CHAPTER 3

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

CLUSTER 1 Exponential Functions

ACTIVITY 3.1

The Summer Job

OBJECTIVES

1. Determine the growth or decay factor of an exponential function.
2. Identify the properties of the graph of an exponential function defined by $y = b^x$, where $b > 0$ and $b \neq 1$.
3. Graph an exponential function.

Your neighbor's son will be attending college in the fall, majoring in mathematics. On July 1, he comes to your house looking for summer work to help pay for college expenses. You are interested since you need some odd jobs done, but you don't have a lot of extra money to pay him. He can start right away and will work all day July 1 for 2 cents. This gets your attention, but you wonder if there is a catch. He says that he will work July 2 for 4 cents, July 3 for 8 cents, July 4 for 16 cents, and so on for *every* day of the month of July.

1. Do you hire him?

For Problems 2–8, assume that you do hire him.

2. How much will he earn on July 5? July 6?

3. What will be his total pay for the first week of July (July 1 through July 7)?

4. a. Complete the following table.

DAY IN JULY (Input)	PAY IN CENTS (Output)
1	
2	
3	
4	
5	
6	
7	
8	

b. Do you notice a pattern in the output values? Describe how you can obtain the pay on a given day knowing the pay on the previous day.

c. Use what you discovered in part b to determine the pay on July 9.

5. a. The pay on any given day can be written as a power of 2. Write each pay entry in the output column of the table in Problem 4 as a power of 2. For example, $2 = 2^1$, $4 = 2^2$.

b. Let n represent the number of days worked. Write an equation for the daily pay, $P(n)$, (in cents) as a function of n, the number of days worked. Note that the number of days worked is the same as the July date.

c. Use the equation from part b to determine how much your neighbor's son will earn on July 20. That is, determine the value of $P(n)$ when $n = 20$. What are the units of measurement of your answer?

d. How much will he earn on July 31? Be sure to indicate the units of your answer.

e. Was it a good idea to hire him?

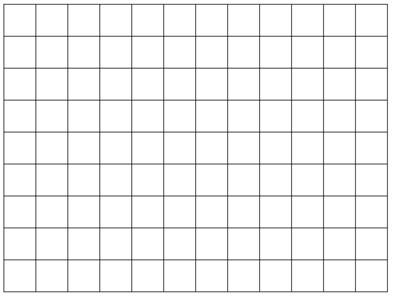
6. a. Determine the average rate of change of $P(n)$ as n increases from $n = 3$ to $n = 4$. What are the units of measurement of your answer?

b. Determine the average rate of change of $P(n)$ as n increases from $n = 7$ to $n = 8$. Include units in your answer.

c. Is the function linear? Explain.

7. a. What is the practical domain of the function defined by $P(n) = 2^n$?

b. Sketch a scatterplot of ordered pairs of the form $(n, P(n))$ from July 1 to July 10 on appropriately scaled and labeled axes.



The function defined by $P(n) = 2^n$ gives the relationship between the pay $P(n)$ (in cents) and the given July date, n, worked. This function belongs to a family of functions called **exponential functions**.

Some **exponential functions** can be defined by equations of the form $y = b^x$, where the base b is a constant such that b is a positive number not equal to 1 ($b > 0$ and $b \neq 1$). Such functions are called **exponential** functions because the independent variable (input) x is the exponent.

EXAMPLE 1 ***Some examples of exponential functions are***

$$g(x) = 10^x, \text{ where } b = 10, \; h(x) = (1.08)^x, \text{ where } b = 1.08,$$

$$V(x) = \left(\frac{1}{2}\right)^x, \text{ where } b = \frac{1}{2}, \text{ and } T(x) = (0.75)^x, \text{ where } b = 0.75.$$

Graphs of Exponential Functions

Because n in $P(n) = 2^n$ (the summer job situation) represents a given day in July, the practical domain (whole numbers from 1 to 31) limits the investigation of the exponential function.

8. a. Consider the general function defined by $f(x) = 2^x$. Use your graphing calculator to sketch a graph of this function. Use the window Xmin $= -10$, Xmax $= 10$, Ymin $= -2$, and Ymax $= 10$.

b. Because the graph of the general function $f(x) = 2^x$ is continuous (it has no holes or breaks), what appears to be the domain of the function f? What is the range of the function f?

c. Determine the y-intercept of the graph of f by substituting 0 for x in the equation $y = 2^x$ and solving for y.

d. Is the function f increasing or decreasing?

DEFINITION

If the base b of an exponential function defined by $y = b^x$ is greater than 1, then b is the **growth factor**. The graph of $y = b^x$ is increasing if $b > 1$. For each increase of 1 of the value of the input, the output increases by a factor of b.

EXAMPLE 2 ***The base 2 of $f(x) = 2^x$ is the growth factor because each time the input, x, is increased by 1, the output is multiplied by 2.***

9. Identify the growth factor, if any, for the given function.

a. $y = 1.08^x$

b. $h(x) = 0.8^x$

c. $y = 8x$

d. $g(x) = 10^x$

10. Return to the graph of $f(x) = 2^x$.

a. Does the graph of $f(x) = 2^x$ appear to have an x-intercept?

b. Use your calculator to complete the following table.

x	-1	-2	-4	-6	-8	-10
$f(x) = 2^x$						

Note: 2^{-10} is equivalent to $\frac{1}{2^{10}} \approx 0.000977$.

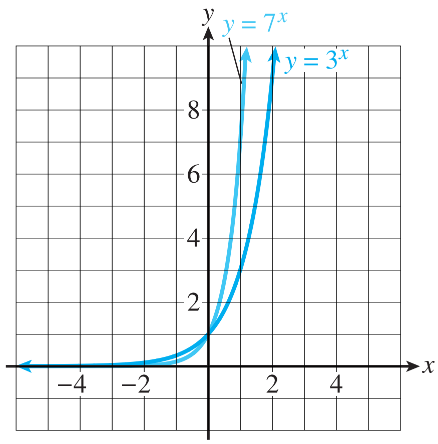
c. As the values of the input variable x decrease, what happens to the output values?

d. Use the trace feature of your graphing calculator to trace the graph of $f(x) = 2^x$ for $x < 0$. What appears to be the relationship between the graph of $y = 2^x$ and the x-axis when x becomes more negative?

DEFINITION

A horizontal axis having equation $y = 0$ is called a **horizontal asymptote** of the graph of a function defined by $y = b^x$, where $b > 0$ and $b \neq 1$. The graph of the function gets closer and closer to the x-axis $(y = 0)$ as the input gets farther from the origin, in the negative direction.

EXAMPLE 3 ***The x-axis is the horizontal asymptote of $y = 3^x$ and $y = 7^x$ because, as x gets more negative, the graph gets closer and closer to the x-axis. See the graph that follows.***



11. a. Complete the following table.

x	−3	−2	−1	0	1	2	3	4	5
$f(x) = 2^x$									
$g(x) = 10^x$									

b. Sketch the graph of the functions f and g on your graphing calculator. Use the window Xmin = −5, Xmax = 5, Ymin = −2, and Ymax = 9.

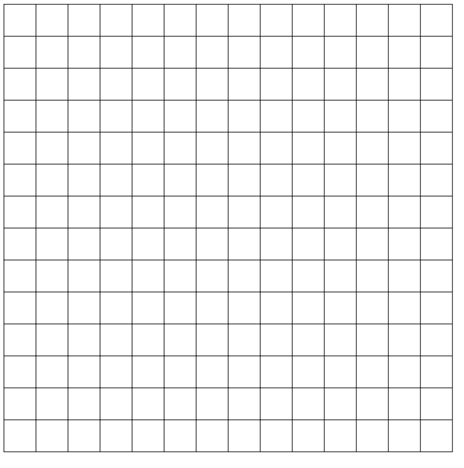
c. Use the results from parts a and b to describe how the graphs of $f(x) = 2^x$ and $g(x) = 10^x$ are similar and how they are different. Be sure to include domain, growth factor, x- and y-intercepts, and horizontal asymptotes. Also discuss whether the graph of g increases faster or slower than the graph of f.

12. a. Complete the following table.

x	-3	-2	-1	0	1	2	3	4	5
$V(x) = \left(\frac{1}{2}\right)^x$									

b. Describe how you can obtain the output value for $x = 6$, using the output value for $x = 5$.

c. Sketch the graph of $V(x) = \left(\frac{1}{2}\right)^x$. Verify your sketch using your graphing calculator.



d. What are the domain and range of the function V?

e. Determine the vertical intercept of the graph of V.

f. Is the function V increasing or decreasing?

DEFINITION

If the base b of an exponential function $y = b^x$ is between 0 and 1, then b is the **decay factor**. The graph of $y = b^x$ is decreasing if $0 < b < 1$. For each increase of 1 of the value of the input, the output decreases by a factor of b.

EXAMPLE 4 ***The base $\frac{1}{2}$ in the function $V(x) = \left(\frac{1}{2}\right)^x$ is the* decay *factor because each time x is increased by* 1, *the output value is multiplied by $\frac{1}{2}$.***

13. Identify the decay factor, if any, for the given function.

a. $y = 0.98^x$

b. $h(x) = 1.8^x$

c. $y = 0.8x$

d. $g(x) = \left(\frac{2}{7}\right)^x$

14. Return to the graph of $V(x) = \left(\frac{1}{2}\right)^x$.

a. Does the graph of $V(x) = \left(\frac{1}{2}\right)^x$ have an x-intercept?

b. Complete the following table.

x	1	3	5	7	10
$V(x) = \left(\frac{1}{2}\right)^x$					

c. As the values of the input variable x get larger, what happens to the output values?

d. Does the graph of V have a horizontal asymptote? Explain.

15. a. For each of the following exponential functions, identify the base, b, and determine whether the base is a growth or decay factor. Graph each function on your graphing calculator, and complete the table below.

FUNCTION	BASE, b	GROWTH OR DECAY FACTOR	x-INTERCEPT	y-INTERCEPT	HORIZONTAL ASYMPTOTE	INCREASING OR DECREASING
$h(x) = (1.08)^x$						
$T(x) = (0.75)^x$						
$f(x) = (3.2)^x$						
$r(x) = \left(\frac{1}{4}\right)^x$						

b. Without graphing, how might you determine which of the functions in part a increase and which decrease? Explain.

16. Examine the output pattern to determine which of the following data sets is linear and which is exponential. For the linear set, determine the slope. For the exponential set, determine the growth or decay factor.

a.

x	-2	-1	0	1	2	3	4
y	-8	-4	0	4	8	12	16

b.

x	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64	256

17. Determine the decay factor of the function represented by the data, and complete the table.

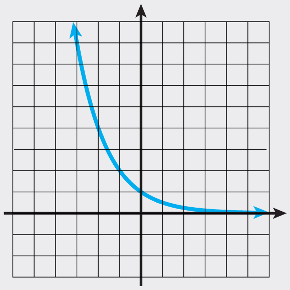
x	-2	-1	0	1	2
$f(x)$	16	4			

SUMMARY
ACTIVITY 3.1

Functions defined by equations of the form $y = b^x$, where $b > 0$ and $b \neq 1$, are called **exponential functions** and have the following properties.

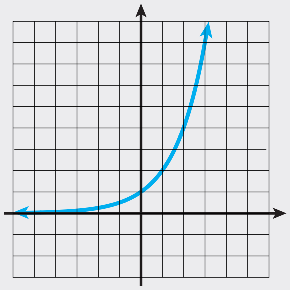
1. The domain is all real numbers.
2. The range is $y > 0$.

3. If $0 < b < 1$, the function is decreasing and has the following general shape.



In this case, b is called the **decay factor**.

4. If $b > 1$, the function is increasing and has the following general shape.



In this case, b is called the **growth factor**.

5. The vertical intercept (y-intercept) is (0, 1).

6. The graph does not intersect the horizontal axis. There is no x-intercept.

7. The line $y = 0$ (the x-axis) is a **horizontal asymptote**.

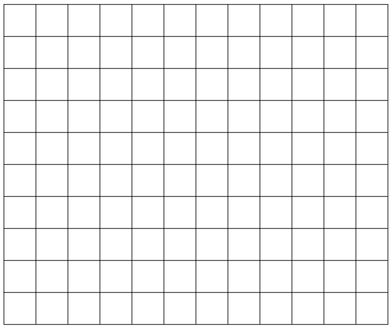
EXERCISES
ACTIVITY 3.1

1. **a.** Complete the following tables.

x	−3	−2	−1	0	1	2	3
$h(x) = 5^x$							

x	−3	−2	−1	0	1	2	3
$g(x) = \left(\frac{1}{5}\right)^x$							

b. Sketch graphs of *h* and *g* on the following grid.



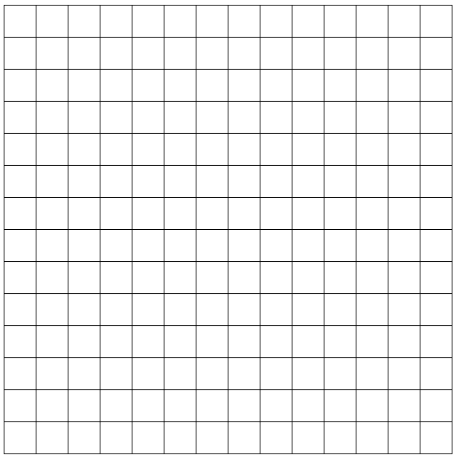
c. Use the tables and graphs in parts a and b to complete the following table.

FUNCTION	BASE, *b*	GROWTH OR DECAY FACTOR	*x*-INTERCEPT	*y*-INTERCEPT	HORIZONTAL ASYMPTOTE	INCREASING OR DECREASING
$h(x) = 5^x$						
$g(x) = \left(\frac{1}{5}\right)^x$						

2. a. Complete the following table.

x	-3	-2	-1	0	1	2	3
$f(x) = 3^x$							
$g(x) = x^3$							
$h(x) = 3x$							

b. Sketch a graph of each of the given functions *f*, *g*, and *h*.



c. Describe any similarities or differences that you observe in the graphs.

3. Using your graphing calculator, investigate the graphs of the following families (groups) of functions. Describe any relationships within each family, including domain and range, growth or decay factors, vertical and horizontal intercepts, and asymptotes. Identify the functions as increasing or decreasing.

a. $f(x) = \left(\frac{3}{4}\right)^x, g(x) = \left(\frac{4}{3}\right)^x$

b. $f(x) = 10^x, g(x) = -10^x$

c. $f(x) = 3^x, g(x) = \left(\frac{1}{3}\right)^x$

4. Determine which of the following data sets are linear and which are exponential. For the linear sets, determine the slope. For the exponential sets, determine the growth factor or the decay factor.

a.

x	−2	−1	0	1	2	3	4
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81

b.

x	−2	−1	0	1	2	3	4
y	2	2.5	3	3.5	4	4.5	5

c.

x	−2	−1	0	1	2	3	4
y	0.75	1.5	3	6	12	24	48

d.

x	−2	−1	0	1	2	3	4
y	6.25	2.5	1	0.4	0.16	.064	.0256

5. Assume that y is an exponential function of x.

a. If the growth factor is 1.08, then complete the following table.

x	0	1	2	3
y	23.1			

b. If the decay factor is 0.75, then complete the following table.

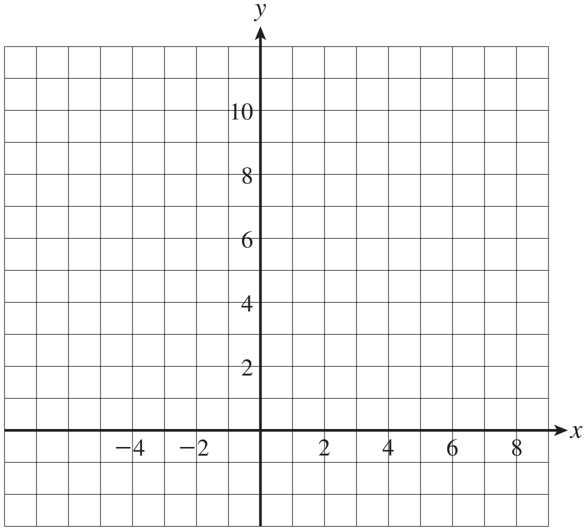
x	0	1	2	3
y	10			

6. a. Would you expect $f(x) = 3^x$ to increase faster or slower than $g(x) = 2.5^x$ for $x > 0$? Explain. (*Hint:* You may want to use your graphing calculator for help.)

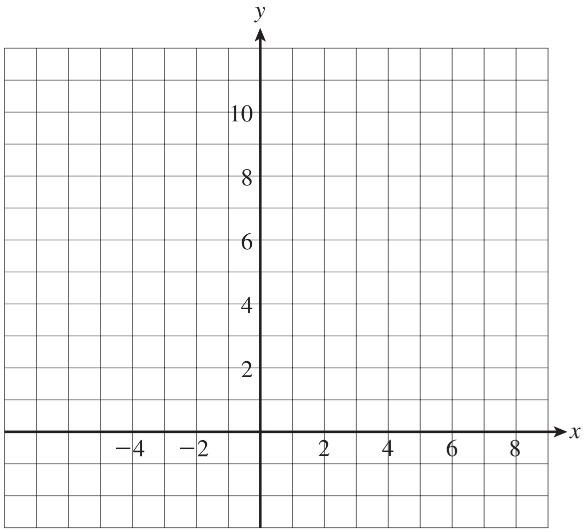
b. Would you expect $f(x) = \left(\frac{1}{2}\right)^x$ to decrease faster or slower than $g(x) = (0.70)^x$ for $x > 0$? Explain.

7. Determine the domain and range of each of the following functions.

a.



b.



8. Take a piece of paper from your notebook. Let x represent the number of times you fold the paper in half and $f(x)$ represent the number of sections the paper is divided into after the folding.

a. Complete the table of values.

x	0	1	2	3	4	5
$f(x)$	1					

b. If you could fold the paper 8 times, how many individual sections will there be on the paper?

c. Does this data represent an exponential function? Explain.

d. What is the practical domain and range in this situation?

ACTIVITY 3.2

Cellular Phones

OBJECTIVES

1. Determine the growth and decay factor for an exponential function represented by a table of values or an equation.
2. Graph exponential functions defined by $y = ab^x$, where $b > 0$ and $b \neq 1$, $a \neq 0$.
3. Determine the doubling and halving time.

During a meeting, you hear the familiar ring of a cell phone. Without hesitation, several of your colleagues reach into their jacket pockets, briefcases, and purses to receive the anticipated call. Although sometimes annoying, cell phones have become part of our way of life.

The following table shows the rapid increase in the number of cellular phones (figures are approximate) in the late 1990's. Note that the input variable (year) increases in steps of 1 unit (year).

Calling All Cells

YEAR	NUMBER OF CELLULAR PHONES AS OF JAN. 1 (in millions)
1996	44.248
1997	55.312
1998	69.140
1999	86.425
2000	108.031

1. Is this a linear function? How do you know?

2. a. Evaluate the indicated ratios to complete the following table.

$\frac{\text{No. of phones in 1997}}{\text{No. of phones in 1996}}$	$\frac{\text{No. of phones in 1998}}{\text{No. of phones in 1997}}$	$\frac{\text{No. of phones in 1999}}{\text{No. of phones in 1998}}$	$\frac{\text{No. of phones in 2000}}{\text{No. of phones in 1999}}$

b. What do you notice about the values of the table?

In an exponential function with base b, equally spaced input values yield output values whose successive ratios are constant. If the input values increase by increments of 1, the common ratio is the base b. If $b > 1$, b is the growth factor; if $0 < b < 1$, b is the decay factor.

3. a. Does the relationship in the table preceding Problem 1 represent an exponential function? Explain.

b. What is the growth factor?

c. As a consequence of the result found in part b, you can start with 44.248, the number of cellular phones in 1996, and obtain the number of cellular phones in 1997 by multiplying by the growth factor, $b = 1.25$. You can then determine the number of cellular phones in 1998 by multiplying the number of cellular phones in 1997 by b, and so on. Verify this with your calculator. Note that because the exponential function is a mathematical model, the results will vary slightly from the actual number of phones given in the table preceding Problem 1.

Once you know the growth factor $(b = 1.25)$, you can determine the equation that gives the number of phones as a function of t, the number of years since 1996. Note that $t = 0$ corresponds to 1996, $t = 1$ to 1997, and so on.

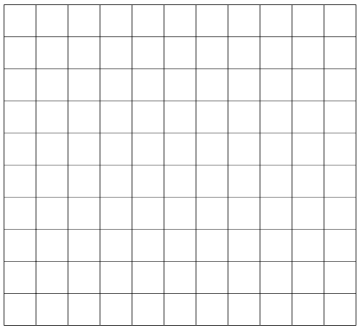
4. a. Complete the table.

t	CALCULATION FOR THE NUMBER OF CELL PHONES	EXPONENTIAL FORM	NUMBER OF CELL PHONES
0	44.248	$44.248(1.25)^0$	44.248
1	(44.248)1.25	$44.248(1.25)^1$	
2			
3			

b. Use the pattern in the preceding table to help you write the equation of the form $N(t) = a \cdot b^t$, where $N(t)$ represents the number of cell phones (in millions) in use at time t, the number of years since 1996.

c. What is the practical domain of the function N?

d. Graph the function N on your graphing calculator, and then sketch the result below on an appropriately scaled and labeled axis.



e. Determine the vertical intercept of the graph of N by substituting 0 for the input, t. What is the practical meaning of the vertical intercept in this situation?

DEFINITION

Many exponential functions can be represented symbolically by $f(t) = a \cdot b^t$, where a is the value of f when $t = 0$ and b is the growth or decay factor. If the input, t, of $y = a \cdot b^t$ represents time, then the coefficient a is called the initial value.

EXAMPLE 1 ***The exponential function defined by $f(x) = 5 \cdot 2^x$ has y-intercept $(0, 5)$ and growth factor $b = 2$. The exponential function defined by $h(x) = \frac{1}{2}(0.75)^x$ has y-intercept $\left(0, \frac{1}{2}\right)$ and decay factor $b = 0.75$.***

5. Use the function defined by $N(t) = 44.248(1.25)^t$ to estimate the number of cell phone users in 2005. Do you think this is a good estimate? Explain.

6. a. Use the graph of the exponential function $N(t) = 44.248(1.25)^t$ and the trace or table feature of your graphing calculator to estimate the number of years it takes for the number of cell phone users to double from 44.248 million to 88.496 million.

b. Estimate the time necessary for the number of cell phone users to double from 88.496 million to 176.992 million. Verify your estimate using your calculator.

c. How long will it take for any given number of cell phone users to double?

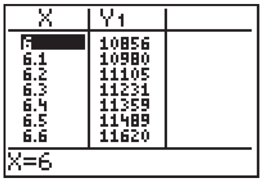
DEFINITION

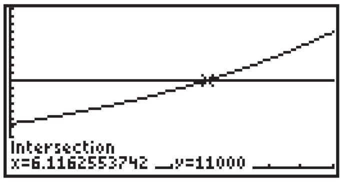
The **doubling time** of an exponential function is the time it takes for an output to double. The doubling time is determined by the growth factor and remains the same for all output values.

EXAMPLE 2 ***The balance $B(t)$, in dollars, of an investment account is defined by $B(t) = 5500(1.12)^t$, where t is the number of years. The initial value for this function is \$5500. Determine the value of t when the balance is doubled or equal to \$11,000.***

SOLUTION

If you use the table feature of your calculator, the doubling time is estimated at 6.1 years (see the following calculator graphic). The intersect feature on the graphing calculator shows the doubling time to be 6.12 years to the nearest hundredth.

