QUADRATIC AND HIGHER-ORDER POLYNOMIAL FUNCTIONS

CLUSTER 1 Introduction to Quadratic Functions

ACTIVITY 4.1

Baseball and the Sears Tower

OBJECTIVES

1. Identify functions of the form $f(x) = ax^2 + bx + c$ as quadratic functions.
2. Explore the role of c as it relates to the graph of $f(x) = ax^2 + bx + c$.
3. Explore the role of a as it relates to the graph of $f(x) = ax^2 + bx + c$.
4. Explore the role of b as it relates to the graph of $f(x) = ax^2 + bx + c$.

Imagine yourself standing on the roof of the 1450-foot-high Sears Tower in Chicago. When you release and drop a baseball from the roof of the tower, the ball's height above the ground, H (in feet), can be described as a function of the time, t (in seconds), since it was dropped. This height function is defined by

$$H(t) = -16t^2 + 1450.$$

1. a. Complete the following table.

TIME, t (sec)	$H(t) = -16t^2 + 1450$
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

b. How far does the baseball fall during the first second?

c. How far does it fall during the second second?

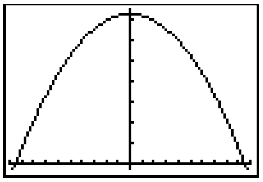
2. Using the height function, $H(t) = -16t^2 + 1450$, determine the average rate of change of H with respect to t over the given interval. Remember: average rate of change $= \dfrac{\text{change in output}}{\text{change in input}}$.

a. $0 \le t \le 1$

b. $1 \le t \le 2$

c. Based on the results of parts a and b, do you believe that $H(t) = -16t^2 + 1450$ is a linear function? Explain.

3. Graph the height function setting the window parameters at Xmin $= -10$ and Xmax $= 10$ for the input and Ymin $= -50$ and Ymax $= 1500$ for the output. Your graph should appear as follows.



a. Describe the important features of the graph of $H(t) = -16t^2 + 1450$. Discuss the shape, symmetry, and intercepts.

b. What are the practical domain and range of the height function?

c. Is the graph of the height function the actual path of the object when the ball is dropped? Explain.

The graph of the height function is a parabola. The graph of a **parabola** is a ∪-shaped figure that opens upward, ∪, or downward, ∩.

DEFINITION

Any function defined by an equation of the form $y = ax^2 + bx + c$ or $f(x) = ax^2 + bx + c$, where a, b, and c represent real numbers and $a \neq 0$, is called a **quadratic function**. The output variable y is defined by an expression having three terms: the **quadratic term**, ax^2, the **linear term**, bx, and the **constant term**, c. The numerical factors of the quadratic and linear terms, a and b, are called the **coefficients** of the terms.

EXAMPLE 1 ***$H(t) = -16t^2 + 100$ defines a quadratic function. The quadratic term is $-16t^2$. The linear term is $0t$, although it is not written as part of the expression defining $H(t)$. The constant term is 100. The numbers -16 and 0 are the coefficients of the quadratic and linear terms, respectively. Therefore, $a = -16$, $b = 0$, and $c = 100$.***

4. For each of the following quadratic functions, identify the value of a, b, and c.

QUADRATIC FUNCTION	a	b	c
$y = 3x^2$			
$y = -2x^2 + 3$			
$y = x^2 + 2x - 1$			
$y = -x^2 + 4x$			

The Constant Term c

Consider once again the height function $H(t) = -16t^2 + 1450$ from the beginning of the activity.

5. a. What is the vertical intercept of the graph?

b. What is the practical meaning of the vertical intercept in this situation?

c. Predict what the graph of $h(t) = -16t^2 + 1450$ would look like if the constant term 1450 were changed to 800. Verify your prediction by graphing $y = -16t^2 + 800$. What does the constant term tell you about the graph of the parabola?

The constant term c of a quadratic function $f(x) = ax^2 + bx + c$ *always* indicates the vertical intercept of the parabola. The vertical intercept of any quadratic function is $(0, c)$.

6. Graph the parabolas defined by the following quadratic equations. Note the similarities and differences among the graphs, especially the vertical intercepts. Be careful in your choice of a window.

a. $f(x) = 1.5x^2$

b. $g(x) = 1.5x^2 + 7$

c. $q(x) = 1.5x^2 + 4$

d. $s(x) = 1.5x^2 - 4$

The Effects of the Coefficient *a*

7. **a.** Graph the quadratic function defined by $g(t) = 16t^2 + 1450$ on the same screen as $H(t) = -16t^2 + 1450$. Use the window settings Xmin $= -10$, Xmax $= 10$, Ymin $= -50$, and Ymax $= 3000$.

b. What effect does the sign of the coefficient of t^2 appear to have on the graph of the parabola?

8. Graph the functions $h(t) = -16t^2 + 100$, $f(t) = -6t^2 + 100$, $g(t) = -40t^2 + 100$ in the same window. What effect does the magnitude of the coefficients of t^2 (namely, $|-16| = 16$, $|-6| = 6$, and $|-40| = 40$) appear to have on the graph of that particular parabola? Use window settings Xmin $= -15$, Xmax $= 15$, Ymin $= -200$, and Ymax $= 200$.

The results from Problems 7 and 8 regarding the effects of the coefficient *a* can be summarized as follows.

> The graph of a quadratic function defined by $f(x) = ax^2 + bx + c$ is called a parabola.
>
> - If $a > 0$, the parabola opens upward.
> - If $a < 0$, the parabola opens downward.
> - The magnitude of *a* affects the width of the parabola. The larger the absolute value of *a*, the narrower the parabola.

9. **a.** Is the graph of $h(x) = 0.3x^2$ wider or narrower than the graph of $f(x) = x^2$?

b. How do the output values of *h* and the output values of *f* compare for the same input value?

c. Is the graph of $g(x) = 3x^2$ wider or narrower than the graph of $f(x) = x^2$?

d. How do the output values of g and f compare for the same input value?

e. Describe the effect of the magnitude of the coefficient a on the width of the graph of the parabola.

f. Describe the effect of the magnitude of the coefficient of a on the output value.

The Effects of the Coefficient *b*

Assume for the time being that you are back on the roof of the 1450-foot Sears Tower. Instead of merely releasing the ball, suppose you *throw it down* with an initial velocity of 40 feet per second. Then the function describing its height above ground as a function of time is modeled by

$$H_{\text{down}}(t) = -16t^2 - 40t + 1450.$$

If you tossed the ball straight up with an initial velocity of 40 feet per second, then the function describing its height above ground as a function of time is modeled by

$$H_{\text{up}}(t) = -16t^2 + 40t + 1450.$$

10. Predict what features of the graphs of H_{down} and H_{up} have in common with

$$H(t) = -16t^2 + 1450.$$

11. a. Graph the three functions $H(t)$, $H_{\text{down}}(t)$, and $H_{\text{up}}(t)$ using the same window setting.

b. What effect do the $-40t$ and $40t$ terms seem to have upon the graphs?

If $b = 0$, the turning point of the parabola is located on the vertical axis. If $b \neq 0$, the turning point will not be on the vertical axis.

12. Set the window of your calculator to Xmin $= -8$, Xmax $= 8$, Ymin $= -20$, and Ymax $= 20$, and graph the following quadratic functions. For each function determine whether or not the coefficient b is zero. For each graph determine whether or not the turning point is on the y-axis.

a. $y = x^2$

b. $y = x^2 - 4x$

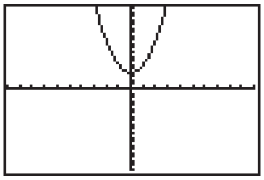
c. $y = x^2 + 4$

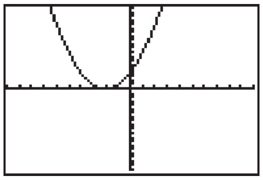
d. $y = x^2 + x$

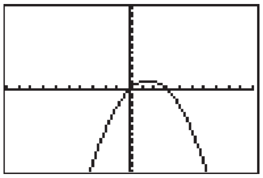
e. $y = x^2 - 3$

13. Match each function with its corresponding graph below, and then verify using your graphing calculator.

a. $f(x) = x^2 + 4x + 4$ **b.** $g(x) = 0.2x^2 + 4$ **c.** $h(x) = -x^2 + 3x$







SUMMARY
ACTIVITY 4.1

1. The equation of a **quadratic function** with x as the input variable and y as the output variable has the standard form

$$y = ax^2 + bx + c,$$

where a, b, and c are real numbers and $a \neq 0$.

2. The graph of a quadratic function is called a **parabola**.

3. For the quadratic function defined by $f(x) = ax^2 + bx + c$:

- If $a > 0$, the parabola opens upward.
- If $a < 0$, the parabola opens downward.

The magnitude of a affects the width of the parabola. The larger the absolute value of a, the narrower the parabola.

4. If $b = 0$, the turning point of the parabola is located on the vertical axis. If $b \neq 0$, the turning point will not be on the vertical axis.

5. The constant term, c, of a quadratic function $f(x) = ax^2 + bx + c$ always indicates the vertical intercept of the parabola. The vertical intercept of any quadratic function is $(0, c)$.

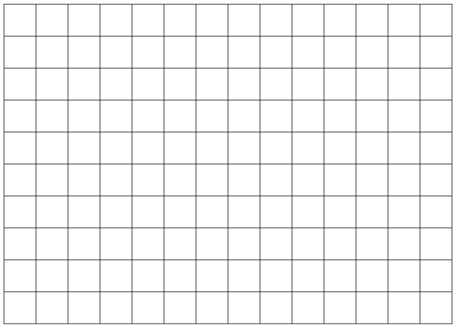
EXERCISES
ACTIVITY 4.1

1. a. Complete the following input/output table for $y = x^2$.

INPUT	OUTPUT
−3	
−2	
−1	
0	
1	
2	
3	

Exercise numbers appearing in color are answered in the Selected Answers appendix.

b. Use the results of part a to sketch a graph $y = x^2$.



c. Use your graphing calculator to compare and check the information for parts a and b. Are the graphs the same?

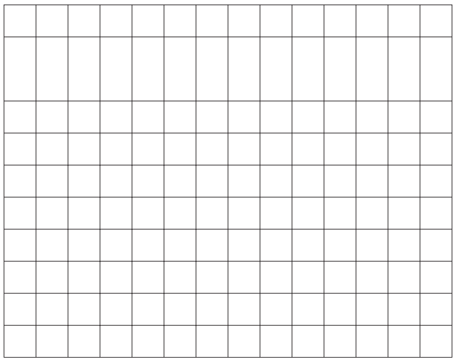
d. What is the coefficient of the term x^2?

e. From the graph, determine the domain and range of the function.

2. a. Create a table similar to the one in Exercise 1a to show the output for $y = -x^2$.

INPUT	−3	−2	−1	0	1	2	3
OUTPUT							

b. Sketch the graph of $y = -x^2$.



c. Use your graphing calculator to compare and check the information for parts a and b. Are the graphs the same?

d. What is the coefficient of the term $-x^2$?

3. In each of the following functions defined by an equation of the form $y = ax^2 + bx + c$, identify the value of a, b, and c.

a. $y = -2x^2$

b. $y = \frac{2}{5}x^2 + 3$

c. $y = -x^2 + 5x$

d. $y = 5x^2 + 2x - 1$

4. Predict what the graph of each of the following quadratic functions will look like. Use your graphing calculator to verify your prediction.

a. $f(x) = 3x^2 + 5$

b. $g(x) = -2x^2 + 1$

c. $h(x) = 0.5x^2 - 3$

5. Graph the following pairs of functions, and describe any similarities as well as any differences that you observe in the graphs.

a. $f(x) = 3x^2$, $g(x) = -3x^2$

b. $h(x) = \frac{1}{2}x^2$, $f(x) = 2x^2$

c. $g(x) = 5x^2$, $h(x) = 5x^2 + 2$

d. $f(x) = 4x^2 - 3$, $g(x) = 4x^2 + 3$

e. $f(x) = 6x^2 + 1$, $h(x) = -6x^2 - 1$

6. Use your graphing calculator to graph the two functions $y_1 = 3x^2$ and $y_2 = 3x^2 + 2x - 2$.

a. What is the vertical intercept of the graph of each function?

b. Compare the two graphs to determine the effect of the linear term $2x$ and the constant term -2 on the graph of $y_1 = 3x^2$.

For Exercises 7–11, determine

a. whether the parabola opens upward or downward and

b. the vertical intercept.

7. $f(x) = -5x^2 + 2x - 4$

a.

b.

8. $g(t) = \frac{1}{2}t^2 + t$

a.

b.

9. $h(v) = 2v^2 + v + 3$

a.

b.

10. $r(t) = 3t^2 + 10$

a.

b.

11. $f(x) = -x^2 + 6x - 7$

a.

b.

12. Does the graph of $y = -2x^2 + 3x - 4$ have any horizontal intercepts? Explain.

13. **a.** Is the graph of $y = \frac{3}{5}x^2$ wider or narrower than the graph of $y = x^2$?

b. For the same input value, which graph would have a larger output value?

14. Put the following in order from narrowest to widest.

a. $y = 0.5x^2$ **b.** $y = 8x^2$ **c.** $y = -2.3x^2$

ACTIVITY 4.2

The Shot Put

OBJECTIVES

1. Determine the vertex or turning point of a parabola.
2. Determine the axis of symmetry of a parabola.
3. Identify the domain and range.
4. Determine the vertical intercept of a parabola.
5. Determine the horizontal intercept(s) of a parabola graphically.

Parabolas are good models for a variety of situations that you encounter in everyday life. Examples include the path of a golf ball after it is struck, the arch (cable system) of a bridge, the path of a baseball thrown from the outfield to home plate, the stream of water from a drinking fountain, and the path of a cliff diver.

Consider the 2004 men's Olympic shot put event, which was won by Ukraine's Yuriy Bilonog with a throw of 69 feet $4\frac{3}{4}$ inches. The path of his winning throw can be approximately modeled by the quadratic function

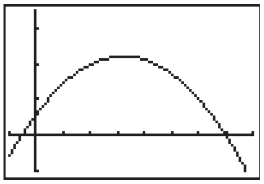
$$H(x) = -0.015663x^2 + x + 6,$$

where x is the horizontal distance in feet from the point of the throw and $H(x)$ is the vertical height in feet of the shot above the ground.

1. a. After inspecting the equation for the path of the winning throw, which way do you expect the parabola to open? Explain.

   b. What is the vertical intercept of the graph of the parabola? What practical meaning does this intercept have in this situation?

2. Use your graphing calculator to produce a plot of the path of the winning throw. Be sure to adjust your window settings so that all of the important features of the parabola (including horizontal intercepts) appear on the screen. Your graph should resemble the following.



3. a. Use the trace feature of your graphing calculator to estimate the practical domain of the function.

   b. What does the practical domain mean in the shot put situation?

   c. Use the trace feature of your graphing calculator to estimate the practical range of the function.

   d. What does the practical range mean in the shot put situation?

4. Use the table feature of your graphing calculator to complete the following table.

x	10	20	30	40	50
H(*x*)					

Vertex of a Parabola

An important feature of the *graph* of any quadratic function defined by $f(x) = ax^2 + bx + c$ is its **turning point**, also called the **vertex**. The turning point of a parabola that opens downward or upward is the point at which the parabola changes direction from increasing to decreasing or decreasing to increasing.

5. Use the TRACE feature of your graphing calculator to approximate the vertex of the shot put function H.

6. The vertex is often very important in a situation. What is the significance of the coordinates of the turning point in this problem?

The coordinates of the vertex of a parabola having equation $y = ax^2 + bx + c$ can be determined from the values of a and b in the equation, but first you need an equivalent way of writing the equation $y = ax^2 + bx + c$.

> An equivalent form of the equation $y = f(x) = ax^2 + bx + c$ with vertex (h, k) is $y = f(x) = a(x - h)^2 + k$. The vertex of the parabola is (h, k), where $h = -\dfrac{b}{2a}$ and $k = f(h)$.

Appendix A: For a discussion of why this is true see "Solving Quadratic Equations by Completing the Square" in Appendix A.

7. a. Identify the values of a, b, and c in the quadratic equation $y = 2x^2 - 12x + 11$.

b. Using the values of a, b, and c from part a, determine the values of $h = -\dfrac{b}{2a}$ and $k = f(h)$

$$k = f(3) = 2(3)^2 - 12(3) + 11 = -7$$

c. Write the equation $y = 2x^2 - 12x + 11$ in the form $y = a(x - h)^2 + k$.

d. Enter the equation $y = 2x^2 - 12x + 11$ into your graphing calculator as Y1 and the equation from part c into your graphing calculator as Y2. Graph both equations in the same screen. What do you notice about the graphs of the two equations?

e. Use the TRACE feature of your graphing calculator to determine the vertex of the parabola.

f. Describe the relationship between the coordinates of the vertex and the quadratic equation written in the form $y = a(x - h)^2 + k$.

DEFINITION

The **vertex** or turning point of a parabola having equation $y = f(x) = ax^2 + bx + c$ has coordinates

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right),$$

where a is the coefficient of the x^2 term and b is the coefficient of the x term.

Note that the y-coordinate (output) of the vertex is determined by substituting the x-coordinate of the vertex into the equation of the parabola and evaluating the resulting expression.

EXAMPLE 1 ***Determine the vertex of the parabola defined by the equation*** $\boldsymbol{y = -3x^2 + 12x + 5}$.

SOLUTION

Step 1. Determine the x-coordinate of the vertex by substituting the values of a and b into the formula $x = \frac{-b}{2a}$.

Because $a = -3$ and $b = 12$, you have

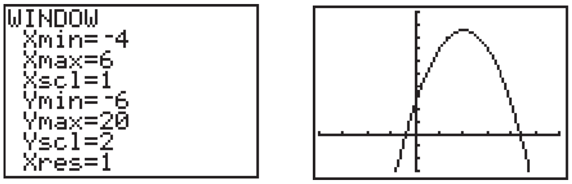
$$x = \frac{-(12)}{2(-3)} = \frac{-12}{-6} = 2.$$

Step 2. The y-value of the vertex is the corresponding output value for $x = 2$. Substituting 2 for x in the equation, you have

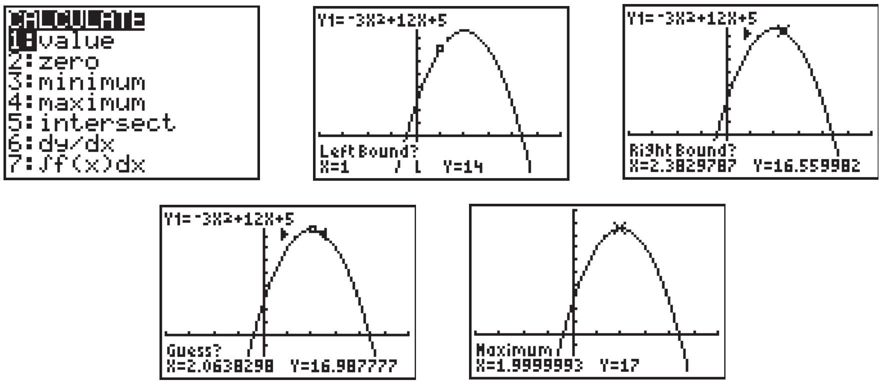
$$y = -3(2)^2 + 12(2) + 5 = 17.$$

Therefore, the vertex is (2, 17).

Because the parabola in Example 1 opens downward ($a = -3 < 0$), the vertex is the high point (maximum) of the parabola as demonstrated by the following graph of the parabola.



Rather than using the TRACE feature to approximate the vertex of a parabola, you can determine the vertex of a parabola opening downward by selecting the maximum option in the CALC menu of your graphing calculator. Follow the prompts to obtain the coordinates of the maximum point (vertex).



