Chapter 2 Vectors

"COROLLARY I. A body, acted on by two forces simultaneously, will describe the diagonal of a parallelogram in the same time as it would describe the sides by those forces separately."

Isaac Newton - "Principia"

2.1 Introduction

Of all the varied quantities that are observed in nature, some have the characteristics of scalar quantities while others have the characteristics of vector quantities. *A **scalar** quantity is a quantity that can be completely described by a magnitude, that is, by a number and a unit.* Some examples of scalar quantities are mass, length, time, density, and temperature. The characteristic of scalar quantities is that they add up like ordinary numbers. That is, if we have a mass m_1 = 3 kg and another mass m_2 = 4 kg then the sum of the two masses is

$$m = m_1 + m_2 = 3 \text{ kg} + 4 \text{ kg} = 7 \text{ kg} \tag{2.1}$$

*A **vector** quantity, on the other hand, is a quantity that needs both a magnitude and a direction to completely describe it.* Some examples of vector quantities are force, displacement, velocity, and acceleration. The velocity of a car moving at 50 km per hour (km/hr) due east can be represented by a vector. Velocity is a vector because it has a magnitude, 50 km/hr, and a direction, due east.

A vector is represented in this text book by boldface script, that is, **A.** Because we cannot write in boldface script on note paper or a blackboard, a vector is written there as the letter with an arrow over it. A vector can be represented on a diagram by an arrow. A picture of this vector can be obtained by drawing an arrow from the origin of a cartesian coordinate system, figure 2.1. *The length of the arrow represents the magnitude of the vector, while the direction of the arrow represents the direction of the vector.* The direction is specified by the angle θ that the vector makes with an axis, usually the x-axis, and is shown in figure 2.1. The magnitude of vector **A** is written as the absolute value of **A** namely |**A**|, or simply by the letter A without boldfacing. One of the defining characteristics of vector quantities is that they must be added in a way that takes their direction into account.

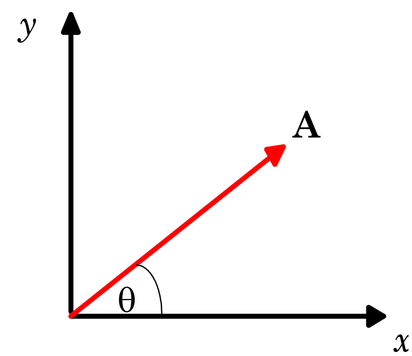


Figure 2.1 Representation of a vector.

2.2 The Displacement

Probably the simplest vector that can be discussed is the displacement vector. Whenever a body moves from one position to another it undergoes a displacement. *The displacement can be represented as a vector that describes how far and in what direction the body has been displaced from its original position.* The tail of the displacement vector is located at the position where the displacement started, and its tip is located at the position at which the displacement ended. For example, if you walk 3 km due east, this walk can be represented as a vector that is 3 units long and points due east. It is shown as $\mathbf{d}_1$ in figure 2.2. This is an example of a displacement vector. Suppose you now walk 4 km due north. This distance of 4 km in a northerly direction can be represented as another displacement vector $\mathbf{d}_2$, which is also shown in figure 2.2. The result of these two displacements is a final displacement vector **d** that shows the total displacement from the original position.

We now ask how far did you walk? Well, you walked 3 km east and 4 km north and hence you have walked a total distance of 7 km. But how far are you from where you started? Certainly not 7 km, as we can easily see using a little high school geometry. In fact the final displacement **d** is a vector of magnitude d and that distance d can be immediately determined by simple geometry. Applying the Pythagorean theorem to the right triangle of figure 2.2 we get

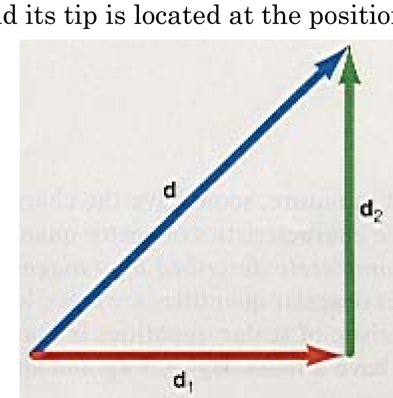


Figure 2.2 The displacement vector.

$$d = \sqrt{d_1^2 + d_2^2} \tag{2.2}$$

$$d = \sqrt{(3 \text{ km})^2 + (4 \text{ km})^2} = \sqrt{25 \text{ km}^2}$$

and thus,

$$d = 5 \text{ km}$$

Even though you have walked a total distance of 7 km, you are only 5 km away from where you started. Hence, when these vector displacements are added

$$\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2 \tag{2.3}$$

we do not get 7 km for the magnitude of the final displacement, but 5 km instead. *The displacement is thus a change in the position of a body from its initial position to its final position. Its magnitude is the distance between the initial position and the final position of the body.*

It should now be obvious that vectors do not add like ordinary scalar numbers. In fact, all the rules of algebra and arithmetic that you were taught in school are the rules of scalar algebra and scalar arithmetic, although the word scalar was probably never used at that time. To solve physical problems associated with vectors it is necessary to deal with vector algebra.

2.3 Vector Algebra - The Addition of Vectors

Let us now add any two arbitrary vectors **a** and **b.** The result of adding the two vectors **a** and **b** forms a new **resultant** vector **R,** which is the sum of **a** and **b.** This can be shown graphically by laying off the first vector **a** in the horizontal direction and then placing the tail of the second vector **b** at the tip of vector **a,** as shown in figure 2.3.

The resultant vector **R** is drawn from the origin of the first vector to the tip of the last vector. The resultant vector is written mathematically as

$$\mathbf{R} = \mathbf{a} + \mathbf{b} \tag{2.4}$$

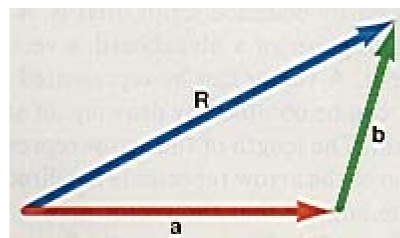


Figure 2.3 The addition of vectors.

Remember that in this sum we do not mean scalar addition. The resultant vector is the vector sum of the individual vectors **a** and **b.**

We can add these vectors graphically, with the aid of a ruler and a protractor. First, we need to choose a scale such that a unit distance on the graph paper corresponds to a unit of magnitude of the vector. Using this scale, we lay off the length that corresponds to the magnitude of vector **a** in the x-direction with a ruler. Then, at the tip of vector **a,** place the center of the protractor and measure the angle ϕ that vector **b** makes with the x-axis. Mark that direction on the paper. Using the ruler, measure off the length of vector **b** in the marked direction, as shown in figure 2.4. Now draw a line from the tail of vector **a** to the tip of vector **b.** This is the resultant vector **R.** Take the ruler and measure the length of vector **R** from the diagram. This length R is the magnitude of vector **R.** Using the protractor, measure the angle θ between **R** and the x-axis — this angle θ is the direction of the resultant vector **R.**

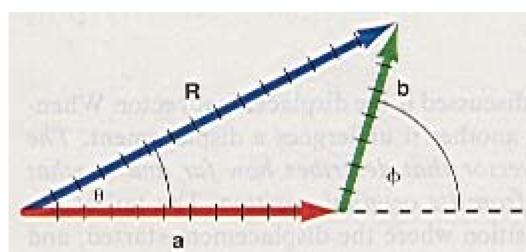


Figure 2.4 The graphical addition of vectors.

Although a vector is a quantity that has both magnitude and direction, it does not have a position. Consequently a vector may be moved parallel to itself without changing the characteristics of the vector. Because the magnitude of the moved vector is still the same, and its direction is still the same, the vector is the same.