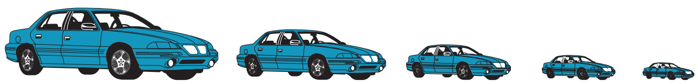




Decreasing Exponential Functions, Decay Factor, and Halving Time

You have just purchased a new automobile for \$22,000. Much to your dismay, you have just learned that you should expect the value of your car to depreciate by 30% per year! The following table shows the book value of the car for the next several years, where V is the value in thousands of dollars.

DEPRECIATION: TAKING ITS TOLL



t (year)	0	1	2	3	4
$V(t)$ (in thousands of dollars)	22	15.4	10.8	7.5	5.3

When a quantity is increased or decreased by a constant percent rate, it can be modeled by an exponential function. In this situation, the car value is decreased by 30% per year. A decreasing exponential function has a decay factor, b, with $0 < b < 1$. For consecutive values of the input, an output value is determined by multiplying the previous output value by b.

7. As the input, t, increases from 0 to 1, the output, V, decreases from 22 to 15.4 (in \$1000).

a. Determine the value that 22 is multiplied by to get 15.4.

b. Use the result from part a to complete the following table.

t	CALCULATION OF THE VALUE OF THE CAR	EXPONENTIAL FORM	VALUE (in \$1000)
0	22	$22(0.7)^0$	22
1	22(0.7)	$22(0.7)^1$	15.4
2	22(0.7)(0.7)	$22(0.7)^2$	
3			
4			

c. Use the pattern in the preceding table to write an equation in the form $V = a \cdot b^t$ that gives the car value, V, as a function years, t.

d. Input the function into Y_1 on your calculator and sketch the result below.

e. Determine the value of the vertical intercept of the graph. Write the result as an ordered pair.

f. What is the practical meaning of the vertical intercept in this situation?

8. a. Estimate the number of years, t, it takes for the value of the car to be \$11,000, half the original value. (*Hint:* Put 11000 in Y_2 and find the point of intersection.)

b. How many years will it take for the car value to be halved again, that is from \$11,000 to \$5500?

DEFINITION

The **half-life** of an exponential function is the time it takes for an output to decay by one-half. The half-life is determined by the decay factor and remains the same for all output values.

EXAMPLE 3 ***The population of Buffalo, New York, can be modeled by the equation $B(t) = 1102(0.995)^t$ with $t = 0$ representing the year 1970 and $B(t)$ representing the population in thousands. If the population of Buffalo continues to decline at the same rate, determine the number of years it will take for the population of Buffalo to be one half of the 1970 population.***

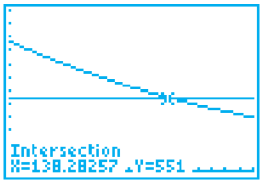
SOLUTION

The equation indicates that the population of Buffalo, NY, in 1970 ($t = 0$) is $B(0) = 1102(0.995)^0 = 1102$ thousand people. Therefore, half of that is 551 thousand people. Use the table feature of your graphing calculator; the halving time is estimated at 138 years.

X	Y_1	Y_2
135	560.14	551
136	557.34	551
137	554.55	551
138	551.78	551
139	549.02	551
140	546.28	551
141	543.55	551

X=138

Use the intersect feature of your graphing calculator and you will also find the halving time to be 138 years rounded to the nearest year.



SUMMARY
ACTIVITY 3.2

1. For **exponential functions** defined by $f(x) = ab^x$, a is the value of f when $x = 0$ (sometimes called the initial value), and b is the growth or decay factor.

2. The vertical intercept of these functions is $(0, a)$.

3. In an exponential function, equally spaced input values yield output values whose successive ratios are constant. If the input values increase by 1 unit, then

   a. the constant ratio is the **growth factor** if the output values are increasing

   b. the constant ratio is the **decay factor** if the output values are decreasing

4. The **doubling time** of an increasing exponential function is the time it takes for an output to double. The doubling time is set by the growth factor and remains the same for all output values.

5. The **half-life** of a decreasing exponential function is the time it takes for an output to decay by one-half. The half-life is determined by the decay factor and remains the same for all output values.

EXERCISES
ACTIVITY 3.2

1. The population of Russia in selected years can be approximated by the following table.

YEAR	1995	1996	1997	2000
POPULATION (in millions)	148.0	147.6	146.9	146.0

a. Let 1995 correspond to $t = 0$. Let b be the ratio between the population of Russia in 1996 and 1995. Determine an exponential function of the form $y = a \cdot b^t$ to represent the population of Russia symbolically. Round to four decimal places.

b. Does the function in part a give an accurate value of the population of Russia in 2000? Explain.

c. Use your model in part b to predict the population of Russia in 2007.

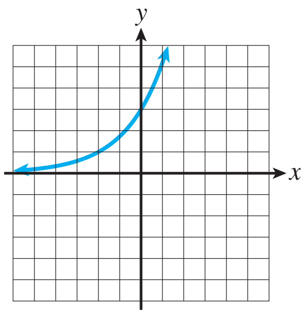
Exercise numbers appearing in color are answered in the Selected Answers appendix.

2. Without using your graphing calculator, match each graph with its equation. Then check your answer using your graphing calculator.

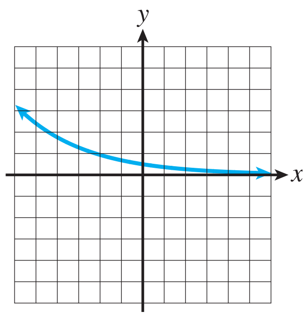
a. $f(x) = 0.5(0.73)^x$

b. $g(x) = 3(1.73)^x$

i.



ii.



a.

b.

3. Which of the following tables represent exponential functions? Indicate the growth or decay factor for the data that is exponential.

a.

x	0	1	2	3	4
y	0	2	16	54	128

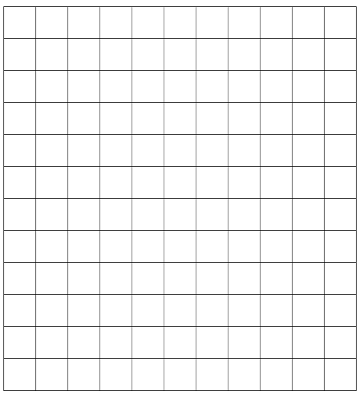
b.

x	0	1	2	3	4
y	1	4	16	64	256

c.

x	1	2	3	4	5
y	1750	858	420	206	101

4. a. Sketch a graph of $f(x) = 2^x$ and $g(x) = 3 \cdot 2^x$ on the same coordinate axes.



b. Describe how the graphs of f and g are similar and how they are different.

5. If $f(x) = 3 \cdot 4^x$, determine the exact value of each of the following, when possible. Otherwise, use your calculator to approximate the value to the nearest hundredth.

a. $f(-2)$

b. $f\left(\frac{1}{2}\right)$

c. $f(2)$

d. $f(1.3)$

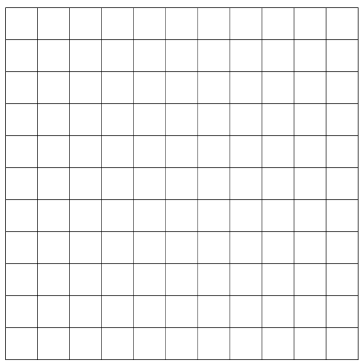
6. In 1995, the United States emitted approximately 1400 million tons of carbon into the atmosphere. This represented about one-fourth of the world total. The U.S. emissions were increasing at about 1.3% per year. If t represents the number of years since 1995 and $A(t)$ represents the amount of carbon (in millions of tons) emitted in a given year, then $A(t) = 1400(1.013)^t$.

a. Complete the following table.

t, NUMBER OF YEARS SINCE 1995	0	1	2	3	4	5
$A(t)$, AMOUNT OF U.S. CARBON EMISSIONS (in millions of tons)						

b. Determine the growth factor for carbon emissions.

c. Sketch a graph of this exponential function. Use $0 \le t \le 25$ and $0 \le A(t) \le 2500$.



d. Use the equation $A(t) = 1400(1.013)^t$ to determine the amount of carbon emission in 2010. Include the units of measurement in your answer.

e. Use the graph and trace features of your graphing calculator to approximate the year in which carbon emissions in the United States will exceed 2000 million tons.

7. Chlorine is used to disinfect swimming pools. The chlorine concentration should be between 1.5 and 2.5 parts per million (ppm). On sunny, hot days, 30% of the chlorine dissipates into the air or combines with other chemicals. Therefore, chlorine concentration, $A(x)$, (in parts per million) in a pool after x sunny days can be modeled by

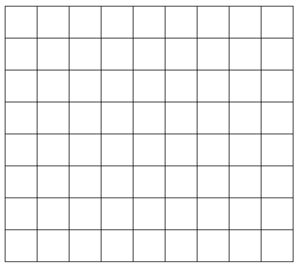
$$A(x) = 2.5(0.7)^x.$$

a. What is the initial concentration of chlorine in the pool?

b. Complete the following table.

x	0	1	2	3	4	5
$A(x)$						

c. Sketch the graph of the chlorine function.



d. What is the chlorine concentration in the pool after 3 days?

e. Approximate graphically and numerically the number of days before chlorine should be added.

8. In the nineties, there was a rapid growth in the number of investment clubs in the United States. An investment club is a group of people who meet on a regular basis to invest in the stock market. By joining a club, members are able to share in a diverse portfolio and therefore reduce the risk of losing money.

The following table shows the rapid growth in the number of clubs from 1990 to 1996 (figures for the number of clubs are approximate). Note that the input variable (year) increases in steps of 1 unit.

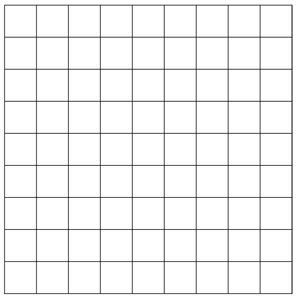
YEAR	NUMBER OF CLUBS AS OF JAN. 1
1990	5820
1991	7180
1992	8860
1993	10,930
1994	13,480
1995	16,630
1996	20,510

a. Does the relationship in the table represent an exponential function? Explain.

b. What is the growth factor?

c. Determine the equation that gives the number of clubs, $N(t)$, as a function of t, the number of years since 1990. Note that $t = 0$ corresponds to 1990.

d. Graph the function.



e. What is the vertical intercept? What is the practical meaning of this intercept in this situation?

f. Use the equation to estimate the number of clubs in 2000. Do you think this is a good estimate? Explain.

g. Use the graph of the exponential function and the trace or table feature of your grapher to estimate the number of years it takes for the number of clubs to double from 5820 to 11,640.

h. How long will it take for any given number of clubs to double?

9. Homemade chocolate chip cookies lose their freshness over time. Let the taste quality be 1 when the cookies are fresh. The taste quality decreases according to the function

$$Q = 0.8^x,$$

where x is the number of days since the cookies were baked.

Determine when the taste quality will be one-half of its value. Use the intersect feature of your calculator to determine when $f(x)$ is $\frac{1}{2}$ of 1 or 0.5.

ACTIVITY 3.3

Population Growth

OBJECTIVES

1. Determine annual growth or decay factor of an exponential function represented by a table of values or an equation.
2. Graph an exponential function having equation $y = a(1 + r)^x$, $a \neq 0$.

According to the 2000 U.S. Census, the city of Charlotte, North Carolina, had a population of approximately 541,000.

1. a. Assuming that the population increases at a constant rate of 3.2%, determine the population of Charlotte (in thousands) in 2001.

b. Determine the population of Charlotte (in thousands) in 2002.

c. Divide the population in 2001 by the population in 2000, and record this ratio.

d. Divide the population in 2002 by the population in 2001, and record this ratio.

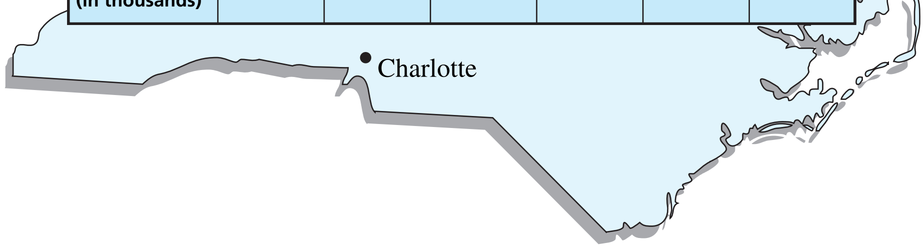
e. What do you notice about the ratios in parts c and d? What do these ratios represent?

Linear functions represent quantities that change at a constant rate (slope). Exponential functions represent quantities that change by a constant growth or decay factor.

EXAMPLE 1 ***Population growth, sales and advertising trends, compound interest, spread of disease, and concentration of a drug in the blood are examples of quantities that increase or decrease by a constant growth factor.***

2. a. Let t represent the number of years since 2000 ($t = 0$ corresponds to 2000). Use the growth factor from Problem 1 to complete the following table.

***t*, YEARS** (since 2000)	0	1	2	3	4	5
***P*, POPULATION** (in thousands)	541	558.3	576.2			



Once you know the growth factor, b, and the initial value, a, you can write the exponential equation. In this situation, the initial value is the population, in 1000s, in 2000 ($t = 0$) and the growth factor is $b = 1.032$.

b. Write the exponential equation, $P = a \cdot b^t$, for the population of Charlotte.

3. a. Write the growth rate, $r = 3.2\%$, as a decimal.

b. Add 1 to the decimal form of the growth rate r.

The growth factor, b, is determined from the growth rate, r, by writing r in decimal form and adding 1: $b = 1 + r$.

EXAMPLE 2 ***Determine the growth factor, b, for a growth rate of r = 8%.***

SOLUTION

$$r = 8\% = 0.08,\ b = 1 + r = 1 + 0.08 = 1.08$$

c. Solve the equation for the growth factor, $b = 1 + r$, for r.

The growth rate, r, is determined from the growth factor, b, by subtracting 1 from b and writing the result in percent form.

EXAMPLE 3 ***Determine the growth rate, r, for a growth factor of b = 1.054.***

SOLUTION

$$r = b - 1 = 1.054 - 1 = 0.054 = 5.4\%$$

The growth rate is 5.4%

4. a. Complete the following table.

t	CALCULATION FOR POPULATION (in thousands)	EXPONENTIAL FORM	$P(t)$, POPULATION (in thousands)
0	541	$541(1.032)^0$	541
1	(541)1.032	$541(1.032)^1$	
2	(541)(1.032)(1.032)		
3			

b. Use the pattern in the table in part a to help you write the equation for $P(t)$, the population of Charlotte (in thousands), using t, the number of years since 2000, as the input value. How does your result compare to the equation obtained in Problem 2?

The equation $P(t) = 541(1.032)^t$ has the general form $P = P_0(1 + r)^t$, where r is the annual **growth rate**, $(1 + r)$ is the **growth factor** or the base, b, of the exponential function, t is the time in years, and P_0 is the initial value, the population when $t = 0$.

EXAMPLE 4

a. ***Determine the growth factor and the growth rate of the function defined by $f(x) = 250(1.7)^x$.***

SOLUTION

The growth factor $1 + r$ is the base 1.7. To determine the growth rate, solve the equation $1 + r = 1.7$ for r.

$$r = 0.7, \text{ or } 70\%$$

b. ***If the growth rate of a function is 5%, determine the growth factor.***

SOLUTION

If $r = 5\%$ or 0.05, the growth factor is $1 + r = 1 + 0.05 = 1.05$.

5. a. Determine the growth factor in the Charlotte population function $P(t) = 541(1.032)^t$.

b. Determine the growth rate. Express your answer as a percent.

6. a. Using the function defined by $P(t) = 541(1.032)^t$, determine the population of Charlotte in 2006. That is, determine $P(t)$ when $t = 6$.

b. Graph the population function with your graphing calculator. Set the window to Xmin = −50, Xmax = 100, Ymin = 0, and Ymax = 13,000, and display the graph.

c. Determine $P(0)$. What is the graphical and the practical meaning of $P(0)$?

7. **a.** Use the model to predict Charlotte's population in 2010.

b. Verify your prediction on the graph.

8. **a.** Use the graph to estimate when Charlotte's population will reach 700,000, assuming it continues to grow at the same rate. Remember, $P(t)$ is the number of thousands.

b. Evaluate $P(32)$ and describe what it means.

9. Use the population model to estimate the population of Charlotte in 2002 and in 2020. In which prediction are you more confident? Why?

10. **a.** Assuming the growth rate remains constant, how long will it take for the population of Charlotte to double its 2000 population?

b. Explain how you reached your conclusion in part a.

Wastewater Treatment Facility

You are working at a wastewater treatment facility. You are presently treating water contaminated with 18 micrograms (μg) of pollutant per liter. Your process is designed to remove 20% of the pollutant during each treatment. Your goal is to reduce the pollutant to less than 3 micrograms per liter.

11. **a.** What percent of pollutant present at the start of a treatment remains at the end of the treatment?

b. The concentration of pollutant is 18 micrograms per liter at the start of the first treatment. Use the result of part a to determine the concentration of pollutant at the end of the first treatment.

c. Complete the following table. Round the results to the nearest tenth.

n, NUMBER OF TREATMENTS	0	1	2	3	4	5
$C(n)$, CONCENTRATION OF POLLUTANT, IN μg/l, AT THE END OF THE *n*TH TREATMENT	18	14.4				

d. Write an equation for the concentration, $C(n)$, of the pollutant as a function of the number of treatments, n.

The equation $C(n) = 18(0.80)^n$ has the general form $C = C_0(1 - r)^n$, where r is the **decay rate**, $(1 - r)$ is the **decay factor** or the base of the exponential function, n is the number of treatments, and C_0 is the initial value, the concentration when $n = 0$.

EXAMPLE 5 **a.** ***Determine the decay factor and the decay rate of the function defined by*** $h(x) = 123(0.43)^x$.

SOLUTION

The decay factor $1 - r$ is the base, 0.43. To determine the decay rate, solve the equation $1 - r = 0.43$ for r.

$$r = 0.57, \text{ or } 57\%$$

b. ***If the decay rate of a function is 5%, determine the decay factor.***

SOLUTION

If $r = 5\%$, or 0.05, the decay factor is $1 - r = 1 - 0.05 = 0.95$.

12. **a.** If the decay rate is 2.5%, what is the decay factor?

b. If the decay factor is 0.76, what is the decay rate?

13. **a.** Use the function defined by $C(n) = 18(0.8)^n$ to predict the concentration of contaminants at the wastewater treatment facility after seven treatments.

b. Sketch a graph of the concentration function on your graphing calculator. Use the table in Problem 11c to set a window. Does the graph look like you expected it would? Explain.

c. What is the vertical intercept? What is the practical meaning of the intercept in this situation?

d. Reset the window of your graphing calculator to Xmin $= -5$, Xmax $= 15$, Ymin $= -10$, and Ymax $= 50$. Does the graph have a horizontal asymptote? Explain what this means in this situation.

14. Use the table or trace feature of your graphing calculator to estimate the number of treatments necessary to bring the concentration of pollutant below 3 micrograms per liter.

SUMMARY
ACTIVITY 3.3

1. **Exponential functions** are used to describe phenomena that grow or decay by a constant percent rate per unit time.
2. If r represents the **annual growth rate**, the exponential function that models the quantity, P, can be written as

$$P(t) = P_0(1 + r)^t,$$

where P_0 is the initial amount, t represents the amount of elapsed time, and $1 + r$ is the growth factor.

3. If r represents the **annual decay rate**, the exponential function that models the amount remaining can be written as

$$P(t) = P_0(1 - r)^t,$$

where $1 - r$ is the decay factor.

EXERCISES
ACTIVITY 3.3

1. Determine the growth and decay factors and growth and decay rates in the following tables.

GROWTH FACTOR	GROWTH RATE
1.02	
	2.9%
2.23	
	34%
1.0002	

DECAY FACTOR	DECAY RATE
0.77	
	68%
0.953	
	19.7%
0.9948	

2. The 2000 U.S. Census reports the populations of Bozeman, Montana, as 27,509 and Butte, Montana, as 32,370. Since the 1990 Census, Bozeman's population has been increasing at approximately 1.96% per year. Butte's population has been decreasing at approximately 0.29% per year. Assume that the growth and decay rates stay constant.

   a. Let $P(t)$ represent the population t years after 2000. Determine the exponential functions that model the populations of both cities.

   b. Use your models to predict the populations of both cities in 2005.

Exercise numbers appearing in color are answered in the Selected Answers appendix.

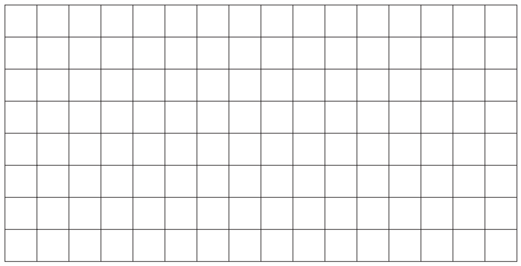
c. Estimate the number of years necessary for the population of Bozeman, Montana, to double.

d. Using the table and/or graphs of these functions, predict when the populations will be equal.

3. You have just taken over as the city manager of a small city. The personnel expenses were \$8,500,000 in 2007. Over the previous 5 years, the personnel expenses have increased at a rate of 3.2% annually.

a. Assuming that this rate continues, write an equation describing personnel costs, $C(t)$, in millions of dollars, where $t = 0$ corresponds to 2007.

b. Sketch a graph of this function up to the year 2017 ($t = 10$).



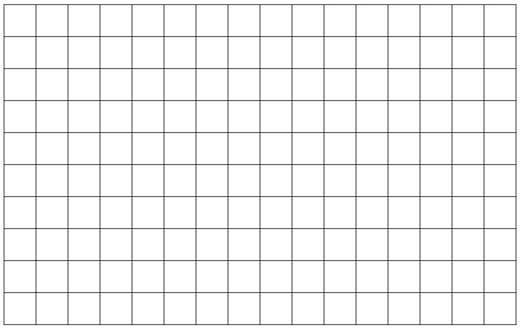
c. What are your projected personnel costs in the year 2012?

d. What is the vertical intercept? What is the practical meaning of the intercept in this situation?

e. In what year will the personnel expenses be double the 2007 personnel expenses?

4. According to the U.S. Bureau of the Census, the population of the United States from 1930 to 2000 can be modeled by $P(t) = 120.6 \cdot 1.0125^t$, where t represents the number of years since 1930.

a. Sketch a graph of the U.S. population model from 1930 to 2000.



b. Determine the annual growth rate and the growth factor from the equation.

c. Use the population equation to determine the population (in millions) of the United States in 2000. How does your answer compare to the actual population of 281.4 million?

5. You have recently purchased a new car for $20,000 by arranging financing for the next 5 years. You are curious to know what your new car will be worth when the loan is completely paid off.

a. Assuming that the value depreciates at a constant rate of 15%, write an equation that represents the value, $V(t)$, of the car t years from now.

b. What is the decay rate in this situation?

c. What is the decay factor in this situation?

d. Use the equation from part a to estimate the value of your car 5 years from now.

e. Use the trace and table features of your graphing calculator to check your results in part d.

f. Use the trace or table features of your graphing calculator to determine when your car will be worth $10,000.

PROJECT ACTIVITY 3.4

Photocopying Machines

OBJECTIVES

1. Generate data given the growth or decay rate of an exponential function.
2. Write exponential functions given the growth or decay rate.
3. Graph exponential functions from data.
4. Determine doubling and halving times from exponential functions.

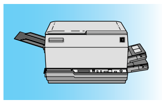
Most photocopy machines allow for enlarging or reducing the size of the original. Suppose you have a chart you wish to photocopy for a report you need to submit to your supervisor. The chart is 10 inches wide by 7 inches high. To reduce the size of the copy 20%, you set the machine to reduce by taking 80% of the original dimensions.

