Chapter 11 Simple Harmonic Motion

"We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances." Isaac Newton

11.1 Introduction to Periodic Motion

Periodic motion is any motion that repeats itself in equal intervals of time. The uniformly rotating earth represents a periodic motion that repeats itself every 24 hours. The motion of the earth around the sun is periodic, repeating itself every 12 months. A vibrating spring and a pendulum also exhibit periodic motion. The period of the motion is defined as the time for the motion to repeat itself. A special type of periodic motion is simple harmonic motion and we now proceed to investigate it.

11.2 Simple Harmonic Motion

An example of simple harmonic motion is the vibration of a mass *m,* attached to a spring of negligible mass, as the mass slides on a frictionless surface, as shown in figure 11.1. We say that the mass, in the unstretched position, figure 11.1(a), is in its equilibrium position. If an applied force $\mathbf{F_A}$ acts on the mass, the mass will be displaced to the right of its equilibrium position a distance *x,* figure 11.1(b). The distance that the spring stretches, obtained from Hooke's law, is

$$F_A = kx$$

The ***displacement*** *x is defined as the distance the body moves from its equilibrium position.* Because F_A is a force that pulls the mass to the right, it is also the force that pulls the spring to the right. By Newton's third law there is an equal but opposite elastic force exerted by the spring on the mass pulling the mass toward the left. Since this force tends to restore the mass to its original position, we call it the restoring force F_R. Because the restoring force is opposite to the applied force, it is given by

$$F_R = -F_A = -kx \tag{11.1}$$

When the applied force F_A is removed, the elastic restoring force F_R is then the only force acting on the mass *m,* figure 11.1(c), and it tries to restore *m* to its equilibrium position. We can then find the acceleration of the mass from Newton's second law as

$$ma = F_R$$
$$= -kx$$

Thus,

$$a = -\frac{k}{m}x \tag{11.2}$$

Equation 11.2 is the defining equation for simple harmonic motion. ***Simple harmonic motion*** *is motion in which the acceleration of a body is directly proportional to its displacement from the equilibrium position but in the opposite direction.* A vibrating system that executes simple harmonic motion is sometimes called a *harmonic oscillator. Because the acceleration is directly proportional to the displacement x*

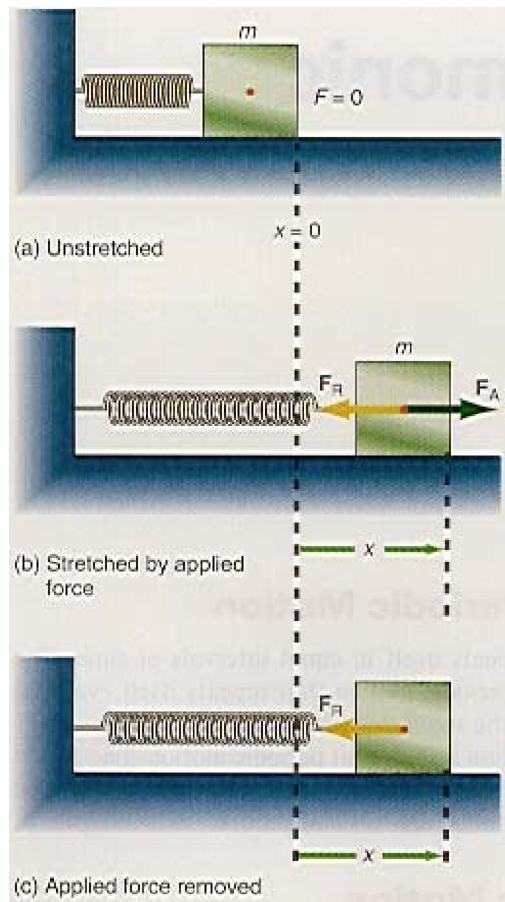

Figure 11.1 The vibrating spring.

in simple harmonic motion, the acceleration of the system is not constant but varies with x. At large displacements, the acceleration is large, at small displacements the acceleration is small. Describing the vibratory motion of the mass *m* requires some new techniques because the kinematic equations derived in chapter 3 were based on the assumption that the acceleration of the system was a constant. As we can see from equation 11.2, this assumption is no longer valid. We need to derive a new set of kinematic equations to describe simple harmonic motion, and we will do so in section 11.3. However, let us first look at the motion from a physical point of view. The mass *m* in figure 11.2(a) is pulled a distance $x = A$ to the right, and is then released. The maximum restoring force on m acts to the left at this position because

$$F_{Rmax} = -kx_{max} = -kA$$

The maximum displacement A is called the ***amplitude of the motion.*** At this position the mass experiences its maximum acceleration to the left. From equation 11.2 we obtain

$$a = -\frac{k}{m}A$$

The mass continues to move toward the left while the acceleration continuously decreases. At the equilibrium position, figure 11.2(b), $x = 0$ and hence, from equation 11.2, the acceleration is also zero. However, because the mass has inertia it moves past the equilibrium position to negative values of x, thereby compressing the spring. The restoring force F_R now points to

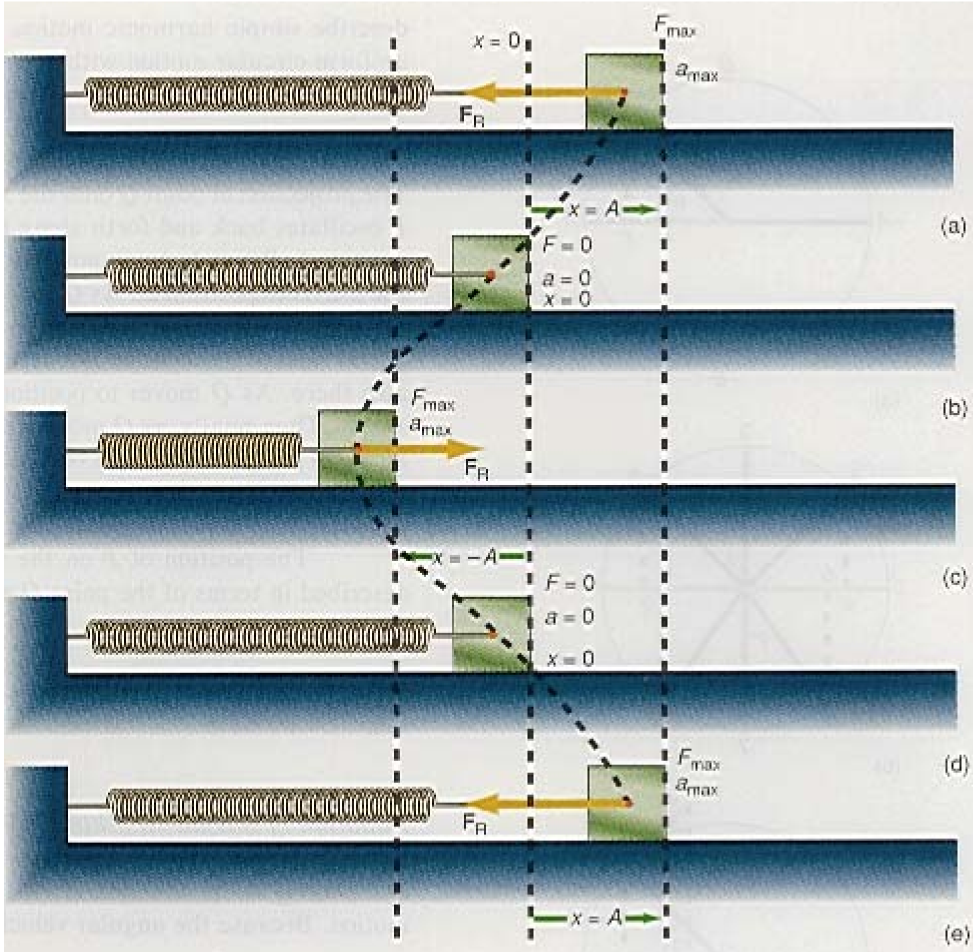

Figure 11.2 Detailed motion of the vibrating spring.

the right, since for negative values of x,

$$F_R = -k(-x) = kx$$

The force acting toward the right causes the mass to slow down, eventually coming to rest at $x = -A$. At this point, figure 11.2(c), there is a maximum restoring force pointing toward the right

$$F_{Rmax} = -k(-A)_{max} = kA$$

and hence a maximum acceleration

$$a_{max} = -\frac{k}{m}(-A) = \frac{k}{m}A$$

also to the right. The mass moves to the right while the force F_R and the acceleration a decreases with x until x is again equal to zero, figure 11.2(d). Then F_R and a are also zero. Because of the inertia of the mass, it moves past the equilibrium position to positive values of x. The restoring force again acts toward the left, slowing down the mass. When the displacement x is equal to A, figure 11.2(e), the mass momentarily comes to rest and then the motion repeats itself. *One complete motion from $x = A$ and back to $x = A$ is called a* ***cycle or an oscillation. The period T*** *is the time for one complete oscillation, and the* ***frequency f*** *is the number of complete oscillations or cycles made in unit time.* The period and the frequency are reciprocal to each other, that is,

$$f = \frac{1}{T} \tag{11.3}$$

The unit for a period is the second, while the unit for frequency, called a hertz, is one cycle per second. The hertz is abbreviated, Hz. Also note that a cycle is a number not a dimensional quantity and can be dropped from the computations whenever doing so is useful.

