Answers to Even-Numbered Exercises

CHAPTER 1

Section 1.1, page 11

2. The solution is $(x_1, x_2) = (12, -7)$, or simply $(12, -7)$.

4. The point of intersection is $(9/4, 1/4)$

6. Replace Row4 by its sum with -3 times Row3. After that, scale Row4 by $-1/5$.

8. $(0, 0, 0)$ **10.** $(-3, -5, 6, -3)$

12. Inconsistent **14.** $(2, -1, 1)$ **16.** Consistent

18. Calculations show the system is inconsistent, so the three planes have no point in common.

20. All h **22.** $h = -5/3$

23. **a.** True. See the remarks following the box titled "Elementary Row Operations."
b. False. A 5×6 matrix has five rows.
c. False. The description applies to a single solution. The solution *set* consists of all possible solutions. Only in special cases does the solution set consist of exactly one solution. A statement should be marked True only if the statement is *always* true.
d. True. See the box before Example 2.

24. **a.** True. See the box preceding the subsection titled "Existence and Uniqueness Questions."
b. False. The definition of *row equivalent* requires that there exist a sequence of row operations that transforms one matrix into the other.
c. False. By definition, an inconsistent system has *no* solution.
d. True. This definition of *equivalent systems* is in the second paragraph after equation (2).

26. Answers may vary. The systems corresponding to the following matrices each have the solution set $x_1 = -2$, $x_2 = 1$, $x_3 = 0$. (The tildes represent row equivalence.)

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & -3 \\ 2 & 0 & 1 & -4 \end{bmatrix}$$

28. $d - c(b/a) \neq 0$, or $ad - bc \neq 0$.

30. Scale Row2 by $-1/2$; scale Row2 by -2.

32. Replace Row3 by Row3 + (3)Row2; replace Row3 by Row3 + (−3)Row2.

34. $(20, 27.5, 30, 22.5)$

Section 1.2, page 25

2. Reduced echelon form: **a**. Echelon form: **b** and **d**. Not echelon: **c**.

4. $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Pivot cols 1, 2, and 4:

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}.$$

6. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

8. $\begin{cases} x_1 = -9 \\ x_2 = 4 \\ x_3 \text{ is free} \end{cases}$ **10.** $\begin{cases} x_1 = -4 + 2x_2 \\ x_2 \text{ is free} \\ x_3 = -7 \end{cases}$

12. $\begin{cases} x_1 = 5 + 7x_2 - 6x_4 \\ x_2 \text{ is free} \\ x_3 = -3 + 2x_4 \\ x_4 \text{ is free} \end{cases}$ **14.** $\begin{cases} x_1 = -9 - 7x_3 \\ x_2 = 2 + 6x_3 + 3x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$

16. **a.** A unique solution
b. Consistent, with many solutions

18. $h \neq -15$

20. **a.** Inconsistent when $h = 9$ and $k \neq 6$
b. Unique solution when $h \neq 9$
c. Many solutions when $h = 9$ and $k = 6$

21. **a.** False. See Theorem 1.
b. False. See the second paragraph of the section.
c. True. Basic variables are defined after equation (4).
d. True. This statement is at the beginning of "Parametric Descriptions of Solution Sets."
e. False. The row shown corresponds to the equation $5x_4 = 0$, which does not by itself lead to a contradiction. So the system might be consistent or it might be inconsistent.

22. **a.** False. See the statement preceding Theorem 1. Only the *reduced* echelon form is unique.
b. False. See the beginning of the subsection "Pivot Positions." The pivot positions in a matrix are determined completely by the positions of the leading entries in the nonzero rows of any echelon form obtained from the matrix.
c. True. See the paragraph after Example 3.
d. False. The existence of at least one solution is not related to the presence or absence of free variables. If the system is inconsistent, the solution set is empty. See the solution of Practice Problem 2.
e. True. See the paragraph just before Example 4.

24. The system is inconsistent because the pivot in column 5 means that there is a row of the form [0 0 0 0 1]. Since the matrix is the *augmented* matrix for a system, Theorem 2 shows that system has no solution.

26. Since there are three pivots (one in each row), the augmented matrix must reduce to the form

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix} \text{ and so } \begin{aligned} x_1 \quad\quad &= a \\ x_2 \quad &= b \\ x_3 &= c \end{aligned}$$

No matter what the values of a, b, and c, the solution exists and is unique.

28. Every column in the augmented matrix *except the rightmost column* is a pivot column, and the rightmost column is *not* a pivot column.

30. Example:
$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + 2x_3 &= 5 \end{aligned}$$

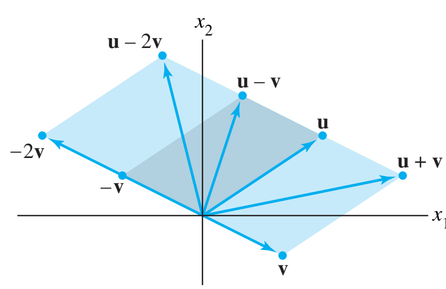
32. When $n = 30$, the backward phase requires about 5% of the total flops. When $n = 300$, the backward phase requires about 0.5% of the total flops. The *Instructor's Solution Manual* has the details.

34. **[M]** $p(t) = 1.7125t - 1.1948t^2 + .6615t^3 - .0701t^4 + .0026t^5$, and $p(7.5) = 64.6$ hundred lb. [*Note:* $p(7.5) = 64.8$ when the coefficients of $p(t)$ are retained as originally computed.] If a polynomial of lower degree is used, the resulting system of equations is overdetermined. The augmented matrix for such a system is the same as the one used to find p, except that at least column 6 is missing. When the augmented matrix is row reduced, the sixth row of the augmented matrix will be entirely zero except for a nonzero entry in the augmented column, indicating that no solution exists.

Section 1.3, page 37

2. $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$

4.



6.
$$\begin{aligned} -2x_2 + 8x_2 + x_3 &= 0 \\ 3x_1 + 5x_2 - 6x_3 &= 0 \end{aligned}$$

8. $\mathbf{w} = 2\mathbf{v} - \mathbf{u}$, $\mathbf{x} = -2\mathbf{u} + 2\mathbf{v}$, $\mathbf{y} = -2\mathbf{u} + 3.5\mathbf{v}$, $\mathbf{z} = -3\mathbf{u} + 4\mathbf{v}$

10. $x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$

12. No, $\mathbf{b}$ is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

14. Yes, $\mathbf{b}$ is a linear combination of the columns of A.

16. Noninteger weights are acceptable, of course, but some simple choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix}3\\0\\2\end{bmatrix},\ 0\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix}-2\\0\\3\end{bmatrix}$$

$$1\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix}1\\0\\5\end{bmatrix},\ 1\cdot\mathbf{v}_1 - 1\cdot\mathbf{v}_2 = \begin{bmatrix}5\\0\\-1\end{bmatrix}$$

18. $h = -7/2$ **20.** Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ *is* the xz-plane.

22. Construct any 3×4 matrix in echelon form that corresponds to an inconsistent system. Perform sufficient row operations on the matrix to eliminate all zero entries in the first three columns.

23. **a.** False. The alternative notation for a (column) vector is $(-4, 3)$, using parentheses and commas.

b. False. Plot the points to verify this. Or, see the statement preceding Example 3. If $\begin{bmatrix}-5\\2\end{bmatrix}$ were on the line through $\begin{bmatrix}-2\\5\end{bmatrix}$ and the origin, then $\begin{bmatrix}-5\\2\end{bmatrix}$ would have to be a multiple of $\begin{bmatrix}-2\\5\end{bmatrix}$, which is not the case.

c. True. See the line displayed just before Example 4.

d. True. See the box that discusses the matrix in (5).

e. False. The statement is often true, but Span$\{\mathbf{u}, \mathbf{v}\}$ is not a plane when $\mathbf{v}$ is a multiple of $\mathbf{u}$, or when $\mathbf{u}$ is the zero vector.

24. **a.** True. See the beginning of the subsection "Vectors in $\mathbb{R}^n$."

b. True. Use Fig. 7 to draw the parallelogram determined by $\mathbf{u} - \mathbf{v}$ and $\mathbf{v}$.

c. False. See the first paragraph of the subsection "Linear Combinations."

d. True. See the statement that refers to Fig. 11.

e. True. See the paragraph following the definition of Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

26. **a.** Yes, $\mathbf{b}$ is a linear combination of the columns of A; that is, $\mathbf{b}$ is in W.

b. The third column of A is in W because $\mathbf{a}_3 = 0\cdot\mathbf{a}_1 + 0\cdot\mathbf{a}_2 + 1\cdot\mathbf{a}_3$.

28. **a.** The amount of heat produced when the steam plant burns x_1 tons of anthracite and x_2 tons of bituminous coal is $27.6x_1 + 30.2x_2$ million Btu.

b. The vector $x_1\begin{bmatrix}27.6\\3100\\250\end{bmatrix} + x_2\begin{bmatrix}30.2\\6400\\360\end{bmatrix}$ gives the total output produced by x_1 tons of anthracite and x_2 tons of bituminous coal.

c. **[M]** Solve $x_1\begin{bmatrix}27.6\\3100\\250\end{bmatrix} + x_2\begin{bmatrix}30.2\\6400\\360\end{bmatrix} = \begin{bmatrix}162\\23{,}610\\1{,}623\end{bmatrix}$.
The steam plant burned 3.9 tons of anthracite coal and 1.8 tons of bituminous coal.

30. Let m be the total mass of the system. By definition,

$$\mathbf{v} = \frac{1}{m}(m_1\mathbf{v}_1 + \cdots + m_k\mathbf{v}_k) = \frac{m_1}{m}\mathbf{v}_1 + \cdots + \frac{m_k}{m}\mathbf{v}_k$$

The second expression displays $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$, which shows that $\mathbf{v}$ is in Span$\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$.

32. On the figure, draw a parallelogram to show that the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + 0\cdot\mathbf{v}_3 = \mathbf{b}$ has a solution. Draw another parallelogram to show that the equation $x_1\mathbf{v}_1 + 0\cdot\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ also has a solution. (See the *Instructor's Solution Manual.*) Thus the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ has at least two solutions, not just one unique solution.

34. **a.** For $j = 1, \dots, n$, $u_j + (-1)u_j = (-1)u_j + u_j = 0$, by properties of $\mathbb{R}$. By vector equality,
$\mathbf{u} + (-\mathbf{u}) = \mathbf{u} + (-1)\mathbf{u} = (-1)\mathbf{u} + \mathbf{u} = -\mathbf{u} + \mathbf{u} = \mathbf{0}$

b. For scalars c and d, the jth entries of $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are $c(du_j)$ and $(cd)u_j$, respectively. These entries in $\mathbb{R}$ are equal, so the vectors $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are equal.

Section 1.4, page 47

2. The product is not defined because the number of columns (1) in the 3×1 matrix does not match the number of entries (2) in the vector.

4. $$A\mathbf{x} = \begin{bmatrix}8 & 3 & -4\\5 & 1 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = 1\cdot\begin{bmatrix}8\\5\end{bmatrix} + 1\cdot\begin{bmatrix}3\\1\end{bmatrix} + 1\cdot\begin{bmatrix}-4\\2\end{bmatrix}$$

$$= \begin{bmatrix}7\\8\end{bmatrix}, \text{ and } A\mathbf{x} = \begin{bmatrix}8 & 3 & -4\\5 & 1 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}8\cdot1 + 3\cdot1 + (-4)\cdot1\\5\cdot1 + 1\cdot1 + 2\cdot1\end{bmatrix} = \begin{bmatrix}7\\8\end{bmatrix}$$

6. $$-2\cdot\begin{bmatrix}7\\2\\9\\-3\end{bmatrix} - 5\cdot\begin{bmatrix}-3\\1\\-6\\2\end{bmatrix} = \begin{bmatrix}1\\-9\\12\\-4\end{bmatrix}$$

8. $\begin{bmatrix} 4 & -4 & -5 & 3 \\ -2 & 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$

10. $x_1 \begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

and $\begin{bmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}$

14. No. The equation $A\mathbf{x} = \mathbf{u}$ has no solution.

16. The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $b_1 + 2b_2 + b_3 = 0$. The set of such $\mathbf{b}$ is a plane through the origin in $\mathbb{R}^3$.

18. Only three rows contain a pivot position. The equation $B\mathbf{x} = \mathbf{y}$ does *not* have a solution for each $\mathbf{y}$ in $\mathbb{R}^4$, by Theorem 4.

20. The work in Exercise 18 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in $\mathbb{R}^4$ can be written as a linear combination of the columns of B. The columns of B certainly do *not* span $\mathbb{R}^3$, because each column of B is in $\mathbb{R}^4$, not $\mathbb{R}^3$.

22. The matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ has a pivot in each row, so the columns of the matrix span $\mathbb{R}^3$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

23. **a.** False. See the paragraph following equation (3). The text calls $A\mathbf{x} = \mathbf{b}$ a *matrix equation.*
b. True. See the box before Example 3.
c. False. See the warning following Theorem 4.
d. True. See Example 4.
e. True. See parts (c) and (a) in Theorem 4.
f. True. In Theorem 4, statement (a) is false if and only if statement (d) is also false.

24. **a.** True. This is part of Theorem 3.
b. True. See Example 2.
c. True, by Theorem 3.
d. True. See the box before Example 2. Saying that $\mathbf{b}$ is not in the set spanned by the columns of A is the same as saying that $\mathbf{b}$ is not a linear combination of the columns of A.
e. False. See the warning that follows Theorem 4.
f. True. In Theorem 4, statement (c) is false if and only if statement (a) is also false.

26. $3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$ can be rewritten as $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$. So, a solution is $x_1 = 3$, $x_2 = -5$.

28. The matrix equation can be written as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{v}_6$, where $c_1 = -3$, $c_2 = 2$, $c_3 = 4$, $c_4 = -1$, $c_5 = 2$, and

$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \end{bmatrix},$

$\mathbf{v}_4 = \begin{bmatrix} 9 \\ -2 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 7 \\ -4 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$

30. Start with any nonzero 3×3 matrix B in echelon form that has fewer than three pivot positions. Perform a row operation that creates a matrix A that is *not* in echelon form. Then A has the desired property. Since A does not have a pivot position in every row, the columns of A do not span $\mathbb{R}^3$, by Theorem 4.

32. A set of three vectors in $\mathbb{R}^4$ cannot span $\mathbb{R}^4$. Reason: The matrix A whose columns are these three vectors has four rows. To have a pivot in each row, A would have to have at least four columns (one for each pivot), which is not the case. Since A does not have a pivot in every row, its columns do not span $\mathbb{R}^4$, by Theorem 4. In general, a set of n vectors in $\mathbb{R}^m$ cannot span $\mathbb{R}^m$ when n is less than m.

34. If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the associated system of equations does not have any free variables. If every variable is a basic variable, then each column of A is a pivot column. So the reduced echelon form of A must be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Now it is clear that A has a pivot position in each *row*. By Theorem 4, the columns of A span $\mathbb{R}^3$.

36. Suppose $\mathbf{y}$ and $\mathbf{z}$ satisfy $A\mathbf{y} = \mathbf{z}$. Then $4\mathbf{z} = 4A\mathbf{y}$. By Theorem 5(b), $4A\mathbf{y} = A(4\mathbf{y})$. So $4\mathbf{z} = A(4\mathbf{y})$, which shows that $4\mathbf{y}$ is a solution of $A\mathbf{x} = 4\mathbf{z}$. Thus the equation $A\mathbf{x} = 4\mathbf{z}$ is consistent.

38. **[M]** The matrix has pivots in only three rows. So, the columns of the original matrix do not span $\mathbb{R}^4$, by Theorem 4.

40. **[M]** The matrix has a pivot in every row, so its columns span $\mathbb{R}^4$, by Theorem 4.

42. **[M]** Delete column 3 of the matrix in Exercise 40. (It is also possible to delete column 2 instead of column 3.) No, you cannot delete more than one column.

Section 1.5, page 55

2. There is no free variable; the system has only the trivial solution.

4. The variable x_3 is free; the system has nontrivial solutions.

6. $\mathbf{x} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$ **8.** $\mathbf{x} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$

10. $\mathbf{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

12. $\mathbf{x} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

14. $\mathbf{x} = \begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix} = \mathbf{p} + x_4\mathbf{q}$. The solution set is the line through $\mathbf{p}$ parallel to $\mathbf{q}$.

16. $\mathbf{x} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} = \mathbf{p} + x_3\mathbf{q}$. The solution set is the line through $\mathbf{p}$ parallel to $\mathbf{q}$.

18. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$. The solution of the homogeneous equation is $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$, the plane through the origin spanned by $\mathbf{u}$ and $\mathbf{v}$. The solution of the nonhomogeneous system is $\mathbf{x} = \mathbf{p} + x_2\mathbf{u} + x_3\mathbf{v}$, the plane through $\mathbf{p}$ parallel to the solution of the homogeneous equation.

20. $\mathbf{x} = \mathbf{a} + t\mathbf{b}$, where t represents a parameter, or
$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + t \begin{bmatrix} -7 \\ 8 \end{bmatrix}$, or $\begin{cases} x_1 = 3 - 7t \\ x_2 = -4 + 8t \end{cases}$

22. $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} -6 \\ 3 \end{bmatrix} + t \begin{bmatrix} 6 \\ -7 \end{bmatrix}$

23. **a.** True. See the first paragraph of the subsection titled "Homogeneous Linear Systems."
b. False. The equation $A\mathbf{x} = \mathbf{0}$ gives an *implicit* description of its solution set. See the subsection titled "Parametric Vector Form."
c. False. The equation $A\mathbf{x} = \mathbf{0}$ *always* has the trivial solution. The box before Example 1 uses the word "nontrivial" instead of "trivial."
d. False. The line goes through $\mathbf{p}$ parallel to $\mathbf{v}$. See the paragraph that precedes Fig. 5.
e. False. The solution set could be *empty*! The statement (from Theorem 6) is true only when there exists a vector $\mathbf{p}$ such that $A\mathbf{p} = \mathbf{b}$.

24. **a.** False. A nontrivial solution of $A\mathbf{x} = \mathbf{0}$ is any nonzero $\mathbf{x}$ that satisfies the equation. See the sentence before Example 2.
b. True. See Example 2 and the paragraph following it.
c. True. If the zero vector is a solution, then $\mathbf{b} = A\mathbf{x} = A\mathbf{0} = \mathbf{0}$.
d. True. See the paragraph following Example 3.
e. False. The statement is true only when the solution set of $A\mathbf{x} = \mathbf{b}$ is nonempty. Theorem 6 applies only to a consistent system.

26. (*Geometric argument using Theorem 6*) Since the equation $A\mathbf{x} = \mathbf{b}$ is consistent, its solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, by Theorem 6. So the solution set of $A\mathbf{x} = \mathbf{b}$ is a single vector if and only if the solution set of $A\mathbf{x} = \mathbf{0}$ is a single vector, and that happens if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*Proof using free variables*) If $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution is unique if and only if there are no free variables in the corresponding system of equations, that is, if and only if every column of A is a pivot column. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

28. No. If the solution set of $A\mathbf{x} = \mathbf{b}$ contained the origin, then $\mathbf{0}$ would satisfy $A\mathbf{0} = \mathbf{b}$, which is not true since $\mathbf{b}$ is not the zero vector.

30. **a.** Yes **b.** No **32.** **a.** Yes **b.** Yes

34. One answer: $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

36. Construct any 3×3 matrix such that the sum of the first and third columns is twice the second column.

38. No. If $A\mathbf{x} = \mathbf{y}$ has no solution, then A cannot have a pivot in each row. Since A is 3×3, it has at most two pivot positions. So the equation $A\mathbf{x} = \mathbf{z}$ for any $\mathbf{z}$ has at most two basic variables and at least one free variable. Thus the solution set for $A\mathbf{x} = \mathbf{z}$ is either empty or has infinitely many elements.

40. Suppose $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Then, since $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ by Theorem 5(a) in Section 1.4,

$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$

Now, let c and d be scalars. Using both parts of Theorem 5,

$A(c\mathbf{u}+d\mathbf{v}) = A(c\mathbf{u}) + A(d\mathbf{v}) = cA\mathbf{u} + dA\mathbf{v} = c\mathbf{0} + d\mathbf{0} = \mathbf{0}$

Section 1.6, page 63

2. Take some other value for p_S, say, 200 million dollars. The other equilibrium prices are then $p_C = 188$ million, $p_E = 170$ million. Any constant nonnegative multiple of these prices is a set of equilibrium prices, because the solution set of the system of equations consists of all multiples of one vector. Changing the unit of measurement to, say, Japanese yen has the same effect as multiplying all equilibrium prices by a constant. The *ratios* of the prices remain the same, no matter what currency is used.

4. a.

	Distribution of Output From:					
	Agric.	Energy	Manuf.	Transp.		Purchased
Output	↓	↓	↓	↓	Input	By:
	.65	.30	.30	.20	→	Agric.
	.10	.10	.15	.10	→	Energy
	.25	.35	.15	.30	→	Manuf.
	0	.25	.40	.40	→	Transp.

b. The data probably justify at most two significant figures. One solution is $p_{\text{Agric.}} = 200$, $p_{\text{Energy}} = 53$, $p_{\text{Manuf.}} = 120$, and $p_{\text{Transp.}} = 100$.

6. $2Na_3PO_4 + 3Ba(NO_3)_2 \to Ba_3(PO_4)_2 + 6NaNO_3$

8. $2KMnO_4 + 3MnSO_4 + 2H_2O \to 5MnO_2 + K_2SO_4 + 2H_2SO_4$

10. $16MnS + 13As_2Cr_{10}O_{35} + 374H_2SO_4 \to 16HMnO_4 + 26AsH_3 + 130CrS_3O_{12} + 327H_2O$

12. a. $\begin{cases} x_1 = 100 + x_3 - x_5 \\ x_2 = 100 - x_3 + x_5 \\ x_3 \text{ is free} \\ x_4 = 60 - x_5 \\ x_5 \text{ is free} \end{cases}$

b. $\begin{cases} x_1 = 40 + x_3 \\ x_2 = 160 - x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \\ x_5 = 60 \end{cases}$

c. Minimum value of x_1 is 40 cars/minute.

14. a. $\begin{cases} x_1 = 100 + x_6 \\ x_2 = x_6 \\ x_3 = 50 + x_6 \\ x_4 = -70 + x_6 \\ x_5 = 80 + x_6 \\ x_6 \text{ is free} \end{cases}$

b. Minimum value of x_6 is 70.

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Answers to Exercises 1–20 are justified in the *Instructor's Solution Manual*.

2. Lin. indep. **4.** Lin. indep. **6.** Lin. indep.

8. Lin. depen. **10. a.** No h **b.** All h

12. All h **14.** $h = -10$ **16.** Lin. depen.

18. Lin. depen. **20.** Lin. depen.

21. a. False. A homogenous system *always* has the trivial solution. See the box before Example 2.
b. False. See the warning after Theorem 7.
c. True. See Fig. 3, after Theorem 8.
d. True. See the remark following Example 4.

22. a. True. See Fig. 1.
b. False. For instance, the set consisting of $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$ is linearly dependent. See the warning after Theorem 8.
c. True. See the remark following Example 4.
d. False. See Example 3(a).

24. $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

26. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$

28. It must have 5 pivot columns. (By Theorem 4 in Section 1.4, A has a pivot in each of its five rows. Since each pivot position is in a different column, A has five pivot columns.)

30. **a.** n

b. The columns of A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens if and only if $A\mathbf{x} = \mathbf{0}$ has no free variables, which in turn happens if and only if every variable is a basic variable, that is, if and only if every column of A is a pivot column.

32. $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ **34.** True, by Theorem 9.

36. False. Counterexample: Take $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$ all to be multiples of one vector. Take $\mathbf{v}_3$ to be *not* a multiple of that vector.

38. True. If the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ had a nontrivial solution (with at least one of x_1, x_2, x_3 nonzero), then so would the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0\cdot\mathbf{v}_4 = \mathbf{0}$. But that can't happen because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be linearly independent. This problem can also be solved using Exercise 37.

40. An $m \times n$ matrix with n pivot columns has a pivot in each column. So the equation $A\mathbf{x} = \mathbf{b}$ has no free variables. If there is a solution, it must be unique.

42. **[M]** Using pivot columns, $B = \begin{bmatrix} 12 & 10 & -3 & 10 \\ -7 & -6 & 7 & 5 \\ 9 & 9 & -5 & -1 \\ -4 & -3 & 6 & 9 \\ 8 & 7 & -9 & -8 \end{bmatrix}$.

Other choices are possible.

44. **[M]** Each column of A that is not a column of B is in the set spanned by the columns of B.

Section 1.8, page 79

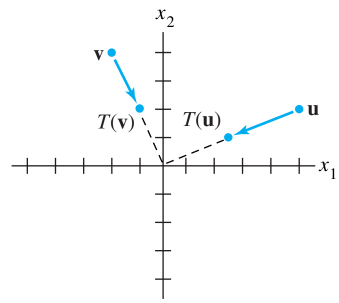
2. $\mathbf{x} = \begin{bmatrix} .5 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} .5a \\ .5b \\ .5c \end{bmatrix}$ **4.** $\mathbf{x} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$, unique solution

6. $\mathbf{x} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$, not unique **8.** 5 rows and 4 columns

10. $x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

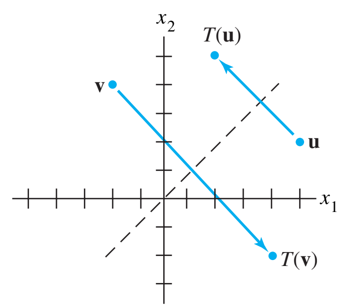
12. No, because the system represented by $[A \quad \mathbf{b}]$ is inconsistent.

14.



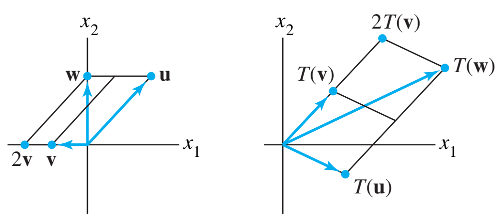
A contraction by the factor .5.

16.



A reflection through the line $x_2 = x_1$.

18.



20. $\begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$

21. **a.** True. Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ are defined before Fig. 2. A linear transformation is a function with certain properties.

b. False. The domain is $\mathbb{R}^5$. See the paragraph before Example 1.

c. False. The range is the set of all linear combinations of the columns of A. See the paragraph before Example 1.

d. False. See the paragraph after the definition of a linear transformation.

e. True. See the paragraph following the box that contains equation (4).

22. **a.** True. See the paragraph following the definition of a linear transformation.

b. False. If A is an $m \times n$ matrix, the codomain is $\mathbb{R}^m$. See the paragraph before Example 1.

c. False. The question is an existence question. See the remark about Example 1(d), following the solution of Example 1.

d. True. See the discussion following the definition of a linear transformation.

e. True. See the paragraph following equation (5).

24. Given any $\mathbf{x}$ in $\mathbb{R}^n$, there are constants $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, because $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$. Then, from property (5) of a linear transformation,

$$T(\mathbf{x}) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = c_1\mathbf{0} + \cdots + c_p\mathbf{0} = \mathbf{0}$$

26. Any point $\mathbf{x}$ on the plane P satisfies the parametric equation $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ for some values of s and t. By linearity, the image $T(\mathbf{x})$ satisfies the parametric equation

$$T(\mathbf{x}) = sT(\mathbf{u}) + tT(\mathbf{v}) \quad (s, t \text{ in } \mathbb{R}) \qquad (*)$$

The set of images is just Span $\{T(\mathbf{u}), T(\mathbf{v})\}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly independent, Span $\{T(\mathbf{u}), T(\mathbf{v})\}$ is a plane through $T(\mathbf{u})$, $T(\mathbf{v})$, and $\mathbf{0}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent and not both zero, then Span $\{T(\mathbf{u}), T(\mathbf{v})\}$ is a line through $\mathbf{0}$. If $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}$, then Span $\{T(\mathbf{u}), T(\mathbf{v})\}$ is $\{\mathbf{0}\}$.

28. Consider a point $\mathbf{x}$ in the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, say $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$ for $0 \le a \le 1$, $0 \le b \le 1$. By linearity of T, the image of $\mathbf{x}$ is

$$\begin{aligned} T(\mathbf{x}) &= T(a\mathbf{u} + b\mathbf{v}) \\ &= aT(\mathbf{u}) + bT(\mathbf{v}), \text{ for } 0 \le a \le 1, 0 \le b \le 1 \qquad (*) \end{aligned}$$

This image point lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$. Special "degenerate" cases arise when $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. See the *Instructor's Solution Manual*.

30. Let $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for $\mathbf{x}$ in $\mathbb{R}^n$. If $\mathbf{b}$ is not zero, $T(\mathbf{0}) = A\mathbf{0} + \mathbf{b} = \mathbf{b} \ne \mathbf{0}$. (Actually, one can show that T fails both properties of a linear transformation.)

32. Take any vector (x_1, x_2) with $x_2 \ne 0$, and use a negative scalar. For instance, $T(0, 1) = (-2, 3)$, but $T(-1 \cdot (0, 1)) = T(0, -1) = (2, 3) \ne (-1) \cdot T(0, 1)$.

34. Suppose $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set in $\mathbb{R}^n$, and yet $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. Then there exist weights c_1, c_2, not both zero, such that

$$c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$$

Because T is linear, $T(c_1\mathbf{u} + c_2\mathbf{v}) = \mathbf{0}$. That is, the vector $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$ satisfies $T(\mathbf{x}) = \mathbf{0}$. Furthermore, $\mathbf{x}$ cannot be the zero vector, since that would mean that a nontrivial linear combination of $\mathbf{u}$ and $\mathbf{v}$ is zero, which is impossible because $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Thus, the equation $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution.

36. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ and let c and d be scalars. Then

$$c\mathbf{u} + d\mathbf{v} = (cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3)$$

The transformation T is linear because

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= (cu_1 + dv_1, 0, cu_3 + dv_3) \\ &= (cu_1, 0, cu_3) + (dv_1, 0, dv_3) \\ &= c(u_1, 0, u_3) + d(v_1, 0, v_3) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

38. **[M]** All multiples of $(-3, -5, 7, 4)$

40. **[M]** Yes. One choice for $\mathbf{x}$ is $(-2, -4, 5, 1)$

Section 1.9, page 90

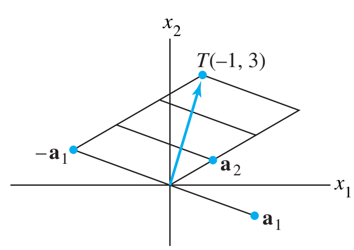
2. $\begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$ **4.** $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

6. $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ **8.** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

12. The transformation T in Exercise 8 maps $\mathbf{e}_1$ into $\mathbf{e}_2$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_1$. A rotation about the origin through $\pi/2$ radians also maps $\mathbf{e}_1$ into $\mathbf{e}_2$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_1$. Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as T on every vector in $\mathbb{R}^2$.

14.



16. $\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix}$ **18.** $\begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

20. $\begin{bmatrix} 2 & 0 & 3 & -4 \end{bmatrix}$ **22.** $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

23. **a.** True. See Theorem 10.
b. True. See Example 3.
c. False. See the paragraph before Table 1.
d. False. See the definition of *onto*. *Any* function from $\mathbb{R}^n$ to $\mathbb{R}^m$ maps each vector onto another vector.
e. False. See Example 5.

24. **a.** False. See the paragraph preceding Example 2.
b. True. See Theorem 10.
c. True. See Table 1.
d. False. See the definition of *one-to-one*. Any *function* from $\mathbb{R}^n$ to $\mathbb{R}^m$ maps a vector onto a single (unique) vector.
e. True. See the solution of Example 5.

26. The transformation in Exercise 2 is not one-to-one, by Theorem 12, because the standard matrix is 2×3 and so has linearly dependent columns. However, the matrix has a pivot in each row and so the columns span $\mathbb{R}^2$. By Theorem 12, the transformation maps $\mathbb{R}^3$ onto $\mathbb{R}^2$.

28. The standard matrix A of the transformation T in Exercise 14 has linearly independent columns, because Figure 6 shows that $\mathbf{a}_1$ and $\mathbf{a}_2$ are not multiples. So T is one-to-one, by Theorem 12. Also, A must have a pivot in each column because the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. Thus, the echelon form of A is $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}$. Since A has a pivot in each row, the columns of A span $\mathbb{R}^2$. So T maps $\mathbb{R}^2$ onto $\mathbb{R}^2$. An alternative argument for the second part is to observe directly from Fig. 6 that $\mathbf{a}_1$ and $\mathbf{a}_2$ span $\mathbb{R}^2$. This is more or less evident, based on experience with grids such as those in Fig. 8 and the figure with Exercises 7 and 8 in Section 1.3.

30. $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix},$
$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$

32. A has m pivot columns if and only if A has a pivot position in each row. By Theorem 4 in Section 1.4, this happens if and only if the columns of A span $\mathbb{R}^m$, and this in turn happens (by Theorem 12) if and only if T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

34. The transformation T maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$ if and only if for each $\mathbf{y}$ in $\mathbb{R}^m$ *there exists* an $\mathbf{x}$ in $\mathbb{R}^n$ such that $\mathbf{y} = T(\mathbf{x})$.

36. Take any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^p$, and let c and d be any scalars. Then

$$\begin{aligned} T(S(c\mathbf{u} + d\mathbf{v})) &= T(c \cdot S(\mathbf{u}) + d \cdot S(\mathbf{v})) && \text{because } S \text{ is linear} \\ &= c \cdot T(S(\mathbf{u})) + d \cdot T(S(\mathbf{v})) && \text{because } T \text{ is linear} \end{aligned}$$

This calculation shows that the mapping $\mathbf{x} \mapsto T(S(\mathbf{x}))$ is linear. See equation (4) in Section 1.8.

38. **[M]** No. There is no pivot in the third column of the standard matrix A, so the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. By Theorem 11, the transformation T is *not* one-to-one.

40. **[M]** No. There is not a pivot in every row, so the columns of the standard matrix do not span $\mathbb{R}^5$. By Theorem 12, the transformation T does *not* map $\mathbb{R}^5$ onto $\mathbb{R}^5$.

Section 1.10, page 99

2. **a.** $B = [\text{COB} \quad \text{Crp}] = \begin{bmatrix} 110 & 110 \\ 3 & 2 \\ 21 & 25 \\ 3 & .4 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

b. **[M]** The system of equations for this problem is inconsistent. The desired nutrients cannot be provided exactly. However, a matrix program can generate an approximate "least-squares" solution, .2443 serving of Oat Bran and .7556 serving of Crispix, that supplies nutrients close to the desired values. (Least-squares solutions are studied in Section 6.5.) *Note:* .25 serving of Oat Bran plus .75 serving of Crispix supplies 110 calories, 2.25 g of protein, 24 g of carbohydrate, and 1.05 g of fat. (The fat is high by 5%.)