1. What will be the dimensions of your photocopy?

2. What is the percentage reduction in *area* of the photocopy?

3. If your chart must fit in a space only 4 inches high in your report, how many times would you need to reduce the original? (Assume the photocopier is set at 80% reduction.)

4. a. Complete the following table showing the dimensions of the chart after x 80% reductions (photocopies of photocopies). Record each length to the nearest hundredth of an inch.

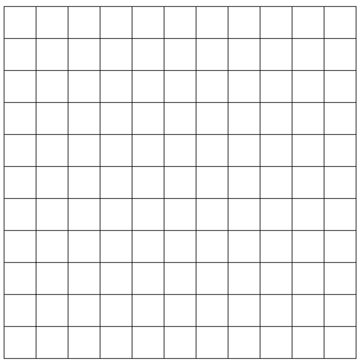


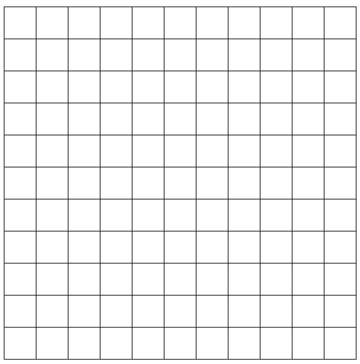
Smaller and Smaller

x, THE NUMBER OF 80% REDUCTIONS	0	1	2	3	4	5	6	7	8	9	10
$h(x)$, HEIGHT (in.)	7	5.6									
$w(x)$, WIDTH (in.)	10	8									

b. If you were actually to perform ten 80% reductions (photocopies of photocopies), what do you think your results would look like?

5. Use the data from your table to plot points for two curves on separate axes. Use the number of reductions, x, versus the heights of the reduced copies for one curve, and the number of reductions versus the widths of the reduced copies for a second curve. Describe how the two graphs are similar and how they are different.





6. a. Notice that each entry in the table for $h(x)$ or $w(x)$ can be obtained by multiplying the previous entry by a constant factor. What is the constant factor in this situation?

b. What is the practical meaning of the constant factor in this situation?

7. Determine the equations that define the functions h and w, where x is the number of 80% reductions. Enter these functions on your grapher to verify your work in Problems 4 and 5.

8. The 80% reduction resulted in a decreasing function. What kind of percentage would you need to enter on the photocopier to result in an increasing function?

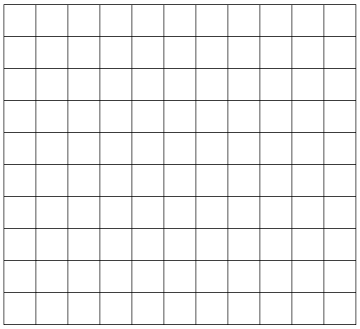
9. a. Complete the following table showing the width of a 10-inch-wide chart in which x represents the number of 20% enlargements.

***x*, NUMBER OF 20% ENLARGEMENTS**	0	1	2	3	4	5
***w*(*x*), WIDTH (in.)**						

b. What is the growth factor? What is the practical meaning of the growth factor in this situation?

c. Write an equation for w in terms of x, where x is the number of 20% enlargements.

d. Use your graphing calculator to graph the function.



10. **a.** If the machine you are using will enlarge only at 20%, how many times would you need to copy the 10-inch width to make it at least 20 inches wide?

b. Will the chart ever be exactly 20 inches wide? Explain.

c. Describe in detail how you found your answers.

11. Assuming a constant 20% enlargement, how many copies would it take to get your original 10-inch width to grow to at least 40 inches? At least 80 inches?

12. Complete the following table with your results from Problems 10 and 11.

x, NUMBER OF 20% ENLARGEMENTS	0	4	8	12
WIDTH (in.)				

13. Use your graphing calculator to determine how many copies are needed to double the output.

14. Examine the height and width functions given in Problem 7, $h(x) = 7(0.8)^x$ and $w(x) = 10(0.8)^x$.

a. Complete the following table.

x, NUMBER OF 80% REDUCTIONS	0	3	6	9
h(*x*), HEIGHT (in.)				

x, NUMBER OF 80% REDUCTIONS	0	3	6	9
w(*x*), WIDTH (in.)				

b. What is the half-life of each function?

ACTIVITY 3.5

Compound Interest

OBJECTIVE

1. Apply the compound interest and continuous compounding formulas to a given situation.

Congratulations, you have inherited \$20,000! Your grandparents suggest that you use half of the inheritance to start a retirement fund. Your grandfather claims that an investment of \$10,000 could grow to over half a million dollars by the time of retirement. You are intrigued by this statement and decide to investigate if this could possibly happen.

Suppose you deposit \$10,000 in the bank at a 6.5% annual interest rate. After 1 year, your balance is

$$10{,}000 + 0.065(10{,}000) = 10{,}000 + 650 = 10{,}650.$$

The interest, \$650, earned during the year becomes part of the new balance. At the end of the second year, your balance is

$$10{,}650 + 0.065(10{,}650) = 10{,}650 + 692.25 = 11{,}342.25.$$

Note that you made interest on the original deposit, plus interest on the first year's interest. In this situation, we say that interest is compounded. Usually, the compounding occurs at fixed intervals (typically at the end of every year, quarter, month, or day). In the preceding situation, interest is compounded annually.

If interest is compounded, then the current balance is given by the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

where A is the current amount, or balance, in the account;

P is the principal (the original amount deposited);

r is the annual interest rate (annual percentage rate in decimal form);

n is the number of times per year that interest is compounded; and

t is the time in years the money has been invested.

This formula is referred to as the **compound interest formula**.

EXAMPLE 1 ***You invest \$100 at 4% compounded quarterly. How much money do you have after 5 years?***

SOLUTION

The principal is \$100, so $P = 100$. The annual interest rate is 4%, so $r = 0.04$. Interest is compounded quarterly, that is, 4 times per year, so $n = 4$. The money is invested for 5 years, so $t = 5$. Substituting the values for the $P, r, n,$ and t in the compound interest formula, you have

$$A = 100\left(1 + \frac{0.04}{4}\right)^{4 \cdot 5} = \$122.02.$$

1. a. Suppose you deposit \$10,000 in an account that has a 6.5% annual interest rate (usually referred to as APR, for annual percentage rate), and whose interest is compounded annually ($n = 1$). Substitute the appropriate values for P, n, and r into the compound interest formula to get the balance, A, as a function of time, t.

b. Use the compound interest formula from part a to determine your balance, A, at the end of the first year $(t = 1)$.

c. What will be the amount of interest earned in the first year?

d. Use the compound interest formula developed in part a and complete the following table.

t, YEAR	0	1	2	3	4
A, BALANCE	10,000.00	10,650.00			

e. The compound interest formula in part a defines A as an exponential function of t. Identify the base.

f. Is the base a growth or decay factor? Explain.

2. a. Suppose you deposit the $10,000 into an account that has the same interest rate (APR) of 6.5%, with compounding quarterly $(n = 4)$ rather than annually $(n = 1)$. Write a new formula for your balance, A, as a function of time.

b. What would be your balance after the first year?

c. Use the table feature of your calculator to determine the balance at the end of each year for 10 years, and record the values in the table in Problem 3 under $n = 4$ (compounded quarterly).

d. What is the base of this exponential function?

3. Now deposit your $10,000 into a 6.5% APR account with *monthly* compounding $(n = 12)$ and then in an account with *daily* compounding $(n = 365)$. Use your graphing calculator and the appropriate formula to complete the following table.

COMPARISON OF $10,000 PRINCIPAL IN 6.5% APR ACCOUNTS WITH VARYING COMPOUNDING PERIODS			
t	$n = 4$	$n = 12$	$n = 365$
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

4. In Problem 3, you calculated the balance on a deposit of $10,000 at an annual interest rate of 6.5% that was compounded at different intervals. After 10 years, which account has the higher balance? Does this seem reasonable? Explain.

Continuous Compounding

You could extend this problem so that interest is compounded every hour or every minute or even every second. However, compounding more frequently than every hour does not increase the balance very much.

To discover why this happens, take a closer look at the exponential functions from Problems 1–3.

$$A = 10{,}000\left(1 + \tfrac{0.065}{1}2\right)^{1\cdot t} = 10{,}000\left[(1 + 0.065)^{1}\right]^{t}$$

$$A = 10{,}000\left(1 + \tfrac{0.065}{4}\right)^{4\cdot t} = 10{,}000\left[\underline{\left(1 + \tfrac{0.065}{4}\right)^{4}}\right]^{t}$$

$$A = 10{,}000\left(1 + \tfrac{0.065}{12}\right)^{12\cdot t} = 10{,}000\left[\underline{\left(1 + \tfrac{0.065}{12}\right)^{12}}\right]^{t}$$

$$A = 10{,}000\left(1 + \tfrac{0.065}{365}\right)^{365\cdot t} = 10{,}000\left[\underline{\left(1 + \tfrac{0.065}{365}\right)^{365}}\right]^{t}$$

Can you discover a pattern in the form of the underlined expressions?

Each formula can be expressed as $A = 10{,}000b^{t}$, where $b = \left(1 + \frac{0.065}{n}\right)^{n}$ for $n = 1, 4, 12,$ and 365. The number b is called the **growth** factor, and n is the number of compounding periods per year.

EXAMPLE 2 *If $n = 4$ in the formula $b = \left(1 + \frac{0.065}{n}\right)^n$, then $b = \left(1 + \frac{0.065}{4}\right)^4 = 1.06660$. The number 1.06660 is the growth factor.*

5. Determine the value of b in the following table, where $b = \left(1 + \frac{0.065}{n}\right)^n$. Round to five decimal places.

n, NUMBER OF COMPOUNDING PERIODS	1	4	12	365
b, GROWTH FACTOR				

The growth rate is the percentage by which the balance grows by in 1 year. It is called the **effective yield, r_e.** Notice that as the number of compounding periods increases, the effective yield increases. This means that with the same annual interest rate (APR), your investment will earn more with more compounding periods.

PROCEDURE

To calculate the effective yield

1. Determine the growth factor $b = \left(1 + \frac{r}{n}\right)^n$.
2. Subtract 1 from b and write the result as a decimal.

$$r_e = b - 1 = \left(1 + \frac{r}{n}\right)^n - 1$$

EXAMPLE 3 *Determine the effective yield for an annual percentage rate (APR) of 4.5% compounded monthly.*

SOLUTION

$r = 4.5\% = 0.045,\ n = 12$

$$r_e = b - 1 = \left(1 + \frac{r}{n}\right)^n - 1 = \left(1 + \frac{0.045}{12}\right)^{12} - 1 = 1.04594 - 1 = 0.04594$$

$r_e = 4.594\%$

6. a. If interest is compounded hourly, then $n = 365 \cdot 24 = 8760$. Compute the growth factor, b, for compounding hourly, using an APR of 6.5%.

b. Determine the effective yield associated with each of the growth factors in the following table.

n	1	4	12	365
GROWTH FACTOR, *b*	1.065	1.0666	1.06697	1.06715
EFFECTIVE YIELD r_e				

c. Write a sentence comparing the growth factor b for compounding hourly, $n = 8760$, to that for daily compounding, $n = 365$.

If the compounding periods become shorter and shorter (compounding every hour, every minute, every second), n gets larger and larger. If you consider the period to be so short that it's essentially an instant in time, you have what is called **continuous compounding**. Some banks use this method for compounding interest.

The compound interest formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ is no longer used when interest is compounded continuously. The following develops a formula for continuous compounding.

Step 1. Rewrite the given formula as indicated using properties of exponents.

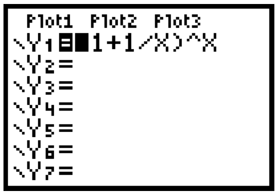
$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt}, \text{ since } \frac{n}{r} \cdot rt = nt$$

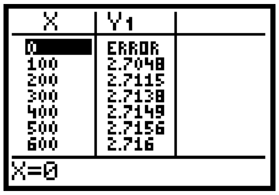
Step 2. Let $\frac{n}{r} = x$. It follows that $\frac{r}{n} = \frac{1}{x}$. Note that as n gets very large, the value of x also gets very large.

Step 3. Substituting x for $\frac{n}{r}$ and $\frac{1}{x}$ for $\frac{r}{n}$, in the rewritten formula in step 1, you have

$$A = P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = P\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt}.$$

7. a. Now take a closer look at the expression $\left(1 + \frac{1}{x}\right)^x$. Enter $\left(1 + \frac{1}{x}\right)^x$ into your calculator as a function of x. Display a table that starts at 0 and is incremented by 100. The results are displayed below.





b. In the table of values, why is there an error at $x = 0$?

c. Scroll down in the table and describe what happens to the output, $\left(1 + \frac{1}{x}\right)^x$, as the input, x, gets very large.

The letter e is used to represent the number that $\left(1 + \frac{1}{x}\right)^x$ approaches as x gets very large. This notation was devised by mathematician Leonhard Euler (1707–1783). Euler used the letter e to denote this number. The number is irrational and its decimal representation never ends and never repeats.

8. The number e is a very important number in mathematics. Find it on your calculator, and write its decimal approximation below. How does this approximation compare to the result in Problem 7a?

You are now ready to complete the compound interest formula for continuous compounding. Substituting e for $\left(1 + \frac{1}{x}\right)^x$ in $A = P\left[\left(1 + \frac{1}{x}\right)^x\right]^{rt}$, you obtain the continuous compounding formula

$$A = Pe^{rt},$$

where A is the current amount, or balance, in the account;

P is the principal;

r is the annual interest rate (annual percentage rate in decimal form);

t is the time in years that your money has been invested; and

e is the base of the continuously compounded exponential function.

EXAMPLE 4 ***You invest $100 at a rate of 4% compounded continuously. How much money will you have after 5 years?***

SOLUTION

The principal is \$100, so $P = 100$. The annual interest rate is 4%, so $r = 0.04$. The money is invested for 5 years, so $t = 5$. Because interest is compounded continuously, you use the formula for continuous compounding as follows.

$$A = 100e^{0.04 \cdot 5} = \$122.14$$

9. **a.** Calculate the balance of your \$10,000 investment in 10 years with an annual interest rate of 6.5% compounded continuously.

The formula used for the preceding result was $A = 10{,}000e^{0.065t}$. Comparing $A = 10{,}000e^{0.065t}$ with $A = 10{,}000b^t$ shows that the growth factor is $b = e^{0.065}$.

b. Determine the growth factor in this situation.

c. What is the effective yield of an annual interest rate of 6.5% compounded continuously?

10. **a.** Historically, investments in the stock market have yielded an average rate of 11.7% per year. Suppose you invest \$10,000 in an account at an 11% annual interest rate that compounds continuously. Use the formula $A = Pe^{rt}$ to determine the balance after 35 years.

b. What is the balance after 40 years?

c. Your grandfather claimed that $10,000 could grow to more than half a million dollars by retirement time (40 years). Is your grandfather correct in his claim?

SUMMARY
ACTIVITY 3.5

1. The formula for **compounding interest** is $A = P\left(1 + \frac{r}{n}\right)^{nt}$.
2. The formula for **continuous compounding** is $A = Pe^{rt}$.
3. If the number of **compounding periods** is large, $A = P\left(1 + \frac{r}{n}\right)^{nt}$ is approximated by $A = Pe^{rt}$.

EXERCISES
ACTIVITY 3.5

1. You inherit $25,000 and deposit it into an account that earns 4.5% annual interest compounded quarterly.

   a. Write an equation that gives the amount of money in the account after t years.

   b. How much money will be in the account after 10 years?

   c. You want to have approximately $65,000 in the bank when your first child begins college. Use your graphing calculator to determine in how many years you will reach this goal.

   d. If the interest were to be compounded continuously at 4.5%, how much money would be in the account after 10 years?

   e. Use your graphing calculator to determine in how many years you would reach your goal of $65,000, if the interest is compounded continuously.

   f. Should you look for an investment account that will be compounded continuously?

Exercise numbers appearing in color are answered in the Selected Answers appendix.

2. You deposit $2000 in an account that earns 5% annual interest compounded monthly.

a. What will be your balance after 2 years?

b. Estimate how long it would take for your investment to double.

c. Identify the annual growth rate and the growth factor.

3. Your friend deposits $1900 in an account that earns 6% compounded continuously.

a. What will be her balance after 2 years?

b. Estimate how long it will take for your friend's investment to double.

4. You are 25 years old and begin to work for a large company that offers you two different retirement options.

Option 1. You will be paid a lump sum of $20,000 for each year you work for the company.

Option 2. The company will deposit $10,000 into an account that will pay you 12% annual interest compounded monthly. When you retire, the money will be given to you.

Let A represent the amount of money you will have for retirement after t years.

a. Write an equation that represents option 1.

b. Write an equation that represents option 2.

c. Use your graphing calculator to sketch a graph of the two options on the same axis.

d. If you plan to retire at age 65, which would be the better plan? Explain.

e. If you decide to retire at age 55, which would be the better plan? Explain.

f. Use your graphing calculator to determine at what age it would not make a difference which plan you choose.

5. The compound interest formula that gives the balance in an account with a principal of $1500 that earns interest at the rate of 4.8% compounded monthly is $A = 1500 \cdot \left(1 + \frac{0.048}{12}\right)^{12t}$. Compare this formula to the exponential equation $A = 1500 \cdot b^t$.

a. What takes the place of b in the compound interest formula?

b. What is the value of the growth factor b?

c. What is the effective rate?

6. The compound interest formula that gives the balance in an account with a principal of $1500 that earns interest at the rate of 4.8% compounded continuously is $A = 1500 \cdot e^{0.048t}$. Compare this formula to the exponential equation $A = 1500 \cdot b^t$.

a. What takes the place of b in the compound interest formula?

b. What is the value of the growth factor b?

c. What is the effective rate?

ACTIVITY 3.6

Continuous Growth and Decay

OBJECTIVES

1. Discover the relationship between the equations of exponential functions defined by $y = ab^t$ and the equations of continuous growth and decay exponential functions defined by $y = ae^{kt}$.
2. Solve problems involving continuous growth and decay models.
3. Graph base e exponential functions.

The U.S. Bureau of the Census reported that the U.S. population on April 1, 2000, was 281,421,906. The U.S. population on April 1, 2001, was 284,236,125.

1. Assuming exponential growth, the U.S. population y can be modeled by the equation $y = ab^t$, where t is the number of years since April 1, 2000. Therefore, $t = 0$ corresponds to April 1, 2000.

a. What is the initial value, a?

b. Determine the annual growth factor, b, for the U.S. population.

c. What is the annual growth rate?

d. Write the equation for the U.S. population as a function of t.

The U.S. population did not remain constant at 281,421,906 from April 1, 2000, to March 31, 2001, and then jump to 284,236,125 on April 1, 2001. The population grew continuously throughout the year. The exponential function used to model continuous growth is the same function used to model continuous compounding for an investment.

Recall from Activity 3.5 that the formula for continuous compounding is $A = Pe^{rt}$, where A (output) is the amount of the investment, P is the initial principal, r is the compounding rate, t (input) is time, and e is the constant irrational number. When this function is used more generally, it is written as $y = ae^{kt}$, where

A has been replaced by y, the output;
P replaced by a, the initial value; and
r replaced by k, the continuous growth rate.

Now, the exponential growth model for the U.S. population determined in Problem 1d, written in the form $y = ab^t$, can be rewritten equivalently in the continuous growth form, $y = ae^{kt}$. Since y represents the same output value in each case, $ab^t = ae^{kt}$. Since a represents the same initial value in each model, it follows that $b^t = e^{kt}$.

2. a. Notice that e^{kt} can be written as $(e^k)^t$. How are b and e^k related?

b. Set the value of b determined in Problem 1 ($b = 1.01$) equal to e^k and solve for k, the continuous growth rate. Solve the equation graphically by entering $Y_1 = e^x$ and $Y_2 = 1.01$. Use the window Xmin = 0, Xmax = 0.02, Ymin = 1, and Ymax = 1.02.

c. Rewrite the U.S. population function from Problem 1 ($y = 281{,}421{,}906(1.01)^t$) in the form $y = a \cdot e^{kt}$.

Notice that an annual growth rate of $r = 0.01 = 1\%$ is equivalent to a continuous growth rate of $k = 0.00995 = 0.995\%$.

Whenever growth is continuous at a constant rate, the exponential model used to describe it is $y = ae^{kt}$, where k is the constant continuous growth rate, a is the amount present initially (when $t = 0$) and e is the constant irrational number approximately equal to 2.718.

EXAMPLE 1

a. Rewrite the equation $y = 42(1.23)^t$ into a continuous growth equation of the form $y = ae^{kt}$.

SOLUTION

$$1.23 = e^k,\ k = 0.207,\ y = 42e^{0.207t}$$

b. What is the continuous growth rate?

SOLUTION

$$0.207 = 20.7\%$$

c. What is the initial amount present (when $t = 0$)?

SOLUTION

$$42$$

Now, consider a situation which involves continuous decay at a constant percentage rate. Tylenol (acetaminophen) is metabolized in your body and eliminated at the rate of 24% per hour. You take two Tylenol tablets (1000 milligrams) at 12 noon.

3. Assume that the amount of Tylenol in your body can be modeled by an exponential function $Q = ab^t$, where t is the number of hours from 12 noon.

a. What is the initial value, a, in this situation?

b. Determine the decay factor, b, for the amount of Tylenol in your body.

c. Write an exponential equation for the amount of Tylenol in your body as a function of t.

Of course, the amount of Tylenol in your body does not decrease suddenly by 24% at the end of each hour; it is metabolized and eliminated continuously. The equation $y = ae^{kt}$ can also be used to model a quantity that decreases at a continuous rate.

Recall in Problem 2a, you compared $y = ab^t$ to $y = ae^{kt}$ and established that $b = e^k$.

4. a. The value of b for the Tylenol equation is 0.76. Set $b = 0.76$ equal to e^k and solve for k graphically as in Problem 2b. Use the window Xmin $= -1$, Xmax $= 0$, Ymin $= 0$, and Ymax $= 1$.

b. Write the equation for the amount of Tylenol in your body, $Q = 1000(0.76)^t$, in the form $Q = ae^{kt}$.

Notice that the value of k in Problem 4 is negative. Whenever $0 < b < 1$, then b is a decay factor and the value of k will be negative. A decreasing exponential function written in the form $y = ae^{kt}$ will have $k < 0$, and $|k|$ is the continuous rate of decrease.

For exponential decrease (decay) at a continuous constant rate, the model $y = ae^{kt}$ is used, where $k < 0$, $|k|$ is the constant continuous decrease (decay) rate, a is the amount present initially, when $t = 0$, and e is the constant irrational number approximately equal to 2.718.

EXAMPLE 2

a. Rewrite the decay equation $y = 12.5(0.83)^t$ as a function of the form $y = ae^{kt}$.

SOLUTION

$$0.83 = e^k,\ k = -0.186,\ y = 12.5e^{-0.186t}$$

b. What is the continuous percentage rate of decay?

SOLUTION

$$0.186 = 18.6\%$$

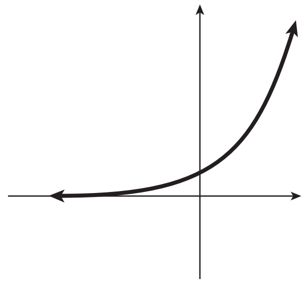
c. What is the initial amount present when $t = 0$?

