SOURCES OF MAGNETIC FIELD

28

? The immense cylinder in this photograph is actually a current-carrying coil, or solenoid, that generates a uniform magnetic field in its interior as part of an experiment at CERN, the European Laboratory for Particle Physics. If two such solenoids were joined end to end, how much stronger would the magnetic field become?

LEARNING GOALS

By studying this chapter, you will learn:

- The nature of the magnetic field produced by a single moving charged particle.
- How to describe the magnetic field produced by an element of a current-carrying conductor.
- How to calculate the magnetic field produced by a long, straight, current-carrying wire.
- Why wires carrying current in the same direction attract, while wires carrying opposing currents repel.
- How to calculate the magnetic field produced by a current-carrying wire bent into a circle.
- What Ampere's law is, and what it tells us about magnetic fields.
- How to use Ampere's law to calculate the magnetic field of symmetric current distributions.

In Chapter 27 we studied the forces exerted on moving charges and on current-carrying conductors in a magnetic field. We didn't worry about how the magnetic field got there; we simply took its existence as a given fact. But how are magnetic fields *created?* We know that both permanent magnets and electric currents in electromagnets create magnetic fields. In this chapter we will study these sources of magnetic field in detail.

We've learned that a charge creates an electric field and that an electric field exerts a force on a charge. But a *magnetic* field exerts a force only on a *moving* charge. Is it also true that a charge *creates* a magnetic field only when the charge is moving? In a word, yes.

Our analysis will begin with the magnetic field created by a single moving point charge. We can use this analysis to determine the field created by a small segment of a current-carrying conductor. Once we can do that, we can in principle find the magnetic field produced by *any* shape of conductor.

Then we will introduce Ampere's law, which plays a role in magnetism analogous to the role of Gauss's law in electrostatics. Ampere's law lets us exploit symmetry properties in relating magnetic fields to their sources.

Moving charged particles within atoms respond to magnetic fields and can also act as sources of magnetic field. We'll use these ideas to understand how certain magnetic materials can be used to intensify magnetic fields as well as why some materials such as iron act as permanent magnets.

28.1 Magnetic Field of a Moving Charge

Let's start with the basics, the magnetic field of a single point charge q moving with a constant velocity $\vec{v}$. In practical applications, such as the solenoid shown in the photo that opens this chapter, magnetic fields are produced by tremendous numbers of charged particles moving together in a current. But once we understand how to calculate the magnetic field due to a single point charge, it's a small leap to calculate the field due to a current-carrying wire or collection of wires.

As we did for electric fields, we call the location of the moving charge at a given instant the **source point** and the point P where we want to find the field the **field point.** In Section 21.4 we found that at a field point a distance r from a point charge q, the magnitude of the *electric* field $\vec{E}$ caused by the charge is proportional to the charge magnitude $|q|$ and to $1/r^2$, and the direction of $\vec{E}$ (for positive q) is along the line from source point to field point. The corresponding relationship for the *magnetic* field $\vec{B}$ of a point charge q moving with constant velocity has some similarities and some interesting differences.

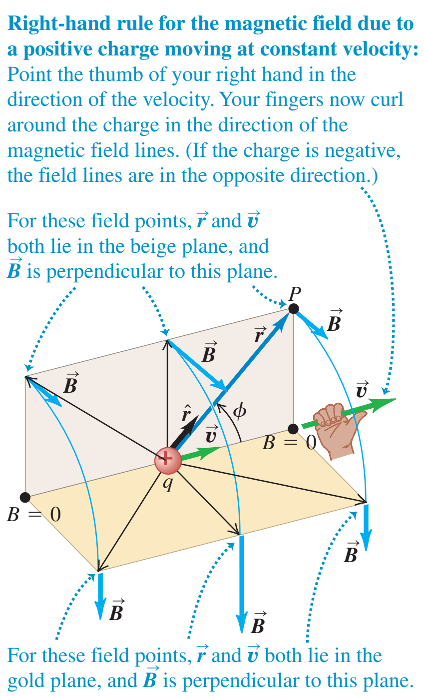
Experiments show that the magnitude of $\vec{B}$ is also proportional to $|q|$ and to $1/r^2$. But the *direction* of $\vec{B}$ is *not* along the line from source point to field point. Instead, $\vec{B}$ is perpendicular to the plane containing this line and the particle's velocity vector $\vec{v}$, as shown in Fig. 28.1. Furthermore, the field *magnitude* B is also proportional to the particle's speed v and to the sine of the angle ϕ. Thus the magnetic field magnitude at point P is given by

$$B = \frac{\mu_0}{4\pi}\frac{|q|v\sin\phi}{r^2} \tag{28.1}$$

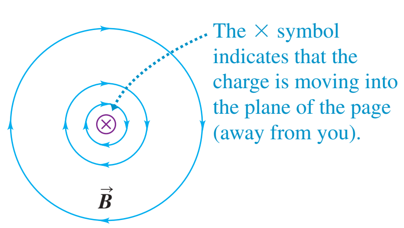
where $\mu_0/4\pi$ is a proportionality constant (μ_0 is read as "mu-nought" or "mu-sub-zero"). The reason for writing the constant in this particular way will emerge shortly. We did something similar with Coulomb's law in Section 21.3.

28.1 (a) Magnetic-field vectors due to a moving positive point charge q. At each point, $\vec{B}$ is perpendicular to the plane of $\vec{r}$ and $\vec{v}$, and its magnitude is proportional to the sine of the angle between them. (b) Magnetic field lines in a plane containing a moving positive charge.

(a) Perspective view



(b) View from behind the charge



Moving Charge: Vector Magnetic Field

We can incorporate both the magnitude and direction of $\vec{B}$ into a single vector equation using the vector product. To avoid having to say "the direction from the source q to the field point P" over and over, we introduce a *unit* vector $\hat{r}$ ("r-hat") that points from the source point to the field point. (We used $\hat{r}$ for the same purpose in Section 21.4.) This unit vector is equal to the vector $\vec{r}$ from the source to the field point divided by its magnitude: $\hat{r} = \vec{r}/r$. Then the $\vec{B}$ field of a moving point charge is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2} \quad \text{(magnetic field of a point charge with constant velocity)} \tag{28.2}$$

Figure 28.1 shows the relationship of $\hat{r}$ to P and also shows the magnetic field $\vec{B}$ at several points in the vicinity of the charge. At all points along a line through the charge parallel to the velocity $\vec{v}$, the field is zero because $\sin\phi = 0$ at all such points. At any distance r from q, $\vec{B}$ has its greatest magnitude at points lying in the plane perpendicular to $\vec{v}$ because at all such points, $\phi = 90°$ and $\sin\phi = 1$. If the charge q is negative, the directions of $\vec{B}$ are opposite to those shown in Fig. 28.1.

Moving Charge: Magnetic Field Lines

A point charge in motion also produces an *electric* field, with field lines that radiate outward from a positive charge. The *magnetic* field lines are completely different. The above discussion shows that for a point charge moving with velocity $\vec{v}$, the magnetic field lines are *circles* centered on the line of $\vec{v}$ and lying in planes perpendicular to this line. The field-line directions for a positive charge are given by the following *right-hand rule,* one of several that we will encounter in this chapter for determining the direction of the magnetic field caused by different sources. Grasp the velocity vector $\vec{v}$ with your right hand so that your right thumb points in the direction of $\vec{v}$; your fingers then curl around the line of $\vec{v}$ in the same sense as the magnetic field lines, assuming q is positive. Figure 28.1a shows parts of a few field lines; Fig. 28.1b shows some field lines in a plane through q, perpendicular to $\vec{v}$, as seen by looking in the direction of $\vec{v}$. If the point charge is negative, the directions of the field and field lines are the opposite of those shown in Fig. 28.1.

Equations (28.1) and (28.2) describe the $\vec{B}$ field of a point charge moving with *constant* velocity. If the charge *accelerates,* the field can be much more complicated. We won't need these more complicated results for our purposes. (The moving charged particles that make up a current in a wire accelerate at points where the wire bends and the direction of $\vec{v}$ changes. But because the magnitude v_d of the drift velocity in a conductor is typically very small, the acceleration v_d^2/r is also very small, and the effects of acceleration can be ignored.)

As we discussed in Section 27.2, the unit of B is one tesla (1 T):

$$1\text{ T} = 1\text{ N}\cdot\text{s}/\text{C}\cdot\text{m} = 1\text{ N}/\text{A}\cdot\text{m}$$

Using this with Eq. (28.1) or (28.2), we find that the units of the constant μ_0 are

$$1\text{ N}\cdot\text{s}^2/\text{C}^2 = 1\text{ N}/\text{A}^2 = 1\text{ Wb}/\text{A}\cdot\text{m} = 1\text{ T}\cdot\text{m}/\text{A}$$

In SI units the numerical value of μ_0 is exactly $4\pi \times 10^{-7}$. Thus

$$\begin{aligned}\mu_0 &= 4\pi \times 10^{-7}\text{ N}\cdot\text{s}^2/\text{C}^2 = 4\pi \times 10^{-7}\text{ Wb}/\text{A}\cdot\text{m}\\ &= 4\pi \times 10^{-7}\text{ T}\cdot\text{m}/\text{A}\end{aligned} \tag{28.3}$$

It may seem incredible that μ_0 has *exactly* this numerical value! In fact this is a *defined* value that arises from the definition of the ampere, as we'll discuss in Section 28.4.

We mentioned in Section 21.3 that the constant $1/4\pi\epsilon_0$ in Coulomb's law is related to the speed of light c:

$$k = \frac{1}{4\pi\epsilon_0} = (10^{-7}\text{ N}\cdot\text{s}^2/\text{C}^2)c^2$$

When we study electromagnetic waves in Chapter 32, we will find that their speed of propagation in vacuum, which is equal to the speed of light c, is given by

$$c^2 = \frac{1}{\epsilon_0\mu_0} \tag{28.4}$$

If we solve the equation $k = 1/4\pi\epsilon_0$ for ϵ_0, substitute the resulting expression into Eq. (28.4), and solve for μ_0, we indeed get the value of μ_0 stated above. This discussion is a little premature, but it may give you a hint that electric and magnetic fields are intimately related to the nature of light.

Example 28.1 Forces between two moving protons

Two protons move parallel to the x-axis in opposite directions (Fig. 28.2) at the same speed v (small compared to the speed of light c). At the instant shown, find the electric and magnetic forces on the upper proton and determine the ratio of their magnitudes.

SOLUTION

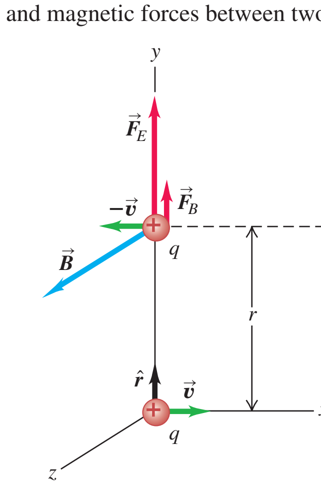
IDENTIFY: The electric force is given by Coulomb's law. To find the magnetic force, we must first find the magnetic field that the lower proton produces at the position of the upper proton.

SET UP: We use Eq. (21.2) for Coulomb's law. Equation (28.2) gives us the magnetic field due to the lower proton, and the magnetic force law, Eq. (27.2), gives us the resulting magnetic force on the upper proton.

EXECUTE: From Coulomb's law, the magnitude of the electric force on the upper proton is

$$F_E = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}$$

28.2 Electric and magnetic forces between two moving protons.



Continued

The forces are repulsive, and the force on the upper proton is vertically upward (in the $+y$-direction).

From the right-hand rule for the cross product $\vec{v} \times \hat{r}$ in Eq. (28.2), the $\vec{B}$-field due to the lower proton at the position of the upper proton is in the $+z$-direction (see Fig. 28.2). From Eq. (28.2), the magnitude of $\vec{B}$ is

$$B = \frac{\mu_0}{4\pi}\frac{qv}{r^2}$$

since $\phi = 90°$. Alternatively, from Eq. (28.2),

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q(v\hat{\imath}) \times \hat{\jmath}}{r^2} = \frac{\mu_0}{4\pi}\frac{qv}{r^2}\hat{k}$$

The velocity of the upper proton is $-\vec{v}$ and the magnetic force on it is $\vec{F} = q(-\vec{v}) \times \vec{B}$. Combining this with the expressions for $\vec{B}$, we find

$$F_B = \frac{\mu_0}{4\pi}\frac{q^2v^2}{r^2} \quad \text{or}$$

$$\vec{F}_B = q(-\vec{v}) \times \vec{B} = q(-v\hat{\imath}) \times \frac{\mu_0}{4\pi}\frac{qv}{r^2}\hat{k} = \frac{\mu_0}{4\pi}\frac{q^2v^2}{r^2}\hat{\jmath}$$

The magnetic interaction in this situation is also repulsive. The ratio of the magnitudes of the two forces is

$$\frac{F_B}{F_E} = \frac{\mu_0 q^2 v^2/4\pi r^2}{q^2/4\pi\epsilon_0 r^2} = \frac{\mu_0 v^2}{1/\epsilon_0} = \epsilon_0\mu_0 v^2$$

Using the relationship $\epsilon_0\mu_0 = 1/c^2$, Eq. (28.4), we can express our result very simply as

$$\frac{F_B}{F_E} = \frac{v^2}{c^2}$$

When v is small in comparison to c, the speed of light, the magnetic force is much smaller than the electric force.

EVALUATE: Note that it is essential to use the same frame of reference in this entire calculation. We have described the velocities and the fields as they appear to an observer who is stationary in the coordinate system of Fig. 28.2. In a coordinate system that moves with one of the charges, one of the velocities would be zero, so there would be *no* magnetic force. The explanation of this apparent paradox provided one of the paths that led to the special theory of relativity.

Test Your Understanding of Section 28.1 (a) If two protons are traveling parallel to each other in the *same* direction and at the same speed, is the magnetic force between them (i) attractive or (ii) repulsive? (b) Is the net force between them (i) attractive, (ii) repulsive, or (iii) zero? (Assume that the protons' speed is much slower than the speed of light.)

28.2 Magnetic Field of a Current Element

Just as for the electric field, there is a **principle of superposition of magnetic fields:**

The total magnetic field caused by several moving charges is the vector sum of the fields caused by the individual charges.

We can use this principle with the results of Section 28.1 to find the magnetic field produced by a current in a conductor.

We begin by calculating the magnetic field caused by a short segment $d\vec{l}$ of a current-carrying conductor, as shown in Fig. 28.3a. The volume of the segment is $A\,dl$, where A is the cross-sectional area of the conductor. If there are n moving charged particles per unit volume, each of charge q, the total moving charge dQ in the segment is

$$dQ = nqA\,dl$$

The moving charges in this segment are equivalent to a single charge dQ, traveling with a velocity equal to the *drift* velocity $\vec{v}_d$. (Magnetic fields due to the *random* motions of the charges will, on average, cancel out at every point.) From Eq. (28.1) the magnitude of the resulting field $d\vec{B}$ at any field point P is

$$dB = \frac{\mu_0}{4\pi}\frac{|dQ|v_d \sin\phi}{r^2} = \frac{\mu_0}{4\pi}\frac{n|q|v_d A\,dl \sin\phi}{r^2}$$

But from Eq. (25.2), $n|q|v_d A$ equals the current I in the element. So

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl \sin\phi}{r^2} \tag{28.5}$$

Current Element: Vector Magnetic Field

In vector form, using the unit vector $\hat{r}$ as in Section 28.1, we have

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l} \times \hat{r}}{r^2} \quad \text{(magnetic field of a current element)} \tag{28.6}$$

where $d\vec{l}$ is a vector with length dl, in the same direction as the current in the conductor.

Equations (28.5) and (28.6) are called the **law of Biot and Savart** (pronounced "Bee-oh" and "Suh-var"). We can use this law to find the total magnetic field $\vec{B}$ at any point in space due to the current in a complete circuit. To do this, we integrate Eq. (28.6) over all segments $d\vec{l}$ that carry current; symbolically,

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l} \times \hat{r}}{r^2} \tag{28.7}$$

In the following sections we will carry out this vector integration for several examples.

Current Element: Magnetic Field Lines

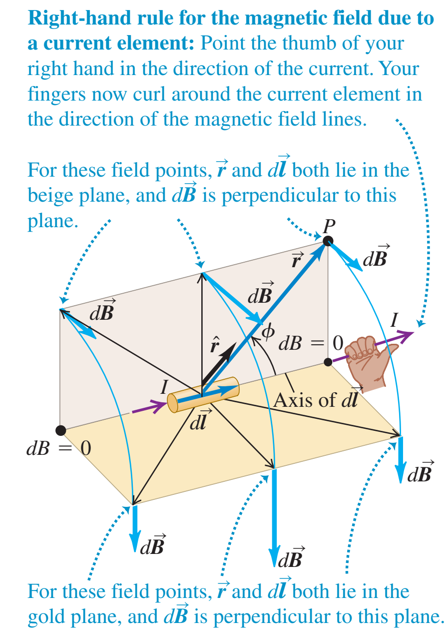
As Fig. 28.3 shows, the field vectors $d\vec{B}$ and the magnetic field lines of a current element are exactly like those set up by a positive charge dQ moving in the direction of the drift velocity $\vec{v}_d$. The field lines are circles in planes perpendicular to $d\vec{l}$ and centered on the line of $d\vec{l}$. Their directions are given by the same right-hand rule that we introduced for point charges in Section 28.1.

We can't verify Eq. (28.5) or (28.6) directly because we can never experiment with an isolated segment of a current-carrying circuit. What we measure experimentally is the *total* $\vec{B}$ for a complete circuit. But we can still verify these equations indirectly by calculating $\vec{B}$ for various current configurations using Eq. (28.7) and comparing the results with experimental measurements.

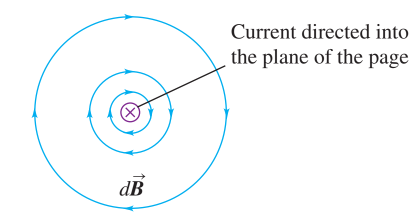
If matter is present in the space around a current-carrying conductor, the field at a field point P in its vicinity will have an additional contribution resulting from the *magnetization* of the material. We'll return to this point in Section 28.8. However, unless the material is iron or some other ferromagnetic material, the additional field is small and is usually negligible. Additional complications arise if time-varying electric or magnetic fields are present or if the material is a superconductor; we'll return to these topics later.

28.3 (a) Magnetic-field vectors due to a current element $d\vec{l}$. (b) Magnetic field lines in a plane containing the current element $d\vec{l}$. Compare this figure to Fig. 28.1 for the field of a moving point charge.

(a) Perspective view



(b) View along the axis of the current element



Problem-Solving Strategy 28.1 Magnetic-Field Calculations

IDENTIFY *the relevant concepts:* The law of Biot and Savart allows you to calculate the magnetic field due to a current-carrying wire of any shape. The idea is to calculate the field due to a representative current element in the wire and then combine the contributions from all such elements to find the total field.

SET UP *the problem* using the following steps:

1. Make a diagram showing a representative current element and the point P at which the field is to be determined (the field point).
2. Draw the current element $d\vec{l}$, being careful that it points in the direction of the current.
3. Draw the unit vector $\hat{r}$. Note that it is always directed *from* the current element (the source point) to the field point P.
4. Identify the target variables. Usually they will be the magnitude and direction of the magnetic field $\vec{B}$.

EXECUTE *the solution* as follows:

1. Use Eq. (28.5) or (28.6) to express the magnetic field $d\vec{B}$ at P from the representative current element.
2. Add up all the $d\vec{B}$'s to find the total field at point P. In some situations the $d\vec{B}$'s at point P have the same direction for all the current elements; then the magnitude of the total $\vec{B}$ field is the sum of the magnitudes of the $d\vec{B}$'s. But often the $d\vec{B}$'s have different directions for different current elements. Then you have to set up a coordinate system and represent each $d\vec{B}$ in terms of its components. The integral for the total $\vec{B}$ is then expressed in terms of an integral for each component.
3. Sometimes you can use the symmetry of the situation to prove that one component of $\vec{B}$ must vanish. Always be alert for ways to use symmetry to simplify the problem.

Continued

4. Look for ways to use the principle of superposition of magnetic fields. Later in this chapter we'll determine the fields produced by certain simple conductor shapes; if you encounter a conductor of a complex shape that can be represented as a combination of these simple shapes, you can use superposition to find the field of the complex shape. Examples include a rectangular loop and a semicircle with straight line segments on both sides.

EVALUATE *your answer:* Often your answer will be a mathematical expression for $\vec{B}$ as a function of the position of the field point. Check the answer by examining its behavior in as many limits as you can.

Example 28.2 Magnetic field of a current segment

A copper wire carries a steady current of 125 A to an electroplating tank. Find the magnetic field caused by a 1.0-cm segment of this wire at a point 1.2 m away from it, if the point is (a) point P_1, straight out to the side of the segment, and (b) point P_2, on a line at 30° to the segment, as shown in Fig. 28.4.

SOLUTION

IDENTIFY: Although Eqs. (28.5) and (28.6) are strictly to be used with infinitesimal current elements only, we may use them here since the segment's 1.0-cm length is much smaller than the 1.2-m distance to the field point.

SET UP: The current element is shown in red in Fig. 28.4 and points in the $-x$-direction (the direction of the current). The unit vector $\hat{r}$ for each field point is directed from the current element toward that point: $\hat{r}$ is in the $+y$-direction for point P_1 and at an angle of 30° above the $-x$-direction for point P_2.

