MOTION IN TWO OR THREE DIMENSIONS

3

? If a car is going around a curve at constant speed, is it accelerating? If so, in what direction is it accelerating?

LEARNING GOALS

By studying this chapter, you will learn:

- How to represent the position of a body in two or three dimensions using vectors.
- How to determine the vector velocity of a body from a knowledge of its path.
- How to find the vector acceleration of a body, and why a body can have an acceleration even if its speed is constant.
- How to interpret the components of a body's acceleration parallel to and perpendicular to its path.
- How to describe the curved path followed by a projectile.
- The key ideas behind motion in a circular path, with either constant speed or varying speed.
- How to relate the velocity of a moving body as seen from two different frames of reference.

What determines where a batted baseball lands? How do you describe the motion of a roller coaster car along a curved track or the flight of a circling hawk? If you throw a water balloon horizontally from your window, will it take the same amount of time to hit the ground as a balloon that you simply drop?

We can't answer these kinds of questions using the techniques of Chapter 2, in which particles moved only along a straight line. Instead, we need to extend our descriptions of motion to two- and three-dimensional situations. We'll still use the vector quantities displacement, velocity, and acceleration, but now these quantities will no longer lie along a single line. We'll find that several important kinds of motion take place in two dimensions only—that is, in a *plane.* These motions can be described with two components of position, velocity, and acceleration.

We also need to consider how the motion of a particle is described by different observers who are moving relative to each other. The concept of *relative velocity* will play an important role later in the book when we study collisions, when we explore electromagnetic phenomena, and when we introduce Einstein's special theory of relativity.

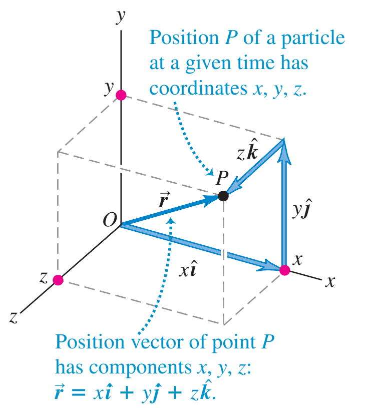
This chapter merges the vector mathematics of Chapter 1 with the kinematic language of Chapter 2. As before, we are concerned with describing motion, not with analyzing its causes. But the language you learn here will be an essential tool in later chapters when we study the relationship between force and motion.

3.1 Position and Velocity Vectors

To describe the *motion* of a particle in space, we must first be able to describe the particle's *position*. Consider a particle that is at a point P at a certain instant. The **position vector** $\vec{r}$ of the particle at this instant is a vector that goes from the origin of the coordinate system to the point P (Fig. 3.1). The Cartesian coordinates x, y, and z of point P are the x-, y-, and z-components of vector $\vec{r}$. Using the unit vectors we introduced in Section 1.9, we can write

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad \text{(position vector)} \qquad (3.1)$$

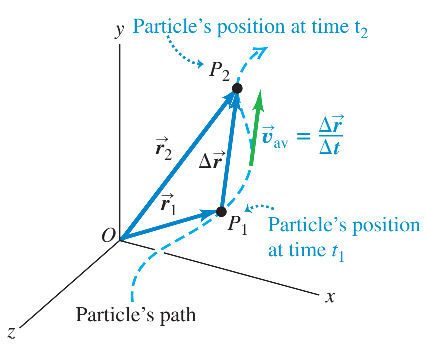
3.1 The position vector $\vec{r}$ from the origin to point P has components x, y, and z. The path that the particle follows through space is in general a curve (Fig. 3.2).



During a time interval Δt the particle moves from P_1, where its position vector is $\vec{r}_1$, to P_2, where its position vector is $\vec{r}_2$. The change in position (the displacement) during this interval is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$. We define the **average velocity** $\vec{v}_{av}$ during this interval in the same way we did in Chapter 2 for straight-line motion, as the displacement divided by the time interval:

$$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t} \qquad \text{(average velocity vector)} \qquad (3.2)$$

3.2 The average velocity $\vec{v}_{av}$ between points P_1 and P_2 has the same direction as the displacement $\Delta\vec{r}$.



Dividing a vector by a scalar is really a special case of *multiplying* a vector by a scalar, described in Section 1.7; the average velocity $\vec{v}_{av}$ is equal to the displacement vector $\Delta\vec{r}$ multiplied by $1/\Delta t$, the reciprocal of the time interval. Note that the x-component of Eq. (3.2) is $v_{av\text{-}x} = (x_2 - x_1)/(t_2 - t_1) = \Delta x/\Delta t$. This is just Eq. (2.2), the expression for average x-velocity that we found in Section 2.1 for one-dimensional motion.

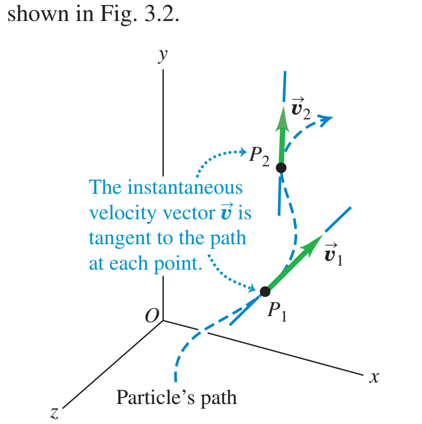
We now define **instantaneous velocity** just as we did in Chapter 2: It is the limit of the average velocity as the time interval approaches zero, and it equals the instantaneous rate of change of position with time. The key difference is that position $\vec{r}$ and instantaneous velocity $\vec{v}$ are now both vectors:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \qquad \text{(instantaneous velocity vector)} \qquad (3.3)$$

The *magnitude* of the vector $\vec{v}$ at any instant is the *speed* v of the particle at that instant. The *direction* of $\vec{v}$ at any instant is the same as the direction in which the particle is moving at that instant.

Note that as $\Delta t \to 0$, points P_1 and P_2 in Fig. 3.2 move closer and closer together. In this limit, the vector $\Delta\vec{r}$ becomes tangent to the path. The direction of $\Delta\vec{r}$ in the limit is also the direction of the instantaneous velocity $\vec{v}$. This leads to an important conclusion: *At every point along the path, the instantaneous velocity vector is tangent to the path at that point* (Fig. 3.3).

3.3 The vectors $\vec{v}_1$ and $\vec{v}_2$ are the instantaneous velocities at the points P_1 and P_2 shown in Fig. 3.2.



It's often easiest to calculate the instantaneous velocity vector using components. During any displacement $\Delta\vec{r}$, the changes Δx, Δy, and Δz in the three coordinates of the particle are the *components* of $\Delta\vec{r}$. It follows that the components v_x, v_y, and v_z of the instantaneous velocity $\vec{v}$ are simply the time derivatives of the coordinates x, y, and z. That is,

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt} \qquad \text{(components of instantaneous velocity)} \qquad (3.4)$$

The x-component of $\vec{v}$ is $v_x = dx/dt$, which is the same as Eq. (2.3)—the expression for instantaneous velocity for straight-line motion that we obtained in Section 2.2. Hence Eq. (3.4) is a direct extension of the idea of instantaneous velocity to motion in three dimensions.

We can also get this result by taking the derivative of Eq. (3.1). The unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are constant in magnitude and direction, so their derivatives are zero, and we find

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k} \qquad (3.5)$$

This shows again that the components of $\vec{v}$ are dx/dt, dy/dt, and dz/dt.

The magnitude of the instantaneous velocity vector $\vec{v}$—that is, the speed—is given in terms of the components v_x, v_y, and v_z by the Pythagorean relation

$$|\vec{v}| = v = \sqrt{v_x^2 + v_y^2 + v_z^2} \qquad (3.6)$$

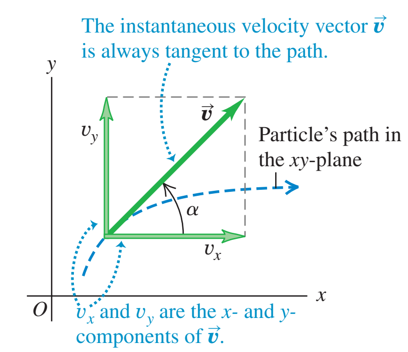
Figure 3.4 shows the situation when the particle moves in the xy-plane. In this case, z and v_z are zero. Then the speed (the magnitude of $\vec{v}$) is

$$v = \sqrt{v_x^2 + v_y^2}$$

and the direction of the instantaneous velocity $\vec{v}$ is given by the angle α in the figure. We see that

$$\tan\alpha = \frac{v_y}{v_x} \qquad (3.7)$$

3.4 The two velocity components for motion in the xy-plane.



(We always use Greek letters for angles. We use α for the direction of the instantaneous velocity vector to avoid confusion with the direction θ of the *position* vector of the particle.)

The instantaneous velocity vector is usually more interesting and useful than the average velocity vector. From now on, when we use the word "velocity," we will always mean the instantaneous velocity vector $\vec{v}$ (rather than the average velocity vector). Usually, we won't even bother to call $\vec{v}$ a vector; it's up to you to remember that velocity is a vector quantity with both magnitude and direction.

Example 3.1 Calculating average and instantaneous velocity

A robotic vehicle, or rover, is exploring the surface of Mars. The landing craft is the origin of coordinates, and the surrounding Martian surface lies in the xy-plane. The rover, which we represent as a point, has x- and y-coordinates that vary with time:

$$x = 2.0\ \text{m} - (0.25\ \text{m/s}^2)t^2$$
$$y = (1.0\ \text{m/s})t + (0.025\ \text{m/s}^3)t^3$$

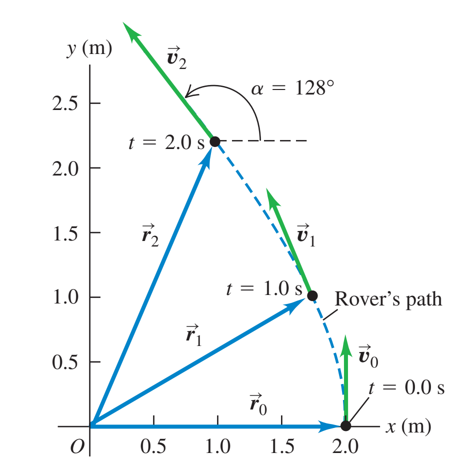
(a) Find the rover's coordinates and its distance from the lander at $t = 2.0$ s. (b) Find the rover's displacement and average velocity vectors during the interval from $t = 0.0$ s to $t = 2.0$ s. (c) Derive a general expression for the rover's instantaneous velocity vector. Express the instantaneous velocity at $t = 2.0$ s in component form and also in terms of magnitude and direction.

SOLUTION

IDENTIFY: This problem involves motion in two dimensions—that is, in a plane. Hence we must use the expressions for the displacement, average velocity, and instantaneous velocity *vectors* obtained in this section. (The simpler expressions in Sections 2.1 and 2.2 don't involve vectors; they apply only to motion along a straight line.)

SET UP: Figure 3.5 shows the rover's path. We'll use Eq. (3.1) for position $\vec{r}$, the expression $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ for displacement, Eq. (3.2) for average velocity, and Eqs. (3.5) and (3.6) for instantaneous velocity and its direction. The target variables are stated in the problem.

3.5 At $t = 0$ the rover has position vector $\vec{r}_0$ and instantaneous velocity vector $\vec{v}_0$. Likewise, $\vec{r}_1$ and $\vec{v}_1$ are the vectors at $t = 1.0$ s; $\vec{r}_2$ and $\vec{v}_2$ are the vectors at $t = 2.0$ s.



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EXECUTE: (a) At time $t = 2.0$ s the rover's coordinates are

$$x = 2.0\text{ m} - (0.25\text{ m/s}^2)(2.0\text{ s})^2 = 1.0\text{ m}$$
$$y = (1.0\text{ m/s})(2.0\text{ s}) + (0.025\text{ m/s}^3)(2.0\text{ s})^3 = 2.2\text{ m}$$

The rover's distance from the origin at this time is

$$r = \sqrt{x^2 + y^2} = \sqrt{(1.0\text{ m})^2 + (2.2\text{ m})^2} = 2.4\text{ m}$$

(b) To find the displacement and average velocity, we express the position vector $\vec{r}$ as a function of time t. From Eq. (3.1), this is

$$\begin{aligned}\vec{r} &= x\hat{\imath} + y\hat{\jmath}\\ &= [2.0\text{ m} - (0.25\text{ m/s}^2)t^2]\hat{\imath}\\ &\quad + [(1.0\text{ m/s})t + (0.025\text{ m/s}^3)t^3]\hat{\jmath}\end{aligned}$$

At time $t = 0.0$ s the position vector $\vec{r}_0$ is

$$\vec{r}_0 = (2.0\text{ m})\hat{\imath} + (0.0\text{ m})\hat{\jmath}$$

From part (a) the position vector $\vec{r}_2$ at time $t = 2.0$ s is

$$\vec{r}_2 = (1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath}$$

Therefore the displacement from $t = 0.0$ s to $t = 2.0$ s is

$$\begin{aligned}\Delta\vec{r} = \vec{r}_2 - \vec{r}_0 &= (1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath} - (2.0\text{ m})\hat{\imath}\\ &= (-1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath}\end{aligned}$$

During the time interval from $t = 0.0$ s to $t = 2.0$ s, the rover moves 1.0 m in the negative x-direction and 2.2 m in the positive y-direction. From Eq. (3.2), the average velocity during this interval is the displacement divided by the elapsed time:

$$\begin{aligned}\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} &= \frac{(-1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath}}{2.0\text{ s} - 0.0\text{ s}}\\ &= (-0.50\text{ m/s})\hat{\imath} + (1.1\text{ m/s})\hat{\jmath}\end{aligned}$$

The components of this average velocity are

$$v_{av\text{-}x} = -0.50\text{ m/s} \qquad v_{av\text{-}y} = 1.1\text{ m/s}$$

(c) From Eq. (3.4), the components of instantaneous velocity are the time derivatives of the coordinates:

$$v_x = \frac{dx}{dt} = (-0.25\text{ m/s}^2)(2t)$$
$$v_y = \frac{dy}{dt} = 1.0\text{ m/s} + (0.025\text{ m/s}^3)(3t^2)$$

Then we can write the instantaneous velocity vector $\vec{v}$ as

$$\begin{aligned}\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} &= (-0.50\text{ m/s}^2)t\hat{\imath}\\ &\quad + [1.0\text{ m/s} + (0.075\text{ m/s}^3)t^2]\hat{\jmath}\end{aligned}$$

At time $t = 2.0$ s, the components of instantaneous velocity are

$$v_x = (-0.50\text{ m/s}^2)(2.0\text{ s}) = -1.0\text{ m/s}$$
$$v_y = 1.0\text{ m/s} + (0.075\text{ m/s}^3)(2.0\text{ s})^2 = 1.3\text{ m/s}$$

The magnitude of the instantaneous velocity (that is, the speed) at $t = 2.0$ s is

$$\begin{aligned}v = \sqrt{v_x^2 + v_y^2} &= \sqrt{(-1.0\text{ m/s})^2 + (1.3\text{ m/s})^2}\\ &= 1.6\text{ m/s}\end{aligned}$$

The direction of $\vec{v}$ with respect to the positive x-axis is given by the angle α, where, from Eq. (3.7),

$$\tan\alpha = \frac{v_y}{v_x} = \frac{1.3\text{ m/s}}{-1.0\text{ m/s}} = -1.3 \qquad \text{so} \qquad \alpha = 128^\circ$$

Your calculator will tell you that the inverse tangent of -1.3 is -52°. But as we learned in Section 1.8, you have to examine a sketch of a vector to decide on its direction. Figure 3.5 shows that the correct answer for α is $-52^\circ + 180^\circ = 128^\circ$.

EVALUATE: Take a moment to compare the components of *average* velocity that we found in part (b) for the interval from $t = 0.0$ s to $t = 2.0$ s ($v_{av\text{-}x} = -0.50$ m/s, $v_{av\text{-}y} = 1.1$ m/s) with the components of *instantaneous* velocity at $t = 2.0$ s that we found in part (c) ($v_x = -1.0$ m/s, $v_y = 1.3$ m/s). The comparison shows that, just as in one dimension, the average velocity vector $\vec{v}_{av}$ over an interval is in general *not* equal to the instantaneous velocity $\vec{v}$ at the end of the interval (see Example 2.1).

You should calculate the position vector, instantaneous velocity vector, speed, and direction of motion at $t = 0.0$ s and $t = 1.0$ s. Figure 3.5 shows the position vectors $\vec{r}$ and instantaneous velocity vectors $\vec{v}$ at $t = 0.0$ s, 1.0 s, and 2.0 s. Notice that at every point, $\vec{v}$ is tangent to the path. The magnitude of $\vec{v}$ increases as the rover moves, which shows that its speed is increasing.

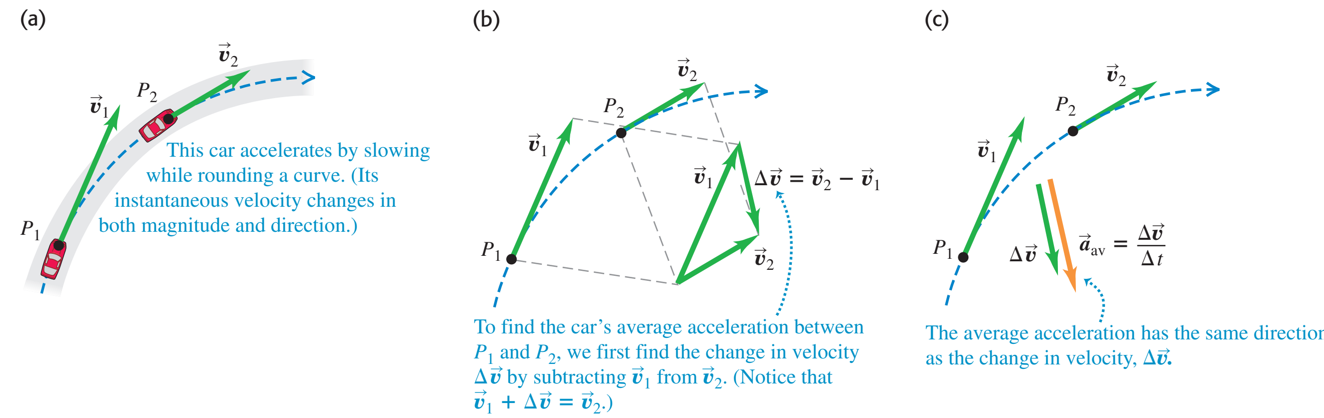
Test Your Understanding of Section 3.1 In which of these situations *would* the average velocity vector $\vec{v}_{av}$ over an interval be equal to the instantaneous velocity $\vec{v}$ at the end of the interval? (i) a body moving along a curved path at constant speed; (ii) a body moving along a curved path and speeding up; (iii) a body moving along a straight line at constant speed; (iv) a body moving along a straight line and speeding up.

3.2 The Acceleration Vector

Now let's consider the *acceleration* of a particle moving in space. Just as for motion in a straight line, acceleration describes how the velocity of the particle changes. But since we now treat velocity as a vector, acceleration will describe changes in the velocity magnitude (that is, the speed) *and* changes in the direction of velocity (that is, the direction in which the particle is moving).

In Fig. 3.6a, a car (treated as a particle) is moving along a curved road. The vectors $\vec{v}_1$ and $\vec{v}_2$ represent the car's instantaneous velocities at time t_1, when the car is at point P_1, and at time t_2, when the car is at point P_2. The two velocities may differ in both magnitude and direction. During the time interval from t_1 to t_2, the *vector change in velocity* is $\vec{v}_2 - \vec{v}_1 = \Delta\vec{v}$ (Fig. 3.6b). We define the **average acceleration** $\vec{a}_{av}$ of the car during this time interval as the velocity change divided by the time interval $t_2 - t_1 = \Delta t$:

3.6 (a) A car moving along a curved road from P_1 to P_2. (b) Obtaining $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ by vector subtraction. (c) The vector $\vec{a}_{av} = \Delta\vec{v}/\Delta t$ represents the average acceleration between P_1 and P_2.



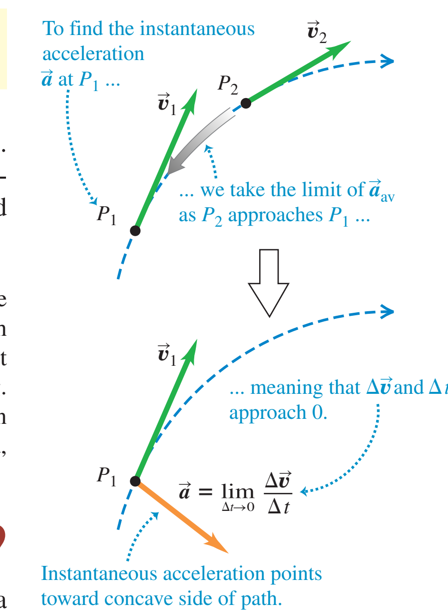
$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t} \qquad \text{(average acceleration vector)} \qquad (3.8)$$

Average acceleration is a *vector* quantity in the same direction as the vector $\Delta\vec{v}$ (Fig. 3.6c). Note that $\vec{v}_2$ is the vector sum of the original velocity $\vec{v}_1$ and the change $\Delta\vec{v}$ (Fig. 3.6b). The x-component of Eq. (3.8) is $a_{av\text{-}x} = (v_{2x} - v_{1x})/(t_2 - t_1) = \Delta v_x/\Delta t$, which is just Eq. (2.4) for the average acceleration in straight-line motion.

As in Chapter 2, we define the **instantaneous acceleration** $\vec{a}$ at point P_1 as the limit of the average acceleration when point P_2 approaches point P_1 and $\Delta\vec{v}$ and Δt both approach zero. The instantaneous acceleration is also equal to the instantaneous rate of change of velocity with time. Because we are not restricted to straight-line motion, instantaneous acceleration is now a vector (Fig. 3.7):

$$\vec{a} = \lim_{\Delta t \to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \qquad \text{(instantaneous acceleration vector)} \qquad (3.9)$$

3.7 Instantaneous acceleration $\vec{a}$ at point P_1 in Fig. 3.6.



The velocity vector $\vec{v}$, as we have seen, is tangent to the path of the particle. But Figs. 3.6c and 3.7 show that if the path is curved, the instantaneous acceleration vector $\vec{a}$ always points toward the concave side of the path—that is, toward the inside of any turn that the particle is making.

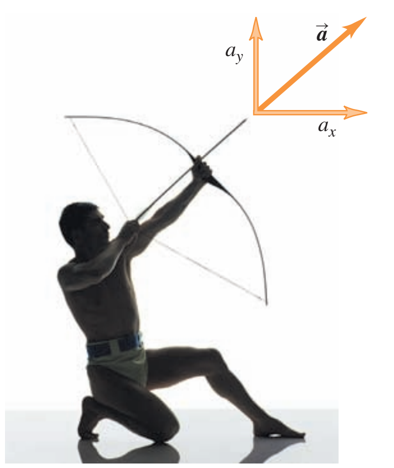
CAUTION **Any particle following a curved path is accelerating** When a particle is moving in a curved path, it always has nonzero acceleration, even when it moves with constant speed. This conclusion may seem contrary to your intuition, but it's really just contrary to the everyday use of the word "acceleration" to mean that speed is increasing. The more precise definition given in Eq. (3.9) shows that there is a nonzero acceleration whenever the velocity vector changes in any way, whether there is a change of speed, direction, or both.