For further help with the TI-83/TI-84 Plus, see Appendix C.

8. Use the method demonstrated in Example 1 to determine the vertex of the parabola defined by $H(x) = -0.015663x^2 + x + 6$ (the shot put function).

9. a. Use your graphing calculator to determine the vertex of the parabola having equation $H(x) = -0.015663x^2 + x + 6$ (the shot put function H).

b. How do the coordinates you determined using your graphing calculator compare with your results in Problem 8?

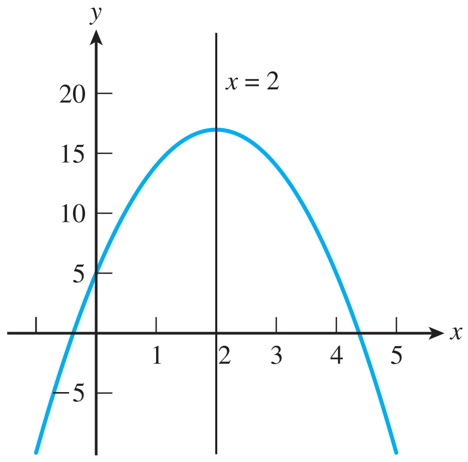
c. What is the practical meaning of the coordinates of the vertex in this situation?

Axis of Symmetry of a Parabola

DEFINITION

The **axis of symmetry** of a parabola is a vertical line that divides the parabola into two symmetrical parts that are mirror images in the line.

EXAMPLE 2 *Consider the parabola from Example 1. The axis of symmetry of the parabola is* $x = 2$*. Note that the line of symmetry passes through the vertex of the parabola.*



Because the vertex (turning point) of a parabola lies on the axis of symmetry, the equation of the axis of symmetry is

$$x = \frac{-b}{2a}.$$

10. What is the axis of symmetry of the shot put function, H?

Intercepts of the Graph of a Parabola

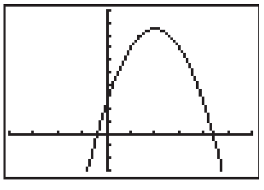
The y-intercept (vertical intercept) of the graph of the parabola defined by $y = -3x^2 + 12x + 5$ (see Example 1) can be determined directly from the equation. If $x = 0$, then

$$y = -3(0)^2 + 12(0) + 5 = 5$$

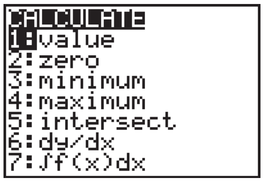
and the y-intercept is $(0, 5)$.

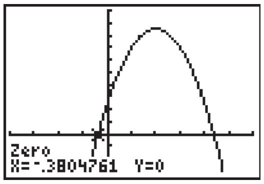
In general, the y-intercept of the parabola defined by $y = ax^2 + bx + c$ is $(0, c)$.

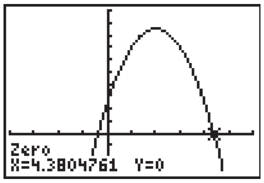
Because the vertex $(2, 17)$ of the parabola having equation $y = -3x^2 + 12x + 5$ is a point above the x-axis (the y-coordinate is positive) and the parabola opens downward, the parabola must intersect the horizontal axis in two places. This is verified by the following graph.



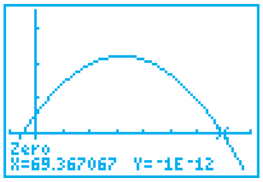
The x-intercepts of $y = -3x^2 + 12x + 5$ can be determined using the zero option in the CALC menu of your graphing calculator. Follow the prompts to obtain one x-intercept at a time. The screens should appear as follows.







11. a. Use your graphing calculator to determine the x-intercept(s) for the shot put function having equation $H(x) = -0.015663x^2 + x + 6$. The right-most intercept appears in the following screen.



b. Is either x-intercept determined in part a significant to the problem situation? Explain.

The graph of $H(x) = -0.015663x^2 + x + 6$ has two x-intercepts. Does the graph of every parabola have x-intercepts? Problems 12 and 13 will help answer this question.

12. a. Use the values of a, b, and c to determine the coordinates of the vertex of the graph of $y = x^2 + 6x + 12$.

b. Use the y-coordinate of the vertex to determine if the vertex is above or below the x-axis.

c. Use the value of a in $y = x^2 + 6x + 12$ to determine if the parabola opens upward or downward.

d. Use the results from parts b and c to determine if the parabola has x-intercepts.

e. Use your graphing calculator to verify your answer to part d.

13. a. Use the values of a, b, and c to determine the coordinates of the vertex of the graph of $y = -x^2 + 8x - 21$.

b. Use the y-coordinate of the vertex to determine if the vertex is above or below the x-axis.

c. Use the value of a in $y = -x^2 + 8x - 21$ to determine if the parabola opens upward or downward.

d. Use the results from parts b and c to determine if the parabola has x-intercepts.

e. Use your graphing calculator to verify your answer to part d.

If a parabola opens upward and the vertex is above the x-axis, there are no x-intercepts. If a parabola opens downward and the vertex is below the x-axis, there are no x-intercepts.

14. a. Use your result from Problem 11 to determine the practical domain of the shot put function. How does this compare with your answer in Problem 3?

b. Sketch the path of the winning throw of the shot put. Be sure to label all key points, including the vertex and intercepts.

c. From the graph of the winning throw, over what horizontal distance (x-interval) is the height of the shot put increasing?

d. Determine the x-interval over which the height of the shot put is decreasing.

e. What is the practical range?

15. Now consider the function $H(x) = -0.015663x^2 + x + 6$ as a general function that is not restricted by the physical situation in the activity.

a. What is the domain of the general function?

b. Over what x-interval does the general function increase?

c. Over what x-interval does the general function decrease?

d. What is the range?

SUMMARY
ACTIVITY 4.2

The following characteristics are commonly used in analyzing the quadratic function defined by $f(x) = ax^2 + bx + c$, $a \neq 0$, and its graph.

1. The **axis of symmetry** is a vertical line that separates the parabola into two mirror images. The equation of the vertical axis of symmetry is given by $x = \frac{-b}{2a}$.
2. The **vertex** (turning point) always falls on the axis of symmetry. The x-coordinate of the vertex is given by $\frac{-b}{2a}$. Its y-coordinate is determined by evaluating the function at this value. In other words, the y-coordinate of the vertex is given by $f(-\frac{b}{2a})$.
3. The ***y*-intercept**, the point where the parabola crosses the y-axis (that is, where its x-coordinate is zero), is always given by $(0, c)$.
4. The ***x*-intercept** is the point or points (if any) where the parabola crosses the x-axis (that is, where its y-coordinate is zero).
5. If a parabola opens upward and the vertex is above the x-axis, there are no x-intercepts. If a parabola opens downward and the vertex is below the x-axis, there are no x-intercepts.
6. The **domain** of the general quadratic function is the set of all real numbers.
7. If the parabola opens upward, the **range** is all real numbers greater than or equal to the output value of the vertex. If the parabola opens downward, the *range* is all real numbers less than or equal to the output value of the vertex.

EXERCISES
ACTIVITY 4.2

For Exercises 1–8, determine the following characteristics of each quadratic function:

a. The direction in which the graph opens

b. The axis of symmetry

c. The turning point (vertex)

d. The y-intercept

1. $f(x) = x^2 - 3$
   a.
   b.
   c.
   d.

2. $g(x) = x^2 + 2x - 8$
   a.
   b.
   c.
   d.

3. $y = x^2 + 4x - 3$
   a.
   b.
   c.
   d.

4. $f(x) = 3x^2 - 2x$
   a.
   b.
   c.
   d.

5. $h(x) = x^2 + 3x + 4$
   a.
   b.
   c.
   d.

6. $g(x) = -x^2 + 7x - 6$
   a.
   b.
   c.
   d.

7. $y = 2x^2 - x - 3$
   a.
   b.
   c.
   d.

8. $f(x) = x^2 + x + 3$
   a.
   b.
   c.
   d.

Exercise numbers appearing in color are answered in the Selected Answers appendix.

For Exercises 9–16, use your graphing calculator to sketch the graphs of the functions, and then determine each of the following:

a. The coordinates of the x*-intercepts for each function, if they exist*

b. The domain and range for each function

c. The horizontal interval over which each function is increasing

d. The horizontal interval over which each function is decreasing

9. $g(x) = -x^2 + 7x - 6$

a.

b.

c.

d.

10. $h(x) = 3x^2 + 6x + 4$

a.

b.

c.

d.

11. $y = x^2 - 12$

a.

b.

c.

d.

12. $f(x) = x^2 + 4x - 5$

a.

b.

c.

d.

13. $g(x) = -x^2 + 2x + 3$

a.

b.

c.

d.

14. $h(x) = x^2 + 2x - 8$

a.

b.

c.

d.

15. $y = -5x^2 + 6x - 1$

a.

b.

c.

d.

16. $f(x) = 3x^2 - 2x + 1$

a.

b.

c.

d.

17. You shoot an arrow vertically into the air from a height of 5 feet with an initial velocity of 96 feet per second. The height, h, in feet above the ground, at any time, t (in seconds), is modeled by

$$h(t) = 5 + 96t - 16t^2.$$

a. Determine the maximum height the arrow will attain.

b. Approximately when will the arrow reach the ground?

c. What is the significance of the vertical intercept?

d. What are the practical domain and practical range in this situation?

e. Use your graphing calculator to determine the horizontal intercepts. Determine the practical meaning of these intercepts in this situation.

18. As part of a recreational waterfront grant, the city council plans to enclose a rectangular area along the waterfront of Lake Erie and create a park and swimming area. The budget calls for the purchase of 3000 feet of fencing. (*Note:* There is no fencing along the lake.)

a. Draw a picture of the planned recreational area. Let x represent the length of one of the two equal sides that are perpendicular to the water.

b. Write an expression that represents the width (side opposite the water) in terms of x. (*Note:* You have 3000 feet of fencing.)

c. Write an equation that expresses the area, $A(x)$, of this rectangular site as a quadratic function of x.

d. Determine the value of x for which $A(x)$ is a maximum.

e. What is the maximum area that can be enclosed?

f. What are the dimensions of the maximum enclosed area?

g. Use your graphing calculator to graph the area function. What point on the graph represents the maximum area?

h. What is the vertical intercept? Does this point have any practical meaning in this situation?

i. From the graph, determine the horizontal intercepts. Do they have any practical meaning in this situation? Explain.

19. The average cost to produce metal statues for local parks is given by

$$\overline{C}(x) = 2x^2 - 120x + 2000,$$

where x represents the number of statues produced and $\overline{C}(x)$ is the average cost of producing x statues.

a. Use your graphing calculator to graph the average cost function and determine the coordinates of the turning point.

b. Determine the vertex algebraically.

c. How do your answers in parts a and b compare?

d. Is the vertex a minimum or maximum point?

e. What is the practical meaning of the vertex in this situation?

f. What is the vertical intercept? What is the practical meaning of this intercept?

20. You are manufacturing ceramic lawn ornaments. After several months, your accountant tells you that your profit, $P(n)$, can be modeled by

$$P(n) = -0.002n^2 + 5.5n - 1200,$$

where n is the number of ornaments sold each month.

a. Use your graphing calculator to produce a graph of this function. Use the table feature set at TblStart = 0 and ΔTbl = 500 to help you set your window. Include the x-intercepts and the vertex.

b. Determine the x-intercepts of the graph of the profit function.

c. Determine the practical domain of the profit function.

d. Determine the practical range of the profit function.

e. How many ornaments must be sold to maximize the profit?

f. Write the equation that must be solved to determine the number of ornaments that must be sold to produce a profit of $2300.

g. Solve the equation in part f graphically.

ACTIVITY 4.3

Per Capita Personal Income

OBJECTIVES

1. Solve quadratic equations numerically.
2. Solve quadratic equations graphically.
3. Solve quadratic inequalities graphically.

According to statistics from the U.S. Department of Commerce, the per capita personal income (or the average annual income) of each resident of the United States from 1960 to 2000 can be modeled by the equation

$$P(t) = 15.1442t^2 + 98.7687t + 1831.6909,$$

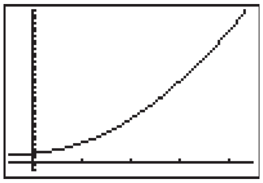
where $P(t)$ represents the per capita income and t represents the number of years since 1960.

1. What is the practical domain for the model represented by the function P?

2. Let $t = 0$ correspond to the year 1960. Use your graphing calculator to complete the following table of values for t, the number of years since 1960, and P(t), the per capita income. Round your output to the nearest dollar.

YEAR	1960	1965	1970	1975	1980	1985	1990	1995	2000
t									
$P(t)$ (\$)									

3. Sketch a graph of the function using your graphing calculator using the window Xmin = −5, Xmax = 45, Ymin = −2000, and Ymax = 35,000. The graph should appear as follows.



4. Estimate the per capita personal income in the year 1989 ($t = 29$).

5. You want to determine in which year the per capita personal income reached \$20,500. Write an equation to determine the value of t when $P(t) = 20{,}500$.

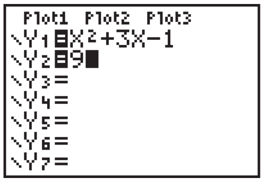
The equation in Problem 5 is called a **quadratic equation**. Such equations involve polynomial expressions of degree 2. The standard form of a quadratic equation is $ax^2 + bx + c = 0, a \neq 0$. Examples of quadratic equations include $x^2 + 3x - 1 = 9$, $2x^2 - 4x + 1 = 0$, and $6x^2 = 18$.

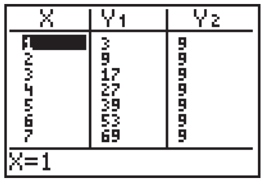
One method for approximating a solution of an equation is numerical, using a table of appropriate data points.

EXAMPLE 1 ***Solve the quadratic equation $x^2 + 3x - 1 = 9$ numerically (using tables of data).***

SOLUTION

Create a table in which x is the input and $y = x^2 + 3x - 1$ is the output. The solution is the x-value corresponding to a y-value of 9.





Using the graphing calculator, the solution is $x = 2$.

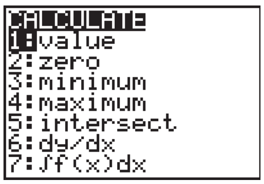
6. Determine the solution to $20{,}500 = 15.1442t^2 + 98.7687t + 1831.6909$ numerically using a table of appropriate data points (see Problem 2). What is your approximation using this approach?

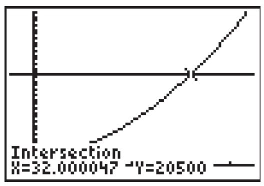
Solving Quadratic Equations Graphically

A second method of solving the quadratic equation in Problem 5 is graphical, using your graphing calculator. Recall from Chapter 1 that you can solve the equation $15.1442x^2 + 98.7687x + 1831.6909 = 20{,}500$ by solving the following system of equations graphically.

$$y_1 = 15.1442x^2 + 98.7687x + 1831.6909$$
$$y_2 = 20{,}500$$

The expression for y_1 gives the per capita personal income in any given year. The value y_2 is the specific per capita personal income in which you are interested. The solution to the equation is the x-value for which $y_1 = y_2$. To do this, determine the point of intersection of these two graphs. If you use the intersect option under the CALC menu, the graph should appear as follows.





Another graphical method for solving the problem is to rearrange the quadratic equation

$$20{,}500 = 15.1442t^2 + 98.7687t + 1831.6909$$

so that the left-hand side is equal to zero. Subtracting 20,500 from each side, you have

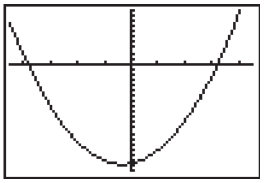
$$0 = 15.1442t^2 + 98.7687t - 18{,}668.3091 \qquad (1)$$

If you let $y = 15.1442t^2 + 98.7687t - 18{,}668.3091$, then the solution to equation 1 is the x-value for which $y = 0$, if it exists. This is the x-value of the x-intercept (zero) of the graph.

7. a. Use your graphing calculator to sketch a graph of

$$y = 15.1442x^2 + 98.7687x - 18{,}668.3091.$$

The screen should appear as follows.



b. Use the ZERO option of the CALC menu to determine the x-intercepts of the new function defined by $y = 15.1442t^2 + 98.7687t - 18{,}668.3091$?

c. Using the results from part b, determine the solutions to the equation $20{,}500 = 15.1442t^2 + 98.7687t + 1831.6909$. Are both of the values relevant to our problem? Explain.

8. Describe two different ways to solve the equation $2x^2 - 4x + 3 = 2$ using a graphing approach. Solve the equation using each graphing method. How do your answers compare?

a.

b.

c.

Solving Quadratic Inequalities Graphically

You are interested in determining in which years the per capita personal income was more than \$15,000. To answer this problem, you need to solve the inequality

$$15.1442t^2 + 98.7687t + 1831.6909 > 15{,}000,$$

where t equals the number of years since 1960.

The following example demonstrates a procedure for solving an inequality similar to the preceding one.

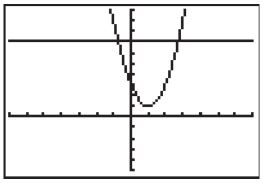
EXAMPLE 2 ***Solve the inequality $2x^2 - 4x + 3 > 7$ using a graphing approach.***

SOLUTION

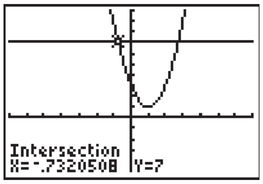
Form the following system of equations

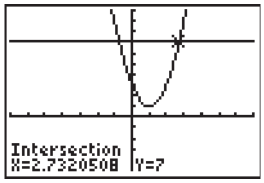
$$Y_1 = 2x^2 - 4x + 3$$
$$Y_2 = 7$$

and graph each equation. The screen should resemble the following.



Use the intersection option on the CALC menu to determine where $Y_1 = Y_2$.





The solutions to the inequality are the values of x where $Y_1 > Y_2$, that is, the x-values of points where the graph of Y_1 is above the graph of Y_2.

Therefore, the solutions are $x < -0.732$ or $x > 2.732$.

If the problem had been the reverse inequality $2x^2 - 4x + 3 < 7$, then the solution would be x-values of points where the graph of Y_1 is below the graph of Y_2. The solution would be $-0.732 < x < 2.732$.

9. a. Solve the inequality $15.1442t^2 + 98.7687t + 1831.6909 > 15{,}000$ using a graphing approach. Be careful. The practical domain is $t \geq 0$.

b. In which years was the per capita personal income more than \$15,000?

c. There are negative values of x for which Y1 $> 15{,}000$. Determine them graphically.

d. Explain why the values determined in part c are not relevant to the original problem situation.

10. Solve the following quadratic inequalities using a graphing approach.

a. $x^2 - x - 6 < 0$

b. $x^2 - x - 6 > 0$

SUMMARY
ACTIVITY 4.3

1. A **quadratic equation** is an equation involving polynomial expressions of degree 2. The standard form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.
2. To solve $f(x) = c$ **numerically**, construct a table, and determine the x-values that produce c as an output.
3. To solve $f(x) = c$ **graphically**,
   a. graph $y = f(x)$, graph $y = c$, and determine the x-values of the points of intersection.
   b. or graph $y = f(x) - c$, and determine the x-intercepts.
4. To solve $f(x) > c$ graphically,
   a. graph $y = f(x)$, graph $y = c$, and determine all x-values for which the graph of f is above the graph of $y = c$.
   b. or graph $y = f(x) - c$, and determine all x-values for which the graph of $f(x) - c$ is above the x-axis.
5. To solve $f(x) < c$ graphically,
   a. graph $y = f(x)$, graph $y = c$, and determine all x-values for which the graph of f is below the graph of $y = c$.
   b. or graph $y = f(x) - c$, and determine all x-values for which the graph of $f(x) - c$ is below the x-axis.

