POTENTIAL ENERGY AND ENERGY CONSERVATION

7

? As this diver enters the water, is the force of gravity doing positive or negative work on him? Is the water doing positive or negative work on him?

LEARNING GOALS

By studying this chapter, you will learn:

- How to use the concept of gravitational potential energy in problems that involve vertical motion.
- How to use the concept of elastic potential energy in problems that involve a moving body attached to a stretched or compressed spring.
- The distinction between conservative and nonconservative forces, and how to solve problems in which both kinds of forces act on a moving body.
- How to calculate the properties of a conservative force if you know the corresponding potential-energy function.
- How to use energy diagrams to understand the motion of an object moving in a straight line under the influence of a conservative force.

When a diver jumps off a high board into a swimming pool, he hits the water moving pretty fast, with a lot of kinetic energy. Where does that energy come from? The answer we learned in Chapter 6 was that the gravitational force (his weight) does work on the diver as he falls. The diver's kinetic energy—energy associated with his *motion*—increases by an amount equal to the work done.

However, there is a very useful alternative way to think about work and kinetic energy. This new approach is based on the concept of *potential energy,* which is energy associated with the *position* of a system rather than its motion. In this approach, there is *gravitational potential energy* even while the diver is standing on the high board. Energy is not added to the earth–diver system as the diver falls, but rather a storehouse of energy is *transformed* from one form (potential energy) to another (kinetic energy) as he falls. In this chapter we'll see how the work–energy theorem explains this transformation.

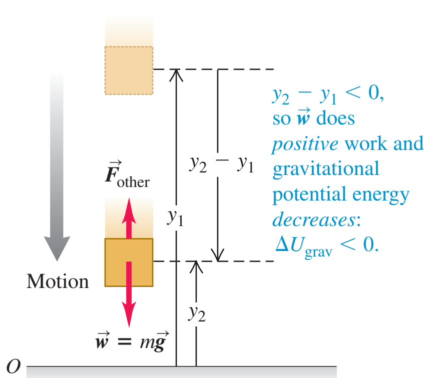
If the diver bounces on the end of the board before he jumps, the bent board stores a second kind of potential energy called *elastic potential energy.* We'll discuss elastic potential energy of simple systems such as a stretched or compressed spring. (An important third kind of potential energy is associated with the positions of electrically charged particles relative to each other. We'll encounter this potential energy in Chapter 23.)

We will prove that in some cases the sum of a system's kinetic and potential energy, called the *total mechanical energy* of the system, is constant during the motion of the system. This will lead us to the general statement of the *law of conservation of energy,* one of the most fundamental and far-reaching principles in all of science.

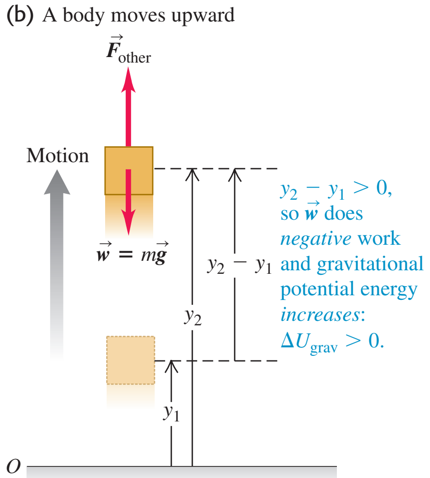
7.1 As a basketball descends, gravitational potential energy is converted to kinetic energy and the basketball's speed increases.

7.2 When a body moves vertically from an initial height y_1 to a final height y_2, the gravitational force $\vec{w}$ does work and the gravitational potential energy changes.

(a) A body moves downward



(b) A body moves upward



7.1 Gravitational Potential Energy

We learned in Chapter 6 that a particle gains or loses kinetic energy because it interacts with other objects that exert forces on it. During any interaction, the change in a particle's kinetic energy is equal to the total work done on the particle by the forces that act on it.

In many situations it seems as though energy has been stored in a system, to be recovered later. For example, you must do work to lift a heavy stone over your head. It seems reasonable that in hoisting the stone into the air you are storing energy in the system, energy that is later converted into kinetic energy when you let the stone fall.

This example points to the idea of an energy associated with the *position* of bodies in a system. This kind of energy is a measure of the *potential* or *possibility* for work to be done; when a stone is raised into the air, there is a potential for work to be done on it by the gravitational force, but only if the stone is allowed to fall to the ground. For this reason, energy associated with position is called **potential energy.** Our discussion suggests that there is potential energy associated with a body's weight and its height above the ground. We call this *gravitational potential energy* (Fig. 7.1).

We now have *two* ways to describe what happens when a body falls without air resistance. One way is to say that gravitational potential energy decreases and the falling body's kinetic energy increases. The other way, which we learned in Chapter 6, is that a falling body's kinetic energy increases because the force of the earth's gravity (the body's weight) does work on the body. Later in this section we'll use the work–energy theorem to show that these two descriptions are equivalent.

To begin with, however, let's derive the expression for gravitational potential energy. Suppose a body with mass m moves along the (vertical) y-axis, as in Fig. 7.2. The forces acting on it are its weight, with magnitude $w = mg$, and possibly some other forces; we call the vector sum (resultant) of all the other forces $\vec{F}_{other}$. We'll assume that the body stays close enough to the earth's surface that the weight is constant. (We'll find in Chapter 12 that weight decreases with altitude.) We want to find the work done by the weight when the body moves downward from a height y_1 above the origin to a lower height y_2 (Fig. 7.2a). The weight and displacement are in the same direction, so the work W_{grav} done on the body by its weight is positive;

$$W_{grav} = Fs = w(y_1 - y_2) = mgy_1 - mgy_2 \qquad (7.1)$$

This expression also gives the correct work when the body moves *upward* and y_2 is greater than y_1 (Fig. 7.2b). In that case the quantity $(y_1 - y_2)$ is negative, and W_{grav} is negative because the weight and displacement are opposite in direction.

Equation (7.1) shows that we can express W_{grav} in terms of the values of the quantity mgy at the beginning and end of the displacement. This quantity, the product of the weight mg and the height y above the origin of coordinates, is called the **gravitational potential energy,** U_{grav}:

$$U_{grav} = mgy \qquad \text{(gravitational potential energy)} \qquad (7.2)$$

Its initial value is $U_{grav,1} = mgy_1$ and its final value is $U_{grav,2} = mgy_2$. The change in U_{grav} is the final value minus the initial value, or $\Delta U_{grav} = U_{grav,2} - U_{grav,1}$. We can express the work W_{grav} done by the gravitational force during the displacement from y_1 to y_2 as

$$W_{grav} = U_{grav,1} - U_{grav,2} = -(U_{grav,2} - U_{grav,1}) = -\Delta U_{grav} \qquad (7.3)$$

The negative sign in front of ΔU_{grav} is *essential.* When the body moves up, y increases, the work done by the gravitational force is negative, and the gravitational potential energy increases $(\Delta U_{grav} > 0)$. When the body moves down, y decreases, the gravitational force does positive work, and the gravitational potential energy decreases $(\Delta U_{grav} < 0)$. It's like drawing money out of the bank (decreasing U_{grav}) and spending it (doing positive work). As Eq. (7.3) shows, the unit of potential energy is the joule (J), the same unit as is used for work.

CAUTION **To what body does gravitational potential energy "belong"?** It is *not* correct to call $U_{grav} = mgy$ the "gravitational potential energy of the body." The reason is that gravitational potential energy U_{grav} is a *shared* property of the body and the earth. The value of U_{grav} increases if the earth stays fixed and the body moves upward, away from the earth; it also increases if the body stays fixed and the earth is moved away from it. Notice that the formula $U_{grav} = mgy$ involves characteristics of both the body (its mass m) and the earth (the value of g).

Conservation of Mechanical Energy (Gravitational Forces Only)

To see what gravitational potential energy is good for, suppose the body's weight is the *only* force acting on it, so $\vec{F}_{other} = \mathbf{0}$. The body is then falling freely with no air resistance, and can be moving either up or down. Let its speed at point y_1 be v_1 and let its speed at y_2 be v_2. The work–energy theorem, Eq. (6.6), says that the total work done on the body equals the change in the body's kinetic energy: $W_{tot} = \Delta K = K_2 - K_1$. If gravity is the only force that acts, then from Eq. (7.3), $W_{tot} = W_{grav} = -\Delta U_{grav} = U_{grav,1} - U_{grav,2}$. Putting these together, we get

$$\Delta K = -\Delta U_{grav} \qquad \text{or} \qquad K_2 - K_1 = U_{grav,1} - U_{grav,2}$$

which we can rewrite as

$$K_1 + U_{grav,1} = K_2 + U_{grav,2} \qquad \text{(if only gravity does work)} \qquad (7.4)$$

or

$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2 \qquad \text{(if only gravity does work)} \qquad (7.5)$$

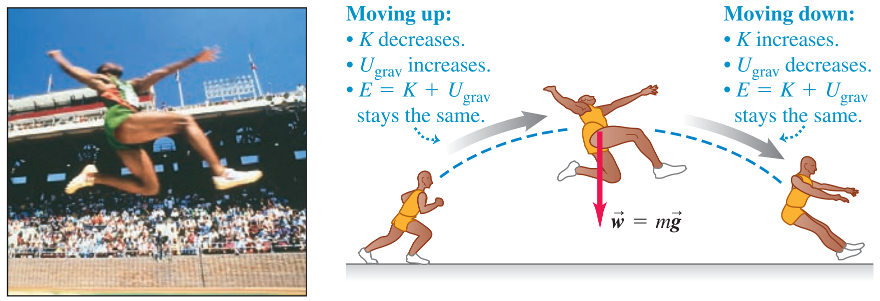
The sum $K + U_{grav}$ of kinetic and potential energy is called E, the **total mechanical energy of the system.** By "system" we mean the body of mass m and the earth considered together, because gravitational potential energy U is a shared property of both bodies. Then $E_1 = K_1 + U_{grav,1}$ is the total mechanical energy at y_1 and $E_2 = K_2 + U_{grav,2}$ is the total mechanical energy at y_2. Equation (7.4) says that when the body's weight is the only force doing work on it, $E_1 = E_2$. That is, E is constant; it has the same value at y_1 and y_2. But since the positions y_1 and y_2 are arbitrary points in the motion of the body, the total mechanical energy E has the same value at *all* points during the motion:

$$E = K + U_{grav} = \text{constant} \qquad \text{(if only gravity does work)}$$

A quantity that always has the same value is called a *conserved* quantity. *When only the force of gravity does work, the total mechanical energy is constant—that is, is conserved* (Fig. 7.3). This is our first example of the **conservation of mechanical energy.**

When we throw a ball into the air, its speed decreases on the way up as kinetic energy is converted to potential energy; $\Delta K < 0$ and $\Delta U_{grav} > 0$. On the way back down, potential energy is converted back to kinetic energy and the ball's speed increases; $\Delta K > 0$ and $\Delta U_{grav} < 0$. But the *total* mechanical energy (kinetic plus potential) is the same at every point in the motion, provided that no force other than gravity does work on the ball (that is, air resistance must be

7.3 While this athlete is in midair, only gravity does work on him (if we neglect the minor effects of air resistance). Mechanical energy E—the sum of kinetic and gravitational potential energy—is conserved.



negligible). It's still true that the gravitational force does work on the body as it moves up or down, but we no longer have to calculate work directly; keeping track of changes in the value of U_{grav} takes care of this completely.

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5.2 Upward-Moving Elevator Stops
5.3 Stopping a Downward-Moving Elevator
5.6 Skier Speed

CAUTION **Choose "zero height" to be wherever you like** When working with gravitational potential energy, we may choose any height to be $y = 0$. If we shift the origin for y, the values of y_1 and y_2 change, as do the values of $U_{grav,1}$ and $U_{grav,2}$. But this shift has no effect on the *difference* in height $y_2 - y_1$ or on the *difference* in gravitational potential energy $U_{grav,2} - U_{grav,1} = mg(y_2 - y_1)$. As the following example shows, the physically significant quantity is not the value of U_{grav} at a particular point, but only the *difference* in U_{grav} between two points. So we can define U_{grav} to be zero at whatever point we choose without affecting the physics. ▮

Example 7.1 Height of a baseball from energy conservation

You throw a 0.145-kg baseball straight up in the air, giving it an initial upward velocity of magnitude 20.0 m/s. Find how high it goes, ignoring air resistance.

SOLUTION

IDENTIFY: After the ball leaves your hand, the only force doing work on the ball is gravity. Hence we can use conservation of mechanical energy.

SET UP: We'll use Eqs. (7.4) and (7.5), taking point 1 to be where the ball leaves your hand and point 2 to be where it reaches its maximum height. As in Fig. 7.2, we take the positive y-direction to be upward. The ball's speed at point 1 is $v_1 = 20.0$ m/s; at its maximum height the ball is instantaneously at rest, so $v_2 = 0$.

We want to know how far the ball moves vertically between the two points, so our target variable is the displacement $y_2 - y_1$. If we take the origin to be where the ball leaves your hand (point 1), then $y_1 = 0$ (Fig. 7.4) and the target variable is just y_2.

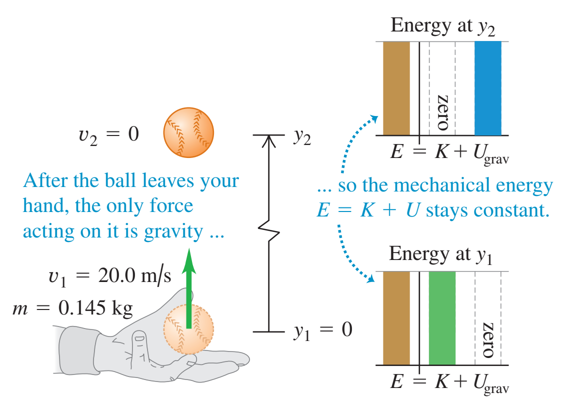
EXECUTE: Since $y_1 = 0$, the potential energy at point 1 is $U_{grav,1} = mgy_1 = 0$. Furthermore, since the ball is at rest at point 2, the kinetic energy at that point is $K_2 = \frac{1}{2}mv_2^2 = 0$. Hence Eq. (7.4), which says that $K_1 + U_{grav,1} = K_2 + U_{grav,2}$, becomes

$$K_1 = U_{grav,2}$$

As the energy bar graphs in Fig. 7.4 show, the kinetic energy of the ball at point 1 is completely converted to gravitational potential energy at point 2. At point 1 the kinetic energy is

$$K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(0.145\text{ kg})(20.0\text{ m/s})^2 = 29.0\text{ J}$$

7.4 After a baseball leaves your hand, mechanical energy $E = K + U$ is conserved.



This equals the gravitational potential energy $U_{grav,2} = mgy_2$ at point 2, so

$$y_2 = \frac{U_{grav,2}}{mg} = \frac{29.0\text{ J}}{(0.145\text{ kg})(9.80\text{ m/s}^2)} = 20.4\text{ m}$$

We can also solve the equation $K_1 = U_{grav,2}$ algebraically for y_2:

$$\frac{1}{2}mv_1^2 = mgy_2$$

$$y_2 = \frac{v_1^2}{2g} = \frac{(20.0\text{ m/s})^2}{2(9.80\text{ m/s}^2)} = 20.4\text{ m}$$

EVALUATE: The mass divides out, as we should expect; we learned in Chapter 2 that the motion of a body in free fall doesn't depend on its mass. Indeed, we could have derived the result $y_2 = v_1^2/2g$ using Eq. (2.13).

In our calculation we chose the origin to be at point 1, so $y_1 = 0$ and $U_{grav,1} = 0$. What happens if we make a different choice? As an example, suppose we choose the origin to be 5.0 m below point 1, so $y_1 = 5.0$ m. Then the total mechanical energy at point 1 is part kinetic and part potential, while at point 2 it's purely potential energy. If you work through the calculation again with this choice of origin, you'll find $y_2 = 25.4$ m; this is 20.4 m above point 1, just as with the first choice of origin. In problems like this, the choice of height at which $U_{grav} = 0$ is up to you; don't agonize over the choice, though, because the physics of the answer doesn't depend on your choice.

When Forces Other Than Gravity Do Work

If other forces act on the body in addition to its weight, then $\vec{F}_{other}$ in Fig. 7.2 is *not* zero. For the pile driver described in Example 6.4 (Section 6.2), the force applied by the hoisting cable and the friction with the vertical guide rails are examples of forces that might be included in $\vec{F}_{other}$. The gravitational work W_{grav} is still given by Eq. (7.3), but the total work W_{tot} is then the sum of W_{grav} and the work done by $\vec{F}_{other}$. We will call this additional work W_{other}, so the total work done by all forces is $W_{tot} = W_{grav} + W_{other}$. Equating this to the change in kinetic energy, we have

$$W_{other} + W_{grav} = K_2 - K_1 \qquad (7.6)$$

Also, from Eq. (7.3), $W_{grav} = U_{grav,1} - U_{grav,2}$, so

$$W_{other} + U_{grav,1} - U_{grav,2} = K_2 - K_1$$

which we can rearrange in the form

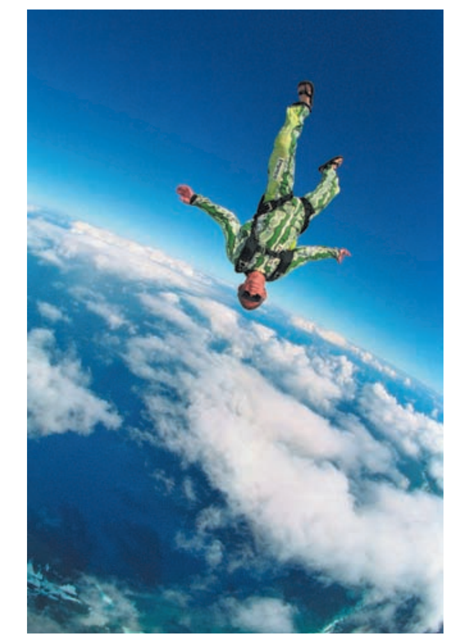
$$K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2} \quad \text{(if forces other than gravity do work)} \qquad (7.7)$$

Finally, using the appropriate expressions for the various energy terms, we obtain

$$\frac{1}{2}mv_1^2 + mgy_1 + W_{other} = \frac{1}{2}mv_2^2 + mgy_2 \quad \text{(if forces other than gravity do work)} \qquad (7.8)$$

The meaning of Eqs. (7.7) and (7.8) is this: *The work done by all forces* other than the gravitational force *equals the change in the total mechanical energy $E = K + U_{grav}$ of the system, where U_{grav} is the gravitational potential energy.* When W_{other} is positive, E increases, and $K_2 + U_{grav,2}$ is greater than $K_1 + U_{grav,1}$. When W_{other} is negative, E decreases (Fig. 7.5). In the special case in which no forces other than the body's weight do work, $W_{other} = 0$. The total mechanical energy is then constant, and we are back to Eq. (7.4) or (7.5).

7.5 As this skydiver moves downward, the upward force of air resistance does negative work W_{other} on him. Hence the total mechanical energy $E = K + U$ decreases: The skydiver's speed and kinetic energy K stay the same, while the gravitational potential energy U goes down.



Problem-Solving Strategy 7.1 Problems Using Mechanical Energy I

MP

IDENFITY *the relevant concepts:* Decide whether the problem should be solved by energy methods, by using $\sum\vec{F} = m\vec{a}$ directly, or by a combination of these. The energy approach is best when the problem involves varying forces, motion along a curved path (discussed later in this section), or both. If the problem involves elapsed time, the energy approach is usually *not* the best choice, because it doesn't involve time directly.

SET UP *the problem* using the following steps:

1. When using the energy approach, first decide what the initial and final states (the positions and velocities) of the system are. Use the subscript 1 for the initial state and the subscript 2 for the final state. It helps to draw sketches showing the initial and final states.
2. Define your coordinate system, particularly the level at which $y = 0$. You will use it to compute gravitational potential energies. We suggest that you always choose the positive y-direction to be upward because this is what Eq. (7.2) assumes.
3. Identify all forces that do work that can't be described in terms of potential energy. (So far this means any forces other than gravity. But later in this chapter we'll see that the work done by an ideal spring can also be expressed as a change in potential energy.) A free-body diagram is always helpful.

Continued

4. List the unknown and known quantities, including the coordinates and velocities at each point. Decide which unknowns are your target variables.

EXECUTE *the solution:* Write expressions for the initial and final kinetic and potential energies—that is, K_1, K_2, $U_{grav,1}$, and $U_{grav,2}$. Then relate the kinetic and potential energies and the work done by other forces, W_{other}, using Eq. (7.7). (You will have to calculate W_{other} in terms of these forces.) If no other forces do work, this expression becomes Eq. (7.4). It's helpful to draw bar graphs showing the initial and final values of K, U_{grav}, and $E = K + U_{grav}$. Then solve to find whatever unknown quantity is required.

EVALUATE *your answer:* Check whether your answer makes physical sense. Keep in mind, here and in later sections, that the work done by each force must be represented either in $U_{grav,1} - U_{grav,2} = -\Delta U_{grav}$ or as W_{other}, but *never* in both places. The gravitational work is included in ΔU_{grav}, so make sure you did not include it again in W_{other}.