4. $x_1 \begin{bmatrix} 10 \\ 50 \\ 30 \end{bmatrix} + x_2 \begin{bmatrix} 20 \\ 40 \\ 10 \end{bmatrix} + x_3 \begin{bmatrix} 20 \\ 10 \\ 40 \end{bmatrix} = \begin{bmatrix} 100 \\ 300 \\ 200 \end{bmatrix}$, where x_1, x_2, and x_3 are the numbers of units of foods 1, 2, and 3, respectively, needed for one meal.

[M] The solution is $x_1 = 150/33 \approx 4.55$, $x_2 = 50/33 \approx 1.52$, $x_3 = 40/33 \approx 1.21$.

6. $\begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 6 & -2 & 0 \\ 0 & -2 & 10 & -3 \\ 0 & 0 & -3 & 12 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ 10 \end{bmatrix}$

[M]: $\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 12.11 \\ 8.44 \\ 4.26 \\ 1.90 \end{bmatrix}$

8. $\begin{bmatrix} 15 & -5 & 0 & -5 & -1 \\ -5 & 15 & -5 & 0 & -2 \\ 0 & -5 & 15 & -5 & -3 \\ -5 & 0 & -5 & 15 & -4 \\ -1 & -2 & -3 & -4 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 40 \\ -30 \\ 20 \\ -10 \\ 0 \end{bmatrix}$

[M]: $\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 3.37 \\ .11 \\ 2.27 \\ 1.67 \\ 1.70 \end{bmatrix}$

10. $\mathbf{x}_{k+1} = M\mathbf{x}_k$, for $k = 0, 1, 2, \ldots$, where $M = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 800{,}000 \\ 500{,}000 \end{bmatrix}$. The population in 2002 (when $k = 2$) is $\mathbf{x}_2 = \begin{bmatrix} 722{,}100 \\ 577{,}900 \end{bmatrix}$.

12. $\mathbf{x}_0 \approx \begin{bmatrix} 304 \\ 48 \\ 98 \end{bmatrix}$, $\mathbf{x}_1 \approx \begin{bmatrix} 307 \\ 48 \\ 95 \end{bmatrix}$, $\mathbf{x}_2 \approx \begin{bmatrix} 310 \\ 48 \\ 92 \end{bmatrix}$

14. **[M]** Here are the solution temperatures (in degrees) for the two figures shown:

(a) (10, 10.0, 10, 10.0)

(b) (10, 17.5, 20, 12.5)

a. At each interior point in the figure for Exercise 34 in Section 1.1, the temperature is the sum of the temperatures at the corresponding interior points here in (a) and (b).

b. If the boundary temperatures in (a) are changed by a factor of 3, the interior temperatures should also change by a factor of 3.

c. The correspondence from the list of eight boundary temperatures to the list of four interior temperatures is a linear transformation. A verification of this statement is not expected.

Chapter 1 Supplementary Exercises, page 102

1. a. False. (The word "reduced" is missing.) Counterexample:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

The matrix A is row equivalent to matrices B and C, both in echelon form.

b. False. Counterexample: Let A be any $n \times n$ matrix with fewer than n pivot columns. Then the equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

c. True. If a linear system has more than one solution, it is a consistent system and has a free variable. By the Existence and Uniqueness Theorem in Section 1.2, the system has infinitely many solutions.

d. False. Counterexample: The following system has no free variables and no solution:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_2 &= 5 \\ x_1 + x_2 &= 2 \end{aligned}$$

e. True. See the box after the definition of elementary row operations, in Section 1.1. If $[A \quad \mathbf{b}]$ is transformed into $[C \quad \mathbf{d}]$ by elementary row operations, then the two augmented matrices are row equivalent.

f. True. Theorem 6 in Section 1.5 essentially says that when $A\mathbf{x} = \mathbf{b}$ is consistent, the solution sets of the nonhomogeneous equation and the homogeneous equation are translates of each other. In this case, the two equations have the same number of solutions.

g. False. For the columns of A to span $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ must be consistent for *all* $\mathbf{b}$ in $\mathbb{R}^m$, not for just one vector $\mathbf{b}$ in $\mathbb{R}^m$.

h. False. *Any* matrix can be transformed by elementary row operations into reduced echelon form.

i. True. If A is row equivalent to B, then A can be transformed by elementary row operations first into B and then further transformed into the reduced echelon form U of B. Since the reduced echelon form of A is unique, it must be U.

j. False. Every equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution whether or not some variables are free.

k. True, by Theorem 4 in Section 1.4. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$, then A must have a pivot position in every one of its m rows. If A has m pivot positions, then A has m pivot columns, each containing one pivot position.

l. False. The word "unique" should be deleted. Let A be any $m \times n$ matrix with m pivot columns but more than m columns altogether. Then the equation $A\mathbf{x} = \mathbf{b}$ is consistent and has m basic variables and at least one free variable. Thus the equation does not have a unique solution.

m. True. If A has n pivot positions, it has a pivot in each of its n columns and in each of its n rows. The reduced echelon form has a 1 in each pivot position, so the reduced echelon form is the $n \times n$ identity marix.

n. True. Both matrices A and B can be row reduced to the 3×3 identity matrix, as discussed in the previous question. Since the row operations that transform B into I_3 are reversible, A can be transformed first into I_3 and then into B.

o. True. The reason is essentially the same as that given for question f.

p. True. If the columns of A span $\mathbb{R}^m$, then the reduced echelon form of A is a matrix U with a pivot in each row, by Theorem 4 in Section 1.4. Since B is row equivalent to A, B can be transformed by row operations first into A and then further transformed into U. Since U has a pivot in each row, so does B. By Theorem 4, the columns of B span $\mathbb{R}^m$.

q. False. See Example 5 in Section 1.7.

r. True. Any set of three vectors in $\mathbb{R}^2$ would have to be linearly dependent, by Theorem 8 in Section 1.7.

s. False. If a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ were to span $\mathbb{R}^5$, then the matrix $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]$ would have a pivot position in each of its five rows, which is impossible since A has only four columns.

t. True. The vector $-\mathbf{u}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, namely, $-\mathbf{u} = (-1)\mathbf{u} + 0\mathbf{v}$.

u. False. If nonzero $\mathbf{u}$ and $\mathbf{v}$ are multiples, then Span$\{\mathbf{u}, \mathbf{v}\}$ is a line, and $\mathbf{w}$ need not be on that line.

v. False. Let $\mathbf{u}$ and $\mathbf{v}$ be any linearly independent pair of vectors and let $\mathbf{w} = 2\mathbf{v}$. Then $\mathbf{w} = 0\mathbf{u} + 2\mathbf{v}$, so $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. However, $\mathbf{u}$ cannot be a linear combination of $\mathbf{v}$ and $\mathbf{w}$ because if it were, $\mathbf{u}$ would be a multiple of $\mathbf{v}$. That is not possible since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

w. False. The statement would be true if the condition $\mathbf{v}_1$ is not zero were present. See Theorem 7 in Section 1.7. However, if $\mathbf{v}_1 = \mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, no matter what else might be true about $\mathbf{v}_2$ and $\mathbf{v}_3$.

x. True. "Function" is another word used for "transformation" (as mentioned in the definition of "transformation" in Section 1.8), and a linear transformation is a special type of transformation.

y. True. For the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to map $\mathbb{R}^5$ onto $\mathbb{R}^6$, the matrix A would have to have a pivot in every row and hence have six pivot columns. This is impossible because A has only five columns.

z. False. For the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to be one-to-one, A must have a pivot in each column. Since A has n columns and m pivots, m might be less than n.

2. If $a \neq 0$, then $x = b/a$; the solution is unique. If $a = 0$, and $b \neq 0$, the solution set is empty, because $0x = 0 \neq b$. If $a = 0$ and $b = 0$, the equation $0x = 0$ has infinitely many solutions.

4. Since there are three pivots (one in each column of A), the *augmented* matrix must reduce to the form

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}.$$ A solution of $A\mathbf{x} = \mathbf{b}$ exists for all $\mathbf{b}$ because there is a pivot in each row of A. Each solution is unique because there are no free variables.

6. a. Set $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. "Determine if $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$." Or, "Determine if $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$." To do this, compute

$$\begin{bmatrix} 4 & -2 & 7 & -5 \\ 8 & -3 & 10 & -3 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 & 7 & -5 \\ 0 & 1 & -4 & 7 \end{bmatrix}$$

The system is consistent, so $\mathbf{b}$ *is* in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

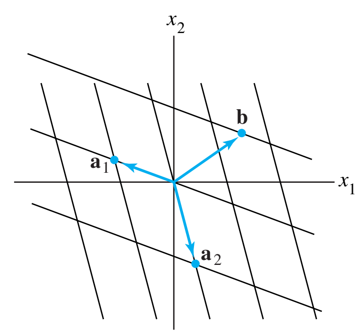
b. Set $A = \begin{bmatrix} 4 & -2 & 7 \\ 8 & -3 & 10 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. "Determine if $\mathbf{b}$ is a linear combination of the columns of A."

c. Define $T(\mathbf{x}) = A\mathbf{x}$. "Determine if $\mathbf{b}$ is in the range of T."

8. a. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \end{bmatrix}$, $\begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \end{bmatrix}$, $\begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

b. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

10. The line through $\mathbf{a}_1$ and the origin and the line through $\mathbf{a}_2$ and the origin determine a "grid" on the x_1x_2-plane, as shown below. Every point in $\mathbb{R}^2$ can be described uniquely in terms of this grid. Thus $\mathbf{b}$ can be reached from the origin by traveling a certain number of units in the $\mathbf{a}_1$-direction and a certain number of units in the $\mathbf{a}_2$-direction.



12. A solution set is a plane where there are two free variables. If the coefficient matrix is 2×3, then only one column can be a pivot column. The echelon form will have all zeros in the second row. Use a row replacement to create a matrix not in echelon form, for instance: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

14. The vectors are linearly independent for all a except $a = 2$ and $a = -1$.

16. Denote the columns from right to left by $\mathbf{v}_1, \ldots, \mathbf{v}_4$. The "first" vector $\mathbf{v}_1$ is nonzero, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ (because the third entry of $\mathbf{v}_2$ is nonzero), and $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ (because the second entry of $\mathbf{v}_3$ is nonzero). Finally, by looking at first entries in the vectors, $\mathbf{v}_4$ cannot be a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. By Theorem 7 in Section 1.7, the columns are linearly independent.

18. Suppose c_1 and c_2 are constants such that

$$c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0} \qquad (*)$$

Then $(c_1 + c_2)\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, both $c_1 + c_2 = 0$ and $c_2 = 0$. It follows that both c_1 and c_2 in $(*)$ must be zero, which shows that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2\}$ is linearly independent.

20. If $T(\mathbf{u}) = \mathbf{v}$, then since T is linear, $T(-\mathbf{u}) = T((-1)\mathbf{u}) = (-1)T(\mathbf{u}) = -\mathbf{v}$.

22. By Theorem 12 in Section 1.9, the columns of A span $\mathbb{R}^3$. By Theorem 4 in Section 1.4, A has a pivot in each of its three rows. Since A has three columns, each column must be a pivot column. So the equation $A\mathbf{x} = \mathbf{0}$ has no free variables, and the columns of A are linearly independent. By Theorem 12 in Section 1.9, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

24. $a = 1/\sqrt{5}$, $b = -2/\sqrt{5}$

CHAPTER 2

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2. $A + 2B = \begin{bmatrix} 16 & -10 & 1 \\ 6 & -13 & -4 \end{bmatrix}$, $3C - E$ is not defined, $CB = \begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix}$, EB is not defined.

4. $A - 5I_3 = \begin{bmatrix} 4 & -1 & 3 \\ -8 & 2 & -6 \\ -4 & 1 & 3 \end{bmatrix}$,

$(5I_3)A = \begin{bmatrix} 45 & -5 & 15 \\ -40 & 35 & -30 \\ -20 & 5 & 40 \end{bmatrix}$

6. a. $A\mathbf{b}_1 = \begin{bmatrix} 0 \\ -3 \\ 13 \end{bmatrix}$, $A\mathbf{b}_2 = \begin{bmatrix} 14 \\ -9 \\ 4 \end{bmatrix}$, $AB = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$

b. $AB = \begin{bmatrix} 4\cdot 1 - 2\cdot 2 & 4\cdot 3 - 2(-1) \\ -3\cdot 1 + 0\cdot 2 & -3\cdot 3 + 0(-1) \\ 3\cdot 1 + 5\cdot 2 & 3\cdot 3 + 5(-1) \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$

8. B has 3 rows. **10.** $AB = AC = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$

12. By inspection of A, a suitable column for B is any multiple of (2, 1). For example: $B = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$.

14. By definition, $UQ\ [U\mathbf{q}_1 \quad \cdots \quad U\mathbf{q}_4]$. From Example 6 of Section 1.8, the first column of UQ lists the total costs for materials, labor, and overhead used to manufacture products B and C during the first quarter of the year. Columns 2, 3, and 4 of UQ list the total amounts spent to manufacture B and C during the 2nd, 3rd, and 4th quarters, respectively.

15. a. False. See the definition of AB.
b. False. The roles of A and B should be reversed in the second half of the statement. See the box after Example 3.
c. True. See Theorem 2(b), read right to left.
d. True. See Theorem 3(b), read right to left.
e. False. The phase "in the same order" should be "in the reverse order." See the box after Theorem 3.

16. a. False. AB must be a 3×3 matrix, but the formula for AB implies that it is 3×1. The plus signs should be just spaces (between columns). This is a common mistake.
b. True. See the box after Example 6.
c. False. The left-to-right order of B and C cannot be changed, in general.
d. False. See Theorem 3(d).
e. True. This general statement follows from Theorem 3(b).

18. The first two columns of AB are $A\mathbf{b}_1$ and $A\mathbf{b}_2$. They are equal because $\mathbf{b}_1$ and $\mathbf{b}_2$ are equal.

20. The second column of AB is also all zeros because $A\mathbf{b}_2 = A\mathbf{0} = \mathbf{0}$.

22. If the columns of B are linearly dependent, then there exists a nonzero vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. From this, $A(B\mathbf{x}) = A\mathbf{0}$ and $(AB)\mathbf{x} = \mathbf{0}$ (by associativity). Since $\mathbf{x}$ is nonzero, the columns of AB must be linearly dependent.

24. Take any $\mathbf{b}$ in $\mathbb{R}^m$. By hypothesis, $AD\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$. Rewrite this equation as $A(D\mathbf{b}) = \mathbf{b}$. Thus, the vector $\mathbf{x} = D\mathbf{b}$ satisfies $A\mathbf{x} = \mathbf{b}$. This proves that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$. By Theorem 4 in Section 1.4, A

has a pivot position in each row. Since each pivot is in a different column, A must have at least as many columns as rows.

26. Write $I_3 = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$ and $D = [\mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3]$. By definition of AD, the equation $AD = I_3$ is equivalent to the three equations $A\mathbf{d}_1 = \mathbf{e}_1$, $A\mathbf{d}_2 = \mathbf{e}_2$, and $A\mathbf{d}_3 = \mathbf{e}_3$. Each of these equations has at least one solution because the columns of A span $\mathbb{R}^3$. (See Theorem 4 in Section 1.4.) Select one solution of each equation, and use them for the columns of D. Then $AD = I_3$.

28. Since the inner product $\mathbf{u}^T\mathbf{v}$ is a real number, it equals its transpose. That is, $\mathbf{u}^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})^T = \mathbf{v}^T(\mathbf{u}^T)^T = \mathbf{v}^T\mathbf{u}$, by Theorem 3(d) regarding the transpose of a product of matrices and by Theorem 3(a). The outer product $\mathbf{u}\mathbf{v}^T$ is an $n \times n$ matrix. By Theorem 3, $(\mathbf{u}\mathbf{v}^T)^T = (\mathbf{v}^T)^T\mathbf{u}^T = \mathbf{v}\mathbf{u}^T$.

30. The (i, j)-entries of $r(AB)$, $(rA)B$, and $A(rB)$ are all equal, because

$$r\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n}(ra_{ik})b_{kj} = \sum_{k=1}^{n} a_{ik}(rb_{kj})$$

32. Let $\mathbf{e}_j$ and $\mathbf{a}_j$ denote the jth columns of I_n and A, respectively. By definition, the jth column of AI_n is $A\mathbf{e}_j$, which is simply $\mathbf{a}_j$ because $\mathbf{e}_j$ has 1 in the jth position and 0's elsewhere. Thus corresponding columns of AI_n and A are equal. Hence $AI_n = A$.

34. By Theorem 3(d), $(AB\mathbf{x})^T = \mathbf{x}^T(AB)^T = \mathbf{x}^TB^TA^T$.

36. **[M]** The answer will depend on the choice of matrix program. In MATLAB, the command `rand(6,4)` creates a 6×4 matrix with random entries uniformly distributed between 0 and 1. The command

```
round(19*(rand(6,4) - .5))
```

creates a random 6×4 matrix with integer entries between -9 and 9. The same result is produced by the command `randomint(6,4)` in the Laydata Toolbox on the text web site. On the TI-86 calculator, the corresponding command is `randM(6,4)`.

38. **[M]** The equality $(AB)^T = A^TB^T$ is very likely to be false for 4×4 matrices selected at random.

40. **[M]** $A^5 = \begin{bmatrix} .3318 & .3346 & .3336 \\ .3346 & .3323 & .3331 \\ .3336 & .3331 & .3333 \end{bmatrix}$,

$$A^{10} = \begin{bmatrix} .333337 & .333330 & .333333 \\ .333330 & .333336 & .333334 \\ .333333 & .333334 & .333333 \end{bmatrix}$$

The entries in A^{20} all agree with .3333333333 to 9 or 10 decimal places. The entries in A^{30} all agree with .33333333333333 to at least 14 decimal places. The matrices appear to approach the matrix

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

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2. $\begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix}$

4. $\frac{1}{4}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -7/4 & 3/4 \end{bmatrix}$

6. $x_1 = 2$ and $x_2 = -5$

8. $AD = I \Rightarrow A^{-1}AD = A^{-1}I \Rightarrow ID = A^{-1} \Rightarrow D = A^{-1}$. Parentheses are routinely suppressed because of the associative property of matrix multiplication.

9. **a.** True, by definition of *invertible.*
b. False. See Theorem 6(b).
c. False. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $ab - cd = 1 - 0 \neq 0$, but Theorem 4 shows that this matrix is not invertible, because $ad - bc = 0$.
d. True. This follows from Theorem 5, which also says that the solution of $A\mathbf{x} = \mathbf{b}$ is unique for each $\mathbf{b}$.
e. True, by the box just before Example 6.

10. **a.** False. The product matrix is invertible, but the product of inverses should be in the reverse order. See Theorem 6(b).
b. True, by Theorem 6(a).
c. True, by Theorem 4.
d. True, by Theorem 7.
e. False. The last part of Theorem 7 is misstated here.

12. Since A is invertible, it can be row reduced to I (Theorem 7). Thus $[A \quad B]$ can be reduced to a matrix of the form $[I \quad X]$ for some X. Let $E_1, \ldots, E_k$ be elementary matrices that implement this row reduction. Then

$$(E_k \cdots E_1)A = I \quad \text{and} \quad (E_k \cdots E_1)B = X \qquad (*)$$

Right-multiply each side of the first equation by A^{-1}, and obtain

$$(E_k \cdots E_1)AA^{-1} = IA^{-1}, \quad (E_k \cdots E_1)I = A^{-1}, \quad \text{and} \quad E_k \cdots E_1 = A^{-1}$$

The second equation in $(*)$ then shows that $A^{-1}B = X$.

14. Right-multiply each side of the equation $(B - C)D = 0$ by D^{-1}, and obtain

$$(B - C)DD^{-1} = 0D^{-1}, \quad (B - C)I = 0$$

Thus $B - C = 0$, and $B = C$.

16. Let $C = AB$. Since B is invertible, use B^{-1} to solve for A:

$$CB^{-1} = ABB^{-1}, \quad CB^{-1} = AI = A$$

This shows that A is the product of invertible matrices and hence is invertible, by Theorem 6.

18. Left-multiply each side of $A = PBP^{-1}$ by P^{-1}:

$$P^{-1}A = P^{-1}PBP^{-1}, \quad P^{-1}A = IBP^{-1}, \quad P^{-1}A = BP^{-1}$$

Then right-multiply each side of the result by P:

$$P^{-1}AP = BP^{-1}P, \quad P^{-1}AP = BI, \quad P^{-1}AP = B$$

20. a. Left-multiply both sides of $(A - AX)^{-1} = X^{-1}B$ by X to see that B is invertible because it is the product of invertible matrices.

b. $X = (A + B^{-1})^{-1}A$. A careful proof should justify $A - AX = B^{-1}X$, and show that $A + B^{-1}$ is invertible.

22. Suppose A is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{b}$ has a solution (in fact, a unique solution) for each $\mathbf{b}$. By Theorem 4 in Section 1.4, the columns of A span $\mathbb{R}^n$.

24. If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then A has a pivot position in each row, by Theorem 4 in Section 1.4. Since A is square, the pivots must be on the diagonal of A. It follows that A is row equivalent to I_n. By Theorem 7, A is invertible.

26.
$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da - bc & 0 \\ 0 & -cb + ad \end{bmatrix}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & -cb + da \end{bmatrix}$$

Divide both sides of each equation by $ad - bc$ to get $CA = I$ and $AC = I$.

28. When row 3 of A is replaced by $\text{row}_3(A) - 4\cdot\text{row}_1(A)$, the result may be written as

$$\begin{bmatrix} \text{row}_1(A) \\ \text{row}_2(A) \\ \text{row}_3(A) - 4\cdot\text{row}_1(A) \end{bmatrix} = \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ \text{row}_3(I)\cdot A - 4\cdot\text{row}_1(I)\cdot A \end{bmatrix}$$
$$= \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ [\text{row}_3(I) - 4\cdot\text{row}_1(I)]\cdot A \end{bmatrix}$$
$$= \begin{bmatrix} \text{row}_1(I) \\ \text{row}_2(I) \\ \text{row}_3(I) - 4\cdot\text{row}_1(I) \end{bmatrix} A = EA$$

Here E is obtained by replacing $\text{row}_3(I)$ by $\text{row}_3(I) - 4\cdot\text{row}_1(I)$.

30. $\begin{bmatrix} -7/5 & 2 \\ 4/5 & -1 \end{bmatrix}$

32. Not invertible

34.
$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1/2 & 1/2 & 0 & & \\ 0 & -1/3 & 1/3 & & \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & & -1/n & 1/n \end{bmatrix}$$

The *Instructor's Solutions Manual* has a proof that this matrix is the desired inverse.

36. [M] Write $B = [A \quad F]$, where F consists of the last two columns of I_3, and row reduce. The last two columns of A^{-1} are

$$\begin{bmatrix} 1.5000 & -4.5000 \\ -72.1667 & 219.5000 \\ 22.6667 & -69.0000 \end{bmatrix}$$

38. $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

There is *no* 4×2 matrix C such that $CA = I_4$. If this were true, then $CA\mathbf{x}$ would equal $\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^4$. This cannot happen because the columns of A are linearly dependent and so $A\mathbf{x} = \mathbf{0}$ for some nonzero vector $\mathbf{x}$. For such an $\mathbf{x}$, $CA\mathbf{x} = C(\mathbf{0}) = \mathbf{0}$. Or, see Exercise 23 or 25 in Section 2.1.

40. [M] $D^{-1} = 125\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} 0 \\ -5 \\ 10 \end{bmatrix}$ pounds

42. [M] The forces at the four points are -104, 167, -113, and 56.0 newtons, respectively (to three significant digits).

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The abbreviation IMT (here and in the *Study Guide*) denotes the Invertible Matrix Theorem (Theorem 8).

2. Not invertible, by Theorem 4 in Section 2.2, because the determinant is zero. Less obvious is the fact that the columns are linearly dependent—the second column is $-3/2$ times the first column. From this and the IMT, it follows that the matrix is singular.

4. The matrix obviously has linearly dependent columns (because one column is zero), and so the matrix is not invertible, by (e) of the IMT.

6. Not invertible, by the IMT. The matrix row reduces to $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ and is not row equivalent to I_3.

8. The 4×4 matrix is invertible, by the IMT, because it is already in echelon form and has four pivot columns.

10. **[M]** The 5×5 matrix is invertible because it has five pivot positions, by the IMT.

11. **a.** True, by the IMT. If statement (d) of the IMT is true, then so is statement (b).

b. True. If statement (h) of the IMT is true, then so is statement (e).

c. False. Statement (g) of the IMT is true only for invertible matrices.

d. True, by the IMT. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then statement (d) of the IMT is false. In this case, all the lettered statements in the IMT are false, including statement (c), which means that A must have fewer than n pivot positions.

e. True, by the IMT. If A^T is not invertible, then statement (l) of the IMT is false, and hence statement (a) must also be false.

12. **a.** True. If statement (k) of the IMT is true, then so is statement (j).

b. True. If statement (e) of the IMT is true, then so is statement (h).

c. True. See the remark immediately following the proof of the IMT.

d. False. The first part of the statement is not part (i) of the IMT. In fact, if A is any $n \times n$ matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, yet not every such matrix has n pivot positions.

e. True, by the IMT. If there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then statement (g) of the IMT is false, and hence statement (f) is also false. That is, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

Note: The answers below for Exercises 14–30 refer mostly to the IMT. In many cases, part or all of an acceptable answer could also be based on various results that were used to establish the IMT.

14. If A is lower triangular with nonzero entries on the diagonal, then these n diagonal entries can be used as pivots to produce zeros below the diagonal. Thus A has n pivots and so is invertible, by the IMT. If one of the diagonal entries in A is zero, A will have fewer than n pivots and hence will be singular.

16. No, because statement (h) of the IMT is then false. A 5×5 matrix cannot be invertible when its columns do not span $\mathbb{R}^5$.

18. By (g) of the IMT, C is invertible. Hence each equation $C\mathbf{x} = \mathbf{v}$ has a unique solution, by Theorem 5 in Section 2.2.

20. By the box following the IMT, E and F are invertible and are inverses. So $FE = I = EF$. Thus E and F commute.

22. Statement (g) of the IMT is false for H, so statement (d) is false, too. That is, the equation $H\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

24. The equation $L\mathbf{x} = \mathbf{0}$ *always* has the trivial solution. This fact gives no information about the columns of L.

26. If the columns of A are linearly independent, then A is invertible, by the IMT. So A^2, which is the product of invertible matrices, is invertible. By the IMT, the columns of A^2 span $\mathbb{R}^n$.

28. Let W be the inverse of AB. Then $WAB = I$ and $(WA)B = I$. By statement (j) of the IMT, applied to B in place of A, the matrix B is invertible, since it is square.

30. Since the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, statement (f) of the IMT is true. Then statement (i) is also true and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Also, A is invertible, which implies that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible, by Theorem 9.

32. If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A must have a pivot in each of its n columns. Since A is square, there must be a pivot in each *row* of A. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

34. The standard matrix of T is $A = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}$, which is invertible because $\det A = 2 \neq 0$. By Theorem 9, T is invertible, and $T^{-1}(\mathbf{x}) = B\mathbf{x}$, where $B = A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}$.

36. If T maps $\mathbb{R}^n$ onto $\mathbb{R}^n$, then the columns of its standard matrix A span $\mathbb{R}^n$, by Theorem 12 in Section 1.9. By the IMT, A is invertible. Hence, by Theorem 9 in Section 2.3, T is invertible, and A^{-1} is the standard matrix of T^{-1}. Since A^{-1} is also invertible, by the IMT, its columns are linearly independent and span $\mathbb{R}^n$. Applying Theorem 12 in Section 1.9 to the transformation T^{-1}, we conclude that T^{-1} is a one-to-one mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

38. Let A be the standard matrix of T. By hypothesis, T is not a one-to-one mapping. So, by Theorem 12 in Section 1.9, the standard matrix A of T has linearly dependent columns.

Since A is square, the columns of A do not span $\mathbb{R}^n$, by the IMT. By Theorem 12, again, T cannot map $\mathbb{R}^n$ onto $\mathbb{R}^n$.

40. Given $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^n$, let $\mathbf{x} = S(\mathbf{u})$ and $\mathbf{y} = S(\mathbf{v})$. Then $T(\mathbf{x}) = T(S(\mathbf{u})) = \mathbf{u}$ and $T(\mathbf{y}) = T(S(\mathbf{v})) = \mathbf{v}$, by equation (2). Hence

$$\begin{aligned} S(\mathbf{u}+\mathbf{v}) &= S(T(\mathbf{x}) + T(\mathbf{y})) \\ &= S(T(\mathbf{x}+\mathbf{y})) && \text{Because } T \text{ is linear} \\ &= \mathbf{x}+\mathbf{y} && \text{By equation (1)} \\ &= S(\mathbf{u}) + S(\mathbf{v}) \end{aligned}$$

So S preserves sums. For any scalar r,

$$\begin{aligned} S(r\mathbf{u}) &= S(rT(\mathbf{x})) = S(T(r\mathbf{x})) && \text{Because } T \text{ is linear} \\ &= r\mathbf{x} && \text{By equation (1)} \\ &= rS(\mathbf{u}) \end{aligned}$$

So S preserves scalar multiples. Thus S is a linear transformation.

42. **[M]** $\text{cond}(A) \approx 23683$, which is approximately 10^4. If you make several trials with MATLAB, which records 16 digits accurately, you should find that $\mathbf{x}$ and $\mathbf{x}_1$ agree to at least 12 or 13 significant digits. So about 4 significant digits are lost.

44. **[M]** Solve $A\mathbf{x} = (0, 0, 0, 0, 1)$. MATLAB shows that $\text{cond}(A) \approx 4.8 \times 10^5$. With MATLAB, the entries in the computed value of $\mathbf{x}$ should be accurate to at least 11 digits.

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2. $\begin{bmatrix} EA & EB \\ FC & FD \end{bmatrix}$ **4.** $\begin{bmatrix} A & B \\ -XA + C & -XB + D \end{bmatrix}$

6. Assume that A, X, C, and Z are square. Then $X = A^{-1}$, and $Z = C^{-1}$, by the IMT, and $Y = -C^{-1}BA^{-1}$.

8. Assume that A and X are square. Then $X = A^{-1}$, by the IMT, $Y = 0$, and $Z = -A^{-1}B$.

10. $X = -A + BC$, $Y = -B$, $Z = -C$

11. **a.** True. See the definition (1) in the paragraph preceding Example 4.
b. False. See Example 3. The number of columns of A_{11} and A_{12} must match the number of rows of B_1 and B_2, respectively.

12. **a.** True. See the definition (1) in the paragraph preceding Example 4.
b. False. Both BA and AB are defined, although they have different dimensions. In fact, $AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{bmatrix}$, which is the block analogue of an outer product. See Example 4.

14. The calculations in Example 5 showed that if A is invertible, then both A_{11} and A_{22} are invertible. Conversely, suppose A_{11} and A_{22} are both invertible, and define B to be the matrix that Example 5 says should be the inverse of A. A routine calculation shows that $AB = I$. Since A is square, the IMT implies that A is invertible. (Alternatively, one could also show that $BA = I$.)

16. The inverse of $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ is $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$. Similarly, $\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$ has an inverse. From equation (7), one obtains

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \qquad (*)$$

If A is invertible, then the matrix on the right side of $(*)$ is a product of invertible matrices and hence is invertible. By Exercise 13, A_{11} and S must be invertible.

18. The Schur complement of X^TX is $\mathbf{x}_0^T\mathbf{x}_0 - (\mathbf{x}_0^TX)(X^TX)^{-1}(X^T\mathbf{x}_0) = \mathbf{x}_0^T(I_m - X(X^TX)^{-1}X^T)\mathbf{x}_0 = \mathbf{x}_0^TM\mathbf{x}_0$.

20. The Schur complement of $A - BC - sI_n$ is $I_m + C(A - BC - sI_n)^{-1}B$. *Note:* The proof that this function actually is the inverse of the $W(s)$ in Exercise 19 involves only matrix algebra, but it is a little tricky. The following algebraic identity is needed:

$$\begin{aligned} CU^{-1}B - CV^{-1}B &= C(U^{-1} - V^{-1})B \\ &= CU^{-1}(V - U)V^{-1}B \end{aligned}$$

for any invertible $n \times n$ matrices U and V and any B and C such that the multiplication is well defined.

22. Let C be any nonzero 2×3 matrix, and define $A = \begin{bmatrix} I_3 & O \\ C & -I_2 \end{bmatrix}$.

24. Let

$$A_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & & 0 \\ 1 & 1 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

$$B_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

By direct computation, $A_2B_2 = I_2$. Assume that for $n = k$, the matrix $A_kB_k = I_k$, and write

$$A_{k+1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{v} & A_k \end{bmatrix} \text{ and } B_{k+1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{w} & B_k \end{bmatrix}$$

where $\mathbf{v}$ and $\mathbf{w}$ are in $\mathbb{R}^k$, $\mathbf{v}^T = [1 \quad 1 \quad \cdots \quad 1]$, and $\mathbf{w}^T = [-1 \quad 0 \quad \cdots \quad 0]$. Then

$$A_{k+1}B_{k+1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{v} & A_k \end{bmatrix}\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{w} & B_k \end{bmatrix} = \begin{bmatrix} 1 + \mathbf{0}^T\mathbf{w} & \mathbf{0}^T + \mathbf{0}^T B_k \\ \mathbf{v} + A_k\mathbf{w} & \mathbf{v}\mathbf{0}^T + A_kB_k \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & I_k \end{bmatrix} = I_{k+1}$$

The (2, 1) entry is $\mathbf{0}$ because $\mathbf{v}$ equals the first column in A_k, and $A_k\mathbf{w}$ is -1 times the first column of A_k.

26. [M] The commands to be used in these exercises will depend on the matrix program. See the *Instructor's Solution Manual.*

Section 2.5, page 149

2. $L\mathbf{y} = \mathbf{b} \Rightarrow \mathbf{y} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, U\mathbf{x} = \mathbf{y} \Rightarrow \mathbf{x} = \begin{bmatrix} 1/4 \\ 2 \\ 1 \end{bmatrix}$

4. $\mathbf{y} = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$

6. $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}$

8. $LU = \begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ 0 & -1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix}\begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -4 & 2 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

18. $L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix},$

$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix},$

$U^{-1} = \frac{1}{6}\begin{bmatrix} 3 & -1 & -2 \\ 0 & -2 & 8 \\ 0 & 0 & 6 \end{bmatrix}$, and

$A^{-1} = U^{-1}L^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -3 & -2 \\ -14 & 6 & 8 \\ -6 & 6 & 6 \end{bmatrix}$

20. Since L is unit lower triangular, it is invertible and may be row reduced to I by adding suitable multiples of a row to the rows below it, beginning with the top row. If elementary matrices $E_1, \ldots, E_p$ implement these row operations, then

$$E_p \cdots E_1A = (E_p \cdots E_1L)U = IU = U$$

This shows that A may be row reduced to U using only row-replacement operations.

22. $B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 2 & -1 \\ -3 & -3 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$; if $A = LU$, with only three nonzero rows in U, use the first three columns of L for B and the top three rows of U for C.