To convince yourself that a particle has a nonzero acceleration when moving on a curved path with constant speed, think of your sensations when you ride in a car. When the car accelerates, you tend to move inside the car in a direction *opposite* to the car's acceleration. (We'll discover the reason for this behavior in Chapter 4.) Thus you tend to slide toward the back of the car when it accelerates forward (speeds up) and toward the front of the car when it accelerates backward (slows down). If the car makes a turn on a level road, you tend to slide toward the outside of the turn; hence the car has an acceleration toward the inside of the turn.

We will usually be interested in the instantaneous acceleration, not the average acceleration. From now on, we will use the term "acceleration" to mean the instantaneous acceleration vector $\vec{a}$.

Each component of the acceleration vector is the derivative of the corresponding component of velocity:

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \qquad a_z = \frac{dv_z}{dt} \qquad \text{(components of instantaneous acceleration)} \tag{3.10}$$



3.8 When the archer shoots the arrow, it accelerates both forward and upward. Thus its acceleration vector has both a horizontal component (a_x) and a vertical component (a_y).

In terms of unit vectors,

$$\vec{a} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} + \frac{dv_z}{dt}\hat{k} \tag{3.11}$$

The x-component of Eqs. (3.10) and (3.11), $a_x = dv_x/dt$, is the expression from Section 2.3 for instantaneous acceleration in one dimension, Eq. (2.5). Figure 3.8 shows an example of an acceleration vector that has both x- and y-components.

Since each component of velocity is the derivative of the corresponding coordinate, we can express the components a_x, a_y, and a_z of the acceleration vector $\vec{a}$ as

$$a_x = \frac{d^2x}{dt^2} \qquad a_y = \frac{d^2y}{dt^2} \qquad a_z = \frac{d^2z}{dt^2} \tag{3.12}$$

The acceleration vector $\vec{a}$ itself is

$$\vec{a} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{k} \tag{3.13}$$

Example 3.2 Calculating average and instantaneous acceleration

Let's return to the motions of the robotic rover in Example 3.1. We found that the components of instantaneous velocity at any time t are

$$v_x = \frac{dx}{dt} = (-0.25\ \text{m/s}^2)(2t)$$

$$v_y = \frac{dy}{dt} = 1.0\ \text{m/s} + (0.025\ \text{m/s}^3)(3t^2)$$

and that the velocity vector is

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} = (-0.50\ \text{m/s}^2)t\hat{\imath} + [1.0\ \text{m/s} + (0.075\ \text{m/s}^3)t^2]\hat{\jmath}$$

(a) Find the components of the average acceleration in the interval from $t = 0.0$ s to $t = 2.0$ s. (b) Find the instantaneous acceleration at $t = 2.0$ s.

SOLUTION

IDENTIFY: This example uses the vector relationships among velocity, average acceleration, and instantaneous acceleration.

SET UP: In part (a) we first determine the values of v_x and v_y at the beginning and end of the interval, and then use Eq. (3.8) to calculate the components of the average acceleration. In part (b) we determine the instantaneous acceleration components at any time t by taking the time derivatives of the velocity components as in Eq. (3.10).

EXECUTE: (a) If we substitute $t = 0.0$ s or $t = 2.0$ s into the expressions for v_x and v_y, we find that at the beginning of the interval $(t = 0.0\ \text{s})$ the velocity components are

$$v_x = 0.0\ \text{m/s} \qquad v_y = 1.0\ \text{m/s}$$

and that at the end of the interval $(t = 2.0\ \text{s})$ the components are

$$v_x = -1.0\ \text{m/s} \qquad v_y = 1.3\ \text{m/s}$$

(The values at $t = 2.0$ s are the same as we found in Example 3.1.) Thus the components of average acceleration in this interval are

$$a_{\text{av-}x} = \frac{\Delta v_x}{\Delta t} = \frac{-1.0\ \text{m/s} - 0.0\ \text{m/s}}{2.0\ \text{s} - 0.0\ \text{s}} = -0.5\ \text{m/s}^2$$

$$a_{\text{av-}y} = \frac{\Delta v_y}{\Delta t} = \frac{1.3\ \text{m/s} - 1.0\ \text{m/s}}{2.0\ \text{s} - 0.0\ \text{s}} = 0.15\ \text{m/s}^2$$

(b) Using Eq. (3.10), we find

$$a_x = \frac{dv_x}{dt} = -0.50\ \text{m/s}^2 \qquad a_y = \frac{dv_y}{dt} = (0.075\ \text{m/s}^3)(2t)$$

We can write the instantaneous acceleration vector $\vec{a}$ as

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = (-0.50\ \text{m/s}^2)\hat{\imath} + (0.15\ \text{m/s}^3)t\hat{\jmath}$$

At time $t = 2.0$ s, the components of instantaneous acceleration are

$$a_x = -0.50\ \text{m/s}^2 \qquad a_y = (0.15\ \text{m/s}^3)(2.0\ \text{s}) = 0.30\ \text{m/s}^2$$

The acceleration vector at this time is

$$\vec{a} = (-0.50\ \text{m/s}^2)\hat{\imath} + (0.30\ \text{m/s}^2)\hat{\jmath}$$

The magnitude of acceleration at this time is

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.50\ \text{m/s}^2)^2 + (0.30\ \text{m/s}^2)^2} = 0.58\ \text{m/s}^2$$

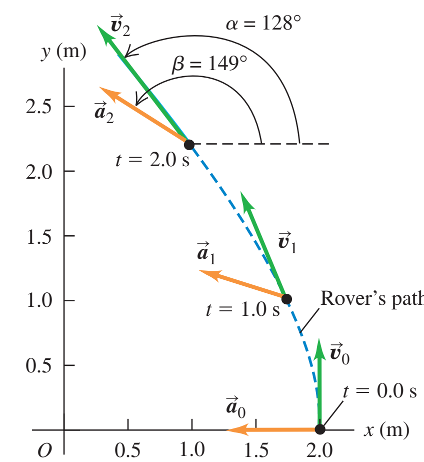
The direction of $\vec{a}$ with respect to the positive x-axis is given by the angle β, where

$$\tan\beta = \frac{a_y}{a_x} = \frac{0.30\ \text{m/s}^2}{-0.50\ \text{m/s}^2} = -0.60$$

$$\beta = 180° - 31° = 149°$$

EVALUATE: You should use the results of part (b) to calculate the instantaneous acceleration at $t = 0.0$ s and $t = 1.0$ s. Figure 3.9 shows the rover's path and the velocity and acceleration vectors at $t = 0.0$ s, 1.0 s, and 2.0 s. Note that $\vec{v}$ and $\vec{a}$ are *not* in the same direction at any of these times. The velocity vector $\vec{v}$ is tangent to the path at each point, and the acceleration vector $\vec{a}$ points toward the concave side of the path.

3.9 The path of the robotic rover, showing the velocity and acceleration at $t = 0.0$ s ($\vec{v}_0$ and $\vec{a}_0$), $t = 1.0$ s ($\vec{v}_1$ and $\vec{a}_1$), and $t = 2.0$ s ($\vec{v}_2$ and $\vec{a}_2$).



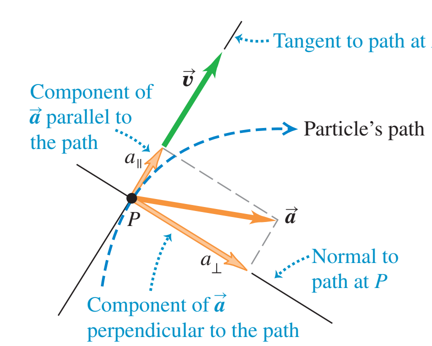
Parallel and Perpendicular Components of Acceleration

The acceleration vector $\vec{a}$ for a particle can describe changes in the particle's speed, its direction of motion, or both. It's useful to note that the component of acceleration *parallel* to a particle's path—that is, parallel to the velocity—tells us about changes in the particle's *speed,* while the acceleration component *perpendicular* to the path—and hence perpendicular to the velocity—tells us about changes in the particle's *direction of motion.* Figure 3.10 shows these components, which we label $a_\parallel$ and $a_\perp$. To see why the parallel and perpendicular components of $\vec{a}$ have these properties, let's consider two special cases.

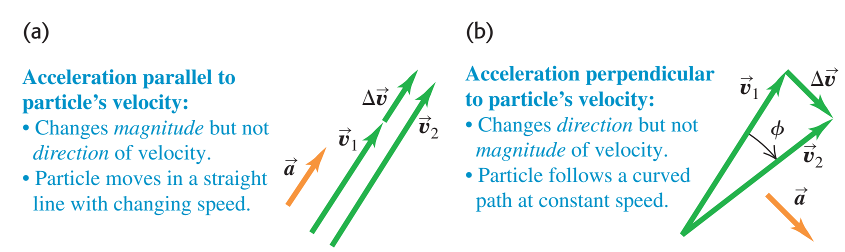
In Fig. 3.11a the acceleration vector is in the same direction as the velocity $\vec{v}_1$, so $\vec{a}$ has only a parallel component $a_\parallel$ (that is, $a_\perp = 0$). The velocity change $\Delta\vec{v}$ during a small time interval Δt is in the same direction as $\vec{a}$ and hence in the same direction as $\vec{v}_1$. The velocity $\vec{v}_2$ at the end of Δt, given by $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$, is in the same direction as $\vec{v}_1$ but has greater magnitude. Hence during the time interval Δt the particle in Fig. 3.11a moved in a straight line with increasing speed.

In Fig. 3.11b the acceleration is *perpendicular* to the velocity, so $\vec{a}$ has only a perpendicular component $a_\perp$ (that is, $a_\parallel = 0$). In a small time interval Δt, the velocity change $\Delta\vec{v}$ is very nearly perpendicular to $\vec{v}_1$. Again $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$, but in this case $\vec{v}_1$ and $\vec{v}_2$ have different directions. As the time interval Δt

3.10 The acceleration can be resolved into a component $a_\parallel$ parallel to the path (that is, along the tangent to the path) and a component $a_\perp$ perpendicular to the path (that is, along the normal to the path).



3.11 The effect of acceleration directed (a) parallel to and (b) perpendicular to a particle's velocity.

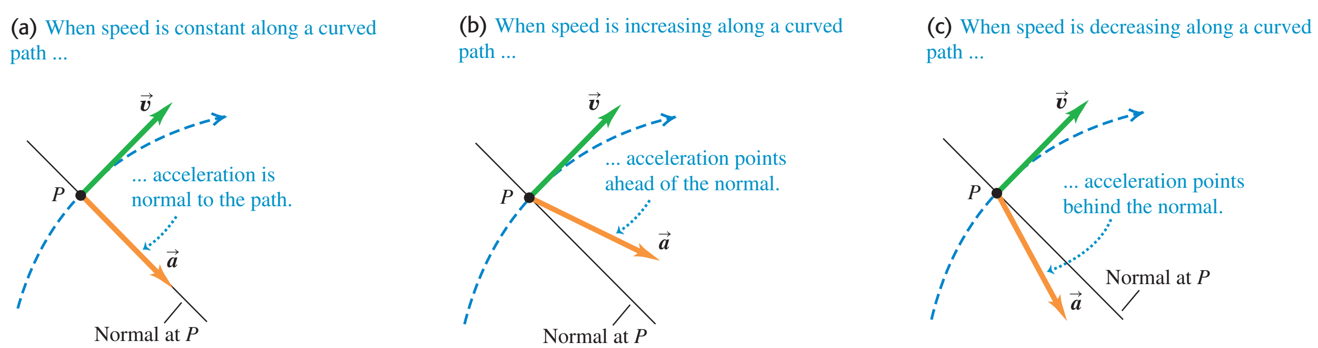


approaches zero, the angle ϕ in the figure also approaches zero, $\Delta\vec{v}$ becomes perpendicular to *both* $\vec{v}_1$ and $\vec{v}_2$, and $\vec{v}_1$ and $\vec{v}_2$ have the same magnitude. In other words, the speed of the particle stays the same, but the direction of motion changes and the path of the particle curves.

In the most general case, the acceleration $\vec{a}$ has components *both* parallel and perpendicular to the velocity $\vec{v}$, as in Fig. 3.10. Then the particle's speed will change (described by the parallel component $a_\parallel$) *and* its direction of motion will change (described by the perpendicular component $a_\perp$) so that it follows a curved path.

Figure 3.12 shows a particle moving along a curved path for three different situations: constant speed, increasing speed, and decreasing speed. If the speed is constant, $\vec{a}$ is perpendicular, or *normal,* to the path and to $\vec{v}$ and points toward the concave side of the path (Fig. 3.12a). If the speed is increasing, there is still a perpendicular component of $\vec{a}$, but there is also a parallel component having the same direction as $\vec{v}$ (Fig. 3.12b). Then $\vec{a}$ points ahead of the normal to the path. (This was the case in Example 3.2.) If the speed is decreasing, the parallel component has the direction opposite to $\vec{v}$, and $\vec{a}$ points behind the normal to the path (Fig. 3.12c). We will use these ideas again in Section 3.4 when we study the special case of motion in a circle.

3.12 Velocity and acceleration vectors for a particle moving through a point P on a curved path with (a) constant speed, (b) increasing speed, and (c) decreasing speed.



Example 3.3 Calculating parallel and perpendicular components of acceleration

For the rover of Examples 3.1 and 3.2, find the parallel and perpendicular components of the acceleration at $t = 2.0$ s.

SOLUTION

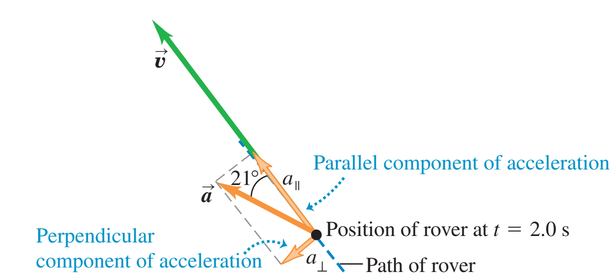
IDENTIFY: We want to find the components of the acceleration vector $\vec{a}$ that are parallel and perpendicular to the velocity vector $\vec{v}$.

SET UP: We found the directions of $\vec{a}$ and $\vec{v}$ in Examples 3.2 and 3.1, respectively. This will allow us to find the angle between the two vectors and hence the components of $\vec{a}$.

EXECUTE: In Example 3.2 we found that at $t = 2.0$ s the particle has an acceleration of magnitude $0.58\ \text{m/s}^2$ at an angle of 149° with respect to the positive x-axis. From Example 3.1, at this same time the velocity vector is at an angle of 128° with respect to the positive x-axis. So Fig. 3.9 shows that the angle between $\vec{a}$ and $\vec{v}$ is $149° - 128° = 21°$ (Fig. 3.13). The parallel and perpendicular components of acceleration are then

$$a_\parallel = a\cos 21° = (0.58\ \text{m/s}^2)\cos 21° = 0.54\ \text{m/s}^2$$
$$a_\perp = a\sin 21° = (0.58\ \text{m/s}^2)\sin 21° = 0.21\ \text{m/s}^2$$

3.13 The parallel and perpendicular components of the acceleration of the rover at $t = 2.0$ s.



EVALUATE: The parallel component $a_\parallel$ is in the same direction as $\vec{v}$, which means that the speed is increasing at this instant; the value of $a_\parallel = 0.54\ \text{m/s}^2$ means that the speed is increasing at a rate of 0.54 m/s per second. The perpendicular component $a_\perp$ is not zero, which means that at this instant the rover is changing direction and following a curved path; in other words, the rover is turning.

Conceptual Example 3.4 Acceleration of a skier

A skier moves along a ski-jump ramp as shown in Fig. 3.14a. The ramp is straight from point A to point C and curved from point C onward. The skier picks up speed as she moves downhill from point A to point E, where her speed is maximum. She slows down after passing point E. Draw the direction of the acceleration vector at points B, D, E, and F.

SOLUTION

Figure 3.14b shows our solution. At point B the skier is moving in a straight line with increasing speed, so her acceleration points downhill, in the same direction as her velocity.

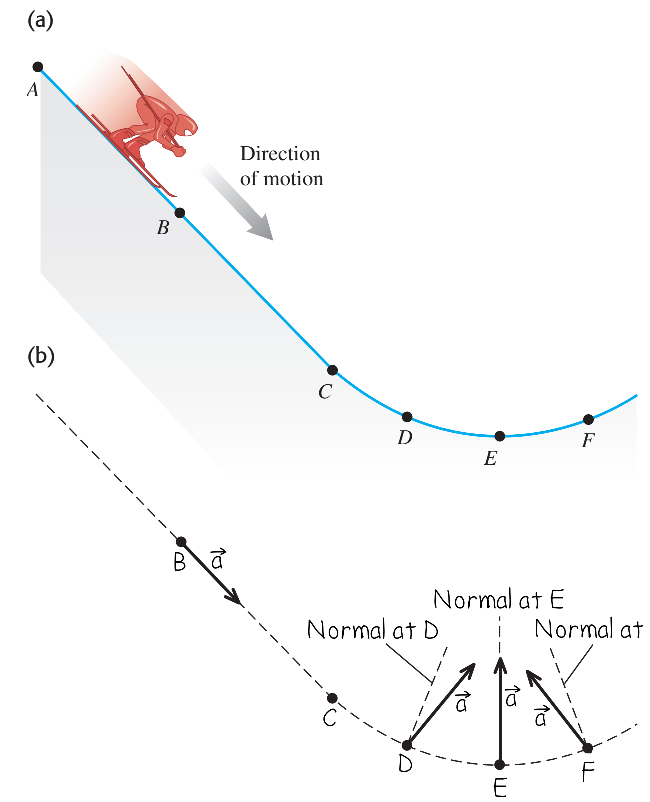
At point D the skier is moving along a curved path, so her acceleration has a component perpendicular to the path. There is also a component in the direction of her motion because she is still speeding up at this point. So the acceleration vector points *ahead* of the normal to her path at point D.

The skier's speed is instantaneously not changing at point E; the speed is maximum at this point, so its derivative is zero. There is no parallel component of $\vec{a}$, and the acceleration is perpendicular to her motion.

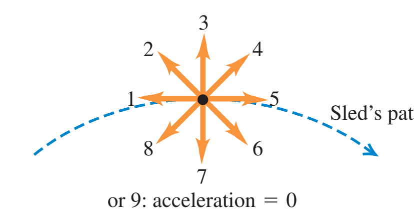
Finally, at point F the acceleration has a perpendicular component (because her path is curved at this point) and a parallel component *opposite* to the direction of her motion (because she's slowing down). So at this point, the acceleration vector points *behind* the normal to her path.

In the next section we'll examine the skier's acceleration after she flies off the ramp.

3.14 (a) The skier's path. (b) Our solution.



Test Your Understanding of Section 3.2 A sled travels over the crest of a snow-covered hill. The sled slows down as it climbs up one side of the hill and gains speed as it descends on the other side. Which of the vectors (1 through 9) in the figure correctly shows the direction of the sled's acceleration at the crest? (Choice 9 is that the acceleration is zero.)



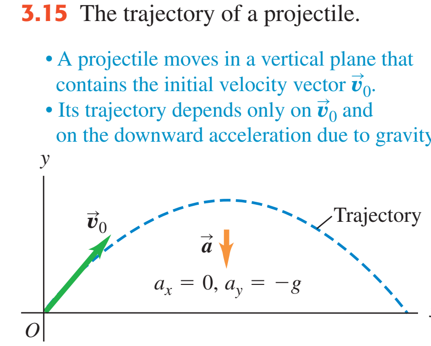
3.3 Projectile Motion

A **projectile** is any body that is given an initial velocity and then follows a path determined entirely by the effects of gravitational acceleration and air resistance. A batted baseball, a thrown football, a package dropped from an airplane, and a bullet shot from a rifle are all projectiles. The path followed by a projectile is called its **trajectory.**

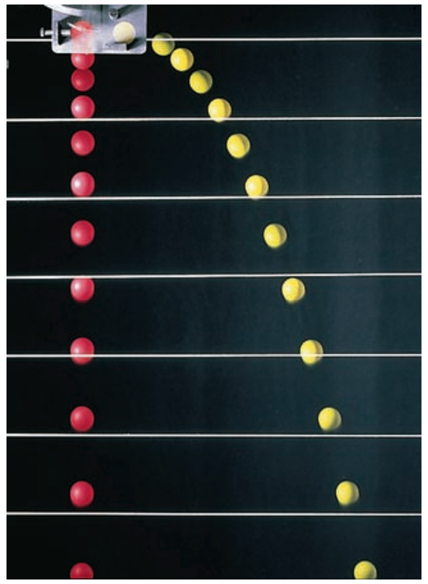
To analyze this common type of motion, we'll start with an idealized model, representing the projectile as a single particle with an acceleration (due to gravity) that is constant in both magnitude and direction. We'll neglect the effects of air resistance and the curvature and rotation of the earth. Like all models, this one has limitations. Curvature of the earth has to be considered in the flight of long-range missiles, and air resistance is of crucial importance to a sky diver. Nevertheless, we can learn a lot from analysis of this simple model. For the remainder of this chapter the phrase "projectile motion" will imply that we're ignoring air resistance. In Chapter 5 we will see what happens when air resistance cannot be ignored.

Projectile motion is always confined to a vertical plane determined by the direction of the initial velocity (Fig. 3.15). This is because the acceleration due to

3.15 The trajectory of a projectile.



3.16 The red ball is dropped from rest, and the yellow ball is simultaneously projected horizontally; successive images in this stroboscopic photograph are separated by equal time intervals. At any given time, both balls have the same y-position, y-velocity, and y-acceleration, despite having different x-positions and x-velocities.



gravity is purely vertical; gravity can't move the projectile sideways. Thus projectile motion is *two-dimensional.* We will call the plane of motion the xy-coordinate plane, with the x-axis horizontal and the y-axis vertically upward.

The key to analyzing projectile motion is that we can treat the x- and y-coordinates separately. The x-component of acceleration is zero, and the y-component is constant and equal to $-g$. (By definition, g is always positive; with our choice of coordinate directions, a_y is negative.) So *we can analyze projectile motion as a combination of horizontal motion with constant velocity and vertical motion with constant acceleration.* Figure 3.16 shows two projectiles with different x-motion but identical y-motion; one is dropped from rest and the other is projected horizontally, but both projectiles fall the same distance in the same time.

We can then express all the vector relationships for the projectile's position, velocity, and acceleration by separate equations for the horizontal and vertical components. The components of $\vec{a}$ are

$$a_x = 0 \qquad a_y = -g \qquad \text{(projectile motion, no air resistance)} \tag{3.14}$$

Since the x-acceleration and y-acceleration are both constant, we can use Eqs. (2.8), (2.12), (2.13), and (2.14) directly. For example, suppose that at time $t = 0$ our particle is at the point (x_0, y_0) and that at this time its velocity components have the initial values v_{0x} and v_{0y}. The components of acceleration are $a_x = 0$, $a_y = -g$. Considering the x-motion first, we substitute 0 for a_x in Eqs. (2.8) and (2.12). We find

$$v_x = v_{0x} \tag{3.15}$$

$$x = x_0 + v_{0x}t \tag{3.16}$$

For the y-motion we substitute y for x, v_y for v_x, v_{0y} for v_{0x}, and $a_y = -g$ for a_x:

$$v_y = v_{0y} - gt \tag{3.17}$$

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \tag{3.18}$$

It's usually simplest to take the initial position (at $t = 0$) as the origin; then $x_0 = y_0 = 0$. This might be the position of a ball at the instant it leaves the thrower's hand or the position of a bullet at the instant it leaves the gun barrel.