Example 7.2 Work and energy in throwing a baseball

In Example 7.1, suppose your hand moves up 0.50 m while you are throwing the ball, which leaves your hand with an upward velocity of 20.0 m/s. Again ignore air resistance. (a) Assuming that your hand exerts a constant upward force on the ball, find the magnitude of that force. (b) Find the speed of the ball at a point 15.0 m above the point where it leaves your hand.

SOLUTION

IDENTIFY: In Example 7.1 we used conservation of mechanical energy because only gravity did work. In this example, however, we must also include the nongravitational work done by your hand.

SET UP: Figure 7.6 shows a diagram of the situation, including a free-body diagram for the ball while it is being thrown. We let point 1 be where your hand first starts to move, point 2 be where the ball leaves your hand, and point 3 be where the ball is 15.0 m above point 2. The nongravitational force $\vec{F}$ of your hand acts only between points 1 and 2. Using the same coordinate system as in Example 7.1, we have $y_1 = -0.50$ m, $y_2 = 0$, and $y_3 = 15.0$ m. The ball starts at rest at point 1, so $v_1 = 0$, and we are given that the ball's speed as it leaves your hand is $v_2 = 20.0$ m/s. Our target variables are (a) the magnitude F of the force of your hand and (b) the speed v_3 at point 3.

EXECUTE: (a) To determine the magnitude of $\vec{F}$, we'll first use Eq. (7.7) to calculate the work W_{other} done by this force. We have

$$K_1 = 0$$

$$U_{grav,1} = mgy_1 = (0.145\text{ kg})(9.80\text{ m/s}^2)(-0.50\text{ m}) = -0.71\text{ J}$$

$$K_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(0.145\text{ kg})(20.0\text{ m/s})^2 = 29.0\text{ J}$$

$$U_{grav,2} = mgy_2 = (0.145\text{ kg})(9.80\text{ m/s}^2)(0) = 0$$

The initial potential energy $U_{grav,1}$ is *negative* because the ball was initially below the origin. (Don't worry about having a potential energy that's less than zero. Remember, all that matters is the *difference* in potential energy from one point to another.) According to Eq. (7.7), $K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2}$, so

$$W_{other} = (K_2 - K_1) + (U_{grav,2} - U_{grav,1}) = (29.0\text{ J} - 0) + (0 - (-0.71\text{ J})) = 29.7\text{ J}$$

The kinetic energy of the ball increases by $K_2 - K_1 = 29.0$ J, and the potential energy increases by $U_{grav,2} - U_{grav,1} = 0.71$ J; the sum is $E_2 - E_1$, the change in total mechanical energy, which is equal to W_{other}.

Assuming the upward force $\vec{F}$ that your hand applies is constant, the work W_{other} done by this force is equal to the magnitude F of the force multiplied by the upward displacement $y_2 - y_1$ over which it acts:

$$W_{other} = F(y_2 - y_1)$$

$$F = \frac{W_{other}}{y_2 - y_1} = \frac{29.7\text{ J}}{0.50\text{ m}} = 59\text{ N}$$

This is about 40 times greater than the weight of the ball.

(b) To find the speed at point 3, note that between points 2 and 3, total mechanical energy is conserved; the force of your hand no longer acts, so $W_{other} = 0$. We can then find the kinetic energy at point 3 using Eq. (7.4):

$$K_2 + U_{grav,2} = K_3 + U_{grav,3}$$

$$U_{grav,3} = mgy_3 = (0.145\text{ kg})(9.80\text{ m/s}^2)(15.0\text{ m}) = 21.3\text{ J}$$

$$K_3 = (K_2 + U_{grav,2}) - U_{grav,3}$$

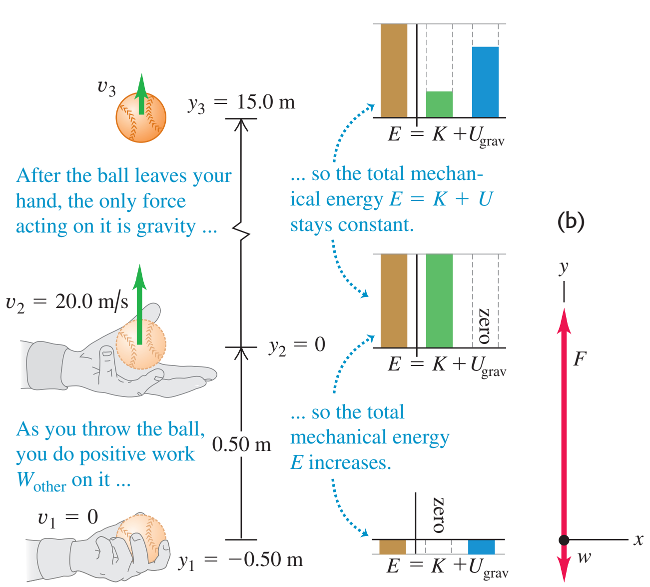
$$= (29.0\text{ J} + 0\text{ J}) - 21.3\text{ J} = 7.7\text{ J}$$

Since $K_3 = \frac{1}{2}mv_{3y}^2$, where v_{3y} is the y-component of the ball's velocity at point 3, we have

$$v_{3y} = \pm\sqrt{\frac{2K_3}{m}} = \pm\sqrt{\frac{2(7.7\text{ J})}{0.145\text{ kg}}} = \pm 10\text{ m/s}$$

The significance of the plus-or-minus sign is that the ball passes point 3 *twice,* once on the way up and again on the way down. The total mechanical energy E is constant and equal to 29.0 J while the ball is in free fall, and the potential energy at point 3 is $U_{grav,3} = 21.3$ J whether the ball is moving up or down. So at point 3, the ball's kinetic energy K_3 and *speed* don't depend on the direction the ball is moving. The velocity v_{3y} is positive (+10 m/s) when the ball is moving up and negative (−10 m/s) when it is moving down; the speed v_3 is 10 m/s in either case.

EVALUATE: As a check on our result, recall from Example 7.1 that the ball reaches a maximum height $y = 20.4$ m. At that point all of the kinetic energy that the ball had when it left your hand at $y = 0$ has been converted to gravitational potential energy. At $y = 15.0$ m, the ball is about three-fourths of the way to its maximum height, so about three-fourths of its mechanical energy should be in the form of potential energy. (This is shown in the energy bar graphs in Fig. 7.6a.) Can you show that this is true from our results for K_3 and $U_{grav,3}$?



7.6 (a) Applying energy ideas to a ball thrown vertically upward. (b) Free-body diagram for the ball as you throw it.

Gravitational Potential Energy for Motion Along a Curved Path

In our first two examples the body moved along a straight vertical line. What happens when the path is slanted or curved (Fig. 7.7a)? The body is acted on by the gravitational force $\vec{w} = m\vec{g}$ and possibly by other forces whose resultant we call $\vec{F}_{other}$. To find the work done by the gravitational force during this displacement, we divide the path into small segments $\Delta\vec{s}$; Fig. 7.7b shows a typical segment. The work done by the gravitational force over this segment is the scalar product of the force and the displacement. In terms of unit vectors, the force is $\vec{w} = m\vec{g} = -mg\hat{j}$ and the displacement is $\Delta\vec{s} = \Delta x\hat{\imath} + \Delta y\hat{j}$, so the work done by the gravitational force is

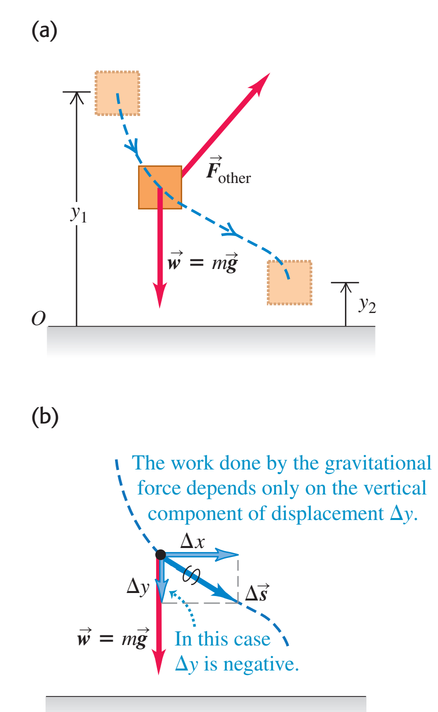
$$\vec{w}\cdot\Delta\vec{s} = -mg\hat{j}\cdot(\Delta x\hat{\imath} + \Delta y\hat{j}) = -mg\Delta y$$

The work done by gravity is the same as though the body had been displaced vertically a distance Δy, with no horizontal displacement. This is true for every segment, so the *total* work done by the gravitational force is $-mg$ multiplied by the *total* vertical displacement $(y_2 - y_1)$:

$$W_{grav} = -mg(y_2 - y_1) = mgy_1 - mgy_2 = U_{grav,1} - U_{grav,2}$$

This is the same as Eq. (7.1) or (7.3), in which we assumed a purely vertical path. So even if the path a body follows between two points is curved, the total work done by the gravitational force depends only on the difference in height between the two points of the path. This work is unaffected by any horizontal motion that may occur. So *we can use the same expression for gravitational potential energy whether the body's path is curved or straight.*

7.7 Calculating the change in gravitational potential energy for a displacement along a curved path.



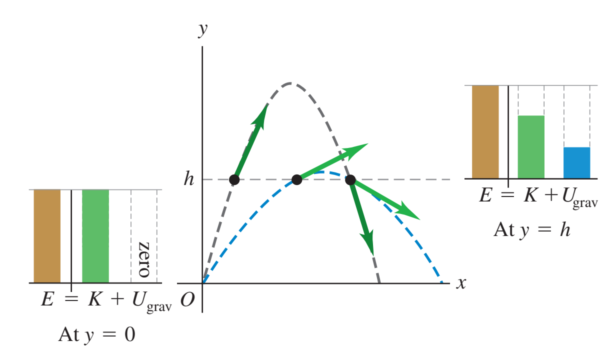
Conceptual Example 7.3 Energy in projectile motion

A batter hits two identical baseballs with the same initial speed and height but different initial angles. Prove that at a given height h, both balls have the same speed if air resistance can be neglected.

SOLUTION

If there is no air resistance, the only force acting on each ball after it is hit is its weight. Hence the total mechanical energy for each ball is constant. Figure 7.8 shows the trajectories of two balls batted at the same height with the same initial speed, and thus the same total mechanical energy, but with different initial angles. At all points at the same height the potential energy is the same. Thus the kinetic energy at this height must be the same for both balls, and the speeds are the same.

7.8 For the same initial speed and initial height, the speed of a projectile at a given elevation h is always the same, neglecting air resistance.



Example 7.4 Calculating speed along a vertical circle

Your cousin Throckmorton skateboards down a curved playground ramp. If we treat Throcky and his skateboard as a particle, he moves through a quarter-circle with radius $R = 3.00$ m (Fig. 7.9). The total mass of Throcky and his skateboard is 25.0 kg. He starts from rest and there is no friction. (a) Find his speed at the bottom of the ramp. (b) Find the normal force that acts on him at the bottom of the curve.

SOLUTION

IDENTIFY: We can't use the constant-acceleration equations because Throcky's acceleration isn't constant; the slope decreases as he descends. Instead, we'll use the energy approach. Since Throcky moves along a circular arc, we'll also use what we learned about circular motion in Section 5.4.

SET UP: Since there is no friction, the only force other than Throcky's weight is the normal force $\vec{n}$ exerted by the ramp (Fig. 7.9b). Although this force acts all along the path, it does *zero* work because $\vec{n}$ is perpendicular to Throcky's displacement at every point. Hence $W_{other} = 0$ and mechanical energy is conserved.

We take point 1 at the starting point and point 2 at the bottom of the curved ramp, and we let $y = 0$ be at the bottom of the ramp (Fig. 7.9a). Then $y_1 = R$ and $y_2 = 0$. (We are treating Throcky as if his entire mass were concentrated at his center.) Throcky starts at rest at the top, so $v_1 = 0$. Our target variable in part (a) is his speed at the bottom, v_2. In part (b) we want to find the magnitude n of the normal force at point 2. Because this force does no work, it doesn't appear in the energy equation, so we'll use Newton's second law instead.

EXECUTE: (a) The various energy quantities are

$$K_1 = 0 \qquad U_{grav,1} = mgR$$
$$K_2 = \frac{1}{2}mv_2^2 \qquad U_{grav,2} = 0$$

From conservation of mechanical energy,

$$K_1 + U_{grav,1} = K_2 + U_{grav,2}$$
$$0 + mgR = \frac{1}{2}mv_2^2 + 0$$
$$v_2 = \sqrt{2gR}$$
$$= \sqrt{2(9.80\ \text{m/s}^2)(3.00\ \text{m})} = 7.67\ \text{m/s}$$

Notice that this answer doesn't depend on the ramp being circular; no matter what the shape of the ramp, Throcky will have the same speed $v_2 = \sqrt{2gR}$ at the bottom. This would be true even if the wheels of his skateboard lost contact with the ramp during the ride, because only the gravitational force would still do work. In fact, the speed is the same as if Throcky had fallen vertically through a height R. The answer is also independent of his mass.

(b) To find n at point 2 using Newton's second law, we need the free-body diagram at that point (Fig. 7.9b). At point 2, Throcky is moving at speed $v_2 = \sqrt{2gR}$ in a circle of radius R; his acceleration is toward the center of the circle and has magnitude

$$a_{rad} = \frac{v_2^2}{R} = \frac{2gR}{R} = 2g$$

If we take the positive y-direction to be upward, the y-component of Newton's second law is

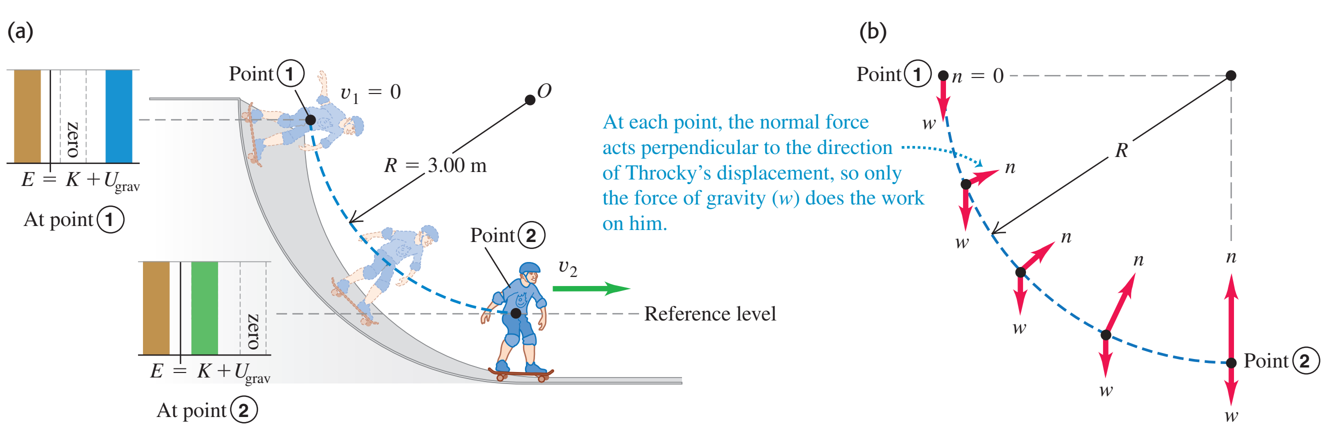
$$\sum F_y = n + (-w) = ma_{rad} = 2mg$$
$$n = w + 2mg = 3mg$$
$$= 3(25.0\ \text{kg})(9.80\ \text{m/s}^2) = 735\ \text{N}$$

At point 2 the normal force is three times Throcky's weight. This result is independent of the radius of the circular ramp. We learned in Example 5.9 (Section 5.2) and Example 5.24 (Section 5.4) that the magnitude of n is the *apparent weight,* so Throcky feels as though he weighs three times his true weight mg. But as soon as he reaches the horizontal part of the ramp to the right of point 2, the normal force decreases to $w = mg$ and Throcky feels normal again. Can you see why?

EVALUATE: This example shows a general rule about the role of forces in problems in which we use energy techniques: What matters is not simply whether a force *acts,* but whether that force *does work.* If the force does no work, like the normal force $\vec{n}$ in this example, then it does not appear at all in Eq. (7.7), $K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2}$.

Notice we had to use *both* the energy approach and Newton's second law to solve this problem; energy conservation gave us the speed and $\sum \vec{F} = m\vec{a}$ gave us the normal force. For each part of the problem we used the technique that most easily led us to the answer.

7.9 (a) Throcky skateboarding down a frictionless circular ramp. The total mechanical energy is constant. (b) Free-body diagrams for Throcky and his skateboard at various points on the ramp.



Example 7.5 A vertical circle with friction

In Example 7.4, suppose that the ramp is not frictionless and that Throcky's speed at the bottom is only 6.00 m/s. What work was done by the friction force acting on him?

SOLUTION

IDENTIFY: Figure 7.10 shows that again the normal force does no work, but now there is a friction force $\vec{f}$ that *does* do work. Hence the nongravitational work done on Throcky between points 1 and 2, W_{other}, is not zero.

SET UP: We use the same coordinate system and the same initial and final points as in Example 7.4 (see Fig. 7.10). Our target variable is the work done by friction, W_f; since friction is the only force other than gravity that does work, this is just equal to W_{other}. We'll find W_f using Eq. (7.7).

EXECUTE: The energy quantities are

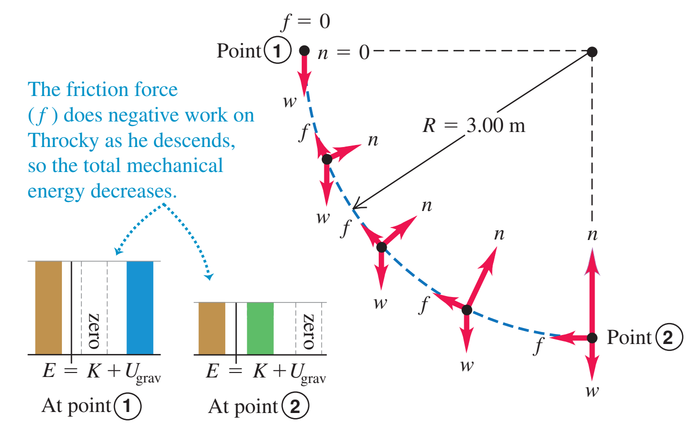
$$K_1 = 0$$
$$U_{grav,1} = mgR = (25.0\ \text{kg})(9.80\ \text{m/s}^2)(3.00\ \text{m}) = 735\ \text{J}$$
$$K_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(25.0\ \text{kg})(6.00\ \text{m/s})^2 = 450\ \text{J}$$
$$U_{grav,2} = 0$$

From Eq. (7.7),

$$W_f = K_2 + U_{grav,2} - K_1 - U_{grav,1}$$
$$= 450\ \text{J} + 0 - 0 - 735\ \text{J} = -285\ \text{J}$$

The work done by the friction force is -285 J, and the total mechanical energy *decreases* by 285 J. Do you see why W_f has to be negative?

7.10 Free-body diagram and energy bar graphs for Throcky skateboarding down a ramp with friction.



EVALUATE: Throcky's motion is determined by Newton's second law, $\sum \vec{F} = m\vec{a}$. But it would be very difficult to apply the second law directly to this problem because the normal and friction forces and the acceleration are continuously changing in both magnitude and direction as Throcky moves. The energy approach, by contrast, relates the motions at the top and bottom of the ramp without involving the details of what happens in between. Many problems are easy if energy considerations are used but very complex if we try to use Newton's laws directly.

Example 7.6 An inclined plane with friction

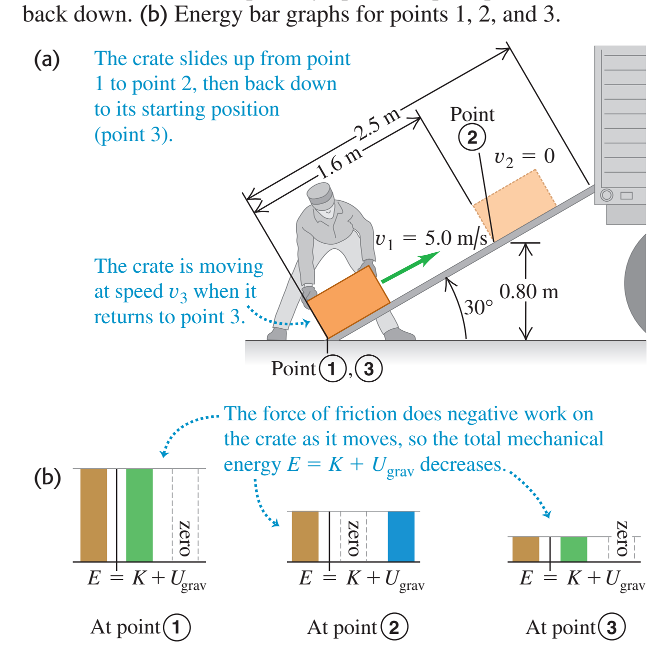
We want to load a 12-kg crate into a truck by sliding it up a ramp 2.5 m long, inclined at 30°. A worker, giving no thought to friction, calculates that he can get the crate up the ramp by giving it an initial speed of 5.0 m/s at the bottom and letting it go. But friction is *not* negligible; the crate slides 1.6 m up the ramp, stops, and slides back down (Fig. 7.11). (a) Assuming that the friction force acting on the crate is constant, find its magnitude. (b) How fast is the crate moving when it reaches the bottom of the ramp?

