CHAPTER 5

RATIONAL AND RADICAL FUNCTIONS

CLUSTER 1 Rational Functions

ACTIVITY 5.1

Speed Limits

OBJECTIVES

1. Determine the domain and range of a function defined by $y = \frac{k}{x}$, where k is a nonzero real number.
2. Determine the vertical and horizontal asymptotes of the graph of $y = \frac{k}{x}$.
3. Sketch a graph of functions of the form $y = \frac{k}{x}$.
4. Determine the properties of graphs having equation $y = \frac{k}{x}$.

The speed limit on the New York State Thruway is 65 miles per hour.

1. If you maintain an average speed of 65 miles per hour, how long will it take you to make a 200-mile trip on the thruway? Recall that $distance = rate \cdot time$, $time = \frac{distance}{rate}$.

2. Complete the following table, in which the input variable r represents the average speed in miles per hour and the output variable t represents the time in hours to complete a 200-mile trip.

r (mph)	20	30	40	50	60	70	80
t (hr.)							

3. Write an equation that defines travel time, t, as a function of the average speed, r.

4. As the average speed, r, increases, what happens to the travel time, t? What does this mean in practical terms?

5. During a winter storm, a combination of drifting snow and icy conditions reduces your average speed to almost a standstill. Complete the following table for a 200-mile trip on the New York State Thruway.

r (mph)	10	7	5	3	2	1
t (hr.)						

6. As the average speed r gets closer to zero, what happens to the travel time t? Explain what this means in practical terms.

7. Can zero be used as an input value? Explain.

8. **a.** What is the practical domain of the function given in Problem 3?

b. Sketch a graph of this function using the table values in Problem 2 and 5.

9. **a.** What are the horizontal and vertical intercepts of the graph?

b. Describe the relationship between the horizontal axis $(t = 0)$ and the graph of the function as the values of r get very large.

In this situation, the horizontal line $t = 0$ is called a **horizontal asymptote.** Recall that a horizontal asymptote is a horizontal line that a graph approaches as the input values of r get very large in a positive direction (or get very large in a negative direction).

c. Describe the relationship between the vertical axis $(r = 0)$ and the graph of the function as the values of r get close to zero.

In this situation, the vertical line $r = 0$ is called a **vertical asymptote.** A vertical asymptote is a vertical line, $x = a$, that the graph of a function defined by $y = f(x)$ becomes very close to but never touches. As the x-values get closer and closer to $x = a$, the y-values get larger and larger in magnitude. That is, the y-values become very large positive or very large negative values.

10. Using the graph in Problem 8, approximate your average speed if the 200-mile trip takes 5 hours.

Functions Defined by $y = \frac{k}{x}$, where k is a Nonzero Constant

The function rule $t = \frac{200}{r}$ gives the relationship between the average speed, r, the time, t, and the given value for distance (200). This function belongs to a family of functions having a general equation of the form $f(x) = \frac{k}{x}$, where k represents some nonzero constant.

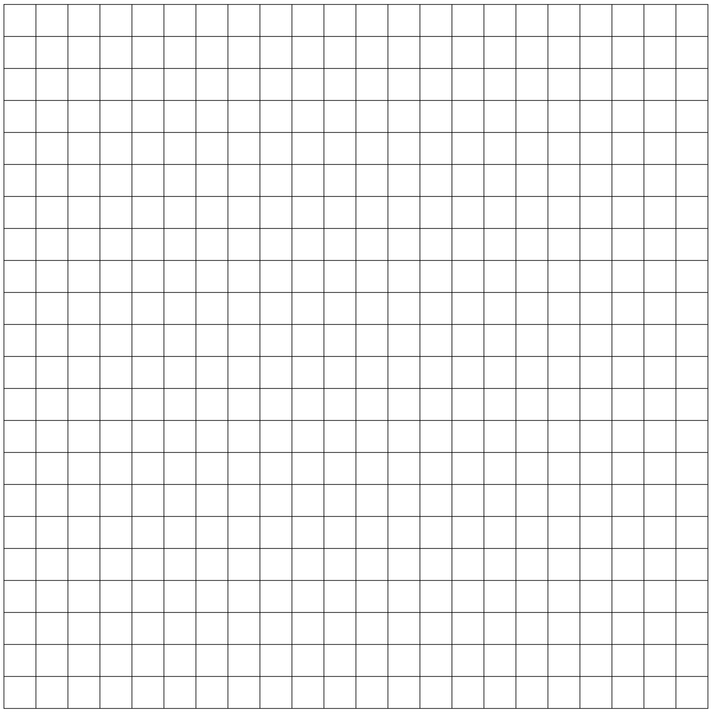
EXAMPLE 1 ***Examples of this type of function are*** $f(x) = \frac{1}{x}$, $g(x) = \frac{5}{x}$, ***and*** $h(x) = \frac{10}{x}$.

11. a. What is the domain of functions f, g, and h defined in Example 1?

b. Complete the following table.

x	-20	-10	-5	-1	-0.5	-0.1	0	0.1	0.5	1	5	10	20
$f(x)$													
$g(x)$													
$h(x)$													

c. Sketch graphs of f, g, and h on the same coordinate system. Verify using your graphing calculator with window Xmin = −5, Xmax = 5, Ymin = −25, and Ymax = 25.



12. Using the table and graphs in Problem 11, answer each of the following questions.

a. What happens to the y-values as the x-values increase in magnitude infinitely (without bound) in both the positive and negative directions?

b. What is the horizontal asymptote for each graph?

c. What happens to the y-values as the positive x-values get closer to zero?

d. What happens to the y-values as the negative x-values get closer to zero?

e. What is the vertical asymptote for each graph?

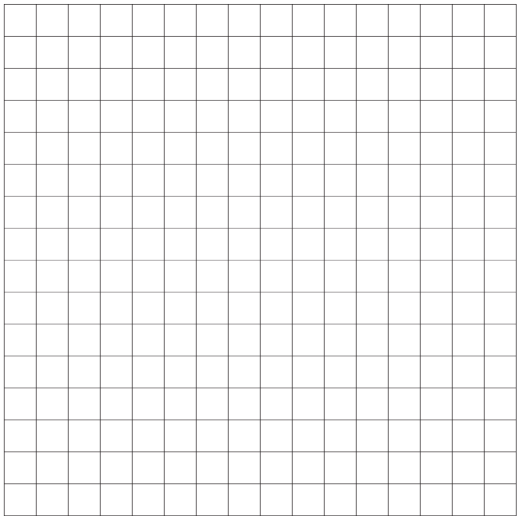
13. **a.** Do the graphs of f, g, or h in Problem 11 have x- or y-intercepts?

b. Do the functions f, g, and h have a maximum function value or a minimum function value? Explain.

14. **a.** Complete the following table, where $Q(x) = \frac{-1}{x}$.

x	−10	−5	−1	−0.5	−0.1	0	0.1	0.5	1	5	10
$Q(x)$											

b. Sketch a graph of Q. Verify using your graphing calculator with window Xmin = −4, Xmax = 4, Ymin = −4, and Ymax = 4.



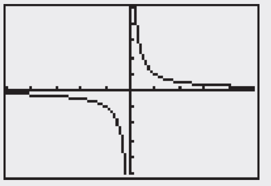
c. Add the graph of $f(x) = \frac{1}{x}$ to the grid in part b.

d. Describe how the graph of $Q(x) = \frac{-1}{x}$ can be obtained from the graph of $f(x) = \frac{1}{x}$.

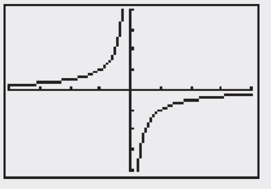
SUMMARY
ACTIVITY 5.1

Functions defined by $f(x) = \frac{k}{x}$, where k represents some nonzero constant, have the following properties.

1. The domain and the range consist of all real numbers except zero.
2. If $k > 0$, the graph of $f(x)$ has the following general shape.



3. If $k < 0$, the graph of $f(x)$ has the following general shape.



4. The vertical line $x = 0$ is the vertical asymptote.
5. The horizontal line $y = 0$ is the horizontal asymptote.
6. The graph does not intersect either axis (there are no intercepts).
7. There is no maximum or minimum y-value.

EXERCISES
ACTIVITY 5.1

1. You are a member of a group of distance runners who compete in races ranging in length from 5 to 25 kilometers. In these races, each runner who finishes is told his or her time. Given the time and the length of the race, you can calculate your average running speed.

   a. If you finish a 20-kilometer race in 1 hour 15 minutes, what is your average speed?

   b. Complete the following table for a 20-kilometer race.

t (hr.)	1.00	1.25	1.50	1.75	2.00	2.25	2.50
s (km/hr.)							

Exercise numbers appearing in color are answered in the Selected Answers appendix.

c. Write an equation that expresses the average speed, s, as a function of time t in a 20-kilometer race.

d. **i.** What is the domain of the function?

ii. What is the practical domain?

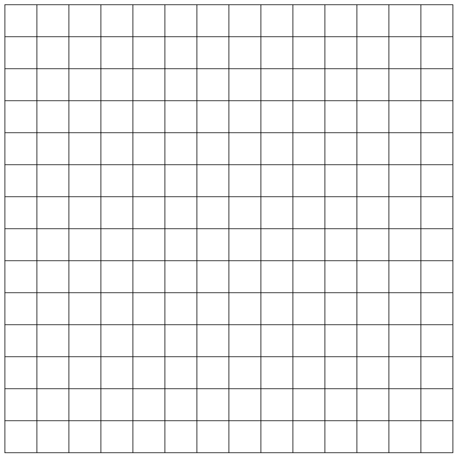
iii. Sketch a graph of the function using the practical domain.

e. As the running time, t, gets longer, what happens to the average speed, s?

f. As the running time, t, is reduced (gets closer to zero), what happens to the average speed, s?

2. a. Sketch the graphs of the following pair of functions on the same coordinate system. Use labels or colors to differentiate the graphs. Verify your sketches using your graphing calculator.

$$f(x) = \frac{5}{x}, \qquad g(x) = \frac{-5}{x}$$



b. Describe how the graph of g can be obtained from the graph of function f.

3. A commercial refrigerator has an initial cost, C, and a scrap value, V. If the life of the refrigerator is N years, then the amount, D, that can be depreciated each year is modeled by the formula

$$D = \frac{C - V}{N}.$$

a. If the initial cost is \$1400 and the scrap value is \$200, write an equation for D as a function of N.

b. Complete the following table.

N	1	2	3	6	12	24
D						

c. Sketch a graph of the function.

d. If the refrigerator is well constructed, it should have a long, useful life. Will an increase in the useful life of the refrigerator increase or decrease the amount, D, that can be depreciated each year? Explain.

4. The speed limit on Route 66 in Arizona is 75 miles per hour.

a. If you maintain an average speed of 75 miles per hour, how long will it take you to make a 350-mile trip on Route 66?

b. Write an equation that defines t as a function of r, in which r represents the average speed in miles per hour and t represents the time in hours to complete the 350-mile trip.

c. Complete the following table for the equation you determined in part b.

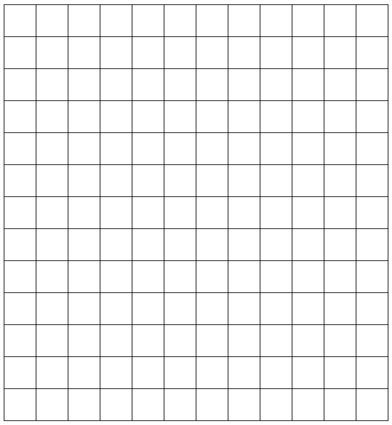
INPUT, r, mph	25	35	45	55	65	75	85
OUTPUT, $t = f(r)$, hr.							

d. As your average speed increases, what happens to the time it takes to complete the trip?

e. As your average speed for the trip gets closer to zero, what happens to the time it takes to complete the 350-mile trip?

f. What is the practical domain?

g. Sketch a graph.



h. What are the vertical and horizontal asymptotes of the graph of the function?

ACTIVITY 5.2

Loudness of a Sound

OBJECTIVES

1. Graph an inverse variation function defined by an equation of the form $y = \frac{k}{x^n}$, where n is any positive integer and k is a nonzero real number.
2. Describe the properties of graphs having equation $y = \frac{k}{x^n}$.
3. Determine the constant of proportionality (also called the constant of variation).

The loudness (or intensity) of any sound is a function of the listener's distance from the source of the sound. In general, the relationship between the intensity, I, and the distance, d, can be modeled by an equation of the form

$$I = \frac{k}{d^2}$$

where I is measured in decibels, d is measured in feet, and k is a constant determined by the source of the sound and the nature of the surroundings.

1. The intensity, I, of a human voice can be given by the formula $I = \frac{1500}{d^2}$. Complete the following table.

d **(ft.)**	0.1	0.5	1	2	5	10	20	30
I **(dB)**								

2. **a.** What is the practical domain of the function?

   b. Sketch a graph that shows the relationship between intensity of sound and distance from the source of the sound. Use the table in Problem 1 to help determine an appropriate scale.

3. As you move closer to the person speaking, what happens to the intensity of the sound?

4. As you move away from the person speaking, what happens to the intensity of the sound?

Functions Defined by $y = \frac{k}{x^2}$, where k is a Nonzero Constant

The function defined by $I = \dfrac{1500}{d^2}$ belongs to a family of functions having an equation of the form $y = \dfrac{k}{x^2}$, where k represents some nonzero constant.

EXAMPLE 1 ***Examples of functions defined by equations of the form*** $y = \dfrac{k}{x^2}$ ***are*** $f(x) = \dfrac{1}{x^2}$ ***and*** $g(x) = \dfrac{10}{x^2}$.

5. a. What is the domain of functions f and g defined in Example 1?

b. Complete the following table.

x	−20	−10	−5	−1	−0.5	−0.1	0	0.1	0.5	1	5	10	20
$f(x)$													
$g(x)$													

c. Sketch the graphs of f and g on the same coordinate system. Verify your sketch using your graphing calculator.

d. Explain why no part of each graph appears below the *x*-axis.

e. What happens to the *y*-values as the *x*-values increase infinitely in the positive direction or decrease infinitely in the negative direction?

f. What is the horizontal asymptote for each graph?

g. What happens to the *y*-values as the positive *x*-values get closer to zero?

h. What happens to the *y*-values as the negative *x*-values get closer to zero?

i. What is the vertical asymptote for each graph?

j. Do the functions have a maximum function value or a minimum function value?

k. For a given *x*-value, how is the output of *g* related to the output of *f*? Describe this relationship graphically.

6. Describe how the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ are similar and how they are different.

7. a. Complete the following table.

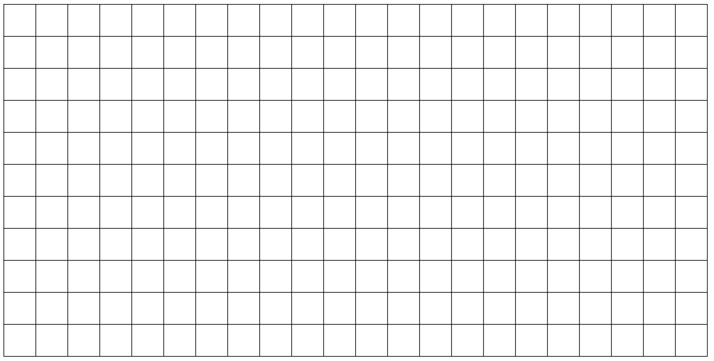
x	−20	−10	−5	−1	−0.5	−0.1	0	0.1	0.5	1	5	10	20
$g(x) = \frac{10}{x^2}$													
$h(x) = \frac{-10}{x^2}$													

b. Sketch graphs of functions g and h on the same coordinate system. Use labels or different colors to differentiate the graphs. Verify your sketch using your graphing calculator.

c. Describe how to obtain the graph of function h using the graph of function g.

8. Sketch a graph of $h(x) = \frac{1}{x^3}$. Is the graph similar to the graph of $f(x) = \frac{1}{x}$ or $g(x) = \frac{1}{x^2}$?

9. Sketch a graph $R(x) = \frac{1}{x^4}$. Is the graph similar to the graph of $f(x) = \frac{1}{x}$ or $g(x) = \frac{1}{x^2}$?



Inverse Variation Functions

DEFINITION

Functions defined by equations of the form $y = \frac{k}{x^n}$, where k is a nonzero constant and n is a positive integer, belong to the family of functions called **rational functions.** The rational functions of the form $y = \frac{k}{x^n}$ are also called **inverse variation functions.** The number k is called the **constant of variation** or **constant of proportionality.**

EXAMPLE 2 ***For the inverse variation function given by $y = \frac{4}{x}$, y varies inversely as x, or y is inversely proportional to x. 4 is the constant of variation. The following table demonstrates that as x doubles in value, the corresponding y-values are reduced by half.***

x	2	4	8	16
$y = \frac{4}{x}$	2	1	$\frac{1}{2}$	$\frac{1}{4}$

10. For the function defined by $I = \frac{1500}{d^2}$ (see Problem 1), answer the following questions.

a. I varies inversely as what quantity?

b. What is the constant of proportionality?

c. If d is doubled, what is the effect on I?

In many applications, the constant of proportionality, k, is unknown. The procedure for determining k for an inverse variation is demonstrated in Example 3.

EXAMPLE 3 ***y varies inversely as the square of x and y = 5 when x = 2. Determine the equation for y.***

SOLUTION

Step 1. Write the equation relating x to y using k for the constant of proportionality.

$$y = \frac{k}{x^2}$$

Step 2. Substitute the given values of x and y ($y = 5$, $x = 2$) into the equation from Step 1.

$$5 = \frac{k}{2^2} \quad \text{or} \quad 5 = \frac{k}{4}$$

Step 3. Solve the equation in Step 2 for k.

$$5 \cdot 4 = \frac{k}{4} \cdot 4$$
$$20 = k$$

Step 4. Rewrite the equation from Step 1 by substituting the value of k determined in Step 3.

$$y = \frac{20}{x^2}$$

11. In this activity, the relationship between the intensity, I, of a human voice and the distance, d, from the individual was given by $I = \frac{1500}{d^2}$, where I is measured in decibels, d is measured in feet, and 1500 is the constant of proportionality. The constant of proportionality depends on the source of the sound and the nature of the surroundings. If the source of the sound changes, the value of the constant of proportionality will also change.

a. The intensity of the sound made by a heavy truck 60 feet away is 90 decibels. Determine the constant of proportionality.

b. Write a formula for the intensity, I, of the sound made by the truck when it is d feet away.

c. Use the formula from part b to determine the intensity of the sound made by the truck when it is 100 feet away.