Figure 3.17 shows the path of a projectile that starts at (or passes through) the origin at time $t = 0$. The position, velocity, and velocity components are shown

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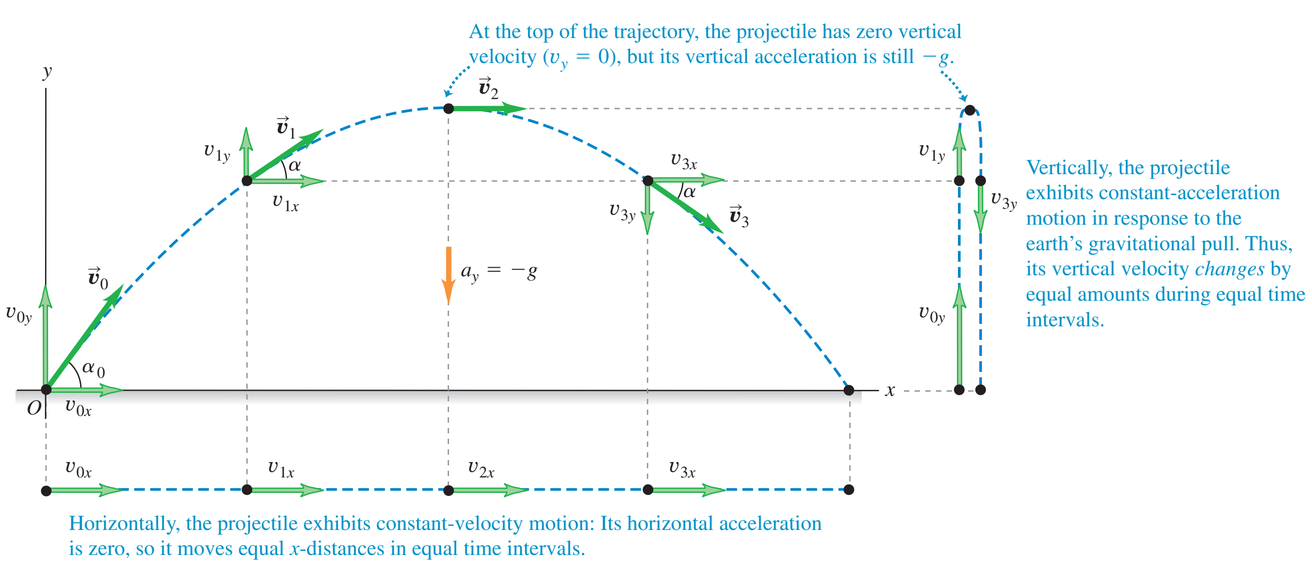
3.1 Solving Projectile Motion Problems
3.2 Two Balls Falling
3.3 Changing the x-velocity
3.4 Projectile x-y-Accelerations

3.17 If air resistance is negligible, the trajectory of a projectile is a combination of horizontal motion with constant velocity and vertical motion with constant acceleration.



at equal time intervals. The x-component of acceleration is zero, so v_x is constant. The y-component of acceleration is constant and not zero, so v_y changes by equal amounts in equal times, just the same as if the projectile were launched vertically with the same initial y-velocity. At the highest point in the trajectory, $v_y = 0$.

We can also represent the initial velocity $\vec{v}_0$ by its magnitude v_0 (the initial speed) and its angle α_0 with the positive x-axis (Fig. 3.18). In terms of these quantities, the components v_{0x} and v_{0y} of the initial velocity are

$$v_{0x} = v_0\cos\alpha_0 \qquad v_{0y} = v_0\sin\alpha_0 \tag{3.19}$$

Using these relationships in Eqs. (3.15) through (3.18) and setting $x_0 = y_0 = 0$, we find

$$x = (v_0\cos\alpha_0)t \qquad \text{(projectile motion)} \tag{3.20}$$

$$y = (v_0\sin\alpha_0)t - \frac{1}{2}gt^2 \qquad \text{(projectile motion)} \tag{3.21}$$

$$v_x = v_0\cos\alpha_0 \qquad \text{(projectile motion)} \tag{3.22}$$

$$v_y = v_0\sin\alpha_0 - gt \qquad \text{(projectile motion)} \tag{3.23}$$

These equations describe the position and velocity of the projectile in Fig. 3.17 at any time t.

We can get a lot of information from these equations. For example, at any time the distance r of the projectile from the origin (the magnitude of the position vector $\vec{r}$) is given by

$$r = \sqrt{x^2 + y^2} \tag{3.24}$$

The projectile's speed (the magnitude of its velocity) at any time is

$$v = \sqrt{v_x^2 + v_y^2} \tag{3.25}$$

The *direction* of the velocity, in terms of the angle α it makes with the positive x-direction (see Fig. 3.17), is given by

$$\tan\alpha = \frac{v_y}{v_x} \tag{3.26}$$

The velocity vector $\vec{v}$ is tangent to the trajectory at each point.

We can derive an equation for the trajectory's shape in terms of x and y by eliminating t. From Eqs. (3.20) and (3.21), which assume $x_0 = y_0 = 0$, we find $t = x/(v_0\cos\alpha_0)$ and

$$y = (\tan\alpha_0)x - \frac{g}{2v_0^2\cos^2\alpha_0}x^2 \tag{3.27}$$

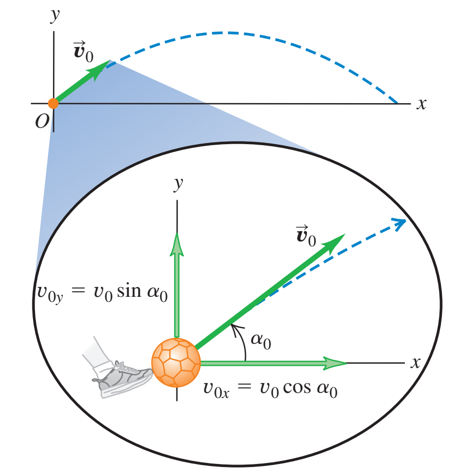
Don't worry about the details of this equation; the important point is its general form. The quantities v_0, $\tan\alpha_0$, $\cos\alpha_0$, and g are constants, so the equation has the form

$$y = bx - cx^2$$

where b and c are constants. This is the equation of a *parabola.* In projectile motion, with our simple model, the trajectory is always a parabola (Fig. 3.19).

When air resistance *isn't* always negligible and has to be included, calculating the trajectory becomes a lot more complicated; the effects of air resistance

3.18 The initial velocity components v_{0x} and v_{0y} of a projectile (such as a kicked soccer ball) are related to the initial speed v_0 and initial angle α_0.



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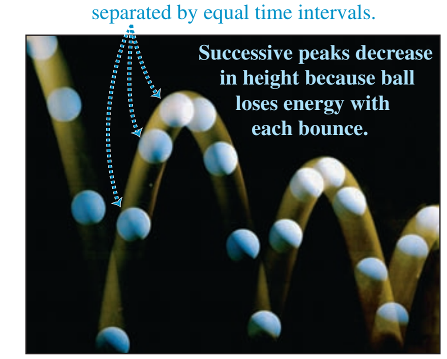
3.5 Initial Velocity Components
3.6 Target Practice I
3.7 Target Practice II

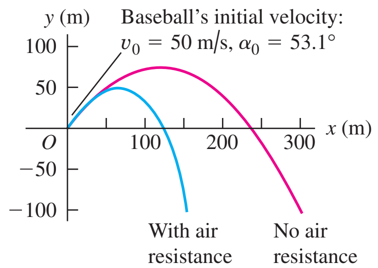
3.19 The nearly parabolic trajectories of (a) a bouncing ball and (b) blobs of molten rock ejected from a volcano.

(a) Successive images of ball are separated by equal time intervals.



(b)

3.20 Air resistance has a large cumulative effect on the motion of a baseball. In this simulation we allow the baseball to fall below the height from which it was thrown (for example, the baseball could have been thrown from a cliff).



depend on velocity, so the acceleration is no longer constant. Figure 3.20 shows a computer simulation of the trajectory of a baseball both without air resistance and with air resistance proportional to the square of the baseball's speed. We see that air resistance has a very large effect; the maximum height and range both decrease, and the trajectory is no longer a parabola. (If you look closely at Fig. 3.19b, you'll see that the trajectories of the volcanic blobs deviate in a similar way from a parabolic shape.)

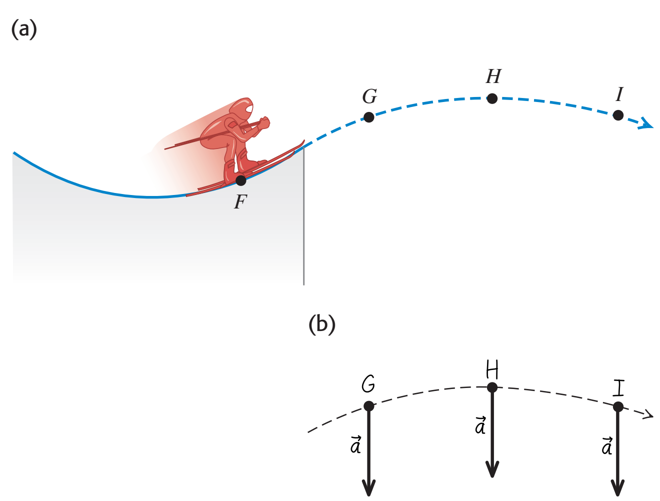
Conceptual Example 3.5 Acceleration of a skier, continued

Let's consider again the skier in Conceptual Example 3.4. What is her acceleration at points G, H, and I in Fig. 3.21a *after* she flies off the ramp? Neglect air resistance.

SOLUTION

Figure 3.21b shows our answer. The skier's acceleration changed from point to point while she was on the ramp. But as soon as she leaves the ramp, she becomes a projectile. So at points G, H, and I, and indeed at *all* points after she leaves the ramp, the skier's acceleration points vertically downward and has magnitude g. No matter how complicated the acceleration of a particle before it becomes a projectile, its acceleration as a projectile is given by $a_x = 0$, $a_y = -g$.

3.21 (a) The skier's path during the jump. (b) Our solution.



Problem Solving Strategy 3.1 Projectile Motion

NOTE: The strategies we used in Sections 2.4 and 2.5 for straight-line, constant-acceleration problems are also useful here.

IDENTIFY *the relevant concepts:* The key concept to remember is that throughout projectile motion, the acceleration is downward and has a constant magnitude g. Note that the projectile-motion equations don't apply to throwing a ball, because during the throw the ball is acted on by both the thrower's hand and gravity. These equations come into play only after the ball leaves the thrower's hand.

SET UP *the problem* using the following steps:

1. Define your coordinate system and make a sketch showing your axes. Usually it's easiest to take the x-axis as being horizontal and the y-axis as being upward and to place the origin at the initial $(t = 0)$ position where the body first becomes a projectile (such as where a ball leaves the thrower's hand). Then the components of the (constant) acceleration are $a_x = 0$, $a_y = -g$, and the initial position is $x_0 = 0$, $y_0 = 0$.

2. List the unknown and known quantities, and decide which unknowns are your target variables. For example, you might be given the initial velocity (either the components or the magnitude and direction) and asked to find the coordinates and velocity components at some later time. In any case, you'll be using Eqs. (3.20) through (3.23). (Certain other equations given in Section 3.3 may be useful as well.) Make sure that you have as many equations as there are target variables to be found.
3. State the problem in words and then translate those words into symbols. For example, *when* does the particle arrive at a certain point? (That is, at what value of t?) *Where* is the particle when its velocity has a certain value? (That is, what are the values of x and y when v_x or v_y has the specified value?) Since $v_y = 0$ at the highest point in a trajectory, the question "When does the projectile reach its highest point?" translates into "What is the value of t when $v_y = 0$?" Similarly, "When does the projectile return to its initial elevation?" translates into "What is the value of t when $y = y_0$?"

EXECUTE *the solution:* Use Eqs. (3.20) through (3.23) to find the target variables. Resist the temptation to break the trajectory into segments and analyze each segment separately. You don't have to start all over when the projectile reaches its highest point! It's almost always easier to use the same axes and time scale throughout the problem. Use the value $g = 9.8\ \text{m/s}^2$.

EVALUATE *your answer:* As always, look at your results to see whether they make sense and whether the numerical values seem reasonable.

Example 3.6 A body projected horizontally

A motorcycle stunt rider rides off the edge of a cliff. Just at the edge his velocity is horizontal, with magnitude 9.0 m/s. Find the motorcycle's position, distance from the edge of the cliff, and velocity after 0.50 s.

SOLUTION

IDENTIFY: Once the rider leaves the cliff, he is in projectile motion. His velocity at the edge of the cliff is therefore his initial velocity.

SET UP: Figure 3.22 shows our sketch. We place the origin of our coordinate system at the edge of the cliff, where the motorcycle first becomes a projectile, so $x_0 = 0$ and $y_0 = 0$. The initial velocity is purely horizontal (that is, $\alpha_0 = 0$), so the initial velocity components are $v_{0x} = v_0 \cos\alpha_0 = 9.0$ m/s and $v_{0y} = v_0 \sin\alpha_0 = 0$. To find the motorcycle's position at time $t = 0.50$ s, we use Eqs. (3.20) and (3.21), which give x and y as functions of time. We then find the distance from the origin using Eq. (3.24). Finally, we use Eqs. (3.22) and (3.23) to find the velocity components v_x and v_y at $t = 0.50$ s.

EXECUTE: Where is the motorcycle at $t = 0.50$ s? From Eqs. (3.20) and (3.21), the x- and y-coordinates are

$$x = v_{0x}t = (9.0\ \text{m/s})(0.50\ \text{s}) = 4.5\ \text{m}$$
$$y = -\frac{1}{2}gt^2 = -\frac{1}{2}(9.8\ \text{m/s}^2)(0.50\ \text{s})^2 = -1.2\ \text{m}$$

The negative value of y shows that at this time the motorcycle is below its starting point.

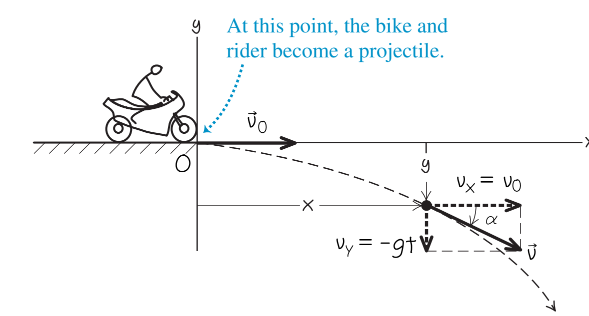
What is the motorcycle's distance from the origin at this time? From Eq. (3.24),

$$r = \sqrt{x^2 + y^2} = \sqrt{(4.5\ \text{m})^2 + (-1.2\ \text{m})^2} = 4.7\ \text{m}$$

What is the velocity at time $t = 0.50$ s? From Eqs. (3.22) and (3.23), the components of velocity at this time are

$$v_x = v_{0x} = 9.0\ \text{m/s}$$
$$v_y = -gt = (-9.8\ \text{m/s}^2)(0.50\ \text{s}) = -4.9\ \text{m/s}$$

3.22 Our sketch for this problem.



The motorcycle has the same horizontal velocity v_x as when it left the cliff at $t = 0$, but in addition there is a downward (negative) vertical velocity v_y. If we use unit vectors, the velocity at $t = 0.50$ s is

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} = (9.0\ \text{m/s})\hat{\imath} + (-4.9\ \text{m/s})\hat{\jmath}$$

We can also express the velocity in terms of magnitude and direction. From Eq. (3.25), the speed (magnitude of the velocity) at this time is

$$v = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{(9.0\ \text{m/s})^2 + (-4.9\ \text{m/s})^2} = 10.2\ \text{m/s}$$

From Eq. (3.26), the angle α of the velocity vector is

$$\alpha = \arctan\frac{v_y}{v_x} = \arctan\left(\frac{-4.9\ \text{m/s}}{9.0\ \text{m/s}}\right) = -29°$$

At this time the velocity is 29° below the horizontal.

EVALUATE: Just as shown in Fig. 3.17, the horizontal aspect of the motion is unchanged by gravity; the motorcycle continues to move horizontally at 9.0 m/s, covering 4.5 m in 0.50 s. The motorcycle initially has zero vertical velocity, so it falls vertically just like a body released from rest and descends a distance $\frac{1}{2}gt^2 = 1.2$ m in 0.50 s.

Example 3.7 Height and range of a projectile I: A batted baseball

A batter hits a baseball so that it leaves the bat at speed $v_0 = 37.0$ m/s at an angle $\alpha_0 = 53.1°$, at a location where $g = 9.80$ m/s^2. (a) Find the position of the ball, and the magnitude and direction of its velocity, at $t = 2.00$ s. (b) Find the time when the ball reaches the highest point of its flight and find its height h at this point. (c) Find the *horizontal range R*—that is, the horizontal distance from the starting point to where the ball hits the ground.

SOLUTION

IDENTIFY: As Fig. 3.20 shows, the effects of air resistance on the motion of a baseball aren't really negligible. For the sake of simplicity, however, we'll ignore air resistance for this example and use the projectile-motion equations to describe the motion.

SET UP: Figure 3.23 shows our sketch. We use the same coordinate system as in Fig. 3.17 or 3.18 so we can use Eqs. (3.20) through (3.23) without any modifications. Our target variables are (1) the position and velocity of the ball 2.00 s after it leaves the bat, (2) the elapsed time after leaving the bat when the ball is at its maximum height—that is, when $v_y = 0$—and the y-coordinate at this time, and (3) the x-coordinate at the time when the y-coordinate is equal to the initial value y_0.

The ball leaves the bat a meter or so above ground level, but we neglect this distance and assume that it starts at ground level $(y_0 = 0)$. The initial velocity of the ball has components

$$v_{0x} = v_0 \cos\alpha_0 = (37.0\ \text{m/s})\cos 53.1° = 22.2\ \text{m/s}$$
$$v_{0y} = v_0 \sin\alpha_0 = (37.0\ \text{m/s})\sin 53.1° = 29.6\ \text{m/s}$$

EXECUTE: (a) We want to find x, y, v_x, and v_y at time $t = 2.00$ s. From Eqs. (3.20) through (3.23),

$$x = v_{0x}t = (22.2\ \text{m/s})(2.00\ \text{s}) = 44.4\ \text{m}$$
$$y = v_{0y}t - \frac{1}{2}gt^2$$
$$= (29.6\ \text{m/s})(2.00\ \text{s}) - \frac{1}{2}(9.80\ \text{m/s}^2)(2.00\ \text{s})^2$$
$$= 39.6\ \text{m}$$
$$v_x = v_{0x} = 22.2\ \text{m/s}$$
$$v_y = v_{0y} - gt = 29.6\ \text{m/s} - (9.80\ \text{m/s}^2)(2.00\ \text{s})$$
$$= 10.0\ \text{m/s}$$

The y-component of velocity is positive, which means that the ball is still moving upward at this time (Fig. 3.23). The magnitude and direction of the velocity are found from Eqs. (3.25) and (3.26):

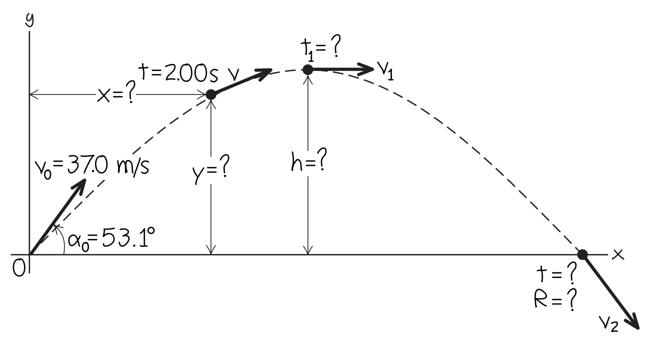
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(22.2\ \text{m/s})^2 + (10.0\ \text{m/s})^2}$$
$$= 24.3\ \text{m/s}$$
$$\alpha = \arctan\left(\frac{10.0\ \text{m/s}}{22.2\ \text{m/s}}\right) = \arctan 0.450 = 24.2°$$

The direction of the velocity (that is, the direction of motion) is 24.2° above the horizontal.

(b) At the highest point, the vertical velocity v_y is zero. When does this happen? Call the time t_1; then

$$v_y = v_{0y} - gt_1 = 0$$
$$t_1 = \frac{v_{0y}}{g} = \frac{29.6\ \text{m/s}}{9.80\ \text{m/s}^2} = 3.02\ \text{s}$$

3.23 Our sketch for this problem.



The height h at this time is the value of y when $t = t_1 = 3.02$ s:

$$h = v_{0y}t_1 - \frac{1}{2}gt_1^2$$
$$= (29.6\ \text{m/s})(3.02\ \text{s}) - \frac{1}{2}(9.80\ \text{m/s}^2)(3.02\ \text{s})^2$$
$$= 44.7\ \text{m}$$

(c) We'll find the horizontal range in two steps. First, *when* does the ball hit the ground? This occurs when $y = 0$. Call this time t_2; then

$$y = 0 = v_{0y}t_2 - \frac{1}{2}gt_2^2 = t_2\left(v_{0y} - \frac{1}{2}gt_2\right)$$

This is a quadratic equation for t_2. It has two roots:

$$t_2 = 0 \qquad \text{and} \qquad t_2 = \frac{2v_{0y}}{g} = \frac{2(29.6\ \text{m/s})}{9.80\ \text{m/s}^2} = 6.04\ \text{s}$$

There are two times at which $y = 0$; $t_2 = 0$ is the time the ball *leaves* the ground, and $t_2 = 2v_{0y}/g = 6.04$ s is the time of its return. This is exactly twice the time to reach the highest point that we found in part (b), $t_1 = v_{0y}/g = 3.02$ s, so the time of descent equals the time of ascent. This is *always* true if the starting and end points are at the same elevation and air resistance can be neglected.

The horizontal range R is the value of x when the ball returns to the ground—that is, at $t = 6.04$ s:

$$R = v_{0x}t_2 = (22.2\ \text{m/s})(6.04\ \text{s}) = 134\ \text{m}$$

The vertical component of velocity when the ball hits the ground is

$$v_y = v_{0y} - gt_2 = 29.6\ \text{m/s} - (9.80\ \text{m/s}^2)(6.04\ \text{s})$$
$$= -29.6\ \text{m/s}$$

That is, v_y has the same magnitude as the initial vertical velocity v_{0y} but the opposite direction (down). Since v_x is constant, the angle $\alpha = -53.1°$ (below the horizontal) at this point is the negative of the initial angle $\alpha_0 = 53.1°$.

EVALUATE: It's often useful to check results by getting them in a different way. For example, we can check our answer for the maximum height in part (b) by applying the constant-acceleration formula Eq. (2.13) to the y-motion:

$$v_y^2 = v_{0y}^2 + 2a_y(y - y_0) = v_{0y}^2 - 2g(y - y_0)$$

At the highest point, $v_y = 0$ and $y = h$. Substituting these, along with $y_0 = 0$, we find

$$0 = v_{0y}^2 - 2gh$$
$$h = \frac{v_{0y}^2}{2g} = \frac{(29.6\ \text{m/s})^2}{2(9.80\ \text{m/s}^2)} = 44.7\ \text{m}$$

which is the same height we obtained in part (b).

It's interesting to note that $h = 44.7$ m in part (b) is comparable to the 52.4-m height above the playing field of the roof of the Hubert H. Humphrey Metrodome in Minneapolis, and the horizontal range $R = 134$ m in part (c) is greater than the 99.7-m distance from home plate to the right-field fence at Safeco Field in Seattle. (The ball's height when it crosses the fence is more than enough to clear it, so this ball is a home run.)