Hence, when adding vectors **a** and **b,** we can move vector **a** parallel to itself until the tip of **a** touches the tip of **b.** Similarly, we can move vector **b** parallel to itself until the tip of **b** touches the tail end of the top vector **a.** In moving the vectors parallel to themselves we have formed a parallelogram, as shown in figure 2.5.

From what was said before about the resultant of **a** and **b,** we can see that the resultant of the two vectors is the main diagonal of the parallelogram formed by the vectors **a** and **b,** hence we call this process the **parallelogram method of vector addition.** It is sometimes stated as part of the definition of a vector, that vectors obey the parallelogram law of addition. Note from the diagram that

$$\mathbf{R} = \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \tag{2.5}$$

that is, vectors can be added in any order. Mathematicians would say vector addition is commutative.

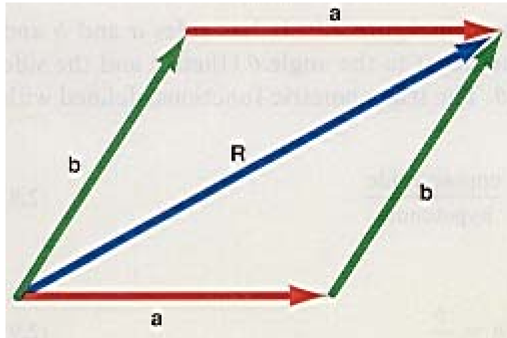


Figure 2.5 The addition of vectors by the parallelogram method.

2.4 Vector Subtraction -- The Negative of a Vector

If we are given a vector **a,** as shown in figure 2.6, then the vector minus **a** (–**a**) is a vector of the same magnitude as **a** but in the opposite direction. That is, if vector **a** points to the right, then the vector –**a** points to the left. Vector –**a** is called the negative of the vector **a.** By defining the negative of a vector in this way, we can now determine the process of vector subtraction. The subtraction of vector **b** from vector **a** is defined as

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) \tag{2.6}$$

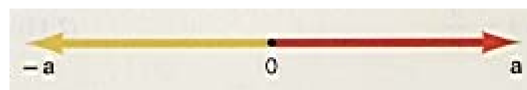
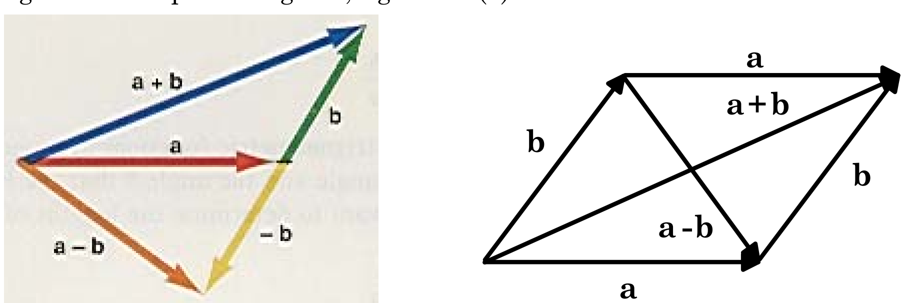


Figure 2.6 The negative of a vector.

In other words, the subtraction of **b** from **a** is equivalent to adding vector **a** and the negative vector (–**b**). This is shown graphically in figure 2.7(a) as the vector **a** – **b**. If we complete the parallelogram for the addition of **a** + **b**, we see that we can move the vector **a** – **b** parallel to itself until it becomes the minor diagonal of the parallelogram, figure 2.7(b).



(a) (b)

Figure 2.7 The subtraction of vectors.

2.5 Addition of Vectors by the Polygon Method

To find the sum of any number of vectors graphically, we use the polygon method. In the polygon method, we add each vector to the preceding vector by placing the tail of one vector to the head of the previous vector, as shown in figure 2.8. The resultant vector **R** is the sum of all these vectors. That is,

$$\mathbf{R} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} \tag{2.7}$$

We find **R** by drawing the vector from the origin of the coordinate system to the tip of the final vector, as shown in figure 2.8. Although this set of vectors could represent forces, velocities, and the like, it is sometimes easier for the beginning student to visualize them as though they were displacement vectors. It is easy to see from the figure that if **a**, **b**, **c**, and **d** were individual displacements, **R** would certainly be the resultant displacement of all the individual displacements.

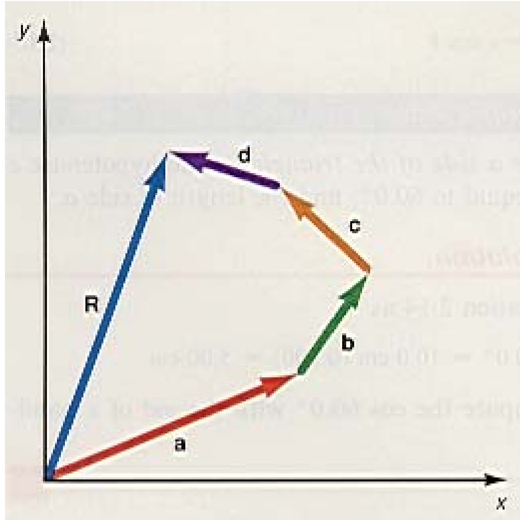


Figure 2.8 Addition of vectors by the polygon method.

Vectors are usually added analytically or mathematically. In order to do that, we need to define the components of a vector. However, to discuss the components of a vector, we first need a brief review of trigonometry.

2.6 Review of Trigonometry

Although we assume that everybody reading this book has been exposed to the fundamentals of trigonometry, the essential ideas and definitions of trigonometry will now be reviewed.

Consider the right triangle shown in figure 2.9. It has sides a and b and hypotenuse c. Side a is called the side adjacent to the angle θ (theta), and the side b is called the side opposite to the angle θ. The trigonometric functions, defined with respect to this triangle, are nothing more than ratios of the different sides of the triangle. The **sine function** is defined as the ratio of the opposite side of the triangle to the hypotenuse of the triangle, that is

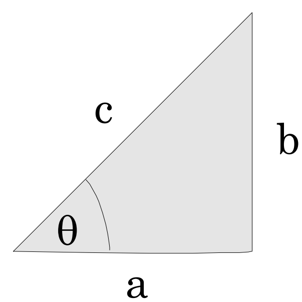


Figure 2.9 A simple right triangle.

$$\text{sine}\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \tag{2.8}$$

or

$$\sin\theta = \frac{b}{c} \tag{2.9}$$

The **cosine function** is defined as the ratio of the adjacent side of the triangle to the hypotenuse of the triangle,

$$\text{cosine}\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \tag{2.10}$$

or

$$\cos\theta = \frac{a}{c} \tag{2.11}$$

The **tangent function** is defined as the ratio of the opposite side of the triangle to the adjacent side of the triangle,

$$\text{tangent}\theta = \frac{\text{opposite side}}{\text{adjacent side}} \tag{2.12}$$

or

$$\tan\theta = \frac{b}{a} \tag{2.13}$$

Let us now review how these simple trigonometric functions are used. Assuming that the hypotenuse c of the right triangle and the angle θ that the hypotenuse makes with the x-axis are known, we want to determine the lengths of sides a and b of the triangle. From the definition of the cosine function,

$$\cos\theta = \frac{a}{c} \tag{2.11}$$

we can find the length of side a by multiplying both sides of equation 2.11 by c, that is,

$$a = c \cos\theta \tag{2.14}$$

Example 2.1

Using the cosine function to determine a side of the triangle. If the hypotenuse c is equal to 10.0 cm and the angle θ is equal to 60.0^0, find the length of side a.

Solution

The length of side a is found from equation 2.14 as

$$a = c \cos\theta = (10.0 \text{ cm}) \cos 60.0^0 = (10.0 \text{ cm})(0.500) = 5.00 \text{ cm}$$

(We assume here that anyone can compute the cos 60.0^0 with the aid of a hand-held calculator.)

To go to this interactive example click on this sentence.

