Mourad Barkat

# signal detection and estimation

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SECOND EDITION

# Signal Detection and Estimation

**Second Edition**

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*To my wife and my children*

# Contents

# Preface

This book provides an overview and introduction to signal detection and estimation. The book contains *numerous* examples solved in detail. Since some material on signal detection could be very complex and require a lot of background in engineering math, a chapter and various sections to cover such background are included, so that one can easily understand the intended material. Probability theory and stochastic processes are prerequisites to the fundamentals of signal detection and parameter estimation. Consequently, Chapters 1, 2, and 3 carefully cover these topics. Chapter 2 covers the different distributions that may arise in radar and communication systems. The chapter is presented in such a way that one may not need to use other references.

In a one-semester graduate course on "Signal Detection and Estimation," the material to cover should be:

Chapter 5 Statistical Decision Theory
Chapter 6 Parameter Estimation
Chapter 8 Representation of Signals
Chapter 9 The General Gaussian Problem
Chapter 10 Detection and Parameter Estimation

and perhaps part of Chapter 7 on filtering. The book can also be used in a two-semester course on "Signal Detection and Estimation" covering in this case: Chapters 5 to 8 for the first semester and then Chapters 9 to 12 for the second semester.

Many graduate courses on the above concepts are given in two separate courses; one on probability theory and random processes, and one on signal detection and estimation. In this case, for the first graduate course on "Probability Theory, Random Variables, and Stochastic Processes," one may cover:

Chapter 1 Probability Concepts
Chapter 2 Distributions
Chapter 3 Random Processes
Chapter 4 Discrete-Time Random Process

and part of Chapter 7 on filtering, while Chapters 5, 6, 8, and 9 can be covered in the course on "Signal Detection and Estimation" in the second semester. The different distributions, which are many, can be discussed on a selective basis.

Chapters 3 and 4, and part of Chapter 7 on filtering, can also be studied in detail for a *graduate course* on "Stochastic Processes."

Chapters 11 and 12 are applications of some aspects of signal detection and estimation, and hence they can be presented in a short graduate course, or in a course of special topics.

The chapters on probability theory, random variables, and stochastic processes contain numerous examples solved in detail, and hence they can be used for *undergraduate courses*. In this case, Chapter 1 and part of Chapter 2 will be covered in a one-semester course on "Probability and Random Variables.". Chapter 3 and part of Chapter 4 can be covered in a second semester course on "Random Processes" for seniors. It is clear that different combinations of the different chapters can used for the different intended courses.

Since the material is essential in many applications of radar, communications, and signal processing, this book can be used as a reference by practicing engineers and physicists. The detailed examples and the problems presented at the end of each chapter make this book suitable for self-study and facilitate teaching a class.

# Acknowledgments

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# Chapter 1

# Probability Concepts

## 1.1 INTRODUCTION

This book is primarily designed for the study of statistical signal detection and parameter estimation. Such concepts require a good knowledge of the fundamental notions on probability, random variables, and stochastic processes. In Chapter 1, we present concepts on probability and random variables. In Chapter 2, we discuss some important distributions that arise in many engineering applications such as radar and communication systems. Probability theory is a prerequisite for Chapters 3 and 4, in which we cover stochastic processes and some applications. Similarly, the fundamentals of stochastic processes will be essential for proper understanding of the subsequent topics, which cover the fundamentals of signal detection and parameter estimation. Some applications of adaptive thresholding radar constant false alarm rate (CFAR) detection will be presented in Chapter 11. In Chapter 12, we consider the concepts of adaptive CFAR detection using multiple sensors and data fusion. This concept of adaptive thresholding CFAR detection will also be introduced in spread spectrum communication systems.

We start this chapter with the set theory, since it provides the most fundamental concepts in the theory of probability. We introduce the concepts of random variables and probability distributions, statistical moments, two- and higher-dimensional random variables, and the transformation of random variables. We derive some basic results, to which we shall refer throughout the book, and establish the notation to be used.

## 1.2 SETS AND PROBABILITY

### 1.2.1 Basic Definitions

A set may be defined as a collection of objects. The individual objects forming the set are the "elements" of the set, or "members" of the set. In general, sets are

denoted by capital letters as $A$, $B$, $C$, and elements or particular members of the set by lower case letters as $a$, $b$, $c$. If an element $a$ "belongs" to or is a "member" of $A$, we write

$$a \in A \tag{1.1}$$

Otherwise, we say that $a$ is not a member of $A$ or does not belong to $A$, and write

$$a \notin A \tag{1.2}$$

A set can be described in three possible ways. The first is listing all the members of the set. For example, $A = \{1, 2, 3, 4, 5, 6\}$. It can also be described in words. For example, we say that $A$ consists of integers between 1 and 6, inclusive. Another method would be to describe the set in the form shown here.

$$A = \left\{ a \,\middle|\, a \quad \text{integer and} \quad 1 \le a \le 6 \right\} \tag{1.3}$$

The symbol | is read as "such that," and the above expression is read in words as "the set of all elements $a$, such that $a$ is an integer between 1 and 6 inclusive."

A set is said to be *countable* if its elements can be put in a one-to-one correspondence with the integers 1, 2, 3, and so forth. Otherwise, it is called *uncountable*.

A *finite* set has a number of elements equal to zero or some specified positive integer. If the number is greater than any conceivable positive integer, then it is considered *infinite*.

The set of all elements under consideration is called the *universal* set and is denoted by $U$. The set containing no elements is called the *empty* set or *null* set and is denoted by $\varnothing$.

Given two sets $A$ and $B$, if every element in $B$ is also an element of $A$, then $B$ is a *subset* of $A$. This is denoted as

$$B \subseteq A \tag{1.4}$$

and is read as "$B$ is a subset of $A$." If at least one element in $A$ is not in $B$, then $B$ is a *proper subset* of $A$, denoted by

$$B \subset A \tag{1.5}$$

On the other hand, if every element in $B$ is in $A$, and every element in $A$ is in $B$, so that $B \subseteq A$ and $A \subseteq B$, then

$$A = B \tag{1.6}$$

If the sets $A$ and $B$ have no common element, then they are called *disjoint* or *mutually exclusive*.

**Example 1.1**

In this example, we apply the definitions that we have just discussed above. Consider the sets $A$, $B$, $C$, $D$, and $E$ as shown below.

$A$ = {numbers that show in the upper face of a rolling die}

$B = \{x \mid x \text{ odd integer and } 1 \leq x \leq 6\}$

$C = \{x \mid x \text{ real and } x \geq 1\}$

$D$ = {2, 4, 6, 8, 10}

$E$ = {1, 3, 5}

$F$ = {1, 2, 3, 4, …}

$G$ = {0}

*Solution*

Note that the sets $A$ and $B$ can be written as $A$ = {1, 2, 3, 4, 5, 6} and $B$ = {1, 3, 5}. $A$, $B$, $D$, $E$, and $G$ are countable and finite. $C$ is uncountable and infinite. $F$ is countable but infinite. Since the elements in $A$ are the numbers that show in the upper face of a rolling die, and if the problem under consideration (game of chance) is the numbers on the upper face of the rolling die, then the set $A$ is actually the universal set $U$.

$A \subset F$, $B \subset F$, $D \subset F$, and $E \subset F$. $B \subset A$ and $E \subset A$. If $B \subseteq E$ and $E \subseteq B$, then $E = B$. $D$ and $E$ are mutually exclusive. Note that $G$ is not the empty set but a set with element zero. The empty set is a subset of all sets. If the universal set has $n$ elements, then there are $2^n$ subsets. In the case of the rolling die, we have $2^6 = 64$ subsets.

### 1.2.2 Venn Diagrams and Some Laws

In order to provide a geometric intuition and a visual relationship between sets, sets are represented by Venn diagrams. The universal set, $U$, is represented by a rectangle, while the other sets are represented by circles or some geometrical figures.

*Union* Set of all elements that are members of $A$ or $B$ or both, and is denoted as $A \cup B$. This is shown in Figure 1.1.

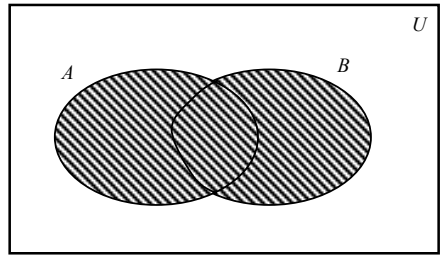


**Figure 1.1** Union.

*Intersection* Set of all elements that belong to both $A$ and $B$, and is denoted as $A \cap B$. This is shown in Figure 1.2.

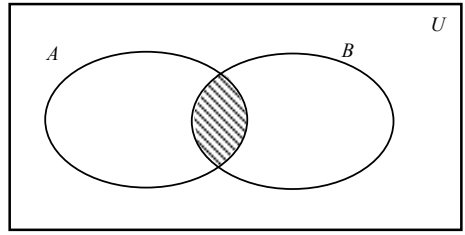


**Figure 1.2** Intersection.

*Difference* Set consisting of all elements in $A$ that are not in $B$, and is denoted as $A - B$. This is shown in Figure 1.3.

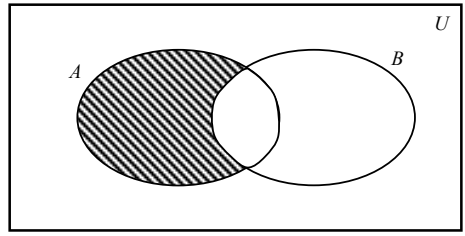


**Figure 1.3** *A–B*.

*Complement* The set composed of all members in $U$ not in $A$ is the complement of $A$, and is denoted as $\overline{A}$. This is shown in Figure 1.4.

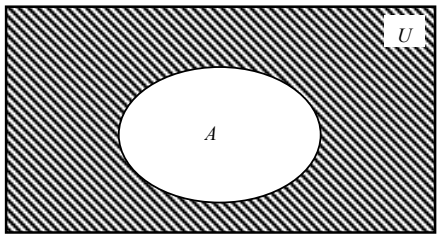


**Figure 1.4** Complement of $A$.

*Partitions* A group of mutually exclusive sets covering the entire universal set $U$ form a partition. This is shown in Figure 1.5.

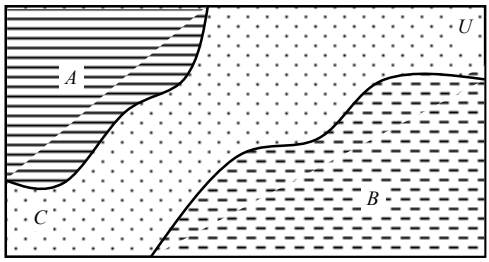


**Figure 1.5** Partitions.

*Cartesian Product* The Cartesian product of sets $A$ and $B$, denoted $A \times B$, is the set of all ordered pairs where the first element of the pairs is taken from set $A$ and the second element from set $B$. That is, if set $A = \{a_1, a_2, \ldots, a_n\}$ and set $B = \{b_1, b_2, \ldots, b_m\}$, then the Cartesian product $A \times B = \{(a_1, b_1), (a_1, b_2), \ldots, (a_1, b_m), (a_2, b_1), (a_2, b_2), \ldots, (a_2, b_m), \ldots, (a_n, b_1), (a_n, b_2), \ldots, (a_n, b_m)\}$. It should be noted that the Cartesian product $A \times B$ is generally not equal to $B \times A$.

*Some Laws and Theorems*

1. If $A$ and $B$ are sets, then $A \cup B$ and $A \cap B$ are sets.
2. There is only one set $\varnothing$ and one universal set $U$, such that $A \cup \varnothing = A$ and $A \cap U = A$ for any $A$.

3. Commutative laws: $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
4. Associative laws: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.
5. Distributive laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
6. $A \cup \overline{A} = U$ and $A \cap \overline{A} = \varnothing$.
7. De Morgan's laws: $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
8. If $A = B$, then $\overline{A} = \overline{B}$. If $A = B$ and $C = D$, then $A \cup C = B \cup D$ and $A \cap C = B \cap D$.
9. $\overline{\overline{A}} = A$.

### 1.2.3 Basic Notions of Probability

Originally, the theory of probability was developed to serve as a model of games of chance, such as rolling a die, spinning a roulette wheel, or dealing from a deck of cards. Later, this theory developed to model scientific physical experiments.

In building the relationship between the set theory and the notion of probability, we call the set of all possible distinct outcomes of interest in a particular experiment as the *sample space S*. An event is a particular outcome or a combination of outcomes. According to the set theory, the notion of an event is a subset of the sample space.

If a basic experiment can lead to $N$ mutually exclusive and equally likely outcomes, and if $N_A$ is the possible outcomes in the occurrence of the event $A$, then the probability of the event $A$ is defined by

$$\text{probability of } A = \frac{N_A}{N} \tag{1.7}$$

However, the most popular definition among engineers is a second definition referred to as *relative frequency*. If an experiment is repeated $n$ times under the same conditions, and if $n_A$ is the number of occurrences of event $A$, then the probability of $A$, $P(A)$, is defined by

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n} \tag{1.8}$$

Note that in the second definition, which is based on an experiment, the concept of equally likely events is not necessary, but in practice $n$ is really finite. Because of its a priori nature, the concept of probability also has a subjective definition, that is, the degree of confidence in a certain outcome of a particular experiment, or in a

certain state in the sample space. Subjective theory of probability, as treated by De Finetti [1], solves the lack of synthesis of the "relative frequency" limit and the combinatory limitation of the "ratio of outcomes."

We now formalize the concept of obtaining an outcome lying in a specified subset $A$ of the sample space $S$ into a definition of probability.

*Definition*. Given the sample space $S$ and an event $A$, a probability function, $P(\cdot)$, associates to the event $A$ a real number such that

1. $P(A) \geq 0$ for every event $A$;
2. $P(S) = 1$;
3. If there exist some countable events $A_1, A_2, \ldots, A_n$, mutually exclusive $\left(A_i \cap A_j = \varnothing,\ i \neq j\right)$, then

$$P\left(A_1 \cup A_2 \cup \Lambda \cup A_n\right) = P\left(A_1\right) + P\left(A_2\right) + \Lambda + P\left(A_n\right).$$

**Example 1.2**

Consider the experiment of two six-sided dice, and that each die has its sides marked 1 through 6. The sample space, $S$, in this case is

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

Let the event $A$ be "the sum is 7," the event $B$ is "one die shows an even number and the other an odd number." The events $A$ and $B$ are

$$A = \{ (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) \}$$

$$B = \begin{Bmatrix} (2,1) & (4,1) & (6,1) \\ (1,2) & (3,2) & (5,2) \\ (2,3) & (4,3) & (6,3) \\ (1,4) & (3,4) & (5,4) \\ (2,5) & (4,5) & (6,5) \\ (1,6) & (3,6) & (5,6) \end{Bmatrix}$$

We can obtain the probability of events $A$, $B$, $A \cap B$, and $\overline{A}$ to be $P(A) = 6/36$, $P(B) = 18/36 = 1/2$, $P(A \cap B) = P(A) = 1/6$, and $P(\overline{A}) = 30/36 = 5/6$.

Example 1.2 illustrates the fact that counting plays an important role in probability theory. However, as the number of possible outcomes becomes large, the counting process becomes very difficult, and thus it may be necessary to divide the counting into several steps, as illustrated in the following section.

### 1.2.4 Some Methods of Counting

One strategy of counting is breaking the task into a finite sequence of subtasks, such that the number of ways of doing a particular task is not dependent on the previous tasks in the sequence. Suppose that there are $n_1$ ways of doing step 1, and for each way of step 1, there are $n_2$ ways of doing step 2. For each way to do step 1 and step 2, there are $n_3$ ways of doing step 3, and so on until step $k$. Then, the number of ways to perform the procedure is $n_1 n_2 \ldots n_k$. The classical example of this principle is the number of ways to write a 5-digit word. The word is ———— —. We observe that there are $n_1 = 26$ ways for step 1, $n_2 = 26$ ways for step 2, and so on, until we have the 5-letter word. The total number of such ways is $26^5 = 11{,}881{,}376$ ways. Note that if no letter can be repeated, then for step 1 we have all 26 letters of the alphabet. Step 2, however, will have 25 ways, until step 5 with $n_5 = 22$. The number of such words becomes now $26 \times 25 \times 24 \times 23 \times 22 = 7{,}893{,}600$.

Suppose that we have now $r$ distinct objects (particles) to be placed in $n$ slots. From Figure 1.6, we observe that we have $r$ ways of placing the objects in the first slot. After choosing the first object, there are $r-1$ ways of placing an object in the second slot, and so on, until the $r$th slot, which will be filled in $n-r+1$ ways. Thus, the total number of ways of arranging $r$ objects in $n$ slots is $n(n-1)\ldots(n-r+1)$. This is called *permutations* or *arrangements* of $r$ objects among $n$ and denoted ${}_nP_r$, which can be written as

$$ {}_nP_r = \frac{n!}{(n-r)!} \tag{1.9}$$

Note that if $r = n$, that is, we have permutations of $n$ distinct objects out of $n$, then following the 1 is filled, w same reasoning as before, we have $n$ ways to fill slot 1. After slot e have $(n-1)$ ways to fill slot 2, and so on, until the $n$th slot which can be filled in just one way. Then, ${}_nP_n = n(n-1)(n-2) \ldots 1 = n!$.

| 1 | 2 | …….. | $n$ |
|---|---|---|---|

**Figure 1.6** $n$ slots.

Substitution of $r = n$ in (1.9) means 0!=1, which is an adopted convention, and we conclude that the permutations of $n$ objects is $n!$.

Note that in the case just discussed above, the *order* in the arrangements of objects is important. However, when the order is not relevant and the problem is always counting the number of ways of choosing $r$ objects out of $n$, we speak not of permutations but of *combinations*. For example, if we have $n = 3$ objects $a$, $b$, and $c$, and we select $r = 2$ objects without regard to the order, the possible cases are $ab$, $ac$, and $bc$. Note that in this case $ab$ and $ba$ are the same combination. The total number of combinations of $r$ objects out of $n$ is given by

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} \tag{1.10}$$

The notation $\binom{n}{r} = {}_nC_r$ also can be used. The numbers $\binom{n}{r}$ are called *binomial coefficients*. It can easily be shown that

$$\binom{n}{r} = \binom{n}{n-r} \tag{1.11}$$

and

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \tag{1.12}$$

If the $n$ objects are not all distinct, such that $n_1$ is of one type, $n_2$ of a second type, and so on, until $n_k$ of a $k$th type, where $n_1 + n_2 + \mathrm{K} + n_k$, then, the number of different permutations of these $n$ objects is given by

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\Lambda\binom{n-n_1-n_2-\mathrm{K}-n_{k-2}}{n_{k-1}} = \frac{n!}{n_1!\,n_2!\mathrm{K}\;n_k!} \tag{1.13}$$

The numbers defined in (1.13) are known as *multinomial coefficients*, and they may also be denoted as ${}_nP_{n_1, n_2, \mathrm{K}, n_k}$. We now solve some examples applying the different strategies of counting.

**Example 1.3** (*Tree Diagram*)

Urn $A$ contains five red balls and two white balls. Urn $B$ contains three red balls

and two white balls. An urn is selected at random, and two balls are drawn successively without replacing the first drawn ball. Each urn is assumed to have the same likelihood of selection.

(a) Draw the tree diagram.
(b) What is the probability of drawing two white balls?

*Solution*

(a) The experiment consists of selecting an urn and then drawing two balls from the selected urn. Note also that the sample size changes after the first ball is drawn, and thus the events are not independent. Since the sample size is small, we introduce the concept of a tree diagram in this example. The whole experiment with all possible outcomes is as shown in Figure 1.7, with $R$ denoting drawing a red ball and $W$ drawing a white ball.
(b) We observe that two branches $AWW$ and $BWW$ marked by an * indicate the possible cases of obtaining two white balls. Hence,

$$P(2W) = \frac{1}{2}\frac{2}{7}\frac{1}{6} + \frac{1}{2}\frac{2}{5}\frac{1}{4} = \frac{1}{42} + \frac{1}{20} = 0.0738$$

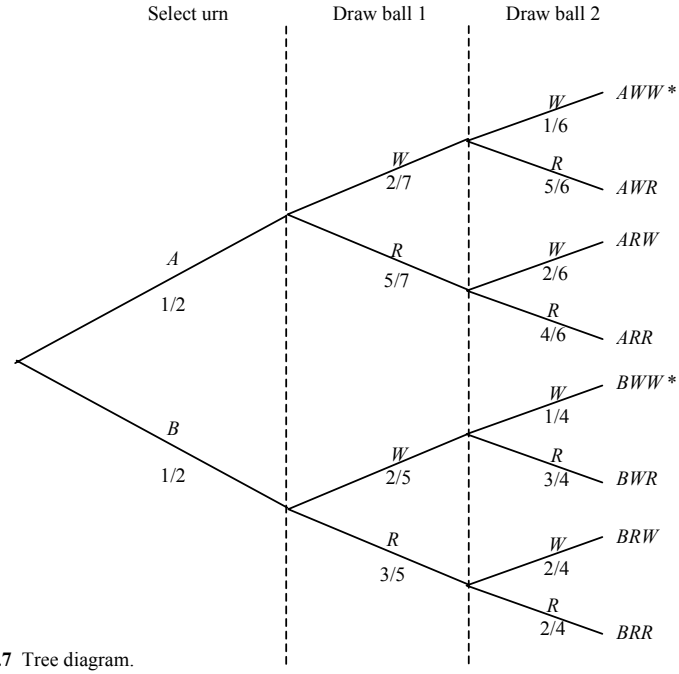


**Figure 1.7** Tree diagram.

**Example 1.4**

An urn contains five red, three green, four blue, and two white balls. What is the probability of selecting a sample size of six balls containing two red, one green, two blue, and one white ball? In this case, the probability is given by

$$\frac{\binom{5}{2}\binom{3}{1}\binom{4}{2}\binom{2}{1}}{\binom{14}{6}} = 0.080$$

**Example 1.5**

A box contains 10 black balls and 15 white balls. One ball at a time is drawn at random, its color is noted, and the ball is then replaced in the box for the next draw.

(a) Find the probability that the first white ball is drawn on the third draw.
(b) Find the probability that the second and third white balls are drawn on the fifth and eighth draws, respectively.

*Solution*

(a) Note that the events are independent, since the ball is replaced in the box and thus the sample space does not change. Let $B$ denote drawing a black ball and $W$ drawing a white ball. The total number of balls in the sample space is 25. Hence, we have

1st draw $\rightarrow B$
2nd draw $\rightarrow B$
3rd draw $\rightarrow W$

Thus,

$$P(\text{first white ball drawn in the 3rd draw}) = \frac{\binom{10}{1}\binom{10}{1}\binom{15}{1}}{\binom{25}{1}\binom{25}{1}\binom{25}{1}}$$

$$= \left(\frac{10}{25}\right)^2 \left(\frac{15}{25}\right) = 0.096$$

To illustrate the experiment that the second and third white balls are drawn on the fifth and eighth draws, we do the following.

1st draw
2nd draw
3rd draw
4th draw
} $1W$ and $3B$, there are four ways of obtaining this: $\binom{4}{1} = \frac{4!}{1!\,3!} = 4$

5th draw → $W$ (the 2nd white)
6th draw → $B$
7th draw → $B$
8th draw → $W$ (the 3rd white)

Note that the sixth and seventh draws would have to be a black ball. Thus, computing the probability, we obtain

$$P = 4\left(\frac{15}{30}\right)\left(\frac{10}{30}\right)^3\left[\left(\frac{15}{30}\right)\left(\frac{10}{30}\right)^2\left(\frac{15}{30}\right)\right] = 0.00206$$

### 1.2.5 Properties, Conditional Probability, and Bayes' Rule

Now that we have defined the concept of probability, we can state some useful properties.

*Properties*

1. For every event $A$, its probability is between 0 and 1.

$$0 \le P(A) \le 1 \tag{1.14}$$

2. The probability of the impossible event is zero.

$$P(\varnothing) = 0 \tag{1.15}$$

3. If $\overline{A}$ is the complement of $A$, then

$$P(\overline{A}) = 1 - P(A) \tag{1.16}$$

4. If $A$ and $B$ are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{1.17}$$

5. If the sample space consists of $n$ mutually exclusive events such that $S = A_1 \cup A_2 \cup \Lambda \cup A_n$, then

$$P(S) = P(A_1) + P(A_2) + \Lambda + P(A_n) = 1 \tag{1.18}$$

*Conditional Probability and Independent Events*

Let $A$ and $B$ be two events, such that $P(B) \geq 0$. The probability of event $B$ *given that* event $A$ has occurred is

$$P(A \mid B) = \frac{P(A \text{ I } B)}{P(B)} \tag{1.19}$$

$P(A \mid B)$ is the probability that $A$ will occur given that $B$ has occurred, and is called the *conditional probability* of $A$ given $B$. However, if the occurrence of event $B$ has no effect on $A$, we say that $A$ and $B$ are *independent* events. In this case,

$$P(A \mid B) = P(A) \tag{1.20}$$

which is equivalent, after substitution of (1.20) in (1.19), to

$$P(A \text{ I } B) = P(A)\, P(B) \tag{1.21}$$

For any three events $A_1$, $A_2$, $A_3$, we have

$$P(A_1 \text{ I } A_2 \text{ I } A_3) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \text{ I } A_2) \tag{1.22}$$

If the three events are independent, then they must be pairwise independent

$$P(A_i \text{ I } A_j) = P(A_i)\, P(A_j) \quad i \neq j \quad \text{and} \quad i, j = 1, 2, 3 \tag{1.23}$$

and

$$P(A_1 \text{ I } A_2 \text{ I } A_3) = P(A_1)\, P(A_2)\, P(A_3) \tag{1.24}$$

Note that both conditions (1.23) and (1.24) must be satisfied for $A_1$, $A_2$, and $A_3$ to be independent.

*Bayes' Rule*

If we have $n$ mutually exclusive events $A_1$, $A_2$, …, $A_n$, the union of which is the sample space $S$, $S = A_1 \text{ Y } A_2 \text{ Y K Y } A_n$, then for every event $A$, Bayes' rule says that

$$P(A_k \mid A) = \frac{P(A_k \text{ I } A)}{P(A)} \tag{1.25}$$

where

$$P(A_k \text{ I } A) = P(A_k)\,P(A \mid A_k), \quad k = 1, 2, \text{K}\ , n \tag{1.26}$$

since $P(A \mid A_k) = P(A_k \mid A) / P(A_k)$, and the *total probability* of $A$ is defined as

$$P(A) = P(A \mid A_1)P(A_1) + P(A \mid A_2)P(A_2) + \text{ K } + P(A \mid A_n)P(A_n) \tag{1.27}$$

**Example 1.6**

A digital communication source transmits symbols of 0s and 1s independently, with probability 0.6 and 0.4, respectively, through some noisy channel. At the receiver, we obtain symbols of 0s and 1s, but with the chance that any particular symbol was garbled at the channel is 0.2. What is the probability of receiving a zero?

*Solution*

Let the probability to transmit a 0 be $P(0) = 0.6$, and the probability to transmit a 1 be $P(1) = 0.4$. The probability that a particular symbol is garbled is 0.2; that is, the probability to receive a 1 when a 0 is transmitted and the probability to receive a 0 when a 1 is transmitted is $P$(receive 0 | 1 transmitted) = $P$(receive 1 | 0 transmitted) = 0.2. Hence, the probability to receive a 0 is

$$P(\text{receive a zero}) = P(0 \mid 1)\,P(1) + P(0 \mid 0)\,P(0) = (0.2)\,(0.4) + (0.8)\,(0.6) = 0.56$$

**Example 1.7**

A ball is drawn at random from a box containing seven white balls, three red balls, and six green balls.

(a) Determine the probability that the ball drawn is
(1) white, (2) red, (3) green, (4) not red, and (5) red or white.

(b) Three balls are drawn successively from the box instead of one. Find the probability that they are drawn in the order red, white, and green, if each ball is (1) replaced in the box before the next draw, and (2) not replaced.

*Solution*

Let $W$, $R$, and $G$ denote the events of drawing a white ball, a red ball, and a green ball. The total number of balls in the sample space is 7 + 3 + 6 = 16.

(a) 1. $P(W) = 7/16 = 0.4375$

2. $P(R) = 3/16 = 0.1875$

3. $P(G) = 6/16 = 3/8 = 0.375$

4. $P(\overline{R}) = 1 - P(R) = 1 - 7/16 = 9/16 = 0.5625$

5. $P(\text{red or white}) = P(R \cup W) = P(R) + P(W) - P(R \cap W)$

Since the events $R$ and $W$ are mutually exclusive, then $P(R \cap W) = 0$, and

$$P(R \cup W) = P(R) + P(W) = \frac{7+3}{16} = \frac{5}{8} = 0.625$$

(b) In this case the order becomes a factor. Let the events $R_1$, $W_2$, and $G_3$ represent "red on first draw," "white on second draw," and "green on third draw," respectively.

1. Since each ball is replaced before the next draw, the sample space does not change, and thus the events are independent. From (1.24), we can write

$$\begin{aligned} P(R_1 \cap W_2 \cap G_3) &= P(R_1)\, P(W_2 \mid R_1)\, P(G_3 \mid R_1 \cap W_2) \\ &= P(R_1)\, P(W_2)\, P(G_3) \\ &= \left(\frac{3}{16}\right)\left(\frac{7}{16}\right)\left(\frac{3}{8}\right) = 0.0308 \end{aligned}$$

2. When the ball is not replaced in the box before the next draw, the sample space changes, and the events are then dependent. Thus,

$$P(R_1 \cap W_2 \cap G_3) = P(R_1)\, P(W_2 \mid R_1)\, P(G_3 \mid R_1 \cap W_2)$$

but $P(W_2 \mid R_1) = 7/(7+2+6) = 7/15 = 0.467$, and $P(G_3 \mid R_1 \cap W_2) = 6/(6+2+6)$ $= 3/7 \Rightarrow P(R_1 \cap W_2 \cap G_3) = 0.0375.$

**Example 1.8**

In three urns, there are balls as shown in Table 1.1. An experiment consists of first randomly selecting an urn, and then drawing a ball from the chosen urn. Each urn is assumed to have the same likelihood of selection.

(a) What is the probability of drawing a white ball, given that Urn $A$ is selected?

**Table 1.1**
Content of Urns *A*, *B*, and *C*

| *Balls* | *Urn A* | *Urn B* | *Urn C* | *Totals* |
|---|---|---|---|---|
| Red | 5 | 3 | 6 | 14 |
| Green | 6 | 3 | 2 | 11 |
| White | 2 | 4 | 1 | 7 |
| Totals | 13 | 10 | 9 | 32 |

(b) If a white ball is drawn, what is the probability it came from Urn *B*?

*Solution*

(a) Given that Urn *A* is selected, we can write the probability of drawing a white ball to be

$$P(1W \mid \text{Urn } A) = \frac{2}{13} = 0.1538$$

(b) In this case, we want to determine the conditional probability of selecting Urn *B*, given that a white ball is drawn; that is, $P(\text{Urn } B \mid 1W)$. Hence,

$$P(\text{Urn } B \mid 1W) = \frac{P(\text{Urn } B \text{ I } 1W)}{P(1W)}$$

The conditional probability of drawing a white ball, given that Urn *B* is selected, is given by

$$P(1W \mid \text{Urn } B) = \frac{P(1W \text{ I Urn } B)}{P(\text{Urn } B)}$$

Thus, $P(1W \text{ I Urn } B) = P(1W \mid \text{Urn } B)P(\text{Urn } B)$

$$\Rightarrow P(\text{Urn } B \mid 1W) = \frac{P(1W \mid \text{Urn } B)P(\text{Urn } B)}{P(1W)}$$

where $P(1W)$ is the total probability of drawing a white ball. Hence,

$$P(1W) = P(1W \mid \text{Urn } A)P(\text{Urn } A) + P(1W \mid \text{Urn } B)P(\text{ Urn } B)$$
$$+ P(1W \mid \text{Urn } C)P(\text{ Urn } C) = \frac{2}{13}\frac{1}{3} + \frac{4}{10}\frac{1}{3} + \frac{1}{9}\frac{1}{3} = 0.2217$$

$$P(1W \mid \text{Urn } B)P(\text{Urn } B) = (4/10)(1/3) = 0.133 \text{ and then } P(\text{ Urn } B \mid 1W) = 0.6013$$

## 1.3 RANDOM VARIABLES

We define a *random variable* as a *real function* that maps the elements of the sample space $S$ into points of the real axis. Notice that the random variable is neither random nor a variable, but is a function, and thus the name may be a little misleading. The random variable is represented by a capital letter ($X$, $Y$, $Z$, …), and any particular real value of the random variable is denoted by a lowercase letter ($x$, $y$, $z$, …). Since we will make use of impulse functions and step functions in characterizing random variables, we first introduce the concepts of impulse and step functions, and then we present the three different types of random variables—discrete, continuous, and mixed.

### 1.3.1 Step and Impulse Functions

The *unit step function*, shown in Figure 1.8, is defined as

$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{1.28}$$

A step function of height $A$ occurring at $x = x_0$ is denoted as

$$Au(x - x_0) = \begin{cases} A, & x \geq x_0 \\ 0, & x < x_0 \end{cases} \tag{1.29}$$


**Figure 1.8** Step function.

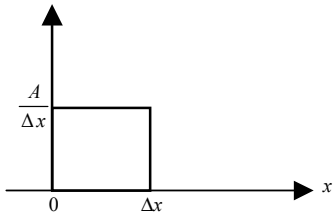


**Figure 1.9** Rectangular pulse function.

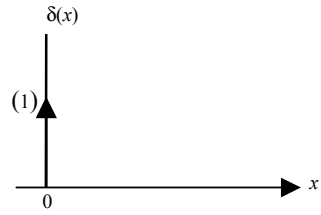


**Figure 1.10** Unit impulse function.

Consider the rectangular function, shown in Figure 1.9, with area $(A/\Delta x)\Delta x = A$. In the limit as $\Delta x \to 0$, the pulse width approaches 0 and the height goes to infinity. However, the area remains constant and equals 1. Thus, the *unit impulse function* (an impulse with unit area) is shown in Figure 1.10 and is denoted by $\delta(x)$. An impulse of area $A$ occurring at $x = x_0$ is denoted by $A\delta(x - x_0)$. Note that the integral of the unit impulse function is the step function, and that the impulse function is the derivative of the step function. An important property of the impulse function is

$$\int_{-\infty}^{\infty} A\,\delta(x - x_0)\, f(x)\, dx = A\, f(x_0) \tag{1.30}$$

### 1.3.2 Discrete Random Variables

If a random variable $X$ can assume only a particular finite or counting infinite set of values, $x_1, x_2, \ldots, x_n$, then $X$ is said to be a *discrete* random variable. If we associate each outcome $x_i$ with a number $P(x_i) = P(X = x_i)$, called the probability of $x_i$, the number $P(x_i)$, sometimes denoted $P_i$ for simplicity, $i = 1, 2, \ldots$, must satisfy the following conditions:

$$P(x_i) \geq 0 \quad \text{for all } i \tag{1.31}$$

and

$$\sum_{i=1}^{\infty} P(x_i) = 1 \tag{1.32}$$

That is, the probability of each value that $X$ can assume must be nonnegative, and the sum of the probabilities over all of the different values must equal 1. If $X$ is a

random variable, its *distribution function* or cumulative distribution function (CDF) is defined as

$$F_X(x) = P(X \le x) \quad \text{for all } x \tag{1.33}$$

The probability density function (PDF) of a discrete random variable that assumes $x_1, x_2, \ldots$, is $P(x_1), P(x_2), \ldots$, where $P(x_i) = P(X = x_i)$, $i = 1, 2, \ldots$. If there is more than one random variable, we denote the PDF of a particular variable $X$ by a subscript $X$ on $P$ as $P_X(x)$.

**Example 1.9**

Consider the experiment of rolling two dice. Let $X$ represent the total number that shows up on the upper faces of the two dice. What is the probability that $X$ is between 4 and 6 inclusive? Determine $P(X \ge 5)$. Sketch the probability density function and the distribution function of $X$.

*Solution*

Since the possible events are mutually exclusive, $P(4 \le X \le 6) = P(X = 4) + P(X = 5) + P(X = 6)$, where $P(X = 4) = 1/12$, $P(X = 5) = 1/9$, and $P(X = 6) = 5/36$. Therefore, $P(4 \le X \le 5) = 12/36 = 0.3333$. Hence, using (1.16)

$$P(X \ge 5) = 1 - P(X \le 4) = 1 - [\, P(X = 2) + P(X = 3) + P(X = 4) \,] = \frac{5}{6} = 0.8333$$

The density function and distribution function of $X$ are shown in Figures 1.11(a, b), respectively.

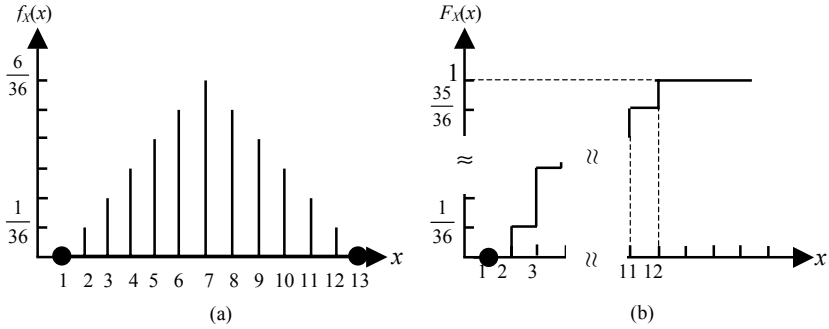


**Figure 1.11** (a) Density function of $X$, and (b) distribution function of $X$.

The density function of $X$ is written as

$$f_X(x) = \frac{1}{36}[\,\delta(x-2)+2\delta(x-3)+3\delta(x-4)+4\delta(x-5)+5\delta(x-6)+6\delta(x-7) + 5\delta(x-8)+4\delta(x-9)+3\delta(x-10)+2\delta(x-11)+\delta(x-12)\,]$$

### 1.3.3 Continuous Random Variables

$X$ is called a continuous random variable if its distribution function $F_X(x)$ may be represented as

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)\, du \tag{1.34}$$

where $f_X(x)$ is a probability density function. By definition, $f_X(x)$ must satisfy

$$f_X(x) \ge 0 \quad \text{for all } x \tag{1.35}$$

and

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \tag{1.36}$$

$f_X(x)$ is often called the *density function*.

**Example 1.10**

(a) Find the constant $c$ such that the function

$$f_X(x) = \begin{cases} c\,x, & 0 < x < 3 \\ 0\ , & \text{otherwise} \end{cases}$$

is a density function.
(b) Compute $P(1 < X < 2)$.
(c) Find the distribution function $F_X(x)$.

*Solution*

(a) $f_X(x)$ is a nonnegative function for the given range of $x$. For $f_X(x)$ to be a

density function, we need to find the constant $c$, such that $\int_0^3 c\,x\,dx = 1$. Solving the integral, we obtain $c = 2/9$, and thus the density function $f_X(x)$, shown in Figure 1.12(a), is

$$f_X(x) = \begin{cases} \frac{2}{9}x, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

(b) $P(1 < X < 2) = \int_1^2 \left(\frac{2}{9}\right) x\,dx = \frac{1}{3} = 0.3333$

(c) $F_X(x) = \int_0^x f_X(u)\,du = \frac{x^2}{9}$ for $0 \le x < 3$, and $F_X(x) = 1$ for $x \ge 3$. Thus, the distribution function, shown in Figure 1.12(b), is

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{9}, & 0 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

The density function can be obtained directly from the distribution function by simply taking the derivative; that is,

$$f_X(x) = \frac{d}{dx} F_X(x) \tag{1.37}$$

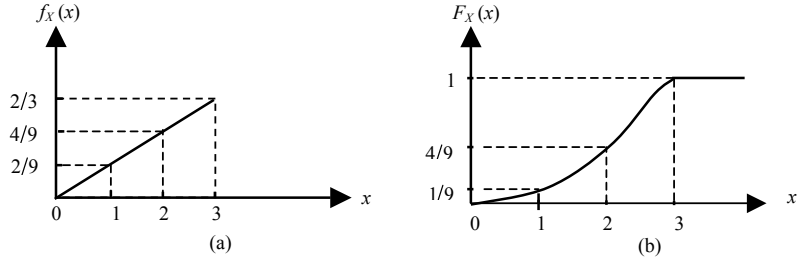


**Figure 1.12** (a) Density function of $X$, and (b) distribution function of $X$.

where $F_X(x) = \int_{-\infty}^{x} f(u)\,du$. This is a special case of *Leibniz's rule* for differentiation of an integral, which is

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(u,x)\,du = \int_{a(x)}^{b(x)} \frac{\partial F}{\partial x}\,du + F[\,b(x),x\,]\frac{db(x)}{dx} - F[\,a(x),x\,]\frac{da(x)}{dx} \quad (1.38)$$

and thus,

$$\frac{d}{dx}\int_{a}^{x} f(u)\,du = f(x) \quad (1.39)$$

### 1.3.4 Mixed Random Variables

The most important random variables that occur in practice are either discrete or continuous. A mixed random variable, however, also may occur in some practical problems. Its density function has both impulses representing probabilities of possible values $x_1, x_2, \ldots, x_n$, and a continuous portion in some interval, say $a < x < b$. A good way to illustrate the mixed random variable is to consider the half-wave rectifier circuit shown in Figure 1.13(a), where $X$ is a random variable with the probability density function as shown in Figure 1.13(b).

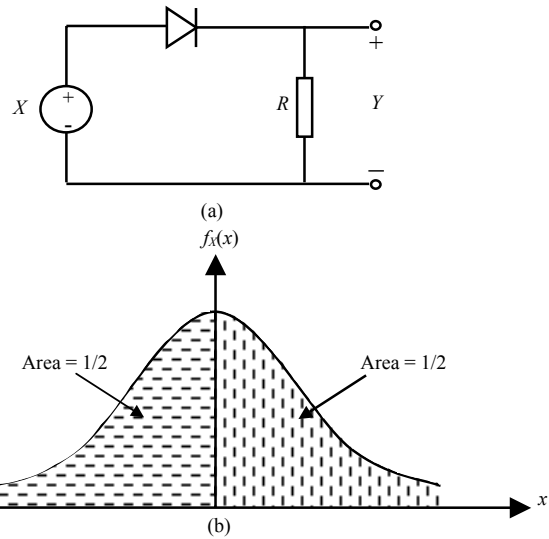


**Figure 1.13** (a) Half-wave rectifier circuit, and (b) density function of $X$.

The diode is assumed to be ideal. The output $Y$ is related to the input $X$ by the equation

$$Y = \begin{cases} X, & x > 0 \\ 0, & x \le 0 \end{cases} \tag{1.40}$$

Thus, $P(Y < 0) = 0$, $P(Y = 0) = \int_{-\infty}^{0^-} f_X(x)dx = 1/2$, and $P(0 < Y < y) = P(0 \le X \le y)$ for all $y > 0$. Hence, the density function is as shown in Figure 1.14. It is composed of a discrete value at zero of $1/2$ represented by the impulse, and a continuous function for $x > 0$, such that the area under the curve is also $1/2$; that is,

$$\int_0^{\infty} f_Y(y)\, dy = 1 = P(0 \le Y < \infty) \tag{1.41}$$

which satisfies condition (1.36), whereas

$$\int_{o^+}^{\infty} f_Y(y)\, dy = \frac{1}{2} = P(0 < Y < \infty) \tag{1.42}$$

## 1.4 MOMENTS

### 1.4.1 Expectations

An important concept in the theory of probability and statistics is the *mathematical expectation*, or *expected value*, or *mean value*, or *statistical average* of a random

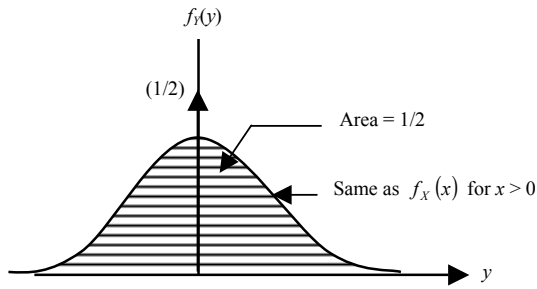


**Figure 1.14** Density function of the output $Y$.

variable $X$. The expected value of a random variable is denoted by $E[X]$ or $\overline{X}$ or $m_x$. If $X$ is a discrete random variable having values $x_1, x_2, \ldots, x_n$, then the expected value of $X$ is defined to be

$$E[X] = \sum_x x\, P(X = x) = \sum_x x\, P(x) \tag{1.43}$$

where the sum is taken over all the appropriate values that $X$ can assume. Similarly, for a continuous random variable $X$ with density function $f_X(x)$, the expectation of $X$ is defined to be

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx \tag{1.44}$$

**Example 1.11**

Find the expected value of the points on the top face in tossing a fair die.

*Solution*

In tossing a fair die, each face shows up with a probability $1/6$. Let $X$ be the points showing on the top face of the die. Then,

$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = 3.5$$

**Example 1.12**

Consider the random variable $X$ with the distribution shown in Figure 1.15. Find $E[X]$.

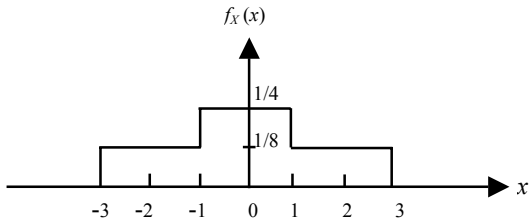


**Figure 1.15** Density function of $X$.

*Solution*

Using (1.44), the expected value of $X$ is

$$E[X] = \int_{-3}^{-1} x\frac{1}{8}\,dx + \int_{-1}^{1} x\frac{1}{4}\,dx + \int_{1}^{3} x\frac{1}{8}\,dx = 0$$

Let $X$ be a random variable. Then, the function $g(X)$ is also a random variable, and its expected value, $E[g(X)]$, is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\,dx \tag{1.45}$$

Equation (1.45) is an important theorem that will be used throughout the book.

*Properties*

1. If $c$ is any constant, then

$$E[cX] = c\,E[X] \tag{1.46}$$

2. If the function $g(X) = X^n$, $n = 0,1, \ldots$, then

$$E[g(X)] = E[X^n] = \int_{-\infty}^{\infty} x^n\, f_X(x)\,dx \tag{1.47}$$

is called the $n$th moment of the random variable $X$ about the origin. For $n = 2$, we obtain the second moment of $X$. Because of its importance, the second moment of $X$, defined as

$$E[X^2] = \int_{-\infty}^{\infty} x^2\, f_X(x)\,dx \tag{1.48}$$

is called the *mean-square value*.

Another quantity of importance is the central moment about the mean. It is called the *variance*, denoted $\sigma_x^2$, and is defined as

$$\sigma_x^2 = E[(X - E[X])^2] = E[X^2] - (E[X])^2 \tag{1.49}$$

The quantity $\sigma_x$ is called the *standard deviation*.

**Example 1.13**

Find the variance of the random variable given in Example 1.12.

*Solution*

The mean was found previously in Example 1.12 to be zero. From (1.48), the mean square value is $E\left[X^2\right] = 2\left[\int_0^1 x^2(1/4)\,dx + \int_1^3 x^2(1/8)\,dx\right] = 7/3 \quad = 2.3333.$ Since the mean is zero, the mean square value is just the variance $\sigma_x^2 = 7/3 = 2.3333.$

### 1.4.2 Moment Generating Function and Characteristic Function

The moment generating function (MGF) $M_x(t)$ of a random variable $X$ is defined by

$$M_x(t) = E\left[e^{tX}\right] \tag{1.50}$$

If $X$ is a discrete random variable with probability distribution $P(x_i) = P(X = x_i)$, $i = 1, 2, \text{K}$, then

$$M_x(t) = \sum_x e^{tx}\ P_X(x) \tag{1.51}$$

If $X$ is a continuous random variable with density function $f_X(x)$, then its MGF is

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx}\ f_x(x)\,dx \tag{1.52}$$

A "nice" advantage of the MGF is its ability to give the moments. Recall that the McLaurin series of the function $e^x$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{K} + \frac{x^n}{n!} + \text{K} \tag{1.53}$$

This is a convergent series. Thus, $e^{tx}$ can be expressed in the series as

$$e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \mathrm{K} + \frac{(tx)^n}{n!} + \mathrm{K} \tag{1.54}$$

By using the fact that the expected value of the sum equals the sum of the expected values, we can write the MGF as

$$M_x(t) = E\left[e^{tX}\right] = E\left[1 + tX + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \mathrm{K} + \frac{(tX)^n}{n!} + \mathrm{K}\right]$$

$$= 1 + tE[X] + \frac{t^2}{2!}E\left[X^2\right] + \frac{t^3}{3!}E\left[X^3\right] + \mathrm{K} + \frac{t^n}{n!}E\left[X^n\right] + \mathrm{K} \tag{1.55}$$

Since $t$ is considered as a constant with respect to the expectation operator, taking the derivative of $M_x(t)$ with respect to $t$, we obtain

$$\frac{dM_x(t)}{dt} = M_x'(t) = E[X] + \frac{2t}{2!}E\left[X^2\right] + \frac{3t^2}{3!}E\left[X^3\right] + \mathrm{K} + \frac{n\,t^{n-1}}{n!}E\left[X^n\right] + \mathrm{K}$$

$$= E[X] + t\,E\left[X^2\right] + \frac{t^2}{2!}E\left[X^3\right] + \mathrm{K} + \frac{t^{n-1}}{(n-1)!}E\left[X^n\right] + \mathrm{K} \tag{1.56}$$

Setting $t = 0$, all terms become zero except $E[X]$. We obtain

$$M_x'(0) = E[X] \tag{1.57}$$

Similarly, taking the second derivative of $M_x(t)$ with respect to $t$ and setting it equal to zero, we obtain

$$M_x''(0) = E\left[X^2\right] \tag{1.58}$$

Continuing in this manner, we obtain all moments to be

$$M_x^{(n)}(0) = E\left[X^n\right] \qquad n = 1, 2, \ldots \tag{1.59}$$

where $M_x^{(n)}(t)$ denotes the $n$th derivative of $M_x(t)$ with respect to $t$.

If we let $t = j\omega$, where $j$ is the complex imaginary unit, in the moment generating function, we obtain the *characteristic function*. Hence, the

characteristic function $E\left[e^{j\omega X}\right]$ and denoted $\Phi_x(x)$ is actually the Fourier transform of the density function $f_X(x)$. It follows that

$$\Phi_x(\omega) = E\left[e^{j\omega X}\right] = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx \tag{1.60}$$

As before, differentiating $\Phi_x(x)$ $n$ times with respect to $\omega$ and setting $\omega = 0$ in the derivative, we obtain the $n$th moment of $X$ to be

$$E\left[X^n\right] = (-j)^n \left.\frac{d^n \Phi_x(\omega)}{d\omega^n}\right|_{\omega=0} \tag{1.61}$$

where $\sqrt{j} = -1$. An important role of the characteristic function is to give the density function of a random variable using the theory of Fourier transform. The inverse Fourier transform of the characteristic function is

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega x}\, \Phi_x(\omega)\, d\omega \tag{1.62}$$

It is preferable to use the characteristic function over the moment generating function because it always exists, whereas the moment generating function may not exist. However, the moment generating function, because of the presence of the exponential term, may exist for a class of functions that is much wider.

If $X$ is a discrete random variable, its characteristic function is defined as

$$m_{10} = E[X] = m_x \tag{1.63}$$

**Example 1.14**

Find the characteristic function of the random variable $X$ having density function $f_X(x) = e^{-\frac{1}{2}|x|}$ for all $x$.

*Solution*

From (1.60), the characteristic function is

$$\Phi_x(\omega) = \int_{-\infty}^{0} e^{j\omega x}\, e^{\frac{1}{2}x}\, dx + \int_{0}^{\infty} e^{j\omega x}\, e^{-\frac{1}{2}x}\, dx = \frac{1}{(0.5 + j\omega)} + \frac{1}{(0.5 - j\omega)} = \frac{4}{1 + 4\omega^2}$$

### 1.4.3 Upper Bounds on Probabilities and Law of Large Numbers

Often when the distributions are not completely specified but the mean and variance are known, we are interested in determining some bounds (upper or lower) on the probabilities. We present the *Chernoff bound*, which is supposed to be a "tighter" bound than the bound provided by the *Tchebycheff inequality*.

*Tchebycheff Inequality*

Let $X$ be any random variable with mean $m_x$ and variance $\sigma_x^2$. Then, for $\varepsilon > 0$, the Tchebycheff inequality states that

$$P\left(|X - m_x| \geq \varepsilon\right) \leq \frac{\sigma_x^2}{\varepsilon^2} \tag{1.64}$$

Choosing $\varepsilon = k\,\sigma_x$, where $k$ is a constant, we obtain

$$P\left(|X - m_x| \geq k\,\sigma_x\right) \leq \frac{1}{k^2} \tag{1.65}$$

or equivalently,

$$P\left(|X - m_x| \geq k\right) \leq \frac{\sigma_x^2}{k^2} \tag{1.66}$$

*Chernoff Bound*

Unlike the Tchebycheff bound, which involves the two sides of the probability density function, the Chernoff bound is applied to only one side of the density function, either in the interval $(\varepsilon, \infty)$ or in the interval $(-\infty, \varepsilon)$. Define

$$Y = \begin{cases} 1, & X \geq \varepsilon \\ 0, & X < \varepsilon \end{cases} \tag{1.67}$$

The expected value of $Y$ is

$$E[Y] = 1 \cdot P\left(X \geq \varepsilon\right) \tag{1.68}$$

and for all $t > 0$, it must be true that

$$Y\, e^{t\varepsilon} \le e^{t\,X} \tag{1.69}$$

then,

$$E\left[Y\, e^{t\varepsilon}\right] = e^{t\varepsilon}\; E[Y] \le E\left[e^{t\,X}\right] \tag{1.70}$$

Substituting (1.68) into (1.70) and rearranging terms, we obtain

$$P\left(X \ge \varepsilon\right) \le e^{-t\varepsilon}\; E\left[e^{t\,X}\right] \tag{1.71}$$

The upper bound of (1.71) is the *Chernoff bound*. Note that in this case more knowledge about the distribution is required to be able to evaluate $E[\, e^{t\,X}]$.

Similarly, if $Y$ is defined to be in the interval $(-\infty, \varepsilon)$ such that

$$Y = \begin{cases} 0\ , & X \ge \varepsilon \\ 1\ , & X < \varepsilon \end{cases} \tag{1.72}$$

The Chernoff bound is given by

$$P\left(X \le \varepsilon\right) \le e^{-t\varepsilon} E\left[e^{tX}\right] \tag{1.73}$$

*Law of Large Numbers*

Let $X_1, X_2, \ldots, X_n$ be $n$ independent random variables, each having mean $E[X_i] = m_x$ and variance $\text{var}\left[X_i\right] = \sigma_x^2$, $i = 1, 2, \ldots, n$. If $S_n = X_1 + X_2 + \ldots + X_n$ is the sum of the $n$ independent random variables, then

$$\lim_{n\to\infty} P\left(\left|\frac{X_1 + X_2 + \text{K} + X_n}{n} - m_x\right| \ge \varepsilon\right) = \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - m_x\right| \ge \varepsilon\right) = 0 \tag{1.74}$$

provided that

$$\sum_{n=1}^{\infty} \frac{\sigma_x^2}{n} \to 0 \quad \text{as} \quad n \to \infty \tag{1.75}$$

This theorem can be proved using the Tchebycheff inequality, and is referred to as the *weak law of large numbers*. In words, this theorem states that the probability

that the arithmetic mean (which could be an estimate) differs from the true mean $m_x$ by more than ε (ε > 0) is zero as $n$ goes to infinity. However, if the probability of $\lim_{n\to\infty}\left[(S_n/n) = m_x\right]$ equals one, we have the *strong law of large numbers*.

## 1.5 TWO- AND HIGHER-DIMENSIONAL RANDOM VARIABLES

In the previous sections, we developed the concept of random variables and other related topics, such as statistical averages, moment generating functions, and characteristic functions.

Often, we are not interested in one random variable, but in the relationship between two or more random variables. We now generalize the above concepts to $N$ random variables. We will mainly consider continuous random variables, since the appropriate modifications for the discrete or mixed cases are easily made by analogy. If $X$ and $Y$ are two continuous random variables, then we define the *joint probability density function* or simply the *joint density function* of $X$ and $Y$ by

$$f_{XY}(x, y) \geq 0 \tag{1.76}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\, dy = 1 \tag{1.77}$$

Geometrically, $f_{XY}(x, y)$ represents a surface, as shown in Figure 1.16. The total volume bounded by this surface and the $xy$-plane is unity, as given in (1.77). The probability that $X$ lies between $x_1$ and $x_2$ and $Y$ lies between $y_1$ and $y_2$, as shown in the shaded area of Figure 1.16, is given by

$$P(x_1 < X < x_2,\ y_1 < Y < y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x, y)\, dx\, dy \tag{1.78}$$

The joint distribution of $X$ and $Y$ is the probability of the joint events $\{X \leq x, Y \leq y\}$ given by

$$F_{XY}(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(u, v)\, du\, dv \tag{1.79}$$

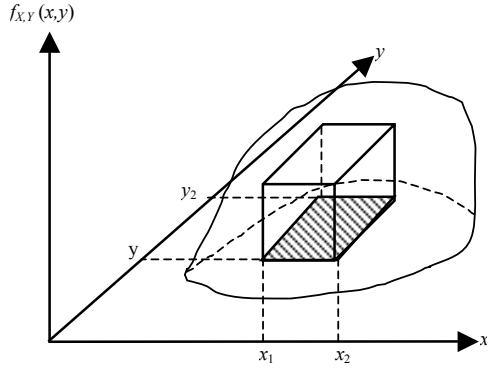


**Figure 1.16** Two-dimensional density function.

The joint distribution $F_{XY}(x, y)$ has the following properties:

1. $0 \le F_{XY}(x, y) \le 1$

2. $F_{XY}(\infty, \infty) = 1$

3. $F_{XY}(-\infty, -\infty) = F_{XY}(x, -\infty) = F_{XY}(-\infty, y) = 0$

4. $P(x_1 < X \le x_2, Y \le y) = F_{XY}(x_2, y) - F_{XY}(x_1, y) \ge 0$

5. $P(X \le x, y_1 < Y \le y_2) = F_{XY}(x, y_2) - F_{XY}(x, y_1) \ge 0$

6. $P(x_1 < X \le x_2, y_1 < Y \le y_2)$
$= F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$

The joint density function can be obtained from the distribution function by taking the derivative of $F_{XY}(x, y)$ with respect to $x$ and $y$ to be

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \, \partial y} F_{XY}(x, y) \tag{1.80}$$

The marginal distribution function of $X$, $F_X(x) = P(X \le x)$, is obtained from (1.79) by integrating $y$ over all possible values. Hence,

$$F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(u, v)\, dv\, du \tag{1.81}$$

Similarly, the marginal distribution of $Y$ is given by

$$F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{XY}(u, v)\, du\, dv \tag{1.82}$$

If we generalize the concepts of the distribution and density functions to $n$ random variables $X_1, X_2, \ldots, X_n$, then the joint probability distribution function is

$$F_{X_1 X_2 \text{K } X_n}(x_1, x_2, \text{K}, x_n) = P(X_1 \le x_1, X_2 \le x_2, \text{K}, X_n \le x_n) \tag{1.83}$$

and the joint probability density function is the $n$th derivative of (1.83) to yield

$$f_{X_1 X_2 \text{K } X_n}(x_1, x_2, \text{K}, x_n) = \frac{\partial^n}{\partial x_1\, \partial x_2\, \text{K}\, \partial x_n} F_{X_1 X_2 \text{K } X_n}(x_1, x_2, \text{K}, x_n) \tag{1.84}$$

### 1.5.1 Conditional Distributions

The marginal density functions of the random variables $X$ and $Y$ are obtained by taking the derivatives of the respective marginal distribution functions $F_X(x)$ and $F_Y(y)$ given in (1.81) and (1.82). Using the joint density function of $X$ and $Y$, the marginal functions $f_X(x)$ and $f_Y(y)$ are

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \tag{1.85}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \tag{1.86}$$

Once the marginal distribution functions are known, it becomes simple to determine the conditional distribution functions. In many practical problems, we are interested in the distribution of the random variable $X$ given that the random variable $Y$ assumes some specific value, or that the random variable $Y$ is between some interval from $y_1$ to $y_2$.

When the random variable assumes some specific value, we say that we have *point conditioning*. To clarify this concept, consider the conditional distribution

function of the random variable $X$ given that $y-\Delta y < Y \le y+\Delta y$, where $\Delta y$ is a small quantity. Hence,

$$F_X\left(x \mid y-\Delta y < Y \le y+\Delta y\right) = \frac{\int_{y-\Delta y}^{y+\Delta y}\int_{-\infty}^{x} f_{XY}(u,v)\,du\,dv}{\int_{y-\Delta y}^{y+\Delta y} f_Y(v)\,dv} \tag{1.87}$$

in the limit, as $\Delta y \to 0$ and for every $y$ such that $f_Y(y)$, we have

$$F_X\left(x \mid Y=y\right) = \frac{\int_{-\infty}^{x} f_{XY}(u,y)\,du}{f_Y(y)} \tag{1.88}$$

where $f_{XY}(x,y)$ is the joint density function of $X$ and $Y$, and $f_Y(y)$ is the marginal density function of $Y$. Differentiating both sides of (1.88) with respect to $x$, we obtain

$$f_X\left(x \mid Y=y\right) = \frac{f_{XY}(x,y)}{f_Y(y)} \tag{1.89}$$

which can also be written as

$$f_X\left(x \mid y\right) = \frac{f_{XY}(x,y)}{f_Y(y)} \tag{1.90}$$

Similarly, we can show that

$$f_Y\left(y \mid x\right) = \frac{f_{XY}(x,y)}{f_X(x)} \tag{1.91}$$

In the *interval conditioning*, the random variable assumes some range of values. The conditional distribution function of $X$ given that $y_1 < Y \le y_2$ is defined as

$$F_X\left(x \mid y_1 < Y \le y_2\right) = \frac{\int_{y_1}^{y_2}\int_{-\infty}^{x} f_{XY}(u,y)\,du\,dy}{\int_{y_1}^{y_2}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy} \tag{1.92a}$$

$$= \frac{F_{XY}(x,y_2) - F_{XY}(x,y_1)}{\int_{y_1}^{y_2} f_Y(y)\,dy} \tag{1.92b}$$

since $\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = f_Y(y)$ is the marginal density function of $Y$. Again, differentiating both sides of (1.92a), we obtain

$$f_X\left(x \mid y_1 < Y \le y_2\right) = \frac{\int_{y_1}^{y_2} f_{XY}(x,y)\,dy}{\int_{y_1}^{y_2} f_Y(y)\,dy} \tag{1.93}$$

Similarly, the conditional density function of $Y$ given that $x_1 < X \le x_2$ is given by

$$f_Y\left(y \mid x_1 < X < x_2\right) = \frac{\int_{x_1}^{x_2} f_{XY}(x,y)\,dx}{\int_{x_1}^{x_2} f_X(x)\,dx} \tag{1.94}$$

where

$$\int_{x_1}^{x_2} f_X(x)\,dx = F_X(x_2) - F_X(x_1) \tag{1.95}$$

If $X$ and $Y$ are independent random variables, then the events $\{X \le x\}$ and $\{Y \le y\}$ are independent events for all $x$ and $y$. This yields

$$P(X \le x, Y \le y) = P(X \le x)\,P(Y \le y) \tag{1.96}$$

that is,

$$F_{XY}(x, y) = F_X(x)\, F_Y(y) \tag{1.97}$$

Equivalently,

$$f_{XY}(x, y) = f_X(x)\, f_Y(y) \tag{1.98}$$

where $f_X(x)$ and $f_Y(y)$ are the marginal density functions of $X$ and $Y$. If the joint distribution functions or the joint density functions cannot be written in a product form as given in (1.97) and (1.98), then the random variables $X$ and $Y$ are not independent. Note that if the random variables $X$ and $Y$ are independent, using (1.97) in (1.98) results in $f_{X|Y}(x|y) = f_X(x)$ and $f_{Y|X}(y|x) = f_Y(y)$, as expected.

The above results can be modified accordingly for discrete random variables. Suppose $X$ and $Y$ are both discrete random variables with values $x_i$, $i = 1, 2, \ldots, n$, and $y_j$, $j = 1, 2, \ldots, m$, having probabilities $P(X = x_i) = P(x_i) = P_i$, $i = 1, 2, \ldots, n$, and $P(Y = y_j) = P(y_j) = P_j$, $j = 1, 2, \ldots, m$, respectively. The joint probability of occurrence of $x_i$ and $y_j$, denoted $P(X = x_i, Y = y_j) = P(x_i, y_j) = P_{ij}$, is given by

$$f_{XY}(x, y) = \sum_{j=1}^{m} \sum_{i=1}^{n} P(x_i, y_j)\, \delta(x - x_i)\, \delta(y - y_j) \tag{1.99}$$

where $\delta(x - x_0)\delta(y - y_0)$ is the impulse function of volume (1) and occurring at $x = x_0$ and $y = y_0$, as shown in Figure 1.17. Note that we wrote 1 in parentheses to indicate that it represents a volume and not a height. Based on the following properties of the two-dimensional impulse function:

1. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\, A\delta(x - x_0)\, \delta(y - y_0)\, dx\, dy = A\, g(x_0, y_0)$

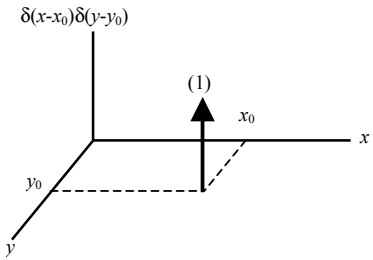


**Figure 1.17** Two-dimensional impulse function.

2. $\int_{-\infty}^{\infty} g(x,y)\, A\, \delta(x-x_0)\,\delta(y-y_0)\,dx = A\, g(x_0,y)\,\delta(y-y_0)$

3. $\int_{-\infty}^{\infty} g(x,y)\, A\, \delta(x-x_0)\,\delta(y-y_0)\,dy = A\, g(x,y_0)\,\delta(x-x_0)$

we can show that the marginal density functions are

$$f_X(x_i) = \int_{-\infty}^{\infty} f_{XY}(x_i, y)\, dy \tag{1.100}$$

Substituting (1.99) into (1.100), and using the above properties of the two-dimensional impulse function, we obtain

$$\begin{aligned} f_X(x_i) &= \int_{-\infty}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j)\,\delta(x-x_i)\,\delta(y-y_j)\,dy \\ &= [\,P(x_i,y_1) + P(x_i,y_2) + \text{K} + P(x_i,y_m)\,]\,\delta(x-x_i) \\ &= P(x_i)\,\delta(x-x_i) \end{aligned} \tag{1.101}$$

since $\sum_{j=1}^{m} P(x_i, y_j) = P(x_i)$. Similarly, we can show that

$$f_Y(y_j) = P(y_j)\,\delta(y-y_j) \tag{1.102}$$

Note that $f_X(x)$ will be all *j*s in (1.99) to obtain

$$f_X(x) = \sum_{j=1}^{m} P(y_j)\,\delta(y-y_j) \tag{1.103}$$

and $f_Y(y)$ will be all *i*s to give

$$f_Y(y) = \sum_{i=1}^{n} P(x_i)\,\delta(x-x_i) \tag{1.104}$$

The conditional density function $f_X(x \mid y = y_j)$ is given by

$$f_X\left(x \mid y = y_j\right) = \sum_{i=1}^{n} \frac{P\left(x_i, y_j\right)}{P\left(y_j\right)} \delta\left(x - x_i\right) \tag{1.105}$$

and the conditional distribution function, which is the integral of (1.105), becomes

$$F_X\left(x \mid y = y_j\right) = \sum_{i=1}^{n} \frac{P\left(x_i, y_j\right)}{P\left(y_j\right)} u\left(x - x_i\right) \tag{1.106}$$

where $u(x - x_i)$ is the unit step function, such that $u(x - x_i)$ is one for $x \geq x_i$ and zero otherwise. The derivative of the unit step function yields the unit impulse function, as discussed in Section 1.3.1.

**Example 1.15**

Let $X$ and $Y$ be two random variables with the joint density function

$$f_{XY}(x, y) = \begin{cases} x^2 + \dfrac{x\,y}{3} & , \quad 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Check that $f_{XY}(x, y)$ is a density function.

(b) Find the marginal density functions $f_X(x)$ and $f_Y(y)$.

(c) Compute $P(X > 1/2)$, $P(Y < X)$, and $P\left(Y < 1/2 \mid X < 1/2\right)$.

*Solution*

(a) For $f_{XY}(x, y)$ to be a density function, it must satisfy (1.76) and (1.77). The first is easily verified, while the second says that the integral over all possible values of $x$ and $y$ must be one. That is,

$$\int_0^2 \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx\, dy = \int_0^2 \left(\frac{1}{3} + \frac{1}{6} y\right) dy = 1$$

(b) The marginal density functions of $X$ and $Y$ are direct applications of (1.85) and (1.86). Thus,

$$f_X(x) = \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy = 2x^2 + \frac{2}{3} x \quad \text{for} \quad 0 < x < 1$$

and

$$f_Y(y) = \int_0^1 \left( x^2 + \frac{xy}{3} \right) dx = \frac{1}{6} y + \frac{1}{3} \quad \text{for} \quad 0 < y < 2$$

(c) Computing the different probabilities, we have

$$P\left( X > \frac{1}{2} \right) = \int_{1/2}^1 f_X(x)\, dx = \int_{1/2}^1 \left( 2x^2 + \frac{2}{3} x \right) dx = \frac{5}{6} = 0.8333$$

$$P(Y < X) = \int_0^1 \int_0^x \left( x^2 + \frac{xy}{3} \right) dy\, dx = \frac{7}{24} = 0.2917$$

$$P\left( Y < \frac{1}{2} \middle| X < \frac{1}{2} \right) = \frac{P\left( Y < \frac{1}{2}, X < \frac{1}{2} \right)}{P\left( X < \frac{1}{2} \right)}$$

We have already found $P(X > 1/2)$ to be $5/6$. Hence, $P(X < 1/2) = 1 - P(X > 1/2) = 1/6 = 0.1667$. We now need only find $P(Y < 1/2, X < 1/2)$, which is

$$P\left( Y < \frac{1}{2}, X < \frac{1}{2} \right) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left( x^2 + \frac{xy}{3} \right) dx\, dy = \frac{5}{192} = 0.0260$$

Hence,

**Table 1.2**
Joint Probabilities of $X$ and $Y$

| | $X$ | |
|---|---|---|
| $Y$ | 1 | 2 |
| 0 | 1/4 | 1/4 |
| 1 | 0 | 1/8 |
| 2 | 1/4 | 1/8 |

$$P\left(Y < \frac{1}{2} \middle| X < \frac{1}{2}\right) = \frac{5/192}{1/6} = \frac{5}{32} = 0.1563$$

**Example 1.16**

($X$,$Y$) is a two-dimensional random variable with joint probability density function as shown in Table 1.2.

(a) Sketch $f_{XY}(x, y)$.
(b) Compute $f_X(1)$ and $f_Y(2)$.
(c) Are $X$ and $Y$ independent?

*Solution*

(a) The joint density function $f_{XY}(x, y)$ is shown in Figure 1.18. Note that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy = 1$$

(b) From (1.100), $f_X(1)$ is the sum of the probabilities at $x$ = 1 along all $y$. We have

$$f_X(1) = \int_{-\infty}^{\infty} f_{X,Y}(1, y)\, dy = \frac{1}{4}\delta(x-1) + \frac{1}{4}\delta(x-1) = \frac{1}{2}\delta(x-1)$$

and $f_Y(2)$ is the sum of the probabilities at $y$ = 2 along all $x$. Hence,

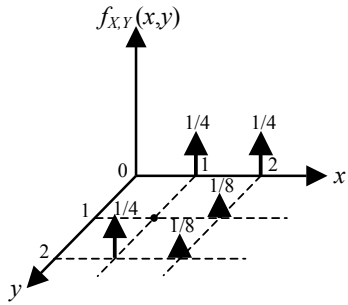


**Figure 1.18** Joint distribution of ($X$,$Y$).

$$f_Y(2) = \int_{-\infty}^{\infty} f_{X,Y}(x,2)\,dx = \frac{1}{4}\delta(y-2) + \frac{1}{8}\delta(y-2) = \frac{3}{8}\delta(y-2)$$

(c) $X$ and $Y$ are independent if $P(x_i, y_j) = P(x_i)\,p(y_j)$ for all $x_i$ and $y_j$. Note that we just need a counterexample to show that the above identity is not verified. Using the results of (b), we see that $P(X = 1,\ Y = 2) = 1/4$, $P(X = 1) = 1/2$, and $P(Y = 2) = 3/8$. Since $P(X = 1,\ Y = 2) = 1/4 \neq P(X = 1)\,P(Y = 2) = 3/16$, then $X$ and $Y$ are not independent.

### 1.5.2 Expectations and Correlations

We have seen in Section 1.4 that, if $X$ is a continuous random variable having density function $f_X(x)$, then the expected value of $g(X)$, a function of the random variable $X$, is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\,f_X(x)\,dx \tag{1.107}$$

This concept is easily generalized to functions of two random variables. In fact, if $X$ and $Y$ are two random variables with joint density function $f_{XY}(x, y)$, then

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f_{XY}(x,y)\,dx\,dy \tag{1.108}$$

If we have $n$ functions of random variables $g_1(X,Y)$, $g_2(X,Y)$, … , $g_n(X,Y)$, then

$$\begin{aligned} &E[g_1(X,Y) + g_2(X,Y) + \mathrm{K} + g_n(X,Y)] \\ &\quad = E[g_1(X,Y)] + E[g_2(X,Y)] + \mathrm{K} + E[g_n(X,Y)] \end{aligned} \tag{1.109}$$

Hence, for the simple case of the sum of two random variables $X$ and $Y$, the expected value of the sum of the random variables is the sum of the individual expected values. Specifically,

$$E[X + Y] = E[X] + E[Y] \tag{1.110}$$

The expected value of the product of the random variables of $X$ and $Y$, $E[XY]$, is known as the correlation, $R_{xy}$, between $X$ and $Y$. The correlation between $X$ and $Y$ is actually a particular case of the joint moments defined to be

$$m_{k\lambda} = E\left[X^k\ Y^\lambda\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k\, y^\lambda\, f_{XY}(x,y)\,dx\,dy \tag{1.111}$$

Note that the *order* of the moment is $n = k + \lambda$. The correlation $R_{xy}$ is then the moment $m_{11}$ of order 2 with $k = 1$ and $\lambda = 1$. It is also known as the *second order moment*. Note also that if $k$ is zero or $\ell$ is zero, we obtain the expected value of a one-dimensional random variable defined in (1.43)

$$m_{10} = E[X] = m_x \tag{1.112}$$

and

$$m_{01} = E[Y] = m_y \tag{1.113}$$

where $m_x$ is the mean of the random variable $X$, and $m_y$ is the mean of the random variable $Y$.

The general form of the central moment is given by

$$\begin{aligned}\mu_{k\lambda} &= E\left[(X - m_x)^k\ (Y - m_y)^\lambda\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - m_x)^k\ (y - m_y)^\lambda\, f_{XY}(x,y)\,dx\,dy\end{aligned} \tag{1.114}$$

When $k = 2$ and $\lambda = 0$, or when $k = 0$ and $\lambda = 2$, we obtain the specific variances $\sigma_x^2$ and $\sigma_y^2$ of the random variables $X$ and $Y$, respectively. Hence,

$$\mu_{20} = E[(X - m_x)^2] = \sigma_x^2 \tag{1.115}$$

and

$$\mu_{02} = E[(Y - m_y)^2] = \sigma_y^2 \tag{1.116}$$

When $X$ and $Y$ are not independent, we often try to determine the "degree of relation" between $X$ and $Y$ by some meaningful parameter. This parameter is the *correlation coefficient*, defined as

$$\rho_{xy} = \frac{E[(X - m_x)(Y - m_y)]}{\sigma_x\,\sigma_y} \tag{1.117}$$

where $\rho_{xy}$ is the correlation coefficient between $X$ and $Y$, $m_x$ is the mean of $X$, $m_y$ is the mean of $Y$, and $\sigma_x$ and $\sigma_y$ are the standard deviations of $X$ and $Y$, respectively. The degree of correlation, which is the value of the coefficient $\rho$, is between $-1$ and $+1$ inclusive:

$$-1 \le \rho \le 1 \tag{1.118}$$

If $X$ and $Y$ are uncorrelated, then the expected value of the product of $X$ and $Y$ can be expressed as the product of expected values. That is,

$$E[XY] = E[X]\,E[Y] \tag{1.119}$$

Observe that $R_{xy} = E[X]\,E[Y]$ means that $\rho_{xy}$ in (1.117) is zero. The numerator of (1.117), given by

$$C_{xy} = E[(X - m_x)\,(Y - m_y)] \tag{1.120}$$

and known as the *covariance* of $X$ and $Y$, becomes equal to zero. Observe that the covariance corresponds to the second order central moment with $k = \lambda = 1$; that is, $\mu_{11} = C_{xy}$. The correlation coefficient can be written in terms of the covariance as

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y} \tag{1.121}$$

Note also that the variance of $X + Y$ is the sum of the variances of $X$ and $Y$; that is,

$$\text{var}[X + Y] = \text{var}[X] + \text{var}[Y] \tag{1.122}$$

or,

$$\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2 \tag{1.123}$$

It should be noted that if the random variables $X$ and $Y$ are independent. They are also uncorrelated, but the inverse is not true. If

$$E[XY] = 0 \tag{1.124}$$

we say that $X$ and $Y$ are *orthogonal.*

When the random variables $X$ and $Y$ are not independent, we can define the conditional expectation of one random variable in terms of its conditional density function. The conditional expectation of $X$ given that $Y = y$ is defined as

$$E[X|y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx \tag{1.125}$$

It can also be easily shown that

$$E\{E[X|Y]\} = E[X] \tag{1.126}$$

and

$$E\{E[Y|X]\} = E[Y] \tag{1.127}$$

where

$$E[X|Y] = \int_{-\infty}^{\infty} E[X|y]\, f_Y(y)\, dy \tag{1.128}$$

Note that if $X$ and $Y$ are independent, then $E[X|Y] = E[X]$ and $E[Y|X] = E[Y]$. In general, the expected value of a function of random variables $X$ and $Y$, given that $X$ equals some value $x$, is given by

$$E[g(X,Y)|\, X = x] = \int_{-\infty}^{\infty} g(x,y)\, f_Y(y|\, X = x)\, dx \tag{1.129}$$

where $f_Y(y|\, X = x) = f_{XY}(x,y)/f_X(x)$. Another important result is

$$E\{E[g(X,Y)|\, X]\} = E[g(X,Y)] \tag{1.130}$$

### 1.5.3 Joint Characteristic Functions

We have seen in Section 1.4.2 that the characteristic functions and moment generating functions are functions that give moments of random variables. We now extend the concept to more than one random variable. The joint characteristic function of two random variables $X$ and $Y$ is defined as

$$\Phi_{xy}(\omega_1, \omega_2) = E\left[e^{j(\omega_1 X + \omega_2 Y)}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j(\omega_1 x + \omega_2 y)} f_{XY}(x, y)\, dxdy \quad (1.131)$$

where $\omega_1$ and $\omega_2$ are real numbers. Thus, $\Phi_{xy}(\omega_1, \omega_2)$ is the double Fourier transform of $f_{X,Y}(x, y)$. The inverse Fourier transform is then

$$f_{XY}(x, y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j(\omega_1 x + \omega_2 y)} \Phi_{xy}(\omega_1, \omega_2)\, d\omega_1\, d\omega_2 \quad (1.132)$$

The marginal characteristic functions are obtained by setting either $\omega_1 = 0$ or $\omega_2 = 0$. Hence,

$$\Phi_{xy}(\omega_1, 0) = E\left[e^{j\omega_1 X}\right] = \Phi_x(\omega_1) \quad (1.133)$$

and

$$\Phi_{xy}(0, \omega_2) = E\left[e^{j\omega_2 Y}\right] = \Phi_y(\omega_2) \quad (1.134)$$

If $g(X)$ is a function of $X$ and $h(Y)$ is a function of $Y$, then $g(X)$ and $h(Y)$ are independent, provided that $X$ and $Y$ are independent. Consequently, the characteristic function of $(X + Y)$ is the product of the individual characteristic functions of $X$ and $Y$. That is,

$$\Phi_{x+y}(\omega) = E\left[e^{j\omega(X+Y)}\right] = E\left[e^{j\omega X}\right] E\left[e^{j\omega Y}\right] = \Phi_x(\omega)\,\Phi_y(\omega) \quad (1.135)$$

The joint characteristic function also can be expressed in terms of the series to obtain the moments. Hence,

$$\begin{aligned}\Phi_{xy}(\omega_1, \omega_2) &= E\left[e^{j(\omega_1 X + j\omega_2 Y)}\right] = E\left[\sum_{n=0}^{\infty} \frac{\{j(\omega_1 X + \omega_2 Y)\}^n}{n!}\right] \\ &= 1 + j\omega_1 m_x + j\omega_2 m_y - \tfrac{1}{2}\omega_1^2 E\left[X^2\right] - \omega_1\omega_2 E[XY] \\ &\quad - \tfrac{1}{2}\omega_2^2 E\left[Y^2\right] + \Lambda \end{aligned} \quad (1.136)$$

The joint moments $m_{k\lambda}$ can be obtained from (1.136) to be

$$m_{k\lambda} = E\left[X^k Y^\lambda\right] = (-j)^{k+\lambda} \left. \frac{\partial^{k+\lambda} \Phi_{xy}(\omega_1, \omega_2)}{\partial \omega_1^k \partial \omega_2^\lambda} \right|_{\omega_1 = \omega_2 = 0} \tag{1.137}$$

which is the two-dimensional extension of expression of (1.61) found in Section 1.4.2.

**Example 1.17**

Consider the two-dimensional random variable ($X$, $Y$) with joint density

$$f_{XY}(x, y) = \begin{cases} kxy, & x \le y \quad \text{and} \quad 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find

(a) the constant $k$;

(b) $f_{X|Y}(x|y)$;

(c) $E[X \mid Y = y]$.

