Chapter 9 Rotational Motion

In experimental philosophy we are to look upon propositions inferred by general induction from phenomena as accurately or very nearly true, notwithstanding any contrary hypothesis that may be imagined, till such time as other phenomena occur, by which they may either be made more accurate, or liable to exceptions.
Isaac Newton

9.1 Introduction

Up to now, the main emphasis in the description of the motion of a body dealt with the translational motion of that body. But in addition to translating, a body can also rotate about some axis, called the *axis of rotation*. Therefore, for a complete description of the motion of a body we also need to describe any rotational motion the body might have. As a matter of fact, the most general motion of a rigid body is composed of the translation of the center of mass of the body plus its rotation about the center of mass.

In the analysis of rotational motion, we will see a great similarity to translational motion. In fact, this chapter can serve as a review of all the mechanics discussed so far.

9.2 Rotational Kinematics

In the study of translational kinematics the first concept we defined was the position of an object. The position of the body was defined as the displacement x from a reference point. In a similar way, the position of a point on a rotating body is defined by the **angular displacement** θ from some reference line that connects the point to the axis of rotation, as shown in figure 9.1. That is, if the point was originally at P, and a little later it is at the point P', then the body has rotated through the angular displacement θ. If the angular displacement is small it can be represented as a vector that is perpendicular to the plane of the motion.[1] If the angular displacement is a positive quantity, the rotation of the body is counterclockwise and the angular displacement vector points upward. If the angular displacement is a negative quantity, the rotation of the body is clockwise and the angular displacement vector points downward. The magnitude of the angular displacement is the angle θ itself. We measure the angle θ in radians, which we defined in section 6.3.

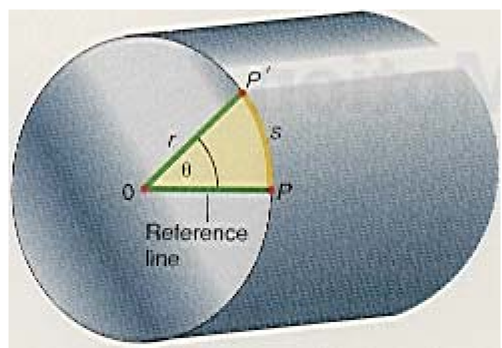


Figure 9.1 The angular displacement.

The linear distance between the points P and P' is given by the arc length s, and is related to the angular displacement by

$$s = r\theta \tag{6.5}$$

The average velocity of a translating body was defined as the displacement of the body divided by the time for that displacement:

$$v_{avg} = \frac{\Delta x}{\Delta t}$$

In the same way, *the average* ***angular velocity*** *of a rotating body is defined as the angular displacement of the body about the axis of rotation divided by the time for that displacement:*

$$\omega_{avg} = \frac{\Delta \theta}{\Delta t} \tag{9.1}$$

The units for angular velocity are radians/second, abbreviated as rad/s. It is important to remember that the radian is a dimensionless quantity, and can be added or deleted from an equation whenever it is convenient. The angular velocity, like the angular displacement, can also be represented as a vector quantity that is also perpendicu-

[1] See question 10 at the end of this chapter

lar to the plane of the motion. It is positive and points upward for counterclockwise rotations and is negative and points downward for clockwise rotations.

The similarities between translational and rotational motion can be seen in table 9.1. The relation between the linear velocity of a point on the rotating body and the angular velocity of the body is found by dividing both sides of equation 6.5 by t, that is,

$$\frac{s}{t} = \frac{r\theta}{t}$$

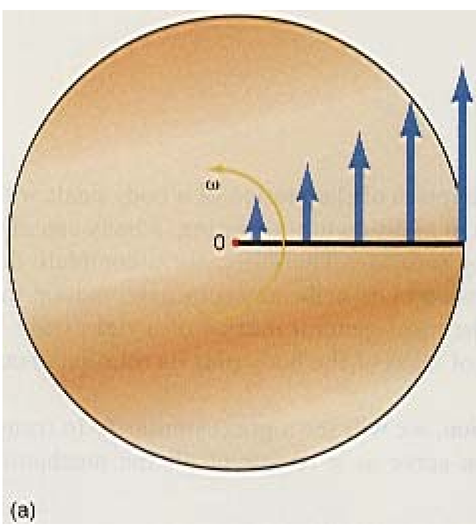
but $s/t = v$ and $\theta/t = \omega$. Therefore,

$$v = r\omega \qquad (9.2)$$

Equation 9.2 says that for a body rotating at an angular velocity ω, the farther the distance r that the body is from the axis of rotation, the greater is its linear velocity. This can be seen in figure 9.2(a). You may recall when you were a child and went on the merry-go-round, you usually wanted to ride on the outside horses because they moved the fastest. You can now see why. They are at the greatest distance from the axis of rotation and hence have the greatest linear velocity. The linear velocity of a point on the rotating body can also be called the *tangential velocity* because the point is moving along the tangential direction at any instant.

Table 9.1
Comparison of Translational and Rotational Motion

Translational Motion	Rotational Motion
$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x}{t}$	$\omega_{avg} = \frac{\Delta\theta}{\Delta t} = \frac{\theta}{t}$
$v = \frac{\Delta x}{\Delta t}$	$\omega = \frac{\Delta\theta}{\Delta t}$
$a = \frac{\Delta v}{\Delta t}$	$\alpha = \frac{\Delta\omega}{\Delta t}$
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2ax$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$KE = \frac{1}{2}mv^2$	$KE = \frac{1}{2}I\omega^2$
$F = ma$	$\tau = I\alpha$
$p = mv$	$L = I\omega$
$F = \frac{\Delta p}{\Delta t}$	$\tau = \frac{\Delta L}{\Delta t}$
$p_f = p_i$	$L_f = L_i$



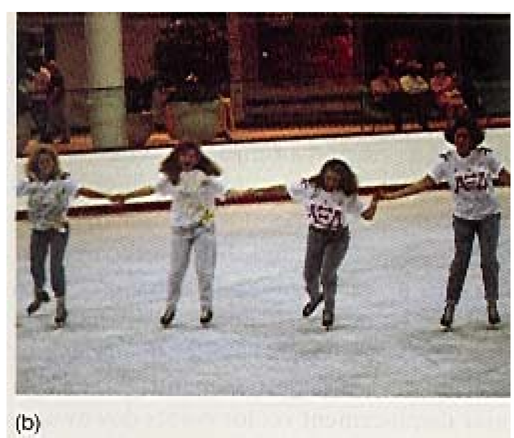


Figure 9.2 The linear velocity varies with the distance r from the center of the rotating body.

Another example of the relation between the tangential velocity and the angular velocity is seen in the old fashioned "whip" that you formed by holding hands while you were ice skating or roller skating, figure 9.2(b). The person at the inside end of the "whip" hardly moved at all ($r = 0$), but the person at the far end of the whip (maximum r) moved at very high speeds.

Example 9.1

The angular velocity of a wheel. A wheel of radius 15.0 cm starts from rest and turns through 2.00 rev in 3.00 s. (a) What is its average angular velocity? (b) What is the tangential velocity of a point on the rim of the wheel?

Solution

a. The average angular velocity, found from equation 9.1, is

$$\omega_{avg} = \frac{\theta}{t}$$
$$= \frac{(2.00 \text{ rev})(2\pi \text{ rad})}{3.00 \text{ s} \quad (1 \text{ rev})}$$
$$= 4.19 \text{ rad/s}$$

Note that we accomplished the conversion from revolutions to radians using the identity that one revolution is equal to 2π radians.

b. The tangential velocity of a point on the rim of the wheel, found from equation 9.2, is

$$v = r\omega$$
$$= (0.150 \text{ m})(4.19 \text{ rad/s})$$
$$= 0.628 \text{ m/s}$$

To go to this Interactive Example click on this sentence.

In the study of kinematics we defined the average translational acceleration of a body in equations 3.7 and 3.9 as the change in the velocity of the body with time, that is

$$a = \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t}$$

Because we considered only problems where motion was at constant acceleration, the average acceleration was equal to the instantaneous acceleration. In the same way, *we now define the average* ***angular acceleration*** α *of the rotating body as the change in the angular velocity of the body with time,* that is,

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t} \tag{9.3}$$

Again, since the only problems that we will consider concern motion at constant angular acceleration, the average angular acceleration is equal to the angular acceleration at any instant of time. We should note that angular acceleration, like angular velocity, can also be represented as a vector that lies along the axis of rotation of the rotating body. If the angular velocity vector is increasing with time, α is positive, and points upward from the plane of the rotation. If the angular velocity is decreasing with time, α is negative, and points downward into the plane of the rotation. The units for angular acceleration are radians/second per second, abbreviated as rad/s^2.

From the definition of the acceleration, equation 3.9, the first of the kinematic equations became

$$v = v_0 + at \tag{3.10}$$

the velocity of the moving body at any instant of time. Similarly, if equation 9.3 is solved for ω, we have

$$\omega = \omega_0 + \alpha t \tag{9.4}$$

the first of the **kinematical equations for rotational motion.** *Equation 9.4 gives the angular velocity of the rotating body at any instant of time for a constant acceleration,* α.

Example 9.2

The angular acceleration of a cylinder. A cylinder rotating at 10.0 rad/s is accelerated to 50.0 rad/s in 10.0 s. What is the angular acceleration of the cylinder?

Solution

The angular acceleration, found from equation 9.4, is

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{50.0 \text{ rad/s} - 10.0 \text{ rad/s}}{10.0 \text{ s}}$$
$$= 4.00 \text{ rad/s}^2$$

To go to this Interactive Example click on this sentence.

Example 9.3

The angular velocity of a crankshaft. A crankshaft rotating at 10.0 rad/s undergoes an angular acceleration of 0.500 rad/s^2. What is the angular velocity of the shaft after 10.0 s?

Solution

The angular velocity, found from equation 9.4, is

$$\omega = \omega_0 + \alpha t$$
$$= 10.0 \text{ rad/s} + (0.500 \text{ rad/s}^2)(10.0 \text{ s})$$
$$= 15.0 \text{ rad/s}$$

To go to this Interactive Example click on this sentence.

The relationship between the magnitude of the tangential acceleration of a point on the rim of the rotating body and the angular acceleration is found by dividing both sides of equation 9.2 by t, that is,

$$v = r\omega \tag{9.2}$$
$$\frac{v}{t} = \frac{r\omega}{t}$$

but $v/t = a$ and $\omega/t = \alpha$, therefore,

$$a = r\alpha \tag{9.5}$$

Equation 9.5 gives the relationship between the magnitude of the tangential acceleration and the angular acceleration.

The next kinematic derivation was the equation giving the position of the moving body as a function of time. Recall that the average velocity was substituted in the equation $x = v_{\mathbf{avg}}t$ to yield the kinematic equation for the position of the moving body as a function of time as

$$x = v_0t + \frac{1}{2}at^2$$

Similarly, to find the angular displacement of a rotating body at any instant of time, we use equation 9.1 in the form

$$\theta = \omega_{\mathbf{avg}}t \tag{9.6}$$

But for a body rotating at constant angular acceleration, the average angular velocity is

$$\omega_{avg} = \frac{\omega + \omega_0}{2} \tag{9.7}$$

where ω_0 is the initial angular velocity and ω is the final angular velocity at some time t. Substituting 9.7 into 9.6 gives

$$\theta = \left(\frac{\omega + \omega_0}{2}\right)t \tag{9.8}$$

Substituting equation 9.4 for the angular velocity ω into equation 9.8 gives

$$\theta = \left[\frac{(\omega_0 + \alpha t) + \omega_0}{2}\right]t$$

Rearranging terms we get

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \tag{9.9}$$

Equation 9.9 gives the angular displacement of the rotating body as a function of time for constant angular acceleration. This is the second kinematic equation for rotational motion.

Example 9.4

The angular displacement of a wheel. A wheel rotating at 15.0 rad/s undergoes an angular acceleration of 10.0 rad/s^2. Through what angle has the wheel turned when t = 5.00 s?

Solution

The angular displacement, found from equation 9.9, is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$
$$= (15.0 \text{ rad/s})(5.00 \text{ s}) + \frac{1}{2}(10.0 \text{ rad/s}^2)(5.00 \text{ s})^2$$
$$= 200 \text{ rad}$$

To go to this Interactive Example click on this sentence.

We obtained the third translational kinematic equation,

$$v^2 = v_0^2 + 2ax \tag{3.16}$$

from the first two translational kinematic equations by eliminating the time t between them. We can find a similar equation for the angular velocity as a function of the angular displacement by eliminating the time between equations 9.4 and 9.9 and we suggest that the student do this as an exercise. We will obtain the third kinematic equation for rotational motion in a slightly different manner, however. Let us start with

$$v^2 = v_0^2 + 2ax \tag{3.16}$$

But we know that a relationship exists between the translational variables and the rotational variables. Those relationships are

$$s = r\theta \quad (6.5)$$
$$v = r\omega \quad (9.2)$$
$$a = r\alpha \quad (9.5)$$

For the rotating body, we replace the linear distance x by the distance s along the arc of the circle. If we substitute the above equations into equation 3.16, we get

$$v^2 = v_0^2 + 2as$$
$$(r\omega)^2 = (r\omega_0)^2 + 2(r\alpha)(r\theta)$$
$$r^2\omega^2 = r^2\omega_0^2 + 2r^2\alpha\theta$$

Dividing each term by r^2, we obtain

$$\omega^2 = \omega_0^2 + 2\alpha\theta \quad (9.10)$$

Equation 9.10 represents the angular velocity of the rotating body at any angular displacement θ for constant angular acceleration α.

Example 9.5

The angular velocity at a particular angular displacement. A wheel, initially rotating at 10.0 rad/s, undergoes an angular acceleration of 5.00 rad/s^2. What is the angular velocity when the wheel has turned through an angle of 50.0 rad?

Solution

The angular velocity, found from equation 9.10, is

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$
$$= (10.0 \text{ rad/s})^2 + 2(5.00 \text{ rad/s}^2)(50.0 \text{ rad})$$
$$= 100 \text{ rad}^2/\text{s}^2 + 500 \text{ rad}^2/\text{s}^2 = 600 \text{ rad}^2/\text{s}^2$$
$$\omega = 24.5 \text{ rad/s}$$

To go to this Interactive Example click on this sentence.

Note in table 9.1 the similarity in the translational and rotational equations. Everywhere there is an x in the translational equations, there is a θ in the rotational equations. Everywhere there is a v in the translational equations, there is an ω in the rotational equations. And finally, everywhere there is an a in the translational equations, there is an α in the rotational equations. We will see additional analogues as we proceed in the discussion of rotational motion.

Another way to express the magnitude of the centripetal acceleration discussed in chapter 6,

$$a_c = \frac{v^2}{r} \quad (6.12)$$

is to use

$$v = r\omega \quad (9.2)$$

to obtain

$$a_c = \frac{\omega^2 r^2}{r}$$

Hence, we can represent the magnitude of the centripetal acceleration in terms of the angular velocity as

$$a_c = \omega^2 r \quad (9.11)$$

For nonuniform circular motion, the resultant acceleration of a point on a rim of a rotating body becomes the vector sum of the tangential acceleration and the centripetal acceleration, as seen in figure 9.3.

Example 9.6

The total acceleration of a point on a rotating body. A cylinder 35.0 cm in diameter is at rest initially. It is then given an angular acceleration of 0.0400 rad/s². Find (a) the angular velocity at 7.00 s, (b) the centripetal acceleration of a point at the edge of the cylinder at 7.00 s, (c) the tangential acceleration at the edge of the cylinder at 7.00 s, and (d) the resultant acceleration of a point at the edge of the cylinder at 7.00 s.

Solution

a. The angular velocity at 7.00 s, found from equation 9.4, is

$$\omega = \omega_0 + \alpha t$$
$$= 0 + (0.0400 \text{ rad/s}^2)(7.00 \text{ s})$$
$$= 0.280 \text{ rad/s}$$

b. The centripetal acceleration, found from equation 9.11, is

$$a_c = \omega^2 r$$
$$= (0.280 \text{ rad/s})^2(17.5 \text{ cm})$$
$$= 1.37 \text{ cm/s}^2$$

c. The tangential acceleration, found from equation 9.5, is

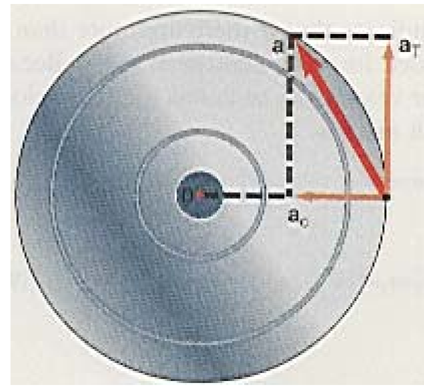


Figure 9.3 The total acceleration of a point on a rotating body is equal to the vector sum of the tangential acceleration and the centripetal acceleration.

$$a_T = r\alpha = (17.5 \text{ cm})(0.0400 \text{ rad/s}^2)$$
$$= 0.700 \text{ cm/s}^2$$

d. The resultant acceleration at 7.00 s, found from figure 9.3, is

$$a = \sqrt{(a_c)^2 + (a_T)^2}$$
$$= \sqrt{(1.37 \text{ cm/s}^2)^2 + (0.700 \text{ cm/s}^2)^2}$$
$$= 1.54 \text{ cm/s}^2$$

To go to this Interactive Example click on this sentence.

9.3 The Kinetic Energy of Rotation

Let us consider the motion of four point masses m_1, m_2, m_3, and m_4 located at distances r_1, r_2, r_3, and r_4, respectively, rotating at an angular speed ω about an axis through the point 0, as shown in figure 9.4.