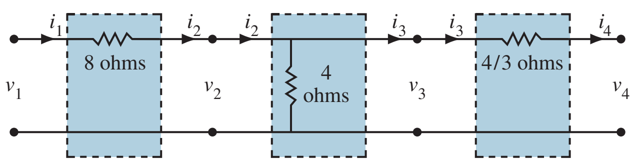
24. Since Q is square and $Q^TQ = I$, Q is invertible and $Q^{-1} = Q^T$, by the Invertible Matrix Theorem. Thus A is the product of invertible matrices and hence is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$. From $A\mathbf{x} = \mathbf{b}$, we have $QR\mathbf{x} = \mathbf{b}$, $Q^TQR\mathbf{x} = Q^T\mathbf{b}$, $R\mathbf{x} = Q^T\mathbf{b}$, and $\mathbf{x} = R^{-1}Q^T\mathbf{b}$. A good algorithm for finding $\mathbf{b}$ is to compute $Q^T\mathbf{b}$ and then row reduce $[\,R \quad Q^T\mathbf{b}\,]$. (See Exercise 12 in Section 2.2.) The reduction is fast because R is triangular.

26. In general, $A^k = PD^kP^{-1}$, where

$D^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2^k & 0 \\ 0 & 0 & 1/3^k \end{bmatrix}.$

28. $\begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} =$ $\begin{bmatrix} 1 & 0 \\ -(1/R_1 + 1/R_2 + 1/R_3) & 1 \end{bmatrix}$. The single shunt resistance is $\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$.

30. The transfer matrix of the circuit below is
$$\begin{bmatrix} 1 + R_3/R_2 & -R_1 - R_3 - R_1R_3/R_2 \\ -1/R_2 & 1 + R_1/R_2 \end{bmatrix}.$$
Set that matrix equal to $A = \begin{bmatrix} 4/3 & -12 \\ -1/4 & 3 \end{bmatrix}$, and solve to find $R_1 = 8$ ohms, $R_2 = 4$ ohms, $R_3 = 4/3$ ohms.



32. [M] a. $L = \begin{bmatrix} 1 & & & & & \\ -1/3 & 1 & & & & \\ & -3/8 & 1 & & & \\ & & -8/21 & 1 & & \\ & & & -21/55 & 1 & \end{bmatrix}$

$U = \begin{bmatrix} 3 & -1 & & & \\ & 8/3 & -1 & & \\ & & 21/8 & -1 & \\ & & & 55/21 & -1 \\ & & & & 144/55 \end{bmatrix}$

b. Let $\mathbf{s}_k$ satisfy $L\mathbf{s}_k = \mathbf{t}_{k-1}$. Then $\mathbf{t}_k$ satisfies $U\mathbf{t}_k = \mathbf{s}_k$.

$\mathbf{t}_0 = \begin{bmatrix} 10 \\ 12 \\ 12 \\ 12 \\ 10 \end{bmatrix}$; $\mathbf{s}_1 = \begin{bmatrix} 10.0000 \\ 15.3333 \\ 17.7500 \\ 18.7619 \\ 17.1636 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 6.5556 \\ 9.6667 \\ 10.4444 \\ 9.6667 \\ 6.5556 \end{bmatrix}$;

$\mathbf{s}_2 = \begin{bmatrix} 6.5556 \\ 11.8519 \\ 14.8889 \\ 15.3386 \\ 12.4121 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 4.7407 \\ 7.6667 \\ 8.5926 \\ 7.6667 \\ 4.7407 \end{bmatrix}$; $\mathbf{s}_3 = \begin{bmatrix} 4.7407 \\ 9.2469 \\ 12.0602 \\ 12.2610 \\ 9.4222 \end{bmatrix}$,

$\mathbf{t}_3 = \begin{bmatrix} 3.5988 \\ 6.0556 \\ 6.9012 \\ 6.0556 \\ 3.5988 \end{bmatrix}$; $\mathbf{s}_4 = \begin{bmatrix} 3.5988 \\ 7.2551 \\ 9.6219 \\ 9.7210 \\ 7.3104 \end{bmatrix}$, $\mathbf{t}_4 = \begin{bmatrix} 2.7922 \\ 4.7778 \\ 5.4856 \\ 4.7778 \\ 2.7922 \end{bmatrix}$

Section 2.6, page 156

Note: Exercises 2, 3, and 4 could be used for students to discover the linearity of the Leontief model.

2. $\begin{bmatrix} 33.33 \\ 35.00 \\ 15.00 \end{bmatrix}$ **4.** $\begin{bmatrix} 73.33 \\ 50.00 \\ 30.00 \end{bmatrix}$ **6.** $\begin{bmatrix} 50 \\ 45 \end{bmatrix}$

8. a. Since $\mathbf{x}$ satisfies $(I - C)\mathbf{x} = \mathbf{d}$ and $\Delta\mathbf{x}$ satisfies $(I - C)\Delta\mathbf{x} = \Delta\mathbf{d}$, linearity of matrix multiplication shows that
$$(I - C)(\mathbf{x} + \Delta\mathbf{x}) = (I - C)\mathbf{x} + (I - C)\Delta\mathbf{x} = \mathbf{d} + \Delta\mathbf{d}$$
which means that $\mathbf{x} + \Delta\mathbf{x}$ satisfies the production equation for a demand of $\mathbf{d} + \Delta\mathbf{d}$.

b. If $\Delta\mathbf{x}$ satisfies $(I - C)\Delta\mathbf{x} = \Delta\mathbf{d}$, then $\Delta\mathbf{x} = (I - C)^{-1}\Delta\mathbf{d}$, which is the first column of $(I - C)^{-1}$ in the case when $\Delta\mathbf{d}$ is the first column of I.

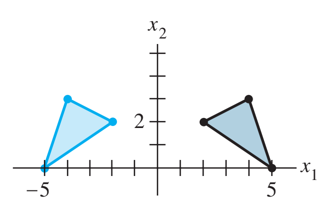
10. By the argument in Exercise 8, the effect of raising the demand for the output of one sector of the economy is given by the entries in the corresponding column of $(I - C)^{-1}$. When these entries are all positive, every sector must increase its output by some positive (though possibly small) quantity. So an increase in demand for *any* sector will increase the demand for *every* sector.

12. $D_{m+1} = I + CD_m$

14. [M] $x = (134034, 131687, 69472, 176912, 66596, 443773, 18431)$. In view of the remarks for Exercise 13, a realistic answer might be $\mathbf{x} = 1000 \times (134, 132, 69, 177, 67, 444, 18)$.

Section 2.7, page 165

2. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -2 & -4 \\ 0 & 2 & 3 \end{bmatrix}$



4. $\begin{bmatrix} .8 & 0 & -1.6 \\ 0 & 1.2 & 3.6 \\ 0 & 0 & 1 \end{bmatrix}$

6. $\begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ -1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 3+2\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2}/2 & 7-5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$

10. D commutes with R but not with T; R does not commute with T.

12. Two identities: $\tan\varphi/2 = \frac{1-\cos\varphi}{\sin\varphi} = \frac{\sin\varphi}{1+\cos\varphi}$. The first identity shows that $1 - (\tan\varphi/2)(\sin\varphi) = \cos\varphi$, and hence

$$\begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\tan\varphi/2 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The second identity shows that

$$(\cos\varphi)(-\tan\varphi/2) - \tan\varphi/2 = -(\cos\varphi + 1)(\tan\varphi/2) = -\sin\varphi$$

Hence

$$\begin{bmatrix} \cos\varphi & -\tan\varphi/2 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

14. The matrix from Exercise 7 may be written as

$$\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 3+4\sqrt{3} \\ \sqrt{3}/2 & 1/2 & 4-3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3+4\sqrt{3} \\ 0 & 1 & 4-3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is a rotation through 60° about the origin, followed by a translation by $(3+4\sqrt{3}, 4-3\sqrt{3})$.

16. Both $(1, -2, 3, 4)$ and $(10, -20, 30, 40)$ are homogeneous coordinates for $(1/4, -1/2, 3/4)$ because of the formulas $x = X/H$, $y = Y/H$, and $z = Z/H$.

18. $\begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 5 \\ -1/2 & \sqrt{3}/2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

20. The triangle with vertices $(6, 2, 0)$, $(15, 10, 0)$, $(2, 3, 0)$.

22. **[M]** $\begin{bmatrix} 1.0031 & .9548 & .6179 \\ .9968 & -.2707 & -.6448 \\ 1.0085 & -1.1105 & 1.6996 \end{bmatrix}\begin{bmatrix} Y \\ I \\ Q \end{bmatrix} = \begin{bmatrix} R \\ G \\ B \end{bmatrix}$

Section 2.8, page 173

2. The set is closed under scalar multiplication, but not sums. For instance, the sum of $(1, 0)$ and $(0, -1)$ is not in the set.

4. The set is closed under sums, but not under multiplication by a negative scalar.

6. No **8.** Yes **10.** Yes, $A\mathbf{u} = \mathbf{0}$.

12. $p = 3$, $q = 4$. Nul A is a subspace of $\mathbb{R}^3$ because solutions of $A\mathbf{x} = \mathbf{0}$ must have three entries, to match the columns of A. Col A is a subspace of $\mathbb{R}^4$ because each column vector has four entries.

14. Nul A: $\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$, or any nonzero multiples of this vector
Col A: any column of A

16. No. One vector is a multiple of the other, so they are linearly dependent and hence cannot be a basis for any subspace.

18. Yes. Let A be the matrix whose columns are the vectors given. Row reduction of A shows three pivots, so A is invertible by the IMT and its columns form a basis for $\mathbb{R}^3$ (as pointed out in Example 5).

20. No. The vectors are linearly dependent because there are more vectors in the set than entries in each vector (Theorem 8 in Section 1.7). So the vectors cannot be a basis for any subspace.

21. **a.** False. See the definition at the beginning of the section. The critical phrases "for each" are missing. (This is a common student error!)
b. True. See the paragraph before Example 4.
c. False. See Theorem 12. The null space is a subspace of $\mathbb{R}^n$, not $\mathbb{R}^m$.
d. True. See Example 5.
e. True. See the first part of the solution of Example 8.

22. **a.** False. See the definition at the beginning of the section. The condition about the zero vector is only one of the conditions for a subspace.
b. True. See Example 3.
c. True. See Theorem 12.
d. False. See the paragraph after Example 4.
e. False. See the Warning that follows Theorem 13.

24. Basis for Col A: $\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$

Basis for Nul A: $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.5 \\ 0 \\ -1.25 \\ 1 \end{bmatrix}$

26. Basis for Col A: $\begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix}$

Basis for Nul A: $\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2.5 \\ -1.5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

28. The easiest construction is to write a 3×3 matrix in echelon form that has only two pivots, and let $\mathbf{b}$ be any vector in $\mathbb{R}^3$ whose third entry is nonzero.

30. Since Col A is the set of all linear combinations of $\mathbf{a}_1, \dots, \mathbf{a}_p$, the set $\{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ spans Col A. Because $\{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ is also linearly independent, it is a basis for Col A. (There is no need to discuss pivot columns and Theorem 13, though a proof could be given using this information.)

32. If Nul R contains nonzero vectors, then the equation $R\mathbf{x} = \mathbf{0}$ has nontrivial solutions. Since R is square, the IMT shows that R is not invertible and the columns of R do not span $\mathbb{R}^6$. So Col R is a subspace of $\mathbb{R}^6$, but Col $R \neq \mathbb{R}^6$.

34. If Nul $P = \{\mathbf{0}\}$, then the equation $P\mathbf{x} = \mathbf{0}$ has only the trivial solution. Since P is square, the IMT shows that P is invertible and the equation $P\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^5$. Also, each solution is unique, by Theorem 5 in Section 2.2.

36. If the columns of A form a basis, they are linearly independent. This means that A cannot have more columns than rows. Since the columns also span $\mathbb{R}^m$, A must have a pivot in each row, which means that A cannot have more rows than columns. As a result, A must be a square matrix.

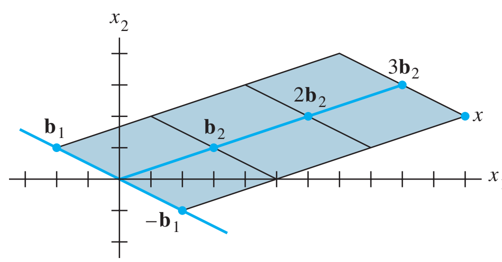
38. **[M]** Display the reduced echelon form of A, and select the pivot columns of A as a basis for Col A. For Nul A, write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

Basis for Col A: $\begin{bmatrix} 5 \\ 4 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 6 \end{bmatrix}$

Basis for Nul A: $\begin{bmatrix} -60 \\ 154 \\ 47 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -122 \\ 309 \\ 94 \\ 0 \\ 1 \end{bmatrix}$

Section 2.9, page 180

2. $\mathbf{x} = (-1)\mathbf{b}_1 + 3\mathbf{b}_2 = (-1)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$



4. $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ **6.** $\begin{bmatrix} -5/2 \\ 1/2 \end{bmatrix}$

8. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, [\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}, [\mathbf{z}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -.5 \end{bmatrix}$

10. Basis for Col A: $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix}$; dim Col $A = 3$

Basis for Nul A: $\begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$; dim Nul $A = 2$

12. Col A: $\begin{bmatrix} 1 \\ 5 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -9 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ -6 \end{bmatrix}$; dim Col $A = 3$

Basis for Nul A: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$; dim Nul $A = 2$

14. Let A be the matrix whose columns are given, and let $H = \text{Col } A$. Columns 1 and 2 of A form a basis for H, so $\dim H = 2$.

16. Col A *cannot* be $\mathbb{R}^3$ because the columns of A have 4 entries. (In fact, Col A is a three-dimensional subspace of $\mathbb{R}^4$, because the 3 pivot columns of A form a basis for Col A.) Since A has 7 columns and 3 pivot columns, the equation $A\mathbf{x} = \mathbf{0}$ has 4 free variables. So, $\dim \text{Nul } A = 4$.

17. **a.** True. This is the definition of a $\mathcal{B}$-coordinate vector.
b. False. A line must be through the origin in $\mathbb{R}^n$ to be a subspace of $\mathbb{R}^n$.
c. True. The sentence before Example 1 concludes that the number of pivot columns of A is the rank of A, which is the dimension of Col A by definition.
d. True. This is equivalent to the Rank Theorem because rank A *is* the dimension of Col A.
e. True, by the Basis Theorem. In this case, the spanning set is automatically a linearly independent set.

18. **a.** True. This fact is justified in the second paragraph of this section.
b. True. See the second paragraph after Fig. 1.
c. False. The dimension of Nul A is the number of *free* variables in the equation $A\mathbf{x} = \mathbf{0}$. See Example 2.
d. True, by the definition of *rank*.
e. True, by the Basis Theorem. In this case, the linearly independent set is automatically a spanning set.

20. A 4×5 matrix A has five columns. By the Rank Theorem, rank $A = 5 - \dim \text{Nul } A$. Since the null space is three-dimensional, rank $A = 2$.

22. Let H be a four-dimensional subspace spanned by a set S of five vectors. If S were linearly independent, it would be a basis for H. This is impossible, by the statement just before the definition of *dimension* in Section 2.9, which essentially says that *every* basis of a p-dimensional subspace consists of p vectors. Thus, S must be linearly dependent.

24. A rank 1 matrix has a one-dimensional column space. Every column is a multiple of some fixed vector. To construct a 4×3 matrix, choose any nonzero vector in $\mathbb{R}^4$, and use it for one column. Choose any multiples of the vector for the other two columns.

26. If columns $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5$, and $\mathbf{a}_6$ of A are linearly independent and if $\dim \text{Col } A = 4$, then $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5, \mathbf{a}_6\}$ is a linearly independent set in a four-dimensional column space. By the Basis Theorem, this set of four vectors is a basis for the column space.

28. If $\mathcal{A}$ contained more vectors than $\mathcal{B}$, then $\mathcal{A}$ would be linearly dependent, by Exercise 27, because $\mathcal{B}$ spans W. Repeat the argument with $\mathcal{B}$ and $\mathcal{A}$ interchanged to conclude that $\mathcal{B}$ cannot contain more vectors than $\mathcal{A}$.

30. **[M]** The first three columns of $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{x}]$ are pivot columns and so form a basis for H. The fourth column of the reduced echelon form of this matrix shows that the $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is (3, 5, 2).

Chapter 2 Supplementary Exercises, page 183

1. **a.** True. If A and B are $m\times n$, then B^T has as many rows as A has columns, so AB^T is defined. Also, A^TB is defined because A^T has m columns and B has m rows.
b. False. B must have two columns. A has as many columns as B has rows.
c. True. The ith row of A has the form $(0, \ldots, d_i, \ldots, 0)$. So the ith row of AB is $(0, \ldots, d_i, \ldots, 0)B$, which is d_i times the ith row of B.
d. False. Take the zero matrix for B. Or, construct a matrix B such that the equation $B\mathbf{x} = \mathbf{0}$ has nontrivial solutions, and construct C and D so that $C \neq D$ and the columns of $C - D$ satisfy the equation $B\mathbf{x} = \mathbf{0}$. Then $B(C - D) = 0$ and $BC = BD$.
e. False. Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
f. False. $(A + B)(A - B) = A^2 - AB + BA - B^2$. This equals $A^2 - B^2$ if and only if A commutes with B.
g. True. An $n\times n$ replacement matrix has $n + 1$ nonzero entries. The $n\times n$ scale and interchange matrices have n nonzero entries.
h. True. The transpose of an elementary matrix is an elementary matrix of the same type.
i. True. An $n\times n$ elementary matrix is obtained by a row operation on I_n.
j. False. Elementary matrices are invertible, so a product of such matrices is invertible. But not every square matrix is invertible.
k. True. If A is 3×3 with three pivot positions, then A is row equivalent to I_3.
l. False. A must be square in order to conclude from the equation $AB = I$ that A is invertible.
m. False. AB is invertible, but $(AB)^{-1} = B^{-1}A^{-1}$, and this product is not always equal to $A^{-1}B^{-1}$.
n. True. Given $AB = BA$, left-multiply by A^{-1} to get $B = A^{-1}BA$, and then right-multiply by A^{-1} to obtain $BA^{-1} = A^{-1}B$.

o. False. The correct equation is $(rA)^{-1} = r^{-1}A^{-1}$, because $(rA)(r^{-1}A^{-1}) = (rr^{-1})(AA^{-1}) = 1 \cdot I = I$.

p. True. If the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, then there are no free variables in this equation, which means that A must have three pivot positions (since A is 3×3). By the Invertible Matrix Theorem, A is invertible.

2. $C = (C^{-1})^{-1} = \dfrac{1}{-2}\begin{bmatrix} 7 & -5 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} -7/2 & 5/2 \\ 3 & -2 \end{bmatrix}$

4. $I + A + A^2 + \cdots + A^{n-1}$

6. By computation, $A^2 = I$, $B^2 = I$, and

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -BA$$

7. See Exercise 12 in Section 2.2.

8. By definition of matrix multiplication, the matrix A satisfies

$$A\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

Right-multiply both sides by the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$:

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}$$

10. Since A is invertible, so is A^T, by the Invertible Matrix Theorem. Then A^TA is the product of invertible matrices and so is invertible. Thus, the formula $(A^TA)^{-1}A^T$ makes sense. By Theorem 6 in Section 2.2,

$$(A^TA)^{-1} \cdot A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}I = A^{-1}$$

An alternative calculation: $(A^TA)^{-1}A^T \cdot A = (A^TA)^{-1}(A^TA) = I$. Since A is invertible, this equation shows that its inverse is $(A^TA)^{-1}A^T$.

11. c. When $x_1, \ldots, x_n$ are distinct, the columns of V are linearly independent, by (b). By the Invertible Matrix Theorem, V is invertible and its columns span $\mathbb{R}^n$. So, for every $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, there is a vector $\mathbf{c}$ such that $V\mathbf{c} = \mathbf{y}$. Let p be the polynomial whose coefficients are listed in $\mathbf{c}$. Then, by (a), p is an interpolating polynomial for $(x_1, y_1), \ldots, (x_n, y_n)$.

12. If $A = LU$, then $\text{col}_1(A) = L \cdot \text{col}_1(U)$. Since $\text{col}_1(U)$ has a zero in every entry except possibly the first, $L \cdot \text{col}_1(U)$ is a linear combination of the columns of L in which all weights except possibly the first are zero. So $\text{col}_1(A)$ is a multiple of $\text{col}_1(L)$.

Similarly, $\text{col}_2(A) = L \cdot \text{col}_2(U)$, which is a linear combination of the columns of L using the first two entries in $\text{col}_2(U)$ as weights, because the other entries in $\text{col}_2(U)$ are zero. Thus $\text{col}_2(A)$ is a linear combination of the first two columns of L.

14. $P\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$, $Q\mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$

16. Since A is not invertible, there is a nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$. Place n copies of $\mathbf{v}$ into an $n \times n$ matrix B. Then $AB = A[\mathbf{v} \ \cdots \ \mathbf{v}] = [A\mathbf{v} \ \cdots \ A\mathbf{v}] = 0$.

18. Suppose $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$. Then $CA\mathbf{x} = C\mathbf{b}$. Since $CA = I$, $\mathbf{x}$ must be $C\mathbf{b}$. This shows that $C\mathbf{b}$ is the only solution of $A\mathbf{x} = \mathbf{b}$.

20. **[M]** If J denotes the $n \times n$ matrix of 1's, then

$$A_n = J - I_n \quad \text{and} \quad A_n^{-1} = \frac{1}{n-1} \cdot J - I_n$$

Proof: Observe that $J^2 = nJ$ and $A_nJ = (J - I)J = J^2 - J = (n-1)J$. Now compute $A_n((n-1)^{-1}J - I) = (n-1)^{-1}A_nJ - A_n = J - (J - I) = I$. Since A_n is square, A_n is invertible and its inverse is $(n-1)^{-1}J - I$.

CHAPTER 3

Section 3.1, page 190

2. 2 **4.** 20 **6.** 1 **8.** −11

10. −6. Start with row 2.

12. 36. Start with row 1 or column 4.

14. 9. Start with row 4 or column 5.

16. 2 **18.** 20

20. $ad - bc$, $a(kd) - b(kc) = k(ad - bc)$. Scaling a row by k multiplies the determinant by k.

22. $ad - bc$, $(ad + kcd) - (bc + kdc) = ad - bc$. Row replacement does not change a determinant.

24. $2a - 6b + 3c$, $-2a + 6b - 3c$. Interchanging two rows reverses the sign of the determinant.

26. 1 **28.** k **30.** −1

32. k. A scaling matrix is diagonal, with k on the diagonal and with 1's as the other diagonal entries. The determinant is the product of the diagonal entries.

34. $\det EA = \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = akd - bkc = k(ad - bc)$

$= (\det E)(\det A)$

36. $\det EA = \begin{vmatrix} a & b \\ ka + c & kb + d \end{vmatrix} = a(kb + d) - b(ka + c)$

$= akb + ad - bka - bc = (+1)(ad - bc)$

$= (\det E)(\det A)$

38. $\det kA = k^2 \cdot \det A$

39. **a.** True. See the paragraph preceding the definition of det A.

b. False. See the definition of cofactor, preceding Theorem 1.

40. **a.** False. See Theorem 1.

b. False. See Theorem 2.

42. The area of the parallelogram and the determinant of $[\mathbf{v} \quad \mathbf{u}]$ are both bc. The determinant of $[\mathbf{u} \quad \mathbf{v}]$ is $-bc$. Both matrices determine the same parallelogram, with base of length c and height b.

44. **[M]** Theorem 6 in Section 3.2 will show that $\det AB = (\det A)(\det B)$.

46. **[M]** If A is invertible, then $\det A \neq 0$, by Theorem 4 in Section 3.2. Students will be asked in Exercise 31 of Section 3.2 to prove that $\det A^{-1} = 1/(\det A)$.

Section 3.2, page 199

2. A constant may be factored out of one row.

4. A row replacement operation does not change the determinant.

6. -18 **8.** 0 **10.** 24 **12.** 114

14. 0 **16.** 21 **18.** 7 **20.** 7

22. Not invertible **24.** Linearly independent

26. Linearly dependent

27. **a.** True. Theorem 3(a).

b. False. If scaling operations are used to produce U, then the formula described may not give det A. See the paragraph following Example 2.

c. True. See the remark following Theorem 4.

d. False. See the warning after Example 5.

28. **a.** True. By Theorem 3(b), the first interchange changes only the sign of the determinant, so the second interchange restores the original sign of the determinant.

b. False. True when A is triangular (Theorem 2 in Section 3.1).

c. False. The conditions described provide only some cases when det A is zero. See the paragraph after Theorem 4.

d. False. See Theorem 5.

30. If two rows are equal, interchange them. This doesn't change the matrix, but the sign of the determinant is reversed. This is possible only if the determinant is zero. The result about columns can be explained the same way, or one can remark that if A has two equal columns, then A^T has two equal rows. In this case, $\det A^T = 0$. So $\det A = 0$, too, by Theorem 5.

32. $\det(rA) = r^n \cdot \det A$

34. $\det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1})$ By Theorem 6

$= (\det P)(\det A)(\det P)^{-1}$ By Exercise 31

$= \det A$

36. $0 = \det A^4 = (\det A)^4$, by Theorem 6. So $\det A = 0$, which implies that A is not invertible, by Theorem 4.

38. $\det AB = \det \begin{bmatrix} 6 & 0 \\ -2 & 0 \end{bmatrix} = 0$

$(\det A)(\det B) = (-6 + 6)(-4 + 2) = 0$

40. **a.** -2

b. 32

c. -16

d. 1

e. -1

42. $\det(A + B) = \det \begin{bmatrix} 1 + a & b \\ c & 1 + d \end{bmatrix} =$

$1 + a + d + ad - bc$. Also $\det A + \det B = 1 + (ad - bc)$. Since $\det(A + B) - (\det A + \det B) = a + d$, we have $\det(A + B) = \det A + \det B$ if and only if $a + d = 0$.

44. $\det AE = \det(AE)^T$ Theorem 5

$= \det E^T A^T$ Section 2.1

$= (\det E^T)(\det A^T)$ Theorem 6

$= (\det E)(\det A)$ Theorem 5 used twice

46. **[M]** For A as in Exercise 9 of Section 2.3, $\det A = 1$ and cond $A = 23683$. Although A is nearly singular, it has an inverse:

$$A^{-1} = \begin{bmatrix} -19 & -14 & 0 & 7 \\ -549 & -401 & -2 & 196 \\ 267 & 195 & 1 & -95 \\ -278 & -203 & -1 & 99 \end{bmatrix}$$

The determinant is sensitive to scaling, but the condition number does not change:

$\det(10A) = 10^4(-1)$, $\det(0.1A) = 10^{-4}(-1)$, but

$\operatorname{cond}(10A) = \operatorname{cond}(0.1A) = \operatorname{cond} A$

The same things happen when $A = I_4$.

Section 3.3, page 209

2. $\begin{bmatrix} 5/3 \\ -2/3 \end{bmatrix}$ **4.** $\begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$ **6.** $\begin{bmatrix} -4 \\ 13 \\ -1 \end{bmatrix}$

8. All real s; $x_1 = \dfrac{3s+2}{3(s^2+3)}$, $x_2 = \dfrac{2s-9}{5(s^2+3)}$

10. $s \neq 0, 1/4$; $x_1 = \dfrac{6s-2}{3s(4s-1)}$, $x_2 = \dfrac{1}{3(4s-1)}$

12. $\operatorname{adj} A = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}$, $A^{-1} = \dfrac{1}{5}\begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}$

14. $\operatorname{adj} A = \begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix}$, $A^{-1} = (-1)\begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix}$

16. $\operatorname{adj} A = \begin{bmatrix} -9 & -6 & 14 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix}$,

$A^{-1} = -\dfrac{1}{9}\begin{bmatrix} -9 & -6 & 14 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix}$

18. Each cofactor in A is an integer because it is just a sum of products of entries of A. Hence all the entries in adj A are integers. Since $\det A = 1$, the inverse formula in Theorem 8 shows that all the entries in A^{-1} are integers.

20. 7 **22.** 21 **24.** 15

26. By definition, $\mathbf{p} + S$ is the set of all vectors of the form $\mathbf{p} + \mathbf{v}$, where $\mathbf{v}$ is in S. Applying T to a typical vector in $\mathbf{p} + S$, we have $T(\mathbf{p} + \mathbf{v}) = T(\mathbf{p}) + T(\mathbf{v})$. This vector is in the set denoted by $T(\mathbf{p}) + T(S)$. This proves that T maps the set $\mathbf{p} + S$ into the set $T(\mathbf{p}) + T(S)$.

Conversely, any vector in $T(\mathbf{p}) + T(S)$ has the form $T(\mathbf{p}) + T(\mathbf{v})$ for some $\mathbf{v}$ in S. This vector may be written as $T(\mathbf{p} + \mathbf{v})$. This shows that every vector in $T(\mathbf{p}) + T(S)$ is the image under T of some point in $\mathbf{p} + S$.

28. Use Theorem 10. Or, compute the vectors that determine the image, namely, the columns of

$$A[\mathbf{b}_1 \quad \mathbf{b}_2] = \begin{bmatrix} 14 & 2 \\ -3 & 1 \end{bmatrix}$$

The determinant of this matrix is 20.

30. Let $\mathbf{p} = (x_3, y_3)$ and let $R' = R - \mathbf{p}$. The vertices of R' are $\mathbf{v} = (x_1 - x_3, y_1 - y_3)$, $\mathbf{v}_2 = (x_2 - x_3, y_2 - y_3)$, and the origin. Then

$$\begin{aligned} \{\text{area of } R\} &= \{\text{area of } R'\} \\ &= \frac{1}{2}\left\{\begin{matrix}\text{area of parallelogram} \\ \text{determined by } \mathbf{v}_1 \text{ and } \mathbf{v}_2\end{matrix}\right\} \\ &= \frac{1}{2}\left|\det\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}\right| \qquad (1) \end{aligned}$$

Also, using row operations, we get

$$\begin{aligned} \det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} &= \det\begin{bmatrix} x_1 - x_3 & y_1 - y_3 & 0 \\ x_2 - x_3 & y_2 - y_3 & 0 \\ x_3 & y_3 & 1 \end{bmatrix} \\ &= \det\begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix} \\ &= \det\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} \end{aligned}$$

This calculation and (1) give the desired result.

32. From the formula in the exercise,

$\{\text{volume of } S\} = \frac{1}{3}\{\text{area of base}\}\cdot\{\text{height}\} = \frac{1}{6}$

because the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ have unit length. The tetrahedron S' with vertices at $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is the image of S under the linear transformation T such that $T(\mathbf{e}_1) = \mathbf{v}_1$, $T(\mathbf{e}_2) = \mathbf{v}_2$, and $T(\mathbf{e}_3) = \mathbf{v}_3$. The standard matrix for T is $A = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,]$. By Theorem 10,

$\{\text{volume of } S'\} = |\det A|\cdot\frac{1}{6} = \frac{1}{6}|\det[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,]|$

34. **[M]** MATLAB:

```
x2 = det([A(:,1)  b  A(:,3:4)])/det(A)
```

Chapter 3 Supplementary Exercises, page 211

1. **a.** True. The columns of A are linearly dependent.
b. True. See Exercise 30 in Section 3.2.
c. False. See Theorem 3(c); in this case $\det 5A = 5^3 \det A$.
d. False. Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, and $A + B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$.
e. False. By Theorem 6, $\det A^3 = 2^3$.
f. False. See Theorem 3(b).
g. True. See Theorem 3(c).
h. True. See Theorem 3(a).
i. False. See Theorem 5.
j. False. See Theorem 3(c); this statement is false for $n \times n$ invertible matrices with n an even integer.
k. True. See Theorems 6 and 5; $\det A^TA = (\det A)^2$.
l. False. The coefficient matrix must be invertible.

m. False. The area of the triangle is 5.
n. True. See Theorem 6; det $A^3 = (\det A)^3$.
o. False. See Exercise 31 in Section 3.2.
p. True. See Theorem 6.

The solutions for Exercises 2 and 4 are based on the fact that if a matrix contains two rows (or two columns) that are multiples of each other, then the determinant of the matrix is zero, by Theorem 4, because the matrix cannot be invertible.

2. $\begin{vmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{vmatrix} = \begin{vmatrix} 12 & 13 & 14 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$

4. $\begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & x & x \\ y & y & y \end{vmatrix}$

$$= xy \begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

6. 12

8. $\det \begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 0 & 1 & m \end{bmatrix} = 0$. When the determinant is expanded by cofactors of the first row, the equation has the form $1\cdot(mx_1 - y_1) - x(m) + y\cdot 1 = 0$, which can be written as $y - y_1 = m(x - x_1)$.

10. An expansion of the determinant along the top row of V will show that $f(t)$ has the form $f(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$ where, by Exercise 9,

$$c_3 = \det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$
$$= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2) \neq 0$$

So $f(t)$ is a cubic polynomial in t. The points $(x_1, 0)$, $(x_2, 0)$, and $(x_3, 0)$ are on the graph of f, because when x_1, x_2, or x_3 are substituted for t in V, the matrix V will have two rows the same and hence have a zero determinant. That is, $f(x_1) = 0$ for $i = 1, 2, 3$.

