5

# Modeling with Difference Equations

## 5.1 Eliminating Drugs from the Body

Prozac is one of the most widely prescribed drugs used to combat extreme depression. Typically, a patient takes a Prozac tablet once a day. As we discussed in Section 2.5, once a medication has been absorbed into the bloodstream, it is washed out of the system by the kidneys, which purify the blood by filtering out foreign chemicals.

For now, let's assume that a person takes a single 80 mg dose of Prozac and that it has been completely absorbed into the blood. During any 24-hour time period, the kidneys eliminate approximately 25% of the Prozac in the bloodstream, so that 75% of the drug remains. As we showed in Section 2.5, the amount of Prozac in the bloodstream based on a single 80 mg dose can be modeled by the exponential decay function

$$D(t) = 80(0.75)^t,$$

where $t$ measures the number of 24 hour periods since the Prozac was taken. The graph of this function is shown in Figure 5.1.

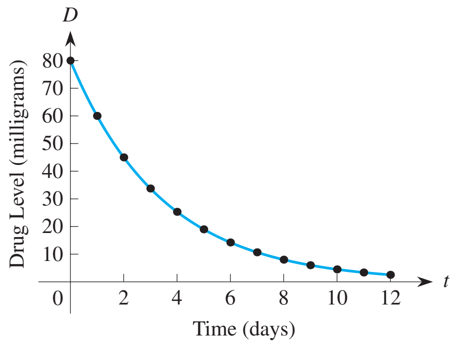



FIGURE 5.1

In particular, $D(0) = 80$. After one 24 hour period, assuming no additional Prozac is taken, $D(1) = 80(0.75) = 60$. After 2 days, $D(2) = 80(0.75)^2 = 45$, and so on. These particular points are highlighted in Figure 5.1.

The collection of values so obtained,

$$\{80, 60, 45, 33.75, 25.3125, \ldots\},$$

is called a **sequence**—it is a set of numbers in a particular order. Thus, a sequence is a function that is defined on a set of nonnegative integers such as $n = 0, 1, 2, \ldots$ The corresponding range is some set of real numbers, such as the levels of Prozac, determined by the rule for the sequence. In other words, for each positive integer, $n = 0, 1, 2, \ldots,$ there corresponds a real number $D(n)$ that is the $n$th term in the sequence. We can write the sequence for the levels of Prozac in the blood as $D(0), D(1), D(2), \ldots$ to emphasize that it is a function.

Although sequences are functions, a special notation is used for them instead of the usual functional notation. For instance, with the Prozac model, we write $D_0 = 80$ instead of $D(0) = 80$, $D_1 = 60$ instead of $D(1) = 60$, and so on. The general, or $n$th, term in the sequence is then written as $D_n$ instead of $D(n)$, where $n = 0, 1, 2, \ldots$ Unless we're working in a context where some other letters are more appropriate (such as $D_n$ for the level of a drug), we denote the $n$th term in a sequence by $x_n$, which represents $\{x_0, x_1, x_2, \ldots\}$.

Keep in mind that, when you work with a sequence such as $D_n$ for the level of Prozac in the system, you are looking only at the value of the function at the start of each 24-hour period. What happens in between isn't considered because the process is basically discrete. Graphs such as the one shown in Figure 5.1 with a smooth curve superimposed over the points in the sequence are convenient for visualizing a trend, but can sometimes be misleading. Such a curve can completely miss anything else that may happen from one value of $n$ to the next.

## Repeated Drug Dosages

In practice people often use a drug such as Prozac on a maintenance basis—they take a fixed dose of the medication every time period, rather than a single initial dose. It might be a repeated daily dose of Prozac or some high-blood-pressure medication or a repeated dose of a cold medication every 4 hours. In such cases, the exponential decay model is not realistic. What can we then say about the amount of medication in the body as a function of time?

Suppose that a person takes 80 mg of Prozac each day, starting on some particular day. After the first 24 hours, 25% of the Prozac, or 20 mg, is eliminated, leaving 75%, or 60 mg and then the next day's dose adds 80 mg to that amount. Therefore, after the first 24 hours, the amount of Prozac in the body is

$$D_1 = 60 + 80 = 140 \text{ mg},$$

or considerably more than the original dose!

During the second 24 hour period, the kidneys eliminate 25% of the 140 mg of Prozac present, or 35 mg, leaving 75% of the 140 mg of Prozac present, or 105 mg. Then the person takes the next dose of 80 mg. Thus, after two 24 hour periods, the amount of Prozac in the body is

$$D_2 = 0.75(140) + 80 = 105 + 80 = 185 \text{ mg}.$$

After 3 days, the level of Prozac is

$$D_3 = 0.75(185) + 80 = 218.75 \text{ mg},$$

and so on, indefinitely. The corresponding sequence of Prozac levels is

$$\{80, 140, 185, 218.75, 244.0625, 263.0469, \ldots\}.$$

Note how the level of the drug keeps rising, but in a concave down manner, as illustrated in Figure 5.2.

**Think About This** Does the foregoing explanation suggest that the level of Prozac in the bloodstream keeps rising indefinitely? If so, is that reasonable?

Let's look at a slightly different way to describe this process. The initial dose is $D_0 = 80$ mg. During the first 24 hour time period, 25% of this amount is removed from the blood, leaving 75% of $D_0$, and the person then takes the next dose of 80 mg. Thus

$$D_1 = 0.75D_0 + 80.$$

Similarly, during the second day, 25% of the Prozac is eliminated, leaving 75% of $D_1$, and the person then takes another 80 mg dose, so that

$$D_2 = 0.75D_1 + 80.$$

Again, after the third day,

$$D_3 = 0.75D_2 + 80.$$

In general, at the end of $n + 1$ days, for *any* value of $n$,

$$D_{n+1} = 0.75D_n + 80.$$

This equation shows the relationship between the level of Prozac on any two successive days. In particular, if we know the amount of Prozac in the body after $n$ days, this equation allows us to calculate the amount of Prozac present the following day.

We call an equation such as $D_{n+1} = 0.75D_n + 80$ that relates the successive terms in a sequence a **difference equation.** As you will see, it is an effective way to model drug concentrations in the body. (Note that some authors reserve the term *difference equation* only for equations that explicitly involve the difference $x_{n+1} - x_n$ between successive values in a sequence; we adopt the more common, broader use of the term for any relationship between $x_{n+1}$ and $x_n$.)

In the drug model, the initial dose of Prozac was $D_0 = 80$ mg. Therefore, if we set $n = 0$ in the difference equation $D_{n+1} = 0.75D_n + 80$, we determine $D_1$ to be

$$D_1 = 0.75D_0 + 80 = 0.75(80) + 80 = 140 \text{ mg}.$$

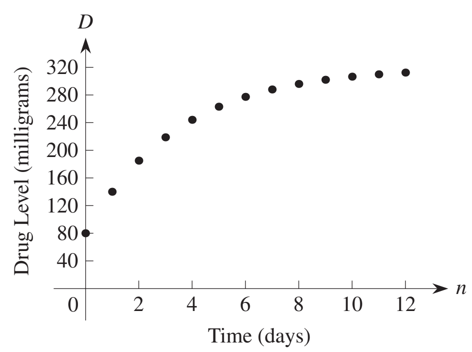



FIGURE 5.2

When $n = 1$ we obtain,

$$D_2 = 0.75D_1 + 80 = 0.75(140) + 80 = 185 \text{ mg}.$$

When $n = 2$ we get,

$$D_3 = 0.75D_2 + 80 = 0.75(185) + 80 = 218.75 \text{ mg}.$$

Continuing in this way, we get

$$D_4 = 0.75D_3 + 80 = 244.0625 \text{ mg};$$
$$D_5 = 0.75D_4 + 80 = 263.0469 \text{ mg};$$
$$D_6 = 0.75D_5 + 80 = 277.2852 \text{ mg};$$

and so on. Over time, the amount of the drug in the body is given by the sequence of numbers

$$\{80, 140, 185, 218.75, 244.0625, 263.0469, 277.2852, \ldots\}.$$

Figure 5.3 shows the graph of these points, connected by a smooth curve. This sequence of numbers is the **solution** to the difference equation; we refer to it as the **solution sequence.**

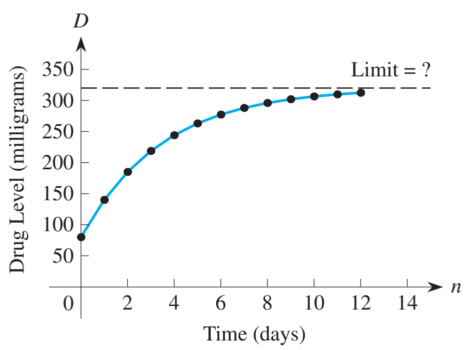



FIGURE 5.3

Note that each successive term, although larger than the preceding value, has grown by somewhat less than the term before. That is, the change from $D_0$ to $D_1$ is 60 mg; the change from $D_1$ to $D_2$ is 45 mg; the change from $D_2$ to $D_3$ is 33.75 mg, and so on. The curve drawn through the points in Figure 5.3 is concave down, and the successive values seem to be leveling off. That is, the drug level continues to rise but at a less steep rate. The overall pattern is characteristic of an upside-down exponential decay process. Rather than gradually dying out toward the horizontal axis as a horizontal asymptote (as an exponential decay function does), this process gradually rises toward the limiting amount of drug in the body as a horizontal asymptote. By continuing this process numerically, we find that the limiting amount $L$ appears to be very close to 320 mg. We say that the terms of the sequence *converge* to this limiting value in the sense that they get closer and closer to $L$ the farther we go in the sequence.

Generating the successive terms of this type of iteration process on either a calculator or a spreadsheet is quite simple. On a calculator, simply enter the starting value for the sequence and press `Enter`. Then enter the iteration formula, using `2nd ANS` as the variable. For instance, using the Prozac model, you would enter the initial dose 80 and then the formula for the difference equation $D_{n+1} = 0.75D_n + 80$ in the form

```
0.75*ANS+80.
```

When you press Enter, you get the next value of the sequence, or 140. Then each time you press Enter, you get the following value. Try this process to see how simple it really is.

In terms of the original problem, the limiting value $L$ represents the maximum level of Prozac that will be reached in the body. This horizontal asymptote is known as the *maintenance level* for the drug. Once that level of Prozac has been reached, the amount in the body will remain constant at that level every 24 hours, so long as the same dose is taken repeatedly every 24 hours.

**Think About This**

The curve shown in Figure 5.3 is actually incorrect; it was obtained by simply connecting the points to demonstrate the overall pattern. However, it completely ignores what happens during the 24 hours between successive doses. Sketch a more detailed curve that accurately reflects what happens.

In practice, researchers determine the specific level $L$ of a medication that is most effective, considering factors of both safety and effectiveness. An initial dose of, say, 80 mg of the medication means that for some period of time, the amount in the bloodstream is below the optimal level. As a result, doctors often prescribe an initial dose higher than the normal dose so that the drug level approaches the maintenance level $L$ more rapidly. For instance, an initial dose of 240 mg followed by daily doses of 80 mg will achieve the desired level quickly. However, the safety of prescribing such a large dosage, especially as the first dose of the drug, must be considered.

## EXAMPLE 1

Suppose that the initial dose of Prozac is 160 mg (instead of 80 mg) but that all subsequent doses are 80 mg.

**a.** Find the first six terms of the solution sequence.

**b.** How does this solution compare to the one we had before?

***Solution***

**a.** We still have the same difference equation model,

$$D_{n+1} = 0.75D_n + 80,$$

but now the initial dose is $D_0 = 160$ mg. Therefore

if $n = 0$: $D_1 = 0.75D_0 + 160 = 0.75(160) + 80 = 200$ mg;

if $n = 1$: $D_2 = 0.75D_1 + 80 = 0.75(200) + 80 = 230$ mg;

if $n = 2$: $D_3 = 0.75D_2 + 80 = 0.75(230) + 80 = 252.5$ mg.

Continuing in this manner,

$$D_4 = 0.75D_3 + 80 = 269.375 \text{ mg};$$
$$D_5 = 0.75D_4 + 80 = 282.031 \text{ mg};$$
$$D_6 = 0.75D_5 + 80 = 291.523 \text{ mg};$$

and so on. Thus the levels of Prozac in the bloodstream are now given by the solution sequence

$$\{160, 200, 230, 252.5, 269.375, 282.031, 291.523, 298.643, 303.982, \ldots\}.$$

**b.** By changing the initial dose $D_0$, we get a different sequence of values. That is, the difference equation has a different solution sequence that depends on the initial value we choose. However, the behavior of this solution is essentially the same—it is an increasing, concave down function that rises toward the same limiting value of $L = 320$ mg, as shown in Figure 5.4.

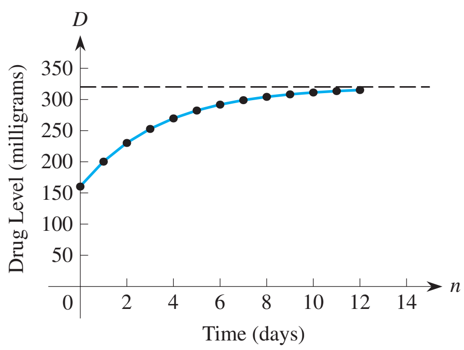



FIGURE 5.4

What happens if the level of drug in the system gets very high? We explore this situation in Example 2.

## EXAMPLE 2

Suppose that a person takes an initial dose of 500 mg of Prozac and thereafter takes the usual 80 mg daily.

a. Find the corresponding solution sequence.
b. Discuss the behavior of the solution.

### Solution

a. We again use the same difference equation,

$$D_{n+1} = 0.75D_n + 80,$$

but now the initial dose is $D_0 = 500$, so that

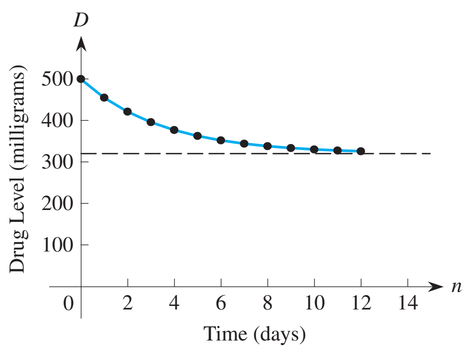



FIGURE 5.5

if $n = 0$: $D_1 = 0.75D_0 + 80 = 455$ mg;
if $n = 1$: $D_2 = 0.75D_1 + 80 = 421.25$ mg;
if $n = 2$: $D_3 = 0.75D_2 + 80 = 395.938$ mg;

and so on. Thus the levels of the drug in the bloodstream are now given by the solution sequence

$$\{500, 455, 421.25, 395.938, 376.953, 362.715, 352.036, 344.027, 338.020, \ldots\}.$$

**b.** Figure 5.5 shows these points, which fall in a decreasing, concave up pattern that apparently approaches the same limiting value $L = 320$ mg, but from above rather than from below.

Consequently, if the drug level rises too high, some counteracting effects reduce the level. Thus the process that we have described can't lead to an infinite drug level in the body. In fact, our difference equation model predicts that, if the level ever exceeds the maintenance level for the drug, the drug level will decrease as subsequent daily doses are taken.

Again, note that by changing the initial condition $D_0$ in the difference equation, we obtain a different solution sequence.

## Determining the Maintenance Level L

We estimated the maintenance level for Prozac as being about 320 mg by looking at the successive values we calculated in the solution sequence. We now determine the limiting value $L$ for Prozac precisely by using the following argument.

Suppose that, for some value of $n$, $D_n$ reaches the limit $L$ so that all successive levels of Prozac are the same. Thus for a large enough $n$, we assume that both $D_{n+1} = L$ and $D_n = L$. Substituting these values into the difference equation for the Prozac drug model,

$$D_{n+1} = 0.75D_n + 80,$$

we obtain

$$L = 0.75L + 80 \quad \text{so that} \quad 0.25L = 80.$$

Hence the limiting value is

$$L = \frac{80}{0.25} = 320 \text{ mg}.$$

We developed all these ideas in the context of a single drug whose level in the bloodstream decreases at a particular rate. In actuality, different drugs are "washed out" of the blood at different rates. For example, aspirin is removed quite rapidly: Its level is reduced by about 50% every 29 minutes. In fact, all drugs are rated by how long it takes for 50% to be eliminated from the body, which is known as the *biological half-life* or simply the **half-life** of the drug. Thus the half-life of aspirin is 29 minutes.

**Think About This**

In the usual 4-hour time period between successive 325 mg doses of aspirin, how much of the aspirin in the blood would be removed by the kidneys?

## Finding a Formula for the Solution

In the preceding situations, we worked with the terms in the solution sequences of the difference equation

$$D_{n+1} = 0.75D_n + 80$$

for different values of the initial dose $D_0$. Thus, if the initial dose is $D_0 = 80$, the corresponding solution sequence is $D_n = \{80, 140, 185, \ldots\}$. But we did not find a formula for this solution as a function of $n$. (In Supplementary Chapter 12 we develop a powerful technique for finding formulas for the solution sequences to such difference equations. For now, we construct such a solution by using some of our previous ideas about functions.)

### EXAMPLE 3

Find a formula for $D_n$, the solution sequence to the drug model difference equation for Prozac, based on an initial dose $D_0 = 80$.

***Solution*** We can think of the values in the solution sequence as a set of data values.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $D_n$ | 80 | 140 | 185 | 218.75 | 244.0625 | 263.0469 | 277.2852 |

(We could use more values, but these will suffice.) Recall that the shape of the curve shown in Figure 5.2 is like an upside-down exponential decay function. Because the values for $D_n$ approach a limiting value of $L = 320$ mg as a horizontal asymptote, we can shift each $D_n$ value to obtain the corresponding values of $320 - D_n$, as shown in the following table and in Figure 5.6.

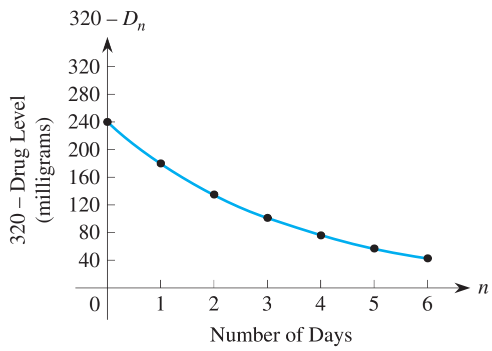



FIGURE 5.6

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $D_n$ | 80 | 140 | 185 | 218.75 | 244.0625 | 263.0469 | 277.2852 |
| $320 - D_n$ | 240 | 180 | 135 | 101.25 | 75.9375 | 56.9531 | 42.7148 |

Note how the values of $320 - D_n$ in the bottom row decay as $n$ increases. If we continue the process further, these differences approach 0 as $n$ gets larger (because $D_n$ approaches 320 as $n$ increases). It therefore makes sense to fit a decaying exponential function to

$320 - D_n$ as a function of $n$. (We would not use a decaying power function because the data start with a finite value of 240 when $n = 0$.)

Using a calculator, we find that the exponential function that best fits these data is

$$320 - D_n = 239.99978(0.7500007)^n.$$

The corresponding correlation coefficient is $r = -1$, which suggests a virtually perfect fit. If we solve for the level of Prozac $D_n$, we obtain

$$D_n = 320 - 239.99978(0.7500007)^n.$$

The numbers in this expression suggest that the "correct" formula for the solution might be

$$D_n = 320 - 240(0.75)^n.$$

We later show that this formula holds for every possible value of $n$, not just for the few particular values of $n$ we used in constructing the best-fitting exponential function.

## Constructing the Solution in General

We now extend the preceding solution formula to solve the comparable difference equation for any medication with any fixed periodic dose. Suppose that the kidneys remove a fixed percentage of a medication every time period, leaving a fraction $a$, $0 < a < 1$, in the bloodstream. Also, suppose that the repeated dose is an amount $B$. The corresponding difference equation then is

$$D_{n+1} = aD_n + B.$$

(In our previous development with Prozac, we had $a = 0.75$ and $B = 80$.) For any value of $a$ between 0 and 1 and any positive value of $B$, the successive terms in the solution sequence for $D_n$ have behavior comparable to that shown in Figure 5.2 assuming $D_0$ is less than the maintenance level: The solution sequence is an increasing concave down function, and approaches a horizontal asymptote. If your graphing calculator displays solutions of difference equations, select some typical values for $a$ and $B$ and check out the behavior of the solution.

Because the level of the medication in the blood rises toward the maintenance level $L$, we can solve for $L$ in this general case by realizing that, should this level actually be achieved, then both $D_n = L$ and $D_{n+1} = L$. Therefore

$$L = aL + B, \quad \text{so that} \quad L - aL = L \cdot (1 - a) = B.$$

Hence the maintenance level is

$$L = \frac{B}{1 - a}.$$

The formula for $D_n$ that we constructed previously for Prozac was $D_n = 320 - 240(0.75)^n$, based on $a = 0.75$ and $B = 80$, with an initial value $D_0 = 80$. In this expression, 320 is the limiting value $L = B/(1 - a)$ and 0.75 is the decay factor $a$. The coefficient 240 is $320 - 80$, the difference between the limiting value $L$ and the initial dose $D_0$. These results suggest that the general formula for the solution sequence is

$$D_n = L - (L - D_0)a^n$$

for any value of $n$, with parameters $a$, $B$, $L = B/(1 - a)$, and $D_0$.

## EXAMPLE 4

Verify that the preceding expression for $D_n$ is indeed a formula for the solution sequence of the difference equation $D_{n+1} = aD_n + B$ for every value of $n$.

**Solution** To verify that the expression for $D_n$ is actually a formula for the solution sequence, we must show that it *satisfies* the difference equation

$$D_{n+1} = aD_n + B$$

for every value of $n$. Thus we substitute $D_n = L - (L - D_0)a^n$ and the corresponding expression $D_{n+1} = L - (L - D_0)a^{n+1}$ when $n$ is replaced by $n + 1$, into the difference equation. The left-hand side of the difference equation becomes

$$D_{n+1} = L - (L - D_0)a^{n+1}.$$

The right-hand side becomes

$$\begin{aligned} aD_n + B &= a[L - (L - D_0)a^n] + B \\ &= aL - a\cdot(L - D_0)a^n + B \\ &= aL - (L - D_0)a^{n+1} + B. \end{aligned}$$

However,

$$L = \frac{B}{1-a} \quad \text{so that} \quad B = L\cdot(1 - a).$$

Therefore the right-hand side becomes

$$\begin{aligned} aD_n + B &= aL - (L - D_0)a^{n+1} + B \\ &= aL - (L - D_0)a^{n+1} + L\cdot(1 - a) \\ &= aL - (L - D_0)a^{n+1} + L - aL \\ &= L - (L - D_0)a^{n+1}, \end{aligned}$$

which is identical to the left-hand side. Thus the expression

$$D_n = L - (L - D_0)a^n$$

is a formula for the solution sequence $D_n$ and it holds for all possible values of $n$.

We summarize the preceding results as follows.

### Level of Medication in the Bloodstream

*Assumptions*

- The kidneys remove a fixed proportion, $1 - a$, of a medication from the bloodstream every time period.
- The repeated dosage of this medication every time period is $B$.

*Mathematical Model*

- Difference equation: $D_{n+1} = aD_n + B$
- Maintenance level for the medication: $L = B/(1 - a)$
- Solution: $D_n = L - (L - D_0)a^n$

We developed this mathematical model by simplifying the situation considerably. First, when a person takes a medication, a certain amount of time is needed for it to be completely absorbed into the blood, as well as to reach the intended part of the body. Second, the rate at which a particular drug is washed out of the blood depends on many factors, including a person's weight, metabolism, and the state of the kidneys and liver, all of which are involved in the elimination process. You definitely should not make any medical judgments about the effectiveness of a drug based on the simplified models of this section.

## Summary

Let's summarize some of the fundamental ideas about difference equations.