Let us assume that the masses are connected to the center of rotation by rigid, massless rods. (A massless rod is one whose mass is so small compared to the mass at the end of the rod that we can neglect it in the analysis.) Let us determine the total kinetic energy of these rotating masses. The total energy is equal to the sum of the kinetic energy of each mass. That is,

$$KE_{total} = KE_1 + KE_2 + KE_3 + KE_4 + \ldots$$

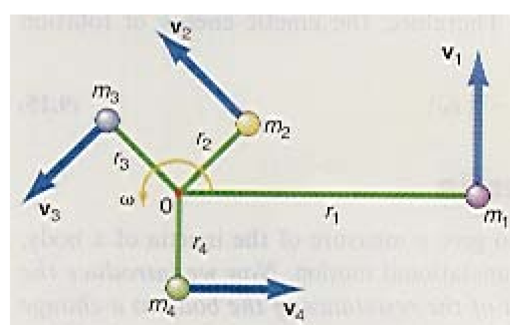


Figure 9.4 Rotational kinetic energy.

The plus sign and dots after the last term indicate that if there are more than the four masses considered, another term is added for each additional mass. Because each mass is rotating with the same angular velocity ω, each has a linear velocity v, as shown. Since the kinetic energy of each mass is

$$KE = \frac{1}{2}mv^2$$

the total kinetic energy is

$$KE_{tot} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \frac{1}{2}m_4v_4^2 + \ldots \tag{9.12}$$

but from equation 9.2,

$$v = r\omega$$

hence, for each mass

$$\begin{aligned} v_1 &= r_1\omega \\ v_2 &= r_2\omega \\ v_3 &= r_3\omega \\ v_4 &= r_4\omega \end{aligned} \tag{9.13}$$

Substituting equations 9.13 back into equation 9.12, gives

$$\begin{aligned} KE_{tot} &= \frac{1}{2}m_1(r_1\omega)^2 + \frac{1}{2}m_2(r_2\omega)^2 + \frac{1}{2}m_3(r_3\omega)^2 + \frac{1}{2}m_4(r_4\omega)^2 + \ldots \\ &= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + \frac{1}{2}m_4r_4^2\omega^2 + \ldots \end{aligned}$$

Note that there is a 1/2 and an ω^2 in every term, so let us factor them out:

$$KE_{tot} = \frac{1}{2}(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2 + \ldots)\omega^2$$

Looking at the form of the equation for the translational kinetic energy (½ mv^2), and remembering all the symmetry in the translational-rotational equations, it is reasonable to expect that the equation for the rotational kinetic energy might have an analogous form. That symmetry is maintained by defining the term in parentheses as the moment of inertia, the rotational analogue of the mass m. That is, the moment of inertia about the axis of rotation for these four masses is

$$I = (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2 + \ldots) \tag{9.14}$$

We will discuss the concept of the moment of inertia in more detail in section 9.4. For now, we see that the equation for the total energy of the four rotating masses is

$$KE_{tot} = \frac{1}{2}I\omega^2$$

And finally let us note that the total kinetic energy of the rotating masses can simply be called the **kinetic energy of rotation.** Therefore, the kinetic energy of rotation about a specified axis is

$$KE_{rot} = \frac{1}{2}I\omega^2 \tag{9.15}$$

9.4 The Moment of Inertia

The concept of mass m was introduced to give a measure of the inertia of a body, that is, its resistance to a change in its translational motion. *Now we introduce the **moment of inertia** to give a measurement of the resistance of the body to a change in its rotational motion.* For example, the larger the moment of inertia of a body, the more difficult it is to put that body into rotational motion. Conversely, the larger the moment of inertia of a body, the more difficult it is to stop its rotational motion.

For the particular configuration studied in section 9.3, the moment of inertia about the axis of rotation was defined as

$$I = (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2 + \ldots) \tag{9.14}$$

For any number of masses, this definition can be generalized to

$$I = \sum_{i=1}^{n} m_i r_i^2 \tag{9.16}$$

where the Greek letter sigma, Σ, means the "sum of," as used before. The subscript i on m and r means that the index i takes on the values from 1 up to the number n. So when $n = 4$ the identical result found in equation 9.14 is obtained.

For the very special case of *the moment of inertia of a single mass m rotating about an axis,* equation 9.16 reduces to ($i = n = 1$),

$$I = mr^2 \tag{9.17}$$

Thus, the significant feature for rotational motion is not the mass of the rotating body, but rather the square of the distance of that body from the axis of rotation. A small mass m, at a great distance r from the axis of rotation, has a greater moment of inertia than a large mass, very close to the axis of rotation.

For continuous mass distributions, the moments of inertia are given in figure 9.5. More extensive tables of moments of inertia are found in various handbooks, such as the *Handbook of Chemistry and Physics* (published by the Chemical Rubber Co. Press, Cleveland, Ohio), if the need for them arises.

It is important to note here that when we ask for the moment of inertia of a body, we must specify about what axis the rotation will occur. Because r is different for each axis, and since I varies as r^2, I is also different for each axis. As an example, consider the slender rod in figure 9.5. When the axis is taken through the center of the rod, as shown, $I = 1/12\ ml^2$, while if the axis of rotation is at the end of the rod, then $I = 1/3\ ml^2$. The unit for the moment of inertia is kg m^2 and has no special name.

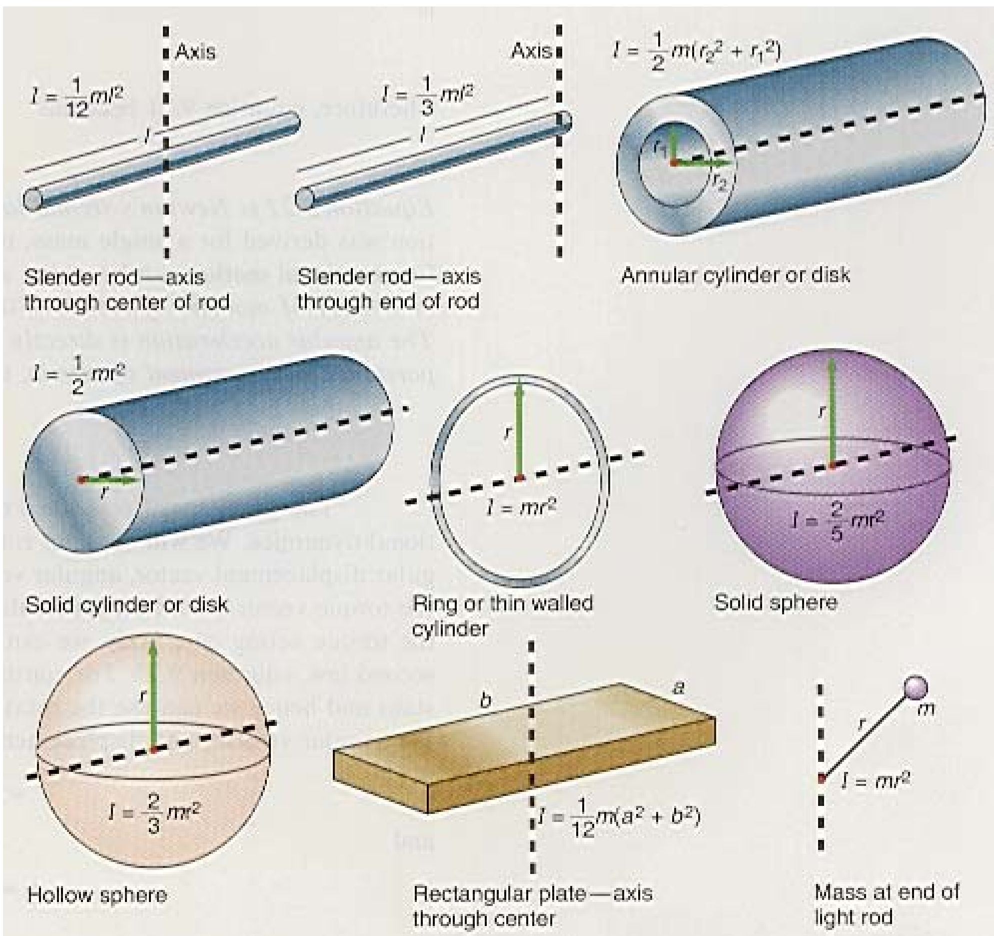


Figure 9.5 Moments of inertia for various mass distributions.

9.5 Newton's Laws for Rotational Motion

Let us consider a single mass m connected by a rigid rod, of negligible mass, to an axis passing through the point 0, as shown in figure 9.6. Let us apply a tangential force F, in the plane of the page, to the body of mass m. The force acting on the constrained body causes a torque, given by

$$\tau = rF \tag{9.18}$$

This torque causes the body to rotate about the axis through 0. The force F acting

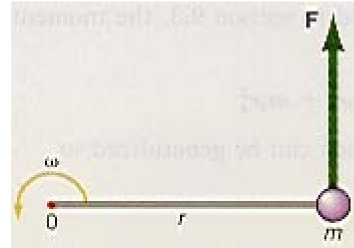


Figure 9.6 Torque causes a body to rotate.

on the mass m causes a tangential acceleration given by Newton's second law as

$$F = ma \tag{9.19}$$

If we substitute equation 9.19 into 9.18, we have

$$\tau = rma \tag{9.20}$$

But the tangential acceleration a is related to the angular acceleration by

$$a = r\alpha \tag{9.5}$$

Substituting this into equation 9.20, gives

$$\tau = rm(r\alpha) = mr^2\alpha \tag{9.21}$$

But, as already seen, the moment of inertia of a single mass rotating about an axis is

$$I = mr^2 \tag{9.17}$$

Therefore, equation 9.21 becomes

$$\tau = I\alpha \tag{9.22}$$

Equation 9.22 is Newton's second law for rotational motion. Although this equation was derived for a single mass, it is true in general, and **Newton's second law for rotational motion** can be stated as: *When an unbalanced external torque acts on a body of moment of inertia I, it gives that body an angular acceleration,* α. *The angular acceleration is directly proportional to the torque and inversely proportional to the moment of inertia,* that is,

$$\alpha = \frac{\tau}{I} \tag{9.23}$$

The problems of rotational dynamics are very similar to those in translational dynamics. We will consider rotational motion only in the x-y plane. The angular displacement vector, angular velocity vector, angular acceleration vector, and the torque vector are all perpendicular to the plane of the rotation. By determining the torque acting on a body, we can find the angular acceleration from Newton's second law, equation 9.23. For constant torque, the angular acceleration is a constant and hence we can use the rotational kinematic equations. Therefore, we find the angular velocity and displacement at any time from the kinematic equations

$$\omega = \omega_0 + \alpha t \tag{9.4}$$

and

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \tag{9.9}$$

To determine Newton's first law for rotational motion, we note that

$$\tau = I\alpha = I\frac{\Delta\omega}{\Delta t}$$

and if there is no external torque (i.e., if $\tau = 0$), then

$$\Delta\omega = 0$$

$$\omega_f - \omega_i = 0$$

or

$$\omega_f = \omega_i \tag{9.24}$$

That is, equation 9.24 says that if there is no external torque acting on a body, then a body rotating at an initial angular velocity ω_i will continue to rotate at that same angular velocity forever.

Stated in more formal terms, ***Newton's first law for rotational motion*** *is A body in motion at a constant angular velocity will continue in motion at that same angular velocity, unless acted on by some unbalanced external torque.*

One of the most obvious examples of Newton's first law for rotational motion is the earth itself. Somehow, someway in its creation, the earth was given an initial angular velocity ω_i of 7.27×10^{-5} rad/s. Since there is no external torque acting on the earth it continues to rotate at this same angular velocity.

For completeness, we can state **Newton's third law of rotational motion** as *If body A and body B have the same common axis of rotation, and if body A exerts a torque on body B, then body B exerts an equal but opposite torque on body A.* That is, if body *A* exerts a torque on body *B* that tends to rotate body *B* in a clockwise direction, then body *B* will exert a torque on body *A* that will tend to rotate body *A* in a counterclockwise direction. An application of this principle is found in a helicopter (see figure 9.7). As the main rotor blades above the helicopter turn counterclockwise, the helicopter itself would start to turn clockwise. To prevent this rotation of the helicopter, a second but smaller set of rotor blades are located at the side and end of the helicopter to furnish a countertorque to prevent the helicopter from turning.

Figure 9.7 Newton's third law for rotational motion and the helicopter.

9.6 Rotational Dynamics

Now let us look at some examples of the use of Newton's laws in solving problems in rotational motion.

Example 9.7

Rotational dynamics of a cylinder. Consider a solid cylinder of mass m = 3.00 kg and radius r = 0.500 m, which is free to rotate about an axis through its center, as shown in figure 9.8. The cylinder is initially at rest when a constant force of 8.00 N is applied tangentially to the cylinder. Find (a) the moment of inertia of the cylinder, (b) the torque acting on the cylinder, (c) the angular acceleration of the cylinder, (d) its angular velocity after 10.0 s, and (e) its angular displacement after 10.0 s.

Solution

a. The equation for the moment of inertia of a cylinder about its main axis, found in figure 9.5, is

$$I = \frac{1}{2}mr^2$$
$$= \frac{1}{2}(3.00\text{ kg})(0.500\text{ m})^2$$
$$= 0.375\text{ kg m}^2$$

b. The torque acting on the cylinder is the product of the force times the lever arm. From figure 9.8, we see that the lever arm is just the radius of the cylinder. Therefore,

$$\tau = rF \quad (9.18)$$
$$= (0.500 \text{ m})(8.00 \text{ N})$$
$$= 4.00 \text{ m N}$$

c. The angular acceleration of the cylinder, determined by Newton's second law, is

$$\alpha = \frac{\tau}{I} \quad (9.23)$$
$$= \frac{4.00 \text{ m N}}{0.375 \text{ kg m}^2} = \frac{10.7 \text{ m kg m/s}^2}{\text{kg m}^2}$$
$$= 10.7 \text{ rad/s}^2$$

Note that in the solution all the units cancel out except the s^2 in the denominator. We then introduced the unit radian in the numerator to

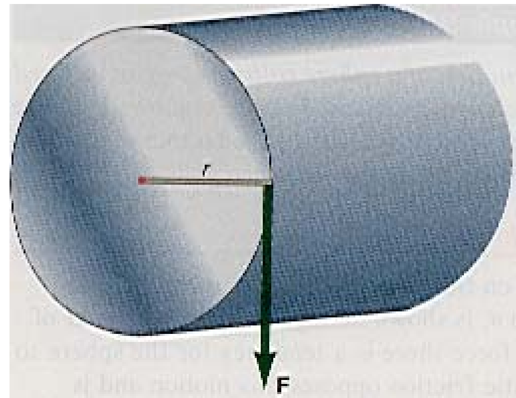


Figure 9.8 Rotational motion of a cylinder.

give us the desired unit for angular acceleration, namely, rad/s^2. Recall that the radian is a unit that can be multiplied by or divided into an equation at will, because the radian is a dimensionless quantity. It was defined as the ratio of the arc length to the radius of the circle,

$$\theta = \frac{s}{r} = \frac{\text{meter}}{\text{meter}} = 1 = \text{radian}$$

d. To determine the angular velocity of the rotating cylinder we use the kinematic equation for the angular velocity, namely

$$\omega = \omega_0 + \alpha t \quad (9.4)$$
$$= 0 + 10.7 \frac{\text{rad}}{\text{s}^2}(10.0 \text{ s})$$
$$= 107 \text{ rad/s}$$

e. The angular displacement, found by the kinematic equation, is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \quad (9.9)$$
$$= 0 + \frac{1}{2}(10.7 \text{ rad/s}^2)(10.0 \text{ s})^2$$
$$= 535 \text{ rad}$$

To go to this Interactive Example click on this sentence.

Example 9.8

Combined translational and rotational motion of a sphere rolling down an inclined plane. A solid sphere of 1.00 kg mass rolls down an inclined plane of angle $\theta = 30^0$, as shown in figure 9.9. Find (a) the acceleration of the sphere, (b) its velocity at the bottom of the 1.00 m long plane, and (c) the frictional force acting on the sphere.