In real life, a batted ball with the initial speed and angle we've used here won't go as high or as far as we've calculated. (If it did, home runs would be far more common and baseball would be a far less interesting game.) The reason is that air resistance, which we neglected in this example, is actually an important factor at the typical speeds of pitched and batted balls (see Fig. 3.20).

Example 3.8 Height and range of a projectile II: Maximum height, maximum range

For a projectile launched with speed v_0 at initial angle α_0 (between 0° and 90°), derive general expressions for the maximum height h and horizontal range R (Fig. 3.23). For a given v_0, what value of α_0 gives maximum height? What value gives maximum horizontal range?

SOLUTION

IDENTIFY: This is really the same exercise as parts (b) and (c) of Example 3.7. The difference is that we are looking for general expressions for h and R. We'll also be looking for the values of α_0 that give the maximum values of h and R.

SET UP: In part (b) of Example 3.7 we found that the projectile reaches the high point of its trajectory (so that $v_y = 0$) at time $t_1 = v_{0y}/g$, and in part (c) of Example 3.7 we found that the projectile returns to its starting height (so that $y = y_0$) at time $t_2 = 2v_{0y}/g$. (As we saw in Example 3.7, $t_2 = 2t_1$.) To determine the height h at the high point of the trajectory, we use Eq. (3.21) to find the y-coordinate at t_1. To determine R, we substitute t_2 into Eq. (3.20) to determine the x-coordinate at t_2. We'll express our answers in terms of the launch speed v_0 and launch angle α_0 using Eq. (3.19).

EXECUTE: From Eq. (3.19), $v_{0x} = v_0\cos\alpha_0$ and $v_{0y} = v_0\sin\alpha_0$. Hence we can write the time t_1 when $v_y = 0$ as

$$t_1 = \frac{v_{0y}}{g} = \frac{v_0\sin\alpha_0}{g}$$

Then, from Eq. (3.21), the height at this time is

$$h = (v_0\sin\alpha_0)\left(\frac{v_0\sin\alpha_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0\sin\alpha_0}{g}\right)^2$$
$$= \frac{v_0^2\sin^2\alpha_0}{2g}$$

For a given launch speed v_0, the maximum value of h occurs when $\sin\alpha_0 = 1$ and $\alpha_0 = 90°$—that is, when the projectile is launched straight up. That's what we should expect. If it is launched horizontally, as in Example 3.6, $\alpha_0 = 0$ and the maximum height is zero!

The time t_2 when the projectile returns to the ground is

$$t_2 = \frac{2v_{0y}}{g} = \frac{2v_0\sin\alpha_0}{g}$$

The horizontal range R is the value of x at this time. From Eq. (3.20),

$$R = (v_0\cos\alpha_0)t_2 = (v_0\cos\alpha_0)\frac{2v_0\sin\alpha_0}{g}$$

We can now use the trigonometric identity $2\sin\alpha_0\cos\alpha_0 = \sin 2\alpha_0$ to rewrite this as

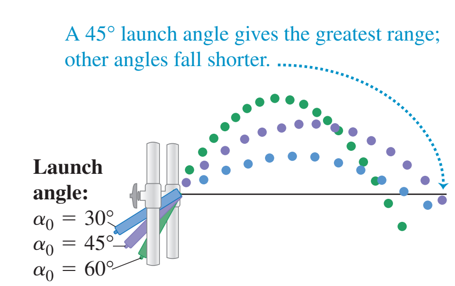
$$R = \frac{v_0^2\sin 2\alpha_0}{g}$$

The maximum value of $\sin 2\alpha_0$ is 1; this occurs when $2\alpha_0 = 90°$, or $\alpha_0 = 45°$. This angle gives the maximum range for a given initial speed.

EVALUATE: Figure 3.24 is based on a composite photograph of three trajectories of a ball projected from a spring gun at angles of 30°, 45°, and 60°. The initial speed v_0 is approximately the same in all three cases. The horizontal ranges are nearly the same for the 30° and 60° angles, and the range for 45° is greater than either. Can you prove that for a given value of v_0 the range is the same for both an initial angle α_0 and an initial angle $90° - \alpha_0$?

CAUTION **Height and range of a projectile** We don't recommend memorizing the above expressions for h and R. They are applicable only in the special circumstances we have described. In particular, the expression for the range R can be used *only* when launch and landing heights are equal. There are many end-of-chapter problems to which these equations do *not* apply.

3.24 A launch angle of 45° gives the maximum horizontal range. The range is shorter with launch angles of 30° and 60°.



Example 3.9 Different initial and final heights

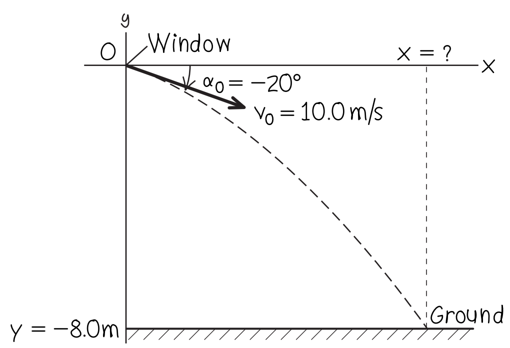
You toss a ball from your window 8.0 m above the ground. When the ball leaves your hand, it is moving at 10.0 m/s at an angle of 20° below the horizontal. How far horizontally from your window will the ball hit the ground? Ignore air resistance.

SOLUTION

IDENTIFY: As in our calculation of the horizontal range in Examples 3.7 and 3.8, we are trying to find the horizontal coordinate of a projectile when it is at a given value of y. The difference here is that this value of y is *not* equal to the initial y-coordinate.

SET UP: Once again we choose the x-axis to be horizontal and the y-axis to be upward, and we place the origin of coordinates at the point where the ball leaves your hand (Fig. 3.25). We have $v_0 = 10.0\text{ m/s}$ and $\alpha_0 = -20°$; the angle is negative because the initial velocity is below the horizontal. Our target variable is the value of x at the point where the ball reaches the ground—that is, when $y = -8.0\text{ m}$. Because the initial and final heights of the ball are different, we can't simply use the expression for the horizontal range found in Example 3.8. Instead, we first use Eq. (3.21) to find the time t when the ball reaches $y = -8.0\text{ m}$ and then calculate the value of x at this time using Eq. (3.20).

3.25 Our sketch for this problem.



EXECUTE: To determine t, we rewrite Eq. (3.21) in the standard form for a quadratic equation for t:

$$\frac{1}{2}gt^2 - (v_0\sin\alpha_0)t + y = 0$$

The roots of this equation are

$$t = \frac{v_0\sin\alpha_0 \pm \sqrt{(-v_0\sin\alpha_0)^2 - 4\left(\frac{1}{2}g\right)y}}{2\left(\frac{1}{2}g\right)}$$

$$= \frac{v_0\sin\alpha_0 \pm \sqrt{v_0^2\sin^2\alpha_0 - 2gy}}{g}$$

$$= \frac{\left[\begin{array}{l}(10.0\text{ m/s})\sin(-20°)\\ \pm\sqrt{(10.0\text{ m/s})^2\sin^2(-20°) - 2(9.80\text{ m/s}^2)(-8.0\text{ m})}\end{array}\right]}{9.80\text{ m/s}^2}$$

$$= -1.7\text{ s} \quad \text{or} \quad 0.98\text{ s}$$

We can discard the negative root, since it refers to a time before the ball left your hand. The positive root tells us that the ball takes 0.98 s to reach the ground. From Eq. (3.20), the ball's x-coordinate at that time is

$$x = (v_0\cos\alpha_0)t = (10.0\text{ m/s})[\cos(-20°)](0.98\text{ s})$$
$$= 9.2\text{ m}$$

The ball hits the ground a horizontal distance of 9.2 m from your window.

EVALUATE: The root $t = -1.7\text{ s}$ is an example of a "fictional" solution to a quadratic equation. We discussed these in Example 2.8 in Section 2.5; you should review that discussion.

With our choice of origin we had initial and final heights $y_0 = 0$ and $y = -8.0\text{ m}$. Can you use Eqs. (3.16) and (3.18) to show that you get the same answers for t and x if you choose the origin to be at the point on the ground directly below where the ball leaves your hand?

Example 3.10 The zookeeper and the monkey

A monkey escapes from the zoo and climbs a tree. After failing to entice the monkey down, the zookeeper fires a tranquilizer dart directly at the monkey (Fig. 3.26). The clever monkey lets go at the same instant the dart leaves the gun barrel, intending to land on the ground and escape. Show that the dart *always* hits the monkey, regardless of the dart's muzzle velocity (provided that it gets to the monkey before he hits the ground).

SOLUTION

IDENTIFY: In this example we have *two* bodies in projectile motion: the tranquilizer dart and the monkey. The dart and the monkey have different initial positions and initial velocities, but they go into projectile motion at the same time. To show that the dart hits the monkey, we have to prove that at some time the monkey and the dart have the same x-coordinate and the same y-coordinate.

SET UP: We make the usual choice for the x- and y-directions, and place the origin of coordinates at the end of the barrel of the tranquilizer gun (Fig. 3.26). We'll first use Eq. (3.20) to find the time t when the x-coordinates x_{monkey} and x_{dart} are the same. Then we'll use Eq. (3.21) to check whether y_{monkey} and y_{dart} are also equal at this time; if they are, the dart hits the monkey.

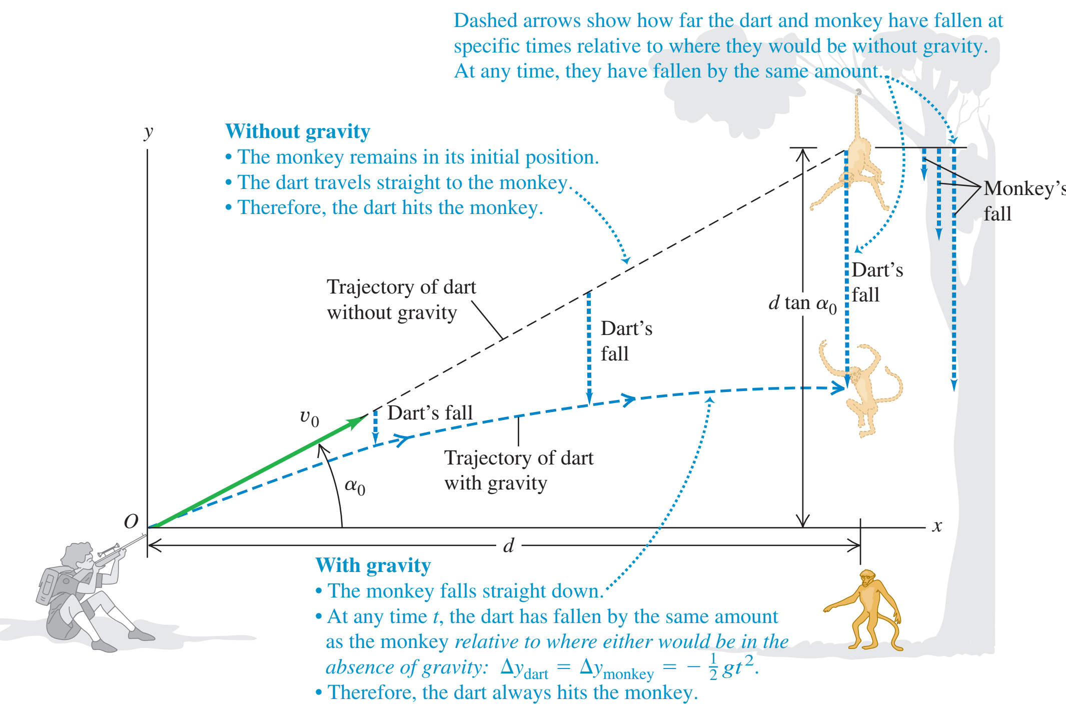
EXECUTE: The monkey drops straight down, so $x_{monkey} = d$ at *all* times. For the dart, Eq. (3.20) tells us that $x_{dart} = (v_0\cos\alpha_0)t$. When these x-coordinates are equal, $d = (v_0\cos\alpha_0)t$, or

$$t = \frac{d}{v_0\cos\alpha_0}$$

To have the dart hit the monkey, it must be true that $y_{monkey} = y_{dart}$ at this same time. The monkey is in one-dimensional free fall; his position at any time is given by Eq. (2.12), with appropriate symbol changes. Figure 3.26 shows that the monkey's initial height is $d\tan\alpha_0$ (the opposite side of a right triangle with angle α_0 and adjacent side d), and we find

$$y_{monkey} = d\tan\alpha_0 - \frac{1}{2}gt^2$$

3.26 The tranquilizer dart hits the falling monkey.



For the dart we use Eq. (3.21):

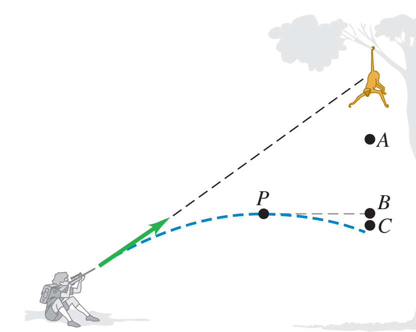
$$y_{dart} = (v_0\sin\alpha_0)t - \frac{1}{2}gt^2$$

So we see that if $d\tan\alpha_0 = (v_0\sin\alpha_0)t$ at the time when the two x-coordinates are equal, then $y_{monkey} = y_{dart}$, and we have a hit. To prove that this happens, we replace t with $d/(v_0\cos\alpha_0)$, the time when $x_{monkey} = x_{dart}$. Sure enough, we find that

$$(v_0\sin\alpha_0)t = (v_0\sin\alpha_0)\frac{d}{v_0\cos\alpha_0} = d\tan\alpha_0$$

EVALUATE: We have proved that at the time the x-coordinates are equal, the y-coordinates are also equal; a dart aimed at the initial position of the monkey *always* hits it, no matter what v_0 is. This result is also independent of the value of g, the acceleration due to gravity. With no gravity $(g = 0)$, the monkey would remain motionless, and the dart would travel in a straight line to hit him. With gravity, both "fall" the same distance $\left(\frac{1}{2}gt^2\right)$ below their $g = 0$ positions, and the dart still hits the monkey (Fig. 3.26).

Test Your Understanding of Section 3.3 In Example 3.10, suppose the tranquilizer dart has a relatively low muzzle velocity so that the dart reaches a maximum height at a point P before striking the monkey, as shown in the figure. When the dart is at point P, will the monkey be (i) at point A (higher than P), (ii) at point B (at the same height as P), or (iii) at point C (lower than P)? Ignore air resistance.



3.4 Motion in a Circle

When a particle moves along a curved path, the direction of its velocity changes. As we saw in Section 3.2, this means that the particle *must* have a component of acceleration perpendicular to the path, even if its speed is constant (see Fig. 3.11b). In this section we'll calculate the acceleration for the important special case of motion in a circle.

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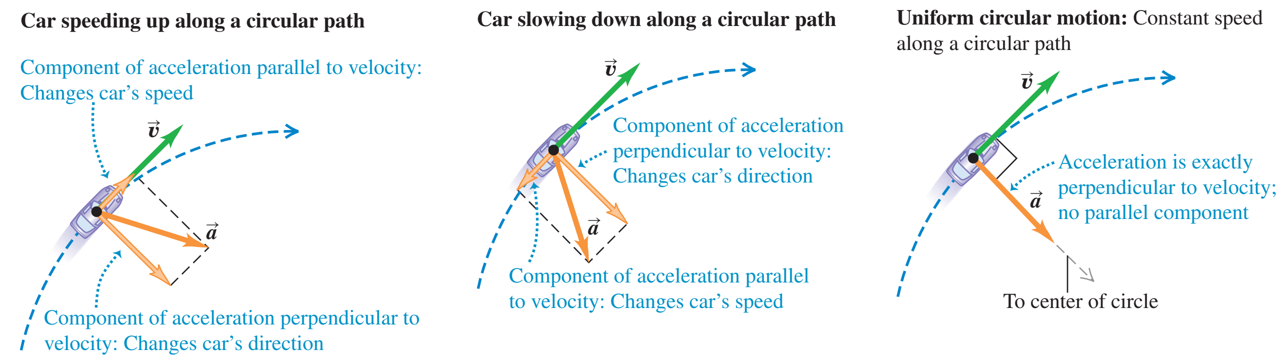
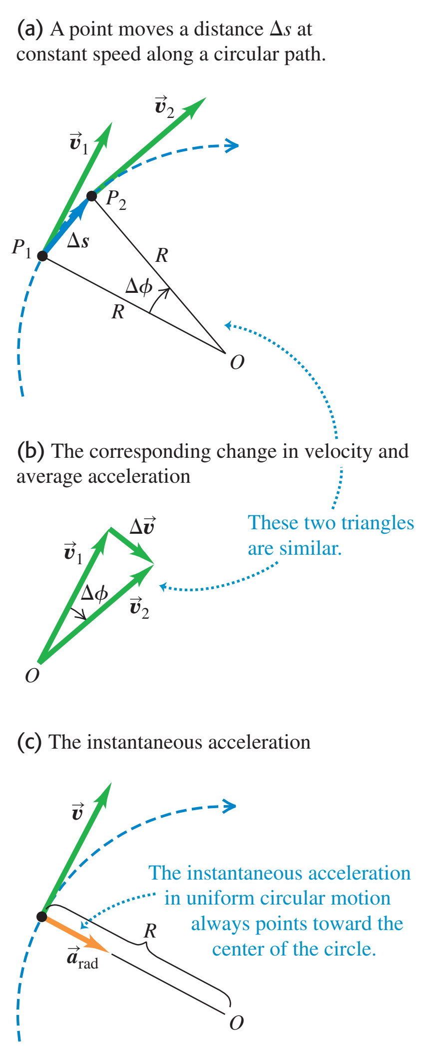
4.1 Magnitude of Centripetal Acceleration

3.27 A car in uniform circular motion. The speed is constant and the acceleration is directed toward the center of the circular path.



3.28 Finding the velocity change $\Delta\vec{v}$, average acceleration $\vec{a}_{av}$, and instantaneous acceleration $\vec{a}_{rad}$ for a particle moving in a circle with constant speed.



Uniform Circular Motion

When a particle moves in a circle with *constant speed,* the motion is called **uniform circular motion.** A car rounding a curve with constant radius at constant speed, a satellite moving in a circular orbit, and an ice skater skating in a circle with constant speed are all examples of uniform circular motion (Fig. 3.27; compare Fig. 3.12). There is no component of acceleration parallel (tangent) to the path; otherwise, the speed would change. The acceleration vector is perpendicular (normal) to the path and hence directed inward (never outward!) toward the center of the circular path. This causes the direction of the velocity to change without changing the speed. Our next project is to show that the magnitude of the acceleration in uniform circular motion is related in a simple way to the speed of the particle and the radius of the circle.

Figure 3.28a shows a particle moving with constant speed in a circular path of radius R with center at O. The particle moves from P_1 to P_2 in a time Δt. The vector change in velocity $\Delta\vec{v}$ during this time is shown in Fig. 3.28b.

The angles labeled $\Delta\phi$ in Figs. 3.28a and 3.28b are the same because $\vec{v}_1$ is perpendicular to the line OP_1 and $\vec{v}_2$ is perpendicular to the line OP_2. Hence the triangles in Figs. 3.28a and 3.28b are *similar.* The ratios of corresponding sides of similar triangles are equal, so

$$\frac{|\Delta\vec{v}|}{v_1} = \frac{\Delta s}{R} \qquad \text{or} \qquad |\Delta\vec{v}| = \frac{v_1}{R}\Delta s$$

The magnitude a_{av} of the average acceleration during Δt is therefore

$$a_{av} = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v_1}{R}\frac{\Delta s}{\Delta t}$$

The magnitude a of the *instantaneous* acceleration $\vec{a}$ at point P_1 is the limit of this expression as we take point P_2 closer and closer to point P_1:

$$a = \lim_{\Delta t \to 0} \frac{v_1}{R}\frac{\Delta s}{\Delta t} = \frac{v_1}{R} \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

But the limit of $\Delta s/\Delta t$ is the speed v_1 at point P_1. Also, P_1 can be any point on the path, so we can drop the subscript and let v represent the speed at any point. Then

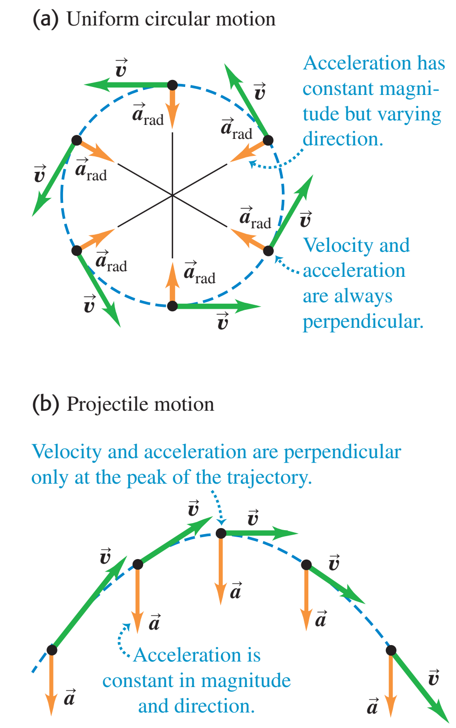
$$a_{rad} = \frac{v^2}{R} \quad \text{(uniform circular motion)} \qquad (3.28)$$

We have added the subscript "rad" as a reminder that the direction of the instantaneous acceleration at each point is always along a radius of the circle, toward its center. Because the speed is constant, the acceleration is always perpendicular to the instantaneous velocity. This is shown in Fig. 3.28c; compare with the right-hand illustration in Fig. 3.27.

We have found that *in uniform circular motion, the magnitude a of the instantaneous acceleration is equal to the square of the speed v divided by the radius R of the circle. Its direction is perpendicular to $\vec{v}$ and inward along the radius.*

Because the acceleration is always directed toward the center of the circle, it is sometimes called **centripetal acceleration.** The word "centripetal" is derived from two Greek words meaning "seeking the center." Figure 3.29a shows the directions of the velocity and acceleration vectors at several points for a particle moving with uniform circular motion.

3.29 Acceleration and velocity (a) for a particle in uniform circular motion and (b) for a projectile with no air resistance.



CAUTION **Uniform circular motion vs. projectile motion** The acceleration in uniform circular motion has some similarities to the acceleration in projectile motion without air resistance, but there are also some important differences. In both uniform circular motion (Fig. 3.29a) and projectile motion (Fig. 3.29b) the *magnitude* of acceleration is the same at all times. However, in uniform circular motion the *direction* of $\vec{a}$ changes continuously so that it always points toward the center of the circle. (At the top of the circle the acceleration points down; at the bottom of the circle the acceleration points up.) In projectile motion, by contrast, the direction of $\vec{a}$ remains the same at all times. ∎

We can also express the magnitude of the acceleration in uniform circular motion in terms of the **period** T of the motion, the time for one revolution (one complete trip around the circle). In a time T the particle travels a distance equal to the circumference $2\pi R$ of the circle, so its speed is

$$v = \frac{2\pi R}{T} \qquad (3.29)$$

When we substitute this into Eq. (3.28), we obtain the alternative expression

$$a_{rad} = \frac{4\pi^2 R}{T^2} \quad \text{(uniform circular motion)} \qquad (3.30)$$

Example 3.11 Centripetal acceleration on a curved road

An Aston Martin V8 Vantage sports car has a "lateral acceleration" of $0.96g$, which is $(0.96)(9.8\ \text{m/s}^2) = 9.4\ \text{m/s}^2$. This represents the maximum centripetal acceleration that the car can attain without skidding out of the circular path. If the car is traveling at a constant 40 m/s (about 89 mi/h, or 144 km/h), what is the minimum radius of curve it can negotiate? (Assume that the curve is unbanked.)