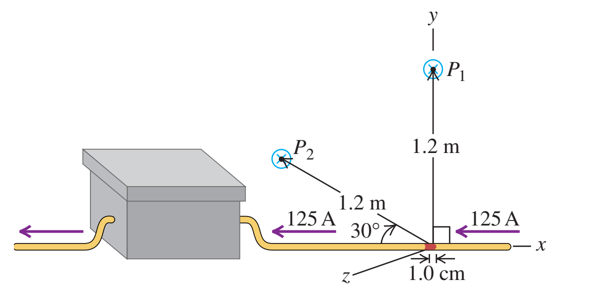
EXECUTE: (a) From the right-hand rule, the direction of $\vec{B}$ at P_1 is *into* the xy-plane of Fig. 28.4. Or, using unit vectors, we note that $d\vec{l} = dl(-\hat{\imath})$. At point P_1, $\hat{r} = \hat{\jmath}$, so in Eq. (28.6),

$$d\vec{l} \times \hat{r} = dl(-\hat{\imath}) \times \hat{\jmath} = dl(-\hat{k})$$

The negative z-direction is *into* the plane.

To find the magnitude of $\vec{B}$, we use Eq. (28.5). At point P_1, the angle between $d\vec{l}$ and $\hat{r}$ is 90°, so

$$B = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\phi}{r^2}$$
$$= (10^{-7}\ \text{T}\cdot\text{m/A})\frac{(125\ \text{A})(1.0\times10^{-2}\ \text{m})(\sin 90°)}{(1.2\ \text{m})^2}$$
$$= 8.7\times10^{-8}\ \text{T}$$



28.4 Finding the magnetic field at two points due to a 1.0-cm segment of current-carrying wire (not shown to scale).

(b) At point P_2 the direction of $\vec{B}$ is again into the xy-plane of the figure. The angle between $d\vec{l}$ and $\hat{r}$ is 30°, and

$$B = (10^{-7}\ \text{T}\cdot\text{m/A})\frac{(125\ \text{A})(1.0\times10^{-2}\ \text{m})(\sin 30°)}{(1.2\ \text{m})^2}$$
$$= 4.3\times10^{-8}\ \text{T}$$

EVALUATE: You can check our results for the direction of $\vec{B}$ by comparing them with Fig. 28.3. The xy-plane in Fig. 28.4 corresponds to the beige plane in Fig. 28.3. However, in the present example the direction of the current and hence of $d\vec{l}$ is the reverse of the direction shown in Fig. 28.3, so the direction of the magnetic field is reversed as well. Hence the field at points in the xy-plane in Fig. 28.4 must point *into,* not out of, that plane. This is just what we concluded above.

Note that these magnetic-field magnitudes are very small; for comparison the magnetic field of the earth is of the order of 10^{-4} T. Note also that the values are not the *total* fields at points P_1 and P_2, but only the contributions from the short segment of conductor described.

Test Your Understanding of Section 28.2 An infinitesimal current element located at the origin $(x = y = z = 0)$ carries current I in the positive y-direction. Rank the following locations in order of the strength of the magnetic field that the current element produces at that location, from largest to smallest value. (i) $x = L, y = 0, z = 0$; (ii) $x = 0, y = L, z = 0$; (iii) $x = 0, y = 0, z = L$; (iv) $x = L/\sqrt{2}, y = L/\sqrt{2}, z = 0$.

28.3 Magnetic Field of a Straight Current-Carrying Conductor

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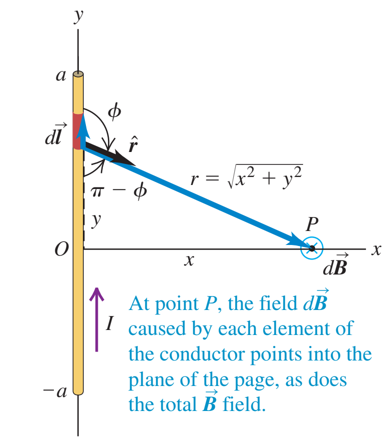
13.1 Magnetic Field of a Wire

An important application of the law of Biot and Savart is finding the magnetic field produced by a straight current-carrying conductor. This result is useful because straight conducting wires are found in essentially all electric and electronic devices. Fig. 28.5 shows such a conductor with length $2a$ carrying a current I. We will find $\vec{B}$ at a point a distance x from the conductor on its perpendicular bisector.

We first use the law of Biot and Savart, Eq. (28.5), to find the field $d\vec{B}$ caused by the element of conductor of length $dl = dy$ shown in Fig. 28.5. From the figure, $r = \sqrt{x^2 + y^2}$ and $\sin\phi = \sin(\pi - \phi) = x/\sqrt{x^2 + y^2}$. The right-hand rule for the vector product $d\vec{l} \times \hat{r}$ shows that the *direction* of $d\vec{B}$ is into the plane of the figure, perpendicular to the plane; furthermore, the directions of the $d\vec{B}$'s from *all* elements of the conductor are the same. Thus in integrating Eq. (28.7), we can just add the *magnitudes* of the $d\vec{B}$'s, a significant simplification.



28.5 Magnetic field produced by a straight current-carrying conductor of length $2a$.

Putting the pieces together, we find that the magnitude of the total $\vec{B}$ field is

$$B = \frac{\mu_0 I}{4\pi}\int_{-a}^{a}\frac{x\,dy}{(x^2 + y^2)^{3/2}}$$

We can integrate this by trigonometric substitution or by using an integral table. The final result is

$$B = \frac{\mu_0 I}{4\pi}\frac{2a}{x\sqrt{x^2 + a^2}} \qquad (28.8)$$

When the length $2a$ of the conductor is very great in comparison to its distance x from point P, we can consider it to be infinitely long. When a is much larger than x, $\sqrt{x^2 + a^2}$ is approximately equal to a; hence in the limit $a \to \infty$, Eq. (28.8) becomes

$$B = \frac{\mu_0 I}{2\pi x}$$

The physical situation has axial symmetry about the y-axis. Hence $\vec{B}$ must have the same *magnitude* at all points on a circle centered on the conductor and lying in a plane perpendicular to it, and the *direction* of $\vec{B}$ must be everywhere tangent to such a circle. Thus, at all points on a circle of radius r around the conductor, the magnitude B is

$$B = \frac{\mu_0 I}{2\pi r} \quad \text{(near a long, straight, current-carrying conductor)} \qquad (28.9)$$

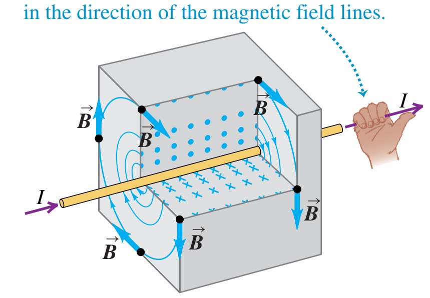
Part of the magnetic field around a long, straight, current-carrying conductor is shown in Fig. 28.6.

The geometry of this problem is similar to that of Example 21.11 (Section 21.5), in which we solved the problem of the *electric* field caused by an infinite line of charge. The same integral appears in both problems, and the field magnitudes in both problems are proportional to $1/r$. But the lines of $\vec{B}$ in the magnetic problem have completely different shapes than the lines of $\vec{E}$ in the analogous electrical problem. Electric field lines radiate outward from a positive line charge distribution (inward for negative charges). By contrast, magnetic field lines *encircle* the current that acts as their source. Electric field lines due to charges begin and end at those charges, but magnetic field lines always form closed loops and *never* have end points, irrespective of the shape of the current-carrying conductor that sets up the field. As we discussed in Section 27.3, this is a consequence of Gauss's law for magnetism, which states that the total magnetic flux through *any* closed surface is always zero:

$$\oint \vec{B}\cdot d\vec{A} = 0 \quad \text{(magnetic flux through any closed surface)} \qquad (28.10)$$

This implies that there are no isolated magnetic charges or magnetic monopoles. Any magnetic field line that enters a closed surface must also emerge from that surface.

28.6 Magnetic field around a long, straight, current-carrying conductor. The field lines are circles, with directions determined by the right-hand rule.



Example 28.3 Magnetic field of a single wire

A long, straight conductor carries a current of 1.0 A. At what distance from the axis of the conductor is the magnetic field caused by the current equal in magnitude to the earth's magnetic field in Pittsburgh (about 0.5×10^{-4} T)?

SOLUTION

IDENTIFY: The straight conductor is described as being long, which means that its length is much greater than the distance from the conductor at which we measure the field. Hence we can use the ideas of this section.

SET UP: The geometry is the same as that in Fig. 28.6, so we use Eq. (28.8). All of the quantities in this equation are known except the target variable, the distance r.

EXECUTE: We solve Eq. (28.8) for r and insert the appropriate numbers:

$$r = \frac{\mu_0 I}{2\pi B} = \frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(1.0\ \text{A})}{(2\pi)(0.5 \times 10^{-4}\ \text{T})} = 4 \times 10^{-3}\ \text{m} = 4\ \text{mm}$$

EVALUATE: Currents of an ampere or so are typical of those found in the wiring of home appliances. This example shows that the magnetic fields produced by these appliances are very weak even at points very close to the wire. At greater distances the field becomes even weaker; for example, at five times the distance $(r = 20\ \text{mm} = 2\ \text{cm} = 2 \times 10^{-2}\ \text{m})$ the field is one-fifth as great $(B = 0.1 \times 10^{-4}\ \text{T})$.

Example 28.4 Magnetic field of two wires

Fig. 28.7a is an end-on view of two long, straight, parallel wires perpendicular to the xy-plane, each carrying a current I but in opposite directions. (a) Find the magnitude and direction of $\vec{B}$ at points P_1, P_2, and P_3. (b) Find the magnitude and direction of $\vec{B}$ at any point on the x-axis to the right of wire 2 in terms of the x-coordinate of the point.

SOLUTION

IDENTIFY: We can find the magnetic fields $\vec{B}_1$ and $\vec{B}_2$ due to each wire using the ideas of this section. The principle of superposition of magnetic fields says that the total magnetic field $\vec{B}$ is the vector sum of $\vec{B}_1$ and $\vec{B}_2$.

SET UP: We use Eq. (28.9) to find the magnitude of the fields $\vec{B}_1$ (due to wire 1) and $\vec{B}_2$ (due to wire 2) at any point. We find the directions of these fields using the right-hand rule. The total magnetic field at the point in question is $\vec{B}_{\text{total}} = \vec{B}_1 + \vec{B}_2$.

EXECUTE: (a) Point P_1 is closer to wire 1 (distance $2d$) than to wire 2 (distance $4d$), so at this point the field magnitude B_1 is greater than the magnitude B_2:

$$B_1 = \frac{\mu_0 I}{2\pi(2d)} = \frac{\mu_0 I}{4\pi d} \qquad B_2 = \frac{\mu_0 I}{2\pi(4d)} = \frac{\mu_0 I}{8\pi d}$$

The right-hand rule shows that $\vec{B}_1$ is in the negative y-direction and $\vec{B}_2$ is in the positive y-direction. Since B_1 is the larger magnitude, the total field $\vec{B}_{\text{total}} = \vec{B}_1 + \vec{B}_2$ is in the negative y-direction, with magnitude

$$B_{\text{total}} = B_1 - B_2 = \frac{\mu_0 I}{4\pi d} - \frac{\mu_0 I}{8\pi d} = \frac{\mu_0 I}{8\pi d} \qquad (\text{point } P_1)$$

At point P_2, a distance d from both wires, $\vec{B}_1$ and $\vec{B}_2$ are both in the positive y-direction, and both have the same magnitude:

$$B_1 = B_2 = \frac{\mu_0 I}{2\pi d}$$

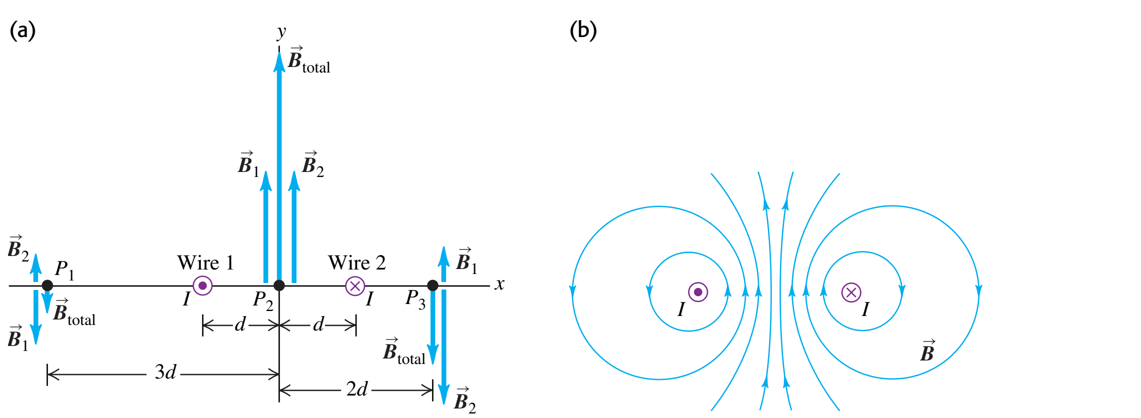
so $\vec{B}_{\text{total}}$ is also in the positive y-direction and has magnitude

$$B_{\text{total}} = B_1 + B_2 = \frac{\mu_0 I}{\pi d} \qquad (\text{point } P_2)$$

Finally, at point P_3 the right-hand rule shows that $\vec{B}_1$ is in the positive y-direction and $\vec{B}_2$ is in the negative y-direction. This point is farther from wire 1 (distance $3d$) than from wire 2 (distance d), so B_1 is less than B_2:

$$B_1 = \frac{\mu_0 I}{2\pi(3d)} = \frac{\mu_0 I}{6\pi d} \qquad B_2 = \frac{\mu_0 I}{2\pi d}$$

28.7 (a) Two long, straight conductors carrying equal currents in opposite directions. The conductors are seen end-on. (b) Map of the magnetic field produced by the two conductors. The field lines are closest together between the conductors, where the field is strongest.



The total field is in the negative y-direction, the same as $\vec{B}_2$, and has magnitude

$$B_{\text{total}} = B_2 - B_1 = \frac{\mu_0 I}{2\pi d} - \frac{\mu_0 I}{6\pi d} = \frac{\mu_0 I}{3\pi d} \qquad (\text{point } P_3)$$

You should be able to use the right-hand rule to verify for yourself the directions of $\vec{B}_1$ and $\vec{B}_2$ for each point.

The fields $\vec{B}_1$, $\vec{B}_2$, and $\vec{B}_{\text{total}}$ at each of the three points are shown in Fig. 28.7a. The same technique can be used to find $\vec{B}_{\text{total}}$ at any point; for points off the x-axis, caution must be taken in vector addition, since $\vec{B}_1$ and $\vec{B}_2$ need no longer be simply parallel or antiparallel (see Problem 28.60). Figure 28.7b shows some of the magnetic field lines due to this combination of wires.

(b) At any point to the right of wire 2 (that is, for $x > d$), $\vec{B}_1$ and $\vec{B}_2$ are in the same directions as at P_3. As x increases, both $\vec{B}_1$ and $\vec{B}_2$ decrease in magnitude, so $\vec{B}_{\text{total}}$ must decrease as well. The magnitudes of the fields due to each wire are

$$B_1 = \frac{\mu_0 I}{2\pi(x+d)} \qquad \text{and} \qquad B_2 = \frac{\mu_0 I}{2\pi(x-d)}$$

At any field point to the right of wire 2, wire 2 is closer than wire 1, and so $B_2 > B_1$. Hence $\vec{B}_{\text{total}}$ is in the negative y-direction, the same as $\vec{B}_2$, and has magnitude

$$B_{\text{total}} = B_2 - B_1 = \frac{\mu_0 I}{2\pi(x-d)} - \frac{\mu_0 I}{2\pi(x+d)} = \frac{\mu_0 I d}{\pi(x^2 - d^2)}$$

where we combined the two terms using a common denominator.

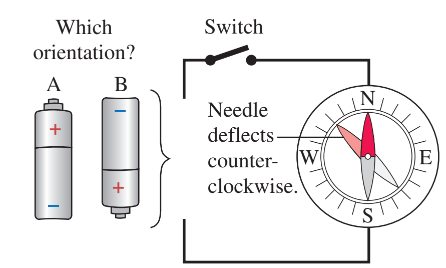
EVALUATE: At points very far from the wires, so that x is much larger than d, the d^2 term in the denominator can be neglected, and

$$B_{\text{total}} = \frac{\mu_0 I d}{\pi x^2}$$

The magnetic-field magnitude for a single wire decreases with distance in proportion to $1/x$, as shown by Eq. (28.9); for two wires carrying opposite currents, $\vec{B}_1$ and $\vec{B}_2$ partially cancel each other, and so the magnitude of $\vec{B}_{\text{total}}$ decreases more rapidly, in proportion to $1/x^2$. This effect is used in communication systems such as telephone or computer networks. The wiring is arranged so that a conductor carrying a signal in one direction and the conductor carrying the return signal are side by side, as in Fig. 28.7a, or twisted around each other (Fig. 28.8). As a result, the magnetic field caused *outside* the conductors by these signals is greatly reduced and is less likely to exert unwanted forces on other information-carrying currents.

28.8 Computer cables, or cables for audio-video equipment, create little or no magnetic field. This is because within each cable, closely spaced wires carry current in both directions along the length of the cable. The magnetic fields from these opposing currents cancel each other.

Test Your Understanding of Section 28.3 The figure at right shows a circuit that lies on a horizontal table. A compass is placed on top of the circuit as shown. A battery is to be connected to the circuit so that when the switch is closed, the compass needle deflects counterclockwise. In which orientation, A or B, should the battery be placed in the circuit?



28.4 Force Between Parallel Conductors

In Example 28.4 (Section 28.3) we showed how to use the principle of superposition of magnetic fields to find the total field due to two long current-carrying conductors. Another important aspect of this configuration is the *interaction force* between the conductors. This force plays a role in many practical situations in which current-carrying wires are close to each other, and it also has fundamental significance in connection with the definition of the ampere. Figure 28.9 shows segments of two long, straight, parallel conductors separated by a distance r and carrying currents I and I' in the same direction. Each conductor lies in the magnetic field set up by the other, so each experiences a force. The diagram shows some of the field lines set up by the current in the lower conductor.

From Eq. (28.9) the lower conductor produces a $\vec{B}$ field that, at the position of the upper conductor, has magnitude

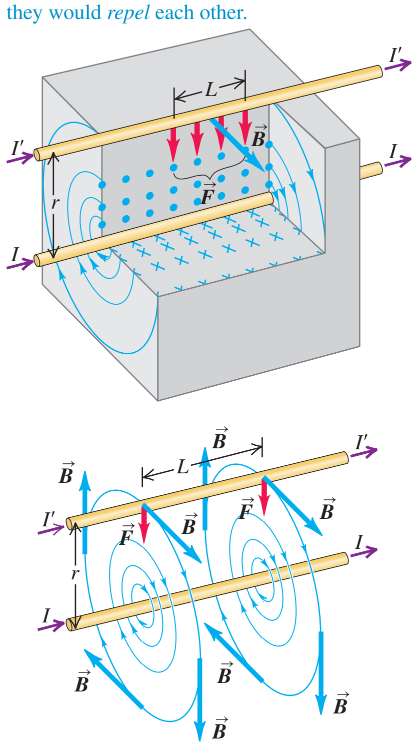
$$B = \frac{\mu_0 I}{2\pi r}$$

From Eq. (27.19) the force that this field exerts on a length L of the upper conductor is $\vec{F} = I'\vec{L} \times \vec{B}$, where the vector $\vec{L}$ is in the direction of the current I' and

28.9 Parallel conductors carrying currents in the same direction attract each other. The diagrams show how the magnetic field $\vec{B}$ caused by the current in the lower conductor exerts a force $\vec{F}$ on the upper conductor.

The magnetic field of the lower wire exerts an attractive force on the upper wire. By the same token, the upper wire attracts the lower one.

If the wires had currents in *opposite* directions, they would *repel* each other.



has magnitude L. Since $\vec{B}$ is perpendicular to the length of the conductor and hence to $\vec{L}$, the magnitude of this force is

$$F = I'LB = \frac{\mu_0 I I' L}{2\pi r}$$

and the force *per unit length* F/L is

$$\frac{F}{L} = \frac{\mu_0 I I'}{2\pi r} \quad \text{(two long, parallel, current-carrying conductors)} \quad (28.11)$$

Applying the right-hand rule to $\vec{F} = I'\vec{L} \times \vec{B}$ shows that the force on the upper conductor is directed *downward.*

The current in the upper conductor also sets up a field at the position of the lower one. Two successive applications of the right-hand rule for vector products (one to find the direction of the $\vec{B}$ field due to the upper conductor, as in Section 28.2, and one to find the direction of the force that this field exerts on the lower conductor, as in Section 27.6) show that the force on the lower conductor is *upward.* Thus *two parallel conductors carrying current in the same direction attract each other.* If the direction of either current is reversed, the forces also reverse. *Parallel conductors carrying currents in opposite directions repel each other.*

Magnetic Forces and Defining the Ampere

The attraction or repulsion between two straight, parallel, current-carrying conductors is the basis of the official SI definition of the **ampere:**

> ***One ampere* is that unvarying current that, if present in each of two parallel conductors of infinite length and one meter apart in empty space, causes each conductor to experience a force of exactly 2×10^{-7} newtons per meter of length.**

From Eq. (28.11) you can see that this definition of the ampere is what leads us to choose the value of $4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A}$ for μ_0. It also forms the basis of the SI definition of the coulomb, which is the amount of charge transferred in one second by a current of one ampere.

This is an *operational definition;* it gives us an actual experimental procedure for measuring current and defining a unit of current. In principle we could use this definition to calibrate an ammeter, using only a meter stick and a spring balance. For high-precision standardization of the ampere, coils of wire are used instead of straight wires, and their separation is only a few centimeters. Even more precise measurements of the standardized ampere are possible using a version of the Hall effect (see Section 27.9).