Solution

a. First, we draw all the forces acting on the sphere. The component of the weight acting down the plane, $w \sin \theta$, is shown acting through the center of mass of the sphere. Because of this force there is a tendency for the sphere to slide down the plane. A force of static friction opposes this motion and is directed up the plane, as shown. This frictional force can not be shown as acting at the center of the body as was done in problems with "blocks" sliding on the inclined plane. It is this frictional force acting at the point of contact of the sphere that creates the necessary torque to rotate the sphere so that it rolls down the plane. The motion is therefore composed of two motions, the translation of the center of mass of the sphere, and the rotation about the center of mass of the sphere. Applying Newton's second law for the translational motion of the center of mass of the sphere gives

$$F = ma$$

$$w \sin\theta - f_s = ma \qquad (9.25)$$

Applying the second law for the rotation of the sphere about its center of mass gives

$$\tau = I\alpha \qquad (9.22)$$

But the torque is the product of the frictional force f_s and the radius of the sphere. Therefore,

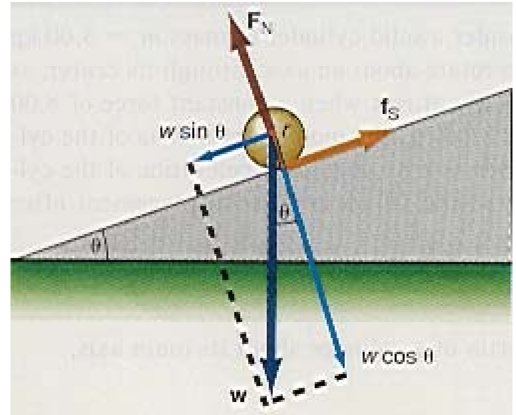


Figure 9.9 A sphere rolling down an inclined plane.

$$f_s r = I\alpha \qquad (9.26)$$

Now we eliminate the frictional force f_s between the two equations 9.25 and 9.26. That is, from 9.26,

$$f_s = \frac{I\alpha}{r}$$

Substituting this into equation 9.25, we get

$$w \sin\theta - \frac{I\alpha}{r} = ma$$

The moment of inertia of a solid sphere, found from figure 9.5, is

$$I = \frac{2}{5} mr^2 \qquad (9.27)$$

Therefore,

$$ma = w \sin\theta - \left(\frac{2}{5} mr^2\right)\frac{\alpha}{r}$$

$$= w \sin\theta - \frac{2}{5} mr\alpha$$

But recall that

$$a = r\alpha \qquad (9.5)$$

Therefore,

$$ma = w \sin\theta - \frac{2}{5} ma$$

$$ma + \frac{2}{5} ma = mg \sin\theta$$

$$\frac{7}{5} a = g \sin\theta$$

Solving for the acceleration of the sphere, we get

$$a = \frac{5}{7} g \sin\theta \qquad (9.28)$$

$$a = \frac{5}{7}(9.80 \text{ m/s}^2)\sin 30.0^0$$

$$= 3.50 \text{ m/s}^2$$

b. The velocity of the center of mass of the sphere at the bottom of the plane is found from the kinematic equation

$$v^2 = v_0^2 + 2ax \qquad (3.16)$$

Because the sphere starts from rest, $v_0 = 0$. Therefore,

$$v = \sqrt{2ax}$$
$$= \sqrt{(2)(3.50 \text{ m/s}^2)(1.00 \text{ m})}$$
$$= 2.65 \text{ m/s}$$

c. The frictional force can be determined from equation 9.25, that is,

$$w \sin\theta - f_s = ma$$
$$f_s = w \sin\theta - ma$$
$$= mg \sin\theta - m\left(\frac{5}{7} g \sin\theta\right)$$
$$= \frac{2}{7} mg \sin\theta$$
$$= \frac{2}{7}(1.00 \text{ kg})(9.80 \text{ m/s}^2)\sin 30.0^0$$
$$= 1.40 \text{ N}$$

To go to this Interactive Example click on this sentence.

As we can see the general motion of a rigid body can become quite complicated. We will see in section 9.8 how these problems can be simplified by the use of the law of conservation of energy.

**Combined Translational and Rotational Motion Treated by Newton's Second Law*

It is appropriate here to return to some of the problems discussed in chapter 4, in which we assumed that the tension in the rope on both sides of a pulley are equal. Let us analyze these problems taking the rotational motion of the pulley into account. Consider the problem of a block moving on a rough horizontal surface, as shown in figure 9.10. What is the acceleration of each block in the system?

Applying Newton's second law to block A, we obtain

$$T_1 - w_A = -m_A a \qquad (9.29)$$

Applying the second law to block B, we get

$$T_2 - f_k = m_B a \qquad (9.30)$$

We find the frictional force f_k from

$$f_k = \mu_k F_N = \mu_k w_B$$

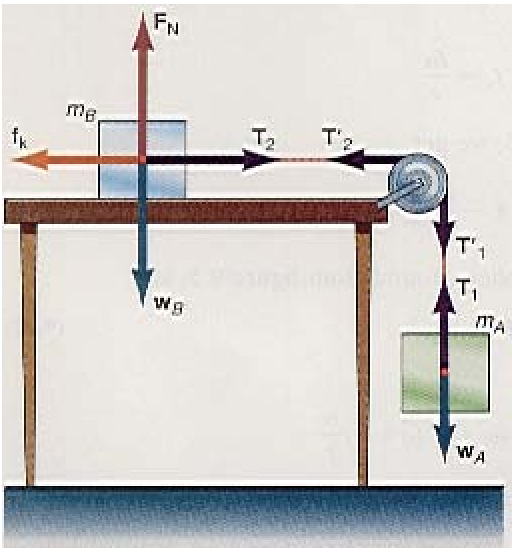


Figure 9.10 Combined motion taking the rotational motion of the pulley into account.

Substituting this into equation 9.30, gives

$$T_2 - \mu_k w_B = m_B a \qquad (9.31)$$

It was at this point in chapter 4 that we made the assumption that the tension $T_1 = T_2$, and then determined the acceleration of each block of the system. Let us now look a little more closely at the assumption of the equality of tensions. The string exerts a force T_1 upward on weight w_A, but by Newton's third law the weight w_A exerts a force down on the string, call it T_1'. Figure 9.11 shows the pulley with the appropriate tensions in the string. The force T_1' acting on the pulley causes a torque

$$\tau_1 = T_1' R$$

which tends to rotate the pulley clockwise. The radius of the pulley is R.

Similarly, the string exerts a tension force T_2 on mass m_B. But by Newton's third law, block B exerts a force on the string, which we call T_2'. The force T_2' causes a counterclockwise torque about the axis of the pulley, given by

$$\tau_2 = T_2'R$$

Because the motion of the system causes the pulley to rotate in a clockwise direction, the net torque on the pulley is equal to the difference in these two torques, namely,

$$\tau = \tau_1 - \tau_2$$

$$\tau = T_1'R - T_2'R \quad (9.32)$$

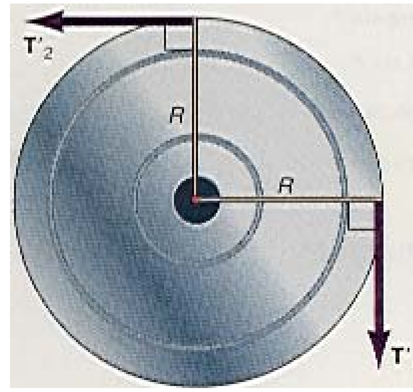


Figure 9.11 Forces acting on the pulley.

But by Newton's second law for rotational motion,

$$\tau = I\alpha \quad (9.22)$$

Substituting equation 9.22 into equation 9.32, gives

$$I\alpha = T_1'\text{R} - T_2'R$$

$$I\alpha = (T_1' - T_2')R \quad (9.33)$$

From figure 9.10 and Newton's third law, we have

$$T_1' = T_1 \quad (9.34)$$

$$T_2' = T_2 \quad (9.34)$$

Substituting equations 9.34 into equation 9.33, gives

$$I\alpha = (T_1 - T_2)R \quad (9.35)$$

However, the angular acceleration $\alpha = a/R$. Therefore, equation 9.35 becomes

$$I\frac{a}{R} = (T_1 - T_2)R \quad (9.36)$$

The pulley resembles a disk, whose moment of inertia, found from figure 9.5, is $I_{\text{Disk}} = ½\, MR^2$, where M is the mass of the pulley and R is the radius of the pulley. Substituting this result into equation 9.36, gives

$$\left(\frac{1}{2}MR^2\right)\frac{a}{R} = (T_1 - T_2)R$$

Simplifying,

$$\frac{1}{2}Ma = (T_1 - T_2) \quad (9.37)$$

There are now three equations 9.29, 9.31, and 9.37 in terms of the three unknowns a, T_1, and T_2. Solving equation 9.29 for T_1, gives

$$T_1 = w_A - m_Aa \quad (9.38)$$

Solving equation 9.31 for T_2, gives

$$T_2 = \mu_k w_B + m_Ba \quad (9.39)$$

Subtracting equation 9.39 from equation 9.38, we get

$$T_1 - T_2 = w_A - m_Aa - \mu_k w_B - m_Ba$$

Substituting for $T_1 - T_2$ from equation 9.37, gives

$$\frac{1}{2}Ma = w_A - m_Aa - \mu_k w_B - m_Ba$$

Gathering the terms with a in them to the left-hand side of the equation, we get

$$\frac{1}{2}Ma + m_A a + m_B a = w_A - \mu_k w_B$$

Factoring out the a, and writing each weight w as mg, we get

$$a\left(\frac{1}{2}M + m_A + m_B\right) = m_A g - \mu_k m_B g$$

Solving for the acceleration of the system, we have

$$a = \frac{(m_A - \mu_k m_B)g}{m_A + m_B + M/2} \tag{9.40}$$

It is immediately apparent in equation 9.40 that the acceleration of the system depends on the mass M of the pulley. If this mass is very small compared to the masses m_A and m_B (i.e., $M \approx 0$), then equation 9.40 would reduce to the simpler problem already found in equation 4.62.

Example 9.9

Combined translational and rotational motion. If $m_A = 2.00$ kg, $m_B = 6.00$ kg, $\mu_k = 0.300$, and $M = 8.00$ kg in figure 9.10, find the acceleration of each block of the system.

Solution

The acceleration of each block in the system, found from equation 9.40, is

$$a = \frac{(m_A - \mu_k m_B)g}{m_A + m_B + M/2}$$
$$= \frac{[2.00 \text{ kg} - (0.300)(6.00 \text{ kg})](9.80 \text{ m/s}^2)}{2.00 \text{ kg} + 6.00 \text{ kg} + 8.00 \text{ kg}/2}$$
$$= 0.163 \text{ m/s}^2$$

If we compare this example with example 4.13 in chapter 4, we see a relatively large difference in the acceleration of the system by assuming M to be negligible.

To go to this Interactive Example click on this sentence.

Example 9.10

The effect of a smaller pulley. Let us repeat example 9.9, but now let us use a much smaller plastic pulley, with M = 25 g. Find the acceleration of each block of the system.

Solution

The acceleration of each block, again found from equation 9.40, is

$$a = \frac{(m_A - \mu_k m_B)g}{m_A + m_B + M/2}$$
$$= \frac{[2.00 \text{ kg} - (0.300)(6.00 \text{ kg})](9.80 \text{ m/s}^2)}{2.00 \text{ kg} + 6.00 \text{ kg} + 0.025 \text{ kg}/2}$$
$$= 0.244 \text{ m/s}^2$$

which agrees very closely to the value found in example 4.13, of chapter 4, when the effect of the pulley was assumed to be negligible.

To go to this Interactive Example click on this sentence.

Example 9.11

The tension in the strings. If the radius of the pulley is 5.00 cm, find the tension in the strings of examples 9.9 and 9.10.

Solution

For example 9.9, the tension T_1, found from equation 9.38, is

$$T_1 = w_A - m_A a = m_A g - m_A a = m_A(g - a)$$
$$= (2.00 \text{ kg})(9.80 \text{ m/s}^2 - 0.163 \text{ m/s}^2)$$
$$= 19.3 \text{ N}$$

Tension T_2, found from equation 9.39, is

$$T_2 = \mu_k w_B + m_B a$$
$$= \mu_k m_B g + m_B a$$
$$= (0.300)(6.00 \text{ kg})(9.80 \text{ m/s}^2) + (6.00 \text{ kg})(0.163 \text{ m/s}^2)$$
$$= 17.6 \text{ N} + 0.978 \text{ N}$$
$$= 18.6 \text{ N}$$

Thus the tensions in the strings on both sides of the pulley are unequal. It is this difference in the tensions that causes the torque,

$$\tau = R(T_1 - T_2)$$
$$= (0.05 \text{ m})(19.3 \text{ N} - 18.6 \text{ N})$$
$$= 3.50 \times 10^{-2} \text{ m N}$$

on the pulley. This torque gives the pulley its angular acceleration.

For example 9.10, the tension T_1 is again found from equation 9.38, only now the acceleration of the system is 0.244 m/s^2. Thus,

$$T_1 = w_A - m_A a = m_A g - m_A a = m_A(g - a)$$
$$= (2.00 \text{ kg})(9.80 \text{ m/s}^2 - 0.244 \text{ m/s}^2)$$
$$= 19.1 \text{ N}$$

Tension T_2, found from equation 9.39, is

$$T_2 = \mu_k w_B + m_B a$$
$$= \mu_k m_B g + m_B a$$
$$= (0.300)(6.00 \text{ kg})(9.80 \text{ m/s}^2) + (6.00 \text{ kg})(0.244 \text{ m/s}^2)$$
$$= 17.6 \text{ N} + 1.46 \text{ N}$$
$$= 19.1 \text{ N}$$

Hence in this case, where the pulley has a small mass, the tensions are equal, at least to three significant figures, and there is no resultant torque to cause the pulley to rotate. The two tensions must be different to cause a net torque to rotate the pulley.

To go to this Interactive Example click on this sentence.

Atwood's Machine

Let us reconsider the Atwood's machine problem solved in chapter 4, only this time we no longer assume the tensions on each side of the pulley to be equal, figure 9.12. We apply Newton's second law to mass m_A to obtain

$$T_1 - w_A = -m_A a \quad (9.41)$$

Applying the second law to mass m_B, we obtain

$$T_2 - w_B = m_B a \quad (9.42)$$

Let us now consider the pulley. The tension T_1' causes a clockwise torque,

$$\tau_1 = T_1'R$$

whereas the tension T_2' causes the counterclockwise torque,

$$\tau_2 = T_2'R$$

The net torque acting on the pulley is

$$\tau = \tau_1 - \tau_2 = T_1'R - T_2'R$$

But by Newton's second law for rotational motion

$$\tau = I\alpha$$

Therefore,

$$I\alpha = (T_1' - T_2')R \quad (9.43)$$

However, by Newton's third law of motion

$$T_2' = T_2$$
$$T_1' = T_1$$

Hence, the second law, equation 9.43, becomes

$$I\alpha = (T_1 - T_2)R \quad (9.44)$$

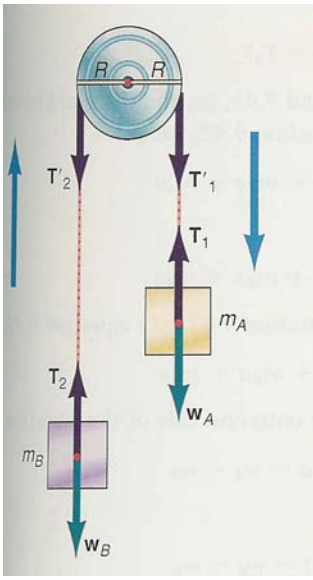


Figure 9.12 Atwood's machine with the rotational motion of the pulley taken into account.

The moment of inertia of the pulley, found in figure 9.5, is

$$I_{\text{disk}} = \frac{1}{2}MR^2$$

and the angular acceleration is given by $\alpha = a/R$. Substituting these two values into equation 9.44, gives

$$\frac{1}{2}MR^2(a/R) = (T_1 - T_2)R$$

or

$$\frac{1}{2}Ma = (T_1 - T_2) \quad (9.45)$$

There are now three equations, 9.41, 9.42, and 9.45, for the three unknowns, a, T_1, and T_2. Subtracting equation 9.41 from equation 9.42, we get

$$T_2 - w_B - T_1 + w_A = m_B a + m_A a$$

or

$$T_2 - T_1 = w_B - w_A + m_B a + m_A a \quad (9.46)$$

Substituting the value for $T_2 - T_1$ from equation 9.45 into equation 9.46, we get

$$-\frac{1}{2}Ma = w_B - w_A + m_B a + m_A a$$

Gathering the terms with the acceleration a onto one side of the equation,

$$\frac{1}{2}Ma + m_Ba + m_Aa = w_A - w_B$$

Factoring out the a, we get,

$$a\left(\frac{1}{2}M + m_B + m_A\right) = w_A - w_B$$

Expressing the weights as $w = mg$, and solving for the acceleration of each mass of the system, we get

$$a = \frac{(m_A - m_B)g}{m_A + m_B + M/2} \tag{9.47}$$

Equation 9.47 is the acceleration of each mass in Atwood's machine, when the rotational motion of the pulley is taken into account. If the mass of the pulley M is very small, then equation 9.47 reduces to the simplified solution in equation 4.38.

Example 9.12

Combined motion in an Atwood's machine. In an Atwood's machine, $m_B = 30.0$ g, $m_A = 50.0$ g, and the mass M of the pulley is 2.00 kg. Find the acceleration of each mass.

Solution

The acceleration, found from equation 9.47, is

$$a = \frac{(m_A - m_B)g}{m_A + m_B + M/2}$$

$$= \frac{(50.0 \text{ g} - 30.0 \text{ g})(9.80 \text{ m/s}^2)}{[50.0 \text{ g} + 30.0 \text{ g} + (2000 \text{ g})/2]}$$

$$= 0.181 \text{ m/s}^2$$

If the pulley were made of light plastic and, therefore, had a negligible mass, then the acceleration of the system would have been, $a = 2.45$ m/s^2, which is a very significant difference.

To go to this Interactive Example click on this sentence.

Example 9.13

Velocity in an Atwood's machine. If block m_A of example 9.12, located a distance $h_A = 2.00$ m above the floor, falls from rest, find its velocity as it hits the floor.

Solution

Because m_A falls at the constant acceleration given by equation 9.47, the kinematic equation can be used to find its velocity at the floor. Thus,

$$v^2 = v_0^2 + 2ay$$

$$v = \sqrt{2ay} = \sqrt{2ah_A}$$

$$= \sqrt{2\left(0.181 \text{ m/s}^2\right)\left(2.00 \text{ m}\right)}$$

$$= 0.850 \text{ m/s}$$

To go to this Interactive Example click on this sentence.

9.7 Angular Momentum and Its Conservation

Just as the linear momentum of a body was defined as the product of its mass and its linear velocity, $p = mv$, *the **angular momentum** of a rotating body is defined as the product of its moment of inertia and its angular velocity.* That is, the angular momentum L, with respect to a given axis, is defined as

$$L = I\omega \tag{9.48}$$

As the concept of momentum led to an alternative form of Newton's second law for the translational case, angular momentum also leads to an alternative form for the rotational case, as shown below.