SOLUTION

IDENTIFY: Because the car is moving at a constant speed along a curve that is a segment of a circle, we can apply the ideas of uniform circular motion.

SET UP: We use Eq. (3.28) to find the target variable R (the radius of the curve) in terms of the given centripetal acceleration a_{rad} and speed v.

EXECUTE: We are given a_{rad} and v, so we solve Eq. (3.28) for R:

$$R = \frac{v^2}{a_{rad}} = \frac{(40\ \text{m/s})^2}{9.4\ \text{m/s}^2} = 170\ \text{m (about 560 ft)}$$

EVALUATE: Our result shows that the required turning radius R is proportional to the *square* of the speed. Hence even a small reduction in speed can make R substantially smaller. For example, reducing v by 20% (from 40 m/s to 32 m/s) would decrease R by 36% (from 170 m to 109 m).

Another way to make the required turning radius smaller is to *bank* the curve. We will investigate this option in Chapter 5.

Example 3.12 Centripetal acceleration on a carnival ride

In a carnival ride, the passengers travel at constant speed in a circle of radius 5.0 m. They make one complete circle in 4.0 s. What is their acceleration?

SOLUTION

IDENTIFY: The speed is constant, so this is a problem involving uniform circular motion.

SET UP: We are given the radius $R = 5.0$ m and the period $T = 4.0$ s, so we can use Eq. (3.30) to calculate the acceleration. Alternatively, we can first calculate the speed v using Eq. (3.29) and then find the acceleration using Eq. (3.28).

EXECUTE: From Eq. (3.30),

$$a_{rad} = \frac{4\pi^2(5.0\text{ m})}{(4.0\text{ s})^2} = 12\text{ m/s}^2$$

We'll check this answer by using Eq. (3.28) after first determining the speed v. From Eq. (3.29), the speed is the circumference of the circle divided by the period T:

$$v = \frac{2\pi R}{T} = \frac{2\pi(5.0\text{ m})}{4.0\text{ s}} = 7.9\text{ m/s}$$

The centripetal acceleration is then

$$a_{rad} = \frac{v^2}{R} = \frac{(7.9\text{ m/s})^2}{5.0\text{ m}} = 12\text{ m/s}^2$$

Happily, we get the same answer for a_{rad} with both approaches.

EVALUATE: As in Example 3.11, the direction of $\vec{a}$ is always toward the center of the circle. The magnitude of $\vec{a}$ is greater than g, the acceleration due to gravity, so this is not a ride for the faint-hearted. (Some roller coasters subject their passengers to accelerations as great as $4g$.)

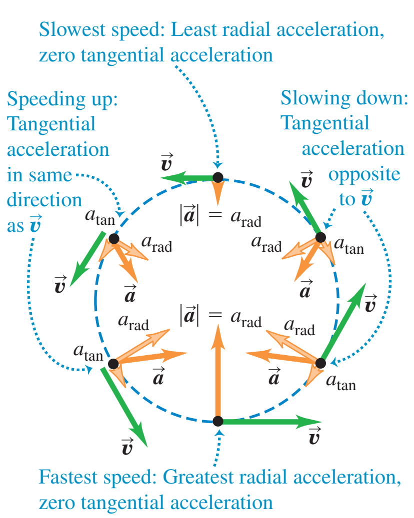
Nonuniform Circular Motion

We have assumed throughout this section that the particle's speed is constant. If the speed varies, we call the motion **nonuniform circular motion.** An example is a roller coaster car that slows down and speeds up as it moves around a vertical loop. In nonuniform circular motion, Eq. (3.28) still gives the *radial* component of acceleration $a_{rad} = v^2/R$, which is always *perpendicular* to the instantaneous velocity and directed toward the center of the circle. But since the speed v has different values at different points in the motion, the value of a_{rad} is not constant. The radial (centripetal) acceleration is greatest at the point in the circle where the speed is greatest.

In nonuniform circular motion there is also a component of acceleration that is *parallel* to the instantaneous velocity. This is the component $a_{\parallel}$ that we discussed in Section 3.2; here we call this component a_{tan} to emphasize that it is *tangent* to the circle. From the discussion at the end of Section 3.2 we see that the tangential component of acceleration a_{tan} is equal to the rate of change of *speed.* Thus

$$a_{rad} = \frac{v^2}{R} \quad \text{and} \quad a_{tan} = \frac{d|\vec{v}|}{dt} \quad \text{(nonuniform circular motion)} \quad (3.31)$$

3.30 A particle moving in a vertical loop with a varying speed, like a roller coaster car.



The vector acceleration of a particle moving in a circle with varying speed is the vector sum of the radial and tangential components of accelerations. The tangential component is in the same direction as the velocity if the particle is speeding up, and in the opposite direction if the particle is slowing down (Fig. 3.30).

In *uniform* circular motion there is no tangential component of acceleration, but the radial component is the magnitude of $d\vec{v}/dt$.

CAUTION **Uniform vs. nonuniform circular motion** Note that the two quantities

$$\frac{d|\vec{v}|}{dt} \quad \text{and} \quad \left|\frac{d\vec{v}}{dt}\right|$$

are *not* the same. The first, equal to the tangential acceleration, is the rate of change of speed; it is zero whenever a particle moves with constant speed, even when its direction of motion changes (such as in *uniform* circular motion). The second is the magnitude of the vector acceleration; it is zero only when the particle's acceleration *vector* is zero—that is, when the particle moves in a straight line with constant speed. In *uniform* circular motion $|d\vec{v}/dt| = a_{rad} = v^2/r$; in *nonuniform* circular motion there is also a tangential component of acceleration, so $|d\vec{v}/dt| = \sqrt{a_{rad}^2 + a_{tan}^2}$.

Test Your Understanding of Section 3.4 Suppose that the particle in Fig. 3.30 experiences four times the acceleration at the bottom of the loop as it does at the top of the loop. Compared to its speed at the top of the loop, is its speed at the bottom of the loop (i) $\sqrt{2}$ times as great; (ii) 2 times as great; (iii) $2\sqrt{2}$ times as great; (iv) 4 times as great; or (v) 16 times as great.

3.5 Relative Velocity

You've no doubt observed how a car that is moving slowly forward appears to be moving backward when you pass it. In general, when two observers measure the velocity of a moving body, they get different results if one observer is moving relative to the other. The velocity seen by a particular observer is called the velocity *relative* to that observer, or simply **relative velocity.** Figure 3.31 shows a situation in which understanding relative velocity is extremely important.

We'll first consider relative velocity along a straight line, then generalize to relative velocity in a plane.

3.31 Airshow pilots face a complicated problem involving relative velocities. They must keep track of their motion relative to the air (to maintain enough airflow over the wings to sustain lift), relative to each other (to keep a tight formation without colliding), and relative to their audience (to remain in sight of the spectators).

Relative Velocity in One Dimension

A passenger walks with a velocity of 1.0 m/s along the aisle of a train that is moving with a velocity of 3.0 m/s (Fig. 3.32a). What is the passenger's velocity? It's a simple enough question, but it has no single answer. As seen by a second passenger sitting in the train, she is moving at 1.0 m/s. A person on a bicycle standing beside the train sees the walking passenger moving at 1.0 m/s + 3.0 m/s = 4.0 m/s. An observer in another train going in the opposite direction would give still another answer. We have to specify which observer we mean, and we speak of the velocity *relative* to a particular observer. The walking passenger's velocity relative to the train is 1.0 m/s, her velocity relative to the cyclist is 4.0 m/s, and so on. Each observer, equipped in principle with a meter stick and a stopwatch, forms what we call a **frame of reference.** Thus a frame of reference is a coordinate system plus a time scale.

Let's use the symbol A for the cyclist's frame of reference (at rest with respect to the ground) and the symbol B for the frame of reference of the moving train. In straight-line motion the position of a point P relative to frame A is given by $x_{P/A}$ (the position of P with respect to A), and the position of P relative to frame B is given by $x_{P/B}$ (see Fig. 3.32b). The position of the origin of A with respect to the origin of B is $x_{B/A}$. Figure 3.32b shows that

$$x_{P/A} = x_{P/B} + x_{B/A} \quad (3.32)$$

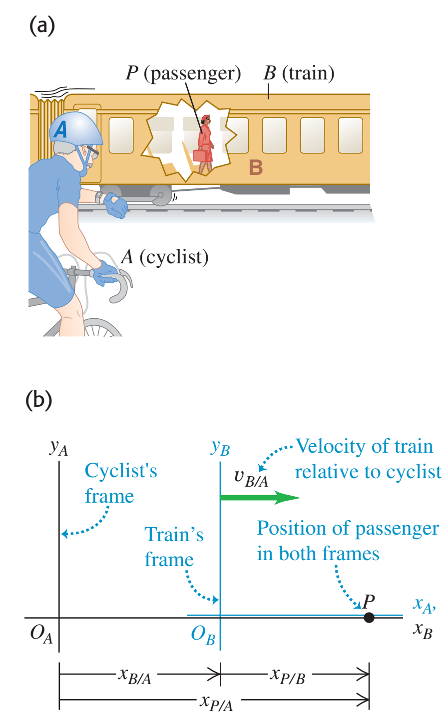
In words, the total distance from the origin of A to point P equals the distance from the origin of B to point P plus the distance from the origin of A to the origin of B.

The x-velocity of P relative to frame A, denoted by $v_{P/A\text{-}x}$, is the derivative of $x_{P/A}$ with respect to time. The other velocities are similarly obtained. So the time derivative of Eq. (3.32) gives us a relationship among the various velocities:

$$\frac{dx_{P/A}}{dt} = \frac{dx_{P/B}}{dt} + \frac{dx_{B/A}}{dt} \quad \text{or}$$

$$v_{P/A\text{-}x} = v_{P/B\text{-}x} + v_{B/A\text{-}x} \quad \text{(relative velocity along a line)} \quad (3.33)$$

3.32 (a) A passenger walking in a train. (b) The position of the passenger relative to the cyclist's frame of reference and the train's frame of reference.



Getting back to the passenger on the train in Fig. 3.32, we see that A is the cyclist's frame of reference, B is the frame of reference of the train, and point P represents the passenger. Using the above notation, we have

$$v_{P/B\text{-}x} = +1.0\text{ m/s} \qquad v_{B/A\text{-}x} = +3.0\text{ m/s}$$

From Eq. (3.33) the passenger's velocity $v_{P/A}$ relative to the cyclist is

$$v_{P/A\text{-}x} = +1.0\text{ m/s} + 3.0\text{ m/s} = +4.0\text{ m/s}$$

as we already knew.

In this example, both velocities are toward the right, and we have taken this as the positive x-direction. If the passenger walks toward the *left* relative to the train, then $v_{P/B\text{-}x} = -1.0\text{ m/s}$, and her x-velocity relative to the cyclist is $v_{P/A\text{-}x} = -1.0\text{ m/s} + 3.0\text{ m/s} = +2.0\text{ m/s}$. The sum in Eq. (3.33) is always an algebraic sum, and any or all of the x-velocities may be negative.

When the passenger looks out the window, the stationary cyclist on the ground appears to her to be moving backward; we can call the cyclist's velocity relative to her $v_{A/P\text{-}x}$. Clearly, this is just the negative of $v_{P/A\text{-}x}$. In general, if A and B are any two points or frames of reference,

$$v_{A/B\text{-}x} = -v_{B/A\text{-}x} \qquad (3.34)$$

Problem Solving Strategy 3.2 Relative Velocity

MP

IDENTIFY *the relevant concepts:* Whenever you see the phrase "velocity relative to" or "velocity with respect to," it's likely that the concepts of relative velocity will be helpful.

SET UP *the problem:* Label each frame of reference in the problem. Each moving body has its own frame of reference; in addition, you'll almost always have to include the frame of reference of the earth's surface. (Statements such as "The car is traveling north at 90 km/h" implicitly refer to the car's velocity relative to the surface of the earth.) Use the labels to help identify the target variable. For example, if you want to find the x-velocity of a car (C) with respect to a bus (B), your target variable is $v_{C/B\text{-}x}$.

EXECUTE *the solution:* Solve for the target variable using Eq. (3.33). (If the velocities are not along the same direction, you'll need to use the vector form of this equation, derived later in this section.) It's important to note the order of the double subscripts in Eq. (3.33): $v_{A/B\text{-}x}$ always means "x-velocity of A relative to B." These subscripts obey an interesting kind of algebra, as Eq. (3.33) shows. If we regard each one as a fraction, then the fraction on the left side is the *product* of the fractions on the right sides: $P/A = (P/B)(B/A)$. This is a handy rule you can use when applying Eq. (3.33) to any number of frames of reference. For example, if there are three different frames of reference A, B, and C, we can write immediately

$$v_{P/A\text{-}x} = v_{P/C\text{-}x} + v_{C/B\text{-}x} + v_{B/A\text{-}x}$$

EVALUATE *your answer:* Be on the lookout for stray minus signs in your answer. If the target variable is the x-velocity of a car relative to a bus $(v_{C/B\text{-}x})$, make sure that you haven't accidentally calculated the x-velocity of the *bus* relative to the *car* $(v_{B/C\text{-}x})$. If you have made this mistake, you can recover using Eq. (3.34).

Example 3.13 Relative velocity on a straight road

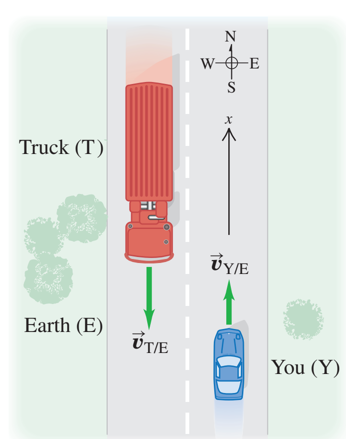
You are driving north on a straight two-lane road at a constant 88 km/h. A truck traveling at a constant 104 km/h approaches you (in the other lane, fortunately). (a) What is the truck's velocity relative to you? (b) What is your velocity with respect to the truck? (c) How do the relative velocities change after you and the truck have passed each other?

SOLUTION

IDENTIFY: This example is about relative velocities along a line.

SET UP: Let you be Y, the truck be T, and the earth's surface be E, and let the positive x-direction be north (Fig. 3.33). Then your x-velocity relative to the earth is $v_{Y/E\text{-}x} = +88\text{ km/h}$. As the truck is initially approaching you, it must be moving south and its x-velocity with respect to the earth is $v_{T/E\text{-}x} = -104\text{ km/h}$. The target variable in part (a) is $v_{T/Y\text{-}x}$; the target variable in part (b) is $v_{Y/T\text{-}x}$. We'll find both target variables by using Eq. (3.33) for relative velocity.

3.33 Reference frames for you and the truck.



EXECUTE: (a) To find $v_{T/Y\text{-}x}$, we first write Eq. (3.33) for the three frames Y, T, and E, and then rearrange:

$$v_{T/E\text{-}x} = v_{T/Y\text{-}x} + v_{Y/E\text{-}x}$$
$$v_{T/Y\text{-}x} = v_{T/E\text{-}x} - v_{Y/E\text{-}x}$$
$$= -104\text{ km/h} - 88\text{ km/h} = -192\text{ km/h}$$

The truck is moving at 192 km/h in the negative x-direction (south) relative to you.

(b) From Eq. (3.34),

$$v_{Y/T\text{-}x} = -v_{T/Y\text{-}x} = -(-192\text{ km/h}) = +192\text{ km/h}$$

You are moving at 192 km/h in the positive x-direction (north) relative to the truck.

(c) The relative velocities do *not* change at all after you and the truck pass each other. The relative positions of the bodies don't matter. The truck is still moving at 192 km/h toward the south relative to you, but it is now moving away from you instead of toward you.

EVALUATE: To check your answer in part (b), try using Eq. (3.33) directly in the form $v_{Y/T\text{-}x} = v_{Y/E\text{-}x} + v_{E/T\text{-}x}$. (Remember that the x-velocity of the earth with respect to the truck is the opposite of the x-velocity of the truck with respect to the earth: $v_{E/T\text{-}x} = -v_{T/E\text{-}x}$.) Do you get the same result?

Relative Velocity in Two or Three Dimensions

We can extend the concept of relative velocity to include motion in a plane or in space by using vector addition to combine velocities. Suppose that the passenger in Fig. 3.32a is walking not down the aisle of the railroad car but from one side of the car to the other, with a speed of 1.0 m/s (Fig. 3.34a). We can again describe the passenger's position P in two different frames of reference: A for the stationary ground observer and B for the moving train. But instead of coordinates x, we use position vectors [&*lcacc*{~bfit~r~normal~}{|Ararr|}&] because the problem is now two-dimensional. Then, as Fig. 3.34b shows,

$$\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A} \qquad (3.35)$$

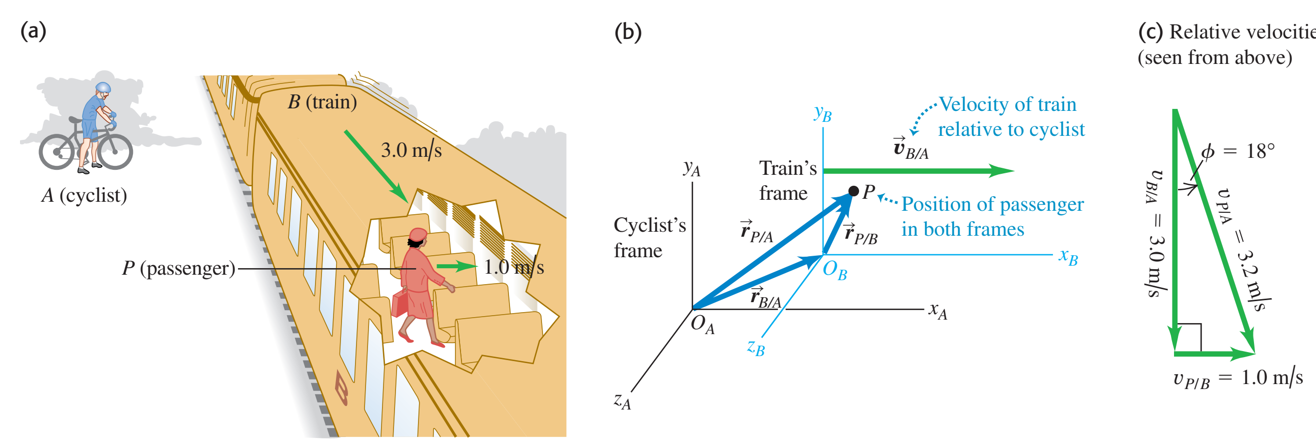
Just as we did before, we take the time derivative of this equation to get a relationship among the various velocities; the velocity of P relative to A is $\vec{v}_{P/A} = d\vec{r}_{P/A}/dt$ and so on for the other velocities. We get

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A} \quad \text{(relative velocity in space)} \qquad (3.36)$$

Equation (3.36) is known as the *Galilean velocity transformation.* It relates the velocity of a body P with respect to frame A and its velocity with respect to frame B ($\vec{v}_{P/A}$ and $\vec{v}_{P/B}$, respectively) to the velocity of frame B with respect to frame A $(\vec{v}_{B/A})$. If all three of these velocities lie along the same line, then Eq. (3.36) reduces to Eq. (3.33) for the components of the velocities along that line.

If the train is moving at $v_{B/A} = 3.0\text{ m/s}$ relative to the ground and the passenger is moving at $v_{P/B} = 1.0\text{ m/s}$ relative to the train, then the passenger's velocity

3.34 (a) A passenger walking across a railroad car. (b) Position of the passenger relative to the cyclist's frame and the train's frame. (c) Vector diagram for the velocity of the passenger relative to the ground (the cyclist's frame), $\vec{v}_{P/A}$.



vector $\vec{v}_{P/A}$ relative to the ground is as shown in Fig. 3.34c. The Pythagorean theorem then gives us

$$v_{P/A} = \sqrt{(3.0\ \text{m/s})^2 + (1.0\ \text{m/s})^2} = \sqrt{10\ \text{m}^2/\text{s}^2} = 3.2\ \text{m/s}$$

Figure 3.34c also shows that the *direction* of the passenger's velocity vector relative to the ground makes an angle ϕ with the train's velocity vector $\vec{v}_{B/A}$, where

$$\tan\phi = \frac{v_{P/B}}{v_{B/A}} = \frac{1.0\ \text{m/s}}{3.0\ \text{m/s}} \quad \text{and} \quad \phi = 18°$$

As in the case of motion along a straight line, we have the general rule that if *A* and *B* are *any* two points or frames of reference,

$$\vec{v}_{A/B} = -\vec{v}_{B/A} \tag{3.37}$$

The velocity of the passenger relative to the train is the negative of the velocity of the train relative to the passenger, and so on.

In the early 20th century Albert Einstein showed in his special theory of relativity that the velocity-addition relationship given in Eq. (3.36) has to be modified when speeds approach the speed of light, denoted by c. It turns out that if the passenger in Fig. 3.32a could walk down the aisle at $0.30c$ and the train could move at $0.90c$, then her speed relative to the ground would be not $1.20c$ but $0.94c$; nothing can travel faster than light! We'll return to the special theory of relativity in Chapter 37.

Example 3.14 Flying in a crosswind

The compass of an airplane indicates that it is headed due north, and its airspeed indicator shows that it is moving through the air at 240 km/h. If there is a wind of 100 km/h from west to east, what is the velocity of the airplane relative to the earth?

SOLUTION

IDENTIFY: This problem involves velocities in two dimensions (northward and eastward), so it is a relative velocity problem using vectors.

SET UP: We are given the magnitude and direction of the velocity of the plane (P) relative to the air (A). We are also given the magnitude and direction of the wind velocity, which is the velocity of the air (A) with respect to the earth (E):

$\vec{v}_{P/A} = 240$ km/h	due north
$\vec{v}_{A/E} = 100$ km/h	due east

Our target variables are the magnitude and direction of the velocity of the plane (P) relative to the earth (E), $\vec{v}_{P/E}$. We'll find these using Eq. (3.36).

EXECUTE: Using Eq. (3.36), we have

$$\vec{v}_{P/E} = \vec{v}_{P/A} + \vec{v}_{A/E}$$

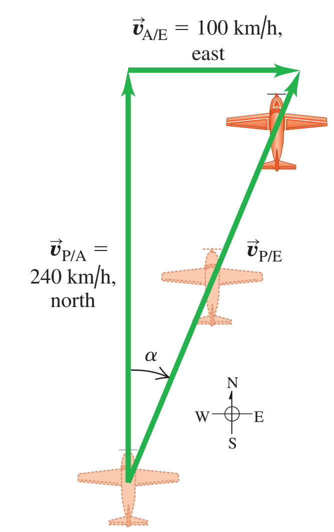
Figure 3.35 shows the three relative velocities and their relationship; the unknowns are the speed $v_{P/E}$ and the angle α. From this diagram we find

$$v_{P/E} = \sqrt{(240\ \text{km/h})^2 + (100\ \text{km/h})^2} = 260\ \text{km/h}$$

$$\alpha = \arctan\left(\frac{100\ \text{km/h}}{240\ \text{km/h}}\right) = 23°\ \text{E of N}$$

EVALUATE: The crosswind increases the speed of the airplane relative to the earth, but at the price of pushing the airplane off course.