12. A 2×2 matrix A is invertible if and only if the parallelogram determined by the columns of A has nonzero area.

14. a. An expansion by cofactors along the last row shows that for $1 \le k \le n$,

$$\det \begin{bmatrix} A & O \\ O & I_k \end{bmatrix}$$
$$= 0 + \cdots + 0 + (-1)^{(n+k)+(n+k)} \cdot 1 \cdot \det \begin{bmatrix} A & O \\ O & I_{k-1} \end{bmatrix}$$

When $k = 1$, we interpret I_0 as having *no* rows or columns. Chaining these equalities together gives

$$\det \begin{bmatrix} A & O \\ O & I_k \end{bmatrix} = \cdots = \det \begin{bmatrix} A & O \\ O & I_2 \end{bmatrix}$$
$$= \det \begin{bmatrix} A & O \\ O & 1 \end{bmatrix} = \det A$$

b. An expansion by cofactors along the first row shows that for $1 \le k \le n$,

$$\det \begin{bmatrix} I_k & O \\ C_k & D \end{bmatrix} = 1 \cdot \det \begin{bmatrix} I_{k-1} & O \\ C_{k-1} & D \end{bmatrix}$$

where $C_n = C$ and C_{k-1} is formed by deleting the first column in C_k. Chaining these equalities together as in (a) produces the desired equation.

c. Observe that

$$\begin{bmatrix} A & O \\ C & D \end{bmatrix} = \begin{bmatrix} A & O \\ O & I \end{bmatrix} \begin{bmatrix} I & O \\ C & D \end{bmatrix}$$

From the multiplicative property of determinants and parts (a) and (b),

$$\det \begin{bmatrix} A & O \\ C & D \end{bmatrix} = \det \begin{bmatrix} A & O \\ O & I \end{bmatrix} \cdot \det \begin{bmatrix} I & O \\ C & D \end{bmatrix}$$
$$= (\det A)(\det D)$$

We have proved that *the determinant of a block lower triangular matrix is the product of the determinants of its diagonal entries* (assuming square diagonal entries). The second part of (c) follows from the first part and the fact that the determinant of a matrix equals the determinant of its transpose:

$$\det \begin{bmatrix} A & B \\ O & D \end{bmatrix} = \det \begin{bmatrix} A & B \\ O & D \end{bmatrix}^T$$
$$= \det \begin{bmatrix} A^T & O \\ B^T & D^T \end{bmatrix}$$
$$= (\det A^T)(\det D^T)$$
$$= (\det A)(\det D)$$

16. a. Row replacement operations do not change the determinant of A. The resulting matrix is

$$\begin{bmatrix} a-b & -a+b & 0 & \cdots & 0 \\ 0 & a-b & -a+b & & 0 \\ 0 & 0 & a-b & & 0 \\ \vdots & & & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix}$$

b. Since column replacement operations are equivalent to row operations on A^T and $\det A^T = \det A$, column replacement operations do not change the determinant of the matrix. The resulting matrix is

$$\begin{bmatrix} a-b & 0 & 0 & \cdots & 0 \\ 0 & a-b & 0 & & 0 \\ 0 & 0 & a-b & & 0 \\ \vdots & & & \ddots & \vdots \\ b & 2b & 3b & \cdots & a+(n-1)b \end{bmatrix}$$

c. Since the preceding matrix is a lower triangular matrix with the same determinant as A,

$\det A = (a-b)^{n-1}(a+(n-1)b)$

18. [M] a. $(3-8)^3[3+(3)8] = -3375$

b. $(8-3)^4[8+(4)3] = 12{,}500$

20. [M] Compute:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 6 \end{vmatrix} = 6, \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 6 & 6 \\ 1 & 3 & 6 & 9 \end{vmatrix} = 18,$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 3 & 6 & 6 & 6 \\ 1 & 3 & 6 & 9 & 9 \\ 1 & 3 & 6 & 9 & 12 \end{vmatrix} = 54 = 18 \cdot 3$$

Conjecture:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 3 & & 3 \\ 1 & 3 & 6 & & 6 \\ \vdots & & & \ddots & \vdots \\ 1 & 3 & 6 & \cdots & 3(n-1) \end{vmatrix} = 2 \cdot 3^{n-2}$$

To confirm the conjecture, use row replacement operations to create zeros below the first pivot and then below the second pivot. The resulting matrix is

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & 2 & 2 & 2 & & 2 \\ 0 & 0 & 3 & 3 & 3 & 3 & & 3 \\ 0 & 0 & 3 & 6 & 6 & 6 & & 6 \\ 0 & 0 & 3 & 6 & 9 & 9 & & 9 \\ \vdots & & & & & \ddots & \vdots & \\ 0 & 0 & 3 & 6 & 9 & 12 & \cdots & 3(n-2) \end{vmatrix}$$

This matrix has the same determinant as the original matrix, and is recognizable as a block matrix of the form

$$\begin{bmatrix} A & B \\ O & D \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & 3 & 3 & 3 & \cdots & 3 \\ 3 & 6 & 6 & 6 & & 6 \\ 3 & 6 & 9 & 9 & & 9 \\ \vdots & & & & \ddots & \vdots \\ 3 & 6 & 9 & 12 & \cdots & 3(n-2) \end{bmatrix}$$

$$= 3\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & 2 & & 2 \\ 1 & 2 & 3 & 3 & & 3 \\ \vdots & & & & \ddots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & n-2 \end{bmatrix}$$

Use Exercise 14(c) to find that the determinant of the matrix $\begin{bmatrix} A & B \\ O & D \end{bmatrix}$ is $(\det A)(\det D) = 2 \det D$, and then use Exercise 32 in Section 3.2 and Exercise 19 above to show that $\det D = 3^{n-2}$.

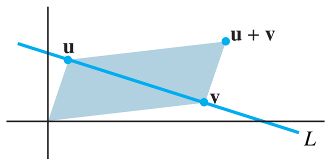
CHAPTER 4

Section 4.1, page 223

2. a. Given $\begin{bmatrix} x \\ y \end{bmatrix}$ in W and any scalar c, the vector $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ is in W because $(cx)(cy) = c^2(xy) \geq 0$, since $xy \geq 0$.

b. *Example:* If $\mathbf{u} = \begin{bmatrix} -1 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $\mathbf{u}$ and $\mathbf{v}$ are in W, but $\mathbf{u} + \mathbf{v}$ is not in W.

4.



u and **v** are on the line, but **u** + **v** is not.

6. No, the zero polynomial is not in the set.

8. Yes. The zero vector is in the set, H. If $\mathbf{p}$ and $\mathbf{q}$ are in H, then $(\mathbf{p}+\mathbf{q})(0) = \mathbf{p}(0) + \mathbf{q}(0) = 0$, so $\mathbf{p}+\mathbf{q}$ is in H. Also, for any scalar c, $(c\mathbf{p})(0) = c\cdot\mathbf{p}(0) = c\cdot 0 = 0$, so $c\mathbf{p}$ is in H.

10. $H = \text{Span}\{\mathbf{v}\}$, where $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. By Theorem 1, H is a subspace of $\mathbb{R}^3$.

12. $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$. By Theorem 1, W is a subspace of $\mathbb{R}^4$.

14. No, because the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}$ has no solution, as revealed by an echelon form of the augmented matrix for this equation.

16. Not a vector space because the zero vector is not in W

18. $S = \left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

20. a. The constant function $\mathbf{f}(t) = 0$ is continuous. The sum of two continuous functions is continuous. A constant multiple of a continuous function is continuous.

22. Yes. See the proof of Theorem 12 in Section 2.8 for a proof that is similar to the one needed here.

23. a. False. The zero vector in V is the function $\mathbf{f}$ whose values $\mathbf{f}(t)$ are zero *for all* t in $\mathbb{R}$. See Example 5.

b. False. See the definition of a vector. An arrow in three-dimensional space is an example of a vector, but not every vector is such an arrow.

c. False. Exercises 1, 2, and 3 each provide an example of a subset that contains the zero vector but is not a subspace.

d. True. See the paragraph before Example 6.

e. False. Digital signals are used. See Example 3.

24. a. True. See the definition of a vector space.

b. True. See statement (3) in the box before Example 1.

c. True. See the paragraph before Example 6.

d. False. See Example 8.

e. False. The second and third parts of the conditions are stated incorrectly. In part (ii) here, for example, there is no statement that **u** and **v** represent all possible elements of H.

26. a. 3 **b.** 5 **c.** 4

28. a. 4 **b.** 7 **c.** 3 **d.** 5 **e.** 4

30. $\mathbf{u} = 1\cdot\mathbf{u}$ Axiom 10
$= c^{-1}c\cdot\mathbf{u} = c^{-1}(c\mathbf{u})$ Axiom 9
$= c^{-1}\mathbf{0} = \mathbf{0}$ Property (2)

32. Both H and K contain the zero vector of V because they are subspaces of V. Hence $\mathbf{0}$ is in $H \cap K$. Take $\mathbf{u}$ and $\mathbf{v}$ in $H \cap K$. Then $\mathbf{u}$ and $\mathbf{v}$ are in both H and K. Since H is a subspace, $\mathbf{u}+\mathbf{v}$ is in H. Likewise, $\mathbf{u}+\mathbf{v}$ is in K. Hence $\mathbf{u}+\mathbf{v}$ is in $H \cap K$. For any scalar c, the vector $c\mathbf{u}$ is in both H and K because they are subspaces. Hence $c\mathbf{u}$ is in $H \cap K$. Thus $H \cap K$ is a subspace.

The union of two subspaces is not, in general, a subspace. In $\mathbb{R}^2$, let H be the x-axis and K the y-axis. The sum of a nonzero vector in H and a nonzero vector in K is not on either the x-axis or the y-axis. So $H \cup K$ is not closed under vector addition, and $H \cup K$ is not a subspace of $\mathbb{R}^2$.

34. A proof that $H + K = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ has two parts. First, one must show that $H + K$ is a subset of $\text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$. Second, one must show that $\text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is a subset of $H + K$.

(1) A typical vector H has the form $c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$ and a typical vector in K has the form $d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q$. The sum of these two vectors is a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_p, \mathbf{v}_1, \dots, \mathbf{v}_q$ and so belongs to $\text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$. Thus $H + K$ is a subset of $\text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$.

(2) Each of the vectors $\mathbf{u}_1, \dots, \mathbf{u}_p, \mathbf{v}_1, \dots, \mathbf{v}_q$ belongs to $H + K$, by Exercise 33(b), and so any linear combination of these vectors belongs to $H + K$, since $H + K$ is a subspace, by Exercise 33(a). Thus, $\text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is a subset of $H + K$.

36. [M] An echelon form of $[A \ \ \mathbf{y}]$ shows that $A\mathbf{x} = \mathbf{y}$ is consistent. In fact, $\mathbf{x} = (5.5, -2, 3.5)$.

38. [M] The functions are $\sin 3t$, $\cos 4t$, and $\sin 5t$.

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2. $\begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $\mathbf{w}$ is in Nul A.

4. $\begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ **6.** $\begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

8. W is not a subspace because $\mathbf{0}$ is not in W. The vector (0, 0, 0) does not satisfy the condition $5r - 1 = s + 2t$.

10. W is a subspace of $\mathbb{R}^4$ by Theorem 2, because W is the set of solutions of the homogeneous system

$$\begin{aligned} a + 3b - c \quad\;\; &= 0 \\ a + \; b + c - d &= 0 \end{aligned}$$

12. If $(b - 5d, 2b, 2d + 1, d)$ were the zero vector, then $2d + 1 = 0$ and $d = 0$, which is impossible. So $\mathbf{0}$ is not in W, and W is not a subspace.

14. $W = \text{Col } A$ for $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -6 \end{bmatrix}$, so W is a vector space by Theorem 3.

16. $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

18. a. 3 **b.** 4 **20. a.** 5 **b.** 1

22. $\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ in Nul A, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in Col A. Other answers are possible.

24. $\mathbf{w}$ is in both Nul A and Col A. $A\mathbf{w} = \mathbf{0}$, and $\mathbf{w} = -\frac{1}{2}\mathbf{a}_1 + \mathbf{a}_2$.

25. a. True, by the definition before Example 1.
b. False. See Theorem 2.
c. True. See the remark just before Example 4.
d. False. The equation $A\mathbf{x} = \mathbf{b}$ must be consistent *for every* $\mathbf{b}$. See #7 in the table on p. 232.
e. True. See Fig. 2. (A subspace is itself a vector space.)
f. True. See the remark after Theorem 3.

26. a. True. See Theorem 2. (A subspace is itself a vector space.)
b. True. See Theorem 3.
c. False. See the box after Theorem 3.
d. True. See the paragraph after the definition of a linear transformation.
e. True. See Fig. 2. (A subspace is itself a vector space.)
f. True. See the paragraph before Example 8.

28. The two systems have the form $A\mathbf{x} = \mathbf{v}$ and $A\mathbf{x} = 5\mathbf{v}$. Since the first system is consistent, $\mathbf{v}$ is in Col A. Since Col A is a subspace of $\mathbb{R}^3$, $5\mathbf{v}$ is also in Col A. Thus the second system is consistent.

30. The zero vector $\mathbf{0}_W$ of W is in the range of T, because the linear transformation maps the zero vector of V to $\mathbf{0}_W$. Typical vectors in the range of T are $T(\mathbf{x})$ and $T(\mathbf{w})$, where $\mathbf{x}$, $\mathbf{w}$ are in V. Since T is a linear transformation,

$T(\mathbf{x}) + T(\mathbf{w}) = T(\mathbf{x} + \mathbf{w})$ In the range of T

Thus the range of T is closed under vector addition. Also, for any scalar c, $c \cdot T(\mathbf{x}) = T(c\mathbf{x})$, since T is a linear transformation. Thus $c \cdot T(\mathbf{x})$ is in the range of T, so the range is closed under scalar multiplication. Hence the range of T is a subspace of W.

32. $\mathbf{p}_1(t) = t$, $\mathbf{p}_2(t) = t^2$. The range of T is $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} : a \text{ real} \right\}$.

34. The kernel of T is $\{\mathbf{0}\}$.

36. Since Z is a subspace of W, the zero vector $\mathbf{0}_W$ of W is in Z. Because T is linear, T maps the zero vector $\mathbf{0}_V$ of V to $\mathbf{0}_W$. Thus $\mathbf{0}_V$ is in $U = \{\mathbf{x} : T(\mathbf{x}) \text{ is in } Z\}$. Now take $\mathbf{u}_1$, $\mathbf{u}_2$ in U. Since T is linear,

$$T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2) \qquad (*)$$

By definition of Z, $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$ are in Z, and so the sum on the right of $(*)$ is in Z because Z is a subspace. This proves that $\mathbf{u}_1 + \mathbf{u}_2$ is in U, so U is closed under vector addition. For any scalar c, $c \cdot T(\mathbf{u}_1)$ is in Z because Z is a subspace. Since T is linear, $T(c\mathbf{u}_1)$ is in Z. Hence $c\mathbf{u}_1$ is in U. Thus U is a subspace of V.

37. [M] $\mathbf{w}$ is in Col A. In fact, $\mathbf{w} = A\mathbf{x}$ for

$\mathbf{x} = (1/95, -20/19, -172/95, 0)$

$\mathbf{w}$ is not in Nul A because $A\mathbf{w} = (14, 0, 0, 0)$.

38. [M] $\mathbf{w}$ is in Col A and in Nul A because $\mathbf{w} = A\mathbf{x}$ for $\mathbf{x} = (-2, 3, 0, 1)$, and $A\mathbf{w} = (0, 0, 0, 0)$.

39. [M] The reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. Most students will row reduce $[\,B \quad \mathbf{a}_3\,]$ and $[\,B \quad \mathbf{a}_5\,]$ to show that the equations $B\mathbf{x} = \mathbf{a}_3$ and $B\mathbf{x} = \mathbf{a}_5$ are consistent. You can use a discussion of this part to lead into Examples 8 and 9 in Section 4.3.

b. The method of Example 3 produces $(-1/3, -1/3, 1, 0, 0)$ and $(-10/3, 26/3, 0, 4, 1)$.

c. This part reviews Section 1.9. An echelon form of A shows that the columns of A are linearly dependent and do not span $\mathbb{R}^4$. By Theorem 12 in Section 1.9, T is not one-to-one and T does not map $\mathbf{R}^5$ onto $\mathbf{R}^4$.

40. **[M]** Row reduction of $[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad -\mathbf{v}_3 \quad -\mathbf{v}_4 \quad \mathbf{0}\,]$ yields

$$\begin{bmatrix} 1 & 0 & 0 & -10/3 & 0 \\ 0 & 1 & 0 & 26/3 & 0 \\ 0 & 0 & 1 & -4 & 0 \end{bmatrix}$$

The general solution is a multiple of $(10, -26, 12, 3)$. One choice for $\mathbf{w}$ is $10\mathbf{v}_1 - 26\mathbf{v}_2$ ($= 12\mathbf{v}_3 + 3\mathbf{v}_4$), which is $(24, -48, -24)$. Another choice is $\mathbf{w} = (1, -2, -1)$.

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2. No, the set is linearly dependent because the zero vector is in the set. The columns of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ do not span $\mathbb{R}^3$, by the Invertible Matrix Theorem.

4. Yes. See Example 5 for an example of a justification.

6. No, $\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$. The matrix does not have a pivot in each row, so its columns do not span $\mathbb{R}^3$ and hence do not form a basis. However, the columns are linearly independent because they are not multiples. (More precisely, neither column is a multiple of the other.)

8. No, the vectors are linearly dependent because there are more vectors than entries in each vector. However, the vectors do span $\mathbb{R}^3$.

10. $\begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

14. Basis for Nul A: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix}$

Basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix}$

16. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ **18.** **[M]** $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$

20. The three simplest answers are $\{\mathbf{v}_1, \mathbf{v}_2\}$ or $\{\mathbf{v}_1, \mathbf{v}_3\}$ or $\{\mathbf{v}_2, \mathbf{v}_3\}$. Other answers are possible.

21. **a.** False. The zero vector by itself is linearly dependent. See the paragraph preceding Theorem 4.
b. False. The set $\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ must also be linearly independent. See the definition of a basis.
c. True. See Example 3.
d. False. See the subsection “Two Views of a Basis.”
e. False. See the box before Example 9.

22. **a.** False. The subspace spanned by the set must also coincide with H. See the definition of a basis.
b. True, by the Spanning Set Theorem, applied to V instead of H. (V is nonzero because the spanning set uses nonzero vectors.)
c. True. See the subsection “Two Views of a Basis.”
d. False. See two paragraphs before Example 8.
e. False. See the warning after Theorem 6.

24. Let $A = [\,\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n\,]$. Since A is square and its columns are linearly independent, its columns also span $\mathbb{R}^n$, by the Invertible Matrix Theorem. So $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$.

26. A basis is $\{\sin t, \sin 2t\}$ because this set is linearly independent (by inspection), and $\sin t \cos t = \frac{1}{2}\sin 2t$, as pointed out in Example 2.

28. $\{e^{-bt}, te^{-bt}\}$. The set is linearly independent because neither function is a *scalar* multiple of the other, and the set spans H.

30. There are more vectors than there are entries in each vector. By Theorem 8 in Section 1.6, the set is linearly dependent and therefore cannot be a basis for $\mathbb{R}^n$.

32. Suppose that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is linearly dependent. Then there exist $c_1, \dots, c_p$, not all zero, such that

$$c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = \mathbf{0}$$

Since T is linear and $\mathbf{0} = T(\mathbf{0})$,

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = T(\mathbf{0})$$

By hypothesis, T is one-to-one, so this equation implies that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$, which shows that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

34. By inspection, $\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2$, or $\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{0}$. By the Spanning Set Theorem, Span$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ = Span$\{\mathbf{p}_1, \mathbf{p}_2\}$. Since neither $\mathbf{p}_1$ nor $\mathbf{p}_2$ is a multiple of the other, they are linearly independent and hence $\{\mathbf{p}_1, \mathbf{p}_2\}$ is a basis for Span$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

36. [M] Row reducing $[\mathbf{u}_1\ \ \mathbf{u}_2\ \ \mathbf{u}_3]$ shows that $\mathbf{u}_1$ and $\mathbf{u}_2$ are the pivot columns of this matrix. Thus $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for H.

Row reducing $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ shows that $\mathbf{v}_1$ and $\mathbf{v}_2$ are the pivot columns of this matrix. Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for K.

Row reducing $[\mathbf{u}_1\ \ \mathbf{u}_2\ \ \mathbf{u}_3\ \ \mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ shows that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{v}_1$ are the pivot columns of this matrix. Thus $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1\}$ is a basis for $H + K$.

38. [M] For example, writing

$$c_1 \cdot 1 + c_2 \cdot \cos t + c_3 \cdot \cos^2 t + c_4 \cdot \cos^3 t + c_5 \cdot \cos^4 t + c_6 \cdot \cos^5 t + c_7 \cdot \cos^6 t = 0$$

with $t = 0, .1, .2, .3, .4, .5, .6$ gives a 7×7 coefficient matrix A for the homogeneous system $A\mathbf{c} = \mathbf{0}$. The matrix A is invertible, so the system $A\mathbf{c} = \mathbf{0}$ has only the trivial solution and $\{1, \cos t, \cos^2 t, \cos^3 t, \cos^4 t, \cos^5 t, \cos^6 t\}$ is a linearly independent set of functions.

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2. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ **4.** $\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$ **6.** $\begin{bmatrix} -6 \\ 2 \end{bmatrix}$ **8.** $\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}$ **12.** $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$ **14.** $\begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$

15. a. True, by definition of the $\mathcal{B}$-coordinate vector.
b. False. See equation (4).
c. False. $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$. See Example 5.

16. a. True. See Example 2.
b. False. By definition, the coordinate mapping goes in the reverse direction.
c. True, when the plane passes through the origin, as in Example 7.

18. Since $\mathbf{b}_1 = 1 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2 + \cdots + 0 \cdot \mathbf{b}_n$, the $\mathcal{B}$-coordinate vector of $\mathbf{b}_1$ is

$$[\mathbf{b}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_1$$

For each k, $\mathbf{b}_k = 0 \cdot \mathbf{b}_1 + \cdots + 1 \cdot \mathbf{b}_k + \cdots + 0 \cdot \mathbf{b}_n$, so $[\mathbf{b}_k]_{\mathcal{B}} = (0, \ldots, 1, \ldots, 0) = \mathbf{e}_k$.

20. For $\mathbf{w}$ in V, there exist scalars $k_1, \ldots, k_4$ such that

$$\mathbf{w} = k_1\mathbf{v}_1 + \cdots + k_4\mathbf{v}_4 \tag{1}$$

because $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ spans V. Also, because the set is linearly dependent, there exist scalars $c_1, \ldots, c_4$, not all zero, such that

$$\mathbf{0} = c_1\mathbf{v}_1 + \cdots + c_4\mathbf{v}_4$$

Adding gives

$$\mathbf{w} = \mathbf{w} + \mathbf{0} = (k_1 + c_1)\mathbf{v}_1 + \cdots + (k_4 + c_4)\mathbf{v}_4$$

At least one of the weights here differs from the corresponding weight in (1) because at least one of the c_i is nonzero. So $\mathbf{w}$ is expressed in more than one way as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_4$.

22. Let $P_{\mathcal{B}} = [\,\mathbf{b}_1\ \ \cdots\ \ \mathbf{b}_n\,]$. Then $P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$ and $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$. As mentioned in the text, the correspondence $\mathbf{x} \mapsto P_{\mathcal{B}}^{-1}\mathbf{x}$ is the coordinate mapping, so the desired matrix is $A = P_{\mathcal{B}}^{-1}$.

24. Given $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, let $\mathbf{u} = y_1\mathbf{b}_1 + \cdots + y_n\mathbf{b}_n$. Then, by definition, $[\mathbf{u}]_{\mathcal{B}} = \mathbf{y}$. So the coordinate mapping transforms $\mathbf{u}$ into $\mathbf{y}$. Since $\mathbf{y}$ was arbitrary, the coordinate mapping is onto.

26. $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ if and only if there exist scalars $c_1, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p \tag{2}$$

Since the coordinate mapping is linear,

$$[\mathbf{w}]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_p[\mathbf{u}_p]_{\mathcal{B}} \tag{3}$$

Conversely, (2) implies (3) because the coordinate mapping is one-to-one. Thus $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ if and only if (3) holds for some $c_1, \ldots, c_p$, which is equivalent to saying that $[\mathbf{w}]_{\mathcal{B}}$ is a linear combination of $[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}$.

Note: Students need to be urged to *write*, not just to compute, in Exercises 27–34. The language in the *Study Guide* solution of Exercise 31 provides a model for the students. In Exercise 32, students may have difficulty distinguishing between the two isomorphic vector spaces, sometimes giving a vector in $\mathbb{R}^3$ as the answer for part (b).

28. Linearly dependent because the coordinate vectors $\begin{bmatrix}1\\0\\-2\\-3\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\3\\-2\\0\end{bmatrix}$ are linearly dependent.

30. Linearly dependent. The coordinate vectors $\begin{bmatrix}1\\-3\\3\\-1\end{bmatrix}, \begin{bmatrix}4\\-12\\9\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\\-4\end{bmatrix}$ are linearly dependent.

32. a. The coordinate vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\2\\-4\end{bmatrix}$ span $\mathbb{R}^3$. Thus these three vectors form a basis for $\mathbb{R}^3$ by the Invertible Matrix Theorem. Because of the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the corresponding polynomials form a basis for $\mathbb{P}_2$.

b. Since $[\mathbf{q}]_{\mathcal{B}} = (-3, 1, 2)$, one may compute

$$\begin{bmatrix}1 & 2 & 1\\0 & -1 & 2\\1 & 3 & -4\end{bmatrix}\begin{bmatrix}-3\\1\\2\end{bmatrix} = \begin{bmatrix}1\\3\\-8\end{bmatrix}$$

and $\mathbf{q} = 1 + 3t - 8t^2$.

34. [M] The coordinate vectors $\begin{bmatrix}5\\-3\\4\\2\end{bmatrix}, \begin{bmatrix}9\\1\\8\\-6\end{bmatrix}, \begin{bmatrix}6\\-2\\5\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ are linearly dependent. Because of the isomorphism between $\mathbb{R}^4$ and $\mathbb{P}_3$, the corresponding polynomials are linearly dependent and therefore cannot form a basis for $\mathbb{P}_3$.

36. [M] Row reduction of $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ shows that there is a pivot in each column, so the columns are linearly independent and hence form a basis for the subspace H which they span.

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}3\\5\\2\end{bmatrix}$$

38. [M] $\begin{bmatrix}1.30\\.75\\1.60\end{bmatrix}$

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2. $\begin{bmatrix}4\\-3\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix}$; dim is 2 **4.** $\begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\-1\end{bmatrix}$; dim is 2

6. $\begin{bmatrix}3\\6\\-9\\-3\end{bmatrix}, \begin{bmatrix}6\\-2\\5\\1\end{bmatrix}$; dim is 2

8. $\begin{bmatrix}3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$; dim is 3

10. 2 **12.** 3 **14.** 3, 3 **16.** 0, 2 **18.** 1, 2

19. a. True. See the box before Example 5.

b. False, unless the plane is through the origin. Read Example 4 carefully.

c. False. The dimension is 5. See Example 1.

d. False. S must have exactly n elements to be a basis for V. See Theorem 10.

e. True. See Practice Problem 2.

20. a. False. The only subspaces of $\mathbb{R}^3$ are listed in Example 4. $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$, because vectors in $\mathbb{R}^3$ have three coordinates. Review Example 8 in Section 4.1.

b. False. The number of *free* variables equals the dimension of Nul A. See the box before Example 5.

c. False. Read carefully the definition before Example 1. *Not* being spanned by a finite set is not the same as being spanned by an infinite set. The space $\mathbb{R}^2$ is finite-dimensional, yet it is spanned by the infinite set S of all vectors of the form (x, y), where x and y are integers. (Of course, the two vectors $(1, 0)$ and $(0, 1)$ in S by themselves span $\mathbb{R}^2$.)

d. False. S must have exactly n elements to be a basis of V. See the Basis Theorem.

e. True. See Example 4.

22. Obviously, none of the Laguerre polynomials is a linear combination of the Laguerre polynomials of lower degree. By Theorem 4 (Section 4.3), the set of polynomials is linearly independent. Since this set contains four vectors, and $\mathbb{P}_3$ is four-dimensional, the set is a basis of $\mathbb{P}_3$, by the Basis Theorem.

24. $[\mathbf{p}]_{\mathcal{B}} = (5, -4, 3)$

26. If $\dim V = 0$, the statement is obvious. Otherwise, H contains a basis, consisting of n linearly independent vectors. By the Basis Theorem applied to V, the vectors form a basis for V.

28. The space $C(\mathbb{R})$ contains the space $\mathbb{P}$ as a subspace. If $C(\mathbb{R})$ were finite-dimensional, $\mathbb{P}$ would be finite-dimensional, too, by Theorem 11. This is not true, by Exercise 27, so $C(\mathbb{R})$ is infinite-dimensional.

30. **a.** False. This is *not* Theorem 9. If $\mathbf{x}$ in V is nonzero, the set $\{\mathbf{0}, \mathbf{x}, 2\mathbf{x}, \dots, (p-1)\mathbf{x}\}$ is linearly dependent, no matter what the dimension of V.

b. True. If $\dim V$ were less than or equal to p, V would have a basis of not more than p elements. Such a set would span V. Since this is not the case, $\dim V$ must be greater than p.

c. False. Counterexample: Take any nonzero vector $\mathbf{v}$, and consider the set $\{\mathbf{v}, 2\mathbf{v}, 3\mathbf{v}, \dots, (p-1)\mathbf{v}\}$.

32. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be a basis for H. Then $\{T(\mathbf{u}_1), \dots, T(\mathbf{u}_p)\}$ spans $T(H)$, as is easily seen. Further, since T is one-to-one, Exercise 32 in Section 4.3 shows that $\{T(\mathbf{u}_1), \dots, T(\mathbf{u}_p)\}$ is linearly independent. So this set of images is a basis for $T(H)$. So $\dim H = p$ and $\dim T(H) = p$.

33. **[M]** **a.** $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_2, \mathbf{e}_3\}$

b. The first k columns of A are pivot columns because, by assumption, the original k vectors are linearly independent. Col $A = \mathbb{R}^n$, because the columns of A include all the columns of the identity matrix.

34. **[M]** The $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$ are the columns of the matrix

$$P = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ & 1 & 0 & -3 & 0 & 5 & 0 \\ & & 2 & 0 & -8 & 0 & 18 \\ & & & 4 & 0 & -20 & 0 \\ & & & & 8 & 0 & -48 \\ & & & & & 16 & 0 \\ & & & & & & 32 \end{bmatrix}$$

a. This problem is an **[M]** exercise because it involves a large matrix. However, one should always think about a problem before rushing to use a matrix program. Actually, neither part of this exercise requires a matrix program. Simply observe that the matrix P is invertible because it is triangular with nonzero entries on the diagonal. So the columns of P are linearly independent. Because the coordinate mapping is an isomorphism, the vectors in $\mathcal{C}$ are linearly independent.

b. $\dim H = 7$, because $\mathcal{B}$ is a basis for H with 7 elements. Since $\mathcal{C}$ is linearly independent, and the vectors in $\mathcal{C}$ lie in H (because of the trig identities), $\mathcal{C}$ is a basis for H, by the Basis Theorem. (Another argument is to use the fact that the $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$ span $\mathbb{R}^7$, so the vectors in $\mathcal{C}$ span H. But you must distinguish between vectors in $\mathbb{R}^7$ and vectors in H.)

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2. rank $A = 3$; dim Nul $A = 2$;

Basis for Col A: $\begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix}$

Basis for Row A: $(1, -3, 0, 5, -7)$, $(0, 0, 2, -3, 8)$, $(0, 0, 0, 0, 5)$

Basis for Nul A: $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$

4. rank $A = 3$; dim Nul $A = 3$;

Basis for Col A: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{bmatrix}$

Basis for Row A: $(1, 1, -3, 7, 9, -9)$, $(0, 1, -1, 3, 4, -3)$, $(0, 0, 0, 1, -1, -2)$

Basis for Nul A: $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

6. 0, 3, 3

8. 2. It is impossible for Col A to be $\mathbb{R}^4$ because the vectors in Col A have 5 entries. Col A is a four-dimensional subspace of $\mathbb{R}^5$.

10. 1 **12.** 2

14. 3, 3. If A is 4×3, its rows are in $\mathbb{R}^3$ and there can be at most three linearly independent vectors in such a set. If A is 4×3, it cannot have more than three linearly independent rows because there are only three rows.

16. 0

17. a. True. The row vectors in A are identified with the columns of A^T. See the paragraph before Example 1.
b. False. See the warning after Example 2.
c. True. See the Rank Theorem.
d. False. See the Rank Theorem. The sum of the two dimensions equals the number of *columns* in A.
e. True. See the Numerical Note before the Practice Problem.

18. a. False. Review the warning after the proof of Theorem 6 in Section 4.3.
b. False. See the warning after Example 2. For instance, a row interchange usually changes dependence relations among the rows.
c. True. See the remark in the proof of the Rank Theorem.
d. True. This fact was noted in the paragraph before Example 4. It also follows from the fact that the rows of a matrix—say, A^T—are the columns of its transpose, and $A^{TT} = A$.
e. True. See Theorem 13.

20. No. The presence of two free variables indicates that the null space of the coefficient matrix A is two-dimensional. Since there are eight unknowns, A has eight columns and therefore must have rank 6, by the Rank Theorem. Since there are only six equations, A has six rows, and Col A is a subspace of $\mathbb{R}^6$. Since rank $A = 6$, we conclude that Col $A = \mathbb{R}^6$, which means that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$.

22. No. The coefficient matrix A is 10×12 and hence has rank at most 10. By the Rank Theorem, dim Nul A will be at least 2, so Nul A cannot be spanned by one vector.

24. The coefficient matrix A in this case is 7×6. It is possible that for some $\mathbf{b}$ in $\mathbb{R}^7$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. In this case, there are no free variables, so the rank of A must equal the number of columns, by the Rank Theorem. However, in any case, the rank of A cannot exceed 6, and so Col A must be a proper subspace of $\mathbb{R}^7$. Thus there exist vectors in $\mathbb{R}^7$ that are not in Col A. For such right-hand sides, the equation $A\mathbf{x} = \mathbf{b}$ will have no solution.

26. When an $m \times n$ matrix A has more rows than columns, A can have at most n pivot columns. So A has *full* rank when all n columns are pivot columns. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, that is, if and only if the columns of A are linearly independent.

28. a. dim Row A = dim Col A = rank A, by the Rank Theorem. So part (a) follows from the second part of that theorem.
b. Apply part (a) with A replaced by A^T and use the fact that Row A^T is just Col A.

30. The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if

rank $[\,A \quad \mathbf{b}\,] =$ rank A

because the two ranks are equal if and only if $\mathbf{b}$ is not a pivot column of $[\,A \quad \mathbf{b}\,]$. The result follows now from Theorem 2 in Section 1.2.

32. $\mathbf{v} = (1, -3, 4) = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$

34. Since A can be reduced to an echelon form U by row operations, there exist invertible $m \times m$ elementary matrices $E_1, \ldots, E_p$, such that $(E_p \cdots E_1)A = U$, and $A = (E_p \cdots E_1)^{-1}U$, since the product of invertible matrices is invertible. Let $E = (E_p \cdots E_1)^{-1}$. Then $A = EU$. Denote the columns of E by $\mathbf{c}_1, \ldots, \mathbf{c}_m$. Since rank $A = r$, its echelon form U has r nonzero rows, which we can denote by $\mathbf{d}_1^T, \ldots, \mathbf{d}_r^T$. By the column–row expansion of EU (Theorem 10 in Section 2.4),

$$A = EU = [\,\mathbf{c}_1 \quad \cdots \quad \mathbf{c}_m\,]\begin{bmatrix} \mathbf{d}_1^T \\ \vdots \\ \mathbf{d}_r^T \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} = \mathbf{c}_1\mathbf{d}_1^T + \cdots + \mathbf{c}_r\mathbf{d}_r^T$$

35. [M] a. Many answers are possible. Here are the “canonical” choices, for $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_7\,]$:

$$C = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_4 \quad \mathbf{a}_6\,], \quad N = \begin{bmatrix} -13/2 & -5 & 3 \\ -11/2 & -1/2 & -2 \\ 1 & 0 & 0 \\ 0 & 11/2 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 13/2 & 0 & 5 & 0 & -3 \\ 0 & 1 & 11/2 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -11/2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

b. $M = [\,2 \quad 41 \quad 0 \quad -28 \quad 11\,]^T$. The matrix $[\,R^T \quad N\,]$ is 7×7 because the columns of R^T and N are in $\mathbb{R}^7$, and dim Row A + dim Nul $A = 7$. The matrix $[\,C \quad M\,]$ is 5×5 because the columns of C and M are in $\mathbb{R}^5$ and dim Col A + dim Nul $A^T = 5$, by Exercise 28(b). The invertibility of these matrices follows from the fact that their columns are linearly independent, which can be proved from Theorem 3 in Section 6.1.