11.3 Analysis of Simple Harmonic Motion -- The Reference Circle

As pointed out in section 11.2, the acceleration of the mass in the vibrating spring system is not a constant, but rather varies with the displacement x. Hence, the kinematic equations of chapter 3 can not be used to describe the motion. (We derived those equations on the assumption that the acceleration was constant.) Thus, a new set of equations must be derived to describe simple harmonic motion.

Simple harmonic motion is related to the uniform circular motion studied in chapter 6. An analysis of uniform circular motion gives us a set of equations to describe simple harmonic motion. As an example, consider a point Q moving in uniform circular motion with an angular velocity ω, as shown in figure 11.3(a). At a particular instant of time t, the angle θ that Q has turned through is

$$\theta = \omega t \tag{11.4}$$

The projection of point Q onto the x-axis gives the point P. As Q rotates in the circle, P oscillates back and forth along the x-axis, figure 11.3(b). That is, when Q is at position 1, P is at 1. As Q moves to position 2 on the circle, P moves to the left along the x-axis to position 2'.As Q moves to position 3, P moves on the x-axis to position 3', which is of course the value of $x = 0$. As Q moves to position 4 on the circle, P moves along the negative x-axis to position 4'.When Q arrives at position 5, P is also there. As Q moves to position 6 on the circle, P moves to position 6' on the x-axis. Then finally, as Q moves through positions 7, 8, and 1, P moves through 7', 8', and 1, respectively. The oscillatory motion of point P on the x-axis corresponds to the simple harmonic motion of a body m moving under the influence of an elastic restoring force, as shown in figure 11.2.

The position of P on the x-axis and hence the position of the mass m is described in terms of the point Q and the angle θ found in figure 11.3(a) as

$$x = A \cos \theta \tag{11.5}$$

Here A is the amplitude of the vibratory motion and

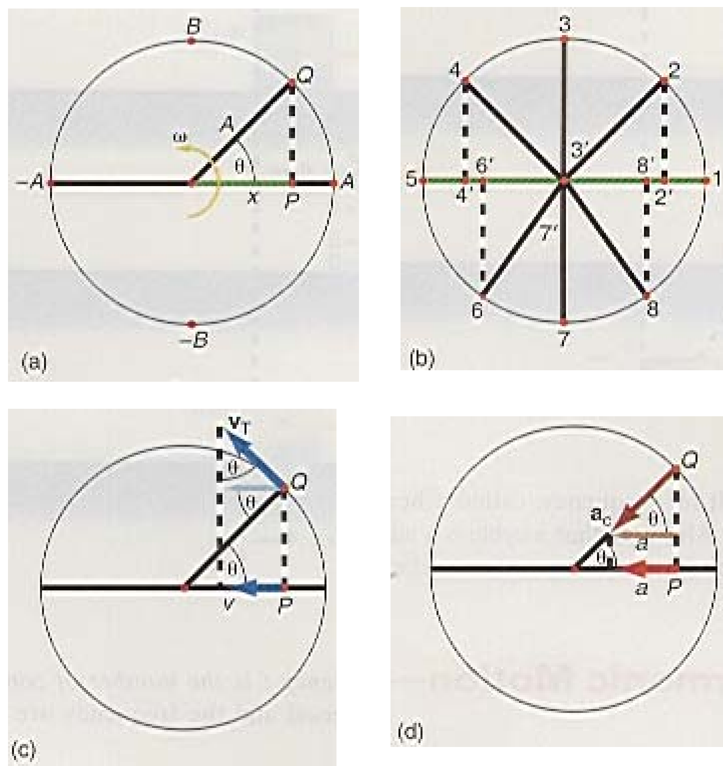

Figure 11.3 Simple harmonic motion and the reference circle.

using the value of θ from equation 11.4 we have

$$x = A \cos \omega t \tag{11.6}$$

Equation 11.6 is the first kinematic equation for simple harmonic motion; it gives the displacement of the vibrating body at any instant of time t. The angular velocity ω of point Q in the **reference circle** is related to the frequency of the simple harmonic motion. Because the angular velocity was defined as

$$\omega = \frac{\theta}{t} \tag{11.7}$$

then, for a complete rotation of point Q, θ rotates through an angle of 2π rad. But this occurs in exactly the time for P to execute one complete vibration. We call this time for one complete vibration the period T. Hence, we can also write the angular velocity, equation 11.7, as

$$\omega = \frac{\theta}{t} = \frac{2\pi}{T} \tag{11.8}$$

Since the frequency f is the reciprocal of the period T (equation 11.3) we can write equation 11.8 as

$$\omega = 2\pi f \tag{11.9}$$

Thus, the ***angular velocity*** *of the uniform circular motion in the reference circle is related to the frequency of the vibrating system.* Because of this relation between the angular velocity and the frequency of the system, we usually call the angular velocity ω the *angular frequency of the vibrating system.* We can substitute equation 11.9 into equation 11.6 to give *another form for the first kinematic equation of simple harmonic motion, namely*

$$x = A\cos(2\pi f t) \tag{11.10}$$

We can find the velocity of the mass m attached to the end of the spring in figure 11.2 with the help of the reference circle in figure 11.3(c). The point Q moves with the tangential velocity $\mathbf{V_T}$. The x-component of this velocity is the velocity of the point P and hence the velocity of the mass m. From figure 11.3(c) we can see that

$$v = -V_T \sin\theta \tag{11.11}$$

The minus sign indicates that the velocity of P is toward the left at this position. The linear velocity V_T of the point Q is related to the angular velocity ω by equation 9.2 of chapter 9, that is

$$v = r\omega$$

For the reference circle, $v = V_T$ and r is the amplitude A. Hence, the tangential velocity V_T is given by

$$V_T = \omega A \tag{11.12}$$

Using equations 11.11, 11.12, and 11.4, the velocity of point P becomes

$$v = -\omega A \sin\omega t \tag{11.13}$$

Equation 11.13 is the second of the kinematic equations for simple harmonic motion and it gives the speed of the vibrating mass at any time t.

A third kinematic equation for simple harmonic motion giving the speed of the vibrating body as a function of displacement can be found from equation 11.13 by using the trigonometric identity

$$\sin^2\theta + \cos^2\theta = 1$$

or

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

From figure 11.3(a) or equation 11.5, we have

$$\cos\theta = \frac{x}{A}$$

Hence,

$$\sin\theta = \pm\sqrt{1-\frac{x^2}{A^2}} \tag{11.14}$$

Substituting equation 11.14 back into equation 11.13, we get

$$v = \pm\,\omega A\sqrt{1-\frac{x^2}{A^2}}$$

or

$$v = \pm\,\omega\sqrt{A^2-x^2} \tag{11.15}$$

Equation 11.15 is the third of the kinematic equations for simple harmonic motion and it gives the velocity of the moving body at any displacement x. The ± sign in equation 11.15 indicates the direction of the vibrating body. If the body is moving to the right, then the positive sign (+) is used. If the body is moving to the left, then the negative sign (–) is used.

Finally, we can find the acceleration of the vibrating body using the reference circle in figure 11.3(d). The point Q in uniform circular motion experiences a centripetal acceleration $\mathbf{a_c}$ pointing toward the center of the circle in figure 11.3(d). The x-component of the centripetal acceleration is the acceleration of the vibrating body at the point P. That is,

$$a = -a_c \cos\theta \tag{11.16}$$

The minus sign again indicates that the acceleration is toward the left. But recall from chapter 6 that the magnitude of the centripetal acceleration is

$$a_c = \frac{v^2}{r} \tag{6.12}$$

where v represents the tangential speed of the rotating object, which in the present case is V_T, and r is the radius of the circle, which in the present case is the radius of the reference circle A. Thus,

$$a_c = \frac{V_T^2}{A}$$

But we saw in equation 11.12 that $V_T = \omega A$, therefore

$$a_c = \omega^2 A$$

The acceleration of the mass m, equation 11.16, thus becomes

$$a = -\omega^2 A \cos \omega t \tag{11.17}$$

Equation 11.17 is the fourth of the kinematic equations for simple harmonic motion. It gives the acceleration of the vibrating body at any time t. This equation has no counterpart in chapter 3, because there the acceleration was always a constant. Also, since $F = ma$ by Newton's second law, the force acting on the mass m, becomes

$$F = -m\omega^2 A \cos \omega t \tag{11.18}$$

Thus, the force acting on the mass m is a variable force.

Equations 11.6 and 11.17 can be combined into the simple equation

$$a = -\omega^2 x \tag{11.19}$$

If equation 11.19 is compared with equation 11.2,

$$a = -\frac{k}{m}x$$

we see that the acceleration of the mass at P, equation 11.19, is directly proportional to the displacement x and in the opposite direction. But this is the definition of simple harmonic motion as stated in equation 11.2. Hence, the projection of a point at Q, in uniform circular motion, onto the x-axis does indeed represent simple harmonic motion. Thus, the kinematic equations developed to describe the motion of the point P, also describe the motion of a mass attached to a vibrating spring.

An important relation between the characteristics of the spring and the vibratory motion can be easily deduced from equations 11.2 and 11.19. That is, because both equations represent the acceleration of the vibrating body they can be equated to each other, giving

$$\omega^2 = \frac{k}{m}$$

or

$$\omega = \sqrt{\frac{k}{m}} \tag{11.20}$$

The value of ω in the kinematic equations is expressed in terms of the force constant k of the spring and the mass m attached to the spring. The physics of simple harmonic motion is thus connected to the angular frequency ω of the vibration.