*Solution*

(a) To find the constant $k$, we solve the integral in (1.77). From Figure 1.19, we see that the integral we have to solve is

$$\int_0^1 \int_0^y kxy \, dx \, dy = 1 \qquad \Rightarrow \quad k = 8$$

(b) In order to use the definition of (1.90), we need to determine $f_Y(y)$.

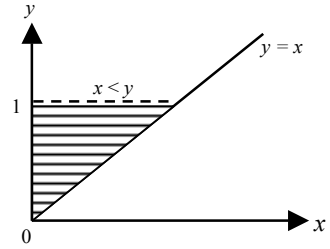


**Figure 1.19** Boundaries of $f_{XY}(x, y)$.

$$f_Y(y) = \int_0^y 8xy\,dx = 4\,y^3 \quad \text{for} \quad 0 \le y \le 1$$

Hence,

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{2x}{y^2} & , \quad \text{for } 0 \le x \le y \\ 0 & , \quad \text{otherwise} \end{cases}$$

(c) $E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx = \int_0^y \dfrac{2x^2}{y^2}\,dx = \dfrac{2}{3}y$.

**Example 1.18**

The probability density function of the two-dimensional random variable ($X$ ,$Y$) in the area shown in Figure 1.20 is given by

$$f_{XY}(x,y) = \frac{2}{\pi} \qquad \text{for} \quad x^2 + y^2 \le 1$$

Find the correlation coefficient $\rho_{xy}$ between the random variables $X$ and $Y$.

*Solution*

The expression of $\rho_{xy}$ is given by (1.117). Hence, we need to determine $E[XY]$, $E[X]$, $E[Y]$, $\sigma_x$, and $\sigma_y$. Using (1.111), the expected value of $XY$ is

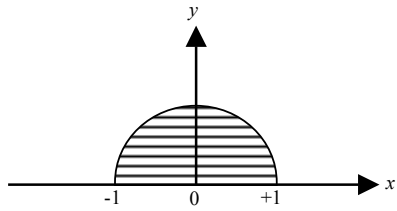


**Figure 1.20** Domain of $f_{XY}(x\,,\,y)$.

where $\rho_{xy}$ is the correlation coefficient between $X$ and $Y$, $m_x$ is the mean of $X$, $m_y$ is the mean of $Y$, and $\sigma_x$ and $\sigma_y$ are the standard deviations of $X$ and $Y$, respectively. The degree of correlation, which is the value of the coefficient $\rho$, is between $-1$ and $+1$ inclusive:

$$-1 \le \rho \le 1 \tag{1.118}$$

If $X$ and $Y$ are uncorrelated, then the expected value of the product of $X$ and $Y$ can be expressed as the product of expected values. That is,

$$E[XY] = E[X]\,E[Y] \tag{1.119}$$

Observe that $R_{xy} = E[X]\,E[Y]$ means that $\rho_{xy}$ in (1.117) is zero. The numerator of (1.117), given by

$$C_{xy} = E[(X - m_x)\,(Y - m_y)] \tag{1.120}$$

and known as the *covariance* of $X$ and $Y$, becomes equal to zero. Observe that the covariance corresponds to the second order central moment with $k = \lambda = 1$; that is, $\mu_{11} = C_{xy}$. The correlation coefficient can be written in terms of the covariance as

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y} \tag{1.121}$$

Note also that the variance of $X + Y$ is the sum of the variances of $X$ and $Y$; that is,

$$\text{var}[X + Y] = \text{var}[X] + \text{var}[Y] \tag{1.122}$$

or,

$$\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2 \tag{1.123}$$

It should be noted that if the random variables $X$ and $Y$ are independent. They are also uncorrelated, but the inverse is not true. If

$$E[XY] = 0 \tag{1.124}$$

we say that $X$ and $Y$ are *orthogonal.*

### 1.6.1 Functions of One Random Variable

Consider the problem of determining the density function of a random variable $Y$, where $Y$ is a function of $X$, $Y = g(X)$, and the density function of $X$, $f_X(x)$, is known. We assume that the function $y = g(x)$ is monotonically increasing and differentiable, as shown in Figure 1.21. The distribution function of $Y$ in terms of $X$ is

$$F_Y(y) = P(Y \leq y) = P\left[X \leq g^{-1}(y)\right] \tag{1.138}$$

where $g^{-1}(x)$ is the inverse transformation. Since we know the density function of $X$, we can then write

$$F_Y(y) = \int_{-\infty}^{g^{-1}(y)} f_X(x)\,dx \tag{1.139}$$

Differentiation of both sides of (1.139) yields

$$f_Y(y) = f_X\left[g^{-1}(y)\right] \frac{d}{dy}\left[g^{-1}(y)\right] \tag{1.140}$$

If the function $g$ were monotonically decreasing, we would have

$$F_Y(y) = \int_{g^{-1}(y)}^{\infty} f_X(x)\,dx \tag{1.141}$$

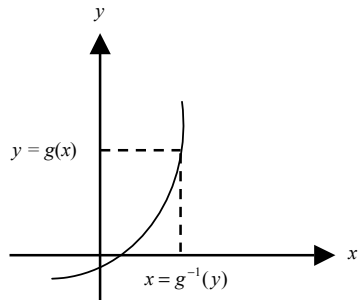


**Figure 1.21** Monotone function of $x$.

and consequently,

$$f_Y(y) = -f_X\left[g^{-1}(y)\right]\frac{d}{dy}\left[g^{-1}(y)\right] \tag{1.142}$$

In this case, the derivative of $d\left[g^{-1}(y)\right]/dy$ is negative. Combining both results of (1.140) and (1.142), the density function of $Y$ is given by

$$f_Y(y) = \left|f_X\left[g^{-1}(y)\right]\right|\left\{\frac{d}{dy}\left[g^{-1}(y)\right]\right\} \tag{1.143}$$

This result can be generalized to the case where the function $g(x)$ has many real roots $x_1, x_2, \text{K}, x_n, \text{K}$, as shown in Figure 1.22. In this case, the density function of the random variable $Y$, $Y = g(X)$, is

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} + \Lambda + \frac{f_X(x_n)}{|g'(x_n)|} + \Lambda \tag{1.144}$$

where $f_X(x)$ is the density function of $X$, and $x_i$, $i = 1, 2, \ldots$, is expressed in terms of $y$, and $g'(x)$ is the derivative of $g(x)$ with respect to $x$. This is known as the *fundamental theorem*. A special case of this fundamental theorem is when $Y = aX + b$. The function $y = g(x) = ax + b$ has one root $x_1 = (y - a)/b$. The derivative of $g(x)$ is just the constant $a$; $g'(x) = a$. Therefore,

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \tag{1.145}$$

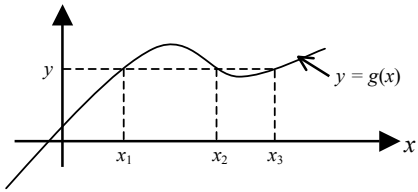


**Figure 1.22** Function $y = g(x)$.

**Example 1.19**

Determine the density function of the random variable $Y$ where $Y = g(X) = aX^2$, given that $a$ is positive and the density function of $X$ is $f_X(x)$.

*Solution*

There are two ways of solving this problem. We either apply directly the fundamental theorem or use the formal derivation starting from the distribution function. We will try both methods and see if the results agree.

*Method 1.* As shown in Figure 1.23, we have two roots, $x_1 = -\sqrt{y/a}$ and $x_2 = +\sqrt{y/a}$.

$$F_Y(y) = P(Y \le y) = P\left(-\sqrt{\frac{y}{a}} \le X \le +\sqrt{\frac{y}{a}}\right) = P\left(X \le \sqrt{\frac{y}{a}}\right) - P\left(X \le -\sqrt{\frac{y}{a}}\right)$$

$$= F_X\left(\sqrt{\frac{y}{a}}\right) - F_X\left(-\sqrt{\frac{y}{a}}\right)$$

Differentiation of both sides of the above relation yields

$$f_Y(y) = \frac{1}{2a\sqrt{y/a}}\left[f_X\left(\sqrt{\frac{y}{a}}\right) + f_X\left(-\sqrt{\frac{y}{a}}\right)\right], \quad y > 0$$

*Method 2.* In this case, we use the fundamental theorem. We have two roots, and consequently the density function of $Y$ is

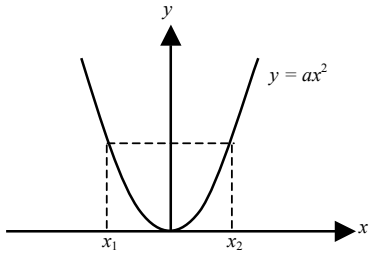


**Figure 1.23** Function $y = g(x) = ax^2$.

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|}$$

where $g'(x) = 2a\,x$, $x_1 = -\sqrt{y/a}$, and $x_2 = +\sqrt{y/a}$. Thus,
$g'(x_1) = 2a\left(-\sqrt{y/a}\right) = -2\sqrt{a\,y}$, and $g'(x_2) = 2a\left(\sqrt{y/a}\right) = 2\sqrt{a\,y} \Rightarrow$

$$f_Y(y) = \frac{f_X\left(-\sqrt{\frac{y}{a}}\right)}{2\sqrt{a\,y}} + \frac{f_X\left(\sqrt{\frac{y}{a}}\right)}{2\sqrt{a\,y}}$$

Both results agree.

### 1.6.2 Functions of Two Random Variables

We shall give some important results for some specific operations. The problem is to determine the density function of $Z$, where $Z$ is a function of the random variables $X$ and $Y$. That is,

$$Z = g(X, Y) \tag{1.146}$$

The joint density function of $(X, Y)$, $f_{XY}(x, y)$, is also known. Let $Z$ be a random variable equal to the sum of two independent random variables $X$ and $Y$,

$$Z = X + Y \tag{1.147}$$

The density function of $Z$ can be shown to be the convolution of the density functions of $X$ and $Y$,

$$f_Z(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_Y(y)\, f_X(z-y)\, dy \tag{1.148}$$

where * denotes convolution, and we used the fact that $f_{XY}(x, y) = f_X(x)\ f_Y(y)$, since $X$ and $Y$ are independent.

**Example 1.20**

Find the density function of $Z = X + Y$, for $X$ and $Y$ independent. The density functions of $X$ and $Y$ for $0 \le x \le a$, $0 \le y \le b$, and $a < b$ are shown in Figure 1.24.

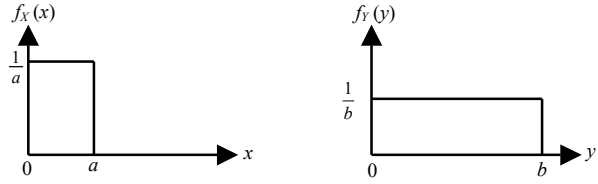


**Figure 1.24** Density functions of (a) $X$ and (b) $Y$.

*Solution*

It is much easier to solve convolutions graphically. For $z < a$, there is no overlap between the areas representing the density functions, as shown in Figure 1.25(a). This yields $f_Z(z) = 0$ for $z < 0$.

For $0 \le z < a$, we have an increasing area as $z$ moves from 0 to $a$, as shown in Figure 1.25(b). Thus,

$$f_Z(z) = \int_0^z \frac{1}{a}\frac{1}{b}dy = \frac{z}{ab}$$

For $a \le z < b$, we have a constant area, as shown in Figure 1.25(c) $\Rightarrow$

$$f_Z(z) = \int_{z-a}^{z} \frac{1}{a}\frac{1}{b}dy = \frac{1}{b}$$

For $b \le z < a + b$, from Figure 1.25(d), we have

$$f_Z(z) = \int_{z-a}^{b} \frac{1}{a}\frac{1}{b}dy = \frac{a+b-z}{ab}$$

For $z \ge a + b$, there is no overlap between the two curves, and consequently $f_Z(z) = 0$ for $z \ge a + b$ $\Rightarrow$

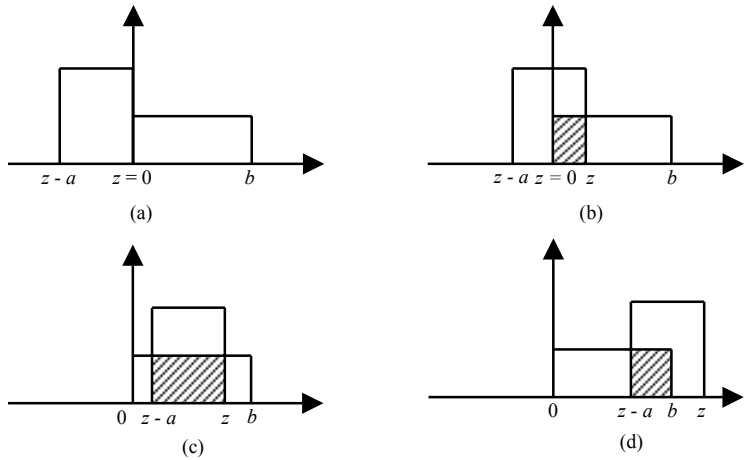


**Figure 1.25** (a–d) Areas of integration for convolution of *X* and *Y*.

$$f_Z(z)=\begin{cases}\dfrac{z}{ab}, & 0\le z<a\\[2ex] \dfrac{1}{b}, & a\le z<b\\[2ex] \dfrac{a+b-z}{ab}, & b\le z<a+b\\[2ex] 0, & z\ge a+b \text{ or } z<0\end{cases}$$

The density function of *Z*, $f_Z(z)$, is shown in Figure 1.26.

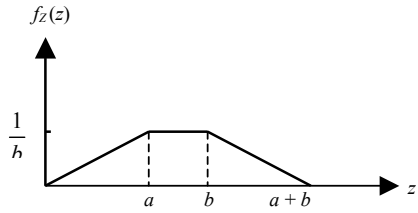


**Figure 1.26** Density function of *Z*.

If $X$ and $Y$ are not independent, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z-y, y)\, dy = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\, dx \tag{1.149}$$

If

$$Z = X - Y \tag{1.150}$$

then the density function of $Z$ is

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z+y, y)\, dy = \int_{-\infty}^{\infty} f_{XY}(x, x-z)\, dx \tag{1.151}$$

If $X$ and $Y$ are independent, then the density function becomes

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z+y)\, f_Y(y)\, dy = \int_{-\infty}^{\infty} f_X(x)\, f_Y(x-z)\, dx \tag{1.152}$$

We obtain similar results for other operations. We assume that the random variables $X$ and $Y$ are independent, with marginal density functions $f_X(x)$ and $f_Y(y)$. Let

$$U = X\,Y \tag{1.153}$$

Hence, we need to determine the density function $f_U(u)$ in the region shown in Figure 1.27. For $u > 0$, the distribution function $F_U(u)$ is given by

$$F_U(u) = \int_0^{\infty} f_Y(y)\, dy \int_{-\infty}^{\frac{u}{y}} f_X(x)\, dx + \int_{-\infty}^{0} f_Y(y)\, dy \int_{\frac{u}{y}}^{\infty} f_X(x)\, dx \tag{1.154}$$

Taking the derivative of (1.154) with respect to $u$ and using Leibniz's rule, we obtain

$$f_U(u) = \frac{dF_U(u)}{du} = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x)\, f_Y\left(\frac{u}{x}\right) dx = \int_{-\infty}^{\infty} \frac{1}{|y|} f_X\left(\frac{u}{y}\right) f_Y(y)\, dy \tag{1.155}$$

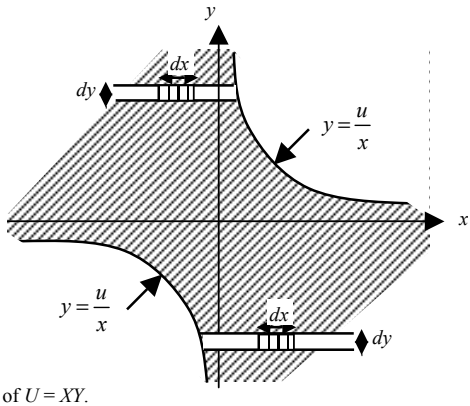


**Figure 1.27** Domain of $U = XY$.

The same results can be obtained for $u < 0$. If we now have

$$V = \frac{X}{Y} \tag{1.156}$$

then the region of integration is as shown in Figure 1.28. The distribution function $F_V(v)$ is given by

$$F_V(v) = \int_0^{\infty} f_Y(y)\,dy \int_{-\infty}^{vy} f_X(x)\,dx + \int_{-\infty}^{0} f_Y(y)\,dy \int_{vy}^{\infty} f_X(x)\,dx \tag{1.157}$$

Differentiating $F_V(v)$ with respect to $v$, we obtain

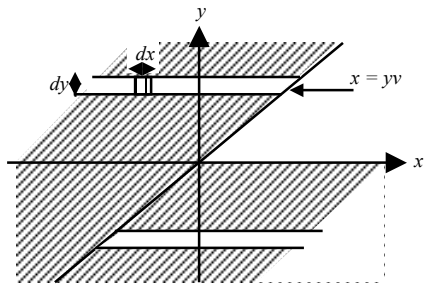


**Figure 1.28** Domain of $V = X / Y$.

$$f_V(v) = \int_0^\infty y\, f_X(v\,y)\, f_Y(y)\, dy - \int_{-\infty}^0 y\, f_X(v\,y)\, f_Y(y)\, dy$$
$$= \int_{-\infty}^\infty |\, y\,|\, f_X(v\,y)\, f_Y(y)\, dy \tag{1.158}$$

If

$$M = \max(X, Y) \tag{1.159}$$

then the region corresponding to max $(x, y) \le m$ is as shown in Figure 1.29. In this case, $F_M(m)$ is given by

$$F_M(m) = F_{XY}(m, m) = F_X(m)\, F_Y(m) \tag{1.160}$$

where $F_X(m)$ and $F_Y(m)$ are the marginal distribution functions of the random variables $X$ and $Y$, respectively. Hence, taking the derivative of $F_M(m)$ with respect to $m$, we obtain

$$f_M(m) = F_X(m)\, f_Y(m) + f_X(m)\, F_Y(m) \tag{1.161}$$

If

$$N = \min(X, Y) \tag{1.162}$$

then the region of min $(x, y) \le n$ in the $x$–$y$ plane is as shown in Figure 1.30. The distribution function $F_N(n)$ of $N$ is given by

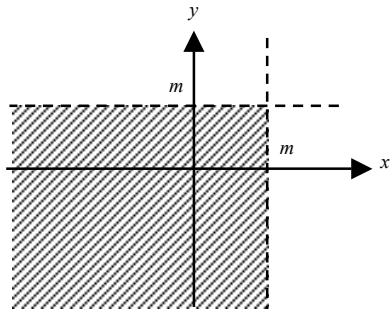


**Figure 1.29** Domain of $M = \max(x, y)$.

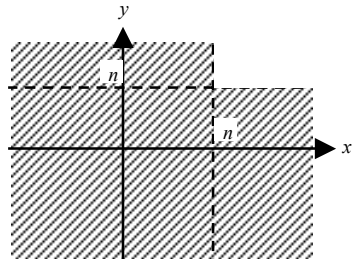


**Figure 1.30** Domain of $N = \min(X,Y)$.

$$\begin{aligned} F_N(n) &= F_{XY}(n,\infty) + F_{XY}(\infty,n) - F_{XY}(n,n) \\ &= F_X(n) + F_Y(n) - F_X(n)\,F_Y(n) \end{aligned} \tag{1.163}$$

where $F_{XY}(x,y)$ is the joint distribution function of $X$ and $Y$, and where the assumption of $X$ and $Y$ being independent is still maintained. Hence, taking the derivation of (1.163) with respect to $n$, we obtain

$$\begin{aligned} f_N(n) &= f_X(n) + f_Y(n) - F_X(n)\,f_Y(n) - f_X(n)\,F_Y(n) \\ &= f_X(n)\left[1 - F_Y(n)\right] + f_Y(n)\left[1 - F_X(n)\right] \end{aligned} \tag{1.164}$$

If now the random variables $X$ and $Y$ are not statistically independent, then the density functions of $U$, $V$, $M$, and $N$ are given by

$$f_U(u) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_{XY}\left(x, \frac{u}{x}\right) dx = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{XY}\left(\frac{u}{y}, y\right) dy \tag{1.165}$$

$$f_V(v) = \int_{-\infty}^{\infty} |y|\, f_{XY}(vy, y)\, dy \tag{1.166}$$

$$f_M(m) = \int_{-\infty}^{m} f_{XY}(m,y)\, dy + \int_{-\infty}^{m} f_{XY}(x,m)\, dx \tag{1.167}$$

and

$$f_N(n) = f_X(n) + f_Y(n) - \int_{-\infty}^{n} f_{XY}(n,y)\, dy - \int_{-\infty}^{n} f_{XY}(x,n)\, dx \tag{1.168}$$

**Example 1.21**

Find the density function of $U = XY$, where $X$ and $Y$ are independent random variables with density functions

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0\ , & \text{otherwise} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} \dfrac{3y^2}{8}, & 0 \le y \le 1 \\ 0\ , & \text{otherwise} \end{cases}$$

*Solution*

Using (1.155) and the given boundaries of $x$ and $y$, we obtain

$$f_U(u) = \int_{u/2}^{1} \frac{1}{x} \cdot 2x \cdot \frac{3}{8}\left(\frac{u}{x}\right)^2 dx = \frac{3}{4}u^2(2-u), \ 0 \le u \le 2$$

### 1.6.3 Two Functions of Two Random Variables

In this section, we extend the concept of the fundamental theorem of one function of a random variable to two functions of two random variables. Let

$$Y_1 = g_1(X_1, X_2) \quad \text{and} \quad Y_2 = g_2(X_1, X_2) \tag{1.169}$$

where $g_1(X_1, X_2)$ and $g_2(X_1, X_2)$ are two functions of two random variables $X_1$ and $X_2$ with real values $x_1$, $x_2$. The joint density function of $X_1$ and $X_2$ is $f_{X_1X_2}(x_1, x_2)$. Our aim is to obtain the joint distribution function $F_{X_1X_2}(x_1, x_2)$ and the joint density function $f_{X_1X_2}(x_1, x_2)$ in terms of the functions $y_1 = g_1(x_1, x_2)$, $y_2 = g_2(x_1, x_2)$, and the joint density function $f_{X_1X_2}(x_1, x_2)$ of $X_1$ and $X_2$. The *Jacobian* of the transformation $(x_1, x_2)$ onto $(y_1, y_2)$ is given by

$$J(x_1, x_2) = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial g_1(x,y)}{\partial x_1} & \dfrac{\partial g_1(x,y)}{\partial x_2} \\ \dfrac{\partial g_2(x,y)}{\partial x_1} & \dfrac{\partial g_2(x,y)}{\partial x_2} \end{vmatrix} \tag{1.170}$$

where $|J(x_1, x_2)| \neq 0$, and $\partial / \partial x$ denotes partial derivative. This Jacobian $J(x_1, x_2)$ is also denoted $J(y_1, y_2/x_1, x_2)$ or $\partial(x_1, x_2)/\partial(y_1, y_2)$. Then, the density function $f_{Y_1Y_2}(y_1, y_2)$ can be shown to be

$$f_{Y_1Y_2}(y_1, y_2) = \frac{f_{X_1X_2}(x_1, x_2)}{|J(x_1, x_2)|} \tag{1.171}$$

However, if the pairs $(x_{11}, x_{21})$, $(x_{12}, x_{22})$, … , $(x_{1n}, x_{2n})$ are all real solutions of equations $y_1 = g_1(x_{1i}, x_{2i})$ and $y_2 = g_2(x_{1i}, x_{2i})$, $i = 1, 2, \ldots, n$ (i.e., the $n$ points in the $x_1$–$x_2$ plane map into one point in the $y_1 - y_2$ plane), then $f_{Y_1Y_2}(y_1, y_2)$ is given by

$$f_{Y_1Y_2}(y_1, y_2) = \frac{f_{X_1X_2}(x_{11}, x_{21})}{|J(x_{11}, x_{21})|} + \frac{f_{X_1X_2}(x_{12}, x_{22})}{|J(x_{12}, x_{22})|} + \text{K} + \frac{f_{X_1X_2}(x_{1n}, x_{2n})}{|J(x_{1n}, x_{2n})|} \tag{1.172a}$$

$$= \sum_{i=1}^{n} \frac{f_{X_1X_2}(x_{1i}, x_{2i})}{|J_i(x_{1i}, x_{2i})|}, \quad i = 1, 2, \text{K}, n \tag{1.172b}$$

$J_i(x_{1i}, x_{2i})$, $i = 1, 2, \ldots, n$, are the Jacobians evaluated at the $i$th root. The Jacobian of the inverse transformation is given by

$$J(y_1, y_2) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} \tag{1.173}$$

where $J(y_1, y_2)$ can also be written as $J(x_1, x_2/y_1, y_2)$ or $\partial(y_1, y_2)/\partial(x_1, x_2)$. For the case of $n$ real roots $(x_{11}, x_{21})$, $(x_{12}, x_{22})$, …, $(x_{1n}, x_{2n})$, the joint density function is then

$$f_{Y_1Y_2}(y_1, y_2) = \sum_{i=1}^{n} |J_i(y_1, y_2)|^{-1} f_{X_1X_2}(x_{1i}, x_{2i}) \tag{1.174}$$

where the subscript $i$, $i = 1, 2, \ldots, n$, indicates that the Jacobians are evaluated at the $i$th root, and that $J_i(y_1, y_2)$ is as defined in (1.173). The distribution function $F_{Y_1Y_2}(y_1, y_2)$ is just

$$F_{Y_1Y_2}(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2) = \iint_D f_{X_1X_2}(x_1, x_2)\, dx_1\, dx_2 \tag{1.175}$$

where $D$ denotes the region in the $x_1$–$x_2$ plane for which $g_1(x_1, x_2) \le y_1$ and $g_2(x_1, x_2) \le y_2$. Note also that

$$f_{Y_1Y_2}(y_1, y_2) = \frac{\partial^2 F_{Y_1Y_2}(y_1, y_2)}{\partial y_1 \partial y_2} \tag{1.176}$$

**Example 1.22**

Consider the standard example given in many references where $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $Y_2 = X_1 / X_2$. The problem is to find the density function $f_{Y_1Y_2}(y_1, y_2)$ in terms of the given density function $f_{X_1X_2}(x_1, x_2)$.

*Solution*

We shall solve this example by giving more details to eliminate all ambiguities. From (1.170), we first need to determine the Jacobian of this transformation $y_1 = g_1(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ and $y_2 = g_2(x_1, x_2) = x_1 / x_2$, which is given by

$$J(x_1, x_2) = \begin{vmatrix} \dfrac{x_1}{(x_1^2 + x_2^2)^{\frac{1}{2}}} & \dfrac{x_2}{(x_1^2 + x_2^2)^{\frac{1}{2}}} \\ \dfrac{1}{x_2} & -\dfrac{x_1}{x_2} \end{vmatrix} = -\left(\frac{x_1}{x_2}\right)^2 \frac{1}{(x_1^2 + x_2^2)^{\frac{1}{2}}} - \frac{1}{(x_1^2 + x_2^2)^{\frac{1}{2}}}$$

$$= -y_2^2 \frac{1}{y_1} - \frac{1}{y_1} = -\frac{y_2^2 + 1}{y_1}$$

Thus, $|J(x_1, x_2)| = (y_2^2 + 1)/y_1$. Solving for the roots of the two functions $y_1$ and $y_2$, we obtain

$$y_1 = (x_1^2 + x_2^2)^{\frac{1}{2}} = \left[x_1^2\left(1 + \frac{x_2^2}{x_1^2}\right)\right]^{\frac{1}{2}} = \pm x_1\left(1 + \frac{x_2^2}{x_1^2}\right)^{\frac{1}{2}} = \pm x_1\left(1 + \frac{1}{y_2^2}\right)^{\frac{1}{2}}$$

$$= \pm x_1\left(\frac{y_2^2 + 1}{y_2^2}\right) \quad \Rightarrow \quad x_1 = \pm \frac{y_1 y_2}{(y_2^2 + 1)^{\frac{1}{2}}}$$

That is, we have two roots, $x_{11} = y_1 y_2 / (y_2^2 + 1)^{\frac{1}{2}}$ and $x_{12} = -y_1 y_2 / (y_2^2 + 1)^{\frac{1}{2}}$. Using the same approach to solve for $x_2$, we obtain $x_2 = \pm y_1 / (y_2^2 + 1)^{\frac{1}{2}}$; that is,

$x_{21} = y_1 \Big/ \left(y_2^2 + 1\right)^{\frac{1}{2}}$ and $x_{22} = -y_1 \Big/ \left(y_2^2 + 1\right)^{\frac{1}{2}}$. Note that in reality we have four possible pairs ($x_{1i}$ , $x_{2i}$), $i$ = 1, 2, 3, 4, but we have to pair the plus signs together and the minus signs together, since $y_1$ must be nonnegative to have real solutions. Hence, using (1.172a), the joint density function of $Y_1$ and $Y_2$ becomes

$$f_{Y_1Y_2}(y_1 , y_2) = \begin{cases} \left[ f_{X_1X_2}\left( \dfrac{y_1y_2}{\left(y_2^2+1\right)^{\frac{1}{2}}} , \dfrac{y_1}{\left(y_2^2+1\right)^{\frac{1}{2}}} \right) + f_{X_2X_2}\left( \dfrac{-y_1y_2}{\left(y_2^2+1\right)^{\frac{1}{2}}} , \dfrac{-y_1}{\left(y_2^2+1\right)^{\frac{1}{2}}} \right) \right] \cdot \dfrac{y_1}{y_2^2+1}, & y_1 > 0 \\ 0 , & y_1 < 0 \end{cases}$$

**Example 1.23**

Let ($X_1$, $X_2$) be a two-dimensional random variable with density function $f_{X_1X_2}(x_1 , x_2) = 2/\pi$ in the region shown in Figure 1.31. Define the transformation ($R$ , $\Theta$) with $X_1 = R \cos \Theta$ and $X_2 = R \sin \Theta$. Determine the joint density function $f_{R\Theta}(r , \theta)$.

*Solution*

In this example, the goal is to use the Jacobian of the inverse transformation, $\left| J^{-1} \right|$, given in (1.173), which could be sometimes confusing. Hence,

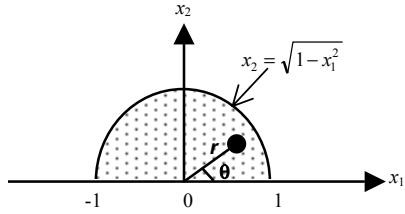


**Figure 1.31** Region of $f_{X_1X_2}(x_1, x_2)$.

$$\left| J^{-1}(r,\theta) \right| = \begin{vmatrix} \dfrac{\partial x_1}{\partial r} & \dfrac{\partial x_1}{\partial \theta} \\ \dfrac{\partial x_2}{\partial r} & \dfrac{\partial x_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \left(\cos^2\theta + \sin^2\theta\right) r = r$$

since $\cos^2\theta + \sin^2\theta = 1$. Thus, using (1.174), the joint density function of $(R, \Theta)$ becomes

$$\begin{aligned} f_{R\Theta}(r,\theta) &= \left| J^{-1}(r,\theta) \right| \; f_{X_1X_2}(x_1, x_2) \\ &= \frac{2r}{\pi} \quad \text{for} \quad 0 \le r \le 1 \quad \text{and} \quad 0 \le \theta \le \pi \end{aligned}$$

Note that the marginal density functions of $R$ and $\Theta$ are easily obtained to be

$$f_R(r) = \int_0^{\pi} \frac{2r}{\pi}\, d\theta = 2r, \quad 0 \le r \le 1$$

and

$$f_\Theta(\theta) = \int_0^{1} \frac{2r}{\pi}\, dr = \frac{1}{\pi}, \quad 0 \le \theta \le \pi$$

$R$ and $\Theta$ are independent random variables, since $f_{R\Theta}(r, \theta) = f_R(r) f_\Theta(\theta)$.

We now generalize the fundamental theorem to vectors of $n$ random variables. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be an $n$-dimensional random variable of the continuous type, with joint probability density function $f_{\boldsymbol{X}}(x_1, x_2, \ldots, x_n)$. Let $Y_1, Y_2, \ldots, Y_n$ be functions of random variables $X_1, X_2, \ldots, X_n$, given by

$$\begin{aligned} Y_1 &= g_1(X_1, X_2, \ldots, X_n) \\ Y_2 &= g_2(X_1, X_2, \ldots, X_n) \\ &\text{M} \\ Y_n &= g_n(X_1, X_2, \ldots, X_n) \end{aligned} \tag{1.177}$$

The functions $g_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, are continuous, and have partial derivatives at all $(x_1, x_2, \ldots, x_n)$. Then, the joint density function of the transformation is given by

$$f_{Y_1,Y_2,\mathrm{K},Y_n}(y_1,y_2,\mathrm{K},y_n)=\frac{1}{\left|J(x_1,x_2,\mathrm{K},x_n)\right|}f_{X_1,X_2,\mathrm{K},X_n}(x_1,x_2,\mathrm{K},x_n) \quad (1.178)$$

where the Jacobian $J(x_1,x_2,\mathrm{K},x_n)=J[(y_1,y_2,\mathrm{K},y_n)/(x_1,x_2,\mathrm{K},x_n)]$ is defined as

$$J(x_1,x_2,\mathrm{K},x_n)=\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \Lambda & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \Lambda & \frac{\partial y_2}{\partial x_n} \\ & & \mathrm{M} & \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \Lambda & \frac{\partial y_n}{\partial x_n} \end{vmatrix} \quad (1.179)$$

If the $n$-tuples $(x_{11}, x_{21}, \ldots, x_{n1})$, $(x_{12}, x_{22}, \ldots, x_{n2})$, $\ldots$, $(x_{1n}, x_{2n}, \ldots, x_{nn})$ are solutions to

$$\begin{aligned} x_1&=g_1^{-1}(y_1,y_2,\mathrm{K},y_n)\\ x_2&=g_2^{-1}(y_1,y_2,\mathrm{K},y_n)\\ &\mathrm{M}\\ x_n&=g_n^{-1}(y_1,y_2,\mathrm{K},y_n) \end{aligned} \quad (1.180)$$

The Jacobian of the inverse transformation is given by

$$J\left(\frac{x_1,x_2,\mathrm{K},x_n}{y_1,y_2,\mathrm{K},y_n}\right)=\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \Lambda & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \Lambda & \frac{\partial x_2}{\partial y_n} \\ & & \mathrm{M} & \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \Lambda & \frac{\partial x_n}{\partial y_n} \end{vmatrix} \quad (1.181)$$

where $J[(x_1, x_2, \text{K}, x_n)/(y_1, y_2, \text{K}, y_n)] = J(y_1, y_2, \text{K}, y_n)$ and $|J(x_1, x_2, \text{K}, x_n)|^{-1} = |J(y_1, y_2, \text{K}, y_n)|$. The joint density function $f_{Y_1 Y_2 \text{K} Y_n}(y_1, y_2, \text{K}, y_n)$ becomes

$$f_{Y_1 Y_2 \text{K} Y_n}(y_1, y_2, \text{K}, y_n) = \sum_{i=1}^{n} |J_i(y_1, y_2, \text{K}, y_n)|^{-1} f_{X_1 X_2 \text{K} X_n}(x_{1i}, x_{2i}, \text{K}, x_{ni}) \tag{1.182}$$

where again the subscript $i$, $i = 1, 2, \ldots, n$, indicates that the Jacobians are evaluated at the $i$th root, and $J_i(y_1, y_2, \Lambda, y_n)$ is as defined in (1.181).

## 1.7 SUMMARY

In this chapter, we have introduced the concepts of sets, probabilities, random variables, and functions of random variables. In most cases, we limited the discussions to two random variables or two functions of two random variables, but the concepts can be extended to $n$ random variables and/or $n$ functions of $n$ random variables. Due to the analogy between continuous random variables and discrete random variables, most mathematical developments were done for continuous functions. However, some points on discrete random variables were given in detail to clarify some fundamentals. In the next chapter, we present some probability distributions.

## PROBLEMS

**1.1** Determine which sets are finite and countable, or infinite and uncountable. $A = \{1, 2, 3, 4\}$, $B = \{x \mid x \text{ integer and } x < 9\}$, $C = \{x \mid x \text{ real and } 1 \le x < 3\}$, $D = \{2, 4, 7\}$, and $E = \{4, 7, 8, 9, 10\}$.

**1.2** Using the sets $A$, $B$, $D$, and $E$ of Problem 1.1, determine the following sets:
(a) $A \text{I } B$
(b) $A \text{Y } B \text{Y } D \text{Y } E$
(c) $(B \text{Y } E) \text{I } D$
(d) $B - E$
(e) $A \text{I } B \text{I } D \text{I } E$

**1.3** Let the universal set be $U = \{x \mid x \text{ integer and } 0 \le x \le 12\}$. For the subsets of $U$ given as $A = \{0, 1, 4, 6, 7, 9\}$, $B = \{x \mid x \text{ even}\}$, and $C = \{x \mid x \text{ odd}\}$, find
(a) $A \text{I } B$

(b) $(A \cup B) \cap C$
(c) $\overline{B \cup C}$
(d) $B - A$
(e) $(A \cup B) \cap (A \cup C)$
(f) $A \cap \overline{C}$
(g) $B - \overline{C}$
(h) $B \cap \overline{C}$

**1.4** Using Venn diagrams for the four sets $A$, $B$, $C$, and $D$ within the universal set $U$, show the areas corresponding to the following sets
(a) $A - B$
(b) $(A \cup B) \cap C$
(c) $A \cap B \cap C \cap D$
(d) $\overline{A}$
(e) $\overline{A \cap B}$

**1.5** Show that if $A \subset B$ and $B \subset C$, then $A \subset C$.

**1.6** Find all mutually exclusive sets defined in Problem 1.3.

**1.7** A ball is drawn at random from a box containing 10 red balls, 3 white balls, and 7 blue balls. Determine the probability that it is
(a) red
(b) white
(c) blue
(d) not red
(e) red or white

**1.8** Assume that three balls are drawn successively from the box of Problem 1.7. Find the probability that they are drawn in the order blue, white, and red, if each ball is
(a) replaced before the next draw
(b) not replaced

**1.9** In addition to Box 1 of Problem 1.7, we have another box, Box 2, containing 2 red balls, 6 white balls, and 1 blue ball. One ball is drawn from each box. Find the probability that
(a) both are red
(b) both are white
(c) one is white and one is blue

**1.10** A small box, $B_1$, contains 4 white balls and 2 black balls; another larger box, $B_2$, contains 3 white balls and 5 black balls. We first select a box, then draw a ball from the selected box. The probability of selecting the larger box is twice that of the smaller box. Find the probability that
(a) a black ball is drawn given box $B_2$
(b) a black ball is drawn given box $B_1$
(c) a black ball is drawn
(d) a white ball is drawn

**1.11** Determine the probability of obtaining three 1s in four tosses of a fair die.

**1.12** In three urns are balls, as shown in Table P1.12. The urns are not of the same size, and thus the probability of selecting Urn $A$ is 0.6, while the probability of selecting Urn $B$ is 0.2.
(a) Find the joint probability of selecting a white ball from Urn $B$.
(b) Find the probability of drawing a green ball, given that Urn $B$ has been selected.
(c) Determine the conditional probability that a red ball is drawn from Urn $C$, $P(\text{Urn } C \mid \text{red ball})$.

**1.13** An urn contains 10 balls marked 0, 1, 2, … , 9. The experiment consists of drawing $k$ balls at random with replacement. The probability of a ball being drawn at each drawing is 1/10, and the drawings are not related. Let $A$ be the event in which neither ball 0 nor ball 1 appear in the sample, and let $B$ be the event in which ball 1 does not appear in the sample but ball 2 does. Find
(a) $P(A)$
(b) $P(B)$
(c) $P(AB)$
(d) $P(A \cup B)$

**Table P1.12**
Content of Urns

| *Balls* | *Urn A* | *Urn B* | *Urn C* | *Totals* |
|---|---|---|---|---|
| Red | 30 | 30 | 40 | 100 |
| White | 50 | 30 | 20 | 100 |
| Green | 20 | 40 | 40 | 100 |
| Totals | 100 | 100 | 100 | 300 |

**1.14** Consider the function

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x} + \frac{1}{2}\delta(x-3), & x \geq 0 \\ 0 \quad , & x < 0 \end{cases}$$

(a) Sketch $f_X(x)$ and verify that it represents a density function.
(b) Calculate $P(X = 1)$, $P(X = 3)$, and $P(X \geq 1)$.

**1.15** Consider the random variable $X$ given in Example 1.12. Find
(a) the distribution function
(b) the probability that $|X| < 1$

**1.16** The density function of the variable $X$ is given by

$$f_X(x) = \begin{cases} \frac{1}{4}, & -2 \leq x \leq 2 \\ 0 \text{ , otherwise} \end{cases}$$

Determine
(a) $P(X \leq x)$
(b) $P(|X| \leq 1)$
(c) the mean and variance
(d) the characteristic function

**1.17** The random variable $X$ has a density function

$$f_X(x) = \begin{cases} x \quad , & 0 < x \leq 1 \\ 2 - x, & 1 \leq x < 2 \\ 0 \quad , & \text{otherwise} \end{cases}$$

(a) What is the probability that $1/3 < X < 3/2$?
(b) Find the mean and variance of $X$.
(c) Obtain the moment generating function of $X$.
(d) Obtain the mean of $X$ from the moment generating function and compare it with the value obtained by direct application of the definition.

**1.18** The random variable $X$ has mean $E[X] = 2/3$ and density function

$$f_X(x) = \begin{cases} \alpha + \beta x^2, & 0 \le x \le 1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

(a) Find $\alpha$ and $\beta$.
(b) Determine $E[X^2]$ and $\sigma_x^2$.

**1.19** The joint probability distribution of the two-dimensional discrete random variable $(X,Y)$ is shown in Table P1.19.
(a) Is $E[XY] = E[X]E[Y]$?
(b) Are the random variables $X$ and $Y$ independent? Justify.

**1.20** The joint density function of two random variables $X$ and $Y$ is given by

$$f_{XY}(x,y) = \begin{cases} k(x+y), & 0 \le x \le 2 \quad \text{and} \quad 0 \le y \le 2 \\ 0 \quad , & \text{otherwise} \end{cases}$$

Find
(a) $k$
(b) The marginal density functions of $X$ and $Y$
(c) $P(X < 1 | Y < 1)$
(d) $E[X]$, $E[Y]$, $E[XY]$, and $\rho_{xy}$
(e) Are $X$ and $Y$ independent?

**1.21** The joint density function of the two random variables $X$ and $Y$ is

$$f_{XY}(x,y) = \begin{cases} kxy, & 1 < x < 5 \quad \text{and} \quad 0 < y < 4 \\ 0 \quad , & \text{otherwise} \end{cases}$$

**Table P1.19**
Joint Probabilities of ( $X$, $Y$ )

| | $X$ | | |
|---|---|---|---|
| $Y$ | −1 | 0 | +1 |
| −1 | 1/12 | 1/6 | 1/12 |
| 0 | 1/6 | 0 | 1/6 |
| +1 | 1/12 | 1/6 | 1/12 |

Find
(a) The constant $k$
(b) $P(X \geq 3,\ Y \leq 2)$ and $P(1 < X < 2,\ 2 < Y < 3)$
(c) $P(1 < X < 2 \mid 2 < Y < 3)$
(d) $E[X \mid Y = y]$

**1.22** The joint density function of the two random variables $X$ and $Y$ is

$$f_{XY}(x, y) = \begin{cases} kxy, & 1 < x < 3 \quad \text{and} \quad 1 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the probability that $X + Y < 3$?
(b) Are $X$ and $Y$ independent?

**1.23** The joint density function of two random variables $X$ and $Y$ is

$$f_{XY}(x, y) = \begin{cases} 16\frac{y}{x^3}, & x > 2 \quad \text{and} \quad 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find $E[X]$ and $E[Y]$, the means of $X$ and $Y$, respectively.

**1.24** The density function of two independent random variables $X$ and $Y$ are

$$f_X(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} kye^{-3y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find
(a) $P(X + Y > 1)$
(b) $P(1 < X < 2,\ Y \geq 1)$
(c) $P(1 < X < 2)$
(d) $P(Y \geq 1)$
(e) $P(1 < X < 2 \mid Y \geq 1)$

**1.25** Find the density function of the random variable $Y = 2X$, where

$$f_X(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Compute $E[Y]$ in two ways:
(a) Directly using $f_X(x)$
(b) Using the density function of $Y$

$$f_{Y_1Y_2}(y_1, y_2) = \frac{\partial^2 F_{Y_1Y_2}(y_1, y_2)}{\partial y_1 \partial y_2} \tag{1.176}$$

**Example 1.22**

Consider the standard example given in many references where $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $Y_2 = X_1 / X_2$. The problem is to find the density function $f_{Y_1Y_2}(y_1, y_2)$ in terms of the given density function $f_{X_1X_2}(x_1, x_2)$.

*Solution*

We shall solve this example by giving more details to eliminate all ambiguities. From (1.170), we first need to determine the Jacobian of this transformation $y_1 = g_1(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ and $y_2 = g_2(x_1, x_2) = x_1 / x_2$, which is given by

$$J(x_1, x_2) = \begin{vmatrix} \dfrac{x_1}{(x_1^2 + x_2^2)^{\frac{1}{2}}} & \dfrac{x_2}{(x_1^2 + x_2^2)^{\frac{1}{2}}} \\ \dfrac{1}{x_2} & -\dfrac{x_1}{x_2} \end{vmatrix} = -\left(\frac{x_1}{x_2}\right)^2 \frac{1}{(x_1^2 + x_2^2)^{\frac{1}{2}}} - \frac{1}{(x_1^2 + x_2^2)^{\frac{1}{2}}}$$

$$= -y_2^2 \frac{1}{y_1} - \frac{1}{y_1} = -\frac{y_2^2 + 1}{y_1}$$

Thus, $|J(x_1, x_2)| = (y_2^2 + 1)/y_1$. Solving for the roots of the two functions $y_1$ and $y_2$, we obtain

$$y_1 = (x_1^2 + x_2^2)^{\frac{1}{2}} = \left[x_1^2\left(1 + \frac{x_2^2}{x_1^2}\right)\right]^{\frac{1}{2}} = \pm x_1\left(1 + \frac{x_2^2}{x_1^2}\right)^{\frac{1}{2}} = \pm x_1\left(1 + \frac{1}{y_2^2}\right)^{\frac{1}{2}}$$

$$= \pm x_1\left(\frac{y_2^2 + 1}{y_2^2}\right) \quad \Rightarrow \quad x_1 = \pm \frac{y_1 y_2}{(y_2^2 + 1)^{\frac{1}{2}}}$$

That is, we have two roots, $x_{11} = y_1 y_2 / (y_2^2 + 1)^{\frac{1}{2}}$ and $x_{12} = -y_1 y_2 / (y_2^2 + 1)^{\frac{1}{2}}$. Using the same approach to solve for $x_2$, we obtain $x_2 = \pm y_1 / (y_2^2 + 1)^{\frac{1}{2}}$; that is,

**1.30** The joint probability density function of $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases} 1, & 0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability density function of $Z = XY$.

**1.31** The joint density function of the two random variables $X$ and $Y$ is given by

$$f_{XY}(x, y) = \frac{\alpha}{\beta} e^{-\alpha x}, \quad 0 \le x < \infty, \quad \text{and} \quad 0 \le y \le \beta$$

where $\alpha$ and $\beta$ are constants.
(a) Find the marginal density $f_X(x)$ of $X$.
(b) Find the marginal density $f_Y(y)$ of $Y$.
(c) Are $X$ and $Y$ statistically independent? Justify.
(d) Determine the density function of $Z$ such that $Z = X + Y$, and sketch it.

**1.32** Let $X$ and $Y$ be two independent random variables with exponential distributions given by

$$f_X(x) = \alpha e^{-\alpha x} u(x) \quad \text{and} \quad f_Y(y) = \beta e^{-\beta y} u(y)$$

where $\alpha > 0$ and $\beta > 0$. Determine the density function of $Z = X/Y$.

**1.33** The joint probability density function of the random variables $X_1$ and $X_2$ is given by

$$f_{X_1X_2}(x_1, x_2) = \begin{cases} kx_1x_2, & 1 \le x_1 \le 3 \quad \text{and} \quad 1 \le x_2 \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Let the random variables $Y_1$ and $Y_2$ be defined as

$$Y_1 = X_1 \text{ and } Y_2 = X_1X_2^2$$

(a) Determine the constant $k$.
(b) Determine the joint density function $f_{Y_1Y_2}(y_1, y_2)$ and sketch the corresponding domain of definition.

**1.34** The joint density function of the random variables $X_1$ and $X_2$ is given by

$$f_{X_1X_2}(x_1, x_2) = \begin{cases} \alpha^2 \, e^{-\alpha(x_1+x_2)}, & x_1 > 0, \, x_2 > 0 \\ 0 \qquad\qquad\quad , & \text{otherwise} \end{cases}$$

(a) Show that $X_1$ and $X_2$ are independent.
(b) Define $Y_1 = X_1 + X_2$ and $Y_2 = X_1 / X_2$. Determine the joint density function $f_{Y_1Y_2}(y_1, y_2)$ of the transformation.

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# Chapter 2

# Distributions

## 2.1 INTRODUCTION

In the previous chapter, we have defined the concepts of probability, random variables, and statistical moments. In this chapter, we shall study some important distribution functions that are frequently encountered. Since these distributions have a wide range of applications, we shall study them in their general form, and in some cases, we give more details for particular applications. Some of the notions defined will be applied to these special distributions, which yield some standard results to be used later. In Sections 2.2 and 2.3, we present some discrete and continuous distribution functions, respectively. Special distribution functions will be presented in Section 2.4.

## 2.2 DISCRETE RANDOM VARIABLES

### 2.2.1 The Bernoulli, Binomial, and Multinomial Distributions

The simplest distribution is one with only two possible events. For example, a coin is tossed, and the events are heads or tails, which must occur with some probability. Tossing the coin $n$ times consists of a series of independent trials, each of which yields one of the two possible outcomes: heads or tails. These two possible outcomes are also referred to as "success" associated with the value 1 and "failure" associated with the value 0. Since all experiments are assumed to be identical, the outcome 1 occurs with probability $p$, whereas the outcome 0 occurs with probability $1-p$, with $0 \le p \le 1$. These are called the *Bernoulli trials*.

A random variable $X$ is said to have a *Bernoulli distribution* if for some $p$, $0 \le p \le 1$, its probability density function is given by

$$P_X(x) = \begin{cases} p^x (1-p)^{1-x}, & x = 0,1 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

$(1-p)$ is often denoted by *q*, such that $p+q=1$. Assume that in the experiment of tossing a coin *n* times, "heads" or "1" occurs in *k* trials, then "tails" or "0" occurs in $(n-k)$ trials. That is, we have

$$\underbrace{111\ldots11}_{k\ times}\ \underbrace{000\ldots00}_{n-k\ times}$$

Note that the order of which comes first, 1 or 0, is not important. What matters is the *k* number of ones and $(n-k)$ number of zeros in the *n* trials. Hence, from Chapter 1, Section 1.2.4, the *n* objects (all the 1s and 0s) can be arranged in *n*! ways. The *k* 1s can be arranged in *k*! ways, whereas the $(n-k)$ 0s can be arranged in $(n-k)$ ways. It follows that there are $n!/(n-k)!k!$ ways of arranging the *k* 1s and $(n-k)$ 0s. Note that $n!/(n-k)!k!$ is the binomial coefficient defined in (1.10). Hence, the probability of occurrence of *k* 1s is

$$\frac{n!}{(n-k)!\,k!}\,p^k q^{n-k} \tag{2.2}$$

In summary, we say that the probability of observing exactly *k* successes in *n* independent Bernoulli trials yields the *binomial distribution*. The probability of success is *p*, and the probability of failure is $q=1-p$. The random variable *X* is said to have a *binomial distribution* with parameters *n* and *p* if

$$P(X=k \text{ in } n \text{ trials})=\binom{n}{k}p^k q^{n-k} \quad \text{for } k=0,1,2,\ldots \tag{2.3}$$

The PDF of the binomial random variable *X* is given by

$$P_X(x)=\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}\,\delta(x-k) \tag{2.4}$$

where $\delta(x-k)$ is the impulse function. The distribution function would be

$$F_X(x)=\int_{-\infty}^{x}P_X(u)\,du=\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}\,u(x-k) \tag{2.5}$$

where $u(x-k)$ is the step function, and the integral of the impulse function is just the unit step function. It should be noted that the binomial power expansion is given by

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k \, q^{n-k} = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} \, p^k \, q^{n-k} \tag{2.6}$$

It can also easily be shown that the mean, variance, and characteristic function of $X$ are given by

$$E[X] = np \tag{2.7}$$

$$\text{var}(X) = npq \tag{2.8}$$

and

$$\Phi_x(\omega) = \left(p\, e^{j\omega} + q\right)^n \tag{2.9}$$

**Example 2.1**

Consider the experiment of rolling a fair die 10 times. What is the probability of obtaining a "6" twice?

*Solution*

Note that the number of rolling a die is $n = 10$, and $k = 2$ is the number of a "6" showing on the top face of the die with probability $p = 1/6$. Hence, using (2.3), the probability of obtaining a "6" twice is

$$P(X = 2) = \binom{10}{2}\left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^8 = 0.2907$$

**Example 2.2**

A receiver receives a string of 0s and 1s transmitted from a certain source. The receiver uses a majority decision rule. In other words, if the receiver acquires three symbols and out of these three symbols two or three are zeros, it will decide that these symbols represent that a 0 was transmitted. The receiver is correct only 80% of the time. What is $P(c)$, the probability of a correct decision, if the probabilities of receiving 0s and 1s are equally likely?

*Solution*

These are Bernoulli trials, with $P(A) = p$ being the probability that event $A$ occurs in a given trial. Define $D$ as the event decide 0 or 1. $P(D) = 0.8$. The number of symbols received is $n = 3$. From (2.3), we have

$$P(\text{ 0 correct decisions }) = \binom{3}{0}(0.8)^0\ (1-0.8)^3 = 0.008$$

$$P(\text{ 1 correct decision }) = \binom{3}{1}(0.8)^1\ (1-0.8)^2 = 0.096$$

$$P(\text{ 2 correct decisions }) = \binom{3}{2}(0.8)^2\ (1-0.8)^1 = 0.384$$

$$P(\text{ 3 correct decisions }) = \binom{3}{3}(0.8)^3\ (1-0.8)^0 = 0.512$$

Therefore, the probability of a correct decision is given by

$$P(c) = P(D=2) + P(D=3) = 0.896$$

In the binomial distribution, the experiment is repeated $n$ times but we only have two possible events. Suppose now that we still repeat the experiment $n$ times independently, but for each experiment we have $k$ mutually exclusive events $A_1, A_2, \ldots, A_k$. Let $P(A_i) = P_i$ and suppose that $P_i$, $i = 1, 2, \ldots, k$, remains constant for all $n$ repetitions, and $P_1 + P_2 + \ldots + P_k = 1$. Define the random variables $X_1, X_2, \ldots, X_k$, such that $X_i = n_i$, $i = 1, 2, \ldots, k$, is the number of times $A_i$ occurs in $n$ repetitions. Then, $n = n_1 + n_2 + \ldots + n_k$, and the joint probability that $X_1 = n_1, X_2 = n_2, \ldots, X_k = n_k$, is given by

$$P(X_1 = n_1, X_2 = n_2, \ldots, X_k = n_k) = \frac{n!}{n_1!\, n_2!\, \ldots\, n_k!} P_1^{n_1}\ P_2^{n_2}\ \ldots\ P_k^{n_k} \quad (2.10)$$

Note that the random variables $X_1, X_2, \cdots, X_k$ are not independent. A random variable $(X_1, X_2, \cdots, X_k)$ with a distribution given as in (2.10) is said to have a *multinomial distribution*.

### 2.2.2 The Geometric and Pascal Distributions

Consider the experiment of tossing a coin as described earlier. The probability of occurrence of some event, say, heads or success, is $P(A) = p$, and the probability of nonoccurrence (or failure) is $P(\overline{A}) = 1 - p = q$. In the binomial distribution, we repeated the experiment $n$ times, and we calculated the probability of occurrence of $k$ successes out of $n$ Bernoulli trials. The experiment now is a little different in

the sense that we continue tossing a coin until we obtain the event $A$ (heads or success) for the first time, then the experiment stops. Hence, the number of trials $n$ in the binomial distribution is fixed, while in this new experiment it is a random variable, since we do not know when we stop the experiment. We now define the *geometric* distribution.

Let $X$ be a random variable representing the repetitions of an experiment until the *first* occurrence of an event $A$ at the $k$th trial. Hence, when $X$ assumes the values 1, 2, … , $k-1$, the results of the repetitions of the experiment are $\overline{A}$. Then, the probability of occurrence of the event $A$ for the first time at the $k$th trial $X = k$ is given by

$$P(X=k) = P_X(k) = \begin{cases} (1-p)^{k-1}\, p, & k = 0, 1, 2, \ldots \\ 0 \quad\quad\quad , & \text{otherwise} \end{cases} \tag{2.11}$$

The random variable $X$ is said to have a *geometric* distribution given by (2.11) with $0 \le p \le 1$ and $1 - p = q$. The mean of $X$ is given by

$$\begin{aligned} E[X] &= \sum_{k=1}^{\infty} k\, P(X=k) = \sum_{k=1}^{\infty} k\, p\, q^{k-1} = p \sum_{k=1}^{\infty} k\, q^{k-1} \\ &= p \sum_{k=1}^{\infty} \frac{d}{dq}\left(q^k\right) = p\, \frac{d}{dq} \sum_{k=1}^{\infty} q^k \end{aligned} \tag{2.12}$$

where $d/dq$ denotes derivative, and the infinite series is known to converge to

$$\sum_{k=1}^{\infty} q^k = \frac{1}{1-q} \quad \text{for} \quad 0 < q < 1 \tag{2.13}$$

Hence, the mean of $X$ becomes

$$E[X] = p\, \frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{p}{(1-q)^2} = \frac{1}{p} \tag{2.14}$$

Similarly, we can show that the variance of $X$ is

$$\text{var}[X] = \frac{q}{p^2} \tag{2.15}$$

The moment generating function of the geometric distribution can be shown to be

$$M_x(t) = \frac{p}{1 - q\,e^t} \tag{2.16}$$

If we now consider the same experiment that gave us the geometric distribution, the experiment does not stop at the first occurrence of the event $A$, but when the event $A$ occurs $r$ times. In this case, at $X = k-1$ trials, we have $r-1$ occurrences of the event $A$, and at $X = r$, the $r$th event occurs. Hence,

$$P(X = k) = \binom{k-1}{r-1} p^r\, q^{k-r}, \quad k = r, r+1, \ldots \tag{2.17}$$

$X$ is said to have the *Pascal* distribution. Note that when $r = 1$ in (2.17), we obtain the geometrical distribution given in (2.11). Often, the Pascal distribution is referred to as the *negative binomial* distribution. In this case, we say we have $x$ failures corresponding to $r-1$ successes at the $(k-1)$th trial. At the $k$th trial, we must have the $r$th success. Hence, the probability of $x$ failures is given by

$$P(X = x) = \binom{x+r-1}{x} p^r q^x, \quad x = 1, 2, \ldots \tag{2.18}$$

or, the probability of the $r$th success at the $k = x + r$ trial, knowing that at $k - 1 = x + r - 1$ we have $r-1$ successes, is

$$P(X = x) = \binom{x+r-1}{r-1} p^r\, q^x, \; x = 0, 1, 2, \ldots \tag{2.19}$$

Note that (2.18) is equivalent to (2.19), since

$$\binom{x+r-1}{r-1} = \binom{x+r-1}{x} \tag{2.20}$$

It should be noted that (2.17) also may be written as

$$P(X = x) = \binom{-r}{x} p^r\, (-q)^x, \; x = 0, 1, 2, \ldots \tag{2.21}$$

which yields that

$$\sum_{x=0}^{\infty} P(X = x) = 1 \tag{2.22}$$

since

$$\sum_{x=0}^{\infty} \binom{-r}{x} (-q)^x = (1-q)^{-r} = p^{-r} \tag{2.23}$$

It is because of the negative exponent $(-r)$ in (2.23) that we call this distribution a *negative binomial*. It is important to observe that in (2.19) we are interested in the distribution of the number of trials required to get $r$ successes with $k = x + r$, whereas in (2.18) we are interested in the number of failures. In other words, the distribution of (2.17) can be defined as $Y = X + r$, with $X$ denoting the number of failures before the $r$th success. Hence,

$$P(Y = y) = \binom{y-1}{r-1} p^r \, q^{y-r}, \quad y = r, r+1, \ldots \tag{2.24}$$

The means of $X$ and $Y$ can be shown to be

$$E[X] = r\frac{q}{p} \tag{2.25}$$

and

$$E[Y] = E[X] + r = \frac{r}{p} \tag{2.26}$$

whereas the variances of $X$ and $Y$ are given by

$$\text{var}[X] = \text{var}[Y] = r\,\frac{q}{p^2} \tag{2.27}$$

The moment generating function of $X$ can be obtained to be

$$\begin{aligned} M_x(t) &= \sum_{x=0}^{\infty} \binom{x+r-1}{x} p^r \, q^x \, e^{tx} = \sum_{x=0}^{\infty} \binom{-r}{x} p^r \, (-q)^x \, e^{tx} \\ &= p^r \sum_{x=0}^{\infty} \binom{-r}{x} \left(-q\,e^t\right)^x = p^r \left(1 - q\,e^t\right)^{-r} \quad \text{for } q\,e^t < 1 \end{aligned} \tag{2.28}$$

whereas the moment generating function of $Y$ can be shown to be

$$M_y(t) = \left( \frac{p\, e^t}{1 - q\, e^t} \right)^r \quad \text{for } q\, e^t < 1 \tag{2.29}$$

We conclude this section by giving the relationship between the binomial distribution and the Pascal distribution. If $X$ is a binomial distribution as defined in (2.4), and $Y$ is a Pascal distribution as defined in (2.17), then

$$P(X \geq r) = P(Y \leq n) \tag{2.30}$$

That is, if there are $r$ or more successes in the first $n$ trials, then the number of trials to obtain the first $r$ successes is at most $n$. Also,

$$P(X < r) = P(Y > n) \tag{2.31}$$

That is, if there are less than $r$ successes in the first $n$ trials, then we need at least $n$ trials to obtain the first $r$ successes.

### 2.2.3 The Hypergeometric Distribution

Suppose an urn containing $N$ balls, $r$ of which are white and the other $N - r$ balls are of other colors. The experiment consists of drawing $n$ balls, where $n \leq N$. As each ball is drawn, its color is noted and *replaced* in the urn. A success is when a white ball is drawn. Let $X$ be the random variable representing white balls drawn (successes) in $n$ trials. Then, the probability of obtaining $k$ successes in $n$ trials is given by

$$P(X = k) = \binom{n}{k} \left( \frac{r}{N} \right)^k \left( \frac{N - r}{N} \right)^{n-k} = \binom{n}{k} \frac{r^k\, (N - r)^{n-k}}{N^n} \tag{2.32}$$

since $p = r / N$ and $q = 1 - p = (N - r) / N$. This is called a *sampling with replacement*. If now, as each ball is drawn, its color is noted but it is *not replaced* in the urn, we have a *sampling without replacement*. In this case, the probability of obtaining $k$ white balls (successes) in $n$ trials is given by

$$P(X = k) = \frac{\binom{r}{k} \binom{N - r}{n - k}}{\binom{N}{n}}, \quad k = 0, 1, 2, \ldots \tag{2.33}$$

A discrete random variable having the distribution given in (2.33) is said to have a *Hypergeometric distribution*. Note that $k$ cannot exceed $n$ or $r$; that is,

$$k \le \min(n,\ r) \tag{2.34}$$

The mean and variance of the Hypergeometric distribution $X$ can be shown to be

$$E[X] = \frac{n}{N} r \tag{2.35}$$

and

$$\text{var}[X] = \frac{n r}{N^2 (N-1)} (N-r)(N-n) \tag{2.36}$$

The mean-square value is also given by

$$E\left[X^2\right] = \frac{r(r-1)}{N(N-1)} n(n-1) + \frac{nr}{N} \tag{2.37}$$

Computing the probability of $k$ white balls in $n$ trials without replacement, given by (2.33), we have

$$\begin{aligned} P(X=k) &= \frac{\binom{r}{k}\binom{N-r}{n-k}}{\binom{N}{n}} = \frac{r!}{k!(r-k)!} \frac{(N-r)!}{(n-k)!(N-r-n+k)!} \frac{n!(N-n)!}{N!} \\ &= \frac{1}{k!} r(r-1)\ldots(r-k+1) \frac{1}{(n-k)!} (N-r)(N-r-1)\ldots(N-r-n+k+1) \\ &\quad \cdot n! \frac{1}{N(N-1)\ldots(N-n+1)} \\ &= \frac{n!}{k!(n-k)!} r(r-1)\ldots(r-k+1) \frac{(N-r)(N-r-1)\ldots(N-r-n+k+1)}{N(N-1)\ldots(N-n+1)} \\ &= \binom{n}{k} \frac{r^k}{N^n} (N-r)^{n-k} \frac{1\left(1-\frac{1}{r}\right)\ldots\left(1-\frac{k-1}{r}\right)}{1\left(1-\frac{1}{N}\right)\ldots\left(1-\frac{n-1}{N}\right)} \frac{1\left(1-\frac{1}{N-r}\right)\ldots\left(1-\frac{n-k-1}{N-r}\right)}{} \end{aligned} \tag{2.38}$$

Let the proportion of white balls in the urn before any drawing be $r/N = p$, and the proportion of the other balls is $1-p=(N-r)/N=q$. Then, (2.38) becomes

$$P(X=k)=\binom{n}{k}p^k\ q^{n-k}\left(1-\frac{1}{r}\right)\left(1-\frac{2}{r}\right)\cdots\left(1-\frac{k-1}{r}\right) \cdot\frac{\left(1-\frac{1}{N-r}\right)\left(1-\frac{2}{N-r}\right)\cdots\left(1-\frac{n-k-1}{N-r}\right)}{\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)} \tag{2.39}$$

The mean and variance in terms of the proportions $p$ and $q$ are given by

$$E[X]=n\,p \tag{2.40}$$

and

$$\mathrm{var}[X]=n\frac{r}{N}\frac{N-r}{N}\frac{N-n}{N-1}=n\ p\ q\ \frac{N-n}{N-1} \tag{2.41}$$

When $N$ goes to infinity ($N$ very large compared to $n$), the mean and variance become

$$E[X]=n\,p \tag{2.42}$$

and

$$\mathrm{var}[X]=n\,p\,q \tag{2.43}$$

whereas the probability of $k$ successes in $n$ trials without replacement given by (2.38) becomes

$$P(X=k)=\binom{n}{k}p^k\ q^{n-k} \tag{2.44}$$

That is, we obtain the result given by (2.32), which is sampling with replacement. This makes sense intuitively, since for a very large $N$, drawing a ball without replacement does not affect the sample size, and the experiment is similar to drawing a ball with replacement.

**Example 2.3**

An urn contains five white balls, three black balls, and three red balls. The experiment is to draw a ball and note its color. Find the probability of obtaining the third white ball in the seventh trial, knowing that the ball drawn is not replaced in the urn.

*Solution*

This is the hypergeometric distribution with $N = 11$ balls; $r = 5$ white balls, and $N - r = 6$ other colors. The probability of obtaining $k = 3$ white balls (successes) in $n = 7$ trials is given by (2.33) to be

$$P(X = 3 \text{ white balls in 7 trials}) = \frac{\binom{5}{3}\binom{6}{4}}{\binom{11}{7}} = \frac{5}{11} = 0.4545$$

### 2.2.4 The Poisson Distribution

In many applications we are concerned about the number of occurrences of an event in a given period of time $t$. Let the occurrence (or nonoccurrence) of the event in any interval be independent of its occurrence (nonoccurrence) in another interval. Furthermore, let the probability of occurrence of the event in a given period be the same, irrespective of the starting or ending of the period. Then we say that the distribution of $X$, the number of occurrences of the event in the time period $t$, is given by a *Poisson distribution*. Applications of such a random phenomenon may include the occurrence of the telephone traffic, random failures of equipment, disintegration of radioactive material, claims in an insurance company, or arrival of customers in a service facility.

Let $X$ be a discrete random variable assuming values $0, 1, 2, \ldots, n, \ldots$ and having parameter $\lambda$, $\lambda > 0$. Then, if

$$P(X = k) = P_X(k) = \begin{cases} e^{-\lambda}\dfrac{\lambda^k}{k!}, & k = 0, 1, 2, \ldots, \text{and } \lambda > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.45}$$

then we say that $X$ has a *Poisson distribution*. The probability density function and the cumulative distribution function are

$$P_X(k) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\delta(x - k) \tag{2.46}$$

and

$$F_X(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} u(x-k) \tag{2.47}$$

where $\delta(x)$ and $u(x)$ are the unit impulse function and the unit step function, respectively.

The mean and variance of $X$ are equal and can be computed to be

$$E[X] = \sigma_x^2 = \lambda \tag{2.48}$$

while the mean-square value is $E[X^2] = \lambda^2 + \lambda$. It can also be shown that the moment generating function and characteristic function of the random variable $X$ are

$$M_x(t) = \exp\left[\lambda\left(e^t - 1\right)\right] \tag{2.49}$$

and

$$\Phi_x(t) = \exp\left[\lambda\left(e^{j\omega} - 1\right)\right] \tag{2.50}$$

**Example 2.4**

Let $X$ and $Y$ be two independent random variables having Poisson distributions with parameters $\lambda_1$ and $\lambda_2$, respectively. Show that the distribution of $X + Y$ is a Poisson distribution, and determine its parameter.

*Solution*

For $n > 0$, the distribution of $X + Y$ is given by

$$\begin{aligned} P(X+Y \le n) &= \sum_{k=0}^{n} P(X=k, Y=n-k) = \sum_{k=0}^{n} P(X=k) P(Y=n-k) \\ &= \sum_{k=0}^{n} e^{-\lambda_1} \frac{\lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{n-k!} = e^{-(\lambda_1+\lambda_2)} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k} \\ &= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!} \end{aligned}$$

where we used the binomial expansion given in (2.6). Hence, the distribution of $X + Y$ is Poisson with parameter $\lambda = \lambda_1+\lambda_2$.

The Poisson distribution is an approximation of the binomial distribution as the number of trials goes to infinity (and, solving the limit, $np = \lambda$ ). Consider a binomial distribution with parameters $n$ and $p$. The probability of $X = k$ in the binomial distribution is given by

$$P(X = k) = \binom{n}{k} p^k \, (1-p)^{n-k} \tag{2.51}$$

with mean $\lambda = np$. Then, taking the limit as $n \to \infty$ and assuming $p = \lambda / n$ to be very small, we have

$$\lim_{n\to\infty} \binom{n}{k} p^k \, (1-p)^{n-k} = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} \tag{2.52}$$

using the result that

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x \tag{2.53}$$

then,

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \tag{2.54}$$

and

$$\begin{aligned} \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k e^{-\lambda} &= \lim_{n\to\infty} \lambda^k \, e^{-\lambda} \, \frac{1}{k!} \frac{n(n-1)\ldots(n-k+1)}{n^k} \\ &= \lim_{n\to\infty} e^{-\lambda} \, \frac{\lambda^k}{k!} \left[ \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right) \right] \\ &= e^{-\lambda} \, \frac{\lambda^k}{k!} \end{aligned} \tag{2.55}$$

since the term between the brackets goes to one. Note also, from Section 2.2.3, the hypergeometric distribution can be approximated to a binomial distribution, and thus to the Poisson distribution.

## 2.3 CONTINUOUS RANDOM VARIABLES

### 2.3.1 The Uniform Distribution

A random variable $X$ is said to be uniformly distributed on the interval from $a$ to $b$, $a < b$, as shown in Figure 2.1, if its density function is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases} \tag{2.56}$$

The distribution function of $X$, shown in Figure 2.2, is given by

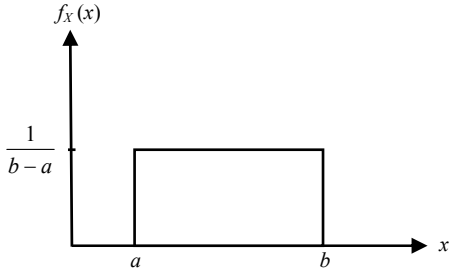


**Figure 2.1** Uniform density function.

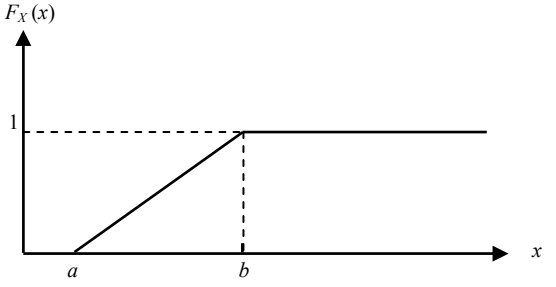


**Figure 2.2** Distribution function of the uniform random variable.

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)\,du = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases} \tag{2.57}$$

The mean, variance, and characteristic function of $X$ are, respectively,

$$E[X] = \frac{1}{2}(a+b) \tag{2.58}$$

$$\sigma_x^2 = \frac{1}{12}(b-a)^2 \tag{2.59}$$

and

$$\Phi_x(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)} \tag{2.60}$$

### 2.3.2 The Normal Distribution

One of the most important continuous random variables of a probability distribution is the normal distribution. Often called the *Gaussian* distribution, it is shown in Figure 2.3. The density function is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] \quad \text{for all } x \tag{2.61}$$

where $m$ and $\sigma$ are, respectively, the mean and standard deviation of $X$ and satisfy

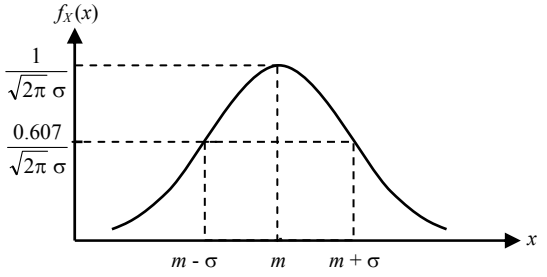


**Figure 2.3** Normal density function.

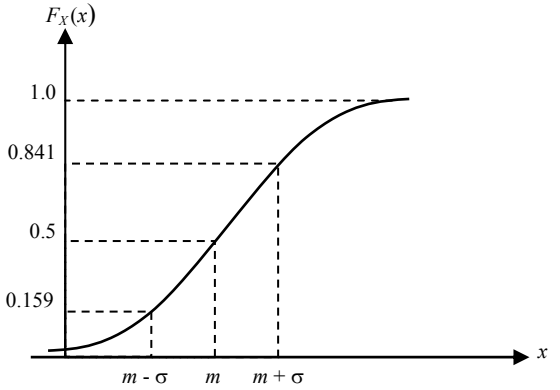


**Figure 2.4** Distribution function of the normal.

the conditions $-\infty < m < \infty$ and $\sigma > 0$. The corresponding distribution function, as shown in Figure 2.4, is given by

$$F_X(x) = P(X \le x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} \exp\left[-\frac{(u-m)^2}{2\sigma^2}\right] du \tag{2.62}$$

The distribution function can be determined in terms of the *error function*. The error function denoted by erf(·) is defined in many different ways in the literature. We define the error function as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^2}\, du \tag{2.63}$$

Additional information on the error function and its tabulated values are given in the Appendix. Let $u = (x - m)/\sigma$ in (2.61); then $du = dx/\sigma$, and the distribution function becomes

$$F_X(x) = P(X \le x) \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du = I(x) \tag{2.64}$$

The values of $I(x)$ are also tabulated in the Appendix. The distribution given in (2.64) can be rewritten as

$$F_X(x) = P(X \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{u^2}{2}}\, du + \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-\frac{u^2}{2}}\, du = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-\frac{u^2}{2}}\, du \tag{2.65}$$

Using the definition of the error function given in (2.63), the distribution function becomes

$$F_X(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \tag{2.66}$$

We define the standard normal distribution as the normal distribution with mean $m = 0$ and variance $\sigma^2 = 1$, denoted $\mathcal{N}(0,1)$, and expressed as

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{2.67}$$

The corresponding distribution is given in (2.66) in terms of the error function. Other important results that we need to define are the *complementary error function* and the *Q-function* given by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-u^2}\, du \tag{2.68}$$

such that

$$\operatorname{erfc}(x) = 1 - \operatorname{erfc}(x) \tag{2.69}$$

and

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{u^2}{2}}\, du \tag{2.70}$$

where

$$Q(0) = \frac{1}{2} \tag{2.71}$$

and

$$Q(-x) = 1 - Q(x) \quad \text{for} \quad x \geq 0 \tag{2.72}$$

Note that, using (2.67) and (2.70), the $Q$-function can be written in terms of the error function to be

$$Q(x) = \frac{1}{2}\left[ 1 - \text{erf}\left( \frac{x}{\sqrt{2}} \right) \right] \tag{2.73a}$$

$$= \frac{1}{2} \text{erfc}\left( \frac{x}{\sqrt{2}} \right) \tag{2.73b}$$

Also note that

$$I(x) + Q(x) = 1 \tag{2.74}$$

and

$$Q(x) \cong \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \text{for } x > 4 \tag{2.75}$$

In some books, $Q(x)$ defined in (2.70) is denoted $\text{erfc}_*(x)$, while $I(x)$ in (2.64) is denoted $\text{erfc}_*(x)$, and thus $\text{erf}_*(x) + \text{erfc}_*(x) = 1$ as in (2.74).

The moment generating function is known to be

$$M_x(t) = E\left[ e^{tX} \right] = \int_{-\infty}^{\infty} f_X(x) e^{tx} \, dx = \exp\left( mt + \frac{\sigma^2 t^2}{2} \right) \tag{2.76}$$

whereas the characteristic function is

$$\Phi_x(\omega) = E\left[ e^{j\omega X} \right] = \exp\left( jm\omega - \frac{\sigma^2 \omega^2}{2} \right) \tag{2.77}$$

The moments can be obtained from the characteristic function to be

$$E\left[ X^n \right] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_x(\omega)\big|_{\omega=0} = n! \sum_{k=0}^{K} \frac{m^{n-2k} \, \sigma^{2k}}{2^k \, k! (n-2k)!} \tag{2.78a}$$

where

$$K = \begin{cases} \dfrac{n}{2} & \text{for } n \text{ even} \\ \dfrac{n-1}{2} & \text{for } n \text{ odd} \end{cases} \tag{2.78b}$$

If the random variable is zero mean, the characteristic function is

$$\Phi_x(x) = e^{-\frac{1}{2}\sigma^2\omega^2} = 1 - \frac{\sigma^2\omega^2}{2} + \frac{1}{2!}\frac{\sigma^4\omega^4}{4} - \frac{1}{3!}\frac{\sigma^6\omega^6}{8} + \ldots \tag{2.79}$$

Therefore, the moments are

$$E\left[X^n\right] = \begin{cases} 0 & \text{for } n \text{ odd} \\ \dfrac{n!\,\sigma^n}{(n/2)!\,2^{n/2}} & \text{for } n \text{ even} \end{cases} \tag{2.80}$$

**Example 2.5**

Suppose that $Y$ has the distribution $\mathcal{N}(m, \sigma^2)$. We want to find the value of $\lambda$, such that $P(Y > \lambda) = \alpha$, where $\alpha$ and $\lambda$ are constants.

*Solution*

The probability of $Y$ greater than $\lambda$ is

$$P(Y > \lambda) = \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2\sigma^2}(y-m)^2\right] dy$$

We need to make a change of variables to obtain the standard normal. Let $x = (y-m)/\sqrt{2}\,\sigma$; then, $dy = \sqrt{2}\,\sigma\, dx$, and the integral becomes

$$\alpha = P(Y > \lambda) = \frac{1}{2}\frac{2}{\sqrt{\pi}} \int_{\frac{\lambda - m}{\sqrt{2}\,\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2} dx$$

thus, $\alpha = (1/2)\,\mathrm{erfc}\left[(\lambda - m)/\sqrt{2}\,\sigma\right]$. Or, letting $x = (y-m)/\sigma \Rightarrow dx = \sigma dy$, we obtain

$$\alpha = P(Y > \lambda) = \int_{\frac{\lambda - m}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = Q\left(\frac{\lambda - m}{\sigma}\right) = 1 - I\left(\frac{\lambda - m}{\sigma}\right)$$

We now give a numerical example to be able to use the tabulated $Q$-function or error function. Suppose that $m = 3$, $\sigma^2 = 4$, and $\lambda = 4$. Then,

$$P(Y > 4) = 1 - P\left(X = \frac{Y - m}{\sigma} \le \frac{4 - m}{\sigma} = \frac{4 - 3}{2}\right)$$
$$= 1 - P\left(X \le \frac{1}{2}\right) = 1 - I\left(\frac{1}{2}\right) = 1 - 0.6915 = 0.3085$$

where $X$ is the standard normal, and $P(X \le 1/2) = I(1/2) = 0.6915$ is read directly from the table in the Appendix. We could have used the result found for $\alpha$ by just substituting the numerical values and using the error function defined in (2.63).

