will

probably take the diver to reach the water from the platform.) Describe the behavior of this function.

4. Repeat Problem 3 by sketching the graph of the diver's height above the diving platform as a function of time. How does the shape of this graph compare to the one you drew in Problem 3?

5. Police sometimes use the formula $s = f(d) = \sqrt{24d}$ as a model to estimate the speed s in miles per hour that a car was going on dry concrete pavement if it left a set of skid marks d feet long. Using this model, estimate the speed of a car whose skid marks stretched

a. 60 ft. **b.** 100 ft.
c. 140 ft. **d.** 200 ft.
e. Suppose that you're driving at 60 mph on dry concrete pavement and slam on your brakes. How long will your skid marks be, according to this model?

6. When a basketball player takes a long shot, the height H of the ball above the floor can be modeled by the equation $H(t) = -16t^2 + 24t + 7$, where t is the number of seconds since the ball was released.

a. Use your calculator to estimate the maximum height that the ball reaches, correct to two decimal places.

b. The rim of the basket is 10 feet above the floor. Use your calculator to estimate all times t when the ball is at the height of the rim.

7. At the beginning of this section, we gave the equation for the height y of a ball thrown vertically upward from ground level with initial velocity 64 feet per second,

$$y = f(t) = 64t - 16t^2,$$

as a function of time t.

a. Suppose that the initial velocity of the ball is 80 feet per second, Write a comparable formula for the height y as a function of time t.

b. If you think of the initial velocity v_0 of the ball as a parameter, write a formula for the height y as a function of time t with any initial velocity v_0.

8. In each expression for a function, identify which letters represent variables, which letters represent functions, and which letters therefore represent parameters.

a. $y = f(x) = ax^3 + bx^2 + cx + d$
b. $z = g(t) = at^b$
c. $z = h(t) = ab^t$
d. $Q = k(m) = \dfrac{am}{bm^2 - c}$

Chapter Summary

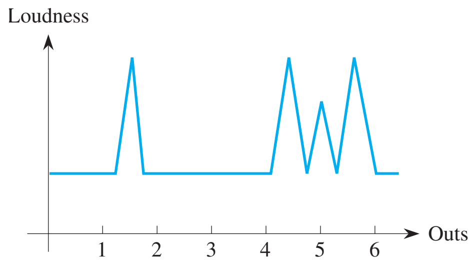
In this chapter we introduced you to functions, their importance, and some of their uses. Specifically we showed you how to work with functions in the following ways.

- How functions arise in the real world.
- How functions can be represented in different ways—by graphs, by tables, by formulas, and in words—and how to move from one representation to another.
- How to identify whether a relationship between two variables given by an equation, a graph, or a table is or is not a function.
- How to decide which is the independent variable and which is the dependent variable.
- What the domain and range of a function are.
- The important characteristics about the behavior of functions—where they increase and decrease, where their turning points are, where they are concave up and concave down, where their points of inflection are, and whether they are periodic.
- How to interpret concavity—whether the growth (increase) or decay (decrease) in a function is speeding up or slowing down.
- How mathematics is used to model phenomena in the real world.

Review Problems

1. In determining the amount of radiation to apply to a tumor site, doctors take into account the depth of the tumor within the body. What is the independent variable and what is the dependent variable in such a relationship? Give reasons for your answer.

2. The accompanying graph describes the loudness of a crowd watching a baseball game during the ninth inning. Write a scenario that might explain what was happening on the field as the inning progressed.



3. An experimental form of insulin is being administered every 4 hours to a person with diabetes. The body uses or excretes about 40% of the drug over the 4-hour period. Draw a graph that shows the amount of the drug in the body as a function of time over a 24-hour period.

4. Populations tend to grow steadily until there are too many members for the space and resources available. Then the population size levels off. Sketch a function that gives population size as a function of time.