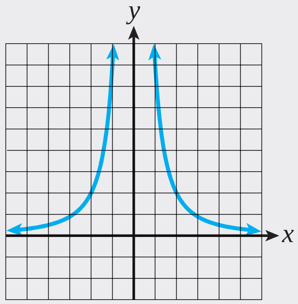
SUMMARY ACTIVITY 5.2

- Functions defined by equations of the form $f(x) = \frac{k}{x^n}$, where k is a nonzero real number, have the following properties.

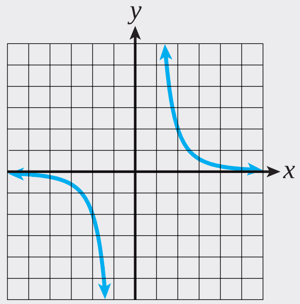
  1. The domain consists of all real numbers except zero.

  2. The graph of f has the following general shape.

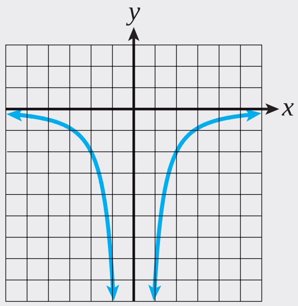
  a. Where $k > 0$, and n is an even integer



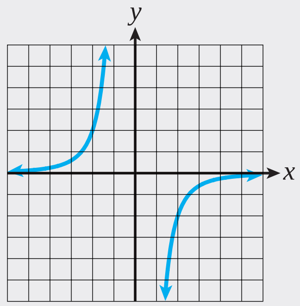
  b. Where $k > 0$, and n is an odd integer



  c. Where $k < 0$, and n is an even integer



  d. Where $k < 0$, and n is an odd integer



  3. The vertical asymptote is the vertical line $x = 0$.

  4. The horizontal asymptote is the horizontal line $y = 0$.

  5. There are no vertical or horizontal intercepts.

  6. There is no maximum or minimum function value.

- Functions defined by $y = \frac{k}{x^n}$ are called **inverse variation functions** in which

  1. y is said to vary inversely as the nth power of x

  2. k is called the constant of variation or constant of proportionality

EXERCISES
ACTIVITY 5.2

1. Doctors sometimes use a patient's body-mass index to determine whether or not the patient should lose weight. The model for the body-mass index, B, is the formula

$$B = \frac{705w}{h^2},$$

where w is the weight in pounds and h is height in inches.

a. What is your body-mass index?

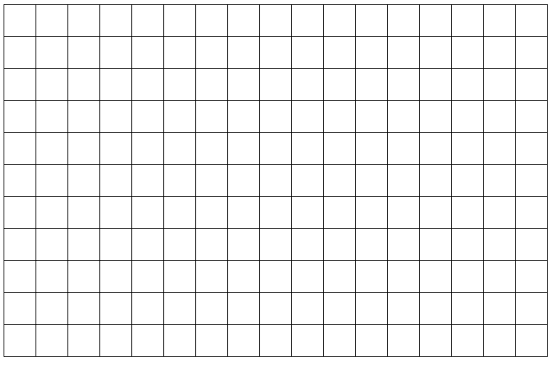
b. Suppose your friend weighs 170 pounds. Substitute this value into the body-mass index formula to obtain an equation for B in terms of height.

c. What is the practical domain of the body-mass index function in part b?

d. Complete the following table using the formula for the body-mass index of a 170-pound person.

***h*, HEIGHT IN INCHES**	60	64	68	72	76	80
B						

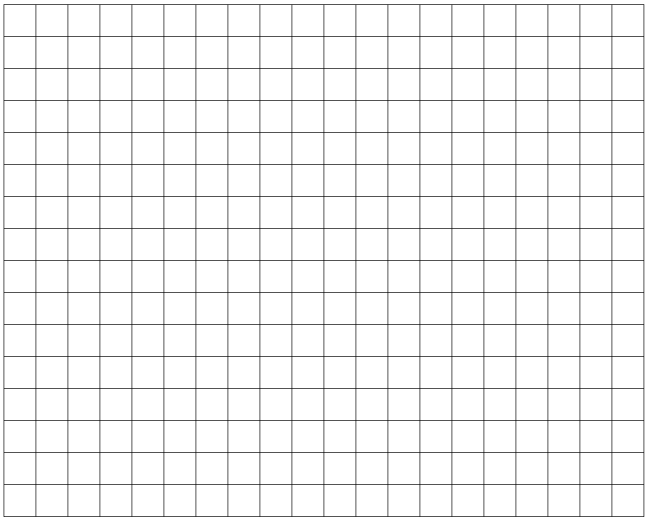
e. Sketch a graph of the function defined by $B = \frac{119{,}850}{h^2}$. Use the data values in part d to help determine an appropriately scaled axis.



f. What happens to the body-mass index as height increases? Does this make sense in the context of the situation? Explain why or why not.

g. It is recommended that a person's body-mass index be between 19 and 25. Use the graph and the trace key on your calculator to approximate the values of h for which $19 < B < 25$.

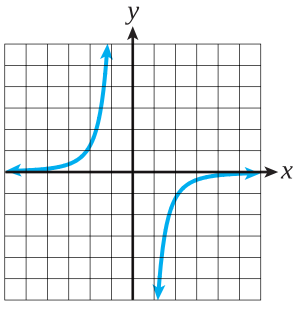
2. Sketch a graph of the functions $f(x) = \frac{3}{x^2}$ and $g(x) = \frac{-3}{x^2}$ on the same coordinate system. Describe how the graph of g is related to the graph of f.

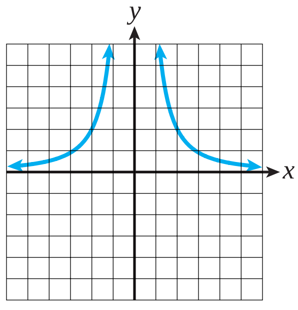


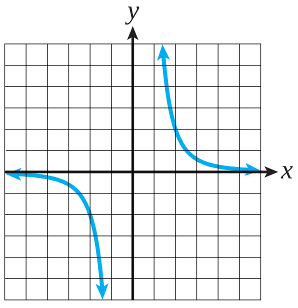
3. Match the following functions with the accompanying graphs.

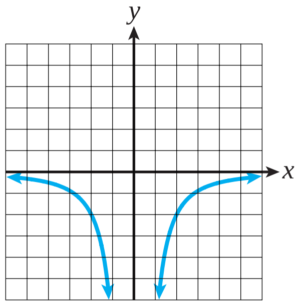
i. $f(x) = \dfrac{10}{x^4}$ ii. $g(x) = \dfrac{100}{x^5}$

iii. $h(x) = \dfrac{-10}{x^3}$ iv. $F(x) = \dfrac{-1}{x^2}$

a.


b.


c.


d.


i. ii.

iii. iv.

4. Describe how the graphs of $y = \dfrac{1}{x^2}$ and $y = \dfrac{1}{x^3}$ are similar and how they are different.

5. Consider the family of functions of the form $f(x) = \dfrac{k}{x^n}$, where k is a nonzero constant and n is a positive integer.

a. What is the domain of f?

b. Use several different values of k and n, where $k > 0$ and n is an odd positive integer, to determine the general shape of the graph of f.

c. Use several different values of k and n, where $k > 0$ and n is an even positive integer, to determine the general shape of the graph of f.

6. How will the general shapes of the graphs in Exercise 5 change if $k < 0$?

7. Complete the following tables of ordered pairs for the given inverse variations.

a. y varies inversely as x.

x	y
$\frac{1}{2}$	
1	2
2	
6	

b. y varies inversely as x^3.

x	y
$\frac{1}{2}$	
1	
2	1
6	

8. If y varies inversely as the cube of x, determine the constant of proportionality if $y = 16$ when $x = 2$.

9. The amount of current, I, in a circuit varies inversely as the resistance, R. A circuit containing a resistance of 10 ohms has a current of 12 amperes. Determine the current in a circuit containing a resistance of 15 ohms.

10. The intensity, I, of light varies inversely as the square of the distance, d, between the source of light and the object being illuminated. A light meter reads 0.25 unit at a distance of 2 meters from a light source. What will the meter read at a distance of 3 meters from the source?

11. You are investigating the relationship between the volume, V, and pressure, P, of a gas. In a laboratory, you conduct the following experiment: While holding the temperature of a gas constant, you vary the pressure and measure the corresponding volume. The data that you collect appears in the following table.

P (psi)	20	30	40	50	60	70	80
V (ft.^{3})	82	54	41	32	27	23	20

a. Sketch a graph of the data.

b. One possible model for the data is that V varies inversely as the square of P. Does the data fit the model $V = \frac{k}{P^2}$? Explain.

c. Another possible model for the data is that V varies inversely as P. Does $V = \frac{k}{P}$ model the data? Explain.

d. Predict the volume of the gas if the pressure is 65 pounds per square inch, using the graph in part a.

e. Verify your answer in part d using the model from part c.

ACTIVITY 5.3

Percent Markup

OBJECTIVES

1. Determine the domain of a rational function defined by an equation of the form $y = \frac{k}{g(x)}$, where k is a nonzero constant and $g(x)$ is a first degree polynomial.
2. Identify the vertical and horizontal asymptotes of $y = \frac{k}{g(x)}$.
3. Sketch graphs of rational functions defined by $y = \frac{k}{g(x)}$.

You are a buyer for a national chain of retail stores. You purchase merchandise at a wholesale cost. The merchandise is then sold at a retail price (called the selling price). The retailer's markup is the difference between the selling price (what the consumer pays) and the wholesale cost.

1. a. You acquire a line of sports jackets at a wholesale cost of \$80 per jacket. If the jackets sell for \$120 each at the retail level, what is the amount of the markup?

b. The markup is what percent of the selling price? (This percent is called the percent markup of the selling price.)

2. The relationship between the selling price, S, the wholesale cost, C, and the percent markup, P, of the selling price (expressed as a decimal) is modeled by

$$S = \frac{C}{1 - P}.$$

If the wholesale cost of a sports jacket is \$80, write an equation for S in terms of P.

3. a. Complete the following table for $S = \frac{80}{1 - P}$.

P (% markup)	0	0.01	0.05	0.10	0.25	0.50	0.75	0.95
S (selling price)								

b. As the values of P approach 1, what happens to the values of S? What does this mean in practical terms?

4. a. Can the percent markup of the selling price be 100% (i.e., can $P = 1$)? Explain.

b. What is the practical domain of this function?

5. Sketch a graph of the function. Use the table of data pairs in Problem 3 to help determine an appropriate scale. Verify your sketch using your graphing calculator.

Graphs of $f(x) = \frac{k}{g(x)}$

Because S and P represent real-world quantities, the practical domain limits our investigation of the function defined by $S = \frac{80}{1 - P}$.

6. Consider the general function f defined by $f(x) = \frac{80}{1 - x}$.

a. What is the domain of function f?

b. Complete the following table.

x	-10	-5	0	0.50	0.75	0.90	1	1.10	1.25	1.50	2	5	10
$f(x)$													

c. Sketch a graph of the function f using your graphing calculator. Use the table in part b to determine an appropriate scale.

d. Does the graph of f have a horizontal asymptote? Explain why or why not. If you answer yes, what is the equation of the horizontal asymptote?

e. Does the graph of f have a vertical asymptote? Explain why or why not. If you answer yes, what is the equation of the vertical asymptote?

f. For what value of x is $f(x)$ maximum?

g. Does the graph have any intercepts?

7. Consider the function defined by $g(x) = \dfrac{80}{x + 1}$.

a. What is the domain of g?

b. Construct a table of data points for g.

x	-4	-3	-2	-1	0	1	2
$g(x)$							

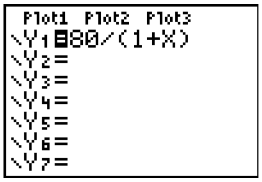
c. Sketch a graph of g using an appropriate scale.

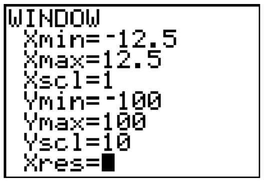
d. Determine the equation of the vertical asymptote. As a graphing aid, if the vertical asymptote is not the y-axis $(x = 0)$, the asymptote is drawn as a dotted vertical line. If you have not done so, draw the vertical asymptote in the graph in part c.

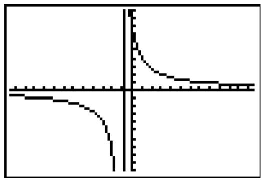
e. Determine the equation of the horizontal asymptote.

f. Does the graph of g have any intercepts?

8. **a.** Sketch a graph of $g(x) = \dfrac{80}{1 + x}$ using your graphing calculator. Your screens should appear as follows.

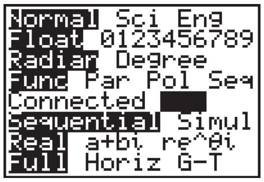


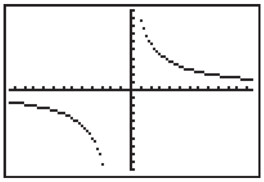




b. The vertical asymptote $(x = -1)$ appears to be part of the graph. What do you think has happened?

The calculator setting in Problem 8 was for connected mode. In this mode, it will connect all the points in plots. If the window is such that the calculator does not try to plot a point at $x = -1$, then it will connect a point for an x-value slightly less than -1 to one slightly greater than -1. This creates the appearance of an asymptote. To avoid this, change the mode to dot mode, as follows:





Note: Some TI-83/TI-84 Plus calculators will not display a vertical asymptote even in connected mode. It depends on the operating system in the calculator and there is no way to tell simply by looking at the calculator.

9. How are the graphs of $f(x) = \dfrac{80}{1 - x}$ and $g(x) = \dfrac{80}{x + 1}$ similar? How are they different?

Rational Functions

A function Q, defined by an equation of the form

$$Q(x) = \frac{k}{g(x)},$$

where k is a nonzero constant and $g(x)$ is a polynomial and $g(x) \neq 0$, belongs to the family of functions known as **rational functions**. The inverse variation function in Activity 5.2 is a special case of a rational function, where $g(x) = x^n$. The only value at which the function is not defined is any value for which the denominator is zero. If $g(a) = 0$, then $x = a$ is a vertical asymptote of the graph of Q. The horizontal asymptote is the x-axis $(y = 0)$.

EXAMPLE 1 ***Determine the domain and the vertical and horizontal asymptotes for each of the following rational functions.***

a. $f(x) = \frac{3}{x}$ **b.** $g(x) = \frac{10}{x - 4}$ **c.** $h(x) = \frac{-5}{2x + 6}$

SOLUTION

FUNCTION	DOMAIN	VERTICAL ASYMPTOTE	HORIZONTAL ASYMPTOTE
a. $f(x) = \frac{3}{x}$	all real numbers except $x = 0$	$x = 0$	$y = 0$
b. $g(x) = \frac{10}{x - 4}$	all real numbers except $x = 4$	$x = 4$	$y = 0$
c. $h(x) = \frac{-5}{2x + 6}$	all real numbers except $x = -3$	$x = -3$	$y = 0$

10. Consider the function defined by $f(x) = \frac{5}{3x - 6}$.

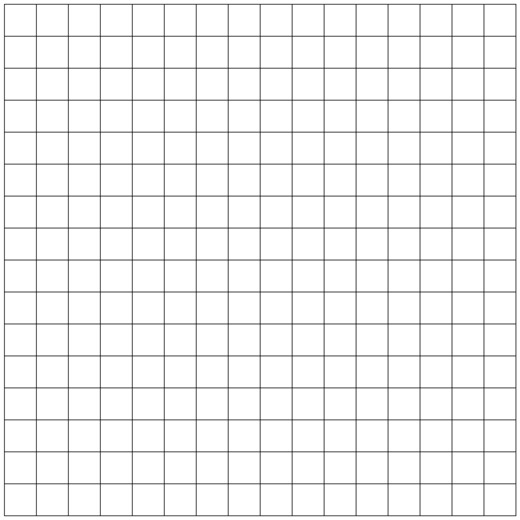
a. Determine the domain of f.

b. Complete the following table.

x	-10	-5	0	1.5	1.9	2	2.1	2.5	3	8	13
$f(x)$											

c. Determine the vertical and horizontal asymptotes of the graph of f.

d. Sketch a graph of *f*.



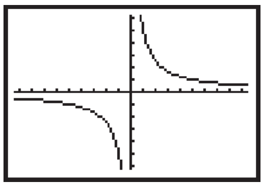
e. Verify the graph using your graphing calculator.

11. Without using your graphing calculator, match the following functions with the accompanying graphs. Use your graphing calculator to verify your matches.

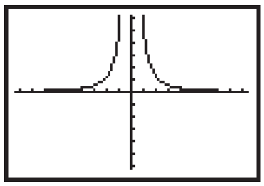
i. $f(x) = \frac{5}{x}$ **ii.** $g(x) = \frac{5}{x^2}$ **iii.** $h(x) = \frac{-5}{x}$

iv. $F(x) = \frac{5}{x + 2}$ **v.** $G(x) = \frac{5}{2x - 4}$

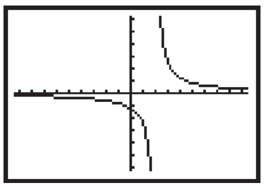
a.



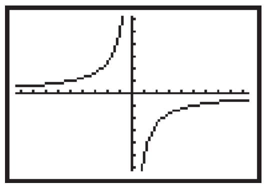
b.



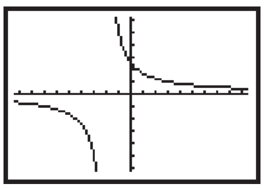
c.



d.



e.



SUMMARY
ACTIVITY 5.3

1. A function Q, defined by an equation of the form

$$Q(x) = \frac{k}{g(x)},$$

where k is any nonzero constant, $g(x)$ is a polynomial and $g(x) \neq 0$, belongs to a family of functions known as **rational functions.** Examples include $f(x) = \frac{10}{x^2}$, $g(x) = \frac{10}{x - 4}$, and $h(x) = \frac{-5}{2x + 6}$.