If $Y$ has a normal distribution with mean $m$ and variance $\sigma^2$, the probability for $Y$ between $a$ and $b$ is

$$P(a \le Y \le b) = P\left(\frac{a - m}{\sigma} \le \frac{Y - m}{\sigma} \le \frac{b - m}{\sigma}\right)$$
$$= F_X\left(\frac{b - m}{\sigma}\right) - F_X\left(\frac{a - m}{\sigma}\right) = I\left(\frac{b - m}{\sigma}\right) - I\left(\frac{a - m}{\sigma}\right) \qquad (2.81)$$

where $X$ is the tabulated standard normal distribution defined in (2.64). Using the definition of the error function, $P(a \le Y \le b)$ given in (2.81) becomes

$$P(a \le Y \le b) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{b - m}{\sigma\sqrt{2}}\right) - \operatorname{erf}\left(\frac{a - m}{\sigma\sqrt{2}}\right)\right] = \frac{1}{2}\left[\operatorname{erfc}\left(\frac{a - m}{\sigma\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b - m}{\sigma\sqrt{2}}\right)\right]$$

For the numerical example above, where $Y = \mathcal{N}(3, 4)$,

$$P(2 < Y < 5) = P\left(\frac{2 - 3}{2} < \frac{Y - 3}{2} \le \frac{5 - 3}{2}\right)$$
$$= \left(-\frac{1}{2} < X \le 1\right) = P(X \le 1) - P\left(X < -\frac{1}{2}\right)$$
$$= I(1) - I\left(-\frac{1}{2}\right) = 0.8413 - 0.3085 = 0.5328$$

In Chapter 1, we defined the law of large numbers. We now give the *central limit theorem* without proof, which essentially says that the sum of *n* independent random variables having the same density function approaches the normal density function as *n* increases.

*The Central Limit Theorem*

Let $X_1, X_2, \ldots, X_k, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables; that is, the corresponding density functions, $f_{X_k}(x), k = 1, 2, \ldots$, are the same. Let $S_n = X_1 + X_2 + \ldots + X_n$, the sum of *n* random variables, with a finite mean $m = m_1 + m_2 + \ldots + m_n$, and variance $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2$, where $m_k = E[X_k]$ and $\sigma_k^2 = \text{var}[X_k]$, $k = 1, 2, \ldots, n$. The density function of $S_n$, given by $f_{S_n}(x) = f_{X_1}(x) * f_{X_2}(x) * \ldots * f_{X_n}(x)$, approaches a normal distribution as *n* increases; that is,

$$f_{S_n}(x) \to \frac{1}{\sqrt{2\pi\,\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] \tag{2.82}$$

If the sum $S_n$ is normalized, such that $S_n = \sum_{k=1}^{n} (X_k - m_k)/\sigma$, then the distribution $S_n$ approaches the standard normal distribution; that is,

$$f_{S_n}(x) \to \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{2.83}$$

In particular, if the means and the variances are equal, $m_1 = m_2 = \ldots = m_n = m$ and $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_n^2 = \sigma^2$, then $S_n$ is $\mathcal{N}(0, 1)$, and

$$\lim_{n\to\infty} P\left[a \le \frac{S_n - n\,m}{\sigma\sqrt{n}} \le b\right] = \lim_{n\to\infty} P\left[a \le \frac{(X_1 - m) + (X_2 - m) + \ldots + (X_n - m)}{\sigma\sqrt{n}} \le b\right]$$

$$\to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{u^2}{2}}\, du \tag{2.84}$$

This theorem is valid for all distributions, but we shall only discuss the binomial and Poisson distributions. For the binomial distribution, if the number of Bernoulli trials *n* is large, then the random variable *U* given by

$$U = \frac{X - n\,p}{\sqrt{npq}} \tag{2.85}$$

where $p$ is the probability of success, approaches the normal distribution; that is,

$$\lim_{n\to\infty} P\left[a \le \frac{U - n\,p}{\sqrt{n\,p\,q}} \le b\right] = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{u^2}{2}}\, du \tag{2.86}$$

Similarly, since the Poisson distribution has mean $\lambda$ and variance $\lambda$, and we showed in Section 2.2.4 that the parameter $\lambda$ in the Poisson distribution is related to $np$ in the binomial distribution ($\lambda = np$ ), then

$$\lim_{\lambda\to\infty} P\left[a \le U = \frac{X - \lambda}{\sqrt{\lambda}} \le b\right] = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{u^2}{2}}\, du \tag{2.87}$$

Although the normal distribution is the most important distribution, there are many applications in which the normal distribution would not be appropriate. We present the different distributions of interest.

### 2.3.3 The Exponential and Laplace Distributions

A random variable $X$ has an *exponential* distribution with parameter $\beta$, $\beta > 0$, if its density function is given by

$$f_X(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{1}{\beta}(x-a)}, & x \ge a\,, -\infty < a < +\infty \\ 0, & \text{otherwise} \end{cases} \tag{2.88}$$

If we set $a = 0$ and $\alpha = 1/\beta$, then $f_X(x)$, shown in Figure 2.5, becomes

$$f_X(x) = \begin{cases} \alpha\, e^{-\alpha x}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.89}$$

The mean and variance of $X$ are

$$E[X] = \beta = \frac{1}{\alpha} \tag{2.90}$$

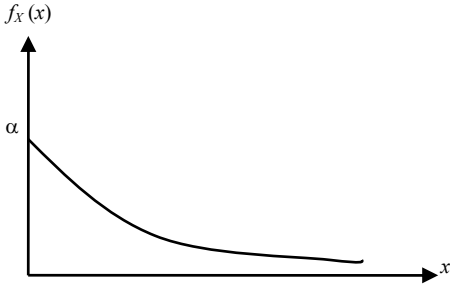


**Figure 2.5** Exponential density function.

and

$$\text{var}\,[X] = \beta^2 = \frac{1}{\alpha^2} \tag{2.91}$$

The moment generating function and characteristic function are

$$M_x(t) = \frac{\alpha}{\alpha - t} = \frac{1}{1 - \beta t}, \quad t < \beta^{-1} \tag{2.92}$$

and

$$\Phi_x(\omega) = \left(1 - \frac{j\omega}{\alpha}\right)^{-1} = \frac{1}{1 - j\beta\omega} \tag{2.93}$$

The *Laplace* density function is defined to be

$$f_X(x) = \frac{1}{2\lambda} \exp\left(-\frac{|x - m|}{\lambda}\right) \quad -\infty < x < \infty, \quad \lambda > 0, \text{ and } -\infty < m < \infty \tag{2.94}$$

If we set the mean $m = 0$ and $\alpha = 1/\lambda$, then the density function becomes

$$f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|} \tag{2.95}$$

and it is shown in Figure 2.6. The moment generating function and the characteristic function of the Laplace distribution defined in (2.94) are

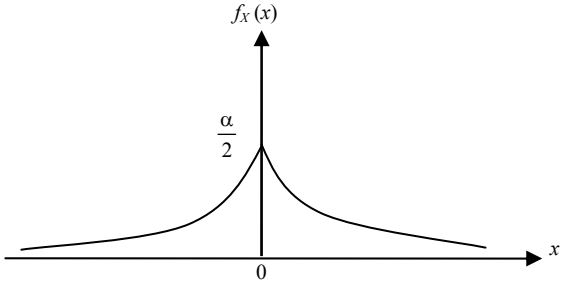


**Figure 2.6** Laplace density function.

$$M_x(t) = \frac{e^{m t}}{1 - \lambda^2 t^2}, \quad |t| < \frac{1}{\lambda} \tag{2.96}$$

and

$$\Phi_x(\omega) = \frac{e^{-jm\omega}}{1 + \lambda^2 \omega^2} \tag{2.97}$$

### 2.3.4 The Gamma and Beta Distributions

In this section, we first describe the gamma function before we introduce the gamma and beta distributions. The gamma function, denoted by Γ, is defined as

$$\Gamma(x) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx, \quad \alpha > 0 \tag{2.98}$$

The above improper integral converges for $\alpha > 0$. Integrating by parts, using $u = x^{\alpha-1}$ and $dv = e^{-x} \, dx$, we obtain

$$\Gamma(\alpha) = (\alpha - 1) \int_0^\infty e^{-x} x^{\alpha-2} \, dx = (\alpha - 1)\Gamma(\alpha - 1) \tag{2.99}$$

Continuing in this manner and letting $\alpha$ be some positive integer, $\alpha = n$, we obtain

$$\Gamma(n) = (n-1)(n-2)\ldots\Gamma(1) = (n-1)! \tag{2.100}$$

where $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$. Another important result about the gamma function is

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx = \sqrt{\pi} \tag{2.101}$$

Now, we are ready to define the Gamma distribution. A random variable $X$ is said to have a *Gamma distribution*, or to be *gamma distributed*, as shown in Figure 2.7, if its density function is given by

$$f_X(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, & x > 0 \\ 0 \qquad\qquad\qquad , & \text{otherwise} \end{cases} \tag{2.102}$$

It is also denoted $X \sim G(\alpha, \beta)$. The mean and variance are, respectively

$$E[X] = m = \alpha\beta \tag{2.103}$$

and

$$\text{var}[X] = \sigma^2 = \alpha\beta^2 \tag{2.104}$$

while the moment generating function and characteristic function are

$$M_x(t) = \frac{1}{(1-\beta t)^\alpha} \tag{2.105}$$

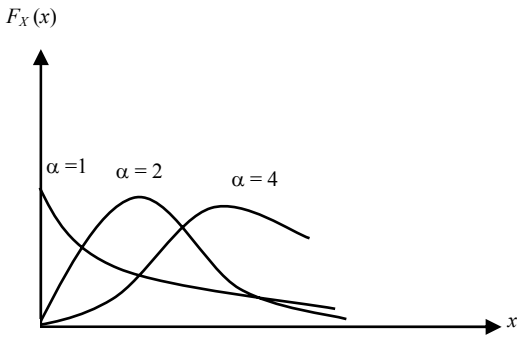


**Figure 2.7** Gamma density function.

and

$$\Phi_x(\omega) = \frac{1}{(1 - j\beta\omega)^{\alpha}} \tag{2.106}$$

Before defining the beta distribution, we define the beta function $B(\alpha, \beta)$, and give its relationship to the gamma function. The beta function is defined to be

$$\begin{aligned} B(\alpha,\beta) &= \int_0^1 u^{\alpha-1} (1-u)^{\beta-1}\, du, \quad \alpha > 0 \text{ and } \beta > 0 \\ &= 2\int_0^1 u^{2\alpha-1} \left(1-u^2\right)^{\beta-1} du \end{aligned} \tag{2.107}$$

The beta function is related to the gamma function by the following

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\beta,\alpha) \tag{2.108}$$

The *beta density function*, shown in Figure 2.8, with parameters $\alpha$ and $\beta$, is defined to be

$$f_X(x) = \begin{cases} \dfrac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1,\ \alpha > 0 \text{ and } \beta > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.109}$$

we write $X \sim B(\alpha, \beta)$. Using (2.108), the beta density function can be written as

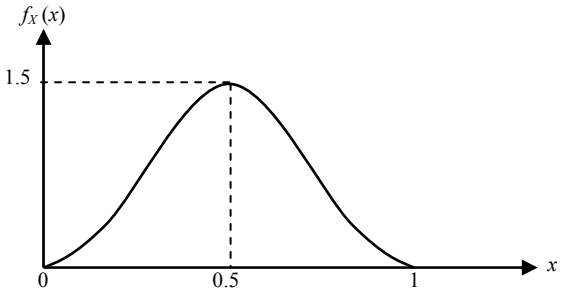


**Figure 2.8** Beta density function; $\alpha = \beta = 2$ and $f_X(x) = 6x(1 - x)$.

$$f_X(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}, & 0<x<1,\ \alpha>0,\ \text{and}\ \beta>0 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.110}$$

Note that for the special case where $\alpha = \beta = 1$, we obtain the uniform distribution for $0 < x < 1$. The mean and variance of the beta distribution for $\alpha > 1$ and $\beta > 1$ are given by

$$E[X] = \frac{\alpha}{\alpha+\beta} \tag{2.111}$$

and

$$\text{var}[X] = \frac{\alpha\beta}{(\alpha+\beta)^2\,(\alpha+\beta+1)} \tag{2.112}$$

whereas the moment generating function and characteristic function are given by

$$M_x(t) = \frac{1}{B(\alpha,\beta)} \int_0^1 e^{tx}\, x^{\alpha-1}(1-x)^{\beta-1}\, dx = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} \frac{\Gamma(\alpha+k)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+k)\Gamma(\alpha)} \tag{2.113}$$

and

$$\Phi_x(\omega) = \sum_{k=0}^{\infty} \frac{(j\omega)^k}{\Gamma(k+1)} \frac{\Gamma(\alpha+k)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+k)\Gamma(\alpha)} \tag{2.114}$$

### 2.3.5 The Chi-Square Distribution

The chi-square distribution is an important distribution function. It may be considered as a special case of the gamma distribution with $\alpha = n/2$ and $\beta = 2$, where $n$ is a positive integer. We say that a random variable $X$ has a *chi-square distribution* with $n$ degrees of freedom, denoted $\chi_n^2$, if its density function is given by

$$f_X(x) = \begin{cases} \dfrac{1}{2^{n/2}\Gamma(n/2)}\, x^{(n/2)-1}\, e^{-x/2}, & x>0 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.115}$$

It should be noted that the chi-square distribution, $\chi_n^2$, represents the distribution of the random variable $X$, where

$$X = X_1^2 + X_2^2 + \ldots + X_n^2 \tag{2.116}$$

and $X_i$, $i = 1, 2, \ldots, n$, is the standard normal random variable $\mathcal{N}(0, 1)$ defined in (2.67); that is, mean zero and variance equal to one. The $X_i$s are i. i. d. (independent and identically distributed). The mean and variance of the chi-square distribution are

$$E[X] = E\left[\chi_n^2\right] = n \tag{2.117}$$

and

$$\text{var}[X] = \text{var}\left[\chi_n^2\right] = 2n \tag{2.118}$$

The moment generating function and characteristic function are given by

$$M_x(t) = \frac{1}{(1-2t)^{n/2}} \qquad \text{for } t < \frac{1}{2} \tag{2.119}$$

and

$$\Phi_x(\omega) = \frac{1}{(1-j2\omega)^{n/2}} \tag{2.120}$$

If we suppose that the $X_i$s are still zero mean but the variances are *not* normalized to one but equal to $\sigma^2$; that is, $E\left[X_i^2\right] = \sigma^2$, $i = 1, 2, \ldots, n$, then the density function of $X$ is

$$f_X(x) = \begin{cases} \dfrac{1}{\sigma^n 2^{n/2}\Gamma(n/2)} x^{(n/2)-1} e^{-\frac{x}{2\sigma^2}}, & x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.121}$$

whereas the characteristic function becomes

$$\Phi_x(\omega) = \frac{1}{\left(1 - j2\omega\sigma^2\right)^{n/2}} \tag{2.122}$$

The mean and variance are

$$E[X] = n\sigma^2 \tag{2.123}$$

and

$$\text{var}[X] = 2n\sigma^4 \tag{2.124}$$

Thus, the second moment is $E[X^2] = 2n\sigma^4 + n^2\sigma^4$. The distribution function is the integral of (2.121), yielding

$$F_X(x) = \int_0^x \frac{1}{\sigma^n 2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-\frac{u}{2\sigma^2}} du \tag{2.125}$$

Using the fact that

$$\int_0^x u^m e^{-\alpha x} dx = \frac{m!}{\alpha^{n+1}} e^{-\alpha x} \sum_{k=0}^{m} \frac{m!}{k!} \frac{x^k}{\alpha^{m-k+1}}, \quad x > 0 \text{ and } \alpha > 0 \tag{2.126}$$

and $m = n/2$ an integer, we obtain the distribution function of $X$ to be

$$F_X(x) = 1 - e^{-\frac{x}{2\sigma^2}} \sum_{k=0}^{m-1} \frac{1}{k!}\left(\frac{x}{2\sigma^2}\right)^k, \quad x \geq 0 \tag{2.127}$$

If we further assume that the $X_i$s, $i = 1, 2, \ldots, n$, in (2.116) are still independent normal variables but with mean $E[X_i] = m_i$ and variance $\sigma^2 = \text{var}[X_i]$, $i = 1, 2, \ldots, n$, then,

$$X = X_1^2 + X_2^2 + \ \ldots \ + X_n^2 \tag{2.128}$$

is said to be a *noncentral chi-square* random variable with $n$ degrees of freedom. The density function of $X$ is given by

$$f_X(x) = \frac{1}{2\sigma^2}\left(\frac{x}{\lambda}\right)^{\frac{n-2}{4}} e^{-\frac{(x+\lambda)}{2\sigma^2}} I_{\frac{n}{2}-1}\left(\frac{\sqrt{x\lambda}}{\sigma^2}\right), \quad x \geq 0 \tag{2.129}$$

where $\lambda$, called the *noncentrality parameter*, is given by

$$\lambda = \sum_{i=1}^{n} m_i^2 \tag{2.130}$$

and $I_\alpha(x)$ is the *modified Bessel function of the first kind of order* $\alpha$ [$\alpha = (n/2) - 1$ is not an integer], and may be written as

$$I_\alpha(x) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\alpha + k + 1)}\left(\frac{x}{2}\right)^{\alpha+2k}, \quad x \geq 0 \tag{2.131}$$

The mean and variance of $X$ are

$$E[X] = n\sigma^2 + \lambda \tag{2.132}$$

and

$$\text{var}[X] = 2n\sigma^4 + 4\sigma^2\lambda \tag{2.133}$$

The moment generating function and characteristic function can be shown to be

$$M_x(t) = \frac{1}{(1 - 2t\sigma^2)^{n/2}} \exp\left(\frac{\lambda t}{1 - 2t\sigma^2}\right), \quad t < \frac{1}{2} \tag{2.134}$$

and

$$\Phi_x(\omega) = \frac{1}{(1 - j2\omega\sigma^2)^{n/2}} \exp\left(\frac{j\omega\lambda}{1 - j2\omega\sigma^2}\right) \tag{2.135}$$

The distribution function of the noncentral chi-square random variable with $n$ degrees of freedom variable does not have a closed form expression. However, we shall study it in order to introduce the $Q$-function, which will be used later in the book. We have

$$F_X(x) = \int_0^x \frac{1}{2\sigma^2}\left(\frac{u}{\lambda}\right)^{\frac{n-2}{4}} e^{-\frac{(u+\lambda)}{2\sigma^2}} I_{\frac{n}{2}-1}\left(\frac{\sqrt{\lambda\, u}}{\sigma^2}\right) du \tag{2.136}$$

If $m = n/2$ is an integer, then

$$F_X(x) = \int_0^x \frac{1}{2\sigma^2}\left(\frac{u}{\lambda}\right)^{\frac{m-1}{2}} e^{-\frac{(u+\lambda)}{2\sigma^2}} I_{m-1}\left(\frac{\sqrt{\lambda\, u}}{\sigma^2}\right) du \tag{2.137}$$

The generalized Marcum's $Q$-function is defined as

$$Q_m(a,b) = \int_b^\infty x\left(\frac{x}{a}\right)^{m-1} e^{-\frac{\left(x^2+a^2\right)}{2}} I_{m-1}(a\,x)\, dx \tag{2.138}$$

Using the fact that

$$\int_0^\infty f_X(x)\, dx = 1 = \int_0^x f_X(u)\, du + \int_x^\infty f_X(u)\, du \tag{2.139}$$

then the distribution function can be written as

$$F_X(x) = 1 - \int_x^\infty \frac{1}{2\,\sigma^2}\left(\frac{u}{\lambda}\right)^{\frac{m-1}{2}} e^{-\frac{(u+\lambda)}{2\,\sigma^2}} I_{m-1}\left(\frac{\sqrt{\lambda\, u}}{\sigma^2}\right) du \tag{2.140}$$

Making the change of variables $v^2 = u/\sigma^2$ and $a^2 = \lambda/\sigma^2$, then $2v\sigma^2 dv = du$, and the distribution function becomes

$$F_X(x) = 1 - \int_{\sqrt{x}/\sigma}^\infty v\left(\frac{v}{a}\right)^{m-1} e^{-\frac{\left(v^2+a^2\right)}{2}} I_{m-1}(a\,v)\, du \tag{2.141}$$

Comparing (2.141) and (2.138), the distribution function is then given in terms of the Marcum's $Q$-function to be

$$F_X(x) = 1 - Q_m\left(\frac{\sqrt{\lambda}}{\sigma}, \frac{\sqrt{x}}{\sigma}\right) \tag{2.142}$$

The noncentral chi-square random variable defined in (2.129) is sometimes called the *noncentral gamma*, while the normalized noncentral chi-square random variable $(\sigma^2 = 1)$ is referred to as the *noncentral chi-square*. For us, we shall call it, as most authors do, the noncentral chi-square random variable. The density function of the normalized noncentral chi-square random variable is obtained directly from (2.129) to be

$$f_X(x) = \frac{1}{2}\left(\frac{x}{\lambda}\right)^{\frac{n-2}{4}} e^{-\frac{(x+\lambda)}{2}} I_{\frac{n}{2}-1}\left(\sqrt{\lambda\, x}\right), x \geq 0 \tag{2.143}$$

where $X = X_1^2 + X_2^2 + \ldots + X_n^2$, and the $X_i$s, $i$ = 1, 2, …, $n$, are independent normal random variables with mean $E[X_i] = m_i$ but have *unit variance*. The noncentrality parameter $\lambda$ is as defined in (2.130). Note also that the chi-square random variable given by (2.115) is just the gamma random variable given by (2.102) with $\alpha = n/2,\ n > 0$ integer, and $\beta = 2$. This leads some authors to refer to the normalized noncentral chi-square random variable as the noncentral gamma random variable.

### 2.3.6 The Rayleigh, Rice, and Maxwell Distributions

The Rayleigh distribution, which is frequently used to model the statistics of signals, finds its application in many radar and communication problems. Let $X = X_1^2 + X_2^2$, where $X_1$ and $X_2$ are statistically independent Gaussian random variables with mean zero and each having variance $\sigma^2$. Then, from (2.116), $X$ has a chi-square distribution with $n$ = 2 degrees of freedom. Substituting $n$ = 2 in (2.121), we obtain the probability density function of $X$ to be

$$f_X(x) = \begin{cases} \dfrac{1}{2\,\sigma^2} \exp\left(-\dfrac{x}{2\,\sigma^2}\right), & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.144}$$

Now, define a new random variable

$$Y = \sqrt{X} = \sqrt{X_1^2 + X_2^2} \tag{2.145}$$

This is a simple transformation of random variables with $Y = g(X) = \sqrt{X}$. Applying the fundamental theorem given in (1.144), we obtain

$$f_Y(y) = \begin{cases} \dfrac{y}{\sigma^2} \exp\left(-\dfrac{y^2}{2\sigma^2}\right), & y \geq 0 \\ 0 \qquad\qquad\qquad , & \text{otherwise} \end{cases} \tag{2.146}$$

*Y* is said to have a *Rayleigh* distribution, as shown in Figure 2.9(a). The distribution function, as shown in Figure 2.9(b), is given by

$$F_Y(y) = \begin{cases} 1 - e^{-\frac{y^2}{2\sigma^2}}, & y \geq 0 \\ 0 \qquad\qquad , & \text{otherwise} \end{cases} \tag{2.147}$$

It can be shown that the moment of order *k* of the Rayleigh distribution is given by

$$E\left[Y^k\right] = \left(2\sigma^2\right)^{\frac{k}{2}} \Gamma\left(1 + \frac{k}{2}\right) \tag{2.148}$$

Thus, the mean and variance of *Y* are given by

$$E[Y] = \sqrt{2}\,\sigma\,\Gamma\left(\frac{3}{2}\right) = \sigma\sqrt{\frac{\pi}{2}} \tag{2.149}$$

since $\Gamma(3/2) = \sqrt{\pi/2}$ and

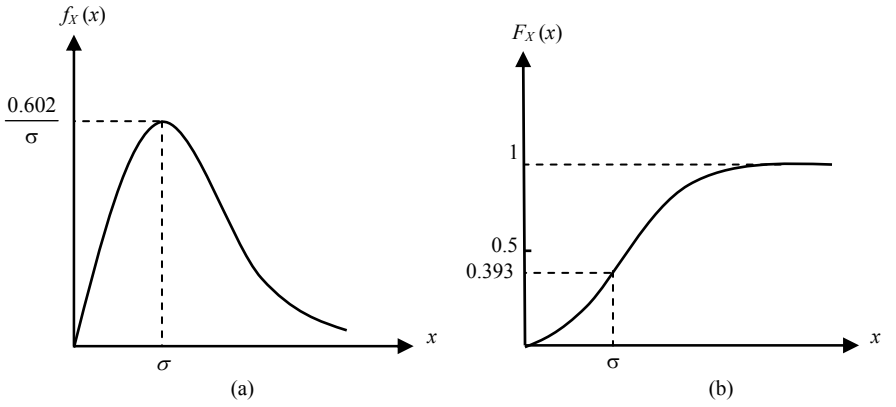


**Figure 2.9** Rayleigh (a) density function and (b) distribution function.

$$\text{var}[Y] = \sigma_y^2 = \left(2 - \frac{\pi}{2}\right)\sigma^2 \tag{2.150}$$

The characteristic function is shown to be

$$\begin{aligned}\Phi_y(\omega) &= \int_0^\infty \frac{y}{\sigma^2} e^{-\frac{y^2}{2\sigma^2}} \; e^{j\omega y} \; dy \\ &= \int_0^\infty \frac{y}{\sigma^2} e^{-\frac{y^2}{2\sigma^2}} \cos \omega y \; dy + j \int_0^\infty \frac{y}{\sigma^2} e^{-\frac{y^2}{2\sigma^2}} \sin \omega y \; dy \\ &= {}_1F_1\left(1; \frac{1}{2}; -\frac{1}{2}\omega^2 \sigma^2\right) + j\sqrt{\frac{\pi}{2}}\, \omega \sigma^2 \, e^{-\frac{\omega^2 \sigma^2}{2}}\end{aligned} \tag{2.151}$$

where ${}_1F_1(a; b; x)$ is the *confluent hypergeometric function*, which is defined to be

$${}_1F_1(a; b; x) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b)}{\Gamma(a)\Gamma(b+k)} \cdot \frac{x^k}{k!} \quad , \; b \neq 0, -1, -2, \ldots \tag{2.152}$$

and

$${}_1F_1\left(1; \frac{1}{2}; -a\right) = e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{(2k-1)k!} \tag{2.153}$$

**Example 2.6**

Using the distribution function $F_X(x) = P(X \leq x)$, determine the density function of

(a) $X = X_1^2 + X_2^2$

(b) $X = \sqrt{X_1^2 + X_2^2}$

where $X_1$ and $X_2$ are identical and independent normal density functions with mean zero and variance $\sigma^2$.

*Solution*

(a) The distribution function of $X$ is

$$F_X(x) = P(X \leq x) = \iint_D f_{X_1 X_2}(x_1, x_2)\, dx_1 \; dx_2, \;\; x \geq 0$$

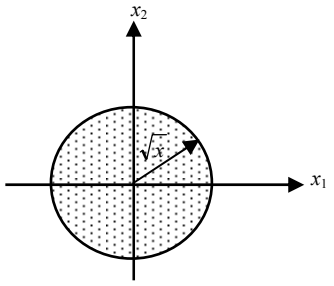


**Figure 2.10** Region of $X_1^2 + X_2^2 \le x$ , $x \ge 0$.

where $D$ is the domain with a definition of $X_1$ and $X_2$, which in this case is the surface in the circle of radius $\sqrt{x}$ , as shown in Figure 2.10. Hence,

$$F_X(x) = \iint_D \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2}\left(x_1^2 + x_2^2\right)\right] dx_1\, dx_2$$

To solve the above integral, we make the transformation to polar coordinates by letting $x_1 = r \cos\theta$ and $x_2 = r \sin\theta$ such that $dx_1\, dx_2 = r\, dr\, d\theta$ and $r^2 = x_1^2 + x_2^2$. Thus,

$$F_X(x) = \frac{1}{2\pi\sigma^2} \int_0^{2\pi} d\theta \int_0^{\sqrt{x}} re^{-\frac{r}{2\sigma^2}}\, dr = 1 - e^{-\frac{x}{2\sigma^2}},\ x \ge 0$$

The density function is

$$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} \dfrac{1}{2\sigma^2} e^{-\frac{x}{2\sigma^2}}, & x \ge 0 \\ 0 \quad , & \text{otherwise} \end{cases}$$

which corresponds to the chi-square distribution with $n = 2$ degrees of freedom, as given in (2.144).

(b) If $X = \sqrt{X_1^2 + X_2^2}$ , then

$$F_X(x) = P\left(\sqrt{X_1^2 + X_2^2} \le x\right) = P\left(X_1^2 + X_2^2 \le x^2\right)$$

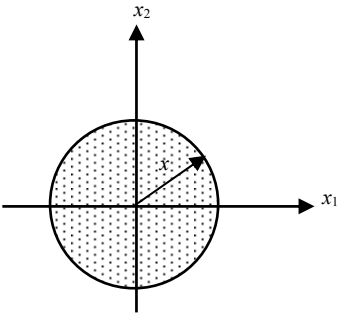


**Figure 2.11** Region of $\sqrt{X_1^2 + X_2^2} \le x,\ x > 0.$

The region of integration is the surface bounded by the circle as shown in Figure 2.11, but the radius is $x$, and not $\sqrt{x}$ as in Figure 2.10. Again making the transformation from Cartesian coordinates to polar coordinates, the distribution function $F_X(x)$ becomes

$$F_X(x) = \frac{1}{2\pi\sigma^2} \int_0^{2\pi} d\theta \int_0^x re^{-\frac{r}{2\sigma^2}}\, dr = 1 - e^{-\frac{x^2}{2\sigma^2}},\ x \ge 0$$

while the density function is

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

which corresponds to the Rayleigh density function given in (2.146). Recall that (2.146) was obtained using the fundamental theorem of transformation of random variables.

**Example 2.7**

Let $X$ be a Rayleigh random variable with density function

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

Define $Y = a + bX^2$, where *a* and *b* are constants. Determine the variance of *Y*.

*Solution*

The variance of *Y* is $\text{var}[Y] = \sigma_y^2 = E[Y^2] + E^2[Y]$. Hence, $E[Y] = E[a + bX^2]$ $= a + b\,E[X^2]$, $E^2[Y] = \left(a + b\,E[X^2]\right)^2 = a^2 + 2abE[X^2] + b^2E^2[X^2]$, and $E[Y^2] = E\left[\left(a + b\,X^2\right)^2\right] = a^2 + 2abE[X^2] + b^2E[X^4]$. Substituting for the expressions of $E[Y^2]$ and $E^2[Y]$ in $\sigma_y^2$, we obtain

$$\sigma_y^2 = b^2\left\{E[X^4] + E^2[X^2]\right\}$$

We know from (2.148) that $E[X^k] = \left(2\sigma^2\right)^{\frac{k}{2}}\Gamma[1 + (k/2)]$. Then, $E[X^4] = 2\left(2\sigma^2\right)^2 = 8\sigma^4$, $E[X^2] = 2\sigma^2$, and the variance of *Y* becomes

$$\sigma_y^2 = b^2\left(8\sigma^4 - 4\sigma^4\right) = 4b^2\sigma^4$$

We now consider $R = \sqrt{X_1^2 + X_2^2}$ but $X_1$ and $X_2$ independent Gaussian random variables with means $m_i$, $i$ = 1, 2, and each having a variance $\sigma^2$. Note that in the definition of (2.145), $X_1$ and $X_2$ were *zero mean*, which gave $X = \sqrt{X_1^2 + X_2^2}$ as a Rayleigh distributed random variable, but now $X_1$ and $X_2$ have means $m_i \neq 0$, $i$ = 1, 2. Hence, from (2.128), the distribution of $R^2 = X_1^2 + X_2^2$ is the noncentral chi-square random variable given in (2.129), with two ($n$ = 2) degrees of freedom and noncentrality parameter $\lambda = m_1^2 + m_2^2$. The distribution function of $R^2 = X_1^2 + X_2^2 = T$ is then

$$f_T(t) = \begin{cases} \dfrac{1}{2\sigma^2}\, e^{-\frac{(\lambda + t)}{2\sigma^2}}\, I_0\left(\dfrac{\sqrt{\lambda\, t}}{\sigma^2}\right), & t \geq 0 \\ 0 \qquad\qquad\qquad\qquad\qquad , & \text{otherwise} \end{cases} \tag{2.154}$$

where, $I_0(x)$ is the *zero-order modified Bessel function* given by

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\, d\theta = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n}(n!)^2} \tag{2.155}$$

Since $R = \sqrt{T} = \sqrt{X_1^2 + X_2^2}$, using the fundamental theorem (1.144) for the transformation of random variables, we obtain the *Rice density function* to be

$$f_R(r) = \begin{cases} \dfrac{r}{\sigma^2}\, e^{-\frac{\left(r^2+\lambda\right)}{2\sigma^2}}\, I_0\left(\dfrac{\sqrt{\lambda}\, r}{\sigma^2}\right), r \geq 0 \\ 0 \qquad\qquad\qquad\qquad , \quad \text{otherwise} \end{cases} \tag{2.156}$$

The *Rician* distribution with $\lambda$ as a parameter is shown in Figure 2.12. The distribution function is known to be

$$F_R(r) = \begin{cases} 1 - Q_1\left(\dfrac{\sqrt{\lambda}}{\sigma}, \dfrac{r}{\sigma}\right), & r \geq 0 \\ 0 \qquad\qquad , \quad \text{otherwise} \end{cases} \tag{2.157}$$

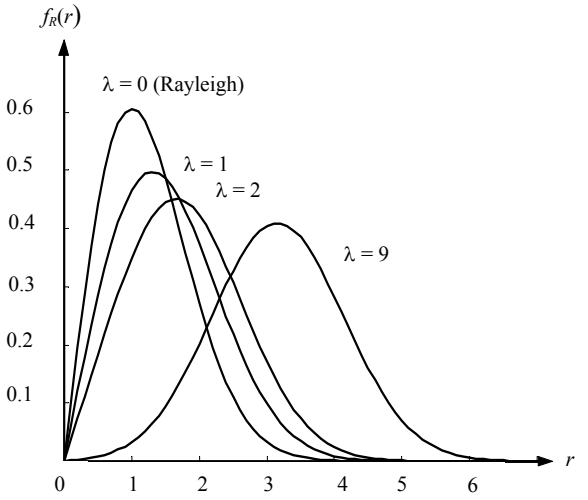


**Figure 2.12** Rice density function.

where $Q_1(a, b)$ is Marcum's $Q$-function defined in (2.138), such that

$$Q_1(a,b) = e^{-\frac{(a^2+b^2)}{2}} \sum_{k=0}^{\infty} \left(\frac{a}{b}\right)^k I_k(ab), \quad b > a > 0 \tag{2.158}$$

Another special case is when $n = 3$. Then, $X = X_1^2 + X_2^2 + X_3^2$, with $X_1$, $X_2$, and $X_3$ Gaussian random variables, each with mean zero and variance $\sigma^2$, is a chi-square distribution with three degrees of freedom. The distribution of $Y = \sqrt{X_1^2 + X_2^2 + X_3^2}$ is known as the *Maxwell's distribution*. It is shown in Figure 2.13, and is given by

$$f_Y(y) = \begin{cases} \dfrac{1}{\sigma^3}\sqrt{\dfrac{2}{\pi}}\, y^2\, e^{-\frac{y^2}{2\sigma^2}}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.159}$$

with mean

$$E[Y] = m_y = 2\sigma\sqrt{\frac{2}{\pi}} \tag{2.160}$$

and variance

$$\text{var}[Y] = \sigma_y^2 = \sigma^2\left(3 - \frac{8}{\pi}\right) \tag{2.161}$$

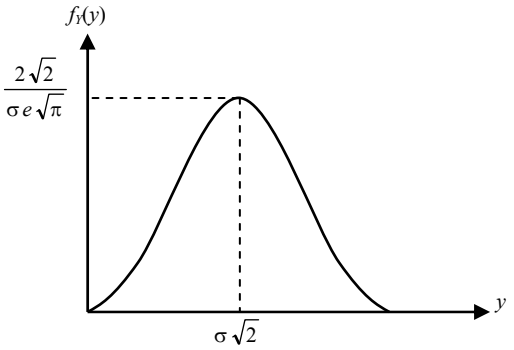


**Figure 2.13** Maxwell density function.

If we generalize the result in (2.159) to $n$ random variables, then $X = X_1^2 + X_2^2 + \ldots + X_n^2$, with $X_i$, $i$ = 1, 2, … , $n$, statistically independent Gaussian random variables with means $m_i$, $i$ = 1, 2, … , $n$, and each having variance $\sigma^2$, has the density function given by

$$f_X(x) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}-1}\,\sigma^n\,\Gamma(n/2)}\; x^{n-1}\; e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0 \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad , & \text{otherwise} \end{cases} \tag{2.162}$$

In general, if $Y = \sqrt{X} = \sqrt{X_1^2 + X_2^2 + \ldots + X_n^2}$, then the density function is given by

$$f_Y(y) = \begin{cases} \dfrac{1}{\sigma^2\left(\sqrt{\lambda}\right)^{\frac{n}{2}-1}}\; y^{\frac{n}{2}}\; \exp\left[-\dfrac{\left(y^2+\lambda\right)}{2\sigma^2}\right]\; I_{\frac{n}{2}-1}\left(\dfrac{\sqrt{\lambda}\,y}{\sigma^2}\right), & y \geq 0 \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , & \text{otherwise} \end{cases} \tag{2.163}$$

and $\lambda = m_1^2 + m_2^2 + \cdots + m_n^2$, while the distribution function is given by

$$F_Y(y) = P(Y \leq y) = P\left(X_1^2 + X_2^2 + \ldots + X_n^2 \leq y^2\right)$$

$$= \begin{cases} 1 - Q_m\left(\dfrac{\sqrt{\lambda}}{\sigma}, \dfrac{y}{\sigma}\right), & y \geq 0 \\ 0 \qquad\qquad\qquad , & \text{otherwise} \end{cases} \tag{2.164}$$

The moment of order $k$ can be obtained to be

$$E\left[Y^k\right] = \int_0^\infty y^k f_Y(y)\,dy = \frac{1}{\sigma^2}\int_0^\infty y^{k+1}\; e^{-\frac{\left(y^2+\lambda\right)}{2\sigma^2}}\; I_0\left(\frac{\sqrt{\lambda}\,y}{\sigma^2}\right)dy$$

$$= \left(2\sigma^2\right)^{\frac{k}{2}}\; e^{-\frac{\sqrt{\lambda}}{2\sigma^2}}\; \frac{\Gamma\left[\frac{1}{2}(n+k)\right]}{\Gamma(n/2)}\; {}_1F_1\left(\frac{n+k}{2}; \frac{n}{2}; \frac{\lambda}{2\sigma^2}\right), \quad y \geq 0 \tag{2.165}$$

where ${}_1F_1(\alpha;\beta;x)$ is the confluent hypergeometric function.

### 2.3.7 The Nakagami *m*-Distribution

The *Nakagami m-distribution* is used in communication systems to characterize the statistics of signals transmitted through multipath fading channels. The density function is given by

$$f_X(x) = \frac{2}{\Gamma(m)}\left(\frac{m}{v}\right)^m x^{2m-1} e^{-\frac{m x^2}{v}} \tag{2.166}$$

where $v$ is the mean-square value of $X$, defined as

$$v = E\left[X^2\right] \tag{2.167}$$

and the parameter $m$ is defined as

$$m = \frac{v^2}{E\left[(X-v)^2\right]}, \quad m \geq \frac{1}{2} \tag{2.168}$$

Notice that the parameter $m$ is a ratio of moments and is referred to as a *fading figure* in communication systems. The moment of order $k$ of $X$ is given by

$$E\left[X^k\right] = \frac{\Gamma\left(m+\frac{k}{2}\right)}{\Gamma(m)}\left(\frac{v}{m}\right)^{\frac{k}{2}} \tag{2.169}$$

Observe that for $m = 1$, we obtain the Rayleigh density function given in (2.146). A plot of $f_X(x)$ with $m$ as a parameter is shown in Figure 2.14.

### 2.3.8 The Student's *t*- and *F*-Distributions

Let $X$ be a Gaussian random variable with mean zero and variance unity $X \sim \mathcal{N}(0,1)$, and let $Y$ be a chi-square random variable with $n$ degrees of freedom $Y \sim \chi_n^2$. If $X$ and $Y$ are statistically independent, then

$$T = \frac{X}{\sqrt{Y/n}} \tag{2.170}$$

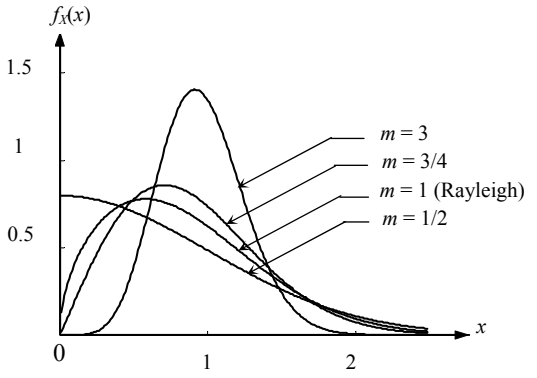


**Figure 2.14** Nakagami *m*-density function.

is said to have a *t-distribution* (*or student's t-distribution*) *with n degrees of freedom* as shown in Figure 2.15, and is given by

$$f_T(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\,\pi}\,\Gamma(n/2)}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}, \quad -\infty < t < \infty \tag{2.171}$$

The mean and variance of the *t*-distribution are

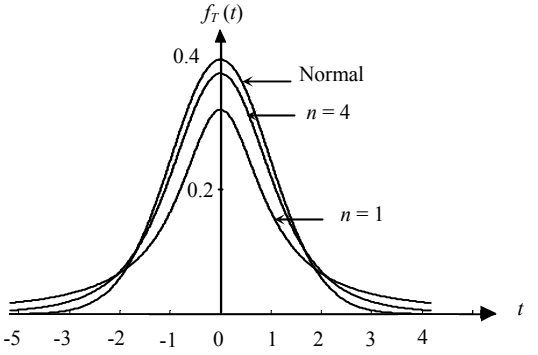


**Figure 2.15** Student's *t*-density function.

$$E[T] = m_t = 0 \tag{2.172}$$

and

$$\text{var}[T] = \sigma_t^2 = \frac{n}{n-2} \quad \text{for } n > 2 \tag{2.173}$$

The moment of order $k$ is given by

$$E[T^k] = \begin{cases} \dfrac{\Gamma\left(\dfrac{k+1}{2}\right)\Gamma\left(\dfrac{n-k}{2}\right)}{\Gamma(1/2)\,\Gamma(n/2)}\, n^{\frac{k}{2}}, & k < n \text{ and } k \text{ even} \\ 0, & k < n \text{ and } k \text{ odd} \end{cases} \tag{2.174}$$

The characteristic function can be shown to be

$$\Phi_t(\omega) = \frac{1}{\pi\,\Gamma(n/2)} \left(\frac{|t|}{2\sqrt{n}}\right)^{\frac{n}{2}} Y_{\frac{n}{2}}\left(\frac{|t|}{\sqrt{n}}\right) \tag{2.175}$$

where $Y_n(x)$ [also denoted $N_n(x)$] is the *Bessel function of the second kind*.

Assume now $X$ does not have zero mean but equals to $m$ [that is, $X \sim \mathcal{N}(m, \sigma^2)$ ], and $Y$ is normalized so that $Y/\sigma^2$ is the chi-square distribution with $n$ degrees of freedom. Then $T$ defined in (2.170) has a *noncentral t-distribution with parameter* (also called *noncentrality parameter*) $\lambda = m/\sigma$ and $n$ degrees of freedom. The density function of the "normalized" noncentral $t$-distribution is given by

$$f_T(t) = \frac{n^{\frac{n}{2}}}{\sqrt{\pi}\,\Gamma(n/2)} \frac{e^{-\frac{\lambda^2}{2}}}{\left(n+t^2\right)^{\frac{n+1}{2}}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(\frac{2t^2}{n+t^2}\right)^{\frac{k}{2}} \Gamma\left(\frac{n+k+1}{2}\right) \tag{2.176}$$

The mean and variance of $T$ are given by

$$E[T] = \lambda \frac{\Gamma\left(\dfrac{n-1}{2}\right)}{\Gamma(n/2)} \sqrt{\frac{n}{2}}, \quad n > 0 \tag{2.177}$$

and

$$\text{var}[T] = \frac{n(1-\lambda^2)}{n-2} - \frac{\lambda^2 n}{2}\left\{\frac{\Gamma\left[(n-1)/2\right]}{\Gamma(n/2)}\right\}^2 , \quad n > 2 \tag{2.178}$$

Let $X$ and $Y$ be two independent chi-square random variables with $n_1$ and $n_2$ degrees of freedom, respectively. Define $U$ to be

$$U = \frac{X/n_1}{Y/n_2} \tag{2.179}$$

Then, $U$ is said to have an *F-distribution*, $F(n_1, n_2)$, with density function

$$f_U(u) = \begin{cases} \dfrac{\Gamma\left(\dfrac{n_1+n_2}{2}\right)}{\Gamma(n_1/2)\Gamma(n_2/2)}\; n_1^{\frac{n_1}{2}}\, n_2^{\frac{n_2}{2}}\, \dfrac{u^{\frac{n_1}{2}-1}}{(n_2+n_1 u)^{\frac{n_1+n_2}{2}}} & , \quad u > 0 \\ 0 & , \quad u \le 0 \end{cases} \tag{2.180}$$

The mean and variance of $U$ are

$$E[U] = m_u = \frac{n_2}{n_2 - k} , \quad n_2 > 2 \tag{2.181}$$

and

$$\text{var}[U] = \sigma_u^2 = \frac{2\, n_2\, (n_1 + n_2 - 2)}{n_1\, (n_2 - 4)(n_2 - 2)^2} , \quad n_2 > 4 \tag{2.182}$$

while the moment of order $k$ is given by

$$E\left[U^k\right] = \left(\frac{n_2}{n_1}\right)^k \frac{\Gamma\left(\dfrac{n_1}{2}+k\right)\Gamma\left(\dfrac{n_2}{2}-k\right)}{\Gamma(n_1/2)\Gamma(n_2/2)} , \quad n_2 > 2k \tag{2.183}$$

The characteristic function is given by

$$\Phi_u(\omega) = M\left(\frac{n_1}{2}, -\frac{n_2}{2}, -j\frac{n_2}{n_1}\omega u\right) \tag{2.184}$$

where $M(a, b, x)$ is the *Kummer's confluent hypergeometric function* given by

$$M(a, b, x) = 1 + \frac{a\,x}{b} + \frac{(a)_2\,x^2}{(b)_2\,2!} + \ \dots\ + \frac{(a)_n\,x^n}{(b)_n\,n!} + \ \dots \tag{2.185}$$

and

$$(a)_n = a\,(a+1)(a+2)\dots(a+n-1) \tag{2.186}$$

$$(a)_0 = 1 \tag{2.187}$$

Let $X$ be a normalized noncentral chi-square random variable with noncentrality parameter $\sqrt{\lambda} = \sum_{i=1}^{n} m_i^2 / \sigma^2$ and $n_1$ degrees of freedom [i.e., $X = \sum_{i=1}^{n} X_i / \sigma^2$, $X_i \sim \mathcal{N}(m_i\ ,\ \sigma^2)$, $i = 1, 2, \dots, n$], and $Y$ a chi-square random variable with $n_2$ degrees of freedom. Then, the random variable

$$Z = \frac{X / n_1}{Y / n_2} \tag{2.188}$$

is said to have a *noncentral F-distribution* with $(n_1, n_2)$ degrees of freedom and a *noncentrality parameter* $\sqrt{\lambda}$. The density function of the noncentral $F$-distribution defined in (2.188) can be shown to be

$$f_Z\left(z; n_1, n_2, \sqrt{\lambda}\right) = \begin{cases} \dfrac{n_1^{\frac{n_1}{2}}\ n_2^{\frac{n_2}{2}}}{\Gamma(n_2/2)}\, e^{-\frac{\sqrt{\lambda}}{2}}\, z^{\frac{n_1}{2}-1} & \\ \qquad \cdot \displaystyle\sum_{k=0}^{\infty} \frac{\Gamma\left(\dfrac{n_1+n_2}{2}+k\right)}{k!\,\Gamma\left(\dfrac{n_1}{2}+k\right)} \frac{\left(\dfrac{n_1\sqrt{\lambda}\,z}{2}\right)^k}{(n_1 z + n_2)^{\frac{n_1+n_2}{2}+k}} & ,\ z > 0 \\ 0 & ,\ z \le 0 \end{cases} \tag{2.189}$$

Note that if the noncentrality parameter is zero $\left(\sqrt{\lambda} = 0\right)$ in (2.189), we obtain the central $F$-distribution $F(n_1, n_2)$ defined in (2.180). The mean and variance of the noncentral $F$-distribution can be shown to be

$$E[Z] = \frac{n_2\left(n_1 + \sqrt{\lambda}\right)}{n_1\left(n_2 - 2\right)} \quad , \; n_2 > 2 \tag{2.190}$$

and

$$\text{var}[Z] = \frac{2\,n_2^2}{n_1^2\left(n_2 - 4\right)\left(n_2 - 2\right)^2}\left[\left(n_1 + \sqrt{\lambda}\right)^2 + \left(n_2 - 2\right)\left(n_1 + 2\sqrt{\lambda}\right)\right] \quad , \; n_2 > 4 \tag{2.191}$$

### 2.3.9 The Cauchy Distribution

A random variable $X$ is said to have a *Cauchy* distribution with parameter $\alpha$, $-\infty \le \alpha \le \infty$, and $\beta$, $\beta > 0$, if its probability density function is given by

$$f_X(x) = \frac{1}{\pi\beta}\left[1 + \left(\frac{x-\alpha}{\beta}\right)^2\right]^{-1} = \frac{\beta}{\pi}\frac{1}{\beta^2 + (x-\alpha)^2} \tag{2.192}$$

It is denoted $C(\alpha, \beta)$. It can be shown that the mean of the Cauchy distribution with parameters $\beta = 1$ and $\alpha = 0$ is zero, but the variance and moments of higher order do not exist. The moment generating function does not exist, but the characteristic function can be shown to be

$$\Phi_x(\omega) = e^{j\alpha\omega - \beta|\omega|} \tag{2.193}$$

Note that if $\alpha = 0$ and $\beta = 1$, then the density function becomes

$$f_X(x) = \frac{1}{\pi}\frac{1}{1 + x^2} \tag{2.194}$$

which is the student's *t*-distribution defined in (2.171) with $n = 1$ degree of freedom. The sum of Cauchy random variables is Cauchy; that is, if $X = X_1 + X_2 + \cdots + X_n$ where $X_k$, $k = 1, 2, \ldots, n$, is Cauchy with parameters $\alpha_k$ and $\beta_k$, $k = 1, 2, \ldots, n$, then $X$ is Cauchy with parameters $\alpha$ and $\beta$, such that $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ and $\beta = \beta_1 + \beta_2 + \ldots + \beta_n$.

## 2.4 SOME SPECIAL DISTRIBUTIONS

### 2.4.1 The Bivariate and Multivariate Gaussian Distributions

Because of the importance of the Gaussian distribution and its many applications, we extend the concepts developed earlier to the two-dimensional and *n*-dimensional Gaussian distribution. Let $X_1$ and $X_2$ be two jointly Gaussian random variables with means $E[X_1] = m_1$ and $E[X_2] = m_2$, and variances $\sigma_1^2$ and $\sigma_2^2$. The bivariate Gaussian density function is defined as

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\,\sigma_2\sqrt{1-\rho^2}} \cdot \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-m_1)^2}{\sigma_1^2} + \frac{(x_2-m_2)^2}{\sigma_2^2} - 2\rho\frac{(x_1-m_1)(x_2-m_2)}{\sigma_1\,\sigma_2}\right]\right\} \quad (2.195)$$

where $\rho$ is the correlation coefficient between $X_1$ and $X_2$. The conditional probability density function $f_{X_2|X_1}(x_2|x_1)$ is given by

$$f_{X_2|X_1}(x_2|x_1) = \frac{f_{X_1X_2}(x_1, x_2)}{f_{X_1}(x_1)} = \frac{1}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}}\exp\left[-\frac{x_2-\alpha}{2\sigma_2^2(1-\rho^2)}\right] \quad (2.196)$$

where

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1X_2}(x_1, x_2)\,dx_2 = \frac{1}{\sigma_1\sqrt{2\pi}}\exp\left(-\frac{(x_1-m_1)^2}{2\sigma_1^2}\right)\left\{\frac{1}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x_2-\alpha)^2}{2\sigma_2^2(1-\rho^2)}\right]dx_2\right\} \quad (2.197)$$

The integrand in (2.197) is a Gaussian density function, with mean $\alpha = m_2 + \rho(\sigma_2/\sigma_1)(x_1 - m_1)$ and variance $\sigma_2^2(1-\rho^2)$. We observe from (2.196) that the conditional density function $f_{X_2|X_1}(x_2|x_1)$ is also Gaussian, with mean $\alpha$ and variance $\sigma_2^2(1-\rho^2)$. The conditional expectation of $X_2$ given $X_1 = x_1$ is

$$E\left[X_2 \mid X_1 = x_1\right] = \alpha = m_2 + \rho \frac{\sigma_2}{\sigma_1}\left(x_1 - m_1\right) \tag{2.198}$$

and

$$\text{var}\left[X_2 \mid X_1 = x_1\right] = \sigma_2^2\left(1-\rho^2\right) \tag{2.199}$$

In a similar manner, we can show that

$$f_{X_1 \mid X_2}\left(x_1 \mid x_2\right) = \frac{1}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left[-\frac{x_1 - \beta}{2\sigma_1^2\left(1-\rho^2\right)}\right] \tag{2.200}$$

is Gaussian, with mean $\beta = m_1 + \rho\left(\sigma_1 / \sigma_2\right)\left(x_2 - m_2\right)$ and variance $\sigma_1^2\left(1-\rho^2\right)$, and the conditional expectation of $X_1$ given $X_2 = x_2$ is

$$E\left[X_1 \mid X_2 = x_2\right] = \beta = m_1 + \rho \frac{\sigma_1}{\sigma_2}\left(x_2 - m_2\right) \tag{2.201}$$

and

$$\text{var}\left[X_1 \mid X_2 = x_2\right] = \sigma_1^2\left(1-\rho^2\right) \tag{2.202}$$

It follows that

$$E\left[X_2^2 \mid X_1 = x_1\right] = \text{var}\left[X_2 \mid X_1 = x_1\right] - E^2\left[X_2 \mid X_1 = x_1\right] \tag{2.203}$$

For the special case in which the means $m_1 = m_2 = 0$, we obtain

$$E\left[X_2^2 \mid X_1 = x_1\right] = \sigma_2^2\left(1-\rho^2\right) - \rho^2\left(\frac{\sigma_2}{\sigma_1}\right)^2 x_1^2 \tag{2.204}$$

The moment generating function and characteristic function of $X_1$ and $X_2$ are

$$\begin{aligned} M_{x_1 x_2}\left(t_1, t_2\right) &= E\left[e^{t_1 X_1 + t_2 X_2}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x_1 + t_2 x_2} f_{X_1 X_2}\left(x_1, x_2\right) dx_1\, dx_2 \\ &= \exp\left[m_1 t_1 + m_2 t_2 + \frac{1}{2}\left(\sigma_1^2 t_1^2 + \sigma_2^2 t_2^2 + 2\rho\sigma_1\sigma_2 t_1 t_2\right)\right] \end{aligned} \tag{2.205}$$

and

$$\Phi_{x_1x_2}(\omega_1,\omega_2)=E\left[e^{j(\omega_1X_1+\omega_2X_2)}\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{j(\omega_1x_1+\omega_2x_2)}\,f_{X_1X_2}(x_1,x_2)\,dx_1\,dx_2$$

$$=\exp\left[-\frac{1}{2}(\sigma_1^2\,\omega_1^2+\sigma_2^2\,\omega_2^2+2\,\rho\,\sigma_1\,\sigma_2\,\omega_1\,\omega_2)+j\left(m_1\omega_1+m_2\omega_2\right)\right] \tag{2.206}$$

The moments are obtained from the characteristic function to be

$$E\left[X_1^n\,X_2^m\right]=(-j)^{n+m}\,\frac{\partial^n}{\partial\omega_1^n}\frac{\partial^m}{\partial\omega_2^m}\Phi_{x_1x_2}(\omega_1,\omega_2)\Big|_{\omega_1=\omega_2=0} \tag{2.207}$$

Sometimes, it is easier to represent the joint density function and characteristic function in matrix form, especially when the number of random variables is greater than two. Let $\boldsymbol{C} = \boldsymbol{C}[X_1, X_2]$ denote the covariance matrix of the two random variables $X_1$ and $X_2$,

$$\boldsymbol{C}=\begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{bmatrix}=\begin{bmatrix}\sigma_1^2 & \rho\,\sigma_1\sigma_2\\ \rho\,\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix} \tag{2.208}$$

where $c_{11}=\sigma_1^2$, $c_{21}=c_{12}=\rho\,\sigma_1\,\sigma_2$, and $c_{22}=\sigma_2^2$. The correlation coefficient is

$$\rho=\frac{c_{12}}{\sqrt{c_{11}\,c_{22}}}=\frac{\rho\,\sigma_1\sigma_2}{\sigma_1\sigma_2} \tag{2.209}$$

The determinant of the covariance matrix $\boldsymbol{C}$ is

$$|\,\boldsymbol{C}\,|=\sigma_1^2\,\sigma_2^2\left(1-\rho^2\right) \tag{2.210}$$

Consequently,

$$\boldsymbol{C}^{-1}=\frac{1}{|\,\boldsymbol{C}\,|}\begin{bmatrix}\sigma_2^2 & -\rho\,\sigma_1\sigma_2\\ -\rho\,\sigma_1\sigma_2 & \sigma_1^2\end{bmatrix} \tag{2.211}$$

Let $\boldsymbol{x}=[x_1 \quad x_2]^T$, $\boldsymbol{\omega}=[\omega_1 \quad \omega_2]^T$, and the mean vector $\boldsymbol{m}=[m_1 \quad m_2]^T$; then the *bivariate* density function is

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{|\boldsymbol{C}|}} \exp\left\{-\frac{1}{2}\left[\left(\boldsymbol{x}^T - \boldsymbol{m}^T\right)\boldsymbol{C}^{-1}\left(\boldsymbol{x} - \boldsymbol{m}\right)\right]\right\} \tag{2.212}$$

where $T$ denotes matrix transpose. The characteristic function becomes

$$\begin{aligned}\Phi_{x_1x_2}(\omega_1, \omega_2) &= \exp\left[-\frac{1}{2}\left(\boldsymbol{\omega}^T\ \boldsymbol{C}\,\boldsymbol{\omega} + j\,\boldsymbol{m}^T\ \boldsymbol{\omega}\right)\right] \\ &= \exp\left(-\frac{1}{2}\sum_{k=1}^{2}\sum_{\ell=1}^{2} C_{k\ell}\omega_k\omega_\ell + j\sum_{k=1}^{2} m_k\omega_k\right)\end{aligned} \tag{2.213}$$

When the correlation coefficient $\rho = 0$, the joint density function becomes

$$\begin{aligned}f_{X_1X_2}(x_1, x_2) &= \frac{1}{2\pi\sigma_1\,\sigma_2}\exp\left\{-\frac{1}{2}\left[(x_1 - m_1)^2 + (x_2 - m_2)^2\right]\right\} \\ &= f_{X_1}(x_1)\,f_{X_2}(x_2)\end{aligned} \tag{2.214}$$

Since the joint density function is the product of the marginal density functions, then $X_1$ and $X_2$ are statistically independent. This is an important characteristic of Gaussian random variables where uncorrelated random variables are necessarily independent. The characteristic function reduces to

$$\begin{aligned}\Phi_{x_1x_2}(\omega_1, \omega_2) &= \exp\left\{-\frac{1}{2}\left[\left(\sigma_1^2\omega_1^2 + \sigma_2^2\omega_2^2\right) + j(m_1\omega_1 + m_2\omega_2)\right]\right\} \\ &= \Phi_{x_1}(\omega_1)\,\Phi_{x_2}(\omega_2)\end{aligned} \tag{2.215}$$

that is, the joint characteristic function equals the product of the marginal characteristic functions when the random variables $X_1$ and $X_2$ are uncorrelated.

*The Standard Ellipse*

The standard ellipse of the bivariate Gaussian density function is obtained from (2.195) by setting the term between brackets in the exponent equal to 1, to yield

$$\frac{(x_1 - m_1)^2}{\sigma_1^2} + \frac{(x_2 - m_2)^2}{\sigma_2^2} - 2\rho\frac{(x_1 - m_1)(x_2 - m_2)}{\sigma_1\,\sigma_2} = 1 \tag{2.216}$$

Equation (2.216) represents the equation of an ellipse centered at $x_1 = m_1$ and $x_2 = m_2$. For simplicity, let $m_1 = m_2 = 0$. The ellipse is easily represented by assuming two independent random variables $U$ and $V$ with zero mean, and respective

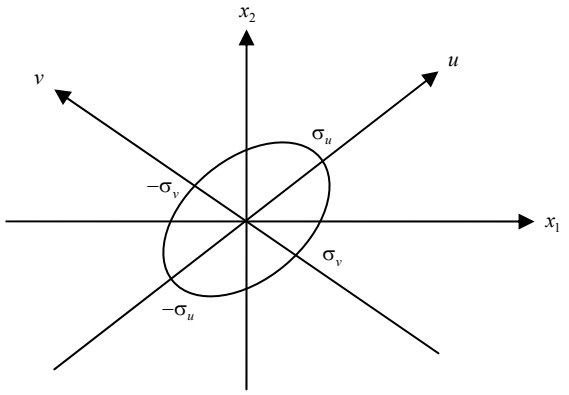


**Figure 2.16** Ellipse centered at $m_1 = m_2 = 0$.

variances $\sigma_u^2$ and $\sigma_v^2$. The standard ellipse, shown in Figure 2.16, is given by

$$\frac{u^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2} = 1 \tag{2.217}$$

The joint density function of $U$ and $V$ is

$$f_{UV}(u,v) = \frac{1}{2\pi\,\sigma_u\,\sigma_v} \exp\left[-\frac{1}{2}\left(\frac{u^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2}\right)\right] \tag{2.218}$$

Applying a rotation by an angle $\theta$ to the $uv$-axes yields the coordinate system $x_1$, $x_2$ given by

$$x_1 = u\cos\theta - v\sin\theta \tag{2.219}$$

$$x_2 = u\sin\theta + v\cos\theta \tag{2.220}$$

The random variables $X_1$ and $X_2$ are obtained by the transformation of (2.219) and (2.220). Specifically,

$$X_1 = U\cos\theta - V\sin\theta \tag{2.221}$$

$$X_2 = U\sin\theta + V\cos\theta \tag{2.222}$$

where

$$\sigma_1^2 = E\left[X_1^2\right] = E\left[U^2 \cos^2\theta - 2UV\cos\theta\sin\theta + V^2\sin^2\theta\right]$$

$$= E\left[U^2\right]\cos^2\theta + E\left[V^2\right]\sin^2\theta$$

$$= \sigma_u^2\cos^2\theta + \sigma_v^2\sin^2\theta \tag{2.223}$$

since $E[U] = E[V] = E[X_1] = E[X_2] = 0$. Note that $\theta$ is the angle at the major axis of the ellipse. Similarly, we can obtain

$$\sigma_2^2 = \sigma_u^2\sin^2\theta + \sigma_v^2\cos^2\theta \tag{2.224}$$

and the covariance between $X_1$ and $X_2$ to be

$$E\left[X_1X_2\right] = E\left[\left(U\cos\theta - V\sin\theta\right)\left(U\sin\theta + V\cos\theta\right)\right]$$

$$= \left(\sigma_u^2 - \sigma_v^2\right)\sin\theta\cos\theta \tag{2.225}$$

The distributions of $U$ and $V$ are derived in a similar manner. Given the distributions of $X_1$ and $X_2$, we obtain

$$\sigma_u^2 = \frac{\sigma_1^2\cos^2\theta - \sigma_2^2\sin^2\theta}{\cos^2\theta - \sin^2\theta} \tag{2.226}$$

$$\sigma_v^2 = \frac{\sigma_2^2\cos^2\theta - \sigma_1^2\sin^2\theta}{\cos^2\theta - \sin^2\theta} \tag{2.227}$$

$$\rho = \frac{E\left[X_1X_2\right]}{\sigma_1\sigma_2} = \frac{1}{2}\left(\frac{\sigma_1}{\sigma_2} - \frac{\sigma_2}{\sigma_1}\right)\tan 2\theta, \quad \theta \neq \pm\frac{\pi}{4}, \pm\frac{3\pi}{4} \tag{2.228}$$

or

$$\theta = \frac{1}{2}\arctan 2\rho\frac{\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2} \tag{2.229}$$

The above results can easily be generalized to $n$ random variables. Let ($X_1$, $X_2$, …, $X_n$) be $n$ jointly Gaussian random variables. We define the means as

$$E[X_k] = m_k, \quad k = 1, 2, \ldots, n \tag{2.230}$$

the covariances as

$$c_{jk} = E[(X_j - m_j)\,(X_k - M_k)], \quad j, k = 1, 2, \ldots, n \tag{2.231}$$

and the correlation coefficients as

$$\rho_{jk} = \frac{c_{jk}}{\sqrt{c_{jj}\; c_{kk}}} \tag{2.232}$$

The variance of $X_k$ is

$$\text{var}\left[ X_k \right] = c_{kk} = \sigma_k^2 \tag{2.233}$$

Let the vectors $\boldsymbol{X}$, $\boldsymbol{x}$, and $\boldsymbol{m}$ be defined as

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix} \tag{2.234}$$

and the covariance matrix $\boldsymbol{C}$ as

$$\begin{aligned} \boldsymbol{C} &= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2 & \rho_{12}\,\sigma_1\,\sigma_2 & \cdots & \rho_{1n}\,\sigma_1\,\sigma_n \\ \rho_{21}\,\sigma_2\,\sigma_1 & \sigma_2^2 & \cdots & \rho_{2n}\,\sigma_2\,\sigma_n \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{n1}\,\sigma_n\,\sigma_1 & \rho_{n2}\,\sigma_n\,\sigma_2 & \cdots & \sigma_n^2 \end{bmatrix} \end{aligned} \tag{2.235}$$

The multivariate Gaussian density function is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{|\boldsymbol{C}|}} \exp\left\{-\frac{1}{2}\left[\left(\boldsymbol{x}^T - \boldsymbol{m}^T\right)\boldsymbol{C}^{-1}\left(\boldsymbol{x} - \boldsymbol{m}\right)\right]\right\} \tag{2.236}$$

The characteristic function corresponding to this $n$-dimensional joint density function is

$$\begin{aligned}\Phi_{\boldsymbol{x}}(\boldsymbol{\omega}) &= E\left[e^{j\boldsymbol{\omega}X}\right] = \exp\left(-\frac{1}{2}\boldsymbol{\omega}^T\,\boldsymbol{C}\,\boldsymbol{\omega} + j\,\boldsymbol{m}^T\,\boldsymbol{\omega}\right) \\ &= \exp\left(-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk}\omega_j\omega_k + j\sum_{k=1}^{n} m_k\,\omega_k\right)\end{aligned} \tag{2.237}$$

If the correlation coefficient $\rho_{jk} = 0$, $j, k = 1, 2, \ldots, n$, then the covariance matrix becomes diagonal to yield

$$\boldsymbol{C} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix} \tag{2.238}$$

Note that the covariance matrix being diagonal is a *necessary and sufficient* condition for the random variables $X_k$, $k = 1, 2, \ldots, n$, to be statistically independent. This will be shown later in detail. The inverse covariance matrix $\boldsymbol{C}^{-1}$ is also diagonal, and is given by

$$\boldsymbol{C}^{-1} = \begin{bmatrix} \sigma_1^{-2} & 0 & \cdots & 0 \\ 0 & \sigma_2^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n^{-2} \end{bmatrix} \tag{2.239}$$

The joint probability density function becomes the product of the marginal density functions to yield

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{k=1}^{n} \frac{1}{\sigma_k\sqrt{2\pi}} \exp\left[-\frac{(x_k - m_k)^2}{2\sigma_k^2}\right] = \prod_{k=1}^{n} f_{X_k}(x_k) \tag{2.240}$$

The joint characteristic function reduces to

$$\Phi_{\boldsymbol{x}}(\omega)=\prod_{k=1}^{n}\exp\left(-\frac{1}{2}\sigma_k^2\omega_k^2+jm_k\omega_k\right)=\prod_{k=1}^{n}\Phi_{x_k}(\omega_k) \tag{2.241}$$

Using the characteristic function, a closed form expression for the joint moments can be obtained. Let $X_1$, $X_2$, … , $X_{2n+1}$ be $(2n + 1)$ zero-mean jointly Gaussian random variables. Then,

$$E\left[\,X_1\;X_2\;\ldots X_{2n+1}\,\right]=\begin{cases}0 & ,\quad (2n+1)\ \text{odd}\\ \sum\prod_{j\neq k}^{n}E\left[\,X_j\;X_k\,\right], & (2n+1)\ \text{even}\end{cases} \tag{2.242}$$

where the summation is taken over all distinct pairs obtained by using each factor once. The number of ways to have such pairs is

$$\frac{(2n)!}{n!\;2^n}=1\cdot3\cdot5\cdot\ldots\cdot(2n-1) \tag{2.243}$$

One of the most frequently used joint moments is the joint moment of order four ($2n = 4$). In this case, $n = 2$ and the number of ways to have the distinct pairs as defined in (2.243) is three. Hence,

$$\begin{aligned}&E\left[X_1X_2X_3X_4\right]\\ &\quad=E\left[X_1X_2\right]E\left[X_3X_4\right]+E\left[X_2X_3\right]E\left[X_1X_4\right]+E\left[X_1X_3\right]E\left[X_2X_4\right]\end{aligned} \tag{2.244}$$

In modern high resolution adaptive thresholding radar CFAR, the clutter (sea clutter, weather clutter, or land clutter) returns may not follow the Gaussian or Rayleigh model, since the amplitude distribution develops a "larger" tail, that may increase the false alarm rate. Some distributions that occur in radar applications and may give a better model in representing the clutter are the Weibull, log-normal, and *K*-distributions.

### 2.4.2 The Weibull Distribution

A random variable $X$ is said to have a *Weibull distribution*, as shown in Figure 2.17, if its probability density function is given by

$$f_X(x)=\begin{cases}a\,b\,x^{b-1}\,e^{-ax^b}, & x>0,\ a>0,\ \text{and}\ b>0\\ 0 & ,\ \text{otherwise}\end{cases} \tag{2.245}$$

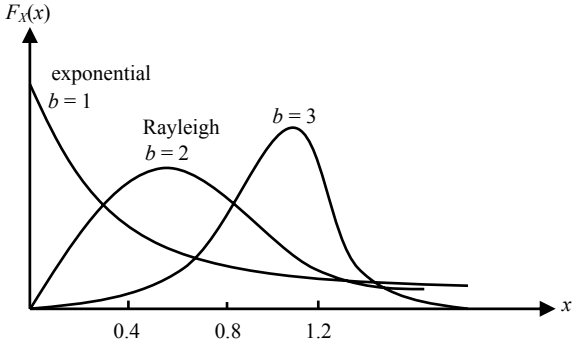


**Figure 2.17** Weibull density function.

where $a$ is referred to as the *scale parameter* and $b$ is the *shape parameter*. Note that for $b = 1$, we obtain $f_X(x) = a\, e^{-ax}$, $x > 0$, and $a > 0$, which is the exponential distribution given in (2.89). When $b = 2$, the Weibull density function becomes

$$f_X(x) = \begin{cases} 2\,a\,x\,e^{-ax^2}, & x > 0 \text{ and } a > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.246}$$

which is the Rayleigh density function defined in (2.146) with $a = 1/2\,\sigma^2$. The distribution function of the Weibull random variable $X$ is then

$$F_X(x) = \begin{cases} 1 - e^{-ax^b}, & x > 0,\ a > 0, \text{ and } b > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.247}$$

The mean and variance of $X$ are given by

$$E[X] = a^{-\frac{1}{b}}\,\Gamma\left(1 + \frac{1}{b}\right) \tag{2.248}$$

and

$$\text{var}[X] = a^{-\frac{2}{b}}\left\{\Gamma\left(1 + \frac{2}{b}\right) - \left[\Gamma\left(1 + \frac{1}{b}\right)\right]^2\right\} \tag{2.249}$$

while the moment of order $k$ is

$$E\left[X^k\right] = a^{-\frac{k}{b}}\,\Gamma\left(1+\frac{k}{b}\right) \tag{2.250}$$

Many authors write the Weibull density function in the form

$$f_X(x) = \begin{cases} \dfrac{c}{b}\left(\dfrac{x}{b}\right)^{c-1} \exp\left[-\left(\dfrac{x}{b}\right)^c\right], & x>0, a>0, b>0, \text{and } c>0 \\ 0, & \text{otherwise} \end{cases} \tag{2.251}$$

where in this case, $b$ is the *scale parameter* and $c$ is the *shape parameter*. Note that (2.251) is equivalent to (2.245) with $a = 1/b^c$. When $X = \ln Z$, the density function of $f_X(x)$ is said to have a *log-Weibull distribution* for the variable $Z$.

### 2.4.3 The Log-Normal Distribution

A random variable $X$ is said to have a *log-normal distribution* if its density function is given by

$$f_X(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\dfrac{\ln^2 \dfrac{x}{x_m}}{2\sigma^2}\right), & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.252}$$

where $x_m$ is the median of $X$ and $\sigma^2$ is the variance of the generating normal distribution. A parameter commonly used to characterize the log-normal distribution is the mean-to-median ratio $\rho$ given by

$$\rho = \frac{E[X]}{X_m} \tag{2.253}$$

Alternatively, the density function of the log-normal random variable $X$ can be written as

$$f_X(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\dfrac{(\ln x - \ln x_m)^2}{2\sigma^2}\right], & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.254}$$

The cumulative distribution function of $X$ is

$$F_X(x) = \frac{1}{2}\left[1 + \text{erf}(u)\right] \tag{2.255}$$

where

$$u = \frac{1}{\sqrt{2}\,\sigma} \ln\left(\frac{x}{x_m}\right) \tag{2.256}$$

The mean and the variance of $X$ are

$$E[X] = x_m \exp\left(\frac{\sigma^2}{2}\right) \tag{2.257}$$

and

$$\text{var}[X] = x_m^2 \, e^{\sigma^2}\left(e^{\sigma^2} - 1\right) \tag{2.258}$$

while the moment of order $k$ is

$$E\left[X^k\right] = x_m^k \exp\left(\frac{k^2\,\sigma^2}{2}\right) \tag{2.259}$$

### 2.4.4 The *K*-Distribution

The *K*-distribution has arisen mainly to represent radar sea clutter. A random variable $X$ with probability density function

$$f_X(x) = \begin{cases} \dfrac{4}{b\,\Gamma(\nu)}\left(\dfrac{x}{b}\right)^{\nu} K_{\nu-1}\left(\dfrac{2}{b}x\right) & , \quad x \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.260}$$

is said to have a *K-distribution*. $K_\nu(x)$ is the *modified Bessel function*, $b$ is the *scale parameter*, and $\nu$ is the *shape parameter*. It is known from radar detection that the *K*-distribution results from a function of two random variables given by

$$X = S\,T \tag{2.261}$$

where $S$, known as *speckle*, obeys a Rayleigh distribution given by

$$f_S(s) = 2\, s\, e^{-s^2}, \quad s > 0 \tag{2.262}$$

and $T$, known as *texture*, is a gamma distribution given by

$$f_T(t) = \frac{2}{b^{\nu}\Gamma(\nu)}\, t^{2\nu-1}\, e^{-\frac{t^2}{b^2}} \tag{2.263}$$

The total probability density function $f_X(x)$ is also known in terms of conditional probabilities to be

$$f_X(x) = \int_0^{\infty} f_{X|T}(x \mid t)\, f_T(t)\, dt \tag{2.264}$$

where

$$f_{X|T}(x|t) = \frac{2x}{t^2}\, e^{-\frac{x^2}{t^2}} \tag{2.265}$$

Substituting (2.265) and (2.263) into (2.264) and solving the integral, we obtain

$$f_X(x) = \int_0^{\infty} \frac{2x}{t^2}\, e^{-\frac{x^2}{t^2}}\, \frac{2}{b^{\nu}\,\Gamma(\nu)}\, t^{2\nu-1}\, e^{-\frac{t^2}{b^2}}\, dt = \frac{4}{b\,\Gamma(\nu)} \left(\frac{x}{b}\right)^{\nu} K_{\nu-1}\left(\frac{2}{b}x\right) \tag{2.266}$$

The moment of order $k$ is given by

$$E\left[X^k\right] = b^k\, \frac{\Gamma\left(\nu + \frac{k}{2}\right)\Gamma\left(1 + \frac{k}{2}\right)}{\Gamma(\nu)} \tag{2.267}$$

From (2.261), it was shown by Anastassopoulos et al. [1] that when the distribution of the speckle $S$ is a generalized gamma and the texture $T$ is also a generalized gamma, the resulting distribution is referred to as *the generalized K-distribution*, and is given by

$$f_X(x; a, b, \nu_1, \nu_2) = \begin{cases} \dfrac{2a}{b\,\Gamma(\nu_1)\Gamma(\nu_2)} \left(\dfrac{x}{b}\right)^{\frac{a}{2}(\nu_1+\nu_2)-1} K_{\nu_1-\nu_2}\left[2\left(\dfrac{x}{b}\right)^{\frac{a}{2}}\right], & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.268}$$

where $b$ is the *scale parameter* and $K_\nu(x)$ is the *modified Bessel function*. The moment of order $k$ is given by

$$E\left[X^k\right] = b^k \frac{\Gamma\left(\nu_1 + \dfrac{k}{a}\right)\Gamma\left(\nu_2 + \dfrac{k}{a}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{2.269}$$

It should be noted that when $a = 2$ and $\nu_1 = 1$ in (2.268), we obtain the *K*-density function given in (2.260). Also, if we let $\nu_1 = 1$ and $\nu_2 = 1/2$, the generalized *K*-density function becomes

$$f_X(x) = \begin{cases} \dfrac{2a}{b\,\Gamma(1/2)} \left(\dfrac{x}{b}\right)^{\frac{3a}{4}-1} K_{-\frac{1}{2}}\left[2\left(\dfrac{x}{b}\right)^{\frac{a}{2}}\right], & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.270}$$

Using the fact that

$$K_{n+\frac{1}{2}}(x) = -K_{-n-\frac{1}{2}}(x), \quad n = 0, 1, 2, \ldots \tag{2.271}$$

and

$$\sqrt{\frac{\pi}{2x}}\, K_{\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right) e^{-x} \tag{2.272}$$

we obtain the Weibull density function to be

$$f_X(x;\rho,b)=\begin{cases}\dfrac{a}{2\rho}\left(\dfrac{x}{\rho}\right)^{\frac{a}{2}-1}\exp\left[-\left(\dfrac{x}{\rho}\right)^{\frac{a}{2}}\right], & x>0\\ 0, & \text{otherwise}\end{cases} \tag{2.273}$$

with $\rho = b\,2^{-2/a}$. The moment of order $k$ is

$$E\left[X^k\right]=\rho^k\ \Gamma\left(1+\frac{2k}{a}\right) \tag{2.274}$$

If again we set $a = 2$ in the Weibull density function given by (2.273), then we obtain the exponential distribution to be

$$f_X(x)=\begin{cases}\dfrac{1}{\rho}e^{-\frac{1}{\rho}x}, & x\geq 0\\ 0, & \text{otherwise}\end{cases} \tag{2.275}$$

and when we set $a = 4$ [in (2.273)], we obtain the Rayleigh density function to be

$$f_X(x)=\begin{cases}\dfrac{2}{\rho}\left(\dfrac{x}{\rho}\right)\exp\left[-\left(\dfrac{x}{\rho}\right)^2\right], & x>0\\ 0, & \text{otherwise}\end{cases} \tag{2.276}$$

### 2.4.5 The Generalized Compound Distribution

As the *K*-distribution, which is a compound distribution, the *generalized compound distribution* is used to represent radar clutter in more severe situations when the distortion of the speckle, usually represented by a Rayleigh density function, has a longer tail. In this case, the distribution of the speckle is the generalized gamma distribution, and the conditional density function is given by

$$f_{X|S}(x\,|\,s)=\begin{cases}\dfrac{a_1}{s\,\Gamma(\nu_1)}\left(\dfrac{x}{s}\right)^{a_1\nu_1-1}\exp\left[-\left(\dfrac{x}{s}\right)^{a_1}\right], & x\geq 0\\ 0, & \text{otherwise}\end{cases} \tag{2.277}$$

whereas the density function of the speckle is

$$f_S(s)=\begin{cases}\dfrac{a_2}{b\,\Gamma(\nu_2)}\left(\dfrac{s}{b}\right)^{a_2\nu_2-1}\exp\left[-\left(\dfrac{s}{b}\right)^{a_2}\right], & s>0\\ 0, & \text{otherwise}\end{cases} \tag{2.278}$$

Thus, the total probability density function of the *generalized compound distribution* is given by

$$\begin{aligned}f_X(x)&=\int_0^\infty f_{X|S}(x|s)\,f_S(s)\,ds\\ &=\frac{a_1\,a_2}{\Gamma(\nu_1)\Gamma(\nu_2)}\frac{x^{a_1\nu_1-1}}{b^{a_2\nu_2}}\int_0^\infty s^{a_2\nu_2-a_1\nu_1-1}\exp\left[-\left(\frac{s}{b}\right)^{a_2}-\left(\frac{x}{s}\right)^{a_1}\right]ds\end{aligned} \tag{2.279}$$

which does not have a closed form. The mean of order $k$ is shown to be

$$E\left[X^k\right]=b^k\,\frac{\Gamma\left(\nu_1+\dfrac{k}{a_1}\right)\Gamma\left(\nu_2+\dfrac{k}{a_2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{2.280}$$

## 2.5 SUMMARY

In this chapter, we defined some distributions and gave the relationships that may exist between them. We started describing the simplest distributions for discrete random variables; namely, Bernoulli and binomial distributions. Then we extended the results to multinomial and hypergeometric distributions. The Poisson distribution, which arises in many applications, was also presented in some detail.

In the second part, we presented some important continuous distributions, and we showed the possible relationships that may exist between them. Many distributions were presented in order to give a more or less complete view of these different distributions. Then we gave some special distributions that arise in many applications of radar and communication systems. These distributions were presented in some detail, since we will discuss their applications in Chapters 11 and 12.