SOLUTION

$$12.5$$

Graphs of Exponential Functions

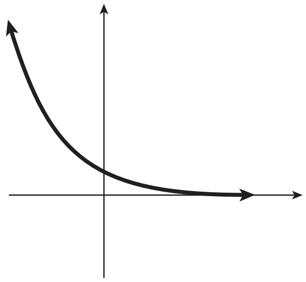
5. The graph of an increasing exponential function has the shape represented below.



a. If the equation for the preceding graph is written as $y = ab^x$, what do you know about the values of a and b?

b. If the equation for the preceding graph is written as $y = ae^{kx}$, what do you know about the values of a and k?

6. The graph of a decreasing exponential function has the shape represented below.



a. If the equation for the preceding graph is written as $y = ab^x$, what do you know about the values of a and b?

b. If the equation for the preceding graph is written as $y = ae^{kx}$, what do you know about the values of a and k?

7. Identify the given exponential function as increasing or decreasing. In each case give the initial value and rate of increase or decrease.

a. $P = 2500e^{0.04t}$

b. $Q = 400(0.86)^t$

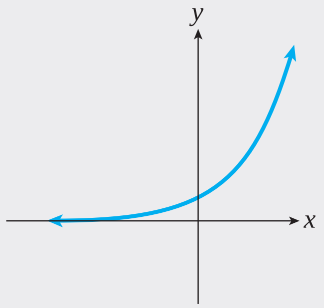
c. $A = 75(1.032)^t$

d. $R = 12e^{-0.12t}$

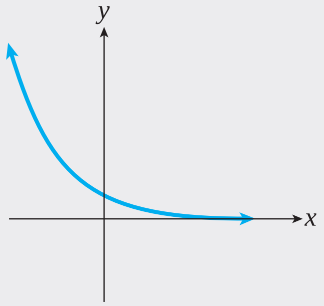
SUMMARY
ACTIVITY 3.6

1. When a quantity increases or decreases continuously at a constant rate, the amount present at time t can be modeled by $y = ae^{kt}$, where a is the initial quantity at $t = 0$. If the quantity is increasing, then $k > 0$ and k is the continuous rate of increase. If the quantity is decreasing, then $k < 0$ and $|k|$ is the continuous rate of decrease.

2. The graph of an increasing exponential function of the form $y = ab^x$ or $y = ae^{kt}$, where $a > 0$ and $b > 1$ or $k > 0$ will be shaped like



3. The graph of a decreasing exponential function of the form $y = ab^x$ or $y = ae^{kt}$, where $a > 0$ and $0 < b < 1$ or $k < 0$ will be shaped like



Notation

The function $y = ae^{kt}$ may be written in other forms, such as $y = y_0e^{kt}$, where the initial value is y_0 (y sub zero) or $Q = Q_0e^{kt}$, where the initial value is Q_0 (Q sub zero).

EXERCISES
ACTIVITY 3.6

1. In 2004, Charlotte, North Carolina, was ranked twentieth in population of all the U.S. cities. The population of Charlotte has been increasing steadily and the exponential function defined by $f(x) = 397.4(1.03)^x$ models this population from 1990 to 2004, where x represents the number of years since 1990 and $f(x)$ represents the total population in thousands. (*Source:* U.S. Bureau of the Census.)

a. Is this an increasing or decreasing exponential function? Explain.

b. Determine the annual growth or decreasing (decay) rate from the model.

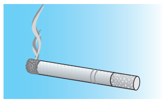
c. According to the model, what is the initial value? What does the initial value mean in this situation?

d. The equation for continuous growth is $y = ae^{kt}$. Set the value of b in your model equal to e^k, and use your graphing calculator to determine the value for k graphically. This will be the continuous growth or decay rate.

e. Rewrite the population function in the form of $y = ae^{kt}$.

f. Use the function from part e and predict the population of Charlotte in 2010.

2. The table shows the smoking prevalence among U.S. male adults (18 years and over) as a percent of the population.



Where There's Smoke

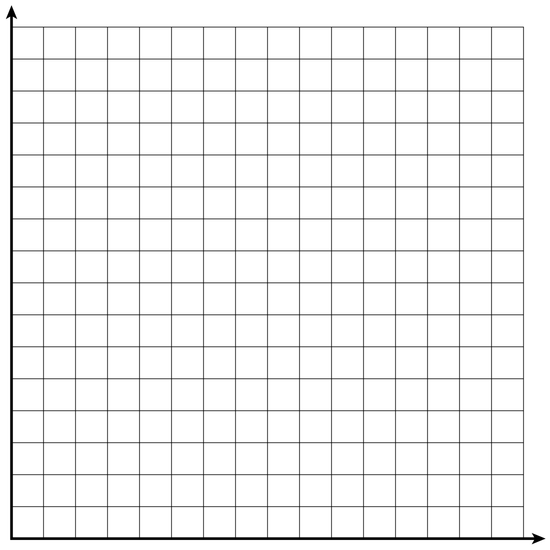
SMOKING PREVALENCE AMONG U.S. MALE ADULTS							
YEAR	1955	1965	1970	1980	1990	2000	2002
YEARS SINCE 1955	0	10	15	25	35	45	47
% OF POPULATION	56.9	51.9	44.1	37.6	28.4	25.7	25.2

Source: The U.S. Center for Disease Control and Prevention.

This data can be modeled by an exponential function $p(t) = 58.8(0.98)^t$, where t equals the number of years since 1955 and p is the smoking prevalence among male adults as a percent of the U.S. population.

a. Is this an increasing or decreasing exponential function? Explain.

b. Sketch a graph of the function using the data in the table. Does the graph reinforce your answer in part a?



c. Determine the growth or decreasing (decay) rate from the model.

d. According to the model, what is the initial value?

e. The equation for the continuous growth or decay is $y = ae^{kt}$. Set the value of b in your model equal to e^k, and use your graphing calculator to determine the value for k graphically. This will be the continuous growth or decay rate.

f. Rewrite the smoking function in the form of $y = ae^{kt}$.

g. Use the function from part f and determine the percent of males in the U.S. population that will be smoking in 2006.

3. Identify the given exponential function as increasing or decreasing. In each case give the initial value and rate of increase or decrease.

a. $R(t) = 33(1.097)^t$

b. $f(x) = 97.8e^{-0.23x}$

c. $S = 3250(0.73)^t$

d. $B = 0.987e^{0.076t}$

4. Sketch a graph of each of the following and verify using your graphing calculator.

a. $y = 20e^{0.08x}$

b. $f(x) = 10e^{-0.3x}$

c. Compare the two graphs. Include shape, direction, and initial values.

5. Strontium 90 is a radioactive material that decays according to the function defined by $y = y_0e^{-0.0244t}$, where y_0 is the amount present initially and t is time in years.

a. If there are 20 grams of strontium 90 present today, how much will be present in 20 years?

b. Use the graph of the function to approximate how long it will take for 20 grams to decay to 10 grams, 10 grams to decay to 5 grams, and 5 grams to decay to 2.5 grams. The length of time is called the *half-life*. In general, a half-life is the time required for half of a radioactive substance to decay.

c. Identify the annual decay rate and the decay factor.

6. When drugs are administered into the bloodstream, the amount present decreases continuously at a constant rate. The amount of a certain drug in the bloodstream is modeled by the function $y = y_0e^{-0.35t}$, where y_0 is the amount of the drug injected (in milligrams) and t is time (in hours).

a. Suppose that 10 milligrams are injected at 10:30 A.M., how much of the drug is still in the bloodstream at 2:00 P.M.?

b. If another dose needs to be administered when there is 1 milligram of the drug present in the bloodstream, approximately when should the next dose be given (to the nearest quarter hour)?

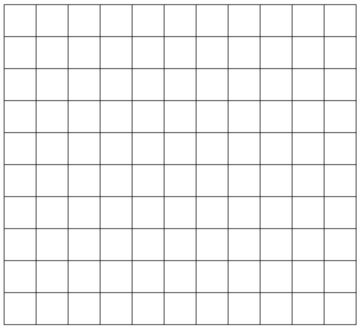
7. The amount of credit-card spending from Thanksgiving to Christmas has increased by 14% per year since 1987. The amount, A, in billions of dollars of credit-card spending during the holiday period in a given year can be modeled by

$$A = f(x) = 36.2e^{0.14x},$$

where x represents the number of years since 1987.

a. How much was spent using credit cards from Thanksgiving to Christmas in 1996?

b. Sketch a graph of the credit-card function.



c. What is the vertical intercept of the graph? What is the practical meaning of the intercept in this situation?

d. Determine, graphically and numerically, the year when credit-card spending reached 75 billion dollars.

e. What is the doubling time?

8. *E. coli* bacteria are capable of very rapid growth, doubling in number approximately every 49.5 minutes. The number, N, of *E. coli* bacteria per milliliter after x minutes can be modeled by the equation

$$N = 500{,}000e^{0.014x}.$$

a. What is the initial number of bacteria per milliliter?

b. How many *E. coli* bacteria would you expect after 99 minutes? (*Hint:* There will be two doublings.) Verify your estimate using the equation.

c. Use a graphing or numerical approach to determine the elapsed time when there would be 20,000,000 *E. coli* bacteria per milliliter.

ACTIVITY 3.7

Bird Flu

OBJECTIVES

1. Determine the regression equation of an exponential function that best fits the given data.
2. Make predictions using an exponential regression equation.
3. Determine whether a linear or exponential model best fits the data.

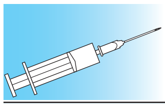
In 2005, the Avian Flu, also known as Bird Flu, received international attention. Although there were very few documented cases of the Avian Flu infecting humans worldwide, world health organizations including the Centers for Diseases Control in Atlanta expressed concern that a mutant strain of the Bird Flu virus that could infect humans via human-to-human contact would develop and produce a worldwide pandemic.

The infection rates, the number of people each infected person will infect, and the incubation period, the time between exposure and the development of symptoms of this flu, cannot be known exactly. However, this information can be approximated by studying the infection rates and incubation periods of existing strains of the virus.

A very conservative infection rate would be 1.5 and a reasonable incubation period would be about 15 days. This means that the first infected person could be expected to infect 1.5 people in roughly a half of a month. This assumes that the spread of the virus is not checked by inoculation or vaccination.

So the total number of infected people 0.5 months after the first person was infected would be 2.5, the sum of the original infected person and the 1.5 newly infected people. During the second half month, the 1.5 newly infected people would infect $1.5 \times 1.5 = 2.25$ new people. This means you have 2.25 people to add to the 2.5 people previously infected, or approximately 5 people infected with Bird Flu at the end of the first month.

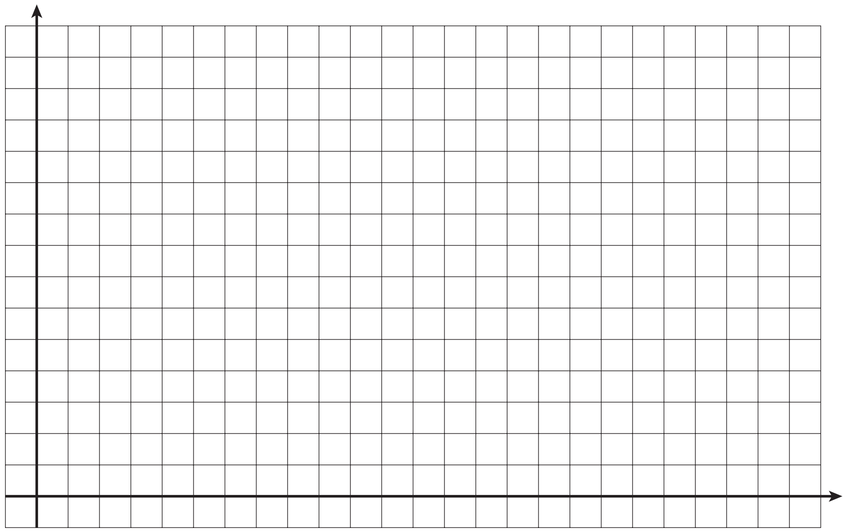
1. The following table represents the total number of people who could be infected with a mutant strain of Bird Flu over a period of 5 months. Complete the table. Round each value to the nearest whole person.



The Spread of Bird Flu

MONTHS SINCE THE FIRST PERSON WAS INFECTED	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
NUMBER OF NEWLY INFECTED		1.5	2.25	3.375	5						
TOTAL NUMBER OF PEOPLE INFECTED	1	2.5	5	8	13						

2. Let t represent the number of months since the first person was infected and T represent the total number of people infected with the Bird Flu virus. Create a scatterplot of the data below.



3. Does the scatterplot indicate a linear relationship between t and T? Explain.

4. a. Use your calculator to model the data with an exponential function. Use option 0:ExpReg in the STAT CALC menu to determine an exponential function that best fits the given data. Record the regression equation of the exponential model below. Round a and b to the nearest 0.001.

   b. Sketch a graph of the exponential model using your calculator and add it to the scatterplot in Problem 2.

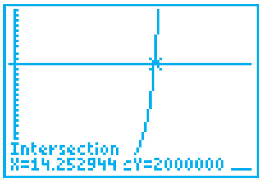
   c. What is the practical domain of this function?

   d. What is the y-intercept of the graph? How does it compare to the actual initial value $(t = 0)$ from the table?

5. a. Use the exponential model to determine the total number of infected people 1 year after the initial infection $(t = 12)$ provided the virus is unchecked. Round your result to the nearest whole person.

   b. Use the exponential function to write an equation that can be used to determine when the virus will first infect 2,000,000 people.

c. Solve the equation in part b using a graphing approach. Use the intersect feature of your calculator; the screen containing the solution should resemble the following.



d. Interpret the meaning of your solution in part c.

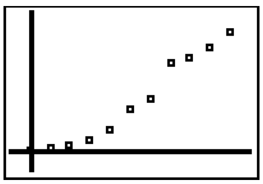
College Graduates

According to the U.S. Department of Education, the number of college graduates increased significantly during the twentieth century. The following table gives the number (in thousands) of college degrees awarded from 1900 through 2000.

College Bound

YEAR	NUMBER OF COLLEGE GRADUATES (thousands)
1900	30
1910	54
1920	73
1930	127
1940	223
1950	432
1960	530
1970	878
1980	935
1990	1017
2000	1180

6. Let t represent the number of years since 1900 ($t = 0$ corresponds to 1900, $t = 10$ to 1910, etc.). Let N represent the number of college graduates (in thousands) at time t. Sketch a scatterplot of the given data on your graphing calculator. Your scatterplot should appear as follows.



7. a. Use your graphing calculator to determine the regression equation of an exponential function that best fits the given data.

b. Sketch a graph of the exponential model using your graphing calculator.

c. What is the practical domain of this exponential function?

d. What is the vertical intercept of the graph? How does it compare to the actual initial value $(t = 0)$ from the table?

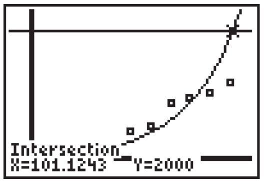
8. a. What is the base of the exponential model? Is the base a growth or decay factor? How do you know?

b. What is the annual growth rate?

9. a. Use the exponential model to determine the number of college graduates in 2010 $(t = 110)$.

b. Use the exponential model to write an equation to determine the year in which there will be 2 million college graduates. Remember that the number of college graduates is measured in thousands.

c. Solve the equation in part a using a graphing approach. Use the intersect feature of your graphing calculator; the screen containing the solution should appear as follows.



10. What is the doubling time for your exponential model? That is, approximately how many years will it take for a given number of college graduates to double?

Decreasing Exponential Model

Students in U.S. public schools have had much greater access to computers in recent years. The following table shows the number of students per computer in selected years.

YEAR	1983	1984	1985	1987	1989	1992	1995	1999
NUMBER OF STUDENTS PER COMPUTER	125	75	50	32	22	16	10	5.7

11. a. Use your graphing calculator to determine the regression equation of an exponential function that models the given data. Let your input, t, represent the number of years since 1983.

b. Sketch a graph of the exponential model using your graphing calculator.

c. What is the base of the exponential model? Is the base a growth or decay factor? How do you know?

d. What is the annual decay rate?

e. Does the graph have a horizontal asymptote? What is the practical meaning of this asymptote in the context of the situation?

EXERCISES
ACTIVITY 3.7

1. The total amount of money spent on health care in the United States is increasing at an alarming rate. The following table gives the total national health care expenditures in billions of dollars in selected years from 1975 through 2003.

Billions for Health

YEAR	1975	1980	1985	1990	1995	2000	2003
TOTAL SPENT (billions of dollars)	129.8	245.8	426.5	695.6	990.2	1309.9	1679.9

Source: National Center for Health Statistics.

Exercise numbers appearing in color are answered in the Selected Answers appendix.

a. Would the data in the preceding table be better modeled by a linear model, $y = mx + b$, or an exponential model, $y = ab^x$? Explain.

b. Sketch a scatterplot of this data.

c. Does the graph reinforce your conclusion in part a? Explain.

d. Use your graphing calculator to determine the exponential regression equation that best fits the health care data in the preceding table. Let your input, t, represent the number of years since 1975.

e. Using the regression equation from part d, determine the predicted total health care expenditures for the year 1995.

f. According to the exponential model, what is the growth factor for the total health care costs per year?

g. What is the growth rate?

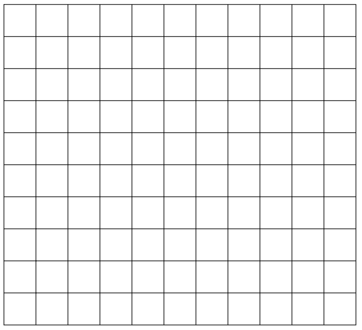
h. According to the exponential model, in what year did the total heath care costs first exceed $1 trillion?

i. What is the doubling time for your exponential model?

2. a. Consider the following data set for the variables *x* and *y*.

x	5	8	11	15	20
y	70.2	50.7	35.1	22.6	9.5

Plot these points on the following grid.



b. Use your graphing calculator to determine both a linear regression and an exponential regression model of the data. Record the equations for these models here.

c. Which model appears to fit the data better? Explain.

d. Use the better model to determine y when $x = 13$ and y when $x = 25$.

e. For the exponential model, what is the decay factor?

f. What does it mean that the decay factor is between 0 and 1?

g. What is the half-life for the exponential model?

3. Use the graph of $y = 5(2)^x$ as a model, and summarize the properties of the exponential function $y = ab^x$, where $a > 0$.

a. What is the domain?

b. What is the range?

c. When is $y = ab^x$ positive?

d. When is $y = ab^x$ negative?

e. What is the vertical intercept of the graph of $y = ab^x$?

CLUSTER 1 What Have I Learned?

1. Consider a linear function defined by $g(x) = mx + b$, $m \neq 0$, and an exponential function defined by $f(x) = ab^x$. Explain how you can determine from the equation whether the function is increasing or decreasing.

2. Suppose you have an exponential function of the form $f(x) = ab^x$, where $a > 0$ and $b > 0$ and $b \neq 1$. By inspecting the graph of f, can you determine if $b > 1$ or if $0 < b < 1$? Explain.

3. You are given a function defined by a table, and the input values are in increments of 1. By looking at the table, can you determine whether or not the function can be approximated by an exponential model? Explain.

4. Explain the difference between growth rate and growth factor.

5. An exponential function $y = ab^x$ passes through the point (0, 2.6). What can you conclude about the values of a and b?

6. You have just received a substantial tax refund of P dollars. You decide to invest the money in a CD for 2 years. You have narrowed your choices to two banks. Bank A will give you 6.75% interest compounded quarterly. Bank B offers you 6.50% compounded continuously. Where do you deposit your money? Explain.

7. Explain why the base in an exponential function cannot equal 1.

CLUSTER 1

How Can I Practice?

1. You are planning to purchase a new car and have your eye on a specific model. You know that new car prices are projected to increase at a rate of 4% per year for the next few years.

   a. Write an equation that represents the projected cost, C, of your dream car t years in the future, given that it costs \$17,000 today.

   b. Identify the growth rate and the growth factor.

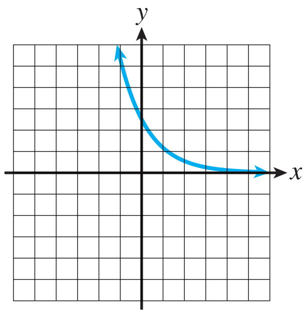
   c. Use your equation in part a to project the cost of your car 3 years from now.

   d. Use your graphing calculator to approximate how long it will take for your dream car to cost \$30,000, if the price continues to increase at 4% per year.

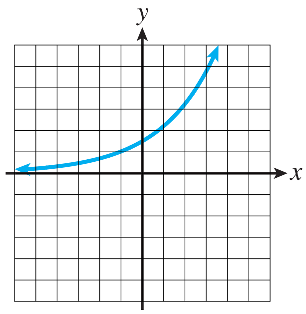
2. Without using your graphing calculator, match the graph with its equation.

   a. $g(x) = 2.5(0.47)^x$ b. $h(x) = 1.5(1.47)^x$

   i.



   ii.



3. Explain the reasons for your choices in Problem 2.

4. Complete the following tables representing exponential functions. Round calculations to two decimal places whenever necessary.

a.

x	0	1	2	3	4
y	2.00	5.10			

b.

x	0	1	2	3	4
y	3.50	2.10			

c.

x	0	1	2	3	4
y	$\frac{1}{6}$	6			

5. Write the equation of the exponential function that represents the data in each table in Problem 4.

a.

b.

c.

6. Without graphing, classify each of the following functions as increasing or decreasing, and determine $f(0)$. (Use your graphing calculator to verify.)

a. $f(x) = 1.3(0.75)^x$

b. $f(x) = 0.6(1.03)^x$

c. $f(x) = 3\left(\frac{1}{5}\right)^x$

7. a. Given the following table, do you believe that it can be approximately modeled by an exponential function?

x	0	1	2	3	4	5	6
y	2	5	12.5	31.3	78.1	195.3	488.3

b. If you answered yes to part a, what is the constant ratio of successive output values?

c. Determine an exponential equation that models this data.

8. Your starting salary for a new job is \$22,000 per year. You are offered two options for salary increases:

Plan 1: an annual increase of \$1000 per year or

Plan 2: an annual percentage increase of 4% of your salary

Your salary is a function of the number of years of employment at your job.

a. Write an equation to determine the salary, S, after x years on the job using plan 1; using plan 2.

b. Complete the following table using the equations from part a.

x	0	1	3	5	10	15
S, PLAN 1						
S, PLAN 2						

c. Which plan would you choose? Explain.

9. The number of victims of a flu epidemic is increasing at a continuous rate of 7.5% per week.

a. If 2000 people are currently infected, write an exponential model of the form $N = f(t) = N_0e^{rt}$, where

N is the number of victims in thousands,

N_0 is the initial number infected in thousands,

r is the weekly percent rate expressed as a decimal, and

t is the number of weeks.

b. Use the exponential model to predict the number of people infected after 8 weeks.

c. Sketch the graph of the flu function using your graphing calculator.

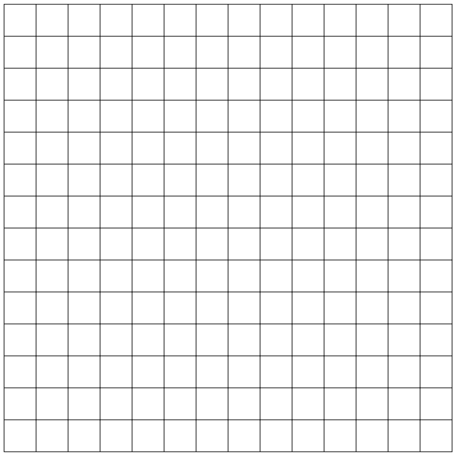
d. Use a graphing approach to predict when the number of victims of the flu will triple.