To find side b of the triangle we use the definition of the sine function:

$$\sin\theta = \frac{b}{c} \tag{2.9}$$

Multiplying both sides of equation 2.9 by c we obtain

$$b = c \sin\theta \tag{2.15}$$

Example 2.2

Using the sine function to determine a side of the triangle. The hypotenuse c of a right triangle is 10.0 cm long, and the angle θ is equal to 60.0^0. Find the length of side b.

Solution

The length of side b is found from equation 2.15 as

$$b = c \sin\theta = 10.0 \text{ cm} \sin 60.0^0 = 10.0 \text{ cm } (0.866) = 8.66 \text{ cm}$$

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Therefore, if the hypotenuse and angle θ of a right triangle are given, the lengths of the sides a and b of that triangle can be determined by simple trigonometry.

Suppose that the lengths of sides a and b of a right triangle are given and we want to find the hypotenuse c and the angle θ of that triangle, as shown in figure 2.9. The hypotenuse is found by the **Pythagorean theorem** from elementary geometry which says that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. Hence

$$c^2 = a^2 + b^2 \tag{2.16}$$

and,

$$c = \sqrt{a^2 + b^2} \tag{2.17}$$

The angle θ is found from the definition of the tangent function,

$$\tan\theta = \frac{b}{a} \tag{2.13}$$

Using the inverse of the tangent function, sometimes called the arctangent, the angle θ becomes

$$\theta = \tan^{-1}\frac{b}{a} \tag{2.18}$$

Example 2.3

Using the Pythagorean theorem and the inverse tangent. The lengths of two sides of a right triangle are a = 3.00 cm and b = 4.00 cm. Find the hypotenuse of the triangle and the angle θ.

Solution

The hypotenuse of the triangle is found from equation 2.17 as

$$c = \sqrt{a^2 + b^2} = \sqrt{(3.00\text{ cm})^2 + (4.00\text{ cm})^2} = 5.00\text{ cm}$$

and the angle θ is found from equation 2.18 as

$$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{4.00\text{ cm}}{3.00\text{ cm}} = \tan^{-1}1.33 = 53.1^0$$

To go to this interactive example click on this sentence.

Therefore, if the lengths of the sides a and b of a right triangle are known we can easily calculate the hypotenuse and angle θ. We will repeatedly use these elementary concepts of trigonometry in the discussion of the components of a vector.

2.7 Resolution of a Vector into Its Components

An arbitrary vector **a** is drawn onto an x,y-coordinate system, as in figure 2.10. Vector **a** makes an angle θ with the x-axis. To find the x-component a_x of vector **a,** we project vector **a** down onto the x-axis, that is, we drop a perpendicular from the tip of **a** to the x-axis. One way of visualizing this concept of a **component of a vector** is to place a light beam above vector **a** and parallel to the y-axis. The light hitting vector **a** will not make it to the x-axis, and will therefore leave a shadow on the x-axis. We call this shadow on the x-axis the x-component of vector **a** and denote it by a_x. The component is shown as the light red line on the x-axis in figure 2.10.

In the same way, we can determine the y-component of vector **a,** a_y, by projecting **a** onto the y-axis in figure 2.10. That is, we drop a perpendicular from the tip of **a** onto the y-axis. Again, we can visualize this by projecting light, which is parallel to the x-axis, onto vector **a.** The shadow of vector **a** on the y-axis is the y-component a_y, shown in figure 2.10 as the light red line on the y-axis.

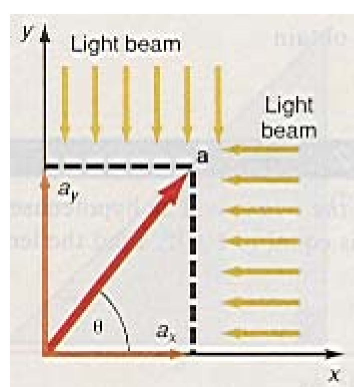

Figure 2.10 Defining the components of a vector.

The components of the vector are found mathematically by noting that the vector and its components constitute a triangle, as seen in figure 2.11. From trigonometry, we find the x-component of **a** from

$$\cos\theta = \frac{a_x}{a} \tag{2.19}$$

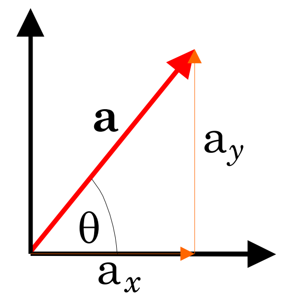

Figure 2.11
Finding the components of a vector mathematically.

Solving for a_x, the x-component of vector **a** obtained is

$$a_x = a\cos\theta \tag{2.20}$$

We find the y-component of vector **a** from

$$\sin\theta = \frac{a_y}{a} \quad 2.21)$$

Hence, the y-component of vector **a** is

$$a_y = a\sin\theta \tag{2.22}$$

Example 2.4

Finding the components of a vector. The magnitude of vector **a** is 15.0 units and the vector makes an angle of 35.0^0 with the x-axis. Find the components of **a**.

Solution

The x-component of vector **a,** found from equation 2.20, is

$$a_x = a\cos\theta = (15.0 \text{ units}) \cos 35.0^0 = 12.3 \text{ units}$$

The y-component of **a,** found from equation 2.22, is

$$a_y = a\sin\theta = (15.0 \text{ units}) \sin 35.0^0 = 8.60 \text{ units}$$

To go to this interactive example click on this sentence.

What do these components of a vector represent physically? If vector **a** is a displacement, then a_x would be the distance that the object is east of its starting point and a_y would be the distance north of it. That is, if you walked a distance of 15.0 km in a direction that is 35.0^0 north of east, you would be 12.3 km east of where you started from and 8.60 km north of where you started from. If, on the other hand, vector **a** were a force of 15.0 N applied at an angle of 35.0^0 to the x-axis, then the x-component a_x is equivalent to a force of 12.3 N in the x-direction, while the y-component a_y is equivalent to a force of 8.60 N in the y-direction.

2.8 Determination of a Vector from Its Components

If the components a_x and a_y of a vector are given, and we want to find the vector **a** itself, that is, its magnitude a and its direction θ, then the process is the inverse of the technique used in section 2.7. The components a_x and a_y of vector **a** are seen in figure 2.12. If we form the triangle with sides a_x and a_y, then the hypotenuse of that triangle is the magnitude a of the vector, and is determined by the Pythagorean theorem as

$$a^2 = a_x^2 + a_y^2 \tag{2.23}$$

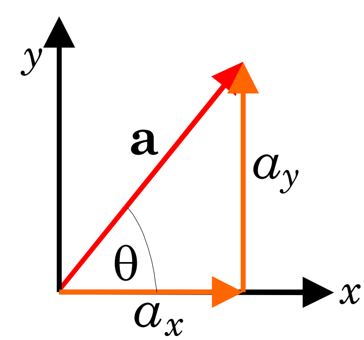

Figure 2.12 Determining a vector from its components.

Hence, the magnitude of vector **a** is

$$a = \sqrt{a_x^2 + a_y^2} \tag{2.24}$$

It is thus very simple to find the magnitude of a vector once its components are known.

To find the angle θ that vector **a** makes with the *x*-axis we use the definition of the tangent, namely

$$\text{tangent}\theta = \frac{\text{opposite side}}{\text{adjacent side}} \tag{2.12}$$

For the simple triangle of figure 2.12, the opposite side is a_y and the adjacent side is a_x. Therefore,

$$\tan\theta = \frac{a_y}{a_x} \tag{2.25}$$

We find the angle θ by using the inverse tangent, as

$$\theta = \tan^{-1}\frac{a_y}{a_x} \tag{2.26}$$

Example 2.5

Finding a vector from its components. The components of a certain vector are given as $a_x = 13.5$ and $a_y = 7.45$. Find the magnitude of the vector and the angle θ that it makes with the *x*-axis.