Mutual forces of attraction exist not only between *wires* carrying currents in the same direction, but also between the longitudinal elements of a single current-carrying conductor. If the conductor is a liquid or an ionized gas (a plasma), these forces result in a constriction of the conductor, as if its surface were acted on by an inward pressure. The constriction of the conductor is called the *pinch effect.* The high temperature produced by the pinch effect in a plasma has been used in one technique to bring about nuclear fusion.

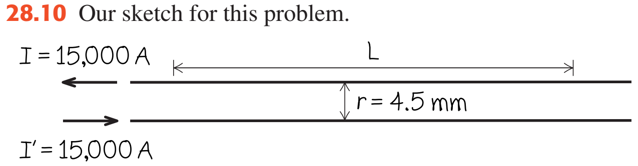
Example 28.5 Forces between parallel wires

Two straight, parallel, superconducting wires 4.5 mm apart carry equal currents of 15,000 A in opposite directions. Should we worry about the mechanical strength of these wires?

SOLUTION

IDENTIFY: Whether or not we need to worry about the wires' mechanical strength depends on how much magnetic force each wire exerts on the other.

28.10 Our sketch for this problem.



SET UP: Figure 28.10 shows the situation. Our target variable is the magnetic force per unit length of wire, which we find using Eq. (28.11).

EXECUTE: Because the currents are in opposite directions, the two conductors repel each other. From Eq. (28.11) the force per unit length is

$$\frac{F}{L} = \frac{\mu_0 I I'}{2\pi r} = \frac{(4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A})(15{,}000\ \text{A})^2}{(2\pi)(4.5 \times 10^{-3}\ \text{m})}$$
$$= 1.0 \times 10^4\ \text{N/m}$$

EVALUATE: This is a large force, more than one ton per meter, so the mechanical strengths of the conductors and insulating materials are certainly a significant consideration. Currents and separations of this magnitude are used in superconducting electromagnets in particle accelerators, and mechanical stress analysis is a crucial part of the design process.

Test Your Understanding of Section 28.4 A solenoid is a wire wound into a helical coil. The figure at right shows a solenoid that carries a current I. (a) Is the *magnetic* force that one turn of the coil exerts on an adjacent turn (i) attractive, (ii) repulsive, or (iii) zero? (b) Is the *electric* force that one turn of the coil exerts on an adjacent turn (i) attractive, (ii) repulsive, or (iii) zero? (c) Is the *magnetic* force between opposite sides of the same turn of the coil (i) attractive, (ii) repulsive, or (iii) zero? (d) Is the *electric* force between opposite sides of the same turn of the coil (i) attractive, (ii) repulsive, or (iii) zero?

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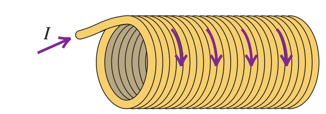


28.5 Magnetic Field of a Circular Current Loop

If you look inside a doorbell, a transformer, an electric motor, or an electromagnet (Fig. 28.11), you will find coils of wire with a large number of turns, spaced so closely that each turn is very nearly a planar circular loop. A current in such a coil is used to establish a magnetic field. So it is worthwhile to derive an expression for the magnetic field produced by a single circular conducting loop carrying a current or by N closely spaced circular loops forming a coil. In Section 27.7 we considered the force and torque on such a current loop placed in an external magnetic field produced by other currents; we are now about to find the magnetic field produced by the loop itself.

28.11 This electromagnet contains a current-carrying coil with numerous turns of wire. The resulting magnetic field can pick up large quantities of steel bars and other iron-bearing items.

Figure 28.12 shows a circular conductor with radius a that carries a current I. The current is led into and out of the loop through two long, straight wires side by side; the currents in these straight wires are in opposite directions, and their magnetic fields very nearly cancel each other (see Example 28.4 in Section 28.3).

We can use the law of Biot and Savart, Eq. (28.5) or (28.6), to find the magnetic field at a point P on the axis of the loop, at a distance x from the center. As the figure shows, $d\vec{l}$ and $\hat{r}$ are perpendicular, and the direction of the field $d\vec{B}$ caused by this particular element $d\vec{l}$ lies in the xy-plane. Since $r^2 = x^2 + a^2$, the magnitude dB of the field due to element $d\vec{l}$ is

$$dB = \frac{\mu_0 I}{4\pi} \frac{dl}{(x^2 + a^2)} \quad (28.12)$$

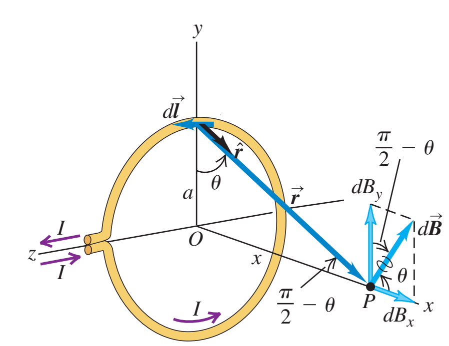
The components of the vector $d\vec{B}$ are

$$dB_x = dB\cos\theta = \frac{\mu_0 I}{4\pi} \frac{dl}{(x^2 + a^2)} \frac{a}{(x^2 + a^2)^{1/2}} \quad (28.13)$$

$$dB_y = dB\sin\theta = \frac{\mu_0 I}{4\pi} \frac{dl}{(x^2 + a^2)} \frac{x}{(x^2 + a^2)^{1/2}} \quad (28.14)$$

The situation has rotational symmetry about the x-axis, so there cannot be a component of the total field $\vec{B}$ perpendicular to this axis. For every element $d\vec{l}$ there is a corresponding element on the opposite side of the loop, with opposite direction. These two elements give equal contributions to the x-component of $d\vec{B}$, given by Eq. (28.13), but *opposite* components perpendicular to the

28.12 Magnetic field on the axis of a circular loop. The current in the segment $d\vec{l}$ causes the field $d\vec{B}$, which lies in the xy-plane. The currents in other $d\vec{l}$'s cause $d\vec{B}$'s with different components perpendicular to the x-axis; these components add to zero. The x-components of the $d\vec{B}$'s combine to give the total $\vec{B}$ field at point P.



Activ Physics
13.2 Magnetic Field of a Loop

x-axis. Thus all the perpendicular components cancel and only the x-components survive.

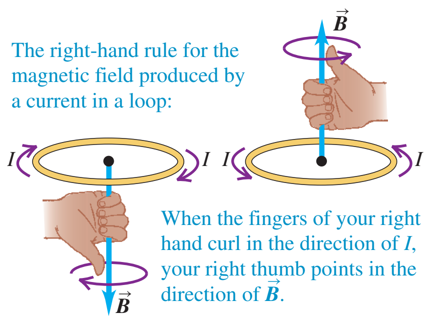
To obtain the x-component of the total field $\vec{B}$, we integrate Eq. (28.13), including all the $d\vec{l}$'s around the loop. Everything in this expression except dl is constant and can be taken outside the integral, and we have

$$B_x = \int \frac{\mu_0 I}{4\pi} \frac{a\,dl}{(x^2 + a^2)^{3/2}} = \frac{\mu_0 I a}{4\pi(x^2 + a^2)^{3/2}} \int dl$$

The integral of dl is just the circumference of the circle, $\int dl = 2\pi a$, and we finally get

$$B_x = \frac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}} \quad \text{(on the axis of a circular loop)} \qquad (28.15)$$

28.13 The right-hand rule for the direction of the magnetic field produced on the axis of a current-carrying coil.



The *direction* of the magnetic field on the axis of a current-carrying loop is given by a right-hand rule. If you curl the fingers of your right hand around the loop in the direction of the current, your right thumb points in the direction of the field (Fig. 28.13).

Magnetic Field on the Axis of a Coil

Now suppose that instead of the single loop in Fig. 28.12 we have a coil consisting of N loops, all with the same radius. The loops are closely spaced so that the plane of each loop is essentially the same distance x from the field point P. Each loop contributes equally to the field, and the total field is N times the field of a single loop:

$$B_x = \frac{\mu_0 N I a^2}{2(x^2 + a^2)^{3/2}} \quad \text{(on the axis of } N \text{ circular loops)} \qquad (28.16)$$

The factor N in Eq. (28.16) is the reason coils of wire, not single loops, are used to produce strong magnetic fields; for a desired field strength, using a single loop might require a current I so great as to exceed the rating of the loop's wire.

28.14 Graph of the magnetic field along the axis of a circular coil with N turns. When x is much larger than a, the field magnitude decreases approximately as $1/x^3$.

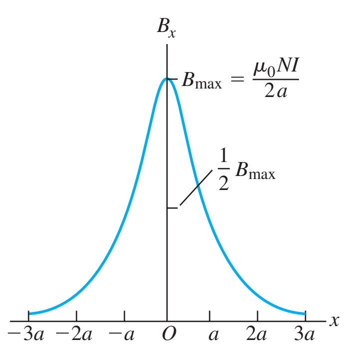


Figure 28.14 shows a graph of B_x as a function of x. The maximum value of the field is at $x = 0$, the center of the loop or coil:

$$B_x = \frac{\mu_0 N I}{2a} \quad \text{(at the center of } N \text{ circular loops)} \qquad (28.17)$$

As we go out along the axis, the field decreases in magnitude.

In Section 27.7 we defined the *magnetic dipole moment* μ (or *magnetic moment*) of a current-carrying loop to be equal to IA, where A is the cross-sectional area of the loop. If there are N loops, the total magnetic moment is NIA. The circular loop in Fig. 28.12 has area $A = \pi a^2$, so the magnetic moment of a single loop is $\mu = I\pi a^2$; for N loops, $\mu = NI\pi a^2$. Substituting these results into Eqs. (28.15) and (28.16), we find that both of these expressions can be written as

$$B_x = \frac{\mu_0 \mu}{2\pi(x^2 + a^2)^{3/2}} \quad \text{(on the axis of any number of circular loops)} \qquad (28.18)$$

We described a magnetic dipole in Section 27.7 in terms of its response to a magnetic field produced by currents outside the dipole. But a magnetic dipole is also a *source* of magnetic field; Eq. (28.18) describes the magnetic field *produced* by a magnetic dipole for points along the dipole axis. This field is directly proportional to the magnetic dipole moment μ. Note that the field along the x-axis is in the same direction as the vector magnetic moment $\vec{\mu}$; this is true on both the positive and negative x-axis.

CAUTION **Magnetic field of a coil** Equations (28.15), (28.16), and (28.18) are valid only on the *axis* of a loop or coil. Don't attempt to apply these equations at other points!

28.15 Magnetic field lines produced by the current in a circular loop. At points on the axis the $\vec{B}$ field has the same direction as the magnetic moment of the loop.

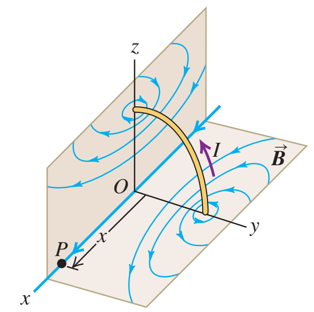


Figure 28.15 shows some of the magnetic field lines surrounding a circular current loop (magnetic dipole) in planes through the axis. The directions of the field lines are given by the same right-hand rule as for a long, straight conductor. Grab the conductor with your right hand, with your thumb in the direction of the current; your fingers curl around in the same direction as the field lines. The field lines for the circular current loop are closed curves that encircle the conductor; they are *not* circles, however.

Example 28.6 Magnetic field of a coil

A coil consisting of 100 circular loops with radius 0.60 m carries a current of 5.0 A. (a) Find the magnetic field at a point along the axis of the coil, 0.80 m from the center. (b) Along the axis, at what distance from the center of the coil is the field magnitude $\frac{1}{8}$ as great as it is at the center?

SOLUTION

IDENTIFY: This problem asks about the magnetic field along the axis of a current-carrying coil, so we can use the ideas of this section.

SET UP: We want the field on the axis of the coil, not necessarily at its center, so we use Eq. (28.16). We are given $N = 100$, $I = 5.0$ A, and $a = 0.60$ m. In part (a) our target variable is the magnetic field at a given value of the coordinate x. In part (b) the target variable is the value of x at which the field has $\frac{1}{8}$ of the magnitude that it has at $x = 0$.

EXECUTE: (a) Using $x = 0.80$ m, from Eq. (28.16) we have

$$B_x = \frac{(4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})(100)(5.0\,\text{A})(0.60\,\text{m})^2}{2[(0.80\,\text{m})^2 + (0.60\,\text{m})^2]^{3/2}}$$
$$= 1.1 \times 10^{-4}\,\text{T}$$

(b) Considering Eq. (28.16), we want to find a value of x such that

$$\frac{1}{(x^2 + a^2)^{3/2}} = \frac{1}{8} \frac{1}{(0^2 + a^2)^{3/2}}$$

To solve this for x, we take the reciprocal of the whole thing and then take the $2/3$ power of both sides; the result is

$$x = \pm\sqrt{3}a = \pm 1.04\,\text{m}$$

At a distance of about 1.7 radii from the center, the field has dropped off to $\frac{1}{8}$ its value at the center.

EVALUATE: We can check our answer in part (a) by first finding the magnetic moment and then substituting the result into Eq. (28.18):

$$\mu = NI\pi a^2 = (100)(5.0\,\text{A})\pi(0.60\,\text{m})^2 = 5.7 \times 10^2\,\text{A}\cdot\text{m}^2$$

$$B_x = \frac{(4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})(5.7 \times 10^2\,\text{A}\cdot\text{m}^2)}{2\pi[(0.80\,\text{m})^2 + (0.60\,\text{m})^2]^{3/2}} = 1.1 \times 10^{-4}\,\text{T}$$

The magnetic moment μ is relatively large, yet this is a rather small field, comparable in magnitude to the earth's magnetic field. This example may give you some idea of the difficulty of producing a field of 1 T or more.

Test Your Understanding of Section 28.5 Figure 28.12 shows the magnetic field $d\vec{B}$ produced at point P by a segment $d\vec{l}$ that lies on the positive y-axis (at the top of the loop). This field has components $dB_x > 0$, $dB_y > 0$, $dB_z = 0$. (a) What are the signs of the components of the field $d\vec{B}$ produced at P by a segment $d\vec{l}$ on the negative y-axis (at the bottom of the loop)? (i) $dB_x > 0$, $dB_y > 0$, $dB_z = 0$; (ii) $dB_x > 0$, $dB_y < 0$, $dB_z = 0$; (iii) $dB_x < 0$, $dB_y > 0$, $dB_z = 0$; (iv) $dB_x < 0$, $dB_y < 0$, $dB_z = 0$; (v) none of these. (b) What are the signs of the components of the field $d\vec{B}$ produced at P by a segment $d\vec{l}$ on the negative z-axis (at the right-hand side of the loop)? (i) $dB_x > 0$, $dB_y > 0$, $dB_z = 0$; (ii) $dB_x > 0$, $dB_y < 0$, $dB_z = 0$; (iii) $dB_x < 0$, $dB_y > 0$, $dB_z = 0$; (iv) $dB_x < 0$, $dB_y < 0$, $dB_z = 0$; (v) none of these.

28.6 Ampere's Law

So far our calculations of the magnetic field due to a current have involved finding the infinitesimal field $d\vec{B}$ due to a current element and then summing all the $d\vec{B}$'s to find the total field. This approach is directly analogous to our *electric*-field calculations in Chapter 21.

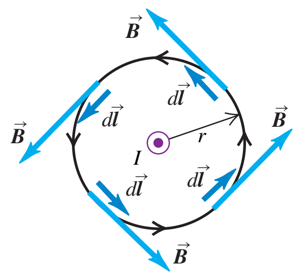
For the electric-field problem we found that in situations with a highly symmetric charge distribution, it was often easier to use Gauss's law to find $\vec{E}$. There is likewise a law that allows us to more easily find the *magnetic* fields caused by highly symmetric *current* distributions. But the law that allows us to do this, called *Ampere's law,* is rather different in character from Gauss's law.

Gauss's law for electric fields involves the flux of $\vec{E}$ through a closed surface; it states that this flux is equal to the total charge enclosed within the surface, divided by the constant ϵ_0. Thus this law relates electric fields and charge distributions. By contrast, Gauss's law for *magnetic* fields, Eq. (28.10), is *not* a relationship between magnetic fields and current distributions; it states that the flux of $\vec{B}$ through *any* closed surface is always zero, whether or not there are currents within the surface. So Gauss's law for $\vec{B}$ can't be used to determine the magnetic field produced by a particular current distribution.

28.16 Three integration paths for the line integral of $\vec{B}$ in the vicinity of a long, straight conductor carrying current I *out* of the plane of the page (as indicated by the circle with a dot). The conductor is seen end-on.

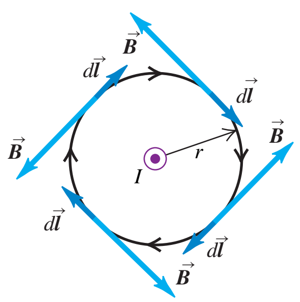
(a) Integration path is a circle centered on the conductor; integration goes around the circle counterclockwise.

Result: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$



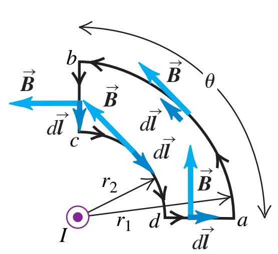
(b) Same integration path as in (a), but integration goes around the circle clockwise.

Result: $\oint \vec{B} \cdot d\vec{l} = -\mu_0 I$



(c) An integration path that does not enclose the conductor.

Result: $\oint \vec{B} \cdot d\vec{l} = 0$



Ampere's law is formulated not in terms of magnetic flux, but rather in terms of the *line integral* of $\vec{B}$ around a closed path, denoted by

$$\oint \vec{B} \cdot d\vec{l}$$

We used line integrals to define work in Chapter 6 and to calculate electric potential in Chapter 23. To evaluate this integral, we divide the path into infinitesimal segments $d\vec{l}$, calculate the scalar product of $\vec{B} \cdot d\vec{l}$ for each segment, and sum these products. In general, $\vec{B}$ varies from point to point, and we must use the value of $\vec{B}$ at the location of each $d\vec{l}$. An alternative notation is $\oint B_\parallel\, dl$, where $B_\parallel$ is the component of $\vec{B}$ parallel to $d\vec{l}$ at each point. The circle on the integral sign indicates that this integral is always computed for a *closed* path, one whose beginning and end points are the same.

Ampere's Law for a Long, Straight Conductor

To introduce the basic idea of Ampere's law, let's consider again the magnetic field caused by a long, straight conductor carrying a current I. We found in Section 28.3 that the field at a distance r from the conductor has magnitude

$$B = \frac{\mu_0 I}{2\pi r}$$

and that the magnetic field lines are circles centered on the conductor. Let's take the line integral of $\vec{B}$ around one such circle with radius r, as in Figure 28.16a. At every point on the circle, $\vec{B}$ and $d\vec{l}$ are parallel, and so $\vec{B} \cdot d\vec{l} = B\,dl$; since r is constant around the circle, B is constant as well. Alternatively, we can say that $B_\parallel$ is constant and equal to B at every point on the circle. Hence we can take B outside of the integral. The remaining integral $\oint dl$ is just the circumference of the circle, so

$$\oint \vec{B} \cdot d\vec{l} = \oint B_\parallel\, dl = B \oint dl = \frac{\mu_0 I}{2\pi r}(2\pi r) = \mu_0 I$$

The line integral is thus independent of the radius of the circle and is equal to μ_0 multiplied by the current passing through the area bounded by the circle.

In Fig. 28.16b the situation is the same, but the integration path now goes around the circle in the opposite direction. Now $\vec{B}$ and $d\vec{l}$ are antiparallel, so $\vec{B} \cdot d\vec{l} = -B\,dl$ and the line integral equals $-\mu_0 I$. We get the same result if the integration path is the same as in Fig. 28.16a, but the direction of the current is reversed. Thus the line integral $\oint \vec{B} \cdot d\vec{l}$ equals μ_0 multiplied by the current passing through the area bounded by the integration path, with a positive or negative sign depending on the direction of the current relative to the direction of integration.

There's a simple rule for the sign of the current; you won't be surprised to learn that it uses your right hand. Curl the fingers of your right hand around the integration path so that they curl in the direction of integration (that is, the direction that you use to evaluate $\oint \vec{B} \cdot d\vec{l}$). Then your right thumb indicates the positive current direction. Currents that pass through the integration path in this direction are positive; those in the opposite direction are negative. Using this rule, you should be able to convince yourself that the current is positive in Fig. 28.16a and negative in Fig. 28.16b. Here's another way to say the same thing: Looking at the surface bounded by the integration path, integrate counterclockwise around the path as in Fig. 28.16a. Currents moving toward you through the surface are positive, and those going away from you are negative.

An integration path that does *not* enclose the conductor is used in Fig. 28.16c. Along the circular arc ab of radius r_1, $\vec{B}$ and $d\vec{l}$ are parallel, and $B_\parallel = B_1 = \mu_0 I/2\pi r_1$; along the circular arc cd of radius r_2, $\vec{B}$ and $d\vec{l}$ are antiparallel and $B_\parallel = -B_2 = -\mu_0 I/2\pi r_2$. The $\vec{B}$ field is perpendicular to $d\vec{l}$ at each point on the straight sections bc and da, so $B_\parallel = 0$ and these sections contribute zero to the line integral. The total line integral is then

$$\oint \vec{B} \cdot d\vec{l} = \oint B_\parallel\, dl = B_1 \int_a^b dl + (0)\int_b^c dl + (-B_2)\int_c^d dl + (0)\int_d^a dl$$
$$= \frac{\mu_0 I}{2\pi r_1}(r_1 \theta) + 0 - \frac{\mu_0 I}{2\pi r_2}(r_2\theta) + 0 = 0$$

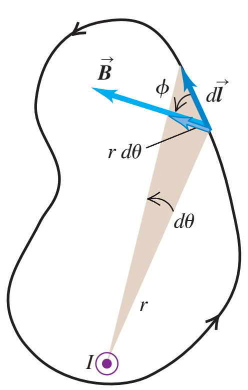
The magnitude of $\vec{B}$ is greater on arc cd than on arc ab, but the arc length is less, so the contributions from the two arcs exactly cancel. Even though there is a magnetic field everywhere along the integration path, the line integral $\oint \vec{B} \cdot d\vec{l}$ is zero if there is no current passing through the area bounded by the path.