1. A **difference equation** relates the successive terms—say, $x_n$ and $x_{n+1}$—of a sequence. For instance, it might be
$$x_{n+1} = 1.2x_n - 3.5 \quad \text{or} \quad x_{n+1} = -0.6x_n + 2n.$$
2. The **solution** to a difference equation is a **sequence** (which is a function of $n$) that satisfies the difference equation. For a given initial value $x_0$ for the sequence, we can always calculate every successive value of the solution sequence $\{x_0, x_1, x_2, \ldots\}$ by using the difference equation directly.
3. For every possible initial value $x_0$, there is a different solution sequence to the difference equation. Figure 5.7 shows the graphs of several different solution sequences to the difference equation $x_{n+1} = ax_n + B$ for the drug model, based on different initial doses. Note that, whenever the initial dose $x_0$ is less than the limiting value $L$, the pattern is increasing and concave down, approaching $L$ as a horizontal asymptote. Note also that, whenever the initial dose $x_0$ is greater than $L$, the pattern is decreasing and concave up and approaches $L$ as a horizontal asymptote.

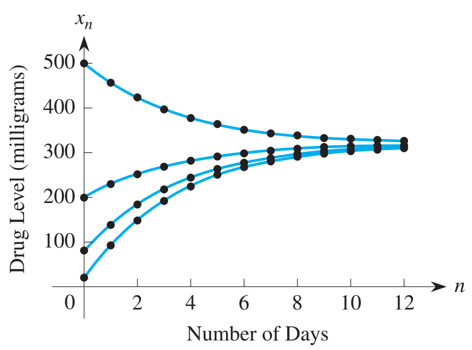



FIGURE 5.7

4. Although the solution sequence can always be expressed in terms of the specific numbers in the sequence, as determined from the difference equation, it is desirable, whenever possible, to write the solution as a formula for $x_n$ in terms of $n$. Such a formula is called a **closed form expression** for the general term $x_n$. Thus a solution sequence $x_n$ can be defined
   *a.* in closed form with a formula for $x_n$ in terms of $n$,
   *b.* by the actual sequence of numbers $\{x_0, x_1, x_2, \ldots, x_n, \ldots\}$, or
   *c.* by a difference equation that relates successive terms of the sequence.

   Think of these ideas in terms of the drug-level model we developed. The same ideas apply to each of the difference equation models we develop throughout the remainder of this chapter.

Finally, the formula $D_n = L - (L - D_0)a^n$ we developed for the solution sequence to the difference equation $D_{n+1} = aD_n + B$ for the drug model can be applied to many other situations. To do so, we rewrite it in a somewhat more general way. Using $x_n$ as the dependent variable gives the difference equation

$$x_{n+1} = ax_n + B.$$

Also, not every difference equation of this form has solutions with a horizontal asymptote. The solutions may grow toward $\infty$, so we write $B/(1 - a)$ in place of $L$. With this terminology, the solution to the difference equation becomes

$$\begin{aligned} x_n &= \frac{B}{1-a} - \left(\frac{B}{1-a} - x_0\right)a^n \\ &= \frac{B}{1-a} + \left(x_0 - \frac{B}{1-a}\right)a^n. \end{aligned}$$

When we multiply out the first equation and collect like terms in a different manner, we get

$$x_n = \frac{B}{1-a} - \frac{B}{1-a}a^n + x_0a^n = \frac{B}{1-a}(1 - a^n) + x_0a^n.$$

In summary we have the following result.

The complete solution to the difference equation

$$x_{n+1} = ax_n + B,$$

where $a$ and $B$ are constants, is

$$x_n = \frac{B}{1-a}(1 - a^n) + x_0a^n.$$

If $0 < a < 1$, the solution can be written as

$$x_n = L + (x_0 - L)a^n,$$

where

$$L = \frac{B}{1-a}$$

is the limiting value for the solution as $n \to \infty$.

This formula for the solution applies only to difference equations of the particular form $x_{n+1} = ax_n + B$. Thus, it applies to the difference equation

$$x_{n+1} = 1.05x_n + 3,$$

but it does not apply to the difference equation

$$x_{n+1} = 1.05x_n + 3n$$

because $3n$ is not a constant.

Furthermore, a difference equation of the form

$$x_{n+1} = ax_n + B, \quad n \geq 0,$$

starting at $x_0$ when $n = 0$, is also known as a *forward difference equation* because it expresses the next value $x_{n+1}$ in a sequence in terms of the current value $x_n$. You can think of it as "looking forward" to where you are going. We can also write a *backward difference equation* in which the current value $x_n$ in a sequence is written in terms of the previous value $x_{n-1}$. Think of it as "looking backward" where you came from. The comparable difference equation is

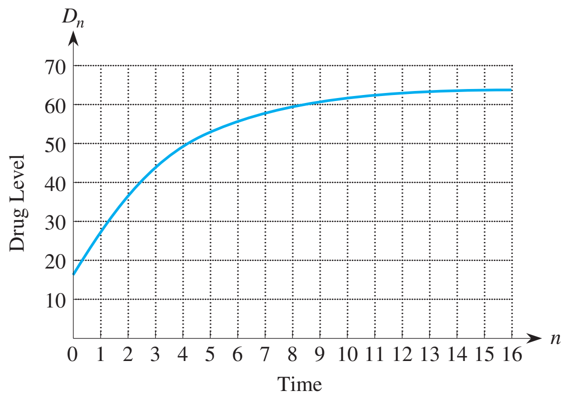
$$x_n = ax_{n-1} + B, \quad n \geq 1.$$

You can convert a forward difference equation to an equivalent backward difference equation by replacing $n$ with $n - 1$ everywhere that $n$ appears, and vice versa. Either way, you get the identical solution sequence, either as a collection of values or as a formula in terms of $n$.

Many calculators have the capability to calculate successive terms in the solution sequence for a difference equation and to plot the solution as part of their `Sequence` Mode. Check your calculator manual for specific instructions.

## Problems

1. Suppose that the kidneys remove 30% of a drug from the bloodstream every 4 hours. If a person takes a single dose of 16 mL, find the amount of the drug in the body after 12 hours and after 24 hours. How long does it take for the level to drop below 1 mL? below 0.01 mL?
2. Suppose that the person in Problem 1 takes repeated doses of 16 mL of the same drug every 4 hours. What is the drug level after 12 hours? after 24 hours? What is the limiting value for the dosage?
3. Suppose that an initial dose of Prozac is $D_0 = 320$ mg followed by daily doses of 80 mg.
   **a.** Calculate the first six terms of the solution sequence. What do you observe about them?
   **b.** Can you explain this result?
4. Suppose that the kidneys remove 25% of a drug in the bloodstream every time period and that the initial dose is 48 mL followed by 16 mL doses every time period thereafter. How long will it be until the drug level in the bloodstream exceeds 60 mL?
5. Recall that Figure 5.3 ignores the removal by the kidneys of the Prozac during each 24-hour time period. Sketch a more accurate graph of the level of Prozac in the bloodstream versus time.
6. The accompanying graph shows the level of a medication in the bloodstream just *after* each repeated dose of 16 mL is taken. Sketch the graph of the drug level of the same medication just *before* the next dose is taken. How do the two graphs compare? Use the two graphs—drawn on the same set of axes—to construct a graph showing the actual level of the medication in the bloodstream at all times, not only at the times just before or just after the medication is taken.




7. The drug dosage for a certain drug is 10 mg per day, and the initial dose is also 10 mg. If the kidneys remove 60% of the drug every 24 hours, find the maintenance level for the medication.
8. Suppose that the daily dosage of the drug in Problem 7 is halved to 5 mg per day. Find the maintenance level. Is the maintenance level also halved?
9. Suppose that the person in Problems 7 and 8 decides to take 10 mg every second day instead of 5 mg each day. Does the patient achieve the same

maintenance level for the medication? Explain why or why not.

**10.** Two 5-grain aspirin tablets contain 650 mg of the drug. With aspirin's half-life of 29 minutes, how much is left in the bloodstream after 2 hours? How long does it take for the level to be equivalent to 10 mg of aspirin? If an individual takes two tablets every 4 hours, what is the maintenance level of the aspirin?

**11.** Some studies have shown that taking one aspirin tablet per day significantly reduces a person's risk of heart attack or stroke. If a person follows this regimen, find the maintenance level of the aspirin in the blood.

**12.** The maintenance level for a certain drug is 600 mg. A patient starts with an initial dose of 100 mg and repeats it daily. Sketch the graph of the level of the drug in the bloodstream as a function of time. Suppose that $D_5 = 400$ and $D_{10} = 520$. Use the graph to determine which values are possible for the drug level and which are impossible.

**a.** $D_7 = 440$ **b.** $D_7 = 460$
**c.** $D_{12} = 540$ **d.** $D_{12} = 560$

**13.** The daily dosage for a certain medication is 200 mL and the maintenance level is 500 mL. A person taking this medication reaches a level of 450 mL in 10 days. Let $r_1$ represent the average daily rate of increase of the drug level over the full 10-day period, let $r_2$ be the average daily rate of increase over the first 5-day period, and let $r_3$ be the average daily rate of increase over the last 5 days. Without calculating their values, list these three rates in increasing order. (See Problem 24 of Section 4.1.)

**14.** Suppose that your car has a 14-gallon gas tank that you fill as soon as the level drops to half-full. Also, every time you fill up, you add one quart ($\frac{1}{4}$ gallon) of an additive that mixes thoroughly with the gas and is then used up along with the gas.

**a.** Write a difference equation that models the amount of the additive $A_n$ in the tank from one fill-up to the next.
**b.** Use the difference equation to calculate the amount of additive in the tank over the first 10 fill-ups.
**c.** Sketch the graph of $A_n$ as a function of $n$ based on the values from part (b). What does the behavior of the function suggest?
**d.** Find the limiting value for the amount of the additive in the tank as $n$ increases indefinitely.
**e.** Find the closed form solution of the difference equation.
**f.** How would the difference equation and the limiting value change if you fill up when the tank is 40% full instead of 50% full?
**g.** How would the limiting value change if your gas tank holds 16 gallons instead of 14 gallons and you fill up when the tank is half full?

Write the first six terms of each sequence whose general term is given.

**15.** $x_n = 4n$ **16.** $x_n = 3n + 5$

**17.** $x_n = \frac{1}{2}n$ **18.** $x_n = n^2 + 5$

**19.** $x_n = n^3 - 10$ **20.** $x_n = \dfrac{n^2+1}{n^2+2}$

**21.** $a_n = \dfrac{2^n}{3^n}$ **22.** $a_n = \dfrac{n^2}{2^n}$

**23.** $y_n = \dfrac{1}{n}, \quad n \geq 1$ **24.** $y_n = \dfrac{\log n}{n}, \quad n \geq 1$

**25.** $p_n = 1 - (0.2)^n$ **26.** $p_n = 1 + (0.2)^n$

**27–38.** Decide which sequences in Problems 15–26 seem to converge and which clearly do not. Give reasons for your decision. For those that you're not sure about, what could you do to come to a decision?

**39–50.** Plot the points $(n, x_n)$ for each of the sequences in Problems 15–26. Decide which appear to be strictly increasing or strictly decreasing and which are concave up or concave down.

**51.** Consider the sequence $E$ whose general term is

$$e_n = \left(1 + \frac{1}{n}\right)^n, \quad \text{for } n = 1, 2, \ldots$$

**a.** Calculate the first 10 terms of this sequence and plot them. Does the graph suggest an eventual limit?
**b.** Calculate $e_{100}$, $e_{500}$, $e_{1000}$, $e_{10,000}$, $e_{100,000}$, and $e_{1,000,000}$. What does the limit of the sequence $e_n$ appear to be if you let $n$ increase indefinitely?

**52.** Consider the sequence $e_n = (1 + 1/n)^n$ again. Use your calculator to evaluate $e_{1000}$, $e_{1,000,000}$, $e_{10,000,000}$, and $e_{100,000,000}$. Keep track of all the results obtained. What do you observe about the terms of this sequence? What is your best estimate for the limiting value? Continue the process of taking larger and larger values for $n$—say, up to $n = 10^{15}$. You will find, depending on your calculator, that the terms eventually jump to 1 instead of continuing as you would expect, owing to calculator round-off. By trial and error, can you find the point where that occurs on your calculator? If so, what is it?

53. Consider the sequence $f_n = (1 - 1/n)^n$. What is the limiting value for the sequence as $n \to \infty$? How is this limiting value related to the one in Problem 51? (*Hint*: There is a simple arithmetic relationship.)

54. Repeat Problem 53, using the sequence $g_n = (1 + 2/n)^n$. How is the limiting value related to the one in Problem 51? Based on this result, conjecture what the limiting value is for $(1 + 5/n)^n$ as $n \to \infty$.

55. What is the limit of the sequence $h_n = (1 + n)^{1/n}$ as $n \to \infty$?

56. Suppose that the successive terms of a sequence are increasing and that the graph drawn through the corresponding points is concave up for all $n$. Can the sequence converge to a limit? Explain your answer.

## 5.2 Modeling with Difference Equations

As we have previously discussed, a *mathematical model* for a process is an expression or an equation that represents that process. Sometimes we simply create a formula to fit a set of data. Often, especially in the sciences, we try to determine the underlying principles or assumptions on which a process is based and then attempt to find a relatively simple mathematical relationship that reflects those principles or assumptions. For instance, in football, when a quarterback throws a long pass down the field, the path of the ball can be represented by a simple mathematical formula involving a quadratic function relating the height $y$ to the horizontal distance $x$—a model based on one of Newton's laws of motion.

Throughout this chapter, the models we develop are difference equations such as $D_{n+1} = 0.75D_n + 80$ that models the level of Prozac in the bloodstream. The solution sequences to such difference equations can be given in two ways: either as a collection of numbers such as $\{80, 140, 185, 218.75, 244.0625, 263.0469, 277.2852, \ldots\}$ based on an initial value $D_0 = 80$, say, or as a formula such as

$$D_n = 320 - 240(0.75)^n,$$

a function of $n$ that holds for every value of $n$.

### Exponential Growth and Decay Models

The population of the world passed 6 billion people in 1999 and was growing at a rate of about 1.5% per year, so the growth factor was 1.015. Assuming that this trend continues, we can model the earth's population with the exponential growth function $P(t) = 6(1.015)^t$, where $t$ is the number of years since 1999 and the population is in billions of people. Alternatively, if we use $n$ to represent the number of years since 1999, we can write this exponential function in sequence notation as $P_n = 6(1.015)^n$. We know that a set of numbers follows an exponential pattern if the successive ratios are all constant. That is, for any $n$,

$$\frac{P_{n+1}}{P_n} = c = 1.015,$$

or when we multiply by $P_n$,

$$P_{n+1} = cP_n = 1.015P_n,$$

starting with $P_0 = 6$ when $n = 0$ in 1999. This difference equation is a model for the world's population. It tells us that each term in the population sequence $P = \{P_n\}$ is precisely the same multiple, 1.015, of the preceding term.

Let's now look at the reverse of the preceding argument. Suppose that each term in a sequence $X = \{x_n\}$ is a constant multiple $c$ of the preceding term. That is, $x_1 = cx_0, x_2 = cx_1, x_3 = cx_2, \ldots$, or, in general $x_{n+1} = cx_n$, which is a difference equation for the sequence for any value of $n$. What does this result mean?

**EXAMPLE 1**

Find a formula for the solution sequence to the difference equation

$$x_{n+1} = cx_n$$

for any constant $c > 0$.

**Solution** Starting with the initial term $x_0$ corresponding to $n = 0$, we have

$$x_1 = cx_0;$$
$$x_2 = cx_1 = c \cdot (cx_0) = c^2 x_0;$$
$$x_3 = cx_2 = c \cdot (c^2 x_0) = c^3 x_0;$$
$$x_4 = cx_3 = c \cdot (c^3 x_0) = c^4 x_0;$$

and so on. This process leads to an obvious formula for the general term of the solution sequence,

$$x_n = c^n x_0, \quad \text{for any } n = 0, 1, 2, \ldots$$

Thus the solution to any difference equation of the form $x_{n+1} = cx_n$ is an exponential function $x_n = c^n x_0$ or $x_n = x_0 c^n$. Such a sequence is known as a **geometric sequence** or an **exponential sequence** in which the ratio of successive terms is a constant, equal to $c$, because each term is the same multiple of the preceding term.

The difference equation for exponential growth or decay

$$x_{n+1} = cx_n$$

has as its solution the exponential sequence

$$x_n = x_0 c^n,$$

where $x_0$ is the starting value for the sequence.

Recall that, whenever the constant multiple $c > 1$, the values for $x_n$ get successively larger; whenever $c$ is between 0 and 1, the subsequent values become successively smaller. We presented some illustrations of this property in Chapter 2.

Think About This

What happens if $c < 0$?

From a somewhat different perspective, the equation $P_{n+1} = cP_n$ is a difference equation for exponential behavior and the formula $P_n = P_0 c^n$ is the solution sequence. Whenever $c > 1$, we have exponential growth, and whenever $0 < c < 1$, we have exponential decay. Thus in any sequence in which each term is a constant multiple of the preceding term,

$$x_{n+1} = cx_n,$$

$x_n$ is an exponential function of $n$ and $x_n = x_0 c^n$.

Alternatively, suppose that $P_{n+1} = cP_n$. If we subtract $P_n$ from both sides of the difference equation, we get

$$P_{n+1} - P_n = cP_n - P_n = (c - 1)P_n = aP_n,$$

where $a = c - 1$. Note that, if $c > 1$, then $a > 0$ and it is the growth rate; if $0 < c < 1$, then $a = c - 1 < 0$ and this gives the decay rate. The expression on the left, $P_{n+1} - P_n$, is simply the difference between successive values, so we can write

$$\Delta P_n = P_{n+1} - P_n = aP_n.$$

Because we can rewrite any such equation as a difference in this way, we call it a *difference* equation.) This difference equation indicates that, for any exponential process, the successive differences $\Delta P_n$ are always a fixed multiple $aP_n$ of the quantity itself.

For reference, we recast the previous difference equation $x_{n+1} = cx_n$ for exponential growth or decay in this alternative format.

> The difference equation for exponential growth or decay
>
> $$\Delta x_n = ax_n$$
>
> has as its solution the exponential sequence
>
> $$x_n = x_0 \cdot (1 + a)^n = x_0 c^n,$$
>
> where $x_0$ is the starting value for the sequence.

Writing difference equations in this form is often helpful because the $\Delta x_n$ formulation lets us think of the *change* in a quantity from one stage to the next instead of how one value depends on the preceding value.

We now consider several examples that demonstrate the use of these ideas relating to difference equations for growth and decay.

## EXAMPLE 2

Suppose that you deposit \$1000 in a bank account paying 5% interest, compounded annually. Write a difference equation to represent the balance in your account after any number of years and find an expression for the balance at any time from the difference equation.

***Solution*** From the discussion of exponential growth in Chapter 2, we know that the balance in the account after any number of years is given by $b(t) = 1000(1.05)^t$. We now look at this situation from the point of view of difference equations.

Let the original balance be $b_0 = 1000$. After 1 year, the original \$1000 balance has earned 5% of \$1000, or $0.05(1000) = \$50$ in interest, so the new balance is

$$b_1 = 1000 + 0.05(1000) = (1 + 0.05)1000 = (1.05)1000 = 1050.$$

Symbolically,

$$b_1 = b_0 + 0.05b_0 = 1.05b_0.$$

By the end of the second year, the balance has grown to

$$b_2 = b_1 + 0.05b_1 = 1.05b_1.$$

Similarly, by the end of third year, the balance is

$$b_3 = b_2 + 0.05b_2 = 1.05b_2.$$

In general, by the end of the $(n + 1)$st year, for any $n$, the balance in the account is

$$b_{n+1} = b_n + 0.05b_n = 1.05b_n,$$

which is the difference equation relating the balance in any year to the balance the next year.

Because this is a difference equation for exponential growth, we immediately know that the solution after $n$ years is given by

$$b_n = b_0 \cdot (1.05)^n.$$

For the initial deposit $b_0 = 1000$, the balance after $n$ years is

$$b_n = 1000(1.05)^n,$$

which is identical to the expression for the exponential growth model.

Before going on, let's recall the formula at the end of Section 5.1.

> For the difference equation
>
> $$x_{n+1} = ax_n + B,$$
>
> a formula for the solution sequence is
>
> $$x_n = \frac{B}{1-a} + \left(x_0 - \frac{B}{1-a}\right)a^n,$$
>
> where $x_0$ is the initial value for the solution sequence.

If the constant $B$ is 0 in the difference equation $x_{n+1} = ax_n + B$, then this equation reduces to $x_{n+1} = ax_n$, which is the difference equation for an exponential process. The formula for the solution sequence similarly reduces to $x_n = x_0a^n$, or the exponential function that we would expect.

*Modeling an IRA Account* We next consider how to use a difference equation to model the growth of an IRA account in which a fixed amount of money is invested each year.

**EXAMPLE 3**

At age 25, Alison sets up an IRA retirement savings account. She invests \$3000 annually into an account that earns 5% interest per year.

**a.** Write a difference equation that models the balance $B_n$ in her account after $n$ years.

**b.** Find a formula for the solution sequence to the difference equation.

**c.** Find the balance in her account on her 60th birthday. How much of this balance is attributed to contributions and how much to accrued interest?

***Solution***

**a.** We denote the balance in the account after $n$ years by $B_n$. During the succeeding year, the balance grows by 5% of $B_n$ and is then augmented by the next contribution of \$3000. Therefore the balance after $n + 1$ years is

$$B_{n+1} = B_n + 5\% \text{ of } B_n + 3000 = B_n + 0.05B_n + 3000 = 1.05B_n + 3000,$$

which is a difference equation model for the balance in the account.

**b.** Note that the difference equation $B_{n+1} = 1.05B_n + 3000$ has the same form as the difference equation $x_{n+1} = ax_n + B$, with $a = 1.05$, $B = 3000$, and initial condition $x_0 = 3000 = B_0$. The solution to this difference equation is

$$\begin{aligned} B_n &= \frac{3000}{1 - 1.05} + \left(3000 - \frac{3000}{1 - 1.05}\right)(1.05)^n \\ &= -\frac{3000}{0.05} + \left(3000 - \frac{3000}{-0.05}\right)(1.05)^n \\ &= -60{,}000 + (3000 + 60{,}000)(1.05)^n \\ &= 63{,}000(1.05)^n - 60{,}000. \end{aligned}$$

Figure 5.8 shows the graph of this solution sequence, with the continuous exponential function superimposed.

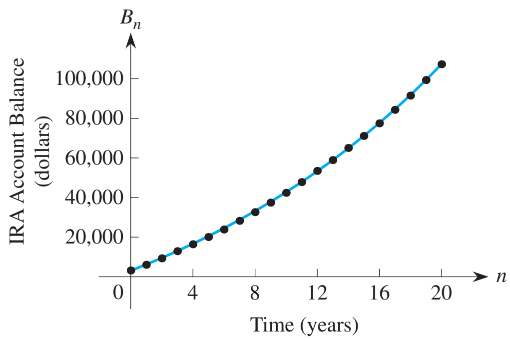



FIGURE 5.8

**c.** At age 60, Alison's IRA account will have been in existence for $n = 35$ years. Therefore the balance in the account will be

$$B_{35} = 63{,}000(1.05)^{35} - 60{,}000 = 287{,}508.97.$$

Of this total, the amount contributed at \$3000 per year is $35(3000) = 105{,}000$, so the account actually earned about \$182,509 in interest.

*Modeling a Population with Harvesting* We now introduce a population growth model in which part of the population is removed each time period.

**EXAMPLE 4**

A poultry farm has 30,000 chickens whose growth rate is 20% per month. Suppose that 5000 chickens are killed (harvested) each month for shipment to stores.

**a.** Write a difference equation to model this situation.

**b.** Write a formula for the solution sequence for the difference equation.

**c.** Discuss the behavior of the solution function and explain the long-term population pattern of the chickens at the farm.

**d.** What would happen if 8000 chickens are harvested monthly?

**e.** Determine the number of chickens that should be harvested monthly to maintain a constant population from one month to the next.