2. The domain of Q is the set of all real numbers except those values of the input x such that $g(x) = 0$.

3. The vertical asymptote is the vertical line that passes through the x-value for which $g(x) = 0$.

4. The horizontal asymptote is the x-axis $(y = 0)$.

EXERCISES
ACTIVITY 5.3

1. To obtain an estimate of the required volume, V, of timber that must be harvested for a logging company to break even, use the model

$$V = \frac{Y + L}{P - S - F - T},$$

where: V is the required annual logging volume (in cubic meters),
Y is the yard cost (in dollars),
L is the loading cost (in dollars),
P is the selling price (in dollars per cubic meter),
S is the skidding cost (in dollars per cubic meter),
F is the felling cost (in dollars per cubic meter), and
T is the transportation cost (in dollars per cubic meter).

a. Suppose a logging company estimates that the yard cost will be \$25,000, the loading cost will be \$55,000, the skidding cost will be \$1.50 per cubic meter, the felling cost will be \$0.40 per cubic meter, and the transportation cost will be \$0.60 per cubic meter. Write a rule for V as a function of P.

b. Complete the following table.

P (\$)	2.50	3.00	5.00	10.00	25.00
V (m^3)					

c. As the selling price per cubic meter increases, what happens to the corresponding required logging volume, V?

d. Determine the value of V when $P = 2$. What is the practical meaning of the negative value of V?

e. What is the practical domain of this function?

f. Sketch a graph of the function. Use the table in part b to determine an appropriate scale.

2. Two functions are defined by $f(x) = \dfrac{10}{x - 5}$ and $g(x) = \dfrac{10}{5 - x}$.

a. Describe how you can determine the vertical asymptote without graphing.

b. Determine the vertical and horizontal asymptote for the graph of each function.

c. Verify your answers by graphing each function on your graphing calculator.

3. Without graphing, determine the domain of each of the following functions. Then determine the equation of the vertical asymptote for each function. Verify your answers using your graphing calculator.

a. $f(x) = \dfrac{6}{x - 7}$

b. $g(x) = \dfrac{20}{25 - x}$

c. $h(x) = \dfrac{3}{2x - 10}$

d. $F(x) = \dfrac{13}{0.5x - 7}$

e. $G(x) = \dfrac{-4}{2x + 5}$

4. Give examples of two different rational functions that have a vertical asymptote at $x = 10$.

5. As the input value of a rational function gets closer to a vertical asymptote, the output becomes larger in magnitude, approaching either positive or negative infinity. Consider the functions $f(x) = \frac{10}{x - 5}$ and $g(x) = \frac{10}{5 - x}$.

a. Determine the equations of the vertical asymptotes for functions f and g.

b. Describe what happens to the output value when x is near the vertical asymptote but to the right of it.

c. Describe what happens to the output value when x is near the vertical asymptote but to the left of it.

ACTIVITY 5.4

Blood-Alcohol Levels

OBJECTIVES

1. Solve an equation involving a rational expression using an algebraic approach.
2. Solve an equation involving a rational expression using a graphing approach.
3. Determine horizontal asymptotes of the graph of $y = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are first-degree polynomials.

The U.S. Department of Transportation has recommended that states adopt 0.08% blood-alcohol concentration as the legal measure of drunk driving. If you assume that a regular 12-ounce beer is 5% alcohol by volume and that the normal bloodstream contains 5 liters (or 169 ounces) of fluid, your maximum blood-alcohol concentration, B, can be approximately modeled by the function having the equation

$$B = \frac{600n}{w(169 + 0.6n)},$$

where n is the number of beers consumed in 1 hour and w is your body weight in pounds.

1. **a.** Replace w with your body weight. Write an equation for B in terms of n.

   b. Complete the following table using your equation from part a.

NUMBER OF BEERS, *n*	1	2	3	4	5	6	7	8	9	10
BLOOD-ALCOHOL CONCENTRATION, *B*										

2. According to this model, how many beers can you consume in 1 hour without exceeding the recommended legal measure of drunk driving?

3. **a.** A football player friend of yours weighs 232 pounds. Rewrite the equation for B in terms of n. What is his maximum blood-alcohol level if he drinks four beers in 1 hour?

   b. Complete the following table using your equation from part a.

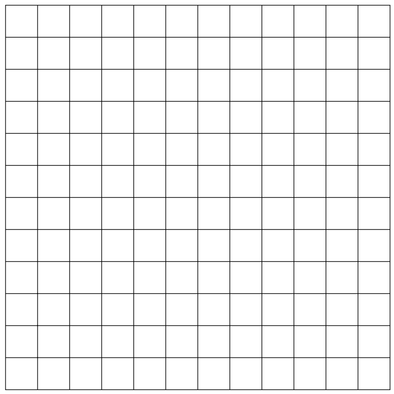
NUMBER OF BEERS, *n*	1	2	3	4	5	6	7	8	9	10
BLOOD-ALCOHOL CONCENTRATION, *B*										

   c. What is the practical domain of the blood-alcohol function of part a?

   d. Does the weight of a person have any impact on the practical domain? Explain.

e. What is the vertical intercept? Does this seem reasonable within the context of the problem?

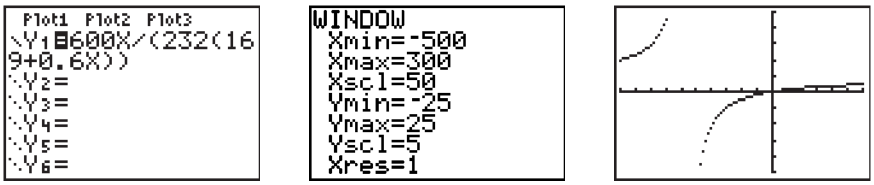
f. Sketch the graph of your blood-alcohol function over the practical domain identified in part c. Use the indicated scale.



Note that the graph of the blood-alcohol function in Problem 3f looks like a line when drawn in the practical domain. The graph of the general function defined by

$$f(x) = \frac{600x}{232(169 + 0.6x)}$$

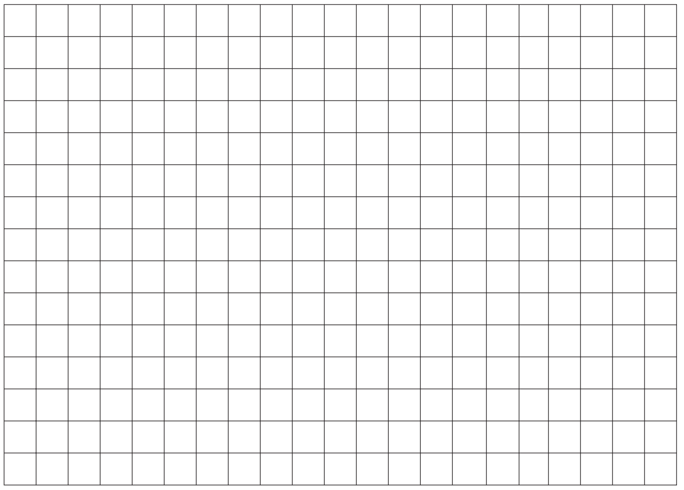
appears as follows.



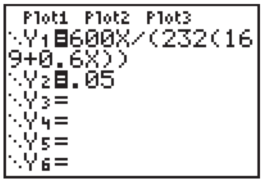
Solving Equations Involving a Rational Expression

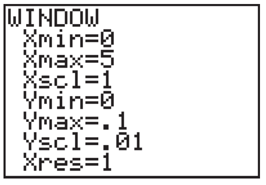
4. a. Your 232-pound football player friend is given a breathalyzer test. The result is a blood-alcohol concentration of 0.05%. Using the blood-alcohol concentration function, write an equation that can be solved to determine the number of beers your friend consumed in the previous hour.

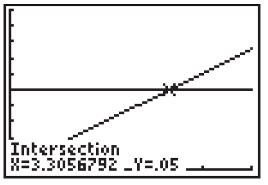
b. Solve the equation in part a graphically on the following grid.



c. Use your graphing calculator to check the answer in part b. Your screen(s) should appear as follows.







The equation in Problem 4 can be solved using an algebraic approach. If a variable appears in the denominator of a fraction, a general approach is to multiply both sides of the equation by the denominator and solve the resulting equation. Example 1 demonstrates this approach as well as two other methods.

EXAMPLE 1 ***Solve the equation*** $\frac{16}{x+3} = 2$**.**

SOLUTION

Method 1. General Case To solve the equation $\frac{16}{x+3} = 2$, first multiply both sides of the equation by the denominator $x + 3$, as follows.

$$(x+3) \cdot \frac{16}{x+3} = 2(x+3)$$
$$16 = 2x + 6$$

Solving for x, you have

$$10 = 2x$$
$$5 = x.$$

Solutions to an equation involving rational functions should always be checked:

$$\frac{16}{5+3} = \frac{16}{8} = 2.$$

Therefore, 5 is a solution.

Method 2. Cross Multiplication

You can also solve the equation $\frac{16}{x+3} = 2$ by applying the following property.

If two ratios $\frac{a}{b}$ and $\frac{c}{d}$ represent the same value, then $\frac{a}{b} = \frac{c}{d}$ is equivalent to $ad = bc$.

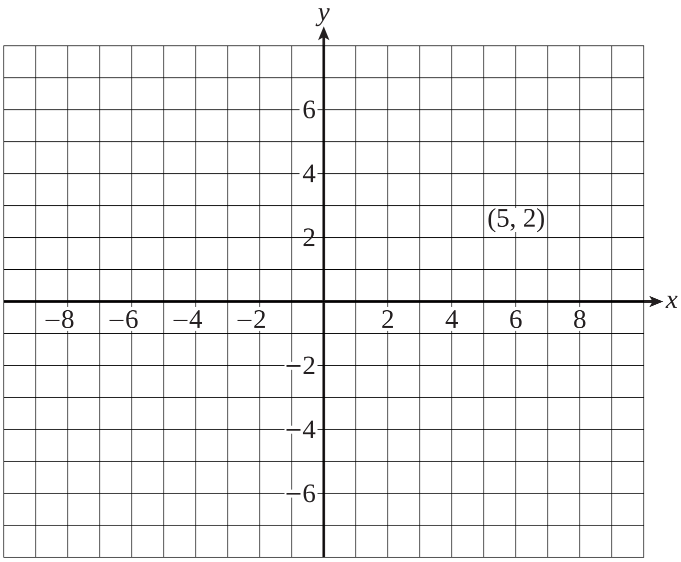
This process is called cross multiplication. Therefore,

$$\frac{16}{x+3} = \frac{2}{1} \quad \text{Cross multiply}$$

$$2(x+3) = 16 \cdot 1$$
$$2x + 6 = 16$$
$$-6 \quad -6$$
$$2x = 10$$
$$x = 5$$

Method 3. Graphical Approach

You can verify that 5 is a solution to the given equation by graphing $y_1 = \frac{16}{x+3}$ and $y_2 = 2$ and determining the x-value of the point of intersection of the two graphs.



5. Solve each of the following equations using an algebraic approach. Verify your answer graphically.

a. $\frac{45}{x} = 9$

b. $\frac{23}{x+2} = 15$

c. $\frac{13}{x} = \frac{2}{5}$

d. $\frac{16}{x^2} = \frac{1}{4}$

6. a. Solve the equation in Problem 4a using an algebraic approach.

b. How does your solution compare with the result in Problem 4c using a graphical approach?

Horizontal Asymptotes

The graphs you have studied so far in this chapter have at least one feature in common. The horizontal asymptote is the horizontal axis. As the input values increase infinitely in the positive direction or decrease infinitely in the negative direction, the output values have always approached zero.

7. Consider the function defined by $y = \frac{2x}{x+5}$. This equation is in the form $y = \frac{f(x)}{g(x)}$, where $f(x) = 2x$ and $g(x) = x + 5$.

a. What is the domain of this function?

b. What is the vertical asymptote?

c. Complete the following table.

x	−100	−50	−10	−5.5	−5.1	−5	−4.9	−4.5	0	50	100
y											

d. What appears to be happening to the *y*-values as the *x*-values increase infinitely to the right or decrease infinitely to the left?

e. What is the horizontal asymptote?

f. Sketch a graph of the function; include horizontal and vertical asymptotes as dotted lines.

g. Verify the graph in part f using your graphing calculator. As a graphing aid, also sketch the graph of $y = 2$.

8. **a.** In the blood-alcohol function $B = \dfrac{600n}{232(169 + 0.6n)}$, as the positive values of n increase in value, what happens to the corresponding values of B?

b. Extend the window of your graphing calculator until you can see the graph leveling off (becoming horizontal) for large values of n. Estimate the equation of the horizontal asymptote of the graph of the blood-alcohol function.

c. Does this asymptote have any practical significance for this application?

9. When comparing smokers and nonsmokers between the ages of 55 and 64, a recent study determined that smokers in the age group had an incidence ratio of 10 for death due to lung cancer. An incidence ratio of 10 means that smokers in this age group are 10 times more likely than nonsmokers to die of lung cancer.

For a given incidence ratio, x, the percent, P, in decimal form of deaths due to a certain illness caused by smoking can be modeled by

$$P = \frac{x - 1}{x}.$$

a. Determine what percent, P, of the deaths due to lung cancer is caused by smoking, x, in the age group (between 55 and 64).

b. Complete the following table.

x	1	2	5	10	30
P					

c. As the incidence ratio, x, increases in value, what happens to the corresponding values of the percent P?

d. Determine the horizontal asymptote. Does this make sense in this situation? Explain.

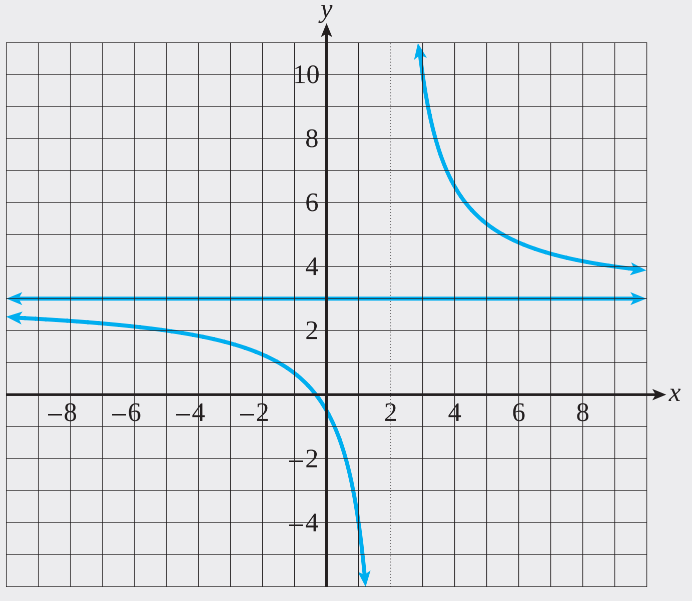
SUMMARY
ACTIVITY 5.4

1. To solve an equation of the form $\frac{f(x)}{g(x)} = \frac{a}{b}$, where $g(x) \neq 0$ and $b \neq 0$, using an algebraic approach,

i. Method 1. Multiply both sides of the equation by the product $b \cdot g(x)$, and solve the resulting equation for x.

ii. Method 2. Cross multiply to obtain $b \cdot f(x) = a \cdot g(x)$, and solve the resulting equation for x.

2. If the y-values of a function R get closer to a number a as the x-values increase infinitely in the positive direction or decrease infinitely in the negative direction, then the graph of the function R has a horizontal asymptote. The equation of the horizontal asymptote is $y = a$. For example, the horizontal asymptote for $R(x) = \frac{3x + 1}{x - 2}$ is $y = 3$. Note that the vertical asymptote is $x = 2$.



EXERCISES
ACTIVITY 5.4

1. You are on the five-year reunion committee for your high school. The committee selects a restaurant that charges $500 to rent a large room to accommodate your group and $50 per person for dinner and a 1-hour open bar. Other expenses include $600 for a DJ; $500 for printing invitations, a program, and name tags; and $400 for decorations.

   a. What are the total fixed costs?

   b. The total cost of the event is a function of the number, *n*, of people who will attend. Write an expression that represents the total cost in terms of *n*.

   c. Your committee decides to divide the total cost evenly among the people attending. Let *m* represent the cost per person if *n* people attend. Write an equation for *m* in terms of *n*.

   d. What is the cost per person if 100 people attend?

   e. What is the practical domain of the cost function?

   f. Complete the following table.

n, NUMBER OF PEOPLE ATTENDING	50	100	150	200	250
m, MEAN COST PER PERSON ($)					

   g. Does the graph of the cost function have a horizontal asymptote? Does it make sense in this situation? Explain.

h. Sketch a graph of the cost function over its practical domain.

2. For each rational function:

i. Determine the domain.

ii. Determine the vertical asymptotes.

iii. Graph the function using your graphing calculator.

iv. Determine the horizontal asymptote by inspecting the graph of your function.

a. $y = \frac{4x}{x + 2}$

i.

ii.

iii.

iv.

b. $y = \frac{1 - x}{x + 1}$

i.

ii.

iii.

iv.

c. $y = \frac{3x}{x - 4}$

i.

ii.

iii.

iv.

d. $y = 12 - \frac{6x}{1 - 2x}$

i.

ii.

iii.

iv.

3. Solve the following equations algebraically and check your results by graphing.

a. $\frac{3x}{2x - 1} = 3$

b. $\frac{x + 1}{5x - 3} = 2$

c. $\frac{-7x}{2.8 + x} = 3.1$

4. In a 20-kilometer race, a runner's average rate (in kilometers per hour) can be expressed as a function of time (in hours) by the equation $r = \frac{20}{t}$.

a. Determine your time to complete the race if you average 16 kilometers per hour.

b. What is your time if you average 18 kilometers per hour?

5. The intensity of the human voice varies inversely to the square of the distance from the source. This is given by the formula from Activity 5.2, Loudness of a Sound: $I = \frac{1500}{d^2}$, where I is decibels and d is distance in feet.

a. Determine the distance from the source when the intensity of the sound is 15 decibels.

b. What is the distance from the source when the intensity is 8000 decibels?