Solution

The magnitude of vector **a,** found from equation 2.24, is

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(13.5)^2 + (7.45)^2}$$
$$= 15.4$$

The angle θ, found from equation 2.26, is

$$\theta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\frac{7.45}{13.5} = \tan^{-1}0.552$$
$$= 28.9^0$$

Therefore, the magnitude of vector **a** is 15.4 and the angle θ is 28.9^0.

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The techniques developed here for finding the components of a vector from its magnitude and direction, and finding the magnitude of a vector and its direction from its components will be very useful later for the addition of any number of vectors.

The components of a vector can also be found along axes other than the traditional horizontal and vertical ones. A coordinate system can be orientated any way we choose. For example, suppose a block is placed on an inclined plane that makes an angle θ with the horizontal, as shown in figure 2.13. Let us find the components of the weight of the block parallel and perpendicular to the inclined plane.

We draw in a set of axes that are parallel and perpendicular to the inclined plane, as shown in figure 2.13, with the positive *x*-axis pointing down the plane and the positive *y*-axis perpendicular to the plane. To find the components parallel and perpendicular to the plane, we draw the weight of the block as a vector pointed toward the center of the earth. The weight vector is therefore perpendicular to the base of the inclined plane. To find the component of **w** perpendicular to the plane, we drop a perpendicular line from the tip of vector **w** onto the negative *y*-axis. This length $w_\perp$ is the perpendicular component of vector **w.** Similarly, to find the parallel component of **w,** we drop a perpendicular line from the tip of **w** onto the positive *x*-axis. This length $w_{||}$ is the parallel component of the vector **w.**

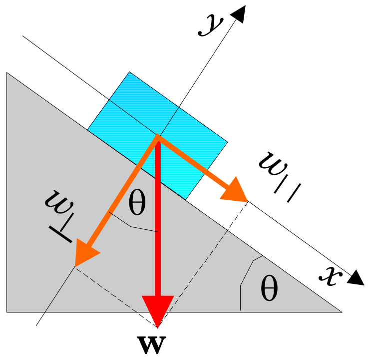

Figure 2.13 Components of the weight parallel and perpendicular to the inclined plane.

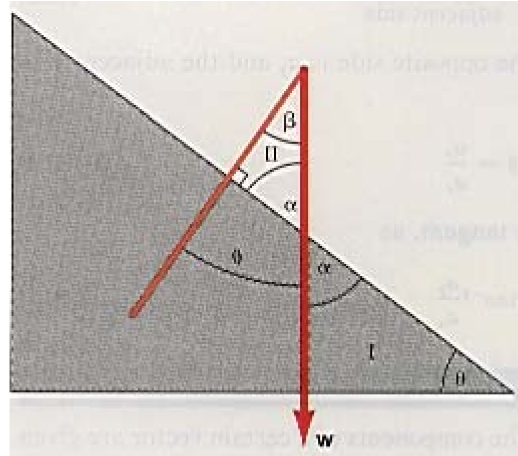

Figure 2.14 Comparison of two triangles.

The angle between vector **w** and the perpendicular axis is also the inclined plane angle θ, as shown in the comparison of the two triangles in figure 2.14. (Figure 2.14 is an enlarged view of the two triangles of figure 2.13). In triangle I, the angles must add up to 180^0. Thus,

$$\theta + \alpha + 90^0 = 180^0 \tag{2.27}$$

while for triangle II

$$\beta + \alpha + 90^0 = 180^0 \tag{2.28}$$

From equations 2.27 and 2.28 we see that

$$\beta = \theta \tag{2.29}$$

This is an important relation that we will use every time we use an inclined plane.

Example 2.6

Components of the weight perpendicular and parallel to the inclined plane. A 100-N block is placed on an inclined plane with an angle $\theta = 50.0^0$, as shown in figure 2.13. Find the components of the weight of the block parallel and perpendicular to the inclined plane.

Solution

We find the perpendicular component of **w** from figure 2.13 as

$$w_\perp = w \cos\theta \tag{2.30}$$
$$= 100 \text{ N} \cos 50.0^0 = 64.3 \text{ N}$$

The parallel component is

$$w_{||} = w \sin\theta \tag{2.31}$$
$$= 100 \text{ N} \sin 50.0^0 = 76.6 \text{ N}$$

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One of the interesting things about this inclined plane is that the component of the weight parallel to the inclined plane supplies the force responsible for making the block slide down the plane. Similarly, if you park your car on a hill with the gear in neutral and the emergency brake off, the car will roll down the hill. Why? You can now see that it is the component of the weight of the car that is parallel to the hill that essentially pushes the car down the hill. That force is just as real as if a person were pushing the car down the hill. That force, as can be seen from equation 2.31, is a function of the angle θ. If the angle of the plane is reduced to zero, then

$$w_{||} = w \sin 0^0 = 0$$

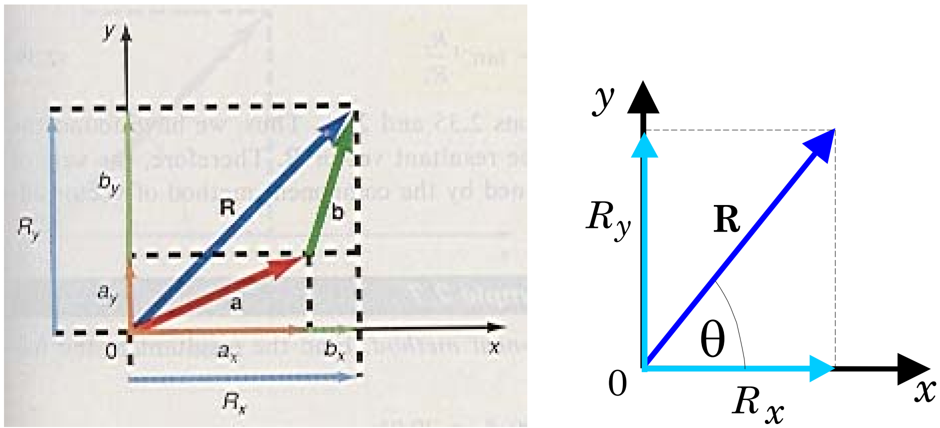
Thus, we can reasonably conclude that when a car is not on a hill (i.e., when $\theta = 0^0$) there is no force, due to the weight of the car, to cause the car to move. Also note that the steeper the hill, the greater the angle θ, and hence the greater the component of the force acting to move the car down the hill.

2.9 The Addition of Vectors by the Component Method

A very important technique for the addition of vectors is **the addition of vectors by the component method.** Let us assume that we are given two vectors, **a** and **b,** and we want to find their vector sum. The sum of the vectors is the resultant vector **R** given by

$$\mathbf{R} = \mathbf{a} + \mathbf{b} \tag{2.32}$$

and is shown in figure 2.15. We determine **R** as follows. First, we find the components a_x and a_y of vector **a** by



(a) (b)

Figure 2.15 The addition of vectors by the component method.

making the projections onto the x- and y-axes, respectively. To find the components of the vector **b,** we again make a projection onto the x- and y-axes, but note that the tail of vector **b** is not at the origin of coordinates, but rather at the tip of **a.** So both the tip and the tail of **b** are projected onto the x-axis, as shown, to get b_x, the x-component of **b.** In the same way, we project **b** onto the y-axis to get b_y, the y-component of **b.** All these components are shown in figure 2.15(a).

The resultant vector **R** is given by equation 2.32, and because **R** is a vector it has components R_x and R_y, which are the projections of **R** onto the x- and y-axes, respectively. They are shown in figure 2.15(b). Now let us go back to the original diagram, figure 2.15(a), and project **R** onto the x-axis. Here R_x is shown a little distance below the x-axis, so as not to confuse R_x with the other components that are already there. Similarly, **R** is projected onto the y-axis to get R_y. Again R_y is slightly displaced from the y-axis, so as not to confuse R_y with the other components already there.