***Solution***

**a.** Let $C_n$ be the chicken population in the $n$th month. During that month, the population grows by 20% and 5000 chickens are harvested. Thus the difference equation model is

$$C_{n+1} = 1.20C_n - 5000, \qquad C_0 = 30{,}000.$$

**b.** We use the formula for the solution sequence with $a = 1.20$ and $B = -5000$ to get

$$\begin{aligned} C_n &= \frac{-5000}{1 - 1.20} + \left(30{,}000 - \frac{-5000}{1 - 1.20}\right)(1.20)^n \\ &= \frac{-5000}{-0.20} + \left(30{,}000 - \frac{-5000}{-0.20}\right)(1.20^n) \\ &= 25{,}000 + (30{,}000 - 25{,}000)(1.20)^n \\ &= 25{,}000 + 5{,}000(1.20)^n. \end{aligned}$$

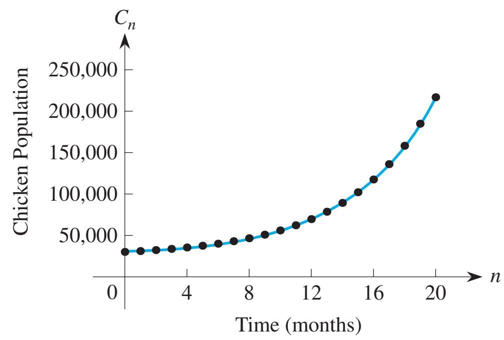



FIGURE 5.9

**c.** The graph of this function with the continuous function superimposed is shown in Figure 5.9. Note how it grows in a concave up pattern. The solution function is a modified exponential function with a vertical shift of 25,000 and a growth factor of 1.20, so the chicken population at the farm is increasing at an increasing rate. Clearly, this growth pattern can't continue indefinitely.

**d.** If 8000 chickens are harvested each month instead of 5000, the formula for the solution becomes

$$\begin{aligned} C_n &= \frac{-8000}{1 - 1.20} + \left(30{,}000 - \frac{-8000}{1 - 1.20}\right)(1.20)^n \\ &= 40{,}000 + (30{,}000 - 40{,}000)(1.20)^n \\ &= 40{,}000 - 10{,}000(1.20)^n. \end{aligned}$$

This solution is an upside-down exponential growth function that has been shifted up by 40,000, as shown in Figure 5.10. Note that it drops ever more quickly, indicating that the chicken population will die out very rapidly.

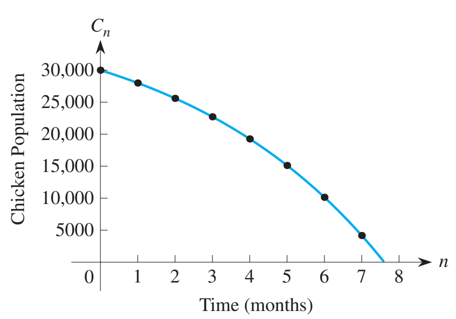



FIGURE 5.10

**e.** To maintain a constant chicken population from one month to the next requires harvesting the number of chickens that will exactly counterbalance their monthly growth. If the farm starts with 30,000 chickens and they grow at a monthly rate of 20%, 6000 new chickens will be hatched. Therefore, if 6000 older chickens are killed monthly, there is no net increase or decrease.

*Modeling the Level of Pollutants* In Examples 5(a)–(e) we consider the level of contaminants in a lake under a variety of circumstances to illustrate how models with difference equations can arise. We also show how the solution to the difference equation model can be used to determine the behavior pattern for the level of contamination over time.

## EXAMPLE 5(a)

Initially, 600 lb of a contaminant are dumped into a lake, and 10% of it is washed away each year. Find a formula for the level of contaminant present after any number of years.

**Solution** Let $C_n$ represent the level of contaminant present after $n$ years; we know that $C_0 = 600$. Because 10% of the contaminant present is washed out of the lake during any year, 90% of the amount present at the start of any year will be left a year later. Therefore the situation is modeled by the difference equation

$$C_{n+1} = 0.9C_n, \qquad C_0 = 600.$$

This is a difference equation for exponential decay, so we know that the solution is

$$C_n = 600(0.9)^n.$$

This exponential decay function tells us that the level of contaminant will slowly fall over time, as shown in Figure 5.11.

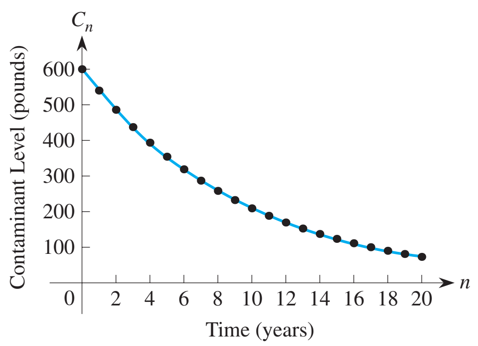



FIGURE 5.11

## EXAMPLE 5(b)

Initially, there are 600 lb of the contaminant in the lake, 10% of it is washed out each year, and a manufacturing plant annually dumps 100 lb of the contaminant into a river that feeds into the lake. Find the level of contaminant present after any number of years.

**Solution** The amount of contaminant present in the lake is reduced by 10% during a year, so 90% of the amount present each year remains in the following year. However, this amount is then increased by an additional 100 lb each year. This situation is modeled by the difference equation

$$C_{n+1} = 0.9C_n + 100, \qquad C_0 = 600.$$

Note that the form of the difference equation is identical to the equations that we developed previously for the Prozac drug level model and the IRA account balance. Using the formula for the solution to such a difference equation with $a = 0.9$ and $B = 100$—starting with an initial level of $C_0 = 600$—gives a formula for the solution sequence

$$C_n = \frac{100}{1-0.9} + \left(600 - \frac{100}{1-0.9}\right)(0.9)^n,$$

or

$$C_n = 1000 - 400(0.9)^n.$$

Its graph is shown in Figure 5.12. To understand the graph, note that the exponential term $400(0.9)^n$ dies out as $n$ increases. But, because this term is subtracted from 1000, the solution rises toward 1000 as a horizontal asymptote, which is the limiting value for the contaminant.

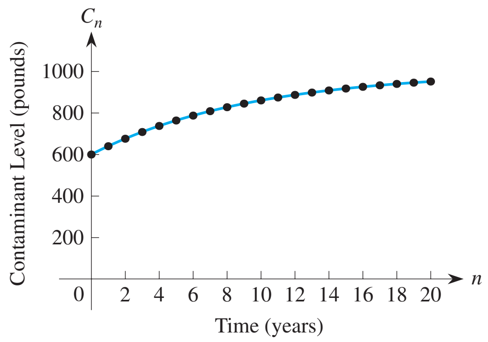



FIGURE 5.12

## EXAMPLE 5(c)

The situation is the same as in Example 5(b), but now the plant increases its annual production and thus increases the amount of the contaminant it dumps by 50 lb each year, starting with the initial level of 600 lb.

***Solution*** The 50-pound per year increase in the amount of contaminant dumped into the river means that 100 lb are dumped the initial year, 150 lb the following year, 200 lb the year after that, and so on, following this pattern of linear growth. Thus during the $n$th year, the company will dump $100 + 50n$ pounds of the contaminant into the river that feeds the lake. As before, 90% of the contaminant in the lake at the start of any year remains at the end of that year and is then augmented by the additional amount dumped into the river. The difference equation that models this situation is

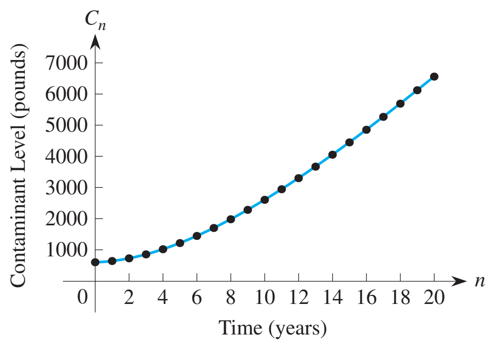



FIGURE 5.13

$$C_{n+1} = 0.9C_n + 100 + 50n, \qquad C_0 = 600.$$

Note that this difference equation has a form different from the preceding two: The term on the right, $100 + 50n$, is no longer a constant but is now a linear function of $n$, so the formula that we previously developed doesn't apply. We show in Supplementary Section 12.3 that a formula for this solution sequence is

$$C_n = 4600(0.9)^n - 4000 + 500n.$$

Note that this formula consists of a decaying exponential term and a linear term with positive slope. Although the exponential term slowly dies out, the linear term continues to increase as time passes. Over the long term, the level of contaminant eventually increases in a roughly linear pattern with a slope of 500 lb per year, as shown in the graph in Figure 5.13.

## EXAMPLE 5(d)

The situation is the same as in Example 5(b), except that the plant now increases the amount of contaminant dumped into the river by 20% per year starting with the initial level of 100 lb.

***Solution*** The yearly increase in contaminant dumped into the river is now given by the exponential function $100(1.20)^n$, so the difference equation modeling this situation is

$$C_{n+1} = 0.9C_n + 100(1.20)^n, \qquad C_0 = 600.$$

We show in Supplementary Section 12.3 that the solution sequence is

$$C_n = \left(\frac{800}{3}\right)(0.9)^n + \left(\frac{1000}{3}\right)(1.2)^n.$$

The first term is an exponential decay function that eventually dies out, while the second term is an exponential growth function. Early on, the decay term makes a contribution, but because its coefficient ($\approx 266.7$) is smaller than the growth term's coefficient ($\approx 333.3$), the contribution is minimal and rather quickly diminishes. The overall behavior pattern is one of roughly exponential growth in the amount of contaminant, as illustrated in Figure 5.14. The eventual exponential growth factor is about 1.2 (verify this result by calculating a pair of successive terms in the solution for moderately large values of $n$—say, $n = 20$ and $n = 21$), so the annual growth rate is eventually about 20%.

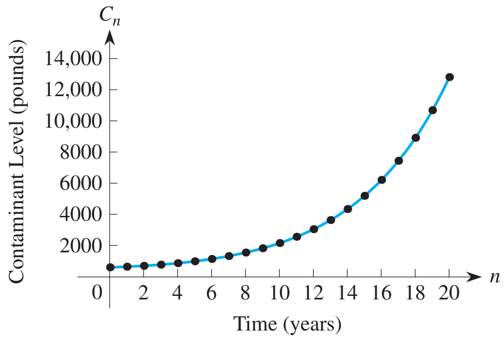



FIGURE 5.14

## EXAMPLE 5(e)

The situation is the same as in Example 5(b), with the plant initially dumping 100 lb of the contaminant, but EPA regulations require that it reduce the level of dumping by 25% per year.

***Solution*** The amount dumped into the river is now represented by the decaying exponential function $100(0.75)^n$, so the corresponding difference equation is

$$C_{n+1} = 0.9C_n + 100(0.75)^n, \qquad C_0 = 600.$$

In Supplementary Section 12.3 we show that the solution sequence is

$$C_n = \left(\frac{3800}{3}\right)(0.9)^n - \left(\frac{2000}{3}\right)(0.75)^n.$$

Both terms decay to zero exponentially, so the level of contaminant will eventually die out. The graph of the solution sequence in Figure 5.15 shows a surprising pattern in which the level of contaminant first increases and then dies out. Let's see why. Suppose that we calculate the first few terms of the solution sequence:

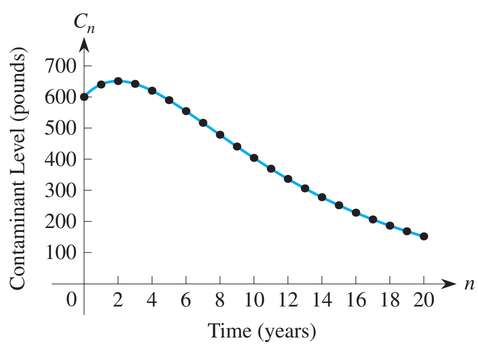



FIGURE 5.15

$$C_1 = \left(\frac{3800}{3}\right)(0.9)^1 - \left(\frac{2000}{3}\right)(0.75)^1 = 1140 - 500 = 640;$$

$$C_2 = \left(\frac{3800}{3}\right)(0.9)^2 - \left(\frac{2000}{3}\right)(0.75)^2 = 1026 - 375 = 651;$$

$$C_3 = \left(\frac{3800}{3}\right)(0.9)^3 - \left(\frac{2000}{3}\right)(0.75)^3 = 923.4 - 281.25 = 642.15;$$

$$C_4 = \left(\frac{3800}{3}\right)(0.9)^4 - \left(\frac{2000}{3}\right)(0.75)^4 = 831.06 - 210.94 = 620.12.$$

Obviously, both $(0.9)^n$ and $(0.75)^n$ decrease as $n$ increases, but the second term decreases more rapidly because its decay factor 0.75 is considerably smaller than the decay factor 0.90 of the first term. As a result, when $n$ is small ($n = 1$ or 2), the second term has a much greater effect on the result than it does when $n$ gets larger—the amount subtracted from the first term is relatively large at first but then quickly diminishes. As less is subtracted away, the solution increases somewhat at first; however, eventually the second term has minimal effect and the first term decays toward zero, as we expect. In particular, $C_{10} = 404.12$, $C_{20} = 151.88$, and $C_{30} = 53.58$.

*The Fibonacci Model for Population Growth* We next consider one of the earliest mathematical models for a biological process, developed by Italian mathematician Fibonacci, who lived about 1200 A.D. Fibonacci constructed a simple model for predicting the local rabbit population based on the following assumptions.

1. Newborn rabbits mature in 1 month.
2. Once they have matured, rabbits have litters monthly.
3. Each litter consists of one male and one female.

The first two assumptions are fairly accurate, but we can argue about the third assumption on a variety of grounds. First, rabbit litters tend to be considerably larger than 2. However, there is a certain mortality rate for newborn rabbits (they can't run fast enough to escape from the predators in the neighborhood) that lowers the number per litter that survive to maturity. Second, expecting one male and one female to survive from each litter is unreasonable. However, if we consider a large population of rabbits, the numbers of males and females for the entire population average out to about a 50–50 split per litter.

Let's start with one pair of newborn rabbits—one male, the other female—on January 1 of some year, By February 1, the original pair are now mature and ready to do what rabbits do best: produce new rabbits. On March 1, there are two pairs of rabbits, the original pair and their first set of offspring. By April 1, the original pair has produced another litter while their first litter has matured and is ready to enter the family business. Thus there are now three pairs of rabbits, two mature and one newborn. However, by this stage, things are starting to get complicated, so we use the diagram shown in Figure 5.16 to keep track of the rabbits. Let the symbol  denote an immature pair and  represent a mature breeding pair.

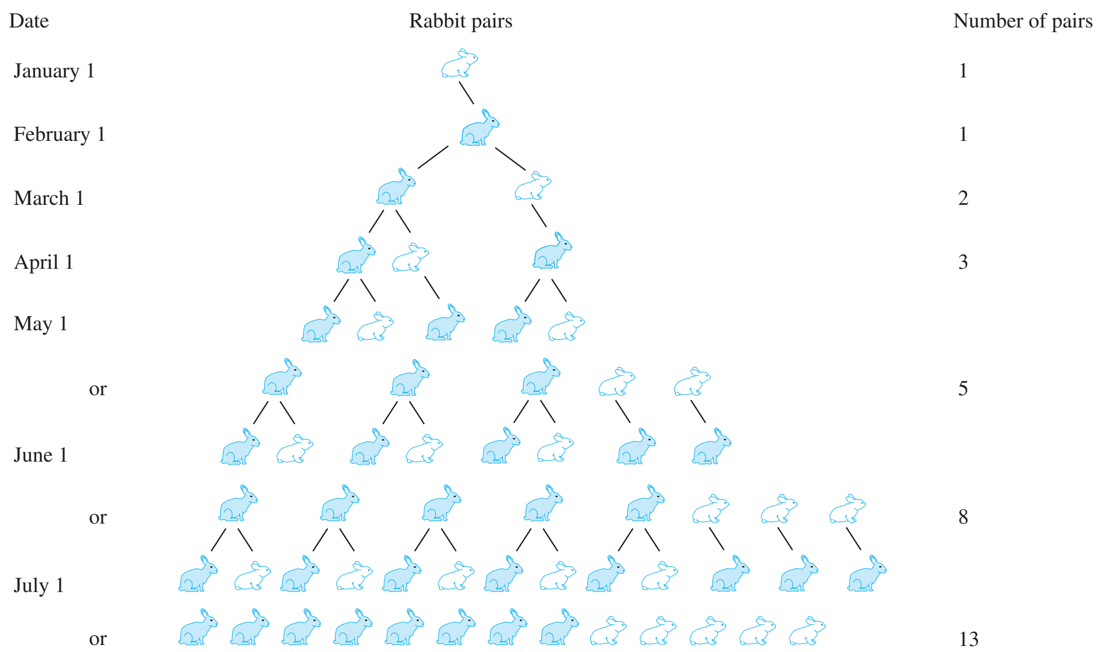



FIGURE 5.16

Consider the rabbit population on July 1, say, which consists of 8 mature pairs and 5 immature pairs. Note that the number of mature pairs, 8, is equal to the total number of pairs in June, the preceding month. Also, the number of immature pairs, 5, is equal to the population 2 months earlier on May 1. Therefore the rabbit population on July 1 is equal to the population on June 1 (mature) plus the population on May 1 (immature). This pattern is not coincidental, and it persists indefinitely—in each month, the rabbit population is equal to the sum of the population values the preceding two months. Let's see why.

The population in the current month $P_n$ consists of breeding pairs and newborn pairs. The number of breeding pairs this month is equal to the total population $P_{n-1}$ last month—all are still alive and all are now mature. Now think about the number of newborn pairs this month. Every rabbit alive two months ago, $P_{n-2}$, was mature last month and so gave birth to a new litter this month. Therefore the number of newborn pairs this month must be equal to the total population two months ago. So for any $n \geq 2$,

$$P_n = P_{n-1} + P_{n-2}.$$

This is a difference equation relating the successive population values.

Alternatively, instead of considering the current population $P_n$ in terms of the preceding two months' populations (a backward difference equation), we look at the population two months ahead $P_{n+2}$, which is determined by the current population $P_n$ and next month's population $P_{n+1}$. As a result, we can rewrite the difference equation in the equivalent forward form

$$P_{n+2} = P_n + P_{n+1}, \qquad n \geq 0,$$

starting with $P_0 = P_1 = 1$. It is known as the **Fibonacci difference equation** or the **Fibonacci model.** Note that with either form we get the population values

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, . . .

This particular sequence of numbers is called the **Fibonacci sequence.**

### Fibonacci Sequence

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\}$$

These numbers arise in a surprising variety of ways—in nature (e.g., the arrangement of petals on sunflowers and the number of rings in seashells), in economics, in human psychology, and in art (see Example 7 of Section 3.2). However, here we continue to focus on the constantly growing rabbit population of old Italy.

## EXAMPLE 6

Construct a table based on Fibonacci's model for the rabbit population over the first 30 months and discuss the growth in this population.

***Solution*** Starting with the initial values $P_0 = P_1 = 1$, the Fibonacci difference equation gives the values presented in the table and the graph shown in Figure 5.17 for the first 30 months.

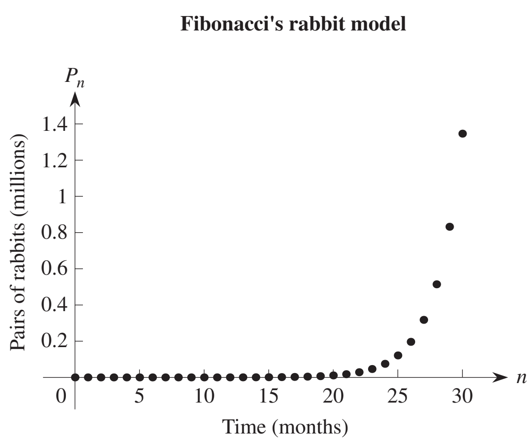



FIGURE 5.17

| Month | Pairs | Month | Pairs | Month | Pairs |
|---|---|---|---|---|---|
| 0 | 1 | 11 | 144 | 21 | 17,711 |
| 1 | 1 | 12 | 233 | 22 | 28,657 |
| 2 | 2 | 13 | 377 | 23 | 46,368 |
| 3 | 3 | 14 | 610 | 24 | 75,025 |
| 4 | 5 | 15 | 987 | 25 | 121,393 |
| 5 | 8 | 16 | 1,597 | 26 | 196,418 |
| 6 | 13 | 17 | 2,584 | 27 | 317,811 |
| 7 | 21 | 18 | 4,181 | 28 | 514,229 |
| 8 | 34 | 19 | 6,765 | 29 | 832,040 |
| 9 | 55 | 20 | 10,946 | 30 | 1,346,269 |
| 10 | 89 | | | | |

Both the table and Figure 5.17 show a population explosion among the rabbits very quickly. After the first few entries in the table, the ratio of successive terms is approximately 1.618. For example, $233/144 \approx 1.618$, $377/244 \approx 1.618$, and so on. These outcomes suggest that, eventually, the growth pattern is roughly exponential with a growth rate of $0.618 = 61.8\%$ per month.

These numbers are, if anything, conservative because each litter will likely contain more than two rabbits. However, deaths among the rabbits have been ignored. Consequently, the Fibonacci model for the rabbit population may not be a particularly good match to the actual population.

Based on Fibonacci's model, Italy clearly would have had a major overpopulation problem with rabbits in his time, let alone by now. Because that hasn't happened, something is wrong either with the mathematical model or the assumptions on which it is based. Actually, the mathematical model is fairly accurate—at least up to a point. So long as the rabbit population remains relatively small, the model gives numbers that reasonably estimate the population. However, it shouldn't be carried too far because no process can continue to grow exponentially indefinitely. Instead, other factors that act to curb the growing population must be taken into account. For example, as the number of rabbits increases, so too will the number of foxes and other predators that live off them. In turn, the larger numbers of predators eventually reduce the rabbit population. Also, when the rabbit population grows too large, they quickly consume most of the available food supply and there won't be enough food to sustain such a large population. The result is starvation until the population decreases to a more sustainable size. We discuss the mathematical details of this more realistic type of scenario in Section 5.3.

A difference equation that relates one term of a sequence $x_{n+1}$ to the preceding term $x_n$ is called a **first order difference equation.** It is the primary type of difference equation we cover in this chapter and in Supplementary Chapter 12. A difference equation such as Fibonacci's that relates one term $x_{n+2}$ to the preceding two terms $x_n$ and $x_{n+1}$ is called a **second order difference equation.**

## Problems

1. In Example 3 about Alison's IRA account, (**a**) what would the account be worth at age 65 instead of age 60? and (**b**) what would it be worth at age 65 if she started the account at age 20 instead of age 25.

2. Repeat the calculations in Example 3 and in Problem 1 if the rate of return for the IRA account is 6% instead of 5%.

3. In Example 4 about the population of chickens, we assumed that there were 30,000 chickens on the farm and a growth rate of 20% per month. We showed that, if the owner kills 8000 chickens a month, the population eventually will die out. Assuming that the owner doesn't notice this developing problem, how long will it be until no chickens are left?

4. Suppose that the population of a certain species of fish in a lake grows exponentially with a growth rate $a$. Write a difference equation to model each situation.
   **a.** The fish population grows exponentially.
   **b.** Fishermen catch and remove 100 fish from the lake each year.
   **c.** Fishermen catch and remove 40% of the fish from the lake each year.
   **d.** The number of fish that fishermen remove from the lake each year is proportional to the square root of the number of fish in the lake.
   **e.** Fishermen remove 40% of the fish each year, and the state's wildlife department restocks the lake with 500 fish each year.

5. Jack and Jill are setting up plans for a retirement fund. Write a difference equation for the balance $b_n$ in this account for each of the scenarios they are contemplating.
   **a.** They deposit $2000 in an account guaranteed to pay 6% interest per year.
   **b.** They deposit $2000 initially in an account that pays 6% interest per year and then deposit an additional $1000 each year.
   **c.** They deposit $2000 initially in the account paying 6% per year and then increase their contribution by $1000 each year.
   **d.** They deposit $2000 initially into the account paying 6% per year and then increase their contribution by 10% each year.

6. Claire has $80,000 in a retirement fund that pays 6% interest per year.
   **a.** If she plans to withdraw $10,000 yearly, write a difference equation for the balance in the account.
   **b.** How long will it take for the balance in the account to be depleted if she withdraws $10,000 every year?
   **c.** Suppose that she plans to withdraw 20% of the account balance every year. Write a difference equation for the balance in the account.
   **d.** How long will it take for the balance to be depleted with this withdrawal plan?
   **e.** Is there a fixed amount she can withdraw from the account every year without diminishing the balance? If so, find it.