6. As a fund-raising project, the international club at your college decides to publish and sell a calendar. The cost of photographs and typesetting is \$450. It costs \$3 to print and assemble each calendar.

a. What is the total cost of printing 200 calendars?

b. What is the average cost per calendar of printing the 200 calendars?

c. Write an expression for the total cost of printing *n* calendars.

d. Let *A* represent the average cost per calendar. Write an equation that gives *A* as a function of *n*.

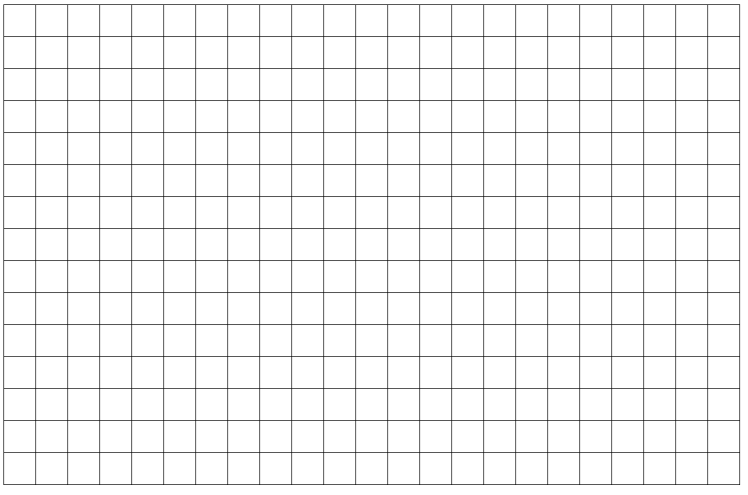
e. Complete the following table.

n **(number)**	50	75	100	500	750	1000
A **(average cost)**						

f. As the input *n* increases, what happens to the output *A*?

g. What is the horizontal asymptote of this function?

h. Verify your answer in part g graphically.



i. Interpret what the horizontal asymptote means in the context of the problem.

j. Suppose you want the average cost to be less than \$3.20. Model this problem with an inequality, solve it algebraically for n, and verify it graphically.

7. The following formula is used by the National Football League (NFL) to calculate quarterback ratings:

$$R = \frac{250C + 12.5Y + 1000T - 1250I + 6.25A}{3A},$$

where

R	=	quarterback rating
A	=	passes attempted
C	=	passes completed
Y	=	passing yardage
T	=	touchdown passes
I	=	number of interceptions

In the 2005–2006 regular season, Tom Brady, quarterback for the New England Patriots, and Brett Favre, for the Green Bay Packers, had the following player statistics:

PLAYER	PASSES ATTEMPTED	PASSES COMPLETED	PASSING YARDAGE	NUMBER OF TOUCHDOWN PASSES	NUMBER OF INTERCEPTIONS
Tom Brady	530	334	4110	26	14
Brett Favre	607	372	3881	20	29

a. Determine the quarterback rating for Tom Brady for the 2005–2006 NFL football season.

b. Determine the quarterback rating for Brett Favre.

c. Suppose an NFL quarterback has the following statistics for the 2006 season: 4000 passing yards, 30 touchdown passes, and 10 interceptions. Write an equation for the quarterback rating *R*, in terms of the number of passes attempted, *A*, and the number of passes completed, *C*.

d. If the quarterback in part c completed 350 passes, how many passes must have been attempted in order to have a quarterback rating of 95?

e. Visit www.nfl.com and select stats to obtain the rating of your favorite quarterback.

8. In a predator-prey model from wildlife biology, the rate, *R*, at which prey are consumed by one predator is approximated by the function

$$R = \frac{0.623n}{1 + 0.046n},$$

measured in prey per week, where *n* is the number of prey available per square mile.

a. If the number of prey available per square mile is 30, what is *R*?

b. Approximately how many prey must be available per square mile for the predator to consume 10 prey per week?

c. Suppose you want to maintain the prey population to ensure that a predator may obtain between 6 and 10 prey per week. One of the necessary equations for this situation was solved in part b above. Write the other equation and solve it.

d. Use the graph of this function to illustrate your solutions to parts b and c.

e. How many prey are required for a predator to consume 20 prey per week? Comment on the practical significance and how this shows up on the graph.

ACTIVITY 5.5

Traffic Flow

OBJECTIVES

1. Determine the least common denominator (LCD) of two or more rational expressions.
2. Solve an equation involving rational expressions using an algebraic approach.
3. Solve a formula for a specific variable.

You are an intern at an architecture firm. The company is designing an auditorium that will be annexed to the local high school building. The rate of traffic flow through the exits is an important consideration. The auditorium will have three exit doors. Two exits are single doors of slightly different sizes. The first exit, by itself, can be used to empty the auditorium in 10 minutes. The second exit can be used to empty the room in 8 minutes. The third exit is a double-wide door that, by itself, can be used to empty the auditorium in 5 minutes.

1. If only the first door is open, how much of the auditorium can be emptied in 5 minutes? 2 minutes? 1 minute?

2. The rate at which a door can be used to empty the auditorium is the fraction of the job that can be completed in 1 minute. In this case, the units of measurement are auditoriums per minute. Determine the rate of emptying for each of the three exits, and record your answers in the following table.

EXIT	RATE OF EMPTYING
First	
Second	
Third	

Your task is to determine the time, T, it takes to empty the auditorium if all three exit doors are open. The relationship between this time, T, and the individual emptying times is given by the formula

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} = \frac{1}{T},$$

where t_1, t_2, and t_3 represent the times that each exit door can be used by itself to empty the auditorium.

Note that three single door rates are added to determine the rate, $\frac{1}{T}$, at which the three doors working together can be used to empty the auditorium.

3. a. Write the equation that can be used to determine the time, T, that it takes for the auditorium to be emptied if all three exits are open.

b. Solve this equation graphically. Use the window Xmin $= 0$, Xmax $= 10$, Ymin $= 0$, Ymax $= 2$, and Yscl $= 0.5$.

Solution Using an Algebraic Approach

An algebraic approach to solving the equation $\frac{1}{10} + \frac{1}{8} + \frac{1}{5} = \frac{1}{T}$ is to eliminate the fractions from the equation. This can be accomplished by first determining the least common denominator (LCD) of the fractions involved in the equation. The following example demonstrates the procedure for determining the LCD.

EXAMPLE 1 ***a. Determine the LCD for $\frac{5}{12}$ and $\frac{7}{45}$.***

SOLUTION

Step 1. Write the prime factorization of each denominator. Express repeated factors as powers.

$$12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$$
$$45 = 3 \cdot 3 \cdot 5 = 3^2 \cdot 5$$

Step 2. Identify the different bases (factors) in step 1.

$$2, 3, 5$$

Step 3. Write the LCD as the product of the highest power of each of the different factors from step 2.

$$\text{LCD} = 2^2 \cdot 3^2 \cdot 5 = 4 \cdot 9 \cdot 5 = 180$$

The smallest number that both 12 and 45 will divide evenly is 180.

b. Determine the LCD for $\frac{11}{6xy^3}$ and $\frac{5a}{9x^2y}$.

SOLUTION

Step 1. $6xy^3 = 2 \cdot 3 \cdot x^1 \cdot y^3$

$9x^2y = 3^2 \cdot x^2 \cdot y$

Step 2. $2, 3, x, y$

Step 3. $\text{LCD} = 2 \cdot 3^2 \cdot x^2 \cdot y^3 = 18x^2y^3$

$18x^2y^3$ is evenly divided by both $6xy^3$ and $9x^2y$.

You are now ready to solve the equation $\frac{1}{10} + \frac{1}{8} + \frac{1}{5} = \frac{1}{T}$ using an algebraic approach.

4. a. Determine the LCD of the rational expressions in the equation $\frac{1}{10} + \frac{1}{8} + \frac{1}{5} = \frac{1}{T}$.

b. Multiply each side of the equation by the LCD, and solve the resulting equation.

c. How does this solution compare to the solution you determined graphically in Problem 3?

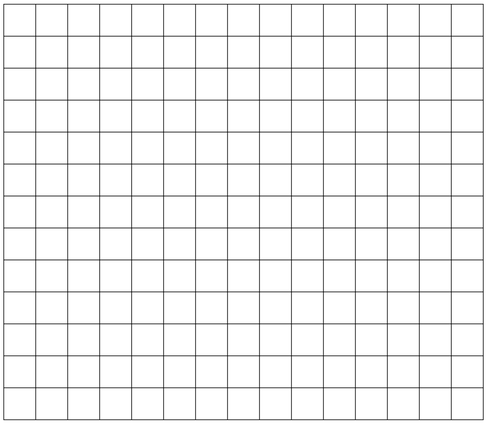
5. An auditorium is to be equipped with two ventilation fans. The first fan can exchange the air in the room in 4 hours. The building code requires a complete exchange of air in the room every 3 hours. To model this situation, use the equation

$$\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$$

where t_1 and t_2 are the exchange times for the fans working alone, and T is the exchange time for the two fans working together.

a. Let x represent the exchange time for the second ventilation fan. Write an equation that can be used to determine x so that the fans working together will satisfy the building code.

b. Solve this equation graphically.



c. Solve the equation in part a algebraically by multiplying both sides by the LCD.

6. Switching to different fans, suppose the first fan can exchange the air in the auditorium twice as fast as the second fan.

a. If x represents the exchange time for the first fan, write an expression that represents the exchange time for the second fan.

b. Working together, the two fans can exchange the air in the room in 4 hours. Using the formula $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$, write an equation that can be solved to determine x.

c. Solve this equation algebraically. Verify your solution graphically.

d. Determine the rate for each fan.

Solving a Formula for a Specified Letter

An alternative approach to solving problems involving formulas of the form $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$ is to use an equivalent formula that has been solved for the variable T.

EXAMPLE 2 ***Solve*** $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$ ***for T.***

SOLUTION

Step 1. Determine the LCD.

$$\text{LCD} = t_1 t_2 T$$

Step 2. Multiply each side of the equation by the LCD.

$$\frac{t_1 t_2 T}{1} \cdot \left(\frac{1}{t_1} + \frac{1}{t_2}\right) = \frac{1}{T} \cdot \frac{t_1 t_2 T}{1}$$

$$\frac{t_1 t_2 T}{t_1} + \frac{t_1 t_2 T}{t_2} = \frac{t_1 t_2 T}{T}$$

Simplifying, you have $t_2 T + t_1 T = t_1 t_2$.

Step 3. Solve the resulting equation for T.

$T(t_2 + t_1) = t_1 t_2$ **T is the common factor on the left side.**

$T = \dfrac{t_1 t_2}{t_2 + t_1}$ **Divide both sides by $t_2 + t_1$.**

7. a. If t_1 and t_2 are the exchange times for the fans working alone, and T is the exchange time for the two fans working together, determine T if $t_1 = 4$ hours and $t_2 = 12$ hours. Use the formula $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$.

b. Repeat part a using the formula $T = \dfrac{t_1 t_2}{t_2 + t_1}$.

c. Compare the results in parts a and b.

8. The following formula is used in work with lenses and mirrors: $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$. Solve the formula for f.

SUMMARY
ACTIVITY 5.5

1. To determine an LCD of two or more rational expressions,

Step 1. Write the prime factorization of each denominator. Express repeated factors as powers.

Step 2. Identify the different bases (factors) in step 1.

Step 3. Write the LCD as the product of the highest power of each of the different factors from step 2.

2. To solve an equation involving rational expressions,

Step 1. Determine the LCD of all denominators in the equation.

Step 2. Multiply each side of the equation by the LCD, and simplify the resulting equation.

Step 3. Solve the resulting equation for the desired variable.

EXERCISES
ACTIVITY 5.5

Many real-life applications involve equations of the form

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c},$$

where a, b, and $c \neq 0$. For example,

$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R}$ is used for resistance of electrical circuits,
$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$ is used in work with lenses and mirrors, and
$\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$ is used to calculate the time it takes to complete a task when two machines or people are working together.

These formulas can also be extended to three or more resistors, lenses, or machines by simply adding additional fractions in each case.

1. Two pumps are working together to empty a gasoline tank.

   a. The emptying times for pump 1 and pump 2 are 30 minutes and 45 minutes, respectively. Determine the time required to empty the tank if both pumps are working. Use the formula $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$, where t_1 and t_2 are the emptying times for pump 1 and pump 2, respectively, and T is the total time required to empty the tank.

   b. Solve the equation $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$ for T.

   c. Using the equation developed in part b, determine T if $t_1 = 20$ minutes and $t_2 = 15$ minutes.

   d. If one pump can empty the tank in 40 minutes, how fast must a second pump work for the pumps working together to empty the tank in 10 minutes? Use $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$.

Exercise numbers appearing in color are answered in the Selected Answers appendix.

e. Suppose three pumps are working together to empty the tank, with emptying times of 25 minutes, 30 minutes, and 50 minutes. How long will it take to empty the tank if all three pumps are working simultaneously? Use the formula $\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} = \frac{1}{T}$.

2. Solve each of the following equations using an algebraic approach. Verify your answers using a graphing approach.

a. $\frac{10}{x + 1} = 4$

b. $\frac{2}{x} + \frac{3}{x} = 1$

c. $\frac{1}{x} + \frac{1}{3x} = \frac{1}{5}$

d. $\frac{3}{x} + \frac{2}{x} = \frac{4}{x}$

In Exercises 3–5, use the formula $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$.

3. The custodian in the mathematics building can buff the main floor 2 minutes faster than his supervisor. If working together they can buff the floor in 35 minutes, how long does it take the supervisor working alone to buff the main floor of the mathematics building? (*Hint:* Let t represent the supervisor's time, and $t - 2$ represent the custodian's time.)

4. It takes you 4 hours working alone to clean your apartment, and your spouse takes 5 hours and 15 minutes. If you begin at noon and work together, will you complete the cleaning in time to leave for the game at 2:30? How late or early will you be?

5. One of your jobs as a work-study student at your college is sending out mailings (stuffing, sealing, and stamping envelopes). You can work twice as fast as your supervisor. If working together you complete a job in 7 hours, how long would it have taken you to complete the job by yourself? (*Hint:* Let t represent your total time working alone and $2t$ the time of your supervisor working alone.)

6. Solve each of the following formulas for the indicated variable. Express your answer as a single fraction.

a. $\frac{1}{a} + \frac{2}{b} = \frac{3}{c}$, solve for a.

b. $\frac{1}{x + y} = \frac{1}{z}$, solve for x.

c. $\frac{1}{a} + \frac{2}{b} = \frac{3}{c}$, solve for c.

ACTIVITY 5.6

Electrical Circuits

OBJECTIVES

1. Multiply and divide rational expressions.
2. Add and subtract rational expressions.
3. Simplify a complex fraction.

In performing some technical work in your new job at a local electronics firm, you need to be familiar with resistors combined in a circuit. The total resistance, R, of two resistors in a parallel circuit is modeled by the formula

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}},$$

where R_1 and R_2 are the two resistors in the circuit, measured in ohms.

1. Calculate the total resistance for each pair of resistors.

R_1 (OHMS)	R_2 (OHMS)	R
10	10	
10	5	
15	5	
20	10	

If one resistor has to be 10 ohms, then the formula

$$R = \frac{1}{\frac{1}{10} + \frac{1}{R_2}}$$

expresses the total resistance as a function of the second resistor's value.

Now the total resistance, R, is a function of R_2. The right side of the equation is a fraction in which fractions also appear in the denominator.

DEFINITION

A fraction that contains fractions in either its numerator or denominator, or both, is called a **complex fraction**.

EXAMPLE 1 ***The following are examples of complex fractions.***

$$\frac{1}{\frac{1}{10} + \frac{1}{R_2}} \qquad \frac{\frac{50}{x}}{\frac{100}{x^2 + 5x}} \qquad \frac{4 + \frac{1}{x}}{\frac{10}{x^2} - \frac{2}{x}}$$

2. To make the equation $R = \frac{1}{\frac{1}{10} + \frac{1}{R^2}}$ less cumbersome to work with, simplify the right side of the equation so that it is written as a single fraction (with only one dividing line). This can be accomplished as follows.

a. Determine the LCD of the fractions $\frac{1}{10}$ and $\frac{1}{R_2}$.

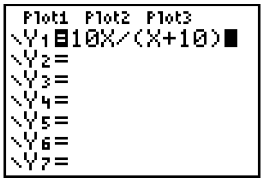
b. Add the fractions $\frac{1}{10}$ and $\frac{1}{R_2}$ by writing each fraction as an equivalent fraction that has the LCD as the denominator.

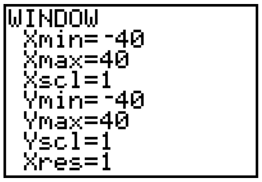
c. Divide the numerator by the denominator, and simplify.

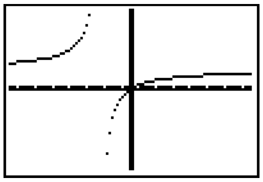
3. a. Using your graphing calculator, sketch a graph of the function defined by

$$R = \frac{10R_2}{R_2 + 10},$$

where R is the total resistance of two resistors in a parallel circuit, one having resistance 10 ohms and the second having a resistance represented by R_2. Your screens should resemble the following.







b. What is the domain of this function?

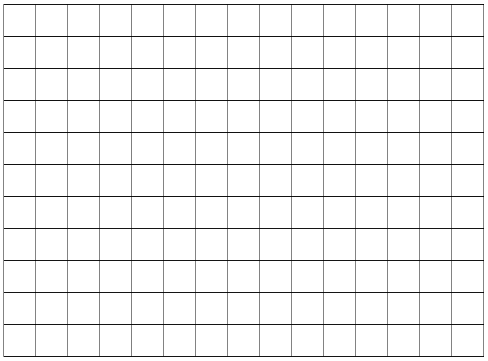
c. What is the practical domain?

d. What is the horizontal asymptote of this function?

e. Interpret its practical meaning.

4. a. Write an equation that you can use to determine the size of the second resistor you would need to add to the circuit to make a total resistance of 7 ohms.

b. Solve this equation graphically.



c. Solve the equation in part a algebraically.

5. If resistors are available only in increments of 0.1 ohm, what size would you use to get as close as possible, but still have a total resistance of at least 7 ohms?

Operations with Rational Expressions

As you discovered in Problem 2, simplifying complex fractions involves a lot of work with rational expressions. The following examples illustrate how to perform operations with algebraic fractions. Appendix A contains several additional examples and practice exercises involving operations with rational expressions.