10. **a.** Complete the following tables.

x	-3	-2	-1	0	1	2	3
$h(x) = 4^x$							

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{4}\right)^x$							

b. Sketch graphs of h and g on the following grid.



c. Use the tables and graphs in part a and b to complete this table.

FUNCTION	BASE, b	GROWTH OR DECAY FACTOR	x-INTERCEPT	y-INTERCEPT	HORIZONTAL ASYMPTOTE	INCREASING OR DECREASING
$h(x) = 4^x$						
$g(x) = \left(\frac{1}{4}\right)^x$						

11. You are a college freshman and have a credit card. You immediately purchase a stereo system for \$415. Your credit limit is \$500. Let's assume that you make no payments and purchase nothing more and there are no other fees. The monthly interest rate is 1.18%.

a. What is your initial credit-card balance?

b. What is the growth rate of your credit-card balance?

c. What is the monthly growth factor of your credit-card balance?

d. Write an exponential function to determine how much you will owe (represented by $f(x)$) after x months with no more purchases or payments.

e. Use your graphing calculator to graph this function. What is the vertical intercept?

f. What is the practical meaning of this intercept in this situation?

g. How much will you owe after 10 months? Use the table feature on your graphing calculator to determine the solution.

h. When you reach your credit limit of $500, the bank will expect a payment. How long do you have before you will have to start paying the money back? Use the trace feature on your grapher to approximate the solution.

12. You are working part-time for a computer company while going to college. The following table shows the hourly wage, $w(t)$, in dollars, that you earn as a function of time, t. Time is measured in years since the beginning of 2000 when you started working.

TIME, *t*, YEARS, SINCE 2000	0	1	2	3	4	5
HOURLY WAGE, *w*(*t*), ($)	12.50	12.75	13.01	13.27	13.53	13.81

a. Calculate the ratios of the outputs to determine if the data in the table is exponential. Round each ratio to the nearest hundredth.

b. What is the growth factor?

c. Write an exponential equation that models the data in the table.

d. What percent raise did you receive each year?

e. If you continue to work for this company, what can you expect your hourly wage to be in 2010?

f. For approximately how many years will you have to work for the company in order for your hourly wage to double? (Assume you will receive the same percentage increase each year.)

13. You deposited \$10,000 in an account that pays 12% annual interest compounded monthly.

a. Write an equation to determine the amount, A, you will have in t years.

b. How much will you have in 5 years?

c. Use your graphing calculator to determine in how many years your investment will double.

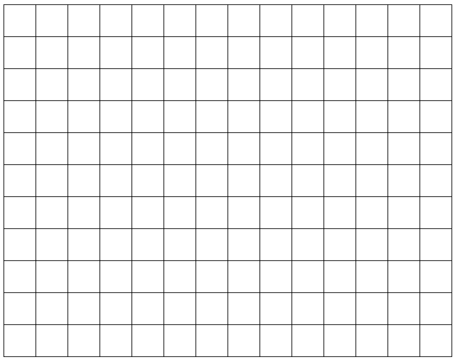
d. Write an equation to determine the amount, A, you will have in t years if the interest is compounded continuously.

e. Use the equation in part d to determine how much you will have in 5 years. Compare your answer to your answer in part b.

14. The number of farms in the United States has declined from 1940 to 2000, as the data in the following table shows. The data is estimated from National Agricultural Statistics Service, U.S. Department of Agriculture.

YEAR	1940	1950	1960	1970	1980	1990	2000
NUMBER OF FARMS, IN MILLIONS	6.2	5.8	4	3	2.5	2.2	2

a. Make a scatterplot of this data.



b. Does the scatterplot show that the data would be better modeled by a linear model or by an exponential model? Explain.

c. Use your graphing calculator to determine the exponential regression equation that best fits the U.S. farm data. Let x represent the number of years since 1940.

d. Use the regression equation to predict the total number of farms in the U.S. in 2010.

e. According to your exponential model, what is the decay factor for the total number of farms in the United States?

f. What is the decay rate?

g. Explain the meaning of the decay rate found in part f in this situation.

h. Use your graphing calculator to determine the halving time for your exponential model.

CLUSTER 2 Logarithmic Functions

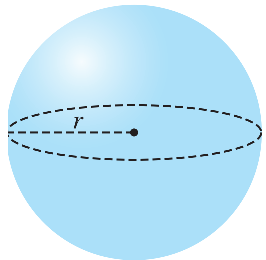
ACTIVITY 3.8

The Diameter of Spheres

OBJECTIVES

1. Define logarithm.
2. Write an exponential statement in logarithmic form.
3. Write a logarithmic statement in exponential form.
4. Determine log and ln values using a calculator.

Spheres are all around you (pardon the pun). You play sports with spheres like baseballs, basketballs, and golf balls. You live on a sphere. Earth is a big ball in space, as are the other planets, the Sun, and the Moon. All spheres have properties in common. For example, the formula for the volume, V, of any sphere is $V = \frac{4}{3}\pi r^3$, and the formula for the surface area, S, of any sphere is $S = 4\pi r^2$ where r represents the radius of the sphere.



However, not all spheres are the same size. The following table gives the diameter, d, of some spheres you all know. Recall that the diameter, d, of a sphere is twice the radius, r.

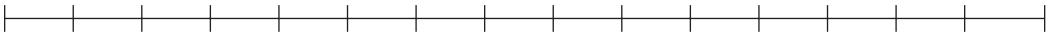
SPHERE	DIAMETER, *d*, IN METERS
Golf ball	0.043
Baseball	0.075
Basketball	0.239
Moon	3,476,000
Earth	12,756,000
Jupiter	142,984,000

If you want to determine either the volume or surface area of any of the spheres in the preceding table, the diameter of the given sphere would be the input value and would be referenced on the horizontal axis. But how would you scale this axis?

1. a. Plot the values in the first three rows of the table. Scale the axis starting at 0 and incrementing by 0.02 meter.

b. Can you plot the values in the last three rows of the table on the same axis? Explain.

2. **a.** Plot the values in the last three rows of the table on a different axis. Scale the axis starting at 0 and incrementing by 10,000,000 meters.

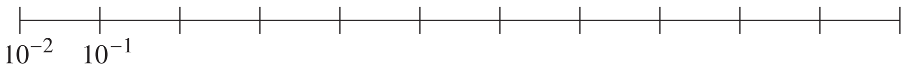


b. Can you plot the values in the first three rows of the table on the axis in part a? Explain.

Logarithmic Scale

3. There is a way to scale the axis so that you can plot all the values in the table on the same axis.

a. Starting with the leftmost tick mark, give the first tick mark a value of 10^{-2} meter. Give the next tick mark a value of 10^{-1} meter. Continue in this way by giving each consecutive tick mark a value that is one power of 10 greater than the preceding tick mark.



b. Complete the following table by writing all of the diameters from the preceding table in scientific notation.

SPHERE	DIAMETER, *d*, IN METERS	*d*, IN SCIENTIFIC NOTATION
Golf ball	0.043	
Baseball	0.075	
Basketball	0.239	
Moon	3,476,000	
Earth	12,756,000	
Jupiter	142,984,000	

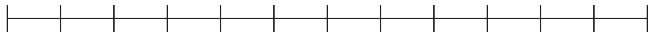
c. To plot the diameter of a golf ball, notice that 0.043 meter is between $10^{-2} = 0.01$ meter and $10^{-1} = 0.1$ meter. Now using the axis in part a, plot 0.043 meter between the tick mark labeled 10^{-2} and 10^{-1} meter, closer to the tick mark labeled 10^{-2} meter.

d. To plot the diameter of Earth, notice that 12,756,000 meters is between $10^7 = 10{,}000{,}000$ meters and $10^8 = 100{,}000{,}000$ meters. Now plot 12,756,000 meters between the tick marks labeled 10^7 and 10^8 meters, closer to the tick mark labeled 10^7.

e. Plot the remaining data in the same way by first determining between which two powers of 10 the number lies.

The scale you used to plot the diameter values is a *logarithmic* or *log scale*. The tick marks on a logarithmic scale are usually labeled with just the exponent of the powers of 10.

4. a. Rewrite the axis from Problem 3a by labeling the tick marks with just the exponents of the powers of 10.



b. The axis looks like a standard axis with tick marks labeled $-2, -1, 0, 1,$ etc. However, it is quite different. Describe the difference between this log scale and a standard axis labeled in the same way. Focus on the values between consecutive tick marks.

DEFINITION

The exponents used to label the tick marks of the preceding axis are **logarithms** or simply **logs**. Since these are exponents of powers of 10, the exponents are logs **base** 10, known as **common logarithms** or common logs.

EXAMPLE 1

a. ***The common logarithm of 10^3 is the exponent to which 10 must be raised to obtain a result of 10^3. Therefore, the common log of 10^3 is 3.***

b. ***The common log of 10^{-2} is -2.***

c. ***The common log of $100 = 10^2$ is 2.***

5. Determine the common log of each of the following.

a. 10^{-1}

b. 10^4

c. 1000

d. 100,000

e. 0.0001

Logarithmic Notation

Remember that a logarithm is an exponent. The common log of x is an exponent, y, to which the base, 10, must be raised to get result x. That is, in the equation $10^y = x$, y is the logarithm. Using log notation $\log_{10} x = y$. Therefore, $\log_{10} 10{,}000 = \log_{10} 10^4 = 4$.

EXAMPLE 2

x, THE NUMBER	y, THE EXPONENT TO WHICH THE BASE, 10, MUST BE RAISED TO GET x	LOG NOTATION $\log_{10} x = y$
10^3	3	$\log_{10} 10^3 = 3$
10^{-2}	-2	$\log_{10} 10^{-2} = -2$
100	2	$\log_{10} 100 = 2$

When using logs base 10, the notation $\log_{10}$ is shortened by dropping the 10. Therefore,

$$\log_{10} 10^3 = \log 10^3 = 3; \log_{10} 100 = \log 100 = 2$$

6. Determine each of the following:

a. $\log 10^{-1}$ **b.** $\log 10^4$ **c.** $\log 1000$

d. $\log 100{,}000$ **e.** $\log 0.0001$

Bases for Logarithms

The logarithmic scale for the diameter of spheres situation was labeled with the exponents of powers of 10. Using 10 as the base for logarithms is common since the number 10 is the base of our number system. However, other numbers could be used as the base for logs. For example, you could use exponents of powers of 5 or exponents of powers of 2.

EXAMPLE 3

Base-5 logarithms: The log base 5 of a number, x, is the exponent to which the base, 5, must be raised to obtain x. For example,

a. $\log_5 5^4 = 4$

b. $\log_5 125 = \log_5 5^3 = 3$

c. $\log_5 \frac{1}{25} = \log_5 5^{-2} = -2$

Base-2 logarithms: The log base 2 of a number, x, is the exponent to which 2 must be raised to obtain x. For example,

a. $\log_2 2^5 = 5$

b. $\log_2 16 = \log_2 2^4 = 4$

c. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

In general, a statement in logarithmic form is $\log_b x = y$, where b is the base of the logarithm, x is a power of b, and y is the exponent. The base b for a logarithm can be any positive number except 1.

7. Determine each of the following.

a. $\log_4 64$

b. $\log_2 \frac{1}{16}$

c. $\log_3 9$

d. $\log_3 \frac{1}{27}$

The examples and problems so far in this activity demonstrate the following property of logarithms.

Property of Logarithms

For any base b ($b > 0$, $b \neq 1$), $\log_b b^n = n$.

8. Determine each of the following:

a. $\log 1$

b. $\log_5 1$

c. $\log_{\frac{1}{2}} 1$

d. $\log 10$

e. $\log_5 5$

f. $\log_{1/2} \left(\frac{1}{2}\right)$

9. **a.** Referring to Problem 8a–c, write a general rule for $\log_b 1$.

b. Referring to Problem 8d–f, write a general rule for $\log_b b$.

Property of Logarithms

In general, $\log_b 1 = 0$ and $\log_b b = 1$, where $b > 0$, $b \neq 1$

Natural Logarithms

Because the base of a log can be any positive number except 1, the base can be the number e. Many applications involve the use of log base e. Log base e is called the **natural** log and has the following special notation:

$\log_e x$ is written as $\ln x$, read simply as el-n-x.

EXAMPLE 4

a. $\ln e^2 = \log_e e^2 = 2$

b. $\ln \frac{1}{e^4} = \ln e^{-4} = -4$

10. Evaluate the following.

a. $\ln e^7$ **b.** $\ln\left(\frac{1}{e^3}\right)$ **c.** $\ln 1$

d. $\ln e$ **e.** $\ln \sqrt{e}$

Logarithmic and Exponential Forms

Because logarithms are exponents, logarithmic statements can be written as exponential statements, and exponential statements can be written as logarithmic statements.

For example, in the statement $3 = \log_5 125$, the base is 5, the exponent (logarithm) is 3, and the result is 125. This relationship can also be written as the equation $5^3 = 125$.

In general, the logarithmic equation $y = \log_b x$ is equivalent to the exponential equation $b^y = x$.

EXAMPLE 5 ***Rewrite the exponential equation $e^{0.5} = x$ as an equivalent logarithmic equation.***

SOLUTION

In the equation $e^{0.5} = x$, the base is e, the result is x, and the exponent (logarithm) is 0.5. Therefore, the equivalent logarithmic equation is $0.5 = \log_e x$, or $0.5 = \ln x$.

11. Rewrite each exponential equation as a logarithmic equation and each log equation as an exponential equation.

a. $3 = \log_2 8$ **b.** $\ln e^3 = 3$ **c.** $\log_2 \frac{1}{16} = -4$

d. $6^3 = 216$ **e.** $e^1 = e$ **f.** $3^{-2} = \frac{1}{9}$

Logarithms and the Calculator

The numbers whose logarithms you have been working with have been exact powers of the base. However, in many situations, you have to evaluate a logarithm where the number is not an exact power of the base. For example, what is log 20 or ln 15? Fortunately, the common log (base 10) and the natural log (base e) are on your calculator.

12. Use your calculator to evaluate the following.

a. log 20

b. ln 15

c. $\ln \frac{1}{2}$

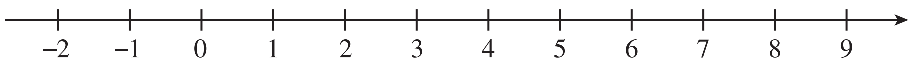
d. log 0.02

e. Use your calculator to check your answers to Problems 6 and 10.

13. a. Use your calculator to complete the following table. Confirm the placement of the diameter values on the log-scaled axis.

SPHERE	DIAMETER, *d*, IN METERS	*d*, IN SCIENTIFIC NOTATION	log(*d*)
Golf ball	0.043	4.3×10^{-2}	
Baseball	0.075	7.5×10^{-2}	
Basketball	0.239	2.39×10^{-1}	
Moon	3,476,000	3.476×10^{6}	
Earth	12,756,000	1.2756×10^{7}	
Jupiter	142,984,000	1.4298×10^{8}	

b. Plot the values from the log column in the preceding table on the following axis.



c. Compare the preceding plot with the plot in Problem 3a and comment.

SUMMARY
ACTIVITY 3.8

1. The notation for logarithms is $\log_b x = y$, where b is the base of the log, x is the resulting power of b, and y is the exponent. The base b can be any positive number except 1; x can be any positive number. The range of y values includes all real numbers.
2. The notation for **common logarithm** or base-10 logarithms is $\log_{10} x = \log x$.
3. The notation for the **natural logarithm** or base e logarithm is $\log_e x = \ln x$.
4. The **logarithmic equation** $y = \log_b x$ is equivalent to the **exponential equation** $b^y = x$.

5. If $b > 0$ and $b \neq 1$,

a. $\log_b 1 = 0$

b. $\log_b b = 1$

c. $\log_b b^n = n$

EXERCISES
ACTIVITY 3.8

1. Use the definition of logarithm to determine the exact value of each of the following.

a. $\log_2 32$ **b.** $\log_3 27$ **c.** $\log 0.1$

d. $\log_2 \left(\frac{1}{64}\right)$ **e.** $\log_5 1$ **f.** $\log_{1/2} \left(\frac{1}{4}\right)$

g. $\log_7 \sqrt{7}$ $\left(\textit{Hint}: \sqrt{7} = 7^{1/2}\right)$ **h.** $\log_{100} 10$ **i.** $\log 1$

j. $\log_2 1$ **k.** $\ln e^5$ **l.** $\ln \left(\frac{1}{e^2}\right)$ **m.** $\ln 1$

2. Evaluate each common logarithm without the use of a calculator.

a. $\log \left(\dfrac{1}{1000}\right) =$ _______ **b.** $\log \left(\dfrac{1}{100}\right) =$ _______

c. $\log \left(\dfrac{1}{10}\right) =$ _______ **d.** $\log 1 =$ _______

e. $\log 10 =$ _______ **f.** $\log 100 =$ _______

g. $\log 1000 =$ _______ **h.** $\ln \sqrt{10}$ _______

3. Rewrite the following equations in logarithmic form.

a. $3^2 = 9$ **b.** $\sqrt{121} = 11$ (*Hint:* First rewrite $\sqrt{121}$ in exponential form.)

c. $4^t = 27$ **d.** $b^3 = 19$

4. Rewrite the following equations in exponential form.

a. $\log_3 81 = 4$ **b.** $\frac{1}{2} = \log_{100} 10$

Exercise numbers appearing in color are answered in the Selected Answers appendix.

c. $\log_9 N = 12$

d. $y = \log_7 x$

e. $\ln \sqrt{e} = \frac{1}{2}$

f. $\ln \left(\frac{1}{e^2}\right) = -2$

5. Estimate between what two integers the solutions for the following equations fall. Then solve each equation exactly by changing it to log form. Use your calculator to approximate your answer to three decimal places.

a. $10^x = 3.25$

b. $10^x = 590$

c. $10^x = 0.0000045$

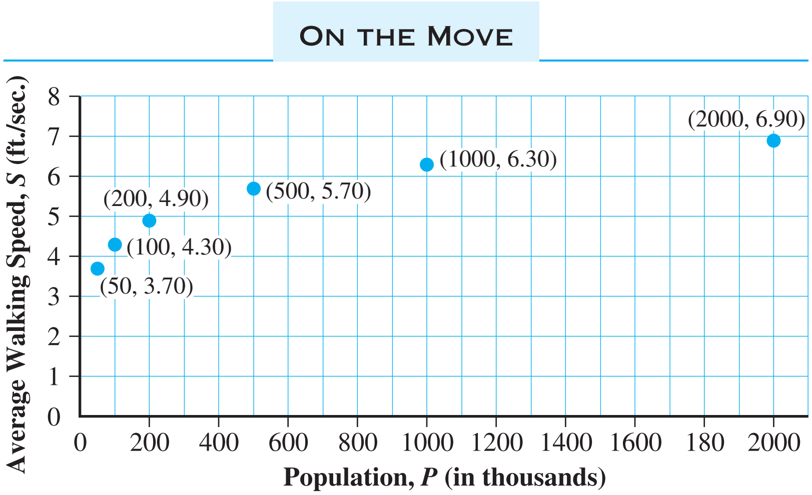
ACTIVITY 3.9

Walking Speed of Pedestrians

OBJECTIVES

1. Determine the inverse of the exponential function.
2. Identify the properties of the graph of a logarithmic function.
3. Graph the natural logarithmic function.

On a recent visit to Boston, you notice that people seem rushed as they move about the city. Upon returning to college, you mention this observation to your psychology instructor. The instructor refers you to a psychology study that investigates the relationship between the average walking speed of pedestrians and the population of the city. The study cites statistics presented graphically as follows.



1. **a.** Does the data appear to be linear? Explain.

   b. Does the data appear to be exponential? Explain.

This data is actually logarithmic. Situations that can be modeled by logarithmic functions will be the focus of this and the following activity.

Introduction to the Logarithmic Function

The logarithmic function base b is defined by $y = \log_b x$, where

b represents the base of the log $(b > 0, b \neq 1)$,

x is the input and represents a power of the base b (x is also called the argument), and y is the output and is the exponent needed on the base b to obtain x.

2. **a.** Evaluate $\log_{10}(-100)$ using your calculator. What do you observe? Does it seem reasonable? Explain.

   b. Is it possible to determine $\log(0)$?

   c. What is the domain for the function defined by $y = \log x$?

d. What is the range? Remember, the output y is an exponent.

3. The exponential function defined by $f(x) = 10^x$ has a special relationship with the corresponding logarithmic function defined by $g(x) = \log_{10} x = \log x$.

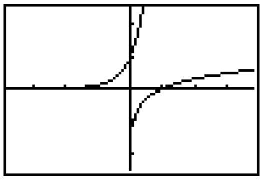
a. Complete the following tables for $f(x) = 10^x$ and $g(x) = \log x$.

x	$f(x) = 10^x$
−2	
−1	
0	
1	
2	

x	$g(x) = \log x$
0.01	
0.1	
1	
10	
100	

b. Compare the input and output values for functions f and g.

c. Sketch the graphs of $Y1 = 10^x$ and $Y2 = \log_{10} x$ using your graphing calculator. Use the window Xmin = −4, Xmax = 4, Ymin = −3, and Ymax = 3. Your screen should appear as follows.



d. Graph $y = x$ on the same coordinate axes as functions f and g. Describe in a sentence or two the symmetry you observe in the graphs of f and g.

Recall the concept of an inverse function from Chapter 2. The inverse function interchanges the domain and range of the original function. Also, the graph of an inverse function is the reflection of the original function about the line $y = x$. Therefore, it appears from the results in Problem 3 that $f(x) = 10^x$ and $g(x) = \log x$ are inverse functions.

You can determine the equation of the inverse function by interchanging the input (x-values) and the output (y-values) in the given equation for the function and solving the new equation for y.

EXAMPLE 1 ***Determine the equation of the inverse of the function defined by $y = 5^x$.***

SOLUTION

Step 1.	Interchange the x and y variables.	$x = 5^y$
Step 2.	Solve the resulting equation for y by writing the statement in logarithmic notation.	$y = \log_5 x$

4. Use the algebraic approach demonstrated in Example 1 to verify that $y = \log x$ is the inverse of $y = 10^x$.

Problems 2, 3, and 4 illustrate the following properties of the common logarithmic function.

Properties of the Common Logarithmic Function $f(x) = \log x$

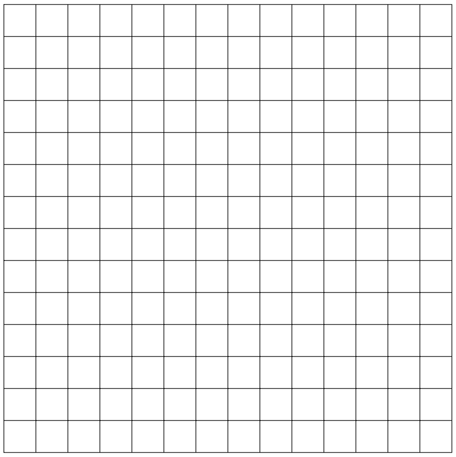
1. The domain of f is the set of all positive real numbers $(x > 0)$.
2. The range of f is all real numbers.
3. f is the inverse of the function defined by $g(x) = 10^x$.