3.35 The plane is pointed north, but the wind blows east, giving the resultant velocity $\vec{v}_{P/E}$ relative to the earth.



Example 3.15 Correcting for a crosswind

In Example 3.14, in what direction should the pilot head to travel due north? What will be her velocity relative to the earth? (Assume that her airspeed and the velocity of the wind are the same as in Example 3.14.)

SOLUTION

IDENTIFY: Like Example 3.14, this is a relative velocity problem with vectors.

SET UP: Figure 3.36 illustrates the situation. The vectors are arranged in accordance with the vector relative-velocity equation, Eq. (3.36):

$$\vec{v}_{P/E} = \vec{v}_{P/A} + \vec{v}_{A/E}$$

As Fig. 3.36 shows, the pilot points the nose of the airplane at an angle β into the wind to compensate for the crosswind. This angle, which tells us the direction of the vector $\vec{v}_{P/A}$ (the velocity of the airplane relative to the air), is one of our target variables. The other target variable is the speed of the airplane over the ground, which is the magnitude of the vector $\vec{v}_{P/E}$ (the velocity of the airplane relative to the earth). Here are the known and unknown quantities:

$\vec{v}_{P/E}$ = magnitude unknown	due north
$\vec{v}_{P/A} = 240$ km/h	direction unknown
$\vec{v}_{A/E} = 100$ km/h	due east

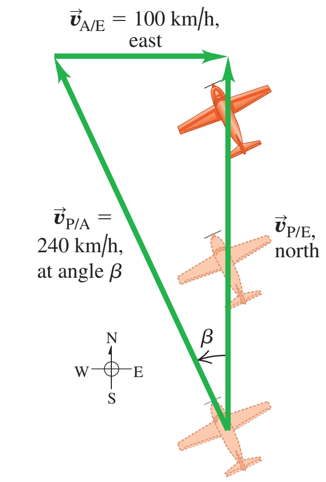
We can solve for the unknown target variables using Fig. 3.36 and trigonometry.

EXECUTE: From the diagram, the speed $v_{P/E}$ and the angle β are given by

$$v_{P/E} = \sqrt{(240\ \text{km/h})^2 - (100\ \text{km/h})^2} = 218\ \text{km/h}^2$$

$$\beta = \arcsin\left(\frac{100\ \text{km/h}}{240\ \text{km/h}}\right) = 25°$$

3.36 The pilot must point the plane in the direction of the vector $\vec{v}_{P/A}$ to travel due north relative to the earth.



The pilot should point the airplane 25° west of north, and her ground speed is then 218 km/h.

EVALUATE: Note that there were two target variables—the magnitude of a vector and the direction of a vector—in both this example and Example 3.14. The difference is that in Example 3.14, the magnitude and direction referred to the *same* vector ($\vec{v}_{P/E}$), whereas in this example they referred to *different* vectors ($\vec{v}_{P/E}$ and $\vec{v}_{P/A}$).

It's no surprise that a headwind reduces an airplane's speed relative to the ground. This example shows that a *crosswind* also slows an airplane down—an unfortunate fact of aeronautical life.

Test Your Understanding of Section 3.5 Suppose the nose of an airplane is pointed due east and the airplane has an airspeed of 150 km/h. Due to the wind, the airplane is moving due *north* relative to the ground and its speed relative to the ground is 150 km/h. What is the velocity of the air relative to the earth? (i) 150 km/h from east to west; (ii) 150 km/h from south to north; (iii) 150 km/h from southeast to northwest; (iv) 212 km/h from east to west; (v) 212 km/h from south to north; (vi) 212 km/h from southeast to northwest; (vii) there is no possible wind velocity that could cause this.

CHAPTER 3 SUMMARY

Position, velocity, and acceleration vectors: The position vector $\vec{r}$ of a point P in space is the vector from the origin to P. Its components are the coordinates x, y, and z.

The average velocity vector $\vec{v}_{av}$ during the time interval Δt is the displacement $\Delta\vec{r}$ (the change in the position vector $\vec{r}$) divided by Δt. The instantaneous velocity vector $\vec{v}$ is the time derivative of $\vec{r}$, and its components are the time derivatives of x, y, and z. The instantaneous speed is the magnitude of $\vec{v}$. The velocity $\vec{v}$ of a particle is always tangent to the particle's path. (See Example 3.1.)

The average acceleration vector $\vec{a}_{av}$ during the time interval Δt equals $\Delta\vec{v}$ (the change in the velocity vector $\vec{v}$) divided by Δt. The instantaneous acceleration vector $\vec{a}$ is the time derivative of $\vec{v}$, and its components are the time derivatives of v_x, v_y, and v_z. (See Example 3.2.)

The component of acceleration parallel to the direction of the instantaneous velocity affects the speed, while the component of $\vec{a}$ perpendicular to $\vec{v}$ affects the direction of motion. (See Examples 3.3 and 3.4.)

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \quad (3.1)$$

$$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t} \quad (3.2)$$

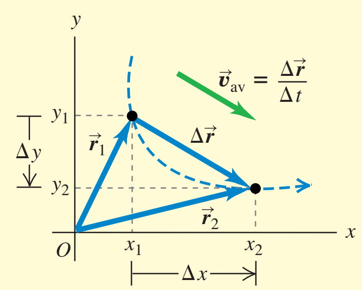
$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \quad (3.3)$$

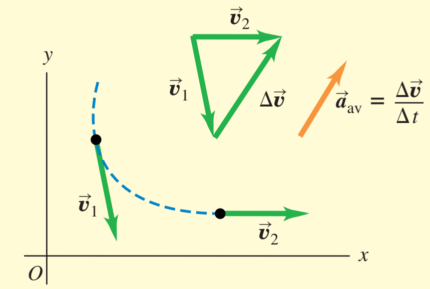
$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt} \quad (3.4)$$

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t} \quad (3.8)$$

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \quad (3.9)$$

$$a_x = \frac{dv_x}{dt} \quad a_y = \frac{dv_y}{dt} \quad a_z = \frac{dv_z}{dt} \quad (3.10)$$





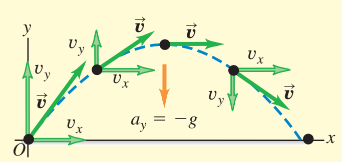
Projectile motion: In projectile motion with no air resistance, $a_x = 0$ and $a_y = -g$. The coordinates and velocity components are simple functions of time, and the shape of the path is always a parabola. We usually choose the origin to be at the initial position of the projectile. (See Examples 3.5–3.10.)

$$x = (v_0 \cos\alpha_0)t \quad (3.20)$$

$$y = (v_0 \sin\alpha_0)t - \frac{1}{2}gt^2 \quad (3.21)$$

$$v_x = v_0 \cos\alpha_0 \quad (3.22)$$

$$v_y = v_0 \sin\alpha_0 - gt \quad (3.23)$$

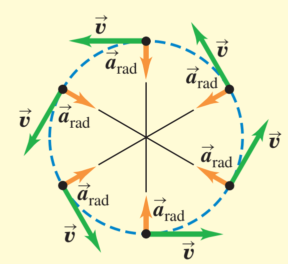


Uniform and nonuniform circular motion: When a particle moves in a circular path of radius R with constant speed v (uniform circular motion), its acceleration $\vec{a}$ is directed toward the center of the circle and perpendicular to $\vec{v}$. The magnitude a_{rad} of the acceleration can be expressed in terms of v and R or in terms of R and the period T (the time for one revolution), where $v = 2\pi R/T$. (See Examples 3.11 and 3.12.)

If the speed is not constant in circular motion (nonuniform circular motion), there is still a radial component of $\vec{a}$ given by Eq. (3.28) or (3.30), but there is also a component of $\vec{a}$ parallel (tangential) to the path. This tangential component is equal to the rate of change of speed, dv/dt.

$$a_{rad} = \frac{v^2}{R} \quad (3.28)$$

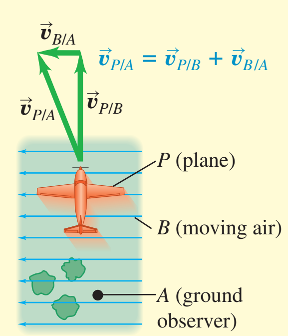
$$a_{rad} = \frac{4\pi^2 R}{T^2} \quad (3.30)$$



Relative velocity: When a body P moves relative to a body (or reference frame) B, and B moves relative to A, we denote the velocity of P relative to B by $\vec{v}_{P/B}$, the velocity of P relative to A by $\vec{v}_{P/A}$, and the velocity of B relative to A by $\vec{v}_{B/A}$. If these velocities are all along the same line, their components along that line are related by Eq. (3.33). More generally, these velocities are related by Eq. (3.36). (See Examples 3.13–3.15)

$$v_{P/A\text{-}x} = v_{P/B\text{-}x} + v_{B/A\text{-}x} \quad (3.33)$$
(relative velocity along a line)

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A} \quad (3.36)$$
(relative velocity in space)



Key Terms

position vector, *72*
average velocity, *72*
instantaneous velocity, *72*
average acceleration, *75*
instantaneous acceleration, *75*
projectile, *79*
trajectory, *79*
uniform circular motion, *88*
centripetal acceleration, *89*
period, *89*
nonuniform circular motion, *90*
relative velocity, *91*
frame of reference, *91*

Answer to Chapter Opening Question ?

A car going around a curve at constant speed has an acceleration directed toward the inside of the curve (see Section 3.2, especially Fig. 3.12a).

Answers to Test Your Understanding Questions

3.1 Answer: (iii) If the instantaneous velocity $\vec{v}$ is constant over an interval, its value at any point (including the end of the interval) is the same as the average velocity $\vec{v}_{av}$ over the interval. In (i) and (ii) the direction of $\vec{v}$ at the end of the interval is tangent to the path at that point, while the direction of $\vec{v}_{av}$ points from the beginning of the path to its end (in the direction of the net displacement). In (iv) $\vec{v}$ and $\vec{v}_{av}$ are both directed along the straight line, but $\vec{v}$ has a greater magnitude because the speed has been increasing.

3.2 Answer: vector 7 At the high point of the sled's path, the speed is minimum. At that point the speed is neither increasing nor decreasing, and the parallel component of the acceleration (that is, the horizontal component) is zero. The acceleration has only a perpendicular component toward the inside of the sled's curved path. In other words, the acceleration is downward.

3.3 Answer: (i) If there were no gravity $(g = 0)$, the monkey would not fall and the dart would follow a straight-line path (shown as a dashed line). The effect of gravity is to make the monkey and the dart both fall the same distance $\frac{1}{2}gt^2$ below their $g = 0$ positions. Point A is the same distance below the monkey's initial position as point P is below the dashed straight line, so point A is where we would find the monkey at the time in question.

3.4 Answer: (ii) At both the top and bottom of the loop, the acceleration is purely radial and is given by Eq. (3.28). The radius R is the same at both points, so the difference in acceleration is due purely to differences in speed. Since a_{rad} is proportional to the square of v, the speed must be twice as great at the bottom of the loop as at the top.

3.5 Answer: (vi) The effect of the wind is to cancel the airplane's eastward motion and give it a northward motion. So the velocity of the air relative to the ground (the wind velocity) must have one 150-km/h component to the west and one 150-km/h component to the north. The combination of these is a vector of magnitude $\sqrt{(150\ \text{km/h})^2 + (150\ \text{km/h})^2} = 212\ \text{km/h}$ that points to the northwest.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q3.1. A simple pendulum (a mass swinging at the end of a string) swings back and forth in a circular arc. What is the direction of the acceleration off the mass at the ends of the swing? At the midpoint? In each case, explain how you obtain your answer.

Q3.2. Redraw Fig. 3.11a if $\vec{a}$ is antiparallel to $\vec{v}_1$. Does the particle move in a straight line? What happens to its speed?

Q3.3. A projectile moves in a parabolic path without air resistance. Is there any point at which $\vec{a}$ is parallel to $\vec{v}$? Perpendicular to $\vec{v}$? Explain.

Q3.4. When a rifle is fired at a distant target, the barrel is not lined up exactly on the target. Why not? Does the angle of correction depend on the distance of the target?

Q3.5. At the same instant that you fire a bullet horizontally from a gun, you drop a bullet from the height of the barrel. If there is no air resistance, which bullet hits the ground first? Explain.

Q3.6. A package falls out of an airplane that is flying in a straight line at a constant altitude and speed. If you could ignore air resistance, what would be the path of the package as observed by the pilot? As observed by a person on the ground?

Q3.7. Sketch the six graphs of the x- and y-components of position, velocity, and acceleration versus time for projectile motion with $x_0 = y_0 = 0$ and $0 < \alpha_0 < 90°$.

Q3.8. An object is thrown straight up into the air and feels no air resistance. How is it possible for it to have an acceleration when it has stopped moving at its highest point?

Q3.9. If a jumping frog can give itself the same initial speed regardless of the direction in which it jumps (forward or straight up), how is the maximum vertical height to which it can jump related to its maximum horizontal range $R_{max} = v_0^2/g$?

Q3.10. A projectile is fired upward at an angle θ above the horizontal with an initial speed v_0. At its maximum height, what are its velocity vector, its speed, and its acceleration vector?

Q3.11. In uniform circular motion, what are the *average* velocity and *average* acceleration for one revolution? Explain.

Q3.12. In uniform circular motion, how does the acceleration change when the speed is increased by a factor of 3? When the radius is decreased by a factor of 2?

Q3.13. In uniform circular motion, the acceleration is perpendicular to the velocity at every instant. Is this still true when the motion is not uniform—that is, when the speed is not constant?

Q3.14. Raindrops hitting the side windows of a car in motion often leave diagonal streaks even if there is no wind. Why? Is the explanation the same or different for diagonal streaks on the windshield?

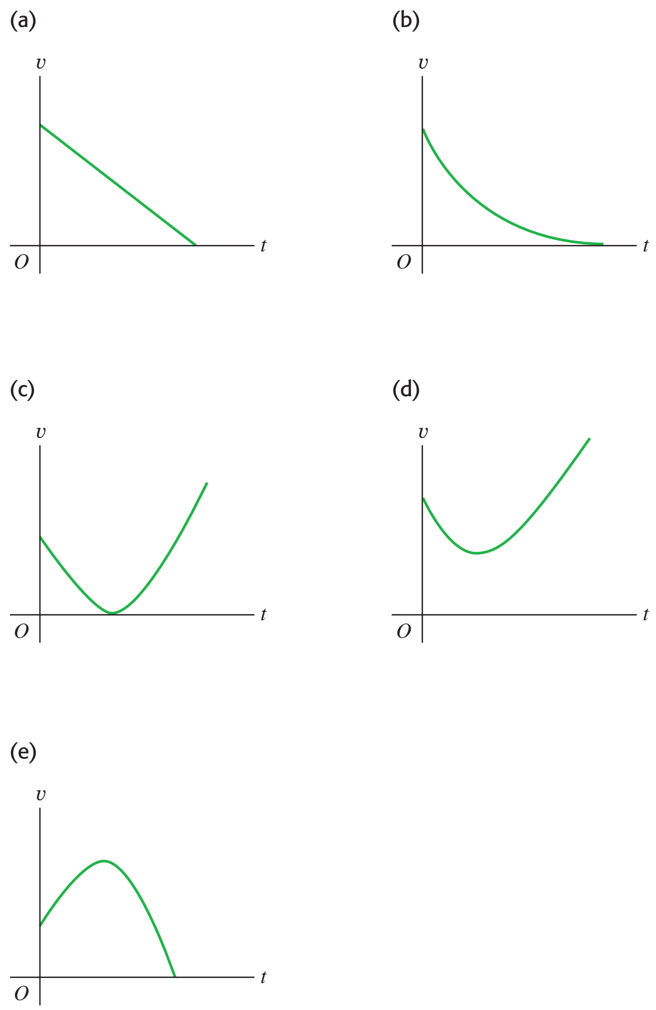
Q3.15. In a rainstorm with a strong wind, what determines the best position in which to hold an umbrella?

Q3.16. You are on the west bank of a river that is flowing north with a speed of 1.2 m/s. Your swimming speed relative to the water is 1.5 m/s, and the river is 60 m wide. What is your path relative to earth that allows you to cross the river in the shortest time? Explain your reasoning.

Q3.17. When you drop an object from a certain height, it takes time T to reach the ground with no air resistance. If you dropped it from three times that height, how long (in terms of T) would it take to reach the ground?

Q3.18. A stone is thrown into the air at an angle above the horizontal and feels negligible air resistance. Which graph in Fig. 3.37 best depicts the stone's *speed* v as a function of time t while it is in the air?

Figure 3.37 Question Q3.18.



Exercises

Section 3.1 Position and Velocity Vectors

3.1. A squirrel has x- and y-coordinates $(1.1\text{ m}, 3.4\text{ m})$ at time $t_1 = 0$ and coordinates $(5.3\text{ m}, -0.5\text{ m})$ at time $t_2 = 3.0\text{ s}$. For this time interval, find (a) the components of the average velocity, and (b) the magnitude and direction of the average velocity.

3.2. A rhinoceros is at the origin of coordinates at time $t_1 = 0$. For the time interval from $t_1 = 0$ to $t_2 = 12.0\text{ s}$, the rhino's average velocity has x-component -3.8 m/s and y-component 4.9 m/s. At time $t_2 = 12.0\text{ s}$, (a) what are the x- and y-coordinates of the rhino? (b) How far is the rhino from the origin?

3.3. A web page designer creates an animation in which a dot on a computer screen has a position of $\vec{r} = [4.0\text{ cm} + (2.5\text{ cm/s}^2)t^2]\hat{\imath} + (5.0\text{ cm/s})t\hat{\jmath}$. (a) Find the magnitude and direction of the dot's average velocity between $t = 0$ and $t = 2.0\text{ s}$. (b) Find the magnitude and direction of the instantaneous velocity at $t = 0$, $t = 1.0\text{ s}$, and $t = 2.0\text{ s}$. (c) Sketch the dot's trajectory from $t = 0$ to $t = 2.0\text{ s}$, and show the velocities calculated in part (b).

3.4. If $\vec{r} = bt^2\hat{\imath} + ct^3\hat{\jmath}$, where b and c are positive constants, when does the velocity vector make an angle of 45.0° with the x- and y-axes?

Section 3.2 The Acceleration Vector

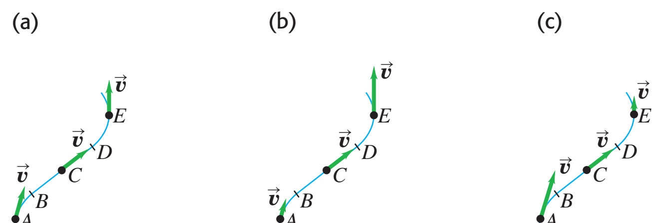
3.5. A jet plane is flying at a constant altitude. At time $t_1 = 0$ it has components of velocity $v_x = 90\text{ m/s}$, $v_y = 110\text{ m/s}$. At time $t_2 = 30.0\text{ s}$ the components are $v_x = -170\text{ m/s}$, $v_y = 40\text{ m/s}$. (a) Sketch the velocity vectors at t_1 and t_2. How do these two vectors differ? For this time interval calculate (b) the components of the average acceleration, and (c) the magnitude and direction of the average acceleration.

3.6. A dog running in an open field has components of velocity $v_x = 2.6\text{ m/s}$ and $v_y = -1.8\text{ m/s}$ at $t_1 = 10.0\text{ s}$. For the time interval from $t_1 = 10.0\text{ s}$ to $t_2 = 20.0\text{ s}$, the average acceleration of the dog has magnitude 0.45 m/s^2 and direction 31.0° measured from the $+x$-axis toward the $+y$-axis. At $t_2 = 20.0\text{ s}$, (a) what are the x- and y-components of the dog's velocity? (b) What are the magnitude and direction of the dog's velocity? (c) Sketch the velocity vectors at t_1 and t_2. How do these two vectors differ?

3.7. The coordinates of a bird flying in the xy-plane are given by $x(t) = \alpha t$ and $y(t) = 3.0\text{ m} - \beta t^2$, where $\alpha = 2.4\text{ m/s}$ and $\beta = 1.2\text{ m/s}^2$. (a) Sketch the path of the bird between $t = 0$ and $t = 2.0\text{ s}$. (b) Calculate the velocity and acceleration vectors of the bird as functions of time. (c) Calculate the magnitude and direction of the bird's velocity and acceleration at $t = 2.0\text{ s}$. (d) Sketch the velocity and acceleration vectors at $t = 2.0\text{ s}$. At this instant, is the bird speeding up, is it slowing down, or is its speed instantaneously not changing? Is the bird turning? If so, in what direction?

3.8. A particle moves along a path as shown in Fig. 3.38. Between points B and D, the path is a straight line. Sketch the acceleration vectors at A, C, and E in the cases in which (a) the particle moves with a constant speed; (b) the particle moves with a steadily increasing speed; (c) the particle moves with a steadily decreasing speed.

Figure 3.38 Exercise 3.8.



Section 3.3 Projectile Motion

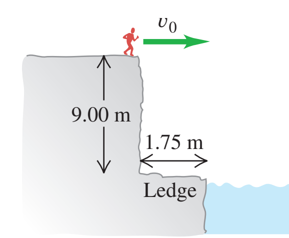
3.9. A physics book slides off a horizontal tabletop with a speed of 1.10 m/s. It strikes the floor in 0.350 s. Ignore air resistance. Find (a) the height of the tabletop above the floor; (b) the horizontal distance from the edge of the table to the point where the book strikes the floor; (c) the horizontal and vertical components of the book's velocity, and the magnitude and direction of its velocity, just before the book reaches the floor. (d) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.10. A military helicopter on a training mission is flying horizontally at a speed of 60.0 m/s and accidentally drops a bomb (fortunately not armed) at an elevation of 300 m. You can ignore air resistance. (a) How much time is required for the bomb to reach the earth? (b) How far does it travel horizontally while falling? (c) Find the horizontal and vertical components of its velocity just before it strikes the earth. (d) Draw x-t, y-t, v_x-t, and v_y-t graphs for the bomb's motion. (e) If the velocity of the helicopter remains constant, where is the helicopter when the bomb hits the ground?

3.11. Two crickets, Chirpy and Milada, jump from the top of a vertical cliff. Chirpy just drops and reaches the ground in 3.50 s, while Milada jumps horizontally with an initial speed of 95.0 cm/s. How far from the base of the cliff will Milada hit the ground?

3.12. A daring 510-N swimmer dives off a cliff with a running horizontal leap, as shown in Fig. 3.39. What must her minimum speed be just as she leaves the top of the cliff so that she will miss the ledge at the bottom, which is 1.75 m wide and 9.00 m below the top of the cliff?

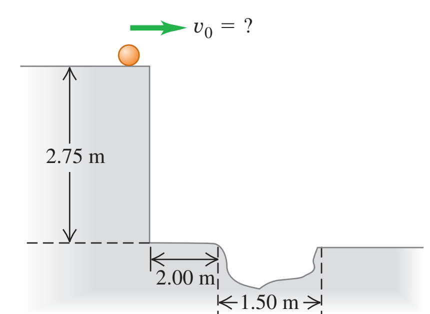
Figure 3.39 Exercise 3.12.



3.13. Leaping the River I. A car comes to a bridge during a storm and finds the bridge washed out. The driver must get to the other side, so he decides to try leaping it with his car. The side of the road the car is on is 21.3 m above the river, while the opposite side is a mere 1.8 m above the river. The river itself is a raging torrent 61.0 m wide. (a) How fast should the car be traveling at the time it leaves the road in order just to clear the river and land safely on the opposite side? (b) What is the speed of the car just before it lands on the other side?