7. Marine biologists estimate that there were about 8000 bowhead whales in the waters near Alaska in 1992 and that they were growing at an annual rate of 3%. Alaskan Eskimos are allowed to catch about 50 whales per year.
   **a.** Write a difference equation giving the population of the whales from one year to the next.
   **b.** Calculate the projected whale population each year until 2005.
   **c.** Determine the largest number of whales that the Eskimos could catch each year without the whale population going into decline.
   **d.** Suppose that the Eskimos are petitioning the government to increase their annual whale harvest and you are acting as their representative before the panel making the decision. What arguments would you use to justify increasing the annual harvest?
   **e.** Suppose that you were representing a conservation group opposed to increasing the annual whale harvest. What arguments would you use to request a denial of the petition?

8. A company expects the productivity of new employees to increase each day as they gain experience. When a new person starts "cold," the company expects the employee to produce $P_0$ items per hour. The following day, hourly production should increase by one item per hour to $P_0 + 1$; the day after that, production should increase by 2 items per hour; then by 3 items per hour; and so on.
   **a.** Write a difference equation to model this situation.
   **b.** Find an expression for the solution to this difference equation for any number of days $n$.

**c.** Write a paragraph discussing whether this expectation for continued improvement seems to be sensible.

**9.** Psychologists have found that, when a person learns a new body of knowledge, the amount of new knowledge gained in any time period is proportional to the amount that the person does not know. That is, it is easier to improve when you know a little than it is to improve when you know a lot. Suppose that Greg, while preparing for the SAT vocabulary test, is trying to learn 400 new words from a set of flash cards.

**a.** Write a difference equation for this learning model based on the number of words $W_n$ that Greg knows out of the 400 total on the $n$th pass through the deck.

**b.** Is the constant of proportionality in the difference equation positive or negative? Is it less than 1 or greater than 1?

**c.** Write a paragraph explaining why it is reasonable to expect Greg to learn more words during the first few passes through the deck of cards than through later passes through it.

**d.** Sketch a graph of the possible number of new words that Greg learns as a function of the number $n$ of passes through the deck based on this learning model.

**10.** Repeat the calculations associated with Fibonacci's rabbit model for the first year, assuming an initial population of 10 pairs of newborn rabbits. How do your values relate to those presented in the text?

**11.** Suppose that a particular breed of rabbits take 2 months to mature instead of 1 month, but that Fibonacci's other assumptions still hold. Calculate the rabbit population each month during the first year.

**12.** Tribbles are adorable, furry little creatures. The only trouble with tribbles is that they breed like tribbles. Specifically, suppose that a tribble matures in 3 days and then reproduces asexually daily by splitting off a new tribble on the fourth day and every day thereafter.

**a.** Construct a difference equation for the tribble population, starting with one newborn tribble, by expressing $T_{n+3}$ in terms of $T_n$, $T_{n+1}$, and $T_{n+2}$. (*Hint*: Let $\triangle$ = newborn tribble, $\square$ = day old immature tribble and $\bigcirc$ = mature tribble and keep track of the number of each over the first 10 days.)

**b.** Use the difference equation to calculate the tribble population during the first 15 days, based on an initial newborn tribble the first day.

**c.** Examine the ratio of successive terms to determine whether the tribble population appears to be growing exponentially. If it does, what is the exponential growth rate?

**d.** Using the result of part (c), estimate the tribble population after a full year.

**e.** Assume that the human population of the Earth is currently 6 billion and growing exponentially at an annual rate of 1.7%. Estimate when every human being alive will have a tribble of his or her own.

**13.** Consider the difference equation $x_{n+1} = x_n + kn$, for any constant multiple $k$. Show that the solution is always a quadratic function of $n$. (*Hint*: Use a result from Section 4.5.)

**14.** Consider the difference equation $x_{n+1} = x_n + kn^2$, for any constant multiple $k$. Show that the solution is always a cubic function of $n$.

## 5.3 The Logistic or Inhibited Growth Model

In Section 5.2 we demonstrated that a population $P_n$ growing exponentially satisfies the difference equation

$$P_{n+1} = cP_n$$

for all values of $n$. A formula for $P_n$ is

$$P_n = c^n P_0,$$

where $c$ is the growth (or decay) factor.

Alternatively, if we subtract $P_n$ from both sides of the difference equation, we get

$$P_{n+1} - P_n = cP_n - P_n = (c - 1)P_n = aP_n,$$

or

$$\Delta P_n = P_{n+1} - P_n = aP_n,$$

where $a$ is the growth (or decay) rate. In this form, the difference equation tells us that, for any exponential process, the successive differences are always a fixed multiple—the growth (or decay) rate—of the quantity itself.

## The Logistic Growth Model

The exponential growth model is effective in predicting the growth of most populations over the short run. However, if any growth process were to continue indefinitely, other factors come into play to slow down, or *inhibit,* the rate of growth. We now modify the exponential growth difference equation $\Delta P_n = aP_n$ to model a situation in which a population starts growing exponentially, then slows, and eventually levels off as the population reaches the maximum size that can be sustained by the environment. This type of behavior is illustrated in Figure 5.18.

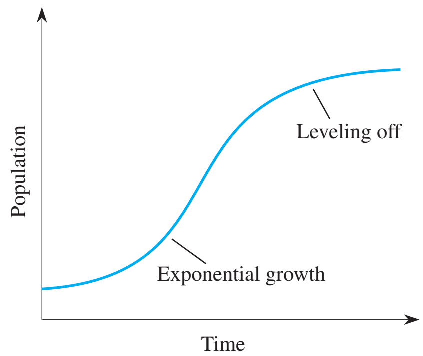



FIGURE 5.18

In the 1830s, the Belgian biologist Verhulst developed a simple extension of the exponential growth model $\Delta P_n = aP_n$ to reflect this situation. He introduced an extra term in the difference equation—one that serves to decrease the rate of growth in the population—to account for this leveling-off effect. Verhulst and other scientists who have since studied such processes have observed that these processes can be modeled by subtracting a term that is proportional to the square of the population. Thus we use the expression for the change in $P_n$,

$$\Delta P_n = P_{n+1} - P_n = aP_n - bP_n^2,$$

where $b$ is the *inhibiting constant,* which typically is much smaller than the growth rate $a$. Alternatively, if we add $P_n$ to both sides, we can rewrite this difference equation as

$$P_{n+1} = (1 + a)P_n - bP_n^2.$$

In either form, the resulting process is known as the **logistic growth model** or the **inhibited growth model.** Note that, if $b = 0$, the logistic growth model reduces to the exponential growth model.

### The Logistic or Inhibited Growth Model

$$\Delta P_n = P_{n+1} - P_n = aP_n - bP_n^2 \quad \text{or} \quad P_{n+1} = (1 + a)P_n - bP_n^2,$$

where $b$ is much smaller than $a$.

To work with this logistic growth model, we must get some feel for appropriate values for the **inhibiting constant** $b$. You may want to experiment, using your graphing calculator (if it has difference equation capabilities) or appropriate computer software. For instance, suppose that the exponential growth rate is $a = 1$ for a species such as rabbits that breeds rapidly and that the inhibiting constant is $b = 0.04$. These values give a graph similar to the one shown in Figure 5.18, and a curve with this shape is called a **logistic curve.** However, if you try $b = 3$ instead, say, the result will be quite different. (We discuss the resulting type of chaotic behavior in Supplementary Section 12.8.) In fact, with a little experimentation, you will find that the model is an effective description of inhibited population growth when $b$ is much smaller than $a$.

## EXAMPLE 1

Consider the logistic model with $a = 1$ and $b = 0.0004$ for a rabbit population, where $n$ represents the number of months. If the initial rabbit population is $P_0 = 1$ pair, use the difference equation to calculate the population over the first 16 months. What does the maximum sustainable population appear to be?

***Solution*** With $a = 1$ and $b = 0.0004$, the logistic difference equation is

$$P_{n+1} = (1 + a)P_n - bP_n^2 = 2P_n - 0.0004P_n^2.$$

Therefore, using $n = 0$ and $P_0 = 1$, we get

$$P_1 = 2P_0 - 0.0004P_0^2 = 2(1) - 0.0004(1^2) \approx 2.$$

Continuing with $n = 2$ and $n = 3$, we get

$$P_2 = 2P_1 - 0.0004P_1^2 = 2(2) - 0.0004(2^2) \approx 4;$$
$$P_3 = 2P_2 - 0.0004P_2^2 = 2(4) - 0.0004(4^2) \approx 8;$$

and so on. Continuing this process, we obtain the following table of values and the graph shown in Figure 5.19.

| Month | Number of Pairs | Month | Number of Pairs |
|---|---|---|---|
| 0 | 1 | 9 | 463 |
| 1 | 2 | 10 | 840 |
| 2 | 4 | 11 | 1398 |
| 3 | 8 | 12 | 2014 |
| 4 | 16 | 13 | 2406 |
| 5 | 32 | 14 | 2496 |
| 6 | 63 | 15 | 2500 |
| 7 | 125 | 16 | 2500 |
| 8 | 243 | | |

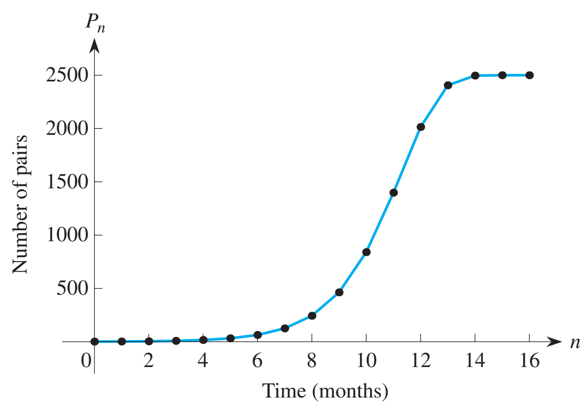



FIGURE 5.19

Note that the population increases to a maximum of 2500 and seems to remain at that level. Note also that the population grows rapidly to about half this maximum during the first 11 or so months and then grows more slowly thereafter until it reaches the 2500 level.

This behavior pattern is typical of the logistic model: An initial spurt in the population is followed by a slower rate of growth and then an eventual leveling off to a constant fixed population known as the **maximum sustainable population** or the **limit to growth**. (This type of horizontal asymptote is similar to what happens with the maintenance level for a drug discussed in Section 5.1.)

The logistic curve changes from concave up to concave down, so it has one point of inflection. We know that one characteristic of a point of inflection is that the function is growing most rapidly or decreasing most rapidly there. To determine roughly where the point of inflection occurs, we can examine the table to estimate where the largest increase in the population occurs by looking at the differences in successive values of the population. In the preceding table the population has its largest increase during the 11th month when it jumps from 1398 to 2014, an increase of 616.

## EXAMPLE 2

In illustrating the logistic model, we used $a = 1$ and $b = 0.0004$. How does the behavior of the solution sequence change if we use $a = 1$ and $b = 0.000032$ instead?

***Solution*** The corresponding results (rounded to the nearest integer) based on the logistic difference equation $P_{n+1} = 2P_n - 0.000032P_n^2$ over the first 20 months are shown in the following table.

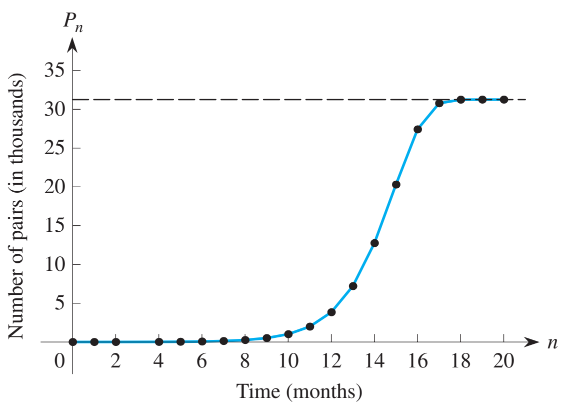



FIGURE 5.20

| Month | Number of Pairs | Month | Number of Pairs |
|---|---|---|---|
| 0 | 1 | 11 | 1,982 |
| 1 | 2 | 12 | 3,839 |
| 2 | 4 | 13 | 7,206 |
| 3 | 8 | 14 | 12,751 |
| 4 | 16 | 15 | 20,299 |
| 5 | 32 | 16 | 27,412 |
| 6 | 64 | 17 | 30,779 |
| 7 | 128 | 18 | 31,243 |
| 8 | 255 | 19 | 31,250 |
| 9 | 508 | 20 | 31,250 |
| 10 | 1,007 | | |

The resulting pattern, shown in Figure 5.20, has the same logistic shape as that shown in Figure 5.19. In fact, the first few population values are the same as in the previous table because we start with the same initial population $P_0 = 1$, the initial growth rate $a$ is the same, and the inhibiting term has minimal impact while the population is small. Also, because $b$ is now smaller than before, the population grows at a faster rate for a longer time and so the population now has a higher maximum sustainable level—31,250—than in Example 1. Further, the point of inflection now occurs during the 14th month when the population grows from 12,751 to 20,299.

## The Maximum Sustainable Population

To understand the behavior of the solution of the logistic difference equation, let's choose a constant $b$ that is much smaller—say, by a factor of 1/1000—than $a$. So long as the population $P_n$ remains relatively small, the inhibiting term, $-bP_n^2$, in the logistic difference equation

$$\Delta P_n = aP_n - bP_n^2$$

is negligible compared to the exponential term, $aP_n$. Therefore, initially, the equation is essentially equivalent to the difference equation

$$\Delta P_n = aP_n$$

for exponential growth. As $P_n$ grows larger, the inhibiting term $-bP_n^2$ becomes larger at a faster rate and so has an ever-greater impact on reducing the value of $\Delta P_n$. Hence the change in $P_n$ begins to decrease. Of course, the fact that the change decreases does not necessarily mean that $P_n$ itself gets smaller; it simply doesn't grow as fast. Moreover, the values for $P_n$ eventually approach a horizontal limit, with no further population growth.

Now let's find a formula for the maximum sustainable population. At this maximum level, there should be no change in $P_n$, so that the difference between successive terms, $\Delta P_n = P_{n+1} - P_n$, should be zero. Therefore we set the right-hand side of the logistic equation $\Delta P_n = aP_n - bP_n^2$ to 0 and obtain

$$aP_n - bP_n^2 = P_n(a - bP_n) = 0.$$

Because $P_n \neq 0$, we must have $a - bP_n = 0$, so that

$$P_n = \frac{a}{b}.$$

> The maximum sustainable population occurs at the level of $P = a/b = L$.

This ratio represents the maximum possible population and is the value, or height, of the horizontal asymptote where the population stabilizes forever. In Example 1 we used $a = 1$ and $b = 0.0004$ so that $L = a/b = 2500$. In Example 2 we used $a = 1$ and $b = 0.000032$ so that $L = a/b = 31{,}250$. These values agree with what we got numerically.

What if the original population $P_0$ is larger than the maximum sustainable population? For instance, in Example 1, if the initial population were greater than 2500, what would the model predict? In general, for any $n$, suppose that $P_n > L = a/b$. When we multiply both sides of the inequality $P_n > a/b$ by the positive constant $b$, we get $bP_n > a$ so that $a - bP_n < 0$. Because $P_n > 0$,

$$\Delta P_n = aP_n - bP_n^2 = (a - bP_n)P_n < 0$$

and the population is decreasing. Thus the phrase *maximum sustainable population* means just what it says: There are too many organisms for the environment to support, and the population will decline due to starvation, predators, and other inhibiting factors. The graph for this scenario is shown in Figure 5.21. You may want to explore this situation further, using a graphing calculator or computer program.

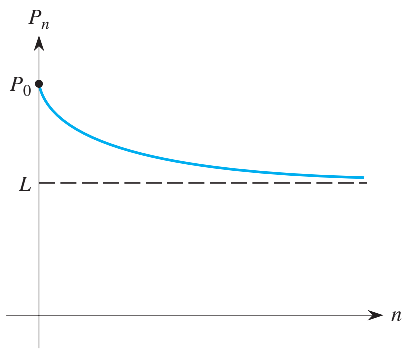



FIGURE 5.21

## The Point of Inflection

We next locate the point of inflection, at which the logistic curve changes concavity. It represents the point at which the population is growing most rapidly, so we want to find where the change in the population, $\Delta P_n = P_{n+1} - P_n$, is greatest, as shown in Figure 5.22. To the left of this point, the logistic curve is concave up, so not only is the population increasing, but it is also increasing at an increasing rate $[\Delta(\Delta P_n) > 0]$. To the right of this point, the logistic curve is concave down, so the population is increasing at a decreasing rate $[\Delta(\Delta P_n) < 0]$. At the point of inflection, the population is growing most rapidly, so $\Delta P_n$ must be a maximum. Biologically, the most vigorous growth in a population occurs just before decline sets in.

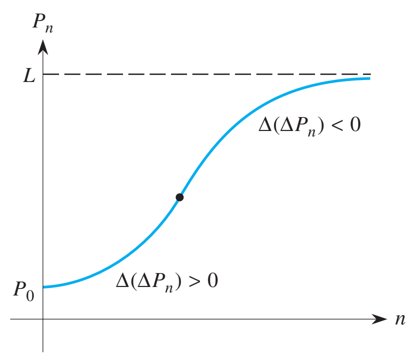



FIGURE 5.22

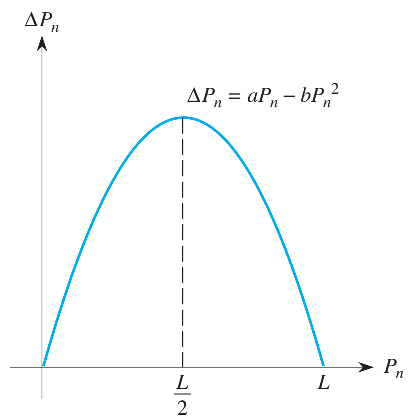



FIGURE 5.23

Let's now find a formula for the location of the point of inflection for the solution sequence of the logistic difference equation,

$$\Delta P_n = aP_n - bP_n^2.$$

Although we typically think of $P_n$ as a function of time $n$, this equation also expresses the change, $\Delta P_n$, in the population as a quadratic function of the population size $P_n$, as shown in Figure 5.23. Note that the parabola opens downward because the leading coefficient, $-b$, is negative. Furthermore, the quadratic equation

$$aP_n - bP_n^2 = P_n(a - bP_n) = 0$$

has two real roots, one when $P_n = 0$ and the other when $P_n = L = a/b$, which is the limit to growth. At both of these extremes, $\Delta P_n$ is zero and the population isn't growing. Moreover, because a parabola is symmetric about its vertex, the maximum value for $\Delta P_n$ occurs at the midpoint of this interval, which is when $P_n$ is half of $L$, or $\frac{1}{2}(a/b)$. This maximum value for $\Delta P_n$ corresponds to the point of inflection.

The point of inflection occurs at the level of $P = \frac{1}{2}(a/b) = \frac{1}{2}L$.

**EXAMPLE 3**

Find the point of inflection for the rabbit population in Example 2. How does it match what we observed by comparing values in the table?

***Solution*** In Example 2, based on $a = 1$ and $b = 0.000032$, the maximum sustainable population was $L = 31{,}250$, so the point of inflection must occur at a height of $\frac{1}{2}L = 15{,}625$. We previously observed that the point of inflection occurred during the 14th month when the population grew most rapidly, jumping from 12,751 to 20,299. Hence the value obtained from the formula agrees well with our previous observation on the data.

The solution sequence of any difference equation consists of a discrete set of numbers, so it isn't reasonable to expect that any one of them will precisely equal $\frac{1}{2}L$. Thus the best we can usually do is to estimate the location of the point of inflection from a table of data values.

We can summarize the logistic growth model as follows.

## Summary of the Logistic Growth Model

The logistic difference equation is

$$\Delta P_n = aP_n - bP_n^2 \quad \text{or} \quad P_{n+1} = (1 + a)P_n - bP_n^2,$$

where $a$ is the initial growth rate and $b$ is the inhibiting constant.

The maximum sustainable population is $L = \frac{a}{b}$.

The point of inflection occurs when

$$P_n = \frac{1}{2}L = \frac{1}{2}\left(\frac{a}{b}\right).$$

We've looked at only one specific application of the logistic model concerned with population growth for a single species. The same mathematical model applies to most other species—only the values of the constants $a$ and $b$ change.

**EXAMPLE 4**

A bacterial culture grows according to the logistic model with $a = 0.4$ and $b = 0.00008$. If there are initially 500 bacteria in the culture, find (**a**) the number present for $n = 1, 2, \ldots, 6$, (**b**) the limiting population for the culture, and (**c**) the location of the point of inflection.

***Solution***

a. Because the bacterial culture satisfies the logistic model, we know that

$$\begin{aligned} P_{n+1} &= (1 + a)P_n - bP_n^2 \\ &= 1.4P_n - 0.00008P_n^2, \end{aligned}$$

starting with $P_0 = 500$. Therefore we find successively that

$$\begin{aligned} P_1 &= 1.4(500) - 0.00008(500)^2 = 700 - 20 = 680; \\ P_2 &= 1.4(680) - 0.00008(680)^2 = 952 - 37 = 915; \\ P_3 &= 1.4(915) - 0.00008(915)^2 = 1214; \\ P_4 &= 1.4(1214) - 0.00008(1214)^2 = 1582; \\ P_5 &= 1.4(1582) - 0.00008(1582)^2 = 2015; \\ P_6 &= 1.4(2015) - 0.00008(2015)^2 = 2496. \end{aligned}$$

We rounded each successive entry to the nearest whole number because we're dealing with the number of bacteria in the culture.

**b.** The maximum sustainable population is

$$\frac{a}{b} = \frac{0.4}{0.00008} = 5000.$$

**c.** The inflection point occurs at

$$\frac{1}{2}\left(\frac{a}{b}\right) = \frac{1}{2}(5000) = 2500,$$

or when the population passes the 2500 level, as depicted in Figure 5.24. Note that the first six terms of the sequence grew to almost one-half the maximum value. However, considerably more than another six terms will be required to get close to the limiting value. For example, $P_{12} = 4680$ and $P_{18} = 4984$.

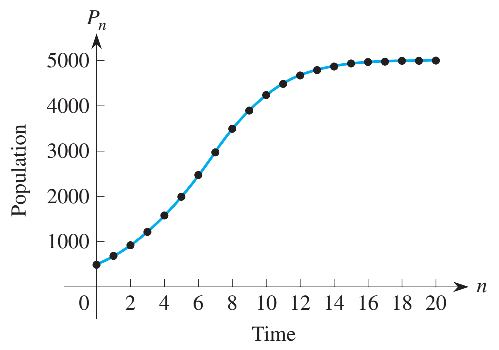



FIGURE 5.24

This type of mathematical modeling is the foundation for most of the projections of limits on world population growth. Moreover, because the growth factor $1 - a$ contains the exponential growth rate $a$, you might also want to interpret some of these ideas in the context of the values given in the population table in Appendix G. In particular, you might explore the results of using some of the growth rates shown for different countries and assume different values for the inhibiting constant $b$ to see the effects on the limits to growth. Remember, though, that $b$ should be much smaller than $a$. We discuss how to estimate values for $a$ and $b$, based on actual population data, later in this section.

## Behavior of the Logistic Function

Let's now examine the behavior of the solution to the logistic difference equation in greater detail. Suppose that the maximum sustainable population for an environment is $L = 1000$. The point of inflection then occurs at the level where $P_n = 500$.

Case 1: If the initial population $P_0$ is less than 500, the population will begin growing in a concave up manner. When it passes a height of 500, the concavity changes and the population continues to grow, but in a concave down manner as it approaches its horizontal asymptote at the level of 1000, as shown in Figure 5.25.