We can also derive these results for more general integration paths, such as the one in Figure 28.17. At the position of the line element $d\vec{l}$, the angle between $d\vec{l}$ and $\vec{B}$ is ϕ, and

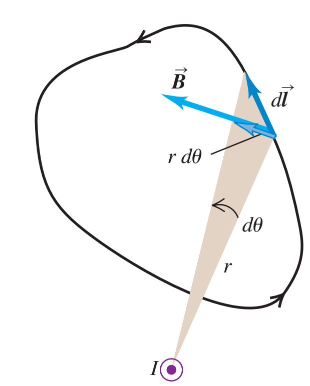
$$\vec{B} \cdot d\vec{l} = B\, dl \cos\phi$$

28.17 (a) A more general integration path for the line integral of $\vec{B}$ around a long, straight conductor carrying current I *out* of the plane of the page. The conductor is seen end-on. (b) A more general integration path that does not enclose the conductor.

(a)



(b)



From the figure, $dl\cos\phi = r\,d\theta$, where $d\theta$ is the angle subtended by $d\vec{l}$ at the position of the conductor and r is the distance of $d\vec{l}$ from the conductor. Thus

$$\oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi r}(r\, d\theta) = \frac{\mu_0 I}{2\pi}\oint d\theta$$

But $\oint d\theta$ is just equal to 2π, the total angle swept out by the radial line from the conductor to $d\vec{l}$ during a complete trip around the path. So we get

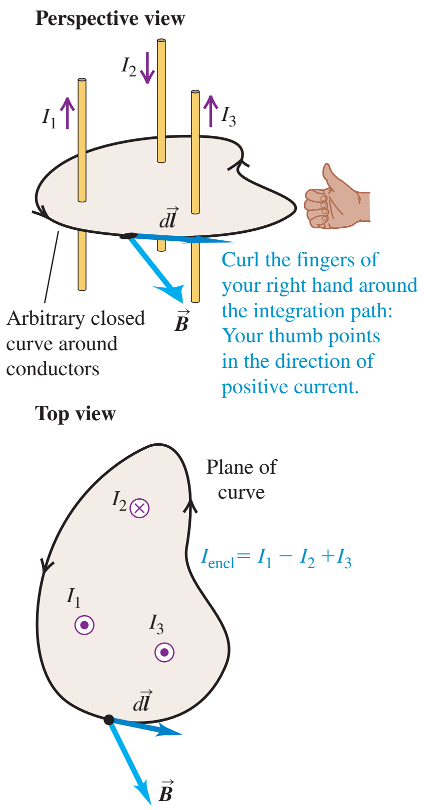
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \qquad (28.19)$$

This result doesn't depend on the shape of the path or on the position of the wire inside it. If the current in the wire is opposite to that shown, the integral has the opposite sign. But if the path doesn't enclose the wire (Fig. 28.17b), then the net change in θ during the trip around the integration path is zero; $\oint d\theta$ is zero instead of 2π and the line integral is zero.

Ampere's Law: General Statement

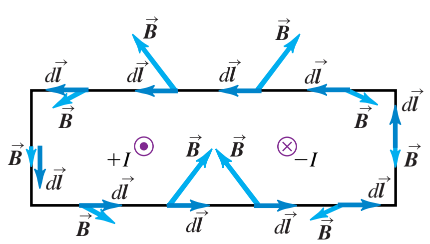
Equation (28.19) is almost, but not quite, the general statement of Ampere's law. To generalize it even further, suppose *several* long, straight conductors pass through the surface bounded by the integration path. The total magnetic field $\vec{B}$ at any point on the path is the vector sum of the fields produced by the individual conductors. Thus the line integral of the total $\vec{B}$ equals μ_0 times the *algebraic sum* of the currents. In calculating this sum, we use the sign rule for currents described above. If the integration path does not enclose a particular wire, the

28.18 Ampere's law.



Ampere's law: If we calculate the line integral of the magnetic field around a closed curve, the result equals μ_0 times the total enclosed current: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$

28.19 Two long, straight conductors carrying equal currents in opposite directions. The conductors are seen end-on, and the integration path is counterclockwise. The line integral $\oint \vec{B} \cdot d\vec{l}$ gets zero contribution from the upper and lower segments, a positive contribution from the left segment, and a negative contribution from the right segment; the net integral is zero.



line integral of the $\vec{B}$ field of that wire is zero, because the angle θ for that wire sweeps through a net change of zero rather than 2π during the integration. Any conductors present that are not enclosed by a particular path may still contribute to the value of $\vec{B}$ at every point, but the *line integrals* of their fields around the path are zero.

Thus we can replace I in Eq. (28.19) with I_{encl}, the algebraic sum of the currents *enclosed* or *linked* by the integration path, with the sum evaluated by using the sign rule just described (Fig. 28.18). Our statement of **Ampere's law** is then

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} \qquad \text{(Ampere's law)} \qquad (28.20)$$

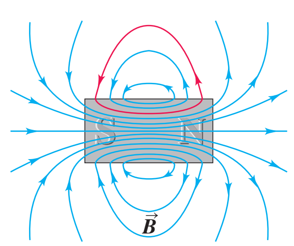
While we have derived Ampere's law only for the special case of the field of several long, straight, parallel conductors, Eq. (28.20) is in fact valid for conductors and paths of *any* shape. The general derivation is no different in principle from what we have presented, but the geometry is more complicated.

If $\oint \vec{B} \cdot d\vec{l} = 0$, it *does not* necessarily mean that $\vec{B} = \mathbf{0}$ everywhere along the path, only that the total current through an area bounded by the path is zero. In Figs. 28.16c and 28.17b, the integration paths enclose no current at all; in Fig. 28.19 there are positive and negative currents of equal magnitude through the area enclosed by the path. In both cases, $I_{encl} = 0$ and the line integral is zero.

CAUTION **Line integrals of electric and magnetic fields** In Chapter 23 we saw that the line integral of the electrostatic field $\vec{E}$ around any closed path is equal to zero; this is a statement that the electrostatic force $\vec{F} = q\vec{E}$ on a point charge q is conservative, so this force does zero work on a charge that moves around a closed path that returns to the starting point. You might think that the value of the line integral $\oint \vec{B} \cdot d\vec{l}$ is similarly related to the question of whether the *magnetic* force is conservative. This isn't the case at all. Remember that the magnetic force $\vec{F} = q\vec{v} \times \vec{B}$ on a moving charged particle is always *perpendicular* to $\vec{B}$, so $\oint \vec{B} \cdot d\vec{l}$ is *not* related to the work done by the magnetic force; as stated in Ampere's law, this integral is related only to the total current through a surface bounded by the integration path. In fact, the magnetic force on a moving charged particle is *not* conservative. A conservative force depends only on the position of the body on which the force is exerted, but the magnetic force on a moving charged particle also depends on the *velocity* of the particle.

In the form we have stated it, Ampere's law turns out to be valid only if the currents are steady and if no magnetic materials or time-varying electric fields are present. In Chapter 29 we will see how to generalize Ampere's law for time-varying fields.

Test Your Understanding of Section 28.6 The figure below shows magnetic field lines through the center of a permanent magnet. The magnet is not connected to a source of emf. One of the field lines is colored red. What can you conclude about the currents inside the permanent magnet within the region enclosed by this field line? (i) There are no currents inside the magnet; (ii) there are currents directed out of the plane of the page; (iii) there are currents directed into the plane of the page; (iv) not enough information given to decide.



28.7 Applications of Ampere's Law

Ampere's law is useful when we can exploit the symmetry of a situation to evaluate the line integral of $\vec{B}$. Several examples are given below. Problem-Solving Strategy 28.2 is directly analogous to Problem Solving Strategy 22.1 (Section 22.4) for applications of Gauss's law; we suggest you review that strategy now and compare the two methods.

Problem-Solving Strategy 28.2 Ampere's Law

MP

IDENTIFY *the relevant concepts:* Like Gauss's law for electricity, Ampere's law is always true but is most useful in situations where the magnetic field pattern is highly symmetrical. In such situations you can use Ampere's law to find a relationship between the magnetic field as a function of position and the current that generates the field.

SET UP *the problem* using the following steps:

1. Select the integration path you will use with Ampere's law. If you want to determine the magnetic field at a certain point, then the path must pass through that point. The integration path doesn't have to be any actual physical boundary. Usually it is a purely geometric curve; it may be in empty space, embedded in a solid body, or some of each. The integration path has to have enough *symmetry* to make evaluation of the integral possible. If the problem itself has cylindrical symmetry, the integration path will usually be a circle coaxial with the cylinder axis.
2. Determine the target variable(s). Usually this will be the magnitude of the $\vec{B}$ field as a function of position.

EXECUTE *the solution* as follows:

1. Carry out the integral $\oint \vec{B} \cdot d\vec{l}$ along your chosen integration path. If $\vec{B}$ is tangent to all or some portion of the integration path and has the same magnitude B at every point, then its line integral equals B multiplied by the length of that portion of the path. If $\vec{B}$ is perpendicular to some portion of the path, that portion makes no contribution to the integral.
2. In the integral $\oint \vec{B} \cdot d\vec{l}$, $\vec{B}$ is always the *total* magnetic field at each point on the path. This field can be caused partly by currents enclosed by the path and partly by currents outside the path. If *no* net current is enclosed by the path, the field at points on the path need not be zero, but the integral $\oint \vec{B} \cdot d\vec{l}$ is always zero.
3. Determine the current I_{encl} enclosed by the integration path. The sign of this current is given by a right-hand rule. Curl the fingers of your right hand so that they follow the integration path in the direction that you carry out the integration. Your right thumb then points in the direction of positive current. If $\vec{B}$ is tangent to the integration at all points along the path and I_{encl} is positive, then the direction of $\vec{B}$ is the same as the direction of the integration path; if instead I_{encl} is negative, $\vec{B}$ is in the direction opposite to that of the integration.
4. Use Ampere's law $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$ to solve for the target variable.

EVALUATE *your answer:* If your result is an expression for the field magnitude as a function of position, you can check it by examining how the expression behaves in different limits.

Example 28.7 Field of a long, straight, current-carrying conductor

In Section 28.6 we derived Ampere's law using Eq. (28.9) for the field of a long, straight, current-carrying conductor. Reverse this process, and use Ampere's law to find the magnitude *and* direction of $\vec{B}$ for this situation.

SOLUTION

IDENTIFY: This situation has cylindrical symmetry, so we can use Ampere's law to find the magnetic field at all points a distance r from the conductor

SET UP: We take as our integration path a circle with radius r centered on the conductor and in a plane perpendicular to it, as in Fig. 28.16a (Section 28.6). At each point, $\vec{B}$ is tangent to this circle.

EXECUTE: With our choice of integration path, Ampere's law [Eq. (28.20)] becomes

$$\oint \vec{B} \cdot d\vec{l} = \oint B_{\parallel}\, dl = B(2\pi r) = \mu_0 I$$

Equation (28.9), $B = \mu_0 I / 2\pi r$, follows immediately.

Ampere's law determines the direction of $\vec{B}$ as well as its magnitude. Since we go around the integration path in the counterclockwise direction, the positive direction for current is out of the plane of Fig. 28.16a; this is the same as the actual current direction in the figure, so I is positive and the integral $\oint \vec{B} \cdot d\vec{l}$ is also positive. Since the $d\vec{l}$'s run counterclockwise, the direction of $\vec{B}$ must be counterclockwise as well, as shown in Fig. 28.16a.

EVALUATE: Our results are consistent with those in Section 28.6, as they must be.

Example 28.8 Field inside a long cylindrical conductor

A cylindrical conductor with radius R carries a current I. (Fig. 28.20). The current is uniformly distributed over the cross-sectional area of the conductor. Find the magnetic field as a function of the distance r from the conductor axis for points both inside $(r < R)$ and outside $(r > R)$ the conductor.

SOLUTION

IDENTIFY: Once again we have a current distribution with cylindrical symmetry. As for a long, straight, skinny current-carrying conductor, the magnetic field lines must be circles concentric with the conductor axis.

SET UP: To find the magnetic field *inside* the conductor, we take as our integration path a circle with radius $r < R$ as shown in Fig. 28.20. *Outside* the conductor, we again use a circle but with a radius $r > R$. In either case, the integration path takes advantage of the circular symmetry of the magnetic field pattern.

EXECUTE: Inside the conductor, $\vec{B}$ has the same magnitude at every point on the circular integration path and is tangent to the path. Thus the magnitude of the line integral is simply $B(2\pi r)$. If we use the right-hand rule for determining the sign of the current, the current through the brown area enclosed by the path is positive; hence $\vec{B}$ points in the same direction as the integration path, as shown. To find the current I_{encl} enclosed by the path, note that the current density (current per unit area) is $J = I/\pi R^2$, so $I_{encl} = J(\pi r^2) = Ir^2/R^2$. Finally, Ampere's law gives

$$B(2\pi r) = \mu_0 \frac{Ir^2}{R^2}$$

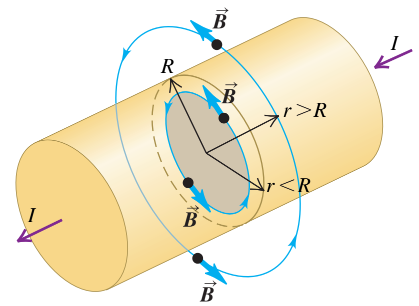
$$B = \frac{\mu_0 I}{2\pi}\frac{r}{R^2} \quad \text{(inside the conductor, } r < R) \tag{28.21}$$

For the circular integration path outside the conductor $(r > R)$, the same symmetry arguments apply and the magnitude of $\oint \vec{B} \cdot d\vec{l}$ is again $B(2\pi r)$. The right-hand rule gives the direction of $\vec{B}$ as shown in Fig. 28.20. For this path, $I_{encl} = I$, the total current in the conductor. Applying Ampere's law gives the same equation as in Example 28.7, with the same result for B:

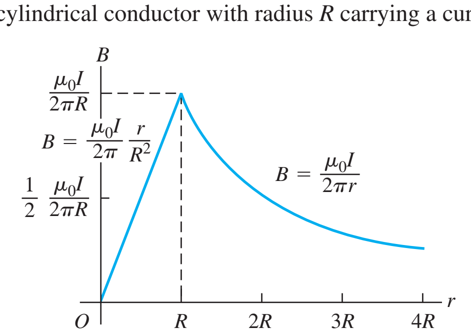
$$B = \frac{\mu_0 I}{2\pi r} \quad \text{(outside the conductor, } r > R) \tag{28.22}$$

Outside the conductor, the magnetic field is the same as that of a long, straight conductor carrying current I, independent of the radius R over which the current is distributed. Indeed, the magnetic field outside *any* cylindrically symmetric current distribution is the same as if the entire current were concentrated along the axis of the distribution. This is analogous to the results of Examples 22.5 and 22.9 (Section 22.4), in which we found that the *electric* field outside a spherically symmetric *charged* body is the same as though the entire charge were concentrated at the center.

EVALUATE: Note that at the surface of the conductor $(r = R)$, Eq. (28.21) for $r < R$ and Eq. (28.22) for $r > R$ agree (as they must). Figure 28.21 shows a graph of B as a function of r, both inside and outside the conductor.



28.20 To find the magnetic field at radius $r < R$, we apply Ampere's law to the circle enclosing the red area. The current through the red area is $(r^2/R^2)I$. To find the magnetic field at radius $r > R$, we apply Ampere's law to the circle enclosing the entire conductor.



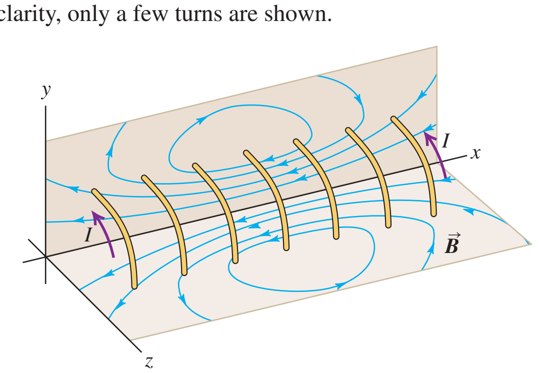
28.21 Magnitude of the magnetic field inside and outside a long, straight cylindrical conductor with radius R carrying a current I.

Example 28.9 Field of a solenoid

A solenoid consists of a helical winding of wire on a cylinder, usually circular in cross section. There can be hundreds or thousands of closely spaced turns, each of which can be regarded as a circular loop. There may be several layers of windings. For simplicity, Fig. 28.22 shows a solenoid with only a few turns. All turns carry the same current I, and the total $\vec{B}$ field at every point is the vector sum of the fields caused by the individual turns. The figure shows field lines in the xy- and xz-planes. We draw a set of field lines that are uniformly spaced at the center of the solenoid. Exact calculations show that for a long, closely wound solenoid, half of these field lines emerge from the ends and half "leak out" through the windings between the center and the end.

The field lines near the center of the solenoid are approximately parallel, indicating a nearly uniform $\vec{B}$; outside the solenoid, the field lines are spread apart, and the magnetic field is weak. If the solenoid is long in comparison with its cross-sectional diameter and the coils are tightly wound, the *internal* field near the midpoint of the solenoid's length is very nearly uniform over the cross section and parallel to the axis, and the *external* field near the midpoint is very small.

Use Ampere's law to find the field at or near the center of such a long solenoid. The solenoid has n turns of wire per unit length and carries a current I.



28.22 Magnetic field lines produced by the current in a solenoid. For clarity, only a few turns are shown.

SOLUTION

IDENTIFY: This is a highly symmetrical situation, with a uniform $\vec{B}$ field inside the solenoid and zero field outside. Hence we can use Ampere's law to find the field inside by using an appropriate choice of integration path.

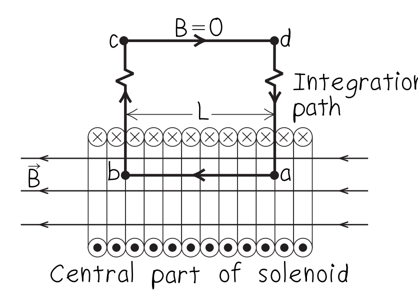
SET UP: Fig. 28.23 shows the situation and our integration path, rectangle $abcd$. Side ab, with length L, is parallel to the axis of the solenoid. Sides bc and da are taken to be very long so that side cd is far from the solenoid; then the field at side cd is negligibly small.

EXECUTE: By symmetry, the $\vec{B}$ field along side ab is parallel to this side and is constant. In carrying out the Ampere's-law integration, we go along side ab in the same direction as $\vec{B}$. So for this side, $B_\parallel = +B$ and

$$\int_a^b \vec{B} \cdot d\vec{l} = BL$$

Along sides bc and da, $B_\parallel = 0$ because $\vec{B}$ is perpendicular to these sides; along side cd, $B_\parallel = 0$ because $\vec{B} = \mathbf{0}$. The integral $\oint \vec{B} \cdot d\vec{l}$ around the entire closed path therefore reduces to BL.

28.23 Our sketch for this problem.



The number of turns in length L is nL. Each of these turns passes once through the rectangle $abcd$ and carries a current I, where I is the current in the windings. The total current enclosed by the rectangle is then $I_{encl} = nLI$. From Ampere's law, since the integral $\oint \vec{B} \cdot d\vec{l}$ is positive, I_{encl} must be positive as well; hence the current passing through the surface bounded by the integration path must be in the direction shown in Fig. 28.23. Ampere's law then gives the magnitude B:

$$BL = \mu_0 nLI$$

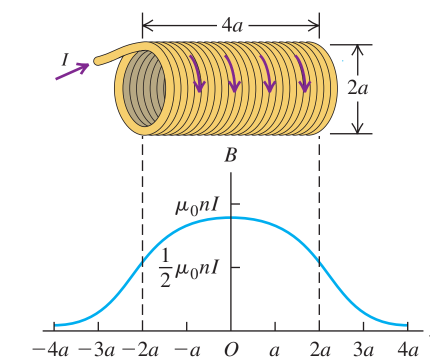
$$B = \mu_0 nI \quad \text{(solenoid)} \tag{28.23}$$

Side ab need not lie on the axis of the solenoid, so this calculation also proves that the field is uniform over the entire cross section at the center of the solenoid's length.

EVALUATE: Note that the *direction* of $\vec{B}$ inside the solenoid is in the same direction as the solenoid's vector magnetic moment $\vec{\mu}$. This is the same result that we found in Section 28.5 for a single current-carrying loop.

For points along the axis, the field is strongest at the center of the solenoid and drops off near the ends. For a solenoid that is very long in comparison to its diameter, the field at each end is exactly half as strong as the field at the center. For a short, fat solenoid the relationship is more complicated. Fig. 28.24 shows a graph of B as a function of x for points on the axis of a short solenoid.

28.24 Magnitude of the magnetic field at points along the axis of a solenoid with length $4a$, equal to four times its radius a. The field magnitude at each end is about half its value at the center. (Compare with Fig. 28.14 for the field of N circular loops.)



Example 28.10 Field of a toroidal solenoid

Figure 28.25a shows a doughnut-shaped **toroidal solenoid,** also called a *toroid,* wound with N turns of wire carrying a current I. In a practical version the turns would be more closely spaced than they are in the figure. Find the magnetic field at all points.

SOLUTION

IDENTIFY: The flow of current around the toroid's circumference produces a magnetic field component perpendicular to the plane of the figure, just as for the current loop discussed in Section 28.5. But if the coils are very tightly wound, we can consider them as circular loops that carry current between the inner and outer radii of the toroidal solenoid; the flow of current around the toroid's circumference is then negligible, and the perpendicular component of $\vec{B}$ is likewise negligible. In this idealized approximation the circular symmetry of the situation tells us that the magnetic field lines must be circles concentric with the axis of the toroid.