The Graph of the Natural Logarithmic Function

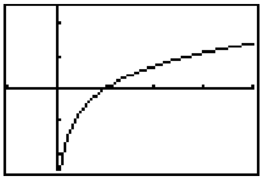
5. a. Using your calculator, complete the following table. Round your answers to three decimal places.

x	0.1	0.5	1	5	10	20	50
$y = \ln x$							

b. Sketch a graph of $y = \ln x$.



c. Verify your graph in part b using your graphing calculator. Using the window Xmin $= -1$, Xmax $= 4$, Ymin $= -2.5$, and Ymax $= 2.5$, your screen should appear as follows.



d. What are the domain and range of the function defined by $y = \ln x$?

e. Does the graph of $y = \ln x$ have a horizontal asymptote? Explain.

f. Does the graph of $y = \ln x$ have a vertical intercept?

g. Complete the following table using your calculator. Round your answers to the nearest tenth.

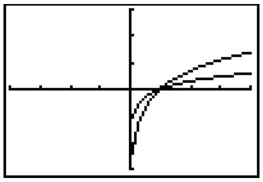
x	1	0.5	0.25	0.1	0.01	0.001
$y = \ln x$						

h. As the input values take on values closer and closer to 0, what happens to the corresponding output values?

DEFINITION

A **vertical asymptote** is a vertical line, $x = a$, that the graph of a function becomes very close to but never touches. As the input values get closer and closer to $x = a$, the output values get larger and larger in magnitude. That is, the output values become very large positive or very large negative values.

EXAMPLE 2 ***The vertical asymptote of the graphs of $y = \log x$ and $y = \ln x$ is the vertical line $x = 0$.***

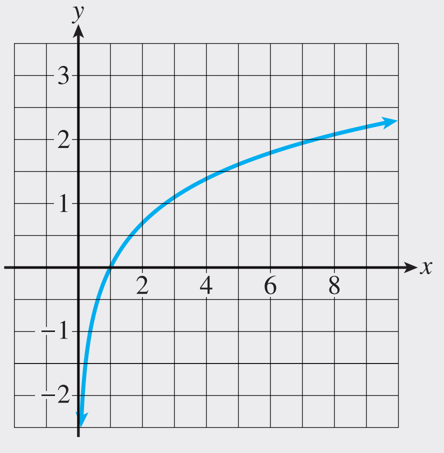


6. a. Graph $y = e^x$, $y = \ln x$, and $y = x$ on the same set of coordinate axes using the window Xmin = −7.5, Xmax = 7.5, Ymin = −5, and Ymax = 5. Describe the symmetry that you observe.

b. Use an algebraic approach to determine the inverse of the exponential function defined by $y = e^x$.

SUMMARY
ACTIVITY 3.9

1. Properties of the log function defined by $y = \log x$.
   a. The domain of f is $x > 0$.
   b. The range of f is all real numbers.
   c. f is the inverse of the function defined by $g(x) = 10^x$.
2. The graph of a logarithmic function defined by $y = \log_b x$, where $b > 1$, resembles the following graph, and the function
   a. is increasing for all $x > 0$
   b. has an x-intercept of $(1, 0)$
   c. has no y-intercept
   d. has a vertical asymptote of $x = 0$, the y-axis



3. The **natural logarithmic function** is defined by

$$y = \ln x = \log_e x.$$

4. The graph of the natural logarithmic function $y = \ln x$

a. is increasing for all $x > 0$

b. has an x-intercept of $(1, 0)$

c. has a vertical asymptote of $x = 0$, the y-axis

5. You can determine the equation of the inverse of the function by interchanging the input (x-values) and the output (y-values) in the given equation for the function and solving the new equation for y.

EXERCISES
ACTIVITY 3.9

1. Using the graph of $y = \log x$ as a check, summarize the following properties of the common logarithmic function.

a. What is the domain?

b. What is the range?

c. For what values of x is $\log x$ positive?

d. For what values of x is $\log x$ negative?

e. For what values of x does $\log x = 0$?

f. For what values of x does $\log x = 1$?

2. a. Complete the following table using your calculator. Round your answers to the nearest tenth.

x	0.001	0.01	0.1	0.25	0.5	1
$y = \log x$						

b. As the positive input values take on values closer and closer to 0, what happens to the corresponding output values?

c. Determine the vertical asymptote of the graph of $y = \log x$.

3. The exponential function defined by $y = 2^x$ has an inverse. Determine the equation of the inverse function. Write your answer in logarithmic form.

4. Using the graph of $y = \ln x$ as a check, summarize the following properties of the natural logarithmic function.

a. What is the domain?

b. What is the range?

c. For what values of x is $\ln x$ positive?

d. For what values of x is $\ln x$ negative?

e. For what values of x does $\ln x = 0$?

f. For what values of x does $\ln x = 1$?

5. The life expectancy for a piece of equipment is the number, n, of years for the equipment to depreciate to a known salvage value, V. The life expectancy, n, is given by the formula

$$n = \frac{\log V - \log C}{\log(1 - r)},$$

where C is the initial cost of the piece of equipment and r is the annual rate of depreciation expressed as a decimal. If a computer costs \$34,000 and has a salvage value of \$1000, what is the life expectancy if the annual rate of depreciation is 40%?

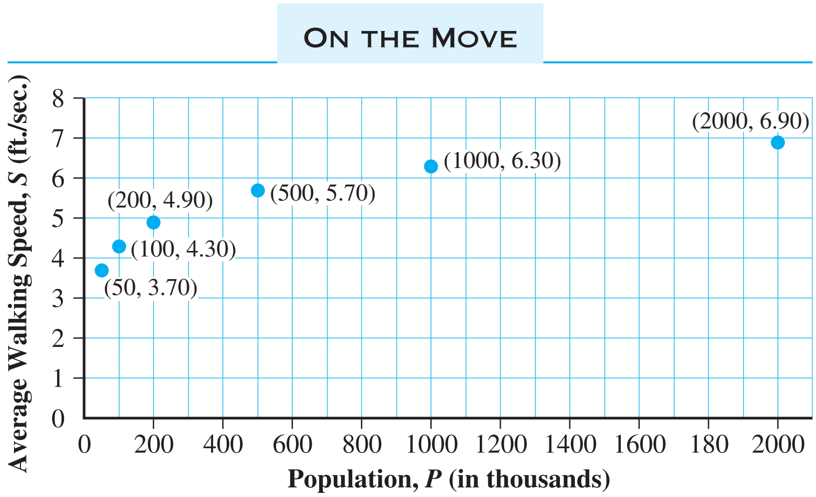
ACTIVITY 3.10

Walking Speed of Pedestrians, continued

OBJECTIVES

1. Compare the average rate of change of increasing logarithmic, linear, and exponential functions.
2. Determine the regression equation of a natural logarithmic function that best fits a set of data.

In Activity 3.9 you looked at a psychology study that investigated the relationship between the average walking speed of pedestrians and the population of the city. Graphically the data was presented as follows.



1. a. Does the data appear to be logarithmic? Explain.

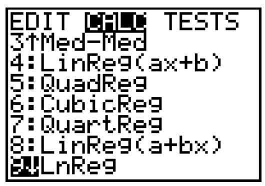
b. Use the data in the graph to complete the following table.

POPULATION, *P* (in thousands)	50	100	200	500	1000	2000
AVERAGE WALKING SPEED, *S*						

The natural logarithmic function can be used to model a variety of scientific and natural phenomena. The natural logarithmic function is so prevalent that on most graphing calculators it has its own built-in regression finder.

2. a. Use your graphing calculator and the table in Problem 1b to produce a scatterplot of the average walking speed data.

b. Use the regression feature of your calculator to produce a natural logarithmic curve that approximates the data in the table. Use option 9 from the STAT CALC menu.



The LnReg option will generate a regression equation of the form $y = a + b \ln x$. Round a and b to the nearest thousandth, and record the function below.

c. Enter the function from part b into your graphing calculator. Verify visually that this function is a good model for your data.

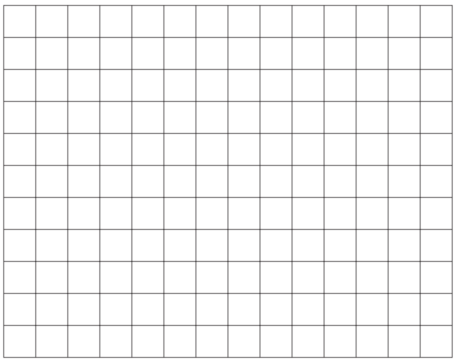
d. What is the practical domain of this function?

e. Use the function from part b to predict the average walking speed in Boston, population 589,121. (*Note:* P is in thousands (589.121 thousands).)

f. Use the model to predict the average walking speed in New York City, population 8,008,278.

3. a. If the average walking speed in a certain city is 5.2 feet per second, write an equation that can be used to estimate the population P of the city.

b. Solve the equation using a graphical approach.



Comparing the Average Rate of Change of Logarithmic, Linear, and Exponential Functions

4. a. Complete the following table using the function defined by $S = 0.303 + 0.868 \ln P$.

***P*, POPULATION (thousands)**	10	20	150	250
***S*, AVERAGE WALKING SPEED (ft./sec.)**				

b. Determine the average rate of change of S as the population increases from

i. 10 to 20 thousand

ii. 20 to 150 thousand

iii. 150 to 250 thousand

c. What can you say in general about the average rate of change in the walking speed as the population increases?

You should have discovered that the average rate of change in this situation is always positive. This means that the walking speed increases as the population increases. Nevertheless, in general, the increase gets smaller as the population increases. This is characteristic of logarithmic functions.

As the input of a logarithmic function with $b > 1$ increases, the output increases at a slower rate (the graph becomes less steep).

5. Complete the following statements by describing the rate at which the output values change.

a. For an increasing linear function, as the input variable increases, the output

____

b. For an increasing exponential function, as the input increases, the output

____

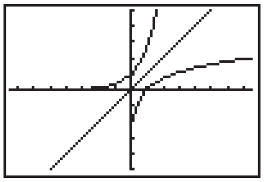
c. For an increasing logarithmic function, as the input increases, the output

____

6. Consider the graphs of

   i. $f(x) = e^x$ ii. $h(x) = x$ iii. $g(x) = \ln x$

   using the window Xmin $= -7.5$, Xmax $= 7.5$, Ymin $= -5$, and Ymax $= 5$.



   a. Which of the functions are increasing?

   b. Which of the functions are decreasing?

   c. As the input values get larger, which of the functions grows fastest?

   d. As the input values get larger, which of the functions grows most slowly?

   e. Do any of these functions have a horizontal asymptote?

   f. Do any of these functions have a vertical asymptote?

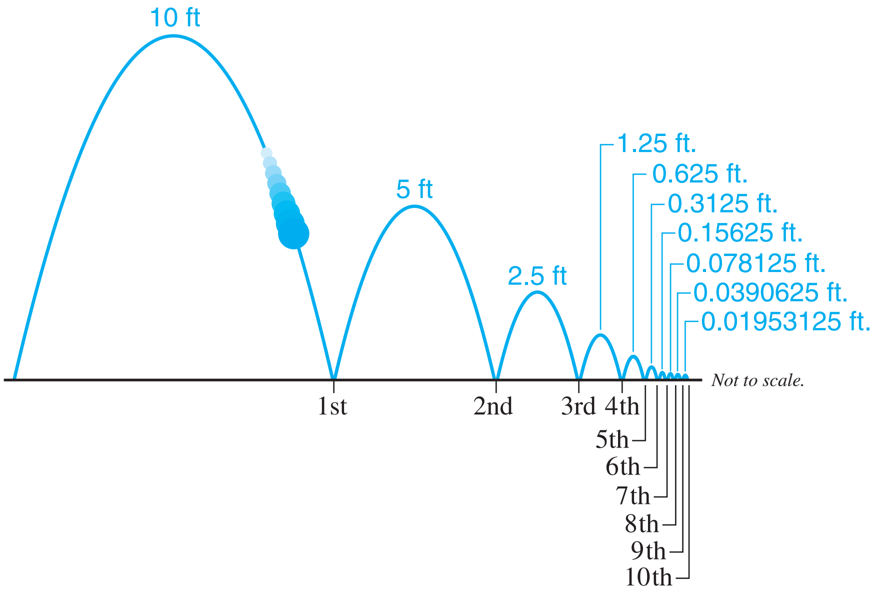
   g. Compare the domains of these functions.

   h. Compare the ranges of these functions.

Problem 6 illustrates some of the relationships between $f(x) = b^x$, where $b > 1$, $g(x) = \log_b x$, where $b > 1$, and $y = mx + b$, where $m > 0$.

Application

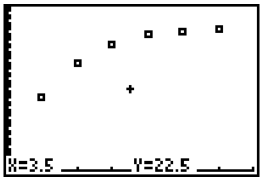
7. You are working on the development of an "elastic" ball for the IBF Toy Company. The question you are investigating is, "If the ball is launched straight up, how far has it traveled vertically when it hits the ground for the 10th time?"



Your launcher will project the ball 10 feet into the air. This means it will travel 20 feet (10 feet up and 10 down) before it hits the ground the first time. Assuming that the ball returns to 50% of its previous height, it will rebound 5 feet and travel 10 feet before it hits the ground again. The following table summarizes this situation.

***N*, times the ball hits the ground**	1	2	3	4	5	6
Distance traveled since last time (ft.)	20	10	5	2.5	1.25	0.625
***T*, Total distance traveled (ft.)**	20	30	35	37.5	38.75	39.375

a. Using the window Xmin = 0, Xmax = 7, Ymin = 0, and Ymax = 45, a plot of N versus T should resemble the following.



b. Do the table and scatterplot indicate the data is linear, exponential, or logarithmic?

c. Use your graphing calculator to produce linear, exponential, and natural log regression equations for the given data.

d. Graph each equation and visually determine which of the regression models best fits the data.

e. Use the equation of best fit to predict the total distance traveled by the ball when it hits the ground for the 10th time.

SUMMARY
ACTIVITY 3.10

1. As the input of a logarithmic function increases, the output increases at a slower rate (the graph becomes less steep).

2. The relationships among the graphs $f(x) = b^x$, where $b > 1$, $g(x) = \log_b x$, where $b > 1$, and $y = mx + b$, where $m > 0$, are identified in the following table.

GRAPHS	INCREASING OR DECREASING	GROWTH RATE	HORIZONTAL OR VERTICAL ASYMPTOTE	DOMAIN	RANGE
$f(x) = b^x$, $b > 1$	increasing	fastest	horizontal asymptote	all real numbers	$y > 0$
$g(x) = \log_b x$, $b > 1$	increasing	slowest	vertical asymptote	$x > 0$	all real numbers
$y = mx + b$, $m > 0$	increasing	constant	none	all real numbers	all real numbers

EXERCISES
ACTIVITY 3.10

1. *Chlamydia trachomatis* infections are the most commonly reported notifiable disease in the United States. These are among the most prevalent of all sexually transmitted diseases. The following data from the Centers for Disease Control and Prevention indicates the reported rates, R, in rates per 100,000 people from 1985 to 2002. Let t represent the number of years since 1985.

YEARS, t, SINCE 1985	1	3	5	9	13	15	17
REPORTED RATES: U.S. (rate per 100,000 population), R	40	90	160	200	250	260	288

Exercise numbers appearing in color are answered in the Selected Answers appendix.

a. Plot the points on the following grid.

b. Does the scatterplot indicate that the data is logarithmic? Explain.

c. Determine the natural log regression equation. Record the regression equation below, and add a sketch of the regression curve to the scatterplot in part a.

d. Is the graph a good fit of the data?

e. Use your model to predict the reported rate of *Chlamydia trachomatis* infections per 100,000 people in 2010.

2. a. Consider a data set for the variables x and y.

x	1	4	7	10	13
$f(x)$	3.0	4.5	5.0	5.2	5.8

Plot these points on the following grid.

b. Does the scatterplot indicate the data is more likely linear, exponential, or logarithmic? Explain.

c. Use your graphing calculator to determine a logarithmic regression model that represents this data.

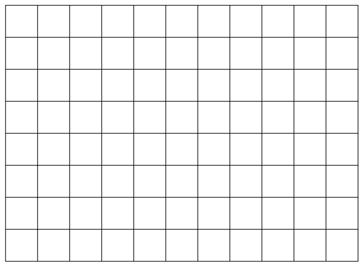
d. Use your model to determine $f(11)$ and $f(20)$.

3. The barometric pressure, P, in inches of mercury at a distance x miles from the eye of a moderate hurricane can be modeled by

$$P = f(x) = 0.48 \ln(x + 1) + 27.$$

a. Determine $f(0)$. What is the practical meaning of the value in this situation?

b. Sketch a graph of this function.



c. Describe how air pressure changes as you move away from the eye of the hurricane.

4. The formula $R = 80.4 - 11 \ln x$ is used to approximate the minimum required ventilation rate, R, as a function of the air space per child in a public school classroom. The rate R is measured in cubic feet per minute, and x is measured in cubic feet.

a. Sketch a graph of the rate function for $100 \le x \le 1500$.

b. Determine the required ventilation rate if the air space per child is 300 cubic feet.

5. You have recently accepted a job working in the coroner's office of a large city. Because of the large numbers of homicides, it has been very difficult for the coroners to complete all of their work. Your job is, in part, to assist them in the paperwork. On one particular day, you are working on a case in which you are attempting to establish the time of death.

The coroner tells you that to establish the time of death, he uses the formula

$$t = 4 \ln \frac{98.6 - T_s}{T_b - T_s},$$

where t is the number of hours the victim has been dead,

T_b represents the temperature of the body when discovered, and

T_s represents the temperature of his surroundings.

The coroner also tells you that the thermostat was set at 68°F in the apartment in which the body was found and that the victim's body temperature was 78°F.

a. Using the preceding formula, determine the number of hours the victim has been deceased. Use your calculator to approximate your answer to one decimal place.

b. If the body was discovered at 10:07 P.M., what do you estimate for the time of death?

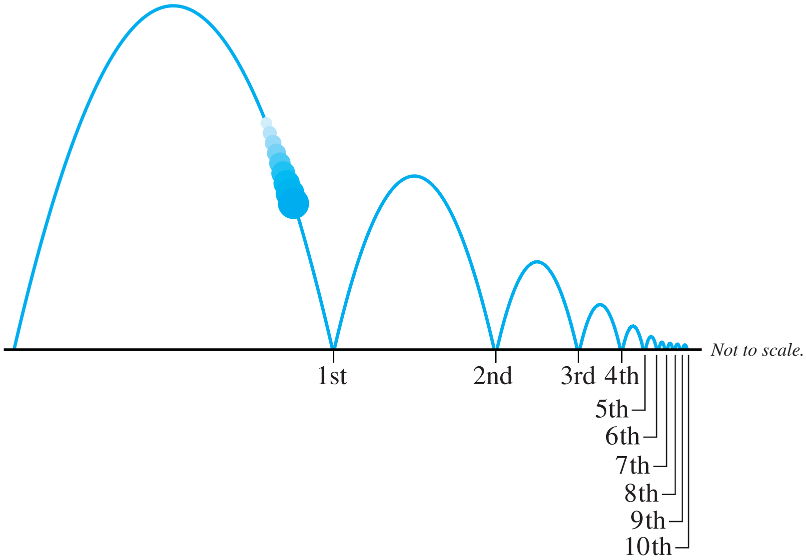
ACTIVITY 3.11

The Elastic Ball

OBJECTIVES

1. Apply the log of a product property.
2. Apply the log of a quotient property.
3. Apply the log of a power property.
4. Discover change of base formula.

You are continuing your work on the development of the elastic ball. You are still investigating the question, "If the ball is launched straight up, how far has it traveled vertically when it hits the ground for the 10th time?" However, your supervisor tells you that you cannot count the initial launch distance. You must calculate only the rebound distance.

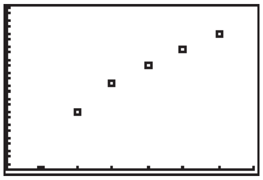


Using some physical properties, timers, and your calculator, you collect the following data.

N, NUMBER OF TIMES THE BALL HITS THE GROUND	1	2	3	4	5	6
T, TOTAL REBOUND DISTANCE (ft.)	0	9.0	13.5	16.3	18.7	21.0

1. Does the data seem reasonable? Explain.

2. Use your graphing calculator to construct a scatterplot of the data with N as the input and T as the output. Using a window of Xmin = 0, Xmax = 7, Ymin = 0, and Ymax = 25, your graph should resemble the following.



3. Do you believe the data can be modeled by a logarithmic function? Explain.

4. This data can be modeled by $T = 26.75 \log N$. Use your graphing calculator to verify visually that this is a reasonable model for the given data.

5. a. Using the log model, complete the following table. Round values to the nearest hundredth.

N	2	5	10
T* = 26.75 log *N			

b. How are the outputs from 2 and 5 related to the output for 10?

c. Using the results from part b, how could you determine the total rebound distance after 10 bounces?

The results from Problem 5 can be written as follows.

$$26.75 = 8.05 + 18.70$$
$$26.75 \log 10 = 26.75 \log 2 + 26.75 \log 5$$
$$26.75 \log (2 \cdot 5) = 26.75 \log 2 + 26.75 \log 5$$

Dividing both sides by 26.75, you have

$$\log (2 \cdot 5) = \log 2 + \log 5.$$

This result illustrates an important property of logarithms.

Property of the Logarithm of a Product

If $A > 0$, $B > 0$, then $\log_b (A \cdot B) = \log_b A + \log_b B$, where $b > 0$, $b \neq 1$.

Expressed verbally, this property states that the logarithm of a product is the sum of the individual logarithms.

EXAMPLE 1

a. $\log_2 32 = \log_2 (4 \cdot 8) = \log_2 4 + \log_2 8 = 2 + 3 = 5.$

b. $\log (5st) = \log (5) + \log (s) + \log (t)$

c. $\ln (xy) = \ln x + \ln y$

6. Use the property of the logarithm of a product to write the following as the sum of two or more logarithms.

a. $\log_b (7 \cdot 13)$

b. $\log_3 (xyz)$

c. $\log 15$ **d.** $\ln(3xy)$

7. Write the following as the logarithm of a single expression.

a. $\ln a + \ln b + \ln c$ **b.** $\log_4 3 + \log_4 9$

Logarithm of a Quotient

Consider the following table from Problem 5.

N	2	5	10
$T = 26.75 \log N$	8.05	18.70	26.75

This table also indicates that the rebound distance after this ball has hit the floor twice (8.05 ft.) is the total rebound distance when the ball has hit the ground 10 times (26.75 ft.) minus the total rebound distance when the ball has hit the ground 5 times (18.70 ft.).

This can be written as

$$\begin{aligned} 8.05 &= 26.75 - 18.70 \\ 26.75 \log 2 &= 26.75 \log 10 - 26.75 \log 5 \\ \log 2 &= \log 10 - \log 5. \end{aligned}$$

Substituting $\log\left(\frac{10}{5}\right)$ for $\log 2$, you have

$$\log\left(\frac{10}{5}\right) = \log 10 - \log 5.$$

This suggests another important property of logarithms. The property is demonstrated further in Problem 8.

8. a. Complete the following table. Round your answers to the nearest thousandth.

x	$Y1 = \log\left(\frac{x}{4}\right)$	$Y2 = \log x - \log 4$
1		
5		
10		
23		

b. Is the expression $\log\left(\frac{x}{4}\right)$ equivalent to $\log x - \log 4$? Explain.

c. Sketch the graph of $y = \log\left(\frac{x}{4}\right)$ and $y = \log x - \log 4$ using your graphing calculator. What do the graphs suggest about the relationship between $\log\left(\frac{x}{4}\right)$ and $\log x - \log 4$?

Property of the Logarithm of a Quotient

If $A > 0$, $B > 0$, then $\log_b\left(\frac{A}{B}\right) = \log_b A - \log_b B$, where $b > 0$, $b \neq 1$.

Expressed verbally, this property states that the logarithm of a quotient is the difference of the logarithm of the numerator and the logarithm of the denominator.