In summary, the kinematic equations for simple harmonic motion are

$$x = A\cos \omega t \tag{11.6}$$
$$v = -\,\omega A \sin \omega t \tag{11.13}$$
$$v = \pm\,\omega\sqrt{A^2 - x^2} \tag{11.15}$$
$$a = -\,\omega^2 A \cos \omega t \tag{11.17}$$
$$F = -\,m\omega^2 A \cos \omega t \tag{11.18}$$

where, from equations 11.9 and 11.20, we have

$$\omega = 2\pi f = \sqrt{\frac{k}{m}}$$

A plot of the displacement, velocity, and acceleration of the vibrating body as a function of time are shown in figure 11.4. We can see that the mathematical description follows the physical description in figure 11.2. When $x = A$, the maximum displacement, the velocity v is zero, while the acceleration is at its maximum value of $-\omega^2 A$. The minus sign indicates that the acceleration is toward the left. The force is at its maximum value of $-m\omega^2 A$, where the minus sign shows that the restoring force is pulling the mass back toward its equilibrium position. At the equilibrium position $x = 0$, $a = 0$, and $F = 0$, but v has its maximum velocity of $-\omega A$ toward the left. As x goes to negative values, the force and the acceleration become positive, slowing down the motion to the left, and hence v starts to decrease. At $x = -A$ the velocity is zero and the force and acceleration take on their maximum values toward the right, tending to restore the mass to its equilibrium position. As x becomes less negative, the velocity to the right increases, until it picks up its maximum value of ωA at $x = 0$, the equilibrium position, where F and a are both zero. Because of this large velocity, the mass passes the equilibrium position in its motion toward the right. However, as soon as x becomes positive, the force and the acceleration become negative thereby slowing down the mass until its velocity becomes zero at the maximum displacement A. One entire cycle has been completed, and the motion starts over again. (We should emphasize here that in this vibratory motion there are two places where the velocity is instantaneously zero, $x = A$ and $x = -A$, even though the instantaneous acceleration is nonzero there.)

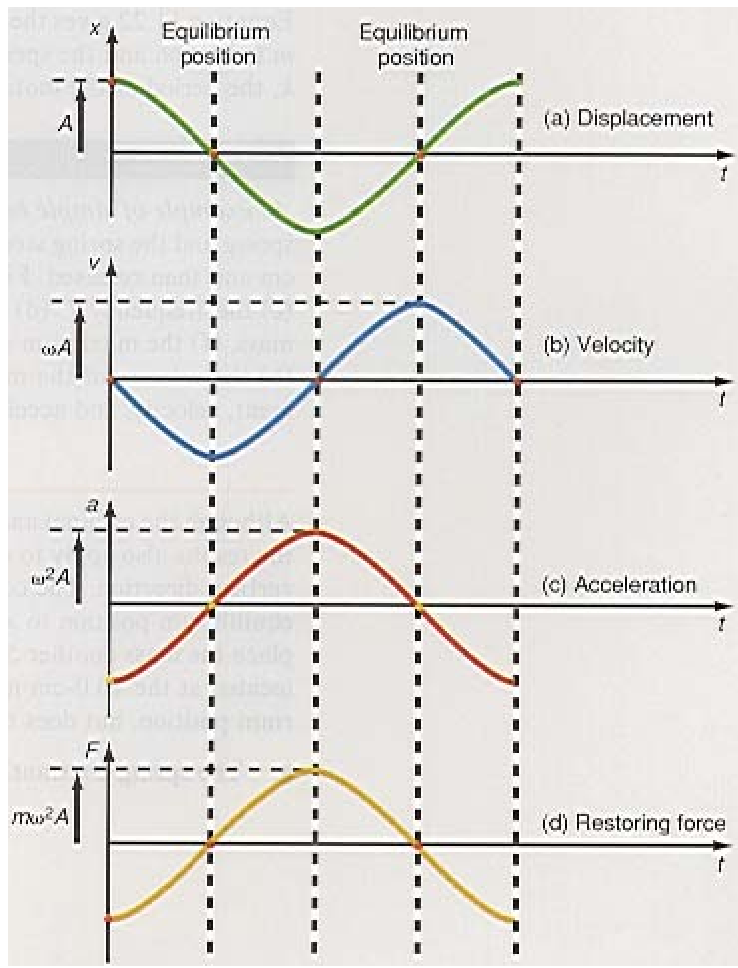

Figure 11.4 Displacement, velocity, and acceleration in simple harmonic motion.

Sometimes the vibratory motion is so rapid that the actual displacement, velocity, and acceleration at every instant of time are not as important as the gross motion, which can be described in terms of the frequency or period of the motion. We can find the frequency of the vibrating mass in terms of the spring constant k and the vibrating mass m by setting equation 11.9 equal to equation 11.20. Thus,

$$\omega = 2\pi f = \sqrt{\frac{k}{m}}$$

Solving for the frequency *f,* we obtain

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{11.21}$$

Equation 11.21 gives the frequency of the vibration. Because the period of the vibrating motion is the reciprocal of the frequency, we get for the period

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{11.22}$$

Equation 11.22 gives the period of the simple harmonic motion in terms of the mass m in motion and the spring constant k. Notice that for a particular value of m and k, the period of the motion remains a constant throughout the motion.

Example 11.1

An example of simple harmonic motion. A mass of 0.300 kg is placed on a vertical spring and the spring stretches by 10.0 cm. It is then pulled down an additional 5.00 cm and then released. Find (a) the spring constant k, (b) the angular frequency ω, (c) the frequency f, (d) the period T, (e) the maximum velocity of the vibrating mass, (f) the maximum acceleration of the mass, (g) the maximum restoring force, (h) the velocity of the mass at $x = 2.00$ cm, and (i) the equation of the displacement, velocity, and acceleration at any time t.

Solution

Although the original analysis dealt with a mass on a horizontal frictionless surface, the results also apply to a mass attached to a spring that is allowed to vibrate in the vertical direction. The constant force of gravity on the 0.300-kg mass displaces the equilibrium position to $x = 10.0$ cm. When the additional force is applied to displace the mass another 5.00 cm, the mass oscillates about the equilibrium position, located at the 10.0-cm mark. Thus, the force of gravity only displaces the equilibrium position, but does not otherwise influence the result of the dynamic motion.

a. The spring constant, found from Hooke's law, is

$$k = \frac{F_A}{x} = \frac{mg}{x}$$

$$= \frac{(0.300 \text{ kg})(9.80 \text{ m/s}^2)}{0.100 \text{ m}}$$

$$= 29.4 \text{ N/m}$$

b. The angular frequency ω, found from equation 11.20, is

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{29.4 \text{ N/m}}{0.300 \text{ kg}}}$$

$$= 9.90 \text{ rad/s}$$

c. The frequency of the motion, found from equation 11.9, is

$$f = \frac{\omega}{2\pi}$$

$$= \frac{9.90 \text{ rad/s}}{2\pi \text{ rad}}$$

$$= 1.58 \frac{\text{cycles}}{\text{s}} = 1.58 \text{ Hz}$$

d. We could find the period from equation 11.22 but since we already know the frequency *f*, it is easier to compute *T* from equation 11.3. Thus,

$$T = \frac{1}{f} = \frac{1}{1.58 \text{ cycles/s}} = 0.633 \text{ s}$$

e. The maximum velocity, found from equation 11.13, is

$$v_{max} = \omega A = (9.90 \text{ rad/s})(5.00 \times 10^{-2} \text{ m})$$
$$= 0.495 \text{ m/s}$$

f. The maximum acceleration, found from equation 11.17, is

$$a_{max} = \omega^2 A = (9.90 \text{ rad/s})^2(5.00 \times 10^{-2} \text{ m})$$
$$= 4.90 \text{ m/s}^2$$

g. The maximum restoring force, found from Hooke's law, is

$$F_{max} = kx_{max} = kA$$
$$= (29.4 \text{ N/m})(5.00 \times 10^{-2} \text{ m})$$
$$= 1.47 \text{ N}$$

h. The velocity of the mass at $x = 2.00$ cm, found from equation 11.15, is

$$v = \pm\, \omega \sqrt{A^2 - x^2}$$
$$v = \pm\, (9.90 \text{ rad/s}) \sqrt{\left(5.00 \times 10^{-2} \text{ m}\right)^2 - \left(2.00 \times 10^{-2} \text{ m}\right)^2}$$
$$= \pm\, 0.454 \text{ m/s}$$

where v is positive when moving to the right and negative when moving to the left.

i. The equation of the displacement at any instant of time, found from equation 11.6, is

$$x = A \cos \omega t$$
$$= (5.00 \times 10^{-2} \text{ m}) \cos(9.90 \text{ rad/s})t$$

The equation of the velocity at any instant of time, found from equation 11.13, is

$$v = -\omega A \sin \omega t$$
$$= -(9.90 \text{ rad/s})(5.00 \times 10^{-2} \text{ m})\sin(9.90 \text{ rad/s})t$$
$$= -(0.495 \text{ m/s})\sin(9.90 \text{ rad/s})t$$

The equation of the acceleration at any time, found from equation 11.17, is

$$a = -\omega^2 A \cos \omega t$$
$$= -(9.90 \text{ rad/s})^2(5.00 \times 10^{-2} \text{ m})\cos(9.90 \text{ rad/s})t$$
$$= -(4.90 \text{ m/s}^2)\cos(9.90 \text{ rad/s})t$$

To go to this Interactive Example click on this sentence.