5. Determine whether each table of values could represent a function. If not, explain why not.

a.

x	1	2	3	4	5	6
$f(x)$	10	10	12	14	18	25

b.

x	11	15	9	20	15	8
$g(x)$	12	13	13	15	16	17

6. The table of values shows the budget and attendance at 15 U.S. zoological parks. Write a short description of how attendance and budget are related.

Budget ($ millions)	10.0	3.4	27.0	6.2	9.7
Attendance (millions)	1.0	0.5	2.0	0.6	1.3
Budget ($ millions)	7.0	4.8	18.0	6.5	13.0
Attendance (millions)	1.0	1.1	4.0	0.6	3.0
Budget ($ millions)	9.0	15.7	7.0	3.2	14.7
Attendance (millions)	0.5	1.3	1.0	0.5	2.7

7. The table of values shows the number, in millions, of prerecorded cassette tapes sold in the United States in various years between 1982 and 1998.
   a. Draw a graph of the number of cassettes sold as a function of the year since 1982.
   b. In approximately what year did the sales of cassettes reach its maximum?
   c. During which year, approximately, did the sale of cassettes change most rapidly? Most slowly?

Year	1982	1985	1990
Cassettes sold	182.3	339.1	442.2
Year	1993	1994	1995
Cassettes sold	339.5	345.4	272.6
Year	1996	1997	1998
Cassettes sold	225.3	172.6	158.5

Source: *2000 Statistical Abstract of the United States*

8. For the function $f(x) = 3x^2 - 2x + 1$, find $f(0)$, $f(1)$, $f(1.1)$, $f(1.01)$, $f(-3)$, and $f(a)$.

9. During the 1990s, the average cost of a new car bought in the United States can be approximated by the function $C = f(t) = 659.7t + 15598$, where C is the cost of the car and t is the number of years since 1990. (Source: *2000 Statistical Abstract of the United States*)
   a. Determine the average cost of a car purchased in 1995.
   b. If this trend continues, estimate the average cost of a car in 2003.
   c. According to this function, the average cost of a new car was increasing, on average, $659.70 each year after 1990. Use this information to calculate

how many years it would take for the average price of a new car to reach $20,000.

10. A study of the relationship between the average longevity (in years) and the gestation period (in days) for a sample of animals shows that the animals' average longevity L can be predicted reasonably well as a function of the gestation period t by the function $L = f(t) = 1.04t^{0.49}$, where t is the gestation period in days.

a. Estimate the lifetime of a chipmunk whose gestation period is 31 days.
b. The gestation times in the study extend from 15 days (opossum) to 645 days (elephant). What would the range of the average longevity be if the given function were a good predictor?
c. Use your function grapher to graph the function. Is it increasing or decreasing? Is it concave up or concave down?
d. Use the graph to estimate the gestation period of an animal whose average longevity is 15 years. (*Hint:* Find the point on the graph where $L = 15$.)
e. The gestation time for humans is 9 months, or about 270 days. What does the formula predict for the average longevity of human beings? Can you think of any reasons why the value you obtained is so inaccurate?

11. The domain of the function $f(x) = 4x^2 + 3x - 5$ is all real numbers.

a. Use your function grapher to estimate the coordinates of the turning point.
b. What is the range of this function?

12. The domain of the function $f(x) = x^3 - 9$ is all real numbers.

a. Use your function grapher to estimate the coordinates of the point of inflection.
b. What is the range of this function?