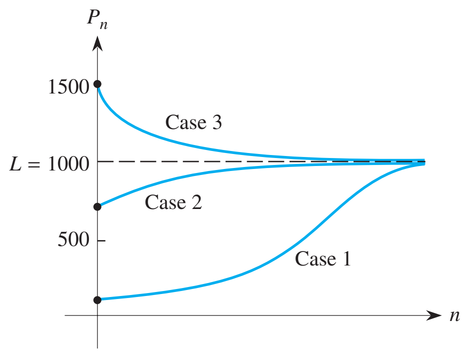



FIGURE 5.25

Case 2: If the initial population $P_0$ is between 500 and 1000, it starts above the point of inflection. The logistic curve grows toward the limit to growth, 1000, in a concave down manner, as shown in Figure 5.25.

Case 3: Suppose that the initial population $P_0$ is greater than the maximum sustainable level of 1000. This can occur if there is a sudden influx of immigrants, an unexpected baby boom, or a change in conditions such as a drought or famine that reduces the maximum sustainable level. The resulting logistic curve starts above the limiting value of 1000 and decreases toward 1000 in a concave up manner, as shown in Figure 5.25.

Case 4: Finally, if the initial population value $P_0$ precisely equals the limiting value $L$, all subsequent values remain the same. To see this, we consider the logistic difference equation

$$\Delta P_n = aP_n - bP_n^2$$

when $P_n = L = a/b$. Then

$$\Delta P_n = a \cdot \left(\frac{a}{b}\right) - b \cdot \left(\frac{a}{b}\right)^2 = \frac{a^2}{b} - b \cdot \left(\frac{a^2}{b^2}\right) = \frac{a^2}{b} - \frac{a^2}{b} = 0.$$

There is no change, so the population remains constant thereafter. For this reason, the limiting value is also considered the *equilibrium* because the population remains balanced at that level. (The maintenance level for a drug is also an equilibrium.)

Although we typically think of a logistic curve as the S-shaped curve corresponding to initial population values below the inflection point, the other cases are also logistic curves because they are solutions to the logistic difference equation.

More generally, suppose that we have the logistic model

$$\Delta P_n = aP_n - bP_n^2 = (a - bP_n)P_n,$$

with limiting value $L = a/b$ and point of inflection at height $\frac{1}{2}L$. If we factor out the coefficient $b$ from the right-hand side of the difference equation, we get

$$\Delta P_n = b \cdot \left(\frac{a}{b} - P_n\right)P_n = b(L - P_n)P_n.$$

Now suppose that, for some value of $n$, $P_n > L$; then $L - P_n < 0$. Therefore $\Delta P_n = b(L - P_n)P_n < 0$. Because $\Delta P_n = P_{n+1} - P_n$, we see that $P_{n+1}$ must be smaller than $P_n$. That is, the solution must be decreasing at such a point. We illustrate this behavior in the top region shown in Figure 5.26 where the arrows are all pointing downward to indicate the direction for the solutions.

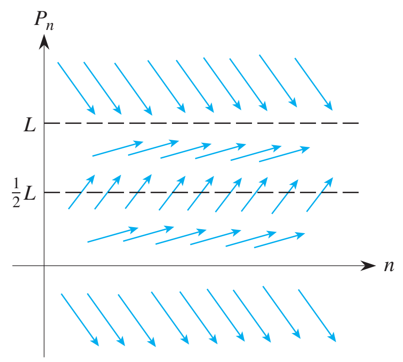



FIGURE 5.26

Next, suppose that, for any value of $n$, $0 < P_n < L$, so $L - P_n > 0$. Therefore $\Delta P_n = b(L - P_n)P_n > 0$ and so $P_{n+1}$ must be larger than $P_n$. That is, the solution must be increasing. We show this behavior in the two middle regions of Figure 5.26 where the arrows are all pointing upward to indicate the direction of the solutions. However, the slopes of the arrows crossing the level for the inflection point are steeper than those below or above that level because the logistic curve increases most rapidly at its inflection point.

Finally, although this case is not realistic as a model for population growth, suppose that $P_n < 0$ for some $n$. Then $L - P_n > 0$ and therefore $\Delta P_n = b(L - P_n)P_n < 0$. Consequently, $P_{n+1}$ must be smaller than $P_n$ and the solution must be decreasing. We show this behavior in the bottom region of Figure 5.26 where all the arrows are pointing downward.

The cases we consider here with the logistic difference equation model are somewhat simplistic. For instance, if the values for $a$ and $b$ are larger than the ones we have used, the solution can overshoot the equilibrium level $L$ for some value of $n$. However, once there, the succeeding term must decrease toward (or actually even past) the equilibrium. Thus it is possible to obtain a solution that oscillates above and below the equilibrium level while still converging to it. In many ways, this scenario is more realistic than the ideal one we portrayed, in which the solution strictly increases toward $L$. A real population is likely to grow beyond its limits, then decrease below the maximum sustainable level, and oscillate in this manner thereafter.

## Applications of Logistic Growth

The mathematical ideas used in the logistic model for population growth can be applied to many other situations, as Examples 5 and 6 illustrate.

### EXAMPLE 5

Suppose that 4000 people are in a university dormitory complex when one student starts a rumor at 9 A.M. Saturday morning. If each person who hears the rumor passes it on to two other people every hour, how long will it be until everyone has heard it?

***Solution*** At first thought, this situation might seem like an exponential growth model because, seemingly, the number of people who have heard the rumor triples every hour.

In practice, the number of people who hear the rumor does grow roughly exponentially for a while, but eventually the people who are passing the rumor on will find it difficult, if not impossible, to encounter two people who haven't yet heard it.

We must change the underlying exponential formulation to introduce an inhibiting term. The number of new people who hear the rumor after $n + 1$ hours (i.e., the change in the number who have heard it during the past hour) depends not just on how many people $P_n$ have already heard it, but also on the number of people left of the original 4000 people, $(4000 - P_n)$, who haven't heard it. That is, $\Delta P_n$, the change in $P_n$, is proportional to both $P_n$ and $(4000 - P_n)$, which leads to the difference equation

$$P_{n+1} - P_n = mP_n \cdot (4000 - P_n) = 4000mP_n - mP_n^2,$$

where $m$ is the constant of proportionality. Adding $P_n$ to both sides of the equation, we get

$$P_{n+1} = (1 + 4000m)P_n - mP_n^2 = (1 + a)P_n - bP_n^2,$$

where we have written $a = 4000m$ and $b = m$ to emphasize that this is a logistic model. Initially, $P_0 = 1$ and, because each person tells two new people, $a = 2$ and $1 + a = 3$. The limiting value is $L = a/b = 4000m/m = 4000$, so

$$b = \frac{a}{L} = \frac{2}{4000} = 0.0005.$$

The resulting difference equation is

$$P_{n+1} = 3P_n - 0.0005P_n^2.$$

The following table shows the number of people (rounded to the nearest whole person) who have heard the rumor after each hour, based on this logistic model. The graph of these solution points is shown in Figure 5.27. Thus we conclude that the 4000 students will have heard the rumor in about 9 hours.

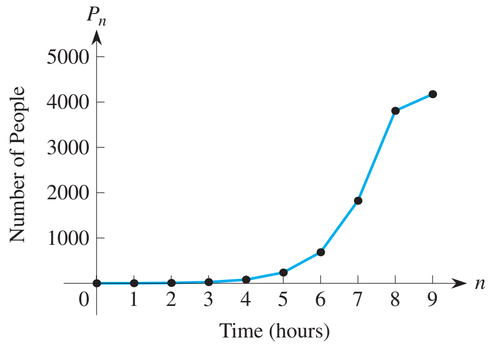



FIGURE 5.27

| Time | Number |
|---|---|
| 0 | 1 |
| 1 | 3 |
| 2 | 9 |
| 3 | 27 |
| 4 | 80 |
| 5 | 238 |
| 6 | 686 |
| 7 | 1823 |
| 8 | 3807 |
| 9 | 4174 |

The final value displayed, 4174 when $n = 9$, actually overshot the limiting value of 4000. As we said before, once a value of a logistic sequence exceeds $L$, the following value determined by the difference equation will be smaller because $\Delta P_n < 0$ and, in fact, will usually overshoot downward to produce a value below $L$.

Think About This

Continue the preceding process repeatedly to verify that the successive values oscillate above and below the limiting value 4000 and eventually converge to it as $n$ increases.

One problem with the logistic difference equation model is that no one has ever been able to discover a closed form solution for $P_n$. However, the fact that no explicit solution is known isn't a major handicap. Other than the aesthetic pleasure of having the solution expressed as a formula, a closed form solution wouldn't likely contribute much to this model. The formula would be used primarily to calculate the values for the population at each time. However, the logistic difference equation itself allows us to calculate the values for the solution recursively by using the previous value at each time. There is a slightly different way to create a continuous logistic model using calculus. That model can be solved to give a closed form solution for a logistic curve of the form

$$f(t) = \frac{C}{1 + Ae^{-Bt}},$$

where $A$, $B$, and $C$ are positive constants and $e = 2.71828\ldots$ is the base of the natural logarithm system, as we discussed in Section 4.9.

Applications of the logistic growth model include the spread of technological innovations—how fast a new product penetrates the marketplace. For example, when vacuum cleaners were first introduced, they were purchased by relatively few people. Based on word of mouth and advertising, more and more people bought them and the number in use grew rapidly. Eventually, the market for vacuum cleaners became saturated as virtually every household had one, so that the rate of growth diminished, and new sales were basically for replacements. The same logistic pattern applies to items such as televisions and stereos. Newer products such as home computers, VCRs, CD players, DVD players, and cellular phones are now at various stages of the logistic growth process.

Wal-Mart reportedly uses this application of logistic growth to maintain its profit levels. It tracks the sales of lines of merchandise, say, a particular style of jeans. Once sales of that item pass the point of inflection on the logistic curve, Wal-Mart stops ordering that item and orders a different one instead. Thus the company only stocks and sells an item while it is "hot." Hence few, if any, items are sold at clearance prices, which would substantially reduce Wal-Mart's profits.

## EXAMPLE 6

The first diesel locomotive was put into service in the United States in 1925. In the early years, the use of diesels among the 25 major railroads then in existence grew at about 25% per year. It took about 30 years for all the railroads to use diesel locomotives. Construct a logistic model for the spread of their use.

***Solution*** Because the early annual growth rate was 25%, we assume that $a = 0.25$ so that $1 + a = 1.25$. Further, because the limiting value must be $L = 25$ railroads,

$$L = 25 = \frac{a}{b}.$$

We find that

$$b = \frac{a}{L} = \frac{0.25}{25} = 0.01.$$

Thus our logistic model is the difference equation

$$R_{n+1} = 1.25R_n - 0.01R_n^2.$$

We begin in 1925 when $n = 0$ and $R_0 = 1$ for the one railroad using a diesel locomotive. We then use the difference equation to obtain the values shown in the table and in Figure 5.28.

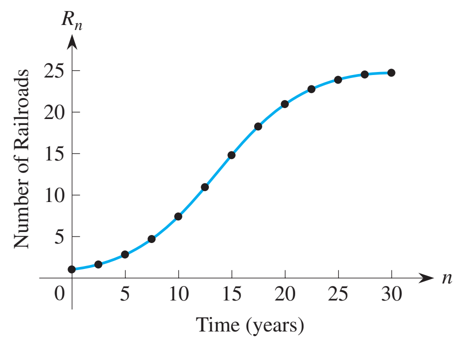



FIGURE 5.28

| Year | Number of Diesels | |
|---|---|---|
| 1 | 1.24 | $=1.25(1) - 0.01(1)^2$ |
| 2 | 1.53 | $=1.25(1.24) - 0.01(1.24)^2$ |
| 3 | 1.89 | $=1.25(1.53) - 0.01(1.53)^2$ |
| 4 | 2.33 | |
| 5 | 2.86 | |
| 10 | 7.31 | |
| ⋮ | ⋮ | |
| 15 | 14.62 | |
| ⋮ | ⋮ | |
| 20 | 21.03 | |
| ⋮ | ⋮ | |
| 25 | 23.88 | |
| ⋮ | ⋮ | |
| 30 | 24.72 | |

The model agrees with the historical fact that it took about 30 years after 1925 (or until about 1955) for all 25 of the major railroads to introduce diesels. Incidentally, the other values from the model match the actual spread in the use of diesels quite well. Incidentally, we didn't round the values calculated to the nearest integer because the numbers involved were so small, even though obviously a fraction of a diesel locomotive makes no sense.

## Estimating $a$ and $b$

Suppose that a set of data points fall in a pattern such as the one shown in Figure 5.29 and that we want to fit a logistic curve to it. The hardest part of constructing a logistic model is determining appropriate values for the growth rate $a$ and the inhibiting constant $b$. To do so, we transform the data so that the resulting points are in a linear pattern, analogous to what we did in Chapter 3.

Suppose that a set of data points $(0, P_0)$, $(1, P_1)$, $(2, P_2)$, . . . appear to follow a logistic pattern so that $P_n$ satisfies the difference equation

$$\Delta P_n = aP_n - bP_n^2 = (a - bP_n)P_n.$$

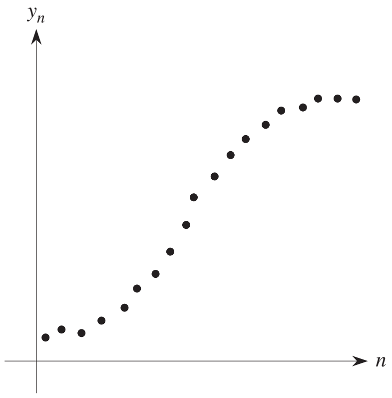



FIGURE 5.29

Dividing both sides by $P_n$, we obtain

$$\frac{\Delta P_n}{P_n} = a - bP_n.$$

Therefore, if $P_n$ follows a logistic pattern, the quantity $(\Delta P_n)/P_n$ is actually a linear function of $P_n$. We plot $(\Delta P_n)/P_n$ against $P_n$ (not against $n$) and find that it falls in a roughly linear pattern, as shown in Figure 5.30. Then we can find the best linear fit to the transformed data. Note that the pattern shown is decreasing, which is the usual case, so the slope of the regression line will be negative. The corresponding regression equation is $Y = a - bX$, which is equivalent to

$$\frac{\Delta P_n}{P_n} = a - bP_n.$$

When we multiply both sides by $P_n$, we get the logistic difference equation

$$\Delta P_n = aP_n - bP_n^2.$$

Once we have this equation, we have our estimates for the logistic coefficients $a$ and $b$. We illustrate how to apply these ideas in Examples 7 and 8.

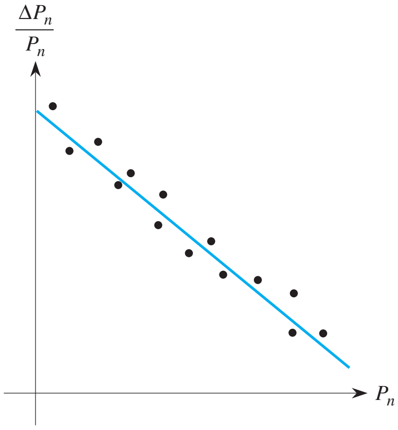



FIGURE 5.30

## EXAMPLE 7

German biologist R. Carlson studied the growth of yeast under controlled conditions and obtained the following set of measurements for the weight of yeast, in milligrams, as a function of time, $n$.

Estimate the parameters $a$ and $b$ to create a logistic model to fit the growth of the yeast population.

| $n$ | $P_n$ |
|---|---|
| 1 | 9.6 |
| 2 | 29.0 |
| 3 | 71.1 |
| 4 | 174.6 |
| 5 | 350.7 |
| 6 | 513.3 |
| 7 | 594.4 |
| 8 | 640.0 |
| 9 | 655.9 |
| 10 | 661.8 |

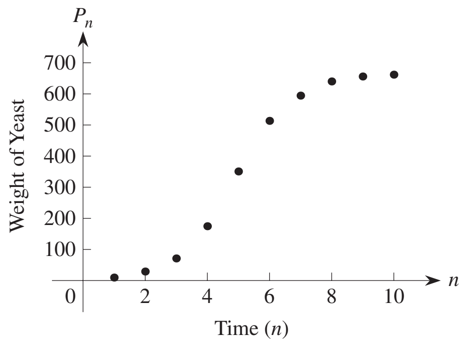



FIGURE 5.31

***Solution*** By inspecting either the entries in the table or the corresponding scatterplot of the points shown in Figure 5.31, we see that they appear to fall in a logistic pattern. To linearize the data, we add two extra columns to the table, one for the differences $\Delta P_n$ of the successive terms and the other for the ratio $(\Delta P_n)/P_n$.

Figure 5.32 shows the plot of the values of $(\Delta P_n)/P_n$ versus $P_n$ (not $n$). They seem to fall in a roughly linear pattern with negative slope. Using the linear regression routine on a calculator, we find the corresponding regression equation:

$$\frac{\Delta P_n}{P_n} = 1.66 - 0.00271P_n.$$

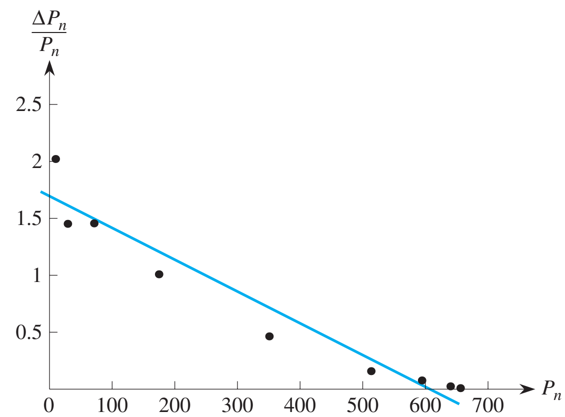



FIGURE 5.32

| $n$ | $P_n$ | $\Delta P_n$ | $(\Delta P_n)/P_n$ | |
|---|---|---|---|---|
| 1 | 9.6 | 19.4 | 2.021 | (=19.4/9.6) |
| 2 | 29.0 | 42.1 | 1.452 | |
| 3 | 71.1 | 103.5 | 1.456 | |
| 4 | 174.6 | 176.1 | 1.009 | |
| 5 | 350.7 | 162.6 | 0.464 | |
| 6 | 513.3 | 81.1 | 0.158 | |
| 7 | 594.4 | 45.6 | 0.077 | |
| 8 | 640.0 | 15.9 | 0.025 | |
| 9 | 655.9 | 5.9 | 0.009 | |
| 10 | 661.8 | | | |

The corresponding correlation coefficient is $r = -0.967$, which suggests a very high degree of negative correlation between $(\Delta P_n)/P_n$ and $P_n$. Multiplying both sides of the regression equation by $P_n$ yields the logistic difference equation

$$\Delta P_n = 1.66P_n - 0.00271P_n^{\,2},$$

giving $a = 1.66$ and $b = 0.00271$.

The sequence of points based on this difference equation superimposed over the original data is shown in Figure 5.33. Note that the curve connecting the points from the logistic model fits the points of the original data reasonably well, especially the first five points. However, the values of the logistic model level out somewhat below the limiting value for the actual yeast population data.

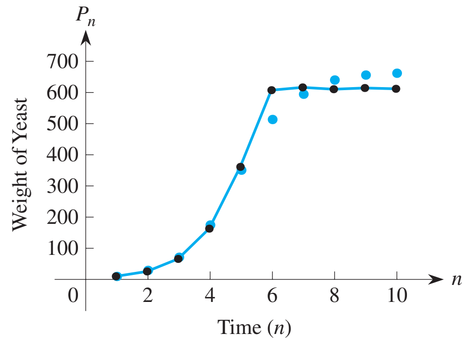



FIGURE 5.33

In Example 1 of Section 3.3, we found that the growth in the U.S. population from 1780 to 1900 closely followed an exponential growth pattern with a growth rate of 32.1% per decade. However, we pointed out that this exponential pattern didn't apply during the twentieth century and, in fact, the rate of growth has slowed considerably since 1900, suggesting a logistic model. In Example 8 we examine the growth in the U.S. population over the entire period since 1780.

## EXAMPLE 8

Find a logistic model that fits the data on the U.S. population since 1780 and find the limiting value for the population using this logistic model.

***Solution*** The data on the U.S. population are presented in the table on the next page and in the scatterplot shown in Figure 5.34.

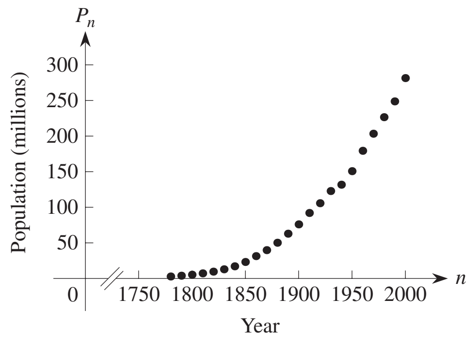



FIGURE 5.34

Both the table and the scatterplot show that the rate of population growth slowed considerably during the twentieth century. Successive ratios also slowly decreased, indicating that the rate of population growth slowed from over 20% per decade at the start of the century to slightly over 10% per decade by the end of the century.

| Year | Population | Ratio |
|---|---|---|
| 1780 | 2.8 | |
| | | 1.39 |
| 1790 | 3.9 | |
| | | 1.36 |
| 1800 | 5.3 | |
| | | 1.36 |
| 1810 | 7.2 | |
| | | 1.33 |
| 1820 | 9.6 | |
| | | 1.34 |
| 1830 | 12.9 | |
| | | 1.33 |
| 1840 | 17.1 | |
| | | 1.36 |
| 1850 | 23.2 | |
| | | 1.35 |
| 1860 | 31.4 | |
| | | 1.27 |
| 1870 | 39.8 | |
| | | 1.26 |
| 1880 | 50.2 | |
| | | 1.25 |
| 1890 | 62.9 | |
| | | 1.21 |
| 1900 | 76.0 | |
| | | 1.21 |
| 1910 | 92.0 | |
| | | 1.15 |
| 1920 | 105.7 | |
| | | 1.16 |
| 1930 | 122.8 | |
| | | 1.07 |
| 1940 | 131.7 | |
| | | 1.14 |
| 1950 | 150.7 | |
| | | 1.19 |
| 1960 | 179.3 | |
| | | 1.13 |
| 1970 | 203.3 | |
| | | 1.11 |
| 1980 | 226.5 | |
| | | 1.10 |
| 1990 | 248.7 | |
| | | 1.13 |
| 2000 | 281.4 | |

To fit a logistic curve to this data, we must first calculate the differences $\Delta P_n = P_{n+1} - P_n$ and then the ratios

$$\frac{\Delta P_n}{P_n} = \frac{P_{n+1} - P_n}{P_n},$$

the results of which we show in the table on the next page.