## PROBLEMS

**2.1** A pair of dice is rolled six times. A success is when the sum of the top appearing faces is seven.
(a) What is the probability that seven will appear twice?
(b) What is the probability that seven will not appear at all?

**2.2** An urn contains 10 white balls, 4 black balls, and 5 red balls. The experiment is to draw a ball and note its color without replacement. Find the probability of obtaining the fourth white ball in the seventh trial.

**2.3** In a special training, a parachutist is expected to land in a specified zone 90% of the time. Ten of them jumped to land in the zone.
(a) Find the probability that at least six of them will land in the specified zone.
(b) Find the probability that none lands in the specified zone.
(c) The training is considered successful if the probability that at least 70% of them land in the prescribed zone is 0.93. Is the training successful?

**2.4** A random variable $X$ is Poisson distributed with parameter $\lambda$ and $P(X=0)=0.2$. Calculate $P(X>2)$.

**2.5** The incoming calls to a particular station have a Poisson distribution with an intensity of 12 per hour. What is the probability that:
(a) More than 15 calls will come in any given hour?
(b) No calls will arrive in a 15-minute break?

**2.6** A random variable $X$ is Poisson distributed with $P(X=2)=\frac{2}{3}P(X=1)$. Calculate $P(X=0)$ and $P(X=3)$.

**2.7** A random variable $X$ has the following exponential distribution with parameter $\alpha$,

$$f_X(x)=\begin{cases}\alpha e^{-\alpha x}, & x>0\\ 0, & \text{otherwise}\end{cases}$$

Show that $X$ has the "lack of memory property." That is, show that

$$P(X\geq x_1+x_2 \mid X>x_1)=P(X\geq x_2)$$

for $x_1, x_2$ positive.

**2.8** Solve Problem 2.5, assuming that $X$ has an exponential distribution.

**2.9** A random variable $X$ is Gaussian with zero mean and variance unity. What is the probability that
(a) $|X| > 1$?
(b) $X > 1$?

**2.10** A random variable $X$ has the distribution $\mathcal{N}(0, 1)$. Find the probability that $X > 3$.

**2.11** Two fair dice are thrown 200 times. Let $X = 7$, the sum of the upper faces, denote a success.
(a) Determine the probability of having success at least 20% of the time.
(b) Use the central limit theorem to evaluate (a).

**2.12** Let $S = X_1 + X_2 + \ldots + X_k + \ldots + X_{100}$, where each $X_k$, $k = 1, 2, \ldots, 100$, is a Poisson distributed random variable with parameter $\lambda = 0.032$.
(a) Determine the probability of $S$ greater than 5.
(b) Use the central limit theorem to evaluate (a).

**2.13** Let $X$ be a normal random variable with mean $E[X] = 1$ and variance $\sigma^2 = 2$. Using tables, evaluate
(a) $P(X > 2)$
(b) $P(1.6 \le X \le 2.2)$

**2.14** Let $X$ be a random variable uniformly distributed between 1 and 6. Determine and sketch the density function $f_Y(y)$ of $Y = 1/X$.

**2.15** Let $X$ be a random variable uniformly distributed between 0 and 1. Find and sketch the density function of
(a) $Y = X^2$
(b) $Z = e^X$

**2.16** Let $X$ and $Y$ be two independent standard normal random variables. Find the density function of
(a) $Z = X/|Y|$
(b) $W = |X|/|Y|$

**2.17** $X_1$ and $X_2$ are two normal random variables with joint density function

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{\left(x_1^2 + x_2^2\right)}{2\sigma^2}\right]$$

Let the transformation be $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $Y_2 = X_1 / X_2$. Determine $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$.

**2.18** Using the distribution function, show the density function of student's *t*-distribution given in (2.171).

**2.19** Show that the characteristic function of the Cauchy distributed random variable of (2.192) with $\alpha = 0$ is given by

$$\Phi_x(\omega) = e^{-\beta\omega}$$

**2.20** For the Weibull distribution, show that the mean and variance are as given by (2.248) and (2.249), respectively.

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# Chapter 3

# Random Processes

## 3.1 INTRODUCTION AND DEFINITIONS

A random process may be viewed as a collection of random variables, with time $t$ as a parameter running through all real numbers. In Chapter 1, we defined a random variable as a mapping of the elements of the sample space $S$ into points of the real axis. For random processes, the sample space would map into a family of time functions. Formally, we say a random process $X(t)$ is a mapping of the elements of the sample space into functions of time. Each element of the sample space is associated with a time function as shown in Figure 3.1.

Associating a time function to each element of the sample space results in a family of time functions called the *ensemble.* Hence, the ensemble is the set of sample functions with the associated probabilities. Observe that we are denoting the random process by $X(t)$, and not $X(t, \xi)$, where the dependence on $\xi$ is omitted. A sample function is denoted by $x(t)$.

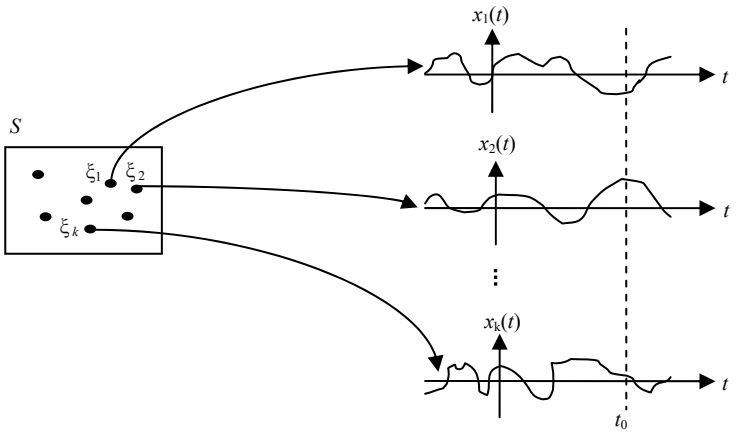


**Figure 3.1** Mapping of sample space into sample functions.

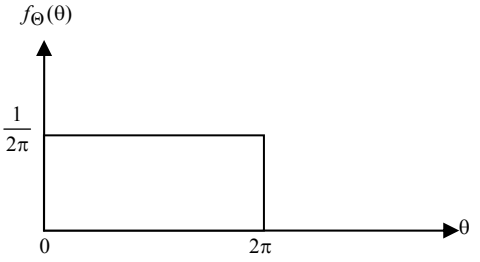


**Figure 3.2** Density function of Θ.

## Example 3.1

Consider a random process $X(t) = A\cos(\omega t + \Theta)$, where Θ is a random variable uniformly distributed between 0 and 2π, as shown in Figure 3.2. That is,

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

some sample functions of this random process are shown in Figure 3.3.

This variation in the sample functions of this particular process is due to the phase only. Such a random process, for which future values are predicted from knowledge of past ones, is said to be predictable or *deterministic*. In fact, fixing the phase to some particular value, π/4, the sample function (corresponding to the particular element $\xi_k$ of the sample space) becomes a *deterministic time function;* that is, $x_k(t) = A\cos[\omega t + (\pi/4)]$.

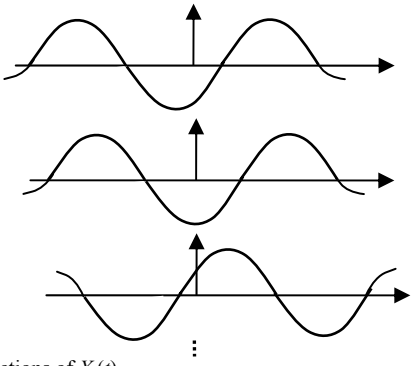

**Figure 3.3** Some sample functions of $X(t)$.

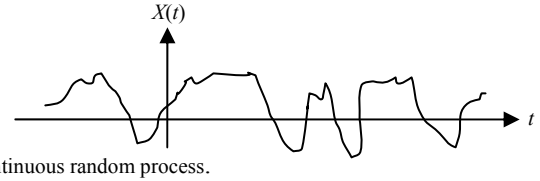


**Figure 3.4** A continuous random process.

When the parameter $t$ is fixed to some instant $t_0$, the random process $X(t)$ becomes the random variable $X(t_0)$, and $x(t_0)$ would be a sample value of the random variable. In general, we are interested in four types of random processes, according to the characteristic of time $t$ and the random variable $X(t) = X$ at time $t$. They are:

1. *Continuous-state and continuous-time.* In this case, both $X(t)$ and $t$ have a continuum of values. $X(t)$ is said to be a *continuous random process*, and is as shown in Figure 3.4.

2. *Discrete-state and continuous-time.* $X(t)$ assumes a discrete set of values while time $t$ is continuous. Such a process is referred to as a *discrete random process*, and is as shown in Figure 3.5.

3. *Continuous-state and discrete-time.* $X(t)$ assumes a continuum of values while $t$ assumes a discrete set of values. Such a process is called a *continuous random sequence*, and is as shown in Figure 3.6.

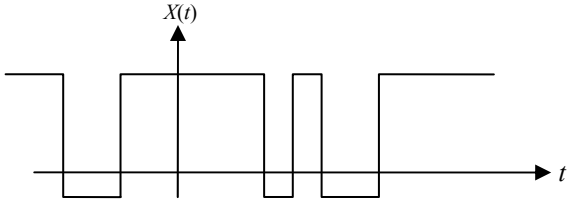


**Figure 3.5** A discrete random process.

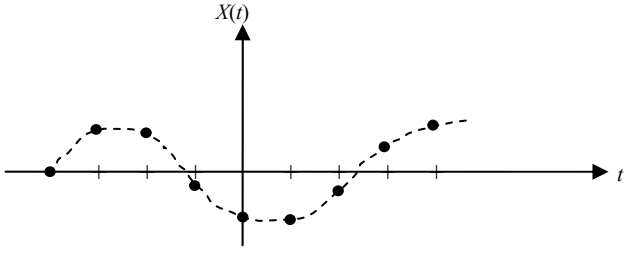


**Figure 3.6** A continuous random sequence.

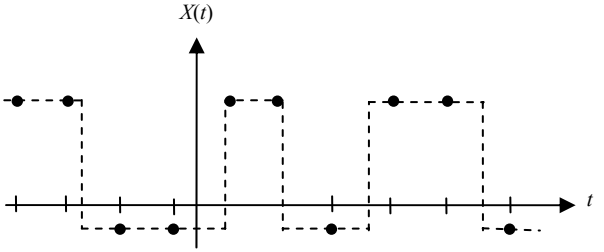


**Figure 3.7** A discrete random sequence.

4. *Discrete-state and discrete-time.* Both $X(t)$ and time $t$ assume a discrete set of values. Such a process is referred to as a *discrete random sequence*, and is as shown in Figure 3.7.

Fixing the time $t$, the random process $X(t)$ becomes a random variable. In this case, the techniques we use with random variables apply. Consequently, we may characterize a random process by the first-order distribution as

$$F_X(x;t) = P\left[X(t_0) \le x\right] \tag{3.1}$$

or by the first-order density function as

$$f_X(x;t) = \frac{d}{dx} F_X(x;t) \tag{3.2}$$

for all possible values of $t$. The second-order distribution function is the joint distribution of the two random variables $X(t_1)$ and $X(t_2)$ for each $t_1$ and $t_2$. This results in

$$F_X(x_1, x_2; t_1, t_2) = P\left[X(t_1) \le x_1 \text{ and } X(t_2) \le x_2\right] \tag{3.3}$$

while the second-order density function is

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2; t_1, t_2) \tag{3.4}$$

Normally, a complete probabilistic description of an arbitrary random process requires the specification of distributions of all orders given by

$$F_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ \ldots\ ,x_n;t_1,t_2,\ \ldots\ ,t_n) = P\left[X(t_1)\le x_1,\, X(t_2)\le x_2,\ \ldots\ ,X(t_n)\le x_n\right] \tag{3.5}$$

or given by the *n*th order density function:

$$f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ \ldots\ ,x_n;t_1,t_2,\ldots,t_n) = \frac{\partial^n F_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ \ldots,x_n;t_1,t_2,\ldots,t_n)}{\partial x_1 \partial x_2 \ldots \partial x_n} \tag{3.6}$$

Fortunately, we are usually interested in processes that may possess some regularity so that they can be described more simply, and knowledge of the first- and second-order density functions may be sufficient to generate higher-order density functions.

## 3.2 EXPECTATIONS

In many problems of interest, only the first- and second-order statistics may be necessary to characterize a random process. Given a real random process $X(t)$, its mean value function is

$$m_x(t) = E\left[X(t)\right] = \int_{-\infty}^{+\infty} x\, f_X(x,t)dx \tag{3.7}$$

The autocorrelation function is defined to be

$$R_{xx}(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_{X_1X_2}(x_1,x_2;t_1,t_2)dx_1dx_2 \tag{3.8}$$

When the autocorrelation function $R_{xx}(t_1, t_2)$ of the random process $X(t)$ varies only with the time difference $|t_1 - t_2|$, and the mean $m_x$ is constant, $X(t)$ is said to be *stationary in the wide-sense,* or *wide-sense stationary.* In this case, the autocorrelation function is written as a function of one argument $\tau = t_1 - t_2$. If we let $t_2 = t$ and $t_1 = t+\tau$, then the autocorrelation function, in terms of $\tau$ only, is

$$R_{xx}(t+\tau,t) = R_{xx}(\tau) \tag{3.9}$$

A random process $X(t)$ is *strictly stationary* or *stationary in the strict sense* if its statistics are unchanged by a time shift in the time origin. Note that a stationary process in the strict sense is stationary in a wide-sense, but not the opposite. The

condition for wide-sense stationary is weaker than the condition for the second-order stationary because, for wide-sense stationary processes, only the second-order statistics, the autocorrelation function, is constrained.

**Example 3.2**

Is the random process given in Example 3.1 wide-sense stationary?

*Solution*

For a random process to be stationary in the wide-sense, it must satisfy two conditions; namely, $E[X(t)]$ = constant and $R_{xx}(t+\tau,t) = R_{xx}(\tau)$. To compute the mean of $X(t)$, we use the concept that

$$E\left[g(\Theta)\right] = \int_{-\infty}^{+\infty} g(\theta) f_{\Theta}(\theta) d\theta$$

where in this case, $g(\theta) = A\cos(\omega t + \theta)$ and $f_{\Theta}(\theta) = 1/2\pi$ in the interval between 0 and $2\pi$. Then

$$E\left[X(t)\right] = \int_{0}^{2\pi} A\cos(\omega t + \theta)\frac{1}{2\pi} d\theta = 0$$

The autocorrelation function is

$$\begin{aligned} E\left[X(t+\tau,t)X(t)\right] &= E\left\{A\cos[\omega(t+\tau)+\theta]A\cos(\omega t+\theta)\right\} \\ &= \frac{A^2}{2} E\left[\cos(\omega\tau) + \cos(2\omega t + \omega\tau + 2\theta)\right] \end{aligned}$$

where we have used the trigonometric identity

$$\cos a \ \cos b = \frac{1}{2}\left[\cos\ (a-b) + \cos\ (a+b)\right]$$

The second term evaluates to zero. Thus, the autocorrelation function is

$$R_{xx}(t+\tau,t) = \frac{A^2}{2}\cos\omega\tau = R_{xx}(\tau)$$

Since the mean is constant and the autocorrelation depends on $\tau$ only, $X(t)$ is a wide-sense stationary process.

When dealing with two random processes $X(t)$ and $Y(t)$, we say that they are *jointly wide-sense stationary* if each process is stationary in the wide-sense, and

$$R_{xy}(t+\tau,t) = E\big[X(t+\tau)Y(t)\big] = R_{xy}(\tau) \tag{3.10}$$

$R_{xy}(t_1,\ t_2)$ represents the cross-correlation function of $X(t)$ and $Y(t)$. We also define the covariance function $C_{xx}(t_1,t_2)$ and cross-covariance function $C_{xy}(t_1,t_2)$ between $X(t)$ and $Y(t)$ as

$$C_{xx}(t_1,t_2) = E\big\{[X(t_1)-m_x(t_1)][X(t_2)-m_x(t_2)]\big\} \tag{3.11}$$

and

$$C_{xy}(t_1,t_2) = E\big\{[X(t_1)-m_x(t_1)][Y(t_2)-m_y(t_2)]\big\} \tag{3.12}$$

If $Z(t)$ is a complex random process such that $Z(t) = X(t) + jY(t)$, the autocorrelation and autocovariance functions of $Z(t)$ are

$$R_{zz}(t_1,t_2) = E[Z(t_1)Z^*(t_2)] \tag{3.13}$$

and

$$C_{zz}(t_1,t_2) = E[\{Z(t_1)-m_z(t_1)\}\{Z(t_2)-m_z(t_2)\}^*] \tag{3.14}$$

where $*$ denotes a complex conjugate and $m_z(t)$ is the mean function of $Z(t)$. The cross-correlation and cross-covariance functions between the complex random process $Z(t)$ and another complex random process $W(t)$, $W(t) = U(t) + jV(t)$, is

$$R_{zw}(t_1,t_2) = E[Z(t_1)W^*(t_2)] \tag{3.15}$$

and

$$C_{zw}(t_1,t_2) = E\big\{[Z(t_1)-m_z(t_1)][W(t_2)-m_w(t_2)]^*\big\} \tag{3.16}$$

**Example 3.3**

Consider an experiment of tossing a coin in an infinite number of interval times. A sample function of the random process $X(t)$ is defined as

$$x(t) = \begin{cases} 1 & \text{for} \quad (n-1)T \le t < nT \text{ if heads at } nth \text{ toss} \\ -1 & \text{for} \quad (n-1)T \le t < nT \text{ if tails at } nth \text{ toss} \end{cases}$$

where $n$ takes all possible integer values. Is the process stationary in the wide-sense?

*Solution*

For the process to be wide-sense stationary, it must be verified that it has a constant mean, and an autocorrelation function which is a function of $\tau$ only.

Let $P(H) = P(\text{head})$ and $P(T) = P(\text{tail})$. Then, from Figure 3.8,

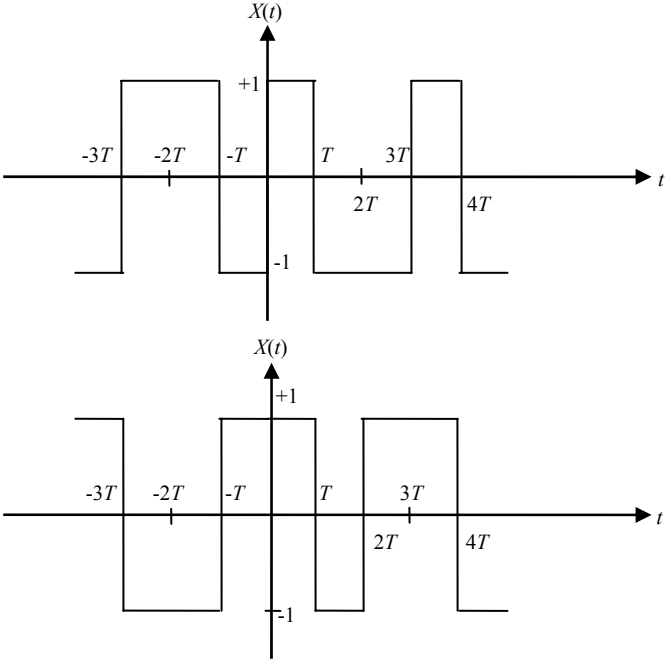


**Figure 3.8** Sample functions of $X(t)$.

$$E[X(t)] = (1)P(H) + (-1)P(T) = (1)\frac{1}{2} + (-1)\frac{1}{2} = 0$$

Since the mean is constant, the process may be wide-sense stationary. The mean-square value is

$$E[X^2(t)] = (1)^2 P(H) + (-1)^2 P(T) = 1$$

We now consider the autocorrelation function

$$R_{xx}(t_1, t_2) = E\left[X(t_1)X(t_2)\right]$$

We have two cases to consider.

*Case 1:* $t_1$ and $t_2$ in the same tossing interval.

In this case, $(n-1)T \le t_1,\ t_2 \le nT$. Hence,

$$R_{xx}(t_1, t_2) = E\left[X(t_1)X(t_2)\right] = E[X^2(t)] = 1$$

*Case 2:* $t_1$ and $t_2$ in different tossing intervals.

We have $(j-1)T \le t_1 \le jT$ and $(k-1)T \le t_2 \le kT$ for $j \ne k$. Since successive tosses are statistically independent, $X(t_1)$ and $X(t_2)$ are also statistically independent. Therefore,

$$R_{xx}(t_1, t_2) = E\left[X(t_1)X(t_2)\right] = E\left[X(t_1)\right]E\left[X(t_2)\right] = 0$$

Since the autocorrelation function is not a function of one variable $\tau = t_1 - t_2$, the process $X(t)$ is not stationary. This process is referred to as *semirandom binary transmission.*

**Example 3.4**

Consider the random process $Y(t) = X(t - \Theta)$, where $X(t)$ is the process of Example 3.3, and $\Theta$ is a random variable uniformly distributed over the interval 0 to $T$. $\Theta$ is statistically independent of $X(t)$. Is $Y(t)$ stationary in the wide-sense?

*Solution*

A sample function of $Y(t)$ is shown in Figure 3.9. As in the previous example, the mean of $Y(t)$ is

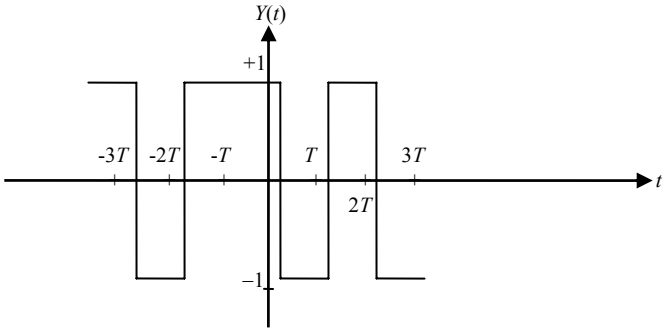


**Figure 3.9** Sample function of $Y(t)$.

$$E[Y(t)] = (1)P(H) + (-1)P(T) = 0$$

which is a constant. The autocorrelation function is given by

$$R_{yy}(t_1, t_2) = E[Y(t_1)Y(t_2)]$$

We have a few possible cases.

*Case 1:* $\tau = t_1 - t_2$ and $|\tau| > T$

In this case, $t_1$ and $t_2$ are in different tossing intervals for each sample function, and hence $Y(t_1)$ and $Y(t_2)$ are statistically independent. Thus,

$$R_{yy}(t_1, t_2) = E[Y(t_1)Y(t_2)] = E[Y(t_1)]E[Y(t_2)] = 0$$

*Case 2:* $|\tau| \le T$

In this case, $t_1$ and $t_2$ may or may not be in the same tossing interval. Let *SI* denote the event that $t_1$ and $t_2$ occur in the same interval, and $SI^c$ (the complementary event of *SI*) be the event that $t_1$ and $t_2$ do not occur in the same interval. Thus,

$$\begin{aligned} R_{yy}(t_1, t_2) &= E[Y(t_1)Y(t_2)] \\ &= E[Y(t_1)Y(t_2) \mid SI]P(SI) + E[Y(t_1)Y(t_2) \mid SI^c]P(SI^c) \end{aligned}$$

Example 3.2 has shown that $E[Y(t_1)Y(t_2) \mid SI] = 1$ and $E[Y(t_1)Y(t_2) \mid SI^c] = 0$. Hence, the autocorrelation function is just the probability that the event *SI* occurs.

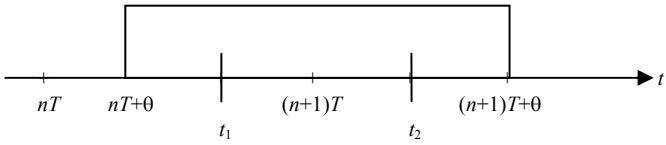


**Figure 3.10** One interval for $-T \le \tau \le 0$.

$$R_{yy}(t_1, t_2) = P(SI)$$

The event *SI* occurs in two possible ways: $t_1 < t_2\,(\tau < 0)$ and $t_2 < t_1\,(\tau > 0)$.

When $t_1 < t_2$, $-T \le \tau \le 0$. The situation is best represented by the diagram of Figure 3.10 representing one interval only. $t_1$ and $t_2$ are in the same interval if $t_1 > nT + \theta$ and $t_2 < (n + 1)T + \theta$, which yields

$$t_2 - (n-1)T < \theta < t_1 - nT$$

Since Θ is uniformly distributed between 0 and $T$, then the probability that $t_1$ and $t_2$ are in the same interval is

$$P(SI) = \int_{t_2-(n+1)T}^{t_1-nT} \frac{1}{T}\, d\theta = 1 + \frac{\tau}{T} \qquad \text{for} \qquad -T \le \tau \le 0$$

Similarly, when $t_2 < t_1$ and $t_1$ and $t_2$ are in the same interval, we have $t_1 < (n + 1)T + \theta$ and $t_2 > nT + \theta$, which yields

$$t_1 - (n+1)T < \theta < t_2 - nT$$

and

$$P(SI) = 1 - \frac{\tau}{T} \quad \text{for} \quad 0 \le \tau \le T$$

Therefore, the autocorrelation function of $Y(t)$ is

$$R_{yy}(t_1, t_2) = \begin{cases} 1 - \dfrac{|\tau|}{T}, & |\tau| \le T \\ 0, & |\tau| > T \end{cases}$$

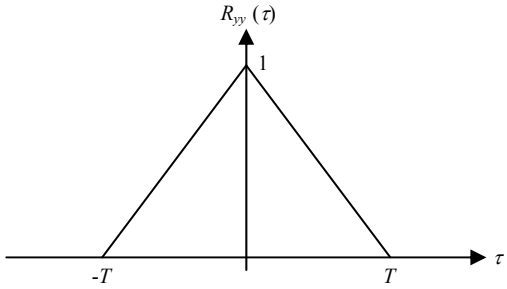


**Figure 3.11** Autocorrelation function of $Y(t)$.

and is shown in Figure 3.11. Because both conditions (the mean is constant and the autocorrelation function is a function of $\tau$ only) are satisfied, the process $Y(t)$ is wide-sense stationary. $Y(t)$ is also referred to as *random binary transmission*.

**Example 3.5**

Let $I(t)$ and $Q(t)$ be two random processes such that

$$I(t) = X\cos\omega t + Y\sin\omega t \quad \text{and} \quad Q(t) = Y\cos\omega t - X\sin\omega t$$

where $X$ and $Y$ are zero mean and uncorrelated random variables. The mean-square values of $X$ and $Y$ are $\mathrm{E}[\mathrm{X}^2] = \mathrm{E}[\mathrm{Y}^2] = \sigma^2$. Derive the cross-correlation function between the processes $I(t)$ and $Q(t)$.

*Solution*

The cross-correlation function between $I(t)$ and $Q(t)$ is

$$\begin{aligned} R_{iq}(t+\tau, t) &= E[I(t+\tau)Q(t)] \\ &= E\{[X\cos(\omega t+\omega\tau) + Y\sin(\omega t+\omega\tau)][Y\cos\omega t - X\sin\omega t]\} \\ &= E[XY][\cos(\omega t+\omega\tau)\cos\omega t - \sin(\omega t+\omega\tau)\sin\omega t] \\ &\quad - E[X^2]\cos(\omega t+\omega\tau)\sin\omega t + E[Y^2]\sin(\omega t+\omega\tau)\cos\omega t \end{aligned}$$

Using trigonometric identities and the fact that $X$ and $Y$ are uncorrelated and zero mean ($E[XY] = E[X]\,E[Y] = 0$), we obtain

$$R_{iq}(t+\tau, t) = -\sigma^2\sin\omega\tau$$

## 3.3 PROPERTIES OF CORRELATION FUNCTIONS

The autocorrelation and the cross-correlation functions introduced in the previous sections are very important concepts in understanding random processes. In this section, we study some of their properties that are most relevant, without giving any formal proof.

### 3.3.1 Autocorrelation Function

Some of the properties of the autocorrelation function are:

$$R_{xx}(t_2, t_1) = R_{xx}^*(t_1, t_2) \tag{3.17}$$

If $X(t)$ is real, then the autocorrelation function is symmetric about the line $t_1 = t_2$ in the $(t_1, t_2)$ plane; that is,

$$R_{xx}(t_2, t_1) = R_{xx}(t_1, t_2) \tag{3.18}$$

The mean-square value function of a random process $X(t)$ is always positive; That is,

$$R_{xx}(t_1, t_1) = E[X(t_1)X^*(t_1)] = E[|X(t)|^2] \geq 0 \tag{3.19}$$

If $X(t)$ is real, the mean-square value $E[X^2(t)]$ is always nonnegative.

$$|R_{xy}(t_1, t_2)| \leq \sqrt{R_{xx}(t_1, t_1)R_{xx}(t_2, t_2)} \tag{3.20}$$

This is known as *Schwarz inequality*, and can be written as

$$|R_{xx}(t_1, t_2)|^2 \leq E[|X(t_1)|^2]E[|X(t_2)|^2] \tag{3.21}$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_i a_j^* R_{xx}(t_i, t_j) \geq 0 \tag{3.22}$$

for any set of constants $a_1, a_2, \ldots, a_n$, and any set of time instants $t_1, t_2, \ldots, t_n$. Therefore, the autocorrelation function is a *nonnegative definite function*.

### 3.3.2 Cross-Correlation Function

Consider $X(t)$ and $Y(t)$ to be two random processes, then

$$R_{xy}(t_1, t_2) = R^*_{yx}(t_2, t_1) \tag{3.23}$$

If the random processes $X(t)$ and $Y(t)$ are real,

$$R_{xy}(t_1, t_2) = R_{yx}(t_2, t_1) \tag{3.24}$$

In general, $R_{xy}(t_1, t_2)$ and $R_{yx}(t_2, t_1)$ are not equal.

$$\begin{aligned} \left|R_{xy}(t_1, t_2)\right| &= \left|E\left[X(t_1)\right]E\left[Y(t_2)\right]\right| \\ &\le \sqrt{R_{xx}(t_1, t_1)R_{yy}(t_2, t_2)} = \sqrt{E[\left|X(t_1)\right|^2]E[\left|Y(t_2)\right|^2]} \end{aligned} \tag{3.25}$$

### 3.3.3 Wide-Sense Stationary

We now consider the processes $X(t)$ and $Y(t)$ to be real and wide-sense stationary.

The autocorrelation function is an even function of $\tau$, that is,

$$R_{xx}(-\tau) = R_{xx}(\tau) \tag{3.26}$$

$$R_{xx}(0) = E[\left|X\right|^2(t)] \tag{3.27}$$

Since $X(t)$ is real,

$$R_{xx}(0) = E[X^2(t)] = \sigma_x^2 + m_x^2 \ge 0 \tag{3.28}$$

The autocorrelation function at $\tau = 0$ is constant and is equal to the mean-square value.

$$\left|R_{xx}(\tau)\right| \le R_{xx}(0) \tag{3.29}$$

The maximum value of the autocorrelation function occurs at $\tau = 0$ and it is nonnegative, as shown in Figure 3.12.

When $X(t)$ has a dc component (or nonzero mean value), then $R_{xx}(\tau)$ has a constant component. This arises from the fact that two observations of a wide-sense stationary process may become uncorrelated as $\tau$ approaches infinity. In this case, the covariance function goes to zero. That is,

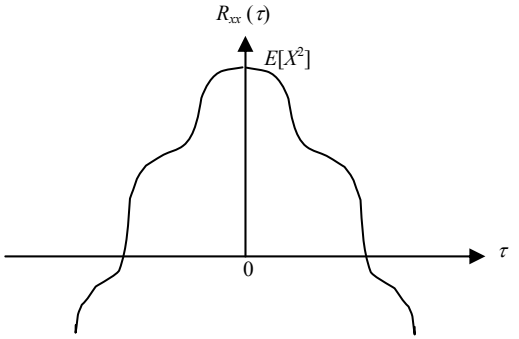


**Figure 3.12** A possible autocorrelation function.

$$\lim_{\tau\to\infty} C_{xx}(\tau) = E\{[X(t+\tau)-m_x][X(t)-m_x]\}$$
$$= R_{xx}(\tau) - m_x^2 = 0 \tag{3.30}$$

or

$$\lim_{\tau\to\infty} R_{xx}(\tau) = |m_x|^2 \tag{3.31}$$

If $X(t)$ and $Y(t)$ are jointly stationary in the wide-sense, similar properties can be obtained. That is,

$$R_{xy}^*(-\tau) = R_{yx}(\tau) \tag{3.32}$$

$$|R_{xy}(\tau)|^2 \le R_{xx}(0)R_{yy}(0) \tag{3.33}$$

$$R_{xy}(0) = R_{yx}^*(0) \tag{3.34}$$

If $X(t)$ and $Y(t)$ are real random processes, then

$$|R_{xy}(\tau)| \le \frac{R_{xx}(0)+R_{yy}(0)}{2} \tag{3.35}$$

## 3.4 SOME RANDOM PROCESSES

In this section, we shall study certain types of random processes that may characterize some applications.

### 3.4.1 A Single Pulse of Known Shape but Random Amplitude and Arrival Time

In radar and sonar applications, a return signal may be characterized as a random process consisting of a pulse with known shape, but with a random amplitude and random arrival time. The pulse may be expressed as

$$X(t) = A\,S(t-\Theta) \tag{3.36}$$

where $A$ and $\Theta$ are statistically independent random variables, and $s(t)$ is a deterministic function. A sample function may be represented, as shown in Figure 3.13. The mean value function of this particular random process is given by

$$E\big[X(t)\big] = E\big[A\,S(t-\Theta)\big] \tag{3.37}$$

Since $A$ and $\Theta$ are statistically independent, we have

$$E\big[X(t)\big] = E\big[A\big]E\big[S(t-\Theta)\big] = E\big[A\big]\int_{-\infty}^{\infty} s(t-\theta)f_{\Theta}(\theta)d\theta \tag{3.38}$$

The integral $\int_{-\infty}^{\infty} s(t-\theta)f_{\Theta}(\theta)d\theta$ is simply the convolution of the pulse $s(t)$ with the density function of $\Theta$. Thus,

$$E[X(t)] = E[A]\,s(t) * f_{\Theta}(\theta) \tag{3.39}$$

Similarly, the autocorrelation function is given by

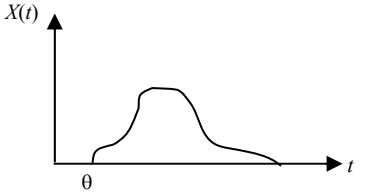


**Figure 3.13** Pulse $X(t)$.

$$R_{xx}(t_1, t_2) = E\left[A^2\right]\int_{-\infty}^{\infty} s(t_1 - \theta)\, s(t_2 - \theta) f_{\Theta}(\theta) d\theta \tag{3.40}$$

If the arrival time is known to be some fixed value $\theta_0$, then the mean and autocorrelation functions of $X(t)$ become

$$E\left[X(t)\right] = E\left[A\right] s(t - \theta_0) \tag{3.41}$$

and

$$R_{xx}(t_1, t_2) = E[A^2]\, s(t_1 - \theta_0)\, s(t_2 - \theta_0) \tag{3.42}$$

Another special case is that the arrival time may be uniformly distributed over the interval from 0 to $T$. The mean and autocorrelation functions are in this case

$$E[X(t)] = \frac{E[A]}{T}\int_0^T s(t - \theta) d\theta \tag{3.43}$$

and

$$R_{xx}(t_1, t_2) = \frac{E[A]^2}{T}\int_0^T s(t_1 - \theta)\, s(t_2 - \theta) d\theta \tag{3.44}$$

### 3.4.2 Multiple Pulses

We now assume that we have a multiple pulse situation. This may be the case in radar applications for a multiple target environment. The random process $X(t)$ can be expressed as

$$X(t) = \sum_{k=1}^{n} A_k S(t - \Theta_k) \tag{3.45}$$

where the $2n$ random variables $A_k$ and $\Theta_k$, $k = 1, 2, \ldots, n$, are mutually and statistically independent. In addition, the amplitudes are independent of the phase shifts, and we assume that the $A_k$s are identically distributed with density function $f_A(a)$, while the $\Theta_k$s are identically distributed with density function $f_{\Theta}(\theta)$. We can easily obtain the mean and autocorrelation functions to be

$$E[X(t)] = E\left[\sum_{k=1}^{n} A_k S(t-\Theta_k)\right] = \sum_{k=1}^{n} E[A_k]E[S(t-\Theta_k)]$$

$$= n\,E[A_k]\int_{-\infty}^{\infty} s(t-\theta) f_\Theta(\theta)d\theta = n\,E[A_k][s(t) * f_\Theta(\theta)] \tag{3.46}$$

and

$$R_{xx}(t_1,t_2) = \left[E\sum_{k=1}^{n} A_k S(t_1-\Theta_k)\sum_{j=1}^{n} A_j S(t_2-\Theta_j)\right]$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{n} E[A_k A_j]\,E[S(t_1-\Theta_k)S(t_2-\Theta_j)]$$

$$= nE[A_k^2]\int_{-\infty}^{\infty} s(t_1-\theta)\,s(t_2-\theta) f_\Theta(\theta)d\theta$$

$$+ (n^2-n)\,(E[A_k])^2 \int_{-\infty}^{\infty} s(t_1-\theta) f_\Theta(\theta)d\theta \int_{-\infty}^{\infty} s(t_2-\theta) f_\Theta(\theta)d\theta \tag{3.47}$$

If the random variable $\Theta$ is uniformly distributed over the interval $(0, T)$, the mean and autocorrelation functions of $X(t)$ become

$$E[X(t)] = nE[A_k]\frac{1}{T}\int_0^T s(t-\theta)d\theta \tag{3.48}$$

and

$$R_{xx}(t_1,t_2) = nE[A_k^2]\frac{1}{T}\int_0^T s(t_1-\theta)\,s(t_2-\theta)d\theta$$

$$+ \frac{(n^2-n)}{T^2}\int_0^T s(t_1-\theta)d\theta \int_0^T s(t_2-\theta)d\theta \tag{3.49}$$

### 3.4.3 Periodic Random Processes

The random process $X(t)$ is said to be periodic with period $T$ if all its sample functions are periodic with period $T$, except those sample functions that occur with probability zero.

*Theorem.* If the random process $X(t)$ is stationary in the wide-sense, then the autocorrelation function is periodic with period $T$, if and only if $X(t)$ is periodic with period $T$, and vice versa.

*Proof.* The first condition says that $R_{xx}(\tau + nT) = R_{xx}(\tau)$ if $X(t)$ is periodic. $X(t)$ periodic means that $X(t + \tau + nT) = X(t + \tau)$. Then,

$$R_{xx}(\tau + nT) = E[X(t + \tau + nT)X(t)] = E[X(t + \tau)X(t)] = R_{xx}(\tau) \quad (3.50)$$

The second condition states that if the autocorrelation function is periodic, then $X(t + nT) = X(t)$, where $X(t)$ is wide-sense stationary. Consider *Tchebycheff's inequality*, which states

$$P[|Y(t) - m_y| > k] \leq \frac{\sigma_y^2}{k^2} \quad (3.51)$$

where $m_y$ and $\sigma_y^2$ are the mean and variance of the process $Y(t)$, respectively, and $k$ is a positive constant.

Let $Y(t) = X(t + T) - X(t)$. Then, the mean and variance of $Y(t)$ are

$$m_y = E[Y(t)] = E[X(t + T) - X(t)] = E[X(t + T)] - E[X(t)] = 0 \quad (3.52)$$

because $X(t)$ is wide-sense stationary (mean is constant). Also,

$$\begin{aligned}\sigma_y^2 &= E[Y^2(t)] = E\{[X(t + T) - X(t)]^2\} \\ &= E[X^2(t + T)] - 2E[X(t + T)X(t)] + E[X^2(t)] \\ &= R_{xx}(0) - 2R_{xx}(T) + R_{xx}(0) = 2[R_{xx}(0) - R_{xx}(T)]\end{aligned} \quad (3.53)$$

The variance $\sigma_y^2$ is zero, due to the fact that the autocorrelation function is periodic with period $T$, and $R_{xx}(0) = R_{xx}(T)$. Consequently, from Tchebycheff's inequality, we have

$$P[|X(t + T) - X(t)| > k] = 0 \qquad \text{for all } t \quad (3.54)$$

Hence, $X(t)$ must be periodic.

*Corollary.* Let $s(t)$ be a deterministic function and periodic with period $T$. The random process $X(t)$, defined as $X(t) = S(t - \Theta)$, where $\Theta$ is a random variable uniformly distributed over the interval $(0, T)$, is stationary in the wide-sense.

*Proof.* For $X(t)$ to be wide-sense stationary, the mean $E[X(t)]$ must be constant, and the autocorrelation function must be a function of the time difference $\tau$. The mean value function of $X(t)$ is

$$E[X(t)] = \int_{-\infty}^{\infty} s(t-\theta) f_{\Theta}(\theta) d\theta = \frac{1}{T} \int_{0}^{T} s(t-\theta) d\theta \tag{3.55}$$

We make a change of variable by letting $u = t - \theta$. Then,

$$E[X(t)] = -\frac{1}{T} \int_{t}^{t-T} s(u) du = \frac{1}{T} \int_{t-T}^{t} s(u) du = \text{constant} \tag{3.56}$$

since we are integrating a periodic function, $s(t)$, over its period. Using the same reasoning, we can easily show that $R_{xx}(t + \tau, t) = R_{xx}(\tau)$.

The process $X(t)$ is *periodically stationary* or *cyclostationary* with period $T$ if its statistics are not changed by a shift of $nT$, $n = \pm 1, \pm 2, \ldots$, from the time origin. That is,

$$f_{X_1,\ldots,X_m}(x_1, \ldots, x_m; t_1, \ldots, t_m) = f_{X_1,\ldots,X_m}(x_1, \ldots, x_m; t_1 + nT, \ldots, t_m + nT) \tag{3.57}$$

for all integers $n$ and $m$.

$X(t)$ is cyclostationary in the wide-sense with period $T$ if its mean and autocorrelation functions are periodic with the same period $T$. That is,

$$m_x(t + kT) = m_x(t) \tag{3.58}$$

and

$$R_{xx}(t_1 + kT, t_2 + kT) = R_{xx}(t_1, t_2) \tag{3.59}$$

for all $t$, $t_1$, $t_2$, and any integer $k$.

*Theorem.* If $X(t)$ is a wide-sense cyclostationary process with period $T$, then the process $Y(t) = X(t - \Theta)$, where $\Theta$ is uniformly distributed over the interval $(0, T)$, is wide-sense stationary.

The proof of this theorem is straightforward and similar to that of the previous theorem. Therefore, we will not show it.

### 3.4.4 The Gaussian Process

A random process $X(t)$ is Gaussian if the random variables $X(t_1)$, $X(t_2)$, …, $X(t_n)$, are jointly Gaussian for all possible values of $n$ and $t_1$, $t_2$, ..., $t_n$. Since the multivariate Gaussian random variable depends only on the mean vector and the covariance matrix of the $n$ random variables, we observe that if $X(t)$ is stationary in the wide-sense, it is also strictly stationary.

If $X(t)$ is a Gaussian random process applied to a linear time-invariant system with impulse response $h(t)$, as shown in Figure 3.14, then the output process

$$Y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \tag{3.60}$$

is also Gaussian. Hence, the output process $Y(t)$ will be completely specified, given the input process $X(t)$ and the impulse response $h(t)$.

**Example 3.6**

Let $X(t)$, a wide-sense stationary, zero-mean Gaussian random process, be the input of a square law detector; that is, *a nonlinear system without memory.*

(a) Verify that the output is no longer Gaussian.

(b) Determine the autocorrelation function $R_{yy}(\tau)$ of the output and its variance.

*Solution*

(a) The system is shown in Figure 3.15. The density function of the input is

$$f_X(x;t) = f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

Using the result given in Example 1.19, the density function of the output is then

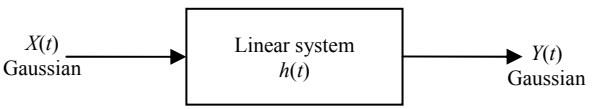


**Figure 3.14** Impulse response $h(t)$.

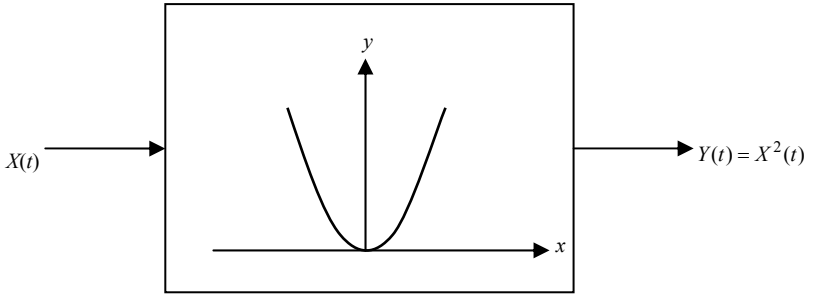


**Figure 3.15** Square law detector.

$$f_Y(y;t) = f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}} e^{-y/2\sigma^2}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

and is shown in Figure 3.16. We observe that the output of the nonlinear system without memory is no longer Gaussian.

(b) The autocorrelation function of the output $Y(t) = X^2(t)$ is given by

$$R_{yy}(t+\tau, t) = E[Y(t+\tau)Y(t)] = E[X^2(t+\tau)X^2(t)] = E[X(t+\tau)X(t+\tau)X(t)X(t)]$$

Using the result given by (2.244), the autocorrelation function of the output process becomes

$$R_{yy}(\tau) = E[X^2(t+\tau)]E[X^2(t)] + 2\{E[X(t+\tau)X(t)]\}^2 = R_{xx}^2(0) + 2R_{xx}^2(\tau)$$

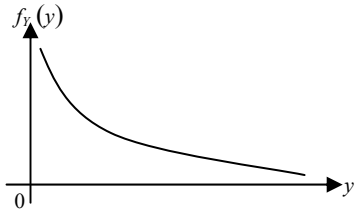


**Figure 3.16** Density function of the output.

Then, the mean-square value of $Y(t)$ is $E[Y^2(t)] = R_{yy}(0) = 3\{E[X^2(t)]\}^2$ $= 3[R_{xx}(0)]^2$, but also $E[Y(t)] = E[X^2(t)] = R_{xx}(0) = \sigma^2$. Hence, the variance of $Y(t)$ is $\sigma_y^2 = E[Y^2(t)] - \{E[Y(t)]\}^2 = 2[R_{xx}(0)]^2 = 2\sigma^4$.

Let the processes $Y_1(t)$ and $Y_2(t)$ be the outputs of two linear time-invariant systems with respective inputs $X_1(t)$ and $X_2(t)$. The processes $Y_1(t)$ and $Y_2(t)$ are jointly Gaussian, provided that $X_1(t)$ and $X_2(t)$ are jointly Gaussian.

### 3.4.5 The Poisson Process

The Poisson process is used for modeling situations, such as alpha particles emitted from a radioactive material, failure times of components of a system, people serviced at a post office, or telephone calls received in an office. These events can be described by a counting function $X(t)$, $t > 0$, such that at time zero, $X(0) = 0$. A typical sample function of the Poisson process $X(t)$, $t > 0$, which is a discrete-amplitude continuous-time process, is as shown in Figure 3.17. The process $X(t)$ is said to be a *Poisson process* if it satisfies the following conditions:

1. $X(t)$ is a nondecreasing step function, as shown in Figure 3.17, with unit jumps (representing the events) at each time $t_k$, and $k$ is a finite and countable number.

2. For any time $t_1$ and $t_2$, $t_2 > t_1$, the number of events (or jumps) that occur in the interval $t_1$ to $t_2$ follow a Poisson distribution, such that

$$P[X(t_2) - X(t_1) = k] = \frac{[\lambda(t_2 - t_1)]^k}{k!} \exp[-\lambda(t_2 - t_1)], \quad k = 0, 1, 2, \ldots \tag{3.61}$$

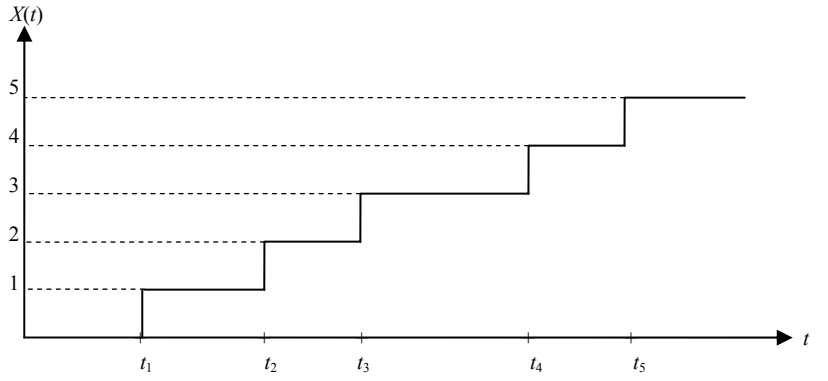


**Figure 3.17** Sample function of a Poisson process.

3. The number of events that occur in any interval of time $t$ is independent of the number of events that occur in any other nonoverlapping interval; that is, $X(t)$ is an independent increment process. Hence,

$$P[X(t) = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2, \ldots \tag{3.62}$$

The Poisson process can also be defined using the concept of *Poisson points*. Let the instant at which the events occur be as depicted in Figure 3.18. We start observing the process at time $t = 0$.

We say that the points $T_i$ are *Poisson points* with parameter $\lambda t$, provided the following properties are satisfied:

1. The number of points $T_i$ in an interval $(t_1, t_2)$, denoted $N(t_1, t_2)$, is a Poisson random variable. That is, the probability of $k$ points in time $t = t_1 - t_2$ is

$$P[N(t_1, t_2) = k] = \frac{e^{-\lambda t} (\lambda t)^k}{k!} \tag{3.63}$$

$\lambda$ is called the *density* or *average arrival rate* of the Poisson process.

2. If the intervals $(t_1, t_2)$ and $(t_3, t_4)$ are nonoverlapping, then the corresponding random variables $N(t_1, t_2)$ and $N(t_3, t_4)$ are independent.

We define the Poisson process as

$$X(t) = N(0\ ,\ t) \tag{3.64}$$

such that

$$X(0) = 0 \tag{3.65}$$

$$P[X(t) = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2, \ldots \tag{3.66}$$

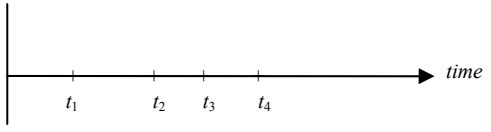


**Figure 3.18** Possible occurring times of particular events.

The first-order distribution of $X(t)$ is

$$F_{X_1}(x_1;t_1) = P[X(t_1) \le x_1] = P[\text{the number of points in interval } (0,t_1) \le x_1]$$
$$= \sum_{k=0}^{x_1} e^{-\lambda t_1} \frac{(\lambda t_1)^k}{k!} \tag{3.67}$$

**Example 3.7**

Let $X(t) = N(0,t)$ be a Poisson process representing the number of events occurring in the interval (0, $t$). Suppose that the first event occurs at $T_1$. Determine

(a) $f_{T_1}(t_1)$.

(b) The mean of $T_1$ and the variance.

*Solution*

From (3.65), $P[X(t) = N(0,t) = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$.

(a) The event $T_1 > t_1$ is equivalent to $N(0\ ,\ t_1) = 0$, since the first event occurs at $t_1$. Hence,

$$P(T_1 > t_1) = P[N(0,t_1) = 0] = e^{-\lambda t_1}, \quad t_1 > 0$$

The distribution function is then

$$F_{T_1}(t_1) = P(T_1 \le t_1) = 1 - P(T_1 > t_1) = 1 - e^{-\lambda t_1}, \quad t_1 > 0$$

and the density function is

$$f_{T_1}(t_1) = \frac{\partial F_{T_1}(t_1)}{\partial t_1} = \lambda e^{-\lambda t_1}, \quad t_1 > 0$$

Note that this is the exponential density function given in (2.88) with $\lambda = 1/\beta$.

(b) The mean of $T_1$ is

$$E[T_1] = \int_0^{t_1} t_1 \lambda e^{-\lambda t_1} dt_1 = \frac{1}{\lambda}$$

while the variance is $\text{var}[T_1] = E[T_1^2] - (E[T_1])^2$ with $E[T_1^2] = \int_0^\infty t_1^2 \lambda e^{-\lambda t_1} dt_1 = 2/\lambda^2$. Hence,

$$\text{var}[T_1] = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

### 3.4.6 The Bernoulli and Binomial Processes

In Chapter 2, we defined the Bernoulli experiment as the "simplest" in the sense that only two outcomes are possible: heads or tails corresponding to one (success) or zero (fail) occurring with probabilities $p$ and $1 - p = q$, respectively. We say $X[n]$, $n = 1, 2, \ldots$, is a *Bernoulli process with parameter p* if $X[1], X[2], \ldots, X[n]$ are independent and identically distributed Bernoulli random variables with parameters $p$. Note that the Bernoulli process is a discrete-time process, as shown in the typical sample function of Figure 3.19. The density function of the Bernoulli process is given by

$$f_{X[n]}(x[n]) = q\,\delta(x[n]) + p\,\delta(x[n]-1), \quad n = 1,\ 2,\ \ldots \tag{3.68}$$

where $\delta(\cdot)$ is the unit impulse function. The second-order density function is given by

$$\begin{aligned} f_{X[n_1]X[n_2]}(x[n_1], x[n_2]) &= q^2 \delta(x[n_1])\,\delta(x[n_2]) \\ &\quad + pq\,\delta(x[n_1]-1)\,\delta(x[n_2]) + qp\,\delta(x[n_1])\,\delta(x[n_2]-1) \\ &\quad + p^2 \delta(x[n_1]-1)\,\delta(x[n_2]-1) \quad \text{for } n_1, n_2 = 1, 2, \ldots \end{aligned} \tag{3.69}$$

The corresponding possible pairs of outcomes are ($X[n_1] = 0$, $X[n_2] = 0$), ($X[n_1] = 1$, $X[n_2] = 0$), ($X[n_1] = 0$, $X[n_2] = 1$), and ($X[n_1] = 1$, $X[n_2] = 1$). Note also

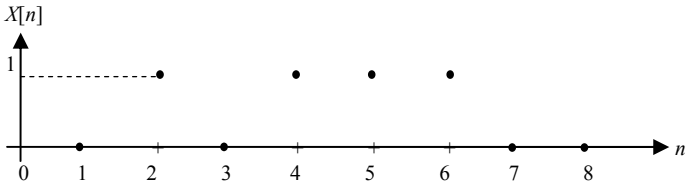


**Figure 3.19** Bernoulli process.

that the sum of probabilities is one; that is, $p^2 + 2pq + q^2 = (p+q)^2 = 1$. Higher-order density functions can be obtained in the same manner.

We define the *Binomial* (*or counting*) *process* as the sum of Bernoulli processes to be

$$S[n] = X[0] + X[1] + \ldots + X[n], \; n = 0, \; 1, \; 2, \; .... \tag{3.70}$$

A typical sample function of the binomial process is shown in Figure 3.20.

We observe that $S[n] = k$ means that $k$ of the Bernoulli variables equals one, while the remaining $(n-k)$ equals zero. Hence, the probability of $S[n] = k$ is given by

$$P(S[n] = k) = \binom{n}{k} p^k q^{n-k} \tag{3.71}$$

while the first-order density function of the binomial process is given by

$$f_{S[n]}(s[n]) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \delta(s[n] - k) \tag{3.72}$$

The Poisson process, which is a continuous time process, can be obtained from the discrete-time process under certain conditions:

1. Let the interval [0, $t$) be subdivided into $n$ very small intervals, $n$ large, of length $\Delta t$, such that $t = n\Delta t$ and only one point can occur in $\Delta t$.

2. Each interval $\Delta t$ is a Bernoulli trial with a probability of a point occurring at $p = \lambda \Delta t$ .

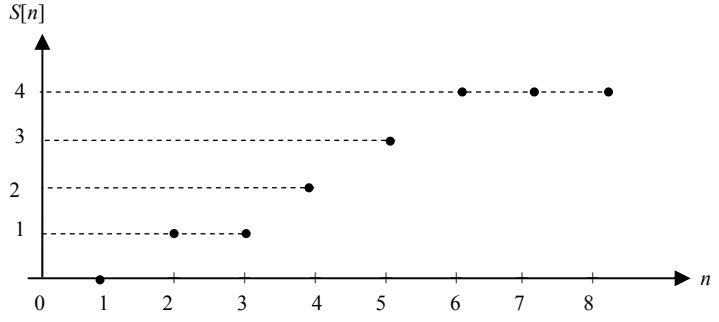


**Figure 3.20** Binomial process.

3. The Bernoulli trials are independent.

Then, $X(t) = N(0, t)$ is equivalent to the binomial process given by (3.70) with parameters $n = t/\Delta t$ and $p = \lambda\Delta t$. In the limit, it can be shown that

$$\lim_{\substack{n\to\infty \\ \Delta t\to 0}} p[X(t) = N(0,t) = k] = \lim_{\substack{n\to\infty \\ \Delta t\to 0}} \binom{n}{k} (\Delta t)^k (1-\lambda\Delta t)^{n-k}$$

$$= \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2, .... \tag{3.73}$$

which is the Poisson distribution, and thus the density function is given by

$$f_{N(0,t)}[n(0,t)] = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \delta[n(0,t) = k] \tag{3.74}$$

### 3.4.7 The Random Walk and Wiener Processes

Consider again the experiment of tossing a fair coin $n$ times every $T$ seconds, such that $t = nT$, $n = 1, 2, 3, \ldots$ . After each tossing, we take a step of length $\Delta$ to the right if heads show, or a step to the left if tails show. A typical sample function is shown in Figure 3.21. $X(t)$ is referred to as the *random walk.*

If $k$ heads show up in the first $n$ tosses, then we have $k$ steps to the right and $(n-k)$ steps to the left, yielding

$$X(nT) = k\Delta - (n-k)\Delta = (2k-n)\,\Delta \tag{3.75}$$

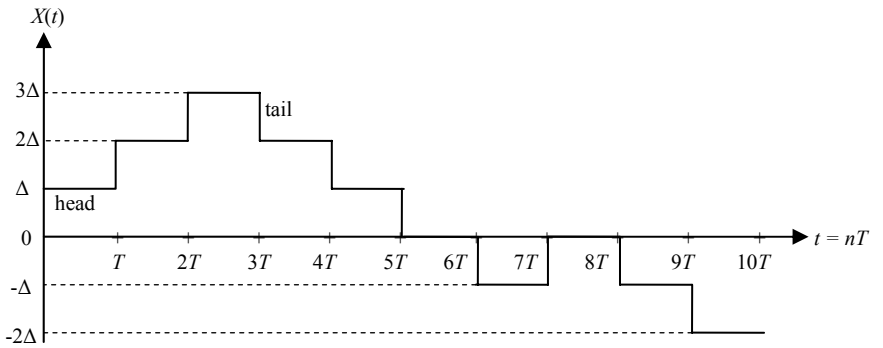


**Figure 3.21** Random walk.

As $k$ varies from 0 to $n$, $X(nT)$ varies from $-n\Delta$ to $+n\Delta$. Since the coin is fair, then $p = q = 1/2$. We can define $X(nT)$ as

$$X(nT) = X_1 + X_2 + \ \ldots \ + X_k \tag{3.76}$$

where $X_k$, $k = 1, 2, \ldots, n$, is referred to as a *symmetric Bernoulli* random variable, since it assumes steps of $+\Delta$ with probability $p = 1/2$ and $-\Delta$ with probability $q = 1/2$. Hence,

$$P[X(nT) = (2k - n)\Delta] = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = \binom{n}{k}\frac{1}{2^n} \tag{3.77}$$

and the density function of the random walk after $n$ steps is

$$f_{X(nT)}[x(nT)] = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^n}\delta[x(nT) - (2k - n)\Delta] \tag{3.78}$$

If we now consider the experiment of independently tossing the same coin two times—the first one $n_1$ times and the second one $n_2$ times, the autocorrelation function of the random walk sequence is given by

$$\begin{aligned} R_{x_1x_2}(n_1, n_2) &= E[X(n_1)X(n_2)] \ = E\{X(n_1)[X(n_2) + X(n_1) - X(n_1)]\} \\ &= E\{X^2(n_1) + X(n_1)[X(n_2) - X(n_1)]\} \ = E[X^2(n_1)] + E\{X(n_1)[X(n_2) - X(n_1)]\} \end{aligned} \tag{3.79}$$

Suppose $n_2 > n_1$, then $X(n_1)$ and $[X(n_2) - X(n_1)]$ are independent random variables, because the number of heads in the first $n_1$ tossing is independent of the number of heads in the $(n_1 + 1)$ tossing to $n_2$ tossing. Consequently, if $n_1 < n_2 \le n_3 < n_4$, the increments $X(n_4T) - X(n_3T)$ and $X(n_2T) - X(n_3T)$ are independent. The autocorrelation function can be written as

$$R_{x_1x_2}(n_1, n_2) = E[X^2(n_1)] + E[X(n_1)]E[X(n_2) - X(n_1)] \tag{3.80}$$

but

$$E[X(n_1)] = \sum_{k=1}^{n_1}\frac{1}{2}(\Delta) + \frac{1}{2}(-\Delta) = 0 \tag{3.81}$$

and

$$E[X^2(n_1)] = \sum_{k=1}^{n_1} \frac{1}{2}\Delta^2 + \frac{1}{2}(-\Delta)^2 = n_1\Delta^2 \tag{3.82}$$

Hence

$$R_{x_1x_2}(n_1, n_2) = n_1\Delta^2 \tag{3.83}$$

Similarly, if $n_1 > n_2$, the autocorrelation function is

$$R_{x_1x_2}(n_1, n_2) = n_2\Delta^2 \tag{3.84}$$

Combining (3.83) and (3.84), we obtain

$$R_{x_1x_2}(n_1, n_2) = \Delta^2 \min(n_1, n_2) \tag{3.85}$$

The *Wiener process*, also called the *Wiener-Levy process* or *Brownian motion*, is a limiting form of the random walk as $n \to \infty$ and $T \to 0$, such that $\lim_{\substack{n\to\infty \\ T\to 0}} (nT) = t$ and the variance remains finite and nonzero. The *Wiener process* $W(t)$ is given by

$$W(t) = \lim_{\substack{n\to\infty \\ T\to 0}} X(t) \tag{3.86}$$

From the central limit theorem, the probability that $X$ given in (3.76), $X = X_1 + X_2 + \ldots + X_n$ with $X_k$ a symmetric binomial, takes $k$ steps to the right is

$$P[X(nT) = (2k-n)\Delta] \approx \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(2k-n)\Delta - m}{2\sigma^2}\right] \tag{3.87}$$

where the mean $m$ and the variance $\sigma^2$ are as derived in (3.81) and (3.82), to yield $m = E[X] = 0$ and $\sigma^2 = \text{var}\,[X] = n\Delta^2$. Substituting for the values of $m$ and $\sigma^2$ in (3.87), after simplification, we obtain

$$P[X(nT) = (2k-n)\Delta] \approx \frac{1}{\sqrt{2\pi n}\Delta} \exp\left[-\frac{(2k-n)^2}{2n}\right] \tag{3.88}$$

At each step of the limiting process $nT = t$, and after setting $\Delta^2 = \alpha T$ to maintain the variance finite and $w = (2k - n)\Delta$, we obtain the first-order density function of the Wiener process to be

$$f_{W(t)}[w(t)] = \frac{1}{\sqrt{2\pi}\sqrt{\alpha t}} \exp\left[-\frac{w^2(t)}{2\alpha t}\right] \tag{3.89}$$

A sample function of the Wiener process is shown in Figure 3.22.

By analogy to the random walk process, the property of independent increments is maintained for the Wiener process. That is, if $t_1 < t_2 \le t_3 < t_4$, the increments $w(t_4) - w(t_3)$ and $w(t_2) - w(t_1)$ are independent.

**Example 3.8**

Determine the autocorrelation function of the Wiener process.

*Solution*

Using the same approach as we did in determining the autocorrelation function of the random walk process, the autocorrelation function of the Wiener process is

$$R_{ww}(t_1, t_2) = E[W(t_1)W(t_2)]$$

in which we have two cases, $t_1 < t_2$ and $t_2 < t_1$.

*Case 1*: $t_1 < t_2$

Using the property of independence of increments, we can write

$$\begin{aligned} E\{W(t_1)[W(t_2) - W(t_1)]\} &= E[W(t_1)]\, E[W(t_2) - W(t_1)] \\ &= E[W(t_1)W(t_2)] - E[W^2(t_1)] \\ &= R_{ww}(t_1, t_2) - E[W^2(t_1)] \end{aligned} \tag{3.90}$$

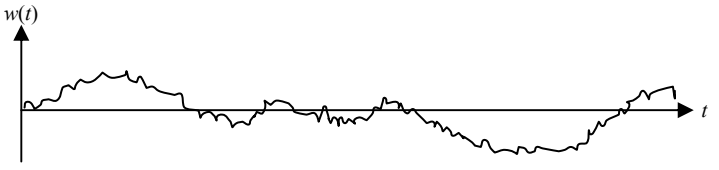


**Figure 3.22** Sample function of the Wiener process.

From (3.89), the Wiener process has a Gaussian distribution with mean zero and variance $\alpha t$. Then, $E[W(t_1)] = 0$ and (3.90) becomes

$$R_{ww}(t_1, t_2) = E[W^2(t_1)] = \alpha\, t_1$$

*Case 2*: $t_2 < t_1$

In the same manner, we can show that

$$R_{ww}(t_1, t_2) = \alpha t_2$$

Combining the results of Cases 1 and 2, we obtain the autocorrelation function of the Wiener process to be

$$R_{ww}(t_1, t_2) = \alpha\ \min(t_1, t_2) = \begin{cases} \alpha t_1, & t_1 < t_2 \\ \alpha t_2, & t_2 < t_1 \end{cases} \tag{3.91}$$

### 3.4.8 The Markov Process

A stochastic process $X(t)$ is said to be a *simple Markov process* (or *first-order Markov*) if for any $n$ and a sequence of increasing times $t_1 < t_2 < \ldots < t_n$, we have

$$P[X(t_n) \le x_n \mid X(t_{n-1}),\ \ldots\ , X(t_1)] = P[X(t_n) \le x_n \mid X(t_{n-1})] \tag{3.92}$$

or equivalently,

$$f_{X_n|X_{n-1},X_{n-2},\ldots,X_1}(x_n \mid x_{n-1}, x_{n-2}, \ldots, x_1) = f_{X_n|X_{n-1}}(x_n \mid x_{n-1}) \tag{3.93}$$

Note that for the simplicity of notation we dropped the subscript $t_k$. We observe that the value at $t_k$ depends only upon the previous value at $t_{k-1}$. The joint density function can be written as

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= f(x_1)\frac{f(x_1, x_2)}{f(x_1)}\frac{f(x_1, x_2, x_3)}{f(x_1, x_2)}\cdots\frac{f(x_1, x_2, \ldots, x_n)}{f(x_1, x_2, \ldots, x_{n-1})} \\ &= f(x_1) f(x_2 \mid x_1) f(x_3 \mid x_2, x_1) \ldots f(x_n \mid x_{n-1}, \ldots, x_2, x_1) \end{aligned} \tag{3.94}$$

Rewriting (3.94), if $X(t)$ is a Markov process, then

$$f(x_1, x_2, \ldots, x_n) = f(x_1)\prod_{k=2}^{n} f(x_k \mid x_{k-1}) \tag{3.95}$$

which means that the process is completely determined by the first-order density function and the conditional density functions. Since the sequence of random variables $X_n, X_{n-1}, \ldots, X_1$ is Markov, it follows from (3.95) that

$$E[X_n \mid X_{n-1}, X_{n-2}, \ldots, X_1] = E[X_n \mid X_{n-1}] \tag{3.96}$$

Also, the Markov process is Markov in reverse time; that is,

$$f(x_n \mid x_{n+1}, x_{n+2}, \ldots, x_{n+k}) = f(x_n \mid x_{n+1}) \tag{3.97}$$

If in a Markov process the present is known, then the past and future are independent; that is, for $m < k < n$ we have

$$f(x_m, x_n \mid x_k) = f(x_m \mid x_k) f(x_n \mid x_k) \tag{3.98}$$

A Markov process is said to be *homogeneous* if $f(X_n = x \mid X_{n-1} = y)$ is invariant to a shift of the origin; that is, it depends on $x$ and $y$ but not $n$. However, the first-order density function $f_{X_n}(x_n)$ might depend on $n$. If the first-order density function does not depend on $n$, $f_{X_n}(x_n) = f_{X_n}(x)$, but depends on $x$ only, the Markov process is said to be *stationary*. In this case, $f(x_n \mid x_{n-1})$ is invariant to a shift of the origin due to the homogeneity of the process, and thus the statistics of the process can be completely determined in terms of the second-order density function, which is given by

$$f(x_1, x_2) = f(x_2 \mid x_1) f(x_1) \tag{3.99}$$

*Chapman-Kolmogorov Equation* For $m < k < n$, the conditional density function $f(x_n \mid x_m)$ can be expressed in terms of the conditional density functions $f(x_n \mid x_k)$ and $f(x_k \mid x_m)$ to be

$$f(x_n \mid x_m) = \int_{-\infty}^{\infty} f(x_n \mid x_k) f(x_k \mid x_m)\, dx_k, \qquad m < k < n \tag{3.100}$$

When the Markov process takes a countable and finite discrete set of values, they are called *Markov chains*. Markov chains will be developed in more detail in the next chapter.

## 3.5 POWER SPECTRAL DENSITY

Given a deterministic signal $s(t)$, its Fourier transform (FT) is

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft}\, dt \tag{3.101}$$

which exists if the integral converges. The function $S(f)$ is sometimes called the *spectrum* of $s(t)$. In going from the time-domain description, $s(t)$, to the frequency domain, $S(f)$, no information about the signal is lost. In other words, $S(f)$ forms a complete description of $s(t)$ and vice versa. Hence, the signal $s(t)$ can be obtained from $S(f)$ by just taking the inverse Fourier transform (IFT); that is,

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{+j2\pi f t} df \tag{3.102}$$

In dealing with random processes, the ensemble is assumed to exist for all time $t$. In general, the sample functions are not absolutely integrable. However, since we are still interested in the notion of spectrum, we proceed in a manner similar to that of deterministic signals with infinite energy. We define $x_T(t)$ as the sample function $x(t)$, truncated between $-T$ and $T$, of the random process $X(t)$. That is,

$$x_T(t) = \begin{cases} x(t), & -T \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{3.103}$$

The truncated Fourier transform of the process $X(t)$ is

$$X_T(f) = \int_{-T}^{T} x_T(t) e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt \tag{3.104}$$

The average power of $x_T(t)$ is

$$P_{ave} = \frac{1}{2T} \int_{-T}^{T} x_T^2(t) dt \tag{3.105}$$

Using *Parseval's theorem*, which says that

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} |X_T(f)|^2 df \tag{3.106}$$

the average power of $x_T(t)$ is

$$P_T = \int_{-\infty}^{\infty} \frac{|X_T(f)|^2}{2T} df \tag{3.107}$$

where the term $|X_T(f)|^2 / 2T$ is the power spectral density of $x_T(t)$. The ensemble average of $P_T$ is given by

$$E[P_T] = \int_{-\infty}^{\infty} E\left[\frac{|X_T(f)|^2}{2T}\right] df \tag{3.108}$$

The power spectral density of the random process $X(t)$ is defined to be

$$S_{xx}(f) = \lim_{T\to\infty} E\left[\frac{|X_T(f)|^2}{2T}\right] \tag{3.109}$$

If $X(t)$ is stationary in the wide-sense, the power spectral density $S_{xx}(f)$ can be expressed as the Fourier transform of the autocorrelation function $R_{xx}(\tau)$. That is,

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f\tau} d\tau \tag{3.110}$$

*Proof.* By definition,

$$\begin{aligned} E\left[\frac{|X_T(f)|^2}{2T}\right] &= E\left[\frac{1}{2T}\int_{-T}^{T} X_T(t_1)e^{-j2\pi f t_1} dt_1 \int_{-T}^{T} X_T(t_2)e^{+j2\pi f t_2} dt_2\right] \\ &= E\left[\frac{1}{2T}\int_{-T}^{T} X(t_1)e^{-j2\pi f t_1} dt_1 \int_{-T}^{T} X(t_2)e^{+j2\pi f t_2} dt_2\right] \\ &= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[X(t_1)X(t_2)]\, e^{-j2\pi f(t_1-t_2)} dt_1 dt_2 \end{aligned} \tag{3.111}$$

where $E[X(t_1)X(t_2)] = R_{xx}(t_1, t_2)$. Since this is a wide-sense stationary process, we would like to express the autocorrelation function in terms of the time difference $\tau = t_1 - t_2$, and consequently, replace the double integral in $t_1$ and $t_2$ to one integral in $\tau$. Let $t_2 = t$ and $t_1 = t_2 + \tau = t + \tau$. The region of integration in the $t_1 - t_2$ plane and $t - \tau$ plane are shown in Figure 3.23. Let $s_1$, $s_2$, $s_3$, and $s_4$ denote the four sides of the square. From Figure 3.23, we see that the change of variables for the four sides will be

$$s_1 \to \tau = T - t_2, \quad s_2 \to \tau = t_1 - T, \quad s_3 \to \tau = -T - t_2, \quad s_4 \to \tau = t_1 + T$$

It follows that

$$\begin{aligned} E\left[\frac{|X_T(f)|^2}{2T}\right] &= \frac{1}{2T}\int_0^{2T} R_{xx}(\tau)e^{-j2\pi f\tau}\left[\int_{-T}^{T-\tau} R_{xx}(\tau)e^{-j2\pi f\tau}dt\right]d\tau \\ &\quad + \frac{1}{2T}\int_{-2T}^{0} R_{xx}(\tau)e^{-j2\pi f\tau}\left[\int_{-T-\tau}^{T} R_{xx}(\tau)e^{-j2\pi f\tau}dt\right]d\tau \\ &= \int_{-2T}^{2T} R_{xx}(\tau)e^{-j2\pi f\tau}\left[1-\frac{|\tau|}{2T}\right]d\tau \end{aligned} \tag{3.112}$$

In the limit, as $T$ approaches infinity, we conclude

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j2\pi f\tau}d\tau \tag{3.113}$$

provided that $R_{xx}(\tau)$ approaches zero at least at the rate $1/|\tau|$ with increasing $\tau$.

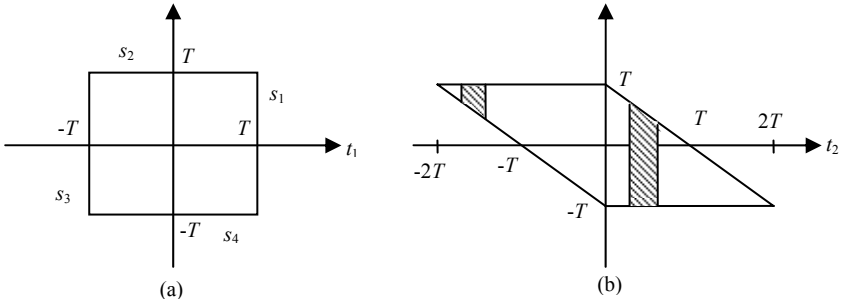


**Figure 3.23** Regions of integration for the autocorrelation function in (a) $t_1 - t_2$ plane and (b) $t - \tau$ plane.

Thus, the power spectral density of a wide-sense stationary process is the Fourier transform of its autocorrelation function. The inverse relationship, using the inverse Fourier transform, is

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(f) e^{j2\pi f\tau} df \tag{3.114}$$

(3.110) and (3.114) are sometimes called the *Wiener-Khinchin* relations. Note that the power spectral density is, from the definition, real, positive, and an even function of *f*. The autocorrelation is an even function of $\tau$.

**Example 3.9**

Consider the random process $X(t) = A\cos(\omega_0 t + \Theta)$, where $\Theta$ is a random variable uniformly distributed over the interval $(0, 2\pi)$, and $A$ and $\omega_0$ are constants. Determine the power spectral density of this process.

*Solution*

Since $X(t)$ is stationary in the wide sense with autocorrelation function $R_{xx}(\tau) = (A^2/2)\cos(2\pi f_0 \tau)$ as shown in Example 3.2, then using (3.113), the power spectral density is

$$S_{xx}(f) = \int_{-\infty}^{\infty} \frac{A^2}{2}\cos(2\pi f_0\tau) e^{-j2\pi f\tau} d\tau = \frac{A^2}{4}\int_{-\infty}^{\infty} [e^{-j2\pi(f-f_0)\tau} + e^{-j2\pi(f+f_0)\tau}] d\tau$$

$$= \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$$

*Cross-Spectral Densities*

Let $X(t)$ and $Y(t)$ be two jointly wide-sense stationary processes. Their cross-spectral densities are defined as

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j2\pi f\tau} d\tau \tag{3.115}$$

and

$$S_{yx}(f) = \int_{-\infty}^{\infty} R_{yx}(\tau) e^{-j2\pi f\tau} d\tau \tag{3.116}$$

By the *Wiener-Khinchin* relations, the cross-correlation functions $R_{xy}(\tau)$ and $R_{yx}(\tau)$ are just the respective inverse Fourier transforms of $S_{xy}(f)$ and $S_{yx}(f)$. From property (4) of Section 3.3.3, we have

$$R_{yx}(\tau) = R_{xy}^{*}(-\tau) \tag{3.117}$$

Consequently, their two cross-spectral densities are related by the following:

$$S_{yx}(f) = S_{xy}^{*}(f) \tag{3.118}$$

It should be observed that, while the power spectral densities $S_{xx}(f)$ and $S_{yy}(f)$ of the respective processes $X(t)$ and $Y(t)$ are always real, their cross-spectral densities $S_{xy}(f)$ and $S_{yx}(f)$ may be complex.

**Example 3.10**

Consider the process $Y(t) = X(t-T)$ ,where $X(t)$ is a real wide-sense stationary process with autocorrelation function $R_{xx}(\tau)$ and power spectral density $S_{xx}(f)$. $T$ is a constant. Express the power spectral density $S_{xy}(f)$ of the process $Y(t)$ in terms of $S_{xx}(f)$ .

*Solution*

The cross-correlation function $R_{xy}(\tau)$ is given by

$$R_{xy}(\tau) = E[X(t+\tau)Y(t)] = E[X(t+\tau)X(t-T)] = R_{xx}(\tau+T)$$

Hence,

$$S_{xy}(f) = S_{xx}(f)e^{j2\pi fT}$$

That is, the delay $T$ appears in the exponent as a phase angle scaled by $2\pi f$.

## 3.6 LINEAR TIME-INVARIANT SYSTEMS

A linear time-invariant system is characterized by its *impulse response* $h(t)$, or by its *system function* $H(f)$, which is the Fourier transform of $h(t)$. That is,

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt \tag{3.119}$$

and

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}\,df \tag{3.120}$$

If $x(t)$, the applied input signal to the linear time-invariant system, is deterministic as shown in Figure 3.24, the output signal is the convolution of $x(t)$ and $h(t)$, yielding

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \tag{3.121}$$

is a sample function of the random process $Y(t)$ corresponding to the sample function of the input process $X(t)$. The time-domain expression for the output process is

$$Y(f) = X(f)H(f) \tag{3.122}$$

where $X(f)$ and $Y(f)$ are the respective Fourier transforms of $x(t)$ and $y(t)$.

The system is realizable, provided the impulse response is *causal*; that is, $h(t) = 0$ for $t < 0$. In this case, the convolution integral becomes

$$y(t) = \int_{0}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau \tag{3.123}$$

### 3.6.1 Stochastic Signals

Consider the linear time-invariant system shown in Figure 3.24. The output signal

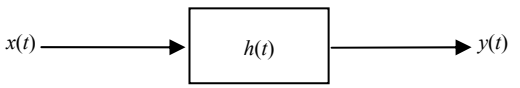


**Figure 3.24** Impulse response $h(t)$.

$$Y(t) = h(t) * X(t) = X(t) * h(t)$$
$$= \int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha = \int_{-\infty}^{\infty} X(\alpha)h(t-\alpha)d\alpha \tag{3.124}$$

*Mean Value Function*

The mean value function of the output process is given by

$$E[Y(t)] = \int_{-\infty}^{\infty} E[X(t-\alpha)]h(\alpha)d\alpha = \int_{-\infty}^{\infty} m_x(t-\alpha)h(\alpha)d\alpha \tag{3.125}$$

where $m_x(t)$ is the mean function of the process $X(t)$. If $X(t)$ is stationary in the wide-sense,

$$m_x(t-\alpha) = m_x(t) = \text{constant} \tag{3.126}$$

Then, the mean function $m_y(t)$ of the process $Y(t)$ is

$$m_y(t) = E[Y(t)] = m_x \int_{-\infty}^{\infty} h(\alpha)d\alpha \tag{3.127}$$

From (3.125), we recall that the system function evaluated at $f = 0$ is just the dc gain, and $\int_{-\infty}^{\infty} h(\alpha)d\alpha = H(0)$. Hence,

$$m_y = m_x H(0) \tag{3.128}$$

*The Mean-Square Value*

The mean-square value of the output process signal is

$$E[Y^2(t)] = E\left[\int_0^{\infty}\int_0^{\infty} X(t-t_1)X(t-t_2)h(t_1)h(t_2)dt_1dt_2\right] \tag{3.129}$$

Simplifying (3.129), the mean-square value function becomes

$$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t-t_1, t-t_2)\, h(t_1)\, h(t_2)\, dt_1 dt_2$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1, t_2)\, h(t-t_1)\, h(t-t_2)\, dt_1 dt_2 \qquad (3.130)$$

Assuming $X(t)$ is stationary in the wide-sense, and making the following change of variables $\alpha = t - t_1$ and $\beta = t - t_2$, the above result reduces to

$$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\alpha-\beta)h(\alpha)h(\beta)\, d\alpha\, d\beta \qquad (3.131)$$

which is independent of the time $t$.

