APPENDIX A

Uniqueness of the Reduced Echelon Form

THEOREM **Uniqueness of the Reduced Echelon Form**

Each $m \times n$ matrix A is row equivalent to a unique reduced echelon matrix U.

PROOF The proof uses the idea from Section 4.3 that the columns of row-equivalent matrices have exactly the same linear dependence relations.

The row reduction algorithm shows that there exists at least one such matrix U. Suppose that A is row equivalent to matrices U and V in reduced echelon form. The leftmost nonzero entry in a row of U is a "leading 1." Call the location of such a leading 1 a pivot position, and call the column that contains it a pivot column. (This definition uses only the echelon nature of U and V and does not assume the uniqueness of the reduced echelon form.)

The pivot columns of U and V are precisely the nonzero columns that are *not* linearly dependent on the columns to their left. (This condition is satisfied automatically by a *first* column if it is nonzero.) Since U and V are row equivalent (both being row equivalent to A), their columns have the same linear dependence relations. Hence, the pivot columns of U and V appear in the same locations. If there are r such columns, then since U and V are in reduced echelon form, their pivot columns are the first r columns of the $m \times m$ identity matrix. Thus, *corresponding pivot columns of U and V are equal*.

Finally, consider any nonpivot column of U, say column j. This column is either zero or a linear combination of the pivot columns to its left (because those pivot columns are a basis for the space spanned by the columns to the left of column j). Either case can be expressed by writing $U\mathbf{x} = \mathbf{0}$ for some $\mathbf{x}$ whose jth entry is 1. Then $V\mathbf{x} = \mathbf{0}$, too, which says that column j of V is either zero or the *same* linear combination of the pivot columns of V to *its* left. Since corresponding pivot columns of U and V are equal, columns j of U and V are also equal. This holds for all nonpivot columns, so $V = U$, which proves that U is unique. ■

APPENDIX B

Complex Numbers

A **complex number** is a number written in the form

$$z = a + bi$$

where a and b are real numbers and i is a formal symbol satisfying the relation $i^2 = -1$. The number a is the **real part** of z, denoted by $\text{Re}\, z$, and b is the **imaginary part** of z, denoted by $\text{Im}\, z$. Two complex numbers are considered equal if and only if their real and imaginary parts are equal. For example, if $z = 5 + (-2)i$, then $\text{Re}\, z = 5$ and $\text{Im}\, z = -2$. For simplicity, we write $z = 5 - 2i$.

A real number a is considered as a special type of complex number, by identifying a with $a + 0i$. Furthermore, arithmetic operations on real numbers can be extended to the set of complex numbers.

The **complex number system**, denoted by $\mathbb{C}$, is the set of all complex numbers, together with the following operations of addition and multiplication:

$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{1}$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \tag{2}$$

These rules reduce to ordinary addition and multiplication of real numbers when b and d are zero in (1) and (2). It is readily checked that the usual laws of arithmetic for $\mathbb{R}$ also hold for $\mathbb{C}$. For this reason, multiplication is usually computed by algebraic expansion, as in the following example.

EXAMPLE 1

$$\begin{aligned} (5 - 2i)(3 + 4i) &= 15 + 20i - 6i - 8i^2 \\ &= 15 + 14i - 8(-1) \\ &= 23 + 14i \end{aligned}$$

That is, multiply each term of $5 - 2i$ by each term of $3 + 4i$, use $i^2 = -1$, and write the result in the form $a + bi$.

Subtraction of complex numbers z_1 and z_2 is defined by

$$z_1 - z_2 = z_1 + (-1)z_2$$

In particular, we write $-z$ in place of $(-1)z$.

The **conjugate** of $z = a + bi$ is the complex number $\overline{z}$ (read as "z bar"), defined by

$$\overline{z} = a - bi$$

Obtain $\overline{z}$ from z by reversing the sign of the imaginary part.

EXAMPLE 2 The conjugate of $-3 + 4i$ is $-3 - 4i$; write $\overline{-3 + 4i} = -3 - 4i$.

Observe that if $z = a + bi$, then

$$z\overline{z} = (a + bi)(a - bi) = a^2 - abi + bai - b^2i^2 = a^2 + b^2 \tag{3}$$

Since $z\overline{z}$ is real and nonnegative, it has a square root. The **absolute value** (or **modulus**) of z is the real number $|z|$ defined by

$$|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$$

If z is a real number, then $z = a + 0i$, and $|z| = \sqrt{a^2}$, which equals the ordinary absolute value of a.

Some useful properties of conjugates and absolute value are listed below; w and z denote complex numbers.