EXERCISES
ACTIVITY 4.3

In Exercises 1–4, solve the quadratic equation numerically (using tables of x- and y-values). Verify your solutions graphically.

1. $-4x = -x^2 + 12$
2. $x^2 + 9x + 18 = 0$
3. $2x^2 = 8x + 90$
4. $x^2 - x - 3 = 0$

In Exercises 5–8, solve the quadratic equation graphically using at least two different approaches. When necessary, give your solutions to the nearest hundredth.

5. $x^2 + 12x + 11 = 0$
6. $2x^2 - 3 = 2x$
7. $16x^2 - 400 = 0$
8. $4x^2 + 12x = -4$

Exercise numbers appearing in color are answered in the Selected Answers appendix.

In Exercises 9–12, solve the equation by using either a numeric or a graphic approach.

9. $x^2 + 2x - 3 = 0$

10. $x^2 + 11x + 24 = 0$

11. $x^2 - 2x - 8 = x + 20$

12. $x^2 - 10x + 6 = 5x - 50$

In Exercises 13–14, solve the given inequality using a graphing approach.

13. **a.** $x^2 - 4x - 1 < 11$

b. $x^2 - 4x - 1 > 11$

14. **a.** $2x^2 + 5x - 3 < 0$

b. $2x^2 + 5x - 3 \geq 0$

15. The stopping distance, d (in feet), for a car moving at a velocity (speed) v miles per hour is modeled by the equation

$$d(v) = 0.04v^2 + 1.1v.$$

a. What is the stopping distance for a velocity of 55 miles per hour?

b. What is the speed of the car if it takes 200 feet to stop?

16. An international rule for determining the number, n, of board feet (usable finished lumber) in a 16-foot log is modeled by the equation

$$n(d) = 0.22d^2 - 0.71d,$$

where d is the diameter of the log in inches.

a. How many board feet can be obtained from a 16-foot log with a 14-inch diameter?

b. Sketch a graph of this function. What is the practical domain of this function?

c. Use the graph to approximate the horizontal intercept(s). What is the practical meaning in this situation?

d. What is the diameter of a 16-foot log that has 200 board feet?

e. What inequality would you solve to determine the diameter when the board feet are at most 200?

f. Solve the inequality by using the graph of the function.

ACTIVITY 4.4

Sir Isaac Newton

OBJECTIVES

1. Factor expressions by removing the greatest common factor.
2. Factor trinomials using trial and error.
3. Use the zero-product principle to solve equations.
4. Solve quadratic equations by factoring.

Sir Isaac Newton XIV, a descendant of the famous physicist and mathematician, takes you to the top of a building to demonstrate a physics property discovered by his famous ancestor. He throws your math book straight up into the air. The book's distance, s, above the ground as a function of time, t, is modeled by

$$s = -16t^2 + 16t + 32.$$

1. When the book strikes the ground, what is the value of s?

2. Write the equation that you must solve to determine when the book strikes the ground.

The quadratic equation in Problem 2 can be solved by using a numerical or a graphical approach. However, an algebraic technique is efficient in this case and will give an exact answer. The algorithm is based on the algebraic principle known as the **zero-product principle**.

If a and b are any numbers and $a \cdot b = 0$, then either a or b, or both, must be equal to zero.

EXAMPLE 1 ***Solve the equation $x(x + 5) = 0$.***

SOLUTION

The two factors in this equation are x and $x + 5$. The zero-product principle says one of these factors must equal zero. That is,

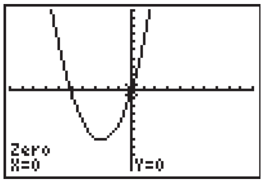
$$x = 0 \quad \text{or} \quad x + 5 = 0.$$

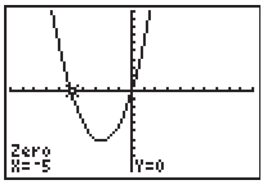
The first equation tells you that $x = 0$ is a solution. To determine a second solution, solve $x + 5 = 0$.

$$\begin{aligned} x + 5 &= 0 \\ -5 \;&\; -5 \\ \hline x &= -5 \end{aligned}$$

There are two solutions, $x = 0$ and $x = -5$.

Your graphing calculator verifies the solutions as follows.





3. Solve each of the following equations using the zero-product principle.

   a. $3x(x - 2) = 0$

   b. $(2x - 3)(x + 2) = 0$

c. $(x + 2)(x + 3) = 0$

For the zero-product principle to be applied, one side of the equation must be zero. Therefore, at first glance, the zero-product principle can be used to solve the quadratic equation $3x^2 - 6x = 0$. However, a second condition must be satisfied. The nonzero side of the equation must be written as a product.

The process of writing an expression such as $3x^2 - 6x$ as a product is called factoring.

DEFINITION

Rewriting an expression as a product is called **factoring**.

Factoring Common Factors

A **common factor** is a number or an expression that is a factor of each term of the entire expression. Whenever you wish to factor a polynomial, look first for a common factor.

PROCEDURE

Removing a Common Factor from a Polynomial: First, identify the common factor, and then apply the distributive property in reverse.

EXAMPLE 2 ***Given the binomial 3x + 6, 3 is a common factor because 3 is a factor of both terms 3x, and 6. Applying the distributive property in reverse, you write***

$$3x + 6 \text{ as } 3(x + 2).$$

You may always check the factored binomial by multiplying:

$$3(x + 2) = 3(x) + 3(2) = 3x + 6$$

When you look for a common factor, determine the largest or **greatest common factor** (or GCF). You can see that 3 is a common factor of $6x + 24$ because 3 is a factor of both 6 and 24. However, there is a larger common factor, 6. Therefore,

$$6x + 24 = 6(x + 4).$$

EXAMPLE 3 ***Given $6x^2 + 14x - 30$, you can see that 2 is a common factor. Is 2 the greatest common factor? Yes, because no larger number is a factor of every term.***

If you divide each term by 2, you obtain $3x^2 + 7x - 15$. The expression $6x^2 + 14x - 30$ can now be written in factored form as $2(3x^2 + 7x - 15)$. Check the factored trinomial by multiplying.

EXAMPLE 4 ***Factor*** $4x^3 - 8x^2 + 28x$.

SOLUTION

Four is a factor of each term, but x is as well. Therefore, the greatest common factor is $4x$. You remove the GCF by dividing each term by $4x$. This leads to the factored form $4x(x^2 - 2x + 7)$.

You can check your factoring by applying the distributive property.

4. Factor the following polynomials by removing the greatest common factor.

a. $9a^6 + 18a^2$

b. $21xy^3 + 7xy$

c. $3x^2 - 21x + 33$

d. $4x^3 - 16x^2 - 24x$

Factoring Trinomials

With patience, you can factor trinomials of the form $ax^2 + bx + c$ by trial and error, using the FOIL method in reverse.

PROCEDURE

Factoring Trinomials by Trial and Error

1. Remove the greatest common factor, GCF.
2. To factor the resulting trinomial into the product of two binomials, try combinations of factors for the first and last terms in two binomials.
3. Check the outer and inner products to match the middle term of the original trinomial.
   - **a.** If the constant term, c, is positive, both of its factors are positive or both are negative.
   - **b.** If the constant term is negative, one factor is positive and one is negative.
4. If the check fails, repeat steps 2 and 3.

EXAMPLE 5 ***Factor*** $6x^2 - 7x - 3$.

SOLUTION

Step 1. There is no common factor, so go to step 2.

Step 2. You could factor the first term, $6x^2$, as $6x(x)$ or as $2x(3x)$. The last term, -3, has factors $3(-1)$ or $-3(1)$. Try $(2x + 1)(3x - 3)$.

Step 3. The outer product is $-6x$. The inner product is $3x$. The sum is $-3x$, not $7x$. The check fails.

Step 4. Try $(2x - 3)(3x + 1)$. The outer product is $2x$. The inner product is $-9x$. The sum is $-7x$. It checks.

5. Factor the following trinomials.

a. $x^2 - 7x + 12$

b. $x^2 - 8x - 9$

c. $x^2 + 14x + 49$

d. $25 + 10w + w^2$

Solving Quadratic Equations by Factoring

The following example demonstrates the procedure for solving quadratic equations written in standard form, $ax^2 + bx + c = 0$, by factoring.

EXAMPLE 6 ***Solve the equation $3x^2 - 2 = -x$ by factoring.***

Step 1. Rewrite the equation in the form $ax^2 + bx + c = 0$ (called standard form).

$$\begin{array}{r} 3x^2 - 2 = -x \\ \underline{+\,x \qquad\quad +\,x} \\ 3x^2 + x - 2 = 0 \end{array}$$

Step 2. Factor the expression on the nonzero side of the equation.

$$(x + 1)(3x - 2) = 0$$

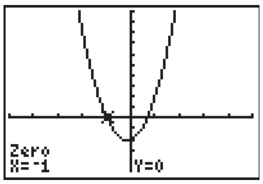
Step 3. Use the zero-product principle to set each factor equal to zero, and then solve each equation.

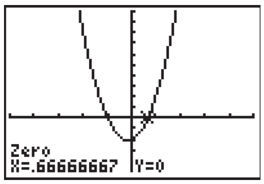
$$(x + 1)(3x - 2) = 0$$

$$\begin{array}{r|r} x + 1 = 0 & 3x - 2 = 0 \\ x = -1 & 3x = 2 \\ & x = \frac{2}{3} \end{array}$$

Therefore, the solutions are $x = -1$ and $x = \frac{2}{3}$.

These solutions can be verified graphically as follows.





6. a. Returning to the math book problem from the beginning of this activity, solve the equation from Problem 2 by factoring.

b. Are both solutions to the equation $(t = 2$ and $t = -1)$ also solutions to the question, "At what time does the book strike the ground"? Explain.

7. a. You want to know at what time the book is 32 feet above the ground. Write a quadratic equation that represents this situation.

b. Solve the quadratic equation in part a by factoring.

8. Solve each of the following quadratic equations by factoring.

a. $2x^2 - x - 6 = 0$ **b.** $3x^2 - 6x = 0$ **c.** $x^2 + 4x = -x - 6$

SUMMARY
ACTIVITY 4.4

1. To remove a **common factor** from a polynomial, first

a. identify the common factor, and then

b. apply the distributive property in reverse.

2. The **zero-product principle** says that if $ab = 0$ is a true statement, then either $a = 0$ or $b = 0$.

3. To factor trinomials of the form $ax^2 + bx + c$ by **trial and error**,

a. remove the greatest common factor.

b. try combinations of factors for the first and last terms in two binomials.

c. check the outer and inner products to match the middle term of the original trinomial.

- If the constant term, c, is positive, both factors of c are positive or both are negative.
- If the constant term is negative, one factor is positive and one is negative.

d. If the check fails, repeat steps 3b and 3c.

4. To solve equations by **factoring**,

a. use the addition principle to remove all terms from one side of the equation; this results in a polynomial being set equal to zero.

b. combine like terms, and then factor the nonzero side of the equation.

c. use the zero-product principle to set each factor containing a variable equal to zero, and then solve the equations.

d. check your solutions in the original equation.

EXERCISES
ACTIVITY 4.4

In Exercises 1–4, factor the polynomials by removing the GCF (greatest common factor).

1. $12x^5 - 18x^8$

2. $14x^6y^3 - 6x^2y^4$

3. $2x^3 - 14x^2 + 26x$

4. $5x^3 - 20x^2 - 35x$

In Exercises 5–13, completely factor the polynomials. Remember to look first for the GCF.

5. $x^2 + x - 6$

6. $p^2 - 16p + 48$

7. $x^2 + 7xy + 10y^2$

8. $x^2 - 4x - 32$

9. $12 + 8x + x^2$

10. $2x^2 + 7x - 15$

11. $3x^2 + 19x - 14$

12. $8x^4 - 47x^3 - 6x^2$

13. $20b^4 - 65b^3 - 60b^2$

In Exercises 14–21, solve each quadratic equation by factoring.

14. $x^2 - 5x + 6 = 0$

15. $x^2 + 2x - 3 = 0$

16. $x^2 - x = 6$

17. $x^2 - 5x = 14$

18. $3x^2 + 11x - 4 = 0$

19. $3x^2 - 12x = 0$

Exercise numbers appearing in color are answered in the Selected Answers appendix.

20. $x^2 - 7x = 18$

21. $3x(x - 6) - 5(x - 6) = 0$

22. Your neighbors have just finished installing a new swimming pool at their home. The pool measures 15 feet by 20 feet. They would like to plant a strip of grass of uniform width around three sides of the pool, the two short sides and one of the longer sides.

a. Sketch a diagram of the pool and the strip of lawn, using x to represent the width of the uniform strip.

b. Write an equation for the area, A, in terms of x that represents the lawn area around the pool.

c. They have enough seed for 168 square feet of lawn. Write an equation that relates the quantity of seed to the area of the uniform strip of lawn.

d. Solve the equation in part c to determine the width of the uniform strip that can be seeded.

ACTIVITY 4.5

Motorcycle Deaths

OBJECTIVE

1. Solve quadratic equations by the quadratic formula.

The following data was contained in a news release from the Insurance Institute for Highway Safety. The following table gives the number of motorcycle deaths of cyclists aged 30–39 from 1991 to 2000.

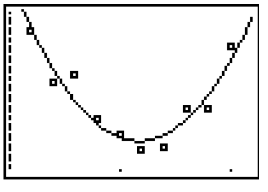
	ANNUAL NUMBER OF DEATHS OF MOTORCYCLISTS AGED 30–39, 1991–2000									
YEAR	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
NUMBER OF MOTORCYCLE DEATHS	711	638	647	584	562	541	547	599	601	687

1. **a.** Use your graphing calculator to create a scatterplot of the data in the preceding table. Let x represent the number of years since 1990. Let n represent the annual number of motorcyclist deaths.

   b. This data can be modeled by the function

$$n = 6.875x^2 - 80.76x + 791,$$

where n represents the number of deaths and x represents the number of years since 1990. Verify that this is a good model by graphing the function on the same screen with your scatterplot. Use the data from the preceding table to determine an appropriate window. Your graph should appear as follows.



   c. You want to know in what years the model predicts the number of deaths of motorcyclists aged 30–39 to be approximately 1000. Write the equation that must be solved to determine these years.

The Quadratic Formula

The equation in Problem 1c can be solved by using a numerical or a graphical approach. However, the algebraic technique of factoring cannot be applied.

The technique of solving quadratic equations by factoring is very limited. In most real-world applications involving quadratic equations, the quadratic is not factorable. In those cases, you can use a formula to solve the quadratic equation.

Beginning with the standard quadratic function defined by $y = ax^2 + bx + c$, $a \neq 0$, set $y = 0$ to obtain the equation

$$0 = ax^2 + bx + c.$$

The quadratic equation $ax^2 + bx + c = 0$ has two solutions,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

These solutions are often written as a single expression,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This formula is known as the **quadratic formula**.

For the details showing the equation $0 = ax^2 + bx + c$ can be solved using the quadratic formula, see Appendix A. The section is called "Derivation of the Quadratic Formula."

The following example demonstrates the procedure for using the quadratic formula to solve an equation of the form $ax^2 + bx + c = 0$.

EXAMPLE 1 ***Solve $x(3x + 4) = 5$ using the quadratic formula.***

SOLUTION

Step 1. Write the equation in standard form, $ax^2 + bx + c = 0$.

$$
\begin{array}{rl}
x(3x + 4) = 5 & \quad \text{Apply the distributive property on the left side.} \\
3x^2 + 4x = 5 & \\
-5 \quad -5 & \quad \text{Subtract 5 from both sides.} \\
\hline
3x^2 + 4x - 5 = 0 &
\end{array}
$$

Step 2. Identify the coefficients a, b, and the constant term, c.

$a = 3, b = 4, c = -5$

Step 3. Substitute the values a, b, and c into the quadratic formula, and simplify.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

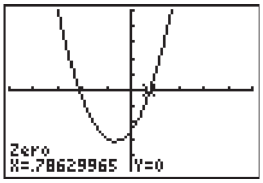
$$x = \frac{-4 \pm \sqrt{4^2 - 4(3)(-5)}}{2(3)}$$

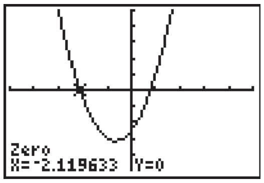
$$x = \frac{-4 \pm \sqrt{16 - (-60)}}{6} = \frac{-4 \pm \sqrt{76}}{6}$$

$$x \approx \frac{-4 \pm 8.7178}{6} = -2.1196 \quad \text{or} \quad 0.7863$$

```
(-4+√(4²-4*3*-5)
)/(2*3)
           .7862996478
(-4-√(4²-4*3*-5)
)/(2*3)
          -2.119632981
```

Step 4. Check your solutions. The following graphs verify the solutions.

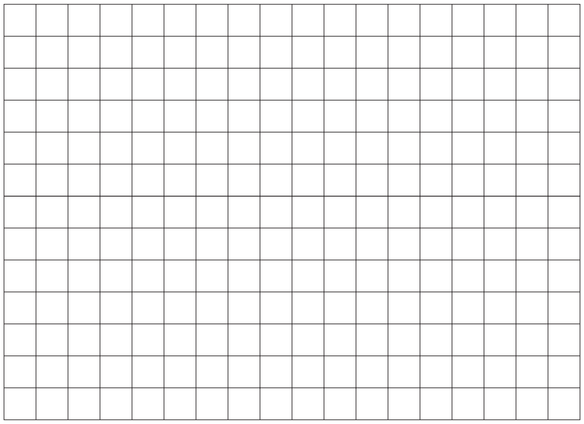




2. a. Solve the equation from Problem 1c, $6.875x^2 - 80.76x + 791 = 1000$, using the quadratic formula. Check your solution graphically.

b. Use the results from part a to predict the years in which the number of motorcycle deaths will be (was) approximately 1000.

3. a. Sketch a graph of $y = 2x^2 + 9x - 5$.



b. Write the equation that you would need to solve to determine the x-intercepts of the graph.

c. Solve the equation $0 = 2x^2 + 9x - 5$ using the quadratic formula.

d. Approximate the x-intercepts of the graph using your graphing calculator.

e. What do you conclude about the solution using the quadratic formula and the x-intercepts determined from the graph?

4. The following data from the National Health and Nutrition Examination Survey indicates that the number of American adults who are overweight or obese is increasing.

YEARS SINCE 1960, t	1	12	18	31	39
PERCENT OF OVERWEIGHT OR OBESE AMERICANS, $P(t)$	45	47	47	56	64.5

This data can be modeled by the equation $P(t) = 0.017t^2 - 0.174t + 45.493$.

a. Use your graphing calculator to create a scatterplot of the data and a graph of the model P.

b. Does the model appear to be a good fit for the data? Explain.

c. Using the quadratic formula, determine the year when the model predicts that the percent of overweight or obese Americans will first exceed 75%.

d. Using your graphing calculator, graph $Y_2 = 75$ on the same axis as the function in part a. Determine the solution from the graph. How does the solution found from the graph compare to the solution found in part c?

Axis of Symmetry Revisited

Recall that the axis of symmetry of a parabola is given by the formula $x = -\frac{b}{2a}$

If you write the quadratic formula in a slightly different form, you obtain

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x_1 = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a},\ x_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

The next problem uses the rewritten form of the quadratic formula to help identify a relationship between the x-intercepts and the axis of symmetry of the graph of f.