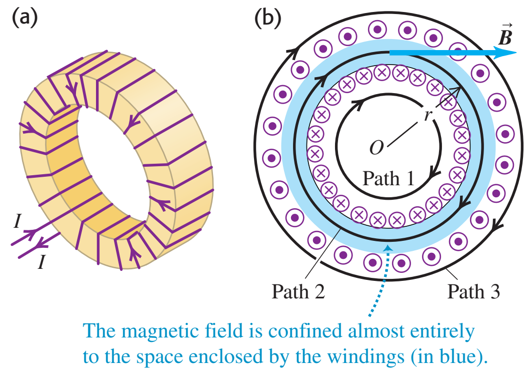
SET UP: To take advantage of this symmetry in finding the field, we choose circular integration paths for use with Ampere's law. Three such paths are shown as black lines in Fig. 28.25b.

EXECUTE: First consider integration path 1 in Fig. 28.25b. If the toroidal solenoid produces any field at all in this region, it must be *tangent* to the path at all points, and $\oint \vec{B} \cdot d\vec{l}$ will equal the product of B and the circumference $l = 2\pi r$ of the path. But the total current enclosed by the path is zero, so from Ampere's law the field $\vec{B}$ must be zero everywhere on this path.

Similarly, if the toroidal solenoid produces any field along path 3, it must also be tangent to the path at all points. Each turn of the winding passes *twice* through the area bounded by this path,

Continued

28.25 (a) A toroidal solenoid. For clarity, only a few turns of the winding are shown. (b) Integration paths (black circles) used to compute the magnetic field $\vec{B}$ set up by the current (shown as dots and crosses).



carrying equal currents in opposite directions. The *net* current I_{encl} enclosed within this area is therefore zero, and hence $\vec{B} = \mathbf{0}$ at all points of the path. Conclusion: *The field of an idealized toroidal solenoid is confined completely to the space enclosed by the windings.* We can think of such an idealized toroidal solenoid as a tightly wound solenoid that has been bent into a circle.

Finally, we consider path 2, a circle with radius r. Again by symmetry we expect the $\vec{B}$ field to be tangent to the path, and $\oint \vec{B} \cdot d\vec{l}$ equals $2\pi rB$. Each turn of the winding passes *once* through the area bounded by path 2. The total current enclosed by the path is $I_{encl} = NI$, where N is the total number of turns in the winding; I_{encl} is positive for the clockwise direction of integration in Fig. 28.25b, so $\vec{B}$ is in the direction shown. Then, from Ampere's law,

$$2\pi rB = \mu_0 NI$$

$$B = \frac{\mu_0 NI}{2\pi r} \quad \text{(toroidal solenoid)} \qquad (28.24)$$

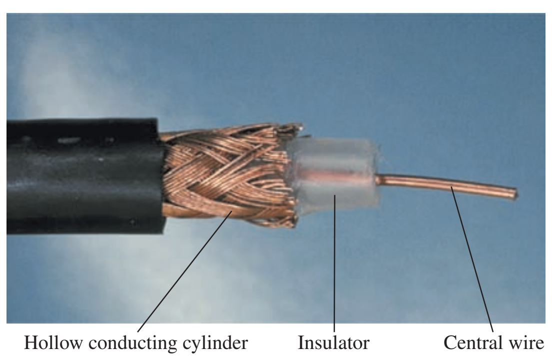
EVALUATE: The magnetic field is *not* uniform over a cross section of the core, because the radius r is larger at the outer side of the section than at the inner side. However, if the radial thickness of the core is small in comparison to r, the field varies only slightly across a section. In that case, considering that $2\pi r$ is the circumferential length of the toroid and that $N/2\pi r$ is the number of turns per unit length n, the field may be written as

$$B = \mu_0 nI$$

just as it is at the center of a long, *straight* solenoid.

In a real toroidal solenoid the turns are not precisely circular loops but rather segments of a bent helix. As a result, the field outside is not strictly zero. To estimate its magnitude, we imagine Fig. 28.25a as being roughly equivalent, for points outside the torus, to a circular loop with a single turn and radius r. Then we can use Eq. (28.17) to show that the field at the *center* of the torus is smaller than the field inside by approximately a factor of N/π.

The equations we have derived for the field in a closely wound straight or toroidal solenoid are strictly correct only for windings in *vacuum.* For most practical purposes, however, they can be used for windings in air or on a core of any nonmagnetic, nonsuperconducting material. In the next section we will show how they are modified if the core is a magnetic material.



Test Your Understanding of Section 28.7 Consider a conducting wire that runs along the central axis of a hollow conducting cylinder. Such an arrangement, called a *coaxial cable,* has many applications in telecommunications. (The cable that connects a television set to a local cable provider is an example of a coaxial cable.) In such a cable a current I runs in one direction along the hollow conducting cylinder and is spread uniformly over the cylinder's cross-sectional area. An equal current runs in the opposite direction along the central wire. How does the magnitude B of the magnetic field outside such a cable depend on the distance r from the central axis of the cable? (i) B is proportional to $1/r$; (ii) B is proportional to $1/r^2$; (iii) B is zero at all points outside the cable.

*28.8 Magnetic Materials

In discussing how currents cause magnetic fields, we have assumed that the conductors are surrounded by vacuum. But the coils in transformers, motors, generators, and electromagnets nearly always have iron cores to increase the magnetic field and confine it to desired regions. Permanent magnets, magnetic recording tapes, and computer disks depend directly on the magnetic properties of materials; when you store information on a computer disk, you are actually setting up an array of microscopic permanent magnets on the disk. So it is worthwhile to examine some aspects of the magnetic properties of materials. After describing the atomic origins of magnetic properties, we will discuss three broad classes of magnetic behavior that occur in materials; these are called *paramagnetism, diamagnetism,* and *ferromagnetism.*

The Bohr Magneton

As we discussed briefly in Section 27.7, the atoms that make up all matter contain moving electrons, and these electrons form microscopic current loops that produce magnetic fields of their own. In many materials these currents are randomly oriented and cause no net magnetic field. But in some materials an external field (a field produced by currents outside the material) can cause these loops to become oriented preferentially with the field, so their magnetic fields *add* to the external field. We then say that the material is *magnetized.*

Let's look at how these microscopic currents come about. Figure 28.26 shows a primitive model of an electron in an atom. We picture the electron (mass m, charge $-e$) as moving in a circular orbit with radius r and speed v. This moving charge is equivalent to a current loop. In Section 27.7 we found that a current loop with area A and current I has a magnetic dipole moment μ given by $\mu = IA$; for the orbiting electron the area of the loop is $A = \pi r^2$. To find the current associated with the electron, we note that the orbital period T (the time for the electron to make one complete orbit) is the orbit circumference divided by the electron speed: $T = 2\pi r/v$. The equivalent current I is the total charge passing any point on the orbit per unit time, which is just the magnitude e of the electron charge divided by the orbital period T:

$$I = \frac{e}{T} = \frac{ev}{2\pi r}$$

The magnetic moment $\mu = IA$ is then

$$\mu = \frac{ev}{2\pi r}(\pi r^2) = \frac{evr}{2} \qquad (28.25)$$

It is useful to express μ in terms of the *angular momentum* L of the electron. For a particle moving in a circular path, the magnitude of angular momentum equals the magnitude of momentum mv multiplied by the radius r, that is, $L = mvr$ (see Section 10.5). Comparing this with Eq. (28.25), we can write

$$\mu = \frac{e}{2m}L \qquad (28.26)$$

Equation (28.26) is useful in this discussion because atomic angular momentum is *quantized;* its component in a particular direction is always an integer multiple of $h/2\pi$, where h is a fundamental physical constant called *Planck's constant.* (We will discuss the quantization of angular momentum in more detail in Chapter 41.) The numerical value of h is

$$h = 6.626 \times 10^{-34}\ \text{J} \cdot \text{s}$$

The quantity $h/2\pi$ thus represents a fundamental unit of angular momentum in atomic systems, just as e is a fundamental unit of charge. Associated with the quantization of $\vec{L}$ is a fundamental uncertainty in the *direction* of $\vec{L}$ and therefore of $\vec{\mu}$. In the following discussion, when we speak of the magnitude of a magnetic moment, a more precise statement would be "maximum component in a given direction." Thus, to say that a magnetic moment $\vec{\mu}$ is aligned with a magnetic field $\vec{B}$ really means that $\vec{\mu}$ has its maximum possible component in the direction of $\vec{B}$; such components are always quantized.

Equation (28.26) shows that associated with the fundamental unit of angular momentum is a corresponding fundamental unit of magnetic moment. If $L = h/2\pi$, then

$$\mu = \frac{e}{2m}\left(\frac{h}{2\pi}\right) = \frac{eh}{4\pi m} \qquad (28.27)$$

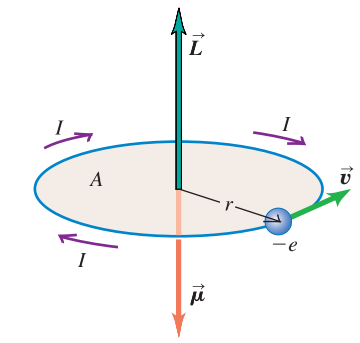
This quantity is called the **Bohr magneton,** denoted by μ_B. Its numerical value is

$$\mu_B = 9.274 \times 10^{-24}\ \text{A} \cdot \text{m}^2 = 9.274 \times 10^{-24}\ \text{J/T}$$

You should verify that these two sets of units are consistent. The second set is useful when we compute the potential energy $U = -\vec{\mu} \cdot \vec{B}$ for a magnetic moment in a magnetic field.

Electrons also have an intrinsic angular momentum, called *spin,* that is not related to orbital motion but that can be pictured in a classical model as spinning

28.26 An electron moving with speed v in a circular orbit of radius r has an angular momentum $\vec{L}$ and an oppositely directed orbital magnetic dipole moment $\vec{\mu}$. It also has a spin angular momentum and an oppositely directed spin magnetic dipole moment.



on an axis. This angular momentum also has an associated magnetic moment, and its magnitude turns out to be almost exactly one Bohr magneton. (Effects having to do with quantization of the electromagnetic field cause the spin magnetic moment to be about 1.001 μ_B.)

Paramagnetism

In an atom, most of the various orbital and spin magnetic moments of the electrons add up to zero. However, in some cases the atom has a net magnetic moment that is of the order of μ_B. When such a material is placed in a magnetic field, the field exerts a torque on each magnetic moment, as given by Eq. (27.26): $\vec{\tau} = \vec{\mu} \times \vec{B}$. These torques tend to align the magnetic moments with the field, the position of minimum potential energy, as we discussed in Section 27.7. In this position, the directions of the current loops are such as to *add* to the externally applied magnetic field.

We saw in Section 28.5 that the $\vec{B}$ field produced by a current loop is proportional to the loop's magnetic dipole moment. In the same way, the additional $\vec{B}$ field produced by microscopic electron current loops is proportional to the total magnetic moment $\vec{\mu}_{total}$ per unit volume V in the material. We call this vector quantity the **magnetization** of the material, denoted by $\vec{M}$:

$$\vec{M} = \frac{\vec{\mu}_{total}}{V} \tag{28.28}$$

The additional magnetic field due to magnetization of the material turns out to be equal simply to $\mu_0\vec{M}$, where μ_0 is the same constant that appears in the law of Biot and Savart and Ampere's law. When such a material completely surrounds a current-carrying conductor, the total magnetic field $\vec{B}$ in the material is

$$\vec{B} = \vec{B}_0 + \mu_0\vec{M} \tag{28.29}$$

where $\vec{B}_0$ is the field caused by the current in the conductor.

To check that the units in Eq. (28.29) are consistent, note that magnetization $\vec{M}$ is magnetic moment per unit volume. The units of magnetic moment are current times area $(A \cdot m^2)$, so the units of magnetization are $(A \cdot m^2)/m^3 = A/m$. From Section 28.1, the units of the constant μ_0 are $T \cdot m/A$. So the units of $\mu_0\vec{M}$ are the same as the units of $\vec{B}$: $(T \cdot m/A)(A/m) = T$.

A material showing the behavior just described is said to be **paramagnetic.** The result is that the magnetic field at any point in such a material is greater by a dimensionless factor K_m, called the **relative permeability** of the material, than it would be if the material were replaced by vacuum. The value of K_m is different for different materials; for common paramagnetic solids and liquids at room temperature, K_m typically ranges from 1.00001 to 1.003.

All of the equations in this chapter that relate magnetic fields to their sources can be adapted to the situation in which the current-carrying conductor is embedded in a paramagnetic material. All that need be done is to replace μ_0 by $K_m\mu_0$. This product is usually denoted as μ and is called the **permeability** of the material:

$$\mu = K_m\mu_0 \tag{28.30}$$

CAUTION **Two meanings of the symbol μ** Equation (28.30) involves some really dangerous notation because we have also used μ for magnetic dipole moment. It's customary to use μ for both quantities, but beware: From now on, every time you see a μ, make sure you know whether it is permeability or magnetic moment. You can usually tell from the context.

The amount by which the relative permeability differs from unity is called the **magnetic susceptibility,** denoted by χ_m:

$$\chi_m = K_m - 1 \tag{28.31}$$

Both K_m and χ_m are dimensionless quantities. Values of magnetic susceptibility for several materials are given in Table 28.1. For example, for aluminum, $\chi_m = 2.2 \times 10^{-5}$ and $K_m = 1.000022$. The first group of materials in the table are paramagnetic; we'll discuss the second group of materials, which are called *diamagnetic,* very shortly.

The tendency of atomic magnetic moments to align themselves parallel to the magnetic field (where the potential energy is minimum) is opposed by random thermal motion, which tends to randomize their orientations. For this reason, paramagnetic susceptibility always decreases with increasing temperature. In many cases it is inversely proportional to the absolute temperature T, and the magnetization M can be expressed as

$$M = C\frac{B}{T} \tag{28.32}$$

This relationship is called *Curie's law,* after its discoverer, Pierre Curie (1859–1906). The quantity C is a constant, different for different materials, called the *Curie constant.*

As we described in Section 27.7, a body with atomic magnetic dipoles is attracted to the poles of a magnet. In most paramagnetic substances this attraction is very weak due to thermal randomization of the atomic magnetic moments. That's why a magnet can't be used to pick up objects made of aluminum (a paramagnetic substance). But at very low temperatures the thermal effects are reduced, the magnetization increases in accordance with Curie's law, and the attractive forces are greater.

Table 28.1 Magnetic Susceptibilities of Paramagnetic and Diamagnetic Materials at $T = 20°C$

Material	$\chi_m = K_m - 1\ (\times 10^{-5})$
Paramagnetic	
Iron ammonium alum	66
Uranium	40
Platinum	26
Aluminum	2.2
Sodium	0.72
Oxygen gas	0.19
Diamagnetic	
Bismuth	−16.6
Mercury	−2.9
Silver	−2.6
Carbon (diamond)	−2.1
Lead	−1.8
Sodium chloride	−1.4
Copper	−1.0

Example 28.11 Magnetic dipoles in a paramagnetic material

Nitric oxide (NO) is a paramagnetic compound. Its molecules have a magnetic moment with a maximum component in any direction of about one Bohr magneton each. In a magnetic field with magnitude $B = 1.5$ T, compare the interaction energy of the magnetic moments with the field to the average translational kinetic energy of the molecules at a temperature of 300 K.

SOLUTION

IDENTIFY: This problem involves both the energy of a magnetic moment in a magnetic field (Chapter 27) and the average translational kinetic energy due to temperature (Chapter 18).

SET UP: In Section 27.7 we derived the equation $U = -\vec{\mu} \cdot \vec{B}$ for the interaction energy of a magnetic moment $\vec{\mu}$ with a $\vec{B}$ field. From Section 18.3 the average translational kinetic energy of a molecule at temperature T is $K = \frac{3}{2}kT$, where k is the Boltzmann constant.

EXECUTE: We can write the interaction energy as $U = -(\mu\cos\phi)B$, where $\mu\cos\phi$ is the component of the magnetic moment $\vec{\mu}$ in the direction of the $\vec{B}$ field. In our case the maximum value of the component $\mu\cos\phi$ is about μ_B, so

$$|U|_{max} \approx \mu_B B = (9.27 \times 10^{-24}\ J/T)(1.5\ T)$$
$$= 1.4 \times 10^{-23}\ J = 8.7 \times 10^{-5}\ eV$$

The average translational kinetic energy K is

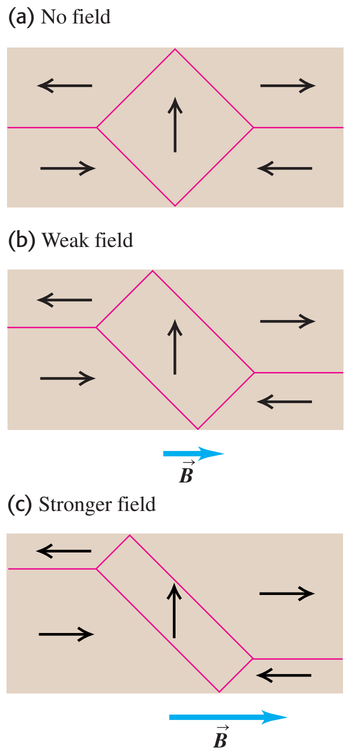
$$K = \frac{3}{2}kT = \frac{3}{2}(1.38 \times 10^{-23}\ J/K)(300\ K)$$
$$= 6.2 \times 10^{-21}\ J = 0.039\ eV$$

EVALUATE: At a temperature of 300 K the magnetic interaction energy is much *smaller* than the random kinetic energy, so we expect only a slight degree of alignment. This is why paramagnetic susceptibilities at ordinary temperature are usually very small.

Diamagnetism

In some materials the total magnetic moment of all the atomic current loops is zero when no magnetic field is present. But even these materials have magnetic effects because an external field alters electron motions within the atoms, causing additional current loops and induced magnetic dipoles comparable to the induced *electric* dipoles we studied in Section 28.5. In this case the additional field caused by these current loops is always *opposite* in direction to that of the external field. (This behavior is explained by Faraday's law of induction, which we will study in Chapter 29. An induced current always tends to cancel the field change that caused it.)

28.27 In this drawing adapted from a magnified photo, the arrows show the directions of magnetization in the domains of a single crystal of nickel. Domains that are magnetized in the direction of an applied magnetic field grow larger.



Such materials are said to be **diamagnetic.** They always have negative susceptibility, as shown in Table 28.1, and relative permeability K_m slightly *less* than unity, typically of the order of 0.99990 to 0.99999 for solids and liquids. Diamagnetic susceptibilities are very nearly temperature independent.

Ferromagnetism

There is a third class of materials, called **ferromagnetic** materials, that includes iron, nickel, cobalt, and many alloys containing these elements. In these materials, strong interactions between atomic magnetic moments cause them to line up parallel to each other in regions called **magnetic domains,** even when no external field is present. Figure 28.27 shows an example of magnetic domain structure. Within each domain, nearly all of the atomic magnetic moments are parallel.

When there is no externally applied field, the domain magnetizations are randomly oriented. But when a field $\vec{B}_0$ (caused by external currents) is present, the domains tend to orient themselves parallel to the field. The domain boundaries also shift; the domains that are magnetized in the field direction grow, and those that are magnetized in other directions shrink. Because the total magnetic moment of a domain may be many thousands of Bohr magnetons, the torques that tend to align the domains with an external field are much stronger than occur with paramagnetic materials. The relative permeability K_m is *much* larger than unity, typically of the order of 1,000 to 100,000. As a result, an object made of a ferromagnetic material such as iron is strongly magnetized by the field from a permanent magnet and is attracted to the magnet (see Fig. 27.38). A paramagnetic material such as aluminum is also attracted to a permanent magnet, but K_m for paramagnetic materials is so much smaller for such a material than for ferromagnetic materials that the attraction is very weak. Thus a magnet can pick up iron nails, but not aluminum cans.

As the external field is increased, a point is eventually reached at which nearly *all* the magnetic moments in the ferromagnetic material are aligned parallel to the external field. This condition is called *saturation magnetization;* after it is reached, further increase in the external field causes no increase in magnetization or in the additional field caused by the magnetization.

28.28 A magnetization curve for a ferromagnetic material. The magnetization M approaches its saturation value M_{sat} as the magnetic field B_0 (caused by external currents) becomes large.

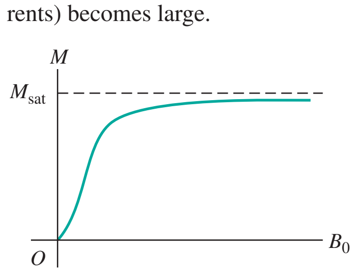


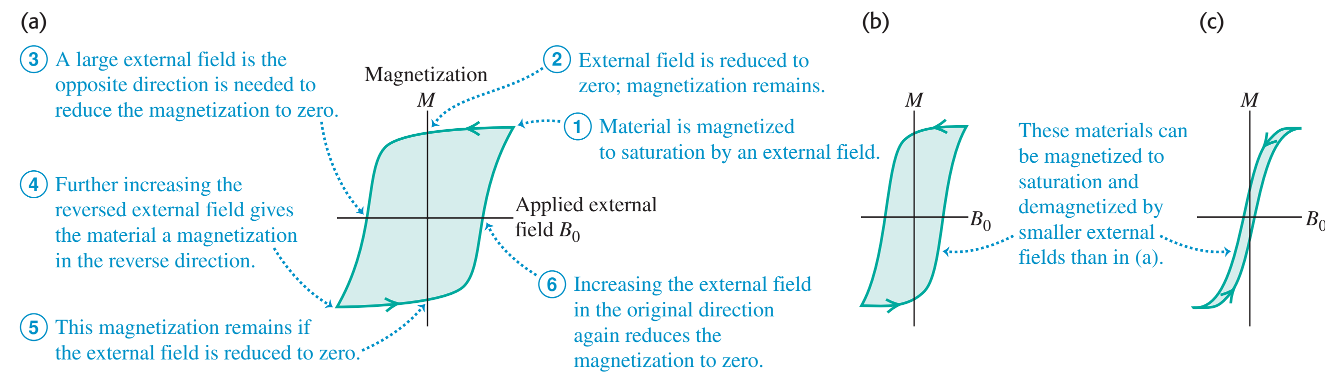
Figure 28.28 shows a "magnetization curve," a graph of magnetization M as a function of external magnetic field B_0, for soft iron. An alternative description of this behavior is that K_m is not constant but decreases as B_0 increases. (Paramagnetic materials also show saturation at sufficiently strong fields. But the magnetic fields required are so large that departures from a linear relationship between M and B_0 in these materials can be observed only at very low temperatures, 1 K or so.)