When we plot these transformed values $(\Delta P_n)/P_n$ against $P_n$, as shown in Figure 5.35, we see that they fall in a predominantly decreasing, roughly linear pattern. However, there does seem to be a fair amount of variation about the regression line. Nevertheless, the corresponding plot of the residuals, as shown in Figure 5.36, indicates that the fit is reasonably accurate (the residuals are small and roughly half are below and roughly half are above the baseline), though there does seem to be a pattern to the residuals. This pattern suggests that the logistic model may not fully explain all the variation in the data. The corresponding correlation coefficient, $r = -0.8803$, represents a high degree of negative correlation for the 22 data pairs. Incidentally, we expect a negative correlation between the transformed values and the actual population values because the slope of the regression line is negative. The equation of the regression line for the transformed data is equivalent to

$$\frac{\Delta P_n}{P_n} = 0.3314 - 0.0011P_n.$$

Multiplying both sides of this equation by $P_n$ gives us the logistic model

$$\Delta P_n = 0.3314P_n - 0.0011P_n^2,$$

| Year | $P_n$ | $\Delta P_n$ | $(\Delta P_n)/P_n$ |
|---|---|---|---|
| 1780 | 2.8 | 1.1 | 0.393 (=1.1/2.8) |
| 1790 | 3.9 | 1.4 | 0.359 |
| 1800 | 5.3 | 1.9 | 0.358 |
| 1810 | 7.2 | 2.4 | 0.333 |
| 1820 | 9.6 | 3.3 | 0.344 |
| 1830 | 12.9 | 4.2 | 0.326 |
| 1840 | 17.1 | 6.1 | 0.357 |
| 1850 | 23.2 | 8.2 | 0.353 |
| 1860 | 31.4 | 8.4 | 0.268 |
| 1870 | 39.8 | 10.4 | 0.261 |
| 1880 | 50.2 | 12.7 | 0.253 |
| 1890 | 62.9 | 13.1 | 0.208 |
| 1900 | 76.0 | 16.0 | 0.211 |
| 1910 | 92.0 | 13.7 | 0.149 |
| 1920 | 105.7 | 17.1 | 0.162 |
| 1930 | 122.8 | 8.9 | 0.072 |
| 1940 | 131.7 | 19.0 | 0.144 |
| 1950 | 150.7 | 28.6 | 0.190 |
| 1960 | 179.3 | 24.0 | 0.134 |
| 1970 | 203.3 | 23.2 | 0.114 |
| 1980 | 226.5 | 22.2 | 0.098 |
| 1990 | 248.7 | 32.7 | 0.131 |
| 2000 | 281.4 | | |

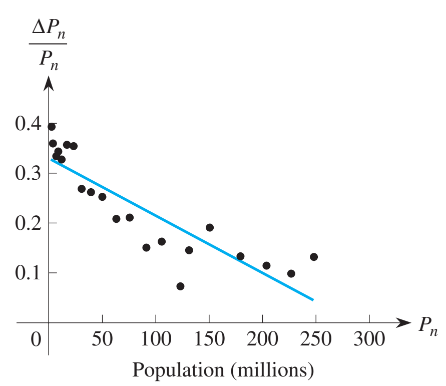



FIGURE 5.35

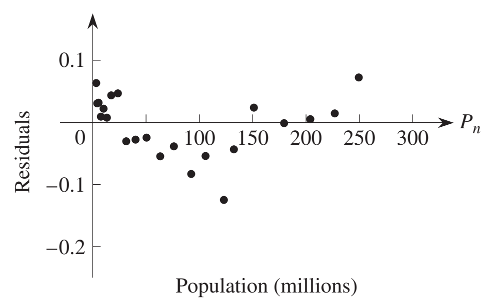



FIGURE 5.36

where $a = 0.3314$ and $b = 0.0011$. Figure 5.37 shows the values predicted by this logistic model, starting from $P_0 = 2.8$; they are the points connected by the smooth curve. For comparison, the original data points for the U.S. population are also shown. The model that we constructed seems to be an excellent fit until about 1930, but the accuracy then seems to break down.

Based on this logistic model, the limiting population for the United States is

$$L = \frac{a}{b} = \frac{0.3314}{0.0011}$$
$$\approx 301.3 \text{ million.}$$

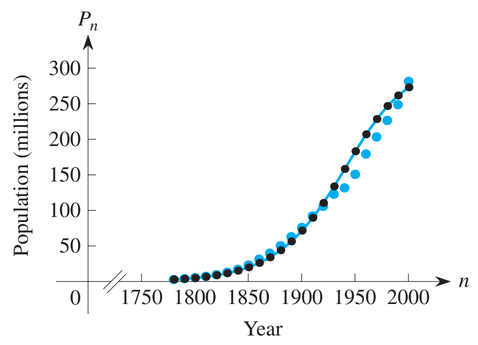



FIGURE 5.37

This prediction is unrealistically low because the U.S. population was fairly close to this level in 2000 and was growing at a rate of about 20 or 30 million people per decade—it isn't reasonable to expect that population growth will slow that much over the next decade. However, recall that, based on the residual plot, we pointed out that the logistic model doesn't seem to account for all the variation in the data. It provides a relatively good fit to the growth of the population, but it isn't an outstanding fit and perhaps a more sophisticated model is needed to give a better match to the population data.

## Problems

Calculate the first 10 values predicted for the logistic solution subject to each set of values and plot the points. What is the limiting value for the population? How close do you come to it during the first five time periods?

1. $a = 0.02, \quad b = 0.0005, \quad P_0 = 10$
2. $a = 0.02, \quad b = 0.0005, \quad P_0 = 3$
3. $a = 0.02, \quad b = 0.0001, \quad P_0 = 5$
4. $a = 0.02, \quad b = 0.002, \quad P_0 = 5$
5. A population grows according to the logistic growth model $\Delta P_n = 0.05P_n - 0.00002P_n^2$. Sketch the behavior of the population if (**a**) $P_0 = 500$, (**b**) $P_0 = 1500$, and (**c**) $P_0 = 3500$.
6. Consider again the rumor spreading through the dorms in Example 2. Suppose that each person repeats it to three new people each hour instead of two people. How long will it be until all 4000 students have heard it?
7. Suppose that a certain population grows according to the logistic model from an initial size of 100 to a final size of 1000.
   - **a.** Sketch the graph of the population as a function of time.

     Use the concavity of your graph from part (a) to answer the following questions.
   - **b.** Suppose that the population is 250 after 10 time periods and 300 after 12 time periods. Use this information to estimate the size of the population after 11 time periods. Is the actual value higher or lower than your estimate? How do you know?
   - **c.** Suppose that the population is 900 after 30 time periods and 910 after 31 time periods. Use this information to estimate the population after 35 time periods. Is the actual value higher or lower than your estimate? How do you know?
8. Suppose that the deer population in a wildlife refuge follows a logistic growth pattern with an annual growth rate of 40% and an inhibiting rate of 0.02%. Write difference equations to model each situation.
   - **a.** The deer population grows according to the logistic model.
   - **b.** The population grows according to the logistic model, but hunters are allowed to eliminate 120 deer from the region each year.
   - **c.** The logistic model applies, and hunters eliminate 30% of the deer in the region each year.
   - **d.** The logistic model applies, and the number of deer that hunters eliminate in the region each year is proportional to the square root of the number of deer living there.
   - **e.** The logistic model applies, hunters eliminate 40% of the deer, and the state's wildlife department moves 75 new deer into the area each year.

**9.** The growth of a population follows a logistic trend. The limiting value seems to be about 16,000 and the first few values are 100, 105, 110.2, 115.17, and 121.5. Use this information to estimate both coefficients in the logistic equation and determine how closely the terms you calculate match these observed values.

**10.** The values for $a$ and $b$ in a logistic model are typically estimated based on a set of observations. As such, they are likely to be somewhat inaccurate. It is important to know how sensitive the results of the logistic model are to slight changes (or errors) in either $a$ or $b$. Suppose that you estimate $a = 0.05$ and $b = 0.00002$ so that $L = a/b = 2500$. What would be the effect on $L$ if $a$ were actually 10% larger than 0.05 (0.055 instead of 0.05) while $b$ remains fixed? 20% larger? 30% larger? 10% smaller? 20% smaller? How does the value of $L$ depend on the estimate for $a$ if $b$ remains fixed?

**11.** Repeat Problem 10 by considering the effect on $L$ of changes in $b$ if $a$ is fixed. In particular, what would be the effect on $L$ if $b$ were actually 10% larger than 0.00002? 20% larger? 30% larger? 10% smaller? 20% smaller? How does the value of $L$ depend on the estimate for $b$ if $a$ is fixed?

**12.** Based on your results in Problems 10 and 11, does the logistic model seem more sensitive to errors or changes in the estimates for $a$ or for $b$? In estimating values for these two parameters based on a set of data, which do you expect to be more accurate? Explain.

**13.** The population of a region can be modeled by the logistic growth model with coefficients $a$ and $b$. A major drought hits the region.

**a.** Which coefficient, $a$ or $b$, is more likely to change because of the drought? Does it become larger or smaller?

**b.** Depending on how close the population was to the maximum sustainable population level before the drought, write a short paragraph to describe the effects of the drought on the long-term behavior of the population.

**14.** Consider the growth model based on the difference equation

$$\Delta P_n = aP_n - bP_n^3,$$

where $b$ is smaller than $a$.

**a.** Sketch the graph of $\Delta P_n$ as a function of $P_n$, for $P_n \geq 0$.

**b.** Determine the maximum sustainable population $L$ in terms of $a$ and $b$ for a species modeled by this difference equation.

**c.** Use your graph from part (a) to determine the sign of $\Delta P_n$ if $P_n$ is between 0 and $L$. What does it tell you about the behavior of $P_n$? Explain.

**d.** Use your graph from part (a) to determine the sign of $\Delta P_n$ if $P_n$ is greater than $L$. What does it tell you about the behavior of $P_n$? Explain.

**e.** Use your graph from part (a) to show that the solution to this difference equation must have a point of inflection if $P_0$ is small enough.

**f.** The turning points of the general cubic curve $y = Ax^3 + Bx^2 + Cx + D$ occur at

$$x = \frac{-B \pm \sqrt{B^2 - 3AC}}{3A}.$$

Use this fact to determine the location of the point of inflection for the solution to this difference equation.

**g.** You know that the point of inflection for the logistic model occurs at half the height of the limiting value. For the model in this problem, does the point of inflection occur at a comparable, a higher, or a lower level? What does that tell you about the kind of behavior for a population that can be well-modeled by this difference equation?

**15.** Repeat parts (a)–(e) of Problem 14 for the difference equation model

$$\Delta P_n = aP_n - bP_n^4.$$

**16.** The table gives the average number of words (both spoken and understood) in the vocabulary of the typical child aged 1–6. Find the best fit to these data from among each of the logistic, linear, exponential, and power functions. From among the best fit in each family, which seems to be the overall best fit to the pattern in the data?

| **Age** | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 | 5.5 | 6 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Vocabulary** | 50 | 200 | 350 | 600 | 880 | 1200 | 1450 | 1800 | 2150 | 2425 | 2750 |

Source: B. A. Moskowitz, Acquisition of Language. *Scientific American,* 1978, vol 279, pp. 92–108.

**17.** The density of human bones increases through childhood and adolescence. By age 18, 95% of bone density has been achieved. The table shows the percentage of maximum bone density achieved at different ages. Find the best fit among each of the logistic, linear, exponential, and power functions for this data. From among the best fit in each family, which seems to be the overall best fit to the pattern in the data?

| **Age** | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| **Percentage of Bone Density** | 43 | 49 | 51 | 56 | 63 | 71 | 82 | 91 | 95 |

Source: Student project.

**18.** The table shows the accumulated total number of reported cases of AIDS in the United States since 1983.

**a.** Determine the logistic fit to this data.

**b.** Use the model to predict the number of AIDS cases in 1990.

**c.** If the trend is indeed logistic, find the total number of deaths from AIDS that the model predicts in the limit.

| **Year** | 1983 | 1984 | 1985 | 1986 | 1987 | 1988 | 1989 | 1990 |
|---|---|---|---|---|---|---|---|---|
| **Number of AIDS Cases** | 4589 | 10,750 | 22,399 | 41,256 | 69,592 | 104,644 | 146,574 | ? |
| **Year** | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 | 1997 | 1998 |
| **Number of AIDS Cases** | 251,638 | 326,648 | 399,613 | 457,280 | 528,144 | 594,641 | 653,084 | 701,353 |

Source: U.S. Centers for Disease Control and Prevention.

**19.** Biologists Reed and Holland studied the growth of sunflower plants. They measured the heights $H$ of a number of sunflowers on the same day of successive weeks $t$ and averaged the readings, which are given in centimeters in the table.

**a.** Estimate the maximum height to which these sunflowers will grow and the point at which the heights pass their point of inflection.

**b.** Estimate the parameters for a logistic difference equation to fit these data.

**c.** Use the difference equation to calculate the predicted heights for these sunflowers. How close do the predictions come to the actual values?

| **Week *t*** | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **Mean Height, *H*** | 17.9 | 36.4 | 67.8 | 98.1 | 131.0 | 169.5 |
| **Week *t*** | 7 | 8 | 9 | 10 | 11 | 12 |
| **Mean Height, *H*** | 205.5 | 228.3 | 247.1 | 250.5 | 253.8 | 254.5 |

Source: H. S. Reed and R. H. Holland, The Growth Rate of an Annual Plant *Helianthus*, Proceedings of the National Academy of Sciences, vol 5, 1919.

**20.** The table on the next page gives measurements on the weight $W$ in grams of a pumpkin on different days $t$ while it is growing.

**a.** Estimate the maximum weight that this pumpkin will reach and the point at which the weight passes its point of inflection.

**b.** Estimate the parameters for a logistic difference equation to fit this data.

**c.** Use the difference equation to calculate the predicted weight for this pumpkin. How close do the predictions come to the actual values?

| $t$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $W$ | 267 | 443 | 658 | 961 | 1498 | 2200 | 2920 | 3366 | 3758 | 4092 | 4488 |
| $t$ | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | |
| $W$ | 4720 | 4864 | 4980 | 5114 | 5176 | 5242 | 5298 | 5352 | 5360 | 5366 | |

Source: Raymond Pearl, *The Biology of Population Growth.* New York: Alfred A. Knopf, 1925.

**21.** When *Penicillium chrysogenum,* a strain of penicillin antibiotic, is grown in a batch fermentation process under carefully controlled conditions, the concentration of the cell as measured by its dry weight at various time intervals is given in the table at the bottom left of this page. Determine the coefficients for the logistic fit to the data.

**22.** From the two tables on the U.S. population in Example 8, compare the entries in the ratio column and the entries in the logistic transformation column $(\Delta P_n)/P_n$ when rounded to two decimal places. What interesting fact do you notice about the two columns of values? Explain how you can account for it mathematically.

**23.** In Example 7 on the yeast experiment, we constructed the best logistic model to fit the data based on the difference equation $\Delta P_n = 1.66P_n - 0.00271P_n^2$ or $P_{n+1} = 2.66P_n - 0.00271P_n^2$.

The initial measurement on the yeast was $P_1 = 9.6$. Use this value and the difference equation to construct a table showing the amount of yeast present over the first 10 time periods. Compare these predictions to the actual values measured in the experiment.

| $t$ | Concentration |
|---|---|
| 0 | 0.40 |
| 22 | 0.99 |
| 46 | 1.95 |
| 70 | 2.52 |
| 94 | 3.09 |
| 118 | 4.06 |
| 142 | 4.48 |
| 166 | 4.25 |
| 190 | 4.36 |

**24.** In Figure 5.32 on the yeast experiment data in Example 7, the scatterplot of $(\Delta P_n)/P_n$ versus $P_n$ may suggest a power function with a negative exponent rather than a linear function.

**a.** Using the transformed data, find the power function that best fits the data.

**b.** Detransform the results to obtain a first order, nonlinear difference equation relating $\Delta P_n$ to $P_n$.

**c.** Use the difference equation from part (b) together with the starting value of $P_0 = 9.6$ grams of yeast to calculate the amount of yeast present for $n = 1, 2, \ldots, 10$. Extend the table you constructed in Problem 23 to include these values. How close do these values come to matching the experimental data shown in Example 7.

**25.** The scatterplot shown in Figure 5.32 may also suggest a decaying exponential function. Repeat Problem 24, using the exponential function that best fits the transformed data.

**26.** The growth pattern in human height or weight development from birth through age 18, say, usually follows a logistic growth pattern. The table below gives the typical height, in centimeters, of a male and a female in the 50th percentile for height at different ages, in years. Use the data to construct a pair of logistic functions that model the heights of boys and girls as a function of age $t$ for people in this 50th percentile group.

| **Age** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Boys** | 50.5 | 76.1 | 87.6 | 96.5 | 102.9 | 109.9 | 116.1 | 125.0 | 127.0 | 132.2 |
| **Girls** | 49.9 | 74.3 | 86.5 | 95.6 | 101.6 | 108.4 | 114.6 | 120.6 | 126.4 | 132.2 |
| **Age** | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | |
| **Boys** | 137.5 | 143.3 | 149.7 | 156.5 | 163.1 | 169.0 | 173.5 | 176.2 | 176.8 | |
| **Girls** | 138.3 | 144.8 | 151.5 | 157.1 | 160.4 | 161.8 | 162.4 | 163.1 | 163.7 | |

Source: NCHS Growth Curves for Children. *Vital and Health Statistics, National Health Survey,* U.S. Department of Health, Education and Welfare.

**27.** The table below shows the worldwide electric generating capacity of nuclear power plants, measured in gigawatts, over time.

**a.** Estimate the parameters for the logistic model that fits these data.

**b.** What is the limiting value for the worldwide electric generating capacity of nuclear power plants?

**c.** From the table, estimate when the worldwide electric generating capacity of nuclear power plants was growing most rapidly.

**d.** From your model in part (a), when was the worldwide electric generating capacity of nuclear power plants growing most rapidly?

| **Year** | 1960 | 1965 | 1970 | 1975 | 1980 | 1985 | 1990 | 1995 | 2000 |
|---|---|---|---|---|---|---|---|---|---|
| **Electric Capacity** | 1 | 5 | 16 | 71 | 135 | 250 | 328 | 340 | 347 |

Source: Lester R. Brown et al., *Vital Signs 2000: The Environmental Trends That Are Shaping Our Future.*

**28.** The table shows the population of the world, in billions, over time. The population appears to be growing more slowly than it was only a few years ago, so it may have passed its point of inflection.

| **Year** | 1950 | 1955 | 1960 | 1965 | 1970 | 1975 |
|---|---|---|---|---|---|---|
| **Population** | 2.556 | 2.780 | 3.039 | 3.345 | 3.707 | 4.086 |
| **Year** | 1980 | 1985 | 1990 | 1995 | 1999 | |
| **Population** | 4.454 | 4.851 | 5.277 | 5.682 | 6 | |

Source: Lester R. Brown et al., *Vital Signs 2000: The Environmental Trends That Are Shaping Our Future.*

**a.** Estimate the parameters for the logistic model that fits these data.

**b.** What is the limiting value for the world's population, based on this model?

**c.** From the table, estimate when the world's population was growing most rapidly.

**d.** From your model in part (a), when was the world's population growing most rapidly?

**e.** Use the difference equation to predict the world's population in 2010.

**f.** Use the difference equation to predict when the world's population will reach 7 billion.

**29.** Recall that the inflection point for the logistic model

$$\Delta P_n = aP_n - bP_n^2$$

occurs at a height of $\frac{1}{2}L$. At this point, the function is growing most rapidly. Show algebraically that the maximum value of $\Delta P_n$ is $\frac{1}{4}bL^2$.

## 5.4 Newton's Laws of Cooling and Heating

In a familiar scene in TV crime shows, the medical examiner studies the homicide victim's body and knowledgeably announces that "Mr. Jones died at approximately 1:30 in the morning." We now develop the mathematics behind this type of conclusion. More generally, we investigate the rate at which any object cools or heats.

### Newton's Law of Cooling

Suppose that you heat a pizza in an oven set at 450°F and then remove it to cool on a kitchen counter where the temperature is a constant 70°F. How fast does the temperature of the pizza drop until it reaches room temperature? Clearly, the temperature of the pizza drops most rapidly at first because of the large difference between the pizza temperature and the room temperature. As the pizza cools, the rate at which the temperature decreases slows. That is, the hotter the pizza, the faster the temperature drops; it drops most slowly when the pizza's temperature is close to room temperature. Geometrically, we expect the graph of the temperature as a function of time to be decreasing and concave up, as shown in Figure 5.38. According to a physical principle developed by Isaac Newton, the change in temperature $\Delta T_n$ after $n$ time periods is proportional to the difference between the temperature of the pizza ($T_n$) and the temperature of the surrounding air (70°F), or $T_n - 70$. Thus

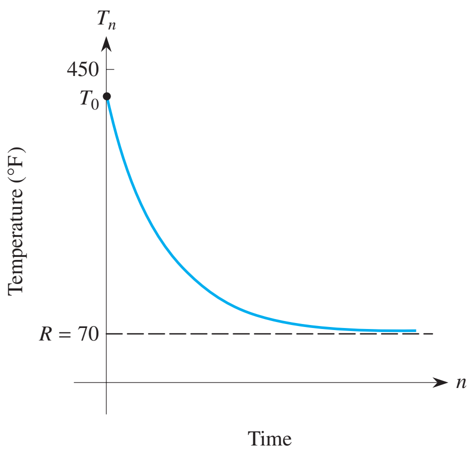



FIGURE 5.38

$$\Delta T_n = -\alpha \cdot (T_n - 70),$$

where the constant of proportionality $\alpha > 0$. Because the temperature change $\Delta T_n$ is negative and both $T_n - 70$ and $\alpha$ are positive, there must be a minus sign in front of $\alpha$. Equivalently, if we add $T_n$ to both sides of the equation, we have the difference equation model

$$\begin{aligned} T_{n+1} &= T_n - \alpha \cdot (T_n - 70) \\ &= (1 - \alpha)T_n + 70\alpha. \end{aligned}$$

More generally, if the room temperature is any constant $R$, the analogous difference equation is

$$T_{n+1} = (1 - \alpha)T_n + \alpha R. \qquad \textbf{(1)}$$

This equation and the principle behind it are both known as **Newton's law of cooling.**

To solve this difference equation, we use the results summarized at the end of Section 5.1. Note that Difference Equation (1) has the same form as

$$x_{n+1} = ax_n + B \qquad \textbf{(2)}$$

with

$$a = 1 - \alpha \quad \text{and} \quad B = \alpha R.$$

The solution to Difference Equation (2) is

$$x_n = L + (x_0 - L)a^n,$$

where $L = B/(1 - a)$ is the limiting value.

For Newton's law of cooling, $\alpha$ is a positive fraction, so $a = 1 - \alpha$ is between 0 and 1. Also, we have

$$L = \frac{B}{1-a} = \frac{\alpha R}{1-(1-\alpha)} = \frac{\alpha R}{\alpha} = R.$$

Therefore, if the initial temperature of the pizza (or any other cooling object) is $T_0$, the solution to Difference Equation (1) is

$$T_n = R + (T_0 - R)(1-\alpha)^n.$$

The constant $\alpha$ usually is determined from an additional temperature measurement, as we illustrate in Examples 1–3 below.

First, let's examine the behavior of this solution function. The term $(T_0 - R)(1-\alpha)^n$ is an exponential decay function because $a = 1 - \alpha$ lies between 0 and 1. This term is added to the constant $R$, so the solution function is decreasing in a concave up manner as it decays to a level of $R$ as $n$ increases. This behavior pattern is precisely what we predicted in Figure 5.38.

We can also see this behavior directly from the formula

$$T_n = R + (T_0 - R)(1-\alpha)^n.$$

As $n$ increases, the exponential decay term $(1-\alpha)^n$ approaches 0 because $1 - \alpha$ lies between 0 and 1. So again $T_n \to R$ as $n \to \infty$.

In Example 1 of Section 4.8, we constructed a function to fit a set of data obtained from an experiment on cooling. A temperature probe was warmed to 42.3°C and plunged into cold water at about 8.6°C to cool. The shifted exponential function that best fit the data was $T(t) = 8.6 + 35.4394(0.8480)^t$. Note that this function has the identical form as the solution sequence for the difference equation model.

The mathematical model for Newton's law of cooling may be summarized as follows.

## Newton's Law of Cooling

*Assumptions*

- The temperature $R$ of the medium remains constant.
- The change in temperature is proportional to the difference between the temperature of the object and the temperature of the medium.

*Mathematical Model*

- Difference equation:
  $$\Delta T_n = -\alpha \cdot (T_n - R) \quad \text{or} \quad T_{n+1} = (1-\alpha)T_n + \alpha R$$
- Solution: $T_n = R + (T_0 - R)(1-\alpha)^n$

### EXAMPLE 1

Suppose that a cake is baking in an oven at 350°F. It is removed when its temperature is 180°F and is left to cool in a kitchen at 70°F. After 10 minutes, the temperature of the cake is 125°F.

a. Find the solution to the difference equation based on Newton's law of cooling.

b. Find the temperature of the cake after 15 minutes.

c. How long does it take the cake to cool to 75°F?