EXAMPLE 2

a. $\log_3\left(\frac{81}{27}\right) = \log_3 81 - \log_3 27 = 4 - 3 = 1$

Note that $\log_3\left(\frac{81}{27}\right) = \log_3(3) = 1$.

b. $\log\left(\frac{2x}{y}\right) = \log(2x) - \log(y) = \log 2 + \log x - \log y$

c. $\ln\left(\frac{x^2}{5}\right) = \ln(x^2) - \ln(5)$

9. Use the properties of logarithms to write the following as the sum or difference of logarithms.

a. $\log_6 \frac{17}{3}$

b. $\ln \frac{x}{23}$

c. $\log_3 \frac{2x}{y}$

d. $\log \frac{3}{2z}$

10. Write the following expressions as the logarithm of a single expression.

a. $\log x - \log 4 + \log z$

b. $\log x - (\log 4 + \log z)$

11. a. Use your graphing calculator to sketch the graphs of $y = \log x + \log 4$ and $y = \log(x + 4)$.

b. How do these graphs compare?

c. What do the graphs suggest about the relationship between $\log(A + B)$ and $\log A + \log B$?

Logarithm of a Power

Before calculators, logarithms were used to help in computing products and quotients of numbers. More importantly, logarithms were used to compute powers such as 734.21^3 and $\sqrt{0.0761} = (0.0761)^{1/2}$. In such a case, the first step was to take the logarithm of the power and rewrite the resulting expression. To determine how to rewrite $\log 734.21^3$, you can investigate the expression $\log x^3$.

12. a. Complete the following table. Round to the nearest thousandth.

x	$Y1 = \log x^3$	$Y2 = 3 \log x$
2		
7		
15		

b. Sketch the graphs of $y = \log x^3$ and $y = 3 \log x$ using your graphing calculator.

c. What do the results of part a and part b demonstrate about the relationship between $\log x^3$ and $3 \log x$?

The results in Problem 12 illustrate another property of logarithms.

Property of the Logarithm of a Power

If $A > 0$ and p is any real number, then $\log_b A^p = p \cdot \log_b A$, where $b > 0$, $b \neq 1$. In words, the property states that the logarithm of a power is equivalent to the exponent times the logarithm of the base.

EXAMPLE 3

a. $\log_3 9^2 = 2 \log_3 9 = 2 \cdot 2 = 4$

b. $\log_5 x^4 = 4 \log_5 x$

c. $\ln (xy)^7 = 7 \ln (xy)$

d. $\ln x^{1/4} = \frac{1}{4} \ln x$

e. $\log \sqrt{63} = \log 63^{1/2} = \frac{1}{2} \log 63$

13. Use the properties of logarithms to write the given logarithms as the sum or difference of two or more logarithms, or as the product of a real number and a logarithm. All variables represent positive numbers.

a. $\log_3 x^{1/2}$

b. $\log_5 x^3$

c. $\ln t^2$

d. $\log \sqrt[3]{50}$ (*Hint:* $\sqrt[3]{50} = 50^{1/3}$)

e. $\log_5 \frac{x^2y^3}{z}$

f. $\log_3 \frac{3x^2}{y^3}$

14. Write each of the following as the logarithm of a single expression with coefficient 1.

a. $2 \log_3 5 + 3 \log_3 2$

b. $\frac{1}{2}\log x^4 - \frac{1}{2}\log y^5$

c. $3 \log_b 10 - 4 \log_b 5 + 2 \log_b 3$

d. $3 \ln 4 - (4 \ln 5 + 2 \ln 3)$

Using the properties of logarithms to solve exponential equations algebraically will be investigated in the next activity.

Change of Base Formula

Because the TI-83/T1-84 Plus has only the log base 10 (log) and the log base e (ln) keys, you cannot graph a logarithmic function such as $y = \log_2 x$ directly. Consider the following argument to rewrite the expression $\log_2 x$ as an equivalent expression using log base 10.

By definition of logs, $y = \log_2 x$ is the same as $x = 2^y$. Taking the log base 10 of both sides of the second equation, $x = 2^y$, you have

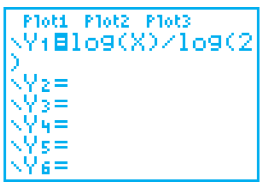
$$\log x = \log 2^y.$$

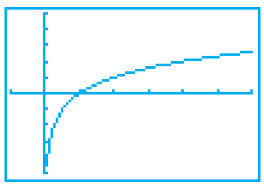
Using the property of the log of a power, $\log x = y \log 2$. Solving for y, you have

$$y = \frac{\log x}{\log 2}.$$

Therefore, the equation $y = \log_2 x$ is equivalent to $y = \frac{\log x}{\log 2}$.

15. To graph $y = \log_2 x$, enter log(X) / log(2) for Y1 in your TI-83/TI-84 Plus calculator. Your graph should resemble the following.





16. **a.** Write $y = \log_6 x$ as an equivalent equation using base 10.

b. Use the result from part a to graph $y = \log_6 x$.

c. What is the domain of the function?

d. What is the x-intercept of the graph?

The formula you used in problems 15 and 16 for graphing log functions of different bases is a special case of the formula

$$\log_b x = \frac{\log_a x}{\log_a b}, \quad \text{where} \quad a > 0_1 \neq 1$$

This is often called the **change of base formula**, where $b > 0$ and $b \neq 1$.

The change of base formula is used to change from base b to base a. Because most calculators have log base 10 (log) and log base e (ln) keys, you usually convert to one of those bases. For those bases,

$$\log_b x = \frac{\log x}{\log b} \text{ or } \log_b x = \frac{\ln x}{\ln b}.$$

EXAMPLE 4 ***Change the equation $y = \log_5 x$ to an equivalent equation in base 10 and/or base e.***

$$y = \log_5 x = \frac{\log x}{\log 5} \text{ or } y = \log_5 x = \frac{\ln x}{\ln 5}$$

17. Use each of the change of base formulas to determine $\log_4 1024$.

a. Using base 10:

b. Using base e:

c. How do the results in parts a and b compare?

SUMMARY
ACTIVITY 3.11

Properties of the Logarithmic Function

If $A > 0$, $B > 0$, $b > 0$, and $b \neq 1$, then

1. $\log_b (A \cdot B) = \log_b A + \log_b B$
2. $\log_b \left(\frac{A}{B}\right) = \log_b A - \log_b B$
3. $\log_b (x + y) \neq \log_b x + \log_b y$
4. $\log_b A^p = p \log_b A$
5. You can use the calculator to change logarithms in base b to common or natural logarithms by

$$\log_b x = \frac{\log x}{\log b} \text{ or } \log_b x = \frac{\ln x}{\ln b}.$$

EXERCISES
ACTIVITY 3.11

1. Use the preceding properties of logarithms to write the following as a sum or difference of two or more logarithms.

   a. $\log_b (3 \cdot 7)$

   b. $\log_3 (3 \cdot 13)$

   c. $\log_7 \frac{13}{17}$

   d. $\log_3 \frac{xy}{3}$

2. Write the following expressions as the logarithm of a single number.

   a. $\log_3 5 + \log_3 3$

   b. $\log 25 - \log 17$

   c. $\log_5 x - \log_5 5 + \log_5 7$

   d. $\ln (x + 7) - \ln x$

3. **a.** Sketch the graphs of $y = \log (2x)$ and $y = \log x + \log 2$ on your graphing calculator.

Exercise numbers appearing in color are answered in the Selected Answers appendix.

b. Are you surprised by the results? Explain.

4. a. Sketch the graphs of $y = \log\left(\frac{3}{x}\right)$ and $y = \log x - \log 3$ on your graphing calculator.

b. Are you surprised by your results? Explain.

c. If your graphs in part a are not identical, can you modify the second function to make the graphs identical? Explain.

5. You have been hired to handle the local newspaper advertising for a large used car dealership in your community. The owner tells you that your predecessor in this position used the formula

$$N(A) = 7.4 \log A$$

to decide how much to spend on newspaper advertising over a 2-week period. The owner admitted that he didn't know much about the formula except that $N(A)$ represented the number of cars that the owner could expect to sell, and A was the amount of money that was spent on local newspaper advertising. He also indicated that the formula seemed to work well. You can purchase small ads in the local paper for \$15 per day, larger ads for \$50 per day, and giant ads for \$750 per day.

a. How many cars do you expect to sell if you purchase one small ad?

b. To understand the relationship between the amount spent on advertising and the number of cars sold, you set up a table. Complete the following table.

AD COST, A	EXPECTED CAR SALES, $N(A)$
15	
50	
750	

c. How do the expected car sales from one small ad and one larger ad compare to the expected car sales from just one giant ad?

d. Are the results in the previous table above consistent with what you know about the properties of logarithms? Explain.

e. What are you going to advise the owner regarding the purchase of a giant ad?

6. Use the properties of logarithms to write the given logarithms as the sum or difference of two or more logarithms or as the product of a real number and a logarithm. Simplify, if possible. All variables represent positive numbers.

a. $\log_3 3^5$

b. $\log_2 2^x$

c. $\log_b \frac{x^3}{y^4}$

d. $\ln \frac{\sqrt[3]{x}\sqrt[4]{y}}{z^2}$

e. $\log_3 (2x + y)$

7. Write each of the following as the logarithm of a single expression with coefficient 1.

a. $2 \log_2 7 + \log_2 5$

b. $\frac{1}{4} \log x^3 - \frac{1}{4} \log z^5$

c. $2 \ln 10 - 3 \ln 5 + 4 \ln z$

d. $\log_5 (x + 2) + \log_5 (x + 1) - 2 \log_5 (x + 3)$

8. Given that $\log_a x = 6$ and that $\log_a y = 25$, determine the numeric value of each of the following.

a. $\log_a \sqrt{y}$

b. $\log_a x^3$

c. $3 + \log_a x^2$

d. $\log_a \frac{x^2 y}{a}$

9. Use the change of base formula and your calculator to determine a decimal approximation of each of the following to the nearest ten thousandth.

a. $\log_7 5$

b. $\log_6 \sqrt{15}$

c. $\log_{13} 47$

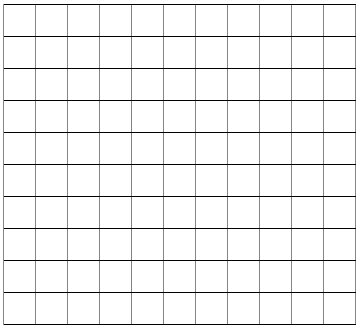
d. $\log_5 \sqrt[3]{31}$

10. The formula

$$P = f(t) = 95 - 30\log_2 t$$

gives the percentage, P, of students who could recall the important content of a classroom presentation as a function of time, t, where t is the number of days that have passed since the presentation was given.

a. Sketch a graph of the function.



b. After 3 days, what percentage of the students will remember the important content of the presentation?

c. According to the model, after how many days do only half $(P = 50)$ of the students remember the important features of the presentation? Use a graphing approach.

ACTIVITY 3.12

Prison Growth

OBJECTIVE

1. Solve exponential equations both graphically and algebraically.

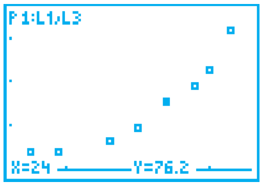
You are a criminal justice major at the local community college. The following statistics appeared in one of your required readings relating to the inmate population of U.S. federal prisons.

YEAR	1975	1979	1986	1990	1994	1998	2000	2003
TOTAL SENTENCED POPULATION, P_T (in thousands)	20.1	21.5	31.8	47.8	76.2	95.5	112.3	158.0
TOTAL SENTENCED DRUG OFFENDERS, P_D (in thousands)	5.5	5.5	12.1	25.0	46.7	56.3	63.93	86.9

You decide to analyze the prison growth situation for a project in your criminology course.

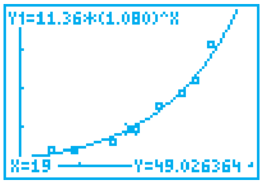
1. Although the years in the table are not evenly spaced, you notice that each of the populations seems to grow rather slowly at first and more quickly later. Do you think the data will be better modeled by linear or exponential functions?

2. Let t represent the number of years since 1970. Use your graphing calculator to produce a scatterplot of the total inmate population, P_T. Your screen should appear as follows.



3. Use your graphing calculator to determine the regression equation of an exponential function that best represents the total inmate population, P_T. Remember that the input variable is t, the number of years since 1970. In your regression equation, $P_T = ab^t$, round the value for a to two decimal places and the value for b to three decimal places. Write your model below.

4. Use your graphing calculator to visually check how well the equation in Problem 3 fits the data. Using the window Xmin = −3, Xmax = 40, Ymin = −5, and Ymax = 200, your graph should resemble the following.



5. Use the exponential regression model from Problem 3 to determine the total federal prison inmate population in 2010.

6. a. Using your model from Problem 3, write an equation that can be used to determine the year in which the total federal inmate population, P_T, is 180,000. Remember, the population in the model is given in thousands.

b. Solve the equation in part a using a graphing approach. Your screen should resemble the following. What is the equation of the horizontal line in the graph?

To solve the equation $11.36(1.080)^t = 180$ for t using an algebraic approach, you need to remove t as an exponent. The following problem guides you through this process. As you will discover, logarithms are essential in this algebraic approach.

7. Solve $11.36(1.086)^t = 180$ for t using an algebraic approach.

a. Isolate the exponential factor $(1.080)^t$ on one side of the equation.

b. Take the log (or ln) of each side of the equation in part a.

c. Apply the appropriate property of logarithms on the left side of the equation to remove t as an exponent.

d. Solve the resulting equation in part c for t.

e. How does your solution in part d compare to the estimate obtained graphically in Problem 6b?

8. You notice that over the years, the number of drug offenders seems to become a bigger percentage of the total population.

a. Determine an exponential regression equation to model the number of total sentenced drug offenders, P_D. Let the input variable t represent the number of years since 1970. In the regression equation $P_D = ab^t$, round the value for a to two decimal places and the value of b to three decimal places.

b. Use the exponential model to predict the total number of sentenced drug offenders in federal prisons in 2010.

c. Write an equation that can be used to determine the year in which the total number of sentenced drug offenders will reach 150,000.

d. Solve the equation in part c using an algebraic approach.

Radioactive Decay

Radioactive substances, such as uranium-235, strontium-90, iodine-131, and carbon-14, decay continuously with time. If P_0 represents the original amount of a radioactive substance, then the amount P present after a time t (usually measured in years) is modeled by

$$P = P_0e^{kt},$$

where k represents the rate of continuous decay.

EXAMPLE 1 ***One type of uranium decays at a rate of 0.35% per day. If 40 pounds of this uranium is available today, how much will be available after 90 days?***

SOLUTION

The uranium decays at a constant rate of 0.35% = 0.0035 per day. The initial amount, the amount available on the first day, is 40 pounds, so the equation for the amount available after t days is

$$P = 40e^{-0.0035t}.$$

To determine the amount available after 90 days, let $t = 90$. The amount available 90 days from now is

$$P = 40e^{-.0035(90)} = 29.2 \text{ lb.}$$

9. Strontium-90 decays continuously at a constant rate of 2.4% per year. Therefore, the equation for the amount P of strontium-90 after t years is

$$P = P_0e^{-0.024t}.$$

a. If 10 grams of strontium-90 are present initially, determine the number of grams present after 20 years.

b. How long will it take for the given quantity to decay to 2 grams?

c. How long would it take for the given amount of strontium-90 to decay to one-half of its original size (called its half-life)? Round to the nearest whole number.

d. Do you think that the half-life of strontium-90 is 29 years regardless of the initial amount? Answer part c using P_0 as the initial amount. (*Hint:* Find t when $P = \frac{1}{2}P_0$.)

SUMMARY
ACTIVITY 3.12

To solve exponential equations of the form $ab^x = c$, where $a > 0$, $b > 0$, $b \neq 1$, and $c > 0$:

1. Isolate the exponential factor on one side of the equation.
2. Take the log (or ln) of each side of the equation.
3. Apply the property $\log b^x = x \log b$ to remove the variable x as an exponent.
4. Solve the resulting equation for the variable.

EXERCISES
ACTIVITY 3.12

1. The number of arrests for possession of marijuana in New York City has increased dramatically. The number of arrests can be modeled by

$$N(t) = 0.37(1.77)^t,$$

where $N(t)$ represents the number of arrests in thousands and t represents the number of years since 1990.

   a. How many arrests were made in 2000?

   b. According to the model, in what year were there 50,000 arrests?

2. The U.S. Department of Transportation recommended that states adopt a 0.08% blood-alcohol concentration as the legal measure of drunk driving. Medical research has shown that as the concentration of alcohol in the blood increases, the risk of having a car accident increases exponentially. The risk, R, expressed as a percentage, is modeled by

$$R(x) = 6e^{12.77x},$$

where x is the blood-alcohol concentration, expressed as a percent.

   a. What is the risk of having a car accident if your blood-alcohol concentration is 0.08% ($x = 0.08$)?

b. What blood-alcohol concentration has a corresponding 25% risk of a car accident?

3. In 1990, the International Panel on Climate Change projected the following future amounts of carbon dioxide (in parts per million or ppm) in the atmosphere.

YEAR	1990	2000	2075	2175	2275
AMOUNT OF CARBON DIOXIDE (ppm)	353	375	590	1090	2000

a. Use your graphing calculator to create a scatterplot of the data. Let t represent the number of years since 1990 and $A(t)$ represent the amount of carbon dioxide (in ppm) in the atmosphere. Do the carbon dioxide levels appear to be growing exponentially?

b. Use your graphing calculator to determine the regression equation of an exponential model that best fits the data.

c. Use the model in part b to determine in what year the 1990 carbon dioxide level is expected to double.

d. Verify your result in part c graphically.

In Exercises 4–9, solve each equation using an algebraic approach. Verify your answers graphically.

4. $2^x = 14$

5. $3^{2x} = 8$

6. $1000 = 500(1.04)^t$

7. $e^{0.05t} = 2$ (*Hint:* Take the natural log of both sides.)

8. $2^{3x+1} = 100$

9. $e^{-0.3t} = 2$

10. **a.** Iodine-131 disintegrates at a continuous constant rate of 8.6% per day. Determine its half-life. Use the model

$$P = P_0 e^{-0.086t},$$

where t is measured in days. Round your answer to the nearest whole number.

b. If dairy cows eat hay containing too much iodine-131, their milk will be unsafe to drink. Suppose that hay contains 5 times the safe level of iodine-131. How many days should the hay be stored before it can be fed to dairy cows?

(*Hint:* Find t when $P = \frac{1}{5}P_0$.)

11. **a.** In 1969 a report written by the National Academy of Sciences (U.S.) estimated that Earth could reasonably support a maximum world population of 10 billion. The world's population was approximately 3.6 billion and growing continuously at 2% per year. If this growth rate remained constant, in what year would the world population reach 10 billion, referred to as Earth's carrying capacity? Use the model

$$P = P_0 e^{kt},$$

where P is the population (in billions), $P_0 = 3.6$, $k = 0.02$, and t is the number of years since 1969.

b. According to your growth model, when would this 1969 population double?

c. The world population in 1995 was approximately 5.7 billion. How does this compare with the population predicted by your growth model in part a?

d. The growth rate in 1995 was 1.5%. Assuming this growth rate remains constant, determine when Earth's carrying capacity will be reached. Use the model $P = P_0e^{kt}$.

ACTIVITY 3.13

Frequency and Pitch

OBJECTIVE

1. Solve logarithmic equations both graphically and algebraically.

Raising a musical note one octave has the effect of doubling the pitch, or frequency, of the sound. However, you do not perceive the note to sound "twice as high," as you might predict. Perceived pitch is given by the function

$$P(f) = 2410 \log(0.0016f + 1),$$

where P is the perceived pitch in mels (units of pitch) and f is the frequency in hertz.

1. Let frequency (input) vary in value from 10 to 100,000 hertz, and let the perceived pitch (output) vary from 0 to 6000 mels. Graph this equation on your graphing calculator, using the following window: Xmin = 0, Xmax = 100,000, Ymin = 0, and Ymax = 6000.

2. What is the perceived pitch, P, for the input value 10,000 hertz?

3. a. Write an equation that can be used to determine what frequency, f, gives an output value of 2000 mels.

   b. Solve the equation in part a using a graphing approach.

To determine the exact answer in Problem 3, you can use an algebraic approach. The following problem guides you through this process.

4. Solve the equation $2410 \log(0.0016f + 1) = 2000$ using an algebraic approach.

   a. Solve the equation for $\log(0.0016f + 1)$. That is, isolate the log on one side of the equation.

   b. The equation in part a is now in the form $\log_b N = E$, where $b = 10$, $N = 0.0016x + 1$, and $E = \frac{2000}{2410}$. Write the equation from part a in exponential form, $b^E = N$.

c. In exponential form, the equation in part a should be

$$0.0016f + 1 = 10^{2000/2410}.$$

Solve this equation for f. Of course, you will need to approximate a value of $10^{2000/2410}$ using your calculator.

d. How does your answer to part c compare to your answer to Problem 3b?

5. a. Use an algebraic approach to determine the frequency, f, that produces a perceived pitch of 3000 mels.

b. Verify your answer in part a using a graphing approach.

6. The formula $W = 0.35 \ln P + 2.74$ is a model for the average walking speed, W, in feet per second for a resident of a city with population P, measured in thousands.

a. Determine the walking speed of a resident of a small city having a population of 500,000.

b. If the average walking speed of a resident is 4.5 feet per second, what is the population of the city? Round your answer to the nearest thousand.

SUMMARY
ACTIVITY 3.13

To solve a logarithmic equation algebraically,

Step 1. Rewrite the equation in the form $\log_b(f(x)) = c$, where $b > 0$, $b \neq 1$, $c > 0$, and $f(x) > 0$.

Step 2. Rewrite the resulting equation from step 1 in exponential form, $f(x) = b^c$.

Step 3. Solve the resulting equation from step 2 algebraically.

Step 4. Check the solutions in the original equation.

EXERCISES
ACTIVITY 3.13

In Exercises 1–6, solve each equation using an algebraic approach. Then verify your answer using a graphical approach.

1. $\log_2 x = 5$

2. $\ln x = 10$

3. $3 \log_5 (x + 2) = 5$

4. $\log_5 (x - 4) = 2$

5. $20 = 3.5 \ln x$

6. $4 + 1.75 \ln x = 31$

7. Stars have been classified into magnitude according to their brightness. Stars in the first six magnitudes are visible to the naked eye; those of higher magnitudes are visible only through a telescope. The magnitude, m, of the faintest star that is visible with a telescope having lens diameter d, in inches, is modeled by

$$m = 8.8 + 5.1 \log d.$$

What is the highest magnitude of a star that is visible with the 200-inch telescope at Mount Palomar, California?

8. Coal consumption in the United States can be modeled by the equation

$$A(x) = 4.95 + 4.67 \ln x,$$

where x is the number of years since 1970, and $A(x)$ is the amount of coal consumed in quadrillions of British thermal units or quads. According to the model, in what year will the consumption of coal in the United States reach 30 quads?

9. The acidity or alkalinity of any solution is determined by the concentration of hydrogen ions, $[H^+]$, in the substance, measured in moles per liter (mol/l). Acidity (or alkalinity) is measured on a pH scale, using the model

$$pH = -\log [H^+].$$

The pH scale ranges from 0 to 14. Values below 7 have progressively greater acidity; values greater than 7 are progressively more alkaline. Normal unpolluted rain has a pH of about 5.6. The acidity of rain over the northeastern United States, caused primarily by sulfur dioxide emissions, has had very damaging effects. One of the most acidic rainfalls on record had a pH of 2.4. What was the concentration of hydrogen ions?