EXAMPLE 2 ***Simplify the following complex fractions.***

a. $\dfrac{\dfrac{50}{x}}{\dfrac{100}{x^2 + 5x}}$

Both the numerator and the denominator of the complex fraction contain single rational expressions. Therefore, write the complex fraction as a division problem, and divide.

$$\frac{\frac{50}{x}}{\frac{100}{x^2 + 5x}} = \frac{50}{x} \div \frac{100}{x^2 + 5x}$$ Divide using the division rule $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

$$= \frac{50}{x} \cdot \frac{x^2 + 5x}{100}$$ Simplify, if possible.

$$= \frac{\cancel{50}}{\cancel{x}} \cdot \frac{\cancel{x}(x + 5)}{2 \cdot \cancel{50}}$$

$$= \frac{x + 5}{2}$$

b. $\dfrac{4 + \dfrac{1}{x}}{\dfrac{10}{x^2} - \dfrac{2}{x}}$

The rational expressions in the numerator and denominator of the complex fraction can each be combined into a single rational expression.

Step 1. $\frac{4}{1} + \frac{1}{x} = \frac{4}{1} \cdot \frac{x}{x} + \frac{1}{x} = \frac{4x}{x} + \frac{1}{x} = \frac{4x + 1}{x}$

Step 2. $\frac{10}{x^2} - \frac{2}{x} = \frac{10}{x^2} - \frac{2x}{xx} = \frac{10}{x^2} - \frac{2x}{x^2} = \frac{10 - 2x}{x^2}$

Step 3. Now divide the numerator of the complex fraction by the denominator.

$$\frac{4 + \frac{1}{x}}{\frac{10}{x^2} - \frac{2}{x}} = \frac{4x + 1}{x} \div \frac{10 - 2x}{x^2} = \frac{(4x + 1)}{\cancel{x}} \cdot \frac{\cancel{x} \cdot x}{(10 - 2x)} = \frac{4x^2 + x}{10 - 2x}$$

6. Simplify the following complex fraction.

$$\frac{\frac{4}{x+3}}{\frac{1}{x+2}+\frac{3}{x}}$$

SUMMARY
ACTIVITY 5.6

1. To **multiply or divide** rational expressions,

a. Factor the numerator and denominator of each fraction completely.

b. Divide out the common factors (cancel).

c. Multiply remaining factors.

d. In division, proceed as above after inverting the divisor (the fraction after the division sign).

2. To **add or subtract** rational expressions,

a. Determine the LCD (least common denominator).

b. Build each fraction to have the LCD.

c. Add or subtract numerators.

d. Place the numerator over the LCD, and simplify if possible.

3. To simplify a complex fraction by simplifying the numerator and denominator,

a. Express the numerator as a single fraction.

b. Express the denominator as a single fraction.

c. Divide the numerator by the denominator.

d. Simplify, if possible.

EXERCISES
ACTIVITY 5.6

1. An EMS vehicle has a single-tone siren with a pitch of approximately 330 hertz (Hz). You are standing on the street corner as the vehicle approaches you at 40 miles per hour. Due to the Doppler effect, the actual pitch you hear is not 330 hertz. The pitch, h, that you hear is modeled by

$$h = \frac{a}{1-\frac{s}{770}},$$

where a is the actual pitch and s is the speed of the source of the sound in miles per hour.

a. Is the pitch you hear due to the Doppler effect lower or higher than the actual pitch?

b. Simplify the original equation by rewriting the complex fraction on the right side as a single fraction (using only one dividing line).

c. Redo part a using the new equation obtained from part b. How do the results compare?

d. What pitch sound would you hear if the EMS vehicle were traveling at 60 miles per hour?

2. If three resistors having resistances R_1, R_2, and R_3 are connected in parallel, their combined resistance, R, is modeled by the formula

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}.$$

a. Determine R if R_1 is 4 ohms, R_2 is 8 ohms, and R_3 is 12 ohms.

b. Simplify the complex fraction on the right side of the original formula.

c. Redo part a using the new formula from part b. How do the answers compare?

3. Simplify the following complex fractions.

a. $\dfrac{\frac{1}{2} - \frac{2}{x}}{x - 4}$

b. $\dfrac{\frac{1}{x} - \frac{1}{2}}{\frac{1}{x^2} - \frac{1}{4}}$

c. $\dfrac{x + \frac{2x - 6}{x - 1}}{\frac{x}{3} - \frac{3}{x}}$

d. $\dfrac{\frac{x}{2} - 1}{x - \frac{4}{x}}$

4. You decide to buy a new car, but you are concerned about the amount of the monthly payments. The amount, A, of each monthly payment is modeled by the formula

$$A = \frac{Pi}{1 - \frac{1}{(1 + i)^n}},$$

where P represents the principal or the amount borrowed,
i is the monthly interest rate, and
n is the number of monthly payments.

a. The car you are looking at costs $16,000 at 1% monthly interest. If you want to pay for the car over 60 months, how much will your monthly payment be?

b. Simplify the complex fraction on the right side of the original formula.

c. Use the simplified formula to determine the monthly payments. How does your answer compare to the answer in part a?

CLUSTER 1 What Have I Learned?

1. Make a list of some of the special features of the graphs of rational functions.

2. Describe the connection, if any, between the domain of a rational function and the equation of its vertical asymptote.

3. Describe the algebraic steps required to determine the vertical asymptote(s) of the graph of a rational function.

4. For what values of k will the graph of $y = \frac{k}{x}$ be in the second and fourth quadrants?

5. **a.** Describe the algebraic steps required to solve $10 = \frac{35}{1 + 5x}$.

   b. Explain the technique for solving the equation in part a graphically.

6. Explain how you would determine the horizontal asymptote of the rational function $f(x) = \frac{6x + 1}{2 - 3x}$.

CLUSTER 1 How Can I Practice?

1. Describe the relationship between the graphs of $f(x) = \frac{1}{x}$ and $g(x) = \frac{-1}{x}$.

2. Describe the relationship between the graphs of $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^3}$.

3. **a.** Suppose you are taking a trip of 145 miles. Assume that you drive the entire distance at a constant speed. Express your time to take this trip as a function of your speed.

   b. What is the practical domain of this function?

   c. Using the equation for the function, determine the domain.

4. Determine the domain of each of the following functions. Then give the equation of the vertical asymptote of each function.

   a. $g(x) = \frac{10}{x + 5}$

   b. $f(x) = \frac{5}{13 - 2x}$

   c. $g(x) = \frac{-3}{5x - 8}$

   d. $h(x) = \frac{0.02}{5.7x - 3.2}$

5. The weight of a body above the surface of Earth varies inversely with the square of the distance from the center of Earth. If an object weighs 100 pounds when it is 4000 miles from the center of Earth, how much will it weigh when it is 4500 miles from the center?

Answers to all How Can I Practice exercises are included in the Selected Answers appendix.

6. A manufacturer of lawn mowers uses the function $C(x) = \dfrac{132x + 75{,}250}{x}$ to model the average cost per lawn mower, in dollars, where x is the number of lawn mowers produced.

a. What is the practical domain of this function?

b. What is the minimum number of lawn mowers that must be manufactured to bring the average cost per lawn mower down to $199? Solve algebraically and verify graphically with your calculator.

7. The concentration of a drug in the bloodstream, measured in milligrams per liter, can be modeled by the function

$$C = \frac{14t}{3t^2 + 2.5}$$

where t is the number of minutes after injection of the drug.

a. How long after injection will it take for the concentration to equal 0.05 milligram per liter? Solve algebraically and check graphically.

b. Use the graph to determine when the drug will be at its highest concentration.

8. Solve each equation algebraically. Verify your answer graphically.

a. $\dfrac{3}{x + 1} = 4$

b. $\dfrac{3x}{2x - 5} = 10$

c. $\dfrac{4}{x + 3} + 12 = 52$

d. $\dfrac{2.4x}{1 + 0.3x} = 5.8$

9. As an object rises, the effect of Earth's gravitational pull on the object is reduced.

If an object weighs E kilograms at sea level, then the weight, W (also in kilograms), of the object at a distance of h kilometers above sea level is modeled by the function

$$W = \frac{E}{\left(1 + \frac{h}{6400}\right)^2}.$$

a. Suppose you are flying in a commercial jetliner 15 kilometers above sea level. Replace E with your body weight, measured in kilograms (1 kilogram = 2.2 pounds), and calculate your weight at 15 kilometers above sea level.

b. If an astronaut weighs 70 kilograms at sea level, write a function equation that expresses the astronaut's weight as a function of his or her distance above sea level.

c. Complete the following table using the function equation from part b.

h	0	10	100	1000	1500	2000	10,000	20,000
W								

d. As the height, h, of the space shuttle increases, what happens to the corresponding weight of the astronaut?

e. As the space shuttle reaches its orbiting altitude of 650 kilometers above sea level, what is the weight of the astronaut?

f. What is the practical domain of the weight function?

g. Use your graphing calculator to sketch a graph of the weight function. Use the window Xmin = 0, Xmax = 40000, Ymin = 0, and Ymax = 80.

h. At what altitude does the astronaut's weight equal one-half of what it is at sea level?

10. An electrical circuit has three resistors. The total resistance of the circuit, R, is related to the individual resistances R_1, R_2, and R_3 by the equation

$$\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{R}.$$

a. You know that $R_1 = 4$ ohms, $R_2 = 6$ ohms, and the total resistance of the circuit is 2 ohms. Determine R_3.

b. Solve the equation $\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{R}$ for R.

c. Using the equation developed in part b, determine R if $R_1 = 4$ ohms, $R_2 = 6$ ohms, and $R_3 = 12$ ohms.

11. Solve each of the following equations using an algebraic approach. Verify your answers using a graphing approach.

a. $\frac{3}{x - 1} = 10$

b. $\frac{2}{x} - \frac{4}{x} = 2$

c. $\frac{1}{x} + \frac{1}{4x} = \frac{1}{4}$

d. $\frac{3}{x} - 4 = \frac{2}{x}$

12. The average speed, s, of your round-trip commute from home to campus is modeled by

$$s = \frac{2d}{\frac{d}{r_1} + \frac{d}{r_2}},$$

where d is the one-way distance from home, r_1 is your average morning commute speed, and r_2 is your average afternoon commute speed.

a. Your one-way commute to campus is 15.3 miles. Your average morning commute speed is 45 miles per hour. Your average afternoon commute speed is 40 miles per hour. What is your average speed, s?

b. Simplify the original equation by rewriting the complex fraction on the right side as a single fraction.

c. Redo part a using the new equation obtained in part b. How do the results compare?

13. Simplify the following complex fractions.

a. $\dfrac{4 + \frac{2}{x}}{1 - \frac{3}{x}}$

b. $\dfrac{\frac{x}{5} - \frac{5}{x}}{\frac{1}{5} + \frac{1}{x}}$

c. $\dfrac{\frac{1}{x+2}}{1 + \frac{1}{x+2}}$

CLUSTER 2 Radical Functions

ACTIVITY 5.7

Hang Time

OBJECTIVES

1. Determine the domain of a radical function defined by $y = \sqrt{g(x)}$, where $g(x)$ is a polynomial.
2. Graph functions having equation $y = \sqrt{g(x)}$ and $y = -\sqrt{g(x)}$.
3. Identify the properties of the graph of $y = \sqrt{g(x)}$ and $y = -\sqrt{g(x)}$.

During the seventeenth century, many mathematicians were interested in projectiles, primarily for their military applications. More recently, studies have been done in the area of sports, where the shot put, the javelin, and a punted football are the projectiles of interest. Dr. Peter Brancazio, a physics professor at Brooklyn College, has spent considerable time studying one of the most famous human projectiles, Michael Jordan. In particular, Brancazio has been interested in Jordan's hang time, the length of time elapsed from the instant that Jordan leaves the floor to the instant that he touches it again. Using basic physics, Brancazio knows that the hang time, T, of any jump (in seconds) is related to the height, H, of that jump (in feet) by the formula

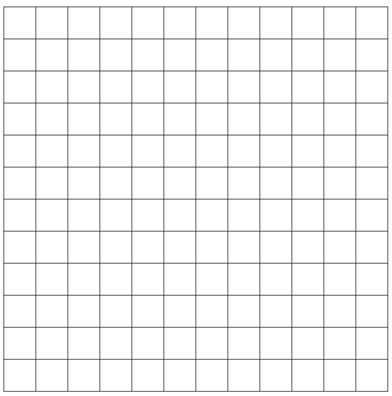
$$H = 4T^2, \qquad T \geq 0.$$

1. Using the formula with the table feature of your calculator, complete the following table.

T **(sec.)**	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
H **(ft.)**									

2. In the formula $H = 4T^2$, the height, H, is a function of the hang time, T. Determine the practical domain and range of this function.

3. Sketch a graph of the function $H = 4T^2$.



Although the formula $H = 4T^2$ is useful, the form of the equation is not entirely satisfactory. The formula implies that the height of the jump, H, depends on hang time, T. Therefore, T is the input and H is the output. The equation may be true in a relational sense, but it may be more logical to say that the hang time depends on the height of the jump. Therefore, it makes more sense to input values for H and find corresponding output values of T.

4. Suppose you measure the height of a jump and obtain a value of 1.44 feet. Determine the corresponding hang time, T.

5. The process of determining hang times for given jump heights would be greatly simplified if you had a formula that expressed T as a function of H.

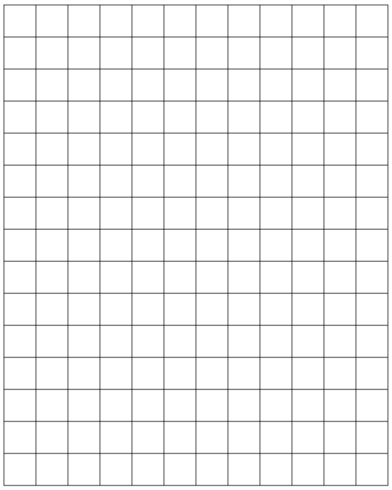
a. Solve the formula $H = 4T^2$ for T, where $T \geq 0$.

b. Complete the following table using the formula you obtained in part a.

H **(ft.)**	0	0.16	0.64	1.44	2.56	4	5.76	7.84	10.24
T **(sec.)**									

c. What are the practical domain and range of this new function?

6. Sketch the graph of $T = \frac{1}{2}\sqrt{H}$, where H is the input, T is the output, and $0 \leq H \leq 4$.



7. a. How are the ordered pairs in the tables in Problems 1 and 5 related?

b. What is the relationship between the domains and ranges of the two functions defined by $H = 4T^2$ and $T = \frac{1}{2}\sqrt{H}$, where $T \geq 0$?

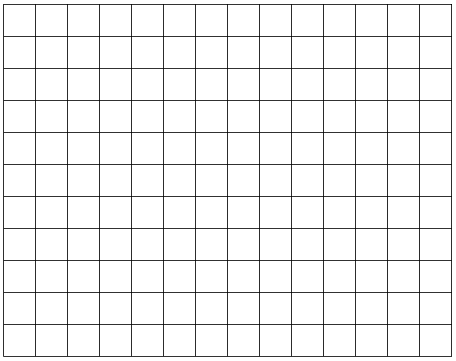
Recall that a fundamental characteristic of *inverse* functions is that their domains and ranges are interchanged and the inputs and outputs are reversed in their tables. Therefore, these functions are inverses.

Radical Functions

The investigation of the properties of the function defined by $T = \frac{1}{2}\sqrt{H}$ is somewhat limited because of the restrictions placed on the variables T and H, which represent real-world quantities.

8. a. Consider the general function defined by $F(x) = \frac{1}{2}\sqrt{x}$. What is the domain of F?

b. Sketch a graph of F.



c. What is the range of F?

Recall that the square root of a negative number is not a real number. Therefore, the domain of a function defined by an equation of the form $y = \sqrt{g(x)}$, where $g(x)$ is a polynomial, is the set of all real numbers for x such that $g(x) \geq 0$.

EXAMPLE 1 ***Determine the domain of the function defined by $f(x) = \sqrt{2x - 10}$.***

SOLUTION

You need to determine all values of x such that $2x - 10 \geq 0$. Therefore,

$$2x - 10 \geq 0$$
$$2x \geq 10$$
$$x \geq 5.$$

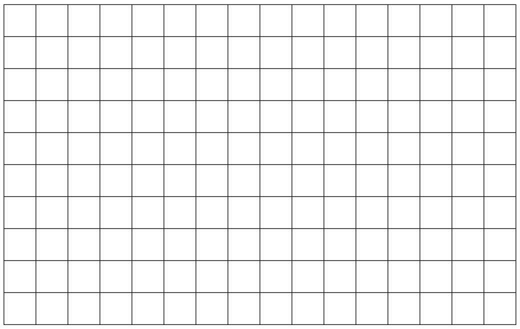
The domain is all real numbers greater than or equal to 5.

9. Consider the functions defined by the following equations.

$$f(x) = \sqrt{x} \qquad g(x) = \sqrt{x + 2} \qquad h(x) = \sqrt{x - 3}$$

a. Determine the domain of each function.

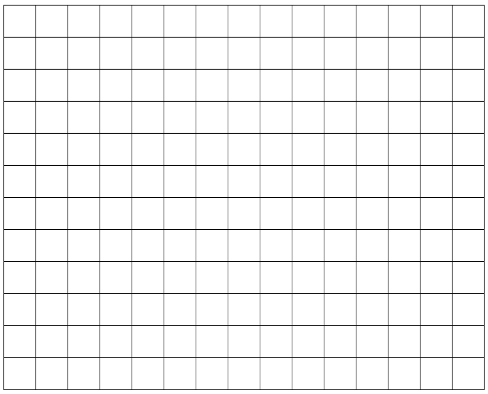
b. Sketch graphs of the functions f, g, and h on the same coordinate system.



c. Are the functions f, g, and h increasing or decreasing?

10. a. Determine the domain and range of the functions defined by $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{x}$.

b. Sketch graphs of f and g on the same coordinate system.



c. How would you obtain the graph of g from the graph of f?