*Cross-Correlation Function Between Input and Output*

Assume that the input process $X(t)$ is wide-sense stationary. The cross-correlation function between the input and output is

$$R_{yx}(t+\tau, t) = E[Y(t+\tau)X^*(t)] \qquad (3.132)$$

Using (3.124) in (3.132), and making a change of variables, the cross-correlation function can be rewritten as

$$R_{yx}(t+\tau, t) = \int_{-\infty}^{\infty} R_{xx}(\tau-\alpha)h(\alpha)d\alpha = R_{xx}(\tau) * h(\tau) \qquad (3.133)$$

Observe that this result does not depend on $t$, and hence $R_{yx}(t+\tau) = R_{yx}(\tau)$. Similarly, it can be shown that the cross-correlation function between the input and output process signals is

$$R_{xy}(\tau) = R_{xx}(\tau) * h(-\tau) \qquad (3.134)$$

If the processes $X(t)$ and $Y(t)$ are jointly wide-sense stationary, their cross-spectral density is the Fourier transform of their cross-correlation function. Since a convolution in time domain is equivalent to a multiplication in frequency domain, taking the Fourier transform of (3.133) and (3.134), we obtain

$$S_{yx}(f) = S_{xx}(f)\, H(f) \qquad (3.135)$$

and

$$S_{xy}(f) = S_{xx}(f)\, H^*(f) \tag{3.136}$$

*Autocorrelation Function and Spectrum of Output*

The autocorrelation function of the output process is

$$R_{yy}(t+\tau, t) = E[Y(t+\tau)Y(t)] \tag{3.137}$$

Using the fact that

$$Y(t+\tau) = \int_{-\infty}^{\infty} X(t+\tau-\alpha)h(\alpha)d\alpha \tag{3.138}$$

and

$$Y(t) = \int_{-\infty}^{\infty} X(t-\beta)h(\beta)d\beta \tag{3.139}$$

substituting (3.138) and (3.139) in (3.137), and making a change of variables $(\alpha = -\beta)$, we obtain

$$R_{yy}(\tau) = R_{yx}(\tau) * h(-\tau) = R_{xy}(\tau) * h(\tau) = R_{xx}(\tau) * h(\tau) * h(-\tau) \tag{3.140}$$

Taking the Fourier transform of the above equations results in

$$\begin{aligned} S_{yy}(f) = S_{yx}(f)H^*(f) = S_{xy}(f)H(f) &= S_{xx}(f)H(f)H^*(f) \\ &= S_{xx}(f)\left|H(f)\right|^2 \end{aligned} \tag{3.141}$$

**Example 3.11**

A white noise process with autocorrelation function $R_{xx}(\tau) = (N_0/2)\delta(\tau)$ is applied to a filter with impulse response

$$h(t) = \begin{cases} \alpha e^{-\alpha t}, & t \geq 0 \text{ and } \alpha > 0 \\ 0, & t < 0 \end{cases}$$

Determine the autocorrelation function, $R_{yy}(\tau)$, of the output process.

*Solution*

The problem can be solved in two ways. We can directly solve the convolution integral of (3.140), or obtain the power spectral density $S_{yy}(f)$ using (3.141), and then take the inverse Fourier transform of $S_{yy}(f)$. We shall solve this problem using both methods.

*Method 1:* For $\tau < 0$, we have from Figure 3.25,

$$h(\tau) * h(-\tau) = \int_{-\infty}^{\tau} \alpha\, e^{-\alpha(\tau-\lambda)} \alpha\, e^{+\alpha\lambda} d\lambda = \alpha^2 e^{-\alpha\tau} \int_{-\infty}^{\tau} e^{2\alpha\lambda} d\lambda = \frac{\alpha}{2} e^{\alpha\tau}$$

For $\tau > 0$, we have

$$h(\tau) * h(-\tau) = \int_{-\infty}^{0} \alpha e^{-\alpha(\tau-\lambda)} \alpha e^{\alpha\lambda} d\lambda = \frac{\alpha}{2} e^{-\alpha\tau}$$

Hence,

$$g(\tau) = h(\tau) * h(-\tau) = \begin{cases} \dfrac{\alpha}{2} e^{\alpha\tau}, & \tau \le 0 \\ \dfrac{\alpha}{2} e^{-\alpha\tau}, & \tau \ge 0 \end{cases}$$

Consequently,

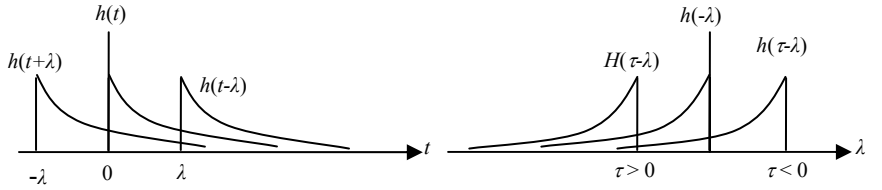


**Figure 3.25** Impulse response with $\tau$ as parameter.

$$R_{yy}(\tau) = R_{xx}(\tau) * g(\tau) = \begin{cases} \dfrac{N_0\alpha}{4} e^{\alpha\tau}, & \tau \le 0 \\ \dfrac{N_0\alpha}{4} e^{-\alpha\tau}, & \tau \ge 0 \end{cases}$$

or

$$R_{yy}(\tau) = \frac{N_0\alpha}{4} e^{-\alpha|\tau|}$$

*Method 2:* From (3.141), we see that we first need to determine the Fourier transform $H(f)$ of the impulse response $h(t)$. Thus,

$$H(f) = \int_0^\infty \alpha e^{-\alpha t} e^{-j2\pi ft} dt = \alpha \int_0^\infty e^{-(j2\pi f + \alpha)t} dt = \frac{\alpha}{j2\pi f + \alpha}$$

The magnitude of $H(f)$ squared is

$$|H(f)|^2 = \frac{\alpha^2}{4\pi^2 f^2 + \alpha^2}$$

while the output power spectral density is

$$S_{yy}(f) = S_{xx}(f)|H(f)|^2 = \frac{N_0\alpha}{4} \frac{2\alpha^2}{\omega^2 + \alpha^2}$$

where $\omega = 2\pi f$. Taking the inverse Fourier transform of $S_{yy}(f)$, we obtain the autocorrelation function, shown in Figure 3.26, to be

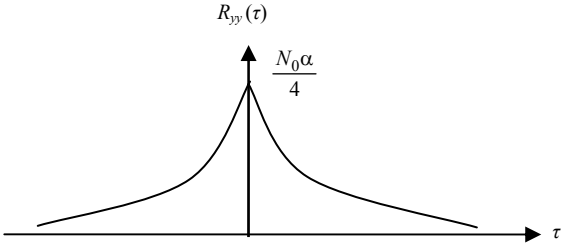


**Figure 3.26** Autocorrelation function of $Y(t)$.

$$R_{yy}(\tau) = \frac{N_0 \alpha}{4} e^{-\alpha|\tau|}$$

The results of both Method 1 and Method 2 agree.

### 3.6.2 Systems with Multiple Terminals

Linear time-invariant systems may have more than one input and/or output. A simple case would be a system with one input and two outputs, as shown in Figure 3.27. The relationship between the output processes $Y(t)$ and $Z(t)$ may be obtained from (3.141) as

$$S_{yy}(f) = |H_1(f)|^2 S_{xx}(f) \tag{3.142}$$

and

$$S_{zz}(f) = |H_2(f)|^2 S_{xx}(f) = S_{yy}(f) \left| \frac{H_2(f)}{H_1(f)} \right|^2, \quad H_1(f) \neq 0 \tag{3.143}$$

In a similar manner, we can obtain the cross-spectral densities of the random processes $Y(t)$ and $Z(t)$ to be

$$S_{zy}(f) = S_{xy}(f) H_2(f) = S_{xx}(f) H_1^*(f) H_2(f) \tag{3.144}$$

and

$$S_{yz}(f) = S_{yx}(f) H_2^*(f) = S_{xx}(f) H_1(f) H_2^*(f) \tag{3.145}$$

In time domain, we have

$$R_{zy}(\tau) = R_{xx}(\tau) * h_1(-\tau) * h_2(\tau) \tag{3.146}$$

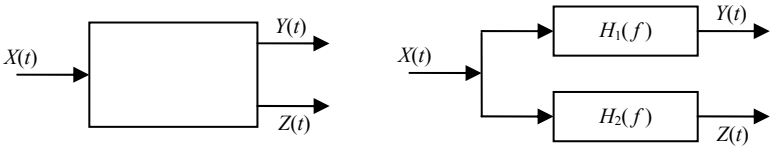


**Figure 3.27** System with multiple terminals.

and

$$R_{yz}(\tau) = R_{xx}(\tau) * h_1(\tau) * h_2(-\tau) \tag{3.147}$$

If $Y(t)$ and $Z(t)$ are *orthogonal*, then $R_{zy}(\tau) = R_{yz}(\tau) = 0$. In this case, the system is said to be *disjoint* because their transfer functions do not overlap; that is, $H_1(f)H_2(f) = 0$.

## 3.7 ERGODICITY

A random process $X(t)$ is ergodic if all of its statistics can be determined (with probability one) from a sample function of the process. That is, the ensemble averages equal the corresponding time averages with probability one. This is a more restrictive process, as shown by the Venn diagrams of Figure 3.28.

Usually, we are not interested in estimating all the ensemble averages of a random process, but rather we are concerned with weaker forms of ergodicity, such as ergodicity in the mean and ergodicity in the autocorrelation.

### 3.7.1 Ergodicity in the Mean

A random process $X(t)$ is *ergodic in the mean* if the time-averaged mean value of a sample function $x(t)$ is equal to the ensemble-averaged mean value function. That is,

$$E[X(t)] = < x(t) > \tag{3.148}$$

where the symbol $< \cdot >$ denotes time-average, and $< x(t) >$ is defined to be

$$< x(t) > = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt \tag{3.149}$$

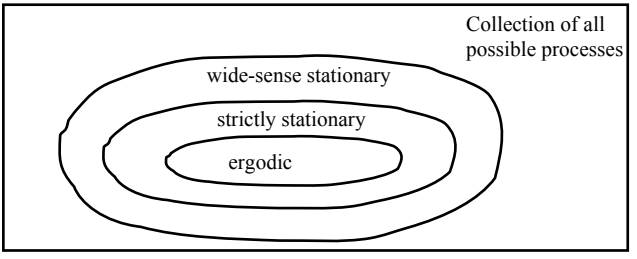


**Figure 3.28** Sets of different classes of processes.

The necessary and sufficient condition under which the process $X(t)$ is ergodic in the mean is

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(\tau)d\tau = m_x^2 \tag{3.150}$$

where $m_x = E[X(t)]$ is the mean value of $X(t)$.

### 3.7.2 Ergodicity in the Autocorrelation

The random process $X(t)$ is *ergodic in the autocorrelation* if

$$R_{xx}(\tau) = < x(t+\tau)x(t) > \tag{3.151}$$

$< x(t+\tau)x(t) >$ denotes the time-averaged autocorrelation function of the sample function $x(t)$, and is defined as

$$< x(t+\tau)x(t) > = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t+\tau)x(t)dt \tag{3.152}$$

The necessary and sufficient condition for ergodicity in the autocorrelation is that the random variables $X(t+\tau)X(t)$ and $X(t+\tau+\alpha)X(t+\alpha)$ become uncorrelated for each $\tau$ as $\alpha$ approaches infinity.

**Example 3.12**

Consider the random process $X(t) = A\cos(2\pi f_c t + \Theta)$, where $A$ and $f_c$ are constants, and $\Theta$ is a random variable uniformly distributed over the interval $[0, 2\pi]$.

*Solution*

It was shown in Example 3.2 that the mean and autocorrelation functions of $X(t)$ are $E[X(t)] = 0$ and $R_{xx}(\tau) = (A^2/2)\cos(2\pi f_c \tau)$. Let the sample function of the process $X(t)$ be

$$x(t) = A\cos(2\pi f_c t + \theta)$$

The time-averaged mean and the time-averaged autocorrelation are

$$< x(t) > = \lim_{T\to\infty} \frac{A}{2T} \int_{-T}^{T} \cos(2\pi f_c t + \theta) dt = 0$$

and

$$< x(t+\tau)x(t) > = \lim_{T\to\infty} \frac{A^2}{2T} \int_{-T}^{T} \cos[2\pi f_c (t+\tau) + \theta] \cos(2\pi f_c t + \theta) dt$$

$$= \frac{A^2}{2} \cos(2\pi f_c \tau)$$

Hence, the process $X(t)$ is ergodic in the mean and in the autocorrelation.

### 3.7.3 Ergodicity of the First-Order Distribution

Let $X(t)$ be a stationary random process. Define the random process $Y(t)$ as

$$Y(t) = \begin{cases} 1, & X(t) \le x_t \\ 0, & X(t) > x_t \end{cases} \tag{3.153}$$

We say that the random process $X(t)$ is *ergodic in the first-order distribution* if

$$F_X(x;t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t) dt \tag{3.154}$$

where $F_X(x\,;\,t) = P[\,X(t) \le x(t)]$ and $y(t)$ is a sample function of the process $Y(t)$.

The necessary and sufficient condition under which the process is ergodic in the first-order distribution is that $X(t + \tau)$ and $X(t)$ become statistically independent as $\tau$ approaches infinity.

### 3.7.4 Ergodicity of Power Spectral Density

A wide-sense stationary process $X(t)$ is ergodic in power spectral density if, for any sample function $x(t)$,

$$S_{xx}(f) = \lim_{T\to\infty} \frac{1}{2T} \left| \int_{-T}^{T} x(t) e^{-j2\pi f t} dt \right|^2 \tag{3.155}$$

except for a set of sample functions that occur with zero probability.

## 3.8 SAMPLING THEOREM

We first give a brief description of the sampling theorem for deterministic signals. Let $g(t)$ be a bandlimited signal to a frequency $f_m$ Hz, where $f_m$ is the highest frequency such that its Fourier transform $G(f) = 0$ for $|f| > f_m$, as shown in Figure 3.29. Ideally, sampling the signal $g(t)$ is multiplying it by $p(t)$ train of impulses, as shown in Figure 3.30, to yield

$$g_s(t) = g(t)p(t) \tag{3.156}$$

where $g_s(t)$ is the sampled signal, as shown in Figure 3.31. Since the sampling function $p(t)$ is periodic, it can be represented by its Fourier series to yield

$$p(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{j2\pi nt}{T}} \tag{3.157}$$

where $c_n$ is the $n$th Fourier coefficient given by

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} p(t)\, e^{-j\frac{2\pi nt}{T}}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-j\frac{2\pi nt}{T}}\, dt = \frac{1}{T} \tag{3.158}$$

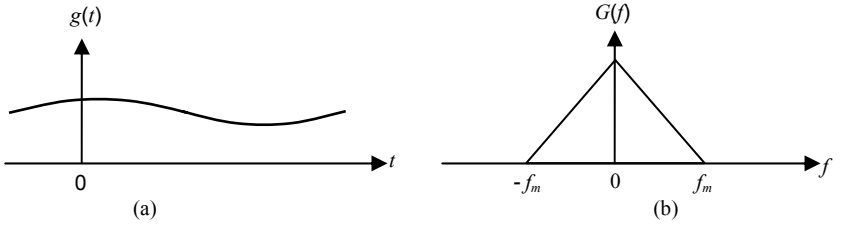


**Figure 3.29** (a) Signal $g(t)$, and (b) spectrum of $g(t)$.

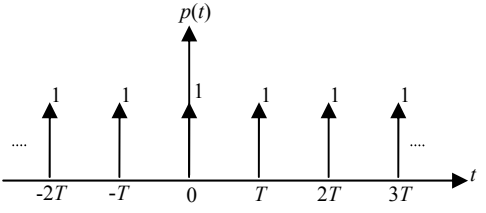


**Figure 3.30** Sampling function.

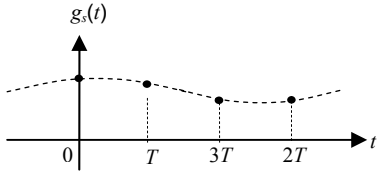


**Figure 3.31** Sampled signal.

$1/T$ is the fundamental frequency of the periodic signal $p(t)$, which is also the sampling frequency $f_s = 1/T$ Hz. Substituting (3.157) and (3.158) in (3.156), we obtain

$$g_s(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} g(t) e^{j2\pi n f_s t} \tag{3.159}$$

The spectrum of the sampled signal, from the definition of the Fourier transform, is given by

$$G_s = f_s \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g(t) e^{-j2\pi(f - f_s)t} dt = f_s \sum_{n=-\infty}^{\infty} G(f - n f_s) \tag{3.160}$$

and is shown in Figure 3.32. We observe that the original signal can be recovered by just using a lowpass filter as shown in dashed lines. We observe also that the sampling rate is at least $2f_m$ per second. The minimum frequency, $f_s = 2f_m$ samples per second, is called the *Nyquist rate.* Sampling with a frequency below the Nyquist rate results in *aliasing* error as shown in Figure 3.33, and the original signal cannot be recovered. We see from (3.157) that sampling introduces a scaling constant of $f_s = 1/T$, and thus to remove it we select the filter to be of

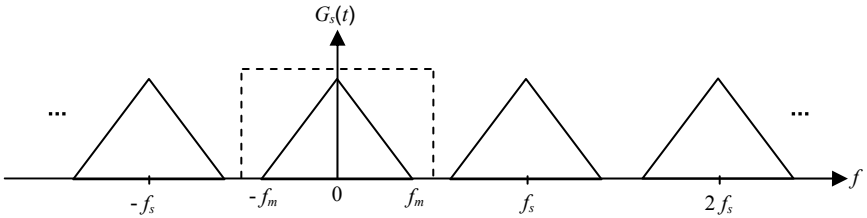


**Figure 3.32** Spectrum of the sampled signal.

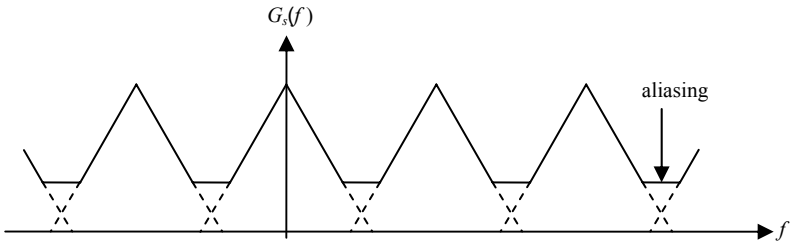


**Figure 3.33** Aliasing.

height $T$. Assuming the filter bandwidth is one-half the sampling frequency, the impulse response of the ideal lowpass filter with gain $T$ is

$$h(t) = T \int_{-f_s/2}^{f_s/2} e^{j\,2\,\pi f t}\, df = \text{sinc}\ \ f_s t \tag{3.161}$$

The output of the lowpass reconstruction filter is the sum of all output samples, as shown in Figure 3.34, to yield

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT)\,\text{sinc}\left(\frac{t}{T} - n\right) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2f_m}\right) \frac{\sin 2\pi f_m\left(t - \dfrac{n}{2f_m}\right)}{2\pi f_m\left(t - \dfrac{n}{2f_m}\right)} \tag{3.162}$$

where sinc $x = \sin\pi x/\pi x$, $T = 1/2f_m$, and $g\left(n/2f_m\right)$ are samples of $g(t)$ taken at $t = n/2f_m$, $n = 0,\ \pm 1,\ \pm 2,\ \ldots$.

*Theorem.* A bandlimited signal of finite energy with no frequency higher than $f_m$ Hz may be completely recovered from its samples taken at the rate of $2f_m$ per second.

If now $X(t)$ is a wide-sense stationary random process with a bandlimited power spectrum density such that $S_{xx}(f) = 0$ for $|f| > f_m$, the inverse Fourier

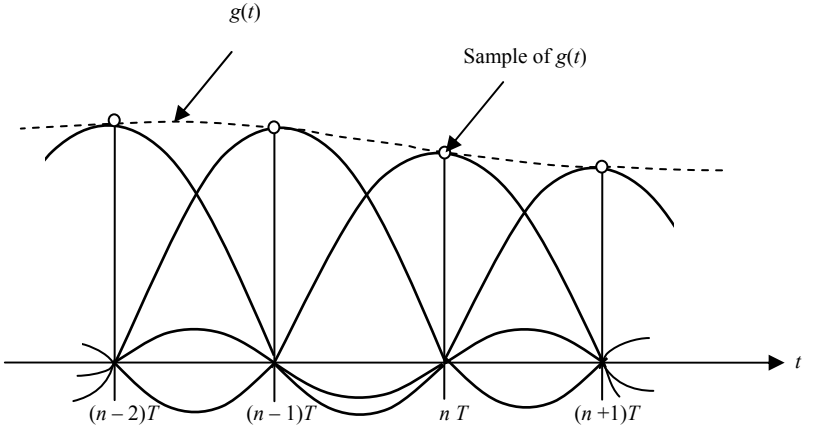


**Figure 3.34** Reconstructed signal.

transform of $S_{xx}(f)$ is just the autocorrelation function $R_{xx}(\tau)$, and thus from (3.162), $R_{xx}(\tau)$ can be expressed as

$$R_{xx}(\tau) = \sum_{n=-\infty}^{\infty} R_{xx}(nT)\,\mathrm{sinc}\left(\frac{\tau}{T} - n\right) = \sum_{n=-\infty}^{\infty} R_{xx}\left(\frac{n}{2f_m}\right) \frac{\sin 2\pi f_m\left(\tau - \dfrac{n}{2f_m}\right)}{2\pi f_m\left(\tau - \dfrac{n}{2f_m}\right)} \tag{3.163}$$

where $R_{xx}\left(n/2f_m\right)$ are samples of the autocorrelation function $R_{xx}(\tau)$ taken at $\tau = n/2f_m$, $n = 0, \pm 1, \pm 2, \ldots$. The sampling representation of $R_{xx}(\tau - a)$, $a$ an arbitrary constant, can be written in terms of the shifted sample sequence to yield

$$R_{xx}(\tau - a) = \sum_{n=-\infty}^{\infty} R_{xx}(nT - a)\,\mathrm{sinc}\left(\frac{\tau}{T} - n\right) \tag{3.164}$$

and making the change of variables $\tau$ – a to $\tau$, we obtain

$$R_{xx}(\tau) = \sum_{n=-\infty}^{\infty} R_{xx}(nT - a)\,\mathrm{sinc}\left(\frac{\tau + a}{T} - n\right) \tag{3.165}$$

An analogous sampling theorem may be stated for random processes.

*Theorem.* Let $X(t)$ be a wide-sense stationary random process bandlimited to the frequency $(-f_m, f_m)$; that is, $S_{xx}(f) = 0$ for $|f| > f_m$. Then,

$$X(t) = \sum_{n=-\infty}^{\infty} X(nT)\,\mathrm{sinc}\left(\frac{t}{T} - n\right) = \sum_{n=-\infty}^{\infty} X\left(\frac{n}{2f_m}\right) \frac{\sin 2\pi f_m\left(t - \dfrac{n}{2f_m}\right)}{2\pi f_m\left(t - \dfrac{n}{2f_m}\right)} \tag{3.166}$$

where $T = 1/2f_m$ and $X\left(n/2f_m\right)$ are samples of $X(t)$ taken at $t = n/2f_m$, $n = 0, \pm 1, \pm 2, \ldots$.

The samples $X\left(n/2f_m\right)$ are in this case random variables, and the equality of (3.166) holds in the mean-square sense. That is, the mean-square value of the difference of the two sides of the equation is zero. Hence, we must show that, as $n \to \infty$,

$$E\left[\left|X(t)-\sum_{n=-\infty}^{\infty}X\left(\frac{n}{2f_m}\right)\frac{\sin 2\pi f_m\left(t-\frac{n}{2f_m}\right)}{2\pi f_m\left(t-\frac{n}{2f_m}\right)}\right|^2\right]=0 \tag{3.167}$$

Let

$$\hat{X}(t)=\sum_{n=-\infty}^{\infty}X(nT)\,\mathrm{sinc}\left(\frac{t}{T}-n\right)=\sum_{n=-\infty}^{\infty}X\left(\frac{n}{2f_m}\right)\frac{\sin 2\pi f_m\left(t-\frac{n}{2f_m}\right)}{2\pi f_m\left(t-\frac{n}{2f_m}\right)} \tag{3.168}$$

then,

$$\begin{aligned}E\left[\left|X(t)-\hat{X}(t)\right|^2\right]&=E\left\{[X(t)-\hat{X}(t)][X^*(t)-\hat{X}^*(t)]\right\}\\&=E\left\{[X(t)-\hat{X}(t)]\,X^*(t)\right\}-E\left\{[X(t)-\hat{X}(t)]\,\hat{X}^*(t)\right\}\end{aligned} \tag{3.169}$$

Using (3.168), the first term of (3.169) may be written as

$$E\left\{[X(t)-\hat{X}(t)]X^*(t)\right\}=R_{xx}(0)-\sum_{n=-\infty}^{\infty}R_{xx}(nT-t)\,\mathrm{sinc}\left(\frac{t}{T}-n\right) \tag{3.170}$$

We also have from (3.165), with $\tau = 0$ and $a = t$,

$$R_{xx}(0)=\sum_{n=-\infty}^{\infty}R_{xx}(nT-t)\,\mathrm{sinc}\left(\frac{t}{T}-n\right) \tag{3.171}$$

Hence,

$$E\left\{[X(t)-\hat{X}(t)]X^*(t)\right\}=0 \tag{3.172}$$

We now compute the second terms of (3.169),

$$E\left\{[X(t)-\hat{X}(t)]\hat{X}^*(t)\right\}=\sum_{m=-\infty}^{\infty}E\left\{[X(t)-\hat{X}(t)]X(mT)\right\}\,\mathrm{sinc}\left(\frac{t}{T}-m\right)$$

$$= \left[ \sum_{m=-\infty}^{\infty} R_{xx}(t-mT) - \sum_{m=-\infty}^{\infty} R_{xx}(nT-mT)\,\mathrm{sinc}\left(\frac{t}{T}-n\right) \right] \mathrm{sinc}\left(\frac{t}{T}-m\right) \quad (3.173)$$

Using (3.164) with $\tau = t$ and $a = mT$, we have

$$R_{xx}(t-mT) = \sum_{n=-\infty}^{\infty} R_{xx}(nT-mT)\,\mathrm{sinc}\left(\frac{t}{T}-n\right) \quad (3.174)$$

and hence, after substituting (3.174) in (3.173), we have

$$E\left\{[X(t)-\hat{X}(t)]\hat{X}^*(t)\right\} = 0 \quad (3.175)$$

The results of (3.172) and (3.175) show that the equality of (3.166) holds in the mean-square sense.

## 3.9 CONTINUITY, DIFFERENTIATION, AND INTEGRATION

### 3.9.1 Continuity

We know from calculus that a function $f(x)$ is said to be *continuous* at a point $x = x_0$, if $f(x)$ is defined at $x_0$, $\lim_{x\to x_0} f(x)$ is a real number, and $\lim_{x\to x_0} f(x) = f(x_0)$. Consequently, we say that a real deterministic function $x(t)$ is *continuous at a given point* $t_0$ if

$$\lim_{t\to t_0} x(t) = x(t_0) \quad (3.176)$$

If $t_0$ takes any real value, $-\infty < t_0 < \infty$, then the function $x(t)$ is said to be *continuous.*

This concept of continuity can be extended to random processes, since a random process is a set of sample functions with associated probabilities, making the ensemble of the process. Hence, we say that the random process $X(t)$ is continuous at a given point $t_0$ if all sample functions are continuous at $t_0$. That is,

$$P[\lim_{t\to t_0} X(t) \neq X(t_0)] = 0 \quad (3.177)$$

or

$$P[X(t) \text{ continuous at } t_0] = 1 \tag{3.178}$$

This kind of continuity is called *strict continuity*. However, in many applications we are interested in a less "strong continuity" for many purposes, which is referred to as *stochastic continuity*. The most attractable stochastic continuity is continuity in the mean-square sense.

A random process $X(t)$ is called *mean-square continuous* (*m. s. continuous*), or *continuous in the mean-square sense*, if

$$\lim_{\tau \to 0} E[|X(t+\tau) - X(t)|^2] = 0 \tag{3.179}$$

Note that

$$\lim_{\tau \to 0} E[|X(t+\tau) - X(t)|^2] = \lim_{\tau \to 0} 2[R_{xx}(0) - R_{xx}(\tau)] \tag{3.180}$$

which is equal to zero if

$$\lim_{\tau \to 0} R_{xx}(\tau) = R_{xx}(0) \tag{3.181}$$

Hence, $X(t)$ is continuous in the mean-square sense if and only if its autocorrelation function $R_{xx}(\tau)$ is continuous at $\tau = 0$.

Note that for real wide-sense stationary processes, the autocorrelation function $R_{xx}(\tau)$ is an even function of $\tau$ as given by (3.26). Hence, the continuity at $\tau = 0$ is violated for the three possible cases of Figure 3.35.

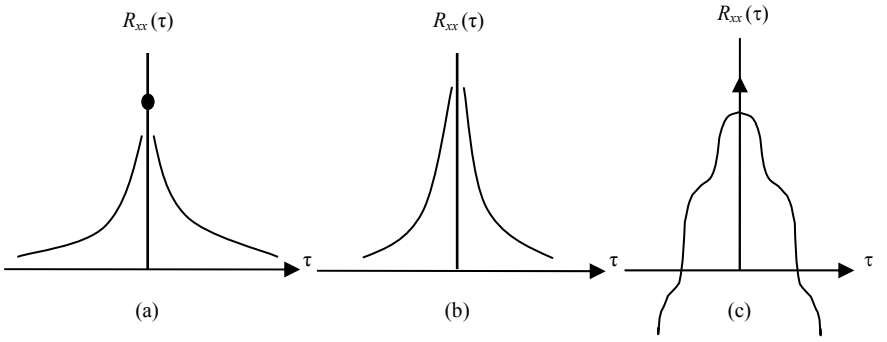


**Figure 3.35** $R_{xx}(\tau)$ not continuous at $\tau = 0$: (a) isolated point, (b) vertical asymptote, and (c) impulse.

**Example 3.13**

Show that if the random process $X(t)$ is mean-square continuous, then the mean $E[X(t)]$ is continuous.

*Solution*

Knowing that $X(t)$ is mean-square continuous, we can write

$$E[|X(t+\tau)-X(t)|^2] \geq E^2[X(t+\tau)-X(t)]$$

We have just shown that the left side of the above inequality goes to zero when $\tau \to 0$ for $X(t)$ to be mean-square continuous. Hence, $E^2[X(t+\tau)-X(t)]$ goes to zero as $\tau \to 0$; that is,

$$\lim_{\tau \to 0} E[X(t+\tau)] = E[X(t)] \tag{3.182}$$

and the proof is complete.

We can also show that if $X(t)$ is mean-square continuous, then

$$\lim_{\tau \to 0} E[X(t+\tau)] = E[\lim_{\tau \to 0} X(t+\tau)] \tag{3.183}$$

that is, we can interchange the expectation and limiting operations.

### 3.9.2 Differentiation

Again, from calculus, if $\lim_{\varepsilon \to 0}[f(x_1+\varepsilon)-f(x_1)]/\varepsilon$ exists, then it is called the *derivative* of $f(x)$ at $x = x_1$. If the function is differentiable at a point $x = x_1$, then it is continuous at $x = x_1$. Similarly, the "ordinary derivative" of a random process $X(t)$ is given by

$$X'(t) = \frac{dX(t)}{dt} = \lim_{\varepsilon \to 0} \frac{X(t+\varepsilon)-X(\varepsilon)}{\varepsilon} \tag{3.184}$$

provided that *all* sample functions of $X(t)$ are differentiable, which is too restrictive. Hence, we prefer talking about the derivative of a random process in the *mean-square sense*. We say that $X(t)$ is *mean-square differentiable* if there exists a random process $Y(t)$, such that

$$\lim_{\varepsilon\to 0}\left\{\left[\frac{X(t+\varepsilon)-X(\varepsilon)}{\varepsilon}-Y(t)\right]^2\right\}=0 \tag{3.185}$$

for every $t$. $Y(t)$ is the *mean-square derivative* process of $X(t)$ and is denoted $X'(t)$.

Assuming $X'(t)$ exists, the cross-correlation function between $X(t)$ and $X'(t)$ is given by

$$\begin{aligned}R_{xx'}(t_1,t_2)=E[X(t_1)X'(t_2)] &= E\left[X(t_1)\lim_{\varepsilon\to 0}\frac{X(t_2+\varepsilon)-X(t_2)}{\varepsilon}\right]\\ &=\lim_{\varepsilon\to 0}E\left[\frac{X(t_1)X(t_2+\varepsilon)-X(t_1)X(t_2)}{\varepsilon}\right]=\lim_{\varepsilon\to 0}\frac{R_{xx}(t_1,t_2+\varepsilon)-R_{xx}(t_1,t_2)}{\varepsilon}\\ &=\frac{\partial R_{xx}(t_1,t_2)}{\partial t_2}\end{aligned} \tag{3.186}$$

Similarly, we can also show that the cross-correlation function between $X'(t)$ and $X(t)$ is directly related to the autocorrelation function of $X(t)$, such that

$$R_{x'x}(t_1,t_2)=\frac{\partial R_{xx}(t_1,t_2)}{\partial t_1} \tag{3.187}$$

The autocorrelation function of $X'(t)$ can now be derived,

$$\begin{aligned}R_{x'x'}(t_1,t_2)=E[X'(t_1)X'(t_2)]&=E\left[\lim_{\varepsilon\to 0}\frac{X(t_1+\varepsilon)-X(t_1)}{\varepsilon}X'(t_2)\right]\\ &=\lim_{\varepsilon\to 0}E\left[\frac{X(t_1+\varepsilon)X'(t_2)-X(t_1)X'(t_2)}{\varepsilon}\right]=\lim_{\varepsilon\to 0}\frac{R_{xx'}(t_1+\varepsilon,t_2)-R_{xx'}(t_1,t_2)}{\varepsilon}\\ &=\frac{\partial R_{xx'}(t_1,t_2)}{\partial t_1}\end{aligned} \tag{3.188}$$

Substituting for the expression of $R_{xx'}(t_1,t_2)$ given in (3.186), we obtain $R_{x'x'}(t_1,t_2)$ to be

$$R_{x'x'}(t_1,t_2)=\frac{\partial^2 R_{xx}(t_1,t_2)}{\partial t_1\partial t_2} \tag{3.189}$$

If $X(t)$ is stationary in the wide sense, then the mean is constant and the mean of $X'(t)$ is zero; that is,

$$E[X'(t)] = 0 \tag{3.190}$$

Also, $R_{xx}(t_1, t_2) = R_{xx}(\tau)$, where $\tau = t_1 - t_2$. Noting that $dt_1 = d\tau$ and $dt_2 = -d\tau$, (3.186), (3.187), and (3.189) can be rewritten as

$$R_{xx'}(\tau) = -\frac{dR_{xx}(\tau)}{d\tau} = -R'_{xx}(\tau) \tag{3.191}$$

$$R_{x'x}(\tau) = \frac{dR_{xx}(\tau)}{d\tau} = R'_{xx}(\tau) \tag{3.192}$$

and

$$R_{x'x'} = -\frac{d^2 R_{xx}(\tau)}{d\tau^2} = -R''_{xx}(\tau) \tag{3.193}$$

At $\tau = 0$, we have

$$R_{x'x'}(0) = E\left\{[X'(t)]^2\right\} = -R''_{xx}(\tau)\Big|_{\tau=0} \tag{3.194}$$

Equation (3.194) is valid assuming $X(t)$ is mean-square differentiable. The above results can be generalized to higher-order derivatives to yield

$$R_{x^{(n)}x^{(m)}}(t_1, t_2) = E\left[\frac{d^n X(t_1)}{dt_1^n}\frac{d^m X(t_2)}{dt_2^m}\right] = \frac{\partial^{n+m} R_{xx}(t_1, t_2)}{\partial t_1^n \partial t_2^m} \tag{3.195}$$

and

$$R_{x^{(n)}y^{(m)}}(t_1, t_2) = E\left[\frac{d^n X(t_1)}{dt_1^n}\frac{d^m Y(t_2)}{dt_2^m}\right] = \frac{\partial^{n+m} R_{xy}(t_1, t_2)}{\partial t_1^n \partial t_2^m} \tag{3.196}$$

where the superscripts in parentheses, ($n$) and ($m$), denote the derivatives of the $n$th order and $m$th order, respectively.

If $X(t)$ and $Y(t)$ are jointly wide-sense stationary, then (3.195) and (3.196) become

$$R_{x^{(n)}x^{(m)}}(\tau) = E\left[\frac{d^n X(t+\tau)}{dt^n}\frac{d^m X(t)}{dt^m}\right] = (-1)^m \frac{d^{n+m}R_{xx}(\tau)}{d\tau^{n+m}} = (-1)^m R_{xx}^{(n+m)}(\tau) \tag{3.197}$$

and

$$R_{x^{(n)}y^{(m)}}(\tau) = E\left[\frac{d^n X(t+\tau)}{dt^n}\frac{d^m Y(t)}{dt^m}\right] = (-1)^m \frac{d^{n+m}R_{xy}(\tau)}{d\tau^{n+m}} = (-1)^m R_{xy}^{(n+m)}(\tau) \tag{3.198}$$

### 3.9.3 Integrals

The *Riemann* integral of an ordinary function $f(x)$ is defined as

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x_k \tag{3.199}$$

where $x_k$ is an arbitrary point in the $k$th subinterval $\Delta x_k$, $k = 1, 2, \ldots, n$.

For a real random process $X(t)$, the existence of the integral

$$I = \int_a^b X(t)dt \tag{3.200}$$

in the strict sense means existence as a Riemann integral for every sample function $x(t)$. In this case, $I$ is a random variable with sample values

$$i = \int_a^b x(t)dt \tag{3.201}$$

where $x(t)$ is a sample function of $X(t)$, and thus (3.201) may not necessarily exist for every sample function. We define the *mean-square integral* of the random process $X(t)$ as

$$I = \int_a^b X(t)dt = \lim_{n\to\infty}\sum_{k=1}^{n} X(t_k)\Delta t_k \tag{3.202}$$

The integral exists when

$$\lim_{\Delta t_k \to 0} E\left[\left| I - \sum_{k=1}^{n} X(t_k)\Delta t_k \right|^2\right] = 0 \tag{3.203}$$

In this case, the mean of $I$ is given by

$$E[I] = E\left[\int_a^b X(t)dt\right] = \int_a^b E[X(t)]dt = \int_a^b m_x(t)dt \tag{3.204}$$

the mean-square value is

$$E[I^2] = E\left[\int_a^b\int_a^b X(t_1)X^*(t_2)dt_1dt_2\right] = \int_a^b\int_a^b R_{xx}(t_1,t_2)dt_1dt_2 \tag{3.205}$$

and the variance is

$$\text{var}[I] = \sigma_i^2 = \int_a^b\int_a^b C_{xx}(t_1,t_2)dt_1dt_2 = \int_a^b\int_a^b R_{xx}(t_1,t_2)dt_1dt_2 - m_x(t_1)m_x(t_2) \tag{3.206}$$

If $X(t)$ is stationary, and we redefine $I$ as a time average to be

$$I = \frac{1}{2T}\int_{-T}^{T} X(t)dt \tag{3.207}$$

Then, the variance $I$ is

$$\text{var}[I] = \sigma_i^2 = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_{xx}(t_1 - t_2)dt_1dt_2 \tag{3.208}$$

Letting $\tau = t_1 - t_2$, and changing the double integral in $t_1$ and $t_2$ to one integral in $\tau$ as we did in Section 3.3, we have

$$\int_{-T}^{T}\int_{-T}^{T} C_{xx}(t_1 - t_2)\,dt_1dt_2 = \int_{-2T}^{2T}(2T - |\tau|)\,C_{xx}(\tau)\,d\tau \tag{3.209}$$

Thus, the variance becomes

$$\text{var}[I] = \sigma_i^2 = \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{xx}(\tau)\,d\tau = \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)[R_{xx}(\tau)-m_x^2]\,d\tau \tag{3.210}$$

## 3.10 HILBERT TRANSFORM AND ANALYTIC SIGNALS

Consider a linear system whose transfer function is given by

$$H(f) = \begin{cases} -j, & f > 0 \\ j, & f < 0 \end{cases} \tag{3.211}$$

Note that $|H(jf)| = 1$ for all frequencies; that is, it is an *all-pass filter*. The phase is given by

$$\arg[H(jf)] = \angle H(jf) = \begin{cases} -\dfrac{\pi}{2}, & f > 0 \\ \dfrac{\pi}{2}, & f < 0 \end{cases} \tag{3.212}$$

The amplitude and phase responses of such a system, called a *quadrature filter*, are shown in Figure 3.36. The impulse response of this filter, shown in Figure 3.37, is given by

$$h(t) = \frac{1}{\pi t} \tag{3.213}$$

If $x(t)$ is the input to the quadrature filter as shown in Figure 3.38, the output $y(t)$ is called the *Hilbert transform* of $x(t)$, $\mathcal{H}\{x(t)\}$, and is given by

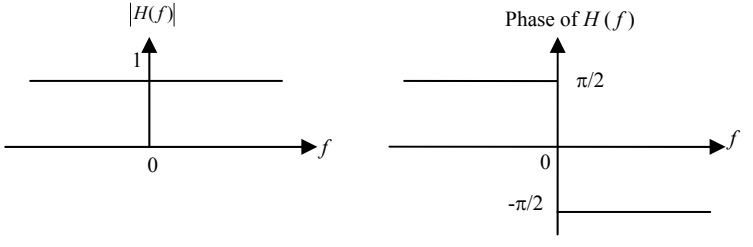


**Figure 3.36** Amplitude and phase response of a quadrature filter.

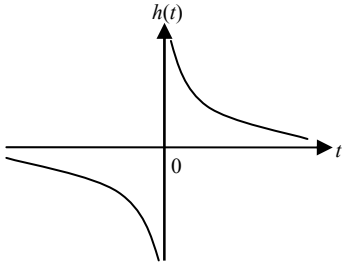


**Figure 3.37** Impulse response of a quadrature filter.

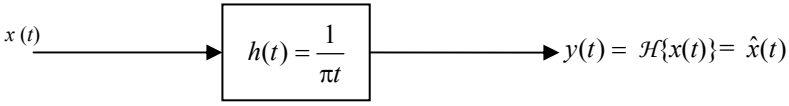


**Figure 3.38** Hilbert transform of $x(t)$.

$$y(t) = \hat{x}(t) = \mathcal{H}\{x(t)\} = x(t) * h(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \tag{3.214}$$

If now $X(t)$, a wide-sense stationary random process, is the input of a quadrature filter with autocorrelation function $R_{xx}(\tau)$ and power spectral density $S_{xx}(f)$, then the output $Y(t)$ is the Hilbert transform of $X(t)$ defined by

$$Y(t) = \hat{X}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X(\tau)}{t-\tau} d\tau \tag{3.215}$$

The power spectral density of the Hilbert transform of $X(t)$ is

$$S_{yy}(f) = S_{\hat{x}\hat{x}}(f) = S_{xx}(f)\left|H(f)\right|^2 = S_{xx}(f) \tag{3.216}$$

Hence, the autocorrelation function of the Hilbert transform is

$$R_{\hat{x}\hat{x}}(\tau) = R_{xx}(\tau) \tag{3.217}$$

From (3.135), the cross-spectral density is

$$S_{\hat{x}x}(f) = S_{xx}(f)H(f) = \begin{cases} -jS_{xx}(f), & f > 0 \\ jS_{xx}(f), & f < 0 \end{cases} \tag{3.218}$$

which is purely imaginary. Hence, using the cross-correlation function, we have by definition

$$S_{\hat{x}x}(f) = \int_{-\infty}^{\infty} R_{\hat{x}x}(\tau) e^{-j2\pi f\tau} d\tau \tag{3.219}$$

$$= \int_{-\infty}^{\infty} R_{\hat{x}x}(\tau) \cos 2\pi f\tau \, d\tau - j \int_{-\infty}^{\infty} R_{\hat{x}x}(\tau) \sin 2\pi f\tau \, d\tau \tag{3.220}$$

Since $S_{\hat{x}x}(f)$ is purely imaginary, then

$$\int_{-\infty}^{\infty} R_{\hat{x}x}(\tau) \cos 2\pi f\tau \, d\tau = 0 \tag{3.221}$$

The cosine is an even function, and thus $R_{\hat{x}x}(\tau)$ is odd, yielding

$$R_{\hat{x}x}(-\tau) = -R_{\hat{x}x}(\tau) \tag{3.222}$$

and

$$R_{\hat{x}x}(0) = 0 \tag{3.223}$$

Since $S_{\hat{x}x}(f) = S_{xx}(f)H(f)$ and $S_{x\hat{x}}(f) = S_{xx}(f)H^*(f)$, it also follows that

$$R_{\hat{x}x}(\tau) = \hat{R}_{xx}(\tau) \tag{3.224}$$

and

$$R_{x\hat{x}}(\tau) = \hat{R}_{xx}(-\tau) = -\hat{R}_{xx}(\tau) \tag{3.225}$$

Finally, we observe that $H(jf)H(jf) = -1$. This implies that the Hilbert transform of a Hilbert transform is the negative of the original signal; that is,

$$\hat{\hat{X}}(t) = -X(t) \tag{3.226}$$

From (3.217) and (3.225), we can write

$$R_{\hat{x}\hat{x}}(\tau) = R_{xx}(\tau) = -\hat{\hat{R}}_{xx}(\tau) \tag{3.227}$$

Consider next a linear system whose transfer function is given by

$$H(f) = \begin{cases} 2, & f > 0 \\ 0, & f < 0 \end{cases} \tag{3.228}$$

as shown in Figure 3.39. The impulse response of this system is

$$h(t) = \delta(t) + \frac{j}{\pi t} \tag{3.229}$$

Such a system changes a real signal into a complex signal. The transfer function can be rewritten as

$$H(f) = 1 + jQ(f) \tag{3.230}$$

where $Q(f)$ is the transfer function of the quadrature filter. If a signal $x(t)$ is applied to this system, as shown in Figure 3.40, the output $y(t)$ is called the *analytic signal* associated with $x(t)$.

Similarly, the analytic process associated with $X(t)$ is defined to be

$$Y(t) = \tilde{X}(t) = X(t) + j\hat{X}(t) \tag{3.231}$$

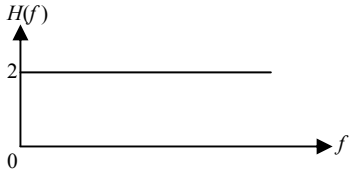


**Figure 3.39** Transfer function.

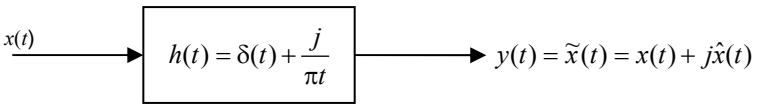


**Figure 3.40** System for analytic signal.

The autocorrelation function of $Y(t)$ is

$$R_{yy}(\tau) = E[Y(t+\tau)\,Y^*(t)] = E\{[X(t+\tau) + j\hat{X}(t)]\ [X(t) - j\hat{X}(t)]\}$$
$$= R_{xx}(\tau) + R_{\hat{x}\hat{x}}(\tau) + j[R_{\hat{x}x}(\tau) - R_{x\hat{x}}(\tau)] \tag{3.232}$$

Knowing $R_{xx}(\tau) = R_{\hat{x}\hat{x}}(\tau)$ and $R_{\hat{x}x}(\tau) = -R_{x\hat{x}}(\tau) = \hat{R}_{xx}(\tau)$, the autocorrelation function of the analytic signal becomes

$$R_{\tilde{x}\tilde{x}}(\tau) = R_{yy}(\tau) = 2[R_{xx}(\tau) + jR_{\hat{x}x}(\tau)] = 2[R_{xx}(\tau) + j\hat{R}_{xx}(\tau)] \tag{3.233}$$

Taking the Fourier transform of (3.233), we have

$$S_{\tilde{x}\tilde{x}}(f) = S_{yy}(f) = 2[S_{xx}(\tau) + jS_{\hat{x}x}(\tau)] \tag{3.234}$$

Using (3.218), we obtain

$$S_{\tilde{x}\tilde{x}}(f) = S_{xx}(f) = \begin{cases} 4S_{xx}(f) & ,\ f > 0 \\ 0 & ,\ f < 0 \end{cases} \tag{3.235}$$

We observe that the power spectral density of an analytic signal is zero for negative frequencies, and from (3.231)

$$X(t) = \Re\mathrm{e}\left\{\tilde{X}(t)\right\} \tag{3.236}$$

that is, $X(t)$ is the real part of the analytic signal. From (3.233),

$$R_{xx}(\tau) = \frac{1}{2}\ \Re\mathrm{e}\,\{R_{\tilde{x}\tilde{x}}(\tau)\} \tag{3.237}$$

## 3.11 THERMAL NOISE

Electrical noise arising from the random motion of electrons in conductors is referred to as *thermal noise*. It has been shown that the power spectral density of thermal noise voltage across the terminals of a resistor $R$ is given by

$$S_{nn}(f) = 2kTR\,\frac{\alpha^2}{\alpha^2 + \omega^2} \tag{3.238}$$

where $k = 1.38 \times 10^{-23}$ J/ K is the *Boltzmann's constant*, and $T$ is the *absolute temperature* in K. A plot of the power spectral density of thermal noise is shown in Figure 3.41. However, $\alpha$ is in the order $10^{14}$ rad/s or $10^{13}$ Hz $=10^4$ GHz, which is greater than most frequencies used in electronic circuit applications. Thus, $(\alpha^2 + \omega^2)/\omega^2 \to 1$, and thermal noise is modeled as a *white noise process* with a flat spectrum of value $2kTR$ $V^2$/Hz as shown in Figure 3.42.

In addition, since the number of electrons in the resistor is very large, with statistically independent random motions, from the central limit theorem, thermal noise is modeled as *Gaussian* with zero mean. Hence, thermal noise voltage is a *zero-mean white Gaussian process*. The resistor can be modeled by the *Thevenin's equivalent circuit*, consisting of a *noiseless* resistor in *series* with a noise *voltage* source, as shown in Figure 3.43(a) of mean-square value

$$E[V_n^2(t)] = 4kTR \tag{3.239}$$

or by *Norton's equivalent circuit*, consisting of a noiseless resistor in *parallel* with a noise *current* source, as shown in Figure 3.43(b), of mean-square value

$$E[I_n^2(t)] = 4kTG \tag{3.240}$$

where $G = 1/R$ is the conductance. The power spectral density of the noise source voltage or the noise source current are, respectively,

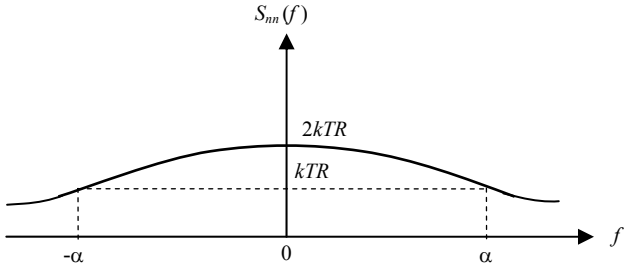


**Figure 3.41** Power spectral density of thermal noise.

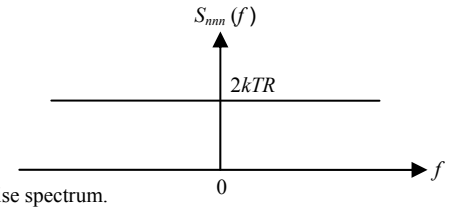


**Figure 3.42** White noise spectrum.

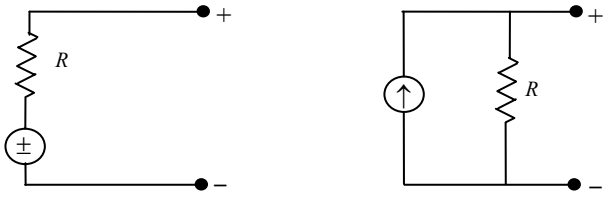


**Figure 3.43** Noisy resistors: (a) Thevenin's equivalent circuit, and (b) Norton's equivalent circuit.

$$S_{v_n v_n}(f) = 2kTR \quad \mathrm{V}^2/\mathrm{H}z \tag{3.241}$$

and

$$S_{i_n i_n}(f) = 2kTG \quad \mathrm{A}^2/\mathrm{Hz} \tag{3.242}$$

*Nyquist's theorem.* Consider a passive RLC one-port network as shown in Figure 3.44. The voltage at the terminal pair in $v(t)$ and $Z(j\omega)$ is the impedance seen looking into the port. Then, the power spectral density of the open-circuit noise voltage at the terminal pair due to *all* thermal noise sources is given by

$$S_{v_n v_n}(f) = 2kT\ \Re\mathrm{e}\{Z(j\omega)\} \tag{3.243}$$

or, the power spectral density of the short-circuit noise current is given by

$$S_{v_i v_i}(f) = 2kT\ \Re\mathrm{e}\{Y(j\omega)\} \tag{3.244}$$

where $Y(j\omega) = 1/Z(j\omega)$ is the network's input admittance, and $\omega = 2\pi f$.

**Example 3.14**

Determine the power spectral density of the voltage $v(t)$ at the terminal pair of the RC network shown in Figure 3.45 due to thermal noise generated in $R$, using:

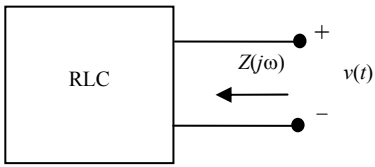


**Figure 3.44** Passive RLC network.

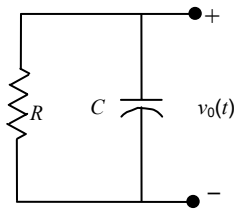


**Figure 3.45** RC network.

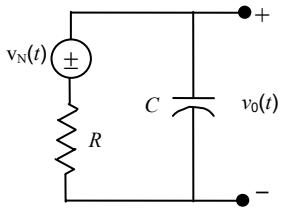


**Figure 3.46** Thevenin's equivalent circuit.

(a) Thevenin's equivalent circuit.
(b) Norton's equivalent circuit.
(c) Nyquist's theorem.

*Solution*

(a) Using Thevenin's equivalent circuit, the resulting circuit is shown in Figure 3.46. The transfer function from the noise source is, by voltage divider,

$$H(j\omega) = \frac{\dfrac{1}{j\omega C}}{R + \dfrac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

It follows that

$$S_{v_0 v_0}(f) = S_{v_n v_n}(\omega)\left|H(j\omega)\right|^2 = \frac{2kTR}{1 + (\omega RC)^2}$$

(b) Using Norton's equivalent circuit, the resulting circuit is shown in Figure 3.47. The transfer function is in this case

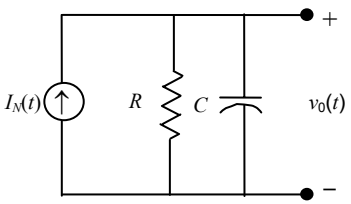


**Figure 3.47** Norton's equivalent circuit.

$$H(j\omega) = \frac{\dfrac{R}{j\omega C}}{R + \dfrac{1}{j\omega C}} = \frac{R}{1 + j\omega RC}$$

The power spectral density of the output voltage is then

$$S_{v_0 v_0}(\omega) = S_{i_n i_n}(\omega)\left|H(j\omega)\right|^2 = \frac{2kT}{R}\frac{R}{1+(\omega RC)^2} = \frac{2kTR}{1+(\omega RC)^2}$$

(c) The impedance seen looking into the terminal pair of the network is

$$H(j\omega) = \frac{\dfrac{R}{j\omega C}}{R + \dfrac{1}{j\omega C}} = \frac{R}{1 + j\omega RC} = \frac{R}{1+(\omega RC)^2} - j\frac{\omega RC}{1+(\omega RC)^2}$$

From Nyquist's theorem, the power spectral density of the resistor noise voltage source is

$$S_{v_0 v_0}(\omega) = 2kT\,\Re\mathrm{e}\left\{Z(j\omega)\right\} = \frac{2kTR}{1+(\omega RC)^2}$$

We observe that the three results of (a), (b), and (c) agree.

Usually, the power spectral density of a white noise process is denoted

$$S_{nn}(f) = \frac{N_0}{2}, \quad -\infty < f < \infty \tag{3.245}$$

The autocorrelation function is then

$$R_{nn}(\tau) = \frac{N_0}{2}\delta(\tau) \tag{3.246}$$

Since the bandwidth of real systems is finite, the integration (3.245) over a finite band of frequencies results in a finite average power.

**Example 3.15** (*Noise Equivalent Bandwidth*)

Suppose that the zero mean white Gaussian noise process with autocorrelation function $R_{nn}(\tau) = (N_0/2)\delta(\tau)$ is applied to a linear time-invariant system with impulse response $h(t)$. Determine the average noise power of the output process.

*Solution*

The power spectral density of the input white Gaussian noise is given by (3.245) to be $S_{nn}(f) = N_0/2$, $-\infty < f < \infty$. Using (3.141), the output noise power spectral density is $S(f) = (N_0/2)|H(f)|^2$, where $H(f)$ is the transfer function of the system. The average power is then

$$P_{ave} = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\,df = N_0\int_0^{\infty}|H(f)|^2\,df \tag{3.247}$$

which is finite if $\int_{-\infty}^{\infty}|H(f)|^2\,df$ converges. In such a situation, the concept of *noise equivalent bandwidth* is considered. To define the noise equivalent bandwidth, consider the same problem of Example 3.15, but the system's function is an ideal lowpass filter of bandwidth $B$ and zero-frequency response $H(0)$. The average noise power at the output is then

$$P_0 = N_0 H^2(0)B \tag{3.248}$$

In equating (3.247) and (3.248), we obtain the *noise equivalent bandwidth*, $B_{eq}$, which is defined to be

$$B_{eq} = \frac{\int_0^{\infty}|H(f)|^2\,df}{H^2(0)} \tag{3.249}$$

Hence, the average noise power at the output of the linear time-invariant system, with a noise equivalent bandwidth $B_{eq}$, and the input white noise process with power spectral density $N_0/2$, is given by

$$P_{ave} = N_0 B_{eq} H^2(0) \tag{3.250}$$

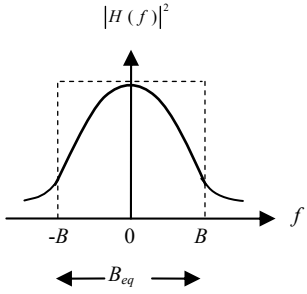


**Figure 3.48** Noise equivalent bandwidth of a lowpass system.

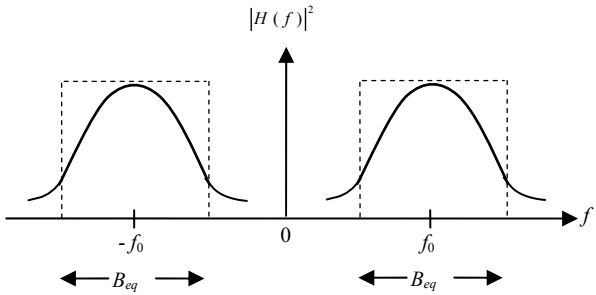


**Figure 3.49** Noise equivalent bandwidth of a bandpass system.

Note that the bandwidth $B_{eq}$ can be interpreted as the bandwidth of a linear time invariant system with a rectangular transfer function of zero-frequency response $H(0)$ and bandwidth $B$, as shown in Figure 3.48. If the system were bandpass, the noise equivalent bandwidth is as shown in Figure 3.49.

## 3.12 SUMMARY

In this chapter, we covered the fundamental concepts of random processes. After defining the random process, which was considered as a random variable (Chapters 1 and 2) with time as a running parameter, we gave the different properties of correlations and power spectral densities. We presented the different type of random processes; namely, the Gaussian process, the Poisson process, the Bernoulli and Binomial processes, the random walk and Winner processes, and a brief description of the Markov process. Markov processes will be developed in more detail in the next chapter on discrete time processes. The sampling theorem and the concepts of continuity, differentiation, and integration, which are well

known for deterministic signals, were developed for stochastic processes. Then, we defined the Hilbert transform of stochastic processes and the corresponding analytic process signals. We concluded the chapter with a discussion on thermal noise and noise equivalent bandwidth. Many examples were solved in some detail to remove the ambiguities that may occur.

## PROBLEMS

**3.1** Consider a random process $X(t)$ defined by

$$X(t) = A\cos(\omega_0 t + \Theta)$$

where $A$ and $\omega_0$ are constants, and $\Theta$ is a random variable with probability density function

$$f_\Theta(\theta) = \begin{cases} \dfrac{4}{\pi}, & |\theta| \le \dfrac{\pi}{8} \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the mean and autocorrelation functions.
(b) Is the process stationary?

**3.2** Let $s(t) = \text{rect}(t) = \begin{cases} 1, & |t| < \dfrac{1}{2} \\ 0, & \text{otherwise} \end{cases}$

Define the process $X(t) = S(t - T_0)$, where $T_0$ is a discrete random variable taking values 0 and 1 with equal probability.
(a) Determine and sketch the distribution function $F_{X_t}(x_t, 0)$.
(b) Determine and sketch the autocorrelation function $R_{xx}(t_1, t_2)$.

**3.3** Consider the random process defined in Problem 3.1.
(a) Is the process ergodic in the mean?
(b) Is the process ergodic in the autocorrelation?

**3.4** Consider the random process defined by $X(t) = A\cos(\omega_0 t + \Theta)$, where $A$ and $\omega_0$ are constants, and $\Theta$ is a random variable uniformly distributed over the interval $(0, 2\pi)$. Let $Y(t)$ be the random process defined as $Y(t) = X^2(t)$.
(a) Find the autocorrelation function of $Y(t)$.
(b) Is $Y(t)$ a stationary process?

**3.5** Consider the random process defined by $X(t) = Ae^{j(\omega t + \Theta)}$, where $A$ is a random variable with density function

$$f_A(a) = \begin{cases} \dfrac{a}{\sigma^2} e^{-\frac{a^2}{2\sigma^2}}, & a \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$\sigma^2$ is constant and $\Theta$ is a random variable uniformly distributed over the interval $(0, 2\pi)$. $A$ and $\Theta$ are statistically independent. Determine

(a) The mean function $E[X(t)]$.

(b) The autocorrelation function $R_{xx}(t_1, t_2)$.

**3.6** Let $X(t)$ and $Y(t)$ be two statistically independent random processes with autocorrelation functions $R_{xx}(\tau) = 2e^{-2|\tau|}\cos\omega\tau$ and $R_{yy}(\tau) = 9 + e^{-3|\tau|}$. Let $Z(t) = AX(t) + Y(t)$, where $A$ is a statistically independent random variable with mean 2 and variance 9.

(a) Determine $R_{zz}(\tau)$, the autocorrelation function of $Z(t)$.

(b) Compute the mean and variance of $Z(t)$.

**3.7** Let $X(t)$ be the random process shown in Figure P3.7. The square wave is periodic with period $T$. The amplitude $A$ is random with zero mean and variance $\sigma^2$. $t_0$ is governed by a random variable $T_0$, which is uniformly distributed over the interval $(0, T)$. $A$ and $T_0$ are statistically independent. Determine the autocorrelation function $R_{xx}(t_1, t_2)$.

**3.8** Let $s(t)$ be the periodic deterministic waveform shown in Figure P3.8. Define the random process $X(t) = S(t - T_0)$, where $T_0$ is a random variable uniformly distributed over the interval $(0, T)$.

(a) Find the autocorrelation function $R_{xx}(t_1, t_2)$. Is the process $X(t)$ stationary in the wide-sense?

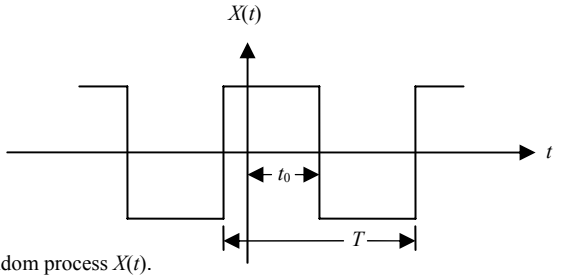


**Figure P3.7** Random process $X(t)$.

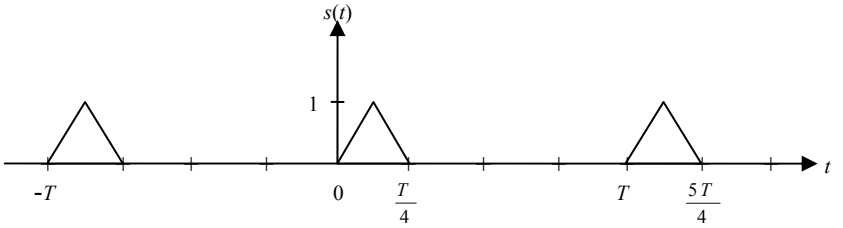


**Figure P3.8** Deterministic signal $s(t)$.

(b) Determine and sketch the distribution function $F_{X_t}(x_t)$.

(c) Determine and sketch the density function $f_{X_t}(x_t)$.

(d) Find $E[X(t)]$, $E[X^2(t)]$, and $\sigma_{x_t}^2$.

(e) Find $< x(t) >$ and $< x^2(t) >$.

**3.9** Let $X(t)$ be a random process with a typical sample function, as shown in Figure P3.9. The sample functions are constant during each second interval. Their values are governed by statistically independent random variables $A_i$, $i = 0, \pm 1, \pm 2, \ldots,$ and uniformly distributed over the interval $[-1, 1)$.

(a) Determine the second-order density function $f_X(0, 0; 1/2, 3/2)$.

(b) Let $Y(t) = X(t - \Theta)$, where $\Theta$ is a uniformly distributed random variable over the interval [0, 1) and statistically independent of the $A_i$s. Determine the second-order density function $f_Y(0, 0; 1/2, 3/2)$.

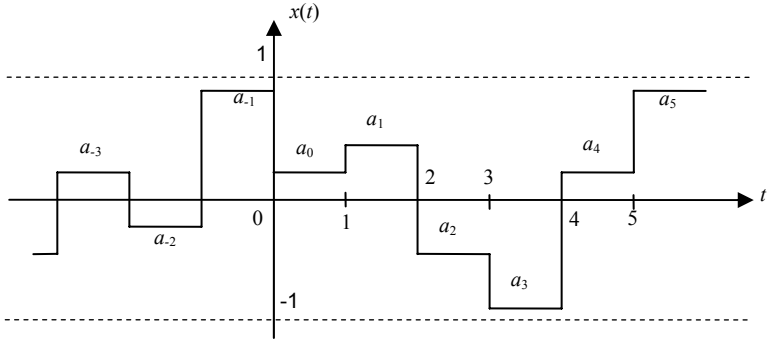


**Figure P3.9** A typical sample function of $X(t)$.

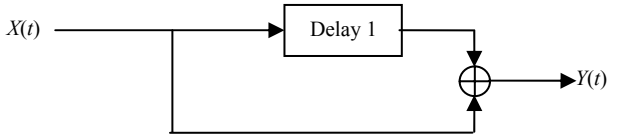


**Figure P3.10** System function for $X(t)$.

**3.10** Let $X(t)$ be a wide-sense stationary process with autocorrelation function

$$R_{xx}(\tau) = \begin{cases} 1 - |\tau|, & |\tau| < 1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

applied to the system shown in Figure P3.10. Determine and sketch the output autocorrelation function $R_{yy}(\tau)$.

**3.11** Let $X(t)$ and $Y(t)$ be two orthogonal processes with power spectral densities

$$S_{xx}(f) = S_{yy}(f) = \begin{cases} 1 - |f|, & |f| < 1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

Define a new process $Z(t) = Y(t) - X(t - 1)$. Determine and sketch the power spectral density $S_{zz}(f)$.

**3.12** Let $X(t)$, a zero-mean, wide-sense stationary real Gaussian random process with power spectral density $S_{xx}(f) = (N_0 / 2)e^{-\alpha|f|}$, be passed through the nonlinear memoryless cubic system given in Figure P3.12. Determine the mean-square value $E[Y^2(t)]$ of the output process in terms of $N_0$ and $\alpha$.

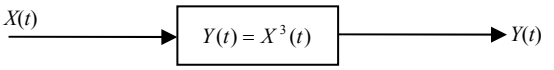


**Figure P3.12** Memoryless cubic system

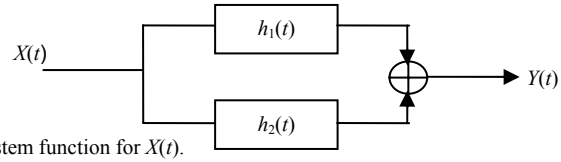


**Figure P3.13** System function for $X(t)$.

**3.13** Let $X(t)$ be a random process with the triangular autocorrelation function

$$R_{xx}(\tau) = \begin{cases} 1-|\tau|, & |\tau| < 1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

If $X(t)$ is applied to the system shown in Figure P3.13 with $h_1(t) = \delta(t-1)$ and $h_2(t) = \delta(t-2)$, then determine and sketch the output autocorrelation function of the output $Y(t)$.

**3.14** (a) Let $N(t)$, a stationary process with power spectral density $S_{nn}(f) = \text{rect}(f)$, be applied to the system shown in Figure P3.14(a). Determine and sketch the output power spectral density $S_{yy}(f)$.

(b) If the process $N(t)$ is added to two stationary processes $U(t)$ and $V(t)$ with cross-spectral density

$$S_{uv}(f) = \begin{cases} \frac{1}{2} - |f|, & |f| < \frac{1}{2} \\ 0 \quad , & \text{otherwise} \end{cases}$$

as shown in Figure P3.14(b), then determine and sketch the cross-spectral density $S_{wz}(f)$. Assume that the noise process is statistically independent of $U(t)$ and $V(t)$, and that $U(t)$ and $V(t)$ are zero mean.

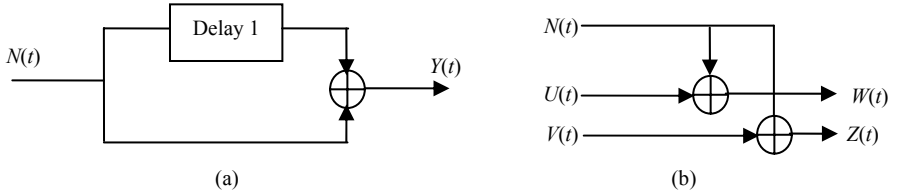


**Figure P3.14** (a) System function for $N(t)$, and (b) system combining $N(t)$,$U(t)$, and $V(t)$.

**3.15** Let $X(t)$ be the input process to a linear system with impulse response

$$h(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0 \quad , & \text{otherwise} \end{cases}$$

$X(t)$ is stationary in the wide-sense with autocorrelation function

$$R_{xx}(\tau) = \begin{cases} 1 - |\tau|, & |\tau| \leq 1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

Determine the autocorrelation function of the output process $R_{yy}(\tau)$.

**3.16** The random process $X(t)$ with autocorrelation function

$$R_{xx}(\tau) = e^{-\alpha|\tau|}, \ \alpha \text{ constant}$$

is applied to the RC filter shown in Figure P3.16. Determine the output power spectral density $S_{yy}(f)$.

**3.17** Let $X(t)$ be the input process to the RLC network shown in Figure P3.17. $X(t)$ is a wide-sense stationary process with mean $m_x(t) = 2$ and autocorrelation function

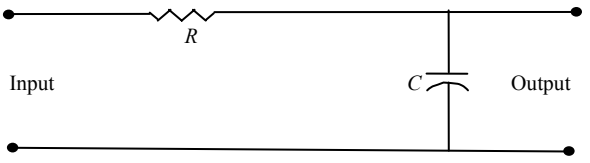


**Figure P3.16** RC filter.

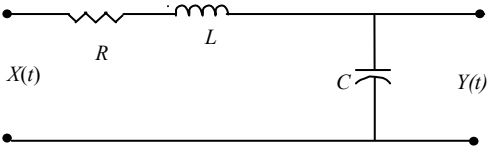


**Figure P3.17** RLC filter.

$$R_{xx}(\tau) = 4 + e^{-2|\tau|}$$

Find the mean $m_y(t)$ of the output process and the output power spectral density $S_{yy}(f)$.

**3.18** Let $N(t)$ be a bandlimited white noise process with power spectral density

$$S_{nn}(f) = \begin{cases} \dfrac{N_0}{2} & , \ |f| \le B \\ 0 & , \ |f| > B \end{cases}$$

where $B$ denotes the highest frequency. Determine the sampling rates for which the noise samples will be uncorrelated.

**3.19** Let $X(t)$ be a wide-sense stationary process with power spectral density

$$S_{xx}(f) = \begin{cases} 1 - |f|, & |f| < 1 \\ 0 & , \ \text{otherwise} \end{cases}$$

and sampled at the Nyquist rate.

(a) What is the interval between the samples?

(b) Determine the correlation coefficient between the samples $X(nT)$ and $X[(n + 1)T]$; $n$ arbitrary.

**3.20** Let $X(t)$ be a stationary random process with autocorrelation function $R_{xx}(\tau)$. Define the stochastic process

$$Y(t) = \int_0^t X(\tau) d\tau$$

Is the process $Y(t)$ stationary, given that

$$R_{xx}(\tau) = 4 + e^{-2|\tau|}$$

**3.21** Let $Y(t)$ be the process defined in Problem 3.20. Obtain the autocorrelation function of $Y(t)$ when $R_{xx}(t_1, t_2) = 2\delta(t_1 - t_2)$.

**3.22** Let $X(t)$ be a stationary, real, zero-mean Gaussian process with autocorrelation function as shown in Figure P3.22. The random variables $I_a$ and $I_b$ are obtained from the integration of the process $X(t)$, such that

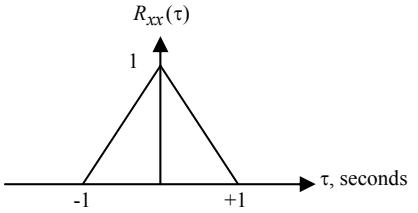


**Figure P3.22** The autocorrelation function.

$I_a = \int_0^1 X(t)dt$ and $I_b = \int_2^3 X(t)dt$, and the mean-square value of $I_a$ is $2/3$.

Determine

(a) $E[I_a^4]$

(b) $E[I_a I_b]$

(c) Let $I_c = \int_0^T X(t)dt$ with $T >> 1$ second. Find the variance of $I_c$.

**3.23** Let $Y(t) = \int_0^t X(\tau)d\tau$, where $X(t)$ is a stationary random process with autocorrelation function $R_{xx}(\tau) = 1 + e^{-2|\tau|}$.

(a) Is the random process $Y(t)$ stationary?

(b) Determine the autocorrelation function of $Y(t)$ in terms of $R_{xx}(\tau)$.

**3.24** Let $X(t)$ be a zero-mean wide-sense stationary process with power spectral density

$$S_{xx}(f) = \begin{cases} 1 - \dfrac{|f|}{f_c}, & |f| \le f_c \\ 0, & \text{otherwise} \end{cases}$$

$\hat{X}(t) = \mathcal{H}\{X(t)\}$ is the Hilbert transform of $X(t)$, and $\widetilde{X}(t)$ is the corresponding analytic signal process. Determine whether the following statements are true, possibly true, or false. Justify your answer.

(a) $X(t)$ and $\widetilde{X}(t)$ are orthogonal processes.

(b) $j\,\mathcal{H}\{\widetilde{X}(t)\} = \widetilde{X}(t)$.

(c) $X(t)e^{j2\pi f_0 t}$ is an analytic signal process.

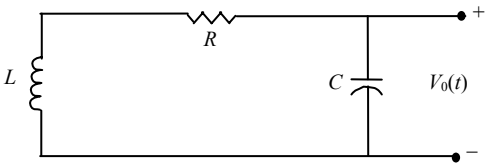


**Figure P3.25** RLC network.

(d) $E[\tilde{X}^2(t)] = 2E[X^2(t)]$.

**3.25** (a) Determine the power spectral density of $V_0(t)$ due to thermal noise for the RLC network shown in Figure P3.25.
(b) Use Nyquist's theorem to verify the result found in (a).

**3.26** Consider the network shown in Figure P3.26. For the noise voltage at the terminal pairs, determine
(a) The power spectral density.
(b) The autocorrelation function.
(c) If $R_1 = 1\,\text{K}\Omega$, $T_1 = 400\,\text{K}$, $R_2 = 2\,\text{K}\Omega$, $T_2 = 300\text{K}$, and $C = 10^{-10}$ F, compute the root mean-square (*rms*) value.

**3.27** Consider the RL network shown in Figure P3.27.
(a) Determine the power spectral density of the mesh current $I(t)$ due to thermal noise.
(b) Check the result found in (a) using Nyquist's theorm.

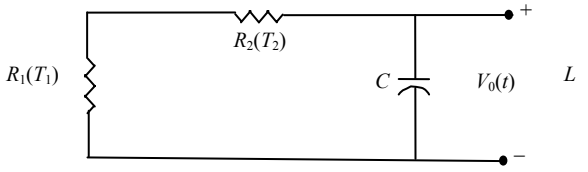


**Figure P3.26** RC network.

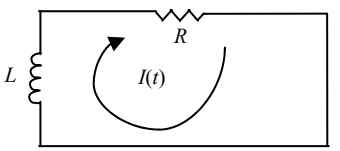


**Figure P3.27** RL network.

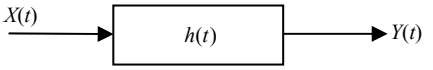


**Figure P3.28** Linear system.

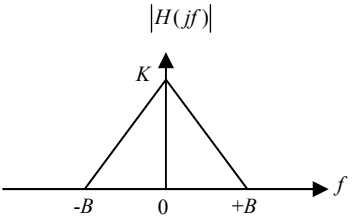


**Figure P3.29** System function.

**3.28** Consider the system shown in Figure P3.28, with impulse $h(t) = e^{-t}u(t)$. The input random process is stationary with mean $m_x$.

(a) Determine the mean of the output process $Y(t)$.

(b) Determine the mean and variance of $Y(t)$ if the input $X(t)$ is a zero mean white noise process.

**3.29** Let $N(t)$, a wide-sense stationary noise with power spectral density

$$S_{nn}(f) = \frac{N_0}{2} \text{ V}^2/\text{Hz}, \quad -\infty < f < \infty$$

be applied to a linear filter with the system function shown in Figure P3.29. Determine the variance of the output filter $Y(t)$.

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# Chapter 4

# Discrete-Time Random Processes

## 4.1 INTRODUCTION

In Chapter 3, we developed the concepts of continuous-time processes and described briefly the Markov process. In this chapter, we consider another class of random processes; namely, the discrete-time stochastic processes. A discrete random process may be a *uniformly sampled* version of a continuous-time process. A discrete random process is a correspondence that maps the sample space into a discrete-domain-functional space; that is, a functional space whose member functions are defined in a discrete set (time samples). Hence, it is a *collection* or an *ensemble* of real or complex discrete sequences of time, also called *realizations*, and denoted $X(n)$. Many authors use the notation $\tilde{x}[n]$. In our case, we keep $X(n)$ to be consistent with the notation $X(t)$ of a continuous-time random process. Note that for the convenience of notation, we normalize the time with respect to the sampling period. Hence, for a fixed $n$, $X(n)$ represents a random variable. One particular ensemble is the *discrete-time series* or just *time series*, where, for example, the sequence $X(n)$, $X(n-1), \ldots, X(n-M+1)$, representing a time series, consists of the present observation $X(n)$ and past $(M-1)$ observation at times $n-1, n-2, \ldots, n-M+1$. In fact, many discrete-time random processes are best approximated by the time series model. In this case, the power spectral density is a function of the model parameters, and thus the selection of the appropriate model and the estimation of the model parameters are necessary. Such an approach is referred to as *parametric*. If $U(n)$ is an input driving sequences and $X(n)$ the output sequence, then a general model of the data may be given by the following linear difference equation

$$X(n) = -\sum_{k=1}^{p} a(k)X(n-k) + \sum_{k=0}^{q} b(k)U(n-k) \tag{4.1}$$

Computing the spectrum using the obtained model parameters is known as *parametric spectrum estimation*. The field of spectrum estimation is wide, and it is not the scope of this book. However, in discussing discrete-time random processes and their applications, we must introduce the autoregressive (AR) processes, the moving average (MA) processes, and the autoregressive moving average (ARMA) processes. In order to have a good grasp of these discrete processes and their applications for spectrum estimation, the fundamental concepts of matrix operations and linear algebra are a prerequisite, and thus they will be given in Section 4.2. Such mathematical concepts will also be needed for later chapters. We conclude the chapter with Markov chains. Markov chains are a special class of Markov processes with discrete states, but with both discrete and continuous times. Note that we present the continuous-time Markov chains in this chapter, which seems to follow logically, after presenting the essential concepts of discrete-time Markov chains, since these concepts must be used when presenting continuous-time Markov chains.

## 4.2 MATRIX AND LINEAR ALGEBRA

In Chapter 2, we briefly used some concepts of matrices to do some operations. We now give, in this section, a review of the fundamentals of matrix and linear algebra.