3.14. A small marble rolls horizontally with speed v_0 off the top of a platform 2.75 m tall and feels no appreciable air resistance. On the level ground, 2.00 m from the base of the platform, there is a gaping hole in the ground (Fig. 3.40.) For what range of marble speeds v_0 will the marble land in the hole?

Figure 3.40 Exercise 3.14.



3.15. Inside a starship at rest on the earth, a ball rolls off the top of a horizontal table and lands a distance D from the foot of the table. This starship now lands on the unexplored Planet X. The commander, Captain Curious, rolls the same ball off the same table with the same initial speed as on earth and finds that it lands a distance $2.76D$ from the foot of the table. What is the acceleration due to gravity on Planet X?

3.16. A rookie quarterback throws a football with an initial upward velocity component of 16.0 m/s and a horizontal velocity component of 20.0 m/s. Ignore air resistance. (a) How much time is required for the football to reach the highest point of the trajectory? (b) How high is this point? (c) How much time (after it is thrown) is required for the football to return to its original level? How does this compare with the time calculated in part (a)? (d) How far has the football traveled horizontally during this time? (e) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.17. On level ground a shell is fired with an initial velocity of 80.0 m/s at 60.0° above the horizontal and feels no appreciable air resistance. (a) Find the horizontal and vertical components of the shell's initial velocity. (b) How long does it take the shell to reach its highest point? (c) Find its maximum height above the ground. (d) How far from its firing point does the shell land? (e) At its highest point, find the horizontal and vertical components of its acceleration and velocity.

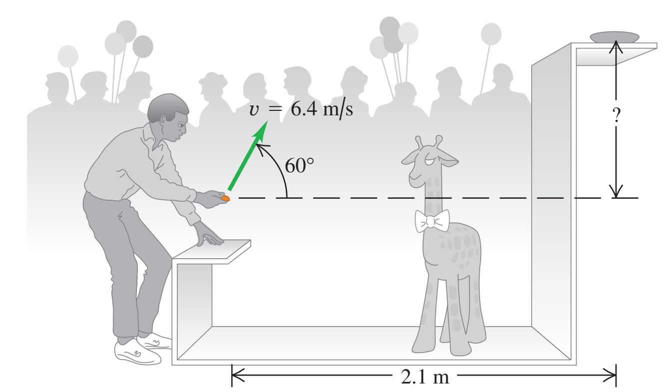
3.18. A pistol that fires a signal flare gives it an initial velocity (muzzle velocity) of 125 m/s at an angle of 55.0° above the horizontal. You can ignore air resistance. Find the flare's maximum height and the distance from its firing point to its landing point if it is fired (a) on the level salt flats of Utah, and (b) over the flat Sea of Tranquility on the Moon, where $g = 1.67\text{ m/s}^2$.

3.19. A major leaguer hits a baseball so that it leaves the bat at a speed of 30.0 m/s and at an angle of 36.9° above the horizontal. You can ignore air resistance. (a) At what *two* times is the baseball at a height of 10.0 m above the point at which it left the bat? (b) Calculate the horizontal and vertical components of the baseball's velocity at each of the two times calculated in part (a). (c) What are the magnitude and direction of the baseball's velocity when it returns to the level at which it left the bat?

3.20. A shot putter releases the shot some distance above the level ground with a velocity of 12.0 m/s, 51.0° above the horizontal. The shot hits the ground 2.08 s later. You can ignore air resistance. (a) What are the components of the shot's acceleration while in flight? (b) What are the components of the shot's velocity at the beginning and at the end of its trajectory? (c) How far did she throw the shot horizontally? (d) Why does the expression for R in Example 3.8 *not* give the correct answer for part (c)? (e) How high was the shot above the ground when she released it? (f) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.21. Win the Prize. In a carnival booth, you win a stuffed giraffe if you toss a quarter into a small dish. The dish is on a shelf above the point where the quarter leaves your hand and is a horizontal distance of 2.1 m from this point (Fig. 3.41). If you toss the coin with a velocity of 6.4 m/s at an angle of 60° above the horizontal, the coin lands in the dish. You can ignore air resistance. (a) What is the height of the shelf above the point where the quarter leaves your hand? (b) What is the vertical component of the velocity of the quarter just before it lands in the dish?

Figure 3.41 Exercise 3.21.



3.22. Suppose the departure angle α_0 in Fig. 3.26 is 42.0° and the distance d is 3.00 m. Where will the dart and monkey meet if the initial speed of the dart is (a) 12.0 m/s? (b) 8.0 m/s? (c) What will happen if the initial speed of the dart is 4.0 m/s? Sketch the trajectory in each case.

3.23. A man stands on the roof of a 15.0-m-tall building and throws a rock with a velocity of magnitude 30.0 m/s at an angle of 33.0° above the horizontal. You can ignore air resistance. Calculate

(a) the maximum height above the roof reached by the rock; (b) the magnitude of the velocity of the rock just before it strikes the ground; and (c) the horizontal range from the base of the building to the point where the rock strikes the ground. (d) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.24. Firemen are shooting a stream of water at a burning building using a high-pressure hose that shoots out the water with a speed of 25.0 m/s as it leaves the end of the hose. Once it leaves the hose, the water moves in projectile motion. The firemen adjust the angle of elevation α of the hose until the water takes 3.00 s to reach a building 45.0 m away. You can ignore air resistance; assume that the end of the hose is at ground level. (a) Find the angle of elevation α. (b) Find the speed and acceleration of the water at the highest point in its trajectory. (c) How high above the ground does the water strike the building, and how fast is it moving just before it hits the building?

3.25. A 124-kg balloon carrying a 22-kg basket is descending with a constant downward velocity of 20.0 m/s. A 1.0-kg stone is thrown from the basket with an initial velocity of 15.0 m/s perpendicular to the path of the descending balloon, as measured relative to a person at rest in the basket. The person in the basket sees the stone hit the ground 6.00 s after being thrown. Assume that the balloon continues its downward descent with the same constant speed of 20.0 m/s. (a) How high was the balloon when the rock was thrown out? (b) How high is the balloon when the rock hits the ground? (c) At the instant the rock hits the ground, how far is it from the basket? (d) Just before the rock hits the ground, find its horizontal and vertical velocity components as measured by an observer (i) at rest in the basket and (ii) at rest on the ground.

3.26. A cannon, located 60.0 m from the base of a vertical 25.0-m-tall cliff, shoots a 15-kg shell at 43.0° above the horizontal toward the cliff. (a) What must the minimum muzzle velocity be for the shell to clear the top of the cliff? (b) The ground at the top of the cliff is level, with a constant elevation of 25.0 m above the cannon. Under the conditions of part (a), how far does the shell land past the edge of the cliff?

3.27. An airplane is flying with a velocity of 90.0 m/s at an angle of 23.0° above the horizontal. When the plane is 114 m directly above a dog that is standing on level ground, a suitcase drops out of the luggage compartment. How far from the dog will the suitcase land? You can ignore air resistance.

Section 3.4 Motion in a Circle

3.28. On your first day at work for an appliance manufacturer, you are told to figure out what to do to the period of rotation during a washer spin cycle to triple the centripetal acceleration. You impress your boss by answering immediately. What do you tell her?

3.29. The earth has a radius of 6380 km and turns around once on its axis in 24 h. (a) What is the radial acceleration of an object at the earth's equator? Give your answer in m/s^2 and as a fraction of g. (b) If a_{rad} at the equator is greater than g, objects would fly off the earth's surface and into space. (We will see the reason for this in Chapter 5.) What would the period of the earth's rotation have to be for this to occur?

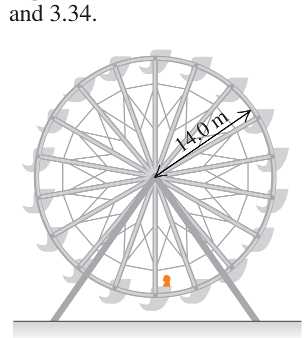
3.30. A model of a helicopter rotor has four blades, each 3.40 m long from the central shaft to the blade tip. The model is rotated in a wind tunnel at 550 rev/min. (a) What is the linear speed of the blade tip, in m/s? (b) What is the radial acceleration of the blade tip expressed as a multiple of the acceleration of gravity, g?

3.31. In a test of a "g-suit," a volunteer is rotated in a horizontal circle of radius 7.0 m. What must the period of rotation be so that the centripetal acceleration has a magnitude of (a) $3.0g$? (b) $10g$?

3.32. The radius of the earth's orbit around the sun (assumed to be circular) is 1.50×10^8 km, and the earth travels around this orbit in 365 days. (a) What is the magnitude of the orbital velocity of the earth, in m/s? (b) What is the radial acceleration of the earth toward the sun, in m/s^2? (c) Repeat parts (a) and (b) for the motion of the planet Mercury (orbit radius $= 5.79 \times 10^7$ km, orbital period $= 88.0$ days).

3.33. A Ferris wheel with radius 14.0 m is turning about a horizontal axis through its center (Fig. 3.42). The linear speed of a passenger on the rim is constant and equal to 7.00 m/s. What are the magnitude and direction of the passenger's acceleration as she passes through (a) the lowest point in her circular motion? (b) The highest point in her circular motion? (c) How much time does it take the Ferris wheel to make one revolution?

Figure 3.42 Exercises 3.33 and 3.34.



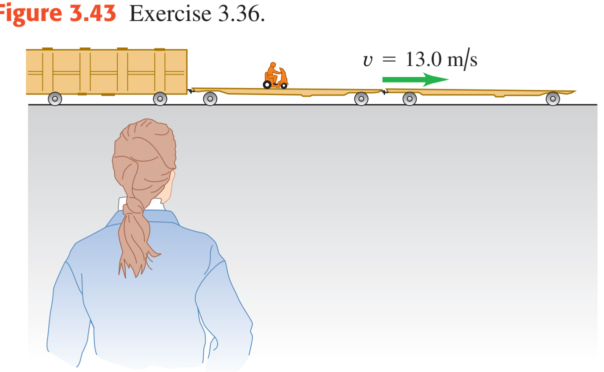
3.34. The Ferris wheel in Fig. 3.42, which rotates counterclockwise, is just starting up. At a given instant, a passenger on the rim of the wheel and passing through the lowest point of his circular motion is moving at 3.00 m/s and is gaining speed at a rate of $0.500\ m/s^2$. (a) Find the magnitude and the direction of the passenger's acceleration at this instant. (b) Sketch the Ferris wheel and the passenger, showing his velocity and acceleration vectors.

3.35. Hypergravity. At its Ames Research Center, NASA uses its large "20-G" centrifuge to test the effects of very large accelerations ("hypergravity") on test pilots and astronauts. In this device, an arm 8.84 m long rotates about one end in a horizontal plane, and the astronaut is strapped in at the other end. Suppose that he is aligned along the arm with his head at the outermost end. The maximum sustained acceleration to which humans are subjected in this machine is typically $12.5g$. (a) How fast must the astronaut's head be moving to experience this maximum acceleration? (b) What is the *difference* between the acceleration of his head and feet if the astronaut is 2.00 m tall? (c) How fast in rpm (rev/min) is the arm turning to produce the maximum sustained acceleration?

Section 3.5 Relative Velocity

3.36. A railroad flatcar is traveling to the right at a speed of 13.0 m/s relative to an observer standing on the ground. Someone is riding a motor scooter on the flatcar (Fig. 3.43). What is the velocity (magnitude and direction) of the motor scooter relative to the flatcar if its velocity relative to the observer on the ground is (a) 18.0 m/s to the right? (b) 3.0 m/s to the left? (c) zero?

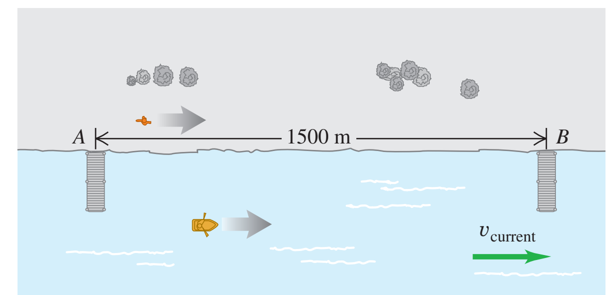
Figure 3.43 Exercise 3.36.



3.37. A "moving sidewalk" in an airport terminal building moves at 1.0 m/s and is 35.0 m long. If a woman steps on at one end and walks at 1.5 m/s relative to the moving sidewalk, how much time does she require to reach the opposite end if she walks (a) in the same direction the sidewalk is moving? (b) In the opposite direction?

3.38. Two piers, A and B, are located on a river: B is 1500 m downstream from A (Fig. 3.44). Two friends must make round trips from pier A to pier B and return. One rows a boat at a constant speed of 4.00 km/h relative to the water; the other walks on the shore at a constant speed of 4.00 km/h. The velocity of the river is 2.80 km/h in the direction from A to B. How much time does it take each person to make the round trip?

Figure 3.44 Exercise 3.38.



3.39. A canoe has a velocity of 0.40 m/s southeast relative to the earth. The canoe is on a river that is flowing 0.50 m/s east relative to the earth. Find the velocity (magnitude and direction) of the canoe relative to the river.

3.40. An airplane pilot wishes to fly due west. A wind of 80.0 km/h (about 50 mi/h) is blowing toward the south. (a) If the airspeed of the plane (its speed in still air) is 320.0 km/h (about 200 mi/h), in which direction should the pilot head? (b) What is the speed of the plane over the ground? Illustrate with a vector diagram.

3.41. Crossing the River I. A river flows due south with a speed of 2.0 m/s. A man steers a motorboat across the river; his velocity relative to the water is 4.2 m/s due east. The river is 800 m wide. (a) What is his velocity (magnitude and direction) relative to the earth? (b) How much time is required to cross the river? (c) How far south of his starting point will he reach the opposite bank?

3.42. Crossing the River II. (a) In which direction should the motorboat in Exercise 3.41 head in order to reach a point on the opposite bank directly east from the starting point? (The boat's speed relative to the water remains 4.2 m/s.) (b) What is the velocity of the boat relative to the earth? (c) How much time is required to cross the river?

3.43. The nose of an ultralight plane is pointed south, and its airspeed indicator shows 35 m/s. The plane is in a 10-m/s wind blowing toward the southwest relative to the earth. (a) In a vector-addition diagram, show the relationship of $\vec{v}_{P/E}$ (the velocity of the plane relative to the earth) to the two given vectors. (b) Letting x be east and y be north, find the components of $\vec{v}_{P/E}$. (c) Find the magnitude and direction of $\vec{v}_{P/E}$.

Problems

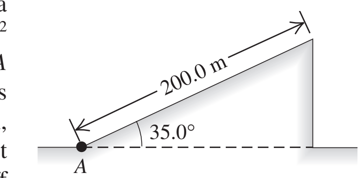
3.44. A faulty model rocket moves in the xy-plane (the positive y-direction is vertically upward). The rocket's acceleration has components $a_x(t) = \alpha t^2$ and $a_y(t) = \beta - \gamma t$, where $\alpha = 2.50\ m/s^4$, $\beta = 9.00\ m/s^2$, and $\gamma = 1.40\ m/s^3$. At $t = 0$ the rocket is at the origin and has velocity $\vec{v}_0 = v_{0x}\hat{\imath} + v_{0y}\hat{\jmath}$ with $v_{0x} = 1.00$ m/s and $v_{0y} = 7.00$ m/s. (a) Calculate the velocity and position vectors as functions of time. (b) What is the maximum height reached by the rocket? (c) Sketch the path of the rocket. (d) What is the horizontal displacement of the rocket when it returns to $y = 0$?

3.45. A rocket is fired at an angle from the top of a tower of height $h_0 = 50.0$ m. Because of the design of the engines, its position coordinates are of the form $x(t) = A + Bt^2$ and $y(t) = C + Dt^3$, where A, B, C, and D are constants. Furthermore, the acceleration of the rocket 1.00 s after firing is $\vec{a} = (4.00\hat{\imath} + 3.00\hat{\jmath})\ m/s^2$. Take the origin of coordinates to be at the base of the tower. (a) Find the constants A, B, C, and D, including their SI units. (b) At the instant after the rocket is fired, what are its acceleration vector and its velocity? (c) What are the x- and y-components of the rocket's velocity 10.0 s after it is fired, and how fast is it moving? (d) What is the position vector of the rocket 10.0 s after it is fired?

3.46. A bird flies in the xy-plane with a velocity vector given by $\vec{v} = (\alpha - \beta t^2)\hat{\imath} + \gamma t\hat{\jmath}$, with $\alpha = 2.4$ m/s, $\beta = 1.6\ m/s^3$, and $\gamma = 4.0\ m/s^2$. The positive y-direction is vertically upward. At $t = 0$ the bird is at the origin. (a) Calculate the position and acceleration vectors of the bird as functions of time. (b) What is the bird's altitude (y-coordinate) as it flies over $x = 0$ for the first time after $t = 0$?

3.47. A test rocket is launched by accelerating it along a 200.0-m incline at $1.25\ m/s^2$ starting from rest at point A (Figure 3.45.) The incline rises at 35.0° above the horizontal, and at the instant the rocket leaves it, its engines turn off and it is subject only to gravity (air resistance can be ignored). Find (a) the maximum height above the ground that the rocket reaches, and (b) the greatest horizontal range of the rocket beyond point A.

Figure 3.45 Problem 3.47.



3.48. Martian Athletics. In the long jump, an athlete launches herself at an angle above the ground and lands at the same height, trying to travel the greatest horizontal distance. Suppose that on earth she is in the air for time T, reaches a maximum height h, and achieves a horizontal distance D. If she jumped in *exactly* the same way during a competition on Mars, where g_{Mars} is 0.379 of its earth value, find her time in the air, maximum height, and horizontal distance. Express each of these three quantities in terms of its earth value. Air resistance can be neglected on both planets.

3.49. Dynamite! A demolition crew uses dynamite to blow an old building apart. Debris from the explosion flies off in all directions and is later found at distances as far as 50 m from the explosion. Estimate the maximum speed at which debris was blown outward by the explosion. Describe any assumptions that you make.

3.50. Spiraling Up. It is common to see birds of prey rising upward on thermals. The paths they take may be spiral-like. You can model the spiral motion as uniform circular motion combined with a constant upward velocity. Assume a bird completes a circle of radius 8.00 m every 5.00 s and rises vertically at a rate of 3.00 m/s. Determine: (a) the speed of the bird relative to the ground; (b) the bird's acceleration (magnitude and direction); and (c) the angle between the bird's velocity vector and the horizontal.

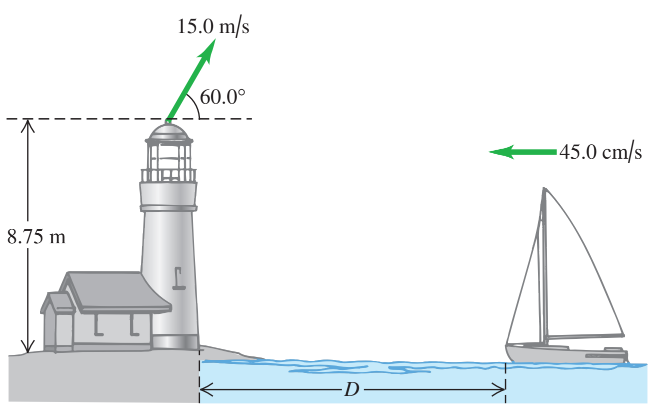
3.51. A jungle veterinarian with a blow-gun loaded with a tranquilizer dart and a sly 1.5-kg monkey are each 25 m above the ground in trees 90 m apart. Just as the hunter shoots horizontally at the monkey, the monkey drops from the tree in a vain attempt to escape being hit. What must the minimum muzzle velocity of the dart have been for the hunter to hit the monkey before it reached the ground?

3.52. A movie stuntwoman drops from a helicopter that is 30.0 m above the ground and moving with a constant velocity whose components are 10.0 m/s upward and 15.0 m/s horizontal and toward the south. You can ignore air resistance. (a) Where on the ground (relative to the position of the helicopter when she drops) should the stuntwoman have placed the foam mats that break her fall? (b) Draw *x-t*, *y-t*, v_x-*t*, and v_y-*t* graphs of her motion.

3.53. In fighting forest fires, airplanes work in support of ground crews by dropping water on the fires. A pilot is practicing by dropping a canister of red dye, hoping to hit a target on the ground below. If the plane is flying in a horizontal path 90.0 m above the ground and with a speed of 64.0 m/s (143 mi/h), at what horizontal distance from the target should the pilot release the canister? Ignore air resistance.

3.54. As a ship is approaching the dock at 45.0 cm/s, an important piece of landing equipment needs to be thrown to it before it can dock. This equipment is thrown at 15.0 m/s at 60.0° above the horizontal from the top of a tower at the edge of the water, 8.75 m above the ship's deck (Fig. 3.46.) For this equipment to land at the front of the ship, at what distance *D* from the dock should the ship be when the equipment is thrown? Air resistance can be neglected.

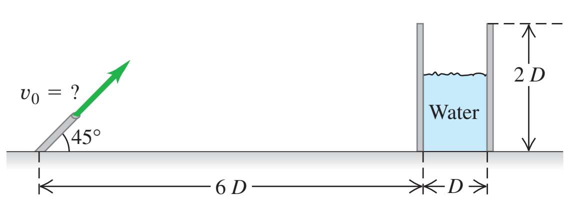
Figure 3.46 Problem 3.54.



3.55. The Longest Home Run. According to the *Guinness Book of World Records,* the longest home run ever measured was hit by Roy "Dizzy" Carlyle in a minor league game. The ball traveled 188 m (618 ft) before landing on the ground outside the ballpark. (a) Assuming the ball's initial velocity was 45° above the horizontal and ignoring air resistance, what did the initial speed of the ball need to be to produce such a home run if the ball was hit at a point 0.9 m (3.0 ft) above ground level? Assume that the ground was perfectly flat. (b) How far would the ball be above a fence 3.0 m (10 ft) high if the fence was 116 m (380 ft) from home plate?

3.56. A water hose is used to fill a large cylindrical storage tank of diameter *D* and height 2*D*. The hose shoots the water at 45° above the horizontal from the same level as the base of the tank and is a distance 6*D* away (Fig. 3.47). For what *range* of launch speeds (v_0) will the water enter the tank? Ignore air resistance, and express your answer in terms of *D* and *g*.

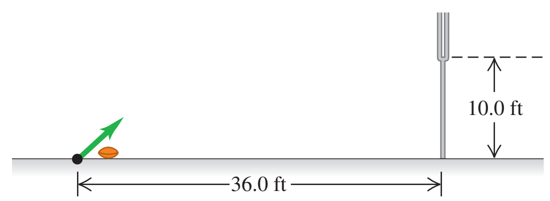
Figure 3.47 Problem 3.56.



3.57. A projectile is being launched from ground level with no air resistance. You want to avoid having it enter a temperature inversion layer in the atmosphere a height *h* above the ground. (a) What is the maximum launch speed you could give this projectile if you shot it straight up? Express your answer in terms of *h* and *g*. (b) Suppose the launcher available shoots projectiles at twice the maximum launch speed you found in part (a). At what maximum angle above the horizontal should you launch the projectile? (c) How far (in terms of *h*) from the launcher does the projectile in part (b) land?