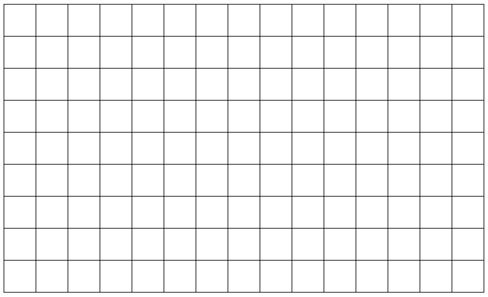
11. **a.** Determine the domains of the functions defined by $h(x) = \sqrt{x - 3}$ and $F(x) = \sqrt{3 - x}$.

b. Complete the following tables.

x	3	4	7	12	19	28
h(*x*)						

x	3	2	−1	−6	−13	−22
F(*x*)						

c. Sketch the graphs of *h* and *F* on the same coordinate system.



d. Determine whether the functions *h* and *F* are increasing or decreasing.

12. Consider the function defined by $f(x) = -\sqrt{3x - 6}$.

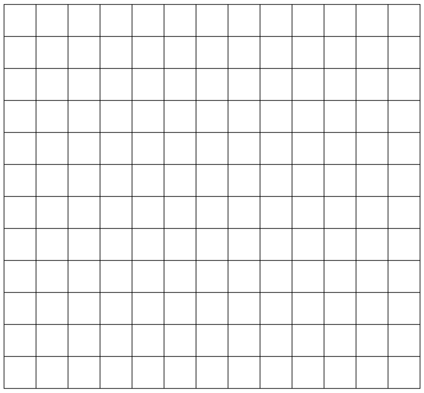
a. Determine the domain of *f*.

b. Determine the *x*-intercept of *f*.

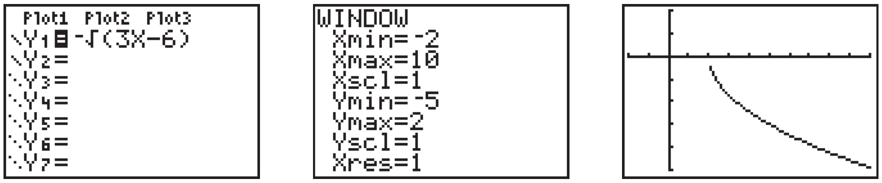
c. Complete the following table. If necessary, approximate *f*(*x*) to the nearest tenth.

x	2	3	4	5	8	10
f(*x*)						

d. Sketch a graph of f.



e. Use your graphing calculator to sketch a graph of f. Your screens should appear as follows.



f. Determine the range of f.

g. Does the graph of f have a maximum value? If so, what is it?

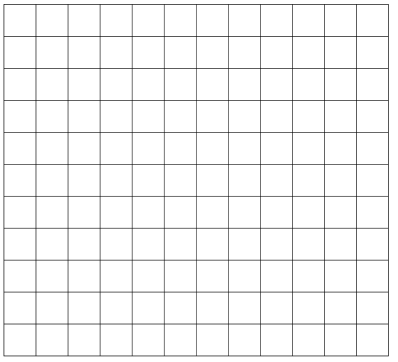
13. Consider the function defined by $g(x) = \sqrt{x^2 + 4}$.

a. Determine the domain of g.

b. Complete the following table.

x	-4	-2	0	2	4	6
$g(x)$						

c. Sketch a graph of g.



d. Verify your graph in part c using your graphing calculator.

e. Determine the range of g.

f. Does the graph of g have a maximum or minimum value? If so, what is it?

14. a. Complete the following table for $f(x) = \sqrt{x^2} + \sqrt{4}$.

x	-4	-2	0	2	4	6
$f(x)$						

b. How do the outputs in the table in part a compare with the outputs for $g(x) = \sqrt{x^2 + 4}$ in Problem 13b?

c. Use your graphing calculator to graph $f(x) = \sqrt{x^2} + \sqrt{4}$.

d. How does the graph of $f(x) = \sqrt{x^2} + \sqrt{4}$ compare to the graph of $g(x) = \sqrt{x^2 + 4}$ in Problem 13c?

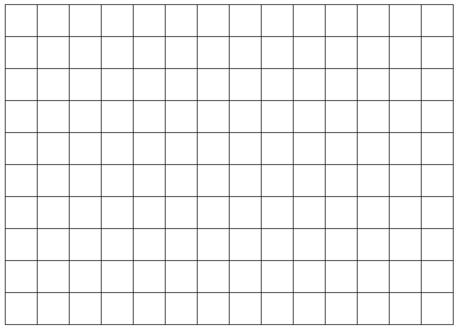
e. Is the expression $\sqrt{x^2 + 4}$ equivalent to $\sqrt{x^2} + \sqrt{4}$? Explain.

Problem 14 demonstrates the following important fact about radicals.

$$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$$

Recall from Chapter 2 that the expression $\sqrt{x}$ can also be written as $x^{1/2}$. The fractional exponent means that you are taking the positive square root of x. Therefore, the expression $\sqrt{3x - 2}$ can also be written as $(3x - 2)^{1/2}$.

15. Sketch the graph of $y = (3x - 2)^{1/2}$. Use your graphing calculator to verify that this is the same as $y = \sqrt{3x - 2}$. What is the domain?



Space and Radicals

16. It is not unreasonable to imagine that some day travel in space will be a common occurrence. According to Einstein's theory of relativity, time would pass more quickly on Earth than it would for someone who is traveling in a spacecraft at a velocity close to the speed of light. As a result, a person on Earth would age more rapidly than a space traveler. The formula

$$A = F\sqrt{1 - \tfrac{v^2}{c^2}}$$

models the relationship between the aging rate, A, of an astronaut and the aging rate, F, of a person on Earth. The variable v represents the astronauts' velocity in miles per second, and c represents the speed of light (approximately 186,000 miles per second).

a. Suppose you are on a spaceship that is traveling 80% of the speed of light. What is your aging rate compared to a person on Earth?

b. If you travel at 80% of the speed of light for 1 year (as you perceive it), approximately how much time has passed for a person on Earth?

c. Suppose you are traveling at a velocity very close to the speed of light. Substitute c (speed of light) for v in the formula, and simplify. Interpret your results.

17. Escape velocity is the minimum speed that an object must attain to escape a planet's pull of gravity. Escape velocity, V, is modeled by the formula

$$V = \sqrt{\frac{2Gm}{r}},$$

where G is the universal gravitational constant,
m is the mass of the planet, and
r is the radius of the planet.

If Earth has mass 5.97×10^{24} kilograms and radius 6.37×10^{6} meters, then determine the escape velocity for Earth. Round your answer to the nearest whole number. Use $G = 6.67 \times 10^{-11}$ m^3/kg · sec.^{2}.

SUMMARY
ACTIVITY 5.7

Square Root Notation and Terminology

1. $\sqrt[2]{n}$, or simply $\sqrt{n}$, represents the square root of a nonnegative number n. The 2 is called the **index.** In general, when you are working with square roots, the 2 is omitted.
2. The symbol $\sqrt{\ }$ is called the **radical sign.** The expression under the radical is called the **radicand.**
3. $\sqrt{n} \geq 0$, where $n \geq 0$
4. $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$, where $a \geq 0$, $b \geq 0$
5. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
6. $\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$

Properties of Radical Functions

1. The function defined by $y = \sqrt{g(x)}$ has domain all real x with $g(x) \geq 0$.
2. The function defined by $y = -\sqrt{g(x)}$ has domain all real x with $g(x) \geq 0$. The graph of $y = -\sqrt{g(x)}$ is the reflection of the graph of $y = \sqrt{g(x)}$ about the x-axis.

EXERCISES
ACTIVITY 5.7

1. Using your calculator, determine the value of each number to the nearest hundredth, if necessary.

   a. $\sqrt{30}$ **b.** $6^{1/2}$ **c.** $(\sqrt{13})^4$ **d.** $(9^{1/2})^3$

2. Determine the domain of each function.

   a. $f(x) = \sqrt{x - 5}$ **b.** $g(x) = \sqrt{3x + 2}$

   c. $h(x) = \sqrt{6 - 2x}$ **d.** $R(x) = -\sqrt{2x}$

3. The following table gives the amount of new money (in billions of dollars) loaned to students in each of the given years.

YEAR	1993	1994	1995	1996	1997	1998	1999	2000
x, NUMBER OF YEARS SINCE 1993	0	1	2	3	4	5	6	7
A(*x*), AMOUNT OF NEW MONEY FOR STUDENT LOANS (in $ billions)	12.0	18.0	22.0	24.0	25.5	27.2	28.7	30.0

This data can be modeled by the function

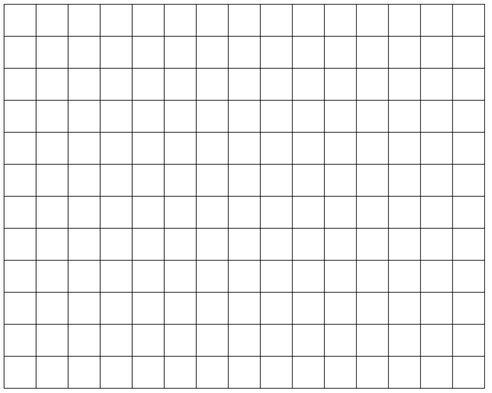
$$A(x) = 6.8\sqrt{x} + 12.$$

a. Determine the *A*-intercept. What does this intercept represent in this situation?

b. Complete the following table using the given equation. How well does the equation represent the actual data?

x	0	1	2	3	4	5	6	7
A(*x*)								

c. Sketch a graph of the student loan function represented by $A(x) = 6.8\sqrt{x} + 12$.



d. Use the model to predict the amount of new money to be loaned to students in 2005.

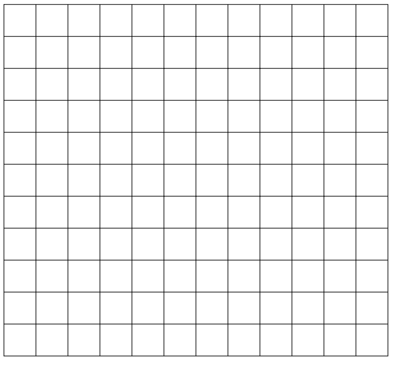
4. Describe how to obtain the graph of the second function from the graph of the first.

a. $g(x) = \sqrt{x}, \quad h(x) = \sqrt{x} + 1$

b. $f(x) = \sqrt{x}, \quad g(x) = -\sqrt{x}$

c. $h(x) = 2\sqrt{x}, \quad H(x) = 2\sqrt{x + 1}$

5. a. Sketch a graph of $f(x) = x^2$ and $g(x) = \sqrt{x}$ on the same coordinate system. Use the graphs to answer parts b and c.



b. Is $x^2 > \sqrt{x}$ for $0 < x < 1$? Explain.

c. Is $x^2 > \sqrt{x}$ for $x > 1$? Explain.

6. Which of the following functions increases more rapidly for $x > 1$: $f(x) = \sqrt{x}$ or $g(x) = \ln(x)$?

7. For each of the given functions,

i. determine the domain.

ii. determine the x- and y-intercepts.

iii. sketch a graph.

a. $f(x) = -\sqrt{4x + 8}$

i.

ii.

iii.

b. $g(x) = \sqrt{5 - x}$

i.

ii.

iii.

c. $h(x) = \sqrt{x^2 + 9}$

i.

ii.

iii.

8. The surface area, S, of a cone is modeled by the formula

$$S = \pi r\sqrt{r^2 + h^2},$$

where r is the radius of the base and h is the height.

An umbrella is in the shape of a cone having radius 4 feet and height 2 feet. Determine the amount of material needed to make the umbrella.

9. A shipping carton in the shape of a rectangular box has dimensions 12 inches $\times$ 24 inches $\times$ 17 inches. The diagonal, d, of a rectangular box is modeled by

$$d = \sqrt{w^2 + l^2 + h^2},$$

where w is the width, l is the length, and h is the height.

Will an umbrella measuring 34 inches long fit in the carton?

10. Law enforcement officers investigating a car accident often use the formula $s = \sqrt{30fl}$ to estimate a car's speed, s, in miles per hour based upon the length, l, of the skid marks. The f in the formula represents the road condition at the time of the accident.

    a. On dry pavement the f-value is 0.85. Write a function equation for speed, s, as a function of the length of the skid marks, l, on dry pavement.

    b. Estimate the speed of a car if the skid marks on dry pavement are 90 feet long.

    c. What is the practical domain for the function in part a?

    d. Graph this function using your graphing calculator.

    e. Use the graph to determine the length of the skid marks on dry pavement if the car was traveling at 70 miles per hour when the brakes were applied.

ACTIVITY 5.8

Falling Objects

OBJECTIVE

1. Solve an equation involving a radical expression using a graphical and algebraic approach.

If an object is dropped from a tall building, the time, t, in seconds, it takes for the object to strike the ground is modeled by

$$t = \frac{\sqrt{d}}{4} = \frac{1}{4}\sqrt{d},$$

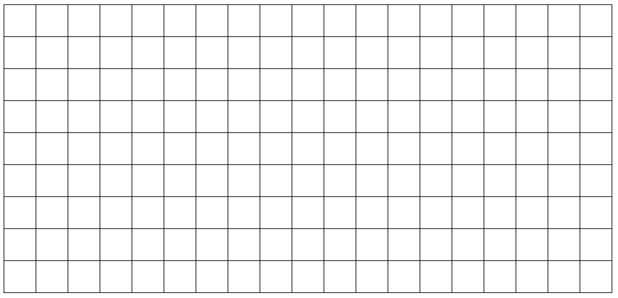
where the input, d, is the distance traveled in feet. The time it takes for the object to hit the ground varies directly to the square root of the distance traveled. The number $\frac{1}{4}$ or 0.25 is the constant of proportionality or constant of variation.

1. a. How long will it take an object to fall from the top of the Sears Tower in Chicago, a distance of 1450 feet? Round to the nearest hundredth of a second.

b. Complete the following table.

d **(ft.)**	0	100	200	300	500	750	1000
t **(sec.)**							

c. Sketch a graph of the given function. Use the table in part b to determine an appropriate scale.



d. How tall must a building be for an object to take 6 seconds to fall to the ground? Use the graph from part c to approximate your answer.

Solving Equations Involving Radical Expressions

Suppose you are interested in determining the value of d for many different values of t. In such a situation, the process could be simplified if you had a rule for d as a function of t. The equation $t = \frac{\sqrt{d}}{4}$ gives t as a function of d. You need to solve this equation for d.

DEFINITION

An equation in which at least one side contains a radical with a variable in the radicand is called a **radical equation**.

EXAMPLE 1 ***Examples of radical equations include*** $t = \frac{\sqrt{d}}{4}$, $\sqrt{2x + 1} = 5$, ***and*** $2\sqrt{3x} = \sqrt{5x - 7}$.

Solving an equation algebraically when the variable appears under a radical involves using the following property of equations.

> If a and b are two quantities such that $a = b$, then $a^n = b^n$, where n is a positive integer.

EXAMPLE 2 ***If*** $t = \sqrt{s}$, ***then apply the preceding property by squaring both sides of the equation.***

$t^2 = (\sqrt{s})^2$ — Rewrite $\sqrt{s}$ as $s^{1/2}$.

$t^2 = (s^{1/2})^2$ — Apply the property of exponents $(a^m)^n = a^{mn}$.

$t^2 = s^{1/2 \cdot 2}$ — Simplify.

$t^2 = s^1$

Therefore, if $t = \sqrt{s}$, then $t^2 = s$.

2. a. Now solve the equation $t = \frac{\sqrt{d}}{4}$ for d by first squaring both sides of the equation.

b. Using the new formula from part a, determine how tall a building must be for an object to take 6 seconds to fall to the ground.

c. How does your answer compare to the result in Problem 1d?

d. You could also answer part b by solving the equation $6 = \frac{\sqrt{d}}{4}$. Solve the equation. How does your answer compare to the result in part b.

The following example demonstrates a general algebraic procedure for solving radical equations.

EXAMPLE 3 ***Solve for x:*** $2\sqrt{3x} - \sqrt{5x+7} = 0.$

SOLUTION

Step 1. If the equation involves more than one radical term, isolate one radical term on one side of the equation.

$$\begin{aligned} 2\sqrt{3x} - \sqrt{5x+7} &= 0 \\ +\sqrt{5x+7} &+ \sqrt{5x+7} \\ \hline 2\sqrt{3x} &= \sqrt{5x+7} \end{aligned}$$

Step 2. Square both sides of the equation.

$$(2\sqrt{3x})^2 = (\sqrt{5x+7})^2$$
$$4 \cdot 3x = 5x + 7$$

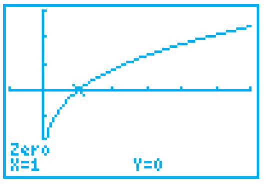
Step 3. If a radical remains, repeat steps 1 and 2. Solve the resulting equation.

$$\begin{aligned} 4 \cdot 3x &= 5x + 7 \\ 12x &= 5x + 7 \\ 7x &= +7 \\ x &= 1 \end{aligned}$$

Step 4. Check all solutions in the original equation.

$$2\sqrt{3(1)} - \sqrt{5(1)+7} = 2\sqrt{3} - \sqrt{12}$$
$$= 2\sqrt{3} - \sqrt{4 \cdot 3} = 2\sqrt{3} - 2\sqrt{3} = 0$$

You can also check your answer by solving the equation graphically.



3. Suppose two different objects are dropped: a marble and a large beach ball. Because of air resistance, the beach ball will take longer than the marble to fall the same distance. Assume that the marble falls according to $t = 0.25\sqrt{d}$, as in Problem 1. The time for the beach ball to hit the ground is modeled by $t = k\sqrt{d}$, where the positive constant, k, is determined by experiment.

The beach ball is dropped from a height of 250 feet, and it takes 4.11 seconds to hit the ground. Determine the constant, k, accurate to the hundredths place. Remember that $(ab)^2 = a^2b^2$.

4. Now suppose the beach ball in Problem 3 is dropped from a height 50 feet lower than the marble. Then $t = 0.26\sqrt{d - 50}$ is the time for the beach ball to drop $d - 50$ feet, where d is the height the marble falls.

a. Write an equation that can be used to determine from what height the marble must be dropped so the beach ball and marble will hit the ground at the same time.

b. Solve this equation using an algebraic approach.

c. Verify your solution in part b using your graphing calculator.