***Solution***

**a.** For $T_0 = 180$ and $R = 70$, the difference equation for the temperature at any time $n$ is

$$T_{n+1} = (1 - \alpha)T_n + 70\alpha.$$

The corresponding solution is

$$\begin{aligned} T_n &= (180 - 70)(1 - \alpha)^n + 70 \\ &= 110(1 - \alpha)^n + 70. \end{aligned}$$

Further, when $n = 10$, the temperature $T_{10} = 125$, so

$$T_{10} = 110(1 - \alpha)^{10} + 70 = 125.$$

Subtracting 70 from both sides gives

$$110(1 - \alpha)^{10} = 125 - 70 = 55, \quad \text{so that} \quad (1 - \alpha)^{10} = \frac{55}{110} = 0.5.$$

Taking the tenth root of both sides of this equation yields

$$1 - \alpha = 0.5^{1/10} = 0.933, \quad \text{and so} \quad \alpha = 1 - 0.933 = 0.067,$$

which is between 0 and 1, as we expected. The solution to the difference equation is

$$\begin{aligned} T_n &= 110(1 - \alpha)^n + 70 \\ &= 70 + 110(0.933)^n. \end{aligned}$$

**b.** After $n = 15$ minutes, the temperature of the cake is

$$\begin{aligned} T_{15} &= 70 + 110(0.933)^{15} \\ &= 70 + 38.87 \approx 108.9°. \end{aligned}$$

**c.** To find the time needed for the cake to cool to 75°F, we need the value of $n$ when $T_n = 75$. We begin with

$$T_n = 110(0.933)^n + 70 = 75 \quad \text{and so} \quad 110(0.933)^n = 5.$$

We divide both sides by 110 to get

$$(0.933)^n = \frac{5}{110} = 0.045.$$

We take logarithms of both sides of this equation to obtain

$$n \log(0.933) = \log(0.045)$$

and then solve for $n$:

$$n = \frac{\log(0.045)}{\log(0.933)} \approx 44.7 \text{ minutes}.$$

Thus it takes about three-quarters of an hour for the cake to cool to 75°F.

## EXAMPLE 2

Mr. Jones's body was found in his kitchen at 9 A.M. by the police who noted that the body temperature was 77.3°F and that the room temperature was 70°F. According to the medical examiner, the body temperature an hour later was 76.1°F. Assuming that Mr. Jones's body temperature was the normal 98.6°F at the time of death, at what time did he die?

***Solution*** Using the solution to the difference equation for cooling, the body temperature after $n$ hours is given by

$$T_n = R + (T_0 - R)(1 - \alpha)^n,$$

where $R = 70°$ and $T_0 = 98.6°$. Therefore

$$\begin{aligned} T_n &= 70 + (98.6 - 70)(1 - \alpha)^n \\ &= 70 + 28.6(1 - \alpha)^n. \end{aligned}$$

Because we don't know the time of death (which corresponds to $n = 0$), we don't know the value of $n$ at 9 A.M. However, at 9 A.M., which is $n$ hours after death, the body temperature was 77.3°, so

$$T_n = 70 + 28.6(1 - \alpha)^n = 77.3,$$

or, after subtracting 70 from both sides,

$$28.6(1 - \alpha)^n = 7.3. \qquad \textbf{(3)}$$

An hour later at 10 A.M., which is $n + 1$ hours after death, the body temperature was

$$T_{n+1} = 70 + 28.6(1 - \alpha)^{n+1} = 76.1,$$

or, again after subtracting 70 from both sides,

$$28.6(1 - \alpha)^{n+1} = 6.1. \qquad \textbf{(4)}$$

Dividing Equation (4) by Equation (3) yields

$$\frac{28.6(1 - \alpha)^{n+1}}{28.6(1 - \alpha)^n} = 1 - \alpha = \frac{6.1}{7.3} \approx 0.8356.$$

Substituting this value into Equation (3) gives

$$28.6(0.8356)^n = 7.3,$$

or, equivalently, when we divide both sides by 28.6,

$$(0.8356)^n = 0.2552.$$

To solve for $n$, we take logs of both sides of this equation and get

$$n \log(0.8356) = \log(0.2552)$$

so that

$$n = \frac{\log(0.2552)}{\log(0.8356)} \approx 7.6.$$

The body was found at 9 A.M., which is $n = 7.6$ hours after death. Thus the murder occurred 7.6 hours (or 7 hours and 36 minutes) before 9 A.M., so we conclude that Mr. Jones was murdered at approximately 1:24 A.M.

## EXAMPLE 3

A cup of hot coffee is left standing on a table in a room where the temperature is 70°F. The temperature $T$, in °F, of the coffee is measured every minute $t$, and the results are shown in the following table. Find a model that best fits these temperature readings as a function of time.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | 186 | 182 | 178 | 175 | 171 | 168 | 165 | 162 | 159 | 156 | 153 | 152 | 148 | 145 | 143 | 141 |

***Solution*** Based on our knowledge of cooling curves, we would expect the best fit to be an exponential function that decays to 70°F. We can't use our data-fitting techniques from Chapter 3 to fit an exponential curve to the data because the function would decay to 0°F rather than 70°F. As in Example 1 of Section 4.8, we must first subtract 70 from each temperature reading, which is equivalent to introducing a vertical shift for the temperature function, to get the following table.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T - 70$ | 116 | 112 | 108 | 105 | 101 | 98 | 95 | 92 | 89 | 86 | 83 | 82 | 78 | 75 | 73 | 71 |

The exponential function that best fits the shifted data is

$$T - 70 = 115.6(0.96783)^t,$$

where $T$ is the temperature. The corresponding correlation coefficient is $r = -0.99945$; it is negative because the temperature readings are decreasing. This value for $r$ is extremely close to $-1$, indicating an extremely good fit. Finally, we undo the vertical shift by adding 70 to both sides of the preceding equation to obtain

$$T = 115.6(0.96783)^t + 70$$

as the best model for the temperature of the coffee, as illustrated in Figure 5.39. Note that this function has the same form as the solution we obtained for the difference equation for Newton's law of cooling.

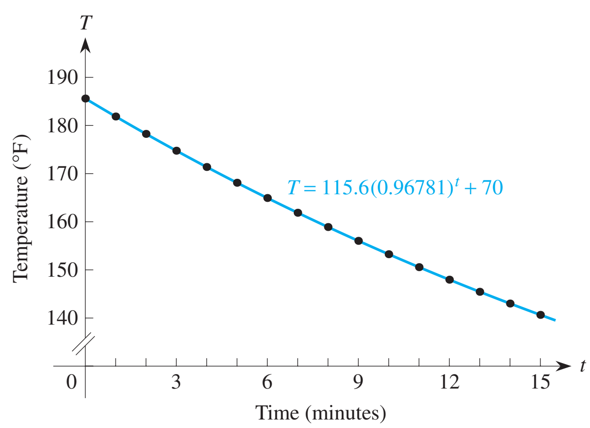



FIGURE 5.39

## Newton's Law of Heating

Similar mathematical methods apply if an object is being warmed rather than cooled. The corresponding principle is known as **Newton's law of heating,** which is based on the assumption that the change in temperature $\Delta T_n$ (now an increase) is proportional to the difference between the temperature $R$ of the medium (the room, the freezer, the oven, etc.) and the temperature of the object $T_n$. This assumption leads to the difference equation

$$\Delta T_n = -\alpha \cdot (T_n - R),$$

which is identical to the difference equation for Newton's law of cooling. Let's see why it is the same. Because the object is being heated, the change in temperature $\Delta T_n$ of the object must be positive. Also, the temperature $T_n$ of the object is always less than $R$, so $T_n - R$ is negative. Finally, because the constant of proportionality $\alpha$ is positive, the product of the terms on the right-hand side will be positive.

We solve this difference equation the same way we solved the one for cooling to get

$$T_n = R + (T_0 - R)(1 - \alpha)^n.$$

### EXAMPLE 4

A chicken is removed from the refrigerator at a temperature of 40°F and placed into an oven kept at a constant temperature of 350°F. After 10 minutes, the temperature of the chicken is 70°F. The chicken is considered cooked when its temperature reaches 180°F. How long must it remain in the oven until it is cooked?

***Solution*** The solution of the difference equation $\Delta T_n = -\alpha \cdot (T_n - 350)$, with $R = 350$ and $T_0 = 40$, is

$$T_n = 350 + (40 - 350)(1 - \alpha)^n = 350 - 310(1 - \alpha)^n.$$

After 10 minutes,

$$T_{10} = 350 - 310(1 - \alpha)^{10} = 70.$$

Therefore

$$310(1 - \alpha)^{10} = 350 - 70 = 280 \quad \text{so that} \quad (1 - \alpha)^{10} = \frac{280}{310} \approx 0.903.$$

Taking the tenth root of both sides, we obtain

$$1 - \alpha \approx 0.9898.$$

Consequently, the solution to the difference equation for the temperature of the chicken is

$$T_n = 350 - 310(0.9898)^n.$$

We must now find how long it takes the temperature to reach 180°F. Doing so requires finding the value of $n$ for which

$$T_n = 350 - 310(0.9898)^n = 180 \quad \text{so that} \quad 310(0.9898)^n = 350 - 180 = 170.$$

Dividing by 310 gives

$$(0.9898)^n = \frac{170}{310} \approx 0.5484.$$

Taking logs of both sides of this equation yields

$$n \log(0.9898) = \log(0.5484),$$

and so

$$n = \frac{\log(0.5484)}{\log(0.9898)} \approx 58.6 \text{ minutes.}$$

Hence the chicken will be ready in just under an hour.

## Problems

1. In Example 2 we assumed that the initial body temperature was a normal 98.6°F. Suppose that Mr. Jones had a slight fever of 100°F when he was murdered. How much difference does that make to the estimated time of death?
2. In Example 2 we also assumed that the room temperature was kept constant at 70°F. Actually, the home heating system probably cycled on and off, so the actual temperature might have oscillated between 67°F and 72°F, say. How might you take this variation into account in predicting the time of death? How much of a difference would it make?
3. In Example 4 we presumed that the temperature of the oven is 350°F for cooking a chicken. Suppose instead that the temperature is set at 325°F. How much difference does that make in the time needed to cook the chicken to 180°F?
4. A pot of bubbling pudding (212°F) is removed from the stove and put immediately into a refrigerator at 40°. After 10 minutes, the temperature of the pudding is 160°. Find the temperature after 1 hour. How long does it take for the temperature to drop to 75°?
5. Sam takes a can of soda at room temperature (70°F) and puts it in a freezer (0°F) to chill quickly. After 10 minutes, the temperature of the soda is 60°. How long does it take the temperature to drop to 40°?
6. A bowl of cold soup is taken out of the refrigerator (36°F) and placed in a heated oven (375°F) to warm. After 10 minutes, the temperature of the soup is 120°. How long does it take for the soup to reach 200°?
7. Professor Smith's body was found in a large walk-in refrigerator in the laboratory at 9 A.M. by the police who noted that the body temperature was 67.3°F and that the refrigerator temperature was 40°. An hour later, with the body still in the refrigerator, the medical examiner found the body temperature to be 63.1°. Assuming that the body temperature at death was 98.6°, at what time did Professor Smith die?
8. A cup of boiling water (100°C) was placed in a refrigerator kept at 7°C at 8 A.M., and the following readings were obtained.

| **Time (minutes)** | 1 | 7 | 21 | 45 |
|---|---|---|---|---|
| **Temperature (°C)** | 89 | 71 | 53 | 36 |
| **Time (minutes)** | 73 | 90 | 123 | 152 |
| **Temperature (°C)** | 25 | 20 | 14 | 11 |

Determine the best exponential fit to these data and use it to predict when the water temperature will be 9°C.

9. The following data are printed on a carton of milk to indicate how many days the milk will last without spoiling at different temperatures.

| **Temperature (°F)** | 32 | 40 | 45 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|
| **Time (days)** | 24 | 11 | 5.5 | 2 | 1 | 0.5 |

   **a.** Determine the linear, exponential, and power functions that best fit these data.

   **b.** Which of the three functions seems to be the best fit to the data?

   **c.** Using the model that you think is the best fit, how long should milk last in a refrigerator kept at 35°F?

10. A cool potato at temperature 60°F is placed in an oven kept at a constant 350°.

   **a.** Sketch the graph of the temperature of the potato as a function of time.

   Use the concavity of your graph from part (a) to answer the questions in parts (b)–(d).

   **b.** Suppose that you measure the temperature of the potato after 5 minutes and find that it is 109° and that, after 7 minutes, it is 127°. Use this information to estimate the temperature after 10 minutes.

Is the actual temperature higher or lower than your estimate? How do you know?

**c.** Suppose that you are told that the temperature of the potato after 12 minutes is 150°. Use this information and the temperature after 5 minutes to estimate the temperature after 10 minutes. Is the actual temperature higher or lower than your estimate? How do you know?

**d.** How might you use the results from parts (b) and (c) to come up with a better estimate of the temperature after 10 minutes?

**11.** Your Thanksgiving turkey is taken from a refrigerator at 40°F and is cooked in an oven kept at a constant temperature of 350°. The temperature $T$ of the bird is 70° after 30 minutes and is 96° after 60 minutes. From your knowledge of the pattern of temperature rise, decide which of the following temperature readings are possible and which are impossible.

**a.** $T(45) = 80°$ **b.** $T(45) = 85°$
**c.** $T(75) = 105°$ **d.** $T(75) = 115°$

180°F

**12.** A cup of hot chocolate (temperature ) is placed on a table where the air temperature is 70°. Suppose that it takes 12 minutes for the drink to cool to 100°. Let $r_1$ represent the average rate of decrease in temperature per minute over the full 12-minute period, let $r_2$ be the average rate of decrease over the first 6 minutes, and let $r_3$ be the average rate of decrease over the last 6 minutes. List these three rates in increasing order.

**13.** Suppose that it takes $t_1$ minutes for a raw potato, starting at 70°F, to reach 200° in an oven. At that time, it is removed from the oven and put on the table where it cools. Suppose that the potato takes $t_2$ minutes to reach 70°. Is $t_1 < t_2$, $t_1 = t_2$, or $t_1 > t_2$? Explain.

**14.** A potato at room temperature $(T_0)$ is placed in an oven at temperature $R$. Construct the actual solution to the corresponding difference equation for heating in terms of the various parameters. What is the formula for the temperature of the potato if $T = 70°$F and $R = 350°$F?

**15.** In Section 4.8, we presented a set of data from a cooling experiment. The temperature readings, in degrees Celsius, were as follows.

| Time | Temp | Time | Temp | Time | Temp | Time | Temp |
|---|---|---|---|---|---|---|---|
| 1 | 42.30 | 10 | 14.77 | 19 | 10.17 | 28 | 9.04 |
| 2 | 36.03 | 11 | 13.82 | 20 | 9.92 | 29 | 8.91 |
| 3 | 30.85 | 12 | 13.11 | 21 | 9.80 | 30 | 8.83 |
| 4 | 26.77 | 13 | 12.51 | 22 | 9.67 | 31 | 8.78 |
| 5 | 23.58 | 14 | 11.91 | 23 | 9.54 | 32 | 8.78 |
| 6 | 20.93 | 15 | 11.54 | 24 | 9.42 | 33 | 8.78 |
| 7 | 18.79 | 16 | 11.17 | 25 | 9.29 | 34 | 8.78 |
| 8 | 17.08 | 17 | 10.67 | 26 | 9.16 | 35 | 8.66 |
| 9 | 15.82 | 18 | 10.42 | 27 | 9.16 | 36 | 8.66 |

According to Newton's law of cooling, the change in temperature $\Delta T$ is proportional to the difference between the temperature of the object and the temperature of the medium (in this case the temperature of the cool water is 8.6°C).

**a.** Extend the table, with additional columns for $T - 8.6$ and also for the differences $\Delta T$ between successive temperature readings.

**b.** Plot the points $(T - 8.6, \Delta T)$ that you calculated. In what type of pattern do they fall?

**c.** Find the equation of the line that best fits these points and write a difference equation for $\Delta T$ for the temperature data.

**d.** Solve the difference equation for the temperature $T$ as a function of time.

## 5.5 Geometric Sequences and Their Sums

Consider the difference equation for exponential growth

$$x_{n+1} = rx_n$$

whose solution is given by

$$x_n = x_0 r^n,$$

where $x_0$ is any initial value. Using the notation for sequences, we can write this solution as

$$\{x_0, x_0 r, x_0 r^2, x_0 r^3, x_0 r^4, \ldots, x_0 r^n, \ldots\}.$$

Recall that such a sequence is called a *geometric sequence* or *exponential sequence.* For simplicity, suppose that $x_0 = 1$, so that the sequence reduces to

$$\{1, r, r^2, r^3, r^4, \ldots, r^n, \ldots\}.$$

For instance, if $r = 5$, we have the geometric sequence

$$\{1, 5, 5^2, 5^3, 5^4, \ldots, 5^n, \ldots\} = \{1, 5, 25, 125, 625, \ldots, 5^n, \ldots\}.$$

The difference equation $x_{n+1} = rx_n$ reinforces the fact that each term in any geometric sequence is a constant multiple of the preceding term, or, equivalently, there is a common ratio $r$ between successive terms. Alternatively, $x_n$ is an exponential function of $n$, and we know that the ratio of successive values is a constant—namely, the growth or decay factor $r$. In the preceding sequence, each term is five times the preceding term, so the common ratio $r$ of each pair of successive terms is 5.

We can deduce a considerable amount of information about the behavior of the terms in a geometric sequence $\{1, r, r^2, r^3, \ldots, r^n, \ldots\}$ from the value of the common ratio $r$. We summarize this information as follows.

1. If $r > 1$, the terms are successively larger and approach infinity. We say that they *increase monotonically.*
2. If $r = 1$, all the terms are equal to 1.
3. If $0 < r < 1$, the terms are successively smaller and approach zero. We say that they *decrease monotonically.*
4. If $r = 0$, all terms after the initial term, 1, are 0.
5. If $-1 < r < 0$, the terms oscillate between positive and negative values, each is numerically smaller than the preceding term, and they approach 0.
6. If $r = -1$, the terms oscillate between 1 and $-1$.
7. If $r < -1$, the terms oscillate between positive and negative values, and each term is numerically larger than the preceding term.

If the terms of a sequence approach a single value as $n$ approaches $\infty$, written $x_n \to \infty$, we say that the sequence *converges* to that value. Otherwise, we say that the sequence *diverges.*

The following seven sequences illustrate these properties.

1. $r = 5$: $\{1, 5, 5^2, 5^3, 5^4, \ldots, 5^n, \ldots\} \to \infty$, so this sequence diverges as $n \to \infty$. (Note that the terms in this sequence increase monotonically.)
2. $r = 1$: $\{1, 1, 1, 1, 1, \ldots\} \to 1$, so this sequence converges to 1 as $n \to \infty$.

3. $r = \frac{1}{2}$: $\{1, \frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3, (\frac{1}{2})^4, \ldots, (\frac{1}{2})^n, \ldots\} = \{1, 0.5, 0.25, 0.125, 0.0625, \ldots\} \to 0$, so this sequence converges to 0 as $n \to \infty$. (Note that the terms in this sequence decrease monotonically.)
4. $r = 0$: $\{1, 0, 0, 0, 0, \ldots\} \to 0$, so this sequence converges to 0 as $n \to \infty$
5. $r = -\frac{1}{4}$: $\{1, -\frac{1}{4}, (-\frac{1}{4})^2, (-\frac{1}{4})^3, (-\frac{1}{4})^4, \ldots, (-\frac{1}{4})^n, \ldots\} = \{1, -0.25, 0.0625, -0.015625, 0.00390625, \ldots\} \to 0$, so this sequence converges to 0 as $n \to \infty$. (Note that the terms in this sequence oscillate between positive and negative values, but that each term is numerically smaller than the previous term; the terms are not monotonic.)
6. $r = -1$: $\{1, -1, (-1)^2, (-1)^3, (-1)^4, \ldots, (-1)^n, \ldots\} = \{1, -1, 1, -1, 1, -1, \ldots\}$, so this sequence does not converge to a single value but diverges as $n \to \infty$.
7. $r = -2$: $\{1, -2, (-2)^2, (-2)^3, (-2)^4, \ldots, (-2)^n, \ldots\} = \{1, -2, 4, -8, 16, -32, 64, -128, \ldots\} \to \pm\infty$, so this sequence diverges as $n \to \infty$. The terms oscillate between positive and negative values, but each term is numerically larger than the previous term; the terms are not monotonic.

## The Sum of the Terms in a Geometric Sequence

Geometric sequences arise in a great variety of applications, and usually they are accompanied by the related question: What is the sum of the terms? That is, for any geometric sequence with constant ratio $r$, what is

$$1 + r + r^2 + r^3 + r^4 + \cdots + r^n$$

for any given value of $n$?

For instance, there is a very old puzzle in which gold coins are placed on a chessboard according to the pattern: one coin on the first square, two on the second, four on the third, and so on to $2^{63}$ on the 64th square. The problem then is: Find the total number of coins. That is, find the sum

$$1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{63}.$$

Before attempting to solve this specific problem, let's look at the more general problem of finding a formula for the sum of the terms in any geometric sequence. We introduce the following notation. Let

$$S_0 = 1, \qquad S_1 = 1 + r, \qquad S_2 = 1 + r + r^2,$$

and, in general, for the sum of the $n$ terms $1, r, r^2, r^3, r^4, \ldots, r^n$,

$$S_n = 1 + r + r^2 + \cdots + r^n.$$

We want to find a formula for the sum $S_n$ for any $n$. We multiply the expression for $S_n$ by the common ratio $r$ to obtain

$$rS_n = r(1 + r + r^2 + \cdots + r^n) = r + r^2 + r^3 + \cdots + r^n + r^{n+1}.$$

When we subtract the expression for $rS_n$ from the expression for $S_n$, all the intermediate terms $r, r^2, r^3, \ldots, r^n$ cancel out and we are left with

$$S_n - rS_n = 1 - r^{n+1}.$$

We factor out the term $S_n$ on the left-hand side and get

$$(1 - r)S_n = 1 - r^{n+1}.$$

If $r \neq 1$, we can divide both sides by $1 - r$ to get

$$S_n = \frac{1 - r^{n+1}}{1 - r}.$$

The sum of the terms $1, r, r^2, \ldots, r^n$ of a finite geometric sequence is

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r},$$

provided that $r \neq 1$.

**EXAMPLE 1**

What is the total number of gold coins that would be needed on a chessboard according to the pattern previously described?

***Solution*** The number of coins is

$$1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{63}.$$

The common ratio in this sum of terms of a geometric sequence is $r = 2$. So, for $n = 63$, we get

$$\begin{aligned} S_{63} &= 1 + 2 + 2^2 + \cdots + 2^{63} \\ &= \frac{1 - 2^{64}}{1 - 2} \\ &= 2^{64} - 1 = 1.844674 \times 10^{19}, \end{aligned}$$

which is more than 18 quintillion.

In two of the seven possible cases for a geometric sequence—namely, $0 < r < 1$ and $-1 < r < 0$—the *terms of the sequence* approach 0. Also, when $r = 0$, all terms after the initial term 1 are 0. In the other 4 cases listed, the terms do not approach 0. Let's see what this means for the *sum of the terms* of the entire geometric sequence, not just the sum of the first $n$ terms.

Using the formula for the sum of the terms from 1 to $r^n$,

$$S_n = \frac{1 - r^{n+1}}{1 - r},$$

we see that, whenever $-1 < r < 1$, the expression $r^{n+1}$ becomes ever smaller and approaches 0 as $n$ increases. As a result, even though we are adding more and more terms, eventually the additional terms are all so small that together they contribute virtually nothing to the sum. This fact enables us to give meaning to the sum of *all* the terms of an infinite geometric sequence, provided that the terms decrease rapidly enough. In other words,

$$1 + r + r^2 + r^3 + \cdots + r^n + \cdots = \frac{1 - 0}{1 - r} = \frac{1}{1 - r},$$

provided that $-1 < r < 1$, or, equivalently, $|r| < 1$.

However, if $r \geq 1$ or if $r \leq -1$, the values $r^{n+1}$ do *not* approach 0 as $n$ approaches $\infty$, so no finite value for the sum of the infinite sequence is possible. Thus the sum of an infinite geometric sequence makes sense only when the common ratio $r$ is strictly between $-1$ and 1.

> The sum of the terms of an infinite geometric sequence is
>
> $$1 + r + r^2 + r^3 + \cdots + r^n + \cdots = \frac{1}{1-r},$$
>
> provided that $-1 < r < 1$.