5. Consider the function $f(x) = 2x^2 + 9x - 5$.

a. Determine the equation of the axis of symmetry of the graph of f.

b. What is the value of $\frac{\sqrt{b^2 - 4ac}}{2a}$?

c. Sketch a graph of the function f, and label the axis of symmetry. Show where the value computed in part b is located graphically. What are the x-intercepts of the graph?

d. What is the relationship between the axis of symmetry and the x-intercepts of a parabola?

6. For each of the following quadratic functions, determine the x-intercepts and the axis of symmetry of the graph. Solve the appropriate equation using the quadratic formula. Round your answers to the nearest hundredth.

a. $f(x) = 2x^2 - 6x - 3$

b. $h(x) = x^2 - 8x + 16$

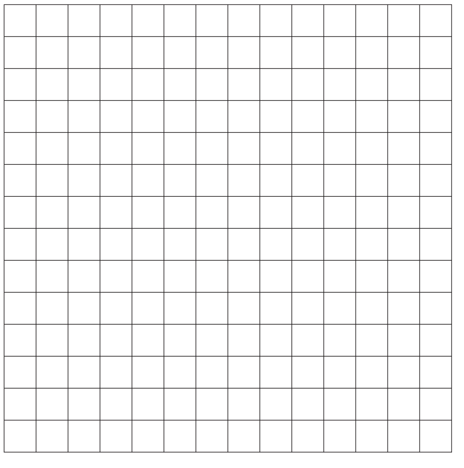
SUMMARY
ACTIVITY 4.5

1. To solve a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$, using the **quadratic formula** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,
   a. rewrite the quadratic equation (if necessary) so that one side is equal to zero.
   b. identify the coefficients a and b, and the constant term, c.
   c. substitute these values into the formula, and simplify.
   d. check your solutions graphically.
2. For a parabola with x-intercepts, the axis of symmetry is always midway between its x-intercepts.
3. The distance from the axis of symmetry to either x-intercept is $\frac{\sqrt{b^2 - 4ac}}{2a}$.

EXERCISES
ACTIVITY 4.5

1. The height of a bridge arch located in the Thousand Islands is modeled by the function $f(x) = -0.04x^2 + 28$, where x is the distance, in feet, from the center of the arch and $f(x)$ is the height of the arch.

   a. Sketch a picture of this arch on a grid using the vertical axis as the center of the arch.



   b. Determine the vertical intercept. What is the practical meaning of this intercept in this situation?

   c. Determine the x-intercepts algebraically using the quadratic formula.

Exercise numbers appearing in color are answered in the Selected Answers appendix.

d. Graph the function on your graphing calculator and check the accuracy of the intercepts you found in part c.

e. If the arch straddles the river exactly, how wide is the river?

f. A sailboat is approaching the bridge. The top of the mast is 30 feet above the water. Will the boat clear the bridge? Explain.

g. You want to install a flagpole on the bridge at an arch height of 20 feet. Write the equation that you must solve to determine how far to the right or left of center the arch height is 20 feet.

h. Solve the equation in part g using the quadratic formula. Use your graphing calculator to check your result.

In Exercises 2–8, identify the values of a, b, *and* c, *and then solve the equations using the quadratic formula. Round your answers to the nearest hundredth. Verify your solutions graphically.*

2. $x^2 + 6x - 3 = 0$

3. $4x^2 + 4x + 1 = 0$

4. $x^2 + 5x = 13$

5. $2x^2 - 6x + 3 = 0$

6. $2x^2 - 3x = 5$

7. $(2x - 1)(x + 2) = 1$

8. $(x + 2)^2 + x^2 = 44$

In Exercises 9–11, determine the x-intercept(s) of the graph algebraically. Then check your results graphically.

9. $y = 3x^2 + 6x$

10. $y = x^2 - x - 6$

11. $f(x) = 2x^2 - x - 5$

12. The number, n (in millions), of cellular phone subscribers in the United States from 1990 to 1999 is given in the following table.

YEAR	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
NUMBER OF SUBSCRIBERS (millions)	5.28	7.56	11.03	16.01	24.13	33.79	44.04	55.31	69.20	86.05

This data can be approximated by the quadratic model

$$n(t) = 0.846t^2 + 1.32t + 5.20,$$

where $t = 0$ corresponds to the year 1990.

a. Use your graphing calculator to sketch a graph of the function.

b. Use the graph in part a to estimate the year in which there will be 120 million cellular phone subscribers.

c. Use the quadratic formula to answer part b. How does your answer compare to the estimate you obtained using a graphical approach?

d. How confident are you in your prediction? Explain.

13. The quadratic function defined by the equation

$$d = 2r^2 - 16r + 34$$

gives the density of smoke, d, in millions of particles per cubic foot for a certain type of diesel engine. The input variable, r, represents the speed of the engine in hundreds of revolutions per minute.

a. Determine the density of smoke when $r = 3.5$ (350 revolutions per minute).

b. Determine the number of revolutions per minute for minimum smoke. What is the minimum output?

c. If the density of smoke is determined to be 100 million particles per cubic foot, determine the speed of the engine.

ACTIVITY 4.6

Air Quality in Atlanta

OBJECTIVES

1. Determine quadratic regression models using the graphing calculator.
2. Solve problems using quadratic regression models.

The Air Quality Index, or AQI, measures how polluted the air is by measuring five major pollutants: ground-level ozone, particulate matter, carbon monoxide, sulfur dioxide, and nitrogen oxide. Based on the amount of each pollutant in the air, the AQI assigns a numerical value to air quality, as follows.

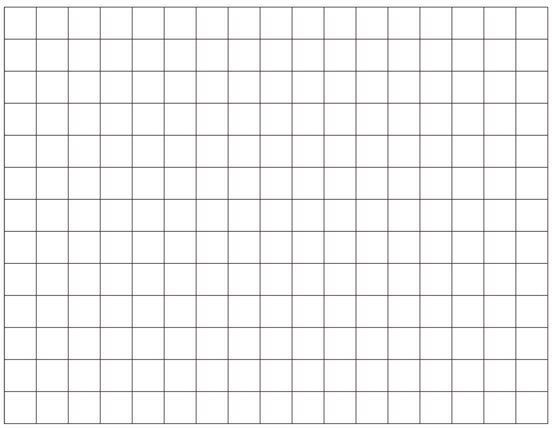
Air Quality Index

0–50	Good
51–100	Moderate
101–150	Unhealthy for sensitive groups
151–200	Unhealthy
201–300	Very unhealthy
301–500	Hazardous

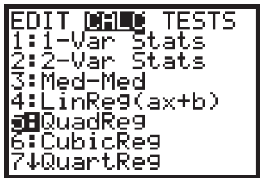
The following table indicates the number of days in which the AQI was greater than 100 in the city of Atlanta, Georgia.

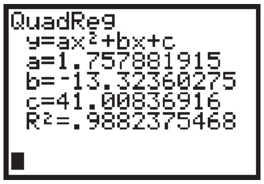
YEAR	1990	1992	1994	1996	1998	1999
NUMBER OF DAYS AQI >100, n	42	20	15	25	50	61

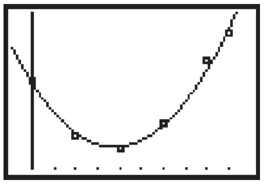
1. Sketch a scatterplot of the data. Let t represent the number of years since 1990. Therefore, $t = 0$ corresponds to the year 1990. Does the data appear to be quadratic? Explain.



2. Use the regression feature of your graphing calculator to determine the regression equation, and plot the quadratic function that best fits this data. Your graph should appear as follows.







3. Do you believe that the quadratic regression model is a good model for the number of days the AQI exceeded 100 in Atlanta from 1990 to 1999?

4. What is the practical domain of this function?

5. a. Use the quadratic regression equation to estimate the number of days the AQI exceeded 100 in Atlanta in each of the following years.

   i. 1995 ii. 1988 iii. 2002

   b. Which, if any, of these estimates do you think is most reliable? Explain.

6. Estimate the years in which the number of days that the AQI exceeded 100 in Atlanta was less than or equal to 30 using

   a. the given table (numerical method).

   b. the graph of the quadratic regression equation (graphical method).

7. Use the quadratic formula (the algebraic method) to estimate the year in which the number of days that the AQI exceeded 100 in Atlanta was equal to 17.

8. Estimate graphically the years between 1990 and 1999 when the number of days that the AQI exceeded 100 in Atlanta was greater than 25.

9. a. The number of days that the AQI exceeded 100 in Atlanta in 1997 was 31. Does this agree with the prediction from the quadratic regression model?

b. Include the data for 1997 from part a in the original data set, and then recalculate the quadratic regression equation.

c. Predict the number of days that the AQI exceeded 100 in Atlanta in 1997 from this new quadratic regression equation from part b. Are the results any better?

10. The following data from the National Health and Nutrition Examination Survey appears in Problem 4 of Activity 4.5. The data indicates that the number of American adults who are overweight or obese is increasing.

YEARS SINCE 1960, t	1	12	18	31	39
PERCENT OF OVERWEIGHT OR OBESE AMERICANS, $P(t)$	45	47	47	56	64.5

Use the regression feature of your graphing calculator to verify that this data can be modeled by the equation $P(t) = 0.017t^2 - 0.174t + 45.493$.

SUMMARY
ACTIVITY 4.6

1. Parabolic data can be modeled by a **quadratic regression equation**.

EXERCISES
ACTIVITY 4.6

1. During one game, the Buffalo Bills punter was called upon to punt the ball 8 times. On one of these punts, the punter struck the ball at his own 30-yard line. The height, h, of the ball above the field in feet as a function of time, t, in seconds can be partially modeled by the following table.

t	0	0.6	1.2	1.8	2.4	3.0
$h(t)$	2.50	28.56	43.10	46.12	37.12	17.60

a. Sketch a scatterplot of the data using your graphing calculator.

b. Use your graphing calculator to obtain a quadratic regression function for this data. Round the values of a, b, and c to four decimal places.

c. Graph the equation from part b on the same coordinate axes as the data points. Does the curve appear to be a good fit for the data? Explain.

d. In this model, what is the practical domain of the quadratic regression function?

e. Estimate the practical range of this model.

f. How long after the ball was struck did the ball reach 35 feet above the field? Explain.

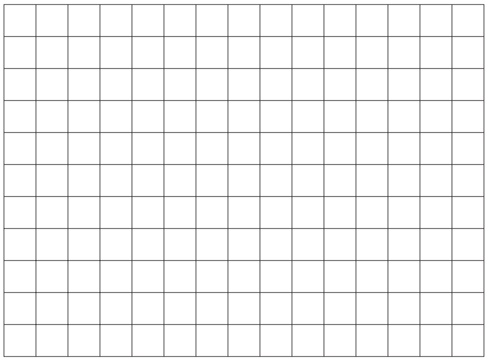
g. How many results did you obtain for part f? Do you think you have all of the solutions? Explain.

2. Use the following data set to perform the tasks in parts a–e.

x	0	3	6	9	12
y	5	28	86	180	310

Exercise numbers appearing in color are answered in the Selected Answers appendix.

a. Determine an appropriate scale, and plot these points.



b. Use your graphing calculator to determine the quadratic regression equation for this data set.

c. Graph the regression equation on the same coordinate axes as the data points in part a.

d. Compare the predicted outputs with the outputs given in the table.

e. Predict the output for $x = 7$ and for $x = 15$.

3. The following table shows the stopping distance for a car at various speeds on dry pavement.

SPEED (mph)	25	35	45	55	65	75
DISTANCE (ft.)	65	108	167	245	340	450

a. Use your graphing calculator to determine a quadratic regression equation that represents this data.

b. Use the regression equation to predict the stopping distance at 90 mph.

c. What speed would produce a stopping distance of 280 feet? (Round to the nearest tenth.) Explain how you arrived at your conclusion.

4. A downturn in the high-tech industry during the early 2000s caused a similar downturn in computer science and engineering enrollments for new undergraduates. The following data from the Computing Research Association represents new enrollments in computer science and engineering from 1995 to 2001.

YEARS SINCE 1995, t	0	1	2	3	4	5	6
NEW COMPUTER SCIENCE AND ENGINEERING UNDERGRADS (in thousands), N	8.2	11.9	16.2	17.1	16.6	18.9	18.0

Source: Computer Research Association

a. Create a scatterplot of the data using your graphing calculator.

b. Use your graphing calculator to obtain a quadratic regression function for this data. Round the values of a, b, and c to four decimal places.

c. Graph the equation from part b on the same coordinate axes as the data points. Does the curve appear to be a good fit for the data? Explain.

d. What does the regression equation predict for the new undergraduate enrollments in computer science and engineering in 2003? Does this seem reasonable?

e. Use the model to determine the year that new undergraduate computer science and engineering enrollments will drop below 15,000.

CLUSTER 1 What Have I Learned?

1. a. In order for the graph of the equation $y = ax^2 + bx + c$ to be a parabola, the value of the coefficient of x^2 cannot be zero. Explain.

b. What is the vertex of a parabola having an equation of the form $y = ax^2$?

c. Describe the relationship between the vertex and the vertical intercept of the graph of $y = ax^2 + c$.

2. Determine if the vertex is a minimum point or a maximum point of $y = ax^2 + bx + c$ in each of the following situations.

a. $a < 0$

b. $a > 0$

3. a. What are the possibilities for the number of vertical intercepts of a quadratic function?

b. What are the possibilities for the number of horizontal intercepts of a parabola?

4. What is the relationship between the vertex and the x-intercept of the graph of $y = x^2 - 4x + 4$?

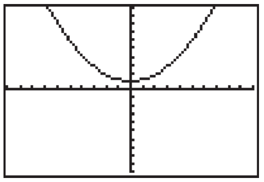
5. a. The vertex of a parabola is (3, 1). Using this information, complete the following table.

x	1	2	3	4	5
y	5	2			

b. If the vertex of a parabola is (2, 4), complete the following table

x	−2	0	2	4	6
y	0	3			

6. a. Given the following graph, explain why choices i, ii, and iii do not fit the curve.



i. $f(x) = ax^2 + bx$ with $a > 0, b < 0$

ii. $g(x) = ax^2 + c$ with $a < 0, c > 0$

iii. $h(x) = ax^2 + bx + c$ with $a < 0, b > 0, c < 0$

b. What restrictions on a, b, and c are necessary to fit $y = ax^2 + bx + c$ to this graph?

7. Review the steps in the following solution. Is the solution correct? Explain why or why not.

$$x^2 - 3x - 4 = 6$$
$$(x - 4)(x + 1) = 6$$
$$x - 4 = 6 \qquad x + 1 = 6$$
$$x = 10 \qquad x = 5$$

8. Describe how you would determine the solutions to $ax^2 + bx + c > 5$ graphically?

CLUSTER 1 How Can I Practice?

1. Complete the following table.

EQUATION OF THE FORM $y = ax^2 + bx + c$	VALUE OF a	VALUE OF b	VALUE OF c
$y = 5x^2$			
$y = \frac{1}{3}x^2 + 3x - 1$			
$y = -2x^2 + x$			

For Exercises 2–7, determine the following characteristics for each graph:

a. The direction in which the parabola opens

b. The equation of the axis of symmetry

c. The vertex

d. The y-intercept

2. $y = -2x^2 + 4$

3. $y = \frac{2}{3}x^2$

4. $f(x) = -3x^2 + 6x + 7$

5. $f(x) = 4x^2 - 4x$

6. $y = x^2 + 6x + 9$

7. $y = x^2 - x + 1$

Answers to all How Can I Practice exercises are included in the Selected Answers appendix.

For Exercises 8–11, use your graphing calculator to sketch the graph of each quadratic function, and then determine the following for each function:

a. The coordinates of the x*-intercepts (if they exist)*

b. The domain and range

c. The horizontal interval over which the function is increasing

d. The horizontal interval over which the function is decreasing

8. $y = -x^2 + 4$

9. $y = x^2 - 5x + 6$

10. $y = -3x^2 - 6x + 8$

11. $y = 0.22x^2 - 0.71x + 2$

12. Use your graphing calculator to approximate the vertex of the graph of the parabola defined by the equation $y = -2x^2 + 3x + 25$.

13. Completely factor the following polynomials.

a. $9a^5 - 27a^2$

b. $24x^3 - 6x^2$

c. $4x^3 - 16x^2 - 20x$

d. $5x^2 - 16x + 6$

e. $x^2 - 5x - 24$

f. $y^2 + 10y + 25$

14. Determine one solution of the following quadratic equations $f(x) = C$ numerically. That is, construct a table of (x, y) ordered pairs, and estimate the value of x (input) that results in the required y-value (output).

a. $5x^2 = 7$

b. $x^2 - 7x + 10 = 5$

c. $3x^2 - 5x = 2$

a.

x	1	1.1	1.2	1.3	1.4	1.5
y						

b.

x	0.5	0.6	0.7	0.8	0.9	1
y						

$x \approx 0.8$

c.

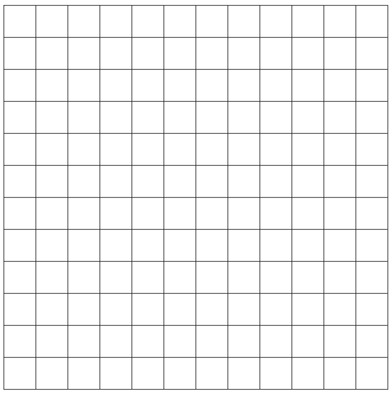
x	0	1	2	3	4	5
y						

15. Solve each of the equations from Exercise 14 using the quadratic formula. When necessary, round your solutions to the nearest tenth. Check your solutions by graphing.

16. Solve the following inequalities using a graphing approach.

a. $x^2 + 6x - 16 < 0$

b. $x^2 + 6x - 16 > 0$



17. Solve each of the following equations by factoring.

a. $4x^2 - 8x = 0$

b. $x^2 - 6 = 7x + 12$

c. $2x(x - 4) = -6$

d. $x^2 - 8x + 16 = 0$

e. $x^2 - 2x - 24 = 0$

f. $y^2 - 2y - 35 = -20$

g. $a^2 + 2a + 1 = 3a + 7$

h. $4x^2 + 4x - 3 = -3x - 1$

18. A fastball is hit straight up over home plate. The ball's height, h (in feet), from the ground is modeled by

$$h(t) = -16t^2 + 80t + 5,$$

where t is measured in seconds.

a. What is the maximum height of the ball above the ground?

b. Write an equation to determine how long it will take for the ball to reach the ground. Solve the equation using the quadratic formula. Check your solution by graphing.

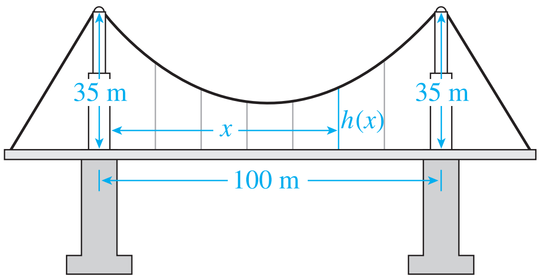
c. Write the equation you would need to determine when the ball is 101 feet above the ground.

d. Solve the equation you determined in part c algebraically to determine the time it will take for the ball to reach a height of 101 feet. Verify your results graphically.

19. A suspension bridge (shown in the accompanying figure) is 100 meters long. The bridge is supported by cables attached to the tops of towers 35 meters high at each end of the bridge. The cables hang from the towers approximately in the shape of a parabola. The height, $h(t)$ (in meters), of the cables above the surface of the roadway is modeled by

$$h(x) = 0.01x^2 - x + 35,$$

where x is the horizontal distance measured from the point where the tower and roadway meet.