11.4 The Potential Energy of a Spring

In chapter 7 we defined the gravitational potential energy of a body as the energy that a body possesses by virtue of its position in a gravitational field. A body can also have elastic potential energy. For example, *a compressed spring has potential energy because it has the ability to do work as it expands to its equilibrium configuration. Similarly, a stretched spring must also contain potential energy because it has the ability to do work as it returns to its equilibrium position.* Because work must be done on a body to put the body into the configuration where it has the elastic potential energy, this work is used as the measure of the potential energy. Thus, the **potential energy of a spring** is equal to the work that you, the external agent, must do to compress (or stretch) the spring to its present configuration. We defined the potential energy as

$$\text{PE} = W = Fx \tag{11.23}$$

However, we can not use equation 11.23 in its present form to determine the potential energy of a spring. Recall that the work defined in this way, in chapter 7, was for a constant force. We have seen in this chapter that the force necessary to compress or stretch a spring is not a constant but is rather a variable force depending on the value of x, ($F = -kx$). We can still solve the problem, however, by using the average value of the force between the value at the equilibrium position and the value at the position x. That is, because the restoring force is directly proportional to the displacement, the average force exerted in moving the mass m from $x = 0$ to the value x in figure 11.5(a) is

$$F_{\mathbf{avg}} = \frac{F_0 + F}{2}$$

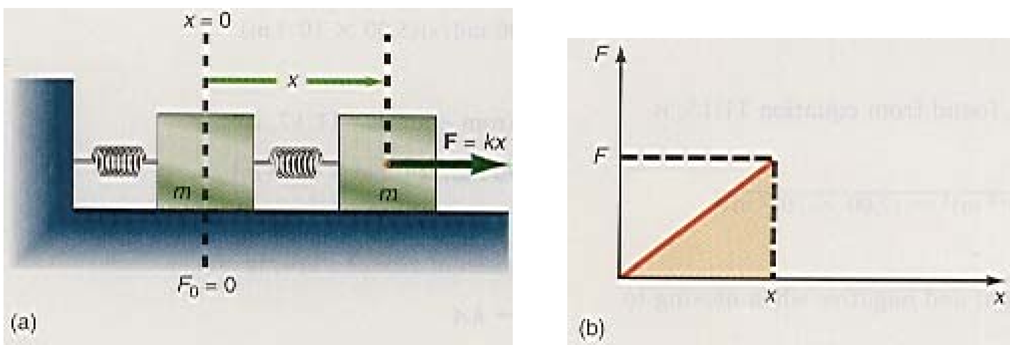

Figure 11.5 The potential energy of a spring.

Thus, we find the potential energy in this configuration by using the average force, that is,

$$\text{PE} = W = F_{\mathbf{avg}}x$$
$$= W = \left(\frac{F_0 + F}{2}\right)x$$
$$= \left(\frac{0 + kx}{2}\right)x$$

Hence,

$$\text{PE} = \frac{1}{2}kx^2 \tag{11.24}$$

Because of the x^2 in equation 11.24, the potential energy of a spring is always positive, whether x is positive or negative. The zero of potential energy is defined at the equilibrium position, $x = 0$.

Note that equation 11.24 could also be derived by plotting the force F acting on the spring versus the displacement x of the spring, as shown in figure 11.5(b). Because the work is equal to the product of the force F and the displacement x, the work is also equal to the area under the curve in figure 11.5(b). The area of that triangle is ½ $(x)(F)$ = ½ $(x)(kx)$ = ½kx^2. (For the more general problem where the force is not a linear function of the displacement x, if the force is plotted versus the displacement x, the work done, and hence the potential energy, will still be equal to the area under the curve.)

Example 11.2

The potential energy of a spring. A spring, with a spring constant of 29.4 N/m, is stretched 5.00 cm. How much potential energy does the spring possess?

Solution

The potential energy of the spring, found from equation 11.24, is

$$\text{PE} = \frac{1}{2}kx^2$$
$$= \frac{1}{2}(29.4\ \text{N/m})(5.00 \times 10^{-2}\ \text{m})^2$$
$$= 3.68 \times 10^{-2}\ \text{J}$$

To go to this Interactive Example click on this sentence.

11.5 Conservation of Energy and the Vibrating Spring

The vibrating spring system of figure 11.2 can also be described in terms of the law of conservation of energy. When the spring is stretched to its maximum displacement A, work is done on the spring, and hence the spring contains potential energy. The mass m attached to the spring also has that potential energy. The total energy of the system is equal to the potential energy at the maximum displacement because at that point, $v = 0$, and therefore the kinetic energy is equal to zero, that is,

$$E_{\text{tot}} = \text{PE} = \frac{1}{2}kA^2 \tag{11.25}$$

When the spring is released, the mass moves to a smaller displacement x, and is moving at a speed v. At this arbitrary position x, the mass will have both potential energy and kinetic energy. The law of conservation of energy then yields

$$E_{\text{tot}} = \text{PE} + \text{KE}$$
$$E_{\text{tot}} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \tag{11.26}$$

But the total energy imparted to the mass m is given by equation 11.25. Hence, the law of conservation of energy gives

$$E_{\text{tot}} = E_{\text{tot}}$$
$$\frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \tag{11.27}$$

We can also use equation 11.27 to find the velocity of the moving body at any displacement x. Thus,

$$\frac{1}{2}mv^2 = \frac{1}{2}kA^2 - \frac{1}{2}kx^2$$
$$v^2 = \frac{k}{m}(A^2 - x^2)$$
$$v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)} \tag{11.28}$$

We should note that this is the same equation for the velocity as derived earlier (equation 11.15). It is informative to replace the values of x and v from their respective equations 11.6 and 11.13 into equation 11.26. Thus,

$$E_{\text{tot}} = \frac{1}{2}k(A\cos\omega t)^2 + \frac{1}{2}m(-\omega A\sin\omega t)^2$$

or

$$E_{tot} = \frac{1}{2}kA^2\cos^2\omega t + \frac{1}{2}m\omega^2A^2\sin^2\omega t$$

but since

$$\omega^2 = \frac{k}{m}$$

$$E_{tot} = \frac{1}{2}kA^2\cos^2\omega t + \frac{1}{2}m\frac{k}{m}A^2\sin^2\omega t$$

$$= \frac{1}{2}kA^2\cos^2\omega t + \frac{1}{2}kA^2\sin^2\omega t \qquad (11.29)$$

These terms are plotted in figure 11.6.

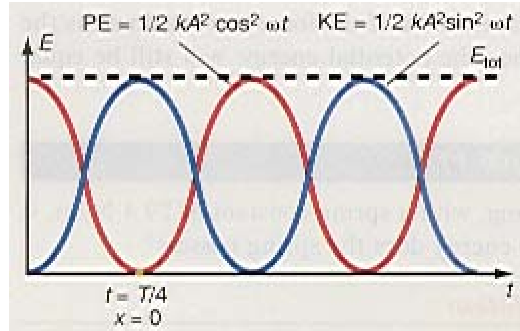

Figure 11.6 Conservation of energy and simple harmonic motion.

The total energy of the vibrating system is a constant and this is shown as the horizontal line, E_{tot}. At $t = 0$ the total energy of the system is potential energy (v is zero, hence the kinetic energy is zero). As the time increases the potential energy decreases and the kinetic energy increases, as shown. However, the total energy remains the same. From equation 11.24 and figure 11.6, we see that at $x = 0$ the potential energy is zero and hence all the energy is kinetic. This occurs when $t = T/4$. The maximum velocity of the mass m occurs here and is easily found by equating the maximum kinetic energy to the total energy, that is,

$$\frac{1}{2}mv_{max}^2 = \frac{1}{2}kA^2$$

$$v_{max} = \sqrt{\frac{k}{m}}A = \omega A \qquad (11.30)$$

When the oscillating mass reaches $x = A$, the kinetic energy becomes zero since

$$\frac{1}{2}kA^2 = \frac{1}{2}kA^2 + \frac{1}{2}mv^2$$

$$\frac{1}{2}mv^2 = \frac{1}{2}kA^2 - \frac{1}{2}kA^2 = 0$$

$$= \text{KE} = 0$$

As the oscillation continues there is a constant interchange of energy between potential energy and kinetic energy but the total energy of the system remains a constant.

Example 11.3

Conservation of energy applied to a spring. A horizontal spring has a spring constant of 29.4 N/m. A mass of 300 g is attached to the spring and displaced 5.00 cm. The mass is then released. Find (a) the total energy of the system, (b) the maximum velocity of the system, and (c) the potential energy and kinetic energy for $x = 2.00$ cm.

Solution

a. The total energy of the system is

$$E_{tot} = \frac{1}{2}kA^2$$

$$= \frac{1}{2}(29.4\text{ N/m})(5.00 \times 10^{-2}\text{ m})^2$$

$$= 3.68 \times 10^{-2}\text{ J}$$

b. The maximum velocity occurs when $x = 0$ and the potential energy is zero. Therefore, using equation 11.30,

$$v_{max} = \sqrt{\frac{k}{m}}A$$

$$v_{max} = \sqrt{\frac{29.4 \text{ N/m}}{3.00 \times 10^{-1} \text{ kg}}}\left(5.00 \times 10^{-2} \text{ m}\right)$$

$$= 0.495 \text{ m/s}$$

c. The potential energy at 2.00 cm is

$$\text{PE} = \frac{1}{2}kx^2 = \frac{1}{2}(29.4 \text{ N/m})(2.00 \times 10^{-2} \text{ m})^2$$

$$= 5.88 \times 10^{-3} \text{ J}$$

The kinetic energy at 2.00 cm is

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{k}{m}(A^2 - x^2)$$

$$= \frac{1}{2}(29.4 \text{ N/m})[(5.00 \times 10^{-2} \text{ m})^2 - (2.00 \times 10{-}2 \text{ m})^2]$$

$$= 3.09 \times 10^{-2} \text{ J}$$

Note that the sum of the potential energy and the kinetic energy is equal to the same value for the total energy found in part a.