Translational Case	Rotational Case
$F = ma = m\frac{\Delta v}{\Delta t}$	$\tau = I\alpha = I\frac{\Delta\omega}{\Delta t}$
$F = m\frac{(v_f - v_i)}{\Delta t}$	$\tau = I\frac{(\omega_f - \omega_i)}{\Delta t}$
$F = \frac{mv_f - mv_i}{\Delta t} = \frac{p_f - p_i}{\Delta t}$	$\tau = \frac{I\omega_f - I\omega_i}{\Delta t} = \frac{L_f - L_i}{\Delta t}$
$F = \frac{\Delta p}{\Delta t}$	$\tau = \frac{\Delta L}{\Delta t}$

Thus, we can write Newton's second law in terms of angular momentum as

$$\tau = \frac{\Delta L}{\Delta t} \tag{9.49}$$

If we apply equation 9.49 to a system of bodies, the total torque τ arises from two sources, external torques and internal torques. Because of Newton's third law for rotational motion, the internal torques will add to zero and equation 9.49 becomes

$$\tau_{ext} = \frac{\Delta L}{\Delta t} \tag{9.50}$$

If the total external torque acting on the system is zero, then

$$0 = \frac{\Delta L}{\Delta t}$$

$$\Delta L = 0 \tag{9.51}$$

$$L_f - L_i = 0$$

Therefore,

$$L_f = L_i \tag{9.52}$$

*Equations 9.51 and 9.52 are a statement of **the law of conservation of angular momentum.** They say: if the total external torque acting on a system is zero, then there is no change in the angular momentum of the system, and the final angular momentum is equal to the initial angular momentum.*

Let us now consider some examples of the conservation of angular momentum.

The Rotating Earth

Because there is no external torque acting on the earth, $\tau = 0$, and there is conservation of angular momentum. Hence,

$$L_f = L_i \tag{9.52}$$

But since the angular momentum is the product of the moment of inertia and the angular velocity, this becomes

$$I_f \omega_f = I_i \omega_i \tag{9.53}$$

However, the moment of inertia of the earth does not change with time and thus, $I_f = I_i$. Therefore,

$$\omega_f = \omega_i$$

That is, the angular velocity of the earth is a constant and will continue to spin forever with the same angular velocity unless it is acted on by some external torque. We also assume that the moment of inertia of the earth does not change.

Figure 9.13 Because there is no torque acting on the earth, its angular momentum is conserved, and it will continue to spin with the same angular velocity forever.

The Spinning Ice Skater

The familiar picture of the spinning ice skater, as shown in figure 9.14, gives another example of the conservation of angular momentum. As the skater (body *A*) pushes against the ice (body *B*), thereby creating a torque, the ice (body *B*) pushes back on the skater (body *A*), creating a torque on her. The net torque on the skater and the ice is therefore zero and angular momentum is conserved.

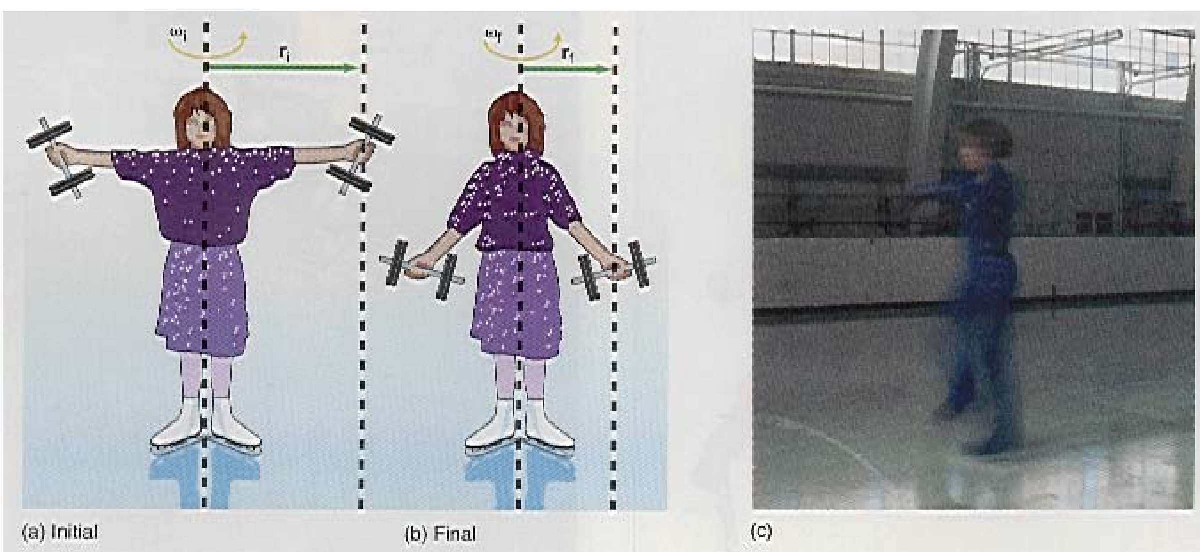


Figure 9.14 The spinning ice skater.

Because the earth is so massive there will be no measurable change in the angular momentum of the earth and we need consider only the skater. The skater first starts spinning relatively slowly with her hands outstretched. We assume that any friction between the skater and the ice is negligible. As the skater draws her arms to her sides, she starts to spin very rapidly. Let us analyze the motion by the law of conservation of angular momentum. The conservation of angular momentum gives

$$L_f = L_i \tag{9.52}$$

or

$$I_f \omega_f = I_i \omega_i \tag{9.53}$$

For simplicity of calculation, let us assume that the skater is holding a set of dumbbells in her hands so that her moment of inertia can be considered to come only from the dumbbells. (That is, we assume that the moment of inertia of the girl's hands and arms can be considered negligible compared to the dumbbells in order to simplify the calculation.) The skater's initial moment of inertia is

$$I_i = mr_i^2$$

where m is the mass of the dumbbells and r_i the distance from the center of the body (the axis of rotation) to the outstretched dumbbells. When the skater pulls her hand down to her side the new moment of inertia is

$$I_f = mr_f^2$$

where r_f is now the distance from the axis of rotation to the dumbbell, as seen in figure 9.14(b). As we can immediately see from the figure, r_f is less than r_i, therefore I_f is less than I_i. But if the moment of inertia is changing, what happens to the skater as a consequence of the conservation of angular momentum? The angular momentum must remain the same, as given by equation 9.53. The final angular momentum must be equal to the initial angular momentum, which is equal to the product of $I_i\omega_i$, which remains a constant. Thus, the final angular momentum $I_f\omega_f$ must equal that same constant. But if I_f has decreased, the only way to maintain the equality is to have the final value of the angular velocity ω_f increase. And this is, in fact, exactly what happens. As the girl's arms are dropped to her side, the spinning increases. When the skater wishes to come out of the spin, she merely raises her arms to the original outstretched position, her moment of inertia increases and her angular velocity decreases.

A Man Diving from a Diving Board

When a man pushes down on a diving board, the board reacts by pushing back on him, as in figure 9.15. As the man leans forward at the start of the dive, the reaction force on him causes a torque to set him into rotational

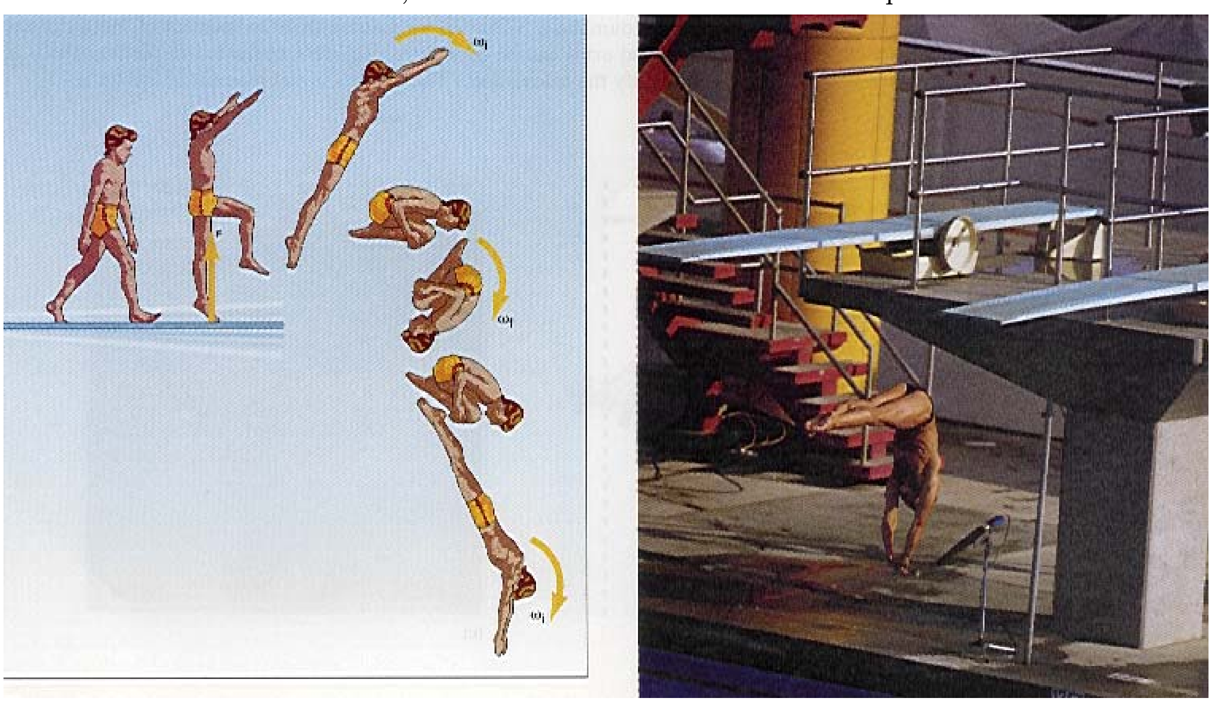


Figure 9.15 A man diving from a diving board.

motion, about an axis through his center of mass, with a relatively small angular velocity ω_i. As the man leaves the board there is no longer a torque acting on him, and his angular momentum must be conserved. His initial moment of inertia is I_i, and he is spinning at an angular velocity ω_i. If he now bends his knees and pulls his legs and arms up toward himself to form a ball, his moment of inertia decreases to a value I_f. But by the conservation of angular momentum

$$I_f\omega_f = I_i\omega_i \tag{9.53}$$

Since I_f has decreased, his angular velocity ω_f must increase to maintain the equality of the conservation of momentum. The man now rotates relatively rapidly for one or two turns. He then stretches his body out to its original configuration with the larger value of the moment of inertia. His angular velocity then decreases to the relatively low value ω_i that he started with. If he has timed his dive properly, his outstretched body will enter the water head first at the end of his dive. The force of gravity acts on the man throughout the motion and causes the center of mass of the man to follow the parabolic trajectory associated with any projectile. Thus, the center of mass of the man is moving under the force of gravity while the man rotates around his center of mass. A trapeze artist uses the same general techniques when she rotates her body while moving through the air from one trapeze to another.

Rotational Collision (an Idealized Clutch)

Consider two disks rotating independently, as shown in figure 9.16(a). The original angular momentum of the two rotating disks is the sum of the angular momentum of each disk, that is,

$$L_i = L_{1i} + L_{2i} \tag{9.54}$$

The initial angular momentum of disk 1 is

$$L_{1i} = I_1\omega_{1i}$$

and the initial angular momentum of disk 2 is

$$L_{2i} = I_2\omega_{2i}$$

Hence, the total initial angular momentum is

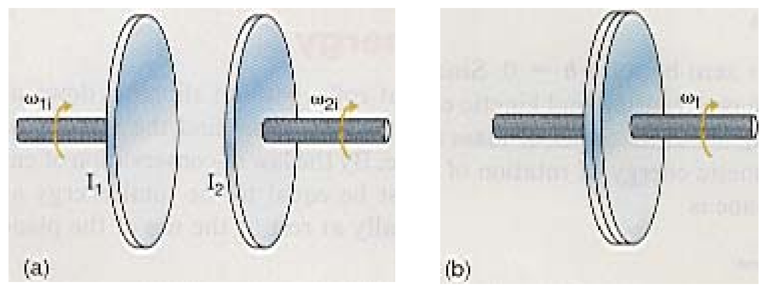


Figure 9.16 Rotational collision—the clutch.

$$L_i = I_1\omega_{1i} + I_2\omega_{2i} \tag{9.55}$$

The two disks are now forced together along their axes. Initially there may be some slipping of the disks but very quickly the two disks are coupled together by friction and spin as one, with one final angular velocity ω_f, as shown in figure 9.16(b). During the coupling process disk 1 exerted a torque on disk 2, while by Newton's third law, disk 2 exerted an equal but opposite torque on disk 1. Therefore, the net torque is zero and angular momentum must be conserved; that is, the final value of the angular momentum must equal the initial value:

$$L_f = L_i \tag{9.52}$$

The final value of the angular momentum is the sum of the angular momentum of each disk:

$$L_f = L_{1f} + L_{2f}$$

The final value of the angular momentum of disk 1 is

$$L_{1f} = I_1\omega_f$$

while for disk 2, we have

$$L_{2f} = I_2\omega_f$$

Note that both disks have the same final angular velocity, since they are coupled together. The final momentum is therefore

$$L_f = I_1\omega_f + I_2\omega_f = (I_1 + I_2)\omega_f \tag{9.56}$$

Substituting equations 9.55 and 9.56 into the conservation of angular momentum, equation 9.52, we get

$$(I_1 + I_2)\omega_f = I_1\omega_{1i} + I_2\omega_{2i} \tag{9.57}$$

Solving for the final angular velocity of the coupled disks, we have

$$\omega_f = \frac{I_1\omega_{1i} + I_2\omega_{2i}}{I_1 + I_2} \quad (9.58)$$

This idealized device is the basis of a clutch. For a real clutch, the first spinning disk could be attached to the shaft of a motor, while the second disk could be connected through a set of gears to the wheels of the vehicle. When disk 2 is coupled to disk 1, the wheels of the vehicle turn. When the disks are separated, the wheels are disengaged.

9.8 Combined Translational and Rotational Motion Treated by the Law of Conservation of Energy

Let us now consider the motion of a ball that rolls, without slipping, down an inclined plane, as shown in figure 9.17. In particular, let us find the velocity of the ball at the bottom of the one meter long incline. By the law of conservation of energy, the total energy at the top of the plane must be equal to the total energy at the bottom of the plane. Because the ball is initially at rest at the top of the plane, all the energy at the top is potential energy:

$$E_{top} = PE_{top} = mgh$$

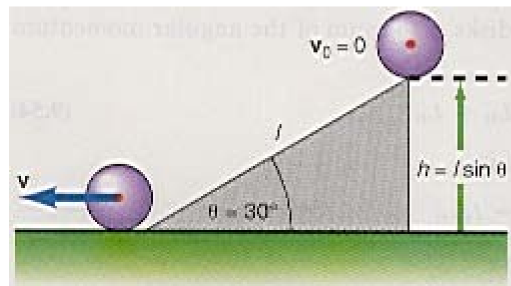


Figure 9.17 Combined translational and rotational motion.

At the bottom of the plane the potential energy is zero because $h = 0$. Since the body is translating at the bottom of the incline, it has a translational kinetic energy of its center of mass of $1/2\ mv^2$. But it is also rotating about its center of mass at the bottom of the plane, and therefore it also has a kinetic energy of rotation of $1/2\ I\omega^2$. Therefore the total energy at the bottom of the plane is

$$E_{bot} = KE_{trans} + KE_{rot}$$

$$E_{bot} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \quad (9.59)$$

Equating the total energy at the bottom to the total energy at the top, we have

$$E_{bot} = E_{top}$$

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh \quad (9.60)$$

The moment of inertia for the ball is the same as a solid sphere,

$$I = \frac{2}{5}mr^2 \quad (9.61)$$

The angular velocity ω of the rotating ball is related to the linear velocity of a point on the surface of the ball by

$$\omega = \frac{v}{r} \quad (9.62)$$

The distance that a point on the edge of the ball moves along the incline is the same as the distance that the center of mass of the ball moves along the incline. Hence, the velocity of the edge of the ball is equal to the velocity of the center of mass of the ball. Substituting equations 9.61 and 9.62 into equation 9.60, we have

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{2}{5}mr^2\right)\left(\frac{v}{r}\right)^2 = mgh$$

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}\tfrac{2}{5}mr^2\frac{v^2}{r^2} = mgh$$

Simplifying,

$$\frac{v^2}{2} + \frac{v^2}{5} = gh$$

$$\frac{5v^2 + 2v^2}{10} = gh$$

$$\frac{7}{10}v^2 = gh$$

$$v = \sqrt{\left(\tfrac{10}{7}\right)gh} \qquad (9.63)$$

the velocity of the ball at the bottom of the plane. The height h, found from the trigonometry of the triangle in figure 9.17, is

$$h = l\sin\theta = (1\text{ m})\sin 30.0^0 = 0.500\text{ m}$$

Therefore the velocity at the bottom of the plane is

$$v = \sqrt{\left(\tfrac{10}{7}\right)\left(9.80\text{ m/s}^2\right)\left(0.500\text{ m}\right)}$$

$$= 2.65\text{ m/s}$$

Note that this is the same result obtained in example 9.8 in section 9.6. The energy approach is obviously much easier.