Look very carefully at figure 2.15(a). Note that the length of R_x is equal to the length of a_x plus the length of b_x. Because components are numbers and hence add like ordinary numbers, this addition can be written simply as

$$R_x = a_x + b_x \tag{2.33}$$

That is, *the x-component of the resultant vector is equal to the sum of the x-components of the individual vectors.*

In the same manner, look at the geometry on the y-axis of figure 2.15(a). The length R_y is equal to the sum of the lengths of a_y and b_y, and therefore

$$R_y = a_y + b_y \tag{2.34}$$

Thus, *the y-component of the resultant vector is equal to the sum of the y-components of the individual vectors.* We demonstrated the addition of vectors for only two vectors because it is easier to see the results in figure 2.15 for two vectors than it would be for many vectors. However, the technique is the same for the addition of any number of vectors. For the general case, where there are many vectors, equations 2.33 and 2.34 for R_x and R_y can be generalized to

$$R_x = a_x + b_x + c_x + d_x + \dots \tag{2.35}$$

and

$$R_y = a_y + b_y + c_y + d_y + \dots \tag{2.36}$$

The plus sign and the dots that appear at the far right in equations 2.35 and 2.36 indicate that additional components can be added for any additional vectors.

We now have R_x and R_y, the components of the resulting vector **R.** But if we know the components of **R,** we can find the magnitude of **R** by using the Pythagorean theorem, that is,

$$R = \sqrt{R_x^2 + R_y^2} \tag{2.37}$$

The angle θ in figure 2.15(b), found from the geometry, is

$$\tan\theta = \frac{R_y}{R_x} \tag{2.38}$$

Thus,

$$\theta = \tan^{-1}\frac{R_y}{R_x} \tag{2.39}$$

where R_x and R_y are given by equations 2.35 and 2.36. Thus, we have found the magnitude R and the direction θ of the resultant vector **R.** Therefore, the sum of any number of vectors can be determined by the component method of vector addition.

Example 2.7

The addition of vectors by the component method. Find the resultant of the following four vectors:

$A = 100,\ \theta_1 = 30.0^0$
$B = 200,\ \theta_2 = 60.0^0$
$C = 75.0,\ \theta_3 = 140^0$
$D = 150,\ \theta_4 = 250^0$

Solution

The four vectors are drawn in figure 2.16. Because any vector can be moved parallel to itself, all the vectors have been moved so that they are drawn as emanating from the origin. Before actually solving the problem, let us first outline the solution. To find the resultant of these four vectors, we must first find the individual components of each vector, then we find the *x*- and *y*-components of the resulting vector from

$$R_x = A_x + B_x + C_x + D_x \tag{2.35}$$
$$R_y = A_y + B_y + C_y + D_y \tag{2.36}$$

We then find the resulting vector from

$$R = \sqrt{R_x^2 + R_y^2} \tag{2.37}$$

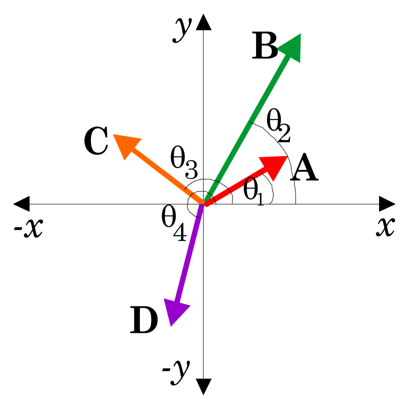


Figure 2.16 Addition of four vectors.

and

$$\theta = \tan^{-1}\frac{R_y}{R_x} \tag{2.39}$$

The actual solution of the problem is found as follows: we find the individual *x*-components as

$A_x = A\cos\theta_1 = 100\cos 30.0^0 = 100(0.866) = 86.6$
$B_x = B\cos\theta_2 = 200\cos 60.0^0 = 200(0.500) = 100.0$
$C_x = C\cos\theta_3 = 75\cos 140^0 = 75(-0.766) = -57.5$
$D_x = D\cos\theta_4 = 150\cos 250^0 = 150(-0.342) = \underline{-51.3}$
$R_x = A_x + B_x + C_x + D_x = 77.8$

whereas the y-components are

$$A_y = A \sin \theta_1 = 100 \sin 30.0^0 = 100(0.500) = 50.0$$
$$B_y = B \sin \theta_2 = 200 \sin 60.0^0 = 200(0.866) = 173.0$$
$$C_y = C \sin \theta_3 = 75 \sin 140^0 = 75(0.643) = 48.2$$
$$D_y = D \sin \theta_4 = 150 \sin 250^0 = 150(-0.940) = \underline{-141.0}$$
$$R_y = A_y + B_y + C_y + D_y = 130.2$$

The x- and y-components of vector **R** are shown in figure 2.17. Because R_x and R_y are both positive, we find vector **R** in the first quadrant. If R_x were negative, **R** would have been in the second quadrant. It is a good idea to plot the components R_x and R_y for any addition so that the direction of **R** is immediately apparent.

We find the magnitude of the resultant vector from equation 2.37 as

$$R = \sqrt{R_x^{\ 2} + R_y^{\ 2}} = \sqrt{(77.8)^2 + (130.2)^2} = \sqrt{23{,}004.8}$$
$$= 152$$

The angle θ that vector **R** makes with the x-axis is found as

$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{130.2}{77.8} = \tan^{-1} 1.674$$
$$= 59.1^0$$

as is seen in figure 2.17.

It is important to note here that the components C_x, D_x, and D_y are negative numbers. This is because C_x and D_x lie along the negative x-axis and D_y lies along the negative y-axis. We should note that in the solution of the components of the vector **C** in this problem, the angle of 140^0 was entered directly into the calculator to give the solution for the cosine and sine of that angle. The calculator automatically gives the correct sign for the components if we always measure the angle from the positive x-axis.[1]

To go to this interactive example click on this sentence.

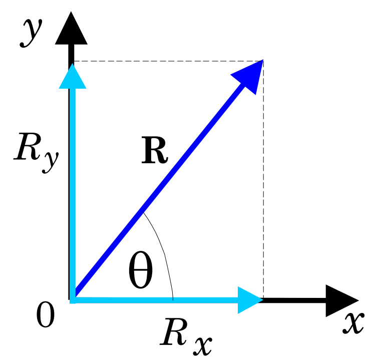


Figure 2.17 The resultant vector.

Example 2.8

The necessity of taking the wind velocity into account when flying an airplane. An airplane is flying due east from city A to city B with an airspeed of 250 km/hr. A wind is blowing from the northwest at 75.0 km/hr. Find the velocity of the airplane with respect to the ground.

Solution

The velocity of the plane with respect to the air is shown as the vector $\mathbf{v}_{PA}$ in figure 2.18. If there were no wind present, the plane would fly in a straight line from city A to city B. However, there is a wind blowing and it is shown as the vector $\mathbf{v}_{AG}$, the velocity of air with respect to the ground. This wind blows the plane away from the straight line motion from A to B. The total velocity of the plane with respect to the ground is the vector sum of $\mathbf{v}_{PA}$ and $\mathbf{v}_{AG}$. That is,

$$\mathbf{v}_{PG} = \mathbf{v}_{PA} + \mathbf{v}_{AG}$$

[1] We can also measure the angle that the vector makes with any axis other than the positive x-axis. For example, instead of using the angle of 140^0 with respect to the positive x-axis, an angle of 40^0 with respect to the negative x-axis can be used to describe the direction of vector **C.** The x-component of vector **C** would then be given by $C_x = C \cos 40^0 = 75.0 \cos 40^0 = 57.5$. Note that this is the same numerical value we obtained before, however the answer given by the calculator is now positive. But as we can see in figure 2.16, C_x is a negative quantity because it lies along the negative x-axis. Hence, if you do not use the angle with respect to the positive x-axis, you must add the positive or negative sign that is associated with that component. In most of the problems that will be covered in this text, we will measure the angle from the positive x-axis because of the simplicity of the calculation. However, whenever it is more convenient to measure the angle from any other axis, we will do so.