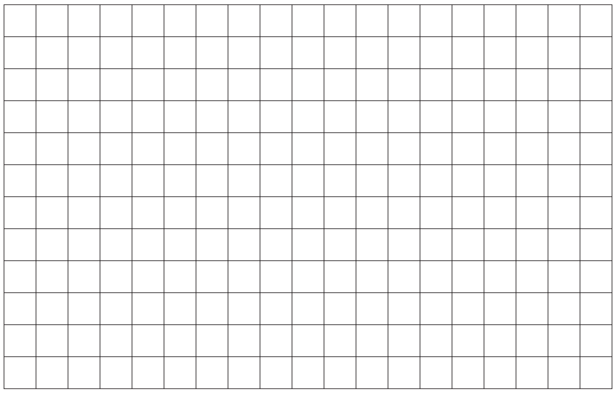
a. Use your graphing calculator to examine the height function. What is the practical domain of this function?

b. What is the minimum distance of the cables from the roadway?

20. Use the following data set to perform the tasks in parts a–f.

x	0	1	3	5	7	8
y	10	4	−18	−54	−107	−145

a. Determine an appropriate scale, and plot these points.



b. Use your graphing calculator to determine the quadratic regression equation for this data set.

c. Graph the regression equation on the same coordinate axes as the data points.

d. Compare the predicted outputs with the given outputs in the table.

e. What is the predicted output for $x = 4$ and for $x = 9$?

f. For what value of x is $y = -40$? Use the quadratic formula.

CLUSTER 2 Complex Numbers and Problem Solving Using Quadratic Functions

ACTIVITY 4.7

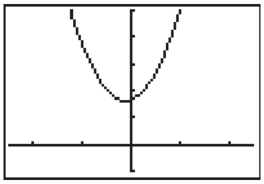
Complex Numbers

OBJECTIVES

1. Identify the imaginary unit $i = \sqrt{-1}$.
2. Identify a complex number.
3. Determine the value of the discriminant $b^2 - 4ac$.
4. Determine the types of solutions to a quadratic equation.
5. Solve a quadratic equation in the complex number system.

Recall that the solutions to $ax^2 + bx + c = 0$ correspond to the x-intercepts of the parabola having equation $y = ax^2 + bx + c$.

Do all parabolas possess x-intercepts? Consider the graph of $y = 2x^2 + x + 5$. If you graph the function in the window Xmin $= -5$, Xmax $= 5$, Ymin $= -3$, and Ymax $= 15$, the graph will resemble the following.



1. a. Based on what you know about parabolas, will the graph of $y = 2x^2 + x + 5$ have any x-intercepts? Explain.

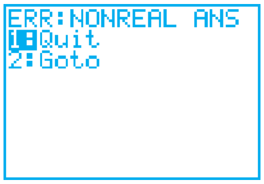
   b. What can you say about the solutions to $2x^2 + x + 5 = 0$?

Your response to Problem 1b should have been, "There are no real solutions." This is consistent with the graph. Because there are no x-intercepts for the graph of $y = 2x^2 + x + 5$, there are no real-valued solutions to the equation $2x^2 + x + 5 = 0$. Would you have discovered this if you had tried to solve the equation $2x^2 + x + 5 = 0$ algebraically using the quadratic formula? Problem 2 addresses this question.

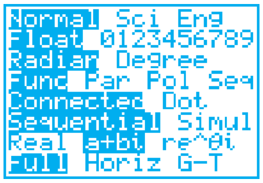
2. Use the quadratic formula to solve $2x^2 + x + 5 = 0$. Where does the solution process break down? Explain.

Complex Numbers

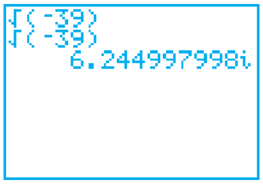
Problem 2 illustrates that the breakdown with the quadratic formula occurs when you are asked to evaluate a radical with a negative radicand. If you try to evaluate $\sqrt{-39}$ using your TI-83/TI-84 Plus, the following screen will appear.



This tells you there is no real number that is the square root of -39. Now change the MODE on the calculator from *real* to $a + bi$.



Now try to evaluate $\sqrt{-39}$ again. This time your calculator returns a value.



This is not a real number. Such a number is a **complex number** (an extension of the real numbers). The distinguishing characteristic of the complex numbers is the **imaginary unit**, $i = \sqrt{-1}$.

The quadratic formula solution to $2x^2 + x + 5 = 0$ uses the values $a = 2$, $b = 1$, and $c = 5$. The solution is

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(5)}}{2(2)} = \frac{-1 \pm \sqrt{1 - 40}}{4} = -\frac{1}{4} \pm \frac{\sqrt{-39}}{4}.$$

The problem is that you cannot evaluate $\sqrt{-39}$ in the real-number system because any real number multiplied by itself is nonnegative. Therefore, you need to introduce the imaginary unit, $i = \sqrt{-1}$, and interpret $\sqrt{-39}$ as

$$\sqrt{39(-1)} = \sqrt{39}\sqrt{-1} = \sqrt{39} \cdot i = i\sqrt{39}.$$

This approach can be used to rewrite any radical expression with a negative radicand.

EXAMPLE 1

a. $\sqrt{-16} = \sqrt{16}\sqrt{-1} = 4i$

b. $\sqrt{-25} = \sqrt{25}\sqrt{-1} = 5i$

c. $\sqrt{-53} = \sqrt{53}\sqrt{-1} = \sqrt{53}i$ or $i\sqrt{53}$

3. Rewrite each of the following in the form bi, where $i = \sqrt{-1}$.

a. $\sqrt{-26}$

b. $\sqrt{-5}$

c. $\sqrt{-64}$

d. $\sqrt{3 \cdot (-7)}$

e. $\sqrt{-18}$

f. $\sqrt{-27}$

g. $\sqrt{-\frac{3}{4}}$

h. $\sqrt{-\frac{15}{27}}$

DEFINITION

Numbers of the form bi, where b is a real number and $i = \sqrt{-1}$, are called **pure imaginary numbers**. Numbers of the form $a \pm bi$, where a and b are real and $i = \sqrt{-1}$, are called **complex numbers**. The term a is called the *real part*. The term bi is called the *imaginary part*. Imaginary numbers are complex numbers of the form $0 + bi$. Real numbers are complex numbers of the form $a + 0i$.

EXAMPLE 2

a. ***The numbers $-3i$, $\frac{2}{3}i$, and $7.4i$ are pure imaginary numbers.***

b. ***The numbers $-4 + 3i$, $\frac{1}{2} - \frac{2}{3}i$, $4i$, $-2i$, and $5 - 6i$ are complex numbers.***

Note that the set of real numbers is contained within the set of complex numbers. A real number a may be thought of as the complex number $a + 0i$.

In the sixteenth century, complex numbers were first used as solutions to polynomial equations. The notation $\sqrt{-1}$ was used during this time. Such numbers were called imaginary because their existence was not clearly understood. In 1777, Leonhard Euler introduced the notation i and wrote complex numbers in the form $a + bi$. Caspar Wessel in 1797 and Carl Friedrich Gauss in 1799 used the geometric interpretation of complex numbers as points in a plane. This made such numbers more concrete and less mysterious. Finally, in 1833, Sir William Hamilton showed that if the number i is defined to have the property

$$i^2 = -1,$$

then the set of real numbers can be extended to include numbers like $\sqrt{-1}$.

Today, complex numbers are used in a variety of applications, including chaos theory (fractals) and engineering.

Operations with Complex Numbers

The operations of addition, subtraction, and multiplication of complex numbers are demonstrated in the following example.

EXAMPLE 3

a. ***To add complex numbers, add the real parts and the imaginary parts.***
$(3 + 2i) + (5 - 7i) = 8 - 5i$

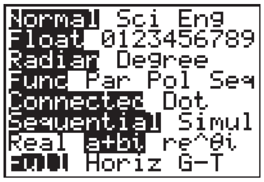
b. ***To subtract complex numbers, add the opposite.***
$(2 - 2i) - (-6 + i) = 2 - 2i + 6 - i = 8 - 3i$

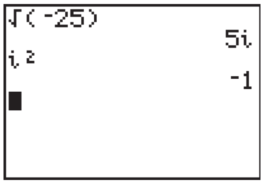
c. ***To multiply complex numbers, multiply each term of the first by each term of the second, and simplify.***

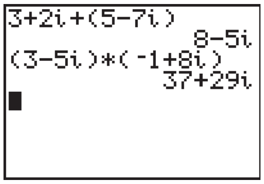
$$\begin{aligned}(3 - 5i)(-1 + 8i) &= -3 + 24i + 5i - 40i^2 \\ &= -3 + 29i + 40 \\ &= 37 + 29i\end{aligned}$$

Remember that $i \cdot i = i^2 = -1$.

The TI-83/TI-84 Plus is capable of operations with complex numbers. You first need to change the mode of the calculator from Real to $a + bi$. Note that the i key is 2nd ⌈ period.







4. Perform the following operations with complex numbers. Use your graphing calculator to check your results.

a. $(3 + 4i) + (-5 + 6i)$

b. $(5 - 7i) - (-2 + 5i)$

c. $5i(2 - 4i)$

d. $(3 - 2i)(4 + 5i)$

Discriminant

In the complex number system, every quadratic equation has at least one solution. In the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

the expression $b^2 - 4ac$ is called the **discriminant** because its value determines the number and type of solutions of a quadratic equation $ax^2 + bx + c = 0$. There are three possible cases, depending on whether the value of the discriminant is positive, zero, or negative. Problems 5, 6, and 7 investigate this relationship.

5. For each of the following quadratic functions, determine the sign of the discriminant. Then sketch a graph using your graphing calculator, and determine the number of x-intercepts.

a. $y = 2x^2 - 7x - 4$

$a =$

$b =$

$c =$

$b^2 - 4ac =$

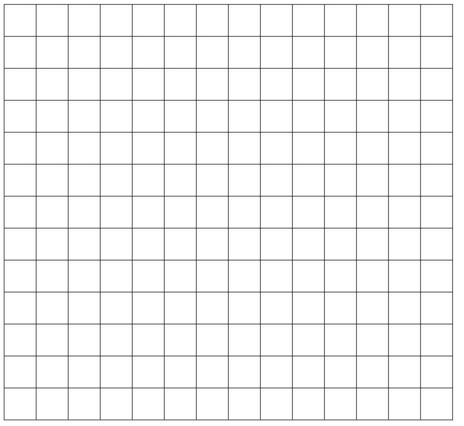
b. $y = 3x^2 + x + 1$

$a =$

$b =$

$c =$

$b^2 - 4ac =$



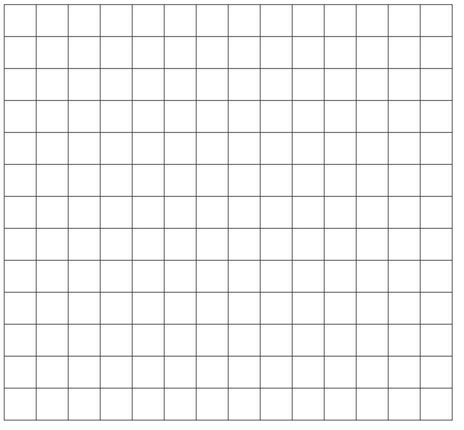
c. $y = x^2 + 2x + 1$

$a =$

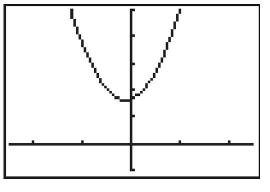
$b =$

$c =$

$b^2 - 4ac =$



Recall that the real solutions to the equation $ax^2 + bx + c = 0$ and the x-intercepts of the graph of $y = ax^2 + bx + c$ are the same. Since the graph of $y = 2x^2 + x + 5$ (see Problem 1) has no x-intercept, it has no real solutions.



But the quadratic equation $2x^2 + x + 5 = 0$ does have exactly two solutions in the complex number system. The two solutions must be complex and not real. Similarly, if the graph of a quadratic function has two x-intercepts, the solutions to the equation $ax^2 + bx + c = 0$ must be two real numbers.

6. If the graph of $y = ax^2 + bx + c$ has exactly one x-intercept, what are the number and type (real or complex) of solutions to the equation $ax^2 + bx + c = 0$?

7. Return to Problem 5. Use the value of the discriminant $b^2 - 4ac$ and the number of x-intercepts in parts a, b, and c to complete the following table.

SOLUTIONS TO $ax^2 + bx + c = 0$ IN THE COMPLEX NUMBER SYSTEM	
$b^2 - 4ac$	NUMBER AND TYPE OF SOLUTIONS
Positive	
Zero	
Negative	

8. a. Evaluate the discriminant for each of the following equations, and indicate the number and type of solutions to the equations.

i. $2x^2 - 7x - 4 = 0$

ii. $3x^2 + x + 1 = 0$

iii. $x^2 - 2x + 1 = 0$

iv. $3x^2 + 2x = -1$

b. Determine the solutions to each of the equations in part a in order to verify your results from part a.

i.

ii.

iii.

iv.

SUMMARY
ACTIVITY 4.7

1. The **imaginary unit** is the number $\sqrt{-1}$. The notation for the imaginary unit is i, where $i^2 = -1$.

2. Any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit, is called a **complex number**. The term a is called the **real part**. The term bi is called the **imaginary part**.

3. In the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

the expression $b^2 - 4ac$ is called the **discriminant**. Its value determines the number and type of solutions of a quadratic equation $ax^2 + bx + c = 0$.

4. Solutions of the quadratic equation $ax^2 + bx + c = 0$ in the complex number system are summarized in the following table.

$b^2 - 4ac$	NUMBER AND TYPE OF SOLUTIONS
Positive	2 real solutions
Zero	1 real solution
Negative	2 complex solutions

EXERCISES
ACTIVITY 4.7

In Exercises 1–8, write each of the following in the form bi, *where* i $= \sqrt{-1}$.

1. $\sqrt{-25}$
2. $\sqrt{-20}$
3. $\sqrt{-36}$
4. $\sqrt{-10}$
5. $\sqrt{-48}$
6. $\sqrt{-80}$
7. $\sqrt{-\frac{9}{16}}$
8. $\sqrt{\frac{-20}{75}}$

In Exercises 9–13, perform the operations, and express your answer in the form $a + bi$. Use your graphing calculator to verify the results.

9. $(2 + 8i) + (-7 + 2i)$
10. $(5 - 3i) - (2 - 6i)$
11. $5i + (3 - 7i)$
12. $3i(-2 + 4i)$
13. $(4 - 3i)(1 + 2i)$

14. Complex numbers are used in electronics to describe the current in an electric circuit. In an alternating current, the resistance, R, in ohms, is the measure of how much the circuit resists (or impedes) the flow of current through it. The resistance, R, is related to the voltage, V, and current, I, by Ohm's Law:

$$V = IR$$

   a. If $I = (0.3 + 2i)$ amperes and $R = (0.5 - 3i)$ ohms, determine the voltage, V.

   b. If $I = (2 - 3i)$ amperes and $R = (3 + 5i)$ ohms, determine the voltage, V.

In Exercises 15–18, solve the quadratic equations in the complex number system using the quadratic formula. Verify your real solutions graphically. Verify that no real solutions mean no x*-intercepts*

15. $3x^2 - 2x + 7 = 0$
16. $x^2 + x = 3$
17. $2x^2 + 5x = 7$
18. $0.5x^2 - x + 3 = 0$

Exercise numbers appearing in color are answered in the Selected Answers appendix.

In Exercises 19–24, determine the number and type of solutions of each equation by examining the discriminant.

19. $2x^2 + 3x - 5 = 0$

20. $6x^2 + x + 5 = 0$

21. $4x^2 - 4x + 1 = 0$

22. $9x^2 + 6x + 1 = 0$

23. $12x^2 = 4x - 3$

24. $3x^2 = 5x + 7$

ACTIVITY 4.8

Airfare

OBJECTIVES

1. Build a quadratic model as a product of linear models.
2. Analyze a model contextually.

You are an assistant to the president of a small commuter airline. You have been asked to develop a strategy for increasing the revenue from your primary route. The current fare for this route is \$160 per person, and each flight averages 40 passengers.

1. What is the average revenue from each flight?

A recent marketing analysis suggests that each \$2 increase in fare will result in one less passenger per flight and that each \$2 reduction in fare will produce one additional passenger per flight. Your job is to use this information to set an airfare that maximizes the revenue from these flights.

You might first decide to adjust the fare up or down by \$2 increments and then determine the projected revenues. Do this by completing the accompanying table, where each positive value in the first column represents the number of upward fare adjustments and each negative value represents the number of downward fare adjustments. (In other words, −3 means you have decreased the fare by \$6; 2 means you have increased the fare by \$4.)

Fare Play

NUMBER AND DIRECTION OF FARE ADJUSTMENTS	FARE (\$)	NUMBER OF PASSENGERS	ANTICIPATED REVENUE (\$)
−3			
−2			
−1			
0			
1			
2			
3			

2. a. Determine an equation for the airfare, $F(x)$, as a function of x, where x represents the number and direction of the \$2 fare adjustments.

   b. Determine the value of $F(4)$. What is its practical meaning?

   c. Determine the value of $F(-5)$. What is its practical meaning?

3. a. Determine an equation for the average number of passengers, $P(x)$, as a function of x.

b. Determine the value of $P(4)$. What is its practical meaning?

c. Determine the value of $P(-5)$. What is its practical meaning?

4. a. The revenue for the flight is the product of the airfare and the number of passengers. Determine an equation that represents the revenue, $R(x)$, as a function of x.

b. Determine the value of $R(2)$. What is its practical meaning?

c. What is the largest value of x for which the revenue function has practical meaning?

d. What is the smallest value of x for which the revenue function has practical meaning?

5. Use your algebraic model for the revenue function in Problem 4a to answer the following.

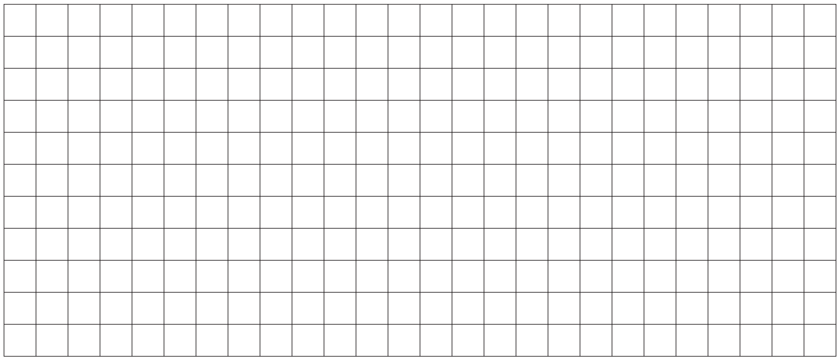
a. What type of function describes the revenue?

b. What is its vertical intercept? What does the intercept signify in terms of the flight revenue?

c. Where is the vertex of this function located? What is the significance of each coordinate of the vertex in terms of the flight revenue?

d. What are the x-intercepts, and how are they significant in terms of the revenue?

6. Set the window of your graphing calculator so that you can see all the features of the revenue function that you examined algebraically in Problem 5. Graphically confirm your results from Problem 5.



7. From your graph, determine what airfares will result in revenue greater than \$6000. Round x to the nearest dollar.

8. Summarize the strategy that you will recommend to your boss for maximizing the revenue from this flight.

EXERCISES
ACTIVITY 4.8

On your job as assistant to the foreman at a construction site, you have been asked to build an enclosure to store valuable materials and equipment. You have a single 500-foot roll of 10-foot-high heavy chain-link fencing. Your boss would like as much storage area as possible. Your job is to figure out the location of the corner posts for this enclosure.