As another example of the combined translational and rotational motion of a rigid body, let us consider the Atwood's machine shown in figure 9.18(a). Using the law of conservation of energy let us find the velocity of the mass m_A as it hits the ground. The total energy of the system in the configuration shown consists only of the potential energy of the two masses m_A and m_B, that is,

$$E_{tot} = m_A g h_A + m_B g h_B \qquad (9.64)$$

When the system is released, m_A loses potential energy as it falls but gains kinetic energy due to its motion. Mass, m_B gains potential energy as it rises and also acquires a kinetic energy. The pulley, when set into rotational motion, also has kinetic energy of rotation. The total energy of the system as m_A strikes the ground, found from figure 9.18(b), is

$$E_{tot} = PE_B + KE_A + KE_B + KE_{pulley}$$

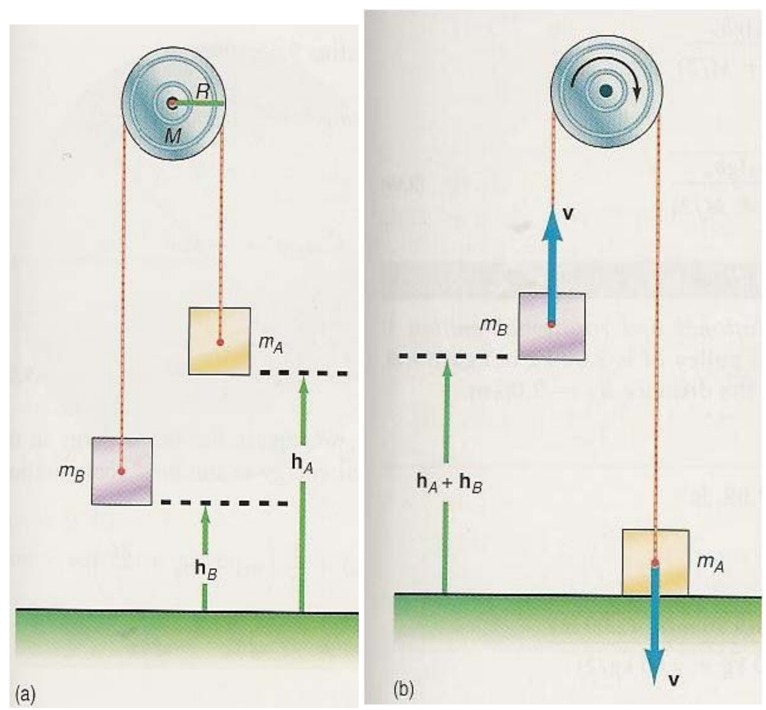


Figure 9.18 Atwood's machine revisited.

$$E_{tot} = m_B g(h_A + h_B) + \frac{1}{2}m_A v^2 + \frac{1}{2}m_B v^2 + \frac{1}{2}I\omega^2 \qquad (9.65)$$

The speed of masses A and B are equal because they are tied together by the string. The moment of inertia of the pulley (disk), found from figure 9.5, is

$$I_{disk} = \frac{1}{2}MR^2 \qquad (9.66)$$

Also, the angular velocity ω of the disk is related to the tangential velocity of the string as it passes over the pulley by

$$\omega = \frac{v}{R} \tag{9.67}$$

Substituting equations 9.66 and 9.67 into equation 9.65, gives

$$E_{\text{tot}} = m_{\text{B}}g(h_{\text{A}} + h_{\text{B}}) + \tfrac{1}{2}m_{\text{A}}v^2 + \tfrac{1}{2}m_{\text{B}}v^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 \tag{9.65}$$

Simplifying,

$$E_{\text{tot}} = m_{\text{B}}g(h_{\text{A}} + h_{\text{B}}) + \tfrac{1}{2}\left(m_{\text{A}} + m_{\text{B}}\right)v^2 + \tfrac{1}{4}Mv^2$$

or

$$E_{\text{tot}} = m_{\text{B}}g(h_{\text{A}} + h_{\text{B}}) + \tfrac{1}{2}\left(m_{\text{A}} + m_{\text{B}} + \frac{M}{2}\right)v^2 \tag{9.68}$$

By the law of conservation of energy, we equate the total energy in the initial configuration, equation 9.64, to the total energy in the final configuration, equation 9.68, obtaining

$$m_{\text{A}}gh_{\text{A}} + m_{\text{B}}gh_{\text{B}} = m_{\text{B}}g(h_{\text{A}} + h_{\text{B}}) + \tfrac{1}{2}\left(m_{\text{A}} + m_{\text{B}} + \frac{M}{2}\right)v^2$$

$$\tfrac{1}{2}\left(m_{\text{A}} + m_{\text{B}} + \frac{M}{2}\right)v^2 = m_{\text{A}}gh_{\text{A}} + m_{\text{B}}gh_{\text{B}} - m_{\text{B}}gh_{\text{A}} - m_{\text{B}}gh_{\text{B}}$$

$$\tfrac{1}{2}\left(m_{\text{A}} + m_{\text{B}} + \frac{M}{2}\right)v^2 = (m_{\text{A}} - m_{\text{B}})gh_{\text{A}}$$

$$v^2 = \frac{(m_A - m_B)gh_A}{\frac{1}{2}(m_A + m_B + M/2)}$$

Solving for v, we get

$$v = \sqrt{\frac{(m_{\text{A}} - m_{\text{B}})gh_{\text{A}}}{\tfrac{1}{2}\left(m_{\text{A}} + m_{\text{B}} + \frac{M}{2}\right)}} \tag{9.69}$$

Example 9.14

Conservation of energy and combined translational and rotational motion. If m_B = 30.0 g, m_A = 50.0 g, and the mass of the pulley M is 2.00 kg in figure 9.18, find the velocity of mass m_A as it falls through the distance h_A = 2.00 m.

Solution

The velocity of block A, found from equation 9.69, is

$$v = \sqrt{\frac{(m_{\text{A}} - m_{\text{B}})gh_{\text{A}}}{\tfrac{1}{2}\left(m_{\text{A}} + m_{\text{B}} + \frac{M}{2}\right)}}$$

$$v = \sqrt{\frac{(0.0500\text{ kg} - 0.0300\text{ kg})\left(9.80\text{ m/s}^2\right)\left(2.00\text{ m}\right)}{\tfrac{1}{2}\left(0.0500\text{ kg} + 0.0300\text{ kg} + \frac{2.00\text{ kg}}{2}\right)}}$$

$$= 0.850\text{ m/s}$$

Note that this is the same result obtained by treating the Atwood's machine by Newton's laws of motion rather than the energy technique.

To go to this Interactive Example click on this sentence.

9.9 Work in Rotational Motion

The work done in translating a body from one position to another was found in chapter 7 as

$$W = Fx \tag{7.1}$$

where F is the force in the direction of the displacement and x is the magnitude of the displacement. We can find the work done in causing a body to rotate from equation 7.1 and figure 9.19. In figure 9.19, a string is wrapped around the disk and pulled with a constant force F, causing the disk to rotate through the angle θ. The rim of the disk moves through the distance s. The work done by the force is

$$W = Fx$$

But $x = s$ and $s = r\theta$. Therefore,

$$W = F\,r\theta \tag{9.70}$$

But F times r is equal to the torque τ acting on the disk, that is

$$Fr = \tau \tag{9.71}$$

Substituting equation 9.71 into equation 9.70 gives the work done to rotate the disk as

$$W = \tau\theta \tag{9.72}$$

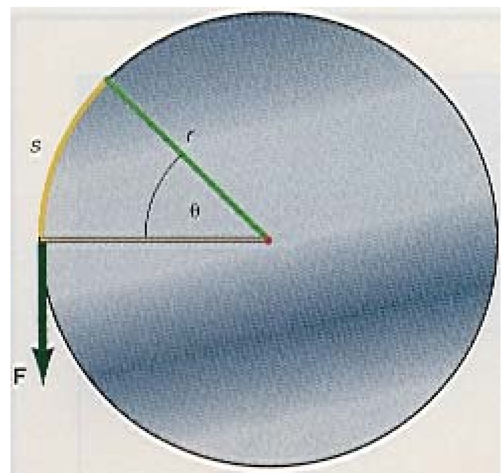


Figure 9.19 Work in rotational motion.

The power expended in rotating the disk for a time t is

$$P = \frac{W}{t} = \frac{\tau\,\theta}{t} \tag{9.73}$$

but $\theta/t = \omega$, the angular velocity. Therefore,

$$P = \tau\omega \tag{9.74}$$

Example 9.15

Work done in rotational motion. A constant force of 5.00 N is applied to a string that is wrapped around a disk of 0.500-m radius. If the wheel rotates through an angle of 2.00 rev, how much work is done?

Solution

The work done, given by equation 9.72, is

$$W = \tau\theta = rF\theta$$

$$= (0.500\text{ m})(5.00\text{ N})(2.00\text{ rev})\left(\frac{2\,\pi\ \text{rad}}{\text{rev}}\right)$$

$$= 31.4\text{ J}$$

To go to this Interactive Example click on this sentence.

Have you ever wondered . . . ?
An Essay on the Application of Physics
Attitude Control of Airplanes and Spaceships

Have you ever wondered how an airplane or space vehicle is able to change its direction of flight? A plane or spacecraft can turn, climb, and dive. But how does it do this?

Attitude Control of Aircraft

An aircraft changes its attitude by the use of control surfaces, figure 1. As we saw in section 6.7, an airplane has three ways of changing the direction of its motion. They are yaw, pitch, and roll. Yaw is a rotation about the vertical axis of the aircraft. The control surface to yaw the aircraft is the rudder, which is located at the rear of the vertical stabilizer. Pitch is a rotation about the lateral axis of the aircraft. The control surface to pitch the aircraft is the elevator, which is located at the rear of the horizontal stabilizer. Roll is a rotation about the longitudinal axis of the aircraft. The control surfaces to roll the aircraft are the ailerons, which are located on the trailing edge of the wings.

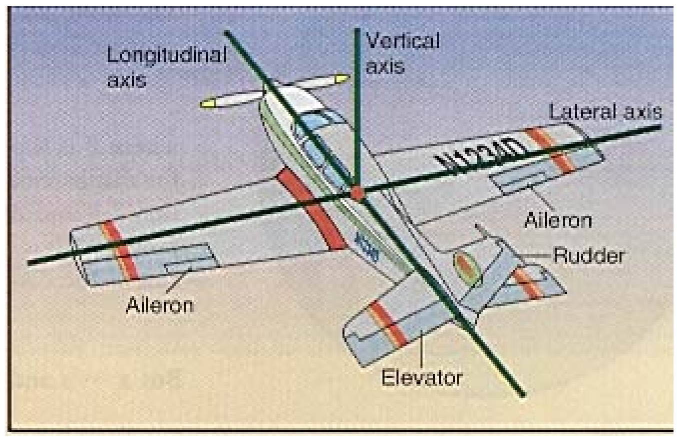


Figure 1 Control surfaces.

1. **Yaw Control:** Yaw is a rotation of the aircraft about a vertical axis that passes through the center of gravity of the aircraft, as shown in figure 2. The aircraft can yaw to the right or left, as seen from the position of the pilot in the aircraft.

Before the pilot presses either rudder pedal in the cockpit, the rudder is aligned with the vertical stabilizer and the air streams past the rudder exerting no unbalanced forces on it. When the pilot presses the right rudder pedal the rudder moves toward the right, as seen from above and behind the aircraft, figure 3(a). In this position the air stream exerts a normal force **F** on the rudder surface, as shown in the figure. If we draw the line r from the center of gravity of the aircraft to the point of application of the force, we see that this force produces a torque about the vertical axis. We find the lever arm for this torque by dropping a perpendicular from the axis of rotation to the line of action of the force. As seen in the figure, the lever arm is $r \sin \psi$. Hence, the torque is

$$\tau = Fr \sin \psi \qquad (9H.1)$$

This torque produces a clockwise torque about the center of gravity causing the aircraft to rotate (yaw) to

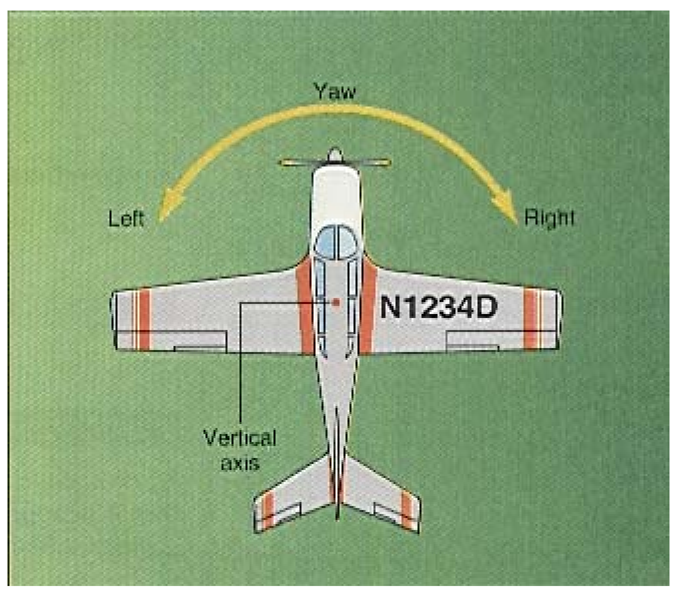


Figure 2 Aircraft yaw.

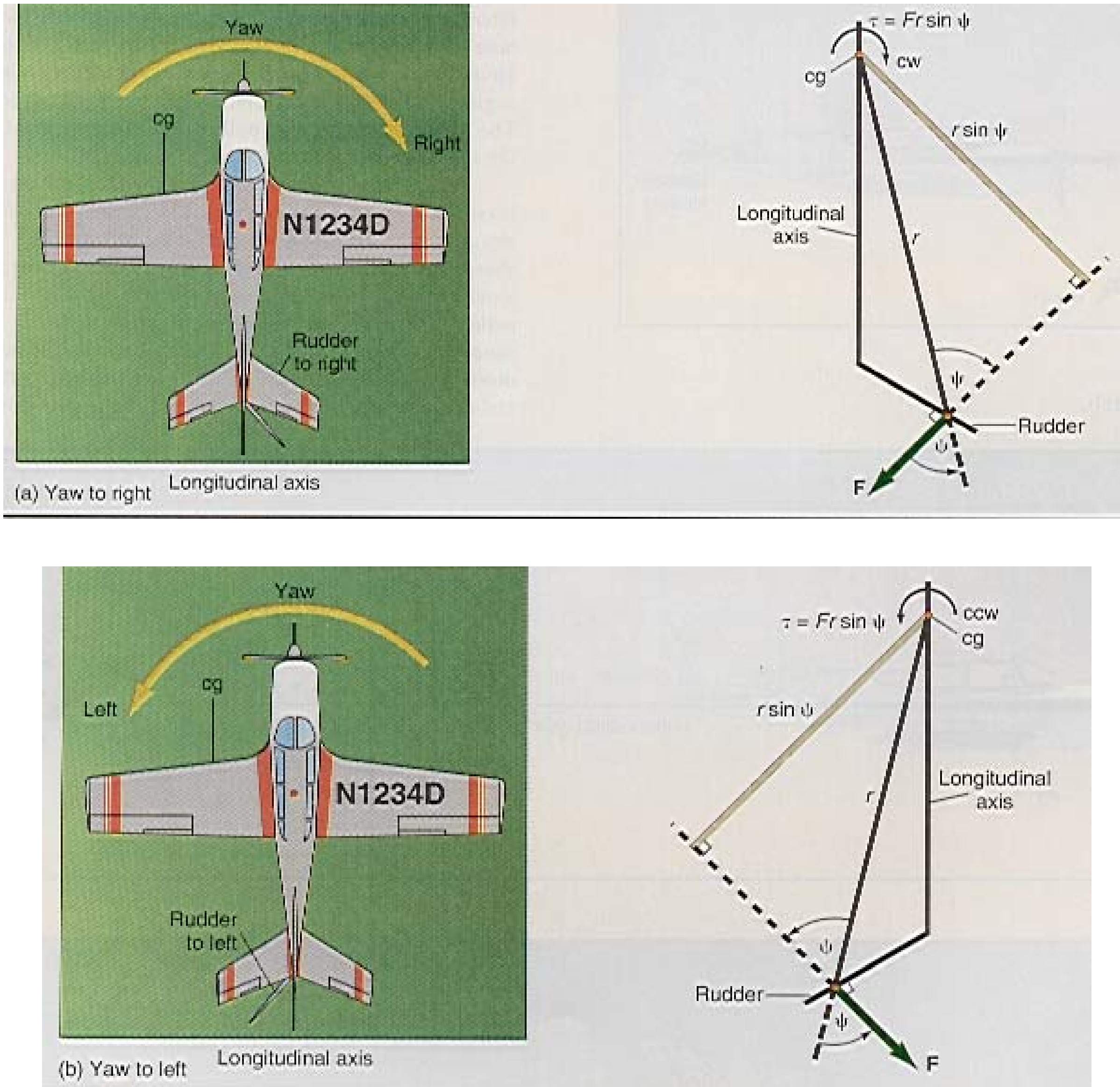


Figure 3 Dynamics of aircraft yaw.

When the pilot presses the left rudder pedal, the rudder moves toward the left, as seen from above and behind the aircraft, figure 3(b). For this case the force of the air on the rudder produces a counterclockwise torque that causes the aircraft to rotate to the left, as seen in the diagram. Thus the rudder is a control surface that produces a torque on the aircraft that causes it to rotate either clockwise or counterclockwise about the vertical axis.

2. **Pitch Control:** Pitch is a rotation of the aircraft about a lateral axis that passes through the center of gravity of the aircraft, figure 4. In straight and level flight, the thrust vector of the aircraft lies along the longitudinal axis of the aircraft and thus the aircraft moves straight ahead.

When the pilot pulls the "stick" backward, the elevator is pushed upward, figure 5(a). The air that hits the elevator exerts a normal force **F** on the elevator, as seen in the diagram. If we draw the line r from the center of gravity of the aircraft to the point of application of the force, we see that this force produces a clockwise torque about the lateral axis of the aircraft. We find the lever arm for this torque by dropping a perpendicular from the axis of rotation to the line of action of the force.

As seen in the figure, the lever arm is $r \sin \theta$. Hence the torque acting on the aircraft is given by

$$\tau = Fr \sin \theta \tag{9H.2}$$

This torque causes the aircraft to rotate (pitch) about the lateral axis, such that the tail goes downward and the nose goes upward, figure 5(a). The thrust vector of the aircraft is no longer horizontal but now makes a positive angle with the horizontal, and hence the plane climbs. The farther back the pilot pulls on the stick the greater the torque and hence the steeper the climb.