A wind from the northwest makes an angle of -45^0 or $+315^0$ with the positive x-axis. We find the x-component of the resulting velocity as

$$(v_{PA})_x = v_{PA} \cos \theta_2 = 250 \text{ km/hr} \cos 0^0 \quad = 250 \text{ km/hr}$$
$$(v_{AG})_x = v_{AG} \cos \theta_1 = 75.0 \text{ km/hr} \cos 315^0 \quad = \underline{\ 53.0 \text{ km/hr}}$$
$$(v_{PG})_x = (v_{PA})_x + (v_{AG})_x = 303 \text{ km/hr}$$

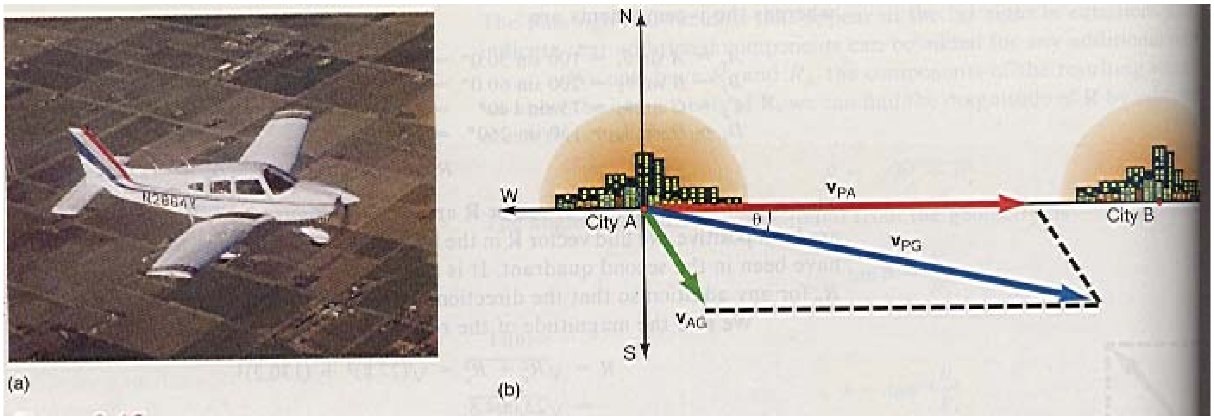


Figure 2.18 When flying an airplane, the velocity of the wind must be taken into account.

While the y-component of the resulting velocity is

$$(v_{PA})_y = v_{PA} \sin \theta_2 = 250 \text{ km/hr} \sin 0^0 \quad = \ 00.0 \text{ km/hr}$$
$$(v_{AG})_y = v_{AG} \sin \theta_1 = 75.0 \text{ km/hr} \sin 315^0 = \underline{-53.0 \text{ km/hr}}$$
$$(v_{PG})_y = (v_{PA})_y + (v_{AG})_y = -53.0 \text{ km/hr}$$

The magnitude of the resulting velocity of the plane with respect to the ground is

$$v_{PG} = \sqrt{\left[(v_{PG})_x\right]^2 + \left[(v_{PG})_y\right]^2}$$
$$= \sqrt{(303 \text{ km/hr})^2 + (-53.0 \text{ km/hr})^2}$$
$$= 308 \text{ km/hr}$$

Even though the aircraft airspeed indicator is reading 250 km/hr, the aircraft is actually moving at 308 km/hr with respect to the ground because of the wind. The angle that the velocity vector $\mathbf{v}_{PG}$ makes with the positive x-axis is

$$\theta = \tan^{-1} \frac{(v_{PG})_y}{(v_{PG})_x}$$
$$\theta = \tan^{-1} \frac{-53.0 \text{ km/hr}}{303 \text{ km/hr}}$$
$$= -9.93^0$$

Thus the direction of the aircraft as it moves over the ground is 9.93^0 south of east. If the pilot does not make a correction, he or she will not arrive at city B as expected.

To go to this interactive example click on this sentence.

Example 2.9

The zero vector. Given the two vectors

$$A = 55.8, \ \theta_1 = 35.0^0$$
$$B = 84.7, \ \theta_2 = 155^0$$

Find the vector **C** that makes the sum of these vectors equal to zero.

Solution

For the sum of all the vectors to be zero, the resultant must be equal to zero. That is,

$$\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C} = 0$$

If **R** is to be zero, then its components must also be zero, hence

$$R_x = A_x + B_x + C_x = 0$$

and hence the x-component of the vector **C** that makes the sum equal to zero is

$$C_x = -(A_x + B_x)$$

Similarly, for the y-component

$$R_y = A_y + B_y + C_y = 0$$

and hence the y-component of the vector **C** that makes the sum equal to zero is

$$C_y = -(A_y + B_y)$$

The x-components are

$$A_x = A\cos\theta_1 = 55.8\cos 35.0^0 = 55.8(0.819) = \quad 45.7$$
$$B_x = B\cos\theta_2 = 84.7\cos 155^0 = 84.7(0.500) = \underline{-76.8}$$
$$C_x = -(A_x + B_x) = -(-31.1) = 31.1$$

whereas the y-components are

$$A_y = A\sin\theta_1 = 55.8\sin 35.0^0 = 55.8(0.574) = 32.0$$
$$B_y = B\sin\theta_2 = 84.7\sin 155^0 = 84.7(0.423) = \underline{35.8}$$
$$C_y = -(A_y + B_y) = -(67.8)$$

Because C_x is positive and C_y is negative, the vector **C** is in the fourth quadrant. We find the magnitude of the vector **C** as

$$C = \sqrt{C_x^{\ 2} + C_y^{\ 2}} = \sqrt{(31.1)^2 + (-67.8)^2} = \sqrt{5564.05}$$
$$= 74.6$$

The angle θ that vector **C** makes with the x-axis is found as

$$\theta = \tan^{-1}\frac{C_y}{C_x} = \tan^{-1}\frac{-67.8}{31.1} = \tan^{-1} - 2.180$$
$$= -65.4^0$$

Hence the vector **C**, that when added to the vectors **A** and **B** gives a resultant of 0, has a magnitude C = 74.6 and is located in the fourth quadrant at an angle of -65.4^0, or $+294.6^0$ with respect to the positive x-axis.

To go to this interactive example click on this sentence.

The Language of Physics

Scalar
A scalar quantity is a quantity that can be completely described by a magnitude, that is, by a number and a unit (p.).

Vector
A vector quantity is a quantity that needs both a magnitude and direction to completely describe it (p.).

Resultant
The vector sum of any number of vectors is called the resultant vector (p.).

Parallelogram method of vector addition
The main diagonal of a parallelogram is equal to the magnitude of the sum of the vectors that make up the sides of the parallelogram (p.).

Sine function
The ratio of the length of the opposite side to the length of the hypotenuse in a right triangle (p.).

Cosine function
The ratio of the length of the adjacent side to the length of the hypotenuse in a right triangle (p.).

Tangent function
The ratio of the length of the opposite side of a right triangle to the length of the adjacent side (p.).

Pythagorean theorem
The sum of the squares of the lengths of two sides of a right triangle is equal to the square of the length of the hypotenuse (p.).

Component of a vector
The projection of a vector onto a specified axis. The length of the projection of the vector onto the x-axis is called the x-component of the vector. The length of the projection of the vector onto the y-axis is called the y-component of the vector (p.).

The addition of vectors by the component method
The x-component of the resultant vector R_x is equal to the sum of the x-components of the individual vectors, while the y-component of the resultant vector R_y is equal to the sum of the y-components of the individual vectors. The magnitude of the resultant vector is then found by the Pythagorean theorem applied to the right triangle with sides R_x and R_y. The direction of the resultant vector is found by trigonometry (p.).