36. [M] In most cases, C will be 6×4, constructed from the first four columns of A, R will be 4×7, N will be 7×3, and M will be 6×2.

37. [M] The C and R given for Exercise 35 work here, and $A = CR$.

38. [M] In general, if A is nonzero, then $A = CR$ because

$$CR = C\,[\,\mathbf{r}_1 \quad \mathbf{r}_2 \quad \cdots \quad \mathbf{r}_n\,] = [\,C\mathbf{r}_1 \quad C\mathbf{r}_2 \quad \cdots \quad C\mathbf{r}_n\,]$$

To explain why the matrix on the right is A itself, consider the pivot columns of A (i.e., the columns of C) and then consider the nonpivot columns of A.

The ith pivot column of R is $\mathbf{e}_i$ (the ith column of the identity matrix). So $C\mathbf{e}_i$ is the ith pivot column of A. Since A and R have pivot columns in the same location, when C multiplies a pivot column of R, the result is a pivot column of A, in the correct location.

A nonpivot column of R—say, $\mathbf{r}_j$—contains the weights needed to construct column j of A from the pivot columns in A, as discussed in Example 9 of Section 4.3 and the paragraph preceding that example. Thus $\mathbf{r}_j$ contains the weights needed to construct column j of A from the columns of C, so $C\mathbf{r}_j = \mathbf{a}_j$.

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2. a. $\begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$ **b.** $\begin{bmatrix} 10 \\ 11 \end{bmatrix}$ **4.** (i)

6. a. $\begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$

8. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$, $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$

10. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$, $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 2 & 3 \\ -5 & -8 \end{bmatrix}$

11. a. False. See Theorem 15.
b. True. See the first paragraph in the subsection "Change of Basis in $\mathbb{R}^n$."

12. a. True. The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are coordinate vectors of the linearly independent set $\mathcal{B}$. See the second paragraph after Theorem 15.
b. False. The row reduction is discussed after Example 2. The matrix P obtained there satisfies $[\mathbf{x}]_\mathcal{C} = P[\mathbf{x}]_\mathcal{B}$.

14. a. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$

b. Solve $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and obtain $t^2 = 3(1 - 3t^2) - 2(2 + t - 5t^2) + (1 + 2t)$.

16. a. $[\mathbf{b}_1]_\mathcal{C} = Q[\mathbf{b}_1]_\mathcal{B} = Q \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Q\mathbf{e}_1$

b. $[\mathbf{b}_k]_\mathcal{C}$
c. $[\mathbf{b}_k]_\mathcal{C} = Q[\mathbf{b}_k]_\mathcal{B} = Q\mathbf{e}_k$

17. a. [M] $P^{-1} =$

$$\frac{1}{32}\begin{bmatrix} 32 & 0 & 16 & 0 & 12 & 0 & 10 \\ & 32 & 0 & 24 & 0 & 20 & 0 \\ & & 16 & 0 & 16 & 0 & 15 \\ & & & 8 & 0 & 10 & 0 \\ & & & & 4 & 0 & 6 \\ & & & & & 2 & 0 \\ & & & & & & 1 \end{bmatrix}$$

b.
$$\begin{aligned} \cos^2 t &= (1/2)[1 + \cos 2t] \\ \cos^3 t &= (1/4)[3\cos t + \cos 3t] \\ \cos^4 t &= (1/8)[3 + 4\cos 2t + \cos 4t] \\ \cos^5 t &= (1/16)[10\cos t + 5\cos 3t + \cos 5t] \\ \cos^6 t &= (1/32)[10 + 15\cos 2t + 6\cos 4t + \cos 6t] \end{aligned}$$

18. [M] Let $\mathcal{C} = \{\mathbf{y}_0, \ldots, \mathbf{y}_6\}$, where $\mathbf{y}_k$ is the function $\cos kt$. Then the $\mathcal{C}$-coordinate vector of $5\cos^3 t - 6\cos^4 t + 5\cos^5 t - 12\cos^6 t$ is $(0, 0, 0, 5, -6, 5, -12)$. Left-multiplication by the inverse of the matrix P in Exercise 17 changes this $\mathcal{C}$-coordinate vector into the $\mathcal{B}$-coordinate vector $(-6, 55/8, -69/8, 45/16, -3, 5/16, -3/8)$. So the integral (8) in this exercise equals

$$\int \left[-6 + \frac{55}{8}\cos t - \frac{69}{8}\cos 2t + \frac{45}{16}\cos 3t - 3\cos 4t + \frac{5}{16}\cos 5t - \frac{3}{8}\cos 6t\right] dt$$

From calculus, the integral equals

$$-6t + \frac{55}{8}\sin t - \frac{69}{16}\sin 2t + \frac{15}{16}\sin 3t - \frac{3}{4}\sin 4t + \frac{1}{16}\sin 5t - \frac{1}{16}\sin 6t + C$$

20. a. $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P} \cdot \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. *Reason:* $[\mathbf{b}_j]_{\mathcal{D}} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}[\mathbf{b}_j]_{\mathcal{C}}$. So, by Theorem 15 and the definition of matrix multiplication:

$$\underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \begin{bmatrix}[\mathbf{b}_1]_{\mathcal{D}} & [\mathbf{b}_2]_{\mathcal{D}}\end{bmatrix} = \begin{bmatrix}\underset{\mathcal{D}\leftarrow\mathcal{C}}{P}[\mathbf{b}_1]_{\mathcal{C}} & \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}[\mathbf{b}_2]_{\mathcal{C}}\end{bmatrix}$$
$$= \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\begin{bmatrix}[\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}}\end{bmatrix} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P} \cdot \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$$

b. Answers will vary.

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2. If $y_k = 3^k$, then $y_{k+1} = 3^{k+1}$ and $y_{k+2} = 3^{k+2}$. Substituting these formulas into the left side of the equation gives

$$\begin{aligned} y_{k+2} - 9y_k &= 3^{k+2} - 9 \cdot 3^k \\ &= 3^k(3^2 - 9) = 3^k(0) = 0 \quad \text{for all } k \end{aligned}$$

Since the difference equation holds for all k, 3^k is a solution. A similar calculation works for $y_k = (-3)^k$:

$$\begin{aligned} y_{k+2} - 9y_k &= (-3)^{k+2} - 9(-3)^k \\ &= (-3)^k[(-3)^2 - 9] \\ &= (-3)^k(0) = 0 \quad \text{for all } k \end{aligned}$$

4. The signals 3^k and $(-3)^k$ are linearly independent because neither is a multiple of the other. If H is the solution space of the difference equation in Exercise 2, then dim $H = 2$, by Theorem 17. So the two linearly independent solutions form a basis for H, by the Basis Theorem in Section 4.5.

6. If $y_k = 5^k \cos\left(\frac{k\pi}{2}\right)$, then

$$\begin{aligned} y_{k+2} + 25y_k &= 5^{k+2}\cos\left(\frac{(k+2)\pi}{2}\right) + 25 \cdot 5^k \cos\left(\frac{k\pi}{2}\right) \\ &= 5^{k+2}\left[\cos\left(\frac{k\pi}{2} + \pi\right) + \cos\left(\frac{k\pi}{2}\right)\right] \\ &= 0 \quad \text{for all } k \end{aligned}$$

because $\cos(t + \pi) = -\cos t$ for all t. A similar calculation holds for $z_k = 5^k \sin\left(\frac{k\pi}{2}\right)$, using the trigonometric identity $\sin(t + \pi) = -\sin t$. Thus y_k and z_k are both solutions of the difference equation $y_{k+2} + 25y_k = 0$. These solutions are obviously linearly independent because neither is a multiple of the other. Since the solution space H is two-dimensional, y_k and z_k form a basis for H, by the Basis Theorem.

8. Yes **10.** Yes

12. No, two signals cannot span a four-dimensional solution space.

14. $3^k, 4^k$ **16.** $\left(\frac{1}{4}\right)^k, \left(-\frac{3}{4}\right)^k$

18. The auxiliary equation is $r^2 - 1.35r + .45 = 0$, with roots .75 and .6. The constant solution of the nonhomogeneous equation is found by solving $T - 1.35T + .45T = 1$, to obtain $T = 10$. The general solution of the nonhomogeneous equation is

$$Y_k = c_1(.75)^k + c_2(.6)^k + 10$$

20. Let $a = -2 + \sqrt{3}$ and $b = -2 - \sqrt{3}$. Then c_1 and c_2 must satisfy

$$\begin{bmatrix} a & b \\ a^N & b^N \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5000 \\ 0 \end{bmatrix}$$

Solving (by row operations or Cramer's rule), we obtain

$$y_k = c_1a^k + c_2b^k = \frac{5000}{ab^N - ba^N}(a^kb^N - a^Nb^k)$$

22. 1.4, 0, −1.4, −2, −1.4, 0, 1.4, 2, 1.4

This signal is 2 times the signal output by the filter when the input (in Example 3) was $\cos(\pi t/4)$. This is what is to be expected since the filter is *linear*. The output should be 2 times the output from $\cos(\pi t/4)$ plus 1 times the (zero) output from $\cos(3\pi t/4)$.

23. b. [M] MATLAB code:

```
pay = 450, y = 10000, m = 0
table = [0 ; y]
while y > 450
     y = 1.01*y - pay
     m = m + 1
     table = [table [m ; y] ]
               %append new column
end
m, y
```

c. [M] At month 26, the last payment is \$114.88. The total paid by the borrower is \$11,364.88.

24. a. $y_{k+1} - 1.005y_k = 200, \quad y_0 = 1{,}000$

b. [M] MATLAB code:

```
pay = 200, y = 1000, table = [0 ; y]
for m = 1:60
     y = 1.005*y + pay
     table = [table  [m ; y] ]
end
interest = y - 60*pay - 1000
```

c. [M] The total is \$6213.55 at $k = 24$, \$12,090.06 at $k = 48$, and \$15,302.86 at $k = 60$. When $k = 60$, the interest earned is \$2302.86.

26. $1 + k + c_1 \cdot 5^k + c_2 \cdot 3^k$ **28.** $2k - 4 + c_1 \cdot (-2)^k + c_2 \cdot 2^{-k}$

30. $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/16 & 0 & 3/4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \end{bmatrix}$$

32. If $a_3 \neq 0$, the order is 3; if $a_3 = 0$ and $a_2 \neq 0$, the order is 2; if $a_3 = a_2 = 0$ and $a_1 \neq 0$, the order is 1; otherwise, the order is 0 (with only the zero signal for a solution).

34. No, the signals could be linearly dependent. *Example:* The following functions are linearly independent when considered as functions on the real line, because they have different periods and no one of the functions is a linear combination of the other two.

$$f(t) = \sin \pi t, \quad g(t) = \sin 2\pi t, \quad h(t) = \sin 3\pi t$$

Since f, g, and h are zero at every integer, the signals are linearly dependent as vectors in $\mathbb{S}$.

36. Given $\mathbf{z}$ in V, suppose that $\mathbf{x}_p$ in V satisfies $T(\mathbf{x}_p) = \mathbf{z}$. Also if $\mathbf{u}$ is in the kernel of T, then $T(\mathbf{u}) = \mathbf{0}$. Since T is linear, $T(\mathbf{u} + \mathbf{x}_p) = T(\mathbf{u}) + T(\mathbf{x}_p) = \mathbf{z}$. So the vector $\mathbf{x} = \mathbf{u} + \mathbf{x}_p$ satisfies the nonhomogeneous equation $T(\mathbf{x}) = \mathbf{z}$.

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2. a.

From: 1	2	3	To:
.5	.25	.25	1
.25	.5	.25	2
.25	.25	.5	3

b. .3125

4. a.

From: G	I	B	To:
.6	.4	.4	Good
.3	.3	.5	Indifferent
.1	.3	.1	Bad

b. 20% **c.** 48%

6. $\begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix}$ **8.** $\begin{bmatrix} 2/5 \\ 1/5 \\ 2/5 \end{bmatrix}$ or $\begin{bmatrix} .4 \\ .2 \\ .4 \end{bmatrix}$

10. No, because P^k has a zero in the lower-left corner for all k.

12. Each food will be preferred equally, because $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ is the steady-state vector.

14. There is a 50% chance of good weather because $\begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix}$ is the steady-state vector.

16. [M] The steady-state vector is approximately (.435, .091, .474). Of the 2000 cars, about 182 will be rented or available from the downtown location.

18. If $\alpha = \beta = 0$, then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the steady-state vectors. Otherwise, $\dfrac{1}{\alpha + \beta}\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$ is the only steady-state vector.

20. Let $P = [\,\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n\,]$, so that

$$P^2 = [\,P\mathbf{p}_1 \quad P\mathbf{p}_2 \quad \cdots \quad P\mathbf{p}_n\,]$$

By Exercise 19(c), the columns of P^2 are probability vectors, so P^2 is a stochastic matrix. Alternatively, $SP = S$, by Exercise 19(b), since P is a stochastic matrix. Right-multiplication by P yields $SP^2 = SP$. The right side is just S, so that $SP^2 = S$. Since the entries in P^2 are obviously nonnegative (they are sums of products of the nonnegative entries in P), this shows that P^2 is also a stochastic matrix.

21. [M]

a. To four decimal places,

$$P^4 = P^5 = \begin{bmatrix} .2816 & .2816 & .2816 & .2816 \\ .3355 & .3355 & .3355 & .3355 \\ .1819 & .1819 & .1819 & .1819 \\ .2009 & .2009 & .2009 & .2009 \end{bmatrix},$$

$$\mathbf{q} = \begin{bmatrix} .2816 \\ .3355 \\ .1819 \\ .2009 \end{bmatrix}$$

Note that, due to round-off, the column sums are not 1.

b. To four decimal places,

$$Q^{80} = \begin{bmatrix} .7354 & .7348 & .7351 \\ .0881 & .0887 & .0884 \\ .1764 & .1766 & .1765 \end{bmatrix},$$

$$Q^{116} = Q^{117} = \begin{bmatrix} .7353 & .7353 & .7353 \\ .0882 & .0882 & .0882 \\ .1765 & .1765 & .1765 \end{bmatrix},$$

$$\mathbf{q} = \begin{bmatrix} .7353 \\ .0882 \\ .1765 \end{bmatrix}$$

c. Let P be an $n \times n$ regular stochastic matrix, $\mathbf{q}$ the steady-state vector of P, and $\mathbf{e}_1$ the first column of the identity matrix. Then $P^k\mathbf{e}_1$ is the first column of P^k. By Theorem 18, $P^k\mathbf{e}_1 \to \mathbf{q}$ as $k \to \infty$. Replacing $\mathbf{e}_1$ by the other columns of the identity matrix, we conclude that each column of P^k converges to $\mathbf{q}$ as $k \to \infty$. Thus $P^k \to [\,\mathbf{q} \quad \mathbf{q} \quad \cdots \quad \mathbf{q}\,]$.

22. **[M]** (Discussion based on MATLAB Student Version 4.0, running on a 100-MHz 486 laptop computer with 32 Mb of memory) Let A be a random 32×32 stochastic matrix.

Method (1): The following command line will construct A and $\mathbf{q}$ but not display them, and it will announce the elapsed computer processing time and the number of flops used: (type all the commands on one line so they can be recalled and rerun several times)

```
A = randomstoc(32); flops(0);
tic, x = nulbasis(A - eye(32));
q = x/sum(x); toc, flops
```

The time ranged from 1.04 to 1.21 seconds, with 35,463 flops.

Method (2):

```
A = randomstoc(32); flops(0);
tic, B = A^100; q = B(:,1); toc, flops
```

The time ranged from 1.37 to 1.48 seconds, with 6,488,082 flops. If only A^{70} is computed, the time is about .94 second, which is faster than method (1), even though it uses about 4,522,000 flops.

Chapter 4 Supplementary Exercises, page 299

1. **a.** True. Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V, and every subspace is itself a vector space.

b. True. Any linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}$ is also a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_p$, using a weight of zero on $\mathbf{v}_p$.

c. False. Take $\mathbf{v}_p = 2\mathbf{v}_1$.

d. False. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Then $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a linearly independent set but is not a basis for $\mathbb{R}^3$.

e. True. See the Spanning Set Theorem (Section 4.3).

f. True. By the Basis Theorem, S must be a basis for V because S contains exactly p vectors, and so must be linearly independent.

g. False. The plane must go through the origin to be a subspace.

h. False. Consider $\begin{bmatrix} 2 & 5 & -2 & 0 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

i. True. This concept is presented before Theorem 13 in Section 4.6.

j. False. Row operations on A do not change the solutions of $A\mathbf{x} = \mathbf{0}$.

k. False. Consider $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

l. False. If U has k nonzero rows, then rank $A = k$ and dim Nul $A = n - k$ by the Rank Theorem.

m. True. Row equivalent matrices have the same number of pivot columns.

n. False. The nonzero rows of A span Row A, but they may not be linearly independent.

o. True. The nonzero rows of the reduced row echelon form E form a basis for the row space of each matrix that is row equivalent to E.

p. True. If H is the zero subspace, let A be the 3×3 zero matrix. If dim $H = 1$, let $\{\mathbf{v}\}$ be a basis for H, and set $A = [\,\mathbf{v} \quad \mathbf{v} \quad \mathbf{v}\,]$. If dim $H = 2$, let $\{\mathbf{u}, \mathbf{v}\}$ be a basis for H, and set $A = [\,\mathbf{u} \quad \mathbf{v} \quad \mathbf{v}\,]$, for example. If dim $H = 3$, then any invertible matrix A will work. Or, let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a basis for H, and set $A = [\,\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}\,]$.

q. False. Consider $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. If rank $A = n$ (the number of columns in A), then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

r. True. If $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then Col $A = \mathbb{R}^m$ and rank $A = m$.

s. True. See the second paragraph after Theorem 15 in Section 4.7.

t. False. The jth column of the change-of-coordinates matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ is $[\mathbf{b}_j]_{\mathcal{C}}$.

2. Any two of these three:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix} \right\}$$

4. The vectors $\mathbf{f}$ and $\mathbf{g}$ are not *scalar* multiples of each other, so $\{\mathbf{f}, \mathbf{g}\}$ is linearly independent.

6. Choose any two polynomials that are not multiples. Since they are linearly independent and belong to a two-dimensional space, they will be a basis for H.

8. The case $n = 0$ is trivial. If $n > 0$, then a basis for H consists of n linearly independent vectors, say, $\mathbf{u}_1, \ldots, \mathbf{u}_n$. These vectors remain linearly independent when considered as elements of V. But any n linearly independent vectors in the n-dimensional space V must form a basis for V, by the Basis Theorem in Section 4.5. So $\mathbf{u}_1, \ldots, \mathbf{u}_n$ span V. Thus $H = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} = V$.

10. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. If S were linearly independent and not a basis for V, then S would not span V. In this case, there would be a vector $\mathbf{v}_{p+1}$ in V that is not in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Let $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{v}_{p+1}\}$. Then S' is linearly independent because none of the vectors in S' is a linear combination of vectors that precede it. Since S' is larger than S, this would contradict the maximality of S. Hence S must be a basis for V.

12. a. Any $\mathbf{y}$ in Col AB has the form $AB\mathbf{x}$ for some $\mathbf{x}$. Then $\mathbf{y} = A(B\mathbf{x})$, which shows that $\mathbf{y}$ is a linear combination of the columns of A. Thus Col AB is a subset of Col A; that is, Col AB is a *subspace* of Col A. By Theorem 11 in Section 4.5, dim Col $AB \le$ dim Col A; that is, rank $AB \le$ rank A.

b. By the Rank Theorem and part (a):

$$\begin{aligned}\text{rank } AB &= \text{rank}(AB)^T = \text{rank } B^T A^T \\ &\le \text{rank } B^T = \text{rank } B\end{aligned}$$

14. Note that $(AQ)^T = Q^T A^T$. Since Q^T is invertible, we can use Exercise 13 to conclude that

$$\text{rank}(AQ)^T = \text{rank } Q^T A^T = \text{rank } A^T$$

Since the ranks of a matrix and its transpose are equal (by the Rank Theorem), rank AQ = rank A.

16. Suppose rank $A = r_1$ and rank $B = r_2$. Let the rank factorizations of A and B be $A = C_1R_1$ and $B = C_2R_2$. Create an $m \times (r_1 + r_2)$ matrix $C = [\,C_1 \quad C_2\,]$ and an $(r_1 + r_2) \times n$ matrix $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$. Then

$$A + B = C_1R_1 + C_2R_2 = CR$$

By Exercise 12, the rank of $A + B$ cannot exceed the rank of C. Since C has $r_1 + r_2$ columns, rank $C \le r_1 + r_2$. Thus the rank of $A + B$ cannot exceed $r_1 + r_2$ = rank A + rank B.

18. a. Using the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$ for $k = 0, 1, 2, 3, 4$, and letting $\mathbf{x}_0 = \mathbf{0}$, we have

$$\begin{aligned}\mathbf{x}_1 &= A\mathbf{x}_0 + B\mathbf{u}_0 = B\mathbf{u}_0 \\ \mathbf{x}_2 &= A\mathbf{x}_1 + B\mathbf{u}_1 = AB\mathbf{u}_0 + B\mathbf{u}_1 \\ \mathbf{x}_3 &= A\mathbf{x}_2 + B\mathbf{u}_2 \\ &= A(AB\mathbf{u}_0 + B\mathbf{u}_1) + B\mathbf{u}_2 \\ &= A^2B\mathbf{u}_0 + AB\mathbf{u}_1 + B\mathbf{u}_2 \\ \mathbf{x}_4 &= A\mathbf{x}_3 + B\mathbf{u}_3 \\ &= A(A^2B\mathbf{u}_0 + AB\mathbf{u}_1 + B\mathbf{u}_2) + B\mathbf{u}_3 \\ &= A^3B\mathbf{u}_0 + A^2B\mathbf{u}_1 + AB\mathbf{u}_2 + B\mathbf{u}_3 \\ &= [\,B \quad AB \quad A^2B \quad A^3B\,]\begin{bmatrix}\mathbf{u}_3 \\ \mathbf{u}_2 \\ \mathbf{u}_1 \\ \mathbf{u}_0\end{bmatrix} \\ &= M\mathbf{u}\end{aligned}$$

where $\mathbf{u}$ is in $\mathbb{R}^8$.

b. If (A, B) is controllable, then the controllability matrix M has rank 4, with a pivot in each row, and the columns of M span $\mathbb{R}^4$. Therefore, for any vector in $\mathbf{v}$ in $\mathbb{R}^4$, there is a vector $\mathbf{u}$ in $\mathbb{R}^8$ such that $\mathbf{v} = M\mathbf{u}$. However, from part (a) we know that $x_4 = M\mathbf{u}$ when $\mathbf{u}$ is partitioned into a control sequence $\mathbf{u}_0, \ldots, \mathbf{u}_3$. This particular control sequence makes $\mathbf{x}_4 = \mathbf{v}$.

20. $[\,B \quad AB \quad A^2B\,] = \begin{bmatrix} 1 & .5 & .19 \\ 1 & .7 & .45 \\ 0 & 0 & 0 \end{bmatrix}$. This matrix has rank less than 3, so the pair (A, B) is not controllable.

22. [M] rank$[\,B \quad AB \quad A^2B \quad A^3B\,] = 4$. The pair (A, B) is controllable.

CHAPTER 5

Section 5.1, page 308

2. Yes **4.** Yes, $\lambda = 3 + \sqrt{2}$ **6.** Yes, $\lambda = -2$

8. Yes, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ **10.** $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

12. $\lambda = 1$: $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$; $\lambda = 5$: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ **14.** $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

16. $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ **18.** 4, 0, −3

20. $\lambda = 0$. Eigenvectors for $\lambda = 0$ have entries that produce linear dependence relations among the columns of A. Any nonzero vector (in $\mathbb{R}^3$) whose entries sum to 0 will work. Find any two such vectors that are not multiples; for example, $(1, 1, -2)$ and $(1, -1, 0)$.

21. a. False. The equation $A\mathbf{x} = \lambda\mathbf{x}$ must have a *nontrivial* solution.

b. True. See the paragraph after Example 5.

c. True. See the discussion of equation (3).

d. True. See Example 2 and the paragraph preceding it. Also, see the Numerical Note.

e. False. See the warning after Example 3.

22. a. False. The vector $\mathbf{x}$ in $A\mathbf{x} = \lambda\mathbf{x}$ must be *nonzero*.

b. False. See Example 4 for a two-dimensional eigenspace, which contains two linearly independent eigenvectors corresponding to the same eigenvalue. The statement given is not at all the same as Theorem 2. In fact, it is the converse of Theorem 2 (for the case $r = 2$).

c. True. See the paragraph after Example 1.

d. False. Theorem 1 concerns a *triangular* matrix. See Examples 3 and 4 for counterexamples.

e. True. See the paragraph following Example 3. The eigenspace of A corresponding to λ is the null space of the matrix $A - \lambda I$.

24. Any triangular matrix with the same number in both diagonal entries, such as $\begin{bmatrix} 4 & 5 \\ 0 & 4 \end{bmatrix}$

26. If $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$, then $A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. However, $A^2\mathbf{x} = \mathbf{0}$ because $A^2 = 0$. Therefore, $\mathbf{0} = \lambda^2\mathbf{x}$. Since $\mathbf{x} \neq \mathbf{0}$, we conclude that λ must be zero. Thus each eigenvalue of A is zero.

28. If A is lower triangular, then A^T is upper triangular and has the same diagonal entries as A. Hence, by the part of Theorem 1 already proved in the text, these diagonal entries are eigenvalues of A^T. By Exercise 27, they are also eigenvalues of A.

30. By Exercise 29 applied to A^T in place of A, we conclude that s is an eigenvalue of A^T. By Exercise 27, s is an eigenvalue of A.

32. Suppose T rotates points about some line L that passes through the origin in $\mathbb{R}^3$. That line consists of all multiples of some nonzero vector $\mathbf{v}$. The points on this line do not move under the action of T. So $T(\mathbf{v}) = \mathbf{v}$. If A is the standard matrix of T, then $A\mathbf{v} = \mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 1. The eigenspace is Span $\{\mathbf{v}\}$.
If the rotation happens to be half of a full rotation, that is, through an angle of 180 degrees, let P be plane through the origin that is perpendicular to the line L. Each point $\mathbf{p}$ in this plane rotates into $-\mathbf{p}$. That is, each point in P is an eigenvector of A corresponding to the eigenvalue -1.

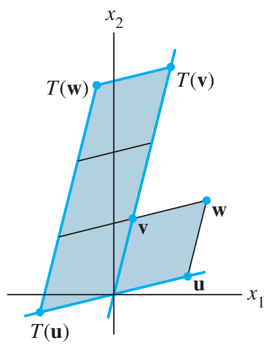
34. You could try to write $\mathbf{x}_0$ as a linear combination of eigenvectors, $\mathbf{v}_1, \ldots, \mathbf{v}_p$, of A. If $\lambda_1, \ldots, \lambda_p$ are corresponding eigenvalues, and if $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then you could *define*

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p$$

In this case, for $k = 0, 1, 2, \ldots,$

$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p) \\ &= c_1\lambda_1^k A\mathbf{v}_1 + \cdots + c_p\lambda_p^k A\mathbf{v}_p && \text{Linearity} \\ &= c_1\lambda_1^{k+1}\mathbf{v}_1 + \cdots + c_p\lambda_p^{k+1}\mathbf{v}_p && \text{The } \mathbf{v}_i \text{ are eigenvectors.} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

36.



38. [M] $\lambda = -12$: $\begin{bmatrix} 2 \\ 7 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$; $\lambda = 13$: $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix}$

40. [M] $\lambda = 3$: $\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$; $\lambda = 2$: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$

The other eigenvalues are the roots of $\lambda^2 - 5\lambda - 362 = 0$, namely

$\lambda = 21.68984106239549$ and $\lambda = -16.68984106239549$

The command `nulbasis(A-λI)` in the Laydata Toolbox needs λ to many (perhaps 12) decimal places in order to compute the corresponding eigenvectors:

$$\begin{bmatrix} -.33333333333333 \\ 2.39082008853296 \\ .33333333333333 \\ .58333333333333 \\ 1.00000000000000 \end{bmatrix} \text{ and } \begin{bmatrix} -.33333333333333 \\ -.80748675519962 \\ .33333333333333 \\ .58333333333333 \\ 1.00000000000000 \end{bmatrix}$$

Section 5.2, page 317

2. $\lambda^2 - 10\lambda + 16$; 8, 2 **4.** $\lambda^2 - 8\lambda + 3$; $4 \pm \sqrt{13}$

6. $\lambda^2 - 11\lambda + 40$; no real eigenvalues

8. $\lambda^2 - 10\lambda + 25$; 5 **10.** $-\lambda^3 + 14\lambda + 12$

12. $-\lambda^3 + 5\lambda^2 - 2\lambda - 8$ **14.** $-\lambda^3 + 4\lambda^2 + 25\lambda - 28$

16. 5, 1, 1, -4 **18.** $h = 6$

20.
$$\begin{aligned} \det(A^T - \lambda I) &= \det(A^T - \lambda I^T) \\ &= \det(A - \lambda I)^T && \text{Transpose property} \\ &= \det(A - \lambda I) && \text{Theorem 3(c)} \end{aligned}$$

21. a. False (although true for a triangular matrix). See Example 1 for a matrix whose determinant is not the product of its diagonal entries.
b. False. However, a row replacement operation does not change the determinant. See Theorem 3.
c. True. See Theorem 3.
d. False. See the solution of Example 4. The monomial $\lambda + 5$ is a factor of the characteristic polynomial if and only if -5 is an eigenvalue of A; it may also happen that 5 is an eigenvalue.

22. a. False. The absolute value of det A equals the volume. See the paragraph before Theorem 3.
b. False. A and A^T have the same determinant. See Theorem 3.
c. True. See the paragraph before Example 4.
d. False. See the warning after Theorem 4.

24. First observe that if P is invertible, then Theorem 3(b) shows that $1 = \det(I) = \det(PP^{-1}) = (\det P)(\det P^{-1})$. Then, if $A = PBP^{-1}$, Theorem 3(b) again shows that

$$\begin{aligned}\det A &= \det(PBP^{-1})\\ &= (\det P)(\det B)(\det P^{-1}) = \det B\end{aligned}$$

26. If $a \neq 0$, then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & b \\ 0 & d - ca^{-1}b \end{bmatrix} = U$, and $\det A = (a)(d - ca^{-1}b) = ad - bc$. If $a = 0$, then $A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = U$ (with one interchange), so $\det A = (-1)^1(cb) = 0 - bc$.

28. [M] In general, the eigenvectors of A are not the same as the eigenvectors of A^T, unless, of course, $A^T = A$.

30. [M] $a = 32$: $\lambda = 1, 1, 2$;
$a = 31.9$: $\lambda = .2958, 1, 2.7042$;
$a = 31.8$: $\lambda = -.1279, 1, 3.1279$;
$a = 32.1$: $\lambda = 1, 1.5 \pm .9747i$;
$a = 32.2$: $\lambda = 1, 1.5 \pm 1.4663i$

Section 5.3, page 325

2. $\frac{1}{16}\begin{bmatrix} 151 & 90 \\ -225 & -134 \end{bmatrix}$ **4.** $\begin{bmatrix} 4 - 3\cdot 2^k & 12\cdot 2^k - 12 \\ 1 - 2^k & 4\cdot 2^k - 3 \end{bmatrix}$

6. $\lambda = 5$: $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $\lambda = 4$: $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

When an answer involves a diagonalization, $A = PDP^{-1}$, the factors P and D are not unique, so your answer may differ from that given here.

8. Not diagonalizable **10.** $P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

12. $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

14. $P = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

16. $P = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

18. $P = \begin{bmatrix} -4 & 1 & -2 \\ 3 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

20. $P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

21. a. False. The symbol D does not automatically denote a diagonal matrix.
b. True. See the remark after the statement of the Diagonalization Theorem.
c. False. The 3×3 matrix in Example 4 has 3 eigenvalues, counting multiplicities, but it is not diagonalizable.
d. False. Invertibility depends on 0 not being an eigenvalue. (See the Invertible Matrix Theorem.) A diagonalizable matrix may or may not have 0 as an eigenvalue. See Examples 3 and 5 for both possibilities.

22. a. False. The n eigenvectors must be linearly independent. See the Diagonalization Theorem.
b. False. The matrix in Example 3 is diagonalizable, but it has only 2 distinct eigenvalues. (The statement given is the *converse* of Theorem 6.)
c. True. This follows from $AP = PD$ and formulas (1) and (2) in the proof of the Diagonalization Theorem.
d. False. See Example 4. The matrix there is invertible because 0 is not an eigenvalue, but the matrix is not diagonalizable.

24. No, by Theorem 7(b). Here is an explanation that does not appeal to Theorem 7: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that span the two one-dimensional eigenspaces. If $\mathbf{v}$ is any other eigenvector, then it belongs to one of the eigenspaces and hence is a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$. So there cannot exist three linearly independent eigenvectors. By the Diagonalization Theorem, A cannot be diagonalizable.

26. Yes, if the third eigenspace is only one-dimensional. In this case, the sum of the dimensions of the eigenspaces will be six, whereas the matrix is 7×7. See Theorem 7(b). An argument similar to that for Exercise 24 can also be given.

28. If A has n linearly independent eigenvectors, then by the Diagonalization Theorem, $A = PDP^{-1}$ for some invertible P and diagonal D. Using properties of transposes,

$$\begin{aligned} A^T &= (PDP^{-1})^T = (P^{-1})^T D^T P^T \\ &= (P^T)^{-1} D P^T = QDQ^{-1} \end{aligned}$$

where $Q = (P^T)^{-1}$. Thus A^T is diagonalizable. By the Diagonalization Theorem, the columns of Q are n linearly independent eigenvectors of A^T.

30. A nonzero multiple of an eigenvector is another eigenvector. To produce P_2, simply multiply one or both columns of P by a nonzero scalar unequal to 1.