When the pilot pushes the stick forward, the elevator is pushed downward, figure 5(b). The air that hits the elevator exerts a normal force **F** on the elevator, as shown. We find the lever arm for this torque by dropping a perpendicular from the axis of rotation to the line of action of the force, as shown. The resulting counterclockwise torque

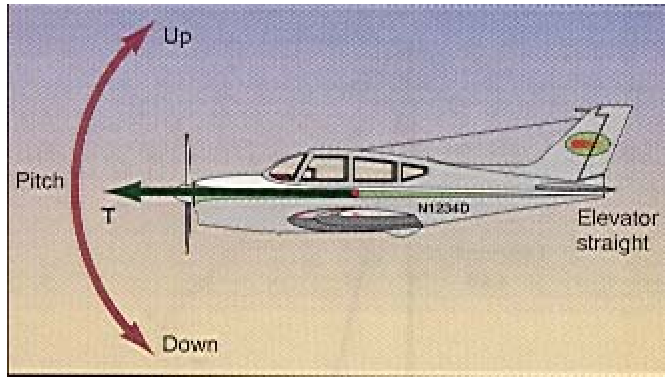


Figure 4 Aircraft pitch.

pushes the tail up and the nose down. The thrust vector now falls below the horizontal and the plane dives. The farther forward the pilot pushes the stick, the greater the torque and hence the steeper the dive. In this way the pilot can make the aircraft climb or dive.

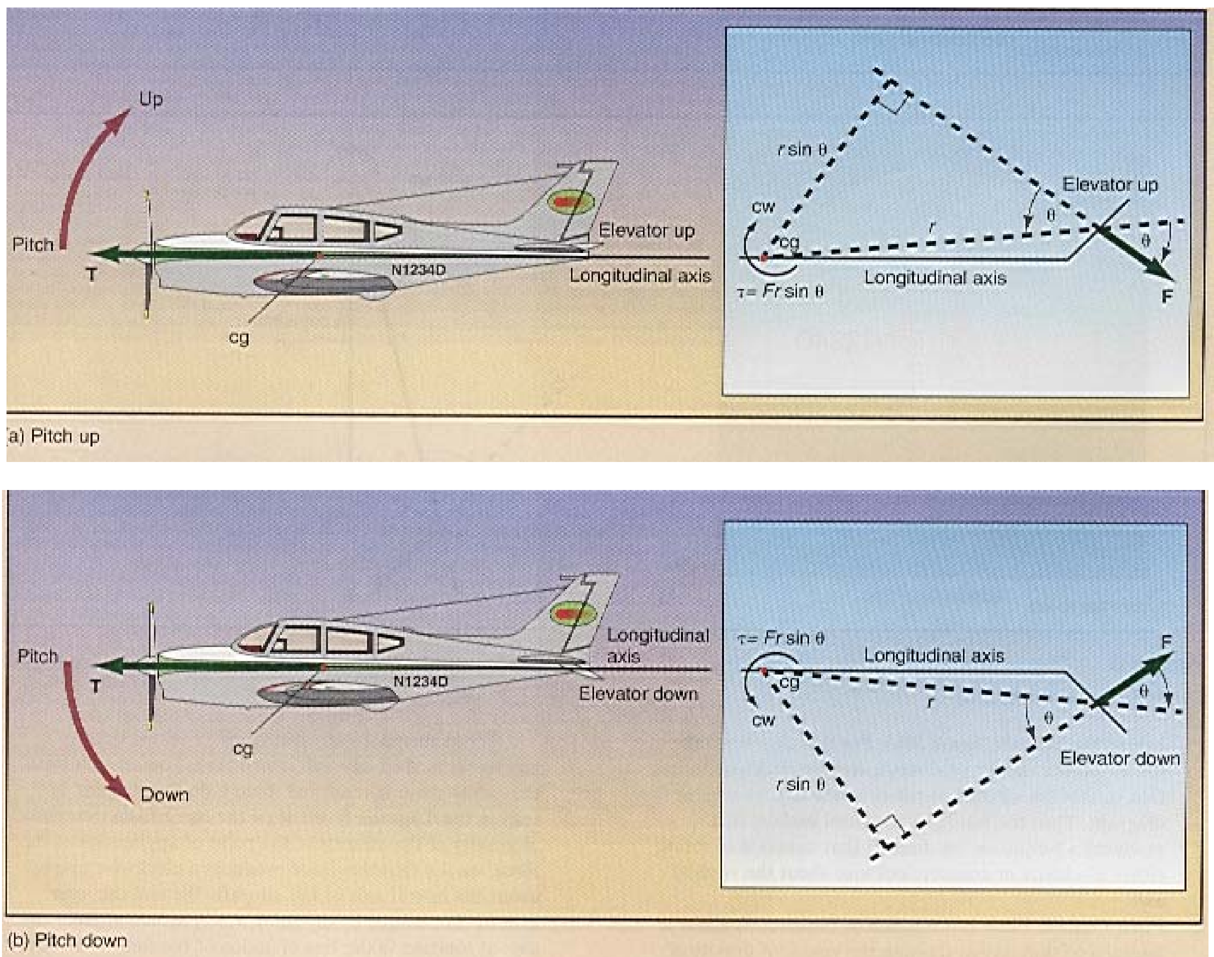


Figure 5 Dynamics of aircraft pitch.

3. **Roll Control:** Roll is a rotation of the aircraft about the longitudinal axis of the aircraft. When the pilot pushes the stick to the left, the plane will roll to the left; when he pushes the stick to the right, the plane will roll to the right, figure 6.

When the pilot pushes the stick to the left, the right aileron is pushed downward and the left aileron is pushed upward, figure 7(c). The wind blowing over the wings exerts a force on the ailerons as shown in figure 7(a,b). The force acting on the raised left aileron pushes the left wing downward, while the force acting on the lowered right aileron pushes the right wing upward. The ailerons act similar to the elevator in that they produce a torque about the lateral axis of the aircraft. However, with one aileron up and one down the torques they produce to pitch the aircraft are equal and opposite and hence have no effect on pitching the aircraft. However, the force up on the right wing and the force down on the left wing cause a counterclockwise torque about the longitudinal axis, as viewed from the rear of the aircraft (the view that is seen by the pilot). Therefore the aircraft rolls to the left, figure 7(c). When the aircraft has rolled to the required bank angle, the pilot places the stick back to the neutral position and the aircraft stays at this angle of bank. To bring the aircraft back to level flight the pilot must push the stick to the right. The aircraft now rolls to the right until the aircraft is level. Then the pilot places the stick in the neutral position.

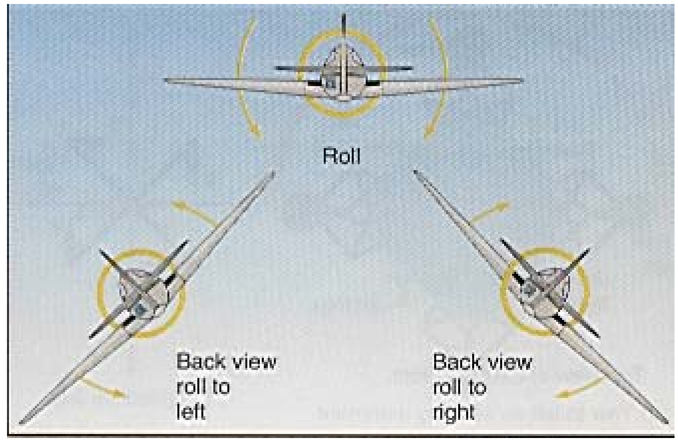


Figure 6 Aircraft roll.

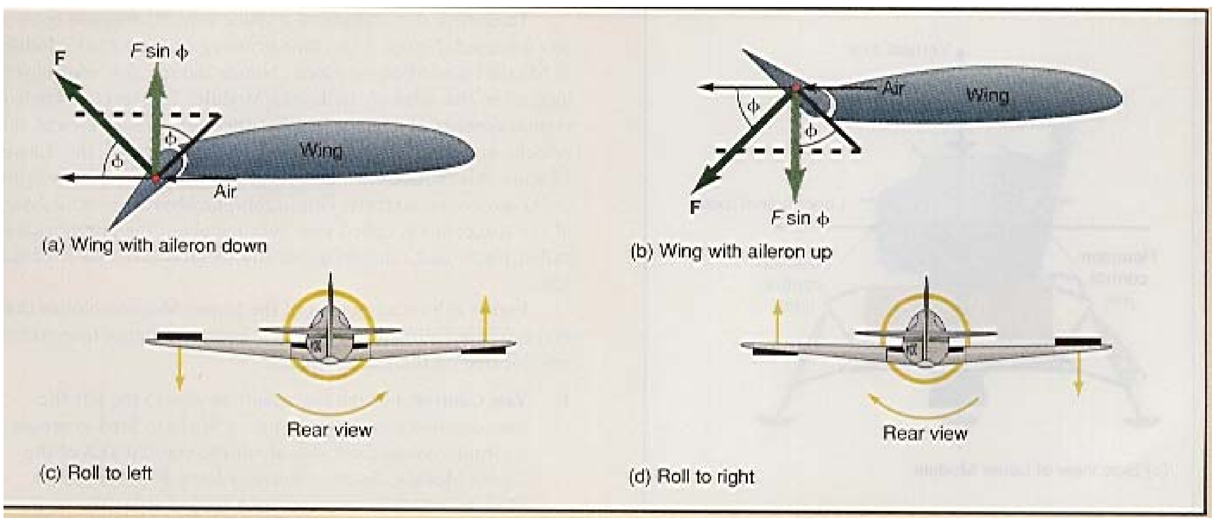


Figure 7 Dynamics of aircraft roll.

To roll the aircraft to the right the pilot pushes the stick to the right. The right aileron now goes up and the left aileron now goes down, figure 7(d). The force down on the right wing and the force up on the left wing causes a clockwise torque about the longitudinal axis. Thus, the aircraft rotates (rolls) to the right.

The force exerted on a control surface by the air creates the necessary torque to rotate the aircraft in any specified direction.

Attitude Control of Space Vehicles

An aircraft will not work in space because there is no air to exert the necessary lift on the wings of the aircraft. Nor can rudders, elevators, or ailerons work in space because there is no air to exert forces on the control surfaces to change the attitude of the vehicle.

To control the attitude of a space vehicle, reaction control jets are used. Figure 8 is a line drawing of the Lunar Module (LM) that landed on the moon. Notice the reaction control jets located on the sides of the Lunar

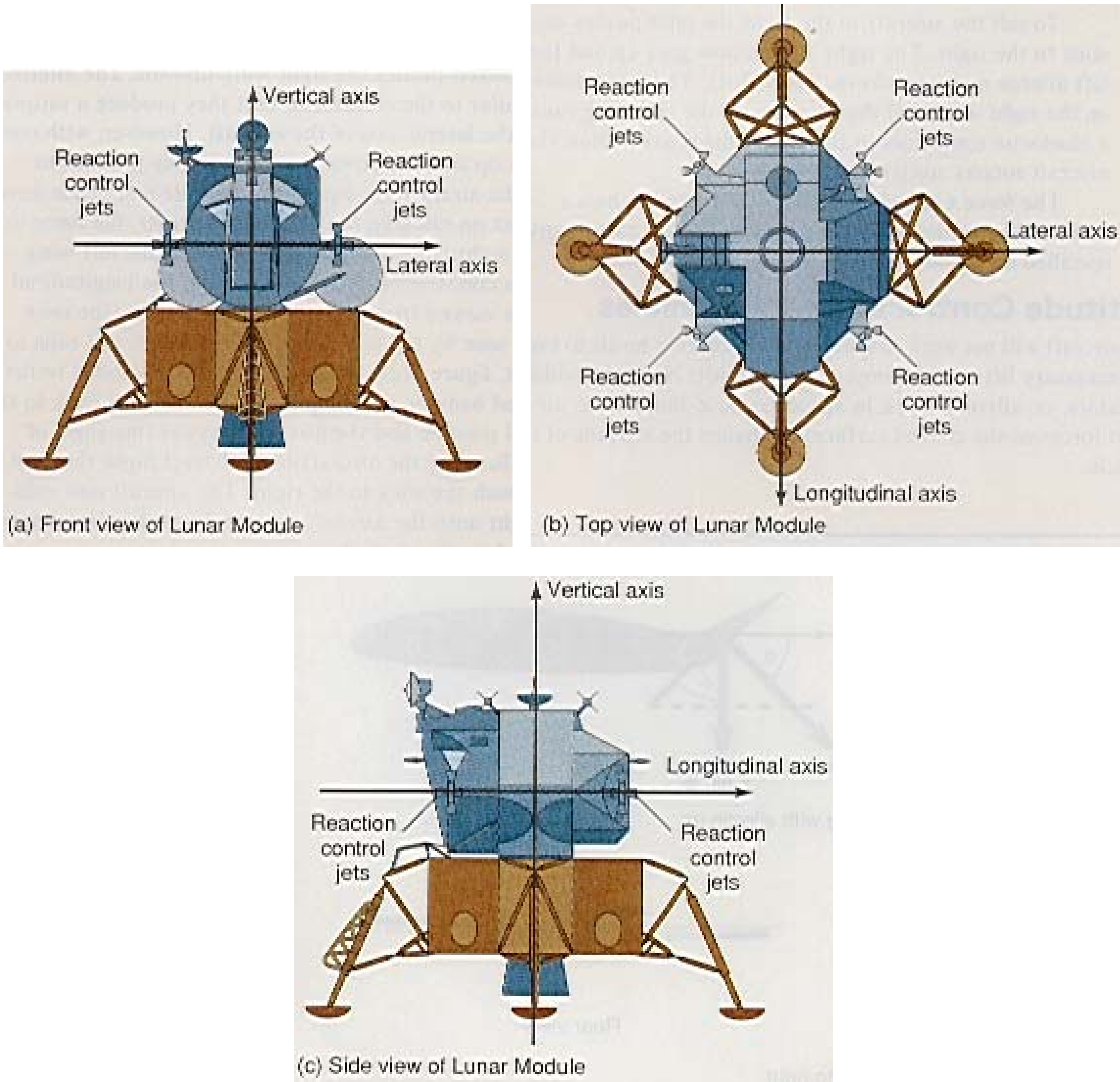


Figure 8 The Lunar Module.

Module. The reaction control system consists of 16 small rocket thrusters placed around the vehicle to control the translation and rotation of the Lunar Module. Also notice that the axes of the spacecraft are the same as the axes of an aircraft. Thus a rotation about the vertical axis of the spacecraft is called yaw, rotation about the lateral axis is called pitch, and rotation about the longitudinal axis is called roll.
Figure 8(b) is a top view of the Lunar Module. Notice that there are four thruster assemblies, each containing four rocket jets, located on the Lunar Module.

1. **Yaw Control:** For the spacecraft to yaw to the left the four reaction jets shown in figure 9(a) are fired to create a torque counterclockwise about the vertical axis of the Lunar Module. Each jet exerts a force **F** on the Lunar Module, which in turn creates a torque about the vertical axis. The total torque is the sum of the four torques. For the spacecraft to yaw to the right the four reaction jets shown in figure 9(b) are fired to create a torque clockwise about the vertical axis of the Lunar Module. Notice that a different set of jets are used to yaw to the right than to yaw to the left.

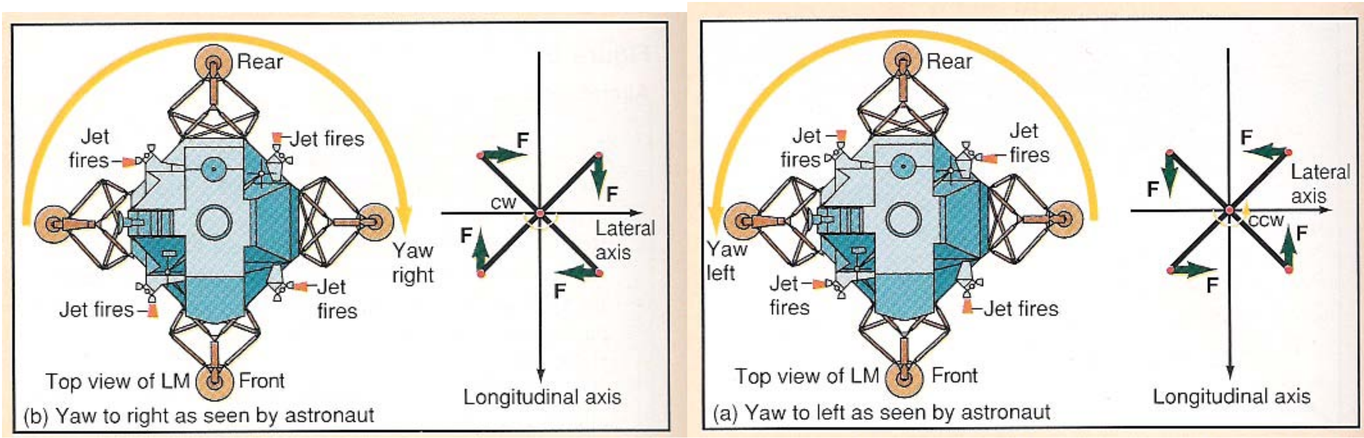


Figure 9 Dynamics of Lunar Module yaw.

2. **Pitch Control:** For the Lunar Module to pitch downward, the two reaction jets on each side of the Lunar Module (total of 4 jets) shown in figure 10(a) are fired to create a torque counterclockwise about the lateral axis of the Lunar Module. For the spacecraft to pitch upward the two reaction jets on each side of the Lunar Module (total of 4 jets) shown in figure 10(b) are fired to create a torque clockwise about the lateral axis of the Lunar Module.