SOLUTION

IDENTIFY: The friction force does work on the crate as it slides. As in Example 7.2, we'll use the energy approach in part (a) to find the magnitude of the nongravitational force that does work (in this case, friction). In part (b) we'll calculate how much nongravitational work this force does as the crate slides back down and then use the energy approach to find the crate's speed at the bottom of the ramp.

SET UP: The first part of the motion is from point 1, at the bottom of the ramp, to point 2, where the crate stops instantaneously. In the second part of the motion, the crate returns to the bottom of the ramp, which we'll also call point 3 (Fig. 7.11a). We take $y = 0$ (and hence $U_{grav} = 0$) to be at ground level, so $y_1 = 0$,

7.11 (a) A crate slides partway up the ramp, stops, and slides back down. (b) Energy bar graphs for points 1, 2, and 3.



Continued

$y_2 = (1.6\text{ m})\sin 30° = 0.80\text{ m}$, and $y_3 = 0$. We are given that $v_1 = 5.0\text{ m/s}$ and $v_2 = 0$ (the crate is instantaneously at rest at point 2). Our target variable in part (a) is f, the magnitude of the friction force. In part (b) our target variable is v_3, the speed at the bottom of the ramp.

EXECUTE: (a) The energy quantities are

$$K_1 = \frac{1}{2}(12\text{ kg})(5.0\text{ m/s})^2 = 150\text{ J}$$
$$U_{grav,1} = 0$$
$$K_2 = 0$$
$$U_{grav,2} = (12\text{ kg})(9.8\text{ m/s}^2)(0.80\text{ m}) = 94\text{ J}$$
$$W_{other} = -fs$$

Here f is the unknown magnitude of the friction force and $s = 1.6$ m. Using Eq. (7.7), we find

$$K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2}$$
$$W_{other} = -fs = (K_2 + U_{grav,2}) - (K_1 + U_{grav,1})$$
$$f = -\frac{(K_2 + U_{grav,2}) - (K_1 + U_{grav,1})}{s}$$
$$= -\frac{(0 + 94\text{ J}) - (150\text{ J} + 0)}{1.6\text{ m}} = 35\text{ N}$$

The friction force of 35 N, acting over 1.6 m, causes the mechanical energy of the crate to decrease from 150 J to 94 J (Fig. 7.11b).

(b) On the way down from point 2 to point 3 at the bottom of the ramp, the friction force and the displacement both reverse direction but have the same magnitudes, so the frictional work has the same negative value as from point 1 to point 2. The total work done by friction between points 1 and 3 is

$$W_{other} = W_{fric} = -2fs = -2(35\text{ N})(1.6\text{ m}) = -112\text{ J}$$

From part (a), $K_1 = 150$ J and $U_{grav,1} = 0$. Equation (7.7) then gives

$$K_1 + U_{grav,1} + W_{other} = K_3 + U_{grav,3}$$
$$K_3 = K_1 + U_{grav,1} - U_{grav,3} + W_{other}$$
$$= 150\text{ J} + 0 - 0 + (-112\text{ J}) = 38\text{ J}$$

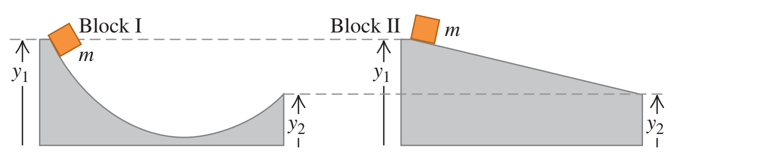
The crate returns to the bottom of the ramp with only 38 J of the original 150 J of mechanical energy (Fig. 7.11b). Using $K_3 = \frac{1}{2}mv_3^2$, we get

$$v_3 = \sqrt{\frac{2K_3}{m}} = \sqrt{\frac{2(38\text{ J})}{12\text{ kg}}} = 2.5\text{ m/s}$$

EVALUATE: The crate's speed when it returns to the bottom of the ramp, $v_3 = 2.5$ m/s, is less than the speed $v_1 = 5.0$ m/s at which it left that point. That's good—energy was lost due to friction.

In part (b) we applied Eq. (7.7) to points 1 and 3, considering the entire round trip as a whole. Alternatively, we could have considered the second part of the motion by itself and applied Eq. (7.7) to points 2 and 3. Try it and see whether you get the same result for v_3.

Test Your Understanding of Section 7.1 The figure shows two different frictionless ramps. The heights y_1 and y_2 are the same for both ramps. If a block of mass m is released from rest at the left-hand end of each ramp, which block arrives at the right-hand end with the greater speed? (i) block I; (ii) block II; (iii) the speed is the same for both blocks.



7.12 The Achilles tendon, which runs along the back of the ankle to the heel bone, acts like a natural spring. When it stretches and then relaxes, this tendon stores and then releases elastic potential energy. This spring action reduces the amount of work your leg muscles must do as you run.

7.2 Elastic Potential Energy

There are many situations in which we encounter potential energy that is not gravitational in nature. One example is a rubber-band slingshot. Work is done on the rubber band by the force that stretches it, and that work is stored in the rubber band until you let it go. Then the rubber band gives kinetic energy to the projectile.

This is the same pattern we saw with the pile driver in Section 7.1: Do work on the system to store energy, which can later be converted to kinetic energy. We'll describe the process of storing energy in a deformable body such as a spring or rubber band in terms of *elastic potential energy* (Fig. 7.12). A body is called *elastic* if it returns to its original shape and size after being deformed.

To be specific, we'll consider storing energy in an ideal spring, like the ones we discussed in Section 6.3. To keep such an ideal spring stretched by a distance x, we must exert a force $F = kx$, where k is the force constant of the spring. The ideal spring is a useful idealization because many elastic bodies show this same direct proportionality between force $\vec{F}$ and displacement x, provided that x is sufficiently small.

We proceed just as we did for gravitational potential energy. We begin with the work done by the elastic (spring) force and then combine this with the work–energy theorem. The difference is that gravitational potential energy is a shared property of a body and the earth, but elastic potential energy is stored just in the spring (or other deformable body).

Figure 7.13 shows the ideal spring from Fig. 6.18, with its left end held stationary and its right end attached to a block with mass m that can move along the x-axis. In Fig. 7.13a the body is at $x = 0$ when the spring is neither stretched nor compressed. We move the block to one side, thereby stretching or compressing the spring, and then let it go. As the block moves from one position x_1 to another position x_2, how much work does the elastic (spring) force do on the block?

We found in Section 6.3 that the work we must do *on* the spring to move one end from an elongation x_1 to a different elongation x_2 is

$$W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2 \qquad \text{(work done } on \text{ a spring)}$$

where k is the force constant of the spring. If we stretch the spring farther, we do positive work on the spring; if we let the spring relax while holding one end, we do negative work on it. We also saw that this expression for work is still correct if the spring is compressed, not stretched, so that x_1 or x_2 or both are negative. Now we need to find the work done *by* the spring. From Newton's third law the two quantities of work are just negatives of each other. Changing the signs in this equation, we find that in a displacement from x_1 to x_2 the spring does an amount of work W_{el} given by

$$W_{el} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \qquad \text{(work done } by \text{ a spring)}$$

The subscript "el" stands for *elastic.* When x_1 and x_2 are both positive and $x_2 > x_1$ (Fig. 7.13b), the spring does negative work on the block, which moves in the $+x$-direction while the spring pulls on it in the $-x$-direction. The spring stretches farther, and the block slows down. When x_1 and x_2 are both positive and $x_2 < x_1$ (Fig. 7.13c), the spring does positive work as it relaxes and the block speeds up. If the spring can be compressed as well as stretched, x_1 or x_2 or both may be negative, but the expression for W_{el} is still valid. In Fig. 7.13d, both x_1 and x_2 are negative, but x_2 is less negative than x_1; the compressed spring does positive work as it relaxes, speeding the block up.

Just as for gravitational work, we can express the work done by the spring in terms of a given quantity at the beginning and end of the displacement. This quantity is $\frac{1}{2}kx^2$, and we define it to be the **elastic potential energy:**

$$U_{el} = \frac{1}{2}kx^2 \qquad \text{(elastic potential energy)} \qquad (7.9)$$

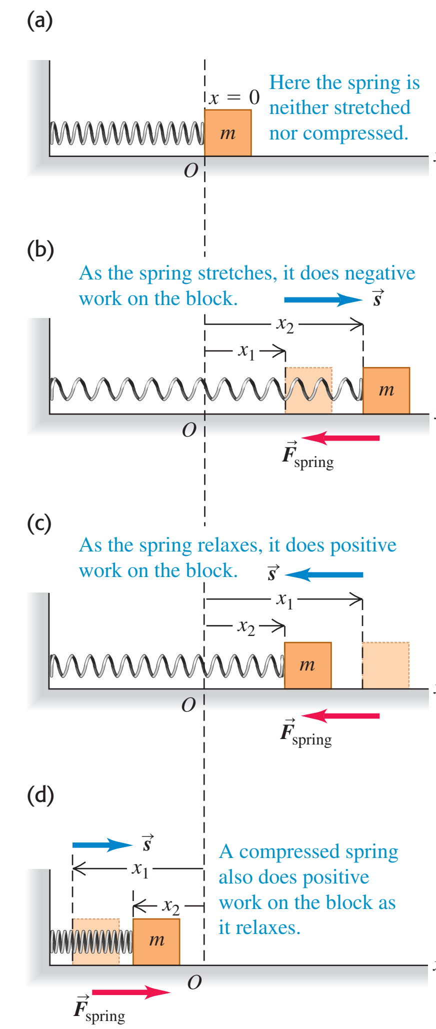
Figure 7.14 is a graph of Eq. (7.9). The unit of U_{el} is the joule (J), the unit used for *all* energy and work quantities; to see this from Eq. (7.9), recall that the units of k are N/m and that $1\text{ N}\cdot\text{m} = 1\text{ J}$.

We can use Eq. (7.9) to express the work W_{el} done on the block by the elastic force in terms of the change in elastic potential energy:

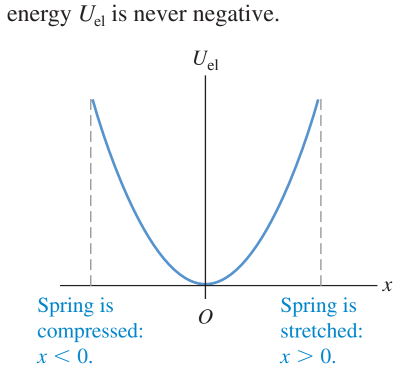
$$W_{el} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 = U_{el,1} - U_{el,2} = -\Delta U_{el} \qquad (7.10)$$

When a stretched spring is stretched farther, as in Fig. 7.13b, W_{el} is negative and U_{el} *increases;* a greater amount of elastic potential energy is stored in the spring. When a stretched spring relaxes, as in Fig. 7.13c, x decreases, W_{el} is positive, and U_{el} *decreases;* the spring loses elastic potential energy. Negative values of x refer

7.13 Calculating the work done by a spring attached to a block on a horizontal surface. The quantity x is the extension or compression of the spring.



7.14 The graph of elastic potential energy for an ideal spring is a parabola: $U_{el} = \frac{1}{2}kx^2$, where x is the extension or compression of the spring. Elastic potential energy U_{el} is never negative.



to a compressed spring. But, as Fig. 7.14 shows, U_{el} is positive for both positive and negative x, and Eqs. (7.9) and (7.10) are valid for both cases. The more a spring is compressed *or* stretched, the greater its elastic potential energy.

CAUTION **Gravitational potential energy vs. elastic potential energy** An important difference between gravitational potential energy $U_{grav} = mgy$ and elastic potential energy $U_{el} = \frac{1}{2}kx^2$ is that we do *not* have the freedom to choose $x = 0$ to be wherever we wish. To be consistent with Eq. (7.9), $x = 0$ *must* be the position at which the spring is neither stretched nor compressed. At that position, its elastic potential energy and the force that it exerts are both zero.

The work–energy theorem says that $W_{tot} = K_2 - K_1$, no matter what kind of forces are acting on a body. If the elastic force is the *only* force that does work on the body, then

$$W_{tot} = W_{el} = U_{el,1} - U_{el,2}$$

The work–energy theorem $W_{tot} = K_2 - K_1$ then gives us

$$K_1 + U_{el,1} = K_2 + U_{el,2} \quad \text{(if only the elastic force does work)} \tag{7.11}$$

Here U_{el} is given by Eq. (7.9), so

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2 \quad \text{(if only the elastic force does work)} \tag{7.12}$$

In this case the total mechanical energy $E = K + U_{el}$—the sum of kinetic and *elastic* potential energy—is *conserved.* An example of this is the motion of the block in Fig. 7.13, provided the horizontal surface is frictionless so that no force does work other than that exerted by the spring.

For Eq. (7.12) to be strictly correct, the ideal spring that we've been discussing must also be *massless.* If the spring has a mass, it also has kinetic energy as the coils of the spring move back and forth. We can neglect the kinetic energy of the spring if its mass is much less than the mass m of the body attached to the spring. For instance, a typical automobile has a mass of 1200 kg or more. The springs in its suspension have masses of only a few kilograms, so their mass can be neglected if we want to study how a car bounces on its suspension.

Situations with Both Gravitational and Elastic Potential Energy

ActivPhysics ONLINE
5.4 Inverse Bungee Jumper
5.5 Spring-Launched Bowler

Equations (7.11) and (7.12) are valid when the only potential energy in the system is elastic potential energy. What happens when we have *both* gravitational and elastic forces, such as a block attached to the lower end of a vertically hanging spring? And what if work is also done by other forces that *cannot* be described in terms of potential energy, such as the force of air resistance on a moving block? Then the total work is the sum of the work done by the gravitational force (W_{grav}), the work done by the elastic force (W_{el}), and the work done by other forces (W_{other}): $W_{tot} = W_{grav} + W_{el} + W_{other}$. Then the work–energy theorem gives

$$W_{grav} + W_{el} + W_{other} = K_2 - K_1$$

The work done by the gravitational force is $W_{grav} = U_{grav,1} - U_{grav,2}$ and the work done by the spring is $W_{el} = U_{el,1} - U_{el,2}$. Hence we can rewrite the work–energy theorem for this most general case as

$$K_1 + U_{grav,1} + U_{el,1} + W_{other} = K_2 + U_{grav,2} + U_{el,2} \quad \text{(valid in general)} \tag{7.13}$$

or, equivalently,

$$K_1 + U_1 + W_{other} = K_2 + U_2 \quad \text{(valid in general)} \tag{7.14}$$

where $U = U_{grav} + U_{el} = mgy + \frac{1}{2}kx^2$ is the *sum* of gravitational potential energy and elastic potential energy. For short, we call U simply "the potential energy."

Equation (7.14) is *the most general statement* of the relationship among kinetic energy, potential energy, and work done by other forces. It says:

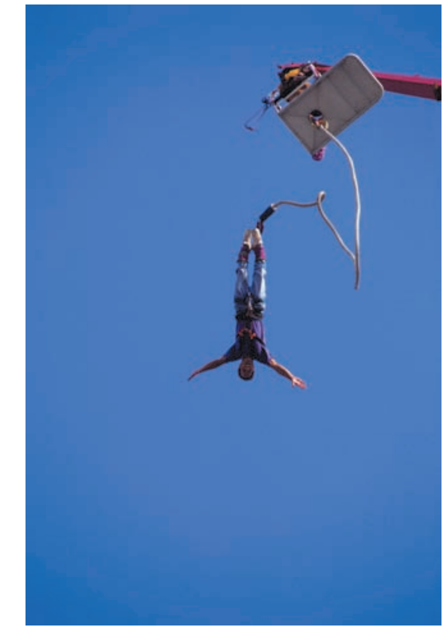
The work done by all forces other than the gravitational force or elastic force equals the change in the total mechanical energy $E = K + U$ of the system, where $U = U_{grav} + U_{el}$ is the sum of the gravitational potential energy and the elastic potential energy.

The "system" is made up of the body of mass m, the earth with which it interacts through the gravitational force, and the spring of force constant k.

If W_{other} is positive, $E = K + U$ increases; if W_{other} is negative, E decreases. If the gravitational and elastic forces are the *only* forces that do work on the body, then $W_{other} = 0$ and the total mechanical energy (including both gravitational and elastic potential energy) is conserved. (You should compare Eq. (7.14) to Eqs. (7.7) and (7.8), which describe situations in which there is gravitational potential energy but no elastic potential energy.)

Bungee jumping (Fig. 7.15) is an example of transformations among kinetic energy, elastic potential energy, and gravitational potential energy. As the jumper falls, gravitational potential energy decreases and is converted into the kinetic energy of the jumper and the elastic potential energy of the bungee cord. Beyond a certain point in the fall, the jumper's speed decreases so that both gravitational potential energy and kinetic energy are converted into elastic potential energy.

7.15 The fall of a bungee jumper involves an interplay among kinetic energy, gravitational potential energy, and elastic potential energy. Due to air resistance and frictional forces within the bungee cord, mechanical energy is not conserved. (If mechanical energy *were* conserved, the bungee jumper would keep bouncing up and down forever!)



Problem-Solving Strategy 7.2 Problems Using Mechanical Energy II

Problem-Solving Strategy 7.1 (Section 7.1) is equally useful in solving problems that involve elastic forces as well as gravitational forces. The only new wrinkle is that the potential energy U now includes the elastic potential energy $U_{el} = \frac{1}{2}kx^2$, where x is the displacement of the spring *from its unstretched length.* The work done by the gravitational and elastic forces is accounted for by their potential energies; the work of the other forces, W_{other}, has to be included separately.

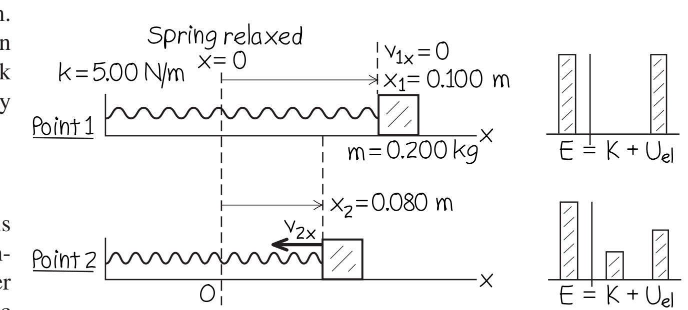
Example 7.7 Motion with elastic potential energy

A glider with mass $m = 0.200$ kg sits on a frictionless horizontal air track, connected to a spring with force constant $k = 5.00$ N/m. You pull on the glider, stretching the spring 0.100 m, and then release it with no initial velocity. The glider begins to move back toward its equilibrium position ($x = 0$). What is its x-velocity when $x = 0.080$ m?

SOLUTION

IDENTIFY: Because the spring force varies with position, this problem can't be solved with the equations for motion with constant acceleration. Instead, we'll use the idea that as the glider starts to move, elastic potential energy is converted into kinetic energy. (The glider remains at the same height throughout the motion, so gravitational potential energy is not a factor. Hence $U = U_{el} = \frac{1}{2}kx^2$.)

7.16 Our sketches and energy bar graphs for this problem.



SET UP: Figure 7.16 shows our sketches. The spring force is the only force doing work on the glider, so $W_{other} = 0$ and we may use

Continued

Eq. (7.11). We designate the point where the glider is released as point 1 and $x = 0.080$ m as point 2. We know the velocity at point 1 ($v_{1x} = 0$); our target variable is the x-velocity at point 2, v_{2x}.

EXECUTE: The energy quantities are

$$K_1 = \frac{1}{2}mv_{1x}^2 = \frac{1}{2}(0.200\text{ kg})(0)^2 = 0$$

$$U_1 = \frac{1}{2}kx_1^2 = \frac{1}{2}(5.00\text{ n/m})(0.100\text{ m})^2 = 0.0250\text{ J}$$

$$K_2 = \frac{1}{2}mv_{2x}^2$$

$$U_2 = \frac{1}{2}kx_2^2 = \frac{1}{2}(5.00\text{ N/m})(0.080\text{ m})^2 = 0.0160\text{ J}$$

Then from Eq. (7.11),

$$K_2 = K_1 + U_1 - U_2 = 0 + 0.0250\text{ J} - 0.0160\text{ J} = 0.0090\text{ J}$$

$$v_{2x} = \pm\sqrt{\frac{2K_2}{m}} = \pm\sqrt{\frac{2(0.0090\text{ J})}{0.200\text{ kg}}} = \pm 0.30\text{ m/s}$$

We choose the negative root because the glider is moving in the $-x$-direction; the answer we want is $v_{2x} = -0.30$ m/s.