1. a. Assuming that the enclosure will be rectangular and that you will use all of the fencing, make a table showing the dimensions (width and length) of some possible enclosures, with the resulting areas.

WIDTH	25	50	75	100	125	150	175	200	225
LENGTH									
AREA									

b. From your table, choose the dimensions that would maximize the area.

c. Letting w represent the width, express the length as a function of w.

Exercise numbers appearing in color are answered in the Selected Answers appendix.

d. Write an equation for the area of the rectangular enclosure as a function of w.

e. With your graphing calculator, graph the area function and determine the maximum possible area of the enclosure. What dimensions yield this largest enclosure?

f. Determine the w-intercepts of the graph. Why must you reject these values in this situation?

g. What are the practical domain and range of this area function?

2. The space for an editorial in a college newspaper is in the shape of a rectangle. The height of the rectangle is 2 inches more than 3 times the base. If the area of the rectangle is 56 square inches, determine its dimensions.

3. A boat leaves Virginia Beach and sails due east for 1 hour and then due north for 2 hours. The boat is then 10 miles from its departure point. If the average rate sailing east was 2 miles per hour faster than the average rate sailing north, determine the rate of speed in each direction.

PROJECT ACTIVITY 4.9

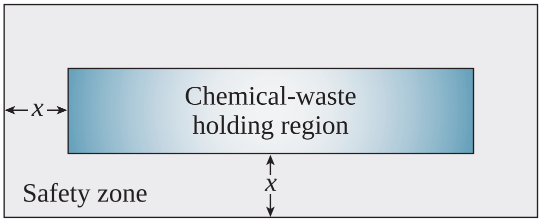
Chemical-Waste Holding Region

OBJECTIVE

1. Solving problems using quadratic functions.

Your architecture firm is designing a rectangular chemical-waste holding region for a local chemical manufacturing company. The holding area is to be located on a rectangular lot that is 200 meters wide and 80 meters long. Federal regulations require that the holding region be 10,000 square meters in area. A safety zone of uniform width around the perimeter of the holding area is also required.

The questions raised here are, can these federal regulations be met if the chemical-waste holding region is constructed on the available rectangular lot? And if so, what would be the width of the safety zone?



1. Let x represent the width of the safety zone.

   a. Because the width of the lot is 200 meters, what is the width of the holding region in terms of x?

   b. Write an expression for the length of the holding region in terms of x.

2. What is the practical domain of the variable x in this situation? That is, what are the smallest and the largest values of x for which the problem makes sense?

3. The width and length of the waste-holding region are both expressed in terms of x. Therefore, the area, A, of the holding region is a function of x. Using the results from Problem 1, write an equation that expresses A as a function of x.

4. In this situation, the area of the holding region is required to be 10,000 square meters.

   a. Write an equation to determine the value of x that gives an output of 10,000. Solve the resulting equation using the quadratic formula. Approximate your solutions to the nearest hundredth.

   b. Verify your results in part a graphically.

5. a. Are both solutions to the equation in Problem 4 practical in this situation? Explain.

   b. What are the dimensions of the chemical-waste holding region?

   c. What is the width of the safety zone?

6. If federal regulations require a safety zone 15 meters wide around the perimeter of the holding region, can the holding region be built on the given rectangular lot? Explain.

7. Consider the holding region area function developed in Problem 3.

   a. What are the horizontal intercepts of the graph of this function? What practical meaning do these intercepts have in this situation?

   b. What is the vertex of the graph (a parabola) of the area function? Does the vertex have any significance in this situation?

CLUSTER 2 What Have I Learned?

1. Which of the following statements are true? In each case, justify your decision.

a. $3 + 2i$ is a pure imaginary number.

b. $\sqrt{-7}$ is a complex number.

c. 0 is a complex number.

2. a. Describe the relationship between the x-intercepts (if they exist) of the graph of $y = ax^2 + bx + c$ and the solutions to the equation $ax^2 + bx + c = 0$.

b. Describe the relationship between the x-intercepts (if they exist) of the graph of $y = ax^2 + bx + c$ and the discriminant $b^2 - 4ac$.

3. Consider the quadratic equation $ax^2 + bx + c = 0$. If the quadratic expression $ax^2 + bx + c$ is factorable, what can you say about the sign of the discriminant, $b^2 - 4ac$? Is it positive, negative, or zero? Explain.

4. For what values of c are the solutions to $2x^2 - 5x + c = 0$ imaginary?

5. For what values of k does $x^2 - kx + k = 0$ have only one solution? (*Hint:* Examine the discriminant.)

CLUSTER 2 How Can I Practice?

Write each of the following in the form bi, where $i = \sqrt{-1}$.

1. $\sqrt{-49}$
2. $\sqrt{-45}$
3. $\sqrt{-121}$
4. $\sqrt{-15}$
5. $\sqrt{-112}$
6. $\sqrt{-125}$
7. $\sqrt{-\frac{16}{25}}$
8. $\sqrt{\frac{-24}{42}}$

Perform the following operations, and express your answer in the form a + bi. *Use your graphing calculator to verify the results.*

9. $(7 + 5i) + (-3 - 2i)$
10. $(3 - 3i) - (8 - 9i)$
11. $3i + (6 - 7i)$
12. $-3i(8 - 4i)$
13. $(3 - 4i)(-1 + 2i)$

14. In parts i-iv, perform the following tasks.
    - **a.** Identify the values of a, b, and c in $ax^2 + bx + c = 0$.
    - **b.** Determine the type of solution by examining the sign of the discriminant.
    - **c.** Solve the given equation using the quadratic formula. If necessary, round your solutions to the nearest hundredth.
    - **d.** Check your solutions by graphing as well as by substitution.

    i. $3x^2 - x = 7$ **ii.** $x^2 - 4x + 10 = 0$

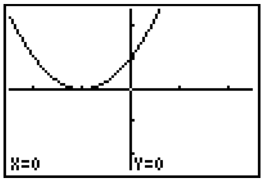
Answers to all How Can I Practice exercises are included in the Selected Answers appendix.

iii. $2x^2 - 3x = 2x + 3$

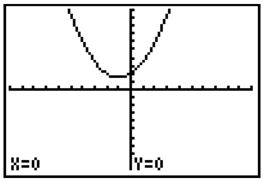
iv. $3x(3x - 2) + 1 = 0$

15. Each of the following graphs represents a quadratic function. For each graph, determine if the discriminant is positive, negative, or zero. Explain your decision.

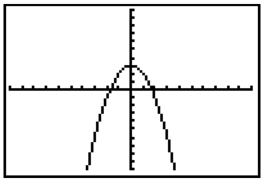
a.



b.



c.



i. graph a

ii. graph b

iii. graph c

16. You are planning to rebuild your garage, which currently measures 20 feet by 30 feet. You want the new garage to be larger than the existing one, but in the same proportions as the original. If you increase the width by x feet, you will have to increase the length by $\frac{3}{2}x$ feet.

a. Write the floor area of the new garage in terms of the variable x.

b. Write a quadratic function for the new floor area using your expression from part b.

c. Which part of the function you determined in part b represents the increase in floor area from the original garage?

d. Use the result of part c to determine the total increase in the floor area if you extend the width by 2 feet and retain the original proportions.

e. You have determined that you want to expand the floor by 264 square feet. Write the equation you need to solve to determine the new dimensions of the garage that are proportional to the original.

f. Solve the equation in part e algebraically. Verify your solution graphically.

g. Are all of your solutions in part g relevant to the garage situation? Explain.

CLUSTER 3 Curve Fitting and Higher-Order Polynomial Functions

ACTIVITY 4.10

The Power of Power Functions

OBJECTIVES

1. Identify a direct variation function.
2. Determine the constant of variation.
3. Identify the properties of graphs of power functions defined by $y = kx^n$, where n is a positive integer, $k \neq 0$.

You are traveling in a hot air balloon when suddenly your binoculars drop from the edge of the balloon's basket. At that moment, the balloon is maintaining a constant height of 500 feet. The distance of the binoculars from the edge of the basket is modeled by

$$s = 16t^2.$$

The following table gives the distance, s (in feet), from the drop point at various times, t (in seconds).

t, TIME	0	1	2	3	4
s, DISTANCE	0	16	64	144	256

As the input values (units of time) increase, the corresponding output values (units of distance) increase. Let us look more closely at how this increase takes place.

Because $s = 16t^2$, you can say that the output, s, varies directly as the square of the input, t. Therefore, as t doubles in value from 1 to 2 or from 2 to 4, the corresponding output values become 4 times as large: increasing from 16 to 64 or 64 to 256.

1. a. As t triples from 1 to 3, the corresponding s-values become ________ times as large.

b. In general, if y varies directly as the square of x, then when x becomes n times as large, the corresponding y-values become ________ times as large.

The volume, V, of a sphere is given by $V = \frac{4}{3}\pi r^3$. In this situation, you can say that the output, V, varies directly as the cube of the radius, r.

2. a. Complete the following table. Leave your answers for V in terms of π.

r	1	2	3	4	8
V					

Note that as r-values double from 2 to 4 the corresponding V-values increase from $\frac{32}{3}\pi$ to $\frac{256}{3}\pi$.

b. As r doubles from 2 to 4 or from 4 to 8, the corresponding V-values become ________ times as large.

c. In general, if y varies directly as the cube of x, then when x becomes n times as large, the corresponding y-values become ________ times as large.

d. Sketch a graph of the volume function. What is the practical domain of this function?

DEFINITION

The equation

$$y = kx^n,$$

where $k \neq 0$ and n is a positive integer, defines a **direct variation** function in which y varies directly as x^n. The constant, k, is called the **constant of variation**.

EXAMPLE 1 ***The constant of variation, k, in the free-falling object situation defined by $s = 16t^2$ is 16.***

3. What is the constant of variation for the direct variation function defined by $V = \frac{4}{3}\pi r^3$?

In the falling binocular situation, you are given the direct variation equation. Suppose you only know that the distance, s, varies directly as the square of t and one data pair. Are you able to determine the direct variation equation? Example 2 demonstrates the process.

EXAMPLE 2 ***Let s vary directly as the square of t. If $s = 64$ when $t = 2$, determine the direct variation equation.***

SOLUTION

Because s varies directly as the square of t, you have

$$s = kt^2,$$

where k is the constant of variation. Substituting 64 for s and 2 for t, you have

$$64 = k(2)^2 \quad \text{or} \quad 64 = 4k \quad \text{or} \quad k = 16.$$

Therefore, the direct variation equation is

$$s = 16t^2.$$

4. For each table, determine the pattern and complete the table. Then write a direct variation equation for each table.

a. y varies directly as x.

x	1	2	4	8	12
y		12			

b. y varies directly as x^3.

x	1	2	3
y		32	

c. y varies directly as x.

x	1	2	3	4	5
y			3		

5. The length, L, of skid distance left by a car varies directly as the square of the initial velocity, v (in miles per hour), of the car.

a. Write a general equation for L as a function of v. Let k represent the constant of variation.

b. Suppose a car traveling at 40 miles per hour leaves skid distance of 60 feet. Use this information to determine the value of k.

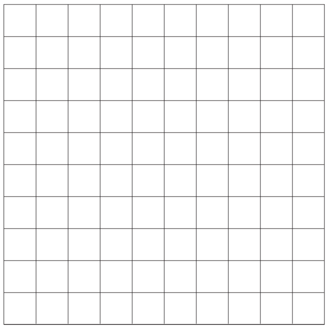
c. Use the function to determine the length of the skid distance left by the car traveling at 60 miles per hour.

Power Functions

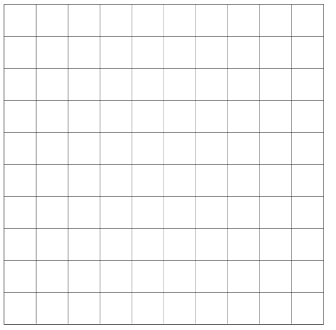
The direct variation functions that have equations of the form $y = kx^n$, where n is a positive integer and $k \neq 0$, are also called **power functions**. The graphs of this family of functions are very interesting and are useful in problem solving.

6. Sketch a graph of each of the following power functions. Use your graphing calculator to verify the graph.

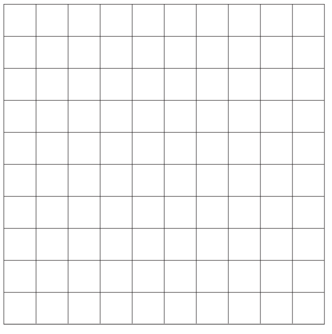
a. $y = x$



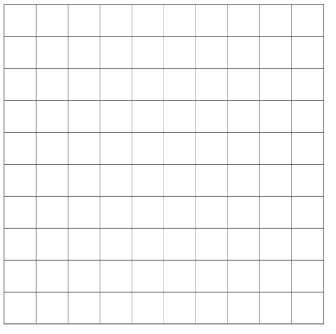
b. $y = x^2$



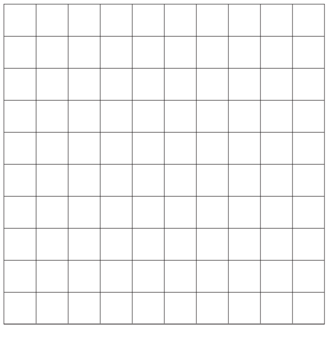
c. $y = x^3$



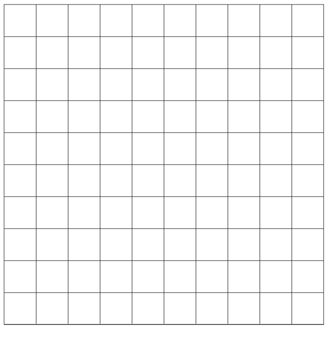
d. $y = x^4$



e. $y = x^5$



f. $y = x^6$



7. Each graph in Problem 6 has an equation of the form $y = x^n$, where n is a positive integer.

a. What is the basic shape of the graph if

i. n is even?

ii. n is odd?

b. If n is even, what happens to the graph as n gets larger in value?

c. If n is odd, is the function increasing or decreasing?

8. Use the patterns from Problem 7 in combination with graphing techniques you have learned previously to sketch a graph of each of the following without using a graphing calculator.

a. $y = x^2 + 1$

b. $y = -2x^4$

c. $y = 3x^8 + 1$

d. $y = -2x^5$

e. $y = x^{10}$

f. $y = 5x^3 + 2$

SUMMARY
ACTIVITY 4.10

1. The equation $y = kx^n$, where $k \neq 0$ and n is a positive integer, defines a **direct variation function**. The constant, k, is called the **constant of variation**.

2. The direct variation functions that have equations of the form $y = kx^n$, where n is a positive integer, are also called **power functions**.

   a. Power functions in which n is even resemble parabolas. As n increases in value, the graph flattens near the vertex.

   b. Power functions in which n is odd resemble the graph of $y = kx^3$. If k is positive, the graph is increasing. If k is negative, the graph is decreasing.

EXERCISES
ACTIVITY 4.10

1. For each table, determine the pattern and complete the table. Then write a direct variation equation for each table.

   a. y varies directly as x.

x	$\frac{1}{4}$	1	4	8
y		8		

   b. y varies directly as x^3.

x	$\frac{1}{2}$	1	3	6
y		1		

2. The area, A, of a circle is given by the function $A = \pi r^2$, where r is the radius of the circle.

   a. Does the area vary directly as the radius? Explain.

   b. What is the constant of variation k?

Exercise numbers appearing in color are answered in the Selected Answers appendix.

3. Assume that y varies directly as the square of x, and that when $x = 2$, $y = 12$. Determine y when $x = 8$.

4. The distance, d, that you drive at a constant speed varies directly as the time, t, that you drive. If you can drive 150 miles in 3 hours, how far can you drive in 6 hours?

5. The number of meters, d, that a skydiver falls before her parachute opens varies directly as the square of the time, t, that she is in the air. A skydiver falls 20 meters in 2 seconds. How far will she fall in 2.5 seconds?

In Exercises 6–10, sketch a graph of the given power function. Verify your graphs using your graphing calculator.

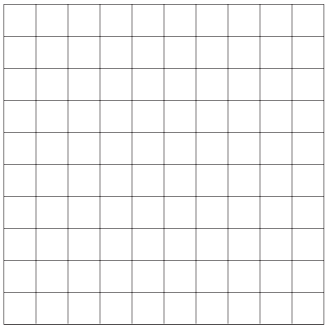
6. $y = -3x^2$

7. $y = x^4 + 1$

8. $y = -2x^5$

9. $f(x) = x^6$

10. $g(x) = 3x^3 - 3$



11. Determine the x-interval over which the function $f(x) = \frac{1}{2}x^4$ is increasing.

12. Does the function $g(x) = -\frac{1}{2}x^6$ have a maximum or a minimum point? Explain.

13. For $x > 1$, is the graph of $y = x^2$ rising faster or slower than the graph of $y = x^3$? Explain.

14. Is the graph of $y = \frac{3}{2}x^4$ wider or narrower than the graph of $y = x^4$?

15. How are the graphs of $y = -2x^3$ and $y = 2x^3 + 1$ different? How are the graphs similar?

16. a. For $x > 0$, is the graph of $y = x^2$ rising faster or slower than the graph of $y = 2^x$? Explain.

b. For $x > 0$, is the graph of $y = x^5$ rising faster or slower than the graph of $y = 2^x$? Explain.

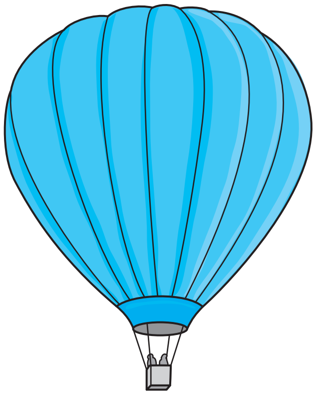
ACTIVITY 4.11

Hot Air Balloon

OBJECTIVES

1. Identify equations that define polynomial functions.
2. Determine the degree of a polynomial function.
3. Determine the intercepts of the graph of a polynomial function.
4. Identify the properties of the graphs of polynomial functions.

Returning to the hot air balloon situation, you are relieved because the binoculars you dropped did not strike anyone on the ground. As you continue your ride in the balloon, you enter into a conversation with the pilot as to why the balloon rises. He says that because the hot air is lighter than the surrounding air, the balloon will rise if the upward lift (force) provided by the hot air is great enough to overcome the downward force on the balloon—its weight.



The upward force varies directly as the volume of the spherical balloon. Because $V = \frac{4}{3}\pi r^3$, the upward force varies directly as the cube of the radius.

1. Write a general equation that represents the upward force, denoted by U, as a function of the radius, r, of the sphere. Let k_1 represent the constant of variation.

The total weight of the balloon itself, the basket, and the heat generator represent the downward force. The weight of the basket and heat generator can be considered a constant, represented by C.

The weight of the balloon material varies directly as the surface area of the balloon when inflated. Because $S = 4\pi r^2$, you can say that the weight of the balloon varies directly as the square of the radius, r.

2. a. Write a general equation that represents the weight of the balloon as a function of the radius, r. Let k_2 represent the constant of variation.

b. The downward force, denoted by D, is the sum of the weight of the balloon material, denoted by W, and the weight of the basket and heat generator, denoted by C. Write an equation that expresses D as a function of r and C.

The total force, F, acting on the balloon is given by

$$F = U - D.$$

Substituting your results from Problem 1 and Problem 2b into this equation gives you

$$F = k_1r^3 - (k_2r^2 + C)$$

or

$$F(r) = k_1r^3 - k_2r^2 - C.$$

This equation expresses the total force acting on the hot air balloon as a function of r. Because the largest exponent on the input variable r is 3, this function is a **third-degree polynomial function** or a **cubic function**.