To go to this Interactive Example click on this sentence.

11.6 The Simple Pendulum

Another example of periodic motion is a pendulum. A **simple pendulum** is a bob that is attached to a string and allowed to oscillate, as shown in figure 11.7(a). The bob oscillates because there is a restoring force, given by

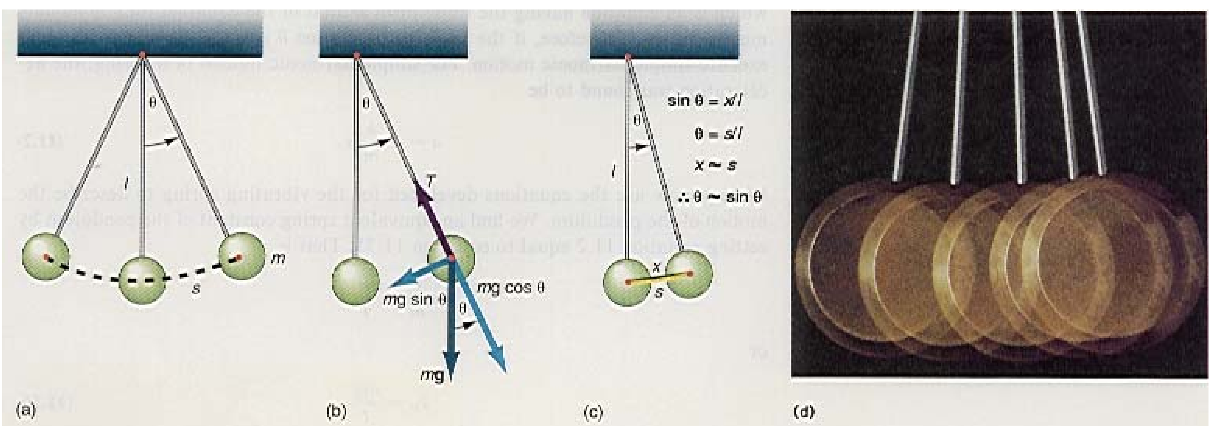

Figure 11.7 The simple pendulum.

$$\text{Restoring force} = -mg \sin \theta \tag{11.31}$$

This restoring force is just the component of the weight of the bob that is perpendicular to the string, as shown in figure 11.7(b). If Newton's second law, $F = ma$, is applied to the motion of the pendulum bob, we get

$$-mg \sin \theta = ma$$

The tangential acceleration of the pendulum bob is thus

$$a = -g \sin \theta \tag{11.32}$$

Note that although this pendulum motion is periodic, it is not, in general, simple harmonic motion because the acceleration is not directly proportional to the displacement of the pendulum bob from its equilibrium position.

However, if the angle θ of the simple pendulum is small, then the sine of θ can be replaced by the angle θ itself, expressed in radians. (The discrepancy in using θ rather than the sin θ is less than 0.2% for angles less than 10 degrees.) That is, for small angles

$$\sin\theta \approx \theta$$

The acceleration of the bob is then

$$a = -g\theta \tag{11.33}$$

From figure 11.7 and the definition of an angle in radians (θ = arc length/radius), we have

$$\theta = \frac{s}{l}$$

where s is the actual path length followed by the bob. Thus

$$a = -g\frac{s}{l} \tag{11.34}$$

The path length s is curved, but if the angle θ is small, the arc length s is approximately equal to the chord x, figure 11.7(c). Hence,

$$a = -g\frac{x}{l} \tag{11.35}$$

which is an equation having the same form as that of the equation for simple harmonic motion. Therefore, if the angle of oscillation θ is small, the pendulum will execute simple harmonic motion. For simple harmonic motion of a spring, the acceleration was found to be

$$a = -\frac{k}{m}x \tag{11.2}$$

We can now use the equations developed for the vibrating spring to describe the motion of the pendulum. We find an equivalent spring constant of the pendulum by setting equation 11.2 equal to equation 11.35. That is

$$\frac{k}{m} = \frac{g}{l}$$

or

$$k_P = \frac{mg}{l} \tag{11.36}$$

Equation 11.36 states that the motion of a pendulum can be described by the equations developed for the vibrating spring by using the equivalent spring constant of the pendulum k_p. Thus, the period of motion of the pendulum, found from equation 11.22, is

$$T_p = 2\pi\sqrt{\frac{m}{k_p}}$$

$$= 2\pi\sqrt{\frac{m}{mg/l}}$$

$$T_p = 2\pi\sqrt{\frac{l}{g}} \tag{11.37}$$

The period of motion of the pendulum is independent of the mass m of the bob but is directly proportional to the square root of the length of the string. If the angle θ is equal to 15^0 on either side of the central position, then the true period differs from that given by equation 11.37 by less than 0.5%.

The pendulum can be used as a very simple device to measure the acceleration of gravity at a particular location. We measure the length l of the pendulum and then set the pendulum into motion. We measure the period by a clock and obtain the acceleration of gravity from equation 11.37 as

$$g = \frac{4\pi^2 l}{T_p^2} \tag{11.38}$$

Example 11.4

The period of a pendulum. Find the period of a simple pendulum 1.50 m long.

Solution

The period, found from equation 11.37, is

$$T_p = 2\pi\sqrt{\frac{l}{g}}$$

$$= 2\pi\sqrt{\frac{1.50 \text{ m}}{9.80 \text{ m/s}^2}}$$

$$= 2.46 \text{ s}$$

To go to this Interactive Example click on this sentence.

Example 11.5

The length of a pendulum. Find the length of a simple pendulum whose period is 1.00 s.

Solution

The length of the pendulum, found from equation 11.37, is

$$l = \frac{T_p^2 g}{4\pi^2}$$

$$= \frac{(1.00 \text{ s})^2 (9.80 \text{ m/s}^2)}{4\pi^2}$$

$$= 0.248 \text{ m}$$

To go to this Interactive Example click on this sentence.

Example 11.6

The pendulum and the acceleration due to gravity. A pendulum 1.50 m long is observed to have a period of 2.47 s at a certain location. Find the acceleration of gravity there.

Solution

The acceleration of gravity, found from equation 11.38, is

$$g = \frac{4\pi^2 l}{T_p^2}$$

$$= \frac{4\pi^2 (1.50 \text{ m})}{(2.47 \text{ s})^2}$$

$$= 9.71 \text{ m/s}^2$$

To go to this Interactive Example click on this sentence.

We can also use a pendulum to measure an acceleration. If a pendulum is placed on board a rocket ship in interstellar space and the rocket ship is accelerated at 9.80 m/s^2, the pendulum oscillates with the same period as it would at rest on the surface of the earth. An enclosed person or thing on the rocket ship could not distinguish between the acceleration of the rocket ship at 9.80 m/s^2 and the acceleration due to gravity of 9.80 m/s^2 on the earth. (This is an example of Einstein's principle of equivalence in general relativity.) An oscillating pendulum of measured length l can be placed in an elevator and the period T measured. We can use equation 11.38 to measure the resultant acceleration experienced by the pendulum in the elevator.

11.7 Springs in Parallel and in Series

Sometimes more than one spring is used in a vibrating system. The motion of the system will depend on the way the springs are connected. As an example, suppose there are three massless springs with spring constants k_1, k_2, and k_3. These springs can be connected in parallel, as shown in figure 11.8(a), or in series, as in figure 11.8(b). The period of motion of either configuration can be found by using an equivalent spring constant k_E.

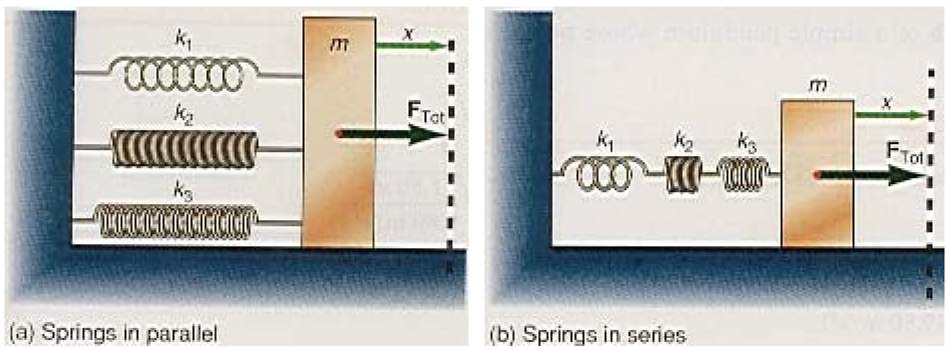

Figure 11.8 Springs in parallel and in series.