For many ferromagnetic materials the relationship of magnetization to external magnetic field is different when the external field is increasing from when it is decreasing. Figure 28.29a shows this relationship for such a material. When the material is magnetized to saturation and then the external field is reduced to zero, some magnetization remains. This behavior is characteristic of permanent magnets, which retain most of their saturation magnetization when the magnetizing field is removed. To reduce the magnetization to zero requires a magnetic field in the reverse direction.

This behavior is called **hysteresis,** and the curves in Fig. 28.29 are called *hysteresis loops.* Magnetizing and demagnetizing a material that has hysteresis involve the dissipation of energy, and the temperature of the material increases during such a process.

Ferromagnetic materials are widely used in electromagnets, transformer cores, and motors and generators, in which it is desirable to have as large a magnetic field as possible for a given current. Because hysteresis dissipates energy, materials that are used in these applications should usually have as narrow a hysteresis loop as possible. Soft iron is often used; it has high permeability without appreciable hysteresis. For permanent magnets a broad hysteresis loop is usually desirable, with large zero-field magnetization and large reverse field needed to demagnetize. Many kinds of steel and many alloys, such as Alnico, are commonly used for permanent magnets. The remaining magnetic field in such a material, after it has been magnetized to near saturation, is typically of the order of 1 T, corresponding to a remaining magnetization $M = B/\mu_0$ of about 800,000 A/m.

28.29 Hysteresis loops. The materials of both (a) and (b) remain strongly magnetized when B_0 is reduced to zero. Since (a) is also hard to demagnetize, it would be good for permanent magnets. Since (b) magnetizes and demagnetizes more easily, it could be used as a computer memory material. The material of (c) would be useful for transformers and other alternating-current devices where zero hysteresis would be optimal.



Example 28.12 A ferromagnetic material

A permanent magnet is made of a ferromagnetic material with a magnetization M of about 8×10^5 A/m. The magnet is in the shape of a cube of side 2 cm. (a) Find the magnetic dipole moment of the magnet. (b) Estimate the magnetic field due to the magnet at a point 10 cm from the magnet along its axis.

SOLUTION

IDENTIFY: This problem uses the relationship between magnetization and magnetic dipole moment, as well as the idea that a magnetic dipole produces a magnetic field.

SET UP: We find the magnetic dipole moment from the magnetization, which equals magnetic moment per unit volume. To estimate the magnetic field, we approximate the magnet as a current loop with the same magnetic moment and use the results of Section 28.5.

EXECUTE: (a) The total magnetic moment is the magnetization multiplied by the volume:

$$\mu_{total} = MV = (8 \times 10^5 \text{ A/m})(2 \times 10^{-2} \text{ m})^3 = 6 \text{ A} \cdot \text{m}^2$$

(b) We found in Section 28.5 that the magnetic field on the axis of a current loop with magnetic moment μ_{total} is given by Eq. (28.18),

$$B = \frac{\mu_0 \mu_{total}}{2\pi(x^2 + a^2)^{3/2}}$$

where x is the distance from the loop and a is its radius. We can use this same expression here, except that a refers to the size of the permanent magnet. Strictly speaking, there are complications because our magnet does not have the same geometry as a circular current loop. But because $x = 10$ cm is fairly large in comparison to the 2-cm size of the magnet, the term a^2 is negligible in comparison to x^2 and can be ignored. So

$$B \approx \frac{\mu_0 \mu_{total}}{2\pi x^3} = \frac{(4\pi \times 10^{-7} \text{ T} \cdot \text{m/A})(6 \text{ A} \cdot \text{m}^2)}{2\pi(0.1 \text{ m})^3}$$
$$= 1 \times 10^{-3} \text{ T} = 10 \text{ G}$$

which is about ten times stronger than the magnetic field of the earth. Such a magnet can easily deflect a compass needle.

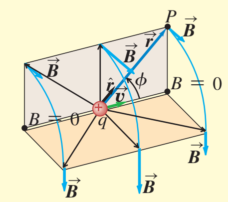
EVALUATE: Note that we used μ_0, not the permeability μ of the magnetic material, in calculating B. The reason is that we are calculating B at a point *outside* the magnetic material. You would substitute permeability μ for μ_0 only if you were calculating B *inside* a material with relative permeability K_m, for which $\mu = K_m\mu_0$.

Test Your Understanding of Section 28.8 Which of the following materials are attracted to a magnet? (i) sodium; (ii) bismuth; (iii) lead; (iv) uranium.

CHAPTER 28 SUMMARY

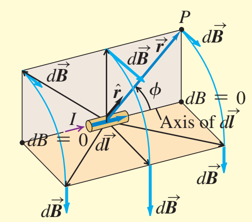
Magnetic field of a moving charge: The magnetic field $\vec{B}$ created by a charge q moving with velocity $\vec{v}$ depends on the distance r from the source point (the location of q) to the field point (where $\vec{B}$ is measured). The $\vec{B}$ field is perpendicular to $\vec{v}$ and to $\hat{r}$, the unit vector directed from the source point to the field point. The principle of superposition of magnetic fields states that the total $\vec{B}$ field produced by several moving charges is the vector sum of the fields produced by the individual charges. (See Example 28.1.)

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2} \qquad (28.2)$$



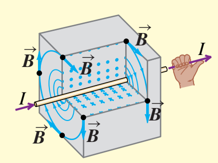
Magnetic field of a current-carrying conductor: The law of Biot and Savart gives the magnetic field $d\vec{B}$ created by an element $d\vec{l}$ of a conductor carrying current I. The field $d\vec{B}$ is perpendicular to both $d\vec{l}$ and $\hat{r}$, the unit vector from the element to the field point. The $\vec{B}$ field created by a finite current-carrying conductor is the integral of $d\vec{B}$ over the length of the conductor. (See Example 28.2.)

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l} \times \hat{r}}{r^2} \qquad (28.6)$$



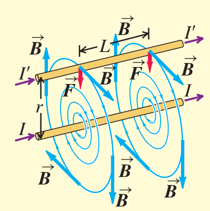
Magnetic field of a long, straight, current-carrying conductor: The magnetic field $\vec{B}$ at a distance r from a long, straight conductor carrying a current I has a magnitude that is inversely proportional to r. The magnetic field lines are circles coaxial with the wire, with directions given by the right-hand rule. (See Examples 28.3 and 28.4.)

$$B = \frac{\mu_0 I}{2\pi r} \qquad (28.9)$$



Magnetic force between current-carrying conductors: Two long, parallel, current-carrying conductors attract if the currents are in the same direction and repel if the currents are in opposite directions. The magnetic force per unit length between the conductors depends on their currents I and I' and their separation r. The definition of the ampere is based on this relationship. (See Example 28.5.)

$$\frac{F}{L} = \frac{\mu_0 I I'}{2\pi r} \qquad (28.11)$$



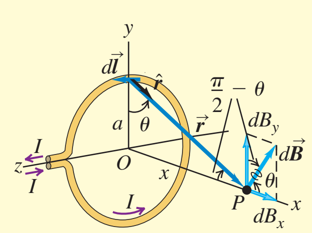
Magnetic field of a current loop: The law of Biot and Savart allows us to calculate the magnetic field produced along the axis of a circular conducting loop of radius a carrying current I. The field depends on the distance x along the axis from the center of the loop to the field point. If there are N loops, the field is multiplied by N. At the center of the loop, $x = 0$. (See Example 28.6.)

$$B_x = \frac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}} \qquad (28.15)$$

(circular loop)

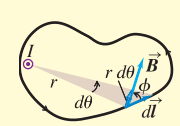
$$B_x = \frac{\mu_0 N I}{2a} \qquad (28.17)$$

(center of N circular loops)



Ampere's law: Ampere's law states that the line integral of $\vec{B}$ around any closed path equals μ_0 times the net current through the area enclosed by the path. The positive sense of current is determined by a right-hand rule. (See Examples 28.7–28.10.)

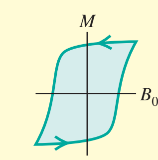
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} \qquad (28.20)$$



Magnetic fields due to current distributions: The table lists magnetic fields caused by several current distributions. In each case the conductor is carrying current I.

Current Distribution	Point in Magnetic Field	Magnetic-Field Magnitude
Long, straight conductor	Distance r from conductor	$B = \frac{\mu_0 I}{2\pi r}$
Circular loop of radius a	On axis of loop	$B = \frac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}}$
	At center of loop	$B = \frac{\mu_0 I}{2a}$ (for N loops, multiply these expressions by N)
Long cylindrical conductor of radius R	Inside conductor, $r < R$	$B = \frac{\mu_0 I}{2\pi}\frac{r}{R^2}$
	Outside conductor, $r > R$	$B = \frac{\mu_0 I}{2\pi r}$
Long, closely wound solenoid with n turns per unit length, near its midpoint	Inside solenoid, near center	$B = \mu_0 n I$
	Outside solenoid	$B \approx 0$
Tightly wound toroidal solenoid (toroid) with N turns	Within the space enclosed by the windings, distance r from symmetry axis	$B = \frac{\mu_0 N I}{2\pi r}$
	Outside the space enclosed by the windings	$B \approx 0$

***Magnetic materials:** When magnetic materials are present, the magnetization of the material causes an additional contribution to $\vec{B}$. For paramagnetic and diamagnetic materials, μ_0 is replaced in magnetic-field expressions by $\mu = K_m \mu_0$, where μ is the permeability of the material and K_m is its relative permeability. The magnetic susceptibility χ_m is defined as $\chi_m = K_m - 1$. Magnetic susceptibilities for paramagnetic materials are small positive quantities; those for diamagnetic materials are small negative quantities. For ferromagnetic materials, K_m is much larger than unity and is not constant. Some ferromagnetic materials are permanent magnets, retaining their magnetization even after the external magnetic field is removed. (See Examples 28.11 and 28.12.)



Key Terms

Answer to Chapter Opening Question ?

There would be *no* change in the magnetic field strength. From Example 28.9 (Section 28.7), the field inside a solenoid has magnitude $B = \mu_0 n I$, where n is the number of turns of wire per unit length. Joining two solenoids end to end doubles both the number of turns and the length, so the number of turns per unit length is unchanged.

Answers to Test Your Understanding Questions

28.1 Answer: (a) (i), (b) (ii) The situation is the same as shown in Fig. 28.2 except that the upper proton has velocity $\vec{v}$ rather than $-\vec{v}$. The magnetic field due to the lower proton is the same as shown in Fig. 28.2, but the direction of the magnetic force $\vec{F} = q\vec{v} \times \vec{B}$ on the upper proton is reversed. Hence the magnetic force is attractive. Since the speed v is small compared to c, the magnetic force is much smaller in magnitude than the repulsive electric force and the net force is still repulsive.

28.2 Answer: (i) and (iii) (tie), (iv), (ii) From Eq. (28.5), the magnitude of the field dB due to a current element of length dl carrying current I is $dB = (\mu/4\pi)(I\,dl\sin\phi/r^2)$. In this expression r is the distance from the element to the field point, and ϕ is the angle between the direction of the current and a vector from the current element to the field point. All four points are the same distance $r = L$ from the current element, so the value of dB is proportional to the value of $\sin\phi$. For the four points the angle is (i) $\phi = 90°$, (ii) $\phi = 0$, (iii) $\phi = 90°$, and (iv) $\phi = 45°$, so the values of $\sin\phi$ are (i) 1, (ii) 0, (iii) 1, and (iv) $1/\sqrt{2}$.

28.3 Answer: A This orientation will cause current to flow clockwise around the circuit. Hence current will flow south through the wire that lies under the compass. From the right-hand rule for the magnetic field produced by a long, straight, current-carrying conductor, this will produce a magnetic field that points to the left at the position of the compass (which lies atop the wire). The combination of the northward magnetic field of the earth and the westward field produced by the current gives a net magnetic field to the northwest, so the compass needle will swing counterclockwise to align with this field.

28.4 Answers: (a) (i), (b) (iii), (c) (ii), (d) (iii) Current flows in the same direction in adjacent turns of the coil, so the magnetic forces between these turns are attractive. Current flows in opposite directions on opposite sides of the same turn, so the magnetic forces between these sides are repulsive. Thus the magnetic forces on the solenoid turns squeeze them together in the direction along its axis but push them apart radially. The *electric* forces are zero because the wire is electrically neutral, with as much positive charge as there is negative charge.

28.5 Answers: (a) (ii), (b) (v) The vector $d\vec{B}$ is in the direction of $d\vec{l} \times \vec{r}$. For a segment on the negative y-axis, $d\vec{l} = -\hat{k}\,dl$ points in the negative z-direction and $\vec{r} = x\hat{\imath} + a\hat{\jmath}$. Hence $d\vec{l} \times \vec{r} = (a\,dl)\hat{\imath} - (x\,dl)\hat{\jmath}$, which has a positive x-component, a negative y-component and zero z-component. For a segment on the negative z-axis, $d\vec{l} = \hat{\jmath}\,dl$ points in the positive y-direction and $\vec{r} = x\hat{\imath} + a\hat{k}$. Hence $d\vec{l} \times \vec{r} = (a\,dl)\hat{\imath} - (x\,dl)\hat{k}$, which has a positive x-component, zero y-component, and a negative z-component.

28.6 Answer: (ii) Imagine carrying out the integral $\oint \vec{B} \cdot d\vec{l}$ along an integration path that goes clockwise around the red magnetic field line. At each point along the path the magnetic field $\vec{B}$ and the infinitesimal segment $d\vec{l}$ are both tangent to the path, so $\vec{B} \cdot d\vec{l}$ is positive at each point and the integral $\oint \vec{B} \cdot d\vec{l}$ is likewise positive. It follows from Ampere's law $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$ and the right-hand rule that the integration path encloses a current directed out of the plane of the page. There are no currents in the empty space outside the magnet, so there must be currents inside the magnet (see Section 28.8).

28.7 Answer: (iii) By symmetry, any $\vec{B}$ field outside the cable must circulate around the cable, with circular field lines like those surrounding the solid cylindrical conductor in Fig. 28.20. Choose an integration path like the one shown in Fig. 28.20 with radius $r > R$, so that the path completely encloses the cable. As in Example 28.8, the integral $\oint \vec{B} \cdot d\vec{l}$ for this path has magnitude $B(2\pi r)$. From Ampere's law this is equal to $\mu_0 I_{encl}$. The net enclosed current I_{encl} is zero because it includes two currents of equal magnitude but opposite direction: one in the central wire and one in the hollow cylinder. Hence $B(2\pi r) = 0$, and so $B = 0$ for any value of r outside the cable. (The field is nonzero *inside* the cable; see Exercise 28.37.)

28.8 Answer: (i), (iv) Sodium and uranium are paramagnetic materials and hence are attracted to a magnet, while bismuth and lead are diamagnetic materials that are repelled by a magnet. (See Table 28.1.)

PROBLEMS

For instructor-assigned homework, go to **www.masteringphysics.com**

Discussion Questions

Q28.1. A topic of current interest in physics research is the search (thus far unsuccessful) for an isolated magnetic pole, or magnetic *monopole*. If such an entity were found, how could it be recognized? What would its properties be?

Q28.2. Streams of charged particles emitted from the sun during periods of solar activity create a disturbance in the earth's magnetic field. How does this happen?

Q28.3. The text discussed the magnetic field of an infinitely long, straight conductor carrying a current. Of course, there is no such thing as an infinitely long *anything*. How do you decide whether a particular wire is long enough to be considered infinite?

Q28.4. Two parallel conductors carrying current in the same direction attract each other. If they are permitted to move toward each other, the forces of attraction do work. From where does the energy come? Does this contradict the assertion in Chapter 27 that magnetic forces on moving charges do no work? Explain.

Q28.5. Pairs of conductors carrying current into or out of the power-supply components of electronic equipment are sometimes twisted together to reduce magnetic-field effects. Why does this help?

Q28.6. Suppose you have three long, parallel wires arranged so that in cross section they are at the corners of an equilateral triangle. Is there any way to arrange the currents so that all three wires attract each other? So that all three wires repel each other? Explain.

Q28.7. In deriving the force on one of the long, current-carrying conductors in Section 28.4, why did we use the magnetic field due to only one of the conductors? That is, why didn't we use the *total* magnetic field due to *both* conductors?

Q28.8. Two concentric, coplanar, circular loops of wire of different diameter carry currents in the same direction. Describe the nature of the force exerted on the inner loop by the outer loop and on the outer loop by the inner loop.

Q28.9. A current was sent through a helical coil spring. The spring contracted, as though it had been compressed. Why?

Q28.10. What are the relative advantages and disadvantages of Ampere's law and the law of Biot and Savart for practical calculations of magnetic fields?

Q28.11. Magnetic field lines never have a beginning or an end. Use this to explain why it is reasonable for the field of a toroidal solenoid to be confined entirely to its interior, while a straight solenoid *must* have some field outside.

Q28.12. If the magnitude of the magnetic field a distance R from a very long, straight, current-carrying wire is B, at what distance from the wire will the field have magnitude $3B$?

Q28.13. Two very long, parallel wires carry equal currents in opposite directions. (a) Is there any place that their magnetic fields completely cancel? If so, where? If not, why not? (b) How would the answer to part (a) change if the currents were in the same direction?

Q28.14. In the circuit shown in Figure 28.30, when switch S is suddenly closed, the wire L is pulled toward the lower wire carrying current I. Which (a or b) is the positive terminal of the battery? How do you know?

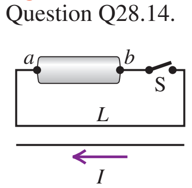


Figure 28.30 Question Q28.14.

Q28.15. A metal ring carries a current that causes a magnetic field B_0 at the center of the ring and a field B at point P a distance x from the center along the axis of the ring. If the radius of the ring is doubled, find the magnetic field at the center. Will the field at point P change by the same factor? Why?

***Q28.16.** Why should the permeability of a paramagnetic material be expected to decrease with increasing temperature?

***Q28.17.** If a magnet is suspended over a container of liquid air, it attracts droplets to its poles. The droplets contain only liquid oxygen; even though nitrogen is the primary constituent of air, it is not attracted to the magnet. Explain what this tells you about the magnetic susceptibilities of oxygen and nitrogen, and explain why a magnet in ordinary, room-temperature air doesn't attract molecules of oxygen *gas* to its poles.

***Q28.18.** What features of atomic structure determine whether an element is diamagnetic or paramagnetic? Explain.

***Q28.19.** The magnetic susceptibility of paramagnetic materials is quite strongly temperature dependent, but that of diamagnetic materials is nearly independent of temperature Why the difference?

***Q28.20.** A cylinder of iron is placed so that it is free to rotate around its axis. Initially the cylinder is at rest, and a magnetic field is applied to the cylinder so that it is magnetized in a direction parallel to its axis. If the direction of the *external* field is suddenly reversed, the direction of magnetization will also reverse and the cylinder will begin rotating around its axis. (This is called the *Einstein-de Haas effect*.) Explain why the cylinder begins to rotate.

***Q28.21.** The discussion of magnetic forces on current loops in Section 27.7 commented that no net force is exerted on a complete loop in a uniform magnetic field, only a torque. Yet magnetized materials that contain atomic current loops certainly *do* experience net forces in magnetic fields. How is this discrepancy resolved?

***Q28.22.** Show that the units $A \cdot m^2$ and J/T for the Bohr magneton are equivalent.

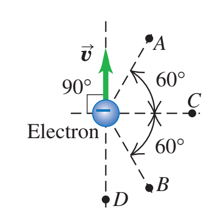
Exercises

Section 26.1 Magnetic Field of a Moving Charge

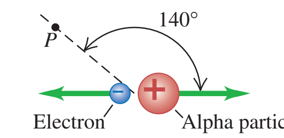
28.1. A $+6.00\text{-}\mu C$ point charge is moving at a constant 8.00×10^6 m/s in the $+y$-direction, relative to a reference frame. At the instant when the point charge is at the origin of this reference frame, what is the magnetic-field vector $\vec{B}$ it produces at the following points: (a) $x = 0.500$ m, $y = 0$, $z = 0$; (b) $x = 0$, $y = -0.500$ m, $z = 0$; (c) $x = 0$, $y = 0$, $z = +0.500$ m; (d) $x = 0$, $y = -0.500$ m, $z = +0.500$ m?

28.2. Fields Within the Atom. In the Bohr model of the hydrogen atom, the electron moves in a circular orbit of radius 5.3×10^{-11} m with a speed of 2.2×10^6 m/s. If we are viewing the atom in such a way that the electron's orbit is in the plane of the paper with the electron moving clockwise, find the magnitude and direction of the electric and magnetic fields that the electron produces at the location of the nucleus (treated as a point).

28.3. An electron moves at $0.100c$ as shown in Fig. 28.31. Find the magnitude and direction of the magnetic field this electon produces at the following points, each 2.00 μm from the electron: (a) points A and B; (b) point C; (c) point D.