For instance, the sum of the terms in the infinite geometric sequence

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{1}{1-(1/2)} = \frac{1}{(1/2)} = 2.$$

**Think About This**

Add enough of the terms from the preceding sequence to convince yourself that this result is reasonable.

The facts and results regarding geometric sequences just described occur frequently in applications of mathematics. We encounter such sequences repeatedly later in this book. For now, let's consider several additional situations in which they arise.

In our discussion of the elimination of a drug from the body in Section 5.1, we saw that the kidneys remove about 25% of any Prozac in the bloodstream every 24 hours and that a person takes the same dose $D_0 = 80$ mg every day. We modeled this situation with the difference equation

$$D_{n+1} = 0.75D_n + 80$$

and created the formula

$$D_n = 320 - 240(0.75)^n$$

for the solution sequence. Let's now find this formula by using our knowledge of the sum of a geometric sequence.

## EXAMPLE 2

a. Find a formula for the level of Prozac in the body after any number of days, using the sum of a geometric sequence.

b. Use the result from part (a) to find the level of Prozac in the bloodstream after 5 days and after 10 days.

c. What is the limiting value $L$ for Prozac in the body?

***Solution***

a. From the difference equation and the initial value $D_0 = 80$, we have

$$D_1 = 0.75D_0 + 80 = 0.75(80) + 80 = (1 + 0.75)(80).$$

Similarly,

$$D_2 = 0.75D_1 + 80$$
$$= 0.75(1 + 0.75)80 + 80 = (1 + 0.75 + 0.75^2)80.$$

Furthermore,

$$D_3 = 0.75D_2 + 80$$
$$= 0.75(1 + 0.75 + 0.75^2)80 + 80 = (1 + 0.75 + 0.75^2 + 0.75^3)80.$$

In general, after $n$ days,

$$D_n = 80(1 + 0.75 + 0.75^2 + 0.75^3 + \cdots + 0.75^n).$$

The expression inside the parentheses is the sum of the terms from 1 to $0.75^n$ in a geometric sequence with common ratio $r = 0.75$. Therefore the sum of these terms is given by

$$\begin{aligned} D_n &= 80\left[\frac{1 - (0.75)^{n+1}}{1 - 0.75}\right] = 80\left[\frac{1 - (0.75)^{n+1}}{0.25}\right] \\ &= 320[1 - (0.75)^{n+1}] = 320 - 320(0.75)(0.75)^n \\ &= 320 - 240(0.75)^n, \end{aligned}$$

which is identical to the formula that we created in Section 5.1 using difference equations.

**b.** After $n = 5$ days, we have

$$D_5 = 320 - 240(0.75)^5 = 263.0469 \text{ mg},$$

which agrees with the value we obtained in Section 5.1. Similarly,

$$D_{10} = 320 - 240(0.75)^{10} = 306.485 \text{ mg}.$$

**c.** Finally, because $r = 0.75 < 1$, the limiting value for the sum of this geometric sequence is

$$\begin{aligned} L &= D_0 \cdot (1 + r + r^2 + r^3 + \cdots) \\ &= 80\left(\frac{1}{1 - r}\right) = \frac{80}{1 - 0.75} \\ &= \frac{80}{0.25} = 320 \text{ mg}, \end{aligned}$$

which is the same value we found in Section 5.1 for the drug maintenance level.

## When Will Our Oil Run Out?

An ongoing political debate having major economic and social implications has to do with energy policy both at home and abroad. One aspect of this debate centers on the use of petroleum, both as a source of energy in vehicles and power plants and as a component of oil-based products such as plastics. Estimated worldwide oil reserves in 2000 were about 2250 billion barrels, and worldwide oil consumption in 2000 was about 26 billion barrels.* How long will the oil last?

*Source: Energy Information Administration, www.eia.doe.gov/ieu/.

If the current rate of consumption remains constant at 26 billion barrels per year, known oil reserves will run out in

$$\frac{2250}{26} \approx 86.5 \text{ years.}$$

But oil consumption hasn't been constant; in fact, some estimates indicate that worldwide oil consumption has been growing at an annual rate of about 2%. How do we then calculate how long the known oil reserves will last?

**EXAMPLE 3**

Calculate how long the estimated 2250 billion barrel oil reserves worldwide will last if oil consumption was 26 billion barrels in 2000 and continues to grow at a 2% annual rate.

***Solution*** Annual oil consumption $C$ can be modeled by the exponential growth function

$$C(t) = 26(1.02)^t,$$

where $t$ is the number of years since 2000. The total oil consumed from 2000 on is then approximately

$$C(0) + C(1) + C(2) + C(3) + \cdots.$$

Let $n$ be the first year when the total passes the 2250 billion barrel level. We need to solve for the value of $n$ that gives

$$C(0) + C(1) + C(2) + \cdots + C(n) = 2250.$$

This expression is equivalent to

$$26(1.02)^0 + 26(1.02)^1 + 26(1.02)^2 + \cdots + 26(1.02)^n = 2250$$

or, when we factor out the common factor of 26,

$$26[1 + (1.02)^1 + (1.02)^2 + \cdots + (1.02)^n] = 2250.$$

The expression in the brackets is the sum of the first $n$ terms of a geometric sequence with common ratio $r = 1.02$, so

$$26\left(\frac{1 - 1.02^{n+1}}{1 - 1.02}\right) = 26\left(\frac{1 - 1.02^{n+1}}{-0.02}\right)$$

$$= 26\left(\frac{1.02^{n+1} - 1}{0.02}\right) = 1300(1.02^{n+1} - 1).$$

We therefore need to solve the equation

$$1300(1.02^{n+1} - 1) = 2250$$

for $n$. We divide through by 1300 to get

$$1.02^{n+1} - 1 = 1.7308$$

or, by adding 1 to both sides, we obtain

$$1.02^{n+1} = 2.7308.$$

To extract $n + 1$ from the exponent, we use logarithms to get

$$\log(1.02^{n+1}) = (n + 1)\log 1.02 = \log 2.7308,$$

so that

$$n + 1 = \frac{\log 2.7308}{\log 1.02} \approx 50.73$$

and therefore $n \approx 49.73$. We therefore conclude that all the world's known oil reserves will be depleted in just under 50 years from 2000 if consumption continues to grow at an annual rate of 2%.

The results in Example 3 are startling because all the industrialized nations depend so heavily on oil. Many people therefore advocate major efforts to discover new oil supplies. Let's see what doing so would gain.

## EXAMPLE 4

Suppose that, as a result of major efforts to find new sources of oil, the worldwide oil reserves are doubled to 4500 billion barrels. How long will it take to use it all if oil consumption continues to grow at the current 2% annual rate?

**Solution** We now have to find the value of $n$ for which

$$C(0) + C(1) + C(2) + \cdots + C(n) = 2(2250) = 4500,$$

which is equivalent to the sum of terms in the geometric sequence

$$26[1 + (1.02)^1 + (1.02)^2 + \cdots + (1.02)^n] = 4500.$$

Following the same algebraic development as in Example 3, we have to solve

$$1300(1.02^{n+1} - 1) = 4500$$

for $n$. Dividing both sides of the equation by 1300, we get

$$1.02^{n+1} - 1 = 3.4615 \quad \text{or} \quad 1.02^{n+1} = 4.4615.$$

Taking logarithms gives

$$(n + 1)\log 1.02 = \log 4.4615,$$

so that

$$n + 1 = \frac{\log 4.4615}{\log 1.02} \approx 75.52$$

and hence $n \approx 74.52$. Note that doubling the total oil reserve didn't double the 50 years we found in Example 3; it merely added less than 25 more years.

Other people advocate reducing the rate of oil consumption by using alternative energy sources (especially renewable sources), by developing more energy efficient vehicles, by encouraging conservation, and by taking other actions. Let's see what such measures can accomplish.

## EXAMPLE 5

Suppose that, as a result of conservation efforts, the annual rate of growth of oil consumption drops to 1.5% from 2%. How long will the current worldwide oil reserve last?

**Solution** Our new exponential model for oil consumption is

$$C(t) = 26(1.015)^t,$$

where $t$ is still the number of years since 2000. We now have to solve

$$26[1 + (1.015)^1 + (1.015)^2 + \cdots + (1.015)^n] = 2250$$

for $n$. When we find the sum of the terms in this geometric sequence and simplify, as we did before, we get

$$1733.33(1.015^{n+1} - 1) = 2250.$$

Dividing by 1733.33, we get

$$1.015^{n+1} - 1 = 1.2981 \quad \text{or} \quad 1.015^{n+1} = 2.2981.$$

Taking logarithms yields

$$(n + 1)\log 1.015 = \log 2.2981,$$

so that

$$n + 1 = \frac{\log 2.2981}{\log 1.015} \approx 55.9,$$

and $n \approx 54.9$ years. Hence decreasing the annual rate at which oil is consumed by one-half percentage point gives the world an additional 5 years of oil reserves.

Think About This

Is it more likely that the rate at which oil consumption is increasing can be reduced through conservation measures or that an amount of undiscovered oil equal to the total known oil reserves can be found?

## A Bouncing Ball

We now consider another application involving the sum of a geometric sequence.

### EXAMPLE 6

Suppose that a properly inflated basketball is designed to bounce back to three-quarters of the height from which it is dropped.

a. If such a ball is initially dropped from a height of 10 feet, find the total vertical distance it travels on the first 10 bounces; the first 20 bounces; the first 30 bounces.

b. What total vertical distance does the ball cover if, theoretically, it keeps bouncing indefinitely?

***Solution***

a. Figure 5.40 shows that the vertical distance the ball travels is 10 feet until the first bounce plus 2 times the distance (up and then down) between the first and second bounces, or

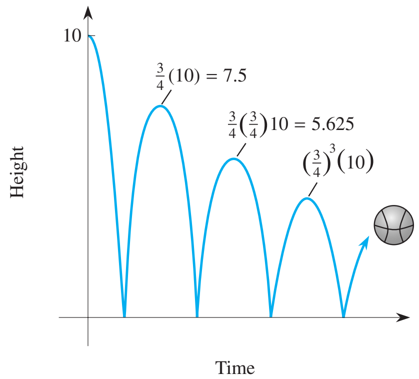



FIGURE 5.40

$$2\left(\frac{3}{4}\cdot 10\right),$$

plus 2 times the distance between the second and third bounces, or

$$2\left[\frac{3}{4}\left(\frac{3}{4}\cdot 10\right)\right] = 2\left(\frac{3}{4}\right)^2\cdot 10,$$

and so on.

The total vertical distance traveled on the first $n$ bounces is therefore

$$\begin{aligned}
D_n &= 10 + 2\left(\frac{3}{4}\right)\cdot 10 + 2\left(\frac{3}{4}\right)^2\cdot 10 + 2\left(\frac{3}{4}\right)^3\cdot 10 + \cdots + 2\left(\frac{3}{4}\right)^n\cdot 10\\
&= 10 + 20\left(\frac{3}{4}\right) + 20\left(\frac{3}{4}\right)^2 + 20\left(\frac{3}{4}\right)^3 + \cdots + 20\left(\frac{3}{4}\right)^n\\
&= 10 + 20\left[\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots + \left(\frac{3}{4}\right)^n\right]\\
&= 10 + 20\left(\frac{3}{4}\right)\left[1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^{n-1}\right]\\
&= 10 + 15\left[1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^{n-1}\right].
\end{aligned}$$

Note that the expression in the brackets on the preceding line is the sum of a finite number of terms of a geometric sequence with common ratio $r = \frac{3}{4}$. Therefore the total vertical distance traveled by the ball during the first $n$ bounces is

$$\begin{aligned}
D_n &= 10 + 15\left(\frac{1-(3/4)^n}{1-(3/4)}\right) = 10 + 15\left(\frac{1-(3/4)^n}{1/4}\right)\\
&= 10 + 60\left[1 - \left(\frac{3}{4}\right)^n\right].
\end{aligned}$$

Consequently, the vertical distance traveled by the ball during the first 10 bounces is

$$\begin{aligned}
D_{10} &= 10 + 60\left[1 - \left(\frac{3}{4}\right)^{10}\right]\\
&= 10 + 56.621 = 66.621,
\end{aligned}$$

or about 66.6 feet. During the first 20 bounces, the ball travels

$$D_{20} = 10 + 60\left[1 - \left(\frac{3}{4}\right)^{20}\right] = 69.810,$$

or about 69.8 feet. And during the first 30 bounces, it travels

$$D_{30} = 10 + 60\left[1 - \left(\frac{3}{4}\right)^{30}\right] = 69.989,$$

or about 69.99 feet.

**b.** Little extra distance is contributed by the 20th through the 30th bounce, and all subsequent bounces will contribute even less. In fact, because the common ratio $r = \frac{3}{4}$ is between $-1$ and 1, we know that $(\frac{3}{4})^n \to 0$ as $n$ increases and so we can sum all of the terms of the infinite geometric sequence to obtain

$$\begin{aligned}
&10 + 15\left[1 + \tfrac{3}{4} + (\tfrac{3}{4})^2 + (\tfrac{3}{4})^3 + \cdots + (\tfrac{3}{4})^n + \cdots\right]\\
&\quad = 10 + 15\left[\tfrac{1}{1-(3/4)}\right] = 10 + 15\left(\tfrac{1}{1/4}\right)
\end{aligned}$$

$$= 10 + 15(4) = 70 \text{ feet.}$$

That is, theoretically, if the ball were to continue bouncing forever, the total vertical distance it would travel would be 70 feet.

## Problems

1. Find the sum $1 + r + r^2 + \cdots + r^n$, with $r = \frac{1}{2}$, for $n = 10$, $n = 20$, and $n = 30$.
2. Repeat Problem 1 with $r = 0.2$.
3. Repeat Problem 1 with $r = 0.8$.
4. Repeat Problem 1 with $r = -0.8$.
5. Repeat Problem 1 with $r = 1.5$.
6. Repeat Problem 1 with $r = -2.5$.
7. Suppose that 6000 new cases of a certain disease occurred in 1960. If the number of new cases diminished 20% per year since then, what is the *total* number of people who contracted this disease from 1960 through 2000? from 2000 through 2010?
8. Repeat Problem 7 if the number of new cases of the disease increased 20% per year since 1960.
9. In 1980, the United States used approximately 2.5 billion kilowatt-hours of electricity. If electric usage grew by 2% per year, find the total amount of electricity used between 1980 and 2000.
10. The United States produced 195,000 metric tons of wheat in 1984. If production increased by 10% per year find the total amount of wheat produced between 1984 and 2002.
11. The United States produced 70,600 metric tons of rice in 1984. If rice production fell by 9% per year, find the total amount of rice produced between 1984 and 2002.
12. At age 22, Ken gets his first job paying $35,000 per year. If he stays with the same employer and gets an average annual increase of 4% each year, what will be his *total* earnings over his entire career by the time he retires at age 65?
13. In 1986, a total of 70,000 pages of new mathematical research was published. If the amount of research grew at the rate of 8% per year, find the total amount of new mathematics research published between 1986 and 2000.
14. Suppose that several immense new oil fields are discovered that increase the worldwide oil reserves tenfold. If oil consumption continues to grow at an annual rate of 2% and 26 billion barrels were consumed in 2000, determine how long the oil reserves will last.
15. Suppose that, as undeveloped countries become more industrialized, the annual growth rate in oil consumption increases to 2.5%. Based on the current 2250 billion barrel estimate for worldwide oil reserves and the estimated worldwide oil consumption of 26 billion barrels in 2000, how long will it be until all the known oil is used?
16. Repeat Example 6 if the initial height of the ball is 6 feet.
17. Repeat Example 6 if the initial height of the ball is 12 feet.
18. Repeat Example 6 if the ball bounces back to 80% of its height. By how much does the total distance traveled by the ball change compared to a 75% bounce?
19. Repeat Example 6 if the ball bounces back $\frac{2}{3}$ of its height.
20. The repeating decimal 0.222222 . . . can be thought of as
$$\frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \cdots.$$
What is the sum of this geometric sequence?
21. The repeating decimal 0.252525 . . . can be thought of as
$$\frac{25}{100} + \frac{25}{10{,}000} + \frac{25}{1{,}000{,}000} + \cdots.$$
What is the sum, as a simple fraction, of this geometric sequence?
22. A geometric sequence is based on the fact that the ratio of successive terms is constant: $x_{n+1}/x_n = r$. Suppose instead that the ratio of successive terms in a sequence is a linear function—say, $x_{n+1}/x_n = rn$, for $n \geq 1$.
    - **a.** How does the growth rate of this sequence compare to that for a geometric sequence?
    - **b.** What is the solution of $x_{n+1}/x_n = n$, for $n \geq 1$?

**c.** What is the solution of $x_{n+1}/x_n = 5n$, for $n \geq 1$?

**d.** What is the solution of $x_{n+1}/x_n = rn$ for $n \geq 1$, for any $r$?

**e.** What is the solution of $x_{n+1}/x_n = rn + b$, for $n \geq 1$?

**23.** How would the solution of $x_{n+1}/x_n = n^2$ compare to the solution of $x_{n+1}/x_n = n$, for $n \geq 1$? Construct a formula for this solution.

**24.** Consider $x_{n+1}/x_n = \frac{1}{n}$. How does its solution behave? How does it compare to the solution of $x_{n+1}/x_n = 1/r$, for any $r > 1$? Construct a formula for this solution.

**25.** The table below shows the box-office gross, in millions of dollars, of the movie *Star Wars: The Phantom Menace* during its first 10 weeks in the theaters.

**a.** Find the exponential function that models the box office gross for the movie as a function of the number of weeks in release.

**b.** Use the exponential function from (a) to estimate the total box office gross for *The Phantom Menace* during its first ten weekends in theaters.

**c.** How does the estimate in part (b) compare to the actual total gross obtained by adding the entries in the table?

**26.** The estimated worldwide reserve of natural gas at the end of 1999 was 5200 billion cubic feet. The worldwide consumption of natural gas in 1999 was about 84 billion cubic feet and was growing at an annual rate of about 2.9%. Source: Energy Information Administration, www.eia.doe. gov/ieu/.

**a.** If this growth rate continues, how long will the known reserves of natural gas last?

**b.** If new discoveries of natural gas triple the known reserves, how long will it take to deplete the natural gas reserves at the current growth rate in consumption?

**c.** How long will the current 5200 billion cubic feet of natural gas last if the annual growth rate of consumption rises to 3.5%?

**d.** How long will the current 5200 billion cubic feet of natural gas last if the annual growth rate of consumption falls to 2%?

| **Weekend** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Gross** | 101.4 | 48.7 | 41.2 | 31.1 | 23.6 | 21.5 | 12.0 | 8.0 | 9.6 | 5.8 |

Source: Internet Movie Database: http://us.imdb.com.

## Chapter Summary

In this chapter we introduced difference equations and their solutions. We also discussed the use of difference equations as mathematical models of population growth and other phenomena. More specifically we showed the following.

- How to present the solution sequence to a difference equation either as a closed-form expression for the $n$th term or as a sequence of numbers generated term-by-term.
- The fact that the solution to a difference equation depends on the initial condition with each different initial condition giving rise to a different solution sequence.
- How to model the level of a drug in the bloodstream with difference equations.
- How to find the maintenance level associated with a drug and what it means.
- How to model exponential growth and decay processes with difference equations.
- How to model population growth with the Fibonacci difference equation.

- How to model inhibited population growth with the logistic model, including finding the maximum sustainable population.
- How to interpret the behavior of a logistic curve.
- How to estimate the logistic coefficients from a set of data.
- How to model temperature decrease with a difference equation based on Newton's law of cooling.
- How to model temperature increase with a difference equation based on Newton's law of heating.
- How to interpret the behavior of the solutions for the heating and cooling models.
- How to sum the first $n$ terms of a finite geometric sequence.
- How to sum all the terms of an infinite geometric sequence, provided that the common ratio is between $-1$ and 1.
- How to apply the formulas for the sum of the terms in a geometric sequence.

## Review Problems

1. Write the first five terms of each sequence.
   **a.** $a_n = 6n - 1$
   **b.** $t_n = \dfrac{3^n}{n}, \quad n > 0$
   **c.** $r_n = 1 - (0.3)^n$

2. Determine the first five terms in the solution sequence of each difference equation.
   **a.** $x_{n+1} = x_n + 8, \quad x_0 = 2$
   **b.** $x_{n+1} = x_n - 8, \quad x_0 = 12$
   **c.** $x_{n+1} = \dfrac{1}{3}x_n, \quad x_0 = 5$
   **d.** $x_{n+1} = x_n + (-3)^n, \quad x_0 = 10$

3. Determine the first five values in each solution sequence.
   **a.** $y_{n+2} = y_{n+1} + y_n, \quad y_0 = 2, \quad y_1 = 7$
   **b.** $y_{n+2} = y_{n+1} + y_n, \quad y_0 = 3, \quad y_1 = 7$

4. A drug is administered every 6 hours. The kidneys eliminate 60% of the drug over that period. If the original dose is 100 mg, how much of the drug remains in the body after 8 days?

5. A drug is administered every 4 hours. The kidneys eliminate about 70% of the drug over the 4-hour period. The initial dose of the medicine is 100 mg. How much should the repeated dosage be to ensure a maintenance level of 30 mg?

6. Chlorine is added to a town's water supply reservoir at the rate of 30 lb/day. An estimated 20% of this amount is lost each day through evaporation or filters. What is the maintenance level of chlorine in the reservoir?

7. The difference equation $v_{n+1} = 1.30v_n - 0.00002v_n^2$ models the number of people in a town who have VCRs as a function of the year $n$. Initially, 40 households had VCRs. Estimate how many people will eventually have VCRs. Determine the year in which half the population has VCRs.

8. A population grows according to the logistic model from an initial size of 1000 to a final size of 12,000. The annual growth rate is 20%. What is the inhibiting constant? Write the difference equation that describes this population for any year $n$.

9. The size of the fish population in a stream grows in accordance with the logistic model at an annual rate of 30% with an inhibiting constant of 0.04%. Write the difference equation for each situation.
   **a.** The fish population grows according to the logistic model.
   **b.** The fish population grows according to the logistic model, but the state game warden allows 2000 fish to be caught and removed from the stream each year.
   **c.** The fish population grows according to the logistic model, and the state game warden stocks the stream with 400 new fish per year.
   **d.** The fish population grows according to the logistic model, but about 10% of the population is caught every year.

10. Write the difference equation for each scenario, draw a graph of the behavior of the solution, and answer the question.
    a. Pancake syrup used to be 100% maple syrup, but over the years the amount of maple syrup has been reduced. Suppose that in 1970 a company began reducing the amount of maple syrup in its product by 15% per year. What percentage of maple syrup is in its product in 2005?
    b. John's investment in the stock market has been growing by 10% per year. He adds $2000 a year to his investment. If he had $50,000 invested in 1995, how much does he have invested in 2005?
    c. Advertising in the print media has been less important recently than it was in the past. In 1998, a company decided to decrease its budget for advertising in the print media by $20,000 per year. Also, the company increased its budget for television by 10% per year. If the budgets were each $2 million in 1998, how much is the company budgeting for print and how much for television at the end of 5 years? How is the total advertising budget changing?
11. Chris's old car has a major oil leak. He estimates that it loses about 25% of the oil in the engine every week, so he adds a quart of oil weekly. The capacity of the engine is 6 quarts of oil.
    a. Write a difference equation for the amount of oil in the engine as a function of time, measured in weeks.
    b. Find the solution to the difference equation.
    c. Use the solution to predict when the level of oil in the engine just after the weekly quart is added will be down to 5 quarts.

For each difference equation in Problems 12–19, decide whether the behavior of the solution is an increasing concave up pattern, a decreasing concave up pattern, an increasing concave down pattern, a decreasing concave down pattern, or none of these patterns. In each case, $x_0 = 50$.

12. $\Delta x_n = 5(1.04)^n$
13. $\Delta x_n = 5n^{2.5}$
14. $\Delta x_n = 8(0.85)^n$
15. $\Delta x_n = -4n^{0.3}$
16. $\Delta x_n = 12n^{-1.2}$
17. $\Delta x_n = 5n + 3$
18. $\Delta x_n = 25 - 2n$
19. $\Delta x_n = 12 - 3(0.90)^n$