Springs in Parallel

If the total force pulling the mass m a distance x to the right is F_{tot}, this force will distribute itself among the three springs such that there will be a force F_1 on spring 1, a force F_2 on spring 2, and a force F_3 on spring 3. If the displacement of each spring is equal to x, then the springs are said to be in parallel. Then we can write the total force as

$$F_{tot} = F_1 + F_2 + F_3 \tag{11.39}$$

However, since we assumed that each spring was displaced the same distance x, Hooke's law for each spring is

$$\begin{aligned} F_1 &= k_1 x \\ F_2 &= k_2 x \\ F_3 &= k_3 x \end{aligned} \tag{11.40}$$

Substituting equation 11.40 into equation 11.39 gives

$$\begin{aligned} F_{tot} &= k_1 x + k_2 x + k_3 x \\ &= (k_1 + k_2 + k_3)x \end{aligned}$$

We now define an equivalent spring constant k_E for springs connected in parallel as

$$k_E = k_1 + k_2 + k_3 \tag{11.41}$$

Hooke's law for the combination of springs is given by

$$F_{tot} = k_E x \tag{11.42}$$

The springs in parallel will execute a simple harmonic motion whose period, found from equation 11.22, is

$$T = 2\pi\sqrt{\frac{m}{k_E}} = 2\pi\sqrt{\frac{m}{k_1 + k_2 + k_3}} \tag{11.43}$$

Springs in Series

If the same springs are connected in series, as in figure 11.8(b), the total force F_{tot} displaces the mass m a distance x to the right. But in this configuration, each spring stretches a different amount. Thus, the total displacement x is the sum of the displacements of each spring, that is,

$$x = x_1 + x_2 + x_3 \tag{11.44}$$

The displacement of each spring, found from Hooke's law, is

$$x_1 = \frac{F_1}{k_1}$$
$$x_2 = \frac{F_2}{k_2}$$
$$x_3 = \frac{F_3}{k_3} \tag{11.45}$$

Substituting these values of the displacement into equation 11.44, yields

$$x = \frac{F_1}{k_1} + \frac{F_2}{k_2} + \frac{F_3}{k_3} \tag{11.46}$$

But because the springs are in series the total applied force is transmitted equally from spring to spring. Hence,

$$F_{tot} = F_1 = F_2 = F_3 \tag{11.47}$$

Substituting equation 11.47 into equation 11.46, gives

$$x = \frac{F_{tot}}{k_1} + \frac{F_{tot}}{k_2} + \frac{F_{tot}}{k_3} \tag{11.46}$$

and

$$x = \left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)F_{tot} \tag{11.48}$$

We now define the equivalent spring constant for springs connected in series as

$$\frac{1}{k_E} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \tag{11.49}$$

Thus, the total displacement, equation 11.48, becomes

$$x = \frac{F_{tot}}{k_E} \tag{11.50}$$

and Hooke's law becomes

$$F_{tot} = k_E x \tag{11.51}$$

where k_E is given by equation 11.49. Hence, the combination of springs in series executes simple harmonic motion and the period of that motion, given by equation 11.22, is

$$T = 2\pi\sqrt{\frac{m}{k_E}} = 2\pi\sqrt{m\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)} \tag{11.52}$$

Example 11.7

Springs in parallel. Three springs with force constants k_1 = 10.0 N/m, k_2 = 12.5 N/m, and k_3 = 15.0 N/m are connected in parallel to a mass of 0.500 kg. The mass is then pulled to the right and released. Find the period of the motion.

Solution

The period of the motion, found from equation 11.43, is

$$T = 2\pi\sqrt{\frac{m}{k_1 + k_2 + k_3}}$$

$$T = 2\pi\sqrt{\frac{0.500 \text{ kg}}{10.0 \text{ N/m} + 12.5 \text{ N/m} + 15.0 \text{ N/m}}}$$
$$= 0.726 \text{ s}$$

To go to this Interactive Example click on this sentence.

Example 11.8

Springs in series. The same three springs as in example 11.7 are now connected in series. Find the period of the motion.

Solution

The period, found from equation 11.52, is

$$T = 2\pi\sqrt{m\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)}$$

$$= 2\pi\sqrt{(0.500 \text{ kg})\left(\frac{1}{10.0 \text{ N/m}} + \frac{1}{12.5 \text{ N/m}} + \frac{1}{15.0 \text{ N/m}}\right)}$$
$$= 2.21 \text{ s}$$

To go to this Interactive Example click on this sentence.

The Language of Physics

Periodic motion
Motion that repeats itself in equal intervals of time (p.).

Displacement
The distance a vibrating body moves from its equilibrium position (p.).

Simple harmonic motion
Periodic motion in which the acceleration of a body is directly proportional to its displacement from the equilibrium position but in the opposite direction. Because the acceleration is directly proportional to the displacement, the acceleration of the body is not constant. The kinematic equations developed in chapter 3 are no longer valid to describe this type of motion (p.).

Amplitude
The maximum displacement of the vibrating body (p.).

Cycle
One complete oscillation or vibratory motion (p.).

Period
The time for the vibrating body to complete one cycle (p.).

Frequency
The number of complete cycles or oscillations in unit time. The frequency is the reciprocal of the period (p.).

Reference circle
A body executing uniform circular motion does so in a circle. The projection of the position of the rotating body onto the x- or y-axis is equivalent to simple harmonic motion along that axis. Thus, vibratory motion is related to motion in a circle, the reference circle (p.).

Angular velocity
The angular velocity of the uniform circular motion in the reference circle is related to the frequency of the vibrating system. Hence, the angular velocity is called the angular frequency of the vibrating system (p.).

Potential energy of a spring
The energy that a body possesses by virtue of its configuration. A compressed spring has potential energy because it has the ability to do work as it expands to its equilibrium configuration. A stretched spring can also do work as it returns to its equilibrium configuration (p.).

Simple pendulum
A bob that is attached to a string and allowed to oscillate to and fro under the action of gravity. If the angle of the pendulum is small the pendulum will oscillate in simple harmonic motion (p.).

Summary of Important Equations

Restoring force in a spring
$$F_R = -kx \qquad (11.1)$$

Defining relation for simple harmonic motion
$$a = -\frac{k}{m}x \qquad (11.2)$$

Frequency
$$f = \frac{1}{T} \qquad (11.3)$$

Displacement in simple harmonic motion
$$x = A\cos\omega t \qquad (11.6)$$

Angular frequency
$$\omega = 2\pi f \qquad (11.9)$$

Velocity as a function of time in simple harmonic motion
$$v = -\omega A \sin\omega t \qquad (11.13)$$

Velocity as a function of displacement
$$v = \pm\,\omega\sqrt{A^2 - x^2} \qquad (11.15)$$

Acceleration as a function of time
$$a = -\omega^2 A\cos\omega t \qquad (11.17)$$

Angular frequency of a spring
$$\omega = \sqrt{\frac{k}{m}} \qquad (11.20)$$

Frequency in simple harmonic motion
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (11.21)$$

Period in simple harmonic motion
$$T = 2\pi\sqrt{\frac{m}{k}} \qquad (11.22)$$

Potential energy of a spring
$$\text{PE} = \frac{1}{2}kx^2 \qquad (11.24)$$

Conservation of energy for a vibrating spring
$$\frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \qquad (11.27)$$

Period of motion of a simple pendulum
$$T_p = 2\pi\sqrt{\frac{l}{g}} \qquad (11.37)$$

Equivalent spring constant for springs in parallel
$$k_E = k_1 + k_2 + k_3 \qquad (11.41)$$

Period of motion for springs in parallel
$$T = 2\pi\sqrt{\frac{m}{k_1 + k_2 + k_3}} \qquad (11.43)$$

Equivalent spring constant for springs in series
$$\frac{1}{k_E} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \qquad (11.49)$$

Period of motion for springs in series
$$T = 2\pi\sqrt{m\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)} \qquad (11.52)$$

Questions for Chapter 11

1. Can the periodic motion of the earth be considered to be an example of simple harmonic motion?
2. Can the kinematic equations derived in chapter 3 be used to describe simple harmonic motion?
3. In the simple harmonic motion of a mass attached to a spring, the velocity of the mass is equal to zero when the acceleration has its maximum value. How is this possible? Can you think of other examples in which a body has zero velocity with a nonzero acceleration?
4. What is the characteristic of the restoring force that makes simple harmonic motion possible?
5. Discuss the significance of the reference circle in the analysis of simple harmonic motion.

6. How can a mass that is undergoing a one-dimensional translational simple harmonic motion have anything to do with an angular velocity or an angular frequency, which is a characteristic of two or more dimensions?

7. How is the angular frequency related to the physical characteristics of the spring and the vibrating mass in simple harmonic motion?

*8. In the entire derivation of the equations for simple harmonic motion we have assumed that the springs are massless and friction can be neglected. Discuss these assumptions. Describe qualitatively what you would expect to happen to the motion if the springs are not small enough to be considered massless.

*9. Describe how a geological survey for iron might be undertaken on the moon using a simple pendulum.

*10. How could a simple pendulum be used to make an accelerometer?

*11. Discuss the assumption that the displacement of each spring is the same when the springs are in parallel. Under what conditions is this assumption valid and when would it be invalid?

Problems for Chapter 11

11.2 Simple Harmonic Motion and 11.3 Analysis of Simple Harmonic Motion

1. A mass of 0.200 kg is attached to a spring of spring constant 30.0 N/m. If the mass executes simple harmonic motion, what will be its frequency?

2. A 30.0-g mass is attached to a vertical spring and it stretches 10.0 cm. It is then stretched an additional 5.00 cm and released. Find its period of motion and its frequency.

3. A 0.200-kg mass on a spring executes simple harmonic motion at a frequency f. What mass is necessary for the system to vibrate at a frequency of $2f$?

4. A simple harmonic oscillator has a frequency of 2.00 Hz and an amplitude of 10.0 cm. What is its maximum acceleration? What is its acceleration at $t = 0.25$ s?

5. A ball attached to a string travels in uniform circular motion in a horizontal circle of 50.0 cm radius in 1.00 s. Sunlight shining on the ball throws its shadow on a wall. Find the velocity of the shadow at (a) the end of its path and (b) the center of its path.