5. Consider the following algebraic solution of the equation $\sqrt{x + 3} + 5 = 0$.

$$\begin{aligned} \sqrt{x + 3} + 5 &= 0 \\ \sqrt{x + 3} &= -5 \\ (\sqrt{x + 3})^2 &= (-5)^2 \\ x + 3 &= 25 \\ x &= 22 \end{aligned}$$

a. It appears that $x = 22$ is a solution to the given equation. Check the solution by substituting 22 for x in the original equation. Does it check?

b. What happened in the solution process to cause an **extraneous solution** (an apparent solution that does not check) to appear?

c. Does the equation $\sqrt{x + 3} + 5 = 0$ have a solution? Include a graph to help support your answer.

6. The following table gives the amount of new money (in billions of dollars) loaned to students in each of the given years.

	YEAR							
	1993	1994	1995	1996	1997	1998	1999	2000
x, NUMBER OF YEARS SINCE 1993	0	1	2	3	4	5	6	7
A(*x*), AMOUNT OF NEW MONEY FOR STUDENT LOANS (in $ billions)	12.0	18.0	22.0	24.0	25.5	27.2	28.7	30.0

This data can be modeled by the function

$$A(x) = 6.8\sqrt{x} + 12.$$

Determine the year in which the amount of new money for student loans will first exceed $40 billion.

SUMMARY
ACTIVITY 5.8

1. If a and b are two quantities such that $a = b$, then $a^n = b^n$, where n is a positive integer.
2. To solve an equation involving one radical expression,
   a. Isolate the radical term on one side of the equation.
   b. Square both sides of the equation.
   c. Solve the resulting equation.
   d. Check all solutions in the original equation.
3. To solve an equation involving more than one radical expression,
   a. If the equation involves more than one radical term, isolate one radical term on one side of the equation.
   b. Square both sides of the equation.
   c. If a radical remains, repeat steps a and b. Solve the resulting equation.
   d. Check all solutions in the original equation.
4. When you raise both sides of an equation to an even power, it is possible to introduce **extraneous solutions** into the process. These are values of the variable that appear by the process to be solutions but do not make the original equation true. It is important to check all potential solutions to determine whether or not they are real solutions or extraneous solutions.

EXERCISES
ACTIVITY 5.8

1. Solve the following equations, or determine which does **not** have a solution. Explain your reasoning, or verify your solution.

   a. $\sqrt{2x + 1} = 3$ **b.** $\sqrt{x + 1} + 5 = 1$

   c. $\sqrt{2 - x} = -x$

2. Solve each of the following equations algebraically. Then verify your answers graphically.

   a. $\sqrt{x} = 2.5$ **b.** $\sqrt{x} - 3 = 0$ **c.** $\sqrt{2x} = 14$

   d. $3\sqrt{x} = 243$ **e.** $4 - 5\sqrt{3x} = 1$ **f.** $\sqrt{x + 1} = 9$

Exercise numbers appearing in color are answered in the Selected Answers appendix.

3. Solve each of the following equations algebraically and by graphing. Be aware of any extraneous roots.

a. $\sqrt{x+5} = 1$

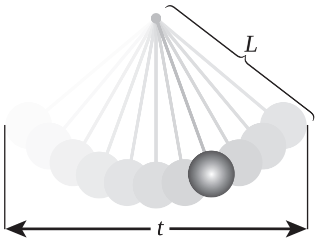
b. $10\sqrt{x+2} = 20$

c. $\sqrt{5-x} = x + 1$

4. Solve algebraically and graphically: $\sqrt{1.4x + 3.2} = \sqrt{3.8x - 1}$.

5. The time, t, in seconds, that it takes for a pendulum to complete one complete period (to swing back and forth one time) is modeled by

$$t = 2\pi\sqrt{\frac{L}{32}},$$



where L is the length of the pendulum, in feet. How long is the pendulum of a clock with a period of 1.95 seconds?

6. In a certain population, there are 28,520 births on a particular day. The number, N, of these people surviving to age x can be modeled by the function $N = 2850\sqrt{100 - x}$.

a. According to this model, how many of the 28,520 babies will survive to age 5?

b. What is the practical domain of this function?

c. When only 5000 of this group are still alive, how old do you expect them to be?

d. After how many years will half of the original population of 28,520 people remain alive?

7. a. A pressure gauge on a bridge indicates a wind pressure, P, of 10 pounds per square foot. What is the velocity, V, of the wind if

$$V = \sqrt{\frac{1000P}{3}},$$

where velocity is measured in miles per hour?

b. What is the wind pressure if the wind is blowing at 70 miles per hour?

8. Artificial gravity can be created in a space station by revolving the station. The number of revolutions required can be modeled by

$$N = \frac{1}{2\pi}\sqrt{\frac{a}{r}}$$

where N is measured in revolutions per second, a is the artificial gravity produced (measured in meters per second squared), and r is the radius of the space station in meters.

a. To produce an artificial acceleration simulating gravity on Earth, a must equal 9.8 meters per second squared. If the space station must revolve at the rate of one revolution every 5 minutes, what must its radius be? Solve both algebraically and graphically. Be careful, N is measured in revolutions per second.

b. Solve the original formula for r.

c. Use the formula in part b to answer part a again. How do the answers compare?

9. The Masteller formula for calculating the adult body surface area, A, is

$$A = \sqrt{\frac{hw}{3131}}$$

where h is the person's height in inches and w is the adult's weight in pounds. A is the surface area in square meters.

a. Determine the body surface area, A, of an adult who is 70 inches tall and weighs 200 pounds.

b. Solve the formula for w.

ACTIVITY 5.9

Propane Tank

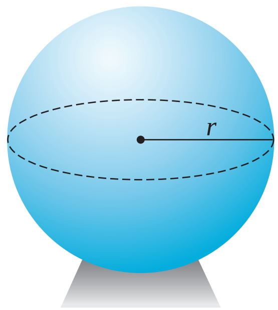
OBJECTIVES

1. Determine the domain of a function defined by an equation of the form $y = \sqrt[n]{g(x)}$, where n is a positive integer and $g(x)$ is a polynomial.
2. Graph $y = \sqrt[n]{g(x)}$.
3. Identify the properties of graphs of $y = \sqrt[n]{g(x)}$.
4. Solve radical equations that contain radical expressions with an index other than 2.

A propane tank is in the shape of a sphere. The radius, r, of the spherical tank is modeled by the formula

$$r = \sqrt[3]{\frac{3V}{4\pi}}$$

where V is the volume of the tank.



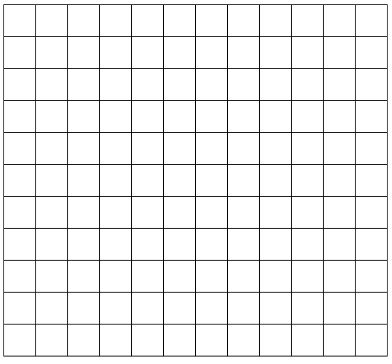
1. a. What is the radius of a propane tank having volume 50 cubic feet (round the answer to the nearest tenth)?

b. What is the practical domain of the radius function?

c. Complete the following table.

V (ft.^{3})	0	5	10	15	20	25	100
r (ft.)							

d. Sketch a graph of the radius function over its practical domain.



Graphs of $y = \sqrt[n]{g(x)}$, $n = 3, 4, 5$

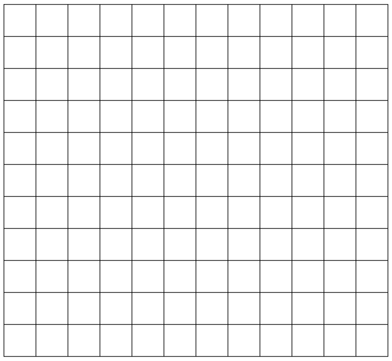
The investigation of the properties of the radius function defined by $r = \sqrt[3]{\frac{3V}{4\pi}}$ is somewhat limited by the restrictions placed on the variables V and r.

2. a. Consider the cube root function defined by $f(x) = \sqrt[3]{x}$. What is the domain of f?

b. Complete the following table.

x	−10	−7	−4	−1	0	1	4	7	10
$f(x)$									

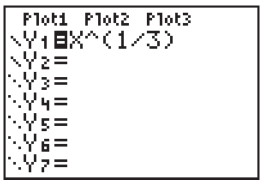
c. Sketch a graph of f. Verify your sketch using your graphing calculator.



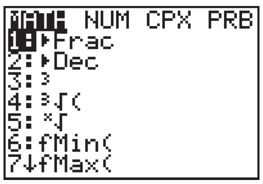
d. Is the function f increasing or decreasing?

There are two different ways to enter the cube root function on your calculator.

Method 1: Using fractional exponents, enter the cube root of x as x^(1/3) in the Y = editor.

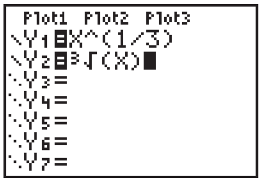


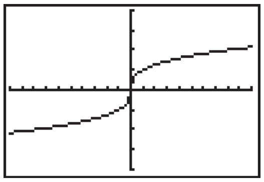
Method 2: Select the Y = editor, highlight the Y_n you want, and then select the MATH menu. Option 4 is the cube root.



Select option 4 and press ENTER. Insert the argument x and the right parenthesis to complete the function.

3. Use your graphing calculator to verify that the graphs of $y = x^{1/3}$ and $y = \sqrt[3]{x}$ are identical. Your screens should appear as follows.





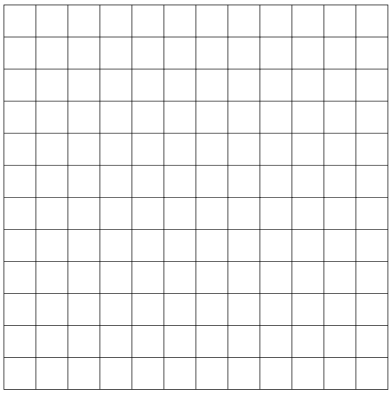
4. Consider functions defined by $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$.

a. Determine the domain of each function.

b. Complete the following table.

x	-10	-7	-4	-1	0	1	4	7	10
$f(x)$									
$g(x)$									

c. Sketch a graph of f and g on the same coordinate system. Verify your sketch using your graphing calculator.

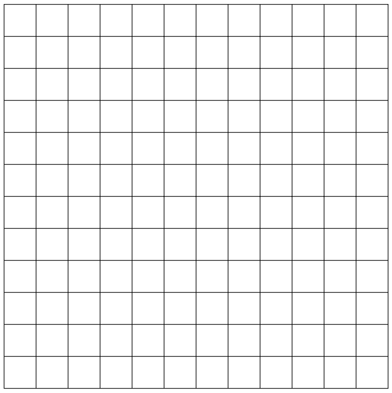


d. Are f and g inverse functions? Explain.

e. Determine the composition of f and g. That is, determine $f(g(x))$ and $g(f(x))$.

5. **a.** Sketch a graph of each of the following.

$$f(x) = \sqrt[3]{x}, \qquad g(x) = \sqrt[3]{x + 1}, \qquad h(x) = \sqrt[3]{x - 2}$$



b. How are the graphs of f, g, and h similar, and how are they different?

6. Given the functions f and g defined by

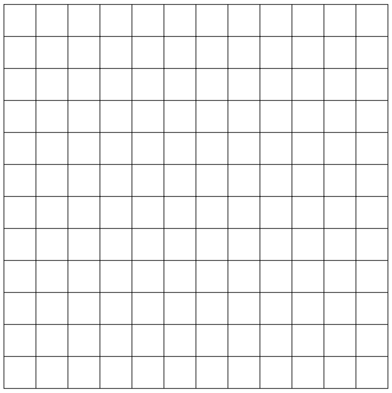
$$f(x) = \sqrt[4]{x - 1} \quad \text{and} \quad g(x) = \sqrt[5]{x}.$$

a. Complete the following table.

x	-10	-5	-1	0	1	5	10
$f(x)$							
$g(x)$							

b. Determine the domains of f and g.

c. Sketch a graph of f and g on the same coordinate axes.



Solving Equations Involving Radical Expressions

Solving an equation such as $\sqrt[3]{x+1}+5=9$ is similar to solving equations involving square roots.

EXAMPLE 1 ***Solve*** $\sqrt[3]{x+1}+5=9$.

SOLUTION

Step 1. Isolate the radical term on one side of the equation.

$$\sqrt[3]{x+1}+5=9$$
$$\sqrt[3]{x+1}=4$$

Step 2. Raise each side of the equation to the power that matches the index of the radical. In this situation, cube each side and simplify.

$$\left(\sqrt[3]{x+1}\right)^3=4^3$$

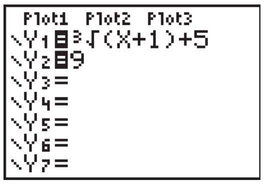
Step 3. If a radical remains, repeat steps 1 and 2. Solve the resulting equation.

$$x+1=64$$
$$x=63$$

Step 4. Check all solutions in the original equation.

$$\sqrt[3]{63+1}+5=9$$
$$\sqrt[3]{64}+5=9$$
$$4+5=9$$
$$9=9$$

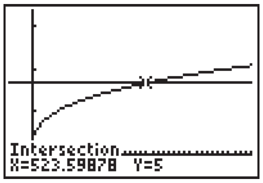
You can also verify your solutions graphically.



7. a. Returning to the propane tank situation, suppose the amount of space available in all directions for a propane tank is 10 feet. Write an equation to determine the maximum volume of a spherical tank that can fit into the given space.

b. Solve this equation using an algebraic approach.

c. Verify the solution using your graphing calculator. The screen should appear as follows.



8. Solve the formula $V = I^3$ for I.

9. The basal metabolic rate (BMR) is the number of calories per day a person needs to maintain life. A person's basal metabolic rate is a function of his or her weight and is modeled by

$$B(w) = 70\sqrt[4]{w^3},$$

where $B(w)$ represents the basal metabolic rate measured in calories per day and w is the person's weight in kilograms.

a. Write the expression $70\sqrt[4]{w^3}$ using fractional exponents.

b. If your friend weighs 50 kilograms (approximately 110 pounds), determine her basal metabolic rate. Round your answer to the nearest calorie.

c. Determine your basal metabolic rate. Be sure to convert your weight to kilograms.

d. Suppose a person is on a 2000-calorie-per-day diet. If the number of calories represents the person's basal metabolic rate, write an equation to determine the weight that is associated with this number of calories per day.

e. Solve the equation in part d. To help determine to what power you need to raise each side, first consider how you simplify the expression $(x^{3/4})^{4/3}$.

f. If the person weighs 210 pounds, is the 2000-calorie diet healthy?

SUMMARY
ACTIVITY 5.9

1. To solve an equation when the variable appears under a radical, use the following two properties.
   i. If a and b are two quantities such that $a = b$, then $a^n = b^n$, for any positive integer.
   ii. $(b^{1/n})^n = b^1$ and $(b^{m/n})^{n/m} = b^1$
2. i. The domain of $f(x) = \sqrt[n]{x}$, where n is a positive odd integer, is all real numbers.
   ii. The domain of $f(x) = \sqrt[n]{x}$, where n is a positive even integer, is $x \geq 0$.
3. To solve equations involving radicals,

   Step 1. Isolate the radical term on one side of the equation.

   Step 2. Raise each side of the equation to the power that matches the index of the radical.

   Step 3. If a radical remains, repeat steps 1 and 2. Solve the resulting equation.

   Step 4. Check all solutions in the original equation.

EXERCISES
ACTIVITY 5.9

1. If possible, determine the exact value of each of the following.

   a. $\sqrt[3]{64}$ b. $\sqrt[4]{16}$ c. $(-27)^{1/3}$ d. $(625)^{1/4}$

   e. $\sqrt{\frac{1}{36}}$ f. $(-81)^{1/4}$ g. $(100{,}000)^{1/5}$ h. $(-1)^{1/6}$

2. If the volume of a cube is 728 cubic centimeters, what is the length of one edge to the nearest tenth of a centimeter?

3. If the volume of a cube is decreased from 1450 cubic inches to 1280 cubic inches (and still remains a cube), by how much has the length of one edge decreased?

4. The volume of a sphere is 520 cubic meters. What is the diameter of the sphere?

5. What is the domain of each function?

a. $y = \sqrt[3]{x + 6}$

b. $f(x) = \sqrt[4]{x - 3}$

c. $g(x) = \sqrt[5]{2 - x}$

d. $f(x) = (2 - x)^{1/6}$

6. Solve each of the following algebraically and graphically.

a. $\sqrt[3]{x + 4} = 3$

b. $\sqrt[4]{x + 5} = 2$

c. $\sqrt[3]{2x - 3} + 4 = 3$

d. $\sqrt[4]{2 - x} = 5$

7. Solve each of the following algebraically, and verify your results graphically.

a. $x^{2/3} = 16$

b. $2x^{3/4} = 54$

8. **a.** The diameter d of a sphere is modeled by the formula

$$d = \sqrt[3]{\frac{6V}{\pi}}$$

where v represents the volume of a sphere. Approximate the diameter of a sphere having a volume of 10 cubic inches.

b. Determine the volume of a sphere having diameter 5 feet.

9. The radius, r, of a sphere is modeled by

$$r = \sqrt[3]{\frac{3V}{4\pi}}$$

where V is the volume of the sphere.

a. Determine the radius of a sphere that has a volume equal to 40 cubic centimeters.

b. Determine the volume of a sphere that has a radius equal to 3.5 feet.

c. Solve the formula for V, expressing volume as a function of the radius.

CLUSTER 2 What Have I Learned?

1. Explain the steps involved in solving the equation $\sqrt{2x + 3} = 5$ using an algebraic approach. Why must you be sure to check your solution?

2. What is the domain for the variable b in each of the following?

a. $\sqrt[n]{b}$, where n is an even positive integer

b. $\sqrt[n]{b}$, where n is an odd positive integer

3. Is it possible for an extraneous solution to appear when the equation $\sqrt[3]{2x + 1} = -3$ is being solved? Explain.

4. Describe two ways to check for extraneous solutions when you are solving an equation by squaring both sides.

5. Determine whether the following statements are true or false. Explain your answer.

a. If two numbers are equal, then their squares are equal.

b. If the squares of two numbers are equal, then the two numbers are equal.

6. a. For a given value of x, which is greater, $\sqrt[5]{x}$ or $x^{1/3}$? Explain how you determine your answer.

b. In part a, did you assume that $x > 0$? Does your answer change if $x < 0$?

CLUSTER 2 How Can I Practice?