Polynomial Functions

Polynomial functions are defined by equations of the form

output = polynomial expression involving the input.

The largest exponent on the input variable, n, is called the **degree** of the function.

If x and y represent the input and output, respectively, then y must equal sums and differences of terms of the form ax^n, where n is a positive integer. The following example gives several types of polynomial functions.

EXAMPLE 1 *Examples of polynomial functions are listed in the following table.*

POLYNOMIAL FUNCTION	DEGREE OF THE POLYNOMIAL	NAME
$y = 3x - 2$	1	linear
$y = 2x^2 + 3x - 4$	2	quadratic
$y = 3x^3 - x - 4$	3	cubic
$y = 0.2x^4 - 2x^2 + 7x - 1$	4	quartic
$y = -2x^5 + 3x^4 + 2x - 6$	5	quintic

Note that the cubic function defined by $y = 3x^3 - x - 4$ can be written as $y = 3x^3 + 0x^2 - x - 4$.

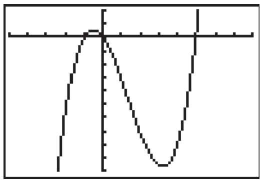
Polynomial Functions of Degree 3 or Greater

You have already studied polynomial functions of degree 1 (linear) and degree 2 (quadratic). What are some of the properties and shapes of the graphs of polynomial functions having degree 3 or greater?

3. a. What is the domain of the cubic function defined by $y = 2x^3 - 8x^2 - 10x$?

   b. Determine the y-intercept of the graph of the cubic function in part a.

c. Use your graphing calculator to sketch a graph. Use the window Xmin = −5, Xmax = 8, Ymin = −50, Ymax = 10, Yscl = 5. The graph should appear as follows.



d. Write an equation to determine the x-intercepts of the graph.

e. Solve the equation in part d using a graphing approach. Use the zero option in the CALC menu of your graphing calculator.

Can the equation $2x^3 - 8x^2 - 10x = 0$ be solved using an algebraic approach? Yes! The solution process is demonstrated in the following example.

EXAMPLE 2 ***Solve the equation*** $2x^3 - 8x^2 - 10x = 0$ ***using an algebraic approach.***

SOLUTION

Step 1. The equation is already written in the form *polynomial expression* = 0.

Step 2. Completely factor the cubic expression on the left side of the equation.

$$2x^3 - 8x^2 - 10x = 0 \quad \text{Factor out the GCF.}$$
$$2x(x^2 - 4x - 5) = 0 \quad \text{Factor the trinomial.}$$
$$2x(x + 1)(x - 5) = 0$$

Step 3. Apply the zero-product principle, and set each factor equal to zero. Solve each resulting equation.

$$2x(x + 1)(x - 5) = 0$$

$2x = 0$ | $x + 1 = 0$ | $x - 5 = 0$

$x = 0$ | $x = -1$ | $x = 5$

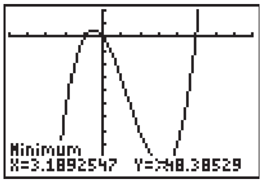
4. Using an algebraic approach (factoring), determine the horizontal intercepts of each of the following polynomial functions. Verify your results using a graphing approach.

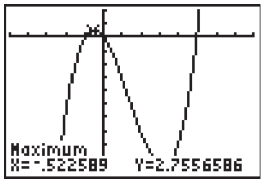
a. $y = 2x^3 + 5x^2 - 12x$

b. $f(x) = x^2(x^2 - 5) + 4$

c. $g(x) = 2x^5 - 18x^3$

5. a. Returning to the cubic function defined by $y = 2x^3 - 8x^2 - 10x$, the graph shows a high point (maximum point) in quadrant II and a low point (minimum point) in quadrant IV. Note that the maximum and minimum points occur at turning points of the graph. Use the maximum and minimum options in the CALC menu of your graphing calculator to approximate the coordinates of each of these points. Your screens should appear as follows.





b. Using the results from part a, determine the interval along the x-axis where the function is

i. increasing.

ii. decreasing.

6. Using your graphing calculator, plot the following third-degree polynomials. Be careful of your choice of windows.

a. $f(x) = x^3$

b. $i(x) = 3x^3 - x - 4$

c. $g(x) = 0.2x^3 - 2x + 7$

d. $j(x) = -5x^3 + 1$

e. $h(x) = -0.6x^3 + 2x^2 - 1$

f. Use the graphs gathered in parts a–e to write a few sentences comparing and contrasting the graph of the general quadratic equation, $y = ax^2 + bx + c$, $a \neq 0$, and the general cubic equation, $y = ax^3 + bx^2 + cx + d, a \neq 0$. Include comments on turning points and general trends, such as increasing and decreasing intervals.

7. Using your graphing calculator, plot the following fourth-degree polynomials. Be careful of your choice of windows.

a. $f(x) = x^4$

b. $i(x) = 3x^4 - x - 4$

c. $g(x) = 0.2x^4 - 2x^2 + 7x - 1$

d. $j(x) = -5x^4 + 1$

e. $h(x) = -0.6x^4 + 2x^3 - x + 1$

f. Use the graphs gathered in parts a–e to write a few sentences comparing and contrasting the graph of the general quadratic equation, $y = ax^2 + bx + c$, $a \neq 0$, and the general quartic equation, $y = ax^4 + bx^3 + cx^2 + dx + e$, $a \neq 0$. Include comments on turning points and general trends, such as increasing and decreasing intervals.

SUMMARY
ACTIVITY 4.11

1. **Polynomial functions** are defined by equations of the form

   output = polynomial expression involving the input.

2. The largest exponent on the input variable, n, is called the **degree** of the function.
3. Polynomial functions are continuous, with the domain of all real numbers.
4. **Polynomial equations** of the form

   polynomial expression = 0

   can be solved graphically by locating the x-intercepts of the graph of the function defined by

   y = polynomial expression

   or algebraically using factoring (if possible) and the zero-product principle.

EXERCISES
ACTIVITY 4.11

In Exercises 1–3, determine the x*-intercept(s) of the graphs of each polynomial function using an algebraic approach (factoring). Verify your answer using your graphing calculator.*

1. $f(x) = x^3 + 3x^2 + 2x$

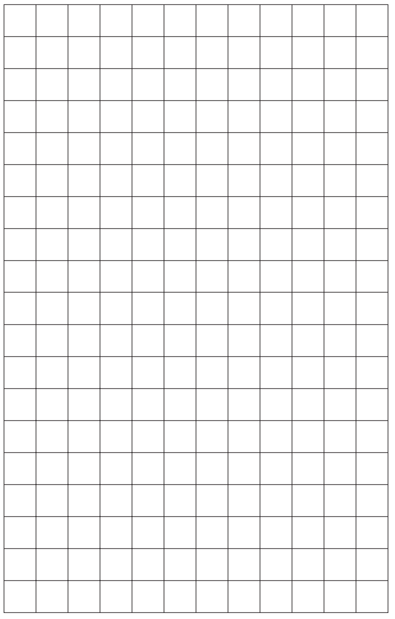
2. $g(x) = 2x^2(x^2 - 4)$

3. $h(x) = x^4 - 13x^2 + 36$

Exercise numbers appearing in color are answered in the Selected Answers appendix.

4. Determine the vertical intercept of each of the functions in Exercises 1–3.

5. Sketch a graph of the function $f(x) = x^4 - 6x^3 + 8x^2 + 1$. Does the function have a maximum and/or a minimum point(s)? If yes, determine these points.

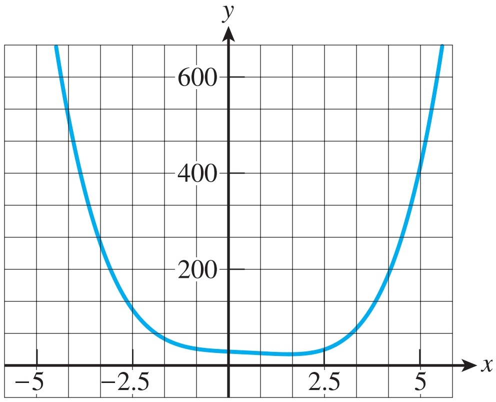


6. Describe any symmetry of the graph of $y = x^4 - 4x^2 - 2$.

7. As the value of the input variable x increases without bound (say 10 to 100 to 1000 and so on), do the output values decrease without bound for the function $y = x^3 + 3x^2 - x - 4$? Use a graph of the function to help answer the question.

8. Is the graph of $y = -1x^3 - x + 3$ increasing or decreasing?

9. Consider the following graph of $y = f(x)$.

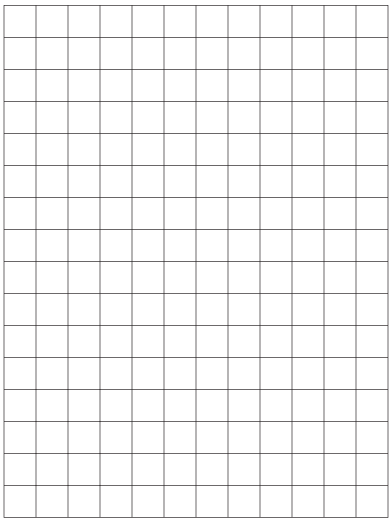


a. As x decreases without bound, the corresponding y-values ____________

b. Is the function f increasing or decreasing for $-2 < x < 2$?

c. How many turning points does the curve have?

10. Sketch a graph of $y = (x - 2)^4$. What is the relationship between the minimum point of the graph and its horizontal intercept?



ACTIVITY 4.12

Stolen Bases

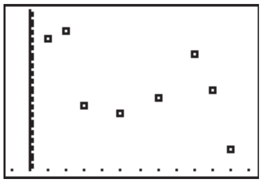
OBJECTIVES

1. Determine the regression equation of a polynomial function that best fits the data.
2. Distinguish between a discrete function and a continuous function.

There is probably no sport in which more statistics are gathered and analyzed by nonparticipants than baseball. Numbers are generated in all kinds of offensive and defensive categories. Some of these categories provide numbers that can be modeled quite nicely by polynomial functions. Consider the following data on stolen base leaders in the National League in recent years.

YEAR	1991	1992	1993	1995	1997	1999	2000	2001
BASES STOLEN	76	78	58	56	60	72	62	46

1. Let x represent the number of years since 1990. Let y represent the number of stolen bases by the National League individual champion. Plot these points on your graphing calculator. Your scatterplot should resemble the following.



2. a. Using your graphing calculator, determine the regression equations of the first-, second-, third-, and fourth-degree curves of best fit.

   First degree (linear):

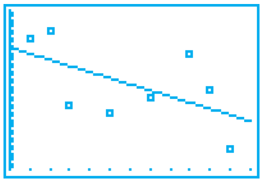
   Second degree (quadratic):

   Third degree (cubic):

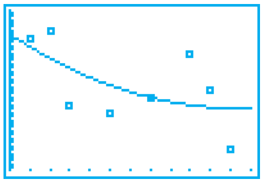
   Fourth degree (quartic):

   b. Use your calculator to fit each of your models from part a to your scatterplot. Your graphs should resemble the following.

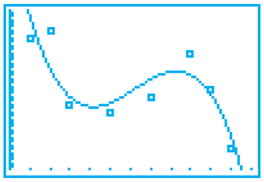
   The linear model:



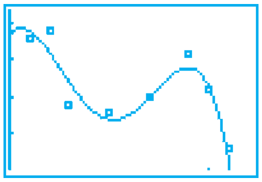
   The quadratic model:



The cubic model:



The quartic model:



c. Which of these curves seems to best represent the data? Explain.

Discrete Versus Continuous Functions

Functions are **discrete** if they are defined only at isolated input values and do not make sense for input values between those values. Functions are **continuous** if they are defined for all input values, with the possible exception of a few isolated values.

EXAMPLE 1 ***A company produces ceramic dolls. The monthly production levels are given in the following table.***

MONTH	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
NUMBER OF DOLLS (in thousands)	8	15	20	25	28	30	28	25	20	15

If month is the input and number of ceramic dolls is the output, this is a discrete function.

The data in Example 1 can be modeled by $y = -0.86x^2 + 8.58x + 7.38$, where x represents the number of months since March. This quadratic function is continuous because any real number can be used as an input. However, when it is used to model this situation, the only input values that make sense are the integers 3, 4, 5, 6, ... , 12 because the production function is discrete.

3. **a.** What is the practical domain of the introductory stolen base situation?

b. Would you consider this a discrete situation (consisting of separate, isolated points) or a continuous situation? Explain.

c. What is the practical range of this problem?

4. a. In 1996, Eric Young led the National League in stolen bases with 53. Is this result consistent with the curve you chose as the best model for the given data? Explain.

b. In 1998, Tony Womack led the National League in stolen bases with 58. Is this result consistent with the curve you chose as the best model for the given data? Explain.

c. Include these two data points on the lists in your graphing calculator, and recalculate your curve of best fit. Describe the changes.

d. Using the function generated in part c, predict the number of stolen bases the National League leader will have in 2005. Is this a realistic prediction? Explain.

SUMMARY
ACTIVITY 4.12

1. The graphing calculator can be used to model data with cubic and quartic polynomial functions as well as linear and quadratic polynomial functions.
2. Functions are **discrete** if they are defined only at isolated input values and do not make sense for input values between those values.
3. Functions are **continuous** if they are defined for all input values, with the possible exception of a few isolated values.
4. Care must be used when modeling discrete situations with continuously defined functions such as polynomials.

EXERCISES
ACTIVITY 4.12

1. According to the U.S. Department of Transportation, the following table summarizes the average number of gallons of fuel consumed per vehicle in the United States from 1960 to 2004.

YEAR	1960	1970	1980	1990	1995	2000	2002	2004
AVERAGE GALLONS OF FUEL CONSUMED PER VEHICLE, g	668	760	576	520	530	547	555	557

Exercise numbers appearing in color are answered in the Selected Answers appendix.

a. Sketch a scatterplot of the data. Let t represent the number of years since 1960. Does the data appear to be linear? Explain.

b. Use your graphing calculator to determine and plot the quadratic, cubic, and quartic regression equations for this data.

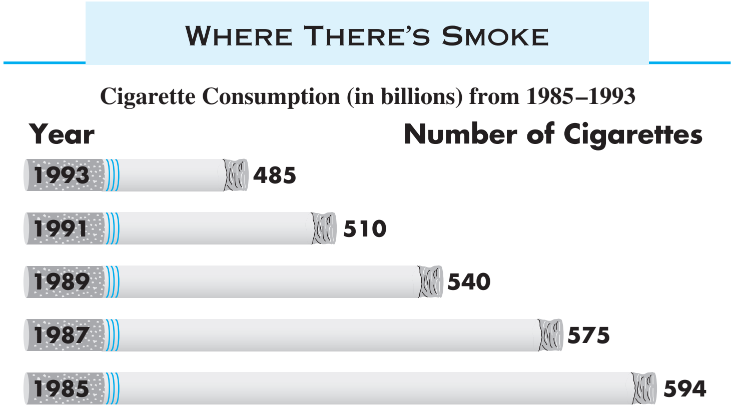
2. a. Using the results from Exercise 1b, select the equation that best models the average number of gallons of fuel consumed per vehicle, g, as a function of the number of years since 1960, t.

b. What is the practical domain of this function?

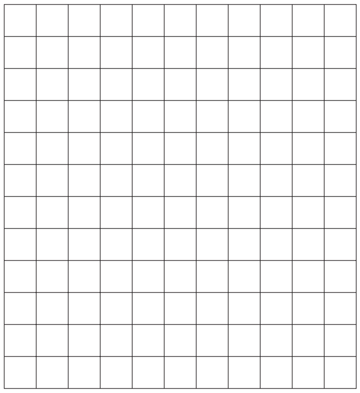
c. What is the practical range of this function? Explain.

d. Use your equation in part a to estimate the average number of gallons of fuel consumed per vehicle in 1955, 1985, and 2008. In which estimates do you have the most confidence? Explain.

3. The following graphic gives the annual consumption of cigarettes (in billions) in the United States for specific years.



a. Let $t = 0$ correspond to the year 1985. Sketch a scatterplot of the data.



b. Determine a linear model (equation) for the data.

c. Determine a quadratic model (equation) for the data.

d. Which model best represents the data? Explain.

e. Use each model to predict the consumption of cigarettes in the year 2005.

f. How confident are you in your predictions? Explain.

PROJECT ACTIVITY 4.13

Finding the Maximum Volume

OBJECTIVE

1. Problem solving using polynomial functions.

You have an 8.5-by-11-inch piece of cardboard that you want to make into an open box (no top). To make the open box, you must cut squares of equal sizes from all four corners of the cardboard and then fold up the sides. Your goal is to obtain the maximum volume of the box.

1. Before doing any cutting or calculating, estimate what size square you think needs to be cut out to make a box with the largest (maximum) volume. Each member of your group should come up with a different estimate.

2. Sketch a diagram that represents the problem.

3. Cut a square from each corner of your cardboard. Measure as carefully as you can to the size you chose in Problem 1. Fold and tape the sides to form your box. Measure the dimensions, and calculate the volume of your box.

4. Make a table showing the size of the cut square, the other two dimensions of the box, and the volume of the box. Enter the data for your box in the first column, followed by the data from other persons in the class.

SQUARE (Height, in.)						
BOX LENGTH (in.)						
BOX WIDTH (in.)						
VOLUME (in.)						

5. Use several other sizes for the cut square, calculate the resulting volume, and enter it in the table above.

6. From the data in your table, what do you think is the maximum volume? What size of cut square results in this largest box?

7. Let x represent the length of each side of the cut squares. Therefore, x is the height of the box. Write two expressions: one for the length of the box in terms of x, and one for the width of the box in terms of x. Then use these expressions to write an equation for the volume, $V(x)$, as a function of x.

8. Graph the volume function on your graphing calculator. Use the maximum command under the CALC menu to determine the maximum volume. How closely does it agree with your answer in Problem 6? How close was your original estimate in Problem 1?

9. As accurately as you can, give the dimensions for the box with maximum volume. Could you reasonably cut out the size square needed to make the maximum volume? Explain.

10. What are the practical domain and range of this volume function?

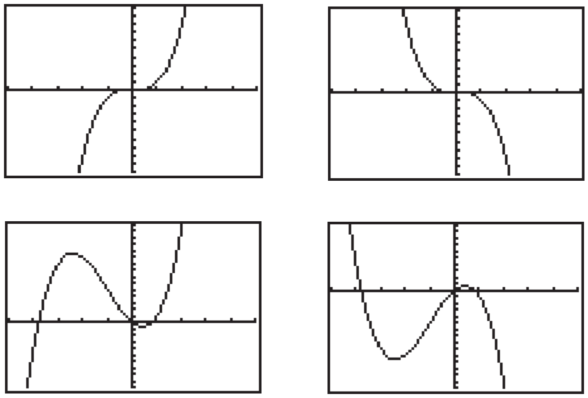
11. If you changed the length of your cardboard from 11 to 14 inches, would the maximum volume change? If so, what would be the value of the cut size needed to obtain the maximum volume? Explain the method you used to get your answer.

CLUSTER 3 What Have I Learned?

1. In a hurricane, the wind pressure varies directly as the square of the wind velocity (speed). If the wind speed doubles in value, what change in the wind pressure do you experience?

2. Is the graph of $y = 3x^4$ narrower or wider than the graph of $y = x^2$? Explain.

3. The graph of any cubic (third-degree polynomial) function must have one of the four following general shapes.



a. Complete the following table, which gives the maximum number of turning points for a given family of polynomial functions.

DEGREE OF POLYNOMIAL FUNCTION	MAXIMUM NUMBER OF TURNING POINTS
1 (linear)	
2 (quadratic)	
3 (cubic)	
4 (quartic)	

b. If n represents the degree, then write an expression that represents the maximum number of turning points.