6. A 50.0-g mass is attached to a spring of force constant 10.0 N/m. The spring is stretched 20.0 cm and then released. Find the displacement, velocity, and acceleration of the mass at 0.200 s.

7. A 25.0-g mass is attached to a vertical spring and it stretches 15.0 cm. It is then stretched an additional 10.0 cm and then released. What is the maximum velocity of the mass? What is its maximum acceleration?

8. The displacement of a body in simple harmonic motion is given by $x = (0.15\text{ m})\cos[(5.00\text{ rad/s})t]$. Find (a) the amplitude of the motion, (b) the angular frequency, (c) the frequency, (d) the period, and (e) the displacement at 3.00 s.

9. A 500-g mass is hung from a coiled spring and it stretches 10.0 cm. It is then stretched an additional 15.0 cm and released. Find (a) the frequency of vibration, (b) the period, and (c) the velocity and acceleration at a displacement of 10.0 cm.

10. A mass of 0.200 kg is placed on a vertical spring and the spring stretches by 15.0 cm. It is then pulled down an additional 10.0 cm and then released. Find (a) the spring constant, (b) the angular frequency, (c) the frequency, (d) the period, (e) the maximum velocity of the mass, (f) the maximum acceleration of the mass, (g) the maximum restoring force, and (h) the equation of the displacement, velocity, and acceleration at any time t.

11.5 Conservation of Energy and the Vibrating Spring

11. A simple harmonic oscillator has a spring constant of 5.00 N/m. If the amplitude of the motion is 15.0 cm, find the total energy of the oscillator.

12. A body is executing simple harmonic motion. At what displacement is the potential energy equal to the kinetic energy?

13. A 20.0-g mass is attached to a horizontal spring on a smooth table. The spring constant is 3.00 N/m. The spring is then stretched 15.0 cm and then released. What is the total energy of the motion? What is the potential and kinetic energy when $x = 5.00$ cm?

14. A body is executing simple harmonic motion. At what displacement is the speed v equal to one-half the maximum speed?

11.6 The Simple Pendulum

15. Find the period and frequency of a simple pendulum 0.75 m long.

16. If a pendulum has a length L and a period T, what will be the period when (a) L is doubled and (b) L is halved?

17. Find the frequency of a child's swing whose ropes have a length of 3.25 m.

18. What is the period of a 0.500-m pendulum on the moon where $g_m = (1/6)g_e$?

19. What is the period of a pendulum 0.750 m long on a spaceship (a) accelerating at 4.90 m/s^2 and (b) moving at constant velocity?

20. What is the period of a pendulum in free-fall?

21. A pendulum has a period of 0.750 s at the equator at sea level. The pendulum is carried to another place on the earth and the period is now found to be 0.748 s. Find the acceleration due to gravity at this location.

11.7 Springs in Parallel and in Series

*22. Springs with spring constants of 5.00 N/m and 10.0 N/m are connected in parallel to a 5.00-kg mass. Find (a) the equivalent spring constant and (b) the period of the motion.

*23. Springs with spring constants 5.00 N/m and 10.0 N/m are connected in series to a 5.00-kg mass. Find (a) the equivalent spring constant and (b) the period of the motion.

Additional Problems

24. A 500-g mass is attached to a vertical spring of spring constant 30.0 N/m. How far should the spring be stretched in order to give the mass an upward acceleration of 3.00 m/s^2?

25. A ball is caused to move in a horizontal circle of 40.0-cm radius in uniform circular motion at a speed of 25.0 cm/s. Its projection on the wall moves in simple harmonic motion. Find the velocity and acceleration of the shadow of the ball at (a) the end of its motion, (b) the center of its motion, and (c) halfway between the center and the end of the motion.

*26. The motion of the piston in the engine of an automobile is approximately simple harmonic. If the stroke of the piston (twice the amplitude) is equal to 20.3 cm and the engine turns at 1800 rpm, find (a) the acceleration at $x = A$ and (b) the speed of the piston at the midpoint of the stroke.

*27. A 535-g mass is dropped from a height of 25.0 cm above an uncompressed spring of k = 20.0 N/m. By how much will the spring be compressed?

28. A simple pendulum is used to operate an electrical device. When the pendulum bob sweeps through the midpoint of its swing, it causes an electrical spark to be given off. Find the length of the pendulum that will give a spark rate of 30.0 sparks per minute.

*29. The general solution for the period of a simple pendulum, without making the assumption of small angles of swing, is given by

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{(\frac{1}{2})^2\sin^2\theta}{2} + \frac{(\frac{1}{2})^2(\frac{3}{4})^2\sin^4\theta}{2} + \dots\right]$$

Find the period of a 1.00-m pendulum for $\theta = 10.0^0$, 30.0^0, and 50.0^0 and compare with the period obtained with the small angle approximation. Determine the percentage error in each case by using the small angle approximation.

30. A pendulum clock on the earth has a period of 1.00 s. Will this clock run slow or fast, and by how much if taken to (a) Mars, (b) Moon, and (c) Venus?

*31. A pendulum bob, 355 g, is raised to a height of 12.5 cm before it is released. At the bottom of its path it makes a perfectly elastic collision with a 500-g mass that is connected to a horizontal spring of spring constant 15.8 N/m, that is at rest on a smooth surface. How far will the spring be compressed?

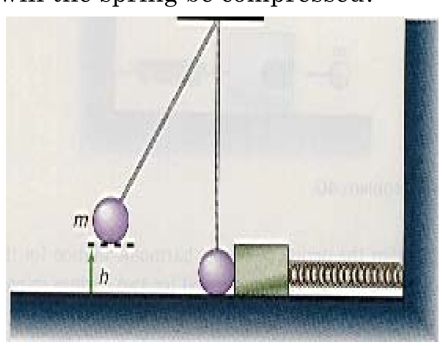

Diagram for problem 31.

*32. A 500-g block is in simple harmonic motion as shown in the diagram. A mass m' is added to the top of the block when the block is at its maximum extension. How much mass should be added to change the frequency by 25%?

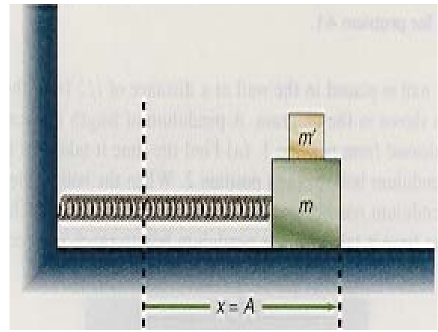

Diagram for problem 32.

*33. A pendulum clock keeps correct time at a location at sea level where the acceleration of gravity is equal to 9.80 m/s^2. The clock is then taken up to the top of a mountain and the clock loses 3.00 s per day. How high is the mountain?

*34. Three people, who together weigh 1880 N, get into a car and the car is observed to move 5.08 cm closer to the ground. What is the spring constant of the car springs?

*35. In the accompanying diagram, the mass m is pulled down a distance of 9.50 cm from its equilibrium position and is then released. The mass then executes simple harmonic motion. Find (a) the total potential energy of the mass with respect to the ground when the mass is located at positions 1, 2, and 3; (b) the total energy of the mass at positions 1, 2, and 3; and (c) the speed of the mass at position 2. Assume m = 55.6 g, k = 25.0 N/m, h_0 = 50.0 cm.

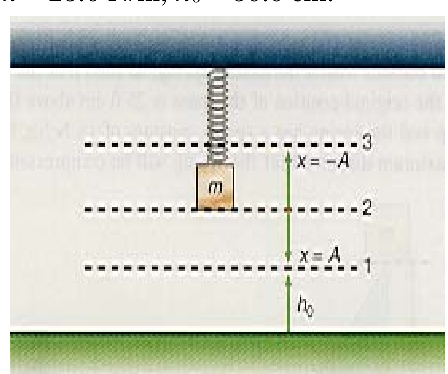

Diagram for problem 35.

*36. A 20.0-g ball rests on top of a vertical spring gun whose spring constant is 20 N/m. The spring is compressed 10.0 cm and the gun is then fired. Find how high the ball rises in its vertical trajectory.

*37. A toy spring gun is used to fire a ball as a projectile. Find the value of the spring constant, such that when the spring is compressed 10.0 cm, and the gun is fired at an angle of 62.5^0, the range of the projectile will be 1.50 m. The mass of the ball is 25.2 g.

*38. In the simple pendulum shown in the diagram, find the tension in the string when the height of the pendulum is (a) h, (b) $h/2$, and (c) $h = 0$. The mass m = 500 g, the initial height h = 15.0 cm, and the length of the pendulum l = 1.00 m.

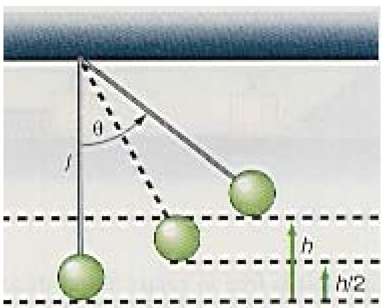

Diagram for problem 38.

*39. A mass is attached to a horizontal spring. The mass is given an initial amplitude of 10.0 cm on a rough surface and is then released to oscillate in simple harmonic motion. If 10.0% of the energy is lost per cycle due to the friction of the mass moving over the rough surface, find the maximum displacement of the mass after 1, 2, 4, 6, and 8 complete oscillations.