### 4.2.1 Algebraic Matrix Operations

*Matrices* are defined as rectangular arrays of real or complex *elements*. The matrices are generally represented by capital boldface letters, whereas the elements of a matrix are denoted by lowercase letters. An $m \times n$ matrix $\boldsymbol{A}$ with elements $a_{ij}$, $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, m$ is a matrix with $m$ rows and $n$ columns, as given by (4.2).

$$\boldsymbol{A} = [\boldsymbol{A}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{4.2}$$

A shorthand notation that is sometimes used in describing matrices is

$$\boldsymbol{A} = [a_{ij}] \tag{4.3}$$

When $m = n$, the matrix is called a *square matrix*. If $m = 1$, the $m \times n$ matrix becomes a $1 \times n$ row matrix called a *row vector*, given by

$$\boldsymbol{a} = [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \tag{4.4}$$

whereas, if $n = 1$, the $m \times n$ matrix becomes an $m \times 1$ column matrix called a *column vector*, given by

$$\boldsymbol{a} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \tag{4.5}$$

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *equal* if $a_{ij} = b_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The sum and difference of two $m \times n$ matrices are performed on an element-by-element basis; that is, if

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A} \tag{4.6}$$

and

$$\boldsymbol{D} = \boldsymbol{A} - \boldsymbol{B} = -\boldsymbol{B} + \boldsymbol{A} \tag{4.7}$$

then,

$$c_{ij} = a_{ij} + b_{ij} \tag{4.8}$$

and

$$d_{ij} = a_{ij} - b_{ij} \tag{4.9}$$

Note that $\boldsymbol{A}$ and $\boldsymbol{B}$ must be of the same dimensions. If $\alpha$ is a scalar, the multiplication of an $m \times n$ matrix $\boldsymbol{A}$ by a scalar amounts to multiplying every element of $\boldsymbol{A}$ by $\alpha$; that is,

$$\alpha \boldsymbol{A} = \boldsymbol{A} \alpha = \alpha \ a_{ij} \tag{4.10}$$

If $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{B}$ is a $p \times q$ matrix, the product

$$\boldsymbol{AB} = \boldsymbol{C} \tag{4.11}$$

is defined when $\boldsymbol{A}$ and $\boldsymbol{B}$ are *conformable*; that is, when the number of columns $n$ of $\boldsymbol{A}$ is equal to the number of rows $p$ of $\boldsymbol{B}$, $n = p$. The product is then given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{4.12}$$

$\boldsymbol{C}$ is an $m \times q$ matrix. The matrix multiplication, if defined, is in general not *commutative*; that is,

$$\boldsymbol{AB} \neq \boldsymbol{BA} \tag{4.13}$$

Unlike scalar algebra, where the product $ab = 0$ means $a = 0$ or $b = 0$ or both, the matrix product $\boldsymbol{AB} = \boldsymbol{0}$ does not necessarily mean $\boldsymbol{A} = \boldsymbol{0}$ or $\boldsymbol{B} = \boldsymbol{0}$, where $\boldsymbol{0}$ is the *null matrix.* However, many operations related to *associative* and *distributive laws* are valid for matrix algebra; namely,

$$\alpha(\boldsymbol{A} + \boldsymbol{B}) = \alpha \boldsymbol{A} + \alpha \boldsymbol{B} \tag{4.14}$$

$$\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} \tag{4.15}$$

$$\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C} \tag{4.16}$$

$$\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC} \tag{4.17}$$

and

$$(\boldsymbol{B} + \boldsymbol{C})\boldsymbol{A} = \boldsymbol{BA} + \boldsymbol{CA} \tag{4.18}$$

The *identity matrix* or *unit matrix* $\boldsymbol{I}$ is an $n \times n$ square matrix all of whose elements are zero, except the elements $a_{ij}$, $i = j$, on the main diagonal, which are ones.

The *transpose* of an $m \times n$ matrix $\boldsymbol{A}$ is an $n \times m$ matrix obtained by interchanging each row with the column of $\boldsymbol{A}$ of the same *index* number, such that

$$\boldsymbol{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \tag{4.19}$$

or

$$\boldsymbol{A}^T = [a_{ji}] \tag{4.20}$$

The superscript *T* indicates matrix transpose. It can be shown that

$$(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T \tag{4.21}$$

$$(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T \tag{4.22}$$

and

$$(\boldsymbol{ABC})^T = \boldsymbol{C}^T \boldsymbol{B}^T \boldsymbol{A}^T \tag{4.23}$$

the *conjugate* of $\boldsymbol{A}$, written $\overline{\boldsymbol{A}}$ or $\boldsymbol{A}^*$, is the matrix obtained from $\boldsymbol{A}$ by changing all of its elements by their complex conjugate, such that

$$\boldsymbol{A}^* = [a_{ij}^*] \tag{4.24}$$

If all elements of $\boldsymbol{A}$ are real, then $\boldsymbol{A}^* = \boldsymbol{A}$. If all elements are purely imaginary, then $\boldsymbol{A}^* = -\boldsymbol{A}$. If the transpose of the conjugate of $\boldsymbol{A}$ is equal to $\boldsymbol{A}$, then $\boldsymbol{A}$ is said to be a *Hermitian matrix*. The order of the two operations, conjugate and transpose, is irrelevant. We write

$$\boldsymbol{A}^H = (\boldsymbol{A}^*)^T = (\boldsymbol{A}^T)^* \tag{4.25}$$

or

$$\boldsymbol{A}^H = [a_{ji}^*] \tag{4.26}$$

or

$$\boldsymbol{A}^H = \begin{bmatrix} a_{11}^* & a_{21}^* & \cdots & a_{m1}^* \\ a_{12}^* & a_{22}^* & \cdots & a_{m2}^* \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n}^* & a_{2n}^* & \cdots & a_{mn}^* \end{bmatrix} \tag{4.27}$$

The superscript *H* denotes Hermitian. If $\boldsymbol{A}$ is real, then $\boldsymbol{A}^H = \boldsymbol{A}^T$, and $\boldsymbol{A}$ is said to be *symmetric*. It can also be shown that

$$(\boldsymbol{A}+\boldsymbol{B})^H = \boldsymbol{A}^H + \boldsymbol{B}^H \tag{4.28}$$

and

$$(\boldsymbol{AB})^H = \boldsymbol{B}^H \boldsymbol{A}^H \tag{4.29}$$

We now show how to compute the determinant of an $n \times n$ square matrix. In order to write the general expression, we need to define the minors and cofactors. If $n=1$, $\boldsymbol{A}=[a_{11}]$, and the determinant of $\boldsymbol{A}$, denoted $|\boldsymbol{A}|$ or $\det(\boldsymbol{A})$, is $\det(\boldsymbol{A}) = a_{11}$. If $n=2$, $\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and $\det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$. If $n=3$,

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and

$$\begin{aligned} \det(\boldsymbol{A}) &= a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

If now $\boldsymbol{A}$ is an $n \times n$ matrix, the *minor* $M_{ij}$ is the determinant of the $(n-1)\times(n-1)$ matrix, formed from $\boldsymbol{A}$ by crossing out the $i$th row and the $j$th column. For example, the minors $M_{12}$, $M_{22}$, and $M_{32}$ for the 3 × 3 matrix above are, respectively,

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\ M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \text{ and } M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Each element $a_{ij}$ of the $n \times n$ matrix $\boldsymbol{A}$ has a *cofactor* $C_{ij}$, which differs from the minor $M_{ij}$ by at most a sign change, such that

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{4.30}$$

The general expression for the determinant of the $n \times n$ matrix $\boldsymbol{A}$ is given by

$$\det(\mathrm{A}) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \tag{4.31}$$

Note that any choice of $i$ for $i = 1, 2, \ldots, n$, yields the same value for the determinant of $\boldsymbol{A}$. This form of computing the determinant of $\boldsymbol{A}$ by the evaluation of a string of $(n-1)\times(n-1)$ determinants is called *Laplace expansion*.

The inverse of an $n \times n$ square matrix $\boldsymbol{A}$ is $\boldsymbol{A}^{-1}$, such that

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} \tag{4.32}$$

The inverse of $\boldsymbol{A}$ exists if the matrix $\boldsymbol{A}$ is *nonsingular*; that is, the determinant of $\boldsymbol{A}$ must be nonzero. The matrix $\boldsymbol{A}$ is singular if and only if $\det(\boldsymbol{A}) = 0$. The inverse of $\boldsymbol{A}$ can be given by

$$\boldsymbol{A}^{-1} = \frac{\boldsymbol{C}^{T}}{\det(\boldsymbol{A})} \tag{4.33}$$

where $\boldsymbol{C}$ is the $n \times n$ square matrix of cofactors of $\boldsymbol{A}$. $\boldsymbol{C}^{T}$ is called the *adjoint matrix* of $A$, and is denoted $\mathrm{Adj}(\boldsymbol{A})$. If $\boldsymbol{A}$, $\boldsymbol{B}$, and the product $\boldsymbol{AB}$ are all nonsingular, it can be shown that

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1} \tag{4.34}$$

and

$$\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) \tag{4.35}$$

We can now define the *rank* of $\boldsymbol{A}$, denoted $r_A$ or $\mathrm{rank}(\boldsymbol{A})$, as being the size of the largest nonzero determinant that can be formed from the matrix $\boldsymbol{A}$. Hence, if the $n \times n$ square matrix is nonsingular, its rank is $n$. The rank of the product of two (or more) matrices is smaller than or equal to the smallest rank of the individual matrices forming the product; that is, if $r_A$ and $r_B$ are the respective ranks of $\boldsymbol{A}$ and $\boldsymbol{B}$, then the rank for $\boldsymbol{C}$, $r_C$, of $\boldsymbol{C} = \boldsymbol{AB}$ is

$$0 \le r_C \le \min(r_A, r_B) \tag{4.36}$$

If $\boldsymbol{A}$ is an $n \times n$ square matrix, the *trace* of $\boldsymbol{A}$, denoted $\mathrm{tr}(\boldsymbol{A})$, is the sum of all the diagonal elements of $\boldsymbol{A}$ given by

$$\text{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii} \tag{4.37}$$

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are conformable square matrices, then

$$\text{tr}(\boldsymbol{A} + \boldsymbol{B}) = \text{tr}(\boldsymbol{A}) + \text{tr}(\boldsymbol{B}) \tag{4.38}$$

and

$$\text{tr}(\boldsymbol{AB}) = \text{tr}(\boldsymbol{BA}) \tag{4.39}$$

Some other useful formulas related to the determinant of an $n \times n$ matrix and its inverse are:

$$(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T \tag{4.40}$$

$$(\boldsymbol{A}^H)^{-1} = (\boldsymbol{A}^{-1})^H \tag{4.41}$$

$$\det(\boldsymbol{A}^T) = \det(\boldsymbol{A}) \tag{4.42}$$

$$\det(\boldsymbol{A}^H) = \det^*(\boldsymbol{A}) \tag{4.43}$$

$$\det(\alpha\, \boldsymbol{A}) = \alpha^n \det(\boldsymbol{A}) \tag{4.44}$$

where $\alpha$ is a constant, and

$$\det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})} \tag{4.45}$$

Another useful formula that is frequently encountered in spectral analysis is the *augmented matrix inversion lemma*, which says

$$(\boldsymbol{A} + \boldsymbol{BCD})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{DA}^{-1}\boldsymbol{B} + \boldsymbol{C}^{-1})^{-1}\boldsymbol{DA}^{-1} \tag{4.46}$$

where the matrix $\boldsymbol{A}$ is $n \times n$, $\boldsymbol{B}$ is $n \times m$, $\boldsymbol{C}$ is $m \times m$, and $\boldsymbol{D}$ is $m \times n$. The inverse of the augmented matrix $(\boldsymbol{A} + \boldsymbol{BCD})$ and the inverse of $\boldsymbol{DA}^{-1}\boldsymbol{B} + \boldsymbol{C}^{-1}$ are assumed to exist. A special case of this lemma, known as the *Woodbury's identity*, is when $\boldsymbol{B}$ is an $n \times 1$ column vector denoted $\boldsymbol{u}$, $\boldsymbol{C}$ is the unity scalar (a $1 \times 1$ matrix), and $\boldsymbol{D}$ is

a conjugate $1 \times n$ row vector denoted $\boldsymbol{u}^H$. Then the inverse of the matrix $\boldsymbol{A}$ augmented with $\boldsymbol{u}\,\boldsymbol{u}^H$ (a rank one matrix) is

$$(\boldsymbol{A}+\boldsymbol{u}\,\boldsymbol{u}^H)^{-1} = \boldsymbol{A}^{-1} - \frac{(\boldsymbol{A}^{-1}\boldsymbol{u})(\boldsymbol{u}^H\boldsymbol{A}^{-1})}{1+\boldsymbol{u}^H\boldsymbol{A}\boldsymbol{u}} \tag{4.47}$$

The *quadratic form* $Q$ associated with a matrix $\boldsymbol{A}$ is a real scalar quantity defined as

$$Q = \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j \tag{4.48}$$

where $\boldsymbol{x} = [x_1 \;\; x_2 \;\ldots\; x_n]^{\boldsymbol{T}}$ and $\boldsymbol{A}$ is an $n \times n$ square matrix with $a_{ij} = a_{ji}$. If $\boldsymbol{A}$ is Hermitian, then

$$Q = \boldsymbol{x}^H\boldsymbol{A}\boldsymbol{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i^* x_j \tag{4.49}$$

with $a_{ji} = a_{ij}^*$.

For $\boldsymbol{A}$ Hermitian, it is *positive semidefinite* if and only if

$$\boldsymbol{x}^H\boldsymbol{A}\boldsymbol{x} \geq 0, \quad \boldsymbol{x} \neq 0 \tag{4.50}$$

It is *positive definite* if

$$\boldsymbol{x}^H\boldsymbol{A}\,\boldsymbol{x} > 0, \quad \boldsymbol{x} \neq 0 \tag{4.51}$$

$\boldsymbol{A}$ is *negative semidefinite* if and only if

$$\boldsymbol{x}^H\boldsymbol{A}\,\boldsymbol{x} \leq 0 \tag{4.52}$$

It is *negative definite* if

$$\boldsymbol{x}^H\boldsymbol{A}\,\boldsymbol{x} < 0 \tag{4.53}$$

However, if $\boldsymbol{x}^H\boldsymbol{A}\,\boldsymbol{x} > 0$ for some $\boldsymbol{x}$, and $\boldsymbol{x}^H\boldsymbol{A}\,\boldsymbol{x} < 0$ for other $\boldsymbol{x}$, then $\boldsymbol{A}$ is said to be *indefinite.*

### 4.2.2 Matrices with Special Forms

We frequently encounter in many applications special matrices. An $n \times n$ square matrix is said to be *diagonal* if all elements $i \neq j$ are zero, except the elements $a_{ij}$, $i = j$, on the main diagonal. We write

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{12} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \text{diag}[a_{11}, a_{22}, \ldots, a_{nn}] \tag{4.54}$$

We observe that the unit matrix is a special case of the diagonal matrix with $a_{ii} = 1$, $i = 1, 2, \ldots, n$. $\boldsymbol{A}^{-1}$ is also a diagonal matrix, given by

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \dfrac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \dfrac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \dfrac{1}{a_{nn}} \end{bmatrix} = \text{diag}\left[\frac{1}{a_{11}}, \frac{1}{a_{22}}, \ldots, \frac{1}{a_{nn}}\right] \tag{4.55}$$

A *block diagonal* matrix is a square matrix that can be partitioned in nonzero square submatrices along the main diagonal, while the other submatrices are zero.

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{A}_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \boldsymbol{A}_k \end{bmatrix} = \text{diag}[\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots., \boldsymbol{A}_k] \tag{4.56}$$

If all $\boldsymbol{A}_i$, $i = 1, 2, \ldots, k$, are nonsingular, then

$$\det(\boldsymbol{A}) = \prod_{i=1}^{k} \det(\boldsymbol{A}_i) \tag{4.57}$$

and

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{A}_1^{-1} & 0 & \cdots & 0 \\ 0 & \boldsymbol{A}_2^{-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \boldsymbol{A}_k^{-1} \end{bmatrix} = \text{diag}[\mathbf{A}_1^{-1}, \mathbf{A}_2^{-1}, \ldots, \mathbf{A}_k^{-1}] \tag{4.58}$$

A square matrix with all of its elements above the main diagonal equal to zero is called a *lower triangular* matrix, and is given by

$$\boldsymbol{L} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{4.59}$$

The determinant of any triangular matrix is the product of its diagonal elements, given by

$$\det(\boldsymbol{L}) = \prod_{i=1}^{n} a_{ii} \tag{4.60}$$

The inverse of the lower triangular matrix is also a lower triangular matrix. If all the elements below the main diagonal are equal to zero, then we have an *upper triangular* matrix, given by

$$\boldsymbol{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \tag{4.61}$$

with a determinant as given by (4.60). The inverse is also an upper triangular matrix.

An $n \times n$ square matrix $\boldsymbol{A}$ is said to be *orthogonal* if

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^{\boldsymbol{T}} \tag{4.62}$$

That is, the columns (and rows) must be orthonormal. If $\boldsymbol{a}_i$ is the $i$th column (or row), then orthogonality means

$$\boldsymbol{a}_i^T \boldsymbol{a}_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{4.63}$$

If

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^H \tag{4.64}$$

then the $n \times n$ complex matrix $\boldsymbol{A}$ is said to be *unitary*; that is,

$$\boldsymbol{a}_i^H \boldsymbol{a}_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{4.65}$$

If

$$\boldsymbol{A}^{-1} = \boldsymbol{A} \tag{4.66}$$

then $\boldsymbol{A}$ is said to be an *involutory matrix*. An *idempotent* matrix is a particular case of a *periodic* matrix; that is, a square matrix such that the matrix power

$$\boldsymbol{A}^k = \boldsymbol{A}^{k+1}, \quad k = 1, 2, 3, \ldots \tag{4.67}$$

The matrix is said to have period $k$ if $k$ is the least such integer. If $k = 1$, then $\boldsymbol{A}^2 = \boldsymbol{A}$, and the matrix is called *idempotent.*

A *persymmetric matrix* is a matrix that is symmetric about its cross diagonal. To be able to see this definition clearly, let $\boldsymbol{R}$ be a $5 \times 5$ matrix, and then

$$\boldsymbol{R} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{25} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{23} & a_{13} \\ a_{41} & a_{42} & a_{32} & a_{22} & a_{12} \\ a_{51} & a_{41} & a_{31} & a_{21} & a_{11} \end{bmatrix} \tag{4.68}$$

An $n \times n$ square matrix $\boldsymbol{A}$ is *circulant* if all of its rows are obtained from the $n$ values $\{a_1, a_2, \ldots, a_n\}$ by introducing a shift to the right on the previous row to obtain

$$\boldsymbol{A} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix} \tag{4.69}$$

A matrix having identical elements along any diagonal, such that $a_{ij} = a_{j-1}$ for all $i$ and $j$, is said to be *Toeplitz*. If $\boldsymbol{A}$ *is* $n \times n$, then

$$\boldsymbol{A} = \begin{bmatrix} a_1 & a_{-2} & a_{-3} & \cdots & a_{-n} \\ a_2 & a_1 & a_{-2} & \cdots & \vdots \\ a_3 & a_2 & a_1 & \ddots & a_{-3} \\ \vdots & \ddots & \ddots & \ddots & a_{-2} \\ a_n & \cdots & a_3 & a_2 & a_1 \end{bmatrix} \tag{4.70}$$

For example, if $n = 4$, we have

$$\boldsymbol{A} = \begin{bmatrix} a_1 & a_{-2} & a_{-3} & a_4 \\ a_2 & a_1 & a_{-2} & a_3 \\ a_3 & a_2 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{bmatrix}$$

If in addition, $a_{-k} = a_k^*$, then $\boldsymbol{A}$ is said to be *Hermitian Toeplitz*. If the matrix is real, then $a_{-k} = a_k$, and $\boldsymbol{A}$ is said to be *symmetric Toeplitz*.

Another special matrix that we may encounter is the $m \times n$ *Vandermonde matrix*, which has the form

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{m-1} & a_2^{m-1} & \cdots & a_n^{m-1} \end{bmatrix} \tag{4.71}$$

### 4.2.3 Eigenvalues and Eigenvectors

In this section, we define eigenvalues and eigenvectors. We present methods of determining eigenvalues and eigenvectors, and some related properties. Eigenvalues and eigenvectors are extremely useful in many applications of signal processing and modern control theory. In the context of this book, eigenvalues and eigenvectors will be used in representing stochastic processes, and solving the general Gaussian problem, which will be covered in a later chapter.

We define a *linear transformation* or *linear operation* or *linear mapping* $\mathcal{T}$ from a vector space $\chi$, called the domain, to a vector space $\mathcal{Y}$, called the range (or codomain), as a correspondence that assigns to every vector $\boldsymbol{x}$ in $\chi$ a vector $\mathcal{T}(\boldsymbol{x})$ in $\mathcal{Y}$, such that

$$\mathcal{T}: \chi \rightarrow \mathcal{Y} \tag{4.72}$$

The transformation $\mathcal{T}$ is said to be *linear* if

$$\mathcal{T}(\alpha \boldsymbol{x}_1 + \beta \boldsymbol{x}_2) = \alpha \mathcal{T}(\boldsymbol{x}_1) + \beta \mathcal{T}(\boldsymbol{x}_2) \tag{4.73}$$

where $\alpha$ and $\beta$ are constants, and $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are vectors in $\chi$.

It can be shown that any equation involving a linear operator on a finite dimensional space can be converted into an equivalent matrix operator. If the transform $\mathcal{T}: v \rightarrow v$ maps elements in $v$ into other elements in $v$, we can define $\mathcal{T}$ by a matrix $\boldsymbol{A}$.

Using the above concept of the linear transformation, we are now ready to define the concept of eigenvalues and eigenvectors. An *eigenvalue* (or *characteristic value*) of a linear operator $\mathcal{T}$ on a vector space $\chi$ is a scalar $\lambda$, such that

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \tag{4.74}$$

for a nonzero vector $\boldsymbol{x}$ in $v$. Every nonzero vector $\boldsymbol{x}$ satisfying the relation $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ is called an *eigenvector* of $\boldsymbol{A}$ associated with the eigenvalue $\lambda$. The matrix representation of (4.74) is

$$(\boldsymbol{A} - \boldsymbol{I}\lambda)\boldsymbol{x} = \boldsymbol{0} \tag{4.75}$$

where $\boldsymbol{I}$ is the identity matrix. If the operator $\mathcal{T}$ acts on a function space, then the eigenvectors associated with the eigenvalues are called *eigenfunctions*.

*Eigenvalues*

If $\boldsymbol{A}$ is an $n \times n$ matrix, a necessary condition for the $n$ homogeneous equations in (4.75) to yield nonzero solutions is that the rank of the matrix $(\boldsymbol{A} - \boldsymbol{I}\lambda)$ must be less than $n$. That is, the determinant

$$\left|\boldsymbol{A} - \boldsymbol{I}\,\lambda\right| = 0 \tag{4.76}$$

Equation (4.76) is called the *characteristic equation of the matrix* $\boldsymbol{A}$ (or of operator $\mathcal{T}$ represented by $\boldsymbol{A}$). Expanding the determinant $\left|\boldsymbol{A} - \boldsymbol{I}\lambda\right|$, we obtain an $n$th degree polynomial in $\lambda$, called, *the characteristic polynomial of* $\boldsymbol{A}$, and is given by

$$\begin{aligned} c(\lambda) &= \left|\lambda\boldsymbol{I} - \boldsymbol{A}\right| = (-1)^n \left|\boldsymbol{A} - \boldsymbol{I}\,\lambda\right| \\ &= \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \ \ldots\ + c_1\lambda + c_0 \end{aligned} \tag{4.77}$$

Solving for $\lambda$ from the characteristic equation results in $n$ roots $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ if all roots are distinct. Consequently, $c(\lambda)$ can be written as

$$c(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) \tag{4.78}$$

However, if the roots are not distinct, then $\lambda_1$ has multiplicity $m_1$, $\lambda_2$ has multiplicity $m_2$, and so on. Then,

$$c(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}\ldots(\lambda - \lambda_p)^{m_p} \tag{4.79}$$

where $m_1 + m_2 + \ldots + m_p = n$.

It should be noted that when all roots are distinct, the following relationships hold:

$$\left|\boldsymbol{A}\right| = \lambda_1\lambda_2\ldots\lambda_n = c_0 \tag{4.80}$$

and

$$\mathrm{tr}\,(\boldsymbol{A}) = \lambda_1 + \lambda_2 + \ \ldots\ + \lambda_n = (-1)^{n+1} c_{n-1} \tag{4.81}$$

*Eigenvectors*

Once the eigenvalues are determined from the characteristic equation, we substitute for $\lambda$ in (4.74) or (4.75), and solve for the corresponding vector $\boldsymbol{x}$.

However, in determining $\lambda$, two possible cases arise: (1) all eigenvalues are distinct, and (2) some eigenvalues have multiplicity greater than one.

1. *Case 1*: All eigenvalues are distinct.

The eigenvectors are solved for directly from (4.74) or (4.75). If $\boldsymbol{x}_k$ is an eigenvector corresponding to the eigenvalues $\lambda_k$, then $\alpha \boldsymbol{x}_k$ is also an eigenvector for any nonzero scalar $\alpha$. Since all eigenvalues and their corresponding eigenvectors satisfy the equation

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x}_1 &= \lambda_1 \boldsymbol{x}_1 \\ \boldsymbol{A}\boldsymbol{x}_2 &= \lambda_2 \boldsymbol{x}_2 \\ &\vdots \\ \boldsymbol{A}\boldsymbol{x}_n &= \lambda_n \boldsymbol{x}_n \end{aligned} \tag{4.82}$$

we can write that

$$\boldsymbol{A}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{\Lambda} \tag{4.83}$$

where the $n \times n$ matrix $\boldsymbol{M}$ is called the *modal matrix*, and defined by

$$\boldsymbol{M} = [\boldsymbol{x}_1 \,\vdots\, \boldsymbol{x}_2 \,\vdots\, \ldots \,\vdots\, \boldsymbol{x}_n] \tag{4.84}$$

the rank of the matrix $\boldsymbol{M}$ is $n$, since the eigenvectors are linearly independent. $\boldsymbol{\Lambda}$ is a diagonal matrix defined by

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n] \tag{4.85}$$

Solving for $\boldsymbol{\Lambda}$ from (4.83), we have

$$\boldsymbol{\Lambda} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M} \tag{4.86}$$

where $\boldsymbol{M}^{-1}$ is the inverse matrix $\boldsymbol{M}$. Equation (4.86) is known as the *similarity transformation*. If the eigenvectors are orthogonal, then $\boldsymbol{M}^{-1} = \boldsymbol{M}^T$, where $T$ denotes transpose, and the matrix $\boldsymbol{A}$ is diagonalized by the *orthogonal transformation*

$$\boldsymbol{\Lambda} = \boldsymbol{M}^T \boldsymbol{A}\boldsymbol{M} \tag{4.87}$$

**Example 4.1**

(a) Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{bmatrix} -3 & 0 & 0 \\ -5 & 2 & 0 \\ -5 & 1 & 1 \end{bmatrix}$$

(b) Find the characteristic polynomial of $\boldsymbol{A}$.
(c) Diagonalize $\boldsymbol{A}$ by the similarity transformation.

*Solution*

(a) The characteristic equation is $|\boldsymbol{A} - \boldsymbol{I}\lambda| = 0 \Rightarrow$

$$\begin{vmatrix} -3-\lambda & 0 & 0 \\ -5 & 2-\lambda & 0 \\ -5 & 1 & 1-\lambda \end{vmatrix} = (\lambda+3)(\lambda-2)(\lambda-1) = 0$$

Thus, the eigenvalues are all distinct, with $\lambda_1 = -3, \lambda_2 = 2$ and $\lambda_3 = 1$. The eigenvectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ are obtained by solving the equations $\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$, $\boldsymbol{A}\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2$, and $\boldsymbol{A}\boldsymbol{x}_3 = \lambda_3 \boldsymbol{x}_3$. For $\lambda = 3$, we have

$$\begin{bmatrix} -3 & 0 & 0 \\ -5 & 2 & 0 \\ -5 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -3 \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where $\boldsymbol{x}^T = [a \quad b \quad c]$. This results in three equations in three unknowns; that is,

$$\begin{aligned} -3a &= -3a \\ -5a + 2b &= -3b \\ -5a + b + c &= -3c \end{aligned}$$

Solving the equations, we obtain $a = b = c$. Thus,

$$\boldsymbol{x}_1^T = \alpha[1 \quad 1 \quad 1]$$

is the eigenvector and $\alpha$ is any constant.

Similarly, we solve for $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ to obtain

$$\boldsymbol{x}_2^T = [0 \quad 0 \quad 1] \qquad \text{and} \qquad \boldsymbol{x}_3^T = [0 \quad 1 \quad 1]$$

(b) The characteristic polynomial of $\boldsymbol{A}$ is $c(\lambda) = |\lambda \boldsymbol{I} - \boldsymbol{A}| \Rightarrow$

$$c(\lambda) = \begin{vmatrix} \lambda + 3 & 0 & 0 \\ -5 & \lambda - 2 & 0 \\ -5 & 1 & \lambda - 1 \end{vmatrix} = \lambda^2 - 7\lambda + 6$$

(c) Using the similarity transformation, $\boldsymbol{\Lambda} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$ , we have

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \boldsymbol{M}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

and

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ -5 & 2 & 0 \\ -5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

2. *Case 2*: All eigenvalues are not distinct.

The corresponding eigenvectors may or may not be linearly independent. If $m_i$ is the order of an eigenvalue, called *algebraic multiplicity*, then the corresponding number of independent vectors, $q_i, q_i \le m_i$, is called *geometric multiplicity* or *degeneracy*. The value of $q_i$ is given by

$$q_i = n - \text{rank}(\boldsymbol{A} - \boldsymbol{I}\lambda_i), \quad 1 \le q_i \le m_i \tag{4.88}$$

If $q_i = m_i$, then all eigenvectors associated with $\lambda_i$ are independent and can be solved for as in Case 1.

If $q_i = 1, \ (m_i > 1)$, then there is one eigenvector associated with $\lambda_i$. The other $(m_i - 1)$ vectors are called *generalized eigenvectors*. A generalized eigenvector of rank *k* is a nonzero vector for which

$$\left(\boldsymbol{A}-\lambda_i \boldsymbol{I}\right)^k \boldsymbol{x}_k = 0 \tag{4.89}$$

and

$$\left(\boldsymbol{A}-\lambda_i \boldsymbol{I}\right)^{k-1} \boldsymbol{x}_{k-1} \neq 0 \tag{4.90}$$

The eigenvector $\boldsymbol{x}_1$ is found as before; that is,

$$\left(\boldsymbol{A}-\lambda_i \boldsymbol{I}\right)\boldsymbol{x}_1 = 0 \tag{4.91}$$

whereas the remaining $(m_i - 1)$ generalized eigenvectors are found by

$$\begin{aligned}
\left(\boldsymbol{A}-\boldsymbol{I}\lambda_i\right)\boldsymbol{x}_2 &= \boldsymbol{x}_1 \\
\left(\boldsymbol{A}-\boldsymbol{I}\lambda_i\right)\boldsymbol{x}_3 &= \boldsymbol{x}_2 \\
&\vdots \\
\left(\boldsymbol{A}-\boldsymbol{I}\lambda_i\right)\boldsymbol{x}_j &= \boldsymbol{x}_{j-1} \\
&\vdots \\
\left(\boldsymbol{A}-\boldsymbol{I}\lambda_i\right)\boldsymbol{x}_{m_i} &= \boldsymbol{x}_{m_{i-1}}
\end{aligned} \tag{4.92}$$

If the modal matrix $\boldsymbol{M}$ is formed as before, then the $m_{i-1}$ eigenvectors are included, and the similarity transformation becomes

$$\boldsymbol{AM} = \boldsymbol{MJ} \tag{4.93}$$

or

$$\boldsymbol{J} = \boldsymbol{M}^{-1}\boldsymbol{AM} \tag{4.94}$$

$\boldsymbol{J}$ is an $n \times n$ diagonal matrix, called the *Jordan form*, such that

$$\boldsymbol{J} = \text{diag}[\boldsymbol{J}_1, \boldsymbol{J}_2, \ldots, \boldsymbol{J}_p] \tag{4.95}$$

and

$$\boldsymbol{J}_i = \begin{bmatrix}
\lambda_i & 1 & 0 & \cdots & 0 & 0 \\
0 & \lambda_i & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_i & 1 \\
0 & 0 & 0 & \cdots & 0 & \lambda_i
\end{bmatrix}, \quad i = 1, 2, \ldots, p \tag{4.96}$$

Equation (4.96) says that each submatrix $\boldsymbol{J}_i$, $i = 1, 2, \ldots, p$, has the same eigenvalue along its main diagonal; ones for all elements in the diagonal above the main diagonal, and zeros for the rest of the elements.

If $1 \le q_i \le m_i$, there may be more than one Jordan block for each eigenvector. Assume that we have a $6 \times 6$ square matrix, such that we have two eigenvalues $\lambda_1$ $(\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5)$ of order 5, and $\lambda_6$ of order 1, and $q_1 = 2$. Then, we have two eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and three generalized eigenvectors for $\lambda_1$, and one eigenvector $\boldsymbol{x}_6$ for $\lambda_6$. The generalized eigenvectors may be associated with $\boldsymbol{x}_1$ or with $\boldsymbol{x}_2$, or with both $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. That is, we may have two Jordan blocks of the form

$$\boldsymbol{J}_1 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \qquad \boldsymbol{J}_2 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \tag{4.97}$$

or

$$\boldsymbol{J}_1 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}, \qquad \boldsymbol{J}_2 = [\lambda_1] \tag{4.98}$$

or vice versa. The approach to determine the Jordan blocks will be shown by an example. Assume that we have the case of (4.97). Then, the corresponding generalized eigenvalues and eigenvectors are determined by

$$\begin{aligned} (\boldsymbol{A} - \boldsymbol{I}\lambda_1)\boldsymbol{x}_{13} &= \boldsymbol{x}_{12} \\ (\boldsymbol{A} - \boldsymbol{I}\lambda_1)\boldsymbol{x}_{12} &= \boldsymbol{x}_1 \\ (\boldsymbol{A} - \boldsymbol{I}\lambda_1)\boldsymbol{x}_1 &= \boldsymbol{0} \\ (\boldsymbol{A} - \boldsymbol{I}\lambda_1)\boldsymbol{x}_2 &= \boldsymbol{x}_{21} \\ (\boldsymbol{A} - \boldsymbol{I}\lambda_1)\boldsymbol{x}_2 &= \boldsymbol{0} \\ (\boldsymbol{A} - \boldsymbol{I}\lambda_6)\boldsymbol{x}_6 &= \boldsymbol{0} \end{aligned} \tag{4.99}$$

The modal matrix $\boldsymbol{M}$ is

$$\boldsymbol{M} = [\boldsymbol{x}_1 \,\vdots\, \boldsymbol{x}_{12} \,\vdots\, \boldsymbol{x}_{13} \,\vdots\, \boldsymbol{x}_2 \,\vdots\, \boldsymbol{x}_{21} \,\vdots\, \boldsymbol{x}_6] \tag{4.100}$$

The similarity transformation is as given by (4.94), where $\boldsymbol{J}$ is

$$\boldsymbol{J} = \left[\begin{array}{ccc|cc|c} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \lambda_1 \end{array}\right] \tag{4.101}$$

If we have the case of (4.98), then the corresponding generalized eigenvalues and eigenvectors are determined by

$$\begin{aligned} (\boldsymbol{A} - \boldsymbol{I}\lambda_1)\boldsymbol{x}_{14} &= \boldsymbol{x}_{13} \\ (\boldsymbol{A} - \boldsymbol{I}\lambda_1)\boldsymbol{x}_{13} &= \boldsymbol{x}_{12} \\ (\boldsymbol{A} - \boldsymbol{I}\lambda_1)\boldsymbol{x}_{12} &= \boldsymbol{x}_1 \\ (\boldsymbol{A} - \boldsymbol{I}\lambda_1)\boldsymbol{x}_1 &= \boldsymbol{0} \\ (\boldsymbol{A} - \boldsymbol{I}\lambda_1)\boldsymbol{x}_2 &= \boldsymbol{0} \\ (\boldsymbol{A} - \boldsymbol{I}\lambda_6)\boldsymbol{x}_6 &= \boldsymbol{0} \end{aligned} \tag{4.102}$$

The modal and Jordan matrices are then given by

$$\boldsymbol{M} = [\boldsymbol{x}_1 \,\vdots\, \boldsymbol{x}_{12} \,\vdots\, \boldsymbol{x}_{13} \,\vdots\, \boldsymbol{x}_{14} \,\vdots\, \boldsymbol{x}_2 \,\vdots\, \boldsymbol{x}_6] \tag{4.103}$$

and

$$\boldsymbol{J} = \left[\begin{array}{cccc|cc} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{array}\right] \tag{4.104}$$

From (4.50) to (4.53), we defined a method for determining the definiteness of a Hermitian matrix. We now give an alternative method in terms of eigenvalues. If all distinct eigenvalues $\lambda_i > 0$, then the matrix is said to be *positive definite*. It is *positive semidefinite* if all eigenvalues $\lambda_i \geq 0$. If all eigenvalues $\lambda_i < 0$, then the matrix is said to be *negative definite*, and the matrix is *negative semidefinite* if all distinct $\lambda_i \leq 0$. However, if some $\lambda_i > 0$ and other $\lambda_i < 0$, then the matrix is *indefinite*.

**Example 4.2**

(a) Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 3 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

(b) Find the Jordan form by the transformation $\boldsymbol{J} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$.

*Solution*

(a) The characteristic equation is given by $|\boldsymbol{A} - \boldsymbol{I}\lambda| = 0$; that is,

$$\begin{vmatrix} 3-\lambda & 0 & 0 & 1 \\ 0 & 2-\lambda & 0 & 0 \\ 1 & 1 & 3-\lambda & 1 \\ -1 & 0 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)^3(3-\lambda) = 0$$

Hence, two eigenvalues $\lambda_1 = 2$ with algebraic multiplicity $m_1 = 3$, and $\lambda_2 = 3$ with multiplicity $m_2 = 1$. We need to determine the number of independent eigenvectors and generalized eigenvectors associated with $\lambda_1$. The rank of $|\boldsymbol{A} - \boldsymbol{I}\lambda|_{\lambda=2} = 2 = r$. Thus, $q_1 = n - r = 4 - 2 = 2$; that is, we have two eigenvectors. Since $m_1 = 3$, there is only $m_1 - q = 1$ generalized eigenvector. Solving for $\boldsymbol{x}_1$ by using the four equations of $\boldsymbol{A}\boldsymbol{x}_1 = 2\boldsymbol{x}_1$, where $\boldsymbol{x}_1 = [a \;\; b \;\; c \;\; d]^T$, we obtain $a = -d$ and $b = -c$. Since we have two eigenvectors corresponding to $\lambda = 2$, we let ($a$ = 1, $b$ = 0) to obtain $\boldsymbol{x}_1 = [1 \;\; 0 \;\; 0 \;\; -1]^T$, and ($a$ = 0, $b$ = 1) to obtain $\boldsymbol{x}_2 = [0 \;\; 1 \;\; -1 \;\; 0]^T$. The generalized eigenvector $\boldsymbol{x}_{12}$ is given by $(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{x}_{12} = \boldsymbol{x}_1$ to yield $\boldsymbol{x}_{12} = [0 \;\; 0 \;\; -1 \;\; -1]^T$. Similarly, we solve for $\boldsymbol{x}_4$ by using $\boldsymbol{A}\boldsymbol{x}_4 = 3\boldsymbol{x}_4$ to obtain $\boldsymbol{x}_4 = [0 \;\; 0 \;\; 1 \;\; 0]^T$.

(b) We form the modal matrix $\boldsymbol{M}$ as

$$\boldsymbol{M} = [\boldsymbol{x}_1 \,\vdots\, \boldsymbol{x}_{12} \,\vdots\, \boldsymbol{x}_2 \,\vdots\, \boldsymbol{x}_4] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

Performing the operation $\boldsymbol{J} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$ results in

$$\boldsymbol{J} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

as expected. The inverse of $\boldsymbol{M}$ is

$$\boldsymbol{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

## 4.3 DEFINITIONS

A *discrete-time random process* or *stochastic process* $X(n)$ is a sequence of real or complex random variables defined for every integer $n$. The *mean value function* of the process $X(n)$ is defined as

$$E[X(n)] = m_x(n) \tag{4.105}$$

and the autocorrelation function is defined as

$$r_{xx}(n_1, n_2) = E[X(n_1)X^*(n_2)] \tag{4.106}$$

where $n_1$ and $n_2$ are two indices, and * denotes a complex conjugate. Note that we use the lowercase letter $r$ to denote correlation. The covariance function is defined as

$$\begin{aligned} c_{xx}(n_1, n_2) &= E\left\{[X(n_1) - m_x(n_1)][X(n_1) - m_x(n_1)]^*\right\} \\ &= r_{xx}(n_1, n_2) - m_x(n_1)m_x^*(n_2) \end{aligned} \tag{4.107}$$

If the process $X(n)$ is stationary in the wide sense, then the mean

$$E\left[X(n)\right] = \text{constant} \tag{4.108}$$

is independent of *n*, and the autocorrelation function

$$r_{xx}(n, n+k) = r_{xx}(k) = c_{xx}(k) + \left|m_x\right|^2 \tag{4.109}$$

depends only on the time difference or lag between the two samples $n_1 = n$ and $n_2 = n + k$.

Similarly, we say two processes $X(n)$ and $Y(n)$ are jointly wide-sense stationary if each is individually wide-sense stationary, and their cross-correlation function is

$$r_{xy}(n, n+k) = r_{xy}(k) = c_{xy}(k) + m_x m_y^* \tag{4.110}$$

where $c_{xy}(k)$ is the cross-covariance function given by

$$c_{xy}(n, n+k) = E\left\{[X(n) - m_x][Y(n+k) - m_y]^*\right\} = r_{xy}(k) - m_x m_y^* \tag{4.111}$$

In light of correlation properties given in Chapter 3, we give the following useful properties for the autocorrelation and cross-covariance functions

$$r_{xx}(0) \geq \left|r_{xx}(k)\right| \tag{4.112}$$

with $r_{xx}(0)$ real and positive.

$$r_{xx}(-k) = r_{xx}^*(k) \tag{4.113}$$

$$r_{xx}(0) r_{yy}(0) \geq \left|r_{xy}(k)\right|^2 \tag{4.114}$$

and

$$r_{xy}(-k) = r_{yx}^*(k) \tag{4.115}$$

Let $\boldsymbol{X}(n)$ be a column vector of $M$ functions of time $X(n)$, $X(n-1)$, … , $X(n-M+1)$, representing a wide-sense stationary discrete-time random process, such that

$$\boldsymbol{X}^T(n) = [X(n), X(n-1), \ldots, X(n-M+1)] \tag{4.116}$$

The *correlation matrix* of this process is defined as

$$\boldsymbol{R}_{XX} = \boldsymbol{R} = E\left[\boldsymbol{X}(n)\boldsymbol{X}^H(n)\right] \tag{4.117}$$

where the superscript $H$ denotes Hermitian. Substituting (4.116) in (4.117), we obtain a Hermitian Toeplitz autocorrelation matrix

$$\boldsymbol{R} = \begin{bmatrix} r(0) & r(-1) & \cdots & r[-(M-1)] \\ r(+1) & r(0) & \cdots & r[-(M-2)] \\ \vdots & \vdots & \vdots & \vdots \\ r(M-1) & r(M-2) & \cdots & r(0) \end{bmatrix} \tag{4.118}$$

where we dropped the index $x$ for the simplicity of notation. This matrix is *positive semidefinite*; that is, all the eigenvalues of the matrix are greater than or equal to zero. For any sequence $a(n)$, we have

$$E\left[\left|\sum_{k=0}^{M-1} a^*(k)X(k)\right|^2\right] = \sum_{\ell=0}^{M-1}\sum_{k=0}^{M-1} a^*(\ell)a(k)r_{xx}(\ell-k) \geq 0 \tag{4.119}$$

In the previous section, we gave some mathematical properties related to a matrix $\boldsymbol{A}$, and its eigenvalues and eigenvectors. If the matrix represents a correlation matrix of a discrete-time stochastic process, from (4.118), this correlation matrix $\boldsymbol{R}$ is Hermitian Toeplitz and positive semidefinite. This will give us some other useful properties.

1. Let $\lambda_1, \lambda_2, \ldots, \lambda_M$ be the distinct eigenvalues of the $M \times M$ correlation matrix $\boldsymbol{R}$. Then, all these eigenvalues are real and negative.

2. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_M$ be the eigenvectors corresponding to the $M$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$ of the $M \times M$ correlation matrix $\boldsymbol{R}$. Then, the eigenvectors are linearly independent.

The eigenvectors are *linearly dependent*, which means that there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_M$, not all zero, such that

$$\sum_{i=1}^{M} \alpha_i \boldsymbol{v}_i = 0 \tag{4.120}$$

If no such scalars exist, then the eigenvectors are *linearly independent*.

3. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_M$ be the eigenvectors corresponding to the $M$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$ of the $M \times M$ correlation matrix $\boldsymbol{R}$. Then, the eigenvectors are orthogonal to each other; that is,

$$\boldsymbol{v}_i^H \boldsymbol{v}_j = 0 \quad , \quad i \neq j \tag{4.121}$$

If the eigenvectors are normalized to have *unit length*, then they are *orthonormal*; that is,

$$\boldsymbol{v}_i^H \boldsymbol{v}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{4.122}$$

4. Let $\lambda_1, \lambda_2, \ldots, \lambda_M$ be the distinct eigenvalues of the correlation matrix $\boldsymbol{R}$. Then, the eigenvalues of $\boldsymbol{R}^k$ are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_M^k$.

Note that for the special case where $k = -1$, the eigenvalues of the inverse correlation matrix $\boldsymbol{R}^{-1}$ are $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_M$.

5. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_M$ be the eigenvectors corresponding to the $M$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$ of the $M \times M$ correlation matrix $\boldsymbol{R}$. Let

$$\boldsymbol{V} = [v_1 \,\vdots\, v_2 \,\vdots\, v_3 \,\vdots\, \ldots \,\vdots\, v_M] \tag{4.123}$$

such that the eigenvectors are orthonormal as defined in (4.122). Then, from (4.83),

$$\boldsymbol{R}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Lambda} \tag{4.124}$$

where

$$\boldsymbol{\Lambda} = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_M] \tag{4.125}$$

Since $\boldsymbol{R}$ is Hermitian,

$$\boldsymbol{V}^{-1} = \boldsymbol{V}^{H} \tag{4.126}$$

The correlation matrix may then be diagonalized by the *unitary similarity transformation*

$$\boldsymbol{V}^{H}\boldsymbol{R}\boldsymbol{V} = \boldsymbol{\Lambda} \tag{4.127}$$

Postmultiplying both sides of (4.124) by $\boldsymbol{R}^{-1}$ and using (4.121), the correlation matrix $\boldsymbol{R}$ may be written as

$$\boldsymbol{R} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{H} = \sum_{i=1}^{M} \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^{H} \tag{4.128}$$

or

$$\boldsymbol{R}^{-1} = \boldsymbol{V}\boldsymbol{\Lambda}^{-1}\boldsymbol{V}^{H} = \sum_{i=1}^{M} \frac{1}{\lambda_i} \boldsymbol{v}_i \boldsymbol{v}_i^{H} \tag{4.129}$$

The decomposition of the correlation matrix $\boldsymbol{R}$ in the form of (4.128) is known as the *spectral theorem.*

6. Let $\lambda_1, \lambda_2, \cdots, \lambda_M$ be the distinct eigenvalues of the $M \times M$ correlation matrix $\boldsymbol{R}$. Then, from (4.83) and (4.128),

$$\text{tr}\,(\boldsymbol{R}) = \sum_{\boldsymbol{i}=1}^{M} \lambda_i = \text{tr}\,(\boldsymbol{V}^{H}\boldsymbol{R}\boldsymbol{V}) \tag{4.130}$$

The Fourier transform of a sequence $r_{xx}(k)$ is

$$S_{xx}(\omega) = \sum_{k=-\infty}^{\infty} r_{xx}(k) e^{-j\omega k}, \quad |\omega| < \pi \tag{4.131}$$

where $\omega = 2\pi f$ is the angular frequency. Since the sequence under consideration is the autocorrelation function, its Fourier transform is then the *power spectral density* or *power spectrum*. Note that for $\ell$ integer,

$$S_{xx}(\omega+2\ell\pi)=\sum_{k=-\infty}^{\infty}r_{xx}(k)e^{-j(\omega+2\ell\pi)k}=\sum_{k=-\infty}^{\infty}r_{xx}(k)e^{-j\omega k}e^{-j\omega 2k\ell\pi}=S_{xx}(\omega)\quad(4.132)$$

since $e^{-j\omega 2k\ell\pi}=1$. Hence, the power spectrum is a *periodic* function with period $2\pi$.

It is known that the Fourier series of a periodic signal, $v(t)$, in exponential form is given by

$$v(t)=\sum_{k=-\infty}^{\infty}v_k e^{jk\omega_0 t}\qquad(4.133)$$

By analogy with (4.133) and (4.131), we note that $-\omega$ is analogous to $t$, $\omega_0$ is one, and $r_{xx}(k)$ is analogous to $v_k$. Therefore, $r_{xx}(k)$ can be interpreted as the Fourier coefficient of $S_{xx}(\omega)$ to yield

$$r_{xx}(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}S_{xx}(\omega)e^{-j\omega k}\,d\omega\qquad(4.134)$$

Equations (4.131) and (4.134) form the *Wiener-Khinchin relations* for discrete-time processes. The mean-square value, which represents the *average power* in the discrete-time random process, is

$$r_{xx}(0)=E\left[|X(n)|^2\right]=\frac{1}{2\pi}\int_{-\pi}^{\pi}S_{xx}(\omega)d\omega=\int_{-f/2}^{f/2}S_{xx}(f)df\qquad(4.135)$$

The power spectrum $S_{xx}(\omega)$ is real, since

$$\begin{aligned}r_{xx}(-k)&=r_{xx}(n,\ n-k)=E[X(n)X^*(n-k)]\\&=E[X^*(n-k)X(n)]=r_{xx}^*(k)\end{aligned}\qquad(4.136)$$

Similarly, the cross-power spectrum is defined as

$$S_{xy}(\omega)=\sum_{k=-\infty}^{\infty}r_{xy}(k)e^{-j\omega k}\qquad(4.137)$$

For more results on the cross-correlation functions and cross-spectrum, consider a discrete-time linear system with impulse response $h(n)$, a wide-sense

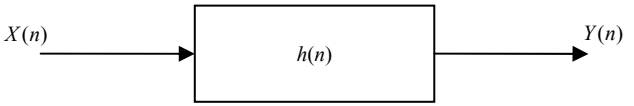


**Figure 4.1** Discrete linear system.

stationary input $X(n)$, and an output $Y(n)$, as shown in Figure 4.1. Then, the cross-correlation functions $r_{xy}(k)$ and $r_{yx}(k)$ are given by

$$r_{xy}(k) = h(k) * r_{xx}(k) = \sum_{\ell=-\infty}^{\infty} h(\ell) r_{xx}(k-\ell) = r_{xx}(k) * h(k) = \sum_{\ell=-\infty}^{\infty} r_{xx}(\ell) h(k-\ell) \tag{4.138}$$

and

$$r_{yx}(k) = h^*(-k) * r_{xx}(k) = \sum_{\ell=-\infty}^{\infty} h^*(\ell) r_{xx}(k-\ell) \tag{4.139}$$

The autocorrelation function of the output process is given by

$$r_{yy}(k) = h(k) * r_{yx}(k) = h(k) * h^*(-k) * r_{xx}(k) \tag{4.140}$$

The corresponding cross-spectrum densities are

$$S_{xy}(\omega) = \sum_{k=-\infty}^{\infty} r_{xy}(k) e^{-j\omega k} \tag{4.141}$$

$$S_{yx}(\omega) = \sum_{k=-\infty}^{\infty} r_{yx}(k) e^{-j\omega k} \tag{4.142}$$

Taking the $Z$-transform of (4.138), (4.139), and (4.140), we obtain

$$S_{xy}(Z) = H(Z) S_{xx}(Z) \tag{4.143}$$

$$S_{yx}(Z) = H^*\left(\frac{1}{Z^*}\right) S_{xx}(Z) \tag{4.144}$$

and

$$S_{yy}(Z) = H(Z)H^*\left(\frac{1}{Z^*}\right)S_{xx}(Z) \tag{4.145}$$

where, $H(Z)$ is the bilateral $Z$-transform of $h(n)$, also denoted $\mathcal{Z}\{h(n)\}$, and is given by

$$H(Z) = \sum_{n=-\infty}^{\infty} h(n)Z^{-n} \tag{4.146}$$

The *frequency response* $H(e^{j\omega})$ of the filter can be deduced from the discrete Fourier transform $H(Z)$ when evaluated on the unit circle in the $z$-plane. For $h(n)$ real, $H^*\left(1/Z^*\right) = H\left(1/Z\right)$, and the output spectral density is then

$$S_{yy}(\omega) = \left|H(e^{j\omega})\right|^2 S_{xx}(\omega) \tag{4.147}$$

**Example 4.3**

Consider the system given in Figure 4.1. Determine the power spectrum of the output if the input process $X(n)$ is a stationary white noise process.

*Solution*

$X(n)$ is a *white noise process* if

$$E\left[X(n_1)X^*(n_2)\right] = \begin{cases} I(n), & n_1 = n_2 \\ 0 \quad , & n_1 \neq n_2 \end{cases} \tag{4.148}$$

The autocorrelation function of the white noise process is

$$r_{xx}(n_1, n_2) = I(n_1)\delta(n_2 - n_1) \tag{4.149}$$

where

$$\delta(n_2 - n_1) = \begin{cases} 1, & n_1 = n_2 \\ 0, & n_1 \neq n_2 \end{cases} \tag{4.150}$$

Since the white noise process is stationary, let

$$I(n) = \sigma_n^2 = \text{constant} \tag{4.151}$$

The autocorrelation function becomes

$$r_{xx}(n, n+k) = r_{xx}(k) = \sigma_n^2 \delta(k) \tag{4.152}$$

where $\delta(k)$ is the impulse function given by

$$\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{4.153}$$

The power spectrum of the input white noise process is

$$S_{xx}(\omega) = \sigma_n^2 \tag{4.154}$$

By definition, the $Z$-transform of the autocorrelation function is

$$S_{xx}(Z) = \sum_{k=-\infty}^{\infty} r_{xx}(k) Z^{-k} \tag{4.155}$$

and thus, the power spectrum of the output is

$$S_{yy}(\omega) = \left| H(e^{j\omega}) \right|^2 S_{xx}(\omega) = \sigma_n^2 \left| H(e^{j\omega}) \right|^2 \tag{4.156}$$

## 4.4 AR, MA, AND ARMA RANDOM PROCESSES

In the previous section, we gave the definitions and properties related to the different correlations and power spectral densities of a stationary discrete-time stochastic process. Determining the power spectral density of a random process is essential in spectrum estimation because the power spectral density provides important information about the structure of the random process. Such information can then be used for different applications, such as modeling, prediction, or filtering of the observed signal, as will be seen in a later chapter. The *nonparametric* approach of spectrum estimation is determined by the transform relationship between the power spectral density and the autocorrelation function, which is the second-order statistics of the random process. However, in the *parametric* approach, a time series model for the random process is assumed, and thus, the power spectral density is a function of the model parameters, which must

be determined or estimated. We now consider the modeling of such random sequences. Consider the system shown in Figure 4.1, with input being the white noise process. The output sequence can be described by a parametric model; that is, the spectrum of the output sequence can be expressed in terms of the parameters of the model considered. Hence, it is necessary to have an appropriate parametric model, and estimate the model parameters. The models considered most frequently are the AR processes, the MA processes, and ARMA processes.

### 4.4.1 AR Processes

An AR process is represented by the following input-output difference equation.

$$X(n) = -\sum_{k=1}^{p} a_k X(n-k) + e(n) = \sum_{k=1}^{p} \omega_k X(n-k) + e(n) \tag{4.157}$$

where $X(n)$ is the observed real random sequences, $a_k$, $k$ = 1, 2, … , $p$, are constants called *parameters*, such that $\omega_k = -a_k$; $e[n]$ is a sequence of independent and identically distributed zero-mean Gaussian random variables with an unknown variance $\sigma_n^2$; and $p$ is the order of the filter. The sequence is referred to as a $p$th order autoregressive model and is abbreviated AR($p$). Note that the term "autoregressive" stems from the fact that $X(n)$, the present value of the process, given by

$$X(n) = -a_1 X(n-1) - a_2 X(n-2) - \ \ldots\ - a_p X(n-p) + e(n) \tag{4.158}$$

is a finite linear combination of $X(n-1)$, $X(n-2)$, … , $X(n-p)$, the past values of the process, and an error term $e(n)$. The $Z$-transform of (4.158) is given by

$$X(Z)(1 + a_1 Z^{-1} + a_2 Z^{-2} + \ \ldots\ + a_p Z^{-p}) = E(Z) \tag{4.159}$$

where $X(Z)$ is the $Z$-transform of $X(n)$, $E(Z)$ is the $Z$-transform of $e(n)$, and

$$\mathcal{Z}[X(n-k)] = Z^{-k} X(Z),\ k = 1, 2,\ \ldots, p \tag{4.160}$$

The *pulse transfer function* of the *all-zero filter* is given by

$$H(Z) = \frac{E(Z)}{X(Z)} = 1 + \sum_{k=1}^{p} a_k Z^{-k} \tag{4.161}$$

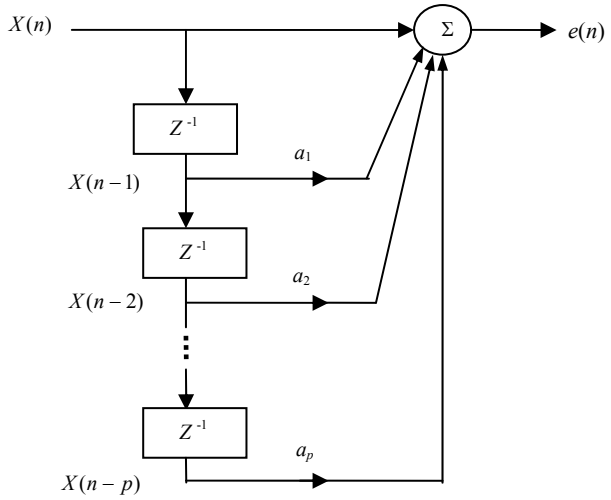


**Figure 4.2** Realization of an AR($p$) filter.

The realization of such a filter, obtained directly from (4.159), is shown in Figure 4.2. However, when the input is the white noise process $e(n)$, and $X(n)$ is the output random sequence, the corresponding AR($p$) filter is an *all-pole* filter with transfer function

$$H(Z) = \frac{1}{1 + \sum_{k=1}^{p} a_k Z^{-k}} = \frac{1}{1 - \sum_{k=1}^{p} \omega_k Z^{-k}} \tag{4.162}$$

The realization of the filter is as shown in Figure 4.3. In order to study the AR($p$) process $X(n)$, we need to determine the mean, autocorrelation function, correlation coefficients, and of course the power spectral density, which will be a function of the parameters of the model. The process is assumed to be stationary. Thus, the mean of $X(n)$, as defined in (4.158), is given by

$$E\big[X(n)\big] = m_x = E\left[ -\sum_{k=1}^{p} a_k X(n-k) + e(n) \right] = -\sum_{k=1}^{p} a_k E[X(n-p)] \tag{4.163}$$

since the white noise process is zero-mean $E[e(n)] = 0$. To obtain a general form with $k = p$ for of all the terms to be determined, we shall first compute them for

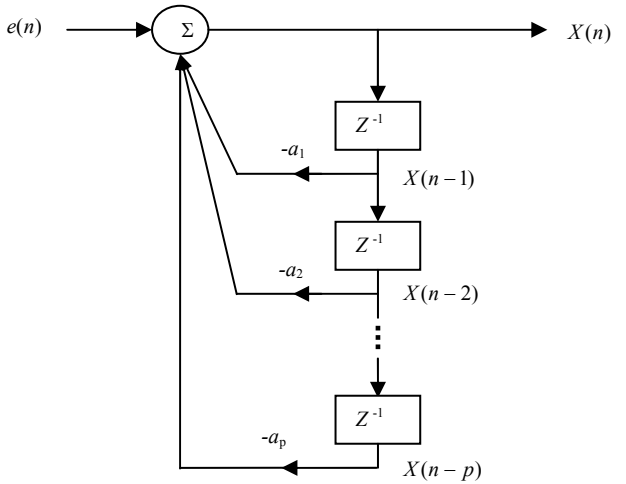


**Figure 4.3** All-pole AR($p$) filter.

$p = 1$, the first-order process, and $p = 2$, the second-order process, and then deduce easily the desired general form.

*AR*(1) *process*

For $p = 1$, the first-order process is

$$X(n) = -a_1 X(n-1) + e(n) \tag{4.164}$$

The first-order mean is then

$$E[X(n)] = m_x = -E[a_1 X(n-1)] = -a_1 m_x \tag{4.165}$$

to yield

$$m_x = 0 \quad \text{for} \quad a_1 \neq 0 \tag{4.166}$$

The corresponding variance is

$$\begin{aligned} \sigma_x^2 = E[X(n)X^*(n)] &= E\left\{[-a_1 X(n-1) + e(n)][-a_1 X^*(n-1) + e^*(n)]\right\} \\ &= E[a_1^2 X^2(n-1) + e^2(n) - 2a_1 X(n-1)e(n)] \end{aligned} \tag{4.167}$$

The initial condition $X(0)$ is assumed to be Gaussian with mean zero, $E[X(0)]=0$, and uncorrelated, thus independent, of the Gaussian noise process. Therefore, $E[X(n-1)e(n)]=0$ and the variance of the AR(1) process is $\sigma_x^2 = a_1^2\sigma_x^2 + \sigma_n^2$, or

$$\sigma_x^2 = \frac{\sigma_n^2}{1-a_1^2} \tag{4.168}$$

Since the variance of the process must be finite and nonnegative, the constant $a_1^2$ must be less than one; that is,

$$-1 < a_1 < +1 \tag{4.169}$$

The autocorrelation function of the AR(1) process is given by

$$r_{xx}(k) = E[X(n)X(n-k)] = E\{[-a_1X(n-1)+e(n)]X(n-k)\} = -a_1 r_{xx}(k-1), k \geq 1 \tag{4.170}$$

By direct substitution for $k = 1, 2, \ldots$ in (4.170), we can see that

$$r_{xx}(k) = (-1)^k a_1^k r_{xx}(0) = (-1)^k a_1^k \sigma_x^2 = \omega_1^k \sigma_x^2 \tag{4.171}$$

The autocorrelation coefficient of the process is defined by

$$\rho_k = \frac{r_{xx}(k)}{r_{xx}(0)} = (-1)^k a_1^k = \frac{1}{\sigma_x^2} r_{xx}(k) \tag{4.172}$$

The power spectrum of the output process in terms of the input noise process is $S_{xx}(f) = |H(f)|^2 S_{ee}(f)$, where $S_{ee}(f)$ is the power spectrum defined in (4.154). The transfer function is given by (4.162), and hence with $Z = e^{j\omega k}$ and $k = 1$, we have

$$H(e^{j\omega}) = \frac{1}{1+a_1e^{-j\omega}}, \quad |\omega| < \pi \tag{4.173}$$

and

$$\left|H(e^{j\omega})\right|^2 = \frac{1}{1+2a_1\cos 2\pi f + a_1^2}, \quad |f| < \frac{1}{2} \tag{4.174}$$

where $\omega = 2\pi f$. Thus, the power spectrum of the AR(1) process is

$$S_{xx}(f) = \frac{\sigma_n^2}{1+2a_1\cos 2\pi f + a_1^2} = \frac{\sigma_x^2(1-a_1^2)}{1+2a_1\cos 2\pi f + a_1^2}, \quad |f| < \frac{1}{2} \tag{4.175}$$

where the value of $\sigma_n^2$ is obtained directly from (4.168).

*AR*(2) *process*

The second-order AR process is given by

$$X(n) = -a_1 X(n-1) - a_2 X(n-2) + e(n) \tag{4.176}$$

The mean of this process is

$$E[X(n)] = m_x = -a_1 m_x - a_2 m_x = -(a_1 + a_2) m_x \tag{4.177}$$

and thus,

$$m_x = 0 \quad \text{if} \quad a_1 + a_2 \neq 1 \tag{4.178}$$

The variance is

$$\begin{aligned}\sigma_x^2 = E[X(n)X^*(n)] &= E\{[-a_1 X(n-1) - a_2 X(n-2) + e(n)]\, X(n)\} \\ &= -a_1 r_{xx}(1) - a_2 r_{xx}(2) + \sigma_n^2\end{aligned} \tag{4.179}$$

Substituting for $r_{xx}(k) = \rho_k \sigma_x^2$ in (4.179), we obtain $\sigma_x^2 = -a_1\rho_1\sigma_x^2 - a_2\rho_2\sigma_x^2 + \sigma_n^2$. Then the variance is

$$\sigma_x^2 = \frac{\sigma_n^2}{1 + a_1\rho_1 + a_2\rho_2} \tag{4.180}$$

which is finite for $a_1\rho_1 + a_2\rho_2 \neq -1$, and nonnegative for

$$a_1\rho_1 + a_2\rho_2 > -1 \tag{4.181}$$

The autocorrelation function is

$$r_{xx}(k) = E[X(n)X(n-k)] = E\{[-a_1X(n-1) - a_2X(n-2) + e(n)]X(n-k)\}$$
$$= -a_1 r_{xx}(k-1) - a_2 r_{xx}(k-2) \tag{4.182}$$

To obtain the constants $a_1$ and $a_2$, we need two equations. Hence, substituting in (4.182) for $k = 1$ and $k = 2$, we have for $k = 1$

$$r_{xx}(1) = -a_1 r_{xx}(0) - a_2 r_{xx}(-1) \tag{4.183}$$

Since the process is stationary, $r_{xx}(1) = r_{xx}(-1)$ and $\sigma_x^2 = r_{xx}(0)$. After substitution in (4.183), we have

$$r_{xx}(1) = \frac{-a_1}{1+a_2}\sigma_x^2 = \rho_1\sigma_x^2 \tag{4.184}$$

where

$$\rho_1 = \frac{-a_1}{1+a_2} \tag{4.185}$$

For $k = 2$, we have

$$r_{xx}(2) = -a_1 r_{xx}(1) - a_2 r_{xx}(0) = \left(\frac{a_1^2}{1+a_2} - a_2\right)\sigma_x^2 = \rho_2\sigma_x^2 \tag{4.186}$$

where

$$\rho_2 = \frac{a_1^2}{1+a_2} - a_2 \tag{4.187}$$

Substituting (4.185) and (4.187) in (4.180), we obtain

$$\sigma_x^2 = \frac{\sigma_n^2(1+a_2)}{(1-a_2)(1+a_1+a_2)(1-a_1+a_2)} \tag{4.188}$$

which is finite if

$$\begin{cases} a_2 \neq 1 \\ a_1 + a_2 \neq -1 \\ a_1 - a_2 \neq 1 \end{cases} \tag{4.189}$$

and positive if

$$\begin{cases} -1 < a_2 < 1 \\ -(a_1 + a_2) < 1 \\ a_1 - a_2 < 1 \end{cases} \tag{4.190}$$

The transfer function is, from (4.162), in this case

$$H(e^{j\omega}) = \frac{1}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}} = \frac{1}{1 + a_1 e^{-j2\pi f} + a_2 e^{-j4\pi f}} \quad , \quad |f| < \frac{1}{2} \tag{4.191}$$

and the power spectrum of the AR(2) process is then

$$S_{xx}(f) = \frac{\sigma_n^2}{\left|1 + a_1 e^{-j2\pi f} + a_2 e^{-j4\pi f}\right|^2}, \ |f| < \frac{1}{2} \tag{4.192}$$

*AR*(*p*) *Process*

We are now ready to determine the general expressions of the AR(*p*) process, which is given by

$$X(n) = -\sum_{k=1}^{p} a_k X(n-k) + e(n) \tag{4.193}$$

with mean

$$E[X(n)] = m_x = 0 \qquad \text{for} \qquad \sum_{k=1}^{p} a_k \neq 1 \tag{4.194}$$

and variance

$$\sigma_x^2 = E[X(n)X^*(n)] = -E\left\{X(n)\left[\sum_{k=1}^{p} a_k X(n-k) + e(n)\right]\right\} = -\sum_{k=1}^{n} a_k r_{xx}(k) + \sigma_n^2 \tag{4.195}$$

In order to determine completely the power spectrum

$$S_{xx}(f) = \frac{\sigma_n^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-j2\pi fk}\right|^2} \tag{4.196}$$

we need the estimates of the AR coefficients, and the noise variance $\sigma_n^2$. Multiplying both sides of (4.193) by $X^*(n-\ell), \quad \ell \geq 0$, and taking the expectation operator, we obtain

$$E[X(n)X^*(n-\ell)] = -\sum_{k=1}^{p} a_k E[X(n-k)X^*(n-\ell)] + E[e(n)X^*(n-\ell)] \tag{4.197}$$

The second term $E[e(n)X^*(n-\ell)]$ is zero for $\ell > 0$, since the terms $X(n-\ell)$ are zero-mean and independent of the noise $e(n)$. For $\ell = 0$, we have the variance $\sigma_n^2$. The first term of (4.197) is the autocorrelation function of the AR process to obtain

$$r_{xx}(\ell) = \begin{cases} -\sum_{k=1}^{p} a_k r_{xx}(\ell - k) & , \quad \ell > 0 \\ -\sum_{k=1}^{p} a_k r_{xx}(\ell - k) + \sigma_n^2, & \ell = 0 \end{cases} \tag{4.198}$$

This set of equations can be expressed, using $\omega_k = -a_k$, in matrix form to yield

$$\begin{bmatrix} r(1) \\ r(2) \\ \vdots \\ r(p) \end{bmatrix} = \begin{bmatrix} r(0) & r(-1) & r(-2) & \cdots & r(-p+1) \\ r(1) & r(0) & r(-1) & \cdots & r(-p+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r(p-1) & r(p-2) & r(p-3) & \cdots & r(0) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_p \end{bmatrix} \tag{4.199}$$

This set of equations (4.199) is called the *Yule-Walker equations*, which may be expressed as

$$\boldsymbol{r} = \boldsymbol{R}\,\boldsymbol{\omega} \tag{4.200}$$

or

$$-\boldsymbol{r} = \boldsymbol{R}\,\boldsymbol{a} \tag{4.201}$$

The parameters $\boldsymbol{a}$ and $\boldsymbol{\omega}$ are the solutions given by

$$\boldsymbol{a} = -\boldsymbol{R}^{-1}\boldsymbol{r} \tag{4.202}$$

or

$$\boldsymbol{\omega} = \boldsymbol{R}^{-1}\boldsymbol{r} \tag{4.203}$$

with the vectors

$$\boldsymbol{a}^T = \begin{bmatrix} a_1, & a_2, & \ldots, & a_p \end{bmatrix} \tag{4.204}$$

and

$$\boldsymbol{\omega}^T = \begin{bmatrix} \omega_1, & \omega_2, & \ldots, & \omega_p \end{bmatrix} \tag{4.205}$$

and noise variance

$$\sigma_n^2 = r_{xx}(0) + \sum_{k=1}^{p} a_k r_{xx}(k) \tag{4.206}$$

### 4.4.2 MA Processes

The MA process is described by

$$X(n) = b_0^* e(n) + b_1^* e(n-1) + \ \ldots\ + b_q^* e(n-q) = \sum_{k=0}^{q} b_k^* e(n-k) \tag{4.207}$$

where $b_0$, $b_1$, $b_2$, … , $b_q$, are constants called the MA *parameters*, such that $\sum_{k=0}^{q} b_k = 1$, and $e(n)$ is the input white noise process. The filter representing the MA process is shown in Figure 4.4. We can write (4.207) with new parameters (new $b_k$s), such that

$$X(n) = e(n) + b_1 e(n-1) + b_1 e(n-1) + \ \ldots\ + b_q e(n-q) \tag{4.208}$$

such that the new $b_1 = 0$ and $b_k \neq 0$ for $k = 1, 2, \ldots, q$. Note that we maintain the notation "$b$" to be consistent with the notation of (4.1). Taking the $Z$-transform of (4.208) and solving for $H(Z)$, we obtain the *all-zero filter* given by

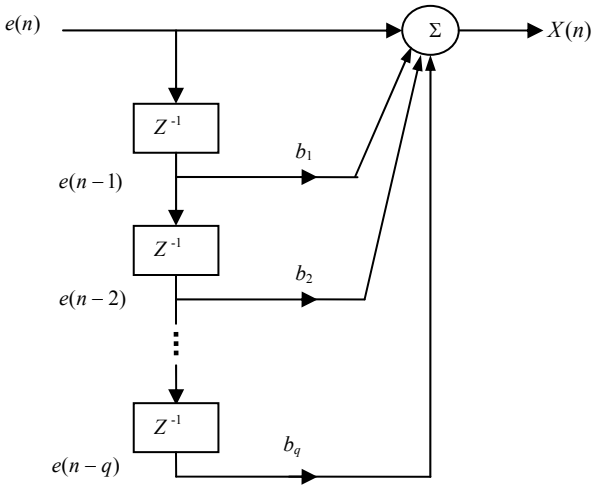


**Figure 4.4** Moving average filter of order $q$.

$$H(Z) = 1 + b_1 Z^{-1} + b_2 Z^{-2} + \ \ldots \ + b_q Z^{-q} \tag{4.209}$$

with transfer function

$$H(e^{j2\pi f}) = 1 + \sum_{k=1}^{q} b_k e^{-j2\pi fk} \tag{4.210}$$

The mean of the process is

$$E[X(n)] = m_x = E\left[ e(n) + \sum_{k=1}^{q} b_k e(n-k) \right] = 0 \tag{4.211}$$

since the white noise process is zero-mean and stationary. The variance is

$$\begin{aligned} \sigma_x^2 = E\big[X(n)X(n)\big] &= E\left\{ \left[ e(n) + \sum_{i=1}^{q} b_i e(n-i) \right] \left[ e(n) + \sum_{j=1}^{q} b_j e(n-j) \right] \right\} \\ &= \sigma_n^2 \left( 1 + \sum_{i=1}^{q} b_i^2 \right) \end{aligned} \tag{4.212}$$

The autocorrelation function can be calculated to be

$$r_{xx}(k) = E\left\{\left[e(n) + \sum_{i=1}^{q} b_i e(n-i)\right]\left[e(n-k) + \sum_{j=1}^{q} b_j e(n-k-j)\right]\right\}$$

$$= r_{ee}(k) + \sum_{j=1}^{q} b_j r_{ee}(k+j) + \sum_{i=1}^{q} b_i r_{ee}(k-i) + \sum_{i=1}^{q}\sum_{j=1}^{q} b_i b_j r_{ee}(k+j-i)$$

$$= \begin{cases} \sigma_n^2\left(1 + \sum_{j=1}^{q} b_j^2\right) & , \quad k = 0 \\ \sigma_n^2\left(b_1 + \sum_{j=2}^{q} b_j b_{j-1}\right), & k = 1 \\ \sigma_n^2\left(b_2 + \sum_{j=3}^{q} b_j b_{j-2}\right), & k = 2 \end{cases} \tag{4.213}$$

and so on, until $k = q$. The general form can then be deduced to be

$$r_{xx}[k] = \begin{cases} \sigma_n^2\left(b_k + \sum_{j=k+1}^{q} b_j b_{j-k}\right) & , \quad k < q \\ \sigma_n^2 b_q & , \quad k = q \\ 0 & , \quad k > q \end{cases} \tag{4.214}$$

Hence, the power spectral density, which is just the Fourier transform of (4.214), can be shown to be

$$S_{xx}(f) = \sigma_n^2 \left| 1 + \sum_{k=1}^{q} b_k e^{-j2\pi k f} \right|^2 \quad , \qquad |f| < \frac{1}{2} \tag{4.215}$$

### 4.4.3 ARMA Processes

An ARMA process for the time series $X(n)$ is given by

$$X(n) + a_1^* X(n-1) + \ \ldots\ + a_p^* X(n-p) = e(n) + b_1^* e(n-1) + \ \ldots\ + b_q e(n-q) \tag{4.216}$$

or

$$X(n) = -\sum_{k=1}^{p} a_k X(n-k) + e(n) + \sum_{\ell=1}^{q} b_\ell e(n-\ell) \tag{4.217}$$

Taking the *Z*-transform of (4.217), and solving for the pulse transfer function, we obtain

$$H(Z) = \frac{1 + \sum_{\ell=1}^{q} b_\ell Z^{-\ell}}{1 + \sum_{k=1}^{p} a_k Z^{-k}} \tag{4.218}$$

which is a filter with both poles and zeros, as shown in Figure 4.5, since it is a combination of AR and MA processes. The *order* of the ARMA process is $(p, q)$.

The autocorrelation function of the ARMA $(p, q)$ process, assuming $p > q$, is

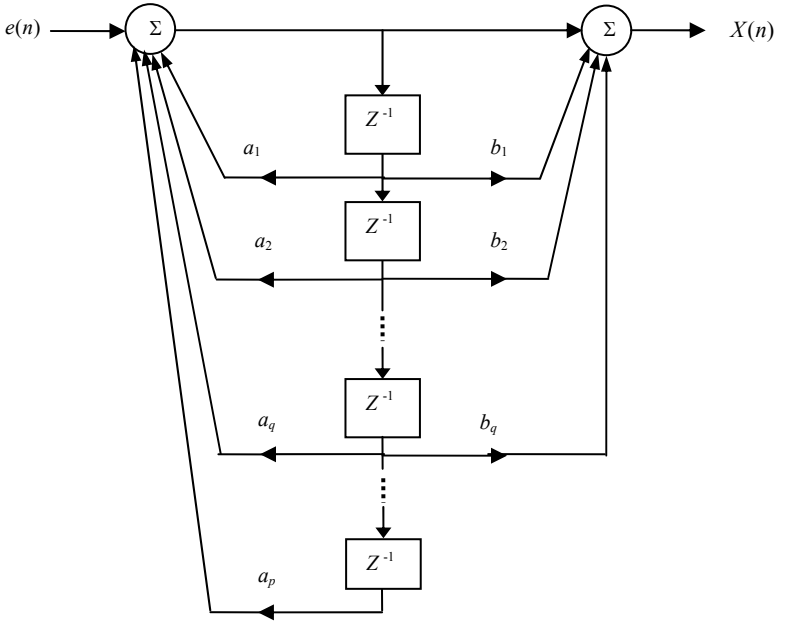


**Figure 4.5** ARMA filter of order $(p, q)$ with $p > q$.

$$r_{xx}(k) = E\{X(n)X(n-k)\} = E\left\{\left[-\sum_{i=1}^{p} a_i X(n-i) + e(n) + \sum_{j=1}^{q} b_j e(n-j)\right] X(n-k)\right\}$$

$$= -\sum_{i=1}^{p} a_i r_{xx}(k-i) + E[e(n)X(n-k)] + \sum_{j=1}^{q} b_j E[e(n-j)X(n-k)] \qquad (4.219)$$

but

$$E[e(n)X(n-k)] = 0 \qquad \text{for} \qquad k \geq 1 \qquad (4.220)$$

and

$$\sum_{j=1}^{q} b_j E[e(n-j)X(n-k)] = 0 \qquad \text{for} \qquad k \geq q+1 \qquad (4.221)$$

Substituting (4.220) and (4.221) in (4.219), the autocorrelation function reduces to

$$r_{xx}(k) = -\sum_{i=1}^{p} a_i r_{xx}(k-i), \qquad k \geq q+1 \qquad (4.222)$$

while the power spectral density of the ARMA ($p$, $q$) process can be shown to be

$$S_{xx}(f) = \sigma_n^2 \frac{\left|1 + \sum_{\ell=1}^{q} b_\ell e^{-j2\pi f\ell}\right|^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-j2\pi fk}\right|^2}, \quad |f| < \frac{1}{2} \qquad (4.223)$$

## 4.5 MARKOV CHAINS

In Section 3.4.8, we defined the concept of Markov processes. When the Markov process is *discrete-valued* (discrete state), it is called a *Markov chain*. To describe a Markov chain, consider a finite set of states $S = \{S_1, S_2, \ldots, S_N\}$. The process starts in one of these states and moves successively from one state to another. The move from one state to another is called a *step.* If the chain is a state $S_i$, it moves to a state $S_j$ in a step with a probability $P_{ij}$, called *transition probability*.

The Markov chain is then a discrete state, but may have a discrete or a continuous time. Both cases will be considered in this section.

### 4.5.1 Discrete-Time Markov Chains

A *discrete-time Markov chain* must satisfy the following *Markov property*:

$$\begin{aligned} P[X(n) = x_n \mid X(n-1) = x_{n-1}, X(n-2) = x_{n-2}, \ldots, X(0) = x_0] \\ = P[X(n) = x_n \mid X(n-1) = x_{n-1}] \end{aligned} \tag{4.224}$$

where we have assumed that the random sequence takes a finite, countable set of values. The values of the process are the *states of the process*, and the conditional probabilities are the *transition probabilities* between the states, defined in the introduction of this section. If $X(n) = i$, we say that the chain is in the "$i$th state at the $n$th step," and write

$$P[X(n) = j] = p_j(n), \quad j = 1, 2, \ldots \tag{4.225}$$

Since the evolution of the chain is described by the transition probability, when we say that the system is in state $j$ at time $t_m$, given that it is in state $i$ at time $t_n$, we write

$$P[X(m) = j \mid X(n) = i] = P_{ij}(n, m) \tag{4.226}$$

Using Bayes' rule, we can write

$$P[X(m) = j, X(n) = i] = P[X(m) = j \mid X(n) = i]P[X(n) = i] \tag{4.227}$$

or, using the new notation

$$P[X(m) = j, X(n) = i] = P_{ij}(n, m)P_i(n) \tag{4.228}$$

Assuming that the finite number of states is $N$, these probabilities must satisfy

$$\sum_{j=1}^{N} P_j(n) = 1 \tag{4.229}$$

and

$$\sum_{j=1}^{N} P_{ij}(n, m) = 1 \tag{4.230}$$

The total probability is

$$P_j(m) = \sum_{i=1}^{N} P_{ij}(n,m) P_i(n) \tag{4.231}$$

In matrix form, the *transition matrix* or *stochastic matrix* $\boldsymbol{P}(n, m)$ can be written as

$$\boldsymbol{P}(n,m) = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1N} \\ P_{21} & P_{22} & \cdots & P_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ P_{N1} & P_{N2} & \cdots & P_{NN} \end{bmatrix} = [P_{ij}(n,m)] \tag{4.232}$$

The entries $P_{ij}$, $i, j = 1, 2, \ldots, N$, are the transition probabilities that the Markov chain, starting in state $S_i$, will be in state $S_j$. The initial state matrix is $\boldsymbol{P}(0) = \boldsymbol{P} = \boldsymbol{W}$, denoted as $\boldsymbol{\Pi}$ in other books.