32. Construct a 2×2 matrix with two distinct eigenvalues, one of which is zero. Simple examples for a and b nonzero:

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}, \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$$

34. [M] $P = \begin{bmatrix} 28 & 1 & -2 & -1 \\ 28 & 1 & 0 & 0 \\ 36 & -2 & 1 & 0 \\ 5 & 1 & 0 & 1 \end{bmatrix}$,

$$D = \begin{bmatrix} 24 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

36. [M] $P = \begin{bmatrix} 1 & 0 & -1 & 4 & -2 \\ 0 & -1 & 1 & -3 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 3 & 0 & -1 & 2 & 0 \\ 3 & 0 & 1 & 0 & 2 \end{bmatrix}$,

$$D = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Section 5.4, page 333

2. $\begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$ **4.** $\begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}$

6. a. $2 - t + 3t^2 - t^3 + t^4$

b. For any $\mathbf{p}$, $\mathbf{q}$ in $\mathbb{P}_2$ and any scalar c,

$$\begin{aligned} T(\mathbf{p}+\mathbf{q}) &= [\mathbf{p}(t)+\mathbf{q}(t)] + t^2[\mathbf{p}(t)+\mathbf{q}(t)] \\ &= [\mathbf{p}(t) + t^2\mathbf{p}(t)] + [\mathbf{q}(t) + t^2\mathbf{q}(t)] \\ &= T(\mathbf{p}) + T(\mathbf{q}) \\ T(c\mathbf{p}) &= [c\cdot\mathbf{p}(t)] + t^2[c\cdot\mathbf{p}(t)] = c\cdot[\mathbf{p}(t) + t^2\mathbf{p}(t)] \\ &= c\cdot T(\mathbf{p}) \end{aligned}$$

c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. $24\mathbf{b}_1 - 20\mathbf{b}_2 + 11\mathbf{b}_3$

10. a. For any $\mathbf{p}$, $\mathbf{q}$ in $\mathbb{P}_3$ and any scalar c,

$$\begin{aligned} T(\mathbf{p}+\mathbf{q}) &= \begin{bmatrix} (\mathbf{p}+\mathbf{q})(-3) \\ (\mathbf{p}+\mathbf{q})(-1) \\ (\mathbf{p}+\mathbf{q})(1) \\ (\mathbf{p}+\mathbf{q})(3) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{p}(-3) \\ \mathbf{p}(-1) \\ \mathbf{p}(1) \\ \mathbf{p}(3) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(-3) \\ \mathbf{q}(-1) \\ \mathbf{q}(1) \\ \mathbf{q}(3) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q}) \end{aligned}$$

$$T(c\cdot\mathbf{p}) = \begin{bmatrix} (c\cdot\mathbf{p})(-3) \\ (c\cdot\mathbf{p})(-1) \\ (c\cdot\mathbf{p})(1) \\ (c\cdot\mathbf{p})(3) \end{bmatrix} = c\cdot\begin{bmatrix} \mathbf{p}(-3) \\ \mathbf{p}(-1) \\ \mathbf{p}(1) \\ \mathbf{p}(3) \end{bmatrix} = c\cdot T(\mathbf{p})$$

b. $\begin{bmatrix} 1 & -3 & 9 & -27 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ **14.** $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

16. $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

18. If there is a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal, then A is similar to a diagonal matrix, by the second paragraph following Example 3. In this case, A would have three linearly independent eigenvectors. However, this is not necessarily the case, because A has only two distinct eigenvalues.

20. If $A = PBP^{-1}$, then $A^2 = (PBP^{-1})(PBP^{-1}) = PB(P^{-1}P)BP^{-1} = PB\cdot I\cdot BP^{-1} = PB^2P^{-1}$. So A^2 is similar to B^2.

22. If A is diagonalizable, then $A = PDP^{-1}$ for some P. Also, if B is similar to A, then $B = QAQ^{-1}$ for some Q. Then

$$\begin{aligned} B &= Q(PDP^{-1})Q^{-1} = (QP)D(P^{-1}Q^{-1}) \\ &= (QP)D(QP)^{-1} \end{aligned}$$

So B is diagonalizable.

24. If $A = PBP^{-1}$, then rank A = rank $P(BP^{-1})$ = rank BP^{-1}, by Supplementary Exercise 13 in Chapter 4. Also, rank BP^{-1} = rank B, by Supplementary Exercise 14 in Chapter 4, since P^{-1} is invertible. Thus rank A = rank B.

26. If $A = PDP^{-1}$ for some P, then the general trace property from Exercise 25 shows that tr A = tr $[(PD)P^{-1}]$ = tr $[P^{-1}PD]$ = tr D. (Or, one can use the result of Exercise 25 that since A is similar to D, tr A = tr D.) Since the eigenvalues of A are on the main diagonal of D, tr D is the sum of the eigenvalues of A.

28. For each j, $I(\mathbf{b}_j) = \mathbf{b}_j$, and $[I(\mathbf{b}_j)]_{\mathcal{C}} = [\mathbf{b}_j]_{\mathcal{C}}$. By formula (4), the matrix for I relative to the bases $\mathcal{B}$ and $\mathcal{C}$ is

$$M = \begin{bmatrix} [\,\mathbf{b}_1\,]_{\mathcal{C}} & [\,\mathbf{b}_2\,]_{\mathcal{C}} & \cdots & [\,\mathbf{b}_n\,]_{\mathcal{C}} \end{bmatrix}$$

In Theorem 15 of Section 4.7, this matrix was denoted by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and was called the *change-of-coordinates matrix from* $\mathcal{B}$ *to* $\mathcal{C}$.

30. **[M]** $P^{-1}AP = \begin{bmatrix} 8 & 3 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix}$

32. **[M]** $\lambda = 2$: $\mathbf{b}_1 = \begin{bmatrix} 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}$; $\lambda = 4$: $\mathbf{b}_2 = \begin{bmatrix} -30 \\ -7 \\ 3 \\ 0 \end{bmatrix}$,

$\mathbf{b}_3 = \begin{bmatrix} 39 \\ 5 \\ 0 \\ 3 \end{bmatrix}$; $\lambda = 5$: $\mathbf{b}_4 = \begin{bmatrix} 11 \\ -3 \\ 4 \\ 4 \end{bmatrix}$;

basis: $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$

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2. $\lambda = 3 + i$, $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$; $\lambda = 3 - i$, $\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

4. $\lambda = 4 + i$, $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$; $\lambda = 4 - i$, $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

6. $\lambda = 4 - 3i$, $\begin{bmatrix} i \\ 1 \end{bmatrix}$; $\lambda = 4 + 3i$, $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

8. $\lambda = \sqrt{3} \pm 3i$, $\varphi = -\pi/3$ radian, $r = \sqrt{12} = 2\sqrt{3}$

10. $\lambda = -5 \pm 5i$, $\varphi = 3\pi/4$ radians, $r = 5\sqrt{2}$

12. $\lambda = \pm .3i$, $\varphi = -\pi/2$ radians, $r = .3$

In Exercises 13–20, other answers are possible. Any P that makes $P^{-1}AP$ equal to the given C or to C^T is a satisfactory answer. First find P; then compute $P^{-1}AP$.

14. $P = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

16. $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$

18. $P = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$

20. $P = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} .28 & -.96 \\ .96 & .28 \end{bmatrix}$

22. $A(\mu\mathbf{x}) = \mu(A\mathbf{x}) = \mu(\lambda\mathbf{x}) = \lambda(\mu\mathbf{x})$

24. $\bar{\mathbf{x}}^T A\mathbf{x} = \bar{\mathbf{x}}^T(\lambda\mathbf{x}) = \lambda \cdot \bar{\mathbf{x}}^T\mathbf{x}$ because $\mathbf{x}$ is an eigenvector. It is easy to see that $\bar{\mathbf{x}}^T\mathbf{x}$ is real (and positive) because $\bar{z}z$ is nonnegative for every complex number z. Since $\bar{\mathbf{x}}^T A\mathbf{x}$ is real, by Exercise 23, so is λ. Next, write $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are real vectors. Then

$$A\mathbf{x} = A(\mathbf{u} + i\mathbf{v}) = A\mathbf{u} + iA\mathbf{v} \quad \text{and} \quad \lambda\mathbf{x} = \lambda\mathbf{u} + i\lambda\mathbf{v}$$

The real part of $A\mathbf{x}$ is $A\mathbf{u}$ because the entries in A, $\mathbf{u}$, and $\mathbf{v}$ are all real. The real part of $\lambda\mathbf{x}$ is $\lambda\mathbf{u}$ because λ and the entries in $\mathbf{u}$ and $\mathbf{v}$ are real. Since $A\mathbf{x}$ and $\lambda\mathbf{x}$ are equal, their real parts are equal, too. (Apply the corresponding statement about complex numbers to each entry of $A\mathbf{x}$.) Thus $A\mathbf{u} = \lambda\mathbf{u}$, which shows that the real part of $\mathbf{x}$ is an eigenvector of A.

26. **a.** If $\lambda = a - bi$, then

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} = (a - bi)(\text{Re}\,\mathbf{v} + i\,\text{Im}\,\mathbf{v}) \\ &= \underbrace{(a\,\text{Re}\,\mathbf{v} + b\,\text{Im}\,\mathbf{v})}_{\text{Re}\,A\mathbf{v}} + i\underbrace{(a\,\text{Im}\,\mathbf{v} - b\,\text{Re}\,\mathbf{v})}_{\text{Im}\,A\mathbf{v}} \end{aligned}$$

By Exercise 25,

$$\begin{aligned} A(\text{Re}\,\mathbf{v}) &= \text{Re}\,A\mathbf{v} = a\,\text{Re}\,\mathbf{v} + b\,\text{Im}\,\mathbf{v} \\ A(\text{Im}\,\mathbf{v}) &= \text{Im}\,A\mathbf{v} = -b\,\text{Re}\,\mathbf{v} + a\,\text{Im}\,\mathbf{v} \end{aligned}$$

b. Let $P = [\,\text{Re}\,\mathbf{v} \quad \text{Im}\,\mathbf{v}\,]$. By (a),

$$A(\text{Re}\,\mathbf{v}) = P\begin{bmatrix} a \\ b \end{bmatrix}, \quad A(\text{Im}\,\mathbf{v}) = P\begin{bmatrix} -b \\ a \end{bmatrix}$$

So

$$\begin{aligned} AP &= [\,A(\text{Re}\,\mathbf{v}) \quad A(\text{Im}\,\mathbf{v})\,] \\ &= \begin{bmatrix} P\begin{bmatrix} a \\ b \end{bmatrix} & P\begin{bmatrix} -b \\ a \end{bmatrix} \end{bmatrix} = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = PC \end{aligned}$$

28. **[M]** $P = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix}$,

$C = \begin{bmatrix} -.4 & -1.0 & 0 & 0 \\ 1.0 & -.4 & 0 & 0 \\ 0 & 0 & -.2 & -.5 \\ 0 & 0 & .5 & -.2 \end{bmatrix}$

Other choices are possible, but C must equal $P^{-1}AP$.

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2. $\mathbf{x}_k = 2\cdot 3^k \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + 1\cdot\left(\frac{4}{5}\right)^k \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + 2\cdot\left(\frac{3}{5}\right)^k \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$

So $\mathbf{x}_k \approx 2\cdot 3^k \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ for all k sufficiently large.

4. $\mathbf{x}_k = c_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix} + c_2(.6)^k \begin{bmatrix} 4 \\ 1 \end{bmatrix} \to c_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, as $k \to \infty$.
Provided that $c_1 > 0$, the owl and wood rat populations each stabilize in size, and eventually the populations are in the ratio of 4 owls to 5 thousand rats. If some aspect of the model were to change slightly, the characteristic equation would change slightly and the perturbed matrix A might not have 1 as an eigenvalue. If the eigenvalue becomes slightly larger than 1, the two populations will grow; if the eigenvalue becomes slightly less than 1, both populations will decline.

6. When $p = .5$, the eigenvalues of A are .9 and .7, both less than 1 in magnitude. The origin is an attractor for the dynamical system, and each trajectory tends toward $\mathbf{0}$. So populations of both owls and squirrels eventually disappear. For any p, the characteristic equation of A is $\lambda^2 - 1.6\lambda + (.48 + .3p) = 0$. The matrix A has an eigenvalue 1 when $p = .4$. In this case, both owl and squirrel populations tend toward constant levels, with 1 spotted owl for every 2 (thousand) flying squirrels.

8. Saddle point (because one or more eigenvalues are greater than 1, and one or more eigenvalues are less than 1, in magnitude); direction of greatest repulsion: the line through (0, 0, 0) and (1, 0, −3); direction of greatest attraction: the line through (0, 0, 0) and (−3, −3, 7)

10. Attractor; eigenvalues: .9, .5; direction of greatest attraction: the line through (0, 0) and (2, 1)

12. Saddle point; eigenvalues: 1.1, .8; greatest repulsion: line through (0, 0) and (1, 1); greatest attraction: line through (0, 0) and (2, 1)

14. Repellor; eigenvalues: 1.3, 1.1; greatest repulsion: line through (0, 0) and (−3, 2)

16. **[M]** $\mathbf{v}_k = c_1 \begin{bmatrix} .435 \\ .091 \\ .474 \end{bmatrix} + c_2(.89)^k \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3(.81)^k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Note: The exact value of the steady-state vector is $\mathbf{q} = (91/209, 19/209, 99/209) \approx (.435, .091, .474)$.

18. **a.** $A = \begin{bmatrix} 0 & 0 & .42 \\ .6 & 0 & 0 \\ 0 & .75 & .95 \end{bmatrix}$

b. **[M]** The long-term growth rate is $\lambda_1 = 1.105$. A corresponding eigenvector is approximately (38, 21, 100). For each 100 adults, there will be approximately 38 calves and 21 yearlings.

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2. $\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + \frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$

4. $\frac{13}{4}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} - \frac{5}{4}\begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-t}$. The origin is a saddle point. The direction of greatest attraction is the line through (−5, 1) and the origin. The direction of greatest repulsion is the line through (−1, 1) and the origin.

6. $5\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}$. The origin is an attractor. The direction of greatest attraction is the line through (2, 3) and the origin.

8. Set $P = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$. Then $A = PDP^{-1}$. Substituting $\mathbf{x} = P\mathbf{y}$ into $\mathbf{x}' = A\mathbf{x}$, we have

$$\begin{aligned} \frac{d}{dt}(P\mathbf{y}) &= A(P\mathbf{y}) \\ P\mathbf{y}' &= PDP^{-1}(P\mathbf{y}) = PD\mathbf{y} \end{aligned}$$

Left-multiplying by P^{-1} gives

$\mathbf{y}' = D\mathbf{y}$, or $\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$

10. (complex): $c_1 \begin{bmatrix} 1+i \\ -2 \end{bmatrix} e^{(2+i)t} + c_2 \begin{bmatrix} 1-i \\ -2 \end{bmatrix} e^{(2-i)t}$

(real): $c_1 \begin{bmatrix} \cos t - \sin t \\ -2\cos t \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} \sin t + \cos t \\ -2\sin t \end{bmatrix} e^{2t}$

The trajectories spiral out, away from the origin.

12. (complex): $c_1 \begin{bmatrix} 3-i \\ 2 \end{bmatrix} e^{(-1+2i)t} + c_2 \begin{bmatrix} 3+i \\ 2 \end{bmatrix} e^{(-1-2i)t}$

(real):
$c_1\begin{bmatrix}3\cos 2t+\sin 2t\\2\cos 2t\end{bmatrix}e^{-t}+c_2\begin{bmatrix}3\sin 2t-\cos 2t\\2\sin 2t\end{bmatrix}e^{-t}$
The trajectories spiral in, toward the origin.

14. (complex): $c_1\begin{bmatrix}1-i\\4\end{bmatrix}e^{2it}+c_2\begin{bmatrix}1+i\\4\end{bmatrix}e^{-2it}$

(real): $c_1\begin{bmatrix}\cos 2t+\sin 2t\\4\cos 2t\end{bmatrix}+c_2\begin{bmatrix}\sin 2t-\cos 2t\\4\sin 2t\end{bmatrix}$
The trajectories are ellipses about the origin.

16. **[M]** $\mathbf{x}(t)=c_1\begin{bmatrix}7\\-2\\3\end{bmatrix}e^{4t}+c_2\begin{bmatrix}3\\-1\\1\end{bmatrix}e^{3t}+c_3\begin{bmatrix}2\\0\\1\end{bmatrix}e^{2t}$

The origin is a repellor. All trajectories curve away from the origin.

18. **[M]** (complex):
$c_1\begin{bmatrix}1\\2\\0\end{bmatrix}e^{-7t}+c_2\begin{bmatrix}6+2i\\9+3i\\10\end{bmatrix}e^{(5+i)t}+c_3\begin{bmatrix}6-2i\\9-3i\\10\end{bmatrix}e^{(5-i)t}$

(real): $c_1\begin{bmatrix}1\\2\\0\end{bmatrix}e^{-7t}+c_2\begin{bmatrix}6\cos t-2\sin t\\9\cos t-3\sin t\\10\cos t\end{bmatrix}e^{5t}+$

$c_3\begin{bmatrix}6\sin t+2\cos t\\9\sin t+3\cos t\\10\sin t\end{bmatrix}e^{5t}$

When $c_2=c_3=0$, the trajectories tend straight toward $\mathbf{0}$. In other cases, the trajectories spiral outward.

20. **[M]** $A=\begin{bmatrix}-2&1/3\\3/2&-3/2\end{bmatrix}$,

$\begin{bmatrix}v_1(t)\\v_2(t)\end{bmatrix}=\frac{5}{3}\begin{bmatrix}1\\3\end{bmatrix}e^{-t}-\frac{2}{3}\begin{bmatrix}-2\\3\end{bmatrix}e^{-2.5t}$

22. **[M]** $A=\begin{bmatrix}0&2\\-.4&-.8\end{bmatrix}$,

$\begin{bmatrix}i_L(t)\\v_C(t)\end{bmatrix}=\begin{bmatrix}30\sin .8t\\12\cos .8t-6\sin .8t\end{bmatrix}e^{-.4t}$

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2. Eigenvector: $\mathbf{x}_4=\begin{bmatrix}-.2520\\1\end{bmatrix}$, or $A\mathbf{x}_4=\begin{bmatrix}-1.2536\\5.0064\end{bmatrix}$;
$\lambda\approx 5.0064$

4. Eigenvector: $\mathbf{x}_4=\begin{bmatrix}1\\.7502\end{bmatrix}$, or $A\mathbf{x}_4=\begin{bmatrix}-.4012\\-.3009\end{bmatrix}$;
$\lambda\approx -.4012$

6. $\mathbf{x}=\begin{bmatrix}-.4996\\1\end{bmatrix}$, $A\mathbf{x}=\begin{bmatrix}-2.0008\\4.0024\end{bmatrix}$; estimated $\lambda=4.0024$

8. **[M]** $\mathbf{x}_k$: $\begin{bmatrix}.5\\1\end{bmatrix}$, $\begin{bmatrix}.2857\\1\end{bmatrix}$, $\begin{bmatrix}.2558\\1\end{bmatrix}$, $\begin{bmatrix}.2510\\1\end{bmatrix}$, $\begin{bmatrix}.2502\\1\end{bmatrix}$

μ_k: 7, 6.14, 6.02, 6.0039, 6.0006

10. **[M]** $\mu_5=9.9319$, $\mu_6=9.9872$; actual value: 10
Note: Starting with $\mathbf{x}_0=(0,0,1)$ produces $\mu_4=9.9993$, $\mu_5=9.9999$.

12. μ_k: $-4.3333, -3.9231, -4.0196, -3.9951$
$R(\mathbf{x}_k)$: $-3.9231, -3.9951, -3.9997, -3.99998$

14. Use the inverse power method, with $\alpha=4$.

16. $\lambda=\alpha+1/\mu$

18. **[M]** $\nu_0=-1.375$, $\nu_1=-1.42623$, $\nu_2=-1.42432$, $\nu_3=-1.42444$. Actual: -1.424429 (accurate to six places)

20. **[M]** **a.** $\mu_8=19.1820=\mu_9$ to four decimal places. To six places, the largest eigenvalue is 19.182037, with eigenvector (.184416, 1, .179615, .407110).

b. $\mu_1^{-1}=.012235$, $\mu_2^{-1}=.012205$. To six places, the smallest eigenvalue is .012206, with eigenvector (1, .222610, $-.917993$, .660483). The other eigenvalues are -2.453128 and -1.741114, to six places.

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1. **a.** True. If A is invertible and if $A\mathbf{x}=1\cdot\mathbf{x}$ for some nonzero $\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x}=A^{-1}\mathbf{x}$, which may be rewritten as $A^{-1}\mathbf{x}=1\cdot\mathbf{x}$, Since $\mathbf{x}$ is nonzero, this shows that 1 is an eigenvalue of A^{-1}.

b. False. If A is row equivalent to the identity matrix, then A is invertible. The matrix in Example 4 of Section 5.3 shows that an invertible matrix need not be diagonalizable. Also, see Exercise 31 in Section 5.3.

c. True. If A contains a row or column of zeros, then A is not row equivalent to the identity matrix and thus is not invertible. By the Invertible Matrix Theorem (as stated in Section 5.2), 0 is an eigenvalue of A.

d. False. Consider a diagonal matrix D whose eigenvalues are 1 and 3; that is, its diagonal entries are 1 and 3. Then D^2 is a diagonal matrix whose eigenvalues (diagonal entries) are 1 and 9. In general, the eigenvalues of A^2 are the *squares* of the eigenvalues of A.

e. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x}=\lambda\mathbf{x}$, then

$$A^2\mathbf{x}=A(A\mathbf{x})=A(\lambda\mathbf{x})=\lambda A\mathbf{x}=\lambda^2\mathbf{x}$$

This shows that $\mathbf{x}$ is also an eigenvector for A^2.

f. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x}=\lambda\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x}=A^{-1}(\lambda\mathbf{x})=\lambda A^{-1}\mathbf{x}$. Since A is invertible, the eigenvalue λ is not zero. So

$\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, which shows that $\mathbf{x}$ is also an eigenvector of A^{-1}.

g. False. Zero is an eigenvalue of each singular square matrix.

h. True. By definition, an eigenvector must be nonzero.

i. False. If the dimension of the eigenspace is at least 2, then there are at least two linearly independent eigenvectors in the same subspace.

j. True. This follows from Theorem 4 in Section 5.2.

k. False. Let A be the 3×3 matrix in Example 3 of Section 5.3. Then A is similar to a diagonal matrix D. The eigenvectors of D are the columns of I_3, but the eigenvectors of A are entirely different.

l. False. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors of A, but $\mathbf{e}_1 + \mathbf{e}_2$ is not. (Actually, it can be shown that if two eigenvectors of A correspond to distinct eigenvalues, then their sum cannot be an eigenvector.)

m. False. *All* the diagonal entries of an upper triangular matrix are the eigenvalues of the matrix (Theorem 1 in Section 5.1). A diagonal entry may be zero.

n. True. Matrices A and A^T have the same characteristic polynomial, because $\det(A^T - \lambda I) = \det(A - \lambda I)^T = \det(A - \lambda I)$, by the determinant transpose property.

o. False. Counterexample: Let A be the 5×5 identity matrix.

p. True. For example, let A be the matrix that rotates vectors through $\pi/2$ radians about the origin. Then $A\mathbf{x}$ is not a multiple of $\mathbf{x}$ when $\mathbf{x}$ is nonzero.

q. False. If A is a diagonal matrix with a zero on the diagonal, then the columns of A are not linearly independent.

r. True. If $A\mathbf{x} = \lambda_1\mathbf{x}$ and $A\mathbf{x} = \lambda_2\mathbf{x}$, then $\lambda_1\mathbf{x} = \lambda_2\mathbf{x}$ and $(\lambda_1 - \lambda_2)\mathbf{x} = \mathbf{0}$. If $\mathbf{x} \neq \mathbf{0}$, then λ_1 must equal λ_2.

s. False. Let A be a singular matrix that is diagonalizable. (For instance, let A be a diagonal matrix with a zero on the diagonal.) Then, by Theorem 8 in Section 5.4, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is represented by a diagonal matrix relative to a coordinate system determined by eigenvectors of A.

t. True. By definition of matrix multiplication,

$$A = AI = A[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n] = [A\mathbf{e}_1 \quad A\mathbf{e}_2 \quad \cdots \quad A\mathbf{e}_n]$$

If $A\mathbf{e}_j = d_j\mathbf{e}_j$ for $j = 1, \dots, n$, then A is a diagonal matrix with diagonal entries $d_1, \dots, d_n$.

u. True. If $B = PDP^{-1}$, where D is a diagonal matrix, and if $A = QBQ^{-1}$, then $A = Q(PDP^{-1})Q^{-1} = (QP)D(QP)^{-1}$, which shows that A is diagonalizable.

v. True. Since B is invertible, AB is similar to $B(AB)B^{-1}$, which equals BA.

w. False. Having n linearly independent eigenvectors makes an $n \times n$ matrix diagonalizable (by the Diagonalization Theorem in Section 5.3), but not necessarily invertible. One of the eigenvalues of the matrix could be zero.

x. True. If A is diagonalizable, then by the Diagonalization Theorem, A has n linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathbb{R}^n$. By the Basis Theorem, $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans $\mathbb{R}^n$. This means that each vector in $\mathbb{R}^n$ can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$.

2. Suppose $B\mathbf{x} \neq \mathbf{0}$ and $AB\mathbf{x} = \lambda\mathbf{x}$ for some λ. Then $A(B\mathbf{x}) = \lambda\mathbf{x}$. Left-multiply each side by B, and obtain $BA(B\mathbf{x}) = B(\lambda\mathbf{x}) = \lambda(B\mathbf{x})$. This equation says that $B\mathbf{x}$ is an eigenvector of BA, because $B\mathbf{x} \neq \mathbf{0}$.

4. Assume that $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$. The desired statement is true for $m = 1$, by the assumption about λ. Suppose the statement holds when $m = k$, for some $k \geq 1$. That is, suppose that $A^k\mathbf{x} = \lambda^k\mathbf{x}$. Then, by the induction hypothesis,

$$A^{k+1}\mathbf{x} = A(A^k\mathbf{x}) = A(\lambda^k\mathbf{x})$$

Continuing, $A^{k+1}\mathbf{x} = \lambda^k A\mathbf{x} = \lambda^{k+1}\mathbf{x}$, because $\mathbf{x}$ is an eigenvector of A corresponding to A. Since $\mathbf{x}$ is nonzero, this equation shows that λ^{k+1} is an eigenvalue of A^{k+1}, with corresponding eigenvector $\mathbf{x}$. Thus the desired statement is true when $m = k + 1$. By the principal of induction, the statement is true for each positive integer m.

6. a. If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$, and

$$\begin{aligned} B &= 5I - 3A + A^2 = 5PIP^{-1} - 3PDP^{-1} + PD^2P^{-1} \\ &= P(5I - 3D + D^2)P^{-1} \end{aligned}$$

Since D is diagonal, so is $5I - 3D + D^2$. Thus B is similar to a diagonal matrix.

b.

$$\begin{aligned} p(A) &= c_0I + c_1PDP^{-1} + c_2PD^2P^{-1} + \cdots + c_nPD^nP^{-1} \\ &= P(c_0I + c_1D + c_2D^2 + \cdots + c_nD^n)P^{-1} \\ &= Pp(D)P^{-1} \end{aligned}$$

This shows that $p(A)$ is diagonalizable, because $p(D)$ is a linear combination of diagonal matrices and hence is diagonal. In fact, because D is diagonal, it is easy to see that

$$p(D) = \begin{bmatrix} p(2) & 0 \\ 0 & p(7) \end{bmatrix}$$

8. a. If λ is an eigenvalue of an $n \times n$ diagonalizable matrix A, then $A = PDP^{-1}$ for an invertible matrix P and an $n \times n$ diagonal matrix D whose diagonal entries are the eigenvalues of A. If the multiplicity of λ is n, then λ must appear in every diagonal entry of D. That is, $D = \lambda I$. In this case, $A = P(\lambda I)P^{-1} = \lambda PIP^{-1} = \lambda PP^{-1} = \lambda I$.

b. Since the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is triangular, its eigenvalues are on the diagonal. Thus 3 is an eigenvalue with multiplicity 2. If the 2×2 matrix A were diagonalizable, then A would be $3I$, by part (a). This is not the case, so A is not diagonalizable.

10. To show that A^k tends to the zero matrix, it suffices to show that each column of A^k can be made as close to the zero vector as desired by taking k sufficiently large. The jth column of A is $A\mathbf{e}_j$, where $\mathbf{e}_j$ is the jth column of the identity matrix. Since A is diagonalizable, there is a basis for $\mathbb{R}^n$ consisting of eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$. So there exist scalars $c_1, \ldots, c_n$, such that

$$\mathbf{e}_j = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \quad \text{(an eigenvector decomposition of } \mathbf{e}_j\text{)}$$

Then, for $k = 1, 2, \ldots,$

$$A^k\mathbf{e}_j = c_1(\lambda_1)^k\mathbf{v}_1 + \cdots + c_n(\lambda_n)^k\mathbf{v}_n \qquad (*)$$

If the eigenvalues are all less than 1 in absolute value, then their kth powers all tend to zero. So $(*)$ shows that $A^k\mathbf{e}_j$ tends to the zero vector, as desired.

12. Let U and V be echelon forms of A and B, obtained with r and s row interchanges, respectively, and no scaling. Then

$$\det A = (-1)^r \det U \quad \text{and} \quad \det B = (-1)^s \det V$$

Using first the row operations that reduce A to U, we can reduce G to a matrix of the form $G' = \begin{bmatrix} U & Y \\ 0 & B \end{bmatrix}$. Then, using the row operations that reduce B to V, we can further reduce G' to $G'' = \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$. There will be $r + s$ row interchanges, and so

$$\det G = \det \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = (-1)^{r+s} \det \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$$

Since $\begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$ is upper triangular, its determinant equals the product of the diagonal entries, and since U and V are upper triangular, this product also equals $(\det U)(\det V)$. Thus

$$\det G = (-1)^{r+s}(\det U)(\det V) = (\det A)(\det B)$$

For any scalar λ, the matrix $G - \lambda I$ has the same partitioned form as G, with $A - \lambda I$ and $B - \lambda I$ as its diagonal blocks. (Here I represents various identity matrices of appropriate sizes.) Hence the result about $\det G$ shows that

$$\det(G - \lambda I) = \det(A - \lambda I) \cdot \det(B - \lambda I)$$

14. 6, -1, -1, -5

16. The 3×3 matrix has eigenvalues $1 - 2$ and $1 + (2)(2)$, that is, -1 and 5. The eigenvalues of the 5×5 matrix are $7 - 3$ and $7 + (4)(3)$, that is, 4 and 19.

18. The eigenvalues of A are 1 and .6. Use this to factor A and A^k.

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .6 \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix}$$

$$\begin{aligned} A^k &= \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & .6^k \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 \cdot (.6)^k & -(.6)^k \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} -2 + 6(.6)^k & -3 + 3(.6)^k \\ 4 - 4(.6)^k & 6 - 2(.6)^k \end{bmatrix} \\ &\to \frac{1}{4} \begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix} \quad \text{as } k \to \infty \end{aligned}$$

20. $C_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24 & -26 & 9 \end{bmatrix}$;

$\det(C_p - \lambda I) = 24 - 26\lambda + 9\lambda^2 - \lambda^3$

22. a. $C_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$

b. Since λ is a zero of p, $a_0 + a_1\lambda + a_2\lambda^2 + \lambda^3 = 0$ and $-a_0 - a_1\lambda - a_2\lambda^2 = \lambda^3$. Thus

$$C_p \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ -a_0 - a_1\lambda - a_2\lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \end{bmatrix}$$

That is, $C_p(1, \lambda, \lambda^2) = \lambda(1, \lambda, \lambda^2)$, which shows that $(1, \lambda, \lambda^2)$ is an eigenvector of C_p corresponding to the eigenvalue λ.

24. [**M**] The MATLAB command `roots(p)` requires as input a row vector p whose entries are the coefficients of a polynomial, with the highest order coefficient listed first. MATLAB constructs a companion matrix C_p whose characteristic polynomial is p, so the roots of p are the eigenvalues of C_p. The numerical values of the eigenvalues (roots) are found by the same QR algorithm used by the command `eig(A)`.

25. [M] The MATLAB command `[P D] = eig(A)` produces a matrix P, whose condition number is 1.6×10^8, and a diagonal matrix D, whose entries are *almost* 2, 2, 1. However, the exact eigenvalues of A are 2, 2, 1, and A is not diagonalizable.

26. [M] This matrix may cause the same sort of trouble as the matrix in Exercise 25. A matrix program that computes eigenvalues by an interative process may indicate that A has four distinct eigenvalues, all close to zero. However, the only eigenvalue is 0, with multiplicity 4, because $A^4 = 0$.

CHAPTER 6

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2. 35, 5, $\frac{1}{7}$ **4.** $\begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$ **6.** $\begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}$

8. 7 **10.** $\begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$ **12.** $\begin{bmatrix} .8 \\ .6 \end{bmatrix}$

14. $2\sqrt{17}$ **16.** Orthogonal **18.** Not orthogonal

19. a. True. See the definition of $\|\mathbf{v}\|$.
b. True. See Theorem 1(c).
c. True. See the discussion of Fig. 5.
d. False. Counterexample: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.
e. True. See the box following Example 6.

20. a. True. See Example 1 and Theorem 1(a).
b. False. The absolute value is missing. See the box before Example 2.
c. True, by definition of the orthogonal complement.
d. True, by the Pythagorean Theorem.
e. True, by Theorem 3.

22. $\mathbf{u}\cdot\mathbf{u} \geq 0$ because $\mathbf{u}\cdot\mathbf{u}$ is a sum of squares of the entries in $\mathbf{u}$. The sum of squares of numbers is zero if and only if all the numbers are themselves zero.

24.
$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 \\ \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot(-\mathbf{v}) - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 \end{aligned}$$
When $\|\mathbf{u}+\mathbf{v}\|^2$ and $\|\mathbf{u}-\mathbf{v}\|^2$ are added, the $\mathbf{u}\cdot\mathbf{v}$ terms cancel, and the result is $2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

26. Theorem 2 in Chapter 4, because W is the null space of the $1 \times n$ matrix $\mathbf{u}^T$. W is a plane through the origin.

28. An arbitrary $\mathbf{w}$ in Span $\{\mathbf{u}, \mathbf{v}\}$ has the form $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$. If $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u}\cdot\mathbf{y} = 0$ and $\mathbf{v}\cdot\mathbf{y} = 0$. By linearity of the inner product [Theorem 1(b) and 1(c)],
$$\mathbf{w}\cdot\mathbf{y} = (c_1\mathbf{u} + c_2\mathbf{v})\cdot\mathbf{y} = c_1\mathbf{u}\cdot\mathbf{y} + c_2\mathbf{v}\cdot\mathbf{y} = c_1 0 + c_2 0 = 0$$

30. a. If $\mathbf{z}$ is in $W^\perp$, $\mathbf{u}$ is in W, and c is any scalar, then $(c\mathbf{z})\cdot\mathbf{u} = c(\mathbf{z}\cdot\mathbf{u}) = c0 = 0$. Since $\mathbf{u}$ is any element of W, $c\mathbf{z}$ is in $W^\perp$.
b. Take any $\mathbf{z}_1, \mathbf{z}_2$ in $W^\perp$. Then, for any $\mathbf{u}$ in W, $(\mathbf{z}_1 + \mathbf{z}_2)\cdot\mathbf{u} = \mathbf{z}_1\cdot\mathbf{u} + \mathbf{z}_2\cdot\mathbf{u} = 0 + 0 = 0$, which shows that $\mathbf{z}_1 + \mathbf{z}_2$ is in $W^\perp$.
c. Obviously $\mathbf{0}$ is in $W^\perp$, because $\mathbf{0}$ is orthogonal to every vector. This fact, together with (a) and (b), shows that $W^\perp$ is a subspace.