EVALUATE: What is the meaning of the second solution, $v_{2x} = +0.30$ m/s? Eventually the spring will compress and push the glider back to the right in the positive x-direction (see Fig. 7.13d). The second solution tells us that when the glider passes through $x = 0.080$ m while moving to the right, its speed will be 0.30 m/s—the same speed as when it passed through this point while moving to the left.

When the glider passes through the point $x = 0$, the spring is relaxed and all of the mechanical energy is in the form of kinetic energy. Can you show that the speed of the glider at this point is 0.50 m/s?

Example 7.8 Motion with elastic potential energy and work done by other forces

For the system of Example 7.7, suppose the glider is initially at rest at $x = 0$, with the spring unstretched. You then apply a constant force $\vec{F}$ in the $+x$-direction with magnitude 0.610 N to the glider. What is the glider's velocity when it has moved to $x = 0.100$ m?

SOLUTION

IDENTIFY: Although the force $\vec{F}$ you apply is constant, the spring force isn't, so the acceleration of the glider won't be constant. Total mechanical energy is not conserved because of the work done by the force $\vec{F}$, so we must use the generalized energy relationship given by Eq. (7.13). (As in Example 7.7, we ignore gravitational potential energy because the glider's height doesn't change. Hence we have only elastic potential energy, and so $U = U_{el} = \frac{1}{2}kx^2$.)

SET UP: Let point 1 be at $x = 0$, where the velocity is $v_{1x} = 0$, and let point 2 be at $x = 0.100$ m. (These points are different from the ones labeled in Fig. 7.16.) Our target variable is v_{2x}, the velocity at point 2.

EXECUTE: The energy quantities are

$$K_1 = 0$$

$$U_1 = \frac{1}{2}kx_1^2 = 0$$

$$K_2 = \frac{1}{2}mv_{2x}^2$$

$$U_2 = \frac{1}{2}kx_2^2 = \frac{1}{2}(5.00\text{ N/m})(0.100\text{ m})^2 = 0.0250\text{ J}$$

$$W_{other} = (0.610\text{ N})(0.100\text{ m}) = 0.0610\text{ J}$$

(To calculate W_{other} we multiplied the magnitude of the force by the displacement, since both are in the $+x$-direction.) Initially, the total mechanical energy is zero; the work done by the force $\vec{F}$ increases the total mechanical energy to 0.0610 J, of which 0.0250 J is elastic potential energy. The remainder is kinetic energy. From Eq. (7.13),

$$K_1 + U_1 + W_{other} = K_2 + U_2$$

$$K_2 = K_1 + U_1 + W_{other} - U_2$$

$$= 0 + 0 + 0.0610\text{ J} - 0.0250\text{ J} = 0.0360\text{ J}$$

$$v_{2x} = \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2(0.0360\text{ J})}{0.200\text{ kg}}} = 0.60\text{ m/s}$$

We choose the positive square root because the glider is moving in the $+x$-direction.

EVALUATE: To test our answer, think what would be different if we disconnected the glider from the spring. Then $\vec{F}$ would be the only force doing work, there would be zero potential energy at all times, and Eq. (7.13) would give us

$$K_2 = K_1 + W_{other} = 0 + 0.0610\text{ J}$$

$$v_{2x} = \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2(0.0610\text{ J})}{0.200\text{ kg}}} = 0.78\text{ m/s}$$

We found a lower velocity than this value because the spring does negative work on the glider as it stretches (see Fig. 7.13b).

If you stop pushing on the glider when it reaches the point $x = 0.100$ m, beyond that point the only force that does work on the glider is the spring force. Hence for $x > 0.100$ m, the total mechanical energy $E = K + U$ is conserved and maintains the same value of 0.0610 J. The glider will slow down as the spring continues to stretch, so the kinetic energy K will decrease as the potential energy increases. The glider will come to rest at a point $x = x_3$; at this point the kinetic energy is zero and the potential energy $U = U_{el} = \frac{1}{2}kx_3^2$ is equal to the total mechanical energy 0.0610 J. You should be able to show that the glider comes to rest at $x_3 = 0.156$ m, which means that it moves an additional 0.056 m after the force $\vec{F}$ is removed at $x_2 = 0.100$ m. (Since there's no friction, the glider will not remain at rest but will start moving back toward $x = 0$ due to the force of the stretched spring.)

Example 7.9 Motion with gravitational, elastic, and friction forces

In a "worst-case" design scenario, a 2000-kg elevator with broken cables is falling at 4.00 m/s when it first contacts a cushioning spring at the bottom of the shaft. The spring is supposed to stop the elevator, compressing 2.00 m as it does so (Fig. 7.17). During the motion a safety clamp applies a constant 17,000-N frictional force to the elevator. As a design consultant, you are asked to determine what the force constant of the spring should be.

SOLUTION

IDENTIFY: We'll use the energy approach to determine the force constant, which appears in the expression for elastic potential energy. Note that this problem involves *both* gravitational and elastic potential energy. Furthermore, total mechanical energy is not conserved because the friction force does negative work W_{other} on the elevator.

SET UP: Since mechanical energy isn't conserved and more than one kind of potential energy is involved, we'll use the most general form of the energy relationship, Eq. (7.13). We take point 1 as the position of the bottom of the elevator when it initially contacts the spring, and take point 2 as its position when it is at rest. We choose the origin to be at point 1, so $y_1 = 0$ and $y_2 = -2.00$ m. With this choice the coordinate of the upper end of the spring is the same as the coordinate of the elevator, so the elastic potential energy at any point between point 1 and point 2 is $U_{el} = \frac{1}{2}ky^2$. (The gravitational potential energy is $U_{grav} = mgy$ as usual.) We know the initial and final speeds of the elevator and the magnitude of the friction force, so the only unknown is the force constant k (our target variable).

EXECUTE: The elevator's initial speed is $v_1 = 4.00$ m/s, so the initial kinetic energy is

$$K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(2000\text{ kg})(4.00\text{ m/s})^2 = 16{,}000\text{ J}$$

The elevator stops at point 2, so $K_2 = 0$. The potential energy at point 1, U_1, is zero; U_{grav} is zero because $y_1 = 0$, and $U_{el} = 0$ because the spring is not yet compressed. At point 2 there is both gravitational and elastic potential energy, so

$$U_2 = mgy_2 + \frac{1}{2}ky_2^2$$

The gravitational potential energy at point 2 is

$$mgy_2 = (2000\text{ kg})(9.80\text{ m/s}^2)(-2.00\text{ m}) = -39{,}200\text{ J}$$

The other force is the 17,000-N friction force, acting opposite to the 2.00-m displacement, so

$$W_{other} = -(17{,}000\text{ N})(2.00\text{ m}) = -34{,}000\text{ J}$$

Putting these terms into $K_1 + U_1 + W_{other} = K_2 + U_2$, we have

$$K_1 + 0 + W_{other} = 0 + \left(mgy_2 + \frac{1}{2}ky_2^2\right)$$

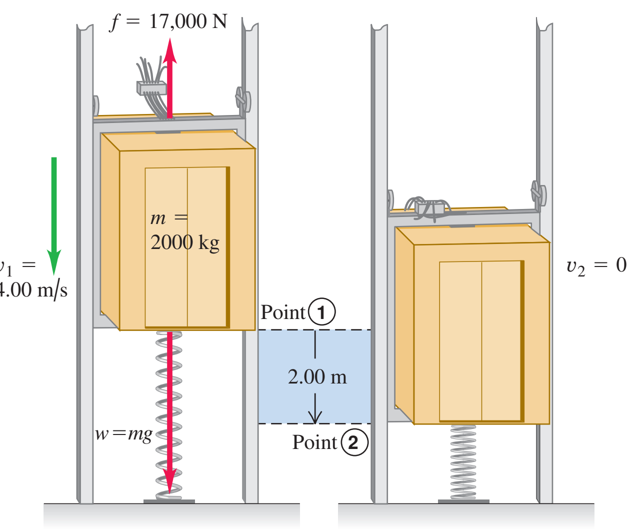
so the force constant of the spring is

$$k = \frac{2(K_1 + W_{other} - mgy_2)}{y_2^2}$$

$$= \frac{2[16{,}000\text{ J} + (-34{,}000\text{ J}) - (-39{,}200\text{ J})]}{(-2.00\text{ m})^2}$$

$$= 1.06 \times 10^4\text{ N/m}$$

7.17 The fall of an elevator is stopped by a spring and by a constant friction force.



This is about one-tenth the force constant of a spring in an automobile suspension.

EVALUATE: Let's note what might seem to be a paradox in this problem. The elastic potential energy in the spring at point 2 is

$$\frac{1}{2}ky_2^2 = \frac{1}{2}(1.06 \times 10^4\text{ N/m})(-2.00\text{ m})^2 = 21{,}200\text{ J}$$

This is *more* than the total mechanical energy at point 1:

$$E_1 = K_1 + U_1 = 16{,}000\text{ J} + 0 = 16{,}000\text{ J}$$

But the friction force caused the mechanical energy of the system to *decrease* by 34,000 J between point 1 and point 2. Does this mean that energy appeared from nowhere? Don't panic; there is no paradox. At point 2 there is also *negative* gravitational potential energy, $mgy_2 = -39{,}200$ J, because point 2 is below the origin. The total mechanical energy at point 2 is

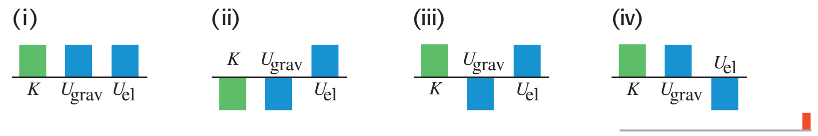
$$E_2 = K_2 + U_2 = 0 + \frac{1}{2}ky_2^2 + mgy_2$$

$$= 0 + 21{,}200\text{ J} + (-39{,}200\text{ J}) = -18{,}000\text{ J}$$

This is just the initial mechanical energy of 16,000 J, minus 34,000 J lost to friction.

Will the elevator stay at the bottom of the shaft? At point 2 the compressed spring exerts an upward force of magnitude $F_{spring} = (1.06 \times 10^4\text{ N/m})(2.00\text{ m}) = 21{,}200$ N, while the downward force of gravity on the elevator is only $w = mg = (2000\text{ kg})(9.80\text{ m/s}^2) = 19{,}600$ N. So if there were no friction, there would be a net upward force of 21,200 N − 19,600 N = 1600 N and the elevator would bounce back upward. However, there *is* friction in the safety clamp, which can exert a force of as much as 17,000 N; hence the clamp can keep the elevator from rebounding.

Test Your Understanding of Section 7.2 Consider the situation in Example 7.9 at the instant when the elevator is still moving downward and the spring is compressed by 1.00 m. Which of the energy bar graphs in the figure most accurately shows the kinetic energy K, gravitational potential energy U_{grav}, and elastic potential energy U_{el} at this instant?



7.3 Conservative and Nonconservative Forces

In our discussions of potential energy we have talked about "storing" kinetic energy by converting it to potential energy. We always have in mind that later we may retrieve it again as kinetic energy. For example, when you throw a ball up in the air, it slows down as kinetic energy is converted into potential energy. But on the way down, the conversion is reversed, and the ball speeds up as potential energy is converted back to kinetic energy. If there is no air resistance, the ball is moving just as fast when you catch it as when you threw it.

Another example is a glider moving on a frictionless horizontal air track that runs into a spring bumper at the end of the track. The glider stops as it compresses the spring and then bounces back. If there is no friction, the glider ends up with the same speed and kinetic energy it had before the collision. Again, there is a two-way conversion from kinetic to potential energy and back. In both cases we can define a potential-energy function so that the total mechanical energy, kinetic plus potential, is constant or *conserved* during the motion.

Conservative Forces

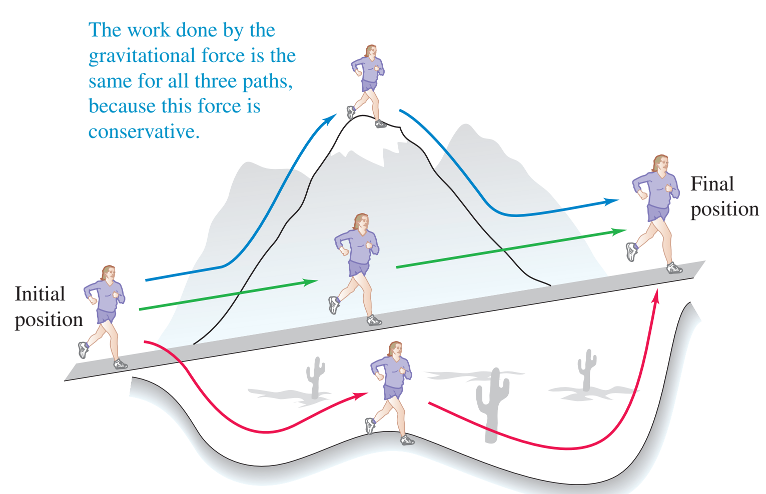
A force that offers this opportunity of two-way conversion between kinetic and potential energies is called a **conservative force.** We have seen two examples of conservative forces: the gravitational force and the spring force. (Later in this book we will study another conservative force, the electric force between charged objects.) An essential feature of conservative forces is that their work is always *reversible.* Anything that we deposit in the energy "bank" can later be withdrawn without loss. Another important aspect of conservative forces is that a body may move from point 1 to point 2 by various paths, but the work done by a conservative force is the same for all of these paths (Fig. 7.18). Thus, if a body stays close to the surface of the earth, the gravitational force $m\vec{g}$ is independent of height, and the work done by this force depends only on the change in height. If the body moves around a closed path, ending at the same point where it started, the *total* work done by the gravitational force is always zero.

The work done by a conservative force *always* has four properties:

1. It can be expressed as the difference between the initial and final values of a *potential-energy* function.
2. It is reversible.
3. It is independent of the path of the body and depends only on the starting and ending points.
4. When the starting and ending points are the same, the total work is zero.

When the *only* forces that do work are conservative forces, the total mechanical energy $E = K + U$ is constant.

7.18 The work done by a conservative force such as gravity depends only on the end points of a path, not on the specific path taken between those points.



Nonconservative Forces

Not all forces are conservative. Consider the friction force acting on the crate sliding on a ramp in Example 7.6 (Section 7.1). When the body slides up and then back down to the starting point, the total work done on it by the friction force is *not* zero. When the direction of motion reverses, so does the friction force, and friction does *negative* work in *both* directions. When a car with its brakes locked skids across the pavement with decreasing speed (and decreasing kinetic energy), the lost kinetic energy cannot be recovered by reversing the motion or in any other way, and mechanical energy is *not* conserved. There is *no* potential-energy function for the friction force.

In the same way, the force of fluid resistance (see Section 5.3) is not conservative. If you throw a ball up in the air, air resistance does negative work on the ball while it's rising *and* while it's descending. The ball returns to your hand with less speed and less kinetic energy than when it left, and there is no way to get back the lost mechanical energy.

A force that is not conservative is called a **nonconservative force.** The work done by a nonconservative force *cannot* be represented by a potential-energy function. Some nonconservative forces, like kinetic friction or fluid resistance, cause mechanical energy to be lost or dissipated; a force of this kind is called a **dissipative force.** There are also nonconservative forces that *increase* mechanical energy. The fragments of an exploding firecracker fly off with very large kinetic energy, thanks to a chemical reaction of gunpowder with oxygen. The forces unleashed by this reaction are nonconservative because the process is not reversible. (The fragments never spontaneously reassemble themselves into a complete firecracker!)

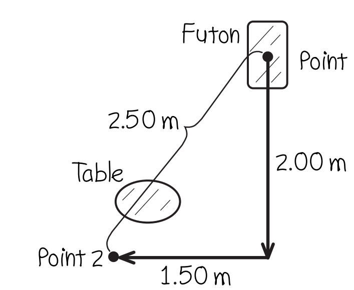
Example 7.10 Frictional work depends on the path

You are rearranging your furniture and wish to move a 40.0-kg futon 2.50 m across the room. However, the straight-line path is blocked by a heavy coffee table that you don't want to move. Instead, you slide the futon in a dogleg path over the floor; the doglegs are 2.00 m and 1.50 m long. Compared to the straight-line path, how much more work must you do to push the futon in the dogleg path? The coefficient of kinetic friction is 0.200.

SOLUTION

IDENTIFY: Here work is done both by you and by the force of friction, so we must use the energy relationship that includes forces other than elastic or gravitational forces. We'll use this relationship to find a connection between the work that *you* do and the work done by *friction.*

7.19 Our sketch for this problem.



SET UP: Figure 7.19 shows our sketch. The futon is at rest at both point 1 and point 2, so $K_1 = K_2 = 0$. There is no elastic potential

Continued

energy (there are no springs), and the gravitational potential energy does not change because the futon moves only horizontally, so $U_1 = U_2$. From Eq. (7.14) it follows that $W_{other} = 0$. The other work done on the futon is the sum of the positive work you do, W_{you}, and the negative work W_{fric} done by the kinetic friction force. Since the sum of these is zero, we have

$$W_{you} = -W_{fric}$$

Thus to determine W_{you}, we'll calculate the work done by friction.

EXECUTE: Because the floor is horizontal, the normal force on the futon equals its weight mg, and the magnitude of the friction force is $f_k = \mu_k n = \mu_k mg$. The work you must do over each path is then

$$\begin{aligned} W_{you} &= -W_{fric} = -(-f_k s) = +\mu_k mgs \\ &= (0.200)(40.0\ \text{kg})(9.80\ \text{m/s}^2)(2.50\ \text{m}) \\ &= 196\ \text{J} \qquad \text{(straight-line path)} \end{aligned}$$

$$\begin{aligned} W_{you} &= -W_{fric} \\ &= (0.200)(40.0\ \text{kg})(9.80\ \text{m/s}^2)(2.00\ \text{m} + 1.50\ \text{m}) \\ &= 274\ \text{J} \qquad \text{(dogleg path)} \end{aligned}$$

The extra work you must do is 274 J − 196 J = 78 J.

EVALUATE: The work done by friction is $W_{fric} = -W_{you} = -196$ J on the straight-line path and −274 J on the dogleg path. The work done by friction depends on the path taken, which illustrates that friction is a *nonconservative* force.

Example 7.11 Conservative or nonconservative?

In a certain region of space the force on an electron is $\vec{F} = Cx\hat{j}$, where C is a positive constant. The electron moves in a counterclockwise direction around a square loop in the xy-plane (Fig. 7.20). The corners of the square are at $(x, y) = (0, 0)$, $(L, 0)$, (L, L), and $(0, L)$. Calculate the work done on the electron by the force $\vec{F}$ during one complete trip around the square. Is this force conservative or nonconservative?

SOLUTION

IDENTIFY: In Example 7.10 the force of friction was constant in magnitude and always opposite to the displacement, so it was easy to calculate the work done. Here, however, the force $\vec{F}$ is not constant and in general is not in the same direction as the displacement.

SET UP: To calculate the work done by the force $\vec{F}$, we'll use the more general expression for work, Eq. (6.14):

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}$$

where $d\vec{l}$ is an infinitesimal displacement. Let's calculate the work done on each leg of the square and then add the results to find the work done on the round trip.

EXECUTE: On the first leg, from $(0, 0)$ to $(L, 0)$, the force varies but is everywhere perpendicular to the displacement. So $\vec{F} \cdot d\vec{l} = 0$, and the work done on the first leg is $W_1 = 0$. The force has the same value $\vec{F} = CL\hat{j}$ everywhere on the second leg from $(L, 0)$ to (L, L). The displacement on this leg is in the $+y$-direction, so $d\vec{l} = dy\hat{j}$ and

$$\vec{F} \cdot d\vec{l} = CL\hat{j} \cdot dy\hat{j} = CL\, dy$$

The work done on the second leg is then

$$W_2 = \int_{(L,0)}^{(L,L)} \vec{F} \cdot d\vec{l} = \int_{y=0}^{y=L} CL\, dy = CL\int_0^L dy = CL^2$$

On the third leg, from (L, L) to $(0, L)$, $\vec{F}$ is again perpendicular to the displacement so $W_3 = 0$. The force is zero on the final leg, from $(0, L)$ to $(0, 0)$, so no work is done and $W_4 = 0$. The work done by the force $\vec{F}$ on the round trip is

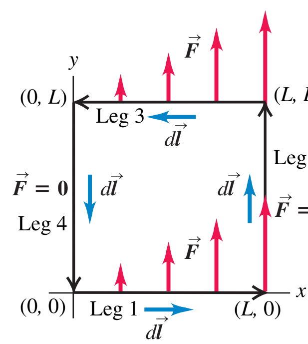
$$W = W_1 + W_2 + W_3 + W_4 = 0 + CL^2 + 0 + 0 = CL^2$$

The starting and ending points are the same, but the total work done by $\vec{F}$ is not zero. This is a nonconservative force; it *cannot* be represented by a potential-energy function.