The column vector

$$\boldsymbol{p}(n) = \begin{bmatrix} \boldsymbol{p}_1(n) \\ \boldsymbol{p}_2(n) \\ \vdots \\ \boldsymbol{p}_N(n) \end{bmatrix} \tag{4.233}$$

is called the *state distribution* vector, with $\boldsymbol{p}(1)$ representing the starting distribution. Note that the stochastic matrix $\boldsymbol{P}$ has nonnegative entries, $P_{ij} > 0$, and the sum across each row is equal to one, $\sum_{j=1}^{N} P_{ij} = 1$, which is (4.230). Hence, (4.231) can be written as

$$\boldsymbol{P}(m) = \boldsymbol{P}(n,m)\boldsymbol{P}(n) \tag{4.234}$$

*Homogeneous Chain*

A Markov chain is called *homogeneous* if the transition probabilities depend only on the difference between states; that is,

$$P_{ij}(m) = P(X_{n+m} = j \mid X_n = i) = P(X_{m+1} = j \mid X_1 = i) \tag{4.235}$$

or

$$\boldsymbol{P}(n, m) = \boldsymbol{P}(m - n) \tag{4.236}$$

If $m = 1$,

$$P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i) = P_{ij}(1) = P_{ij} \tag{4.237}$$

Substituting (4.236) in (4.234), we obtain

$$\boldsymbol{P}(m) = \boldsymbol{P}(m - n)\boldsymbol{P}(n) = \boldsymbol{P}(m - 1)\boldsymbol{P}(1) \tag{4.238}$$

where $\boldsymbol{P}(1) = \boldsymbol{P}$ is the one-step transition matrix. Hence, by direct substitution in (4.238), we have

$$\begin{aligned}
\boldsymbol{P}(2) &= \boldsymbol{P}(1)\boldsymbol{P}(1) = \boldsymbol{P}^2(1) = \boldsymbol{P}^2 \\
\boldsymbol{P}(3) &= \boldsymbol{P}(2)\boldsymbol{P}(1) = \boldsymbol{P}^2(1)\boldsymbol{P}(1) = \boldsymbol{P}^3(1) = \boldsymbol{P}^3 \\
&\vdots \\
\boldsymbol{P}(n) &= \boldsymbol{P}(n-1)\boldsymbol{P}(1) = \boldsymbol{P}^n(1) = \boldsymbol{P}^n
\end{aligned} \tag{4.239}$$

We observe that the *n*-step transition matrix (the matrix of *n*-step transition probabilities) $\boldsymbol{P}(n)$ is

$$\boldsymbol{P}(n) = \boldsymbol{P}^n(1) = \boldsymbol{P}^n \tag{4.240}$$

Observe that the entry $P_{ij}^n$ of the matrix $\boldsymbol{P}^n$ is the probability that the Markov chain, starting in state $S_i$, will be in state $S_j$ after *n* steps, and $\boldsymbol{P}(1) = \boldsymbol{P}$ is the one-step transition matrix with elements $[P_{ij}(1)]$. When the matrix $\boldsymbol{P}(n)$ is independent of *n* and the chain is homogeneous, it is then *stationary*. This implies that $\boldsymbol{P}(n) = \boldsymbol{P}(1) = \boldsymbol{P}$ for all *n*. A "good" way to represent a Markov chain is the *state transition diagram*, which we will show by an example.

**Example 4.4**

Consider the transition matrix $\boldsymbol{P}$ given by

$$\boldsymbol{P} = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}$$

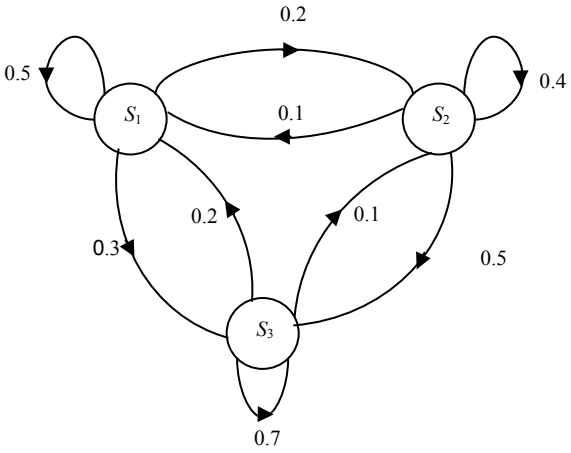


**Figure 4.6** State transition diagram.

The state transition diagram is shown in Figure 4.6. We see, for example, that the probability in going from state $S_1$ to state $S_2$ is $P_{12} = 0.2$, the probability in going from state $S_2$ to state $S_3$ is $P_{23} = 0.5$, and so on. Using (4.239), we have

$$\boldsymbol{P}(1) = \boldsymbol{P} = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}$$

$$\boldsymbol{P}(2) = \boldsymbol{P}^2(1) = \boldsymbol{P}^2 = \begin{bmatrix} 0.33 & 0.21 & 0.46 \\ 0.19 & 0.23 & 0.58 \\ 0.25 & 0.15 & 0.60 \end{bmatrix}$$

$$\boldsymbol{P}(3) = \boldsymbol{P}^3(1) = \boldsymbol{P}^3 = \begin{bmatrix} 0.278 & 0.196 & 0.562 \\ 0.234 & 0.188 & 0.578 \\ 0.260 & 0.170 & 0.579 \end{bmatrix}$$

$$\vdots$$

$$\boldsymbol{P}(10) = \boldsymbol{P}^{10} = \begin{bmatrix} 0.26 & 0.18 & 0.56 \\ 0.26 & 0.18 & 0.56 \\ 0.26 & 0.18 & 0.56 \end{bmatrix}$$

$$\boldsymbol{P}(12) = \boldsymbol{P}^{12} = \begin{bmatrix} 0.26 & 0.18 & 0.56 \\ 0.26 & 0.18 & 0.56 \\ 0.26 & 0.18 & 0.56 \end{bmatrix}$$

$$\vdots$$

We observe that

$$\boldsymbol{P}(20) = \boldsymbol{P}^{20} = \begin{bmatrix} 0.26 & 0.18 & 0.56 \\ 0.26 & 0.18 & 0.56 \\ 0.26 & 0.18 & 0.56 \end{bmatrix} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \omega_1 & \omega_2 & \omega_3 \end{bmatrix}$$

with $\omega_1 = 0.26,\ \omega_2 = 0.18,$ and $\omega_3 = 0.56.$

As *n* increases, we reach the situation where the probabilities that the chain is in states $S_1$, $S_2$, and $S_3$ are 0.26, 0.18, and 0.56, respectively, no matter where the chain started. This type of Markov chain is called a *regular* Markov chain.

In general, by definition, if a set of numbers $\omega_1, \omega_2, \ldots, \omega_N$ exists, such that

$$\lim_{n\to\infty} \boldsymbol{P}^n = \boldsymbol{W} = \begin{bmatrix} \omega_1 & \omega_1 & \cdots & \omega_1 \\ \omega_2 & \omega_2 & \cdots & \omega_2 \\ \vdots & \vdots & \vdots & \vdots \\ \omega_N & \omega_N & \cdots & \omega_N \end{bmatrix} \tag{4.241}$$

we say that *the steady state probabilities* for the chain exist. In this case, the chain is said to be *regular* (as shown in the previous example), and $\boldsymbol{W}$ is called the *stationary distribution matrix* for the Markov chain, with entries $\omega_j,\ j = 1, 2, \ldots, N,$ such that

$$\sum_{i=1}^{N} \omega_i = 1 \tag{4.242}$$

and

$$\omega_i > 0 \qquad \text{for all } i \tag{4.243}$$

We also observe from Example 4.4 that a homogeneous Markov chain reaches a steady state probability after many transitions. That is,

$$\lim_{n\to\infty} \boldsymbol{P}(n) = \lim_{n\to\infty} \boldsymbol{P}^n(1) = \lim_{n\to\infty} \boldsymbol{P}^n = \boldsymbol{W} \tag{4.244}$$

*Theorem*. Let $\boldsymbol{P}$ be the transition matrix of a Markov chain, and let $\boldsymbol{p}^T = \boldsymbol{p}^T(0)$ be the row probability vector representing the starting distribution of the chain. Then, the probability that the chain is in state $S_i$ after $n$ steps is the $i$th entry in the vector, given by

$$\boldsymbol{p}^T(n) = \boldsymbol{p}^{\boldsymbol{T}} \boldsymbol{P}^n \tag{4.245}$$

The important conclusion to draw from this theorem is that the random evolution of the chain is determined by the transition matrix $\boldsymbol{P}$ and the initial distribution vector $\boldsymbol{p}(0)$. Equation (4.245) is just the deduction of the general form given by

$$\boldsymbol{p}(n+m) = \boldsymbol{p}(m)\boldsymbol{P}^n \tag{4.246}$$

The proof of (4.246) is straightforward.

**Example 4.5**

Suppose that we have the same distribution matrix as given in Example 4.4. Let $\boldsymbol{p}^T(0) = \boldsymbol{p}^{\boldsymbol{T}} = [0.20 \quad 0.25 \quad 0.35] = [p_1(0) \quad p_2(0) \quad p_3(0)]$ be the probability vector representing the starting distribution. Using the previous theorem and the results of Example 4.4, we can compute the distribution of the states after $n$ steps to be

$$\begin{aligned}
&\boldsymbol{p}^T(1) = \boldsymbol{p}^T\boldsymbol{P} = [0.225 \quad 0.225 \quad 0.225] = [p_1(1) \quad p_2(1) \quad p_3(1)] \\
&\boldsymbol{p}^T(2) = \boldsymbol{p}^T\boldsymbol{P}^2 = [0.245 \quad 0.19 \quad 0.565] = [p_1(2) \quad p_2(2) \quad p_3(2)] \\
&\boldsymbol{p}^T(3) = \boldsymbol{p}^T\boldsymbol{P}^3 = [0.2545 \quad 0.1815 \quad 0.564] \\
&\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
&\boldsymbol{p}^T(10) = \boldsymbol{p}^T\boldsymbol{P}^{10} = [0.26 \quad 0.18 \quad 0.56] \\
&\qquad\qquad \vdots \\
&\boldsymbol{p}^T(12) = \boldsymbol{p}^T\boldsymbol{P}^{12} = [0.26 \quad 0.18 \quad 0.56] \\
&\qquad\qquad \vdots \\
&\boldsymbol{p}^T(20) = \boldsymbol{p}^T\boldsymbol{P}^{20} = [0.26 \quad 0.18 \quad 0.56] = [p_1(20) \quad p_2(20) \quad p_3(20)]
\end{aligned}$$

We observe that as $n$ becomes large, we reach a steady state value. Hence, we say that if

$$\lim_{n\to\infty} \boldsymbol{p}^T(n) = \lim_{n\to\infty} \boldsymbol{p}^T \boldsymbol{P} = [\omega_1 \quad \omega_2 \quad \ldots \quad \omega_N] \tag{4.247}$$

is a constant distribution, then the column vector $\boldsymbol{p}^T = [\omega_1 \quad \omega_2 \quad \ldots \quad \omega_N]$ is called the *steady state distribution vector.*

**Example 4.6**

Consider the Markov chain given by the two-state transition diagram of Figure 4.7. Then,

$$\boldsymbol{P} = \begin{bmatrix} 0.65 & 0.35 \\ 0.45 & 0.55 \end{bmatrix}$$

Computing the different powers of $\boldsymbol{P}$, we obtain

$$\boldsymbol{P}^2 = \begin{bmatrix} 0.58 & 0.42 \\ 0.54 & 0.46 \end{bmatrix}, \quad \boldsymbol{P}^3 = \begin{bmatrix} 0.566 & 0.434 \\ 0.558 & 0.442 \end{bmatrix}, \quad \boldsymbol{P}^4 = \begin{bmatrix} 0.5632 & 0.4368 \\ 0.5616 & 0.4384 \end{bmatrix}$$

$$\boldsymbol{P}^5 = \begin{bmatrix} 0.5626 & 0.4374 \\ 0.5623 & 0.4377 \end{bmatrix}, \quad \boldsymbol{P}^6 = \begin{bmatrix} 0.5625 & 0.4375 \\ 0.5625 & 0.4375 \end{bmatrix}, \quad \boldsymbol{P}^7 = \begin{bmatrix} 0.5625 & 0.4375 \\ 0.5625 & 0.4375 \end{bmatrix}$$

If $\boldsymbol{p}(0) = \boldsymbol{p} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$ and $\boldsymbol{P}(1) = \boldsymbol{P} = \begin{bmatrix} 0.64 & 0.35 \\ 0.45 & 0.55 \end{bmatrix}$, then

$$\begin{aligned}
\boldsymbol{p}^T(1) &= \boldsymbol{p}^T \boldsymbol{P} = [0.51 \quad 0.49] \\
\boldsymbol{p}^T(2) &= \boldsymbol{p}^T \boldsymbol{P}^2 = [0.552 \quad 0.448] \\
\boldsymbol{p}^T(3) &= \boldsymbol{p}^T \boldsymbol{P}^3 = [0.5604 \quad 0.4396] \\
\boldsymbol{p}^T(4) &= \boldsymbol{p}^T \boldsymbol{P}^4 = [0.5621 \quad 0.4379] \\
\boldsymbol{p}^T(5) &= \boldsymbol{p}^T \boldsymbol{P}^5 = [0.5624 \quad 0.4376] \\
\boldsymbol{p}^T(6) &= \boldsymbol{p}^T \boldsymbol{P}^6 = [0.5625 \quad 0.4375] \\
\boldsymbol{p}^T(7) &= \boldsymbol{p}^T \boldsymbol{P}^6 = [0.5625 \quad 0.4375]
\end{aligned}$$

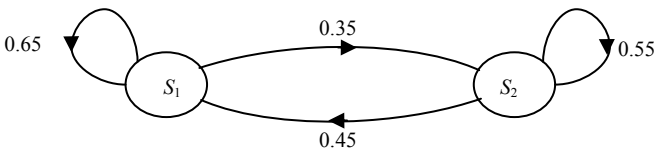


**Figure 4.7** State transition for Example 4.6.

We observe that as $n \to \infty, \boldsymbol{P}^n \to \boldsymbol{W}$, such that the steady state matrix is

$$\boldsymbol{P}^{\boldsymbol{n}} = \boldsymbol{W} = \begin{bmatrix} \omega_1 & \omega_2 \\ \omega_1 & \omega_2 \end{bmatrix}$$

with $\omega_1 = 0.5625$ and $\omega_2 = 0.4375$. Also, $\boldsymbol{p}(n) = \boldsymbol{p}^T \boldsymbol{P}^n \to \boldsymbol{p}^T = [\omega_1 \quad \omega_2]$, with $\omega_1 = 0.5625$ and $\omega_2 = 0.4375$, is the steady state (stationary) distribution vector.

We now give the *Chapman-Kolmogorov equation*, which relates long-term development to short-term development.

*Chapman-Kolmogorov Equation*

For a homogeneous discrete-time Markov chain with $n_1 < n_2 < n_3$,

$$P_{ij}(n_3 - n_1) = \sum_k P_{ik}(n_2 - n_1) P_{kj}(n_3 - n_2) \tag{4.248}$$

In other words,

$$P_{ij}(m+n) = \sum_k P_{ik}(m) P_{kj}(n) \tag{4.249}$$

where $P_{ij}(m) = P[X(m+n) = j \,|\, X(n) = i]$. Hence,

$$\boldsymbol{P}(m+n) = \boldsymbol{P}(m)\boldsymbol{P}(n) \tag{4.250}$$

or

$$\boldsymbol{P}(n_3 - n_1) = \boldsymbol{P}(n_2 - n_1)\boldsymbol{P}(n_3 - n_2) \tag{4.251}$$

*Classification of Chains*

In describing the relation between the states of a Markov chain, we say $S_i$ *communicates with* $S_j$, denoted $S_i \to S_j$, if the chain may visit state $S_j$ starting from $S_i$ with a positive probability. That is, $S_i \to S_j$, if $P_{ij}(m) > 0$ for some $m \geq 0$. If in addition, state $S_j$ communicates with state $S_i$, $S_j \to S_i$, then we say that $S_i$ and $S_j$ *intercommunicate*, and write $S_i \leftrightarrow S_j$. A state $S_i$ is called *persistent* or *recurrent* if

$$P(X_n = i \text{ for some } n \geq 1 | X_0 = i) = 1 \tag{4.252}$$

which means that the probability of an eventual return to state $S_i$, having started from $i$, is one. The state may be visited many times. The state $S_i$ of a Markov chain is called *absorbing* if it is impossible to leave it (i.e., $P_{ii} = 1$). If this probability is strictly less than one, then the state $S_i$ is called *transient*. Hence, every state is either transient or recurrent. The Markov chain is *absorbing* if it has a least one absorbing state, and it is possible to go to an absorbing state from every state (not necessarily in one step). To clarify these concepts, consider the Markov chain shown in Figure 4.8. For example, $S_1$ and $S_5$ are transient states, and $S_2$, $S_4$, and $S_6$ are recurrent states. We do not have an absorbing state, since we can leave any of the states we reach, and thus the chain is not absorbing.

A persistent state is said to be *null* if and only if

$$\lim_{n\to\infty} P_{ii}(n) = 0 \tag{4.253}$$

in this case,

$$\lim_{n\to\infty} P_{ij}(n) = 0 \quad \text{for all } j \tag{4.254}$$

A set of states is called *irreducible* if the states intercommunicate $(S_i \leftrightarrow S_j)$ for all $i$ and $j$ in the set. For example, states $S_2$ and $S_3$ constitute an irreducible set, and so do states $S_4$ and $S_5$ and states $S_3$ and $S_6$.

The number of transitions required for the first return to a state $S_i$ in an irreducible set is a random variable known as the recurrence time. If $P_{ii}(k)$ may be nonzero $k = d, 2d, 3d, \ldots$, with $d$ an integer greater than one, then the irreducible set is called *periodic*. If $d = 1$, the set is called *ergodic*; that is, it is persistent, nonnull and aperiodic. Note that the Markov chain is called an *ergodic* chain if it is possible to go from every state to every state (not necessarily in one move). The period of states $S_4$ and $S_5$ of the previous example is two, and thus the set is periodic.

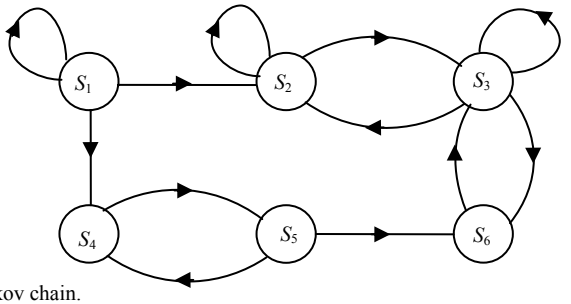


**Figure 4.8** Markov chain.

### 4.5.2 Continuous-Time Markov Chains

Let $X(t)$, $t \geq 0$, be a continuous-time random Markov chain with finite discrete states $S_1, S_2, \ldots, S_N$. By continuous time, we mean that the continuous transition allows changes of states to occur at any instant of time in the continuous time. The transition from state $S_i$ to state $S_j$ $(S_i \leftrightarrow S_j)$, $i \neq j$, occurs in a very small time $\Delta t$. $\Delta t$ is so small that only one transition is possible. The conditional probability that the transition from $S_i$ to $S_j$ occurs in the next $\Delta t$ is $\lambda_{ij}\,\Delta t$. The values of $\lambda_{ij}$, $i \neq j$, are called the *transition probability rates*. For *homogeneous* Markov chains, $\lambda_{ij}$ are positive constants. The transition probability function is

$$P_{ij}(\tau) = P[X(t+\tau) = j \mid X(t) = i] \tag{4.255}$$

with

$$\sum_{j=1}^{N} P_{ij}(\tau) = 1 \tag{4.256}$$

since the system will definitely make a transition from state $i$ to any other state in the chain, and

$$\lim_{\Delta t \to 0} P_{ij}(\Delta t) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{4.257}$$

for $P_{ij}(\tau)$ to be continuous. Hence, the probability that the system makes a transition from state $S_i$ to another state in the chain in a time interval $\Delta t$ is

$$P_{ij}(\Delta t) = \sum_{\substack{i=1 \\ i \neq j}}^{N} \lambda_{ij}\,\Delta t \tag{4.258}$$

We observe that the transition intensities can be defined in terms of the derivatives of the transition probability functions evaluated at $\tau = 0$ to yield

$$\lambda_{ij} = \left.\frac{\partial P_{ij}(\tau)}{\partial \tau}\right|_{\tau=0}, \quad i \neq j \tag{4.259}$$

Note that the transition from state $S_i$ to state $S_i$ $(S_i \to S_i)$ is interpreted as the system remaining in state $S_i$, and thus $\lambda_{ij}$ is undefined in this case. However, taking the derivative of both sides of (4.256), we have

$$\sum_{j=1}^{N} \lambda_{ij} = 0 \tag{4.260}$$

or

$$\lambda_{ii} = -\sum_{\substack{j=1 \\ i \neq j}}^{N} \lambda_{ij} \tag{4.261}$$

From (4.258), the probability that the system remains in the same state is

$$P_{ii}(\Delta t) = 1 - \sum_{\substack{i=1 \\ i \neq j}}^{N} \lambda_{ij} \Delta t \tag{4.262}$$

Using (4.261) in (4.262), we can write that

$$P_{ii}(\Delta t) = 1 + \lambda_{ii} \Delta t \tag{4.263}$$

In order to find the state probabilities, we first give the Chapman-Kolmogorov equation for transition probabilities.

*Chapman-Kolmogorov Equation*

For a Markov chain, the transition probabilities must satisfy the Chapman-Kolmogorov equation for $0 \le t < \tau$, given by

$$P_{ij}(\tau) = \sum_{k=1}^{N} P_{ik}(t) P_{kj}(\tau - t) \tag{4.264}$$

Let $p_i(t) = P\{X(t) = S_i\}$ be the probability that the system is in state $S_i$. The *state distribution vector* is the column vector $\boldsymbol{p}^T(t) = \begin{bmatrix} p_1(t) & p_2(t) & \dots & p_N(t) \end{bmatrix}$, and $\sum_{i=1}^{N} p_i(t) = 1$, since the system must be in some state $S_i$ at time $t$. In the limit, as $t \to \infty$, the probability that the system is in a transient state is zero, and the state distribution vector $\boldsymbol{p}(t)$ becomes *the steady state vector* $\boldsymbol{p}$.

From (4.255), we have

$$P[X(t+\tau) = j] = \sum_{i=1}^{N} P[X(t+\tau) = j \mid X(t) = i] \, P[X(t) = i] \tag{4.265}$$

and using the Markov property, we can write

$$p_j(\tau) = \sum_{i=1}^{N} P_{ij}(\tau) p_i(0) \tag{4.266}$$

$$p_j(t+\Delta t) - p_j(t) = \sum_{\substack{i=1 \\ i\neq j}}^{N} P_{ij}(\Delta t) p_i(t) - \sum_{\substack{k=1 \\ k\neq j}}^{N} P_{jk}(\Delta t) p_j(t) \tag{4.267}$$

Substituting (4.258) in (4.267), we obtain the $N-1$ equations given by

$$p_j(t+\Delta t) = p_j\left[1 - \sum_{\substack{i=1 \\ i\neq j}}^{N} \lambda_{ji}\Delta t\right] + \sum_{\substack{i=1 \\ i\neq j}}^{N} \lambda_{ij}\Delta t \tag{4.268}$$

The $N$th equation is obtained from $\sum_{j=1}^{N} p_j(t) = 1$. Hence, rearranging terms and letting $\Delta t \to 0$, we obtain

$$\frac{dp_j(t)}{dt} = \sum_{\substack{i=1 \\ i\neq j}}^{N} \lambda_{ij} p_i(t) - p_j(t) \sum_{\substack{k=1 \\ k\neq j}}^{N} \lambda_{jk} \tag{4.269}$$

Using (4.263), the set of equations for the $N$-state Markov chain is then given by

$$\left[\frac{dp_1(t)}{dt} \quad \frac{dp_2(t)}{dt} \quad \cdots \quad \frac{dp_N(t)}{dt}\right] = \left[p_1(t) \quad p_2(t) \quad \cdots \quad p_N(t)\right] \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2N} \\ \vdots & \vdots & & \vdots \\ \lambda_{N1} & \lambda_{N2} & \cdots & \lambda_{NN} \end{bmatrix} \tag{4.270}$$

In matrix form,

$$\boldsymbol{p}'(t) = \boldsymbol{p}(t)\boldsymbol{\Lambda} \tag{4.271a}$$

where

$$\boldsymbol{p}'(t) = \left[ \frac{dp_1(t)}{dt} \quad \frac{dp_2(t)}{dt} \quad \cdots \quad \frac{dp_N(t)}{dt} \right] \tag{4.271b}$$

$$\boldsymbol{p}(t) = \left[ p_1(t) \quad p_2(t) \quad \ldots \quad p_N(t) \right] \tag{4.271c}$$

and

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{N1} & \lambda_{2N} & \cdots & \lambda_{NN} \end{bmatrix} \tag{4.271d}$$

Solving the system of equations in (4.271), we obtain the steady state probabilities. If the Markov process $X(t)$ is stationary, then $p_j(t) = p_j =$ constant, and from (4.269) and (4.261), the set of equations to solve is

$$\begin{cases} p_j \lambda_{jj} = \displaystyle\sum_{\substack{i=1 \\ i \neq j}}^{N} \lambda_{ij} p_i \\ \displaystyle\sum_{j=1}^{N} p_j = 1 \end{cases} \tag{4.272}$$

*Birth-Death Process*

A *birth-death* process with intensities $\lambda_{i(i+1)}$ and $\lambda_{i(i-1)}$ is a Markov chain taking values 0, 1, 2, … , and having changes equal to $+1$ or $-1$, such that

$$\begin{cases} \lambda_{i(i+1)} = b_i \quad (\text{birth rate at } S_i, \text{or arrival rate}) \\ \lambda_{i(i-1)} = d_i \quad (\text{death rate at } S_i, \text{or departure rate}) \\ \lambda_{ij} = 0 \qquad j \neq i-1, i, i+1 \\ \lambda_{ii} = -b_i - d_i \end{cases} \tag{4.273}$$

The state diagram of this process is shown in Figure 4.9. Thus,

$$\begin{aligned} P[X(t+\Delta t) = n \mid X(t) = n-1] &= b_{n-1}\Delta t \\ P[X(t+\Delta t) = n \mid X(t) = n+1] &= d_{n+1}\Delta t \\ P[X(t+\Delta t) = n \mid X(t) = n] &= 1-(b_n + d_n)\Delta t \end{aligned} \tag{4.274}$$

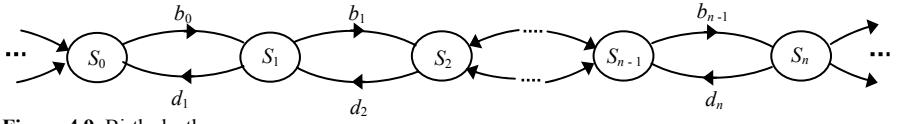


**Figure 4.9** Birth-death process.

Using the notation $P[X(t+\Delta t)=n]=p_n(t+\Delta t)$, and using (4.274), we have

$$p_n(t+\Delta t)=b_{n-1}\Delta t p_{n-1}(t)+p_{n+1}(t)d_{n+1}(t)\Delta t+[1-(b_n+d_n)\Delta t]p_n(t) \quad (4.275)$$

Since $p'_n(t)=\lim_{\Delta t\to 0}[p_n(t+\Delta t)-p_n(t)]/\Delta t$, then

$$\begin{cases} p'_n(t)=b_{n-1}p_{n-1}(t)-(b_n+d_n)p_n(t)+d_{n+1}p_{n+1}(t), & n\geq 1 \\ p'_0(t)=-b_0p_0(t)+d_1p_1(t) \quad , & n=0 \end{cases} \quad (4.276)$$

where we used the fact that $p_{-1}(t)=0$ and $d_0=0$. To determine the steady state probabilities, we set $p'_n(t)=0$ and solve the set of homogeneous equations

$$b_{n-1}p_{n-1}-(b_n+d_n)p_n+d_{n+1}p_{n+1}=0 \quad (4.277a)$$

$$-b_0p_0+d_1p_1=0 \quad (4.277b)$$

and using

$$\sum_{k=0}^{N}p_k=1 \quad (4.278)$$

Hence, from (4.277b)

$$p_1=\frac{b_0}{d_1}p_0 \quad (4.279)$$

For $n=1$,

$$b_0p_0-(b_1+d_1)p_1+d_2p_2=0 \quad (4.280)$$

Solving (4.280), we obtain

$$p_2 = \frac{b_1}{d_2} p_1 \tag{4.281}$$

Using the value of $p_1$ from (2.279), we obtain

$$p_2 = \frac{b_1 b_0}{d_2 d_1} p_0 \tag{4.282}$$

Continuing in this manner, we obtain the general form of $p_n$ to be

$$p_n = \frac{b_{n-1}}{d_{n-1}} p_{n-1} = \frac{b_0 b_1 \ldots b_{n-2} b_{n-1}}{d_1 d_2 \ldots d_{n-1} d_n} p_0 \tag{4.283}$$

From (4.278), $p_0 + \sum_{k=1}^{N} p_k = 1$, and using (4.283), we obtain

$$p_0 = \frac{1}{1 + \sum_{k=1}^{N} \prod_{\ell=0}^{k-1} \left( \frac{b_\ell}{d_{\ell+1}} \right)} \tag{4.284}$$

If we assume that the birth $b_n = \lambda$ and death $d_n = \mu$ are constants, then the system of equations to give the steady state probabilities is obtained from (4.277) to be

$$\begin{cases} \lambda p_{n-1} - (\lambda + \mu) p_n + \mu p_{n+1} = 0 \\ -\lambda p_0 + \mu p_1 = 0 \end{cases} \tag{4.285}$$

Solving the equations in (4.285) [or using (4.283)], we obtain

$$p_n = \left( \frac{\lambda}{\mu} \right)^n p_0 \tag{4.286a}$$

and using $p_0 + p_1 = 1$, we have

$$p_0 = \frac{\mu}{\mu + \lambda} \tag{4.286b}$$

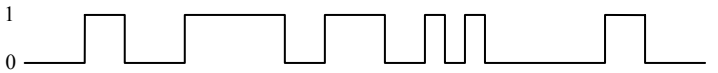


**Figure 4.10** Two-state random waveform.

## Example 4.7

Let $X(t)$ be a random waveform with two states, 0 and 1, as shown in Figure 4.10. The intensities of transitions from state 0 to state 1 and from state 1 to state 0 are $\lambda_{01}$ and $\lambda_{10}$, respectively. The probability to go from state $S_i$ to state $S_j$, $i, j = 0, 1$, is $\lambda_{ij}\Delta t$ . Determine

(a) $P_0(t)$ the probability that the system is in state $S_0$ for $t \geq 0$ .
(b) $P_0(t)$ and $P_1(t)$ if $P_0(0) = 1$.
(c) The steady values of $P_0$ and $P_1$.
(d) The probability that the first transition after a time $t$ will be from $S_1$ to $S_0$.
(e) The probability of a transition, $P(T)$ , occurring in $(t, t + \Delta t)$, $t$ large.

*Solution*

(a) For the simplicity of notation, let $\lambda_{01} = b$ and $\lambda_{10} = d$ . The state diagram of this system is shown in Figure 4.11. Using (4. 270), we have

$$\begin{bmatrix} p'_0(t) & p'_1(t) \end{bmatrix} = \begin{bmatrix} p_0(t) & p_1(t) \end{bmatrix} \begin{bmatrix} -b & b \\ d & -d \end{bmatrix}$$

or

$$\begin{cases} \dfrac{dp_0(t)}{dt} = dp_1(t) - bp_0(t) \\ p_0(t) + p_1(t) = 1 \end{cases}$$

Substituting for $p_1(t) = 1 - p_0(t)$ , we obtain the differential equation

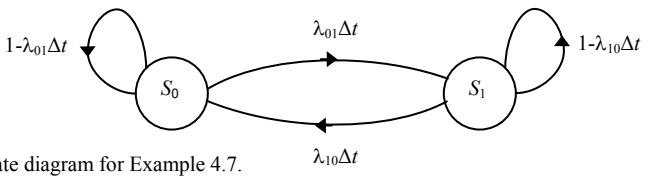


**Figure 4.11** State diagram for Example 4.7.

$$\frac{dp_0(t)}{dt} = -p_0(t)[d+b]+d$$

Solving the differential equation, we obtain the state distribution vector with

$$\begin{cases} p_0(t) = \left[ p_0(0) - \dfrac{d}{b+d} \right] e^{-(b+d)t} + \dfrac{d}{b+d}, & t \geq 0 \\ p_1(t) = 1 - p_0(t) \end{cases}$$

where $\boldsymbol{p}^T(0) = [p_0(0) \quad p_1(0)]$. A plot of $p_0(t)$ is given in Figure 4.12. We observe that this is the birth-death process.

(b) If $p_0(0) = 1$, then $p_1(0) = 1 - p_0(0) = 0$. Thus, after substitution we obtain

$$p_0(t) = \frac{b}{b+d} e^{-(b+d)t} + \frac{d}{b+d}$$
$$p_1(t) = \frac{-b}{b+d} e^{-(b+d)t} + \frac{d}{b+d}$$

(c) As *t* becomes large, the steady-state values are

$$\lim_{t\to\infty} p_0(t) = \frac{d}{b+d} = p_0 \text{ and } \lim_{t\to\infty} p_1(t) = \frac{b}{b+d} = p_1$$

Note that solving the birth-death process equations given by

$$\begin{cases} dp_1 - p_0 b = 0 \\ p_0 + p_1 = 1 \end{cases}$$

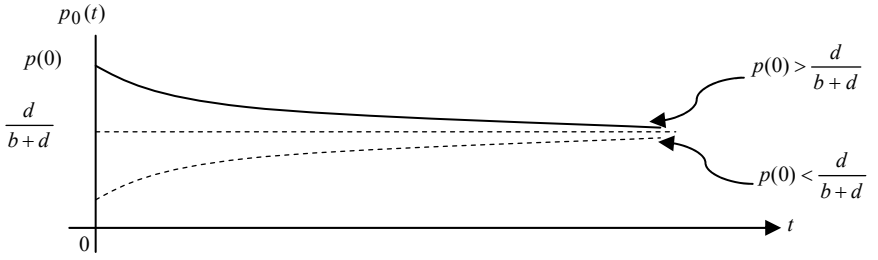


**Figure 4.12** Plot of $p_0(t)$.

with $\lambda = b$ and $\mu = d$ results in

$$p_0 = \frac{d}{b+d} \quad \text{and} \quad p_1 = \frac{b}{b+d}$$

which agrees with the result obtained using the limits.

(d) A transition from $S_1$ to $S_0$ means that the system is in state $S_1$. Hence, the probability is

$$p_1 = \frac{b}{b+d}$$

(e) The probability of a transition $T$ occurring is

$$P(T) = P(S_1 \mid S_0)P(S_0) + P(S_0 \mid S_1)P(S_1) = b\Delta t p_0 + d\Delta t p_1$$

where $P(S_j \mid S_i)$ is the probability of going from state $S_i$ to state $S_j$. For $t$ large, we are in steady state, and thus using the result of (c), we obtain

$$P(T) = \frac{2db\Delta t}{d+b}$$

## 4.6 SUMMARY

In this chapter, we presented discrete-time stochastic processes, namely, the AR, MA, and ARMA processes, and Markov chains. In order to understand the concepts of the AR, MA, and ARMA processes, we first gave a review of matrix operations and linear algebra. We showed how these processes are generated and how the parameters for the correlation functions and power spectrum are computed. Due to the importance of the autocorrelation function, we developed its properties in some detail. This is the realm of parametric spectrum estimation. Spectrum estimation can be very well developed, which is beyond the scope of this book. Then we presented Markov chains. We defined Markov chains and gave a few examples to illustrate the concepts introduced.

## PROBLEMS

**4.1** Find the eigenvalues, the eigenvectors, and the Jordan form by the similarity transformation of the matrix $\boldsymbol{A}$.

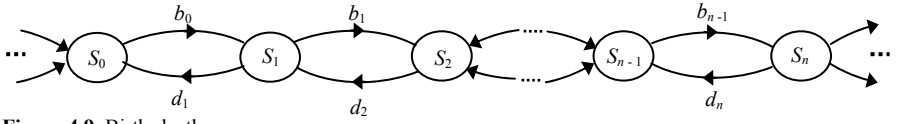


**Figure 4.9** Birth-death process.

Using the notation $P[X(t+\Delta t)=n]=p_n(t+\Delta t)$, and using (4.274), we have

$$p_n(t+\Delta t)=b_{n-1}\Delta t p_{n-1}(t)+p_{n+1}(t)d_{n+1}(t)\Delta t+[1-(b_n+d_n)\Delta t]p_n(t) \quad (4.275)$$

Since $p'_n(t)=\lim_{\Delta t\to 0}[p_n(t+\Delta t)-p_n(t)]/\Delta t$, then

$$\begin{cases} p'_n(t)=b_{n-1}p_{n-1}(t)-(b_n+d_n)p_n(t)+d_{n+1}p_{n+1}(t), & n\geq 1 \\ p'_0(t)=-b_0p_0(t)+d_1p_1(t) \quad , & n=0 \end{cases} \quad (4.276)$$

where we used the fact that $p_{-1}(t)=0$ and $d_0=0$. To determine the steady state probabilities, we set $p'_n(t)=0$ and solve the set of homogeneous equations

$$b_{n-1}p_{n-1}-(b_n+d_n)p_n+d_{n+1}p_{n+1}=0 \quad (4.277a)$$

$$-b_0p_0+d_1p_1=0 \quad (4.277b)$$

and using

$$\sum_{k=0}^{N}p_k=1 \quad (4.278)$$

Hence, from (4.277b)

$$p_1=\frac{b_0}{d_1}p_0 \quad (4.279)$$

For $n=1$,

$$b_0p_0-(b_1+d_1)p_1+d_2p_2=0 \quad (4.280)$$

Solving (4.280), we obtain

$$\boldsymbol{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

**4.8** Consider the second-order difference equation of the AR process.
(a) Obtain the characteristic equation and draw the model.
(b) Determine the possible poles, and specify the stability region in terms of the AR parameters.

**4.9** The autocorrelation function of the AR(2) process given in Problem 4.8 is

$$r_{xx}(k) + a_1 r_{xx}(k-1) + a_2 r_{xx}(k-2) = 0, \quad k > 0$$

(a) Use Yule-Walker equations to obtain the weights $\omega_1$ and $\omega_2$ in terms of the correlations $r_{xx}(0)$, $r_{xx}(1)$, and $r_{xx}(2)$.
(b) Obtain expressions for $r_{xx}(1)$ and $r_{xx}(2)$ in terms of the AR parameters $a_1$ and $a_2$.

**4.10** Consider the discrete-time Markov chain with the following transition matrix

$$\boldsymbol{P} = \begin{bmatrix} 1/3 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Draw the state diagram and classify the states.

**4.11** Suppose that a discrete communication source generates one of the three symbols, 1, 2, and 3. The generation of symbols obeys a homogeneous Markov chain, given by the following transition matrix,

$$\boldsymbol{P} = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.2 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

The initial distribution vector is $\boldsymbol{p}^T(0) = \begin{bmatrix} 0.3 & 0.3 & 0.4 \end{bmatrix}$

(a) Draw the state diagram.
(b) Determine the *n*-step transition matrix, *n* large.
(c) Determine the state probabilities after *n* steps.

**4.12** In the land of Oz , the weather changes a lot [1]. For example, if they have a nice (*N*) day, they may easily have rain (*R*) or snow (*S*) on the next day. Suppose that the weather can be modeled as a Markov chain, whose transition probability matrix is given by

$$\begin{array}{c} \\ R \\ N \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & N & S \end{array} \\ \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

(a) Draw the state diagram.
(b) Compute $\boldsymbol{P}(2)$, $\boldsymbol{P}(3)$, $\boldsymbol{P}(4)$, $\boldsymbol{P}(5)$, and $\boldsymbol{P}(6)$, and comment on the results.
(c) If $\boldsymbol{p}^T(0) = \begin{bmatrix} 0.7 & 0.2 & 0.1 \end{bmatrix}$, then find the steady state distribution vector.

**4.13** Consider a two-state discrete-time Markov chain with probability transition matrix

$$\boldsymbol{P}(1) = \boldsymbol{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

(a) Draw the state diagram.
(b) Verify by induction the limiting state probabilities given by [2]

$$\boldsymbol{P}(n) = \begin{bmatrix} \dfrac{b + a(1-a-b)^n}{a+b} & \dfrac{a - a(1-a-b)^n}{a+b} \\ \dfrac{b - a(1-a-b)^n}{a+b} & \dfrac{a + a(1-a-b)^n}{a+b} \end{bmatrix}$$

(c) Find the limiting state probabilities for the special cases when $a = b = 0$ and $a = b = 1$.

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[2] Shanmugan, K. S., and A. M. Breipohl, *Random Signals: Detection, Estimation and Data Analysis*, New York: John Wiley and Sons, 1988.

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# Chapter 5

# Statistical Decision Theory

## 5.1 INTRODUCTION

In our daily life, we are constantly making decisions. Given some hypotheses, a criterion is selected, upon which a decision has to be made. For example, in engineering, when there is a radar signal detection problem, the returned signal is observed and a decision is made as to whether a target is present or absent. In a digital communication system, a string of zeros and ones may be transmitted over some medium. At the receiver, the received signals representing the zeros and ones are corrupted in the medium by some additive noise and by the receiver noise. The receiver does not know which signal represents a zero and which signal represents a one, but must make a decision as to whether the received signals represent zeros or ones. The process that the receiver undertakes in selecting a decision rule falls under the theory of signal detection.

The situation above may be described by a source emitting two possible outputs at various instants of time. The outputs are referred to as *hypotheses*. The *null hypothesis* $H_0$ represents a zero (target not present) while the *alternate hypothesis* $H_1$ represents a one (target present), as shown in Figure 5.1.

Each hypothesis corresponds to one or more observations that are represented by random variables. Based on the observation values of these random variables, the receiver decides which hypothesis ($H_0$ or $H_1$) is true. Assume that the receiver is to make a decision based on a single observation of the received signal. The range of values that the random variable $Y$ takes constitutes the observation space $Z$. The observation space is partitioned into two regions $Z_0$ and $Z_1$, such that if $Y$

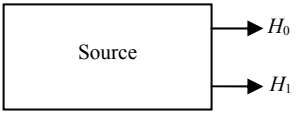


**Figure 5.1** Source for binary hypothesis.

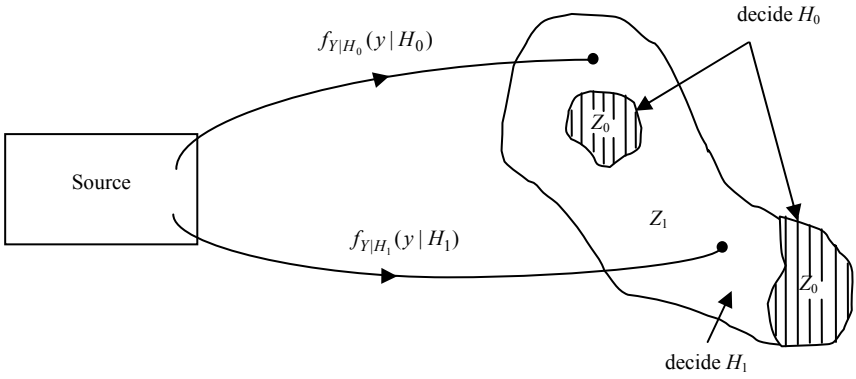


**Figure 5.2** Decision regions.

lies in $Z_0$, the receiver decides in favor of $H_0$; while if $Y$ lies in $Z_1$, the receiver decides in favor of $H_1$, as shown in Figure 5.2. The observation space $Z$ is the union of $Z_0$ and $Z_1$; that is,

$$Z = Z_0 \bigcup Z_1 \tag{5.1}$$

The probability density functions of $Y$ corresponding to each hypothesis are $f_{Y|H_0}(y \mid H_0)$ and $f_{Y|H_1}(y \mid H_1)$, where $y$ is a particular value of the random variable $Y$.

Each time a decision is made, based on some criterion, for this binary hypothesis testing problem, four possible cases can occur:

1. Decide $H_0$ when $H_0$ is true.
2. Decide $H_0$ when $H_1$ is true.
3. Decide $H_1$ when $H_0$ is true.
4. Decide $H_1$ when $H_1$ is true.

Observe that for cases (1) and (4), the receiver makes a correct decision, while for cases (2) and (3), the receiver makes an error. From radar nomenclature, case (2) is called *miss*, case (3) a *false alarm*, and case (4) a *detection*.

In this chapter, we develop the basic principles needed for solving decision problems. The observations are represented by random variables. Extension of these results to time-varying waveforms will be studied in later chapters.

In the next sections, we study some of the criteria that are used in decision theory, and the conditions under which these criteria are useful.

## 5.2 BAYES' CRITERION

### 5.2.1 Binary Hypothesis Testing

In using Bayes' criterion, two assumptions are made. First, the probability of occurrence of the two source outputs is known. They are the a priori probabilities $P(H_0)$ and $P(H_1)$. $P(H_0)$ is the probability of occurrence of hypothesis $H_0$, while $P(H_1)$ is the probability of occurrence of hypothesis $H_1$. Denoting the a priori probabilities $P(H_0)$ and $P(H_1)$ by $P_0$ and $P_1$ respectively, and since either hypothesis $H_0$ or $H_1$ will always occur, we have

$$P_0 + P_1 = 1 \tag{5.2}$$

The second assumption is that a cost is assigned to each possible decision. The cost is due to the fact that some action will be taken based on a decision made. The consequences of one decision are different from the consequences of another. For example, in a radar detection problem, the consequences of miss are not the same as the consequences of false alarm. If we let $D_i$, $i = 0, 1$, where $D_0$ denotes "decide $H_0$" and $D_1$ denotes "decide $H_1$," we can define $C_{ij}$, $i, j = 0, 1$, as the cost associated with the decision $D_i$, given that the true hypothesis is $H_j$. That is,

$$P(\text{incurring cost } C_{ij}) = P(\text{decide } D_i, H_j \text{ true}), \quad i, j = 0, 1 \tag{5.3}$$

In particular, the costs for this binary hypothesis testing problem are $C_{00}$ for case (1), $C_{01}$ for case (2), $C_{10}$ for case (3), and $C_{11}$ for case (4). The goal in Bayes' criterion is to determine the decision rule so that the average cost $E[C]$, also known as *risk* $\Re$, is minimized. The operation $E[C]$ denotes expected value. It is also assumed that the cost of making a wrong decision is greater than the cost of making a correct decision. That is,

$$C_{01} > C_{11} \tag{5.4a}$$

and

$$C_{10} > C_{00} \tag{5.4b}$$

Given $P(D_i, H_j)$, the joint probability that we decide $D_i$, and that the hypothesis $H_j$ is true, the average cost is

$$\Re = E[C] = C_{00}P(D_0, H_0) + C_{01}P(D_0, H_1) + C_{10}P(D_1, H_0) + C_{11}P(D_1, H_1) \quad (5.5)$$

From Bayes' rule, we have

$$P(D_i, H_j) = P(D_i \mid H_j)P(H_j) \quad (5.6)$$

The conditional density functions $P(D_i \mid H_j),\ i, j = 0, 1,$ in terms of the regions shown in Figure 5.2, are

$$P(D_0 \mid H_0) \equiv P(\text{decide } H_0 \mid H_0 \text{ true}) = \int_{Z_0} f_{Y|H_0}(y \mid H_0)dy \quad (5.7)$$

$$P(D_0 \mid H_1) \equiv P(\text{decide } H_0 \mid H_1 \text{ true}) = \int_{Z_0} f_{Y|H_1}(y \mid H_1)dy \quad (5.8)$$

$$P(D_1 \mid H_0) \equiv P(\text{decide } H_1 \mid H_0 \text{ true}) = \int_{Z_1} f_{Y|H_0}(y \mid H_0)dy \quad (5.9)$$

and

$$P(D_1 \mid H_1) \equiv P(\text{decide } H_1 \mid H_1 \text{ true}) = \int_{Z_1} f_{Y|H_1}(y \mid H_1)dy \quad (5.10)$$

The probabilities $P(D_0 \mid H_1)$, $P(D_1 \mid H_0)$, and $P(D_1, H_1)$ represent the *probability of miss*, $P_M$, the *probability of false alarm*, $P_F$, and the *probability of detection*, $P_D$, respectively. We also observe that

$$P_M = 1 - P_D \quad (5.11)$$

and

$$P(D_0 \mid H_0) = 1 - P_F \quad (5.12)$$

Consequently, the probability of a correct decision is given by

$$\begin{aligned} P(\text{correct decision}) &= P(c) = P(D_0, H_0) + P(D_1, H_1) \\ &= P(D_0 \mid H_0)P(H_0) + P(D_1 \mid H_1)P(H_1) \\ &= (1 - P_F)P_0 + P_D P_1 \end{aligned} \quad (5.13)$$

and the probability of error is given by

$$\begin{aligned} P(\text{error}) = P(\varepsilon) &= P(D_0, H_1) + P(D_1, H_0) \\ &= P(D_0 \mid H_1)P(H_1) + P(D_1 \mid H_0)P(H_0) \\ &= P_M P_1 + P_F P_0 \end{aligned} \tag{5.14}$$

A plot of the probability density function of the cost, $P_c(c)$, is illustrated in Figure 5.3. The average cost now becomes

$$\Re = E[C] = C_{00}(1-P_F)P_0 + C_{01}(1-P_D)P_1 + C_{10}P_F P_0 + C_{11}P_D P_1 \tag{5.15}$$

In terms of the decision regions defined in (5.7) to (5.9), the average cost is

$$\begin{aligned} \Re = {} & P_0 C_{00} \int_{Z_0} f_{Y|H_0}(y \mid H_0)dy + P_1 C_{01} \int_{Z_0} f_{Y|H_1}(y \mid H_1)dy \\ & + P_0 C_{10} \int_{Z_1} f_{Y|H_0}(y \mid H_0)dy + P_1 C_{11} \int_{Z_1} f_{Y|H_1}(y \mid H_1)dy \end{aligned} \tag{5.16}$$

Using (5.1) and the fact that

$$\int_Z f_{Y|H_0}(y \mid H_0)dy = \int_Z f_{Y|H_1}(y \mid H_1)dy = 1 \tag{5.17}$$

it follows that

$$\int_{Z_1} f_{Y|H_j}(y \mid H_j)dy = 1 - \int_{Z_0} f_{Y|H_j}(y \mid H_j)dy, \quad j = 0,1 \tag{5.18}$$

where $f_{Y|H_j}(y \mid H_j)$, $j = 0, 1$, is the probability density function of $Y$ corresponding to each hypothesis. Substituting for (5.18) in (5.16), we obtain

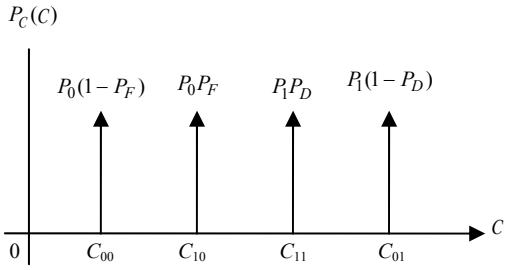


**Figure 5.3** Density function of cost.

$$\Re = P_0 C_{10} + P_1 C_{11} + \int_{Z_0} \{[P_1(C_{01} - C_{11}) f_{Y|H_1}(y \mid H_1)] - [P_0(C_{10} - C_{00}) f_{Y|H_0}(y \mid H_0)]\} dy \tag{5.19}$$

We observe that the quantity $P_0 C_{10} + P_1 C_{11}$ is constant, independent of how we assign points in the observation space, and that the only variable quantity is the region of integration $Z_0$. From (5.4a, b), the terms inside the brackets of (5.19) $\left[P_1(C_{01} - C_{11}) f_{Y|H_1}(y \mid H_1)\right.$ and $\left. P_0(C_{10} - C_{00}) f_{Y|H_0}(y \mid H_0)\right]$, are both positive. Consequently, the risk is minimized by selecting the decision region $Z_0$ to include only those points of $Y$ for which the second term is larger, and hence the integrand is negative. Specifically, we assign to the region $Z_0$ those points for which

$$P_1(C_{01} - C_{11}) f_{Y|H_1}(y \mid H_1) < P_0(C_{10} - C_{00}) f_{Y|H_0}(y \mid H_0) \tag{5.20}$$

All values for which the second term is greater will be excluded from $Z_0$ and assigned to $Z_1$. The values for which the two terms are equal do not affect the risk, and can be assigned to either $Z_0$ or $Z_1$. Consequently, we say if

$$P_1(C_{01} - C_{11}) f_{Y|H_1}(y \mid H_1) > P_0(C_{10} - C_{00}) f_{Y|H_0}(y \mid H_0) \tag{5.21}$$

then we decide $H_1$. Otherwise, we decide $H_0$. Hence, the decision rule resulting from the Bayes' criterion is

$$\frac{f_{Y|H_1}(y \mid H_1)}{f_{Y|H_0}(y \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_0(C_{10} - C_{00})}{P_1(C_{01} - C_{11})} \tag{5.22}$$

The ratio of $f_{Y|H_1}(y \mid H_1)$ over $f_{Y|H_0}(y \mid H_0)$ is called the *likelihood ratio* and is denoted $\Lambda(y)$. That is,

$$\Lambda(y) = \frac{f_{Y|H_1}(y \mid H_1)}{f_{Y|H_0}(y \mid H_0)} \tag{5.23}$$

It should be noted that if we have $K$ observations, for example, $K$ samples of a received waveform, $Y_1$, $Y_1$, … , $Y_k$, based on which we make the decision, the likelihood ratio can be expressed as

$$\Lambda(\boldsymbol{y}) = \frac{f_{\boldsymbol{Y}|H_1}(\boldsymbol{y} \mid H_1)}{f_{\boldsymbol{Y}|H_0}(\boldsymbol{y} \mid H_0)} \tag{5.24}$$

where $\boldsymbol{Y}$, the received vector, is

$$\boldsymbol{Y}^T = [Y_1 \quad Y_2 \quad \ldots \quad Y_K] \tag{5.25}$$

The *likelihood statistic* $\Lambda(\boldsymbol{Y})$ is a random variable since it is a function of the random variable $\boldsymbol{Y}$.

The threshold is

$$\eta = \frac{P_0(C_{10} - C_{00})}{P_1(C_{01} - C_{11})} \tag{5.26}$$

Therefore, Bayes' criterion, which minimizes the average cost, results in the *likelihood ratio test*

$$\Lambda(\boldsymbol{y}) \underset{H_0}{\overset{H_1}{\gtrless}} \eta \tag{5.27}$$

An important observation is that the likelihood ratio test is performed by simply processing the receiving vector to yield the likelihood ratio and comparing it with the threshold. Thus, in practical situations where the a priori probabilities and the cost may change, only the threshold changes, but the computation of likelihood ratio is not affected.

Because the natural logarithm is a monotonically increasing function as shown in Figure 5.4, and since the likelihood ratio $\Lambda(\boldsymbol{y})$ and the threshold $\eta$ are nonnegative, an equivalent decision rule to (5.27) is

$$\ln \Lambda(\boldsymbol{y}) \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta \tag{5.28}$$

We note that if we select the cost of an error to be one and the cost of a correct decision to be zero; that is,

$$C_{01} = C_{10} = 1 \tag{5.29a}$$

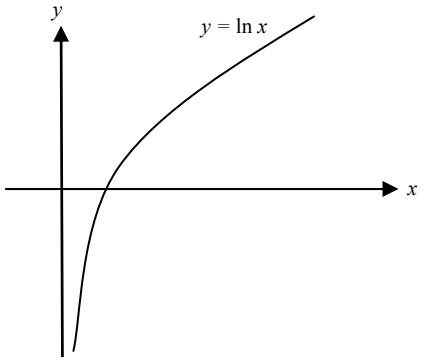


**Figure 5.4** Natural logarithmic function.

and

$$C_{00} = C_{11} = 0 \tag{5.29b}$$

then the risk function of (5.15) reduces to

$$\Re = P_M P_1 + P_F P_0 = P(\varepsilon) \tag{5.30}$$

Thus, in this case, minimizing the average cost is equivalent to minimizing the probability of error. Receivers for such cost assignment are called *minimum probability of error receivers*. The threshold reduces to

$$\eta = \frac{P_0}{P_1} \tag{5.31}$$

If the a priori probabilities are equal, $\eta$ is equal to one, and the log likelihood ratio test uses a zero threshold.

**Example 5.1**

In a digital communication system, consider a source whose output under hypothesis $H_1$ is a constant voltage of value $m$, while its output under $H_0$ is zero. The received signal is corrupted by $N$, an additive white Gaussian noise of zero mean, and variance $\sigma^2$.

(a) Set up the likelihood ratio test and determine the decision regions.
(b) Calculate the probability of false alarm and probability of detection.

*Solution*

(a) The received signals under each hypothesis are

$$H_1 : Y = m + N$$
$$H_0 : Y = \quad N$$

where the noise $N$ is Gaussian with zero mean and variance $\sigma^2$. Under hypothesis $H_0$,

$$f_{Y|H_0}(y \mid H_0) = f_N(n) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

Under hypothesis $H_1$, the mean of $Y$ is $E[Y] = E[m+N] = m$, since $E[N] = 0$. The variance of $Y$ is

$$\text{var}[Y] = \text{var}[m+N] = E[(m+N)^2] - \left(E[m+n]\right)^2 = E[N^2] = \sigma^2$$

Hence,

$$f_{Y|H_1}(y \mid H_1) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\frac{(y-m)^2}{\sigma^2}\right]$$

The likelihood ratio test is

$$\Lambda(y) = \frac{f_{Y|H_1}(y \mid H_1)}{f_{Y|H_0}(y \mid H_0)} = \exp\left(-\frac{m^2 - 2ym}{2\sigma^2}\right)$$

Taking the natural logarithm on both sides of the above equation, the likelihood ratio test becomes

$$\ln \Lambda(y) = \frac{m}{\sigma^2} y - \frac{m^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta$$

Rearranging terms, an equivalent test is

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{m} \ln \eta + \frac{m}{2} = \gamma$$

That is, the received observation is compared with the threshold $\gamma$. The decision regions are as shown in Figure 5.5.

(b) The probabilities of false alarm and detection are

$$P_F = P(\text{decide } H_1 \mid H_0 \text{ true}) = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy = Q\left(\frac{\gamma}{\sigma}\right) = \text{erfc}_*\left(\frac{\gamma}{\sigma}\right)$$

where

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

and denoted $\text{erfc}_*(\cdot)$ in some books.

$$P_D = P(\text{decide } H_1 \mid H_1 \text{ true}) = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-m)^2}{2\sigma^2}} dy = Q\left(\frac{\gamma - m}{\sigma}\right)$$

**Example 5.2**

Suppose that the receiver of Example 5.1 takes $K$ samples, $Y_1$, $Y_2$, … , $Y_K$. The noise samples are independent Gaussian random variables, each with mean zero and variance $\sigma^2$. Obtain the optimum decision rule.

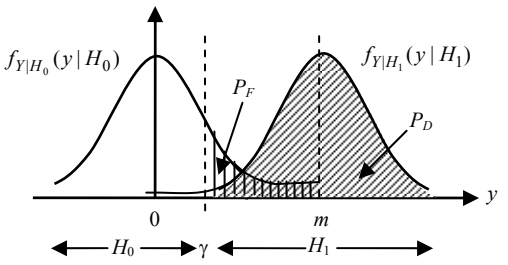


**Figure 5.5** Decision regions.

*Solution*

The received signal under hypothesis $H_0$ and $H_1$ is

$$H_1 : Y_k = m + N, \quad k = 1,2, \ldots, K$$
$$H_0 : Y_k = \quad N, \quad k = 1,2, \ldots, K$$

Under hypothesis $H_0$,

$$f_{Y_k|H_0}(y_k \mid H_0) = f_{N_k}(y_k) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y_k^2}{2\sigma^2}\right)$$

Under hypothesis $H_1$, the $k$th received sample is a Gaussian random variable with mean $m$ and variance $\sigma^2$. Thus,

$$f_{Y_k|H_1}(y_k \mid H_1) = f_{N_k}(y_k - m) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(y_k - m)^2}{2\sigma^2}\right]$$

From (5.24), we need $f_{\boldsymbol{Y}|H_1}(\boldsymbol{y} \mid H_1)$ and $f_{\boldsymbol{Y}|H_0}(\boldsymbol{y} \mid H_0)$. Since the noise samples are statistically independent, the joint density function of the $K$ samples is the product of the individual density functions. This yields

$$f_{\boldsymbol{Y}|H_0}(\boldsymbol{y} \mid H_0) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y_k^2}{2\sigma^2}} \quad \text{and} \quad f_{\boldsymbol{Y}|H_1}(\boldsymbol{y} \mid H_1) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y_k - m)^2}{2\sigma^2}}$$

where $\prod$ denotes product. Using the fact that $\prod_k e^{x_k} = e^{\sum_k x_k}$, the likelihood ratio test is

$$\Lambda(\boldsymbol{y}) = \exp\left[\sum_{k=1}^{K} \frac{y_k^2}{2\sigma^2} - \sum_{k=1}^{K} \frac{(y_k - m)^2}{2\sigma^2}\right] = \exp\left[\frac{m}{\sigma^2} \sum_{k=1}^{K} y_k - \frac{Km^2}{2\sigma^2}\right]$$

Taking the natural logarithm of both sides, the likelihood ratio test becomes

$$\ln \Lambda(\boldsymbol{y}) = \frac{m}{\sigma^2} \sum_{k=1}^{K} y_k - \frac{Km^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta$$

Rearranging terms, an equivalent test is

$$\sum_{k=1}^{K} y_k \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{m} \ln \eta + \frac{Km}{2}$$

That is, the receiver adds the $K$ samples and compares them to the threshold $\frac{\sigma^2}{m} \ln \eta + \frac{Km}{2}$.

*Sufficient Statistic*

A statistic is any random variable that can be computed from observed data. Let $T$ be the value of a statistic given by $T = t(x)$. Let $T'$ be the value of another statistic, with $T$ and $T'$ having a joint density function given by $f(x, y \mid \theta)$. Then,

$$f(x, y \mid \theta) = f_T(x \mid \theta) f_{T'}(y \mid x, \theta) \tag{5.32}$$

where $f_T(x \mid \theta)$ is the probability density function of $T$, and $f_{T'}(y \mid x, \theta)$ is the conditional density function of $T'$, given $T = x$. Note that in (5.32), we have used the fact that $P(A \text{ and } B) = P(A)P(B \mid A)$. Assume the conditional density function $f_{T'}(y \mid x, \theta)$ does not involve θ. Then, if $T$ is known, the conditional density function of $T'$ does not depend on θ, and $T'$ is not relevant in the decision making problem. This can be shown to be the case of all $T'$ for all data. Consequently, $T$ summarizes all data of the experiment relevant to θ, and is called a *sufficient statistic*. From Example 5.2, we observe that only knowledge of the sum $\sum_{k=1}^{K} y_k$ is relevant in making a decision about $\boldsymbol{Y}$. Hence, $T(\boldsymbol{Y}) = \sum_{k=1}^{K} Y_k$ is a sufficient statistic.

**Example 5.3**

Consider the situation where the samples $Y_1, Y_2, \ldots, Y_K$ are independent random variables, each having a Bernoulli distribution with parameter $p$. Assume that the test statistic is

$$T(\boldsymbol{Y}) = \sum_{k=1}^{K} Y_K$$

Is $T(\boldsymbol{Y})$ a sufficient statistic?

*Solution*

From (2.1), a random variable $Y$ is said to have a Bernoulli distribution with parameter $p$ if

$$f_Y(y, p) = p^y (1-p)^{1-y}, \quad y = 0, 1$$

where $0 \le p \le 1$. Since the random variables $Y_1, Y_2, \ldots, Y_K$ are statistically independent, the joint density function is given by

$$\begin{aligned} f_{\boldsymbol{Y}}(\boldsymbol{y}, p) &= [p^{y_1}(1-p)^{1-y_1}][p^{y_2}(1-p)^{1-y_2}]\ldots[p^{y_K}(1-p)^{1-y_K}] \\ &= p^{\sum_{k=1}^{K} y_k} (1-p)^{K-\sum_{k=1}^{K} y_k} \end{aligned}$$

That is, the joint density function of the sample values does not involve the parameter $p$, and depends only on the sum $T(\boldsymbol{y}) = \sum_{k=1}^{K} y_k$. Hence, $T(\boldsymbol{Y}) = \sum_{k=1}^{K} Y_k$ is a sufficient statistic.

**Example 5.4**

Consider the problem where the conditional density functions under each hypothesis are

$$f_{Y|H_0}(y \mid H_0) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma_0^2}\right) \quad \text{and} \quad f_{Y|H_1}(y \mid H_1) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma_1^2}\right)$$

where $\sigma_1^2 > \sigma_0^2$.

(a) Determine the decision rule.
(b) Assuming we have $K$ independent observations, what would the decision rule be?

*Solution*

(a) Applying the likelihood ratio test given in (5.23), we obtain

$$\Lambda(y)=\frac{\dfrac{1}{\sqrt{2\pi}\sigma_1}\exp\left(-\dfrac{y^2}{2\sigma_1^2}\right)}{\dfrac{1}{\sqrt{2\pi}\sigma_0}\exp\left(-\dfrac{y^2}{2\sigma_0^2}\right)}\underset{H_0}{\overset{H_1}{\gtrless}}\eta \quad \text{or} \quad \frac{\sigma_0}{\sigma_1}\exp\left[\frac{y^2}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\right]\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

Taking the logarithm on both sides, we have

$$\ln\frac{\sigma_0}{\sigma_1}+\frac{y^2}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta \quad \text{or} \quad y^2\underset{H_0}{\overset{H_1}{\gtrless}}\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\frac{\eta\sigma_0}{\sigma_1}=\gamma$$

$T(Y)=Y^2$ is the sufficient statistic, and hence the test can be written as

$$T(y)\underset{H_0}{\overset{H_1}{\gtrless}}\gamma$$

(b) Since the random variables $Y_1, Y_2, \ldots, Y_K$ are independent, the joint density function is simply the product of the individual densities. That is,

$$f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\mid H_0)=\prod_{k=1}^{K}\frac{1}{\sqrt{2\pi}\sigma_0}e^{-\frac{y_k^2}{2\sigma_0^2}} \quad \text{and} \quad f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\mid H_1)=\prod_{k=1}^{K}\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{y_k^2}{2\sigma_1^2}}$$

Substituting in (5.24) and taking the logarithm, we have

$$\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_{k=1}^{K}y_k^2+K\ln\frac{\sigma_0}{\sigma_1}\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta \quad \text{or} \quad \sum_{k=1}^{K}y_k^2\underset{H_0}{\overset{H_1}{\gtrless}}\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\left(\ln\eta-K\ln\frac{\sigma_0}{\sigma_1}\right)=\gamma$$

The sufficient statistic is $T(\boldsymbol{Y})=\sum_{k=1}^{K}Y_k^2$, and the test can be written as

$$T(\boldsymbol{y}) = \sum_{k=1}^{K} y_k^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

Note that if $\sigma_1^2 < \sigma_0^2$, then $\sigma_1^2 - \sigma_0^2$ is negative and the inequality is reversed; that is,

$$T(\boldsymbol{y}) \underset{H_1}{\overset{H_0}{\gtrless}} \gamma$$

### 5.2.2 *M*-ary Hypothesis Testing

In the previous section, we considered the choice between two hypotheses, $H_0$ and $H_1$. We now consider the choice of one hypothesis among $M$ hypotheses, $H_0, H_1, \ldots, H_{M-1}$, each time an experiment is conducted. Since any one of the $M$ decisions can be made, there are $M^2$ possible alternatives. Bayes' criterion assigns a cost to each alternative. To the $ij$th alternative, which is the decision $D_i$ given hypothesis $H_j$, the cost $C_{ij}$, $i, j = 0, 1, \ldots, (M-1)$, is assigned. In addition to the hypotheses $H_0, H_1, \ldots, H_{M-1}$, we assign the a priori probabilities $P_0, P_1, \ldots, P_{M-1}$, respectively. The goal is to minimize the risk defined as

$$\Re = \sum_{i=1}^{M-1} \sum_{j=1}^{M-1} P_j C_{ij} P(D_i \mid H_j) \tag{5.33}$$

Using the fact that

$$P(D_i \mid H_j) = \int_{Z_i} f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) d\boldsymbol{y} \tag{5.34}$$

the average cost becomes

$$\Re = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} P_j C_{ij} \int_{Z_i} f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) d\boldsymbol{y} \tag{5.35}$$

The observation space $Z$ is now divided into $M$ subspaces, $Z_0, Z_1, \ldots, Z_{M-1}$, such that

$$Z = Z_0 \cup Z_1 \cup \ldots \cup Z_{M-1} \tag{5.36}$$

In order to find the decision surfaces so that $\Re$ is minimized, we rewrite (5.35) as

$$\Re = \sum_{i=0}^{M-1} \sum_{\substack{j=0 \\ j \neq i}}^{M-1} P_j C_{ij} \int_{Z_i} f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) d\boldsymbol{y} + \sum_{i=1}^{M-1} P_i C_{ii} \int_{Z_i} f_{\boldsymbol{Y}|H_i}(\boldsymbol{y} \mid H_i) d\boldsymbol{y} \tag{5.37}$$

because $\int_Z f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) d\boldsymbol{y} = 1$ ,and the surface $Z_i = Z - \bigcup_{\substack{j=0 \\ j \neq i}}^{M-1} Z_j$ . Substituting in (5.37), the risk becomes

$$\Re = \sum_{i=1}^{M-1} \int_{Z_i} \sum_{\substack{j=0 \\ j \neq i}}^{M-1} P_j \left(C_{ij} - C_{jj}\right) f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) d\boldsymbol{y} + \sum_{i=0}^{M-1} P_i C_{ii} \tag{5.38}$$

Using the same reasoning as before, we observe that the second term of (5.38) is fixed, while the first term determines the cost for the selected decision regions. Hence, the small integral value yields selection of the hypothesis for which

$$I_i(\boldsymbol{y}) = \sum_{\substack{j=0 \\ j \neq i}}^{M-1} P_j \left(C_{ij} - C_{jj}\right) f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) \tag{5.39}$$

is minimum.

Defining the likelihood ratio $\Lambda_i(\boldsymbol{y}),\ i = 1, 2, \ldots, M-1$, as

$$\Lambda_i(\boldsymbol{y}) = \frac{f_{\boldsymbol{Y}|H_i}(\boldsymbol{y} \mid H_i)}{f_{\boldsymbol{Y}|H_0}(\boldsymbol{y} \mid H_0)},\ i = 1, 2, \ldots, M-1 \tag{5.40}$$

and the term $J_i(\boldsymbol{y})$ as

$$J_i(\boldsymbol{y}) = \frac{I_i(\boldsymbol{y})}{f_{\boldsymbol{Y}|H_0}(\boldsymbol{y} \mid H_0)} = \sum_{j=1}^{M-1} P_j (C_{ij} - C_{jj}) \Lambda_j(\boldsymbol{y}) \tag{5.41}$$

the decision rule is to choose the hypothesis for which (5.41) is minimum.

*MAP Criterion*

In communication problems, it is common to have the costs

$$C_{ii} = 0, \quad i = 1, 2, \ldots, M-1 \tag{5.42a}$$

and

$$C_{ij} = 1, \quad i \neq j \text{ and } \quad i, j = 0, 1, \ldots, M-1 \tag{5.42b}$$

In this case, minimizing the risk is equivalent to minimizing the probability of error. After substitution, (5.41) becomes

$$I_i(\boldsymbol{y}) = \sum_{j=0}^{M-1} P(H_j) f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) = \sum_{\substack{j=0 \\ i \neq j}}^{M-1} P(H_j \mid \boldsymbol{Y}) f_{\boldsymbol{Y}}(\boldsymbol{y}) = [1 - P(H_i \mid \boldsymbol{Y})] f_{\boldsymbol{Y}}(\boldsymbol{y}) \tag{5.43}$$

Minimizing (5.43) is equivalent to maximizing $P(H_i \mid \boldsymbol{Y})$, which is the a posteriori probability of hypothesis $H_i$, given the observation vector $\boldsymbol{y}$. If in addition, the a priori probabilities are equal

$$P(H_1) = P(H_2) = \ \ldots \ = P(H_{M-1}) = p \tag{5.44}$$

Equation (5.43) becomes

$$I_i(\boldsymbol{y}) = p \sum_{\substack{j=0 \\ j \neq i}}^{M-1} f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) = p[1 - P(H_i \mid \boldsymbol{Y})] \tag{5.45}$$

We observe that in this decision rule, the receiver computes the a posteriori probabilities $P(H_i \mid \boldsymbol{Y}), i = 0, 1, 2, \ldots, M-1$, and decides in favor of the hypothesis corresponding to the largest a posteriori probability. Such a minimum probability of error receiver is also referred to as the Maximum a posteriori probability (MAP) receiver.

For simplicity, let $M = 3$ and the observation space be $Z = Z_0 \cup Z_1 \cup Z_2$. From (5.35), we obtain

$$\begin{aligned}\Re = {} & P_0C_{00}\int_{Z_0} f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\mid H_0)d\boldsymbol{y} + P_0C_{10}\int_{Z_1} f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\mid H_0)d\boldsymbol{y} \\ & + P_0C_{20}\int_{Z_2} f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\mid H_0)d\boldsymbol{y} + P_1C_{11}\int_{Z_1} f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\mid H_1)d\boldsymbol{y} \\ & + P_1C_{01}\int_{Z_0} f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\mid H_1)d\boldsymbol{y} + P_1C_{21}\int_{Z_2} f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\mid H_1)d\boldsymbol{y} \\ & + P_2C_{22}\int_{Z_2} f_{\boldsymbol{Y}|H_2}(\boldsymbol{y}\mid H_2)d\boldsymbol{y} + P_2C_{02}\int_{Z_0} f_{\boldsymbol{Y}|H_2}(\boldsymbol{y}\mid H_2)d\boldsymbol{y} \\ & + P_2C_{12}\int_{Z_1} f_{\boldsymbol{Y}|H_2}(\boldsymbol{y}\mid H_2)d\boldsymbol{y} \end{aligned} \tag{5.46}$$

Note that

$$\begin{aligned}\int_{Z_i} f_{Y|H_i}(y\mid H_i)dy &= \int_{Z-\sum_{\substack{j\\ j\neq i}} Z_j} f_{\boldsymbol{Y}|H_i}(\boldsymbol{y}\mid H_i)d\boldsymbol{y} \\ &= 1-\sum_{\substack{j\\ j\neq i}}\int_{Z_j} f_{\boldsymbol{Y}|H_i}(\boldsymbol{y}\mid H_i)d\boldsymbol{y}, \quad i,j=0,1,2 \end{aligned} \tag{5.47}$$

Substituting (5.47) for the terms involving $C_{00}, C_{11}$, and $C_{22}$ in (5.46), we obtain

$$\begin{aligned}\Re = {} & P_0C_{00} + P_1C_{11} + P_2C_{22} \\ & + \int_{Z_0}[P_2(C_{02}-C_{22})f_{\boldsymbol{Y}|H_2}(\boldsymbol{y}\mid H_2) + P_1(C_{01}-C_{11})f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\mid H_1)]\,d\boldsymbol{y} \\ & + \int_{Z_1}[P_0(C_{10}-C_{00})f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\mid H_0) + P_2(C_{12}-C_{22})f_{\boldsymbol{Y}|H_2}(\boldsymbol{y}\mid H_2)]\,d\boldsymbol{y} \\ & + \int_{Z_2}[P_0(C_{20}-C_{00})f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\mid H_0) + P_1(C_{21}-C_{11})f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\mid H_1)]\,d\boldsymbol{y} \end{aligned} \tag{5.48}$$

We define $I_0(\boldsymbol{y}), I_1(\boldsymbol{y}),$ and $I_2(\boldsymbol{y})$ as

$$I_0(\boldsymbol{y}) = P_2(C_{02}-C_{22})f_{\boldsymbol{Y}|H_2}(\boldsymbol{y}\mid H_2) + P_1(C_{01}-C_{11})f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\mid H_1) \tag{5.49a}$$

$$I_1(\boldsymbol{y}) = P_0(C_{10}-C_{00})f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\mid H_0) + P_2(C_{12}-C_{22})f_{\boldsymbol{Y}|H_2}(\boldsymbol{y}\mid H_2) \tag{5.49b}$$

and

$$I_2(\boldsymbol{y}) = P_0(C_{20}-C_{00})f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\mid H_0) + P_1(C_{21}-C_{11})f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\mid H_1) \tag{5.49c}$$

To minimize $\Re$, we assign values of $Y$ to the region having the smallest integrands in (5.48), since $I_0(\boldsymbol{y}), I_1(\boldsymbol{y}),$ and $I_2(\boldsymbol{y})$ are nonnegative. Consequently,

$$Z_0 = \{\boldsymbol{y} \mid I_0(\boldsymbol{y}) < I_1(\boldsymbol{y}) \text{ and } I_2(\boldsymbol{y})\}$$

$$Z_1 = \{\boldsymbol{y} \mid I_1(\boldsymbol{y}) < I_0(\boldsymbol{y}) \text{ and } I_2(\boldsymbol{y})\}$$

and

$$Z_2 = \{\boldsymbol{y} \mid I_2(\boldsymbol{y}) < I_0(\boldsymbol{y}) \text{ and } I_1(\boldsymbol{y})\}$$

where | denotes "such that." From (5.40), the likelihood ratios $\Lambda_1(\boldsymbol{y})$ and $\Lambda_2(\boldsymbol{y})$ are

$$\Lambda_1(\boldsymbol{y}) = \frac{f_{\boldsymbol{Y}|H_1}(\boldsymbol{y} \mid H_1)}{f_{\boldsymbol{Y}|H_0}(\boldsymbol{y} \mid H_0)} \tag{5.50a}$$

and

$$\Lambda_2(\boldsymbol{y}) = \frac{f_{\boldsymbol{Y}|H_2}(\boldsymbol{y} \mid H_2)}{f_{\boldsymbol{Y}|H_0}(\boldsymbol{y} \mid H_0)} \tag{5.50b}$$

In order to incorporate the likelihood ratios into the decision rule, we use the following equivalent test

$$I_0(\boldsymbol{y}) \underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{\gtrless}} I_1(\boldsymbol{y}) \tag{5.51a}$$

$$I_0(\boldsymbol{y}) \underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\gtrless}} I_2(\boldsymbol{y}) \tag{5.51b}$$

and

$$I_1(\boldsymbol{y}) \underset{H_0 \text{ or } H_1}{\overset{H_0 \text{ or } H_2}{\gtrless}} I_2(\boldsymbol{y}) \tag{5.51c}$$

Substituting (5.49) and (5.50) into (5.51), we obtain the test

$$P_1(C_{01}-C_{11})\Lambda_1(\boldsymbol{y}) \underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{\gtrless}} P_0(C_{10}-C_{00})+P_2(C_{12}-C_{02})\Lambda_2(\boldsymbol{y}) \quad (5.52a)$$

$$P_2(C_{02}-C_{22})\Lambda_2(\boldsymbol{y}) \underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\gtrless}} P_0(C_{20}-C_{00})+P_1(C_{21}-C_{01})\Lambda_1(\boldsymbol{y}) \quad (5.52b)$$

and

$$P_2(C_{12}-C_{22})\Lambda_2(\boldsymbol{y}) \underset{H_0 \text{ or } H_1}{\overset{H_0 \text{ or } H_2}{\gtrless}} P_0(C_{20}-C_{10})+P_1(C_{21}-C_{11})\Lambda_1(\boldsymbol{y}) \quad (5.52c)$$

Because $M=3$, there are only two likelihood ratios and the decision space is two-dimensional, as shown in Figure 5.6.

For the costs

$$C_{00}=C_{11}=C_{22}=0 \quad (5.53a)$$

$$C_{ij}=1, \quad i \neq j \quad (5.53b)$$

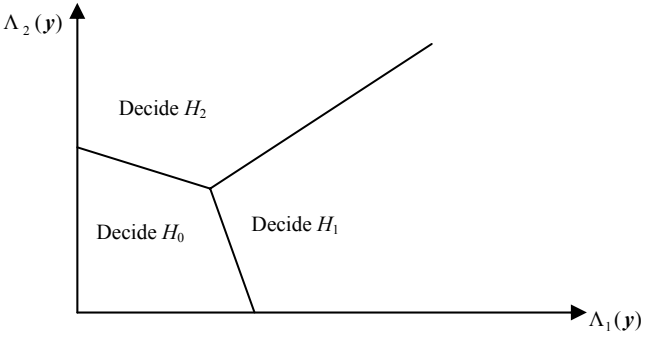


**Figure 5.6** Decision space for $M=3$.

It is easier to observe, in this case of $M = 3$, that minimizing the risk is equivalent to minimizing the probability of error, and the decision rule reduces to

$$\Lambda_1(\boldsymbol{y}) \underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{\gtrless}} \frac{P_0}{P_1} \tag{5.54a}$$

$$\Lambda_2(\boldsymbol{y}) \underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\gtrless}} \frac{P_0}{P_2} \tag{5.54b}$$

and

$$\Lambda_2(\boldsymbol{y}) \underset{H_0 \text{ or } H_1}{\overset{H_0 \text{ or } H_2}{\gtrless}} \frac{P_1}{P_2}\Lambda_1(\boldsymbol{y}) \tag{5.54c}$$

The resulting decision regions are shown in Figure 5.7(a). The overall decision space is given in Figure 5.7(b).