1. Solve each of the following equations algebraically and check graphically.

   a. $\sqrt{x + 2} = 10$

   b. $(x - 5)^{1/2} = 6$

   c. $\sqrt{2x + 1} - 5 = 0$

   d. $\sqrt[3]{x^2 + 3} = 4$

   e. $\sqrt{x} = \sqrt{x + 2}$

   f. $(2 - x)^{1/3} = -2$

   g. $\sqrt[4]{2x - 5} = 2$

   h. $(2.3x + 1.9)^{1/3} = 1.6$

2. Identify the domain of each of the following functions.

   a. $f(x) = \sqrt{6 - x}$

   b. $g(x) = (2x - 9)^{1/3}$

   c. $h(x) = (x^2 - 4)^{1/4}$

3. If the volume of a cube is 458 cubic inches, what is the length of one edge? Determine the value to the nearest hundredth of an inch.

4. If the volume of a sphere is 620 cubic centimeters, what is its radius?

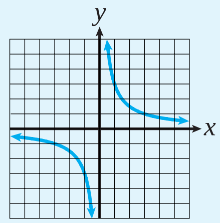
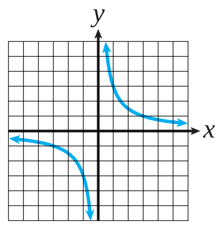
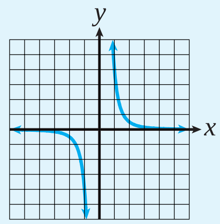
5. When a stone is dropped to the ground, its velocity is modeled by the function $v = \sqrt{64d}$, where d is the distance the stone has fallen, in feet, and v is its velocity in feet per second. If the stone hits the ground at 100 feet per second, from what height was it dropped?

6. A cardboard box with a square bottom has a height of 10 inches and a volume of 422.5 cubic inches. What are the dimensions of the bottom of the box?

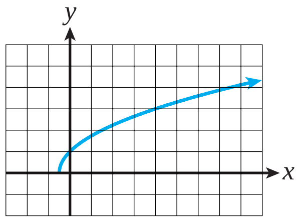
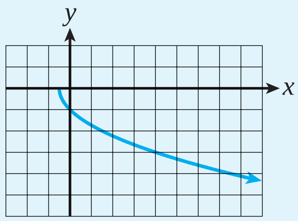
7. Describe the similarities and differences between the graphs of $y = \sqrt{2 - x}$ and $y = \sqrt{x - 2}$.

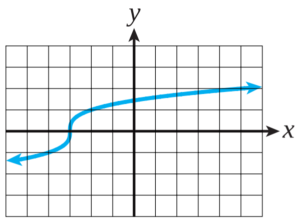
CHAPTER 5 SUMMARY

The bracketed numbers following each concept indicate the activity in which the concept is discussed.

CONCEPT/SKILL	DESCRIPTION	EXAMPLE
Domain and range of $y = \frac{k}{x}$ [5.1]	The domain and range consist of all real numbers except zero.	$y = \frac{3}{x}$
Graph of $y = \frac{k}{x}$ [5.1]	The graph is in the first and third quadrants if $k > 0$ and in the second and fourth if $k < 0$.	$y = \frac{3}{x}$ 
Asymptotes of the graph of $y = \frac{k}{x}$ [5.1]	The y-axis, $x = 0$, is the vertical asymptote. The x-axis, $y = 0$, is the horizontal asymptote.	$y = \frac{3}{x}$ 
Domain of $y = \frac{k}{x^n}$ [5.2]	The domain consists of all real numbers except zero.	$y = \frac{4}{x^3}$
Graph of $y = \frac{k}{x^n}$ [5.2]	The graphs will vary depending on the values of k and n.	See the summary at the end of Activity 5.2.
Asymptotes of the graph of $y = \frac{k}{x^n}$ [5.2]	The y-axis, $x = 0$, is the vertical asymptote. The x-axis, $y = 0$, is the horizontal asymptote.	$y = \frac{4}{x^3}$ 
Inverse variation functions [5.2]	Functions defined by $y = \frac{k}{x^n}$ are called inverse variation functions in which y is said to vary inversely as the nth power of x; k is called the constant of variation.	For the function $y = \frac{4}{x^3}$, y varies inversely as the cube of x, and 4 is the constant of variation.
Rational function [5.3]	A function Q, defined by an equation of the form $Q(x) = \frac{k}{g(x)}$, where k is a nonzero constant and $g(x)$ is a polynomial, belongs to the family of functions known as rational functions.	$f(x) = \frac{10}{x - 3}$

CONCEPT/SKILL	DESCRIPTION	EXAMPLE
Domain of a rational function [5.3]	The domain of $y = \frac{k}{g(x)}$ is the set of all real numbers except those values of the input x such that $g(x) = 0$.	The domain of $f(x) = \frac{10}{x - 3}$ is all real numbers except 3.
Vertical asymptote of a rational function [5.3]	The vertical asymptote is the vertical line that passes through the x-value for which $g(x) = 0$.	The vertical asymptote of $f(x) = \dfrac{10}{x - 3}$ is $x = 3$.
Horizontal asymptote of a rational function [5.3]	The horizontal asymptote of $g = \frac{k}{g(x)}$ is the x-axis $(y = 0)$.	$f(x) = \dfrac{10}{x - 3}$
Rational equations [5.4]	Method 1. To solve an equation of the form $\frac{f(x)}{g(x)} = \frac{a}{b}$, where $g(x) \neq 0$ and $b \neq 0$, multiply both sides of the equation by the product $b \cdot g(x)$, and solve the resulting equation for x.	See Example 1, Activity 5.4, page 574.
Rational equations [5.4]	Method 2. To solve an equation of the form $\frac{f(x)}{g(x)} = \frac{a}{b}$, where $g(x) \neq 0$ and $b \neq 0$, cross multiply to obtain $b \cdot f(x) = a \cdot g(x)$, and solve the resulting equation for x.	See Example 1, Activity 5.4, page 575.
Horizontal asymptotes of rational functions [5.4]	Suppose the output values of a rational function R get closer and closer to a number a as the input values increase infinitely in both the positive and negative directions. Then, the graph of the function R has a horizontal asymptote. The equation of the horizontal asymptote is $y = a$.	The horizontal asymptote of $R(x) = \frac{3x + 1}{x - 2}$ is $y = 3$.
Determine an LCD of two or more expressions [5.5]	1. Write the prime factorization of each denominator. Express repeated factors as powers. 2. Identify the different bases (factors) in step 1. 3. Write the LCD as the product of the highest power of each of the different factors from step 2.	See Example 1, Activity 5.5, page 587.
Solving an equation involving rational expressions [5.5]	1. Determine the LCD of all denominators in the equation. 2. Multiply each side of the equation by the LCD, and simplify the resulting equation. 3. Solve the resulting equation for the desired variable.	See Problem 4, pages 587–588.

CONCEPT/SKILL	DESCRIPTION	EXAMPLE
Simplifying rational expressions [5.6]	1. Factor the numerator and the denominator. 2. Divide the numerator and the denominator by the common factors.	See Appendix A.
Multiplying or dividing rational expressions [5.6]	1. Factor the numerator and denominator of each fraction completely. 2. Divide out the common factors (cancel). 3. Multiply remaining factors. 4. In division, proceed as above after inverting the divisor (the fraction after the division sign).	See Appendix A.
Adding or subtracting rational expressions [5.6]	1. Determine the LCD. 2. Build each fraction to have the LCD. 3. Add or subtract numerators. 4. Place the numerator over the LCD, and simplify if necessary.	See Appendix A.
Simplifying a complex fraction [5.6]	1. Express the numerator as a single fraction. 2. Express the denominator as a single fraction. 3. Divide the numerator by the denominator. 4. Simplify, if possible.	See Example 2, Activity 5.6, page 596.
Radical functions [5.7]	The function defined by $y = \sqrt{g(x)}$ has domain all real x such that $g(x) \geq 0$.	$y = \sqrt{2x + 1}$ 
Radical functions [5.7]	The function defined by $y = -\sqrt{g(x)}$ has domain all real x such that $g(x) \geq 0$.	$y = -\sqrt{2x + 1}$ 
Solving an equation involving one radical expression [5.8]	1. Isolate the radical term on one side of the equation. 2. Square both sides of the equation. 3. Solve the resulting equation. 4. Check all solutions in the original equation.	See Appendix A.

CONCEPT/SKILL	DESCRIPTION	EXAMPLE
Solving an equation involving more than one radical expression [5.8]	1. If the equation involves more than one radical term, isolate one radical term on one side of the equation. 2. Square both sides of the equation. 3. Solve the resulting equation. If a radical remains, repeat steps 1 and 2. 4. Check all solutions in the original equation.	See Example 3, Activity 5.8, page 622.
Domain of a function defined by an equation of the form $y = \sqrt[n]{g(x)}$, where n is a positive integer and $g(x)$ is a polynomial [5.9]	The domain of $y = \sqrt[n]{g(x)}$ is all real numbers if n is an odd positive integer. The domain of $y = \sqrt[n]{g(x)}$ is all real numbers for which $g(x) \geq 0$ if n is an even positive integer.	$y = \sqrt[3]{x + 3}$ 
Solving radical equations that contain radical expressions with an index other than 2 [5.9]	1. Isolate a radical term on one side of the equation. 2. Raise each side of the equation to the power that matches the index of the radical. 3. If a radical remains, repeat steps 1 and 2. Solve the resulting equation. 4. Check all solutions in the original equation.	See Example 1, Activity 5.9, page 633.

GATEWAY REVIEW

1. According to the blueprint, the floor area of the stage in the new auditorium at your college must be rectangular and equal to 1200 square feet. The width of the stage is key to all of the theater productions. Therefore, in this situation, the stage's depth is a function of its width.

   a. Let d represent the depth and w represent the width. Write an equation that expresses d as a function of w.

   b. Complete the following table using the equation from part a.

w	30	35	40	50	60
d					

   c. What happens to the depth as the width increases?

   d. What happens if the width is 100 feet? Is this realistic? Explain.

   e. Can the width be zero? Explain.

   f. What do you think is the practical domain for this function?

   g. What type of a function do you have in this situation?

   h. What is the domain of the general function?

   i. What is the vertical asymptote?

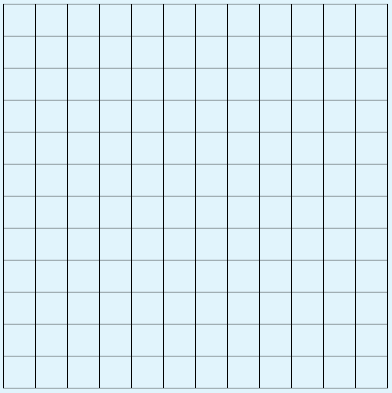
   j. What is the horizontal asymptote? Explain in words how you determined it.

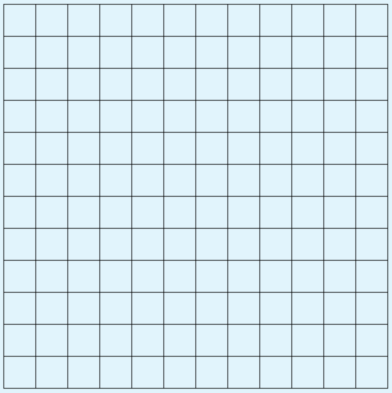
Answers to all Gateway exercises are included in the Selected Answers appendix.

2. Sketch the following graphs without using your graphing calculator.

a. $f(x) = \frac{1}{x^2}$

b. $g(x) = \frac{-1}{x^3}$





c. Describe how the graphs are similar and how they are different.

3. a. If y varies inversely as x, and $x = 10$ when $y = 12$, then determine the value of y when $x = 30$.

b. The loudness, in decibels, of a stereo varies inversely to the square of the distance from the speaker to the person listening. If the loudness is 32 decibels at a distance of 4 feet, then what is the loudness when the listener is 10 feet from the speaker?

c. When the volume of a circular cylinder is constant, the height varies inversely as the square of the radius. If the radius is 2 inches when the height is 8 inches, determine the height when the radius is 5 inches.

4. Determine the horizontal and vertical asymptotes and the intercepts of each of the following. Then sketch a graph of each function.

a. $f(x) = \frac{10}{x^3}$

b. $g(x) = \frac{4}{x - 3}$

c. $f(x) = \frac{2x}{x + 2}$

5. Students from the local community college plan to celebrate their 10-year reunion. They reserve a restaurant for an evening of entertainment. The fee for the band is $600, and food will cost each person $45.

   a. If $f(n)$ represents the total cost for n people to participate in the reunion, write an equation for the total cost.

   b. If 100 people attend, what will be the total cost of the event?

c. Determine a function, $A(n)$, that will represent the average cost per person to attend the event.

d. If 100 people attend the reunion, how much should each person pay?

e. Use your graphing calculator to complete the following table.

n **(no. of people attending)**	50	100	150	200	250
***A*(*n*) (cost per person)**					

f. If the committee thinks that each person should pay at most \$50, how many would have to attend for the cost to be \$50? Show your answer algebraically, and check it using your graphing calculator.

g. Determine the practical domain of your function.

h. From the graph, determine the vertical asymptote. Is there a practical meaning of this asymptote in this situation?

i. From the graph, determine the horizontal asymptote. Is there a practical meaning of this asymptote in this situation?

6. a. Solve the equation $\frac{4}{x-2} + 3 = 9$ using an algebraic approach. Verify your answer graphically.

b. When you graph the function $f(x) = \frac{4}{x-2} - 6$, what do you discover about the solution to the equation in part a and the x-intercept of the graph of the function f? Explain.

7. Solve each of the following equations using an algebraic approach. Verify your solutions graphically.

a. $\frac{3}{x + 2} = 5$

b. $\frac{-2x}{3x - 4} = 2$

8. The local grocery store has just hired you and your friend. You can stock a shelf in 15 minutes. Your friend will take 20 minutes to do the shelf. How long will it take to stock the shelf if you work together?

9. You work in the admissions office at your community college. You must assemble all of the packets for the placement test sessions. You work with a friend who takes twice as long as you do to assemble the packets. If you work together, the packets can be completed in 45 minutes. On the day you must assemble the packets, you have a big exam. How many hours does it take your friend to do the job alone?

10. Solve each of the following equations algebraically. Verify your answers graphically.

a. $\frac{1}{6} - \frac{3}{2x} = \frac{1}{5x}$

b. $\frac{2}{x} + \frac{3}{4x} = \frac{1}{12}$

11. Solve each of the following equations for the indicated variable. Express your answer as a single fraction.

a. Solve $S = \frac{C}{1 - r}$ for r.

b. Solve $\frac{1}{a} + \frac{3}{b} = \frac{4}{c}$ for b.

12. Simplify each of the following complex expressions.

a. $\dfrac{\frac{1}{a} + \frac{2}{b}}{\frac{2}{a} + \frac{1}{b}}$

b. $\dfrac{1 + \frac{1}{x - 2}}{1 - \frac{3}{x + 2}}$

13. A camera lens possesses a measurement called the focal length, f. When an object is in focus, the focal length is related to the distance of the object from the lens, p, and the image distance from the lens, q, by the formula

$$f = \frac{1}{\frac{1}{p} + \frac{1}{q}}.$$

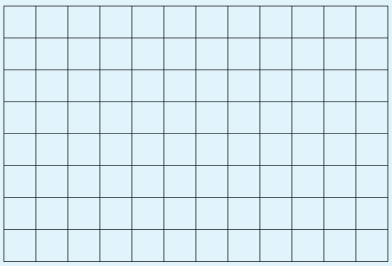
a. Determine f if p is 4 meters and q is 3 meters.

b. Simplify the complex fraction on the right side of the original formula.

c. Redo part a using the new formula from part b. How do the answers compare?

14. **a.** What is the domain of the function defined by $f(x) = \sqrt{x + 4}$.

b. Sketch the graph of the function f.



c. As the input increases, what happens to the output values?

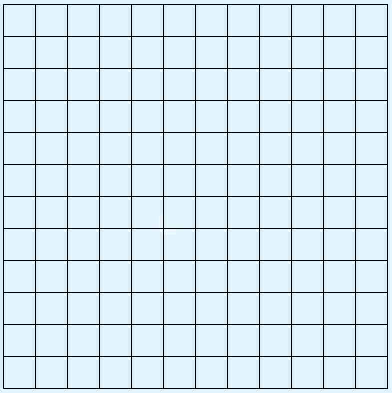
d. What is the range of this function?

e. Are there any intercepts? If so, what are they?

f. How is the function f similar to the function $g(x) = \sqrt{x - 4}$?

g. How is the graph of the function f similar to $h(x) = -\sqrt{x + 4}$?

15. **a.** Draw the graph of $f(x) = \sqrt{x}$.



b. Determine the equation of the inverse of the function f.

c. Sketch the graph of the inverse on the same axes as the graph of f.

d. From the graphs, describe how you know that they are inverses.

e. Show that f and f^{-1} are inverses algebraically.

16. For each of the given functions,

i. determine the domain and range

ii. determine the *x*- and *y*-intercepts

iii. sketch a graph

a. $f(x) = \sqrt{x} + 4$

i.

ii.

iii.

b. $f(x) = \sqrt{x + 4}$

i.

ii.

iii.

17. Solve each of the following equations using an algebraic approach. Verify your solutions graphically.

a. $\sqrt{3x - 2} - 6 = -2$

b. $\sqrt{2x + 1} - \sqrt{x + 7} = 0$

c. $\sqrt[3]{5x + 4} = 3$

d. $\sqrt{4x + 8} - 3 = -5$

e. $x^{4/3} = 81$

18. What is the domain of each function?

a. $y = \sqrt[3]{x + 8}$

b. $y = \sqrt[4]{x - 6}$

c. $y = (x + 1)^{1/6}$

19. A submarine periscope must be a certain distance above the water for it to be used to locate a ship a certain number of miles away. The model for the distance (in miles) that the submarine periscope can see is the formula $d = \sqrt{1.5h}$, where h represents the height (in feet) above the surface of the water.

How far above the surface of the water would the periscope have to be to see a ship that is 6 miles away?

20. A ring is dropped from the American span of the Thousand Island Bridge and hits the water 3.1 seconds later. What is the height of the bridge? Use the formula $T = \sqrt{\frac{d}{16}}$, where d represents the distance in feet and T represents the time in seconds.