1. $\overline{z} = z$ if and only if z is a real number.
2. $\overline{w + z} = \overline{w} + \overline{z}$.
3. $\overline{wz} = \overline{w}\,\overline{z}$; in particular, $\overline{rz} = r\overline{z}$ if r is a real number.
4. $z\overline{z} = |z|^2 \geq 0$.
5. $|wz| = |w||z|$.
6. $|w + z| \leq |w| + |z|$.

If $z \neq 0$, then $|z| > 0$ and z has a multiplicative inverse, denoted by $1/z$ or z^{-1} and given by

$$\frac{1}{z} = z^{-1} = \frac{\overline{z}}{|z|^2}$$

Of course, a quotient w/z simply means $w \cdot (1/z)$.

EXAMPLE 3 Let $w = 3 + 4i$ and $z = 5 - 2i$. Compute $z\bar{z}$, $|z|$, and w/z.

Solution From (3),

$$z\bar{z} = 5^2 + (-2)^2 = 25 + 4 = 29$$

For the absolute value, $|z| = \sqrt{z\bar{z}} = \sqrt{29}$. To compute w/z, first multiply both the numerator and the denominator by $\bar{z}$, the conjugate of the denominator. Because of (3), this eliminates the i in the denominator:

$$\begin{aligned}\frac{w}{z} &= \frac{3+4i}{5-2i} \\ &= \frac{3+4i}{5-2i}\cdot\frac{5+2i}{5+2i} \\ &= \frac{15+6i+20i-8}{5^2+(-2)^2} \\ &= \frac{7+26i}{29} \\ &= \frac{7}{29} + \frac{26}{29}i\end{aligned}$$

Geometric Interpretation

Each complex number $z = a + bi$ corresponds to a point (a, b) in the plane $\mathbb{R}^2$, as in Fig. 1. The horizontal axis is called the **real axis** because the points $(a, 0)$ on it correspond to the real numbers. The vertical axis is the **imaginary axis** because the points $(0, b)$ on it correspond to the **pure imaginary numbers** of the form $0 + bi$, or simply bi. The conjugate of z is the mirror image of z the real axis. The absolute value of z is the distance from (a, b) to the origin.

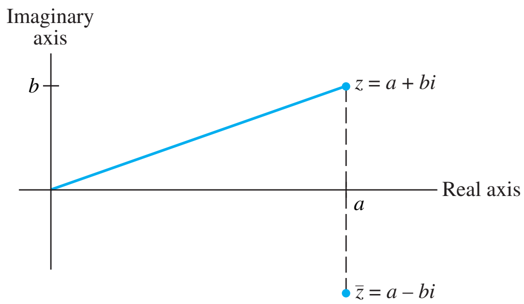


FIGURE 1 The complex conjugate is a mirror image.

Addition of complex numbers $z = a + bi$ and $w = c + di$ corresponds to vector addition of (a, b) and (c, d) in $\mathbb{R}^2$, as in Fig. 2.

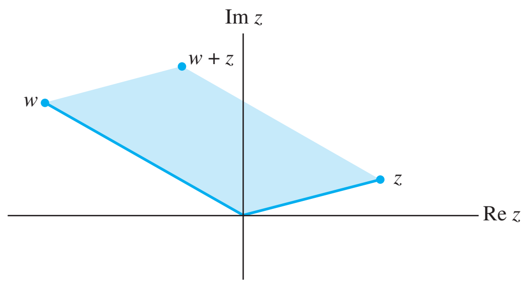


FIGURE 2 Addition of complex numbers.

To give a graphical representation of complex multiplication, we use **polar coordinates** in $\mathbb{R}^2$. Given a nonzero complex number $z = a + bi$, let φ be the angle between the positive real axis and the point (a, b), as in Fig. 3 where $-\pi < \varphi \leq \pi$. The angle φ is called the **argument** of z; we write $\varphi = \arg z$. From trigonometry,

$$a = |z| \cos \varphi, \qquad b = |z| \sin \varphi$$

and so

$$z = a + bi = |z|(\cos \varphi + i \sin \varphi)$$

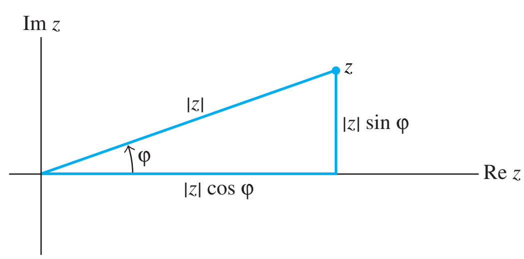


FIGURE 3 Polar coordinates of z.