10. The Richter scale is a well-known method of measuring the magnitude of an earthquake in terms of the amplitude, A (height), of its shock waves. The magnitude of any given earthquake is given by

$$m = \log \left(\frac{A}{A_0}\right),$$

where A_0 is a constant representing the amplitude of an average earthquake.

a. The magnitude of the 1906 San Francisco earthquake was 8.3 on the Richter scale. Write an equation that gives the amplitude, A, of the San Francisco earthquake in terms of A_0.

b. An earthquake with a magnitude of 5.5 will begin to cause serious damage. Write an equation that gives the amplitude, A, of a serious-damage earthquake in terms of A_0.

c. Determine the ratio of the amplitude of the San Francisco earthquake to the amplitude of a serious-damage (magnitude 5.5) earthquake. What is the significance of this number?

CLUSTER 2 What Have I Learned?

1. A logarithm is an exponent. Explain how this fact relates to the following properties of logarithms.

   a. $\log_b(x \cdot y) = \log_b x + \log_b y$

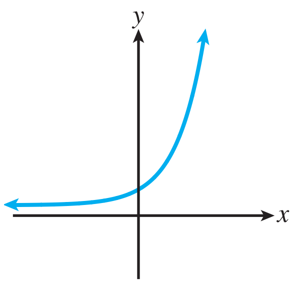
   b. $\log_b \frac{x}{y} = \log_b x - \log_b y$

   c. $\log_b x^n = n \cdot \log_b x$

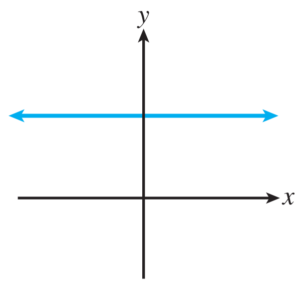
2. You have $20,000 to invest. Your broker tells you that the value of shares of mutual fund A has been growing exponentially for the past 2 years and that shares of mutual fund B have been growing logarithmically over the same period. If you make your decision based solely on the past performances of the funds, in which fund would you choose to invest? Explain.

3. Study the following graphs showing various types of functions you have encountered in this course.

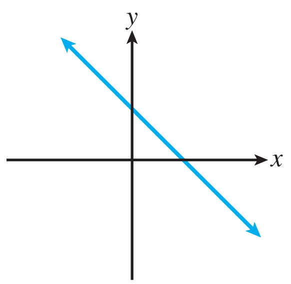
a.



b.



c.



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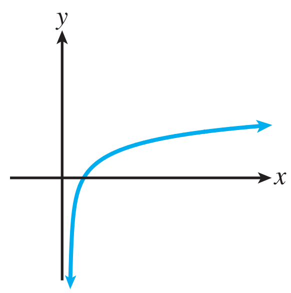
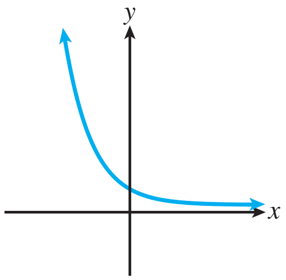
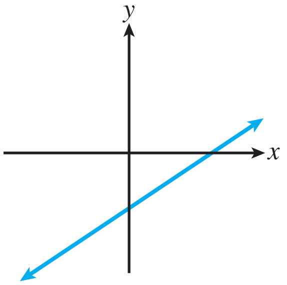


e.



f.



Complete the following table with respect to the preceding graphs.

DESCRIPTION	GRAPH LETTER	GENERAL EQUATION
Constant function		
Linearly decreasing function		
Logarithmically increasing function		
Exponentially decreasing function		
Exponentially increasing function		
Linearly increasing function		

4. The graph of $y = \log_b x$ will never be located in the second or third quadrants. Explain.

5. What function would you enter into Y1 on your graphing calculator to graph the function $y = \log_4 x$?

6. What values of x cannot be inputs in the function $y = \log_b (3x - 2)$?

7. What is the relationship between the functions $y = \log x$ and $y = 10^x$? How are the graphs related?

CLUSTER 2 How Can I Practice?

1. Write each equation in logarithmic form.

   a. $4^2 = 16$ b. $0.0001 = 10^{-4}$ c. $3^{-4} = \frac{1}{81}$

2. Write each equation in exponential form.

   a. $\log_2 32 = 5$ b. $\log_5 1 = 0$

   c. $\log_{10} 0.001 = -3$ d. $\ln e = 1$

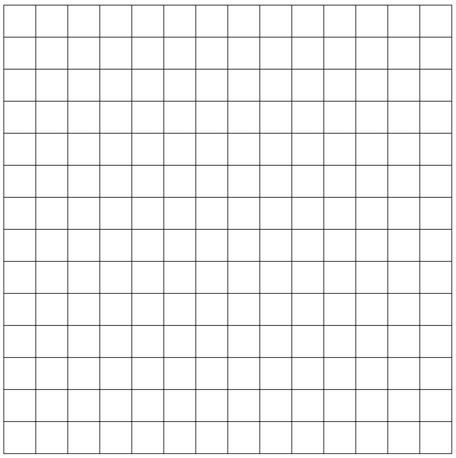
3. Solve each equation for the unknown variable.

   a. $\log_4 x = -3$ b. $\log_b 32 = 5$ c. $\log_5 125 = y$

4. a. Complete the table of values for the function $f(x) = \log_4 x$.

x	0.25	0.5	1	4	16	64
$f(x)$						

   b. Sketch a graph of the function, f.



c. Use your graphing calculator to check your result in parts a and b.

d. Determine the x-intercept.

e. What is the domain of the function?

f. What is the range?

g. Does the graph have a vertical or horizontal asymptote?

h. Use your graphing calculator to determine $f(32)$.

i. Use your graphing calculator to determine x when $f(x) = 3.25$.

5. Write each of the following as a sum, difference, or multiple of logarithms. Assume that x, y, and z are all greater than 0.

a. $\log_b \frac{xy^2}{z}$

b. $\log_3 \frac{\sqrt{x^3 y}}{z}$

c. $\log_5 \left(x\sqrt{x^2 + 4}\right)$

d. $\log_4 \sqrt[3]{\frac{xy^2}{z^2}}$

6. Rewrite the following as the logarithm of a single quantity.

a. $\log x + \frac{1}{3}\log y - \frac{1}{2}\log z$

b. $3 \log_3 (x + 3) + 2 \log_3 z$

c. $\frac{1}{3}\log_3 x - \frac{2}{3}\log_3 y - \frac{4}{3}\log_3 z$

7. Use the change of base formula and your calculator to approximate the following.

a. $\log_5 17$

b. $\log_{13} \sqrt[3]{41}$

8. Solve each of the following using an algebraic approach.

a. $25 + 3 \ln x = 10$

b. $1.5 \log_4 (x - 1) = 7$

9. Solve the following algebraically. Check your solutions using graphs or tables.

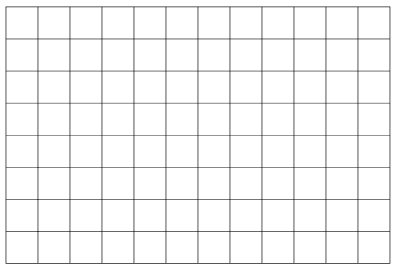
a. $3^x = 17$

b. $42 = 3e^{1.7x}$

10. The following table shows per capita health care expenditures (in dollars) in the United States from 1988 to 1993.

YEAR	1988	1989	1990	1991	1992	1993
EXPENDITURE ($)	2201	2422	2688	2902	3144	3331

a. Plot these points on the following grid.



b. The logarithmic model that fits this data is $E = f(x) = 2090 + 630 \ln x$, where E represents the per capita health care expenditures and x is the number of years since 1987. Add a sketch of this model to the grid in part a.

c. Use this model to predict the per capita health care expenditures in 2005.

d. Use the model in part c to predict in what year health care expenditures will reach $3500. Use a graphing approach.

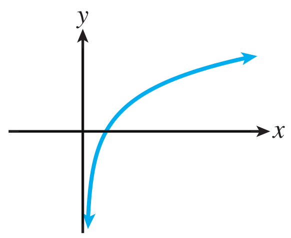
e. Write the equation that you would solve to determine the answer to part d.

f. Solve the equation in part e algebraically, and compare your result with your answer in part d.

CHAPTER 3 SUMMARY

The bracketed numbers following each concept indicate the activity in which the concept is discussed.

CONCEPT / SKILL	DESCRIPTION	EXAMPLE
Exponential functions [3.1]	The exponential functions are defined by $y = b^x$, $b > 0$, $b \neq 1$.	$y = 3^x$
Decay factor of an exponential function [3.1]	If $0 < b < 1$, the function $y = b^x$ is decreasing and b is called the decay factor.	The exponential function $y = \left(\frac{1}{2}\right)^x$ has a decay factor of $\frac{1}{2}$.
Growth factor of an exponential function [3.1]	If $b > 1$, the function $y = b^x$ is increasing and b is called the growth factor.	The exponential function $y = 3^x$ has a growth factor of 3.
Vertical intercept of an exponential function [3.1]	The vertical intercept (y-intercept) of an exponential function $y = b^x$ is (0, 1).	The graph of $y = 2^x$ passes through the point (0, 1).
Horizontal asymptote of an exponential function [3.1]	The line $y = 0$ is a horizontal asymptote of an exponential function $y = b^x$.	As x gets smaller, the output values of $y = 3^x$ approach 0.
Doubling time [3.2]	The doubling time of an increasing exponential function is the time it takes for an output to double. The doubling time is set by the growth factor and remains the same for all output values.	Example 2, Activity 3.2; see pages 300–301
Half-life [3.2]	The half-life of a decreasing exponential function is the time it takes for an output to decay by one-half. The half-life is determined by the decay factor and remains the same for all output values.	Example 3, Activity 3.2; see pages 302–303
Growth model [3.3]	If r represents the annual percentage growth rate, the exponential function that models the quantity P can be written as $P(t) = P_0(1 + r)^t$, where P_0 is the initial amount, t represents the number of elapsed years, and $1 + r$ is the growth factor.	Example 4, Activity 3.3; see page 311
Decay model [3.3]	If r represents the annual percent that decays, the exponential function that models the amount remaining can be written as $P(t) = P_0(1 - r)^t$, where $1 - r$ is the decay factor.	Example 5, Activity 3.3; see page 313
Compound interest [3.5]	The formula for compounding interest is $A = P\left(1 + \frac{r}{n}\right)^{nt}$.	Example 1, Activity 3.5; see page 320

CONCEPT / SKILL	DESCRIPTION	EXAMPLE
Continuous compounding [3.5]	The formula for continuous compounding is $A = Pe^{rt}$.	Example 4, Activity 3.5; see page 325
Continuous growth at a constant percentage rate [3.5], [3.6]	Whenever growth is continuous at a constant percentage rate, the exponential model used is $y = y_0e^{rt}$.	Problem 10, Activity 3.5; see page 325
Continuous decay at a constant percentage rate [3.6]	Whenever decay is continuous at a constant rate, the model used is $y = y_0e^{-rt}$.	Example 2, Activity 3.6; see page 331
Logarithm [3.8]	In the equation $y = b^x$, where $b > 0$ and $b \neq 1$, x is called a logarithm or log.	For the equation $3^4 = 81$, 4 is the logarithm of 81, base 3.
Notation for logarithms [3.8]	The notation for logarithms is $\log_b x = y$, where b is the base of the log, x (a positive number) is the power of b, and y is the exponent, and x is the power of b.	In the equation $\log_2 16 = 4$, 2 is the base, 4 is the log or exponent, and 16 is the power of z.
Common logarithm [3.8]	A common logarithm is a base-10 logarithm. The notation is $\log_{10} x = \log x$.	$1000 = 10^3$. The common logarithm of 10^3 is 3; i.e., $\log 1000 = 3$.
Natural logarithm [3.8]	A natural logarithm is a base-e logarithm. The notation is $\log_e x = \ln x$.	$\log_e e^3 = \ln e^3 = 3$
Logarithmic equation [3.8]	The logarithmic equation $y = \log_b x$ is equivalent to the exponential equation $b^y = x$.	The equations $6 = \log_4 x$ and $x = 4^6$ are equivalent.
Basic properties of logarithms [3.8]	If $b > 0$ and $b \neq 1$, $\log_b 1 = 0$, $\log_b b = 1$, and $\log_b b^n = n$.	$\log_4 1 = 0$, $\log_7 7 = 1$, $\log_6 6^4 = 4$
Logarithmic function [3.9]	If $b > 0$ and $b \neq 1$, the logarithmic function is defined by $y = \log_b x$.	$y = \log_4 x$
Graph of the logarithmic function [3.9]	The graph is increasing for all $x > 0$, has an x-intercept of $(1, 0)$ and has a vertical asymptote of $x = 0$, the y-axis.	
Comparison of the graphs of $f(x) = b^x$, where $b > 1$, and $g(x) = \log_b(x)$, where $b > 1$ [3.9]	Both graphs increase. The exponential function increases faster as x increases; the log function increases slower as x increases. The domain of the exponential function is the range of the log, which is all real numbers; the range of the exponential function is the domain of the log, which is the interval $(0, \infty)$.	Problem 6, Activity 3.9; see page 368

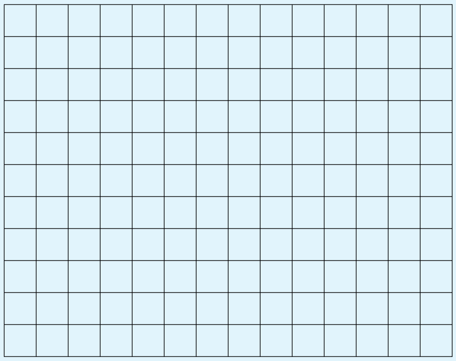
CONCEPT / SKILL	DESCRIPTION	EXAMPLE
If $A > 0$, $B > 0$, $b > 0$, and $b \neq 1$, then $\log_b (A \cdot B) = \log_b A + \log_b B$ [3.11]	The logarithm of a product is the sum of the logarithms.	$\log_2 (4 \cdot 8) = \log_2 (4) + \log_2 (8) = 2 + 3 = 5$
If $A > 0$, $B > 0$, $b > 0$, and $b \neq 1$, then $\log_b \left(\frac{A}{B}\right) = \log_b A - \log_b B$ [3.11]	The logarithm of a quotient is the difference of the logarithms.	$\log_3 \left(\frac{81}{27}\right) = \log_3 81 - \log_3 27 = 4 - 3 = 1$
If $A > 0$, $B > 0$, $b > 0$, and $b \neq 1$, then $\log_b (A + B) \neq \log_b (A) + \log_b (B)$ [3.11]	The logarithm of a sum is not the sum of the logarithms.	$\log 2 + \log 3 = 0.3010 + 0.4771 = 0.7781$ $\log (2 + 3) = \log 5 = 0.6990$
If $A > 0$, p is a real number, $b > 0$, and $b \neq 1$, then $\log_b A^p = p \log_b A$ [3.11]	The logarithm of a power of A is the exponent times the logarithm of A.	$\log_5 x^4 = 4 \log_5 x$ $\log_3 \sqrt{x} = \frac{1}{2} \log_3 x$
Change of base formula [3.11]	The logarithm of any positive number x to any base can be found using the formula $\log_b x = \frac{\log x}{\log b}$ or $\log_b x = \frac{\ln x}{\ln b}$.	$\log_2 (2.5) = \frac{\log (2.5)}{\log 2} = 1.3219$

GATEWAY REVIEW

1. a. Determine some of the output values for the function $f(x) = 8^x$ by completing the following table.

x	-1	$-\frac{1}{3}$	0	1	$\frac{4}{3}$	2	3
$f(x) = 8^x$							

b. Sketch the graph of the function, f.



c. Is this function increasing or decreasing? Explain how you know this by looking at the equation of the function.

d. What is the domain?

e. What is the range?

f. What are the x- and y-intercepts?

g. Are there any asymptotes? If yes, write the equations of the asymptotes.

h. Compare the graph of f to the graph of $g(x) = \left(\frac{1}{8}\right)^x$. What are the similarities and the differences?

i. In what way does the graph of $h(x) = 8^x + 5$ differ from that of $f(x) = 8^x$?

Answers to all Gateway exercises are included in the Selected Answers appendix.

j. Write the equation of the function that is the inverse of the function $f(x)$.

2. Complete the table for each exponential function. Use your graphing calculator to check your work.

FUNCTION	BASE, b	GROWTH OR DECAY FACTOR	x-INTERCEPT	y-INTERCEPT	HORIZONTAL ASYMPTOTE	INCREASING OR DECREASING
$h(x) = 6^x$						
$g(x) = \left(\frac{1}{3}\right)^x$						
$p(x) = 5(2.34)^x$						
$q(x) = 3(0.78)^x$						
$r(x) = 2^x - 4$						

3. Use your graphing calculator to help you determine the domain and range for each function.

FUNCTION	$f(x) = 0.8^x$	$h(x) = 6^x + 2$	$t(x) = 3^x - 5$	$q(x) = \log_4 x$	$r(x) = \ln(x - 3)$
DOMAIN					
RANGE					

4. a. Given the following table, determine whether the given data can be approximately modeled by an exponential function. If it can, what is the growth or decay factor?

x	0	1	2	3	4
y	10	15.5	24	36	55.5

b. Determine an exponential equation that models this data.

5. a. Your salary has increased at the rate of 1.5% annually for the past 5 years, and your boss projects this will remain unchanged for the next 5 years. You were making $15,000 annually in 2002. Complete the following table.

2002	2003	2004	2005	2006	2007

b. Write the exponential growth function that models your annual salary during this period of time. Let x represent the number of years since 2002.

c. If your increase in salary continues at this rate, how much will you make in 2010? Is this realistic?

d. You would like to double your salary. How many years will you have to work before your salary will be twice the salary you made in 2002?

6. a. You just inherited \$5000. You can invest the money at a rate of 6.5% compounded continuously. In 8 years, your oldest child will be going to college. How much will be in the bank for her education? Use the equation $A = A_0 e^{rt}$.

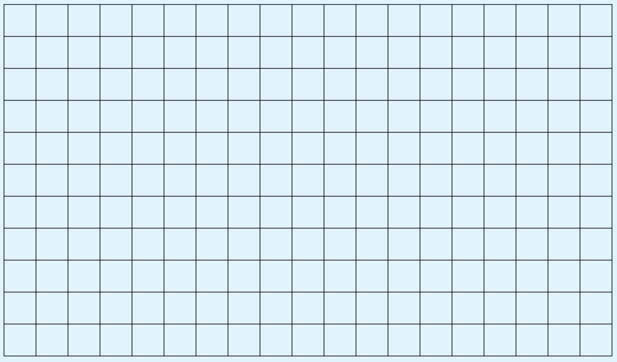
b. You actually need to have \$12,000 for your child's first year of college. For how many years would you have to leave the money in the bank to have the \$12,000?

7. The number of multiple births (triplets or higher) in the United States between 1990 and 1999 is listed in the following table, with 0 representing the year 1990.

NUMBER OF YEARS SINCE 1990	0	2	3	4	5	6	7	8	9
NUMBER OF MULTIPLE BIRTHS	3028	3883	4168	4594	4973	5939	6737	7625	7321

Source: National Center for Health Statistics.

a. Plot the data on an appropriately scaled and labeled coordinate axis.



b. Does the scatterplot show that the data would be better modeled by a linear or an exponential model?

c. Use your graphing calculator to determine the exponential regression equation that best fits the multiple births data.

d. According to your model, what is the growth factor for the multiple births data?

e. Estimate the growth rate (written as a percent) in multiple births each year.

f. Use the regression equation to determine the predicted total number of multiple births in 2012.

g. Use your graphing calculator to determine the doubling time for your exponential model.

8. Determine the value of each of the following without using your calculator.

a. $25^{3/2}$ **b.** $81^{3/4}$ **c.** $64^{-5/6}$

d. $\sqrt[3]{125^2}$ **e.** $\log_3 \frac{1}{9}$ **f.** $\log_5 625$

g. $\log 0.001$ **h.** $\ln e^2$

9. Write each equation in logarithmic form.

a. $6^2 = 36$ **b.** $0.000001 = 10^{-6}$ **c.** $2^{-5} = \frac{1}{32}$

10. Write each equation in exponential form.

a. $\log_3 81 = 4$ **b.** $\log_7 1 = 0$ **c.** $\log_{10} 0.0001 = -4$

d. $\ln e = 1$ **e.** $\log_q y = b$

11. Solve each equation for the unknown variable.

a. $\log_5 x = -3$

b. $\log_b 256 = 4$

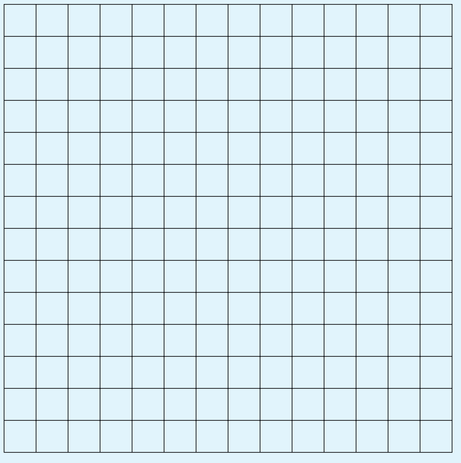
c. $\log_2 64 = y$

d. $\log_4 x = \frac{3}{2}$

12. a. Complete the table of values for the function $f(x) = \log_5 x$.

x	0.008	0.04	0.2	1	5	25
$f(x)$						

b. Sketch a graph of the function.



c. Use your graphing calculator to check your result in parts a and b.

d. Determine the x-intercept.

e. What is the domain of the function?

f. What is the range?

g. Does the graph have a vertical or horizontal asymptote?

h. Use your graphing calculator to determine $f(23)$.

i. Use your graphing calculator to determine x when $f(x) = 2.46$.

13. Use the change of base formula and your calculator to approximate the following.

a. $\log_7 21$

b. $\log_{15} \frac{8}{9}$

14. Write each of the following as a sum, difference, or multiple of logarithms. Assume that x, y, and z are all greater than 0.

a. $\log_2 \dfrac{x^3 y}{z^{1/2}}$

b. $\log \sqrt[3]{\dfrac{x^4 y^3}{z}}$

15. Rewrite the following as the logarithm of a single quantity.

a. $\log x + \frac{1}{4} \log y - 3 \log z$

b. $\frac{1}{3}\left(\log x - 2 \log y - \log z\right)$

16. Solve the following algebraically.

a. $3^{3+x} = 7$

b. $\log_2 (4x + 9) = 4$

c. $50 + 6 \ln x = 85$

17. a. Sketch the graph of the function using the data from the given table.

x	0.1	0.5	1	2	4	16
$f(x)$	−1.66	−0.5	0	.5	1	2

b. Use the table and the graphing feature of your calculator to verify that the equation that defines function f is $f(x) = 0.5 \log_2 x$.

c. Use the function to determine the value of $f(54)$.

d. If $f(x) = 2.319$, determine the value of x.

e. Use your graphing calculator to verify that the function $g(x) = 4^x$ is the inverse of f.

18. The population (in millions) of New York State and Florida can be modeled by the following:

New York State: $P_N = 18.98e^{0.0055t}$

Florida: $P_F = 15.98e^{0.0235t}$

where t represents the number of years since 2000.

a. Determine the population of New York and Florida in 2000 ($t = 0$).

b. Sketch a graph of each function on the same coordinate axes.

c. Determine graphically the year when the population of Florida will equal the population of New York State.

d. Determine algebraically the year when the population of Florida will first exceed 25 million.