32. [M] This exercise anticipates Theorem 7 in Section 6.2. The matrix A has orthonormal columns.

33. [M] The mapping $\mathbf{x} \mapsto T(\mathbf{x}) = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$ is a linear transformation. In Section 6.2, the mapping will be called the orthogonal projection of $\mathbf{x}$ onto Span $\{\mathbf{v}\}$. To verify the linearity, take any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^4$ (or $\mathbb{R}^n$) and any scalar c. Then properties of the inner product (Theorem 1) show that
$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= \frac{(\mathbf{x}+\mathbf{y})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\mathbf{x}\cdot\mathbf{v} + \mathbf{y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} \\ &= \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} + \frac{\mathbf{y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} + \left(\frac{\mathbf{y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} \\ &= T(\mathbf{x}) + T(\mathbf{y}) \\ T(c\mathbf{x}) &= \frac{(c\mathbf{x})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{c(\mathbf{x}\cdot\mathbf{v})}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = c\left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = cT(\mathbf{x}) \end{aligned}$$
Another argument is to view T as the composition of three linear mappings: $\mathbf{x} \mapsto a = \mathbf{x}\cdot\mathbf{v}$, $a \mapsto b = a/(\mathbf{v}\cdot\mathbf{v})$, and $b \mapsto b\mathbf{v}$.

34. [M] $N = \begin{bmatrix} -5 & 1 \\ -1 & 4 \\ 1 & 0 \\ 0 & -1 \\ 0 & 3 \end{bmatrix}$,

$R = \begin{bmatrix} 1 & 0 & 5 & 0 & -1/3 \\ 0 & 1 & 1 & 0 & -4/3 \\ 0 & 0 & 0 & 1 & 1/3 \end{bmatrix}$,

$RN = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

The row–column rule for computing RN produces a 3×2 matrix of zeros, which shows that the rows of R are orthogonal to the columns of N. This is to be expected from Theorem 3, because each row of R is in Row A and each column of N is in Nul A.

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2. Orthogonal **4.** Orthogonal **6.** Not orthogonal

8. Show $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, mention Theorem 4, and observe that two linearly independent vectors in $\mathbb{R}^2$ form a basis. Then obtain

$$\mathbf{x} = -\frac{15}{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{30}{40}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = -\frac{3}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

10. Show $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$, and $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$. Mention Theorem 4, and observe that three linearly independent vectors in $\mathbb{R}^3$ form a basis. Then obtain

$$\mathbf{x} = \tfrac{24}{18}\mathbf{u}_1 + \tfrac{3}{9}\mathbf{u}_2 + \tfrac{6}{18}\mathbf{u}_3 = \tfrac{4}{3}\mathbf{u}_1 + \tfrac{1}{3}\mathbf{u}_2 + \tfrac{1}{3}\mathbf{u}_3$$

12. $\begin{bmatrix} .4 \\ -1.2 \end{bmatrix}$ **14.** $\mathbf{y} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$

16. $\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$, distance is $\sqrt{45} = 3\sqrt{5}$

18. Not orthogonal **20.** $\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$

22. Orthonormal

23. **a.** True. For example, the vectors **u** and **y** in Example 3 are linearly independent but not orthogonal.
b. True. The formulas for the weights are given in Theorem 5.
c. False. See the paragraph following Example 5.
d. False. The matrix must also be square. See the paragraph before Example 7.
e. False. See Example 4. The distance is $\|\mathbf{y} - \hat{\mathbf{y}}\|$.

24. **a.** True. But every orthogonal set of *nonzero vectors* is linearly independent. See Theorem 4.
b. False. To be orthonormal, the vectors in S must be unit vectors as well as being orthogonal to each other.
c. True. See Theorem 7(a).
d. True. See the paragraph before Example 3.
e. True. See the paragraph before Example 7.

26. If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are nonzero and orthogonal, then they are linearly independent, by Theorem 4. By the Invertible Matrix Theorem, $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$. If $W = \text{Span}\,\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, then W must be $\mathbb{R}^n$.

28. If U is an $n \times n$ orthogonal matrix, then $I = UU^{-1} = UU^T$. Since U is the transpose of U^T, Theorem 6 applied to U^T says that U^T has orthonormal columns. In particular, the columns of U^T are linearly independent and hence form a basis for $\mathbb{R}^n$, by the Invertible Matrix Theorem (see Section 4.6). That is, the rows of U form a basis (in fact, an orthonormal basis) for $\mathbb{R}^n$.

30. If U is an orthogonal matrix, its columns are orthonormal. Interchanging the columns does not change their orthonormality, so the new matrix—say, V—still has orthonormal columns. By Theorem 6, $V^TV = I$. Since V is square, $V^T = V^{-1}$ by the Invertible Matrix Theorem.

32. If $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, then by Theorem 1(c) in Section 6.1, $(c_1\mathbf{v}_1)\cdot(c_2\mathbf{v}_2) = c_1[\mathbf{v}_1\cdot(c_2\mathbf{v}_2)] = c_1c_2(\mathbf{v}_1\cdot\mathbf{v}_2) = c_1c_2 0 = 0$.

34. Let $L = \text{Span}\{\mathbf{u}\}$, where **u** is nonzero, and let $T(\mathbf{y}) = \text{refl}_L\,\mathbf{y} = 2\cdot\text{proj}_L\,\mathbf{y} - \mathbf{y}$. By Exercise 33, the mapping $\mathbf{y} \mapsto \text{proj}_L\,\mathbf{y}$ is linear. Thus, for **y** and **z** in $\mathbb{R}^n$ and any scalars c and d,

$$\begin{aligned} T(c\mathbf{y} + d\mathbf{z}) &= 2\cdot\text{proj}_L(c\mathbf{y} + d\mathbf{z}) - (c\mathbf{y} + d\mathbf{z}) \\ &= 2(c\cdot\text{proj}_L\,\mathbf{y} + d\cdot\text{proj}_L\,\mathbf{z}) - c\mathbf{y} - d\mathbf{z} \\ &= 2c\cdot\text{proj}_L\,\mathbf{y} - c\mathbf{y} + 2d\cdot\text{proj}_L\,\mathbf{z} - d\mathbf{z} \\ &= cT(\mathbf{y}) + dT(\mathbf{z}) \end{aligned}$$

Thus T is linear.

35. **[M]** The proof of Theorem 6 shows that the inner products to be checked are actually entries in the matrix product A^TA. A calculation shows that $A^TA = 100I_4$. Since the off-diagonal entries in A^TA are zero, the columns of A are orthogonal.

36. **[M]** **a.** $U^TU = I_4$, but UU^T is an 8×8 matrix which is nothing like I_8. In fact

$$UU^T = (.01)\begin{bmatrix} 82 & 0 & -20 & 8 & 6 & 20 & 24 & 0 \\ 0 & 42 & 24 & 0 & -20 & 6 & 20 & -32 \\ -20 & 24 & 58 & 20 & 0 & 32 & 0 & 6 \\ 8 & 0 & 20 & 82 & 24 & -20 & 6 & 0 \\ 6 & -20 & 0 & 24 & 18 & 0 & -8 & 20 \\ 20 & 6 & 32 & -20 & 0 & 58 & 0 & 24 \\ 24 & 20 & 0 & 6 & -8 & 0 & 18 & -20 \\ 0 & -32 & 6 & 0 & 20 & 24 & -20 & 42 \end{bmatrix}$$

b. The vector $\mathbf{p} = UU^T\mathbf{y}$ is in Col U because $\mathbf{p} = U(U^T\mathbf{y})$. Since the columns of U are simply scaled versions of the columns of A, Col U = Col A. Thus **p** is in Col A.
d. From (c), **z** is orthogonal to each column of A. By Exercise 29 in Section 6.1, **z** must be orthogonal to every vector in Col A; that is, **z** is in $(\text{Col}\,A)^\perp$.

Section 6.3, page 400

2. $\mathbf{v} = 2\mathbf{u}_1 + \frac{3}{7}\mathbf{u}_2 + \frac{12}{7}\mathbf{u}_3 - \frac{8}{7}\mathbf{u}_4$; $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$ **6.** $\begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} = \mathbf{y}$ **8.** $\mathbf{y} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} + \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$

10. $\mathbf{y} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$ **12.** $\begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{bmatrix}$ **16.** 8

18. a. $U^T U = [1] = 1$, $UU^T = \begin{bmatrix} .1 & -.3 \\ -.3 & .9 \end{bmatrix}$

b. $\text{proj}_W\, \mathbf{y} = \frac{-20}{\sqrt{10}}\mathbf{u}_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$,

$(UU^T)\mathbf{y} = \begin{bmatrix} .7 - 2.7 \\ -2.1 + 8.1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

20. Any multiple of $\begin{bmatrix} 0 \\ 4/5 \\ 2/5 \end{bmatrix}$, such as $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

21. a. True. See the calculations for $\mathbf{z}_2$ in Example 1 or the box after Example 6 in Section 6.1.

b. True, by the Orthogonal Decomposition Theorem.

c. False. See the last paragraph in the proof of Theorem 8, or see the second paragraph after the statement of Theorem 9.

d. True. See the box before The Best Approximation Theorem.

e. True. Theorem 10 applies to the column space W of U because the columns of U are linearly independent and hence form a basis for W.

22. a. True. See the proof of the Orthogonal Decomposition Theorem.

b. True. See the subsection "A Geometric Interpretation of the Orthogonal Projection."

c. True, by the uniqueness of the orthogonal decomposition in Theorem 8.

d. False. The Best Approximation Theorem says that the best approximation to $\mathbf{y}$ is $\text{proj}_W\, \mathbf{y}$.

e. False, unless $n = p$, because $UU^T\mathbf{x}$ is only the orthogonal projection of $\mathbf{x}$ onto the column space of U. See the paragraph following the proof for Theorem 10.

24. a. By hypothesis, the vectors $\mathbf{w}_1, \dots, \mathbf{w}_p$ are pairwise orthogonal, and the vectors $\mathbf{v}_1, \dots, \mathbf{v}_q$ are pairwise orthogonal. Also, $\mathbf{w}_i \cdot \mathbf{v}_j = 0$ for any i and j because the $\mathbf{v}$'s are in the orthogonal complement of W.

b. For any $\mathbf{y}$ in $\mathbb{R}^n$, write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ as in the Orthogonal Decomposition Theorem, with $\hat{\mathbf{y}}$ in W and $\mathbf{z}$ in $W^\perp$. Then there exist scalars $c_1, \dots, c_p$ and $d_1, \dots, d_q$ such that

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = c_1\mathbf{w}_1 + \cdots + c_p\mathbf{w}_p + d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q$$

Thus $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ spans $\mathbb{R}^n$.

c. The set $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is linearly independent by (a), spans $\mathbb{R}^n$ by (b), and thus is a basis for $\mathbb{R}^n$. Hence

$$\dim W + \dim W^\perp = p + q = \dim \mathbb{R}^n = n$$

25. [M] U has orthonormal columns, by Theorem 6 in Section 6.2, because $U^T U = I_4$. The closest point to $\mathbf{y}$ in Col U is the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto Col U. From Theorem 10,

$\hat{\mathbf{y}} = UU^T\mathbf{y} = (1.2, .4, 1.2, 1.2, .4, 1.2, .4, .4)$

26. [M] To two decimal places, $\hat{\mathbf{b}} = UU^T\mathbf{b} = (.20, .92, .44, 1.00, -.20, -.44, .60, -.92)$. The distance from $\mathbf{b}$ to Col U is $\|\mathbf{b} - \hat{\mathbf{b}}\| = 2.1166$, to four decimal places.

Section 6.4, page 407

2. $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$ **4.** $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

4. 6. $\begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$ **8.** $\begin{bmatrix} 3/\sqrt{50} \\ -4/\sqrt{50} \\ 5/\sqrt{50} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

10. $\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

14. $R = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$

16. $Q = \begin{bmatrix} 1/2 & -1/\sqrt{8} & 1/2 \\ -1/2 & 1/\sqrt{8} & 1/2 \\ 0 & 2/\sqrt{8} & 0 \\ 1/2 & 1/\sqrt{8} & -1/2 \\ 1/2 & 1/\sqrt{8} & 1/2 \end{bmatrix}, R = \begin{bmatrix} 2 & 8 & 7 \\ 0 & \sqrt{8} & 12/\sqrt{8} \\ 0 & 0 & 6 \end{bmatrix}$

17. **a.** False. Scaling was used in Example 2, but the scale factor was nonzero.
b. True. See (1) in the statement of Theorem 11.
c. True. See the solution of Example 4.

18. **a.** False. The three orthogonal vectors must be *nonzero* to be a basis for a three-dimensional subspace. (This was the case in Step 3 of the solution of Example 2.)
b. True. If $\mathbf{x}$ is not in a subspace W, then $\mathbf{x}$ cannot equal $\text{proj}_W\,\mathbf{x}$, because $\text{proj}_W\,\mathbf{x}$ is in W. This idea was used for $\mathbf{x} = \mathbf{v}_{k+1}$ in the proof of Theorem 11.
c. True, by Theorem 12.

20. If $\mathbf{y}$ is in Col A, then $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$. Then $\mathbf{y} = QR\mathbf{x} = Q(R\mathbf{x})$, which shows that $\mathbf{y}$ is a linear combination of the columns of Q using the entries in $R\mathbf{x}$ as weights. Conversely, suppose $\mathbf{y} = Q\mathbf{x}$ for some $\mathbf{x}$. Since R is invertible, the equation $A = QR$ implies that $Q = AR^{-1}$. So $\mathbf{y} = AR^{-1}\mathbf{x} = A(R^{-1}\mathbf{x})$, which shows that $\mathbf{y}$ is in Col A.

22. We may assume that $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthonormal basis for W, by normalizing the vectors in the original basis given for W, if necessary. Let U be the matrix whose columns are $\mathbf{u}_1, \dots, \mathbf{u}_p$. Then, by Theorem 10 in Section 6.3, $T(\mathbf{x}) = \text{proj}_W\,\mathbf{x} = (UU^T)\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^n$. Thus T is a matrix transformation and hence is a linear transformation, as was shown in Section 1.8.

24. **[M]** $\begin{bmatrix} -10 \\ 2 \\ -6 \\ 16 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 6 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ -5 \end{bmatrix}$

25. **[M]** $Q = \begin{bmatrix} -.5 & .5 & .5774 & 0 \\ .1 & .5 & 0 & .7071 \\ -.3 & -.5 & .5774 & 0 \\ .8 & 0 & .5774 & 0 \\ .1 & .5 & 0 & -.7071 \end{bmatrix}$

$R = \begin{bmatrix} 20 & -20 & -10 & 10 \\ 0 & 6 & -8 & -6 \\ 0 & 0 & 10.3923 & -5.1962 \\ 0 & 0 & 0 & 7.0711 \end{bmatrix}$

26. **[M]** In MATLAB, when A has n columns, suitable commands are

```
Q = A(:,1)/norm(A(:,1)
          % The first column of Q
for j = 2:n
   v = A(:,j) - Q*(Q'*A(:,j))
   Q(:,j) = v/norm(v)
   % Add a new column to Q
end
```

Section 6.5, page 416

2. **a.** $\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

4. **a.** $\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

6. $\hat{\mathbf{x}} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ **8.** $\sqrt{6}$

10. **a.** $\hat{\mathbf{b}} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$

12. **a.** $\hat{\mathbf{b}} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$

14. $A\mathbf{u} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}, A\mathbf{v} = \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix}, \mathbf{b} - A\mathbf{u} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$,
$\mathbf{b} - A\mathbf{v} = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}$. Note that $\|\mathbf{b} - A\mathbf{u}\| = \|\mathbf{b} - A\mathbf{v}\| = \sqrt{24}$, so $A\mathbf{u}$ and $A\mathbf{v}$ are equally close to $\mathbf{b}$. The orthogonal projection is the *unique* closest point in Col A to $\mathbf{b}$, so neither $A\mathbf{u}$ nor $A\mathbf{v}$ can be $\hat{\mathbf{b}}$. That is, neither $\mathbf{u}$ nor $\mathbf{v}$ can be a least-squares solution of $A\mathbf{x} = \mathbf{b}$.

16. $\hat{\mathbf{x}} = \begin{bmatrix} 2.9 \\ .9 \end{bmatrix}$

17. **a.** True. See the beginning of the section. The distance from $A\mathbf{x}$ to $\mathbf{b}$ is $\|A\mathbf{x} - \mathbf{b}\|$.
b. True. See the comments about equation (1).
c. False. The inequality points in the wrong direction. See the definition of a least-squares solution.
d. True. See Theorem 13.
e. True. See Theorem 14.

18. **a.** True. See the paragraph following the definition of a least-squares solution.
b. False. If $\hat{\mathbf{x}}$ is the least-squares solution, then $A\hat{\mathbf{x}}$ is the point in the column space of A closest to $\mathbf{b}$. See Fig. 1 and the paragraph preceding it.
c. True. See the discussion following equation (1).

d. False. The formula applies only when the columns of A are linearly independent. See Theorem 14.
e. False. See the comments after Example 4.
f. False. See the Numerical Note.

20. Suppose that $A\mathbf{x} = \mathbf{0}$. Then $A^TA\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$. Since A^TA is invertible, by hypothesis, $\mathbf{x}$ must be zero. Hence the columns of A are linearly independent.

22. A^TA has n columns because A does. Then

$$\begin{aligned} \text{rank } A^TA &= n - \dim \text{Nul } A^TA && \text{The Rank Theorem} \\ &= n - \dim \text{Nul } A && \text{Exercise 19} \\ &= \text{rank } A && \text{The Rank Theorem} \end{aligned}$$

24. $\hat{\mathbf{x}} = A^T\mathbf{b}$, from the normal equations, because $A^TA = I$.

26. **[M]** $a_0 = a_2 = .35\underline{35}$, $a_1 = .5$
(With .707 in place of .7, $a_0 = a_2 \approx .35355339$, $a_1 = .5$.)

Section 6.6, page 425

2. $y = -.6 + .7x$ **4.** $y = 4.3 - .7x$

6. If the columns of X were linearly dependent, then the same dependence relation would hold for the vectors in $\mathbb{R}^3$ formed from the top three entries of the column. In this case, the *Vandermonde* matrix

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

would be noninvertible. However, it can be shown that since x_1, x_2, and x_3 are distinct, this matrix *is* invertible, which means that the columns of X are, in fact, linearly independent. As in Exercise 5, Theorem 14 implies that there is only one least-squares solution of $\mathbf{y} = X\boldsymbol{\beta}$.

One way to show that the 3×3 matrix above is invertible is to show that its determinant is $(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$. Another way is to appeal to Supplementary Exercise 11(b) in Chapter 2.

8. a. $X = \begin{bmatrix} x_1 & x_1^2 & x_1^3 \\ \vdots & \vdots & \vdots \\ x_n & x_n^2 & x_n^3 \end{bmatrix}$, $\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$

b. **[M]** $y = .5132x - .03348x^2 + .001016x^3$, using four significant figures in the coefficients. *Note:* If you use .001 as the coefficient of x^3, your graph will fall somewhat below the last three or four data points.

10. a. $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{y} = \begin{bmatrix} 21.34 \\ 20.68 \\ 20.05 \\ 18.87 \\ 18.30 \end{bmatrix}$,

$$X = \begin{bmatrix} e^{-.02(10)} & e^{-.07(10)} \\ e^{-.02(11)} & e^{-.07(11)} \\ e^{-.02(12)} & e^{-.07(12)} \\ e^{-.02(14)} & e^{-.07(14)} \\ e^{-.02(15)} & e^{-.07(15)} \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} M_A \\ M_B \end{bmatrix}, \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$$

b. **[M]** $y = 19.94e^{-.02t} + 10.10e^{-.07t}$, $M_A = 19.94$, $M_B = 10.10$

12. **[M]** $p = 18.55 + 19.23 \ln w$ (using text values for $\ln w$). When w is 100, $p \approx 107$. Better: $p = 17.92 + 19.38 \ln w$, and $p \approx 104$.

14. Write the design matrix as $X = [\,\mathbf{1} \quad \mathbf{x}\,]$. Since the residual vector, $\boldsymbol{\epsilon} = \mathbf{y} - X\hat{\boldsymbol{\beta}}$, is orthogonal to Col X, we have (using the notation shown just after Exercise 14)

$$\begin{aligned} 0 &= \mathbf{1}\cdot\boldsymbol{\epsilon} = \mathbf{1}\cdot(\mathbf{y} - X\hat{\boldsymbol{\beta}}) = \mathbf{1}^T\mathbf{y} - (\mathbf{1}^TX)\hat{\boldsymbol{\beta}} \\ &= (y_1 + \cdots + y_n) - [\,n \quad \Sigma x\,]\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} \\ &= \Sigma y - n\hat{\beta}_0 - \hat{\beta}_1 \Sigma x \end{aligned}$$

Divide by $-n$, move the first term to the left side of the equation, and obtain $\bar{y} = \hat{\beta}_0 + \hat{\beta}_1\bar{x}$.

16. The determinant of the coefficient matrix of the equations in (7) is $n\Sigma x^2 - (\Sigma x)^2$. Using the 2×2 formula for the inverse of the coefficient matrix, we have

$$\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \frac{1}{n\Sigma x^2 - (\Sigma x)^2}\begin{bmatrix} \Sigma x^2 & -\Sigma x \\ -\Sigma x & n \end{bmatrix}\begin{bmatrix} \Sigma y \\ \Sigma xy \end{bmatrix}$$

Hence

$$\hat{\beta}_0 = \frac{(\Sigma x^2)(\Sigma y) - (\Sigma x)(\Sigma xy)}{n\,\Sigma x^2 - (\Sigma x)^2}, \quad \hat{\beta}_1 = \frac{n\,\Sigma xy - (\Sigma x)(\Sigma y)}{n\,\Sigma x^2 - (\Sigma x)^2}$$

Note: A simple algebraic calculation shows that $\Sigma y - (\Sigma x)\hat{\beta}_1 = n\hat{\beta}_0$, which provides a simple formula for $\hat{\beta}_0$, once $\hat{\beta}_1$ is known.

18. $X^TX = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \Sigma x \\ \Sigma x & (\Sigma x)^2 \end{bmatrix}$

This matrix is a diagonal matrix when $\Sigma x = 0$.

20. $\|X\hat{\boldsymbol{\beta}}\|^2 = (X\hat{\boldsymbol{\beta}})^T(X\hat{\boldsymbol{\beta}}) = \hat{\boldsymbol{\beta}}^TX^TX\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}^TX^T\mathbf{y}$, because $\hat{\boldsymbol{\beta}}^T$ satisfies the normal equations: $X^TX\boldsymbol{\beta} = X^T\mathbf{y}$. Since $\|X\hat{\boldsymbol{\beta}}\|^2 = \text{SS(R)}$ and $\mathbf{y}^T\mathbf{y} = \|\mathbf{y}\|^2 = \text{SS(T)}$, Exercise 19 shows that

$$\text{SS(E)} = \text{SS(T)} - \text{SS(R)} = \mathbf{y}^T\mathbf{y} - \hat{\boldsymbol{\beta}}^TX^T\mathbf{y}$$

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2. $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x}\rangle = 4(3)(3) + 5(-2)(-2) = 56$
$\|\mathbf{y}\|^2 = \langle \mathbf{y}, \mathbf{y}\rangle = 4(-2)(-2) + 5(1)(1) = 21$
$\|\mathbf{x}\|^2\|\mathbf{y}\|^2 = 56(21) = 1176$
$\langle \mathbf{x}, \mathbf{y}\rangle = 4(3)(-2) + 5(-2)(1) = -34$
$|\langle \mathbf{x}, \mathbf{y}\rangle|^2 = 1156 < 1176 = \|\mathbf{x}\|^2\|\mathbf{y}\|^2$

4. Polynomials: $3t - t^2$ $\quad 3 + 2t^2$

Values: $\begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$

$\langle p, q\rangle = -20 + 0 + 10 = -10$

6. $\|p\| = 2\sqrt{5}$, $\|q\| = \sqrt{59}$

8. $\dfrac{\langle q, p\rangle}{\langle p, p\rangle} p(t) = -\frac{10}{20} p(t) = -\frac{1}{2}(3t - t^2) = -\frac{3}{2}t + \frac{1}{2}t^2$

10. Polynomials: $p_0 \quad p_1 \quad q \quad p(t) = t^3$

Values: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -27 \\ -1 \\ 1 \\ 27 \end{bmatrix}$

$\hat{p}(t) = \frac{0}{4} p_0 + \frac{164}{20} p_1 + \frac{0}{4} q = \frac{41}{5} t$

12. Use Exercise 11 to get $t^3 - \frac{17}{5}t$. Then $5t^3 - 17t$ is also orthogonal to p_0, p_1, p_2, but its vector of values is $(-6, 12, 0, -12, 6)$. Answer: $p_3(t) = \frac{1}{6}(5t^3 - 17t)$.

14. 1. $\langle \mathbf{u}, \mathbf{v}\rangle = T(\mathbf{u})\cdot T(\mathbf{v})$ Definition
$= T(\mathbf{v})\cdot T(\mathbf{u})$ Property of dot product
$= \langle \mathbf{v}, \mathbf{u}\rangle$ Definition

2.
$\langle \mathbf{u} + \mathbf{v}, \mathbf{w}\rangle = T(\mathbf{u} + \mathbf{v})\cdot T(\mathbf{w})$ Definition
$= [T(\mathbf{u}) + T(\mathbf{v})]\cdot T(\mathbf{w})$ Linearity of T
$= T(\mathbf{u})\cdot T(\mathbf{w}) + T(\mathbf{v})\cdot T(\mathbf{w})$ Property of $\cdot$
$= \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$ Definition

3. $\langle c\mathbf{u}, \mathbf{v}\rangle = T(c\mathbf{u})\cdot T(\mathbf{v})$ Definition
$= cT(\mathbf{u})\cdot T(\mathbf{v})$ Linearity of T
$= c\langle \mathbf{u}, \mathbf{v}\rangle$ Definition

4. $\langle \mathbf{u}, \mathbf{u}\rangle = T(\mathbf{u})\cdot T(\mathbf{u}) \geq 0$ Property of dot product
If $\mathbf{u} = \mathbf{0}$, then $T(\mathbf{u}) = \mathbf{0}$, because T is linear, and $\langle \mathbf{u}, \mathbf{u}\rangle = 0$. Conversely, if $\langle \mathbf{u}, \mathbf{u}\rangle = 0$, then $T(\mathbf{u})\cdot T(\mathbf{u}) = 0$, and hence $T(\mathbf{u}) = \mathbf{0}$ by a property of the dot product. Since T is one-to-one, $\mathbf{u} = \mathbf{0}$.

16. $\|\mathbf{u} - \mathbf{v}\|^2$
$= \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle$
$= \langle \mathbf{u}, \mathbf{u} - \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle$ Axioms 2 and 3
$= \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle$ Axioms 1–3
$= \langle \mathbf{u}, \mathbf{u}\rangle - 2\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle$ Axiom 1
$= \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2$
If $\{\mathbf{u}, \mathbf{v}\}$ is orthonormal, then $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 = 1$ and $\langle \mathbf{u}, \mathbf{v}\rangle = 0$. So $\|\mathbf{u} - \mathbf{v}\|^2 = 2$.

18. The calculation in Exercise 16 shows that

$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2$

Similarly,

$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2$

Adding gives $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

20. If $\mathbf{u} = (a, b)$ and $\mathbf{v} = (1, 1)$, then $\|\mathbf{u}\|^2 = a^2 + b^2$, $\|\mathbf{v}\|^2 = 2$, and $|\langle \mathbf{u}, \mathbf{v}\rangle| = |a + b|$. The desired inequality follows when the Cauchy–Schwarz inequality is rewritten as

$$\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{2}\right)^2 \leq \frac{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}{4}$$

22. $\displaystyle\int_0^1 (5t - 3)(t^3 - t^2)\, dt = \int_0^1 (5t^4 - 8t^3 + 3t^2)\, dt = 0$

24. $\displaystyle\int_0^1 (t^3 - t^2)^2\, dt = \int_0^1 (t^6 - 2t^5 + t^4)\, dt = 1/105$,
$\|g\| = 1/\sqrt{105}$

26. $1, t, 3t^2 - 4$

27. **[M]** $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = -2 + t^2$,
$p_3(t) = (-17t + 5t^3)/6$, $p_4(t) = (72 - 155t^2 + 35t^4)/12$

The columns of the following matrix list the values of the respective polynomials at $-2, -1, 0, 1$, and 2:

$$A = \begin{bmatrix} 1 & -2 & 2 & -1 & 1 \\ 1 & -1 & -1 & 2 & -4 \\ 1 & 0 & -2 & 0 & 6 \\ 1 & 1 & -1 & -2 & -4 \\ 1 & 2 & 2 & 1 & 1 \end{bmatrix}$$

28. **[M]** The orthogonal basis is $f_0(t) = 1$, $f_1(t) = \cos t$, $f_2(t) = \cos^2 t - \frac{1}{2}$, and $f_3(t) = \cos^3 t - \frac{3}{4}\cos t$. Note that $2f_2(t) = \cos 2t$ and $4f_3(t) = \cos 3t$.

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2. Let X be the original design matrix, and let $\mathbf{y}$ be the original observation vector. Let W be the weighting matrix for the first method. Then the weighting matrix for the second method is $2W$. The weighted least-squares by the first method is equivalent to the ordinary least-squares for an equation whose normal equation is

$$(WX)^T WX\hat{\boldsymbol{\beta}} = (WX)^T W\mathbf{y} \quad (1)$$

while the second method is equivalent to the ordinary least-squares for an equation whose normal equation is

$$(2WX)^T(2W)X\hat{\boldsymbol{\beta}} = (2WX)^T(2W)\mathbf{y} \quad (2)$$

Since equation (2) can be written as $4(WX)^T WX\hat{\boldsymbol{\beta}} = 4(WX)^T W\mathbf{y}$, it has the same solutions as equation (1).

4. a. The vectors of polynomial values are

$$p_0 \leftrightarrow (1, 1, 1, 1, 1, 1), \quad p_1 \leftrightarrow (-5, -3, -1, 1, 3, 5),$$

$$p_2 \leftrightarrow (5, -1, -4, -4, -1, 5)$$

Verify that these vectors in $\mathbb{R}^6$ are mutually orthogonal.

b. $4p_0 + \frac{5}{7}p_1 + \frac{1}{14}p_2$

6. Use the identity

$$\sin mt \cos nt = \tfrac{1}{2}[\sin(mt + nt) + \sin(mt - nt)]$$

8. $-1 + \pi - 2\sin t - \sin 2t - \frac{2}{3}\sin 3t$

10. $\dfrac{4}{\pi}\sin t + \dfrac{4}{3\pi}\sin 3t$

12. The trigonometric identity $\cos 3t = 4\cos^3 t - 3\cos t$ shows that

$$\cos^3 t = \tfrac{3}{4}\cos t + \tfrac{1}{4}\cos 3t$$

The expression on the right is in the subspace spanned by the trigonometric polynomials of order 3 or less, so this expression *is* the third-order Fourier approximation to $\cos^3 t$.

14. g and h are both in the subspace H spanned by the trigonometric polynomials of order 2 or less. Since h is the second-order Fourier approximation to f, it is closer to f than any other function in the subspace H.

16. [M] $f_4(t) = \dfrac{4}{\pi}\sin t + \dfrac{4}{3\pi}\sin 3t$, $f_5(t) = f_4(t) + \dfrac{4}{5\pi}\sin 5t$

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1. a. False. The length of the zero vector is 0.

b. True. By the displayed equation before Example 2 in Section 6.1, with $c = -1$, $\|-\mathbf{x}\| = \|(-1)\mathbf{x}\| = |-1|\|\mathbf{x}\| = \|\mathbf{x}\|$.

c. True. This is the definition of distance.

d. False. The equation would be true if $r\|\mathbf{v}\|$ were replaced by $|r|\|\mathbf{v}\|$.

e. False. Orthogonal *nonzero* vectors are linearly independent.

f. True. If $\mathbf{x}\cdot\mathbf{u} = 0$ and $\mathbf{x}\cdot\mathbf{v} = 0$, then $\mathbf{x}\cdot(\mathbf{u} - \mathbf{v}) = \mathbf{x}\cdot\mathbf{u} - \mathbf{x}\cdot\mathbf{v} = 0$.

g. True. This is the "only if" part of the Pythagorean Theorem in Section 6.1.

h. True. This is the "only if" part of the Pythagorean Theorem in Section 6.1 when $\mathbf{v}$ is replaced by $-\mathbf{v}$, because $\|-\mathbf{v}\|^2$ is the same as $\|\mathbf{v}\|^2$.

i. False. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ is a scalar multiple of $\mathbf{u}$, not $\mathbf{y}$ (except when $\mathbf{y}$ itself is already a multiple of $\mathbf{u}$).

j. True. The orthogonal projection of any vector $\mathbf{y}$ onto W is always a vector in W.

k. True. This is a special case of the statement in the box following Example 6 in Section 6.1 (and proved in Exercise 30 of Section 6.1).

l. False. The zero vector is in both W and $W^\perp$.

m. True. (See Exercise 32 in Section 6.2.) If $\mathbf{v}_i\cdot\mathbf{v}_j = 0$, then $(c_i\mathbf{v}_i)\cdot(c_j\mathbf{v}_j) = c_ic_j(\mathbf{v}_i\cdot\mathbf{v}_j) = c_ic_j(0) = 0$.

n. False. The statement is true only for a *square* matrix. See Theorem 10 in Section 6.3.

o. False. An orthogonal matrix is square and has ortho*normal* columns.

p. True. See Exercises 27 and 28 in Section 6.2. If U has orthonormal columns, then $U^TU = I$. If U is also square, then the Invertible Matrix Theorem shows that U is invertible and U^T is U^{-1}. In this case, $U^TU = UU^{-1} = I$, which shows that the columns of U^T are orthonormal; that is, the rows of U are orthonormal.

q. True. By the Orthogonal Decomposition Theorem, the vectors $\text{proj}_W\,\mathbf{v}$ and $\mathbf{v} - \text{proj}_W\,\mathbf{v}$ are orthogonal, so the stated equality follows from the Pythagorean Theorem.

r. False. A least-squares solution is a vector $\hat{\mathbf{x}}$ (not $A\hat{\mathbf{x}}$) such that the vector $A\hat{\mathbf{x}}$ is the closest point to $\mathbf{b}$ in Col A.

s. False. The equation $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$ describes the *solution* of the normal equations, not the matrix form of the normal equations. Furthermore, this equation makes sense only when A^TA is invertible.

2. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal set and $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then the vectors $c_1\mathbf{v}_1$ and $c_2\mathbf{v}_2$ are orthogonal (Exercise 32 in Section 6.2). By the Pythagorean Theorem and properties of the norm

$$\begin{aligned}\|\mathbf{x}\|^2 &= \|c_1\mathbf{v}_1 + c_2\mathbf{v}_2\|^2 = \|c_1\mathbf{v}_1\|^2 + \|c_2\mathbf{v}_2\|^2 \\ &= (|c_1|\|\mathbf{v}_1\|)^2 + (|c_1|\|\mathbf{v}_2\|)^2 = |c_1|^2 + |c_2|^2\end{aligned}$$

So the stated equality holds for $p = 2$. Now suppose the equality holds for $p = k$, with $k \geq 2$. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$ be an orthonormal set, and consider

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} = \mathbf{u}_k + c_{k+1}\mathbf{v}_{k+1}$$

where $\mathbf{u}_k = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$. Observe that $\mathbf{u}_k$ and $c_{k+1}\mathbf{v}_{k+1}$ are orthogonal, because $\mathbf{v}_j \cdot \mathbf{v}_{k+1} = 0$ for $j = 1, \ldots, k$. By the Pythagorean Theorem and the assumption that the stated equality holds for k, and because $\|c_{k+1}\mathbf{v}_{k+1}\|^2 = |c_{k+1}|^2\|\mathbf{v}_{k+1}\|^2 = |c_{k+1}|^2$,

$$\|\mathbf{x}\|^2 = \|\mathbf{u}_k\|^2 + \|c_{k+1}\mathbf{v}_{k+1}\|^2 = |c_1|^2 + \cdots + |c_k|^2 + |c_{k+1}|^2$$

Thus the truth of the equality for $p = k$ implies its truth for $p = k + 1$. By the principle of induction, the equality is true for all integers $p \geq 2$.

4. By parts (a) and (c) of Theorem 7 in Section 6.2, $\{U\mathbf{v}_1, \ldots, U\mathbf{v}_n\}$ is an orthonormal set in $\mathbb{R}^n$. Since there are n vectors in this linearly independent set, the set is a basis for $\mathbb{R}^n$.

6. If $U\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$, then by Theorem 7(a) in Section 6.2 and by a property of the norm, $\|\mathbf{x}\| = \|U\mathbf{x}\| = \|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$, which shows that $|\lambda| = 1$ (because $\|\mathbf{x}\| \neq 0$).

8. a. Suppose $\mathbf{x}\cdot\mathbf{y} = 0$. By the Pythagorean Theorem,

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x} + \mathbf{y}\|^2$$

Since T preserves lengths and is linear,

$$\|T(\mathbf{x})\|^2 + \|T(\mathbf{y})\|^2 = \|T(\mathbf{x} + \mathbf{y})\|^2 = \|T(\mathbf{x}) + T(\mathbf{y})\|^2$$

This equation shows that $T(\mathbf{x})$ and $T(\mathbf{y})$ are orthogonal, because of the Pythagorean Theorem. Thus T preserves orthogonality.

b. The standard matrix of T is $[\,T(\mathbf{e}_1) \ \cdots \ T(\mathbf{e}_n)\,]$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of the identity matrix. Then $\{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)\}$ is an orthonormal set because T preserves both orthogonality and lengths (and because the columns of the identity matrix form an orthonormal set). Finally, a square matrix with orthonormal columns is an orthogonal matrix, as was observed in Section 6.2.

10. Use Theorem 14 in Section 6.5. If $c \neq 0$, the least-squares solution of $A\mathbf{x} = c\mathbf{b}$ is given by $(A^TA)^{-1}A^T(c\mathbf{b})$, which equals $c(A^TA)^{-1}A^T\mathbf{b}$, by linearity of matrix multiplication. This solution is c times the least-squares solution of $A\mathbf{x} = \mathbf{b}$.

12. The equation (1) in the exercise has been written as $V\lambda = \mathbf{b}$, where V is a single nonzero column vector $\mathbf{v}$, and $\mathbf{b} = A\mathbf{v}$. The least-squares solution $\hat{\lambda}$ of $V\lambda = \mathbf{b}$ is the exact solution of the normal equations $V^TV\lambda = V^T\mathbf{b}$. In the original notation, this equation is $\mathbf{v}^T\mathbf{v}\lambda = \mathbf{v}^TA\mathbf{v}$. Since $\mathbf{v}^T\mathbf{v}$ is nonzero, the least-squares solution $\hat{\lambda}$ is $\mathbf{v}^TA\mathbf{v}/(\mathbf{v}^T\mathbf{v})$. This expression is the Rayleigh quotient discussed in the exercises for Section 5.8.

14. The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in Col A. By Exercise 13(c), $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is orthogonal to Nul A^T. This happens if and only if $\mathbf{b}$ is orthogonal to all solutions of $A^T\mathbf{x} = \mathbf{0}$.

16. a. If $U = [\,\mathbf{u}_1 \ \ \mathbf{u}_2 \ \ \cdots \ \ \mathbf{u}_n\,]$, then $AU = [\,\lambda_1\mathbf{u}_1 \ \ A\mathbf{u}_2 \ \ \cdots \ \ A\mathbf{u}_n\,]$. Since $\mathbf{u}_1$ is a unit vector and $\mathbf{u}_2, \ldots, \mathbf{u}_n$ are orthogonal to $\mathbf{u}_1$, the first column of U^TAU is $U^T(\lambda_1\mathbf{u}_1) = \lambda_1U^T\mathbf{u}_1 = \lambda_1\mathbf{e}_1$.

b. From (a),

$$U^TAU = \begin{bmatrix} \lambda_1 & * & * & * & * \\ 0 & & & & \\ \vdots & & A_1 & & \\ 0 & & & & \end{bmatrix}$$

View U^TAU as a 2×2 block upper-triangular matrix, with A_1 as the (2, 2)-block. Then, from Supplementary Exercise 12 in Chapter 5,

$$\begin{aligned}\det(U^TAU - \lambda I_n) &= \det((\lambda_1 - \lambda)I_1)\cdot\det(A_1 - \lambda I_{n-1}) \\ &= (\lambda_1 - \lambda)\cdot\det(A_1 - \lambda I_{n-1})\end{aligned}$$

This shows that the eigenvalues of U^TAU, namely, $\lambda_1, \ldots, \lambda_n$, consist of λ_1 and the eigenvalues of A_1. So the eigenvalues of A_1 are $\lambda_2, \ldots, \lambda_n$.

18. [M] $\dfrac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} = .00212$, $\text{cond}(A) \times \dfrac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|} = 3363 \times (.00212) \approx 7.1$. In this case, $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|$ is almost the same as $\|\Delta\mathbf{b}\|/\|\mathbf{b}\|$, even though the large condition number suggests that $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|$ could be much larger.

20. [M] $\text{cond}(A) \times \dfrac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|} = 23683 \times (1.097 \times 10^{-5}) = .2598$. This calculation shows that the relative change in $\mathbf{x}$, for this particular $\mathbf{b}$ and $\Delta\mathbf{b}$, should not exceed .2598. As it turns out, $\|\Delta\mathbf{x}\|/\|\mathbf{x}\| = .2597$. So the theoretical maximum change is almost achieved.

CHAPTER 7

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2. Not symmetric **4.** Symmetric **6.** Not symmetric

8. Orthogonal, $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ **10.** Not orthogonal

12. Orthogonal, $\begin{bmatrix} .5 & -.5 & .5 & -.5 \\ .5 & .5 & .5 & .5 \\ -.5 & -.5 & .5 & .5 \\ -.5 & .5 & .5 & -.5 \end{bmatrix}$

14. $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$

16. $P = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$, $D = \begin{bmatrix} 25 & 0 \\ 0 & -25 \end{bmatrix}$

18. $P = \begin{bmatrix} -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -50 \end{bmatrix}$

20. $P = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$, $D = \begin{bmatrix} 13 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

22. $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$,

$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

24. $P = \begin{bmatrix} -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$, $D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

25. **a.** True. See Theorem 2 and the paragraph preceding the theorem.
b. True. This is a particular case of the statement in Theorem 1, when **u** and **v** are nonzero.
c. False. There are n real eigenvalues (Theorem 3), but they need not be distinct (Example 3).
d. False. See the paragraph following formula (2), in which each **u** is a unit vector.

26. **a.** True, by Theorem 2.
b. True. See the displayed equation in the paragraph before Theorem 2.
c. False. An orthogonal matrix can be symmetric (and hence orthogonally diagonalizable), but not every orthogonal matrix is symmetric. The matrix P in Example 2 is an orthogonal matrix, but it is not symmetric.
d. True, by Theorem 3(b).

28. $(A\mathbf{x})\cdot\mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^TA^T\mathbf{y} = \mathbf{x}^TA\mathbf{y} = \mathbf{x}\cdot(A\mathbf{y})$, because $A^T = A$.

30. If A and B are orthogonally diagonalizable, then A and B are symmetric, by Theorem 2. If $AB = BA$, then $(AB)^T = (BA)^T = A^TB^T = AB$. So AB is symmetric and hence is orthogonally diagonalizable, by Theorem 2.

32. If $A = PRP^{-1}$, then $P^{-1}AP = R$. Since P is orthogonal, $R = P^TAP$. Hence $R^T = (P^TAP)^T = P^TA^TP^{TT} = P^TAP = R$, which shows that R is symmetric. Since R is also upper triangular, its entries above the diagonal must be zeros, to match the zeros below the diagonal. Thus R is a diagonal matrix.

34. $A = 7\mathbf{u}_1\mathbf{u}_1^T + 7\mathbf{u}_2\mathbf{u}_2^T - 2\mathbf{u}_3\mathbf{u}_3^T$, where

$\mathbf{u}_1\mathbf{u}_1^T = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$,

$\mathbf{u}_2\mathbf{u}_2^T = \begin{bmatrix} 1/18 & -4/18 & -1/18 \\ -4/18 & 16/18 & 4/18 \\ -1/18 & 4/18 & 1/18 \end{bmatrix}$, and

$\mathbf{u}_3\mathbf{u}_3^T = \begin{bmatrix} 4/9 & 2/9 & -4/9 \\ 2/9 & 1/9 & -2/9 \\ -4/9 & -2/9 & 4/9 \end{bmatrix}$.

36. Given any $\mathbf{y}$ in $\mathbb{R}^n$, let $\hat{\mathbf{y}} = B\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$. Suppose $B^T = B$ and $B^2 = B$. Then $B^TB = BB = B$.

a. $\mathbf{z}\cdot\hat{\mathbf{y}} = (\mathbf{y} - B\mathbf{y})\cdot(B\mathbf{y}) = \mathbf{y}\cdot(B\mathbf{y}) - (B\mathbf{y})\cdot(B\mathbf{y})$
$= \mathbf{y}^TB\mathbf{y} - (B\mathbf{y})^TB\mathbf{y} = \mathbf{y}^TB\mathbf{y} - \mathbf{y}^TB^TB\mathbf{y} = 0$

So $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.

b. Any vector in $W = \text{Col } B$ has the form $B\mathbf{u}$ for some $\mathbf{u}$. To show that $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to $B\mathbf{u}$, use Exercise 28 since B is symmetric:

$(\mathbf{y} - \hat{\mathbf{y}})\cdot B\mathbf{u} = [B(\mathbf{y} - \hat{\mathbf{y}})]\cdot\mathbf{u} = [B\mathbf{y} - BB\mathbf{y}]\cdot\mathbf{u} = 0$

because $B^2 = B$. So $\mathbf{y} - \hat{\mathbf{y}}$ is in $W^\perp$, and the decomposition $\mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}})$ expresses $\mathbf{y}$ as the sum of a vector in W and a vector in $W^\perp$. By the Orthogonal Decomposition Theorem in Section 6.3, this decomposition is unique, and so $\hat{\mathbf{y}}$ must be $\text{proj}_W \mathbf{y}$.

37. **[M]** $P = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$,

$D = \begin{bmatrix} 18 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix}$

38. **[M]** $P = \begin{bmatrix} .8 & -.2 & .4 & -.4 \\ .4 & -.4 & -.2 & .8 \\ .4 & .4 & -.8 & -.2 \\ .2 & .8 & .4 & .4 \end{bmatrix}$,

$D = \begin{bmatrix} .25 & 0 & 0 & 0 \\ 0 & .30 & 0 & 0 \\ 0 & 0 & .55 & 0 \\ 0 & 0 & 0 & .75 \end{bmatrix}$

39. **[M]** $P = \begin{bmatrix} .7071 & .4243 & -.4 & -.4 \\ 0 & .5657 & -.2 & .8 \\ 0 & .5657 & .8 & -.2 \\ .7071 & -.4243 & .4 & .4 \end{bmatrix}$,

$D = \begin{bmatrix} .75 & 0 & 0 & 0 \\ 0 & .75 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.25 \end{bmatrix}$.

Note: $.4243 \approx 3/\sqrt{50}$ and $.5657 \approx 4/\sqrt{50}$.

40. **[M]** $P =$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & -2/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & 0 & -3/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & 0 & 0 & -4/\sqrt{20} & 1/\sqrt{5} \end{bmatrix},$$

$D = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 32 & 0 & 0 \\ 0 & 0 & 0 & -28 & 0 \\ 0 & 0 & 0 & 0 & 17 \end{bmatrix}$

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2. **a.** $4x_1^2 + 2x_2^2 + x_3^2 + 6x_1x_2 + 2x_2x_3$ **b.** 21 **c.** 5

4. **a.** $\begin{bmatrix} 20 & 7.5 \\ 7.5 & -10 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & .5 \\ .5 & 0 \end{bmatrix}$

6. **a.** $\begin{bmatrix} 5 & 5/2 & -3/2 \\ 5/2 & -1 & 0 \\ -3/2 & 0 & 7 \end{bmatrix}$

b. $\begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

8. $P = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, $\mathbf{y}^T D\mathbf{y} = 15y_1^2 + 9y_2^2 + 3y_3^2$

In Exercises 10–14, other answers (change of variables and new quadratic form) are possible.

10. Positive definite; eigenvalues are 11 and 1

Change of variable: $\mathbf{x} = P\mathbf{y}$, with $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

New quadratic form: $11y_1^2 + y_2^2$

12. Negative definite; eigenvalues are -1 and -6

Change of variable: $\mathbf{x} = P\mathbf{y}$, with $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

New quadratic form: $-y_1^2 - 6y_2^2$

14. Indefinite; eigenvalues are 9 and -1

Change of variable: $\mathbf{x} = P\mathbf{y}$ where $P = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

New quadratic form: $9y_1^2 - y_2^2$

16. **[M]** Positive definite; eigenvalues are 6.5 and 1.5

Change of variable: $\mathbf{x} = P\mathbf{y}$; $P = \frac{1}{\sqrt{50}}\begin{bmatrix} 3 & -4 & 3 & 4 \\ 5 & 0 & -5 & 0 \\ 4 & 3 & 4 & -3 \\ 0 & 5 & 0 & 5 \end{bmatrix}$

New quadratic form: $6.5y_1^2 + 6.5y_2^2 + 1.5y_3^2 + 1.5y_4^2$

18. **[M]** Indefinite; eigenvalues are 17, 1, -1, -7

Change of variable: $\mathbf{x} = P\mathbf{y}$;

$P = \begin{bmatrix} -3/\sqrt{12} & 0 & 0 & 1/2 \\ 1/\sqrt{12} & 0 & -2/\sqrt{6} & 1/2 \\ 1/\sqrt{12} & -1/\sqrt{2} & 1/\sqrt{6} & 1/2 \\ 1/\sqrt{12} & 1/\sqrt{2} & 1/\sqrt{6} & 1/2 \end{bmatrix}$

New quadratic form: $17y_1^2 + y_2^2 - y_3^2 - 7y_4^2$

20. 5

21. **a.** True, by the definition before Example 1, even though a nonsymmetric matrix could be used to compute values of a quadratic form.

b. True. See the paragraph following Example 3.

c. True, because the columns of P in Theorem 4 are eigenvectors of A. Review the Diagonalization Theorem (Theorem 5) in Section 5.3.

d. False. $Q(\mathbf{x}) = 0$ when $\mathbf{x} = \mathbf{0}$.

e. True. Theorem 5(a).

f. True. See the Numerical Note after Example 6.

22. **a.** True. See the paragraph before Example 1.

b. False. The matrix P must be orthogonal and make P^TAP diagonal. See the paragraph before Example 4.

c. False. There are also "degenerate" cases: a single point, two intersecting lines, or no points at all. See the subsection "A Geometric View of Principal Axes."

d. False. See the definition before Theorem 5.

e. True, by Theorem 5(b). If $\mathbf{x}^TA\mathbf{x}$ has only negative values for $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^TA\mathbf{x}$ is negative definite.

24. If $\det A > 0$, then by Exercise 23, $\lambda_1\lambda_2 > 0$, so that λ_1 and λ_2 have the same sign; also, $ad = \det A + b^2 > 0$.

a. If $\det A > 0$ and $a > 0$, then $d > 0$, too (because $ad > 0$). By Exercise 23, $\lambda_1 + \lambda_2 = a + d > 0$. Since λ_1 and λ_2 have the same sign, they are both positive. So Q is positive definite, by Theorem 5.

b. If $\det A > 0$ and $a < 0$, then $d < 0$, too. As in (a), we conclude that λ_1 and λ_2 are both negative and that Q is negative definite.

c. If $\det A < 0$, then by Exercise 23, $\lambda_1\lambda_2 < 0$, which shows that λ_1 and λ_2 have opposite signs. By Theorem 5, Q is indefinite.

26. We may assume $A = PDP^T$, with $P^T = P^{-1}$. The eigenvalues of A are all positive; denote them by $\lambda_1, \dots, \lambda_n$. Let C be the diagonal matrix with $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$ on the diagonal. Then $D = C^2 = C^TC$. If $B = PCP^T$, then B is positive definite because its eigenvalues are the positive numbers on the diagonal of C. Also,

$$\begin{aligned} B^TB &= (PCP^T)^T(PCP^T) = (P^{TT}C^TP^T)(PCP) \\ &= PC^TCP \qquad \text{Because } P^TP = I \\ &= PDP = A \end{aligned}$$

28. The eigenvalues of A are all positive, by Theorem 5. Since the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A (see Exercise 25 in Section 5.1), the eigenvalues of A^{-1} are positive. (Note that A^{-1} is symmetric.) By Theorem 5, the quadratic form $\mathbf{x}^TA^{-1}\mathbf{x}$ is positive definite.

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2. $\mathbf{x} = P\mathbf{y}$, where $P = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$

4. a. 5 **b.** $\pm\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ **c.** 2

6. a. $\frac{15}{2}$ **b.** $\pm\begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$ **c.** $\frac{5}{2}$

8. Any unit vector that is a linear combination of $\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$. Equivalently, any unit vector that is orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

10. $1 + \sqrt{17}$

12. Let $\mathbf{x}$ be a unit eigenvector for the eigenvalue λ. Then $\mathbf{x}^TA\mathbf{x} = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda$, because $\mathbf{x}^T\mathbf{x} = 1$. So λ must satisfy $m \le \lambda \le M$.

14. [M] a. 17 **b.** $\begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix}$ **c.** 13

16. [M] a. 9 **b.** $\begin{bmatrix} -2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$ **c.** 3

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2. 5, 0 **4.** 3, 1

6. $\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

10. $\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

12. $\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

14. From Exercise 7, $A = U\Sigma V^T$ with $V = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$. The first column of V is a unit vector at which $\|A\mathbf{x}\|$ is maximized.

16. a. rank $A = 2$

b. Basis for Col A: $\left\{\begin{bmatrix} -.86 \\ .31 \\ .41 \end{bmatrix}, \begin{bmatrix} -.11 \\ .68 \\ -.73 \end{bmatrix}\right\}$

Basis for Nul A: $\begin{bmatrix} .65 \\ .08 \\ -.16 \\ -.73 \end{bmatrix}, \begin{bmatrix} -.34 \\ .42 \\ -.84 \\ -.08 \end{bmatrix}$

18. The determinant of an orthogonal matrix U is ± 1, because $1 = \det I = \det U^TU = (\det U^T)(\det U) = (\det U)^2$

Suppose A is square and $A = U\Sigma V^T$. Then Σ is square, and

$$\begin{aligned}\det A &= (\det U)(\det \Sigma)(\det V^T)\\ &= \pm\det\Sigma = \pm\sigma_1\cdots\sigma_n\end{aligned}$$

20. If A is positive definite, then $A = PDP^T$, where P is an orthogonal matrix and D is a diagonal matrix. The diagonal entries of D are positive, because they are the eigenvalues of a positive definite matrix. Also, the matrix P^T is an orthogonal matrix because it is invertible, and the inverse and transpose of P^T coincide, since $(P^T)^{-1} = (P^{-1})^{-1} = P = (P^T)^T$. Thus, the factorization $A = PDP^T$ has the properties that make it a singular value decomposition.

22. The right singular vector $\mathbf{v}_1$ is an eigenvector for the largest eigenvalue λ_1 of A^TA. By Theorem 7 in Section 7.3, the second largest eigenvalue, λ_2, is the maximum of $\mathbf{x}^T(A^TA)\mathbf{x}$ over all unit vectors orthogonal to $\mathbf{v}_1$. Since $\mathbf{x}^T(A^TA)\mathbf{x} = \|A\mathbf{x}\|^2$, the square root of λ_2, which is the second singular value of A, is the maximum of $\|A\mathbf{x}\|$ over all unit vectors orthogonal to $\mathbf{v}_1$.

24. From Exercise 23, $A^T = \sigma_1\mathbf{v}_1\mathbf{u}_1^T + \cdots + \sigma_r\mathbf{v}_r\mathbf{u}_r^T$. Then

$$\begin{aligned}A^T\mathbf{u}_j &= (\sigma_j\mathbf{v}_j\mathbf{u}_j^T)\mathbf{u}_j && \text{Because } \mathbf{u}_i^T\mathbf{u}_j = 0 \text{ for } i \neq j\\ &= \sigma_j\mathbf{v}_j && \text{Because } \mathbf{u}_j^T\mathbf{u}_j = 1\end{aligned}$$

26. **[M]** $\begin{bmatrix} .5 & -.5 & -.5 & -.5\\ .5 & .5 & .5 & -.5\\ .5 & -.5 & .5 & .5\\ .5 & .5 & -.5 & .5\end{bmatrix}\begin{bmatrix} 40 & 0 & 0 & 0\\ 0 & 20 & 0 & 0\\ 0 & 0 & 10 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

$\times\begin{bmatrix} -.4 & .8 & -.2 & .4\\ .8 & .4 & .4 & .2\\ .4 & -.2 & -.8 & .4\\ -.2 & -.4 & .4 & .8\end{bmatrix}$

The entries in this exercise are simple, to allow students to check their work mentally or by hand. The *Study Guide* contains a sequence of MATLAB commands that produce this SVD.

28. **[M]** 27.3857, 12.0914, 2.61163, .00115635; $\sigma_1/\sigma_4 = 23{,}683$

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2. $M = \begin{bmatrix} 4\\ 9\end{bmatrix}$; $B = \begin{bmatrix} -3 & 1 & -2 & 2 & 3 & -1\\ -6 & 2 & -3 & -1 & 6 & 2\end{bmatrix}$,

$S = \begin{bmatrix} 5.6 & 8\\ 8 & 18\end{bmatrix}$

4. $\begin{bmatrix} .44\\ .90\end{bmatrix}$ for $\lambda = 21.9$, $\begin{bmatrix} -.90\\ .44\end{bmatrix}$ for $\lambda = 1.7$

6. **[M]** $y_1 = .62x_1 + .60x_2 + .51x_3$, which explains 64.9% of the total variance.

8. $y_1 = .44x_1 + .90x_2$; y_1 explains 92.9% of the variance.

10. **[M]** $c_1 = .41$, $c_2 = .82$, $c_3 = .41$ to two decimal places, or $c_1 = 1/\sqrt{6}$, $c_2 = 2/\sqrt{6}$, $c_3 = 1/\sqrt{6}$. The variance of y is 15.

12. By Exercise 11, the change of variable $\mathbf{X} = P\mathbf{Y}$ changes the covariance matrix S of $\mathbf{X}$ into the covariance matrix P^TSP of $\mathbf{Y}$. The total variance of the data, as described by $\mathbf{Y}$, is $\text{tr}\,(P^TSP)$. However, since P^TSP is similar to S, they have the same trace (Exercise 25 in Section 5.4). Thus the total variance of the data is unchanged by this change of variable.

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1. **a.** True. This is part of Theorem 2 in Section 7.1. The proof appears just before the statement of the theorem.

b. False. Counterexample: $A = \begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}$.

c. True. This is proved in the first part of the proof of Theorem 6 in Section 7.3. It is also a consequence of Theorem 7 in Section 6.2.

d. False. The principal axes of $\mathbf{x}^TA\mathbf{x}$ are the columns of any *orthogonal* matrix P that diagonalizes A. *Note:* When A has an eigenvalue whose eigenspace has dimension greater than 1 (for example, when $A = I$), the principal axes are not uniquely determined.

e. False. Counterexample: $P = \begin{bmatrix} 1 & -1\\ 1 & 1\end{bmatrix}$. The columns here are orthogonal but not orthonormal. If P is a square matrix with ortho*normal* columns, $P^T = P^{-1}$.

f. False. See Example 6 in Section 7.2.

g. False. Counterexample: $A = \begin{bmatrix} 2 & 0\\ 0 & -3\end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1\\ 0\end{bmatrix}$, then $\mathbf{x}^TA\mathbf{x} = 2 > 0$, but $\mathbf{x}^TA\mathbf{x}$ is an indefinite quadratic form.

h. True. This is basically the Principal Axes Theorem (Section 7.2). Any quadratic form can be written as $\mathbf{x}^TA\mathbf{x}$ for some symetric matrix A.

i. False. See Example 3 in Section 7.3.

j. False. The maximum value must be computed over the set of *unit* vectors. Without a restriction on the norm of $\mathbf{x}$, the values of $\mathbf{x}^TA\mathbf{x}$ can be made as large as desired.

k. False. Any orthogonal change of variable $\mathbf{x} = P\mathbf{y}$ changes a positive definite form into another positive definite form. Proof: By Theorem 5 of Section 7.2, the classification of a quadratic form is determined by the eigenvalues of the matrix of the form. Given a form $\mathbf{x}^TA\mathbf{x}$, the matrix of the new quadratic form is $P^{-1}AP$, which is similar to A and therefore has the same eigenvalues as A.

l. False. The term "definite eigenvalue" is undefined and therefore meaningless.

m. True. If $\mathbf{x} = P\mathbf{y}$, then $\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T(P^{-1}AP)\mathbf{y}$.

n. False. Counterexample: Let $U = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. The columns of U must be ortho*normal* to make $UU^T\mathbf{x}$ the orthogonal projection of $\mathbf{x}$ onto Col U.

o. True. This follows from the discussion in Example 2 of Section 7.4, which refers to a proof given in Example 1.

p. True. Theorem 10 in Section 7.4 writes the decomposition in the form $U\Sigma V^T$, where U and V are orthogonal matrices. In this case, V^T is also an orthogonal matrix. [Proof: Because V is orthogonal, V is invertible and $V^{-1} = V^T$. Then $(V^{-1})^T = (V^T)^T$ and $(V^T)^{-1} = (V^T)^T$. Since V^T is square and invertible, the second equality shows that V^T is an orthogonal matrix.]

q. False. Counterexample: The singular values of $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ are 2 and 1, but the singular values of $A^T A$ are 4 and 1.

2. a. Each term in the expansion of A is symmetric, by Exercise 35 in Section 7.1. The fact that $(B + C)^T = B^T + C^T$ implies that any sum of symmetric matrices is symmetric. So A is symmetric. A direct calculation also shows that $A^T = A$.

b.
$$A\mathbf{u}_1 = (\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T)\mathbf{u}_1 + \cdots + (\lambda_n \mathbf{u}_n \mathbf{u}_n^T)\mathbf{u}_1 = \lambda_1 \mathbf{u}_1$$
because $\mathbf{u}_1^T\mathbf{u}_1 = 1$ and $\mathbf{u}_j^T\mathbf{u}_1 = 0$ for $j \neq 1$. Since $\mathbf{u}_1 \neq \mathbf{0}$, λ_1 is an eigenvalue of A. A similar argument shows that for $j = 2, \dots, n$, λ_j is an eigenvalue of A.

4. a. By Theorem 3 in Section 6.1, $(\text{Col } A)^\perp = \text{Nul } A^T = \text{Nul } A$, because $A^T = A$.

b. Take $\mathbf{y}$ in $\mathbb{R}^n$. By the Orthogonal Decomposition Theorem (Section 6.3), $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, with $\hat{\mathbf{y}}$ in Col A and $\mathbf{z}$ in $(\text{Col } A)^\perp$. By part (a), $\mathbf{z}$ is in Nul A, which concludes the proof.

6. Because A is symmetric, there is an orthonormal eigenvector basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ for $\mathbb{R}^n$. Let $r = \text{rank } A$. If $r = 0$, then $A = 0$, and the decomposition of Exercise 4(b) is $\mathbf{y} = \mathbf{0} + \mathbf{y}$ for each $\mathbf{y}$ in $\mathbb{R}^n$; if $r = n$, then the decomposition is $\mathbf{y} = \mathbf{y} + \mathbf{0}$ for each $\mathbf{y}$.

So, assume $0 < r < n$. Then dim Nul $A = n - r$, by the Rank Theorem, and so 0 is an eigenvalue with multiplicity $n - r$. Hence, there are r nonzero eigenvalues, counted according to their multiplicities. Renumber the eigenvector basis, if necessary, so that $\mathbf{u}_1, \dots, \mathbf{u}_r$ are the eigenvectors corresponding to the nonzero eigenvalues.

By Exercise 5, $\mathbf{u}_1, \dots, \mathbf{u}_r$ are in Col A. Also, $\mathbf{u}_{r+1}, \dots, \mathbf{u}_n$ are in Nul A, because these vectors are eigenvectors for $\lambda = 0$. For $\mathbf{y}$ in $\mathbb{R}^n$, there are scalars $c_1, \dots, c_n$ such that

$$\mathbf{y} = \underbrace{c_1\mathbf{u}_1 + \cdots + c_r\mathbf{u}_r}_{\hat{\mathbf{y}}} + \underbrace{c_{r+1}\mathbf{u}_{r+1} + \cdots + c_n\mathbf{u}_n}_{\mathbf{z}}$$

This provides the decomposition in Exercise 4(b).

8. Suppose A is positive definite, and consider a Cholesky factorization $A = R^T R$, with R upper triangular and having positive entries on its diagonal. Let D be the diagonal matrix whose diagonal entries are the entries on the diagonal or R. Since right-multiplication by a diagonal matrix scales the columns of the matrix on its left, the matrix $L = R^T D^{-1}$ is lower triangular with 1's on its diagonal. If $U = DR$, then $A = R^T D^{-1} DR = LU$.

10. If rank $G = r$, then dim Nul $G = n - r$, by the Rank Theorem. Hence 0 is an eigenvalue of multiplicity $n - r$, and the spectral decomposition of G is

$$G = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_r \mathbf{u}_r \mathbf{u}_r^T$$

Also, $\lambda_1, \dots, \lambda_r$ are positive because G is positive semidefinite. Thus

$$G = \left(\sqrt{\lambda_1}\mathbf{u}_1\right)\left(\sqrt{\lambda_1}\mathbf{u}_1\right)^T + \cdots + \left(\sqrt{\lambda_r}\mathbf{u}_r\right)\left(\sqrt{\lambda_r}\mathbf{u}_r\right)^T$$

By the column–row expansion of a matrix product, $G = BB^T$, where B is the $n \times r$ matrix:

$$B = \begin{bmatrix} \sqrt{\lambda_1}\mathbf{u}_1 & \cdots & \sqrt{\lambda_r}\mathbf{u}_r \end{bmatrix}$$

Finally, $G = A^T A$ for $A = B^T$.

12. a. Because the columns of V_r are orthonormal, $AA^+\mathbf{y} = (U_r D V_r^T)(V_r D^{-1} U_r^T)\mathbf{y} = (U_r D D^{-1} U_r^T)\mathbf{y} = U_r U_r^T \mathbf{y}$. Since $U_r U_r^T \mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto Col U_r (by Theorem 10 in Section 6.3), and since Col U_r = Col A by (5) in Example 6 of Section 7.4, $AA^+\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto Col A.

b. $A^+A\mathbf{x} = (V_r D^{-1} U_r^T)(U_r D V_r^T)\mathbf{x} = (V_r D^{-1} D V_r^T)\mathbf{x} = V_r V_r^T \mathbf{x}$. Since $V_r V_r^T \mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto Col V_r, and since Col V_r = Row A by (8) in Example 6 of Section 7.4, $A^+A\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto Row A.

c. Use the reduced singular value decomposition of A, the definition of A^+, and associativity of matrix multiplication:

$$\begin{aligned} AA^+A &= (U_r D V_r^T)(V_r D^{-1} U_r^T)(U_r D V_r^T) \\ &= (U_r D D^{-1} U_r^T)(U_r D V_r^T) = (U_r D D^{-1} D V_r^T) \\ &= U_r D V_r^T = A \end{aligned}$$

$$\begin{aligned} A^+AA^+ &= (V_r D^{-1} U_r^T)(U_r D V_r^T)(V_r D^{-1} U_r^T) \\ &= (V_r D^{-1} D V_r^T)(V_r D^{-1} U_r^T) = (V_r D^{-1} D D^{-1} U_r^T) \\ &= V_r D^{-1} U_r^T = A^+ \end{aligned}$$

14. The least-squares solutions of $A\mathbf{x} = \mathbf{b}$ are precisely the solutions of $A\mathbf{x} = \hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is the orthogonal projection of $\mathbf{b}$ onto Col A. From Exercise 13, the minimum length solution of $A\mathbf{x} = \hat{\mathbf{b}}$ is $A^+\hat{\mathbf{b}}$, so $A^+\hat{\mathbf{b}}$ is the minimum length least-squares solution of $A\mathbf{x} = \mathbf{b}$. However, $\hat{\mathbf{b}} = AA^+\mathbf{b}$, by Exercise 12(a), and hence $A^+\hat{\mathbf{b}} = A^+AA^+\mathbf{b} = A^+\mathbf{b}$, by Exercise 12(c). Thus $A^+\mathbf{b}$ is the minimum length least-squares solution of $A\mathbf{x} = \mathbf{b}$.

16. [**M**] $A^+ = \begin{bmatrix} .5 & 0 & -.05 & -.15 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & .50 & 1.50 \\ .5 & -1 & -.35 & -1.05 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\hat{\mathbf{x}} = \begin{bmatrix} 2.3 \\ 0 \\ 5.0 \\ -.9 \\ 0 \end{bmatrix}$

Basis for Nul A: $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Adding any nonzero vector $\mathbf{u}$ in Nul A to $\hat{\mathbf{x}}$ changes a zero entry to a nonzero entry; in this case the inequality $\|\hat{\mathbf{x}}\| < \|\hat{\mathbf{x}} + \mathbf{u}\|$ is evident.