3.58. Kicking a Field Goal. In U.S. football, after a touchdown the team has the opportunity to earn one more point by kicking the ball over the bar between the goal posts. The bar is 10.0 ft above the ground, and the ball is kicked from ground level, 36.0 ft horizontally from the bar (Fig. 3.48). Football regulations are stated in English units, but convert to SI units for this problem. (a) There is a minimum angle above the ground such that if the ball is launched below this angle, it can never clear the bar, no matter how fast it is kicked. What is this angle? (b) If the ball is kicked at 45.0° above the horizontal, what must its initial speed be if it to just clear the bar? Express your answer in m/s and km/h.

Figure 3.48 Problem 3.58.



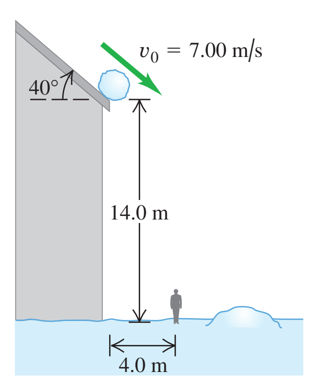
3.59. A projectile is launched with speed v_0 at an angle α_0 above the horizontal. The launch point is a height *h* above the ground. (a) Show that if air resistance is ignored, the horizontal distance that the projectile travels before striking the ground is

$$x = \frac{v_0 \cos\alpha_0}{g}\left(v_0 \sin\alpha_0 + \sqrt{v_0^2 \sin^2\alpha_0 + 2gh}\right)$$

Verify that if the launch point is at ground level so that $h = 0$, this is equal to the horizontal range *R* found in Example 3.8. (b) For the case where $v_0 = 10$ m/s and $h = 5.0$ m, graph *x* as a function of launch angle α_0 for values of α_0 from 0° to 90°. Your graph should show that *x* is zero if $\alpha_0 = 90°$, but *x* is nonzero if $\alpha_0 = 0$; explain why this is so. (c) We saw in Example 3.8 that for a projectile that lands at the same height from which it is launched, the horizontal range is maximum for $\alpha_0 = 45°$. For the case graphed in part (b), is the angle for maximum horizontal distance equal to, less than, or greater than 45°? (This is a general result for the situation where a projectile is launched from a point higher than where it lands.)

3.60. Look Out! A snowball rolls off a barn roof that slopes downward at an angle of 40° (Fig. 3.49). The edge of the roof is 14.0 m above the ground, and the snowball has a speed of 7.00 m/s as it rolls off the roof. Ignore air resistance. (a) How far from the edge of the barn does the snowball strike the ground if it doesn't strike anything else while falling? (b) Draw *x-t*, *y-t*, v_x-*t*, and v_y-*t* graphs for the motion in part (a). (c) A man 1.9 m tall is standing 4.0 m from the edge of the barn. Will he be hit by the snowball?

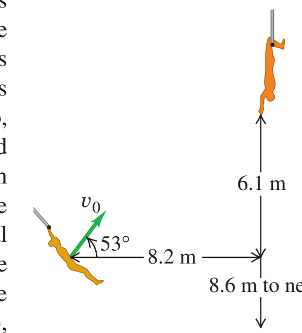
Figure 3.49 Problem 3.60.



3.61. (a) Prove that a projectile launched at angle α_0 has the same horizontal range as one launched with the same speed at angle $(90° - \alpha_0)$. (b) A frog jumps at a speed of 2.2 m/s and lands 25 cm from its starting point. At which angles above the horizontal could it have jumped?

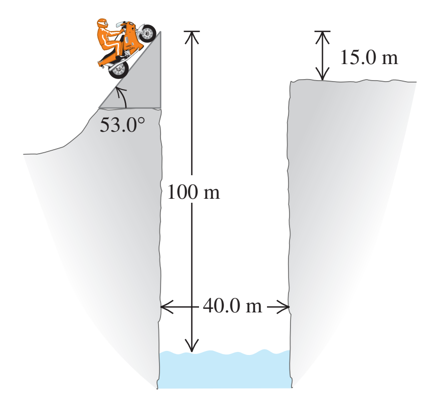
3.62. On the Flying Trapeze. A new circus act is called the Texas Tumblers. Lovely Mary Belle swings from a trapeze, projects herself at an angle of 53°, and is supposed to be caught by Joe Bob, whose hands are 6.1 m above and 8.2 m horizontally from her launch point (Fig. 3.50). You can ignore air resistance. (a) What initial speed v_0 must Mary Belle have just to reach Joe Bob? (b) For the initial speed calculated in part (a), what are the magnitude and direction of her velocity when Mary Belle reaches Joe Bob? (c) Assuming that Mary Belle has the initial speed calculated in part (a), draw *x-t*, *y-t*, v_x-*t*, and v_y-*t* graphs showing the motion of both tumblers. Your graphs should show the motion up until the point where Mary Belle reaches Joe Bob. (d) The night of their debut performance, Joe Bob misses her completely as she flies past. How far horizontally does Mary Belle travel, from her initial launch point, before landing in the safety net 8.6 m below her starting point?

Figure 3.50 Problem 3.62.



3.63. Leaping the River II. A physics professor did daredevil stunts in his spare time. His last stunt was an attempt to jump across a river on a motorcycle (Fig. 3.51). The takeoff ramp was inclined at 53.0°, the river was 40.0 m wide, and the far bank was 15.0 m lower than the top of the ramp. The river itself was 100 m below the ramp. You can ignore air resistance. (a) What should his speed have been at the top of the ramp to have just made it to the edge of the far bank? (b) If his speed was only half the value found in (a), where did he land?

Figure 3.51 Problem 3.63.



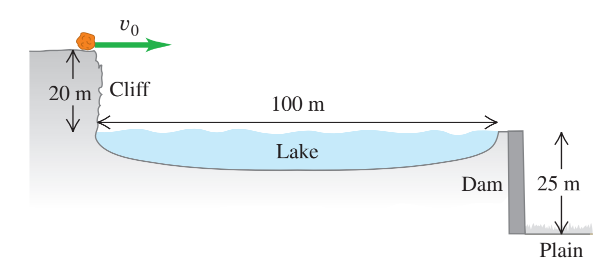
3.64. A rock is thrown from the roof of a building with a velocity v_0 at an angle of α_0 from the horizontal. The building has height *h*. You can ignore air resistance. Calculate the magnitude of the velocity of the rock just before it strikes the ground, and show that this speed is independent of α_0.

3.65. A 5500-kg cart carrying a vertical rocket launcher moves to the right at a constant speed of 30.0 m/s along a horizontal track. It launches a 45.0-kg rocket vertically upward with an initial speed of 40.0 m/s relative to the cart. (a) How high will the rocket go? (b) Where, relative to the cart, will the rocket land? (c) How far does the cart move while the rocket is in the air? (d) At what angle, relative to the horizontal, is the rocket traveling just as it leaves the cart, as measured by an observer at rest on the ground? (e) Sketch the rocket's trajectory as seen by an observer (i) stationary on the cart and (ii) stationary on the ground.

3.66. A 2.7-kg ball is thrown upward with an initial speed of 20.0 m/s from the edge of a 45.0-m-high cliff. At the instant the ball is thrown, a woman starts running away from the base of the cliff with a constant speed of 6.00 m/s. The woman runs in a straight line on level ground, and air resistance acting on the ball can be ignored. (a) At what angle above the horizontal should the ball be thrown so that the runner will catch it just before it hits the ground, and how far does the woman run before she catches the ball? (b) Carefully sketch the ball's trajectory as viewed by (i) a person at rest on the ground and (ii) the runner.

3.67. A 76.0-kg boulder is rolling horizontally at the top of a vertical cliff that is 20 m above the surface of a lake, as shown in Fig. 3.52. The top of the vertical face of a dam is located 100 m from the foot of the cliff, with the top of the dam level with the surface of the water in the lake. A level plain is 25 m below the top of the dam. (a) What must be the minimum speed of the rock just as it leaves the cliff so it will travel to the plain without striking the dam? (b) How far from the foot of the dam does the rock hit the plain?

Figure 3.52 Problem 3.67.



3.68. Tossing Your Lunch. Henrietta is going off to her physics class, jogging down the sidewalk at 3.05 m/s. Her husband Bruce suddenly realizes that she left in such a hurry that she forgot her lunch of bagels, so he runs to the window of their apartment, which is 43.9 m above the street level and directly above the sidewalk, to throw them to her. Bruce throws them horizontally 9.00 s after Henrietta has passed below the window, and she catches them on the run. You can ignore air resistance. (a) With what initial speed must Bruce throw the bagels so Henrietta can catch them just before they hit the ground? (b) Where is Henrietta when she catches the bagels?

3.69. Two tanks are engaged in a training exercise on level ground. The first tank fires a paint-filled training round with a muzzle speed of 250 m/s at 10.0° above the horizontal while advancing toward the second tank with a speed of 15.0 m/s relative to the ground. The second tank is retreating at 35.0 m/s relative to the ground, but is hit by the shell. You can ignore air resistance and assume the shell hits at the same height above ground from which it was fired. Find the distance between the tanks (a) when the round was first fired and (b) at the time of impact.

3.70. Bang! A student sits atop a platform a distance *h* above the ground. He throws a large firecracker horizontally with a speed *v*. However, a wind blowing parallel to the ground gives the firecracker a constant horizontal acceleration with magnitude *a*. This results in the firecracker reaching the ground directly under the student. Determine the height *h* in terms of *v*, *a*, and *g*. You can ignore the effect of air resistance on the vertical motion.

3.71. A rocket is launched vertically from rest with a constant upward acceleration of 1.75 m/s^2. Suddenly 22.0 s after launch, an unneeded fuel tank is jettisoned by shooting it away from the rocket. A crew member riding in the rocket measures that the initial speed of the tank is 25.0 m/s and that it moves perpendicular to the rocket's path. The fuel tank feels no appreciable air resistance and feels only the force of gravity once it leaves the rocket. (a) How fast is the rocket moving at the instant the fuel tank is jettisoned? (b) What are the horizontal and vertical components of the fuel tank's velocity just as it is jettisoned as measured by (i) a crew member in the rocket and (ii) a technician standing on the ground? (c) At what angle with respect to the horizontal does the jettisoned fuel tank initially move, as measured by (i) a crew member in the rocket and (ii) a technician standing on the ground? (d) What maximum height above the launch pad does the jettisoned tank reach?

3.72. When it is 145 m above the ground, a rocket traveling vertically upward at a constant 8.50 m/s relative to the ground launches a secondary rocket at a speed of 12.0 m/s at an angle of 53.0° above the horizontal, both quantities being measured by an astronaut sitting in the rocket. Air resistance is too small to worry about. (a) Just as the secondary rocket is launched, what are the horizontal and vertical components of its velocity relative to (i) the astronaut sitting in the rocket and (ii) Mission Control on the ground? (b) Find the initial speed and launch angle of the secondary rocket as measured by Mission Control. (c) What maximum height above the ground does the secondary rocket reach?

3.73. In a Fourth of July celebration, a firecracker is launched from ground level with an initial velocity of 25.0 m/s at 30.0° from the *vertical.* At its maximum height it explodes in a starburst into many fragments, two of which travel forward initially at 20.0 m/s at ±53.0° with respect to the horizontal, both quantities measured *relative to the original firecracker just before it exploded.* With what angles with respect to the horizontal do the two fragments initially move right after the explosion, as measured by a spectator standing on the ground?

3.74. In an action-adventure film, the hero is supposed to throw a grenade from his car, which is going 90.0 km/h, to his enemy's car, which is going 110 km/h. The enemy's car is 15.8 m in front of the hero's when he lets go of the grenade. If the hero throws the grenade so its initial velocity relative to him is at an angle of 45° above the horizontal, what should the magnitude of the initial velocity be? The cars are both traveling in the same direction on a level road. You can ignore air resistance. Find the magnitude of the velocity both relative to the hero and relative to the earth.

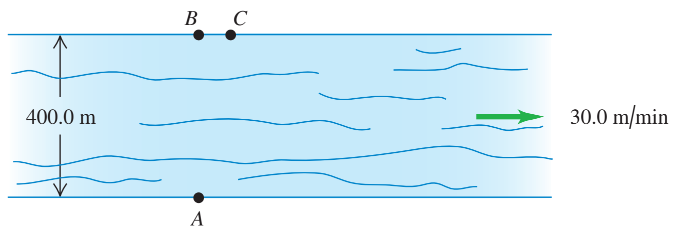
3.75. A rock tied to a rope moves in the *xy*-plane. Its coordinates are given as functions of time by

$$x(t) = R\cos\omega t \qquad y(t) = R\sin\omega t$$

where R and ω are constants. (a) Show that the rock's distance from the origin is constant and equal to R—that is, that its path is a circle of radius R. (b) Show that at every point the rock's velocity is perpendicular to its position vector. (c) Show that the rock's acceleration is always opposite in direction to its position vector and has magnitude $\omega^2 R$. (d) Show that the magnitude of the rock's velocity is constant and equal to ωR. (e) Combine the results of parts (c) and (d) to show that the rock's acceleration has constant magnitude v^2/R.

3.76. A 400.0-m-wide river flows from west to east at 30.0 m/min. Your boat moves at 100.0 m/min relative to the water no matter which direction you point it. To cross this river, you start from a dock at point A on the south bank. There is a boat landing directly opposite at point B on the north bank, and also one at point C, 75.0 m downstream from B (Fig. 3.53). (a) Where on the north shore will you land if you point your boat perpendicular to the water current, and what distance will you have traveled? (b) If you initially aim your boat directly toward point C and do not change that bearing relative to the shore, where on the north shore will you land? (c) To reach point C: (i) at what bearing must you aim your boat, (ii) how long will it take to cross the river, (iii) what distance do you travel, and (iv) and what is the speed of your boat as measured by an observer standing on the river bank?

Figure 3.53 Problem 3.76.



3.77. Cycloid. A particle moves in the *xy*-plane. Its coordinates are given as functions of time by

$$x(t) = R(\omega t - \sin\omega t) \qquad y(t) = R(1 - \cos\omega t)$$

where R and ω are constants. (a) Sketch the trajectory of the particle. (This is the trajectory of a point on the rim of a wheel that is rolling at a constant speed on a horizontal surface. The curve traced out by such a point as it moves through space is called a *cycloid.*) (b) Determine the velocity components and the acceleration components of the particle at any time t. (c) At which times is the particle momentarily at rest? What are the coordinates of the particle at these times? What are the magnitude and direction of the acceleration at these times? (d) Does the magnitude of the acceleration depend on time? Compare to uniform circular motion.

3.78. A projectile is fired from point A at an angle above the horizontal. At its highest point, after having traveled a horizontal distance D from its launch point, it suddenly explodes into two identical fragments that travel horizontally with equal but opposite velocities as measured *relative to the projectile just before it exploded.* If one fragment lands back at point A, how far from A (in terms of D) does the other fragment land?

3.79. Centrifuge on Mercury. A laboratory centrifuge on earth makes n rpm (rev/min) and produces an acceleration of 5.00g at its outer end. (a) What is the acceleration (in g's) at a point halfway out to the end? (b) This centrifuge is now used in a space capsule on the planet Mercury, where $g_{Mercury}$ is 0.378 what it is on earth. How many rpm (in terms of n) should it make to produce $5g_{Mercury}$ at its outer end?

3.80. Raindrops. When a train's velocity is 12.0 m/s eastward, raindrops that are falling vertically with respect to the earth make traces that are inclined 30.0° to the vertical on the windows of the train. (a) What is the horizontal component of a drop's velocity with respect to the earth? With respect to the train? (b) What is the magnitude of the velocity of the raindrop with respect to the earth? With respect to the train?

3.81. An airplane pilot sets a compass course due west and maintains an airspeed of 220 km/h. After flying for 0.500 h, she finds herself over a town 120 km west and 20 km south of her starting point. (a) Find the wind velocity (magnitude and direction). (b) If the wind velocity is 40 km/h due south, in what direction should the pilot set her course to travel due west? Use the same airspeed of 220 km/h.

3.82. An elevator is moving upward at a constant speed of 2.50 m/s. A bolt in the elevator ceiling 3.00 m above the elevator floor works loose and falls. (a) How long does it take for the bolt to fall to the elevator floor? What is the speed of the bolt just as it hits the elevator floor (b) according to an observer in the elevator? (c) According to an observer standing on one of the floor landings of the building? (d) According to the observer in part (c), what distance did the bolt travel between the ceiling and the floor of the elevator?

3.83. Suppose the elevator in Problem 3.82 starts from rest and maintains a constant upward acceleration of 4.00 m/s^2, and the bolt falls out the instant the elevator begins to move. (a) How long does it take for the bolt to reach the floor of the elevator? (b) Just as it reaches the floor, how fast is the bolt moving according to an observer (i) in the elevator? (ii) Standing on the floor landings of the building? (c) According to each observer in part (b), how far has the bolt traveled between the ceiling and floor of the elevator?

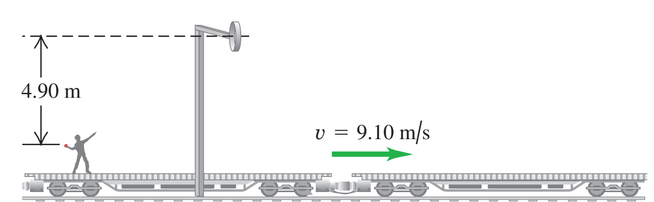
3.84. City A lies directly west of city B. When there is no wind, an airliner makes the 5550-km round-trip flight between them in 6.60 h of flying time while traveling at the same speed in both directions. When a strong, steady 225-km/h wind is blowing from west to east and the airliner has the same airspeed as before, how long will the trip take?

3.85. In a World Cup soccer match, Juan is running due north toward the goal with a speed of 8.00 m/s relative to the ground. A teammate passes the ball to him. The ball has a speed of 12.0 m/s and is moving in a direction of 37.0° east of north, relative to the ground. What are the magnitude and direction of the ball's velocity relative to Juan?

Challenge Problems

3.86. A man is riding on a flatcar traveling at a constant speed of 9.10 m/s (Fig. 3.54). He wishes to throw a ball through a stationary hoop 4.90 m above the height of his hands in such a manner that the ball will move horizontally as it passes through the hoop. He throws the ball with a speed of 10.8 m/s with respect to himself. (a) What must the vertical component of the initial velocity of the ball be? (b) How many seconds after he releases the ball will it pass through the hoop? (c) At what horizontal distance in front of the hoop must he release the ball? (d) When the ball leaves the man's hands, what is the direction of its velocity relative to the frame of reference of the flatcar? Relative to the frame of reference of an observer standing on the ground?

Figure 3.54 Challenge Problem 3.86.

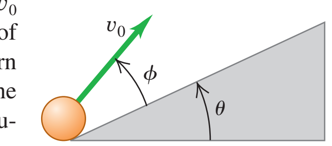


3.87. A shotgun fires a large number of pellets upward, with some pellets traveling very nearly vertically and others as much as 1.0° from the vertical. Assume that the initial speed of the pellets is uniformly 150 m/s, and ignore air resistance. (a) Within what radius from the point of firing will the pellets land? (b) If there are 1000 pellets, and they fall in a uniform distribution over a circle with the radius calculated in part (a), what is the probability that at least one pellet will fall on the head of the person who fires the shotgun? Assume that his head has a radius of 10 cm. (c) Air resistance, in fact, has several effects. It slows down the rising pellets, decreases their horizontal component of velocity, and limits the speed with which they fall. Which of these effects will tend to make the radius larger than calculated in part (a), and which will tend to make it smaller? What do you think the overall effect of air resistance will be? (The effect of air resistance on a velocity component increases as the magnitude of the component increases.)

3.88. A projectile is thrown from a point P. It moves in such a way that its distance from P is always increasing. Find the maximum angle above the horizontal with which the projectile could have been thrown. You can ignore air resistance.

3.89. Projectile Motion on an Incline I. A baseball is given an initial velocity with magnitude v_0 at an angle ϕ above the surface of an incline, which is in turn inclined at an angle θ above the horizontal (Fig. 3.55) (a) Calculate the distance, measured along the incline, from the launch point to where the baseball strikes the incline. Your answer will be in terms of v_0, g, θ, and ϕ. (b) What angle ϕ gives the maximum range, measured along the incline? (*Note:* You might be interested in the three different methods of solution presented by I. R. Lapidus in *Amer. Jour. of Phys.,* Vol. 51 (1983), pp. 806 and 847. See also H. A. Buckmaster in *Amer. Jour. of Phys.,* Vol. 53 (1985), pp. 638–641, for a thorough study of this and some similar problems.)

Figure 3.55 Challenge Problem 3.89.



3.90. Projectile Motion on an Incline II. Refer to Challenge Problem 3.89. (a) An archer on ground that has a constant upward slope of 30.0° aims at a target 60.0 m farther up the incline. The arrow in the bow and the bull's-eye at the center of the target are each 1.50 m above the ground. The initial velocity of the arrow just after it leaves the bow has magnitude 32.0 m/s. At what angle above the *horizontal* should the archer aim to hit the bull's-eye? If there are two such angles, calculate the smaller of the two. You might have to solve the equation for the angle by iteration—that is, by trial and error. How does the angle compare to that required when the ground is level, with 0 slope? (b) Repeat the above for ground that has a constant *downward* slope of 30.0°.

3.91. For no apparent reason, a poodle is running at a constant speed of $v = 5.00$ m/s in a circle with radius $R = 2.50$ m. Let $\vec{v}_1$ be the velocity vector at time t_1, and let $\vec{v}_2$ be the velocity vector at time t_2. Consider $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ and $\Delta t = t_2 - t_1$. Recall that $\vec{a}_{av} = \Delta\vec{v}/\Delta t$. For $\Delta t = 0.5$ s, 0.1 s, and 0.05 s, calculate the magnitude (to four significant figures) and direction (relative to $\vec{v}_1$) of the average acceleration $\vec{a}_{av}$. Compare your results to the general expression for the instantaneous acceleration $\vec{a}$ for uniform circular motion that is derived in the text.

3.92. A rocket designed to place small payloads into orbit is carried to an altitude of 12.0 km above sea level by a converted airliner. When the airliner is flying in a straight line at a constant speed of 850 km/h, the rocket is dropped. After the drop, the airliner maintains the same altitude and speed and continues to fly in a straight line. The rocket falls for a brief time, after which its rocket motor turns on. Once its rocket motor is on, the combined effects of thrust and gravity give the rocket a constant acceleration of magnitude 3.00g directed at an angle of 30.0° above the horizontal. For reasons of safety, the rocket should be at least 1.00 km in front of the airliner when it climbs through the airliner's altitude. Your job is to determine the minimum time that the rocket must fall before its engine starts. You can ignore air resistance.

Your answer should include (i) a diagram showing the flight paths of both the rocket and the airliner, labeled at several points with vectors for their velocities and accelerations; (ii) an *x*-*t* graph showing the motions of both the rocket and the airliner; and (iii) a *y*-*t* graph showing the motions of both the rocket and the airliner. In the diagram and the graphs, indicate when the rocket is dropped, when the rocket motor turns on, and when the rocket climbs through the altitude of the airliner.

3.93. Two students are canoeing on a river. While heading upstream, they accidentally drop an empty bottle overboard. They then continue paddling for 60 minutes, reaching a point 2.0 km farther upstream. At this point they realize that the bottle is missing and, driven by ecological awareness, they turn around and head downstream. They catch up with and retrieve the bottle (which has been moving along with the current) 5.0 km downstream from the turn-around point. (a) Assuming a constant paddling effort throughout, how fast is the river flowing? (b) What would the canoe speed in a still lake be for the same paddling effort?