13. Give the domain of each of the following functions.

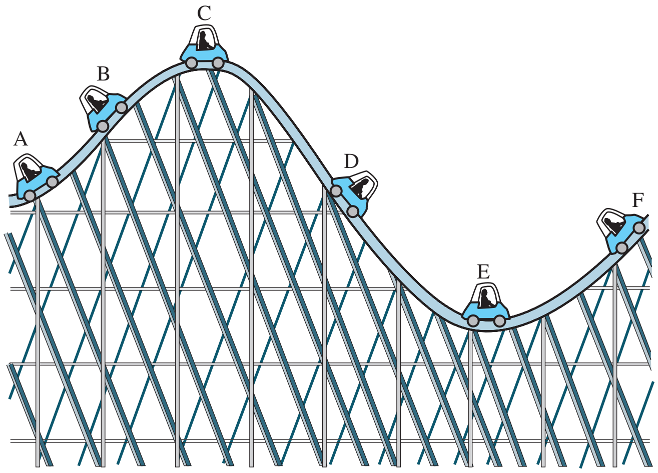
a. $f(x) = \sqrt{x+5}$. **b.** $f(x) = \sqrt{x^2 - 16}$.

c. $g(x) = \dfrac{x^2+4}{x^2-9}$. **d.** $h(x) = \dfrac{x^2-4}{x^2+9}$.

14. The U.S. Postal Service rates for first-class mail in 2003 were 37¢ for the first ounce and 23¢ for every additional ounce.

a. Construct a table showing the cost of postage to mail a first-class letter weighing 0 to 1 oz, 1 to 2 oz, 2 to 3 oz, 3 to 4 oz, and 4 to 5 oz.
b. Sketch a graph of this function $F(w)$, showing the cost of sending in 2003 a first-class letter as a function of the weight w of the letter in ounces.

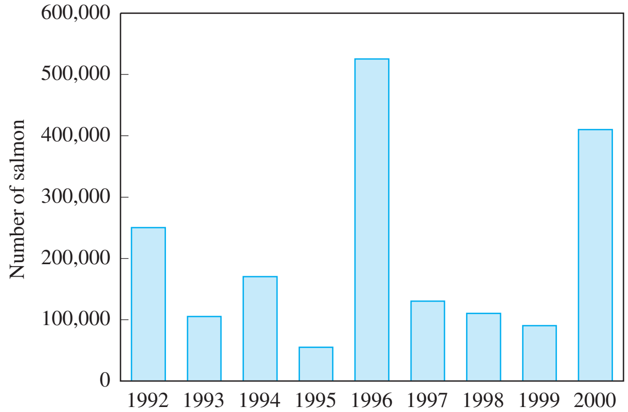
15. Consider the roller coaster shown in the accompanying figure. The points A, B, C, D, E, and F divide the curve representing the track into portions that are increasing and concave up, increasing and concave down, decreasing and concave up, and decreasing and concave down. For each of the five portions of the track, A to B, B to C, and so on, (**a**) identify the mathematical behavior of the curve, and (**b**) describe whether the speed of the cars is increasing at an increasing rate, increasing at a decreasing rate, decreasing at a decreasing rate or decreasing at an increasing rate. In each case, explain your answer.



16. Picture a water slide at an amusement park. The slide starts at an initial height H_0 above the pool and smoothly drops to water level. The slide is first concave down, then concave up, then concave down, and finally concave up.

a. Sketch a graph of the slide's height H above water level as a function of horizontal distance x.
b. Suppose that you go down the slide in a sitting position. Sketch a graph of the height of your eye above the water as a function of horizontal distance x.
c. Sketch the graph of the height of your eye above the water as a function of time t.
d. Sketch the graph of your speed as a function of time t.

17. Pacific coast salmon hatch in rivers and then migrate to the ocean where they live most of their lives. Eventually, they migrate back up the same river to spawn and then die. The accompanying graph shows the number of chinook salmon, in various years, that pass a particular point on one waterway near Seattle as they wend their way upstream to spawn. (The U.S. Department of Fisheries keeps accurate counts in its efforts to maintain a healthy salmon population.) Based on this graph, write a paragraph explaining why the number of salmon who swim upstream here is a roughly periodic function of time. What is the period? What does this graph suggest about the life span of the chinook salmon?



Source: Hiram M. Chittenden Locks data, U.S. Army Corps of Engineers, Seattle, Washington.