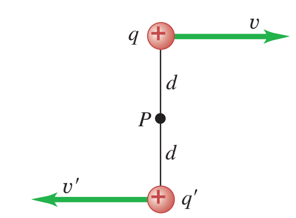
Figure 28.31 Exercise 28.3.


28.4. An alpha particle (charge $+2e$) and an electron move in opposite directions from the same point, each with the speed of 2.50×10^5 m/s (Fig. 28.32). Find the magnitude and direction of the total magnetic field these charges produce at point P, which is 1.75 nm from each of them.

Figure 28.32 Exercise 28.4.


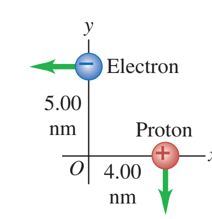
28.5. A $-4.80\text{-}\mu C$ charge is moving at a constant speed of 6.80×10^5 m/s in the $+x$-direction relative to a reference frame. At the instant when the point charge is at the origin, what is the magnetic-field vector it produces at the following points: (a) $x = 0.500$ m, $y = 0$, $z = 0$; (b) $x = 0$, $y = 0.500$ m, $z = 0$; (c) $x = 0.500$ m, $y = 0.500$ m, $z = 0$; (d) $x = 0$, $y = 0$, $z = 0.500$ m?

28.6. Positive point charges $q = +8.00\ \mu C$ and $q' = +3.00\ \mu C$ are moving relative to an observer at point P, as shown in Fig. 28.33. The distance d is 0.120 m, $v = 4.50 \times 10^6$ m/s, and $v' = 9.00 \times 10^6$ m/s. (a) When the two charges are at the locations shown in the figure, what are the magnitude and direction of the net magnetic field they produce at point P? (b) What are the magnitude and direction of the electric and magnetic forces that each charge exerts on the other, and what is the ratio of the magnitude of the electric force to the magnitude of the magnetic force? (c) If the direction of $\vec{v}'$ is reversed, so both charges are moving in the same direction, what are the magnitude and direction of the magnetic forces that the two charges exert on each other?

Figure 28.33 Exercises 28.6 and 28.7.


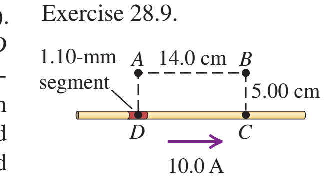
28.7. Figure 28.33 shows two point charges, q and q', moving relative to an observer at point P. Suppose that the lower charge is actually *negative*, with $q' = -q$. (a) Find the magnetic field (magnitude and direction) produced by the two charges at point P if (i) $v' = v/2$; (ii) $v' = v$; (iii) $v' = 2v$. (b) Find the direction of the magnetic force that q exerts on q', and find the direction of the magnetic force that q' exerts on q. (c) If $v = v' = 3.00 \times 10^5$ m/s, what is the ratio of the magnitude of the magnetic force acting on each charge to that of the Coulomb force acting on each charge?

28.8. An electron and a proton are each moving at 845 km/s in perpendicular paths as shown in Fig. 28.34. At the instant when they are at the positions shown in the figure, find the magnitude and direction of (a) the total magnetic field they produce at the origin; (b) the magnetic field the electron produces at the location of the proton; (c) the total electrical force and the total magnetic force that the electron exerts on the proton.

Figure 28.34 Exercise 28.8.


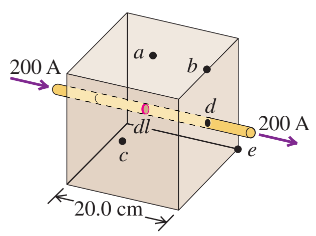
Section 28.2 Magnetic Field of a Current Element

28.9. A straight wire caries a 10.0-A current (Fig. 28.35). $ABCD$ is a rectangle with point D in the middle of a 1.10-mm segment of the wire and point C in the wire. Find the magnitude and direction of the magnetic field due to this segment at (a) point A; (b) point B; (c) point C.

Figure 28.35 Exercise 28.9.


28.10. A long, straight wire, carrying a current of 200 A, runs through a cubical wooden box, entering and leaving through holes in the centers of opposite faces (Fig. 28.36). The length of each side of the box is 20.0 cm. Consider an element dl of the wire 0.100 cm long at the center of the box. Compute the magnitude dB of the magnetic field produced by this element at the points a, b, c, d, and e in Fig. 28.36. Points a, c, and d are at the centers of the faces of the cube; point b is at the midpoint of one edge; and point e is at a corner. Copy the figure and show the directions and relative magnitudes of the field vectors. (*Note:* Assume that the length dl is small in comparison to the distances from the current element to the points where the magnetic field is to be calculated.)

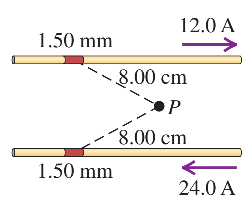
Figure 28.36 Exercise 28.10.



28.11. A long, straight wire lies along the z-axis and carries a 4.00-A current in the $+z$-direction. Find the magnetic field (magnitude and direction) produced at the following points by a 0.500-mm segment of the wire centered at the origin: (a) $x = 2.00$ m, $y = 0$, $z = 0$; (b) $x = 0$, $y = 2.00$ m, $z = 0$; (c) $x = 2.00$ m, $y = 2.00$ m, $z = 0$; (d) $x = 0$, $y = 0$, $z = 2.00$ m.

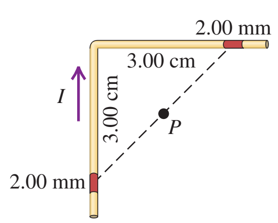
28.12. Two parallel wires are 5.00 cm apart and carry currents in opposite directions, as shown in Fig. 28.37. Find the magnitude and direction of the magnetic field at point P due to two 1.50-mm segments of wire that are opposite each other and each 8.00 cm from P.

Figure 28.37 Exercise 28.12.



28.13. A wire carrying a 28.0-A current bends through a right angle. Consider two 2.00-mm segments of wire, each 3.00 cm from the bend (Fig. 28.38). Find the magnitude and direction of the magnetic field these two segments produce at point P, which is midway between them.

Figure 28.38 Exercise 28.13.



28.14. A square wire loop 10.0 cm on each side carries a clockwise current of 15.0 A. Find the magnitude and direction of the magnetic field at its center due to the four 1.20-mm wire segments at the midpoint of each side.

Section 28.3 Magnetic Field of a Straight Current-Carrying Conductor

28.15. The Magnetic Field from a Lightning Bolt. Lightning bolts can carry currents up to approximately 20 kA. We can model such a current as the equivalent of a very long, straight wire. (a) If you were unfortunate enough to be 5.0 m away from such a lightning bolt, how large a magnetic field would you experience? (b) How does this field compare to one you would experience by being 5.0 cm from a long, straight household current of 10 A?

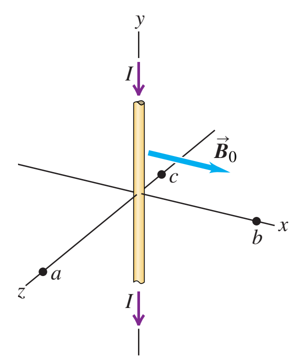
28.16. A very long, straight horizontal wire carries a current such that 3.50×10^{18} electrons per second pass any given point going from west to east. What are the magnitude and direction of the magnetic field this wire produces at a point 4.00 cm directly above it?

28.17. (a) How large a current would a very long, straight wire have to carry so that the magnetic field 2.00 cm from the wire is equal to 1.00 G (comparable to the earth's northward-pointing magnetic field)? (b) If the wire is horizontal with the current running from east to west, at what locations would the magnetic field of the wire point in the same direction as the horizontal component of the earth's magnetic field? (c) Repeat part (b) except the wire is vertical with the current going upward.

28.18. Two long, straight wires, one above the other, are seperated by a distance $2a$ and are parallel to the x-axis. Let the $+y$-axis be in the plane of the wires in the direction from the lower wire to the upper wire. Each wire carries current I in the $+x$-direction. What are the magnitude and direction of the net magnetic field of the two wires at a point in the plane of the wires (a) midway between them; (b) at a distance a above the upper wire; (c) at a distance a below the lower wire?

28.19. A long, straight wire lies along the y-axis and carries a current $I = 8.00$ A in the $-y$-direction (Fig. 28.39). In addition to the magnetic field due to the current in the wire, a uniform magnetic field $\vec{B}_0$ with magnitude 1.50×10^{-6} T is in the $+x$-direction What is the total field (magnitude and direction) at the following points in the xz-plane: (a) $x = 0$, $z = 1.00$ m; (b) $x = 1.00$ m, $z = 0$; (c) $x = 0$, $z = -0.25$ m?

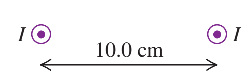
Figure 28.39 Exercise 28.19.



28.20. Effect of Transmission Lines. Two hikers are reading a compass under an overhead transmission line that is 5.50 m above the ground and carries a current of 800 A in a horizontal direction from north to south. (a) Find the magnitude and direction of the magnetic field at a point on the ground directly under the conductor. (b) One hiker suggests they walk on another 50 m to avoid inaccurate compass readings caused by the current. Considering that the magnitude of the earth's field is of the order of 0.5×10^{-4} T, is the current really a problem?

28.21. Two long, straight, parallel wires, 10.0 cm apart, carry equal 4.00-A currents in the same direction, as shown in Fig. 28.40. Find the magnitude and direction of the magnetic field at (a) point P_1, midway between the wires; (b) point P_2, 25.0 cm to the right of P_1; (c) point P_3, 20.0 cm directly above P_1.

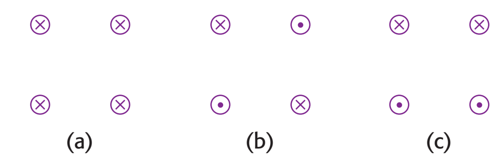
Figure 28.40 Exercise 28.21.



28.22. Two parallel transmission lines, 40.0 cm apart, carry 25.0-A and 75.0-A currents. Find all locations where the net magnetic field of the two wires is zero if these currents are in (a) the same direction and (b) the opposite direction.

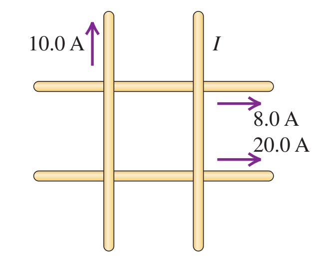
28.23. Four, long, parallel power lines each carry 100-A currents. A cross-sectional diagram of these lines is a square, 20.0 cm on each side. For each of the three cases shown in Fig. 28.41, calculate the magnetic field at the center of the square.

Figure 28.41 Exercise 28.23.



28.24. Four very long, current-carrying wires in the same plane intersect to form a square 40.0 cm on each side, as shown in Fig. 28.42. Find the magnitude and direction of the current I so that the magnetic field at the center of the square is zero.

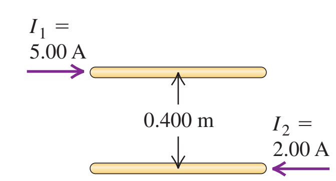
Figure 28.42 Exercise 28.24.



Section 28.4 Force Between Parallel Conductors

28.25. Two long, parallel wires are separated by a distance of 0.400 m (Fig. 28.43). The currents I_1 and I_2 have the directions shown. (a) Calculate the magnitude of the force exerted by each wire on a 1.20-m length of the other. Is the force attractive or repulsive? (b) Each current is doubled, so that I_1 becomes 10.0 A and I_2 becomes 4.00 A. Now what is the magnitude of the force that each wire exerts on a 1.20-m length of the other?

Figure 28.43 Exercise 28.25.

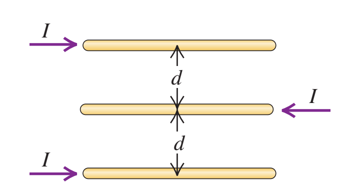


28.26. Two long, parallel wires are separated by a distance of 2.50 cm. The force per unit length that each wire exerts on the other is 4.00×10^{-5} N/m, and the wires repel each other. The current in one wire is 0.600 A. (a) What is the current in the second wire? (b) Are the two currents in the same direction or in opposite directions?

28.27. Lamp Cord Wires. The wires in a household lamp cord are typically 3.0 mm apart center to center and carry equal currents in opposite directions. If the cord carries current to a 100-W light bulb connected across a 120-V potential difference, what force per meter does each wire of the cord exert on the other? Is the force attractive or repulsive? Is this force large enough so it should be considered in the design of lamp cord? (Model the lamp cord as a very long straight wire.)

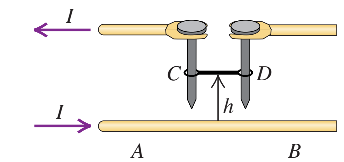
28.28. Three parallel wires each carry current I in the directions shown in Fig. 28.44. If the separation between adjacent wires is d, calculate the magnitude and direction of the net magnetic force per unit length on each wire.

Figure 28.44 Exercise 28.28.



28.29. A long, horizontal wire AB rests on the surface of a table and carries a current I. Horizontal wire CD is vertically above wire AB and is free to slide up and down on the two vertical metal guides C and D (Fig. 28.45). Wire CD is connected through the sliding contacts to another wire that also carries a current I, opposite in direction to the current in wire AB. The mass per unit length of the wire CD is λ. To what equilibrium height h will the wire CD rise, assuming that the magnetic force on it is due entirely to the current in the wire AB?

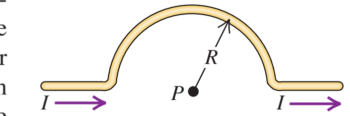
Figure 28.45 Exercise 28.29.



Section 28.5 Magnetic Field of a Circular Current Loop

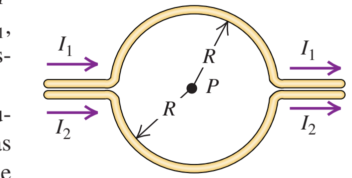
28.30. Calculate the magnitude and direction of the magnetic field at point P due to the current in the semicircular section of wire shown in Fig. 28.46. (*Hint:* Does the current in the long, straight section of the wire produce any field at P?)

Figure 28.46 Exercise 28.30.



28.31. Calculate the magnitude of the magnetic field at point P of Fig. 28.47 in terms of R, I_1, and I_2. What does your expression give when $I_1 = I_2$?

Figure 28.47 Exercise 28.31.



28.32. A closely wound, circular coil with radius 2.40 cm has 800 turns. a) What must the current in the coil be if the magnetic field at the center of the coil is 0.0580 T? b) At what distance x from the center of the coil, on the axis of the coil, is the magnetic field half its value at the center?

28.33. A closely wound, circular coil with a diameter of 4.00 cm has 600 turns and carries a current of 0.500 A. What is the magnitude of the magnetic field (a) at the center of the coil and (b) at a point on the axis of the coil 8.00 cm from its center?

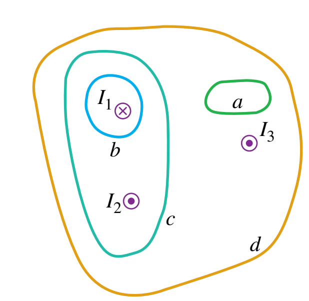
28.34. A closely wound coil has a radius of 6.00 cm and carries a current of 2.50 A. How many turns must it have if, at a point on the coil axis 6.00 cm from the center of the coil, the magnetic field is 6.39×10^{-4} T?

Section 28.6 Ampere's Law

28.35. A closed curve encircles several conductors. The line integral $\oint \vec{B} \cdot d\vec{l}$ around this curve is 3.83×10^{-4} T · m. (a) What is the net current in the conductors? (b) If you were to integrate around the curve in the opposite direction, what would be the value of the line integral? Explain.

28.36. Figure 28.48 shows, in cross section, several conductors that carry currents through the plane of the figure. The currents have the magnitudes $I_1 = 4.0$ A, $I_2 = 6.0$ A, and $I_3 = 2.0$ A, and the directions shown. Four paths, labeled a through d, are shown. What is the line integral $\oint \vec{B} \cdot d\vec{l}$ for each path? Each integral involves going around the path in the counterclockwise direction. Explain your answers.

Figure 28.48 Exercise 28.36.



Section 28.7 Applications of Ampere's Law

28.37. Coaxial Cable. A solid conductor with radius a is supported by insulating disks on the axis of a conducting tube with inner radius b and outer radius c (Fig. 28.49). The central conductor and tube carry equal currents I in opposite directions. The currents are distributed uniformly over the cross sections of each conductor. Derive an expression for the magnitude of the magnetic field (a) at points outside the central, solid conductor but inside the tube $(a < r < b)$ and (b) at points outside the tube $(r > c)$.

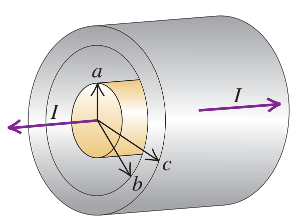


Figure 28.49 Exercise 28.37.

28.38. Repeat Exercise 28.37 for the case in which the current in the central, solid conductor is I_1, the current in the tube is I_2, and these currents are in the same direction rather than in opposite directions.

28.39. A long, straight, cylindrical wire of radius R carries a current uniformly distributed over its cross section. At what location is the magnetic field produced by this current equal to half of its largest value? Consider points inside and outside the wire.

28.40. A 15.0-cm-long solenoid with radius 2.50 cm is closely wound with 600 turns of wire. The current in the windings is 8.00 A. Compute the magnetic field at a point near the center of the solenoid.

28.41. A solenoid is designed to produce a magnetic field of 0.0270 T at its center. It has radius 1.40 cm and length 40.0 cm, and the wire can carry a maximum current of 12.0 A. (a) What minimum number of turns per unit length must the solenoid have? (b) What total length of wire is required?

28.42. As a new electrical technician, you are designing a large solenoid to produce a uniform 0.150 T magnetic field near the center of the solenoid. You have enough wire for 4000 circular turns. This solenoid must be 1.40 m long and 20.0 cm in diameter. What current will you need to produce the necessary field?

28.43. A magnetic field of 37.2 T has been achieved at the MIT Francis Bitter National Magnetic Laboratory. Find the current needed to achieve such a field (a) 2.00 cm from a long, straight wire; (b) at the center of a circular coil of radius 42.0 cm that has 100 turns; (c) near the center of a solenoid with radius 2.40 cm, length 32.0 cm, and 40,000 turns.

28.44. A toroidal solenoid (see Example 28.10) has inner radius $r_1 = 15.0$ cm and outer radius $r_2 = 18.0$ cm. The solenoid has 250 turns and carries a current of 8.50 A. What is the magnitude of the magnetic field at the following distances from the center of the torus: (a) 12.0 cm; (b) 16.0 cm; (c) 20.0 cm?

28.45. A wooden ring whose mean diameter is 14.0 cm is wound with a closely spaced toroidal winding of 600 turns. Compute the magnitude of the magnetic field at the center of the cross section of the windings when the current in the windings is 0.650 A.

*Section 28.8 Magnetic Materials

***28.46.** A toroidal solenoid with 400 turns of wire and a mean radius of 6.0 cm carries a current of 0.25 A. The relative permeability of the core is 80. (a) What is the magnetic field in the core? (b) What part of the magnetic field is due to atomic currents?

***28.47.** A toroidal solenoid with 500 turns is wound on a ring with a mean radius of 2.90 cm. Find the current in the winding that is required to set up a magnetic field of 0.350 T in the ring (a) if the ring is made of annealed iron $(K_m = 1400)$ and (b) if the ring is made of silicon steel $(K_m = 5200)$.

***28.48.** The current in the windings of a toroidal solenoid is 2.400 A. There are 500 turns, and the mean radius is 25.00 cm. The toroidal solenoid is filled with a magnetic material. The magnetic field inside the windings is found to be 1.940 T. Calculate (a) the relative permeability and (b) the magnetic susceptibility of the material that fills the toroid.

***28.49.** A long solenoid with 60 turns of wire per centimeter carries a current of 0.15 A. The wire that makes up the solenoid is wrapped around a solid core of silicon steel $(K_m = 5200)$. (The wire of the solenoid is jacketed with an insulator so that none of the current flows into the core.) (a) For a point inside the core, find the magnitudes of (i) the magnetic field $\vec{B}_0$ due to the solenoid current; (ii) the magnetization $\vec{M}$; (iii) the total magnetic field $\vec{B}$. (b) In a sketch of the solenoid and core, show the directions of the vectors $\vec{B}$, $\vec{B}_0$, and $\vec{M}$ inside the core.

***28.50. Curie's Law.** Experimental measurements of the magnetic susceptibility of iron ammonium alum are given in the table. Graph values of $1/\chi_m$ against Kelvin temperature. Does the material obey Curie's law? If so, what is the Curie constant?

T (°C)	χ_m
−258.15	129×10^{-4}
−173	19.4×10^{-4}
−73	9.7×10^{-4}
27	6.5×10^{-4}

Problems

28.51. A pair of point charges, $q = +8.00\ \mu$C and $q' = -5.00\ \mu$C, are moving as shown in Fig 28.50 with speeds $v = 9.00 \times 10^4$ m/s and $v' = 6.50 \times 10^4$ m/s. When the charges are at the locations shown in the figure, what are the magnitude and direction of (a) the magnetic field produced at the origin and (b) the magnetic force that q' exerts on q?

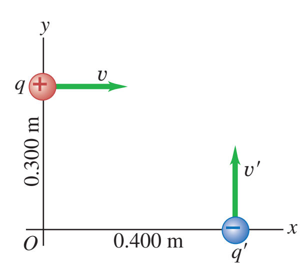


Figure 28.50 Problem 28.51.

28.52. A long, straight wire carries a current of 2.50 A. An electron is traveling in the vicinity of the wire. At the instant when the electron is 4.50 cm from the wire and traveling with a speed of 6.00×10^4 m/s directly toward the wire, what are the magnitude and direction (relative to the direction of the current) of the force that the magnetic field of the current exerts on the electron?