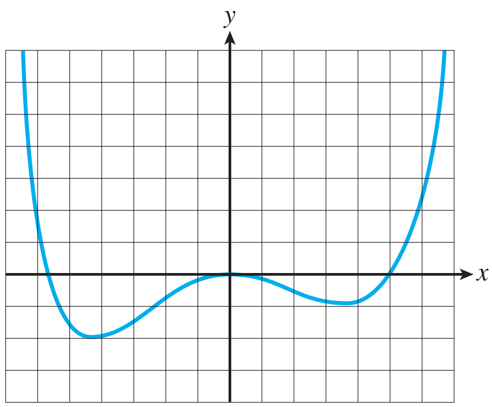
4. a. Sketch a graph of $y = x^4 - 4x^2$. Describe any symmetry that you observe.

b. Do all graphs of quartic (fourth-degree) functions have symmetry? Explain.

5. a. Does the graph of any cubic function have a horizontal intercept? Can the graph have more than one horizontal intercept? Explain.

b. Does the graph of any cubic function have at least one vertical intercept? Explain.

6. Given the following graph, determine whether any of the three functions in parts a–c fit the curve. Explain.



a. $f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a > 0, e = 0$

b. $g(x) = ax^3 + bx^2 + cx + d, \quad a > 0, d = 0$

c. $h(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a < 0, e = 0$

CLUSTER 3 How Can I Practice?

1. y varies directly as x^2. When $x = 3$, $y = 45$. Determine y when $x = 6$.

2. Have you ever noticed that, during a thunderstorm, you see lightning before you hear the thunder? This is true because light travels faster than sound. If d represents the distance (in feet) of the lightning from the observer, then d varies directly as the time, t (in seconds), it takes to hear the thunder. The relationship is modeled by

$$d = 1080t.$$

   a. As the time t doubles (say from 3 to 6), the corresponding d-values

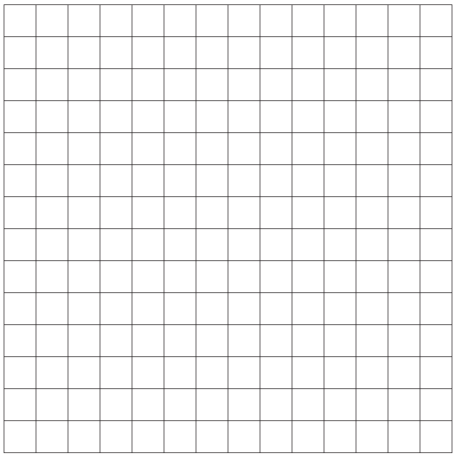
   ____

   b. What is the value of k, the constant of variation, in this situation? What significance does k have in this problem?

3. The velocity, v, of a falling object varies directly as the time, t, of the fall. After 3 seconds, the velocity of the object is 60 feet per second. What will be its velocity after 4 seconds?

4. Sketch a graph of each of the following pairs of functions. Describe the differences and the similarities in the graphs.

   a. $y = 3x^2$, $y = 3x^2 + 5$



Answers to all How Can I Practice exercises are included in the Selected Answers appendix.

b. $y = 5x^4$, $y = -5x^4$

c. $y = 2x^3 + 1$, $y = 2x^3 - 4$

d. $y = 4x^2$, $y = 4(x - 1)^2$

5. Using your graphing calculator, graph each of the following polynomial functions. For each graph,

   i. determine the vertical intercepts.

   ii. approximate the horizontal intercepts (if they exist).

   iii. determine the coordinates of any turning points.

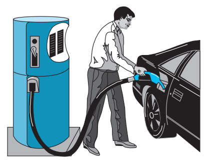
   a. $y = x^3 + 2x^2 - 8x$ b. $y = -1x^4 + 2x + 3$

6. The average miles per gallon (mpg) for U.S. cars has steadily increased over the past several years. The following table gives the average miles per gallon for selected years.

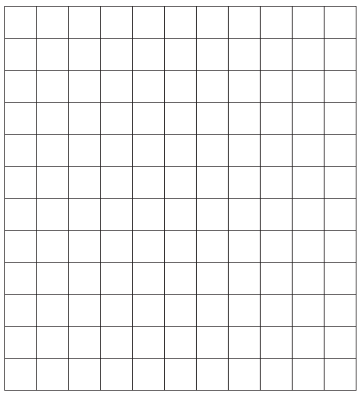
PUMP YOU UP

Average Miles per Gallon for Cars in the United States from 1960 to 2000

Year	Average mpg
2000	21.9
1990	20.2
1980	15.9
1970	13.5
1960	14.3



a. Draw a scatterplot of the data points, where the input, x, represents the number of years since 1960.



b. Determine the equation of a quadratic function that best fits the data.

c. Use the regression equation to predict the average miles per gallon in the year 2010.

d. Use your model to determine the year when the average miles per gallon is 30.

7. Your bathtub is partially filled. You finish filling the tub and settle in for a nice hot bath. The drain plug is broken, and water is slowly leaking out. The amount of water (in gallons) in the bathtub is given by

$$W(t) = 10 + 7t^2 - t^3 \qquad (t \geq 0),$$

where t is time in minutes and $W(t)$ represents the amount of water in gallons.

a. With the aid of your graphing calculator, sketch the graph of $W(t)$. Don't go beyond $t = 10$. Why?

b. How much water was in the bathtub at the time you began to fill it?

c. Determine the maximum amount of water in the tub from the graph. Explain your result.

d. Use the zero features of your graphing calculator to determine when the tub will be completely empty, to the nearest 0.01 minute.

CHAPTER 4 SUMMARY

The bracketed numbers following each concept indicate the activity in which the concept is discussed.

CONCEPT/SKILL	DESCRIPTION	EXAMPLE
Quadratic function [4.1]	The quadratic function with the input variable x has the standard form $y = ax^2 + bx + c$, where a, b, and c represent real numbers and $a \neq 0$.	$y = 2x^2 - 3x - 2$
Graph of a quadratic function (a parabola) [4.1]	For the quadratic function defined by $f(x) = ax^2 + bx + c$, if $a > 0$ the parabola opens upward; if $a < 0$ the parabola opens downward.	The graph of $y = 2x^2 - 3x - 2$ is a parabola that opens upward.
Vertical intercept of the graph of a quadratic function [4.1]	The constant term c of a quadratic function $f(x) = ax^2 + bx + c$ always indicates the vertical intercept of the parabola. The vertical intercept of any quadratic function is $(0, c)$.	The vertical intercept of the graph of $y = 2x^2 - 3x - 2$ is $(0, -2)$.
Axis of symmetry [4.2]	The axis of symmetry of a parabola is a vertical line that separates the parabola into two mirror images. The equation of the vertical axis of symmetry is given by $x = \frac{-b}{2a}$.	The axis of symmetry of the parabola defined by $y = 2x^2 - 3x - 2$ is $x = \frac{3}{4}$.
Vertex (turning point) [4.2]	The vertex of a parabola defined by $f(x) = ax^2 + bx + c$ is the point where the graph changes direction. It is given by $\left(\frac{-b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.	The vertex of the parabola defined by $y = 2x^2 - 3x - 2$ is $\left(\frac{3}{4}, -\frac{25}{8}\right)$.
x-intercept(s) [4.2]	An x-intercept is the point or points (if any) where the graph crosses the x-axis (that is, where its y-coordinate is zero).	The x-intercepts of the parabola defined by $y = 2x^2 - 3x - 2$ are $(2, 0)$ and $(-0.5, 0)$.
Domain of quadratic functions [4.2]	The domain of any quadratic function is all real numbers.	The domain of $y = 2x^2 - 3x - 2$ is all real numbers.
Range of quadratic functions [4.2]	If the parabola opens upward, the range is [output value of turning point, ∞). If the parabola opens downward, the range is ($-\infty$, output value of turning point].	The range of the parabola defined by $y = 2x^2 - 3x - 2$ is $\left[-\frac{25}{8}, \infty\right)$.
Solving $f(x) = c$ graphically [4.3]	Graph $y = f(x)$, graph $y = c$, and determine the x-values of the points of intersection. Or graph $y = f(x) - c$ and determine the x-intercepts.	

CONCEPT/SKILL	DESCRIPTION	EXAMPLE
Solving $f(x) > c$ graphically [4.3]	Graph $y = f(x)$, graph $y = c$, and determine all x-values for which the graph of f is above the graph of $y = c$. Or graph $y = f(x) - c$ and determine all x-values for which the graph of $f(x) - c$ is above the x-axis.	Example 1, Activity 4.3
Solving $f(x) < c$ graphically [4.3]	Graph $y = f(x)$, graph $y = c$, and determine all x-values for which the graph of f is below the graph of $y = c$. Or graph $y = f(x) - c$ and determine all x-values for which the graph of $f(x) - c$ is below the x-axis.	
Greatest common factor (or GCF) [4.4]	The GCF is the largest factor common to all terms in an expression.	The GCF of $3x^4 - 6x^3 + 18x^2$ is $3x^2$.
Zero-product principle [4.4]	If a and b are any numbers and $a \cdot b = 0$, then either a or b, or both, must be equal to zero.	Example 1, Activity 4.4
Factoring trinomials by trial and error [4.4]	To factor trinomials by trial and error, 1. remove the GCF. 2. try combinations of factors for the first and last terms in two binomials. 3. check the outer and inner products to match middle term of the original trinomial. 4. if the check fails, repeat steps 2 and 3.	Example 5, Activity 4.4
Solving quadratic equations by factoring [4.4]	To solve a quadratic equation by factoring, 1. use the addition principle to remove all terms from one side of the equation. This results in a quadratic polynomial being set equal to zero. 2. combine like terms, and then factor the nonzero side of the equation. 3. use the zero-product principle to set each factor containing a variable equal to zero, and then solve the equations. 4. check your solutions in the original equation.	Example 6, Activity 4.4
Quadratic formula [4.5]	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Example 1, Activity 4.5

CONCEPT/SKILL	DESCRIPTION	EXAMPLE
Solving a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$, using the quadratic formula [4.5]	To solve a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$, using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ 1. rewrite the quadratic equation with one side zero. 2. identify the coefficients a and b and the constant term c. 3. substitute these values into the formula, and simplify. 4. check your solutions.	Example 1, Activity 4.5
Imaginary unit [4.7]	The imaginary unit is the number $\sqrt{-1}$. The notation for the imaginary unit is i.	
Complex number [4.7]	Any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit, is called a complex number.	$2 + 6i$
Discriminant [4.7]	In the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ the expression $b^2 - 4ac$ is called the discriminant. Its value determines the number and type of solutions of a quadratic equation $ax^2 + bx + c = 0$.	For the quadratic equation $y = 2x^2 - 7x - 4$, the discriminant is $49 - 4(2)(-4) = 81$. The equation has two real solutions.
Direct variation function [4.10]	The equation $y = kx^n$, where $k \neq 0$ and n is a positive integer, defines a direct variation function in which y varies directly as x^n.	$y = 4x^3$
Constant of variation [4.10]	In the direct variation equation $y = kx^n$, the constant, k, is called the constant of variation.	In $y = 4x^3$, the constant of variation is 4.
Power functions [4.10]	The direct variation function having an equation of the form $y = kx^n$, where n is a positive integer, is also called a power function.	$y = 4x^3$ is a third-power function.
Polynomial functions [4.11]	Polynomial functions are defined by equations of the form output $=$ polynomial expression involving the input.	$y = 5x^4 + 7x^2 - 3x + 1$

CONCEPT/SKILL	DESCRIPTION	EXAMPLE
Degree of a polynomial [4.11]	The largest exponent on the input variable n is called the degree of the function.	$y = 5x^4 + 7x^2 - 3x + 1$ is a fourth-degree polynomial function.
Discrete functions [4.12]	Functions are discrete if they are defined only at isolated input values and do not make sense for input values between those values.	Example 1, Activity 4.12
Continuous functions [4.12]	Functions are continuous if they are defined for all input values, and these graphs can be drawn so that all parts of the graph are connected.	The quadratic function $y = 2x^4 + 3x - 1$ is continuous for all real numbers.

GATEWAY REVIEW

In Exercises 1–8, determine the following characteristics of each quadratic function by inspecting its equation.

a. the direction in which the graph opens
b. the equation of the axis of symmetry
c. the vertex
d. the y-intercept

1. $f(x) = x^2 + 2$

2. $F(x) = -3x^2$

3. $g(x) = -3x^2 + 4$

4. $f(x) = 2x^2 - x$

5. $h(x) = x^2 + 5x + 6$

6. $F(x) = x^2 - 3x + 4$

7. $f(x) = x^2 - 2x + 1$

8. $g(x) = -x^2 + 5x - 6$

In Exercises 9–15, sketch the graph of each quadratic function using your graphing calculator. Then determine each of the following using the graph.

a. the coordinates of the x-intercepts, if they exist
b. the domain and the range of the function
c. the horizontal interval in which the function is increasing
d. the horizontal interval in which the function is decreasing

9. $g(x) = x^2 + 4x + 3$

10. $f(x) = x^2 + 2x - 3$

Answers to all Gateway exercises are included in the Selected Answers appendix.

11. $F(x) = x^2 - 3x + 1$

12. $h(x) = 2x^2 + 8x + 5$

13. $F(x) = -2x^2 + 8$

14. $f(x) = -3x^2 + 4x - 1$

15. $g(x) = 4x^2 + 5$

In Exercises 16–19, solve the quadratic equation numerically (using tables). Verify your solutions graphically.

16. $x^2 + 4x + 4 = 0$

17. $x^2 - 5x + 6 = 0$

18. $3x^2 = 18x + 10$

19. $-x^2 = 3x - 10$

In Exercises 20–21, solve the equation using two different approaches. Round your answer to the nearest tenth when necessary.

20. $8x^2 = 10$

21. $5x^2 + 25x = -5$

22. Completely factor the following polynomials.

a. $9a^5 - 27a^2$

b. $24x^3 - 6x^2$

c. $4x^3 - 16x^2 - 20x$

d. $5x^2 - 16x + 6$

e. $x^2 - 5x - 24$

f. $t^2 + 10t + 25$

In Exercises 23–27, solve each equation by factoring. Verify your answer graphically or by substitution of the solutions in the equations.

23. $x^2 - 9 = 0$

24. $-x^2 + 36 = 0$

25. $x^2 - 7x + 12 = 0$

26. $x^2 - 6x = 27$

27. $x^2 = -x$

In Exercises 28–32, write each of the equations in the form $\mathrm{a}x^2 + \mathrm{b}x + \mathrm{c} = 0$. *Then identify* a, b, *and* c, *and solve the equation using the quadratic formula. Verify your solutions by substitution.*

28. $x^2 + 5x + 3 = 0$

29. $2x^2 - x = -3$

30. $x^2 = 81$

31. $3x^2 + 5x = 12$

32. $2x^2 = 3x + 5$

33. For the quadratic function $f(x) = 2x^2 - 8x + 3$, determine the x-intercepts of the graph, if they exist. First, approximate the intercepts using your graphing calculator. Second, solve the equation using the quadratic formula. Approximate your answers to the nearest hundredth.

34. Write each of the following using the imaginary unit, i.

a. $\sqrt{-49}$

b. $\sqrt{-48}$

c. $\sqrt{-9}$

d. $\sqrt{-23}$

e. $\sqrt{-\frac{5}{9}}$

f. $\sqrt{-\frac{17}{16}}$

35. Perform the following operations with complex numbers. Use your graphing calculator to check your results.

a. $(2 + 7i) + (-7 + 10i)$

b. $(4 - 9i) - (-1 + 7i)$

c. $4i(3 - 8i)$

d. $(4 - i)(6 + 3i)$

In Exercises 36–39, determine the type of solution to each of the equations considering only its discriminant.

36. $2x^2 - 3x + 1 = 0$

37. $4x^2 + 16x = 0$

38. $x^2 - 9 = 0$

39. $3x^2 + 2x + 2 = 0$

40. Solve the equation in Problem 39 in the complex number system using the quadratic formula. Verify your solution graphically.

41. Solve the following inequalities using a graphing approach.

a. $x^2 - x - 6 < 0$

b. $x^2 - x - 6 > 0$

42. **a.** Suppose y varies directly as x. When $x = 3$, $y = 12$. Determine y when $x = 5$.

b. Suppose y varies directly as x^2. When $x = 4$, $y = 8$. Determine y when $x = 8$.

c. Suppose y varies directly as x^3. When $x = 1$, $y = 5$. Determine y when $x = 2$.

In Exercises 43–47, graph the function using your graphing calculator. Then answer the following questions, referring to the graphing calculator.

a. Determine the x*-intercepts of the function, if it has any.*
b. Determine the domain and range of the function.
c. Determine the values of x *for which the function is increasing and the values of* x *for which the function is decreasing.*

43. $y = x^3 - 8$

44. $y = -2x^3 - 2$

45. $y = x^4 - 8$

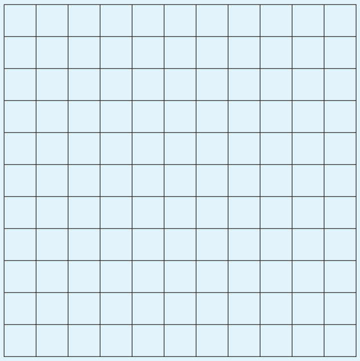
46. $y = x^4 + 2x$

47. $y = x^4 + 5$

48. The height, h (in feet), of a golf ball is a function of the time, t (in seconds), it has been in flight. A golfer strikes a golf ball with an initial upward velocity of 80 feet per second. The flight path of the ball is a parabola. The approximate height of the ball above the ground is modeled by

$$h(t) = -16t^2 + 80t.$$

a. Sketch a graph of the function. What is the practical domain in this situation?



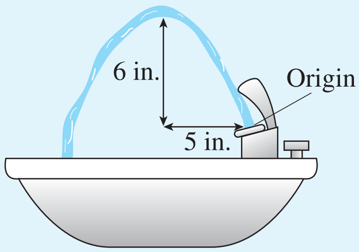
b. Determine the vertex of the parabola. What is the practical meaning of this point?

c. What is the vertical intercept, and what is its practical meaning in this situation?

d. Determine the horizontal intercepts. What is the significance of these intercepts?

e. What assumption are you making in this situation about the elevation of the spot where the ball is struck and the point where the ball lands?

49. To use the regression feature of your calculator to determine the equation of a parabola, you need three distinct points. The stream of water flowing out of a water fountain is in the shape of a parabola. Suppose you let the origin of a coordinate system correspond to the point where the water begins to flow out of the nozzle (see figure).



The maximum height of the water stream occurs approximately 5 inches measured horizontally from the nozzle. The maximum height of the stream of water is measured to be approximately 6 inches.

a. What is the vertex of the parabola?

b. You already have two points that lie on the parabola. What are they? Use symmetry to obtain a third point.

c. Using these three points and the regression feature of your graphing calculator, determine the equation of the stream of water.

50. A fastball is hit straight up over home plate. The ball's height, h (in feet), from the ground is modeled by

$$h(t) = -16t^2 + 80t + 5,$$

where t is measured in seconds.

a. What is the maximum height of the ball above the ground?

b. How long will it take for the ball to reach the ground?

51. Safe automobile spacing, S (in feet), is modeled by

$$S(v) = 0.03125v^2 + v + 18,$$

where v is average velocity in feet per second.

a. Suppose a car is traveling at 44 feet per second. To be safe, how far should it be from the car in front of it?

b. If the car is following 50 feet behind a van, what is a safe speed for the car to be traveling? How fast is this in miles per hour (60 miles per hour $\approx$ 88 feet per second)?