If w is another nonzero complex number, say,

$$w = |w| (\cos \vartheta + i \sin \vartheta)$$

then using standard trigonometric identities for the sine and cosine of the sum of two angles, one can verify that

$$wz = |w|\,|z|\,[\cos(\vartheta + \varphi) + i \sin(\vartheta + \varphi)] \tag{4}$$

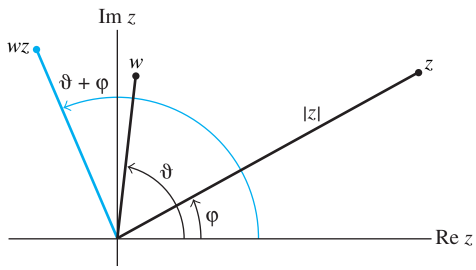
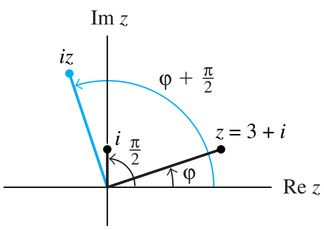


FIGURE 4 Multiplication with polar coordinates.

See Fig. 4. A similar formula may be written for quotients in polar form. The formulas for products and quotients can be stated in words as follows.

The product of two nonzero complex numbers is given in polar form by the product of their absolute values and the sum of their arguments. The quotient of two nonzero complex numbers is given by the quotient of their absolute values and the difference of their arguments.



Multiplication by i.

EXAMPLE 4

a. If w has absolute value 1, then $w = \cos\vartheta + i\sin\vartheta$, where ϑ is the argument of w. Multiplication of any nonzero number z by w simply rotates z through the angle ϑ.

b. The argument of i itself is $\pi/2$ radians, so multiplication of z by i rotates z through an angle of $\pi/2$ radians. For example, $3+i$ is rotated into $(3+i)i = -1+3i$.

Powers of a Complex Number

Formula (4) applies when $z = w = r(\cos\varphi + i\sin\varphi)$. In this case

$$z^2 = r^2(\cos 2\varphi + i\sin 2\varphi)$$

and

$$\begin{aligned} z^3 &= z\cdot z^2 \\ &= r(\cos\varphi + i\sin\varphi)\cdot r^2(\cos 2\varphi + i\sin 2\varphi) \\ &= r^3(\cos 3\varphi + i\sin 3\varphi) \end{aligned}$$

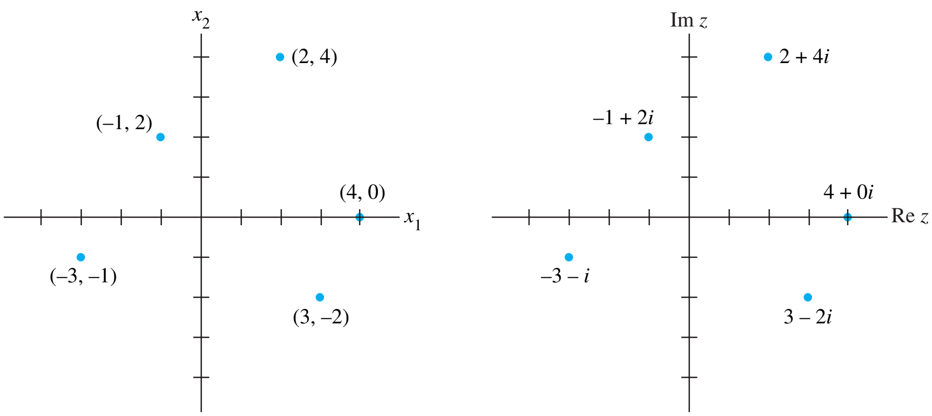
In general, for any positive integer k,

$$z^k = r^k(\cos k\varphi + i\sin k\varphi)$$

This fact is known as *De Moivre's Theorem*.

Complex Numbers and $\mathbb{R}^2$

Although the elements of $\mathbb{R}^2$ and $\mathbb{C}$ are in one-to-one correspondence, and the operations of addition are essentially the same, there is a logical distinction between $\mathbb{R}^2$ and $\mathbb{C}$. In $\mathbb{R}^2$ we can only multiply a vector by a real scalar, whereas in $\mathbb{C}$ we can multiply any two complex numbers to obtain a third complex number. (The dot product in $\mathbb{R}^2$ doesn't count, because it produces a scalar, not an element of $\mathbb{R}^2$.) We use scalar notation for elements in $\mathbb{C}$ to emphasize this distinction.



The real plane $\mathbb{R}^2$. The complex plane $\mathbb{C}$.