28.53. A long, straight wire carries a 25.0-A current. An electron is fired parallel to this wire with a velocity of 250 km/s in the same direction as the current, 2.00 cm from the wire. (a) Find the magnitude and direction of the electron's initial acceleration. (b) What should be the magnitude and direction of a uniform electric field that will allow the electron to continue to travel parallel to the wire? (c) Is it necessary to include the effects of gravity? Justify your answer.

28.54. In Fig. 28.51 the battery branch of the circuit is very far from the two horizontal segments containing two resistors. These horizontal segments are separated by 5.00 cm, and they are much longer than 5.00 cm. A proton (charge $+e$) is fired at 650 km/s from a point midway between the upper two horizontal segments of the circuit. The initial velocity of the proton is in the plane of the circuit and is directed toward the upper wire. Find the magnitude and direction of the initial magnetic force on the proton.

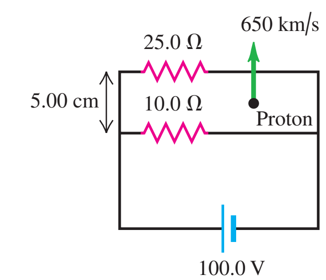


Figure 28.51 Problem 28.54.

28.55. Two identical circular, wire loops 40.0 cm in diameter each carry a current of 1.50 A in the same direction. These loops are parallel to each other and are 25.0 cm apart. Line ab is normal to the plane of the loops and passes through their centers. A proton is fired at 2400 km/s perpendicular to line ab from a point midway between the centers of the loops. Find the magnitude and direction of the magnetic force these loops exert on the proton just after it is fired.

28.56. Two very long, straight wires carry currents as shown in Fig. 28.52. For each case, find all locations where the net magnetic field is zero.

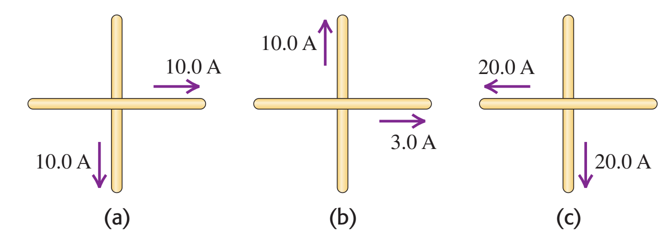


Figure 28.52 Problem 28.56.

28.57. A negative point charge $q = -7.20$ mC is moving in a reference frame. When the point charge is at the origin, the magnetic field it produces at the point $x = 25.0$ cm, $y = 0$, $z = 0$ is $\vec{B} = (6.00\ \mu\text{T})\hat{j}$, and its speed is 800 km/s. (a) What are the x-, y-, and z-components of the velocity $\vec{v}_0$ of the charge? (b) At this same instant, what is the magnitude of the magnetic field that the charge produces at the point $x = 0$, $y = 25.0$ cm, $z = 0$?

28.58. A neophyte magnet designer tells you that he can produce a magnetic field $\vec{B}$ in vacuum that points everywhere in the x-direction and that increases in magnitude with increasing x. That is, $\vec{B} = B_0(x/a)\hat{\imath}$, where B_0 and a are constants with units of teslas and meters, respectively. Use Gauss's law for magnetic fields to show that this claim is *impossible*. (*Hint:* Use a Gaussian surface in the shape of a rectangular box, with edges parallel to the x-, y-, and z-axes.)

28.59. Two long, straight, parallel wires are 1.00 m apart (Fig. 28.53). The wire on the left carries a current I_1 of 6.00 A into the plane of the paper. (a) What must the magnitude and direction of the current I_2 be for the net field at point P to be zero? (b) Then what are the magnitude and direction of the net field at Q? (c) Then what is the magnitude of the net field at S?

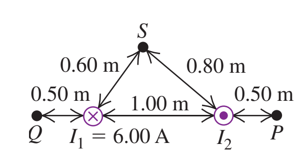


Figure 28.53 Problem 28.59.

28.60. Figure 28.54 shows an end view of two long, parallel wires perpendicular to the xy-plane, each carrying a current I but in opposite directions. (a) Copy the diagram, and draw vectors to show the $\vec{B}$ field of each wire and the net $\vec{B}$ field at point P. (b) Derive the expression for the magnitude of $\vec{B}$ at any point on the x-axis in terms of the x-coordinate of the point. What is the direction of $\vec{B}$? (c) Graph the magnitude of $\vec{B}$ at points on the x-axis. (d) At what value of x is the magnitude of $\vec{B}$ a maximum? (e) What is the magnitude of $\vec{B}$ when $x \gg a$?

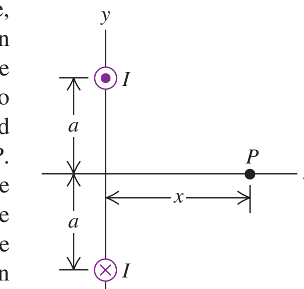


Figure 28.54 Problems 28.60 and 28.61.

28.61. Refer to the situation in Problem 28.60. Suppose that a third long, straight wire, parallel to the other two, passes through point P (see Fig. 28.54) and that each wire carries a current $I = 6.00$ A. Let $a = 40.0$ cm and $x = 60.0$ cm. Find the magnitude and direction of the force per unit length on the third wire, (a) if the current in it is directed into the plane of the figure, and (b) if the current in it is directed out of the plane of the figure.

28.62. A pair of long, rigid metal rods, each of length L, lie parallel to each other on a perfectly smooth table. Their ends are connected by identical, very light conducting springs of force constant k (Fig. 28.55) and negligible unstretched length. If a current I runs through this circuit, the springs will stretch. At what separation will the rods remain at rest? Assume that k is large enough so that the separation of the rods will be much less than L.

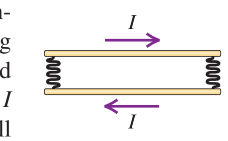


Figure 28.55 Problem 28.62.

28.63. Two long, parallel wires hang by 4.00-cm-long cords from a common axis (Fig. 28.56). The wires have a mass per unit length of 0.0125 kg/m and carry the same current in opposite directions. What is the current in each wire if the cords hang at an angle of 6.00° with the vertical?

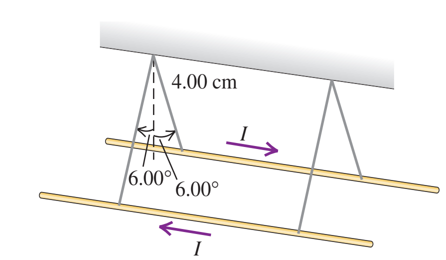


Figure 28.56 Problem 28.63.

28.64. The long, straight wire AB shown in Fig. 28.57 carries a current of 14.0 A. The rectangular loop whose long edges are parallel to the wire carries a current of 5.00 A. Find the magnitude and direction of the net force exerted on the loop by the magnetic field of the wire.

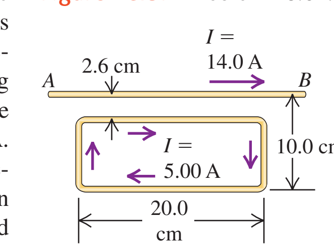
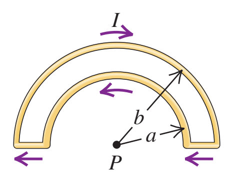


Figure 28.57 Problem 28.64.

28.65. A circular wire loop of radius a has N turns and carries a current I. A second loop with N' turns of radius a' carries current I' and is located on the axis of the first loop, a distance x from the center of the first loop. The second loop is tipped so that its axis is at an angle θ from the axis of the first loop. The distance x is large compared to both a and a'. (a) Find the magnitude of the torque exerted on the second loop by the first loop. (b) Find the potential energy for the second loop due to this interaction. (c) What simplifications result from having x much larger than a? From having x much larger than a'?

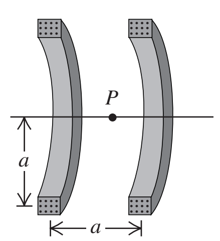
28.66. The wire semicircles shown in Fig. 28.58 have radii a and b. Calculate the net magnetic field (magnitude and direction) that the current in the wires produces at point P.

Figure 28.58 Problem 28.66.



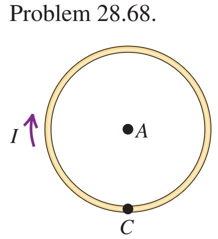
28.67. Helmholtz Coils. Fig. 28.59 is a sectional view of two circular coils with radius a, each wound with N turns of wire carrying a current I, circulating in the same direction in both coils. The coils are separated by a distance a equal to their radii. In this configuration the coils are called Helmholtz coils; they produce a very uniform magnetic field in the region between them. (a) Derive the expression for the magnitude B of the magnetic field at a point on the axis a distance x to the right of point P, which is midway between the coils. (b) Graph B versus x for $x = 0$ to $x = a/2$. Compare this graph to one for the magnetic field due to the right-hand coil alone. (c) From part (a), obtain an expression for the magnitude of the magnetic field at point P. (d) Calculate the magnitude of the magnetic field at P if $N = 300$ turns, $I = 6.00$ A, and $a = 8.00$ cm. (e) Calculate dB/dx and d^2B/dx^2 at P $(x = 0)$. Discuss how your results show that the field is very uniform in the vicinity of P.

Figure 28.59 Problem 28.67.



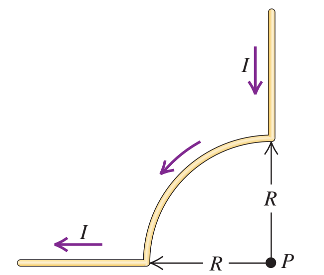
28.68. A circular wire of diameter D lies on a horizontal table and carries a current I. In Fig. 28.60 point A marks the center of the circle and point C is on its rim. (a) Find the magnitude and direction of the magnetic field at point A. (b) The wire is now unwrapped so it is straight, centered on point C, and perpendicular to the line AC, but the same current is maintained in it. Now find the magnetic field at point A. (c) Which field is greater: the one in part (a) or in part (b)? By what factor? Why is this result physically reasonable?

Figure 28.60 Problem 28.68.



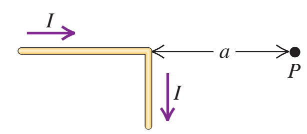
28.69. The wire in Fig. 28.61 carries current I in the direction shown. The wire consists of a very long, straight section, a quarter-circle with radius R, and another long, straight section. What are the magnitude and direction of the net magnetic field at the center of curvature of the quarter-circle section (point P)?

Figure 28.61 Problem 28.69.



28.70. The wire shown in Fig. 28.62 is infinitely long and carries a current I. Calculate the magnitude and direction of the magnetic field that this current produces at point P.

Figure 28.62 Problem 28.70.



28.71. A long, straight wire with a circular cross section of radius R carries a current I. Assume that the current density is not constant across the cross section of the wire, but rather varies as $J = \alpha r$, where α is a constant. (a) By the requirement that J integrated over the cross section of the wire gives the total current I, calculate the constant α in terms of I and R. (b) Use Ampere's law to calculate the magnetic field $B(r)$ for (i) $r \le R$ and (ii) $r \ge R$. Express your answers in terms of I.

28.72. (a) For the coaxial cable of Exercise 28.37, derive an expression for the magnitude of the magnetic field at points inside the central solid conductor $(r < a)$. Compare your result when $r = a$ to the results of part (a) of Exercise 28.37 at that same point. (b) For this coaxial cable derive an expression for the field within the tube $(b < r < c)$. Compare your result when $r = b$ to part (a) of Exercise 28.37 at that same point. Compare your result when $r = c$ to part (b) of Exercise 28.37 at that same point.

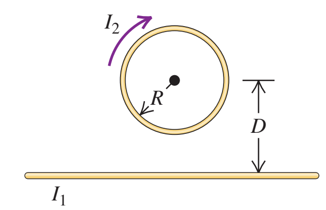
28.73. The electric field of an infinite line of positive charge is directed radially outward from the wire and can be calculated using Gauss's law for the electric field (see Example 22.6 in Section 22.4). Use Gauss's law for magnetism to show that the *magnetic* field of a straight, infinitely long, current-carrying *conductor* cannot have a radial component.

28.74. A conductor is made in the form of a hollow cylinder with inner and outer radii a and b, respectively. It carries a current I uniformly distributed over its cross section. Derive expressions for the magnitude of the magnetic field in the regions (a) $r < a$; (b) $a < r < b$; (c) $r > b$.

28.75. Knowing Magnetic Fields Inside and Out. You are given a hollow copper cylinder with inner radius a and outer radius $3a$. The cylinder's length is $200a$ and its electrical resistance to current flowing down its length is R. To test its suitability for use in a circuit, you connect the ends of the cylinder to a voltage source, causing a current I to flow down the length of the cylinder. The current is spread uniformly over the cylinder's cross section. You are interested in knowing the strength of the magnetic field that the current produces within the solid part of the cylinder, at a radius $2a$ from the cylinder axis. But since it's not easy to insert a magnetic-field probe into the solid metal, you decide instead to measure the field at a point outside the cylinder where the field should be as strong as at radius $2a$. At what distance from the axis of the cylinder should you place the probe?

28.76. A circular loop has radius R and carries current I_2 in a clockwise direction (Fig. 28.63). The center of the loop is a distance D above a long, straight wire. What are the magnitude and direction of the current I_1 in the wire if the magnetic field at the center of the loop is zero?

Figure 28.63 Problem 28.76.



28.77. A long, straight, solid cylinder, oriented with its axis in the z-direction, carries a current whose current density is $\vec{J}$. The current density, although symmetrical about the cylinder axis, is not constant but varies according to the relationship

$$\vec{J} = \frac{2I_0}{\pi a^2}\left[1 - \left(\frac{r}{a}\right)^2\right]\hat{k} \quad \text{for } r \le a$$
$$= 0 \quad \text{for } r \ge a$$

where a is the radius of the cylinder, r is the radial distance from the cylinder axis, and I_0 is a constant having units of amperes. (a) Show that I_0 is the total current passing through the entire cross section of the wire. (b) Using Ampere's law, derive an expression for the magnitude of the magnetic field $\vec{B}$ in the region $r \ge a$. (c) Obtain an expression for the current I contained in a circular cross section of radius $r \le a$ and centered at the cylinder axis. (d) Using Ampere's law, derive an expression for the magnitude of the magnetic field $\vec{B}$ in the region $r \le a$. How do your results in parts (b) and (d) compare for $r = a$?

28.78. A long, straight, solid cylinder, oriented with its axis in the z-direction, carries a current whose current density is $\vec{J}$. The current density, although symmetrical about the cylinder axis, is not constant and varies according to the relationship

$$\vec{J} = \left(\frac{b}{r}\right)e^{(r-a)/\delta}\hat{k} \quad \text{for } r \le a$$
$$= 0 \quad \text{for } r \ge a$$

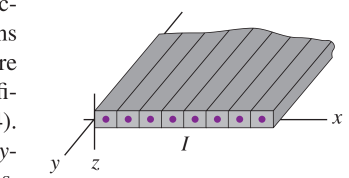
where the radius of the cylinder is $a = 5.00$ cm, r is the radial distance from the cylinder axis, b is a constant equal to 600 A/m, and δ is a constant equal to 2.50 cm. (a) Let I_0 be the total current passing through the entire cross section of the wire. Obtain an expression for I_0 in terms of b, δ, and a. Evaluate your expression to obtain a numerical value for I_0. (b) Using Ampere's law, derive an expression for the magnetic field $\vec{B}$ in the region $r \ge a$. Express your answer in terms of I_0 rather than b. (c) Obtain an expression for the current I contained in a circular cross section of radius $r \le a$ and centered at the cylinder axis. Express your answer in terms of I_0 rather than b. (d) Using Ampere's law, derive an expression for the magnetic field $\vec{B}$ in the region $r \le a$. (e) Evaluate the magnitude of the magnetic field at $r = \delta$, $r = a$, and $r = 2a$.

28.79. Integrate B_x as given in Eq. (28.15) from $-\infty$ to $+\infty$; that is, calculate $\int_{-\infty}^{+\infty} B_x\,dx$. Explain the significance of your result.

28.80. In a region of space where there are no conduction or displacement currents, it is impossible to have a uniform magnetic field that abruptly drops to zero. To prove this statement, use the method of contradiction: Assume that such a case *is* possible, and then show that your assumption contradicts a law of nature. (a) In the bottom half of a piece of paper, draw evenly spaced, horizontal lines representing a uniform magnetic field to your right. Use dashed lines to draw a rectangle $abcda$ with horizontal side ab in the magnetic field region and horizontal side cd in the top half of your paper where $B = 0$. (b) Show that integration around your rectangle contradicts Ampere's law.

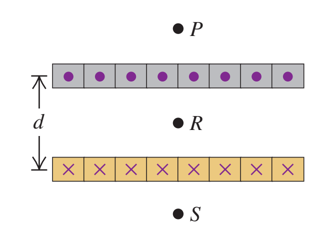
28.81. An Infinite Current Sheet. Long, straight conductors with square cross sections and each carrying current I are laid side by side to form an infinite current sheet (Fig. 28.64). The conductors lie in the xy-plane, are parallel to the y-axis, and carry current in the $+y$-direction There are n conductors per unit length measured along the x-axis. (a) What are the magnitude and direction of the magnetic field a distance a below the current sheet? (b) What are the magnitude and direction of the magnetic field a distance a above the current sheet?

Figure 28.64 Problem 28.81.



28.82. Long, straight conductors with square cross section, each carrying current I, are laid side by side to form an infinite current sheet with current directed out of the plane of the page (Fig. 28.65). A second infinite current sheet is a distance d below the first and is parallel to it. The second sheet carries current into the plane of the page. Each sheet has n conductors per unit length. (Refer to Problem 28.81.) Calculate the magnitude and direction of the net magnetic field at (a) point P (above the upper sheet); (b) point R (midway between the two sheets); (c) point S (below the lower sheet).

Figure 28.65 Problem 28.82.



***28.83.** A piece of iron has magnetization $M = 6.50 \times 10^4$ A/m. Find the average magnetic dipole moment *per atom* in this piece of iron. Express your answer both in $A \cdot m^2$ and in Bohr magnetons. The density of iron is given in Table 14.1, and the atomic mass of iron (in grams per mole) is given in Appendix D. The chemical symbol for iron is Fe.

***28.84.** (a) In Section 27.7 we discussed how a magnetic dipole, such as a current loop or a magnetized object, can be attracted or repelled by a permanent magnet. Use this to explain why *either* pole of a magnet *attracts* both paramagnetic materials and (initially unmagnetized) ferromagnetic materials, but *repels* diamagnetic materials. (b) The force that a magnet exerts on an object is directly proportional to the object's magnetic moment. A particular magnet is just strong enough to pick up a cube of annealed iron $(K_m = 1400)$ 2.00 cm on a side so that the iron sticks to one of the magnet's poles; that is, the magnet exerts an upward force on the iron cube equal to the cube's weight. If you tried to use this magnet to pick up a 2.00-cm cube of aluminum instead, what would be the upward force on the cube? How does this compare to the weight of the cube? Could the magnet pick up the cube? (*Hint:* You will need to use information from Tables 14.1 and 28.1.) (c) If you tried to use the magnet to pick up a 2.00-cm cube of silver, what would be the magnitude and direction of the force on the cube? How does this magnitude compare to the weight of the cube? Would the effects of the magnetic force be noticeable?

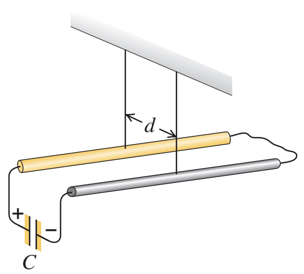
Challenge Problems

28.85. Two long, straight conducting wires with linear mass density λ are suspended from cords so that they are each horizontal, parallel to each other, and a distance d apart. The back ends of the wires are connected to each other by a slack, low-resistance connecting wire. A charged capacitor (capacitance C) is now added to the system; the positive plate of the capacitor (initial charge $+Q_0$) is connected to the front end of one of the wires, and the negative plate of the capacitor (initial charge $-Q_0$) is connected to the front end of the other wire (Fig. 28.66). Both of these connections are also made by slack, low-resistance wires. When the connection is made, the wires are pushed aside by the repulsive force between the wires, and each wire has an initial horizontal velocity of magnitude v_0. Assume that the time constant for the capacitor to discharge is negligible compared to the time it takes for any appreciable displacement in the position of the wires to occur. (a) Show that the initial speed v_0 of either wire is given by

$$v_0 = \frac{\mu_0 Q_0^2}{4\pi\lambda RCd}$$

where R is the total resistance of the circuit. (b) To what height h will each wire rise as a result of the circuit connection?

Figure **28.66** Challenge Problem 28.85.



28.86. A wide, long, insulating belt has a uniform positive charge per unit area σ on its upper surface. Rollers at each end move the belt to the right at a constant speed v. Calculate the magnitude and direction of the magnetic field produced by the moving belt at a point just above its surface. (*Hint:* At points near the surface and far from its edges or ends, the moving belt can be considered to be an infinite current sheet like that in Problem 28.81.)

28.87. A Charged Dielectric Disk. A thin disk of dielectric material with radius a has a total charge $+Q$ distributed uniformly over its surface. It rotates n times per second about an axis perpendicular to the surface of the disk and passing through its center. Find the magnetic field at the center of the disk. (*Hint:* Divide the disk into concentric rings of infinitesimal width.)

28.88. A wire in the shape of a semicircle with radius a is oriented in the yz-plane with its center of curvature at the origin (Fig. 28.67). If the current in the wire is I, calculate the magnetic-field components produced at point P, a distance x out along the x-axis. (*Note:* Do not forget the contribution from the straight wire at the bottom of the semicircle that runs from $z = -a$ to $z = +a$. You may use the fact that the fields of the two antiparallel currents at $z > a$ cancel, but you must explain *why* they cancel.)

Figure **28.67** Challenge Problem 28.88.