Summary of Important Equations

Vector addition is commutative
$$\mathbf{R} = \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \qquad (2.5)$$

Subtraction of vectors
$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) \qquad (2.6)$$

Addition of vectors
$$\mathbf{R} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} \qquad (2.7)$$

Definition of the sine
$$\text{sine}\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad (2.8)$$

Definition of the cosine
$$\text{cosine}\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad (2.10)$$

Definition of the tangent
$$\text{tangent}\theta = \frac{\text{opposite side}}{\text{adjacent side}} \qquad (2.12)$$

Pythagorean theorem
$$c = \sqrt{a^2 + b^2} \qquad (2.17)$$

x-component of a vector
$$a_x = a \cos \theta \qquad (2.20)$$

y-component of a vector
$$a_y = a \sin \theta \qquad (2.22)$$

Magnitude of a vector
$$a = \sqrt{a_x{}^2 + a_y{}^2} \qquad (2.24)$$

Direction of a vector
$$\theta = \tan^{-1} \frac{a_y}{a_x} \qquad (2.26)$$

x-component of resultant vector
$$R_x = a_x + b_x + c_x + d_x \qquad (2.35)$$

y-component of resultant vector
$$R_y = a_y + b_y + c_y + d_y \qquad (2.36)$$

Magnitude of resultant vector
$$R = \sqrt{R_x{}^2 + R_y{}^2} \qquad (2.37)$$

Direction of resultant vector
$$\theta = \tan^{-1} \frac{R_y}{R_x} \qquad (2.39)$$

Questions for Chapter 2

1. Give an example of some quantities that are scalars and vectors other than those listed in section 2.1.
2. Can a vector ever be zero? What does a zero vector mean?
*3. Since time seems to pass from the past to the present and then to the future, can you say that time has a direction and therefore could be represented as a vector quantity?
4. Does the subtraction of two vectors obey the commutative law?
5. What happens if you multiply a vector by a scalar?
6. What happens if you divide a vector by a scalar?
7. If a person walks around a block that is 80 m on each side and ends up at the starting point, what is the person's displacement?
8. How can you add three vectors of equal magnitude in a plane such that their resultant is zero?
9. When are two vectors **a** and **b** equal?
*10. If a coordinate system is rotated, what does this do to the vector? to the components?
*11. Why are all the fundamental quantities scalars?
12. A vector equation is equivalent to how many component equations?

13. If the components of a vector **a** are a_x and a_y, what are the components of the vector $\mathbf{b} = -5\mathbf{a}$?

14. If $\mathbf{a} + \mathbf{b} = \mathbf{a} - \mathbf{b}$, what is the angle between **a** and **b**?

Problems for Chapter 2

2.7- 2.8 Resolution of a Vector into Its Components and Determination of a Vector from Its Components

1. A strong child pulls a sled with a force of 300 N at an angle of 35^0 above the horizontal. Find the vertical and horizontal components of this pull.

2. A 50-N force is directed at an angle of 50^0 above the horizontal. Resolve this force into vertical and horizontal components.

3. A boy wants to hold a 68.0-N sled at rest on a snow-covered hill. The hill makes an angle of 27.5^0 with the horizontal. (a) What force must he exert parallel to the slope? (b) What is the force perpendicular to the surface of the hill that presses the sled against the hill?

4. A displacement vector, at an angle of 35^0 with respect to a specified direction, has a y-component equal to 150 cm. What is the magnitude of the displacement vector?

5. A plane is traveling northeast at 200 km/hr. What is (a) the northward component of its velocity, and (b) the eastward component of its velocity?

6. While taking off, an airplane climbs at an 8^0 angle with respect to the ground. If the aircraft's speed is 200 km/hr, what are the vertical and horizontal components of its velocity?

7. A car that weighs 8900 N is parked on a hill that makes an angle of 43^0 with the horizontal. Find the component of the car's weight parallel to the hill and perpendicular to the hill.

8. A girl pushes a lawn mower with a force of 90 N. The handle of the mower makes an angle of 40^0 with the ground. What are the vertical and horizontal components of this force and what are their physical significances? What effect does raising the handle to 50^0 have?

9. A missile is launched with a speed of 1000 m/s at an angle of 73^0 above the horizontal. What are the horizontal and vertical components of the missile's velocity?

10. When a ladder leans against a smooth wall, the wall exerts a horizontal force **F** on the ladder, as shown in the diagram. If F is equal to 50 N and θ is equal to 63^0, find the component of the force perpendicular to the ladder and the component parallel to the ladder.

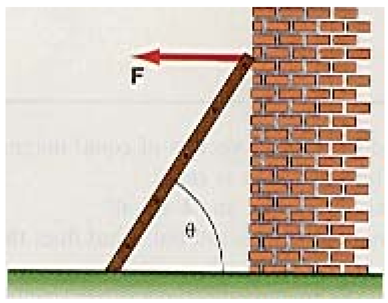


Diagram for problem 10.

2.9 The Addition of Vectors by the Component Method

11. Find the resultant of the following three displacements; 3 km due east, 6 km east-northeast, and 7 km northwest.

12. A girl drives 3 km north, then 12 km to the northwest, and finally 5 km south-southwest. How far has she traveled? What is her displacement?

13. An airplane flies due north at 380 km/hr straight from city A to city B. A southeast wind of 75 km/hr is blowing. (Note that all winds are defined in terms of the direction from which the wind blows. Hence, a southeast wind blows out of the southeast and blows toward the northwest.) What is the resultant velocity of the plane with respect to the ground?

14. Find the resultant of the following forces: (a) 30 N at an angle of 40^0 with respect to the x-axis, (b) 120 N at an angle of 135^0, and (c) 60 N at an angle of 260^0.

15. Find the resultant of the following set of forces. (a) $\mathbf{F}_1$ of 200 N at an angle of 53^0 with respect to the x-axis. (b) $\mathbf{F}_2$ of 300 N at an angle of 150^0 with respect to the x-axis. (c) $\mathbf{F}_3$ of 200 N at an angle of 270^0 with respect to the x-axis. (d) $\mathbf{F}_4$ of 350 N at an angle of 310^0 with respect to the x-axis.

Additional Problems

16. A heavy trunk weighing 800 N is pulled along a smooth station platform by a 210-N force making an angle of 53^0 above the horizontal. Find (a) the horizontal component of the force, (b) the vertical component of the force, and (c) the resultant downward force on the floor.

17. Vector **A** has a magnitude of 15.0 m and points in a direction of 50^0 north of east. What are the magnitudes and directions of the vectors, (a) 2**A**, (b) 0.5**A**, (c) –**A**, (d) –5**A**, (e) **A** + 4**A**, (f) **A** – 4**A**?

18. Given the two force vectors $F_1 = 20.0$ N at an angle of 30.0^0 with the positive x-axis and $F_2 = 40.0$ N at an angle of 150.0^0 with the positive x-axis, find the magnitude and direction of a third force that when added to $\mathbf{F}_1$ and $\mathbf{F}_2$ gives a zero resultant.

19. When vector **A**, of magnitude 5.00 m/s at an angle of 120^0 with respect to the positive x-axis, is added to a second vector **B**, the resultant vector has a magnitude $R = 8.00$ m/s and is at an angle of 85.0^0 with the positive x-axis. Find the vector **B**.

20. A car travels 100 km due west and then 45.0 km due north. How far is the car from its starting point? Solve graphically and analytically.

21. Find the resultant of the following forces graphically and analytically: 25 N at an angle of 53^0 above the horizontal and 100 N at

an angle of 117⁰ counterclockwise from the horizontal.