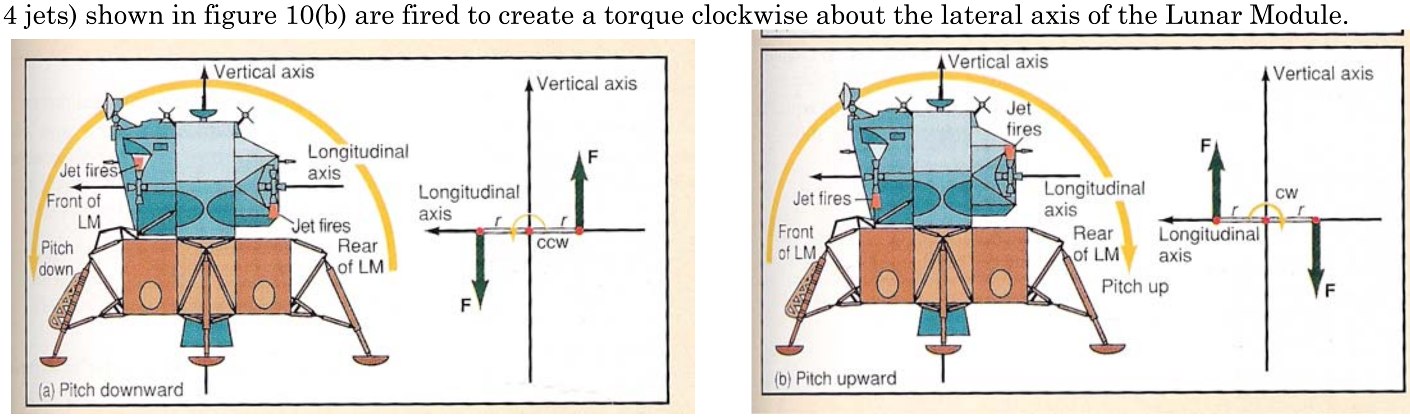


Figure 10 Dynamics of Lunar Module pitch.

3. **Roll Control:** For the spacecraft to roll to the left the two reaction jets on each side of the Lunar Module (total of 4 jets) shown in figure 11(a) are fired to create a torque counterclockwise about the longitudinal axis of the Lunar Module. (Don't forget that left and right are defined from the position of the pilot. Figure 11 shows the Lunar Module from a front view, and hence left and right appear to be reversed.)

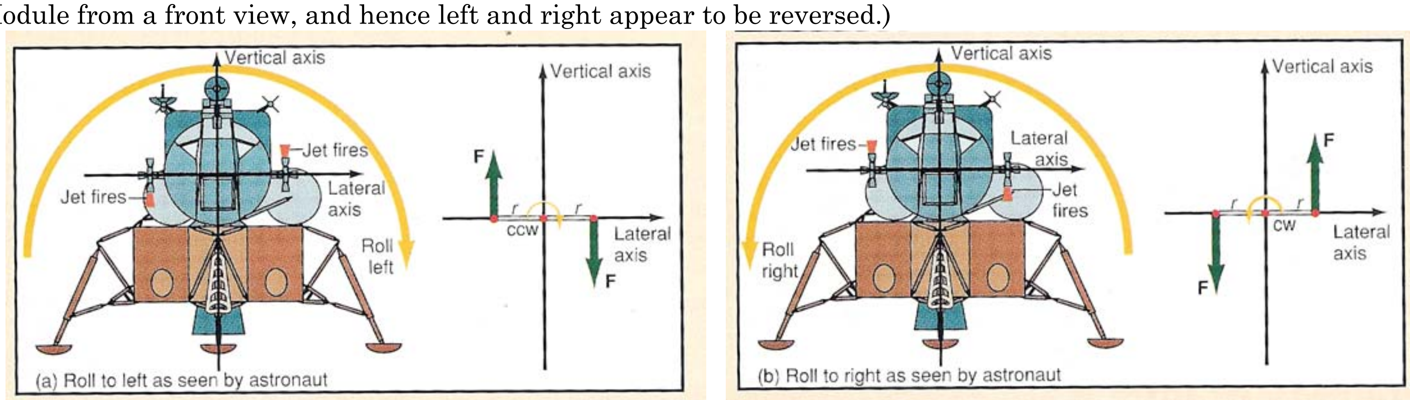


Figure 11 Dynamics of Lunar Module roll.

A roll to the right is accomplished by firing the two reaction jets on each side of the Lunar Module (total of 4 jets) shown in figure 11(b) to create a torque clockwise about the longitudinal axis of the Lunar Module. Thus the Lunar Module, and any spacecraft for that matter, can control its attitude by supplying torques for its rotation by the suitable firing of the different reaction control jets.

The Language of Physics

Angular displacement
The angle that a body rotates through while in rotational motion (p.).

Angular velocity
The change in the angular displacement of a rotating body about the axis of rotation with time (p.).

Angular acceleration
The change in the angular velocity of a rotating body with time (p.).

Kinematic equations for rotational motion
A set of equations that give the angular displacement and angular velocity of a rotating body at any instant of time, and the angular velocity at a particular angular displacement, if the angular acceleration of the body is constant (p.).

Kinetic energy of rotation
The energy that a body possesses by virtue of its rotational motion (p.).

Moment of inertia
The measure of the resistance of a body to a change in its rotational motion. It is the rotational analogue of mass, which is a measure of the resistance of a body to a change in its translational motion. The larger the moment of inertia of a body the more difficult it is to put that body into rotational motion (p.).

Newton's second law for rotational motion
When an unbalanced external torque acts on a body, it gives that body an angular acceleration. The angular acceleration is directly proportional to the torque and inversely proportional to the moment of inertia (p.).

Newton's first law for rotational motion
A body in motion at a constant angular velocity will continue in motion at that same constant angular velocity unless acted upon by some unbalanced external torque (p.).

Newton's third law of rotational motion
If body A and body B have the same axis of rotation, and if body A exerts a torque on body B, then body B exerts an equal but opposite torque on body A (p.).

Angular momentum
The product of the moment of inertia of a rotating body and its angular velocity (p.).

Law of conservation of angular momentum
If the total external torque acting on a system is zero, then there is no change in the angular momentum of the system, and the final angular momentum is equal to the initial angular momentum (p.).

Summary of Important Equations

Angular velocity $\omega = \frac{\Delta\theta}{\Delta t} = \frac{\theta}{t}$ (9.1)

Angular acceleration
$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t} \quad (9.3)$$

Kinematic equations
$$\omega = \omega_0 + \alpha t \quad (9.4)$$
$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \quad (9.9)$$
$$\omega^2 = \omega_0^2 + 2\alpha\theta \quad (9.10)$$

Relations between translational and rotational variables
$$s = r\theta \quad (6.5)$$
$$v = r\omega \quad (9.2)$$
$$a = r\alpha \quad (9.5)$$

Centripetal acceleration
$$a_c = \omega^2 r \quad (9.11)$$

Kinetic energy of rotation
$$KE_{rot} = \frac{1}{2}I\omega^2 \quad (9.15)$$

Moment of inertia
$$I = \sum_{i=1}^{n} m_i r_i^2 \quad (9.16)$$

Moment of inertia for a single mass
$$I = mr^2 \quad (9.17)$$

Newton's second law for rotational motion $\tau = I\alpha$ (9.22)

Angular momentum
$$L = I\omega \quad (9.48)$$

Newton's second law in terms of momentum $\tau = \frac{\Delta L}{\Delta t}$ (9.49)

Law of conservation of angular momentum (no external torques)
$$L_f = L_i \quad (9.52)$$

Work done in rotational motion
$$W = \tau\theta \quad (9.72)$$

Power expended in rotational motion $P = \tau\omega$ (9.74)

Questions for Chapter 9

1. Discuss the similarity between the equations for translational motion and the equations for rotational motion.

2. When moving in circular motion at a constant angular velocity, why does the body at the greatest distance from the axis of rotation move faster than the body closest to the axis of rotation?

3. It is easy to observe the angular velocity of the second hand of a clock. Why is it more difficult to observe the angular velocity of the minute and hour hands of the clock?

*4. If a cylinder, a ball, and a ring are placed at the top of an inclined plane and then allowed to roll down the plane, in what order will they arrive at the bottom of the plane? Why?

*5. How would you go about approximating the rotational kinetic energy of our galaxy?

6. Which would be more difficult to put into rotational motion, a large sphere or a small sphere? Why?

7. Why must the axis of rotation be specified when giving the moment of inertia of an object?

*8. If two balls collide such that the force transmitted lies along a line connecting the center of mass of each body, can either ball be put into rotational motion? If the balls collide in a glancing collision in which there is also friction between the two surfaces as they collide, can either ball be put into rotational motion? Draw a diagram of the collision in both cases and discuss both possibilities.

*9. As long as there are no external torques acting on the earth, the earth will continue to spin forever at its present angular velocity. Discuss the possibility of small perturbative torques that might act on the earth and what effect they might have.

*10. We said that the angular displacement θ could be treated as a vector. Consider a rotation of your book through an angular displacement of 90^0 about the x-axis, then a rotation through an angular displacement of 90^0 about the y-axis, and finally a rotation through an angular displacement of 90^0 about the z-axis. Would you get the same result if you changed the order of the rotations to the y-, x-, and then z-axis? So should angular displacements be treated as vectors? What happens if the angular displacements are infinitesimal or at least very small? Then can angles be treated as vectors? What about the angular velocity $\omega = \Delta\theta/\Delta t$ and the angular acceleration $\alpha = \Delta\omega/\Delta t$? Is it legitimate to consider these quantities as vectors?

*11. If the instantaneous angular velocity can be considered as a vector, should the angular momentum also be considered as a vector? If so, what direction would it have? What would the change in the direction of the angular momentum look like?

*12. It is said that if you throw a cat, upside down, into the air, it will always land on its feet. Discuss this possibility from the point of view of the cat moving his legs and tail and thus changing his moment of inertia and hence his angular velocity.

A falling cat lands on all four legs.

Problems for Chapter 9

9.2 Rotational Kinematics

1. Express the following angular velocities of a phonograph turntable in terms of rad/s. (a) 33 1/3 rpm (revolutions per minute), (b) 45 rpm, and (c) 78 rpm.

2. Determine the angular velocity of the following hands of a clock: (a) the second hand, (b) the minute hand, and (c) the hour hand.

3. A cylinder 15.0 cm in diameter rotates at 1000 rpm. (a) What is

its angular velocity in rad/s? (b) What is the tangential velocity of a point on the rim of the cylinder?

4. A circular saw blade rotating at 3600 rpm is reduced to 3450 rpm in 2.00 s. What is the angular acceleration of the blade?

5. A circular saw blade rotating at 3600 rpm is braked to a stop in 6 s. What is the angular acceleration? How many revolutions did the blade make before coming to a stop?

6. A wheel 50.0 cm in diameter is rotating at an initial angular velocity of 0.010 rad/s. It is given an acceleration of 0.050 rad/s^2. Find (a) the angular velocity at 5.00 s, (b) the angular displacement at 5.00 s, (c) the tangential velocity of a point on the rim at 5.00 s, (d) the tangential acceleration of a point on the rim, (e) the centripetal acceleration of a point on the rim, and (f) the resulting acceleration of a point on the rim.

9.3 The Kinetic Energy of Rotation

7. Find the kinetic energy of a 2.00-kg cylinder, 25.0 cm in diameter, if it is rotating about its longitudinal axis at an angular velocity of 0.550 rad/s.

8. A 3.00-kg ball, 15.0 cm in diameter, rotates at an angular velocity of 3.45 rad/s. Find its kinetic energy.

9.4 The Moment of Inertia

9. Calculate the moment of inertia of a 0.500-kg meterstick about an axis through its center, and perpendicular to its length.

10. Compute the moment of inertia through its center of a 7.27 kg bowling ball of radius 10.2 cm.

11. Find the moment of inertia for the system of point masses shown for (a) rotation about the y-axis and (b) for rotation about the x-axis. Given are m_1 = 2.00 kg, m_2 = 3.50 kg, r_1 = 0.750 m, and r_2 = 0.873 m.

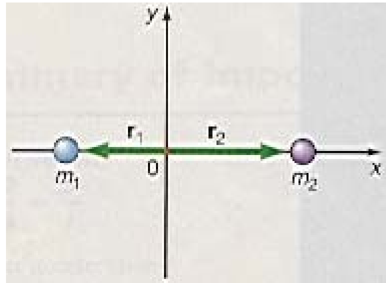


Diagram for problem 11.

*12. Find the moment of inertia for the system shown for rotation about (a) the y-axis, (b) the x-axis, and (c) an axis going through masses m_2 and m_4. Assume m_1 = 0.532 kg, m_2 = 0.425 kg, m_3 = 0.879 kg, and m_4 = 0.235 kg.

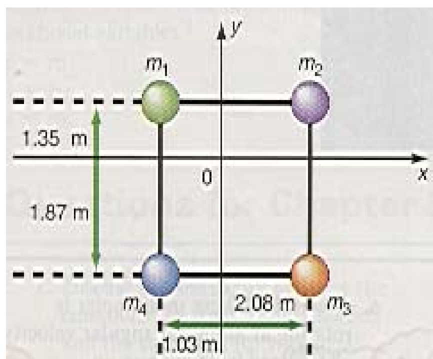


Diagram for problem 12.

9.5 Newton's Laws for Rotational Motion and 9.6 Rotational Dynamics

13. A solid wheel of mass 5.00 kg and radius 0.350 m is set in motion by a constant force of 6.00 N applied tangentially. Determine the angular acceleration of the wheel.

14. A torque of 5.00 m N is applied to a body. Of this torque, 2.00 m N of it is used to overcome friction in the bearings. The body has a resultant angular acceleration of 5.00 rad/s^2. (a) When the applied torque is removed, what is the angular acceleration of the body? (b) If the angular velocity of the body was 100 rad/s when the applied torque was removed how long will it take the body to come to rest?

15. A mass of 200 g is attached to a wheel by a string wrapped around the wheel. The wheel has a mass of 1.00 kg. Find the acceleration of the mass. Assume that the moment of inertia of the wheel is the same as a disk.

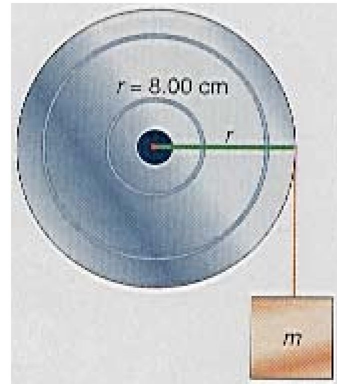


Diagram for problem 15.

16. A mass m_A of 10.0 kg is attached to another mass m_B of 4.00 kg by a string that passes over a pulley of mass M = 1.00 kg. The coefficient of kinetic friction between block B and the table is 0.400. Find (a) the acceleration of each block of the system, (b) the tensions in the cords, and (c) the velocity of block A as it hits the floor 0.800 m below its starting point.

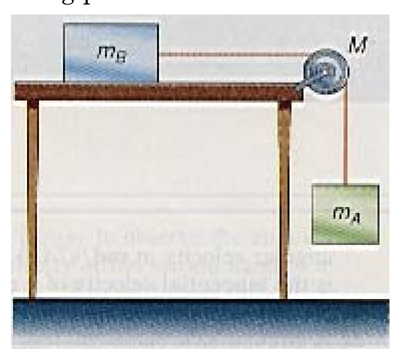


Diagram for problem 16.

17. A mass m_A = 200 g, and another mass m_B = 100 g are attached to an Atwood's machine that has a pulley mass M = 1.00 kg. (a) Find the acceleration of each block of the system. (b) Find the velocity of mass A as it hits the floor 1.50 meters below its starting point.

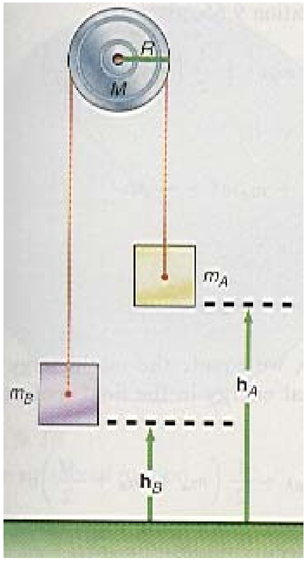


Diagram for problem 17.

9.7 Angular Momentum and Its Conservation

*18. A 75-kg student stands at the edge of a large disk of 150-kg mass that is rotating freely at an angular velocity of 0.800 rad/s. The disk has a radius of R = 3.00 m. (a) Find the initial moment of inertia of the disk and student and its kinetic energy. The student now walks toward the center of the disk. Find the moment of inertia, the angular velocity, and the kinetic energy when the student is at (b) 3R/4, (c) R/2, and (d) R/4.

19. Two disks are to be made into an idealized clutch. Disk 1 has a mass of 3.00 kg and a radius of 20.0 cm, while disk 2 has a mass of 1.00 kg and a radius of 20.0 cm. If disk 2 is originally at rest and disk 1 is rotating at 2000 rpm, what is the final angular velocity of the coupled disks?

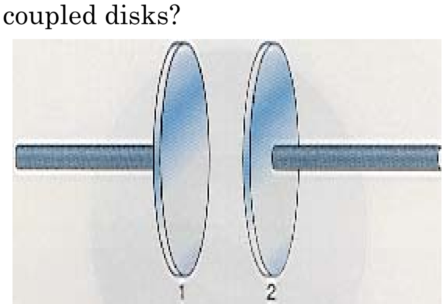


Diagram for problem 19.

*20. Two beads are fixed on a thin long wire on the x-axis at r_1 = 0.700 m and r_2 = 0.800 m, as shown in the diagram. Assume m_1 = 85.0 g and m_2 = 63.0 g. The combination is spinning about the y-axis at an angular velocity of 4.00 rad/s. A catch is then released allowing the beads to move freely to the stops at the end of the wire, which is 1.00 m from the origin. Find (a) the initial moment of inertia of the system, (b) the initial angular momentum of the system, (c) the initial kinetic energy of the system, (d) the final angular momentum of the system, (e) the final angular velocity of the system, and (f) the final kinetic energy of the system.