EVALUATE: Because W is positive, the mechanical energy *increases* as the electron goes around the loop. This is not a mathematical curiosity; it's a description of what happens in an electrical generating plant. A loop of wire is moved through a magnetic field, which gives rise to a nonconservative force similar to the one in this example. Electrons in the wire gain energy as they move around the loop, and this energy is carried via transmission lines to the consumer. (We'll discuss how this works in detail in Chapter 29.)

If the electron went around the loop clockwise instead of counterclockwise, the force $\vec{F}$ would be unaffected but the direction of each infinitesimal displacement $d\vec{l}$ would reverse. Thus the sign of work would also reverse, and the work for a clockwise round trip would be $W = -CL^2$. This is a different behavior than the nonconservative friction force. When a body slides over a stationary surface with friction, the work done by friction is always negative, no matter what the direction of motion (see Example 7.6 in Section 7.1).

7.20 An electron moving around a square loop while being acted on by the force $\vec{F} = Cx\hat{j}$.



The Law of Conservation of Energy

Nonconservative forces cannot be represented in terms of potential energy. But we can describe the effects of these forces in terms of kinds of energy other than kinetic and potential energy. When a car with locked brakes skids to a stop, the tires and the road surface both become hotter. The energy associated with this change in the state of the materials is called **internal energy.** Raising the temperature of a body increases its internal energy; lowering the body's temperature decreases its internal energy.

To see the significance of internal energy, let's consider a block sliding on a rough surface. Friction does *negative* work on the block as it slides, and the change in internal energy of the block and surface (both of which get hotter) is *positive.* Careful experiments show that the increase in the internal energy is *exactly* equal to the absolute value of the work done by friction. In other words,

$$\Delta U_{int} = -W_{other}$$

where ΔU_{int} is the change in internal energy. If we substitute this into Eq. (7.7) or (7.14), we find

$$K_1 + U_1 - \Delta U_{int} = K_2 + U_2$$

Writing $\Delta K = K_2 - K_1$ and $\Delta U = U_2 - U_1$, we can finally express this as

$$\Delta K + \Delta U + \Delta U_{int} = 0 \qquad \text{(law of conservation of energy)} \qquad (7.15)$$

This remarkable statement is the general form of the **law of conservation of energy.** In a given process, the kinetic energy, potential energy, and internal energy of a system may all change. But the *sum* of those changes is always zero. If there is a decrease in one form of energy, it is made up for by an increase in the other forms (Fig. 7.21). When we expand our definition of energy to include internal energy, Eq. (7.15) says: *Energy is never created or destroyed; it only changes form.* No exception to this rule has ever been found.

The concept of work has been banished from Eq. (7.15); instead, it suggests that we think purely in terms of the conversion of energy from one form to another. For example, when you throw a baseball straight up, you convert a portion of the internal energy of your molecules into kinetic energy of the baseball. This is converted into gravitational potential energy as the ball climbs and back to kinetic energy as the ball falls. If there is air resistance, part of the energy is used to heat up the air and the ball and increase their internal energy. Energy is converted back into the kinetic form as the ball falls. If you catch the ball in your hand, whatever energy was not lost to the air once again becomes internal energy; the ball and your hand are now warmer than they were at the beginning.

In Chapters 19 and 20, we will study the relationship of internal energy to temperature changes, heat, and work. This is the heart of the area of physics called *thermodynamics.*

ActivPhysics ONLINE
5.7 Modified Atwood Machine

7.21 When 1 liter of gasoline is burned in an automotive engine, it releases 3.3×10^7 J of internal energy. Hence $\Delta U_{int} = -3.3 \times 10^7$ J, where the minus sign means that the amount of energy stored in the gasoline has decreased. This energy can be converted into kinetic energy (making the car go faster) or into potential energy (enabling the car to climb uphill).

Example 7.12 Work done by friction

Let's look again at Example 7.5 (Section 7.1), in which your cousin Throcky skateboards down a curved ramp. He starts with zero kinetic energy and 735 J of potential energy, and at the bottom he has 450 J of kinetic energy and zero potential energy. So $\Delta K = +450$ J and $\Delta U = -735$ J. The work $W_{other} = W_{fric}$ done by the nonconservative friction forces is −285 J, so the change in internal energy is $\Delta U_{int} = -W_{other} = +285$ J. The wheels, the bearings, and the ramp all get a little warmer as Throcky rolls down. In accordance with Eq. (7.15), the sum of the energy changes equals zero:

$$\Delta K + \Delta U + \Delta U_{int} = +450\ \text{J} + (-735\ \text{J}) + 285\ \text{J} = 0$$

The total energy of the system (including nonmechanical forms of energy) is conserved.

Test Your Understanding of Section 7.3 In a hydroelectric generating station, falling water is used to drive turbines ("water wheels"), which in turn run electric generators. Compared to the amount of gravitational potential energy released by the falling water, how much electrical energy is produced? (i) the same; (ii) more; (iii) less.

7.4 Force and Potential Energy

For the two kinds of conservative forces (gravitational and elastic) we have studied, we started with a description of the behavior of the *force* and derived from that an expression for the *potential energy*. For example, for a body with mass m in a uniform gravitational field, the gravitational force is $F_y = -mg$. We found that the corresponding potential energy is $U(y) = mgy$. To stretch an ideal spring by a distance x, we exert a force equal to $+kx$. By Newton's third law the force that an ideal spring exerts on a body is opposite this, or $F_x = -kx$. The corresponding potential energy function is $U(x) = \frac{1}{2}kx^2$.

In studying physics, however, you'll encounter situations in which you are given an expression for the *potential energy* as a function of position and have to find the corresponding *force*. We'll see several examples of this kind when we study electric forces later in this book: it's often far easier to calculate the electric potential energy first and then determine the corresponding electric force afterward.

Here's how we find the force that corresponds to a given potential-energy expression. First let's consider motion along a straight line, with coordinate x. We denote the x-component of force, a function of x, by $F_x(x)$, and the potential energy as $U(x)$. This notation reminds us that both F_x and U are *functions* of x. Now we recall that in any displacement, the work W done by a conservative force equals the negative of the change ΔU in potential energy:

$$W = -\Delta U$$

Let's apply this to a small displacement Δx. The work done by the force $F_x(x)$ during this displacement is approximately equal to $F_x(x)\,\Delta x$. We have to say "approximately" because $F_x(x)$ may vary a little over the interval Δx. But it is at least approximately true that

$$F_x(x)\,\Delta x = -\Delta U \qquad \text{and} \qquad F_x(x) = -\frac{\Delta U}{\Delta x}$$

You can probably see what's coming. We take the limit as $\Delta x \rightarrow 0$; in this limit, the variation of F_x becomes negligible, and we have the exact relationship

$$F_x(x) = -\frac{dU(x)}{dx} \quad \text{(force from potential energy, one dimension)} \quad (7.16)$$

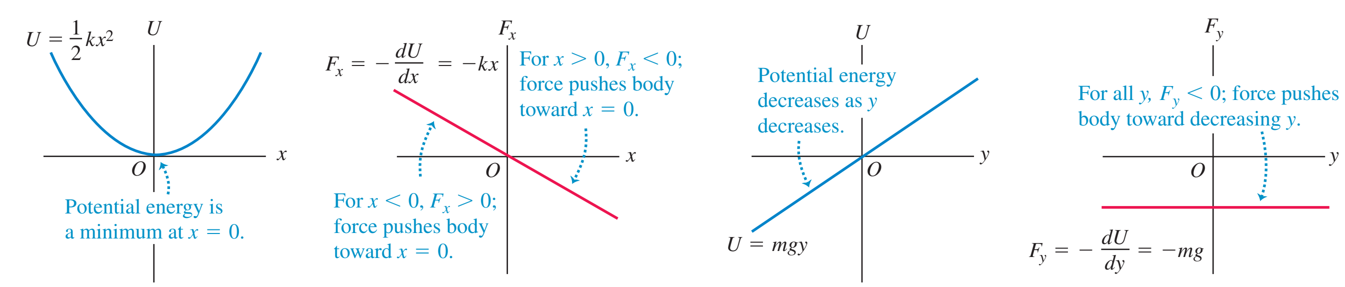
This result makes sense; in regions where $U(x)$ changes most rapidly with x (that is, where $dU(x)/dx$ is large), the greatest amount of work is done during a given displacement, and this corresponds to a large force magnitude. Also, when $F_x(x)$ is in the positive x-direction, $U(x)$ *decreases* with increasing x. So $F_x(x)$ and $dU(x)/dx$ should indeed have opposite signs. The physical meaning of Eq. (7.16) is that *a conservative force always acts to push the system toward lower potential energy.*

As a check, let's consider the function for elastic potential energy, $U(x) = \frac{1}{2}kx^2$. Substituting this into Eq. (7.16) yields

$$F_x(x) = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -kx$$

7.22 A conservative force is the negative derivative of the corresponding potential energy.

(a) Spring potential energy and force as functions of x (b) Gravitational potential energy and force as function of y



which is the correct expression for the force exerted by an ideal spring (Fig. 7.22a). Similarly, for gravitational potential energy we have $U(y) = mgy$; taking care to change x to y for the choice of axis, we get $F_y = -dU/dy = -d(mgy)/dy = -mg$, which is the correct expression for gravitational force (Fig. 7.22b).

Example 7.13 An electric force and its potential energy

An electrically charged particle is held at rest at the point $x = 0$, while a second particle with equal charge is free to move along the positive x-axis. The potential energy of the system is

$$U(x) = \frac{C}{x}$$

where C is a positive constant that depends on the magnitude of the charges. Derive an expression for the x-component of force acting on the movable charged particle, as a function of its position.

SOLUTION

IDENTIFY: We are given the potential-energy function $U(x)$, and we want to find the force function $F_x(x)$.

SET UP: We'll use Eq. (7.16), $F_x(x) = -dU(x)/dx$.

EXECUTE: The derivative with respect to x of the function $1/x$ is $-1/x^2$. So the force on the movable charged particle for $x > 0$ is

$$F_x(x) = -\frac{dU(x)}{dx} = -C\left(-\frac{1}{x^2}\right) = \frac{C}{x^2}$$

EVALUATE: The x-component of force is positive, corresponding to a repulsion between like electric charges. The potential energy is very large when the particles are close together (small x) and approaches zero as the particles move farther apart (large x); the force pushes the movable particle toward large positive values of x, for which the potential energy is less. The force $F_x(x) = C/x^2$ gets weaker as the particles move farther apart (x increases). We'll study electric forces in greater detail in Chapter 21.

Force and Potential Energy in Three Dimensions

We can extend this analysis to three dimensions, where the particle may move in the x-, y-, or z-direction, or all at once, under the action of a conservative force that has components F_x, F_y, and F_z. Each component of force may be a function of the coordinates x, y, and z. The potential-energy function U is also a function of all three space coordinates. We can now use Eq. (7.16) to find each component of force. The potential-energy change ΔU when the particle moves a small distance Δx in the x-direction is again given by $-F_x\,\Delta x$; it doesn't depend on F_y and F_z, which represent force components that are perpendicular to the displacement and do no work. So we again have the approximate relationship

$$F_x = -\frac{\Delta U}{\Delta x}$$

The y- and z-components of force are determined in exactly the same way:

$$F_y = -\frac{\Delta U}{\Delta y} \qquad F_z = -\frac{\Delta U}{\Delta z}$$

To make these relationships exact, we take the limits $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$, and $\Delta z \rightarrow 0$ so that these ratios become derivatives. Because U may be a function of all three coordinates, we need to remember that when we calculate each of these derivatives, only one coordinate changes at a time. We compute the derivative of U with respect to x by assuming that y and z are constant and only x varies, and so on. Such a derivative is called a *partial derivative*. The usual notation for a partial derivative is $\partial U/\partial x$ and so on; the symbol ∂ is a modified d. So we write

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z} \qquad \text{(force from potential energy)} \tag{7.17}$$

We can use unit vectors to write a single compact vector expression for the force $\vec{F}$:

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) \qquad \text{(force from potential energy)} \tag{7.18}$$

The expression inside the parentheses represents a particular operation on the function U, in which we take the partial derivative of U with respect to each coordinate, multiply by the corresponding unit vector, and then take the vector sum. This operation is called the **gradient** of U and is often abbreviated as $\vec{\nabla} U$. Thus the force is the negative of the gradient of the potential-energy function:

$$\vec{F} = -\vec{\nabla} U \tag{7.19}$$

As a check, let's substitute into Eq. (7.19) the function $U = mgy$ for gravitational potential energy:

$$\vec{F} = -\vec{\nabla}(mgy) = -\left(\frac{\partial(mgy)}{\partial x}\hat{\imath} + \frac{\partial(mgy)}{\partial y}\hat{\jmath} + \frac{\partial(mgy)}{\partial z}\hat{k}\right) = (-mg)\hat{\jmath}$$

This is just the familiar expression for the gravitational force.

Example 7.14 Force and potential energy in two dimensions

A puck slides on a level, frictionless air-hockey table. The coordinates of the puck are x and y. It is acted on by a conservative force described by the potential-energy function

$$U(x, y) = \frac{1}{2}k(x^2 + y^2)$$

Derive an expression for the force acting on the puck, and find an expression for the magnitude of the force as a function of position.

SOLUTION

IDENTIFY: Starting with the function $U(x, y)$, we need to find the vector components and magnitude of the corresponding conservative force $\vec{F}$.

SET UP: We'll find the components of the force from $U(x, y)$ using Eq. (7.18). This function doesn't depend on z, so the partial derivative of U with respect to z is $\partial U/\partial z = 0$ and the force has no z-component. We'll then determine the magnitude of the force using the formula for the magnitude of a vector: $F = \sqrt{F_x^2 + F_y^2}$.

EXECUTE: The x- and y-components of the force are

$$F_x = -\frac{\partial U}{\partial x} = -kx \qquad F_y = -\frac{\partial U}{\partial y} = -ky$$

From Eq. (7.18) this corresponds to the vector expression

$$\vec{F} = -k(x\hat{\imath} + y\hat{\jmath})$$

Now $x\hat{\imath} + y\hat{\jmath}$ is just the position vector $\vec{r}$ of the particle, so we can rewrite this expression as $\vec{F} = -k\vec{r}$. This represents a force that at each point is opposite in direction to the position vector of the point—that is, a force that at each point is directed toward the origin. The potential energy is minimum at the origin, so again the force pushes in the direction of decreasing potential energy.

The *magnitude* of the force at any point is

$$F = \sqrt{(-kx)^2 + (-ky)^2} = k\sqrt{x^2 + y^2} = kr$$

where r is the particle's distance from the origin. This is the force that would be exerted on the puck if it were attached to one end of a spring that obeys Hooke's law and has a negligibly small length (compared to the other distances in the problem) when it is not stretched. (The other end is attached to the air-hockey table at the origin.)

EVALUATE: To check our result, note that the potential-energy function can also be expressed as $U = \frac{1}{2}kr^2$. Written this way, U is a function of a single coordinate r, so we can find the force using Eq. (7.16) with x replaced by r:

$$F_r = -\frac{dU}{dr} = -\frac{d}{dr}\left(\frac{1}{2}kr^2\right) = -kr$$

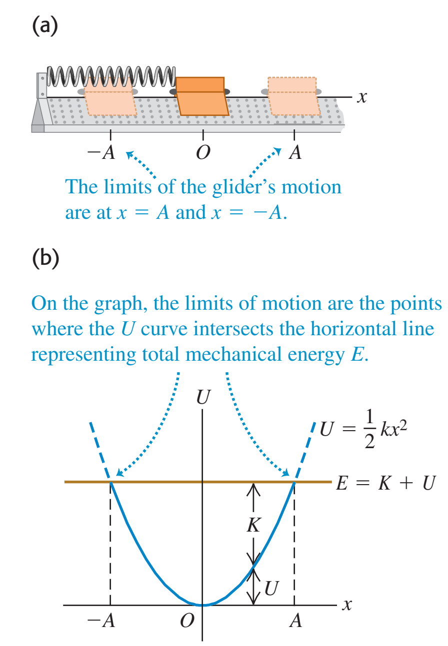
Just as we calculated above, the force has magnitude kr; the minus sign indicates that the force is radially inward (toward the origin).

Test Your Understanding of Section 7.4 A particle moving along the x-axis is acted on by a conservative force F_x. At a certain point, the force is zero. (a) Which of the following statements about the value of the potential-energy function $U(x)$ at that point is correct? (i) $U(x) = 0$; (ii) $U(x) > 0$; (iii) $U(x) < 0$; (iv) not enough information is given to decide. (b) Which of the following statements about the value of the derivative of $U(x)$ at that point is correct? (i) $dU(x)/dx = 0$; (ii) $dU(x)/dx > 0$; (iii) $dU(x)/dx < 0$; (iv) not enough information is given to decide.

7.5 Energy Diagrams

When a particle moves along a straight line under the action of a conservative force, we can get a lot of insight into its possible motions by looking at the graph of the potential-energy function $U(x)$. Figure 7.23a shows a glider with mass m that moves along the x-axis on an air track. The spring exerts on the glider a force with x-component $F_x = -kx$. Figure 7.23b is a graph of the corresponding potential-energy function $U(x) = \frac{1}{2}kx^2$. If the elastic force of the spring is the *only* horizontal force acting on the glider, the total mechanical energy $E = K + U$ is constant, independent of x. A graph of E as a function of x is thus a straight horizontal line. We use the term **energy diagram** for a graph like this, which shows both the potential-energy function $U(x)$ and the energy of the particle subjected to the force that corresponds to $U(x)$.

7.23 (a) A glider on an air track. The spring exerts a force $F_x = -kx$. (b) The potential-energy function.



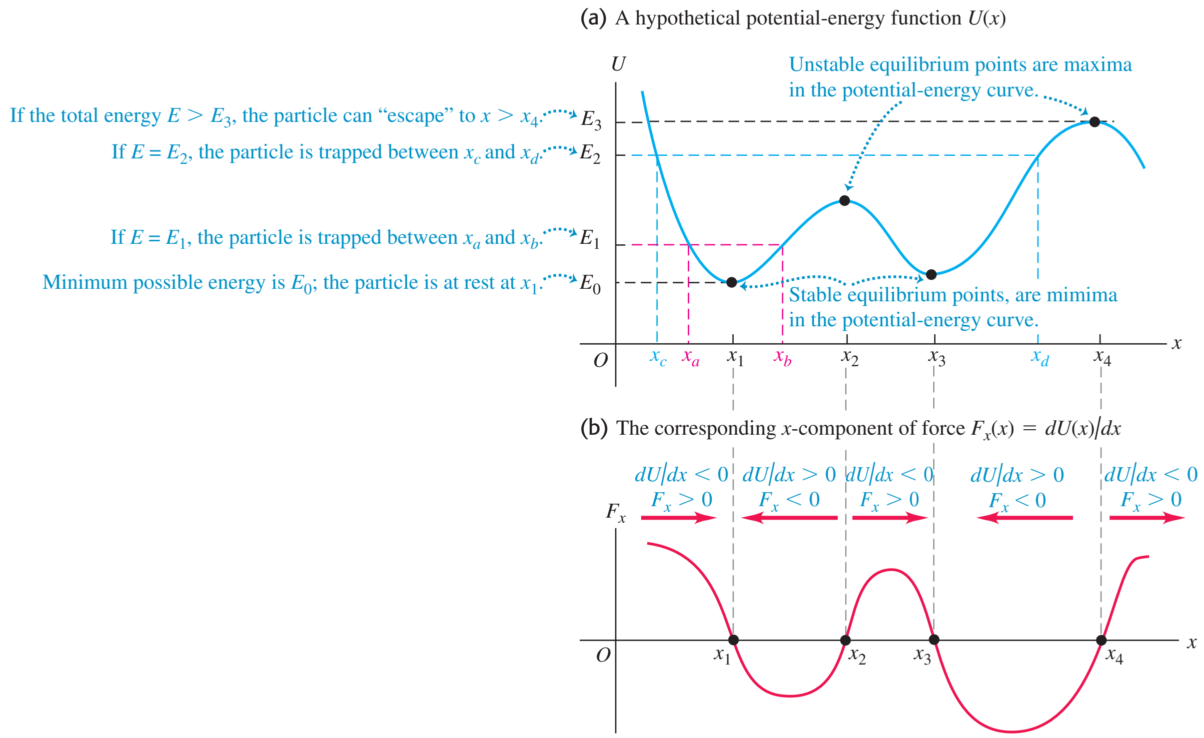
The vertical distance between the U and E graphs at each point represents the difference $E - U$, equal to the kinetic energy K at that point. We see that K is greatest at $x = 0$. It is zero at the values of x where the two graphs cross, labeled A and $-A$ in the diagram. Thus the speed v is greatest at $x = 0$, and it is zero at $x = \pm A$, the points of *maximum* possible displacement from $x = 0$ for a given value of the total energy E. The potential energy U can never be greater than the total energy E; if it were, K would be negative, and that's impossible. The motion is a back-and-forth oscillation between the points $x = A$ and $x = -A$.