*22. The velocity of an aircraft is 200 km/hr due west. A northwest wind of 50 km/hr is blowing. (a) What is the velocity of the aircraft relative to the ground? (b) If the pilot's destination is due west, at what angle should he point his plane to get there? (c) If his destination is 400 km due west, how long will it take him to get there?

23. A plane flies east for 50.0 km, then at an angle of 30.0⁰ north of east for 75.0 km. In what direction should it now fly and how far, such that it will be 200 km northwest of its original position?

*24. The current in a river flows south at 7 km/hr. A boat starts straight across the river at 19 km/hr relative to the water. (a) What is the speed of the boat relative to the land? (b) If the river is 1.5 km wide, how long does it take the boat to cross the river? (c) If the boat sets out straight for the opposite side, how far south will it reach the opposite shore? (d) If we want to have the boat go straight across the river, at what angle should the boat be headed?

*25. Show that if the angle between vectors **a** and **b** is an acute angle, then the sum **a** + **b** becomes the main diagonal of the parallelogram and the difference **a** – **b** becomes the minor diagonal of the parallelogram. Also show that if the angle is obtuse the results are reversed.

26. Find the resultant of the following three vectors. The magnitudes of the vectors are $a = 5.00$ km, $b = 10.0$ km, and $c = 20.0$ km.

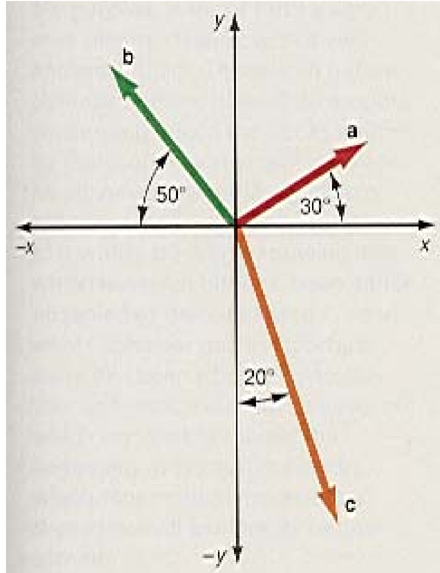


Diagram for problem 26.

27. Find the resultant of the following three forces. The magnitudes of the forces are $F_1 = 2.00$ N, $F_2 = 8.00$ N, and $F_3 = 6.00$ N.

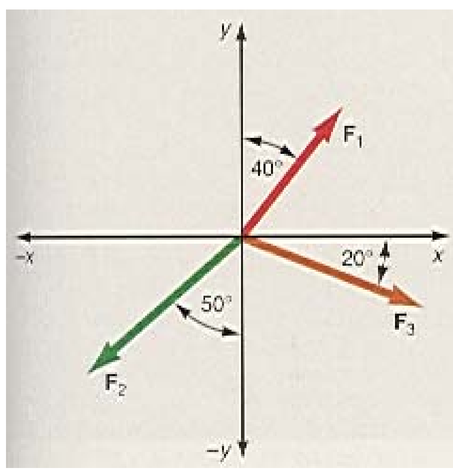


Diagram for problem 27.

*28. Show that for three nonparallel vectors all in the same plane, any one of them can be represented as a linear sum of the other two.

*29. A unit vector is a vector that has a magnitude of one unit and is in a specified direction. If a unit vector **i** is defined to be in the x-direction, and a unit vector **j** is defined to be in the y-direction, show that any vector **a** can be written in the form

$$\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j}$$

*30. Prove that

$$|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|.$$

31. An airplane flies due east at 200 km/hr straight from city A to city B a distance of 200 km. A wind of 40 km/hr from the northwest is blowing. If the pilot doesn't make any corrections, where will the plane be in 1 hr?

32. Given vectors **a** and **b,** where $a = 50$, $\theta_1 = 33^0$, $b = 80$, and $\theta_2 = 128^0$, find (a) **a + b,** (b) **a** – **b,** (c) **a** – 2**b,** (d) 3**a** + **b,** (e) 2**a** – **b,** and (f) 2**b** – **a.**

33. In the accompanying figure the tension T in the cable is 200 N. Find the vertical component T_y and the horizontal component T_x of this tension.

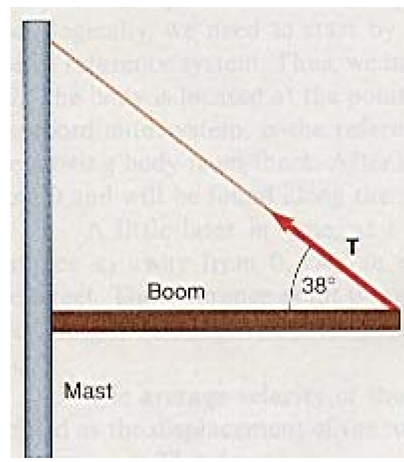


Diagram for problem 33.

*34. In the accompanying diagram w_1 is 5 N and w_2 is 3 N. Find the angle θ such that the component of w_1 parallel to the incline is equal to w_2.

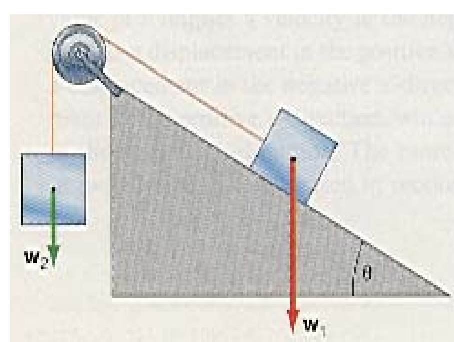


Diagram for problem 34.

*35. In the accompanying diagram $w_1 = 2$ N, $w_2 = 5$ N, and θ =

65^0. Find the angle ϕ such that the components of the two forces parallel to the inclines are equal.

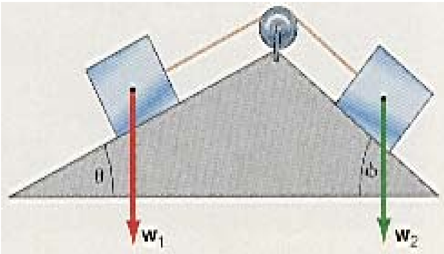


Diagram for problem 35.

*36. In the accompanying diagram w = 50 N, and θ = 10^0. What must be the value of F such that w will be held in place? What happens if the angle is doubled to 20^0?

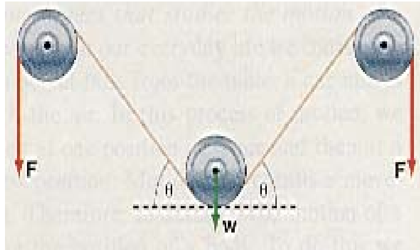


Diagram for problem 36.

*37. In projectile motion in two dimensions the projectile is located by the displacement vector $\mathbf{r}_1$ at the time t_1 and by the displacement vector $\mathbf{r}_2$ at t_2, as shown in the diagram. If r_1 = 20 m, θ_1 = 60^0, r_2 = 25 m, and θ_2 = 25^0, find the magnitude and direction of the vector $\mathbf{r}_2 - \mathbf{r}_1$.

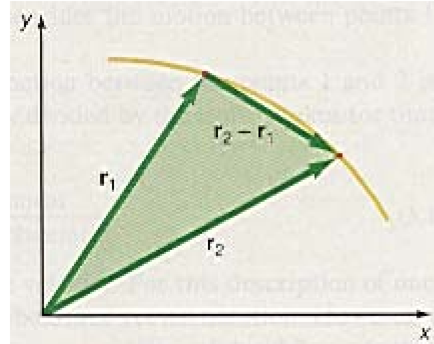


Diagram for problem 37.

Interactive Tutorials

38. *The components of a vector.* A 50.0-N force is directed at an angle of 50^0 above the horizontal. Resolve this force into vertical and horizontal components.

39. *Resultant vector.* Find the resultant of any number of force vectors (up to five vectors).

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