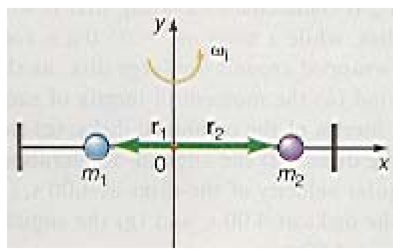


Diagram for problem 20.

9.8 Combined Translational and Rotational Motion Treated by the Law of Conservation of Energy

21. Find the velocity of (a) a cylinder and (b) a ring at the bottom of an inclined plane that is 2.00 m high. The cylinder and ring start from rest and roll down the plane.

22. Compute the velocity of a cylinder at the bottom of a plane 1.5 m high if (a) it slides without rotating on a frictionless plane and (b) it rotates on a rough plane.

23. A 1.50-kg solid ball, 10.0 cm in radius, is rolling on a table at a velocity of 0.500 m/s. (a) What is its angular velocity about its center of mass? (b) What is the translational kinetic energy of its center of mass? (c) What is its rotational kinetic energy about its center of mass? (d) What is its total kinetic energy?

24. Using the law of conservation of energy for the Atwood's machine shown, find the velocity of m_A at the ground, if m_A = 20.0 g, m_B = 10.0 g, M = 1.00 kg, and r = 15.0 cm.

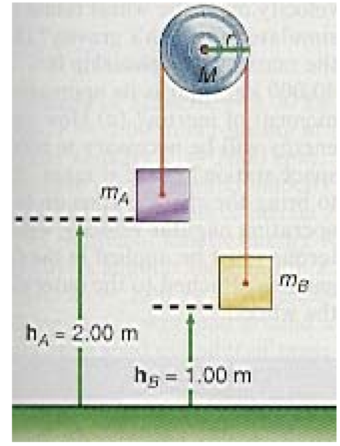


Diagram for problem 24.

9.9 Work in Rotational Motion

25. A constant force of 2.50 N acts tangentially on a cylinder of 12.5-cm radius and the cylinder rotates through an angle of 5.00 rev. How much work is done in rotating the cylinder?

26. An engine operating at 1800 rpm develops 200 hp, what is the torque developed?

Additional Problems

27. Determine (a) the angular velocity of the earth, (b) its moment of inertia, and (c) its kinetic energy of rotation. (d) Compare this with its kinetic energy of translation. (e) Find the angular momentum of the earth.

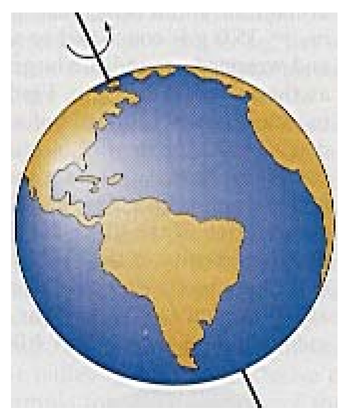


Diagram for problem 27.

28. The earth rotates once in a day. If the earth could collapse into a smaller sphere, what would be the radius of that sphere that would give a point on the equator a linear velocity equal to the velocity of light

$c = 3.00 \times 10^8$ m/s? Use the initial angular velocity of the earth and the moment of inertia determined in problem 27.

29. A disk of 10.0-cm radius, having a mass of 100 g, is set into motion by a constant tangential force of 2.00 N. Determine (a) the moment of inertia of the disk, (b) the torque applied to the disk, (c) the angular acceleration of the disk, (d) the angular velocity at 2.00 s, (e) the angular displacement at 2.00 s, (f) the kinetic energy at 2.00 s, and (g) the angular momentum at 2.00 s.

30. A 3.50-kg solid disk of 25.5 cm diameter has a cylindrical hole of 3.00-cm radius cut into it. The hole is 1.00 cm in from the edge of the solid disk. Find (a) the initial moment of inertia of the disk about an axis perpendicular to the disk before the hole was cut into it and (b) the moment of inertia of the solid disk with the hole in it. State the assumptions you use in solving the problem.

*31. Due to slight effects caused by tidal friction between the water and the land and the nonsphericity of the sun, there is a slight angular deceleration of the earth. The length of a day will increase by approximately 1.5×10^{-3} s in a century. (a) What will be the angular velocity of the earth after one century? (b) What will be the change in the angular velocity of the earth per century? (c) As a first approximation, is it reasonable to assume that there are no external torques acting on the earth and the angular velocity of the earth is a constant?

32. A string of length 1.50 m with a small bob at one end is connected to a horizontal disk of negligible radius at the other end. The disk is put into rotational motion and is now rotating at an angular velocity $\omega = 5.00$ rad/s. Find the angle that the string makes with the vertical.

33. A constant force of 5.00 N acts on a disk of 3.00-kg mass and diameter of 50.0 cm for 10.0 s. Determine (a) the angular acceleration, (b) the angular velocity after 10.0 s, and (c) the kinetic energy after 10.0 s. (d) Compute the work done to cause the disk to rotate and compare with your answer to part c.

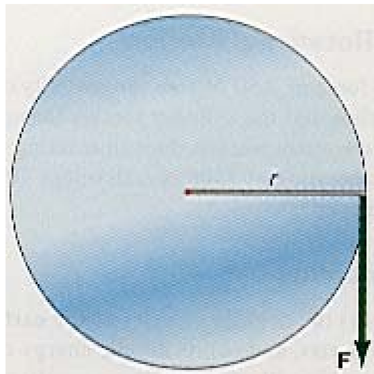


Diagram for problem 33.

*34. A 5.00-kg block is at rest at the top of the inclined plane shown in the diagram. The plane makes an angle of 32.5^0 with the horizontal. A string is attached to the block and tied around the disk, which has a mass of 2.00 kg and a radius of 8.00 cm. Find the acceleration of the block down the plane if (a) the plane is frictionless, and (b) the plane is rough with a value of $\mu_k = 0.54$.

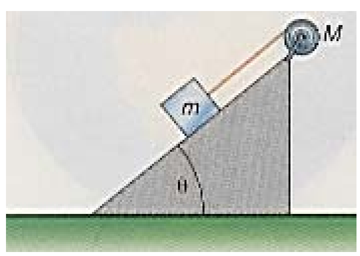


Diagram for problem 34.

35. A large cylinder has a radius of 12.5 cm and it is pressed against a smaller cylinder of radius 4.50 cm such that the two axes of the cylinders are parallel. When the larger cylinder rotates about its axis, it causes the smaller cylinder to rotate about its axis. The larger cylinder accelerates from rest to a constant angular velocity of 20 rad/s. Find (a) the tangential velocity of a point on the surface of the large cylinder, (b) the tangential velocity of a point on the surface of the smaller cylinder, and (c) the angular velocity of the smaller cylinder. Can you think of this setup as a kind of mechanical advantage?

*36. A small disk of $r_1 = 5.00$-cm radius is attached to a larger disk of $r_2 = 15.00$-cm radius such that they have a common axis of rotation, as shown in the diagram. The small disk has a mass $M_1 = 0.250$ kg and the large disk has a mass $M_2 = 0.850$ kg. A string is wrapped around the small disk and a force is applied to the string causing a constant tangential force of 2.00 N to be applied to the disk. Find (a) the applied torque, (b) the moment of inertia of the system, (c) the angular acceleration of the system, and (d) the angular velocity at 4.00 s.

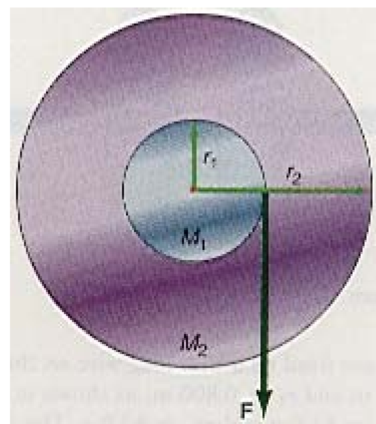


Diagram for problem 36.

*37. Repeat problem 36 with the string wrapped around the large disk instead of the small disk.

*38. A small disk of mass $M_1 = 50.0$ g is connected to a larger disk of mass $M_2 = 200.0$ g such that they have a common axis of rotation. The small disk has a radius $r_1 = 10.0$ cm, while the large disk has a radius of $r_2 = 30.0$ cm. A mass $m_1 = 25.0$ g is connected to a string that is wrapped around the small disk, while a mass $m_2 = 35.0$ g is connected to a string and wrapped around the large disk, as shown in the diagram. Find (a) the moment of inertia of each disk, (b) the moment of inertia of the combined disks, (c) the net torque acting on the disks, (d) the angular acceleration of the disks, (e) the angular velocity of the disks at 4.00 s, (f) the kinetic energy of the disks at 4.00 s, and (g) the angular momentum of the disks at 4.00 s.

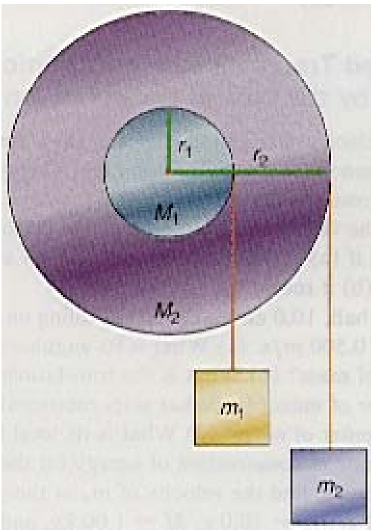


Diagram for problem 38.

*39. One end of a string is wrapped around a pulley and the other end is connected to the ceiling, which is 3.00 m above the floor. The mass of the pulley is 200 g and has a radius of 10.0 cm. The pulley is released from rest and is allowed to fall. Find (a) the initial total energy of the system, and (b) the velocity of the pulley just before it hits the floor.

40. This is essentially the same problem as problem 39 but is to be treated by Newton's second law for rotational motion. Find the angular acceleration of the cylinder and the tension in the string.

*41. A 1.5-kg disk of 0.500-m radius is rotating freely at an angular velocity of 2.00 rad/s. Small 5-g balls of clay are dropped onto the disk at 3/4 of the radius at a rate of 4 per second. Find the angular velocity of the disk at 10.0 s.

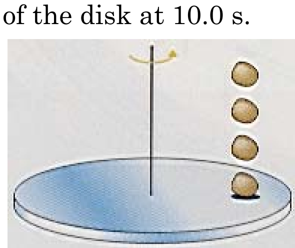


Diagram for problem 41.

*42. A space station is to be built in orbit in the shape of a large wheel of outside radius 100.0 m and inside radius of 97.0 m. The satellite is to rotate such that it will have a centripetal acceleration exactly equal to the acceleration of gravity g on earth. The astronauts will then be able to walk about and work on the rim of the wheel in an environment similar to earth. (a) At what angular velocity must the wheel rotate to simulate the earth's gravity? (b) If the mass of the spaceship is 40,000 kg, what is its approximate moment of inertia? (c) How much energy will be necessary to rotate the space station? (d) If it takes 20.0 rev to bring the space station up to its operating angular velocity, what torque must be applied in the form of gas jets attached to the outer rim of the wheel?

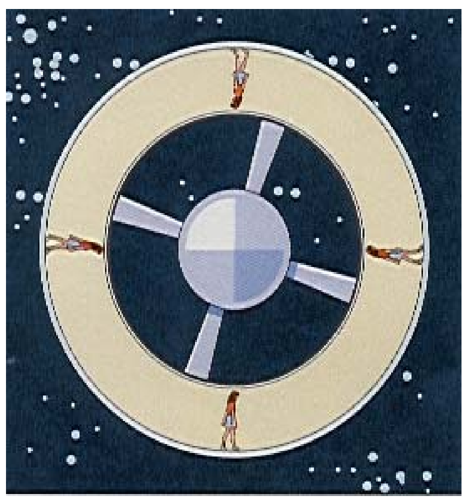


Diagram for problem 42.

*43. At the instant that ball 1 is released from rest at the top of a rough inclined plane a second ball (2) moves past it on the horizontal surface below at a constant velocity of 2.30 m/s. The plane makes an angle $\theta = 35.0^0$ with the horizontal and the height of the plane is 0.500 m. Using Newton's second law for combined translational and rotational motion find (a) the acceleration a of ball 1 down the plane, (b) the velocity of ball 1 at the base of the incline, (c) the time it takes for ball 1 to reach the bottom of the plane, (d) the distance that ball 2 has moved in this time, and (e) at what horizontal distance from the base of the incline will ball 1 overtake ball 2.

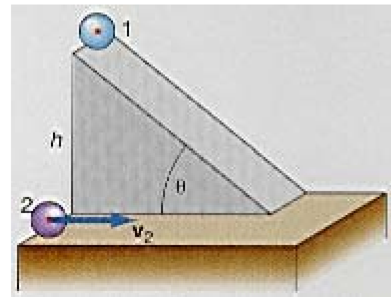


Diagram for problem 43.

Interactive Tutorials

44. A cylinder of mass m = 4.00 kg and radius r = 2.00 m is rotating at an angular velocity ω = 3600 rpm. Calculate (a) its angular velocity ω in rad/s, (b) its moment of inertia I, (c) its rotational kinetic energy KE_{rot}, and (d) its angular momentum L.

45. A mass m = 2.00 kg is attached by a string that is wrapped around a frictionless solid cylinder of mass M = 8.00 kg and radius R = 0.700 m that is free to rotate. Calculate (a) the acceleration a of the mass m and (b) the tension T in the string.

46. *Rotational Dynamics.* A cylinder of mass m = 2.35 kg and radius r = 0.345 m is initially rotating at an angular velocity ω_0 = 1.55 rad/s when a constant force F = 9.25 N is applied tangentially to the cylinder as in figure 9.8. Find (a) the moment of inertia I of the cylinder, (b) the torque τ acting on the cylinder, (c) the angular acceleration α of the cylinder, (d) the angular velocity ω of the cylinder at t = 4.55 s, and (e) the angular displacement θ at t = 4.55 s.

47. *The moment of inertia of a continuous mass distribution.* A meterstick, m = 0.149 kg, lies on the x-axis with the zero of the meterstick at the origin of the coordinate system. Determine the moment of iner-

tia of the meterstick about an axis that passes through the zero of the meterstick and perpendicular to it. Assume that the meterstick can be divided into $N = 10$ equal parts.

48. *This is a generalization of Interactive Tutorial problem 72 of chapter 4 but it also takes the rotational motion of the pulley into account.* Derive the formula for the magnitude of the acceleration of the system shown in the diagram for problem 57 of chapter 4. The pulley has a mass M and the radius R. As a general case assume that the coefficient of kinetic friction between block A and the surface is μ_{kA} and between block B and the surface is μ_{kB}. Solve for all the special cases that you can think of. In all the cases, consider different values for the mass M of the pulley and see the effect it has on the results of the problem.

49. *An Atwood's machine taking the rotational motion of the pulley into account.* Consider the general motion in an Atwood's machine such as the one shown in figure 9.18. Mass $m_A = 0.650$ kg and is at a height $h_A = 2.55$ m above the reference plane and mass $m_B = 0.420$ kg is at a height $h_B = 0.400$ m. The pulley has a mass of $M = 2.00$ kg and a radius $R = 0.100$ m. If the system starts from rest, find (a) the initial potential energy of mass A, (b) the initial potential energy of mass B, and (c) the total energy of the system. When mass m_A has fallen a distance $y_A = 0.750$ m, find (d) the potential energy of mass A, (e) the potential energy of mass B, (f) the speed of each mass at that point, (g) the kinetic energy of mass A, (h) the kinetic energy of mass B, (i) the moment of inertia of the pulley (assume it to be a disk), (j) the angular velocity ω of the pulley, and (k) the rotational kinetic energy of the pulley. (l) When mass A hits the ground, find the speed of each mass and the angular velocity of the pulley.

50. *Combined motion taking the rotational motion of the pulley into account.* Consider the general motion in the combined system shown in the diagram of problem 16. Mass $m_A = 0.750$ kg and is at a height $h_A = 1.85$ m above the reference plane and mass $m_B = 0.285$ kg is at a height $h_B = 2.25$ m, $\mu_k = 0.450$. The pulley has a mass $M = 1.85$ kg and a radius $R = 0.0800$ m. If the system starts from rest, find (a) the initial potential energy of mass A, (b) the initial potential energy of mass B, and (c) the total energy of the system. When m_A has fallen a distance $y_A = 0.35$ m, find (d) the potential energy of mass A, (e) the potential energy of mass B, (f) the energy lost due to friction as mass B slides on the rough surface, (g) the speed of each mass at that point, (h) the kinetic energy of mass A, (i) the kinetic energy of mass B, (j) the moment of inertia of the pulley (assumed to be a disk), (k) the angular velocity ω of the pulley, and (l) the rotational kinetic energy of the pulley. (m) When mass A hits the ground, find the speed of each mass.

51. *Changing the moment of inertia of a rotating disk.* A disk of mass $M = 3.55$ kg and a radius $R = 1.25$ m is rotating freely at an initial angular velocity $\omega_i = 1.45$ rad/s. Small balls of clay of mass $m_b = 0.025$ kg are dropped onto the rotating disk at the radius $r = 0.85$ m at the rate of $n = 5$ ball/s. Find (a) the initial moment of inertia of the disk, (b) the initial angular momentum of the disk, and (c) the angular velocity ω at $t = 6.00$s. (d) Plot the angular velocity ω as a function of the number of balls dropped.

To go to these Interactive Tutorials click on this sentence.

To go to another chapter, return to the table of contents by clicking on this sentence.