*40. Find the maximum amplitude of vibration after 2 periods for a 85.0-g mass executing simple harmonic motion on a rough horizontal surface of $\mu_k = 0.350$. The spring constant is 24.0 N/m and the initial amplitude is 20.0 cm.

41. A 240-g mass slides down a circular chute without friction and collides with a horizontal spring, as shown in the diagram. If the original position of the mass is 25.0 cm above the table top and the spring has a spring constant of 18 N/m, find the maximum distance that the spring will be compressed.

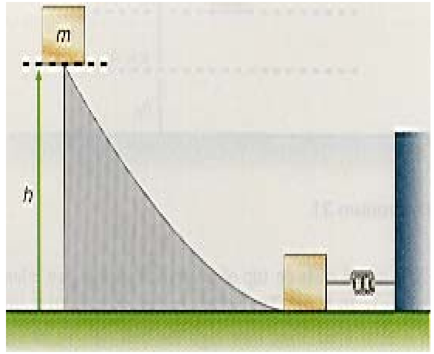

Diagram for problem 41.

*42. A 235-g block slides down a frictionless inclined plane, 1.25 m long, that makes an angle of 34.0^0 with the horizontal. At the bottom of the plane the block slides along a rough horizontal surface 1.50 m long. The coefficient of kinetic friction between the block and the rough horizontal surface is 0.45. The block then collides with a horizontal spring of k = 20.0 N/m. Find the maximum compression of the spring.

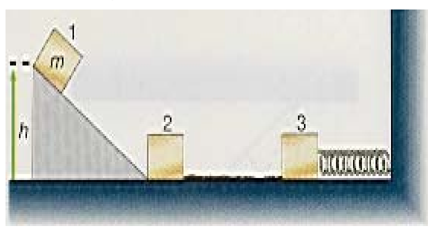

Diagram for problem 42.

*43. A 335-g disk that is free to rotate about its axis is connected to a spring that is stretched 35.0 cm. The spring constant is 10.0 N/m. When the disk is released it rolls without slipping as it moves toward the equilibrium position. Find the speed of the disk when the displacement of the spring is equal to –17.5 cm.

*44. A 25.0-g ball moving at a velocity of 200 cm/s to the right makes an inelastic collision with a 200-g block that is initially at rest on a frictionless surface. There is a hole in the large block for the small ball to fit into. If k = 10 N/m, find the maximum compression of the spring.

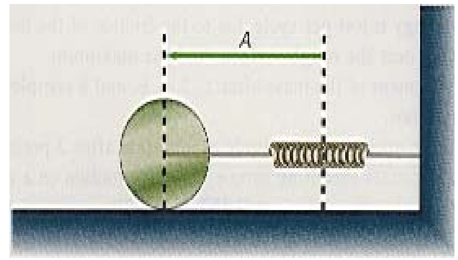

Diagram for problem 43.

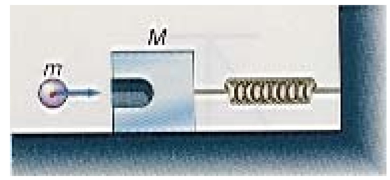

Diagram for problem 44.

*45. Show that the period of simple harmonic motion for the mass shown is equivalent to the period for two springs in parallel.

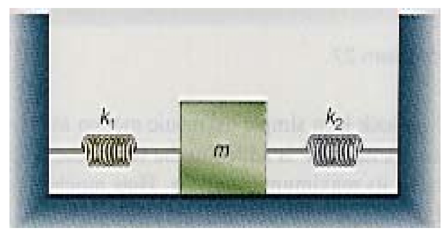

Diagram for problem 45.

*46. A nail is placed in the wall at a distance of $l/2$ from the top, as shown in the diagram. A pendulum of length 85.0 cm is released from position 1. (a) Find the time it takes for the pendulum bob to reach position 2. When the bob of the pendulum reaches position 2, the string hits the nail. (b) Find the time it takes for the pendulum bob to reach position 3.

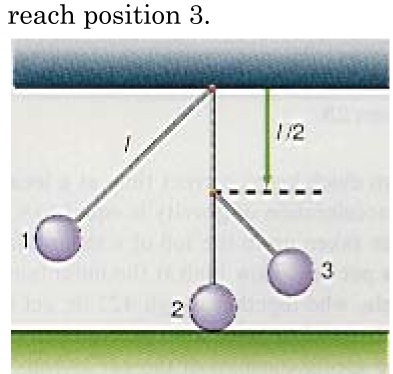

Diagram for problem 46.

*47. A spring is attached to the top of an Atwood's machine as shown. The spring is stretched to A = 10 cm before being released. Find the velocity of m_2 when $x = -A/2$. Assume m_1 = 28.0 g, m_2 = 43.0 g, and k = 10.0 N/m.

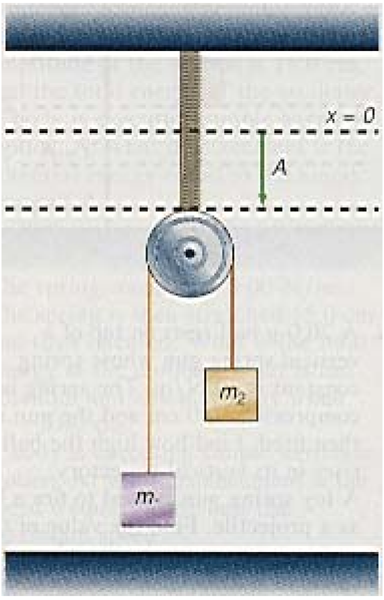

Diagram for problem 47.

*48. A 280-g block is connected to a spring on a rough inclined plane that makes an angle of 35.5^0 with the horizontal. The block is pulled down the plane a distance A = 20.0 cm, and is then released. The spring constant is 40.0 N/m and the coefficient of kinetic friction is 0.100. Find the speed of the block when the displacement $x = -A/2$.

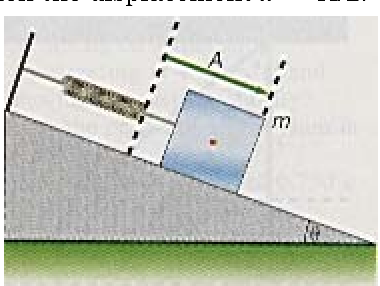

Diagram for problem 48.

49. The rotational analog of simple harmonic motion, is angular simple harmonic motion, wherein a body rotates periodically clockwise and then counterclockwise. Hooke's law for rotational motion is given by

$$\tau = -C\,\theta$$

where τ is the torque acting on the body, θ is the angular displacement, and C is a constant, like the spring constant. Use Newton's second law for rotational motion to show

$$\alpha = \frac{C}{I}\,\theta$$

Use the analogy between the linear result, $a = -\omega^2 x$, to show that the frequency of vibration of an object executing angular simple harmonic motion is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{C}{I}}$$

Interactive Tutorials

50. *Simple Pendulum*. Calculate the period T of a simple pendulum located on a planet having a gravitational acceleration of g = 9.80 m/s², if its length l = 1.00 m is increased from 1 to 10 m in steps of 1.00 m. Plot the results as the period T versus the length l.

51. *Simple Harmonic Motion.* The displacement x of a body undergoing simple harmonic motion is given by the formula $x = A \cos \omega t$, where A is the amplitude of the vibration, ω is the angular frequency in rad/s, and t is the time in seconds. Plot the displacement x as a function of t for an amplitude A = 0.150 m and an angular frequency ω = 5.00 rad/s as t increases from 0 to 2 s in 0.10 s increments.

52. *The Vibrating Spring.* A mass m = 0.500 kg is attached to a spring on a smooth horizontal table. An applied force F_A = 4.00 N is used to stretch the spring a distance x_0 = 0.150 m. (a) Find the spring constant k of the spring. The mass is returned to its equilibrium position and then stretched to a value A = 0.15 m and then released. The mass then executes simple harmonic motion. Find (b) the angular frequency ω, (c) the frequency f, (d) the period T, (e) the maximum velocity v_{max} of the vibrating mass, (f) the maximum acceleration a_{max} of the vibrating mass, (g) the maximum restoring force F_{Rmax}, and (h) the velocity of the mass at the displacement x = 0.08 m. (i) Plot the displacement x, velocity v, acceleration a, and the restoring force F_R at any time t.

53. *Conservation of Energy and the Vibrating Horizontal Spring.* A mass m = 0.350 kg is attached to a horizontal spring. The mass is then pulled a distance $x = A$ = 0.200 m from its equilibrium position and when released the mass executes simple harmonic motion. Find (a) the total energy E_{tot} of the mass when it is at its maximum displacement A from its equilibrium position. When the mass is at the displacement x = 0.120 m find, (b) its potential energy PE, (c) its kinetic energy KE, and (d) its speed v. (e) Plot the total energy, potential energy, and kinetic energy of the mass as a function of the displacement x. The spring constant k = 35.5 N/m.

54. *Conservation of Energy and the Vibrating Vertical Spring.* A mass m = 0.350 kg is attached to a vertical spring. The mass is at a height h_0 = 1.50 m from the floor. The mass is then pulled down a distance A = 0.220 m from its equilibrium position and when released executes simple harmonic motion. Find (a) the total energy of the mass when it is at its maximum displacement A below its equilibrium position, (b) the gravitational potential energy when it is at the displacement x = 0.120 m, (c) the elastic potential energy when it is at the same displacement x, (d) the kinetic energy at the displacement x, and (e) the speed of the mass when it is at the displacement x. The spring constant k = 35.5 N/m.

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