At each point, the force F_x on the glider is equal to the negative of the slope of the $U(x)$ curve: $F_x = -dU/dx$ (see Fig. 7.22a). When the particle is at $x = 0$, the slope and the force are zero, so this is an *equilibrium* position. When x is positive, the slope of the $U(x)$ curve is positive and the force F_x is negative, directed toward the origin. When x is negative, the slope is negative and F_x is positive, again toward the origin. Such a force is called a *restoring force;* when the glider is displaced to either side of $x = 0$, the force tends to "restore" it back to $x = 0$. An analogous situation is a marble rolling around in a round-bottomed bowl. We say that $x = 0$ is a point of **stable equilibrium.** More generally, *any minimum in a potential-energy curve is a stable equilibrium position.*

Figure 7.24a shows a hypothetical but more general potential-energy function $U(x)$. Figure 7.24b shows the corresponding force $F_x = -dU/dx$. Points x_1 and x_3 are stable equilibrium points. At each of these points, F_x is zero because the slope of the $U(x)$ curve is zero. When the particle is displaced to either side, the force pushes back toward the equilibrium point. The slope of the $U(x)$ curve is also zero at points x_2 and x_4, and these are also equilibrium points. But when the particle is displaced a little to the right of either point, the slope of the $U(x)$ curve becomes negative, corresponding to a positive F_x that tends to push the particle still farther from the point. When the particle is displaced a little to the left, F_x is negative, again pushing away from equilibrium. This is analogous to a marble rolling on the top of a bowling ball. Points x_2 and x_4 are called **unstable equilibrium** points; *any maximum in a potential-energy curve is an unstable equilibrium position.*

CAUTION **Potential energy and the direction of a conservative force** The direction of the force on a body is *not* determined by the sign of the potential energy U. Rather, it's the sign of $F_x = -dU/dx$ that matters. As we discussed in Section 7.1, the physically significant quantity is the *difference* in the value of U between two points, which is just

7.24 The maxima and minima of a potential-energy function $U(x)$ correspond to points where $F_x = 0$.



what the derivative $F_x = -dU/dx$ measures. This means that you can always add a constant to the potential-energy function without changing the physics of the situation.

If the total energy is E_1 and the particle is initially near x_1, it can move only in the region between x_a and x_b determined by the intersection of the E_1 and U graphs (Fig. 7.24a). Again, U cannot be greater than E_1 because K can't be negative. We speak of the particle as moving in a *potential well,* and x_a and x_b are the *turning points* of the particle's motion (since at these points, the particle stops and reverses direction). If we increase the total energy to the level E_2, the particle can move over a wider range, from x_c to x_d. If the total energy is greater than E_3, the particle can "escape" and move to indefinitely large values of x. At the other extreme, E_0 represents the least possible total energy the system can have.

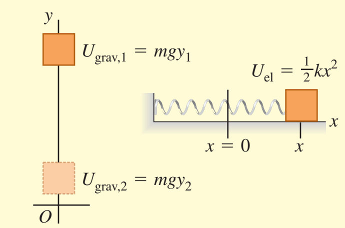
Test Your Understanding of Section 7.5 The curve in Fig. 7.24b has a maximum at a point between x_2 and x_3. Which statement correctly describes what happens to the particle when it is at this point? (i) The particle's acceleration is zero. (ii) The particle accelerates in the positive x-direction; the magnitude of the acceleration is less than at any other point between x_2 and x_3. (iii) The particle accelerates in the positive x-direction; the magnitude of the acceleration is greater than at any other point between x_2 and x_3. (iv) The particle accelerates in the negative x-direction; the magnitude of the acceleration is less than at any other point between x_2 and x_3. (v) The particle accelerates in the negative x-direction; the magnitude of the acceleration is greater than at any other point between x_2 and x_3.

CHAPTER 7 SUMMARY

Gravitational potential energy and elastic potential energy: The work done on a particle by a constant gravitational force can be represented as a change in the gravitational potential energy $U_{grav} = mgy$. This energy is a shared property of the particle and the earth. A potential energy is also associated with the elastic force $F_x = -kx$ exerted by an ideal spring, where x is the amount of stretch or compression. The work done by this force can be represented as a change in the elastic potential energy of the spring, $U_{el} = \frac{1}{2}kx^2$.

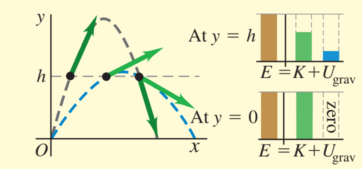
$$W_{grav} = mgy_1 - mgy_2 = U_{grav,1} - U_{grav,2} = -\Delta U_{grav} \quad (7.1), (7.3)$$

$$W_{el} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 = U_{el,1} - U_{el,2} = -\Delta U_{el} \quad (7.10)$$



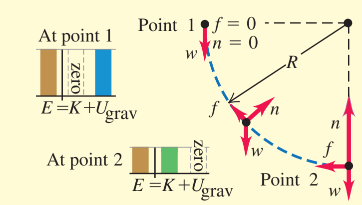
When total mechanical energy is conserved: The total potential energy U is the sum of the gravitational and elastic potential energy: $U = U_{grav} + U_{el}$. If no forces other than the gravitational and elastic forces do work on a particle, the sum of kinetic and potential energy is conserved. This sum $E = K + U$ is called the total mechanical energy. (See Examples 7.1, 7.3, 7.4, and 7.7.)

$$K_1 + U_1 = K_2 + U_2 \quad (7.4), (7.11)$$



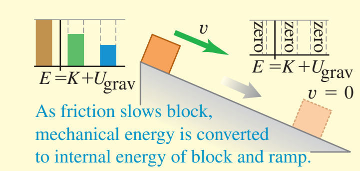
When total mechanical energy is not conserved: When forces other than the gravitational and elastic forces do work on a particle, the work W_{other} done by these other forces equals the change in total mechanical energy (kinetic energy plus total potential energy). (See Examples 7.2, 7.5, 7.6, 7.8, and 7.9.)

$$K_1 + U_1 + W_{other} = K_2 + U_2 \quad (7.14)$$



Conservative forces, nonconservative forces, and the law of conservation of energy: All forces are either conservative or nonconservative. A conservative force is one for which the work–kinetic energy relationship is completely reversible. The work of a conservative force can always be represented by a potential-energy function, but the work of a nonconservative force cannot. The work done by nonconservative forces manifests itself as changes in the internal energy of bodies. The sum of kinetic, potential, and internal energy is always conserved. (See Examples 7.10–7.12.)

$$\Delta K + \Delta U + \Delta U_{int} = 0 \quad (7.15)$$

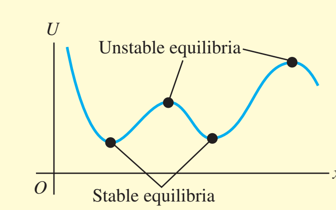


Determining force from potential energy: For motion along a straight line, a conservative force $F_x(x)$ is the negative derivative of its associated potential-energy function U. In three dimensions, the components of a conservative force are negative partial derivatives of U. (See Examples 7.13 and 7.14.)

$$F_x(x) = -\frac{dU(x)}{dx} \quad (7.16)$$

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial y} \quad (7.17)$$

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) \quad (7.18)$$



Key Terms

potential energy, *214*
gravitational potential energy, *214*
total mechanical energy, *215*
conservation of mechanical energy, *215*
elastic potential energy, *223*
conservative force, *228*
nonconservative force, *229*
dissipative force, *229*
internal energy, *231*
law of conservation of energy, *231*
gradient, *234*
energy diagram, *235*
stable equilibrium, *235*
unstable equilibrium, *236*

Answer to Chapter Opening Question ?

Gravity is doing positive work on the diver, since this force is in the same downward direction as his displacement. This corresponds to a decrease in gravitational potential energy. The water is doing negative work on the diver; it exerts an upward force of fluid resistance as he moves downward. This corresponds to an increase in internal energy of the diver and the water (see Section 7.3).

Answers to Test Your Understanding Questions

7.1 Answer: (iii) The initial kinetic energy $K_1 = 0$, the initial potential energy $U_1 = mgy_1$, and the final potential energy $U_2 = mgy_2$ are the same for both blocks. Mechanical energy is conserved in both cases, so the final kinetic energy $K = \frac{1}{2}mv_2^2$ is also the same for both blocks. Hence the speed at the right-hand end is the *same* in both cases!

7.2 Answer: (iii) The elevator is still moving downward, so the kinetic energy K is positive (remember that K can never be negative); the elevator is below point 1, so $y < 0$ and $U_{grav} < 0$; and the spring is compressed, so $U_{el} > 0$.

7.3 Answer: (iii) Because of friction in the turbines and between the water and turbines, some of the potential energy goes into raising the temperatures of the water and the mechanism.

7.4 Answers: (a) (iv), (b) (i) If $F_x = 0$ at a point, then the derivative of $U(x)$ must be zero at that point because $F_x = -dU(x)/dx$. However, this tells us absolutely nothing about the *value* of $U(x)$ at that point.

7.5 Answers: (iii) Figure 7.24b shows the x-component of force, F_x. Where this is maximum (most positive), the x-component of force and the x-acceleration have more positive values than at adjacent values of x.

PROBLEMS

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Discussion Questions

Q7.1. A baseball is thrown straight up with initial speed v_0. If air resistance cannot be ignored, when the ball returns to its initial height its speed is less than v_0. Explain why, using energy concepts.

Q7.2. A projectile has the same initial kinetic energy no matter what the angle of projection. Why doesn't it rise to the same maximum height in each case?

Q7.3. Does an object's speed at the bottom of a frictionless ramp depend on the shape of the ramp or just on its height? Explain. What if the ramp is *not* frictionless?

Q7.4. An egg is released from rest from the roof of a building and falls to the ground. Its fall is observed by a student on the roof of the building, who uses coordinates with origin at the roof, and by a student on the ground, who uses coordinates with origin at the ground. Do the two students assign the same or different values to the initial gravitational potential energy, the final gravitational potential energy, the change in gravitational potential energy, and the kinetic energy of the egg just before it strikes the ground? Explain.

Q7.5. A physics teacher had a bowling ball suspended from a very long rope attached to the high ceiling of a large lecture hall. To illustrate his faith in conservation of energy, he would back up to one side of the stage, pull the ball far to one side until the taut rope brought it just to the end of his nose, and then release it. The massive ball would swing in a mighty arc across the stage and then return to stop momentarily just in front of the nose of the stationary, unflinching teacher. However, one day after the demonstration he looked up just in time to see a student at the other side of the stage *push* the ball away from his nose as he tried to duplicate the demonstration. Tell the rest of the story and explain the reason for the potentially tragic outcome.

Q7.6. Lost Energy? The principle of the conservation of energy tells us that energy is never lost, but only changes from one form to another. Yet in many ordinary situations, energy may appear to be lost. In each case, explain what happens to the "lost" energy. (a) A box sliding on the floor comes to a halt due to friction. How did friction take away its kinetic energy, and what happened to that energy? (b) A car stops when you apply the brakes. What happened to its kinetic energy? (c) Air resistance uses up some of the original gravitational potential energy of a falling object. What type of energy did the "lost" potential energy become? (d) When a returning space shuttle touches down on the runway, it has lost almost all its kinetic energy and gravitational potential energy. Where did all that energy go?

Q7.7. Is it possible for a frictional force to *increase* the mechanical energy of a system? If so, give examples.

Q7.8. A woman bounces on a trampoline, going a little higher with each bounce. Explain how she increases the total mechanical energy.

Q7.9. Fractured Physics. People often call their electric bill a *power* bill, yet the quantity on which the bill is based is expressed in *kilowatt-hours.* What are people really being billed for?

Q7.10. A rock of mass m and a rock of mass $2m$ are both released from rest at the same height and feel no air resistance as they fall. Which statements about these rocks are true? (There may be more than one correct choice.) (a) Both have the same initial gravitational potential energy. (b) Both have the same kinetic energy when they reach the ground. (c) Both reach the ground with the same speed. (d) When it reaches the ground, the heavier rock has twice the kinetic energy of the lighter one. (e) When it reaches the ground, the heavier rock has four times the kinetic energy of the lighter one.

Q7.11. On a friction-free ice pond, a hockey puck is pressed against (but not attached to) a fixed ideal spring, compressing the spring by a distance x_0. The maximum energy stored in the spring is U_0, the maximum speed the puck gains after being released is v_0, and its maximum kinetic energy is K_0. Now the puck is pressed so it compresses the spring twice as far as before. In this case, (a) what is the maximum potential energy stored in the spring (in terms of U_0), and (b) what are the puck's maximum kinetic energy and speed (in terms of K_0 and x_0)?

Q7.12. When people are cold, they often rub their hands together to warm them up. How does doing this produce heat? Where did the heat come from?

Q7.13. You often hear it said that most of our energy ultimately comes from the sun. Trace each of the following energies back to the sun. (a) the kinetic energy of a jet plane; (b) the potential energy gained by a mountain climber; (c) the electrical energy used to run a computer; (d) the electrical energy from a hydroelectric plant.

Q7.14. A box slides down a ramp and work is done on the box by the forces of gravity and friction. Can the work of each of these forces be expressed in terms of the change in a potential-energy function? For each force explain why or why not.

Q7.15. In physical terms, explain why friction is a nonconservative force. Does it store energy for future use?

Q7.16. A compressed spring is clamped in its compressed position and then is dissolved in acid. What becomes of its potential energy?

Q7.17. Since only changes in potential energy are important in any problem, a student decides to let the elastic potential energy of a spring be zero when the spring is stretched a distance x_1. The student decides, therefore, to let $U = \frac{1}{2}k(x - x_1)^2$. Is this correct? Explain.

Q7.18. Figure 7.22a shows the potential-energy function for the force $F_x = -kx$. Sketch the potential-energy function for the force $F_x = +kx$. For this force, is $x = 0$ a point of equilibrium? Is this equilibrium stable or unstable? Explain.

Q7.19. Figure 7.22b shows the potential-energy function associated with the gravitational force between an object and the earth. Use this graph to explain why objects always fall toward the earth when they are released.

Q7.20. For a system of two particles we often let the potential energy for the force between the particles approach zero as the separation of the particles approaches infinity. If this choice is made, explain why the potential energy at noninfinite separation is positive if the particles repel one another and negative if they attract.

Q7.21. Explain why the points $x = A$ and $x = -A$ in Fig. 7.23b are called *turning points.* How are the values of E and U related at a turning point?

Q7.22. A particle is in *neutral equilibrium* if the net force on it is zero and remains zero if the particle is displaced slightly in any direction. Sketch the potential-energy function near a point of neutral equilibrium, for the case of one-dimensional motion. Give an example of an object in neutral equilibrium.

Q7.23. The net force on a particle of mass m has the potential-energy function graphed in Fig. 7.24a. If the total energy is E_1, graph the speed v of the particle versus its position x. At what value of x is the speed greatest? Sketch v versus x if the total energy is E_2.

Q7.24. The potential-energy function for a force $\vec{F}$ is $U = \alpha x^3$, where α is a positive constant. What is the direction of $\vec{F}$?

Exercises

Section 7.1 Gravitational Potential Energy

7.1. In one day, a 75-kg mountain climber ascends from the 1500-m level on a vertical cliff to the top at 2400 m. The next day, she descends from the top to the base of the cliff, which is at an elevation of 1350 m. What is her change in gravitational potential energy (a) on the first day and (b) on the second day?

7.2. A 5.00-kg sack of flour is lifted vertically at a constant speed of 3.50 m/s through a height of 15.0 m. (a) How great a force is required? (b) How much work is done on the sack by the lifting force? What becomes of this work?

7.3. A 120-kg mail bag hangs by a vertical rope 3.5 m long. A postal worker then displaces the bag to a position 2.0 m sideways from its original position, always keeping the rope taut. (a) What horizontal force is necessary to hold the bag in the new position? (b) As the bag is moved to this position, how much work is done (i) by the rope and (ii) by the worker?

7.4. A 72.0-kg swimmer jumps into the old swimming hole from a diving board 3.25 m above the water. Use energy conservation to find his speed just he hits the water (a) if he just holds his nose and drops in, (b) if he bravely jumps straight up (but just beyond the board!) at 2.50 m/s, and (c) if he manages to jump downward at 2.50 m/s.

7.5. A baseball is thrown from the roof of a 22.0-m-tall building with an initial velocity of magnitude 12.0 m/s and directed at an angle of 53.1° above the horizontal. (a) What is the speed of the ball just before it strikes the ground? Use energy methods and ignore air resistance. (b) What is the answer for part (a) if the initial velocity is at an angle of 53.1° *below* the horizontal? (c) If the effects of air resistance are included, will part (a) or (b) give the higher speed?

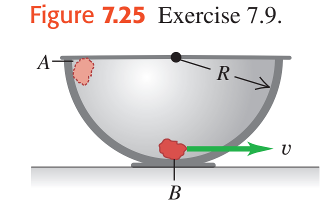
7.6. A crate of mass M starts from rest at the top of a frictionless ramp inclined at an angle α above the horizontal. Find its speed at the bottom of the ramp, a distance d from where it started. Do this in two ways: (a) Take the level at which the potential energy is zero to be at the bottom of the ramp with y positive upward. (b) Take the zero level for potential energy to be at the top of the ramp with y positive upward. (c) Why did the normal force not enter into your solution?

7.7. Answer part (b) of Example 7.6 (Section 7.1) by applying Eq. (7.7) to points 2 and 3, rather than to points 1 and 3 as was done in the example.

7.8. An empty crate is given an initial push down a ramp, starting it with a speed v_0, and reaches the bottom with speed v and kinetic energy K. Some books are now placed in the crate, so that the total mass is quadrupled. The coefficient of kinetic friction is constant and air resistance is negligible. Starting again with v_0 at the top of the ramp, what are the speed and kinetic energy at the bottom? Explain the reasoning behind your answers.

7.9. A small rock with mass 0.20 kg is released from rest at point A, which is at the top edge of a large, hemispherical bowl with radius $R = 0.50$ m (Fig. 7.25). Assume that the size of the rock is small compared to R, so that the rock can be treated as a particle, and assume that the rock slides rather than rolls. The work done by friction on the rock when it moves from point A to point B at the bottom of the bowl has magnitude 0.22 J. (a) Between points A and B, how much work is done on the rock by (i) the normal force and (ii) gravity? (b) What is the speed of the rock as it reaches point B? (c) Of the three forces acting on the rock as it slides down the bowl, which (if any) are constant and which are not? Explain. (d) Just as the rock reaches point B, what is the normal force on it due to the bottom of the bowl?

Figure 7.25 Exercise 7.9.



7.10. A stone of mass m is thrown upward at an angle θ above the horizontal and feels no appreciable air resistance. Use conservation of energy to show that at its highest point, it is a distance $v_0^2(\sin^2\theta)/2g$ above the point where it was launched. (*Hint:* $v_0^2 = v_{0x}^2 + v_{0y}^2$.)

7.11. You are testing a new amusement park roller coaster with an empty car with mass 120 kg. One part of the track is a vertical loop with radius 12.0 m. At the bottom of the loop (point *A*) the car has speed 25.0 m/s, and at the top of the loop (point *B*) it has speed 8.0 m/s. As the car rolls from point *A* to point *B*, how much work is done by friction?

7.12. Tarzan and Jane. Tarzan, in one tree, sights Jane in another tree. He grabs the end of a vine with length 20 m that makes an angle of 45° with the vertical, steps off his tree limb, and swings down and then up to Jane's open arms. When he arrives, his vine makes an angle of 30° with the vertical. Determine whether he gives her a tender embrace or knocks her off her limb by calculating Tarzan's speed just before he reaches Jane. You can ignore air resistance and the mass of the vine.

7.13. A 10.0-kg microwave oven is pushed 8.00 m up the sloping surface of a loading ramp inclined at an angle of 36.9° above the horizontal, by a constant force $\vec{F}$ with a magnitude 110 N and acting parallel to the ramp. The coefficient of kinetic friction between the oven and the ramp is 0.250. (a) What is the work done on the oven by the force $\vec{F}$? (b) What is the work done on the oven by the friction force? (c) Compute the increase in potential energy for the oven. (d) Use your answers to parts (a), (b), and (c) to calculate the increase in the oven's kinetic energy. (e) Use $\sum\vec{F} = m\vec{a}$ to calculate the acceleration of the oven. Assuming that the oven is initially at rest, use the acceleration to calculate the oven's speed after traveling 8.00 m. From this, compute the increase in the oven's kinetic energy, and compare it to the answer you got in part (d).

7.14. Pendulum. A small rock with mass 0.12 kg is fastened to a massless string with length 0.80 m to form a pendulum. The pendulum is swinging so as to make a maximum angle of 45° with the vertical. Air resistance is negligible. (a) What is the speed of the rock when the string passes through the vertical position? (b) What is the tension in the string when it makes an angle of 45° with the vertical? (c) What is the tension in the string as it passes through the vertical?

Section 7.2 Elastic Potential Energy

7.15. A force of 800 N stretches a certain spring a distance of 0.200 m. (a) What is the potential energy of the spring when it is stretched 0.200 m? (b) What is its potential energy when it is compressed 5.00 cm?

7.16. An ideal spring of negligible mass is 12.00 cm long when nothing is attached to it. When you hang a 3.15-kg weight from it, you measure its length to be 13.40 cm. If you wanted to store 10.0 J of potential energy in this spring, what would be its *total* length? Assume that it continues to obey Hooke's law.

7.17. A spring stores potential energy U_0 when it is compressed a distance x_0 from its uncompressed length. (a) In terms of U_0, how much energy does it store when it is compressed (i) twice as much and (ii) half as much? (b) In terms of x_0, how much must it be compressed from its uncompressed length to store (i) twice as much energy and (ii) half as much energy?

7.18. A slingshot will shoot a 10-g pebble 22.0 m straight up. (a) How much potential energy is stored in the slingshot's rubber band? (b) With the same potential energy stored in the rubber band, how high can the slingshot shoot a 25-g pebble? (c) What physical effects did you ignore in solving this problem?

7.19. A spring of negligible mass has force constant $k =$ 1600 N/m. (a) How far must the spring be compressed for 3.20 J of potential energy to be stored in it? (b) You place the spring vertically with one end on the floor. You then drop a 1.20-kg book onto it from a height of 0.80 m above the top of the spring. Find the maximum distance the spring will be compressed.

7.20. A 1.20-kg piece of cheese is placed on a vertical spring of negligible mass and force constant $k = 1800$ N/m that is compressed 15.0 cm. When the spring is released, how high does the cheese rise from this initial position? (The cheese and the spring are *not* attached.)

7.21. Consider the glider of Example 7.7 (Section 7.2) and Fig. 7.16. As in the example, the glider is released from rest with the spring stretched 0.100 m. What is the displacement x of the glider from its equilibrium position when its speed is 0.20 m/s? (You should get more than one answer. Explain why.)

7.22. Consider the glider of Example 7.7 (Section 7.2) and Fig. 7.16. (a) As in the example, the glider is released from rest with the spring stretched 0.100 m. What is the speed of the glider when it returns to $x = 0$? (b) What must the initial displacement of the glider be if its maximum speed in the subsequent motion is to be 2.50 m/s?

7.23. A 2.50-kg mass is pushed against a horizontal spring of force constant 25.0 N/cm on a frictionless air table. The spring is attached to the tabletop, and the mass is not attached to the spring in any way. When the spring has been compressed enough to store 11.5 J of potential energy in it, the mass is suddenly released from rest. (a) Find the greatest speed the mass reaches. When does this occur? (b) What is the greatest acceleration of the mass, and when does it occur?

7.24. (a) For the elevator of Example 7.9 (Section 7.2), what is the speed of the elevator after it has moved downward 1.00 m from point 1 in Fig. 7.17? (b) When the elevator is 1.00 m below point 1 in Fig. 7.17, what is its acceleration?

7.25. You are asked to design a spring that will give a 1160-kg satellite a speed of 2.50 m/s relative to an orbiting space shuttle. Your spring is to give the satellite a maximum acceleration of $5.00g$. The spring's mass, the recoil kinetic energy of the shuttle, and changes in gravitational potential energy will all be negligible. (a) What must the force constant of the spring be? (b) What distance must the spring be compressed?

Section 7.3 Conservative and Nonconservative Forces

7.26. A 75-kg roofer climbs a vertical 7.0-m ladder to the flat roof of a house. He then walks 12 m on the roof, climbs down another vertical 7.0-m ladder, and finally walks on the ground back to his starting point. How much work is done on him by gravity (a) as he climbs up; (b) as he climbs down; (c) as he walks on the roof and on the ground? (d) What is the total work done on him by gravity during this round trip? (e) On the basis of your answer to part (d), would you say that gravity is a conservative or nonconservative force? Explain.

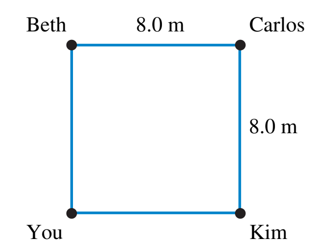
7.27. A 10.0-kg box is pulled by a horizontal wire in a circle on a rough horizontal surface for which the coefficient of kinetic friction is 0.250. Calculate the work done by friction during one complete circular trip if the radius is (a) 2.00 m and (b) 4.00 m. (c) On the basis of the results you just obtained, would you say that friction is a conservative or nonconservative force? Explain.

7.28. In an experiment, one of the forces exerted on a proton is $\vec{F} = -\alpha x^2\hat{\imath}$, where $\alpha = 12$ N/m². (a) How much work does $\vec{F}$ do when the proton moves along the straight-line path from the point $(0.10\text{ m}, 0)$ to the point $(0.10\text{ m}, 0.40\text{ m})$? (b) Along the straight-line path from the point $(0.10\text{ m}, 0)$ to the point $(0.30\text{ m}, 0)$? (c) Along the straight-line path from the point $(0.30\text{ m}, 0)$ to the point $(0.10\text{ m}, 0)$? (d) Is the force $\vec{F}$ conservative? Explain. If $\vec{F}$ is conservative, what is the potential-energy function for it? Let $U = 0$ when $x = 0$.

7.29. A 0.60-kg book slides on a horizontal table. The kinetic friction force on the book has magnitude 1.2 N. (a) How much work is done on the book by friction during a displacement of 3.0 m to the left? (b) The book now slides 3.0 m to the right, returning to its starting point. During this second 3.0-m displacement, how much work is done on the book by friction? (c) What is the total work done on the book by friction during the complete round trip? (d) On the basis of your answer to part (c), would you say that the friction force is conservative or nonconservative? Explain.

7.30. You and three friends stand at the corners of a square whose sides are 8.0 m long in the middle of the gym floor, as shown in Fig. 7.26. You take your physics book and push it from one person to the other. The book has a mass of 1.5 kg, and the coefficient of kinetic friction between the book and the floor is $\mu_k = 0.25$. (a) The book slides from you to Beth and then from Beth to Carlos, along the lines connecting these people. What is the work done by friction during this displacement? (b) You slide the book from you to Carlos along the diagonal of the square. What is the work done by friction during this displacement? (c) You slide the book to Kim who then slides it back to you. What is the total work done by friction during this motion of the book? (d) Is the friction force on the book conservative or nonconservative? Explain.

Figure 7.26 Exercise 7.30.



7.31. A block with mass m is attached to an ideal spring that has force constant k. (a) The block moves from x_1 to x_2, where $x_2 > x_1$. How much work does the spring force do during this displacement? (b) The block moves from x_1 to x_2 and then from x_2 to x_1. How much work does the spring force do during the displacement from x_2 to x_1? What is the total work done by the spring during the entire $x_1 \rightarrow x_2 \rightarrow x_1$ displacement? Explain why you got the answer you did. (c) The block moves from x_1 to x_3, where $x_3 > x_2$. How much work does the spring force do during this displacement? The block then moves from x_3 to x_2. How much work does the spring force do during this displacement? What is the total work done by the spring force during the $x_1 \rightarrow x_3 \rightarrow x_2$ displacement? Compare your answer to the answer in part (a), where the starting and ending points are the same but the path is different.

Section 7.4 Force and Potential Energy

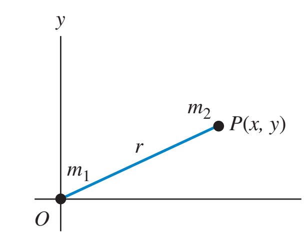
7.32. The potential energy of a pair of hydrogen atoms separated by a large distance x is given by $U(x) = -C_6/x^6$, where C_6 is a positive constant. What is the force that one atom exerts on the other? Is this force attractive or repulsive?

7.33. A force parallel to the x-axis acts on a particle moving along the x-axis. This force produces potential energy $U(x)$ given by $U(x) = \alpha x^4$, where $\alpha = 1.20$ J/m⁴. What is the force (magnitude and direction) when the particle is at $x = -0.800$ m?

7.34. Gravity in One Dimension. Two point masses, m_1 and m_2, lie on the x-axis, with m_1 held in place at the origin and m_2 at position x and free to move. The gravitational potential energy of these masses is found to be $U(x) = -Gm_1m_2/x$, where G is a constant (called the *gravitational constant*). You'll learn more about gravitation in Chapter 12. Find the x-component of the force acting on m_2 due to m_1. Is this force attractive or repulsive? How do you know?

7.35. Gravity in Two Dimensions. Two point masses, m_1 and m_2, lie in the xy-plane, with m_1 held in place at the origin and m_2 free to move a distance r away at a point P having coordinates x and y (Fig. 7.27). The gravitational potential energy of these masses is found to be $U(r) = -Gm_1m_2/r$, where G is the gravitational constant. (a) Show that the components of the force on m_2 due to m_1 are

Figure 7.27 Exercise 7.35.



$$F_x = -\frac{Gm_1m_2x}{(x^2 + y^2)^{3/2}} \qquad \text{and} \qquad F_y = -\frac{Gm_1m_2y}{(x^2 + y^2)^{3/2}}$$

(*Hint:* First write r in terms of x and y.) (b) Show that the magnitude of the force on m_2 is $F = Gm_1m_2/r^2$. (c) Does m_1 attract or repel m_2? How do you know?

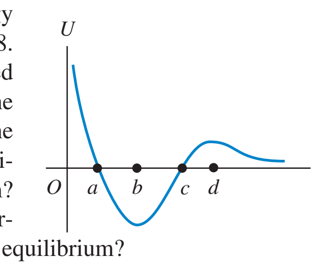
7.36. An object moving in the xy-plane is acted on by a conservative force described by the potential-energy function $U(x, y) = \alpha(1/x^2 + 1/y^2)$, where α is a positive constant. Derive an expression for the force expressed in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

Section 7.5 Energy Diagrams

7.37. The potential energy of two atoms in a diatomic molecule is approximated by $U(r) = a/r^{12} - b/r^6$, where r is the spacing between atoms and a and b are positive constants. (a) Find the force $F(r)$ on one atom as a function of r. Make two graphs, one of $U(r)$ versus r and one of $F(r)$ versus r. (b) Find the equilibrium distance between the two atoms. Is this equilibrium stable? (c) Suppose the distance between the two atoms is equal to the equilibrium distance found in part (b). What minimum energy must be added to the molecule to *dissociate* it—that is, to separate the two atoms to an infinite distance apart? This is called the *dissociation energy* of the molecule. (d) For the molecule CO, the equilibrium distance between the carbon and oxygen atoms is 1.13×10^{-10} m and the dissociation energy is 1.54×10^{-18} J per molecule. Find the values of the constants a and b.

7.38. A marble moves along the x-axis. The potential-energy function is shown in Fig. 7.28. (a) At which of the labeled x-coordinates is the force on the marble zero? (b) Which of the labeled x-coordinates is a position of stable equilibrium? (c) Which of the labeled x-coordinates is a position of unstable equilibrium?

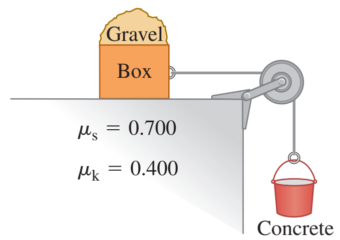
Figure 7.28 Exercise 7.38.



Problems

7.39. At a construction site, a 65.0-kg bucket of concrete hangs from a light (but strong) cable that passes over a light friction-free pulley and is connected to an 80.0-kg box on a horizontal roof (Fig. 7.29). The cable pulls horizontally on the box, and a 50.0-kg bag of gravel rests on top of the box. The coefficients of friction between the box and roof are shown. (a) Find the friction force on the bag of gravel and on the box. (b) Suddenly a worker picks up the bag of gravel. Use energy conservation to find the speed of the bucket after it has descended 2.00 m, from rest. (You can check your answer by solving this problem using Newton's laws.)

Figure 7.29 Problem 7.39.

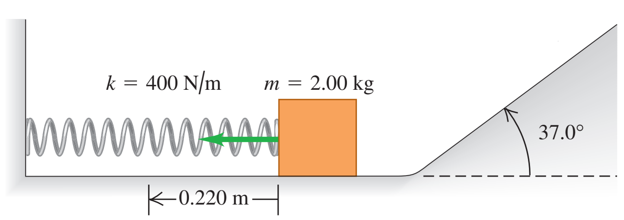


7.40. Two blocks with different mass are attached to either end of a light rope that passes over a light, frictionless pulley that is suspended from the ceiling. The masses are released from rest, and the more massive one starts to descend. After this block has descended 1.20 m, its speed is 3.00 m/s. If the total mass of the two blocks is 15.0 kg, what is the mass of each block?

7.41. Legal Physics. In an auto accident, a car hit a pedestrian and the driver then slammed on the brakes to stop the car. During the subsequent trial, the driver's lawyer claimed that he was obeying the posted 35 mi/h speed limit, but that the legal speed was too high to allow him to see and react to the pedestrian in time. You have been called in as the state's expert witness. Your investigation of the accident found that the skid marks made while the brakes were applied were 280 ft long, and the tread on the tires produced a coefficient of kinetic friction of 0.30 with the road. (a) In your testimony in court, will you say that the driver was obeying the posted speed? You must be able to back up your conclusion with clear reasoning because one of the lawyers will surely cross-examine you. (b) If the driver's speeding ticket were \$10 for each mile per hour he was driving above the posted speed limit, would he have to pay a fine? If so, how much would it be?

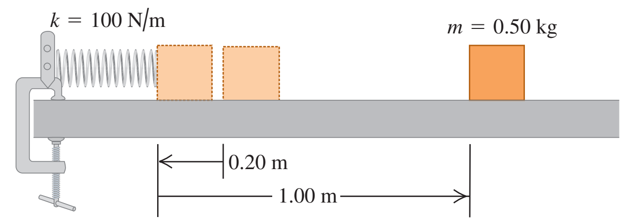
7.42. A 2.00-kg block is pushed against a spring with negligible mass and force constant $k = 400$ N/m, compressing it 0.220 m. When the block is released, it moves along a frictionless, horizontal surface and then up a frictionless incline with slope 37.0° (Fig. 7.30). (a) What is the speed of the block as it slides along the horizontal surface after having left the spring? (b) How far does the block travel up the incline before starting to slide back down?

Figure 7.30 Problem 7.42.



7.43. A block with mass 0.50 kg is forced against a horizontal spring of negligible mass, compressing the spring a distance of 0.20 m (Fig. 7.31). When released, the block moves on a horizontal tabletop for 1.00 m before coming to rest. The spring constant k is 100 N/m. What is the coefficient of kinetic friction μ_k between the block and the tabletop?

Figure 7.31 Problem 7.43.

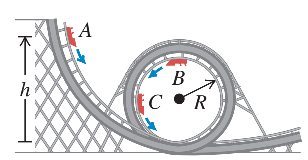


7.44. On a horizontal surface, a crate with mass 50.0 kg is placed against a spring that stores 360 J of energy. The spring is released, and the crate slides 5.60 m before coming to rest. What is the speed of the crate when it is 2.00 m from its initial position?

7.45. Bouncing Ball. A 650-gram rubber ball is dropped from an initial height of 2.50 m, and on each bounce it returns to 75% of its previous height. (a) What is the initial mechanical energy of the ball, just after it is released from its initial height? (b) How much mechanical energy does the ball lose during its first bounce? What happens to this energy? (c) How much mechanical energy is lost during the second bounce?

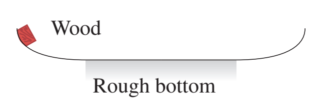
7.46. Riding a Loop-the-Loop. A car in an amusement park ride rolls without friction around the track shown in Fig. 7.32. It starts from rest at point A at a height h above the bottom of the loop. Treat the car as a particle. (a) What is the minimum value of h (in terms of R) such that the car moves around the loop without falling off at the top (point B)? (b) If $h = 3.50R$ and $R = 20.0$ m, compute the speed, radial acceleration, and tangential acceleration of the passengers when the car is at point C, which is at the end of a horizontal diameter. Show these acceleration components in a diagram, approximately to scale.

Figure 7.32 Problem 7.46.



7.47. A 2.0-kg piece of wood slides on the surface shown in Fig. 7.33. The curved sides are perfectly smooth, but the rough horizontal bottom is 30 m long and has a kinetic friction coefficient of 0.20 with the wood. The piece of wood starts from rest 4.0 m above the rough bottom. (a) Where will this wood eventually come to rest? (b) For the motion from the initial release until the piece of wood comes to rest, what is the total amount of work done by friction?

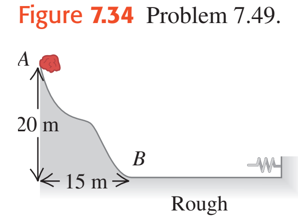
Figure 7.33 Problem 7.47.



7.48. Up and Down the Hill. A 28-kg rock approaches the foot of a hill with a speed of 15 m/s. This hill slopes upward at a constant angle of 40.0° above the horizontal. The coefficients of static and kinetic friction between the hill and the rock are 0.75 and 0.20, respectively. (a) Use energy conservation to find the maximum height above the foot of the hill reached by the rock. (b) Will the rock remain at rest at its highest point, or will it slide back down the hill? (c) If the rock does slide back down, find its speed when it returns to the bottom of the hill.

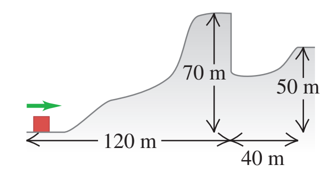
7.49. A 15.0-kg stone slides down a snow-covered hill (Fig. 7.34), leaving point A with a speed of 10.0 m/s. There is no friction on the hill between points A and B, but there is friction on the level ground at the bottom of the hill, between B and the wall. After entering the rough horizontal region, the stone travels 100 m and then runs into a very long, light spring with force constant 2.00 N/m. The coefficients of kinetic and static friction between the stone and the horizontal ground are 0.20 and 0.80, respectively. (a) What is the speed of the stone when it reaches point B? (b) How far will the stone compress the spring? (c) Will the stone move again after it has been stopped by the spring?

Figure 7.34 Problem 7.49.



7.50. A 2.8-kg block slides over the smooth, icy hill shown in Fig. 7.35. The top of the hill is horizontal and 70 m higher than its base. What minimum speed must the block have at the base of the hill so that it will not fall into the pit on the far side of the hill?

Figure 7.35 Problem 7.50.



7.51. Bungee Jump. A bungee cord is 30.0 m long and, when stretched a distance x, it exerts a restoring force of magnitude kx. Your father-in-law (mass 95.0 kg) stands on a platform 45.0 m above the ground, and one end of the cord is tied securely to his ankle and the other end to the platform. You have promised him that when he steps off the platform he will fall a maximum distance of only 41.0 m before the cord stops him. You had several bungee cords to select from, and you tested them by stretching them out, tying one end to a tree, and pulling on the other end with a force of 380.0 N. When you do this, what distance will the bungee cord that you should select have stretched?

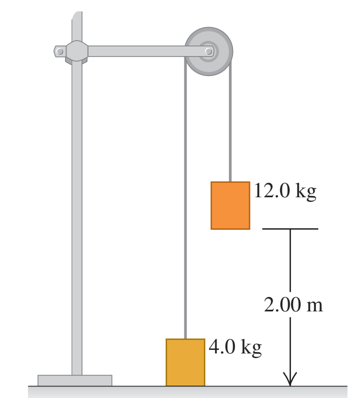
7.52. Ski Jump Ramp. You are designing a ski jump ramp for the next Winter Olympics. You need to calculate the vertical height h from the starting gate to the bottom of the ramp. The skiers push off hard with their ski poles at the start, just above the starting gate, so they typically have a speed of 2.0 m/s as they reach the gate. For safety, the skiers should have a speed of no more than 30.0 m/s when they reach the bottom of the ramp. You determine that for a 85.0-kg skier with good form, friction and air resistance will do total work of magnitude 4000 J on him during his run down the slope. What is the maximum height h for which the maximum safe speed will not be exceeded?

7.53. The Great Sandini is a 60-kg circus performer who is shot from a cannon (actually a spring gun). You don't find many men of his caliber, so you help him design a new gun. This new gun has a very large spring with a very small mass and a force constant of 1100 N/m that he will compress with a force of 4400 N. The inside of the gun barrel is coated with Teflon, so the average friction force will be only 40 N during the 4.0 m he moves in the barrel. At what speed will he emerge from the end of the barrel, 2.5 m above his initial rest position?

7.54. You are designing a delivery ramp for crates containing exercise equipment. The 1470-N crates will move at 1.8 m/s at the top of a ramp that slopes downward at 22.0°. The ramp exerts a 550-N kinetic friction force on each crate, and the maximum static friction force also has this value. Each crate will compress a spring at the bottom of the ramp and will come to rest after traveling a total distance of 8.0 m along the ramp. Once stopped, a crate must not rebound back up the ramp. Calculate the force constant of the spring that will be needed in order to meet the design criteria.

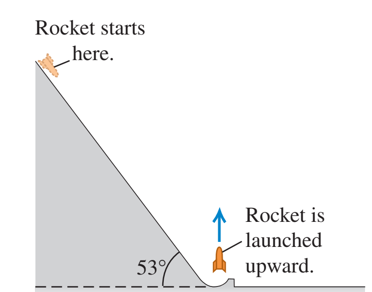
7.55. A system of two paint buckets connected by a lightweight rope is released from rest with the 12.0-kg bucket 2.00 m above the floor (Fig. 7.36). Use the principle of conservation of energy to find the speed with which this bucket strikes the floor. You can ignore friction and the mass of the pulley.

Figure 7.36 Problem 7.55.



7.56. A 1500-kg rocket is to be launched with an initial upward speed of 50.0 m/s. In order to assist its engines, the engineers will start it from rest on a ramp that rises 53° above the horizontal (Fig. 7.37). At the bottom, the ramp turns upward and launches the rocket vertically. The engines provide a constant forward thrust of 2000 N, and friction with the ramp surface is a constant 500 N. How far from the base of the ramp should the rocket start, as measured along the surface of the ramp?

Figure 7.37 Problem 7.56.



7.57. A machine part of mass m is attached to a horizontal ideal spring of force constant k that is attached to the edge of a frictionfree horizontal surface. The part is pushed against the spring, compressing it a distance x_0, and then released from rest. Find the maximum (a) speed and (b) acceleration of the machine part. (c) Where in the motion do the maxima in parts (a) and (b) occur? (d) What will be the maximum extension of the spring? (e) Describe the subsequent motion of this machine part. Will it ever stop permanently?

7.58. A wooden rod of negligible mass and length 80.0 cm is pivoted about a horizontal axis through its center. A white rat with mass 0.500 kg clings to one end of the stick, and a mouse with mass 0.200 kg clings to the other end. The system is released from rest with the rod horizontal. If the animals can manage to hold on, what are their speeds as the rod swings through a vertical position?

7.59. A 0.100-kg potato is tied to a string with length 2.50 m, and the other end of the string is tied to a rigid support. The potato is held straight out horizontally from the point of support, with the string pulled taut, and is then released. (a) What is the speed of the potato at the lowest point of its motion? (b) What is the tension in the string at this point?

7.60. These data are from a computer simulation for a batted baseball with mass 0.145 kg, including air resistance:

t	x	y	v_x	v_y
0	0	0	30.0 m/s	40.0 m/s
3.05 s	70.2 m	53.6 m	18.6 m/s	0
6.59 s	124.4 m	0	11.9 m/s	−28.7 m/s

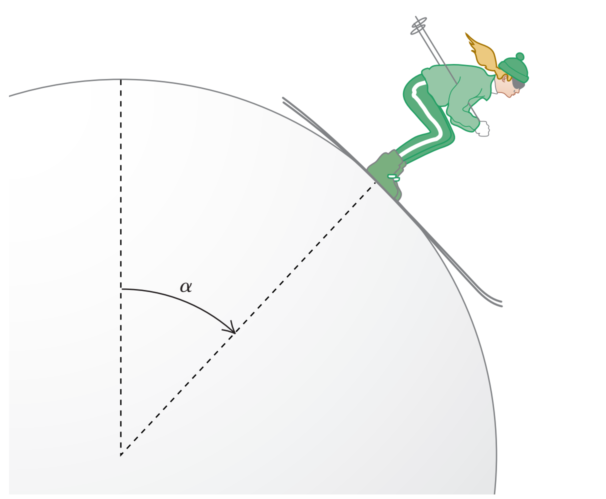
(a) How much work was done by the air on the baseball as it moved from its initial position to its maximum height? (b) How much work was done by the air on the baseball as it moved from its maximum height back to the starting elevation? (c) Explain why the magnitude of the answer in part (b) is smaller than the magnitude of the answer in part (a).

7.61. Down the Pole. A fireman of mass m slides a distance d down a pole. He starts from rest. He moves as fast at the bottom as if he had stepped off a platform a distance $h \le d$ above the ground and descended with negligible air resistance. (a) What average friction force did the fireman exert on the pole? Does your answer make sense in the special cases of $h = d$ and $h = 0$? (b) Find a numerical value for the average friction force a 75-kg fireman exerts, for $d = 2.5$ m and $h = 1.0$ m. (c) In terms of g, h, and d, what is the speed of the fireman when he is a distance y above the bottom of the pole?

7.62. A 60.0-kg skier starts from rest at the top of a ski slope 65.0 m high. (a) If frictional forces do −10.5 kJ of work on her as she descends, how fast is she going at the bottom of the slope? (b) Now moving horizontally, the skier crosses a patch of soft snow, where $\mu_k = 0.20$. If the patch is 82.0 m wide and the average force of air resistance on the skier is 160 N, how fast is she going after crossing the patch? (c) The skier hits a snowdrift and penetrates 2.5 m into it before coming to a stop. What is the average force exerted on her by the snowdrift as it stops her?

7.63. A skier starts at the top of a very large, frictionless snowball, with a very small initial speed, and skis straight down the side (Fig. 7.38). At what point does she lose contact with the snowball and fly off at a tangent? That is, at the instant she loses contact with the snowball, what angle α does a radial line from the center of the snowball to the skier make with the vertical?

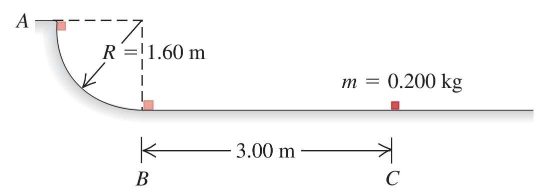
Figure 7.38 Problem 7.63.



7.64. A rock is tied to a cord and the other end of the cord is held fixed. The rock is given an initial tangential velocity that causes it to rotate in a vertical circle. Prove that the tension in the cord at the lowest point exceeds the tension at the highest point by six times the weight of the rock.

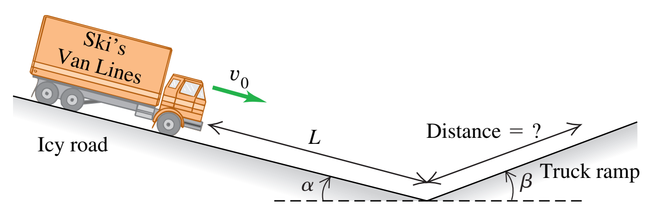
7.65. In a truck-loading station at a post office, a small 0.200-kg package is released from rest at point A on a track that is one-quarter of a circle with radius 1.60 m (Fig. 7.39). The size of the package is much less than 1.60 m, so the package can be treated as a particle. It slides down the track and reaches point B with a speed of 4.80 m/s. From point B, it slides on a level surface a distance of 3.00 m to point C, where it comes to rest. (a) What is the coefficient of kinetic friction on the horizontal surface? (b) How much work is done on the package by friction as it slides down the circular arc from A to B?

Figure 7.39 Problem 7.65.



7.66. A truck with mass m has a brake failure while going down an icy mountain road of constant downward slope angle α (Fig. 7.40). Initially the truck is moving downhill at speed v_0. After careening downhill a distance L with negligible friction, the truck driver steers the runaway vehicle onto a runaway truck ramp of constant upward slope angle β. The truck ramp has a soft sand surface for which the coefficient of rolling friction is μ_r. What is the distance that the truck moves up the ramp before coming to a halt? Solve using energy methods.

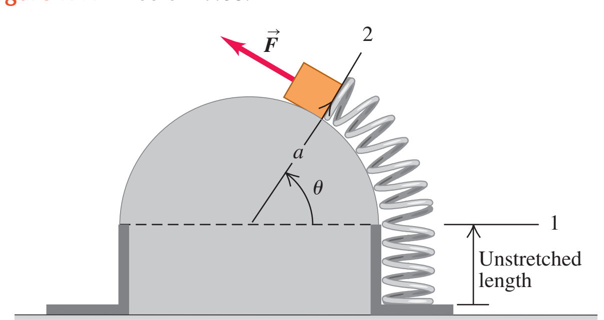
Figure 7.40 Problem 7.66.



7.67. A certain spring is found *not* to obey Hooke's law; it exerts a restoring force $F_x(x) = -\alpha x - \beta x^2$ if it is stretched or compressed, where $\alpha = 60.0$ N/m and $\beta = 18.0$ N/m^2. The mass of the spring is negligible. (a) Calculate the potential-energy function $U(x)$ for this spring. Let $U = 0$ when $x = 0$. (b) An object with mass 0.900 kg on a frictionless, horizontal surface is attached to this spring, pulled a distance 1.00 m to the right (the $+x$-direction) to stretch the spring, and released. What is the speed of the object when it is 0.50 m to the right of the $x = 0$ equilibrium position?

7.68. A variable force $\vec{F}$ is maintained tangent to a frictionless, semicircular surface (Fig. 7.41). By slow variations in the force, a block with weight w is moved, and the spring to which it is attached is stretched from position 1 to position 2. The spring has negligible mass and force constant k. The end of the spring moves in an arc of radius a. Calculate the work done by the force $\vec{F}$.

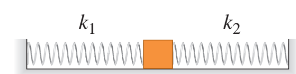
Figure 7.41 Problem 7.68.



7.69. A 0.150-kg block of ice is placed against a horizontal, compressed spring mounted on a horizontal tabletop that is 1.20 m above the floor. The spring has force constant 1900 N/m and is initially compressed 0.045 m. The mass of the spring is negligible. The spring is released, and the block slides along the table, goes off the edge, and travels to the floor. If there is negligible friction between the block of ice and the tabletop, what is the speed of the block of ice when it reaches the floor?

7.70. A 3.00-kg block is connected to two ideal horizontal springs having force constants $k_1 = 25.0$ N/cm and $k_2 = 20.0$ N/cm (Fig. 7.42). The system is initially in equilibrium on a horizontal, frictionless surface. The block is now pushed 15.0 cm to the right and released from rest. (a) What is the maximum speed of the block? Where in the motion does the maximum speed occur? (b) What is the maximum compression of spring 1?

Figure 7.42 Problem 7.70.



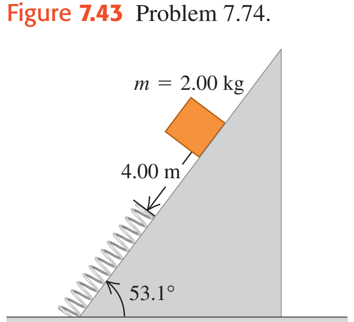
7.71. An experimental apparatus with mass m is placed on a vertical spring of negligible mass and pushed down until the spring is compressed a distance x. The apparatus is then released and reaches its maximum height at a distance h above the point where it is released. The apparatus is not attached to the spring, and at its maximum height it is no longer in contact with the spring. The maximum magnitude of acceleration the apparatus can have without being damaged is a, where $a > g$. (a) What should the force constant of the spring be? (b) What distance x must the spring be compressed initially?

7.72. If a fish is attached to a vertical spring and slowly lowered to its equilibrium position, it is found to stretch the spring by an amount d. If the same fish is attached to the end of the unstretched spring and then allowed to fall from rest, through what maximum distance does it stretch the spring? (*Hint:* Calculate the force constant of the spring in terms of the distance d and the mass m of the fish.)

7.73. A wooden block with mass 1.50 kg is placed against a compressed spring at the bottom of an incline of slope 30.0° (point A). When the spring is released, it projects the block up the incline. At point B, a distance of 6.00 m up the incline from A, the block is moving up the incline at 7.00 m/s and is no longer in contact with the spring. The coefficient of kinetic friction between the block and the incline is $\mu_k = 0.50$. The mass of the spring is negligible. Calculate the amount of potential energy that was initially stored in the spring.

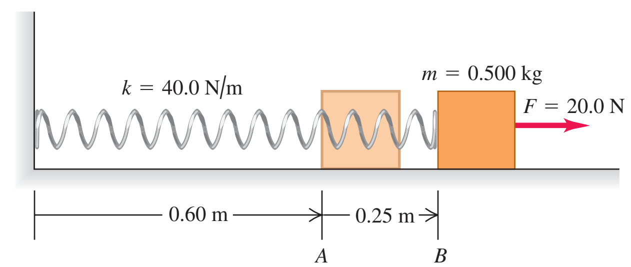
7.74. A 2.00-kg package is released on a 53.1° incline, 4.00 m from a long spring with force constant 120 N/m that is attached at the bottom of the incline (Fig. 7.43). The coefficients of friction between the package and the incline are $\mu_s = 0.40$ and $\mu_k = 0.20$. The mass of the spring is negligible. (a) What is the speed of the package just before it reaches the spring? (b) What is the maximum compression of the spring? (c) The package rebounds back up the incline. How close does it get to its initial position?

Figure 7.43 Problem 7.74.



7.75. A 0.500-kg block, attached to a spring with length 0.60 m and force constant 40.0 N/m, is at rest with the back of the block at point A on a frictionless, horizontal air table (Fig. 7.44). The mass of the spring is negligible. You move the block to the right along the surface by pulling with a constant 20.0-N horizontal force. (a) What is the block's speed when the back of the block reaches point B, which is 0.25 m to the right of point A? (b) When the back of the block reaches point B, you let go of the block. In the subsequent motion, how close does the block get to the wall where the left end of the spring is attached?

Figure 7.44 Problem 7.75.



7.76. Fraternity Physics. The brothers of Iota Eta Pi fraternity build a platform, supported at all four corners by vertical springs, in the basement of their frat house. A brave fraternity brother wearing a football helmet stands in the middle of the platform; his weight compresses the springs by 0.18 m. Then four of his fraternity brothers, pushing down at the corners of the platform, compress the springs another 0.53 m until the top of the brave brother's helmet is 0.90 m below the basement ceiling. They then simultaneously release the platform. You can ignore the masses of the springs and platform. (a) When the dust clears, the fraternity asks you to calculate their fraternity brother's speed just before his helmet hit the flimsy ceiling. (b) Without the ceiling, how high would he have gone? (c) In discussing their probation, the dean of students suggests that the next time they try this, they do it outdoors on another planet. Would the answer to part (b) be the same if this stunt were performed on a planet with a different value of g? Assume that the fraternity brothers push the platform down 0.53 m as before. Explain your reasoning.

7.77. A particle with mass m is acted on by a conservative force and moves along a path given by $x = x_0 \cos \omega_0 t$ and $y = y_0 \sin \omega_0 t$, where x_0, y_0, and ω_0 are constants. (a) Find the components of the force that acts on the particle. (b) Find the potential energy of the particle as a function of x and y. Take $U = 0$ when $x = 0$ and $y = 0$. (c) Find the total energy of the particle when (i) $x = x_0$, $y = 0$ and (ii) $x = 0$, $y = y_0$.

7.78. When it is burned, 1 gallon of gasoline produces 1.3×10^8 J of energy. A 1500-kg car accelerates from rest to 37 m/s in 10 s. The engine of this car is only 15% efficient (which is typical), meaning that only 15% of the energy from the combustion of the gasoline is used to accelerate the car. The rest goes into things like the internal kinetic energy of the engine parts as well as heating of the exhaust air and engine. (a) How many gallons of gasoline does this car use during the acceleration? (b) How many such accelerations will it take to burn up 1 gallon of gas?

7.79. A hydroelectric dam holds back a lake of surface area 3.0×10^6 m^2 that has vertical sides below the water level. The water level in the lake is 150 m above the base of the dam. When the water passes through turbines at the base of the dam, its mechanical energy is converted into electrical energy with 90% efficiency. (a) If gravitational potential energy is taken to be zero at the base of the dam, how much energy is stored in the top meter of the water in the lake? The density of water is 1000 kg/m^3. (b) What volume of water must pass through the dam to produce 1000 kilowatt-hours of electrical energy? What distance does the level of water in the lake fall when this much water passes through the dam?

7.80. How much total energy is stored in the lake in Problem 7.79? As in that problem, take the gravitational potential energy to be zero at the base of the dam. Express your answer in joules and in kilowatt-hours. (*Hint:* Break the lake up into infinitesimal horizontal layers of thickness dy, and integrate to find the total potential energy.)

7.81. Gravity in Three Dimensions. A point mass m_1 is held in place at the origin, and another point mass m_2 is free to move a distance r away at a point P having coordinates x, y, and z. The gravitational potential energy of these masses is found to be $U(r) = -Gm_1m_2/r$, where G is the gravitational constant (see Exercises 7.34 and 7.35). (a) Show that the components of the force on m_2 due to m_1 are

$$F_x = -\frac{Gm_1m_2x}{(x^2+y^2+z^2)^{3/2}} \qquad F_y = -\frac{Gm_1m_2y}{(x^2+y^2+z^2)^{3/2}}$$

$$F_z = -\frac{Gm_1m_2z}{(x^2+y^2+z^2)^{3/2}}$$

(*Hint:* First write r in terms of x, y, and z.) (b) Show that the magnitude of the force on m_2 is $F = Gm_1m_2/r^2$. (c) Does m_1 attract or repel m_2? How do you know?

7.82. (a) Is the force $\vec{F} = Cy^2\hat{j}$, where C is a negative constant with units of N/m^2, conservative or nonconservative? Justify your answer. (b) Is the force $\vec{F} = Cy^2\hat{i}$, where C is a negative constant with units of N/m^2, conservative or nonconservative? Justify your answer.

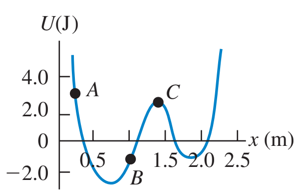
7.83. A cutting tool under microprocessor control has several forces acting on it. One force is $\vec{F} = -\alpha xy^2\hat{j}$, a force in the negative y-direction whose magnitude depends on the position of the tool. The constant is $\alpha = 2.50\ N/m^3$. Consider the displacement of the tool from the origin to the point $x = 3.00$ m, $y = 3.00$ m. (a) Calculate the work done on the tool by $\vec{F}$ if this displacement is along the straight line $y = x$ that connects these two points. (b) Calculate the work done on the tool by $\vec{F}$ if the tool is first moved out along the x-axis to the point $x = 3.00$ m, $y = 0$ and then moved parallel to the y-axis to the point $x = 3.00$ m, $y = 3.00$ m. (c) Compare the work done by $\vec{F}$ along these two paths. Is $\vec{F}$ conservative or nonconservative? Explain.

7.84. An object has several forces acting on it. One force is $\vec{F} = \alpha xy\hat{i}$, a force in the x-direction whose magnitude depends on the position of the object. (See Problem 6.96.) The constant is $\alpha = 2.00\ N/m^2$. The object moves along the following path: (1) It starts at the origin and moves along the y-axis to the point $x = 0$, $y = 1.50$ m; (2) it moves parallel to the x-axis to the point $x = 1.50$ m, $y = 1.50$ m; (3) it moves parallel to the y-axis to the point $x = 1.50$ m, $y = 0$; (4) it moves parallel to the x-axis back to the origin. (a) Sketch this path in the xy-plane. (b) Calculate the work done on the object by $\vec{F}$ for each leg of the path and for the complete round trip. (c) Is $\vec{F}$ conservative or nonconservative? Explain.

7.85. A Hooke's law force $-kx$ and a constant conservative force F in the $+x$-direction act on an atomic ion. (a) Show that a possible potential-energy function for this combination of forces is $U(x) = \frac{1}{2}kx^2 - Fx - F^2/2k$. Is this the *only* possible function? Explain. (b) Find the stable equilibrium position. (c) Graph $U(x)$ (in units of F^2/k) versus x (in units of F/k) for values of x between $-5F/k$ and $5F/k$. (d) Are there any unstable equilibrium positions? (e) If the total energy is $E = F^2/k$, what are the maximum and minimum values of x that the ion reaches in its motion? If the ion has mass m, find its maximum speed if the total energy is $E = F^2/k$. For what value of x is the speed maximum?

7.86. A particle moves along the x-axis while acted on by a single conservative force parallel to the x-axis. The force corresponds to the potential-energy function graphed in Fig. 7.45. The particle is released from rest at point A. (a) What is the direction of the force on the particle when it is at point A? (b) At point B? (c) At what value of x is the kinetic energy of the particle a maximum? (d) What is the force on the particle when it is at point C? (e) What is the largest value of x reached by the particle during its motion? (f) What value or values of x correspond to points of stable equilibrium? (g) Of unstable equilibrium?

Figure 7.45 Problem 7.86.



Challenge Problem

7.87. A proton with mass m moves in one dimension. The potential-energy function is $U(x) = \alpha/x^2 - \beta/x$, where α and β are positive constants. The proton is released from rest at $x_0 = \alpha/\beta$. (a) Show that $U(x)$ can be written as

$$U(x) = \frac{\alpha}{x_0^2}\left[\left(\frac{x_0}{x}\right)^2 - \frac{x_0}{x}\right]$$

Graph $U(x)$. Calculate $U(x_0)$ and thereby locate the point x_0 on the graph. (b) Calculate $v(x)$, the speed of the proton as a function of position. Graph $v(x)$ and give a qualitative description of the motion. (c) For what value of x is the speed of the proton a maximum? What is the value of that maximum speed? (d) What is the force on the proton at the point in part (c)? (e) Let the proton be released instead at $x_1 = 3\alpha/\beta$. Locate the point x_1 on the graph of $U(x)$. Calculate $v(x)$ and give a qualitative description of the motion. (f) For each release point ($x = x_0$ and $x = x_1$), what are the maximum and minimum values of x reached during the motion?