Taking the logarithm of both sides of (5.54a–c), we obtain

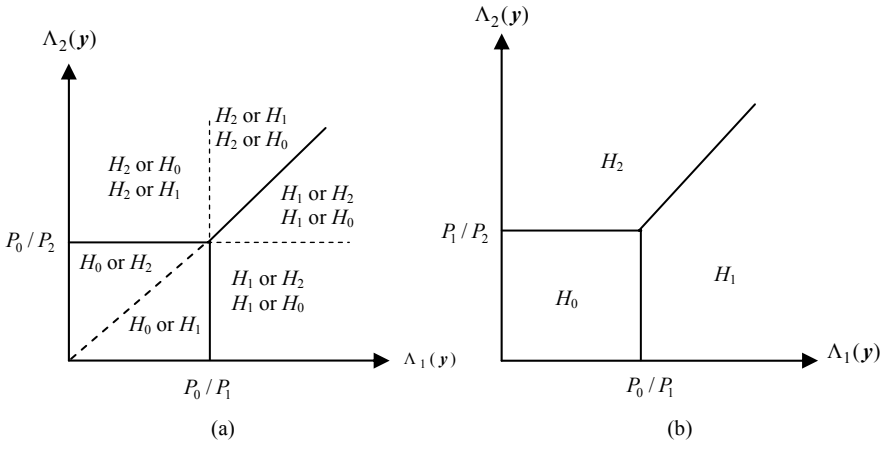


**Figure 5.7** Decision space for $M = 3$: (a) resulting decision regions and (b) overall decision space.

$$\ln \Lambda_1(\boldsymbol{y}) \underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{\gtrless}} \ln \frac{P_0}{P_1} \tag{5.55a}$$

$$\ln \Lambda_2(\boldsymbol{y}) \underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\gtrless}} \ln \frac{P_0}{P_2} \tag{5.55b}$$

and

$$\ln \Lambda_2(\boldsymbol{y}) \underset{H_0 \text{ or } H_1}{\overset{H_0 \text{ or } H_2}{\gtrless}} \ln \frac{P_1}{P_2} \Lambda_1(\boldsymbol{y}) \tag{5.55c}$$

The decision space in the $\ln \Lambda_1(y) - \ln \Lambda_2(y) -$ plane is shown in Figure 5.8. We observe that the decision space now consists of the entire plane.

Furthermore, substituting (5.50) in (5.54), dividing by $f_{\boldsymbol{Y}}(\boldsymbol{y})$ and using $P(A \mid B)P(B),$ we obtain the following decision rule

$$P(H_1 \mid \boldsymbol{y}) \underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{\gtrless}} P(H_0 \mid \boldsymbol{y}) \tag{5.56a}$$

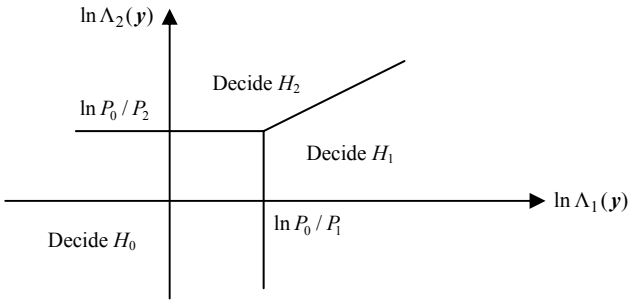


**Figure 5.8** Decision space using logarithm for $M = 3$.

$$P(H_2 \mid \boldsymbol{y}) \underset{H_0 \text{ or } H_1}{\overset{H_1 \text{ or } H_2}{\gtrless}} P(H_0 \mid \boldsymbol{y}) \tag{5.56b}$$

and

$$P(H_2 \mid \boldsymbol{y}) \underset{H_0 \text{ or } H_1}{\overset{H_0 \text{ or } H_2}{\gtrless}} P(H_1 \mid \boldsymbol{y}) \tag{5.56c}$$

Hence, this form shows clearly that the decision amounts to computing the a posteriori probabilities $P(H_0 \mid \boldsymbol{y}), P(H_1 \mid \boldsymbol{y}),$ and $P(H_2 \mid \boldsymbol{y})$, and then selecting the hypothesis corresponding to the largest.

**Example 5.5**

A ternary communication system transmits one of the three amplitude signals $\{1, 2, 3\}$ with equal probabilities. The independent received signal samples under each hypothesis are

$$\begin{aligned} H_1 &: Y_k = 1 + N, \quad k = 1, 2, \ldots, K \\ H_2 &: Y_k = 2 + N, \quad k = 1, 2, \ldots, K \\ H_3 &: Y_k = 3 + N, \quad k = 1, 2, \ldots, K \end{aligned}$$

The additive noise $N$ is Gaussian with mean zero and variance $\sigma^2$. The costs are $C_{ii} = 0$ and $C_{ij} = 1$ for $i \neq j, \quad i, j = 1, 2, 3$. Determine the decision regions.

*Solution*

Since the observation samples are independent, the conditional density function of the observation $\boldsymbol{Y}$ under each hypothesis $H_j, \ j = 1, 2, 3$ is

$$\begin{aligned} f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) &= \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[ -\frac{1}{2\sigma^2} (y_k - m_j)^2 \right] \\ &= \frac{1}{(2\pi\sigma^2)^{K/2}} \exp\left[ -\frac{1}{2\sigma^2} \sum_{k=1}^{K} (y_k - m_j)^2 \right] \end{aligned}$$

$$= \frac{1}{(2\pi\sigma^2)^{K/2}} \exp\left[ -\frac{1}{2\sigma^2} \sum_{k=1}^{K} y_k^2 + \frac{1}{2\sigma^2} \sum_{k=1}^{K} \left( 2y_k m_j - m_j^2 \right) \right]$$

The decision rule is to choose the hypothesis for which $f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j)$ is maximum. Rewriting $f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j)$

$$f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) = \frac{1}{(2\pi\sigma^2)^{K/2}} \exp\left[ -\frac{1}{2\sigma^2} \sum_{k=1}^{K} y_k^2 \right] \exp\left[ \frac{1}{2\sigma^2} \sum_{k=1}^{K} \left( 2y_k m_j - m_j^2 \right) \right]$$

we observe that we choose the hypothesis $H_j$, for which

$$\sum_{k=1}^{K} \left( 2y_k m_j - m_j^2 \right) = \frac{2}{K} \sum_{k=1}^{K} y_k m_j - m_j^2$$

is maximum. That is, we choose the maximum of

$$\frac{2}{K} \sum_{k=1}^{K} 2y_k - 1, \qquad \frac{4}{K} \sum_{k=1}^{K} y_k - 4, \quad \text{and} \quad \frac{6}{K} \sum_{k=1}^{K} y_k - 9$$

where the means $m_1 = 1, m_2 = 2$, and $m_3 = 3$ correspond to hypotheses $H_1$, $H_2$, and $H_3$, respectively. If

$$\frac{2}{K} \sum_{k=1}^{K} y_k - 1 > \frac{4}{K} \sum_{k=1}^{K} y_k - 4$$

we choose $H_1$ for the region

$$\frac{1}{K} \sum_{k=1}^{K} y_k < \frac{3}{2}$$

and so on for all terms. We observe that the test statistic is $T(\boldsymbol{Y}) = (1/K) \sum_{k=1}^{K} Y_k$, which is just the sample mean. Hence, the decision regions are given by

$$\begin{aligned} T(\boldsymbol{y}) &\le 1.5 \quad \text{choose } H_1 \\ 1.5 < T(\boldsymbol{y}) &\le 2.5 \quad \text{choose } H_2 \\ T(\boldsymbol{y}) &> 2.5 \quad \text{choose } H_3 \end{aligned}$$

A plot of conditional density functions showing the decision regions is shown in Figure 5.9.

## 5.3 MINIMAX CRITERION

The Bayes' criterion assigns costs to decisions and assumes knowledge of the a priori probabilities. In many situations, we may not have enough information about the a priori probabilities and consequently, the Bayes' criterion cannot be used. One approach would be to select a value of $P_1$, the a priori probability of $H_1$, for which the risk is maximum, and then minimize that risk function. This principle of minimizing the maximum average cost for the selected $P_1$ is referred to as *minimax criterion*.

From (5.2), we have

$$P_0 = 1 - P_1 \tag{5.57}$$

substituting (5.2) in (5.15), we obtain the risk function in terms of $P_1$ as

$$\Re = C_{00}(1-P_F) + C_{10}P_F + P_1[(C_{11}-C_{00}) + (C_{01}-C_{11})P_M - (C_{10}-C_{00})P_F] \tag{5.58}$$

Assuming a fixed value of $P_1$, $P_1 \in [0,1]$, we can design a Bayes' test. These decision regions are then determined, as are the probabilities of false alarm, $P_F$, and miss, $P_M$. The test results in

$$\Lambda(y) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(1-P_1)(C_{10}-C_{00})}{P_1(C_{01}-C_{11})} \tag{5.59}$$

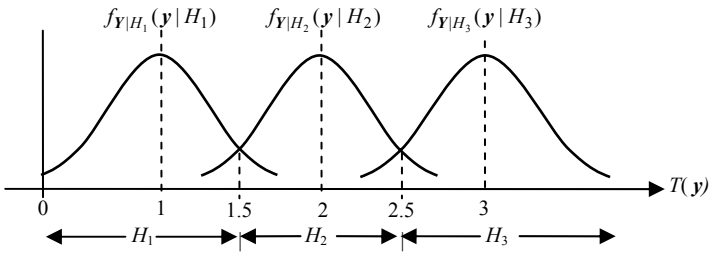


**Figure 5.9** Conditional density functions and decision regions.

As $P_1$ varies, the decision regions change, resulting in a nonoptimum decision rule. This in turn causes a variation in the average cost, which would be larger than the Bayes' costs. The two extreme cases are when $P_1$ is zero or one. If $P_1$ is zero, then the threshold is infinity and the decision rule is

$$\Lambda(y) \underset{H_0}{\overset{H_1}{\gtrless}} \infty \tag{5.60}$$

$H_0$ is always true. The observation is $Z_0$, and the resulting probability of false alarm and probability of miss are

$$P_F = \int_{Z_1} f_{Y|H_0}(y \mid H_0)dy = 0 \tag{5.61}$$

and

$$P_M = \int_{Z_0} f_{Y|H_1}(y \mid H_1)dy = 1 \tag{5.62}$$

Substituting for the values of $P_1, P_F,$ and $P_M$ in (5.58), we obtain that the risk is

$$\Re = C_{00} \tag{5.63}$$

Similarly, when $P_1 = 1$, the threshold of (5.59) is zero and the new decision rule is

$$\Lambda(y) \underset{H_0}{\overset{H_1}{\gtrless}} 0 \tag{5.64}$$

Since $\Lambda(y)$ is nonnegative, we always decide $H_1$. Hence, $P_F = 1$ and $P_M = 0$. The resulting risk is

$$\Re = C_{11} \tag{5.65}$$

If $P_1 = P_1^*$ such that $P_1^* \in (0,1),$ then the risk as a function of $P_1$ is as shown in Figure 5.10. From (5.58), we see that the risk $\Re$ is linear in terms of $P_1$, and the Bayes' test for $P_1 = P_1^*$ gives the minimum risk $\Re_{\min}$. The tangent to $\Re_{\min}$ is

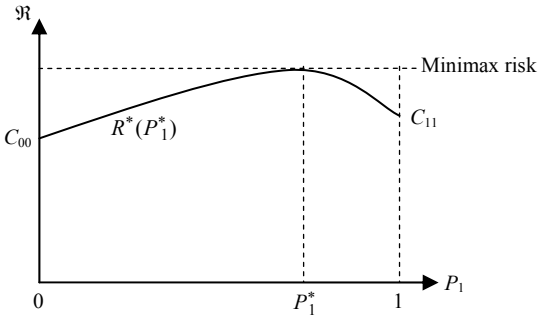


**Figure 5.10** Risk as function of $P_1$.

horizontal, and $\Re^*(P_1)$ at $P_1 = P_1^*$ represents the maximum cost. Observe that the Bayes' curve must be concave downward. Thus, the average cost will not exceed $\Re^*(P_1^*)$. Taking the derivative of $\Re$ with respect to $P_1$ and setting it equal to zero, we obtain the *minimax equation* to be

$$(C_{11} - C_{00}) + (C_{01} - C_{11})P_M - (C_{10} - C_{00})P_F = 0 \tag{5.66}$$

If the cost of a correct decision is zero $(C_{00} = C_{11} = 0)$, then the minimax equation for $P_1 = P_1^*$ reduces to

$$C_{01}P_M = C_{10}P_F \tag{5.67}$$

Furthermore, if the cost of a wrong decision is one $(C_{01} = C_{10} = 1)$, then the probability of false alarm equals the probability of miss. That is,

$$P_F = P_M \tag{5.68}$$

and the minimax cost is

$$\Re = P_F(1 - P_1) + P_1P_M = P_0P_F + P_1P_M \tag{5.69}$$

which is the average probability of error.

**Example 5.6**

Consider the problem of Example 5.1. Calculate the minimum probability of error when:

(a) $P_0 = P_1$.

(b) $P_0$ and $P_1$ are unknown.

*Solution*

(a) From Example 5.1, we found that the decision rule is

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{m} \ln \eta + \frac{m}{2} = \gamma$$

Given $P_0 = P_1 = 1/2$, the probability of error is $P(\varepsilon) = (1/2)(P_F + P_M)$, where

$$P_F = Q\left(\frac{\gamma}{\sigma}\right) = \text{erfc}_*\left(\frac{\gamma}{\sigma}\right)$$

and

$$P_M = 1 - P_D = 1 - Q\left(\frac{\gamma - m}{\sigma}\right) = Q\left(\frac{m - \gamma}{\sigma}\right)$$

(b) In this case, the optimum threshold $\gamma^*$ is obtained when $P_F = P_M$ as given in (5.68). Hence,

$$Q\left(\frac{\gamma^*}{\sigma}\right) = Q\left(\frac{m - \gamma^*}{\sigma}\right)$$

or the threshold $\gamma^*$ is $\gamma^* = m/2$. Consequently, the average probability of error is

$$P(\varepsilon) = P_0 P_F + P_1 P_M = (P_0 + P_1) P_M = Q\left(\frac{m}{2\sigma}\right) = \text{erfc}_*\left(\frac{m}{2\sigma}\right)$$

In order to compare the results of (b) and (a), we normalize the standard deviation of the observation in (a) to one. Let $y' = y/\sigma$, and since $\eta = 1$, the decision rule becomes

$$y' \underset{H_0}{\overset{H_1}{\gtrless}} \frac{m}{2\sigma} = \gamma$$

Let $\alpha = m/\sigma$, and the decision rule reduces to

$$y' \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\alpha}{2}$$

The probability of false alarm and probability of detection are given by

$$P_F = \int_{\alpha/2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y'^2}{2}} dy' = Q\left(\frac{\alpha}{2}\right)$$

$$P_D = \int_{\alpha/2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y'-\alpha)^2}{2}} dy' = Q\left(\frac{\alpha}{2} - \alpha\right) = Q\left(-\frac{\alpha}{2}\right)$$

and thus $P_M = 1 - P_D = 1 - Q(-\alpha/2) = Q(\alpha/2)$. The average probability of error is $P(\varepsilon) = (1/2)\left[Q(\alpha/2) + Q(\alpha/2)\right] = Q(\alpha/2) = Q(m/2\sigma)$. Therefore, both results obtained in (a) and (b) are the same.

## 5.4 NEYMAN-PEARSON CRITERION

In the previous sections, we have seen that for the Bayes' criterion we require knowledge of the a priori probabilities and cost assignments for each possible decision. Then we have studied the minimax criterion, which is useful in situations where knowledge of the a priori probabilities is not possible. In many other physical situations, such as radar detection, it is very difficult to assign realistic costs and a priori probabilities. To overcome this difficulty, we use the conditional probabilities of false alarm, $P_F$, and detection $P_D$. The *Neyman-Pearson* test requires that $P_F$ be fixed to some value $\alpha$ while $P_D$ is maximized. Since $P_M = 1 - P_D$, maximizing $P_D$ is equivalent to minimizing $P_M$.

In order to minimize $P_M$ (maximize $P_D$) subject to the constraint that $P_F = \alpha$, we use the calculus of extrema, and form the objective function $J$ to be

$$J = P_M + \lambda(P_F - \alpha) \tag{5.70}$$

where $\lambda\,(\lambda \geq 0)$ is the *Lagrange multiplier*. We note that given the observation space $Z$, there are many decision regions $Z_1$ for which $P_F = \alpha$. The question is to determine those decision regions for which $P_M$ is minimum. Consequently, we rewrite the objective function $J$ in terms of the decision region to obtain

$$J = \int_{Z_0} f_{Y|H_1}(y \mid H_1)dy + \lambda\left[\int_{Z_1} f_{Y|H_0}(y \mid H_0)dy - \alpha\right] \tag{5.71}$$

Using (5.1), (5.71) can be rewritten as

$$\begin{aligned} J &= \int_{Z_0} f_{Y|H_1}(y \mid H_1)dy + \lambda\left[\int_{Z_0} f_{Y|H_0}(y \mid H_0)dy - \alpha\right] \\ &= \lambda(1-\alpha) + \int_{Z_0} [f_{Y|H_1}(y \mid H_1) - \lambda f_{Y|H_0}(y \mid H_0)]dy \end{aligned} \tag{5.72}$$

Hence, $J$ is minimized when values for which $f_{Y|H_1}(y \mid H_1) > f_{Y|H_0}(y \mid H_0)$ are assigned to the decision region $Z_1$. The decision rule is, therefore,

$$\Lambda(y) = \frac{f_{Y|H_1}(y \mid H_1)}{f_{Y|H_0}(y \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda \tag{5.73}$$

The threshold $\eta$ derived from the Bayes' criterion is equivalent to $\lambda$, the Lagrange multiplier in the Neyman-Pearson (*N-P*) test for which the probability of false alarm is fixed to the value $\alpha$. If we define the conditional density of $\Lambda$ given that $H_0$ is true as $f_{\Lambda|H_0}(\lambda \mid H_0)$, then $P_F = \alpha$ may be rewritten as

$$P_F = \int_{Z_1} f_{Y|H_0}(y \mid H_0)dy = \int_{\lambda}^{\infty} f_{\Lambda(y)|H_0}[\lambda(y) \mid H_0]d\lambda \tag{5.74}$$

The test is called *most powerful of level* $\alpha$ if its probability of rejecting $H_0$ is $\alpha$.

**Example 5.7**

Consider the binary hypothesis problem with received conditional probabilities

$$f_{Y|H_0}(y \mid H_0) = \frac{1}{2(1-e^{-1})} e^{-|y|} \text{ for } |y| \le 1 \text{ and } f_{Y|H_1}(y \mid H_1) = \frac{1}{2}\text{rect}\left(\frac{1}{2}\right)$$

The hypotheses $H_0$ and $H_1$ are equally likely.

(a) Find the decision regions for which the probability of error is minimum.
(b) Calculate the minimum probability of error.
(c) Find the decision rule based on the Neyman-Pearson criterion, such that the probability of false alarm is constrained to be $P_F = 0.5$.
(d) Calculate the probability of detection for the given constraint of $P_F$ in (b).

*Solution*

(a) The minimum probability of error receiver requires that $C_{00} = C_{11} = 0$ and $C_{01} = C_{10} = 1$. Since the a priori probabilities are equal, the likelihood ratio test reduces to

$$\Lambda(y) = \frac{f_{Y|H_1}(y \mid H_1)}{f_{Y|H_0}(y \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

That is, we choose the hypothesis for which $f_{Y|H_j}(y \mid H_j), j = 0, 1,$ is maximum. The decision regions are as shown in Figure 5.11.

Note that we decide $H_1$ for $-1 \le y \le -0.459$ and $0.459 \le y \le 1$, and we decide $H_0$ for $-0.459 < y < 0.459$.

(b) The probability of error is $P(\varepsilon) = P_0 P_F + P_1 P_M$, where

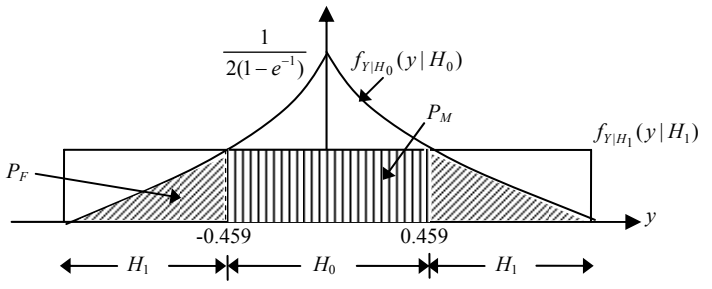


**Figure 5.11** Decision regions for Example 5.7.

$$P_F = P(\text{decide } H_1 \mid H_0 \text{ true}) = \frac{1}{2(1-e^{-1})}\left(\int_{-1}^{-0.459} e^{y}dy + \int_{0.459}^{1} e^{-y}dy\right) = 0.418$$

and $P_M = P(\text{decide } H_0 \mid H_1 \text{ true}) = 2\left[(0.459)(1/2)\right] = 0.459$. Thus, the probability of error is $P(\varepsilon) = (1/2)(0.418 + 0.459) = 0.4385$.

(c) In using the Neyman-Pearson criterion, we have

$$\Lambda(y) = \frac{\frac{1}{2}}{\frac{1}{2(1-e^{-1})}e^{-|y|}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Thus,

$$\frac{1-e^{-1}}{e^{-|y|}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \qquad \text{or} \qquad |y| \underset{H_0}{\overset{H_1}{\gtrless}} -\ln\frac{1-e^{-1}}{\eta} = \gamma$$

$P_F$ is as shown in Figure 5.12(a). Hence, $P_F = P(D_1 \mid H_0)$

$$= \frac{1}{2(1-e^{-1})}\left(\int_{-1}^{-\gamma} e^{y}dy + \int_{-\gamma}^{1} e^{-y}dy\right) = 0.5 \Rightarrow \gamma = 0.38 \text{ is the threshold.}$$

(d) The probability of detection, as shown in Figure 5.12(b), is $P_D = 2\left[(1-0.38)(1/2)\right] = 0.62$.

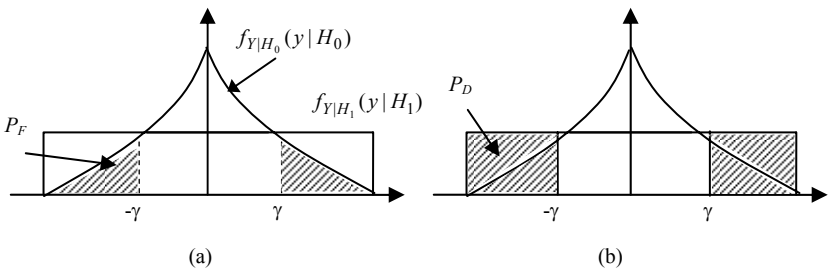


**Figure 5.12** Regions showing: (a) $P_F$ and (b) $P_D$.

*Receiver Operating Characteristic*

A plot of the probability of detection, $P_D$, versus the probability of false alarm with the threshold as a parameter is referred to as receiver operating characteristic (ROC) curves. We note that the ROC depends on the conditional density function of the observed signal under each hypothesis, that is, $f_{Y|H_j}(y \mid H_j), j = 0, 1,$ and not on the assigned costs, or the a priori probabilities. We shall explain the concept of the ROC through an example. From Example 5.2, the decision rule was shown to be

$$T(\boldsymbol{y}) = \sum_{k=1}^{K} y_k \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{m} \ln \eta + \frac{Km}{2}$$

We observe that the sufficient statistic $T(\boldsymbol{Y})$ is Gaussian. Calculating the mean and variance of the sufficient statistic under each hypothesis, we obtain

$$E[T(\boldsymbol{Y}) \mid H_0] = E\left[\sum_{k=1}^{K} Y_k \mid H_0\right] = 0$$

$$\text{var}[T(\boldsymbol{Y}) \mid H_0] = \text{var}\left[\sum_{k=1}^{K} Y_k \mid H_0\right] = K\sigma^2$$

$$E[T(\boldsymbol{Y}) \mid H_1] = E\left[\sum_{k=1}^{K} Y_k \mid H_1\right] = Km$$

and

$$\text{var}[T(\boldsymbol{Y}) \mid H_1] = \text{var}\left[\sum_{k=1}^{K} Y_k \mid H_1\right] = K\sigma^2$$

Hence, to obtain a unit variance under each hypothesis, we need to normalize the test statistic by $\sqrt{K}\sigma$ to yield $\text{var}[T(\boldsymbol{Y}) \mid H_1] = \text{var}[T(\boldsymbol{Y}) \mid H_0] = 1$, $E[T(\boldsymbol{Y}) \mid H_0] = 1$, and $E[T(\boldsymbol{Y}) \mid H_1] = \sqrt{K}m/\sigma$. For the variance of $T(\boldsymbol{Y})$ under $H_0$ equal to one, the distance between the two means is defined as

$$d \triangleq m_1 - m_0 \tag{5.75}$$

where $m_0$ and $m_1$ are the means under hypothesis $H_0$ and $H_1$, respectively. That is, $d = \sqrt{K}m/\sigma$. It should be noted that

$$d^2 = \frac{Km^2}{\sigma^2} = \frac{K^2m^2}{K\sigma^2} = \frac{S_0}{N_0}$$

can be thought of as a signal-to-noise ratio, where the signal power is $S_0 = K^2m^2$ and the noise power is $N_0 = K\sigma^2$. The conditional density functions of the statistic under hypotheses $H_0$ and $H_1$ are

$$f_{T|H_0}(t \mid H_0) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \quad \text{and} \quad f_{T|H_1}(t \mid H_1) = \frac{1}{\sqrt{2\pi}} e^{-(t-d)^2/2}$$

The decision rule becomes

$$T(\boldsymbol{y}) = \frac{1}{\sqrt{K}\sigma} \sum_{k=1}^{K} y_k \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln \eta}{d} + \frac{d}{2}$$

The probabilities of false alarm and detection are

$$P_F = \int_{\frac{\ln\eta}{d}+\frac{d}{2}}^{\infty} f_{T|H_0}(t \mid H_0)dt = Q\left(\frac{\ln\eta}{d} + \frac{d}{2}\right)$$

and

$$P_D = \int_{\frac{\ln\eta}{d}+\frac{d}{2}}^{\infty} f_{T|H_1}(t \mid H_1)dt = Q\left(\frac{\ln\eta}{d} - \frac{d}{2}\right)$$

The conditional density functions $f_{Y|H_j}(y \mid H_j)$, $j = 0, 1,$ and the probabilities of detection and false alarm are as shown in Figure 5.13.

Varying the threshold $\gamma$, the areas representing $P_D$ and $P_F$ vary. The corresponding ROC curves are shown in Figure 5.14. We observe that as $d$ increases, the probability of detection increases for a given probability of false alarm. However, the threshold remains constant for a fixed $P_F$ even as $d$ increases. Thus, $d$ gives a measure of the hypothesis testing, and therefore it is also called the *detection parameter*.

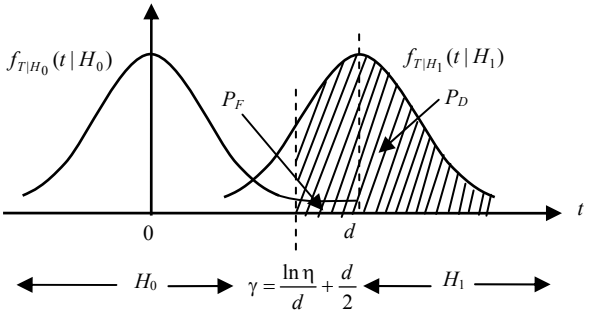


**Figure 5.13** Decision regions showing $P_D$ and $P_F$.

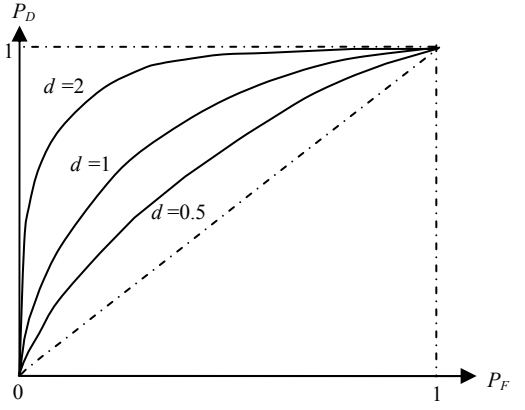


**Figure 5.14** ROC with $d$ as a parameter.

The two extreme points on the ROC for $P_F = P_D = 1$ and $P_F = P_D = 0$ are easily verified. Since both the Neyman-Pearson receiver and the Bayes' receiver employ the likelihood ratio test, and since $\Lambda(y)$ is a random variable, $P_D$ and $P_F$ may be rewritten as

$$P_D = P(\text{decide } H_1 \mid H_1 \text{ true}) = \int_{\eta}^{\infty} f_{\Lambda|H_1}(\lambda \mid H_1) d\lambda \tag{5.76}$$

and

$$P_F = P(\text{decide } H_1 \mid H_0 \text{ true}) = \int_{\eta}^{\infty} f_{\Lambda|H_0}(\lambda \mid H_0) d\lambda \tag{5.77}$$

$\Lambda(y)$ is a ratio of two negative quantities, $f_{Y|H_1}(y \mid H_1)$ and $f_{Y|H_0}(y \mid H_0)$, and thus takes values from zero to infinity. When the threshold $\eta$ is zero ($\eta = 0$ corresponds to $P_0 = 0$), hypothesis $H_1$ is always true, and thus $P_D = P_F = 1$. When the threshold $\eta$ is infinity ($\eta \to \infty$ corresponds to $P_1 = 0$), hypothesis $H_0$ is always true, and thus $P_D = P_F = 0$. This is clearly depicted in Figure 5.14.

The slope of the ROC at a particular point on the curve represents the threshold $\eta$ for the Neyman-Pearson test to achieve $P_D$ and $P_F$ at that point. Taking the derivative of (5.76) and (5.77) with respect to $\eta$, we have

$$\frac{dP_D}{d\eta} = \frac{d}{d\eta}\int_{\eta}^{\infty} f_{\Lambda|H_1}(\lambda \mid H_1)d\lambda = -f_{\Lambda|H_1}(\eta \mid H_1) \tag{5.78}$$

and

$$\frac{dP_F}{d\eta} = \frac{d}{d\eta}\int_{\eta}^{\infty} f_{\Lambda|H_0}(\lambda \mid H_0)d\lambda = -f_{\Lambda|H_0}(\eta \mid H_0) \tag{5.79}$$

Also,

$$P_D(\eta) = P[\Lambda(y) \geq \eta \mid H_1] = \int_{\eta}^{\infty} f_{\Lambda|H_1}[\lambda(y) \mid H_1]d\lambda = \int_{\eta}^{\infty} \Lambda(y) f_{\Lambda|H_0}[\lambda(y) \mid H_0]d\lambda \tag{5.80}$$

Taking the derivative of the above equation with respect to $\eta$, we obtain

$$\frac{dP_D}{d\eta} = -\eta f_{\Lambda|H_1}(\eta \mid H_0) \tag{5.81}$$

Combining (5.78), (5.79), and (5.80) results in

$$\frac{f_{\Lambda|H_1}(\eta \mid H_1)}{f_{\Lambda|H_0}(\eta \mid H_0)} = \eta \tag{5.82}$$

and

$$\frac{dP_D}{DP_F} = \eta \tag{5.83}$$

In the Bayes' criterion, the threshold $\eta$ is determined by the a priori probabilities and costs. Consequently, the probability of detection, $P_D$, and the probability of false alarm, $P_F$, are determined on the point of the ROC curve at which the tangent has a slope of $\eta$.

The minimax equation represents a straight line in the $P_D - P_F$ plane starting at the point $P_D = 0$ and $P_F = 1$, and crosses the ROC curve. The slope of the tangent of the intersection with the ROC is the threshold $\eta$.

**Example 5.8**

Consider a problem with the following conditional density functions

$$f_{Y|H_0}(y \mid H_0) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{Y|H_1}(y \mid H_1) = \begin{cases} \alpha e^{-\alpha y}, & y \geq 0, \alpha > 1 \\ 0, & \text{otherwise} \end{cases}$$

Plot the ROC.

*Solution*

The ROC is a plot of $P_D$, the probability of detection, versus $P_F$, the probability of false alarm, with the threshold $\eta$ as a parameter. The likelihood ratio is

$$\Lambda(y) = \frac{\alpha e^{-\alpha y}}{e^{-y}} = \alpha e^{-(\alpha-1)y} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Taking the logarithm and rearranging terms, the decision rule becomes

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{1-\alpha} \ln \frac{\eta}{\alpha} = \gamma$$

From the Neyman-Pearson test, the probability of detection and probability of false alarm are

$$P_D = P(D_1 \mid H_1) = \int_0^{\gamma} \alpha e^{-\alpha y} dy = 1 - e^{-\alpha\gamma} \quad \text{and} \quad P_F = P(D_1 \mid H_0) = \int_0^{\gamma} e^{-y} dy = 1 - e^{-\gamma}$$

Note that taking the derivative of $P_D$ and $P_F$ with respect to the threshold $\gamma$, and substituting in (5.83), we obtain the threshold $\eta$; that is,

$$\frac{dP_D}{dP_F} = \gamma e^{-(\alpha-1)\gamma} = \eta$$

A plot of the ROC with $\alpha$ as a parameter is shown in Figure 5.15.

## 5.5 COMPOSITE HYPOTHESIS TESTING

In the simple hypothesis testing problem previously considered, the parameters characterizing a hypothesis were all known. In many situations, the parameters characterizing a hypothesis may not be known. In this case, the hypothesis is called a *composite hypothesis.*

**Example 5.9**

Consider the situation where the observations under each hypothesis are given by

$$H_1 : Y = m + N$$
$$H_0 : Y = \quad N$$

where $N$ denotes a white Gaussian noise of zero mean and variance $\sigma^2$, and $m$ is unknown. Then, we say that $H_0$ is a simple hypothesis, and $H_1$ a composite hypothesis.

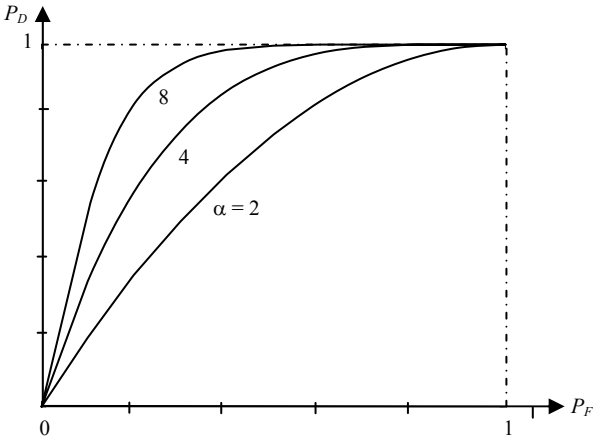


**Figure 5.15** ROC of Example 5.8.


**Figure 5.16** Block diagram showing composite hypothesis.

In the previous sections, we developed the theory of designing good tests for simple hypotheses. We now consider tests for composite hypotheses. The situation may be best described by the following block diagram of Figure 5.16.

Each hypothesis is characterized by a set of *K* parameters, such that

$$\boldsymbol{\theta}^T = [\theta_1 \quad \theta_2 \quad \ldots \quad \theta_K] \tag{5.84}$$

Two cases will be considered. First, $\boldsymbol{\Theta}$ may be a random variable with known density functions $f_{\boldsymbol{\Theta}|H_1}(\boldsymbol{\theta} \mid H_1)$ and $f_{\boldsymbol{\Theta}|H_0}(\boldsymbol{\theta} \mid H_0)$. Second, $\theta$ may not be a random variable but still an unknown constant.

### 5.5.1 $\boldsymbol{\Theta}$ Random Variable

If $\boldsymbol{\Theta}$ is a random variable with known density functions, $f_{\boldsymbol{\Theta}|H_1}(\boldsymbol{\theta} \mid H_1)$ and $f_{\boldsymbol{\Theta}|H_0}(\boldsymbol{\theta} \mid H_0)$, then the decision is obtained by using the Bayes' criterion and minimizing the risk. The analysis is as before. In order to apply the likelihood ratio test, we need $f_{\boldsymbol{Y}|H_1}(\boldsymbol{y} \mid H_1)$ and $f_{\boldsymbol{Y}|H_0}(\boldsymbol{y} \mid H_0)$. They are readily obtained by averaging over all possible values of $\boldsymbol{\Theta}$. That is,

$$f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) = \int f_{\boldsymbol{Y}|\boldsymbol{\Theta},H_j}(\boldsymbol{y} \mid \boldsymbol{\theta}, H_j) f_{\boldsymbol{\Theta}|H_j}(\boldsymbol{\theta} \mid H_j) d\boldsymbol{\theta}, \quad j = 0, 1 \tag{5.85}$$

The likelihood ratio becomes

$$\Lambda(\boldsymbol{y}) = \frac{f_{\boldsymbol{Y}|H_1}(\boldsymbol{y} \mid H_1)}{f_{\boldsymbol{Y}|H_0}(\boldsymbol{y} \mid H_0)} = \frac{\int f_{\boldsymbol{Y}|\boldsymbol{\Theta},H_1}(\boldsymbol{y} \mid \boldsymbol{\theta}, H_1) f_{\boldsymbol{\Theta}|H_1}(\boldsymbol{\theta} \mid H_1) d\boldsymbol{\theta}}{\int f_{\boldsymbol{Y}|\boldsymbol{\Theta},H_0}(\boldsymbol{y} \mid \boldsymbol{\theta}, H_0) f_{\boldsymbol{\Theta}|H_0}(\boldsymbol{\theta} \mid H_0) d\boldsymbol{\theta}} \tag{5.86}$$

**Example 5.10**

Consider the problem of Example 5.9, where the constant *m*, now denoted *M*, is a Gaussian random variable with mean zero and variance $\sigma_m^2$. Determine the optimum decision rule.

*Solution*

Using (5.86), the optimum decision rule can be directly obtained from the likelihood ratio test. Hence,

$$\Lambda(y) = \frac{\int_{-\infty}^{\infty} f_{Y|M,H_1}(y \mid m, H_1) f_{M|H_1}(m \mid H_1) dm}{f_{Y|H_0}(y \mid H_0)}$$

Note that only $H_1$ is a composite hypothesis, and consequently the numerator of $\Lambda(y)$ is integrated over $M$. Since the actual value of $M$ is not important, $M$ is referred to as the "unwanted parameter." The numerator of $\Lambda(y)$, denoted $N(y)$, is

$$\begin{aligned} N(y) &= \frac{1}{2\pi\sigma\sigma_m} \int_{-\infty}^{\infty} \exp\left[ -\frac{(y-m)^2}{2\sigma^2} - \frac{m^2}{2\sigma_m^2} \right] dm \\ &= \frac{1}{2\pi\sigma\sigma_m} \int_{-\infty}^{\infty} \exp\left[ -\frac{\sigma_m^2 + \sigma^2}{2\sigma^2\sigma_m^2} \left( m^2 - \frac{2\sigma_m^2 y}{\sigma_m^2 + \sigma^2} m \right) - \frac{y^2}{2\sigma^2} \right] dm \end{aligned}$$

Completing the square in the exponent, $N(y)$ becomes

$$\begin{aligned} N(y) &= \frac{1}{2\pi\sigma\sigma_m} \int_{-\infty}^{\infty} \exp\left[ -\frac{\sigma_m^2 + \sigma^2}{2\sigma^2\sigma_m^2} \left( m - \frac{\sigma_m^2 y}{\sigma_m^2 + \sigma^2} \right)^2 - \frac{\sigma_m^2 y^2}{2\sigma^2(\sigma_m^2 + \sigma^2)} - \frac{y^2}{2\sigma^2} \right] dm \\ &= \frac{1}{2\pi\sigma\sigma_m} \exp\left[ -\frac{y^2}{2(\sigma_m^2 + \sigma^2)} \right] \int_{-\infty}^{\infty} \exp\left[ -\frac{\sigma_m^2 + \sigma^2}{2\sigma^2\sigma_m^2} \left( m - \frac{\sigma_m^2 y}{\sigma_m^2 + \sigma^2} \right)^2 \right] dm \end{aligned}$$

Because the integral

$$\int_{-\infty}^{\infty} \exp\left[ -\frac{\sigma_m^2 + \sigma^2}{2\sigma^2\sigma_m^2} \left( m - \frac{\sigma_m^2 y}{\sigma_m^2 + \sigma^2} \right)^2 \right] dm = \sqrt{2\pi} \frac{\sigma\sigma_m}{\sqrt{\sigma_m^2 + \sigma}}$$

$N(y)$ becomes

$$f_{Y|H_1}(Y \mid H_1) = \frac{1}{\sqrt{\pi}\sqrt{\sigma_m^2 + \sigma^2}} \exp\left[ -\frac{y^2}{2(\sigma_m^2 + \sigma^2)} \right]$$

The likelihood ratio test reduces to

$$\Lambda(y) = \frac{\dfrac{1}{\sqrt{\pi}\sqrt{\sigma_m^2 + \sigma^2}} \exp\left[ -\dfrac{y^2}{2(\sigma_m^2 + \sigma^2)} \right]}{\dfrac{1}{\sqrt{2\pi}\sigma} \exp\left[ -\dfrac{y^2}{2\sigma^2} \right]} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Taking the natural logarithm on both sides and simplifying the expression, we obtain

$$y^2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{2\sigma^2(\sigma_m^2 + \sigma^2)}{\sigma_m^2} \left[ \ln \eta + \frac{1}{2} \ln\left( 1 + \frac{\sigma_m^2}{\sigma^2} \right) \right]$$

We observe that exact knowledge of "the unwanted parameter" *m* is not important because it does not appear in the decision rule.

### 5.5.2 $\theta$ Nonrandom and Unknown

If $\theta$ is not a random variable but still unknown, the Bayes' test is no longer applicable, since $\theta$ does not have a probability density function over which $f_{Y|\Theta,H_j}(y \mid \theta, H_j), j = 0, 1,$ can be averaged, and consequently the risk cannot be determined. Instead, we use the Neyman-Pearson test. In this case, we maximize the probability of detection, $P_D$, while the probability of false alarm, $P_F$, is fixed, given that the assumed value $\theta$ is the true value.

Performing this test for several values of $\theta$ results in a plot of $P_D$ versus $\theta$, known as the *power function*. A test that maximizes the probability of detection as mentioned above for all possible values of $\theta$ is referred to as a uniformly most powerful (UMP) test. Hence, a UMP test maximizes the probability of detection *irrespective* of the values of $\theta$.

If $H_0$ is a simple hypothesis and $H_1$ is a composite hypothesis, then the test is called UMP (of level $\alpha$) if it is the most powerful of level $\alpha$.

**Example 5.11**

Consider the problem of Example 5.9, where $m$ is a positive constant. Determine the optimum decision rule.

*Solution*

The conditional density functions under hypotheses $H_0$ and $H_1$ are

$$H_0 : f_{Y|H_0}(y \mid H_0) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

$$H_1 : f_{Y|H_1}(y \mid H_1) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(y-m)^2}{2\sigma^2}\right]$$

The exact value of $m$ is not known, but it is known to be positive. Assuming a value of $m$, the likelihood ratio test is given by

$$\Lambda(y) = \frac{\dfrac{1}{\sqrt{2\pi}\sigma} \exp\left[-\left(\dfrac{y^2 - my + m^2}{2\sigma^2}\right)\right]}{\dfrac{1}{\sqrt{2\pi}\sigma} \exp\left(-\dfrac{y^2}{2\sigma^2}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Simplifying the likelihood ratio test and taking the natural logarithm, we obtain

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{m} \ln \eta + \frac{m}{2} = \gamma_1$$

Note that the threshold $\eta$ is determined from the specified value of the probability of false alarm $P_F$. In fact, knowledge of $\eta$ is not necessary to determine $\gamma_1$. Assuming $\gamma_1$, as shown in Figure 5.17, we have

$$P_F = \int_{\gamma_1}^{\infty} f_{Y|H_0}(y \mid H_0)\, dy = \int_{\gamma_1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

Once $\gamma_1$ is determined, the application of the likelihood ratio test

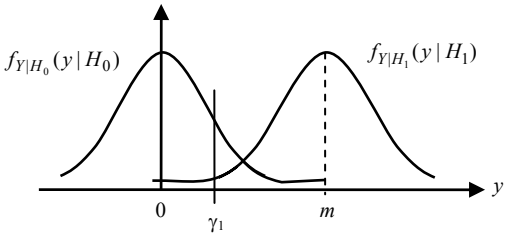


**Figure 5.17** Threshold $\gamma_1$ for composite hypothesis.

$$\begin{matrix} & H_1 & \\ y & \gtrless & \gamma_1 \\ & H_0 & \end{matrix}$$

does not require any knowledge of $m$. That is, a best test can be completely designed independently of $m$. Hence, a UMP test exists.

Similarly, if $m$ were unknown but negative, the likelihood ratio test reduces to

$$\begin{matrix} & H_0 & \\ y & \lessgtr & \dfrac{\sigma^2}{m}\ln\eta + \dfrac{m}{2} = \gamma_2 \\ & H_1 & \end{matrix}$$

$\gamma_2$ is determined from the specified probability of false alarm to be

$$P_F = \int_{-\infty}^{\gamma_2} f_{Y|H_0}(y \mid H_0)dy = \int_{-\infty}^{\gamma_2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy$$

Again, a UMP test exists, since application of the likelihood ratio test is independent of $m$. It should be noted that the probability of detection for both cases, $m < 0$ and $m > 0$, cannot be evaluated because the exact value of $m$ is not known. Nevertheless, the test is optimum for all possible positive or negative values of $m$.

Note that the test designed for positive $m$ is not the same for negative $m$. Consequently, if $m$ were unknown and takes all possible values, positive and negative, a UMP test does not exist. We know from the definition that a UMP test exists if it is optimum for all possible values of $m$. In this case, the test designed for positive (negative) $m$ is not optimum for negative (positive) $m$. This requires

that different tests are to be used, which will be discussed in the coming chapter after we cover maximum likelihood estimation (MLE).

## 5.6 SEQUENTIAL DETECTION

In the previous sections, we considered the theory of hypothesis testing, such that the number of observations on which the test was based was fixed. In many practical situations, observations may be taken in a sequential manner so that the test is performed after each observation. Each time an observation is taken, one of the three possible decisions is made:

1. Decide $H_1$
2. Decide $H_0$
3. Not enough information to decide in favor of either $H_1$ or $H_0$.

If decisions (1) or (2) are made, the hypothesis testing procedure stops. Otherwise, an additional observation is taken, and the test is performed again. This process continues until a decision is made either in favor of $H_1$ or $H_0$. Note that the number of observation $K$ is not fixed, but is a random variable.

The test to be performed after each observation is to compute the likelihood ratio and compare it to two thresholds, $\eta_0$ and $\eta_1$. Such a test that makes one of the three possible decisions mentioned above after the $k$th observation is referred to as *sequential likelihood ratio test.*

Let $Y_k$, $k = 1, 2, \ldots, K$, represent the $k$th observation sample of the vector $\boldsymbol{Y}_K$ defined as

$$\boldsymbol{Y}_K^T = [Y_1 \quad Y_2 \quad \ldots \quad Y_K] \tag{5.87}$$

The likelihood ratio based on the first $K$ observations is

$$\Lambda(\boldsymbol{y}_K) = \frac{f_{\boldsymbol{Y}_K|H_1}(\boldsymbol{y}_K \mid H_1)}{f_{\boldsymbol{Y}_K|H_0}(\boldsymbol{y}_K \mid H_0)} \tag{5.88}$$

To compute the likelihood ratio of (5.88), we need to know the joint density function of these $K$ observations. For simplicity, we assume that the observations are identically distributed, and are taken independently. The likelihood ratio can be written as a product of $K$ likelihood ratios to obtain

$$\Lambda(\boldsymbol{y}_K) = \frac{f_{\boldsymbol{Y}_K|H_1}(\boldsymbol{y}_K \mid H_1)}{f_{\boldsymbol{Y}_K|H_0}(\boldsymbol{y}_K \mid H_0)} = \prod_{k=1}^{K} \frac{f_{Y_k|H_1}(y_K \mid H_1)}{f_{Y_k|H_0}(y_K \mid H_0)} \tag{5.89}$$

The goal is to determine $\eta_0$ and $\eta_1$ in terms of $P_F$, the probability of false alarm, and $P_M$, the probability of a miss. We set

$$P_F = \alpha \tag{5.90}$$

and

$$P_M = \beta \tag{5.91}$$

and perform the following test. If

$$\Lambda(\boldsymbol{y}_K) \geq \eta_1 \tag{5.92}$$

we decide in favor of $H_1$. If

$$\Lambda(\boldsymbol{y}_K) \leq \eta_0 \tag{5.93}$$

we decide in favor of $H_0$. Otherwise, if

$$\eta_0 < \Lambda(\boldsymbol{y}_K) < \eta_1 \tag{5.94}$$

we take an additional observation and perform another test. The probability of detection, $P_D$, in terms of the integral over the observation space is

$$P_D = P(\text{decide } H_1 \mid H_1 \text{ true}) = \int_{Z_1} f_{\boldsymbol{Y}_K|H_1}(\boldsymbol{y}_K \mid H_1) d\boldsymbol{y}_K \tag{5.95}$$

Using (5.88), $P_D$ can be written as

$$P_D = \int_{Z_1} \Lambda(\boldsymbol{y}_K) f_{\boldsymbol{Y}_K|H_0}(\boldsymbol{y}_K \mid H_0) d\boldsymbol{y}_K \tag{5.96}$$

The decision in favor of $H_1$ means that $\Lambda(\boldsymbol{y}_K) \geq \eta_1$. Hence, substituting (5.92) for (5.96), we obtain the inequality

$$P_D \geq \eta_1 \int_{Z_1} f_{\boldsymbol{Y}_K|H_0}(\boldsymbol{y}_K \mid H_0) d\boldsymbol{y}_K \tag{5.97}$$

Note that the integral

$$\int_{Z_1} f_{\boldsymbol{Y}_K|H_0}(\boldsymbol{y}_K \mid H_0) d\boldsymbol{y}_K = P_F = \alpha \tag{5.98}$$

and since $P_D = 1 - P_M = 1 - \beta,$ (5.97) reduces to

$$1 - \beta \geq \eta_1 \alpha \tag{5.99}$$

or, the threshold $\eta_1$ is

$$\eta_1 \leq \frac{1-\beta}{\alpha} \tag{5.100}$$

Similarly, it can be shown that the threshold $\eta_0$ is

$$\eta_0 \geq \frac{\beta}{1-\alpha} \tag{5.101}$$

At this stage, some important questions need to be investigated and answered. What is the probability that the procedure never terminates? What are some of the properties of the distribution of the random variable $K$? In particular, what is the expected value of this sample size $K$?

To answer such questions, it is much easier to use the log likelihood function. Taking the natural logarithm of (5.94), we obtain

$$\ln \eta_0 < \ln \frac{f_{Y_1|H_1}(y_1 \mid H_1)}{f_{Y_1|H_0}(y_1 \mid H_0)} + \ldots + \ln \frac{f_{Y_K|H_1}(y_K \mid H_1)}{f_{Y_K|H_0}(y_K \mid H_0)} < \ln \eta_1 \tag{5.102}$$

Let the $k$th term, $k = 1, 2, \ldots, K,$ of the above sum be denoted as

$$L(y_k) = \ln \frac{f_{Y_k|H_1}(y_k \mid H_1)}{f_{Y_k|H_0}(y_k \mid H_0)} \tag{5.103}$$

then, (5.102) becomes

$$\ln \eta_0 < L(y_1) + \ldots + L(y_k) + \ldots + L(y_K) < \ln \eta_1 \tag{5.104}$$

The sum may be written in a recursive relation as

$$L(\boldsymbol{y}_K) = L(\boldsymbol{y}_{K-1}) + L(y_K) \tag{5.105a}$$

where

$$L(\boldsymbol{y}_{K-1}) = L(y_1) + L(y_2) + \ldots + L(y_{K-1}) = \sum_{k=1}^{K-1} L(y_k) \qquad (5.105b)$$

In order to calculate $E[K]$, the average number of observations under each hypothesis, we assume that the test terminates in $K$ observations with probability one. This assumption implies that $L(\boldsymbol{y}_K)$ takes on two possible values, $\ln \eta_0$ and $\ln \eta_1$. If hypothesis $H_1$ is true, a detection is declared when $L(\boldsymbol{y}_K) \geq \ln \eta_1$ with probability $P_D = 1 - P_M = 1 - \beta$. A miss occurs when $L(\boldsymbol{y}_K) \leq \ln \eta_0$ with probability $P_M = \beta$. Hence, the expected value of $L(\boldsymbol{y}_K)$ under hypothesis $H_1$ is

$$E[L(\boldsymbol{y}_K) \mid H_1] = \beta \ln \eta_0 + (1-\beta) \ln \eta_1 \qquad (5.106)$$

Following the same reasoning, the expected value of $L(\boldsymbol{y}_K)$ under hypothesis $H_0$ is

$$E[L(\boldsymbol{y}_K) \mid H_0] = \alpha \ln \eta_1 + (1-\alpha) \ln \eta_0 \qquad (5.107)$$

Let $B$ be a random variable taking binary numbers zero and one such that

$$B_k = \begin{cases} 1, & \text{no decision made up to } (k-1) \text{ sample} \\ 0, & \text{decision made at an earlier sample} \end{cases} \qquad (5.108)$$

that is, $B_k$ depends on the observations $Y_k$, $k = 1, 2, \ldots, K-1,$ and not $Y_K$. Rewriting the log-likelihood ratio in terms of the random variable $B$, we obtain

$$L(\boldsymbol{y}_K) = \sum_{k=1}^{K} L(y_K) = \sum_{k=1}^{\infty} B_k L(y_K) \qquad (5.109)$$

Since the observations are independent and identically distributed, we have

$$E[L(\boldsymbol{y}_K) \mid H_j] = E[L(y) \mid H_j] \sum_{k=1}^{\infty} E[B_k], \quad j = 0, 1 \qquad (5.110a)$$

where

$$E[L(y) \mid H_j] = E[L(y_1) \mid H_j] = \cdots = E[L(y_K) \mid H_j] \qquad (5.110b)$$

The sum in (5.110a) is just

$$\sum_{k=1}^{\infty} E[B_k] = \sum_{k=1}^{\infty} P(K \geq k) = \sum_{k=1}^{\infty} kP(K = k) = E[K] \tag{5.111}$$

Substituting (5.111) and (5.110b) into (5.106), we obtain

$$E[L(y) \mid H_1]E[K \mid H_1] = \alpha \ln \eta_1 + (1-\alpha)\ln \eta_0 \tag{5.112}$$

or

$$E[K \mid H_1] = \frac{(1-\beta)\ln \eta_1 + \beta \ln \eta_0}{E[L(y) \mid H_1]} \tag{5.113}$$

Similarly, the expected value of $K$ under hypothesis $H_0$ can be expected to be

$$E[K \mid H_0] = \frac{\alpha \ln \eta_1 + (1-\alpha)\ln \eta_0}{E[L(y) \mid H_0]} \tag{5.114}$$

To answer the question that the process terminates with probability one, we need to show that

$$\lim_{k \to \infty} P(K \geq k) = 0 \tag{5.115}$$

which is straightforward. Furthermore, it can be shown that the expected value of the number of the observations $K$ under each hypothesis is minimum for the specified values of $P_F$ and $P_M$.

**Example 5.12**

Suppose that the receiver of Example 5.2 takes $K$ observations sequentially. Let the variance $\sigma^2 = 1$ and mean $m = 1$. Determine

(a) The decision rule such that $P_F = \alpha = 0.1 = P_M = \beta$.

(b) The expected value of $K$ under each hypothesis.

*Solution*

(a) The definition of the decision rule is expressed in (5.92), (5.93), and (5.94). Consequently, we need to solve for the likelihood ratio at the $k$th stage and for the thresholds $\eta_0$ and $\eta_1$. Substituting for $\sigma^2 = 1$ and $m = 1$ in the likelihood ratio of Example 5.2, we obtain the likelihood ratio at the $k$th stage to be

$$\Lambda(\boldsymbol{y}_K) = \exp\left(\sum_{k=1}^{K} y_k - \frac{K}{2}\right)$$

The log likelihood ratio is just

$$L(\boldsymbol{y}_K) = \ln \Lambda(\boldsymbol{y}_K) = \sum_{k=1}^{K} y_k - \frac{K}{2}$$

From (5.100) and (5.101), the two thresholds are

$$\ln \eta_1 = 2.197 \text{ and } \ln \eta_0 = -2.197$$

Hence, the decision rule in terms of the log-likelihood ratio is:

If $L(\boldsymbol{y}_K) \geq 2.197$, decide $H_1$.
If $L(\boldsymbol{y}_K) \leq -2.197$, decide $H_0$.
If $-2.197 \leq L(\boldsymbol{y}_K) \leq 2.197$, take an additional observation $K+1$ and perform another test.

(b) The expected values of $K$ under hypotheses $H_1$ and $H_0$ are given by (5.113) and (5.114), respectively. We observe that we need to obtain $E[L(y) \mid H_1]$ and $E[L(y) \mid H_0]$. Assuming that the observations are identical, we have $E[L(y) \mid H_1] = 1 - (1/2) = 1/2$ and $E[L(y) \mid H_0] = 0 - (1/2) = -1/2$. Substituting for the values of $E[L(Y) \mid H_1]$ and $E[L(Y) \mid H_0]$ in (5.113) and (5.114), we obtain $E[K \mid H_1] = 3.515$ and $E[K \mid H_0] = 3.515$. That is, we need four samples to obtain the performance specified by $P_F = P_M = 0.1$.

## 5.7 SUMMARY

In this chapter, we have developed the basic concepts of hypothesis testing. First, we studied the Bayes' criterion, which assumes knowledge of the a priori probability of each hypothesis, and the cost assignment of each possible decision. The average cost, known as the risk function, was minimized to obtain the optimum decision rule. The Bayes' criterion was considered for the simple binary hypothesis testing and the $M$-ary hypothesis testing. The minimax criterion, which minimizes the average cost for a selected a priori probability, $P_1$, was studied in Section 5.3. The minimax criterion applies to situations where the a priori probabilities are not known, even though realistic cost assignments to the various decisions are possible. In cases where realistic cost assignments are not possible

and the a priori probabilities are not known, we considered the Neyman-Pearson approach. In the Neyman-Pearson criterion, the probability of detection (miss) is maximized (minimized), while the probability of false alarm is fixed to a designated value. The receiver operating characteristic, which is a plot of the probability of detection versus the probability of false alarm, was useful in analyzing the performance of detectors based on the Neyman-Pearson approach.

In Section 5.5, we studied the composite hypothesis testing problem. A composite hypothesis is characterized by an unknown parameter. When the parameter was a random variable with a known density function, we applied the likelihood ratio test by averaging the conditional density function corresponding to the hypotheses, over all possible values of the parameter. However, if the parameter were not random but still unknown, then the Bayes' test was no longer applicable, and instead we used the Neyman-Pearson test. Furthermore, when it was possible to apply the Neyman-Pearson test to all possible values of the parameter, a *uniformly most powerful* test was said to exist. Otherwise, a different approach that estimates the parameter should be considered. This will be described in the next chapter. We concluded this chapter with a brief section on sequential detection.

## PROBLEMS

**5.1** Consider the hypothesis testing problem in which

$$f_{Y|H_1}(y \mid H_1) = \frac{1}{2}\text{rect}\left(\frac{y-1}{2}\right) \quad \text{and} \quad f_{Y|H_0}(y \mid H_0) = e^{-y} \text{ for } y > 0$$

(a) Set up the likelihood ratio test and determine the decision regions.
(b) Find the minimum probability of error when
(1) $P_0 = 1/2$ (2) $P_0 = 3/2$ (3) $P_0 = 1/3$.

**5.2** Consider the hypothesis testing problem in which

$$f_{Y|H_0}(y \mid H_0) = \text{rect}\left(y - \frac{1}{2}\right) \quad \text{and} \quad f_{Y|H_1}(y \mid H_1) = \frac{1}{2}\text{rect}\left(\frac{y-1}{2}\right)$$

(a) Set up the likelihood ratio test and determine the decision regions.
(b) Calculate $P_F$, the probability of false alarm, and $P_M$, the probability of miss.

**5.3** A binary communication system transmits polar signals of values $-A$ and $+A$ under hypotheses $H_0$ and $H_1$, respectively. The received signal is corrupted by an additive Gaussian noise with zero mean and variance $\sigma^2$.

(a) Determine the optimum decision rule for minimum probability of error.
(b) Study the decision rule for $P_1 = P_0/3$, $P_0 = P_1$, and $P_1 = 5P_0/3$.

**5.4** A ternary communication system transmits one of the three signals, $s_1(t) = -A$, $s_0(t) = 0$, and $s_2(t) = +A$, with equal probabilities under hypotheses $H_0$, $H_1$, and $H_2$, respectively. The received signal is corrupted by an additive zero mean Gaussian noise with variance $\sigma^2$. Find
(a) The optimum decision rule (draw the decision regions) assuming minimum probability of error criterion.
(b) The minimum probability of error.

**5.5** Consider the following binary hypothesis testing problem

$$H_1 : Y = S + N$$
$$H_0 : Y = \quad N$$

where $S$ and $N$ are statistically independent random variables with probability density functions

$$f_S(s) = \begin{cases} \frac{1}{2}, & -1 < s < 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_N(n) = \begin{cases} \frac{1}{4}, & -2 < n < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Set up the likelihood ratio test and determine the decision regions when (1) $\eta = 1/4$ (2) $\eta = 1$ (3) $\eta = 2$.
(b) Find the probability of false alarm and the probability of detection for the three values of $\eta$ in part (a).
(c) Sketch the ROC.

**5.6** The output of a receiver consists of a signal voltage $S$ and a noise voltage $N$. The joint density function of the signal and noise is given by

$$P(S \cap N) = f_{SN}(s, n) = \frac{\alpha}{N_0} e^{-\alpha s}, \quad 0 \le s < \infty \text{ and } 0 \le n \le N_0$$

(a) Obtain $f_S(s)$ and $f_N(n)$, the marginal density functions of the signal and noise voltages.
(b) Show that they are statistically independent.
(c) Find the density function of sum voltage $Y = S + N$ and sketch it.

(d) Suppose now that $f_S(s)$ and $f_N(n)$ correspond to the conditional density functions under $H_1$ and $H_0$, respectively; that is, $f_{Y|H_1}(Y \mid H_1) = f_S(s)$ and $f_{Y|H_0}(Y \mid H_0) = f_N(n)$. For $N_0 = 2$ and $\alpha = 1$, obtain the optimum decision rule assuming minimum probability of error criterion.

(e) Find the minimum probability of error for $P_1 = P_0/3$, $P_1 = P_0$, and $P_1 = 2P_0/3$.

**5.7** The conditional density functions corresponding to the hypotheses $H_1$ and $H_0$ are given by

$$f_{Y|H_0}(Y \mid H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad \text{and} \quad f_{Y|H_1}(Y \mid H_1) = \frac{1}{2} e^{-|y|}$$

(a) Find the likelihood ratio and determine the decision regions.

(b) Find the probability of false alarm and the probability of detection assuming minimum probability of error and $P_0 = 2/3$.

(c) Discuss the performance of the minimax text for the cost assignments as in part (b).

(d) Determine the decision rule based on the Neyman-Pearson test for a probability of false alarm of 0.2.

**5.8** In a binary hypothesis problem, the observed random variable under each hypothesis is

$$f_{Y|H_j}(Y \mid H_j) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-m_j)^2}{2}}, \quad j = 0, 1$$

where $m_0 = 0$ and $m_1 = 1$.

(a) Find the decision rule for minimum probability of error and $P_0 = P_1$.

(b) Find the decision rule for a Neyman-Pearson test if $P_F = 0.005$.

(c) Find $P_D$ based on the test of (b).

**5.9** Consider the binary hypothesis testing problem where we are given $K$ independent observations.

$$H_1 : Y_k = m + N_k, \quad k = 1, 2, \ldots, K$$
$$H_0 : Y_k = \quad N_k, \quad k = 1, 2, \ldots, K$$

where $m$ is a constant, and $N_k$ is a zero mean Gaussian random variable with variance $\sigma^2$.

(a) Compute the likelihood ratio.

(b) Obtain the decision rule in terms of the sufficient statistic and the threshold $\gamma$.

**5.10** Repeat Problem 5.9, assuming that $m$ is zero and the variances of $N_k, k = 1, 2, \ldots, K,$ under $H_1$ and $H_0$ are $\sigma_1^2$ and $\sigma_0^2$ $(\sigma_1 > \sigma_0)$, respectively.

**5.11** Consider Problem 5.10.

(a) Obtain an expression for the probabilities of false alarm and miss for $K = 1$.

(b) Plot the ROC if $\sigma_1^2 = 2\sigma_0^2 = 2$.

(c) Determine the threshold for the minimax criterion, assuming $C_{00} = C_{11} = 0$ and $C_{01} = C_{10}$.

**5.12** The conditional density function of the observed random variable under each hypothesis is

$$f_{Y|H_j}(y \mid H_j) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{(y - m_j)^2}{2\sigma_j^2}\right], \quad j = 0, 1, 2$$

(a) Find the decision rule (draw the decision regions), assuming minimum probability of error criterion and equal a priori probabilities.

(b) Determine the decision regions, assuming

$$H_0 : m_0 = 0, \sigma_0 = 1$$
$$H_1 : m_1 = 1, \sigma_1 = 1$$
$$H_2 : m_2 = 0, \sigma_2 = 2$$

(c) Calculate the minimum probability of error for the assumptions of (b).

**5.13** Consider Problem 5.9 where $m$, now denoted $M$, is not a constant, but a zero mean Gaussian random variable with variance $\sigma_m^2$. $M$ and $N_k$, $k = 1, \ldots, K,$ are statistically independent. Determine the optimum decision rule.

**5.14** Consider the following hypothesis testing problem

$$H_1 : Y_k = M_k + N_k, \quad k = 1, 2, \ldots, K$$
$$H_0 : Y_k = \quad N_k, \quad k = 1, 2, \ldots, K$$

where $M_k$ and $N_k, k = 1, 2, \ldots, K$, are statistically independent zero mean Gaussian random variables. Their respective variances are $\sigma_m^2$ and $\sigma_n^2$, where $\sigma_n^2$ is normalized to one, but $\sigma_m^2$ is unknown. Does a UMP test exist?

**5.15** Consider the following composite hypothesis testing problem. The observations are $\boldsymbol{Y} = [Y_1, Y_2, \ldots, Y_K]^T$, where $Y_k, k = 1, 2, \ldots, K$, are independent Gaussian random variables with a known variance $\sigma^2 = 1$. The mean $m_j$, $j = 0, 1$, under each hypothesis is

$$H_1 : m_1 = m, \quad m > 0$$
$$H_0 : m_0 = 0$$

(a) Does a UMP test exist?
(b) If $P_F = 0.05$ and $m_1 = 1$, using a most powerful test, find the smallest value of $K$ that will guarantee a power greater than 0.9.

**5.16** Consider the situation where the conditional density functions under each hypothesis are

$$f_{Y_k|H_0}(y_k \mid H_0) = \frac{1}{\theta_0} \exp\left(-\frac{y_k}{\theta_0}\right) \quad \text{for } y_k \geq 0, \ k = 1, 2, \ldots, K$$

and

$$f_{Y_k|H_1}(y_k \mid H_1) = \frac{1}{\theta_1} \exp\left(-\frac{y_k}{\theta_1}\right) \quad \text{for } y_k \geq 0, \ k = 1, 2, \ldots, K$$

It is known that the signal components under each hypothesis are statistically independent, $\theta_0$ is a constant equal to 10, and $\theta_1 > \theta_0$. Find a UMP test of level $\alpha = 0.05$ and $K = 21$.

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# Chapter 6

# Parameter Estimation

## 6.1 INTRODUCTION

In Chapter 5, we considered the problem of detection theory, where the receiver receives a noisy version of a signal and decides which hypothesis is true among the $M$ possible hypotheses. In the binary case, the receiver had to decide between the null hypothesis $H_0$ and the alternate hypothesis $H_1$.

In this chapter, we assume that the receiver has made a decision in favor of the true hypothesis, but some parameter associated with the signal may not be known. The goal is to estimate those parameters in an optimum fashion based on a finite number of samples of the signal.

Let $Y_1, Y_2, \ldots, Y_K$ be $K$ independent and identically distributed samples of a random variable $Y$, with some density function depending on an unknown parameter $\theta$. Let $y_1, y_2, \ldots, y_K$ be the corresponding values of samples $Y_1, Y_2, \ldots, Y_K$ and $g(Y_1, Y_2, \ldots, Y_K)$, a function (a statistic) of the samples used to estimate the parameter $\theta$. We call

$$\hat{\theta} = g(Y_1, Y_2, \ldots, Y_K) \tag{6.1}$$

the *estimator* of $\theta$. The value that the statistic assumes is called the *estimate* of $\theta$ and is equal to $\hat{\theta} = g(y_1, y_2, \ldots, y_K)$. In order to avoid any confusion between a random variable and its value, it should be noted that $\hat{\theta}$, the estimate of $\theta$, is actually $g(Y_1, Y_2, \ldots, Y_K)$. Consequently, when we speak of the mean of $\hat{\theta}$, $E[\hat{\theta}]$, we are actually referring to $E[g(Y_1, Y_2, \ldots, Y_K)]$.

The parameter to be estimated may be random or nonrandom. The estimation of random parameters is known as the *Bayes' estimation*, while the estimation of nonrandom parameters is referred to as the maximum likelihood estimation (MLE).

In Section 6.2, we present the maximum likelihood estimator, then we use this estimator to compute the likelihood ratio test. This is called the *generalized likelihood ratio test*. In Section 6.4, we present the criteria for a "good" estimator. When the parameter to be estimated is a random variable, we use the Bayes' estimation. Specifically, we study the minimum mean-square estimation, the minimum mean absolute value of error estimation, and the maximum a posteriori estimation. The Cramer-Rao lower bound on the estimator is presented in Section 6.6. Then, we generalize the above concepts to multiple parameter estimation. Based on the fact that sometimes it is not possible to determine the optimum mean-square estimate, even if it exists, we present the best linear unbiased estimator, which is a suboptimum solution, and discuss the conditions under which it becomes optimum. In Section 6.9, we present the least-square estimation, which is different than the above-mentioned methods, in the sense that it is not based on an unbiased estimator with minimum variance, but rather on minimizing the squared difference between the observed data and the signal data. We conclude the chapter with a brief section on recursive least-square estimation for real-time applications.

## 6.2 MAXIMUM LIKELIHOOD ESTIMATION

As mentioned in the previous function, the procedure commonly used to estimate nonrandom parameters is the maximum likelihood (ML) estimation. Let $Y_1, Y_2, \ldots, Y_K$ be $K$ observations of the random variable $\boldsymbol{Y}$, with sample values $y_1, y_2, \ldots, y_K$. These random variables are independent and identically distributed. Let $f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)$ denote the conditional density function of the random variable $Y$. Note that the density function of $\boldsymbol{Y}$ depends on the parameter $\theta$, $\theta \in \Theta$, which needs to be estimated. The likelihood function, $L(\theta)$, is

$$L(\theta) = f_{Y_1,\ldots,Y_K|\Theta}(y_1, y_2, \ldots, y_K\,|\,\theta) = f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta) = \prod_{k=1}^{K} f_{Y_k|\Theta}(y_k\,|\,\theta) \qquad (6.2)$$

The value $\hat{\theta}$ that maximizes the likelihood function is called the *maximum likelihood estimator* of $\theta$. In order to maximize the likelihood function, standard techniques of calculus may be used. Because the logarithmic function $\ln x$ is a monotonically increasing function of $x$, as was shown in Chapter 5, maximizing $L(\theta)$ is equivalent to maximizing $\ln L(\theta)$. Hence, it can be shown that a necessary but not sufficient condition to obtain the ML estimate $\hat{\theta}$ is to solve the *likelihood equation*.

$$\frac{\partial}{\partial\theta} \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta) = 0 \qquad (6.3)$$

*Invariance Property*. Let $L(\theta)$ be the likelihood function of $\theta$ and $g(\theta)$ be a one-to-one function of $\theta$; that is, if $g(\theta_1) = g(\theta_2) \Leftrightarrow \theta_1 = \theta_2$. If $\hat{\theta}$ is an MLE of $\theta$, then $g(\hat{\theta})$ is an MLE of $g(\theta)$.

**Example 6.1**

In Example 5.2, the received signal under hypotheses $H_1$ and $H_0$ was

$$H_1 : Y_k = m + N_k, \quad k = 1, 2, \ldots, K$$
$$H_0 : Y_k = \quad N_k, \quad k = 1, 2, \ldots, K$$

(a) Assuming the constant $m$ is not known, obtain the ML estimate $\hat{m}_{ml}$ of the mean.

(b) Suppose now that the mean $m$ is known, but the variance $\sigma^2$ is unknown. Obtain the MLE of $\theta = \sigma^2$.

*Solution*

Detection theory (Chapter 5) was used to determine which of the two hypotheses was true. In this chapter of estimation theory, we assume that $H_1$ is true. However, a parameter is not known and needs to be estimated using MLE.

(a) The parameter $\hat{\theta}$ to be determined in this example is $\hat{m}_{ml}$, where the mean $m \in M$. Since the samples are independent and identically distributed, the likelihood function, using (6.2), is

$$f_{\boldsymbol{Y}|M}(\boldsymbol{y} \mid m) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[ -\frac{(y_k - m)^2}{2\sigma^2} \right] = \frac{1}{(2\pi)^{K/2} \sigma^K} \exp\left[ -\sum_{k=1}^{K} \frac{(y_k - m)^2}{2\sigma^2} \right]$$

Taking the logarithm on both sides, we obtain

$$\ln f_{\boldsymbol{Y}|M}(\boldsymbol{y} \mid m) = \ln\left[ \frac{1}{(2\pi)^{K/2} \sigma^K} \right] - \sum_{k=1}^{K} \frac{(y_k - m)^2}{2\sigma^2}$$

The ML estimate is obtained by solving the likelihood equation, as shown in (6.3). Hence,

$$\frac{\partial \ln f_{\boldsymbol{Y}|M}(\boldsymbol{y}\,|\,m)}{\partial m} = \sum_{k=1}^{K} \frac{y_k - m}{\sigma^2} = \sum_{k=1}^{K} \frac{y_k}{\sigma^2} - \frac{Km}{\sigma^2} = \frac{K}{\sigma^2}\left(\frac{1}{K}\sum_{k=1}^{K} y_k - m\right) = 0$$

or $m = (1/k)\sum_{k=1}^{K} y_k$ . Thus, the ML estimator is $\hat{m}_{ml} = (1/k)\sum_{k=1}^{K} y_k$.

(b) The likelihood function is

$$L(\sigma^2) = \frac{1}{(2\pi)^{\frac{K}{2}}\sigma^K} \exp\left[-\sum_{k=1}^{K} \frac{(y_k - m)^2}{2\sigma^2}\right]$$

Taking the logarithm, we obtain

$$\ln L(\sigma^2) = -\frac{K}{2}\ln 2\pi - K\ln\sigma - \sum_{k=1}^{K} \frac{(y_k - m)^2}{2\sigma^2}$$

Observe that maximizing $\ln L(\sigma^2)$ with respect to $\sigma^2$ is equivalent to minimizing

$$g(\sigma^2) = K\ln\sigma + \sum_{k=1}^{K} \frac{(y_k - m)^2}{2\sigma^2}$$

Using the invariance property, it is easier to differentiate $g(\sigma^2)$ with respect to $\sigma$ to obtain $\hat{\sigma}_{ml}$ the MLE of $\sigma$, instead of $\hat{\sigma}^2_{ml}$ the MLE of $\sigma^2$. Hence,

$$\frac{dg(\sigma^2)}{d\sigma} = \frac{K}{\sigma} - \sum_{k=1}^{K} \frac{(y_k - m)^2}{\sigma^3} = 0 \quad \text{or} \quad \hat{\sigma} = \sqrt{\frac{1}{K}\sum_{k=1}^{K}(y_k - m)^2}$$

Consequently, the MLE of $\sigma^2$ is $\hat{\sigma}^2_{ml} = (1/K)\sum_{k=1}^{K}(y_k - m)^2$.

## 6.3 GENERALIZED LIKELIHOOD RATIO TEST

In Example 5.9, we solved the hypothesis testing problem where the alternative hypothesis was composite. The parameter $m$ under hypothesis $H_1$ was unknown, although it was known that $m$ was either positive or negative. When $m$ was positive only (negative only), a UMP test existed and the decision rule was

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{m} \ln \eta + \frac{m}{2} = \gamma_1$$

for positive $m$, and

$$y \underset{H_1}{\overset{H_0}{\gtrless}} \frac{\sigma^2}{m} \ln \eta + \frac{m}{2} = \gamma_2$$

for negative $m$. Since the test designed for positive $m$ was not the same as the test designed for negative $m$, we concluded that a UMP test did not exist for all possible values of $m$; that is, positive and negative. This requires that different tests be used. One approach is to use the concepts developed in Section 6.2. That is, we use the required data to estimate $\theta$, as though hypothesis $H_1$ is true. Then, we use these estimates in the likelihood ratio test as if they are the correct values. There are many ways to estimate $\theta$, as will be shown in this chapter. If the estimates used are the maximum likelihood estimates, then the result is called the *generalized likelihood ratio test* and is given by

$$\Lambda_g(y) = \frac{\max_{\theta_1} f_{\boldsymbol{Y}|\boldsymbol{\Theta}_1}(\boldsymbol{y} \mid \boldsymbol{\theta}_1)}{\max_{\theta_0} f_{\boldsymbol{Y}|\boldsymbol{\Theta}_0}(\boldsymbol{y} \mid \boldsymbol{\theta}_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \tag{6.4}$$

$\theta_1$ and $\theta_0$ are the unknown parameters to be estimated under hypotheses $H_1$ and $H_0$, respectively.

**Example 6.2**
Consider the problem of Example 5.9, where $m$ is an *unknown* parameter. Obtain the generalized likelihood ratio test and compare it to the optimum Neyman-Pearson test.

*Solution*

Since the $K$ observations are independent, the conditional density functions under both hypotheses $H_1$ and $H_0$ are

$$H_0 : f_{\mathbf{Y}|M,H_0}(\boldsymbol{y} \mid m, H_0) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y_k^2}{2\sigma^2}\right)$$

$$H_1 : f_{\mathbf{Y}|M,H_1}(\boldsymbol{y} \mid m, H_1) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(y_k - m)^2}{2\sigma^2}\right]$$

where $m$ is an unknown parameter. Since hypothesis $H_0$ does not contain $m$ ($H_0$ is simple), the estimation procedure is applicable to hypothesis $H_1$ only. From the likelihood equation given by (6.3), the ML estimate of $m$ under $H_1$ is given by

$$\frac{\partial \ln f_{\mathbf{Y}|M,H_1}(\boldsymbol{y} \mid m, H_1)}{\partial m} = 0$$

Substituting for $f_{\mathbf{Y}|M,H_1}(\boldsymbol{y} \mid m, H_1)$ in the above equation, we have

$$\frac{\partial}{\partial m}\left[-\sum_{k=1}^{K} \frac{(y_k - m)^2}{2\sigma^2}\right] = 0 \quad \text{or} \quad \hat{m} = \frac{1}{K}\sum_{k=1}^{K} Y_K$$

The details are given in Example 6.1. The likelihood ratio test becomes

$$\Lambda_g(\boldsymbol{y}) = \frac{\prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}(y_k - \hat{m})^2\right]}{\prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y_k}{2\sigma^2}\right)} \begin{matrix} H_1 \\ > \\ < \\ H_0 \end{matrix} \ln \eta$$

Substituting for the obtained value of $\hat{m}$ in the above expression, and simplifying after taking the logarithm, the test becomes

$$\frac{1}{2\sigma^2 K}\left(\sum_{k=1}^{K} y_k\right)^2 \begin{matrix} H_1 \\ > \\ < \\ H_0 \end{matrix} \ln \eta$$

Since $(1/2\sigma^2 K)\left(\sum_{k=1}^{K} y_k\right)^2$ is nonnegative, the decision will always be $H_1$ if $\eta$ is less then one ($\ln \eta$ negative) or $\eta$ is set equal to one. Consequently, $\eta$ can always be chosen greater than or equal to one. Thus, an equivalent test is

$$\left(\frac{1}{\sqrt{K}}\sum_{k=1}^{K} y_k\right)^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\sigma^2 \ln \eta = \gamma_1^2$$

where $\gamma_1 \geq 0$ . Equivalently, we can use the test

$$|Z| = \left|\frac{1}{\sqrt{K}}\sum_{k=1}^{K} y_k\right| \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_1$$

The decision regions are shown in Figure 6.1.

Given the desired probability of false alarm, the value of $\gamma_1$ can be determined. Before we can get an expression for $P_F$, the probability of false alarm, we need to determine the density function of *Z*. Since

$$Z = \frac{1}{\sqrt{K}}\sum_{k=1}^{K} Y_k$$

the mean and variance of *Y* under hypothesis $H_0$ are zero and $\sigma^2$, respectively. All the observations are Gaussian and statistically independent. Thus, the density.function of $Z_1 = \sum_{k=1}^{K} Y_k$ is Gaussian with mean zero and variance $K\sigma^2$. Consequently, *Z* is Gaussian with mean zero and variance $\sigma^2$. That is,

$$f_{Z|H_0}(Z \mid H_0) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{z^2}{2\sigma^2}\right)$$

The probability of false alarm, from Figure 6.2, is

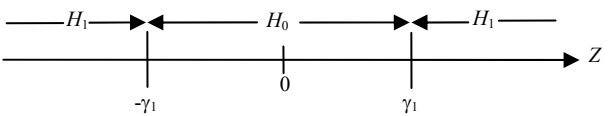


**Figure 6.1** Decision regions of the generalized likelihood ratio test.

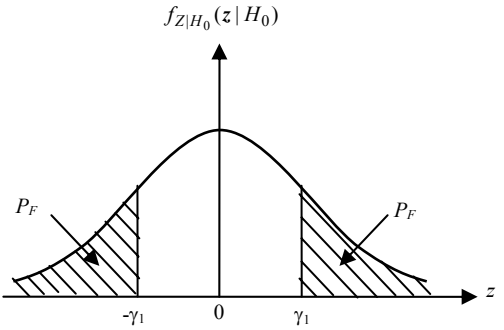


**Figure 6.2** Density function of $Z$ under $H_0$.

$$P_F = P(\text{decide } H_1 \mid H_0 \text{ true})$$
$$= \int_{-\infty}^{-\gamma_1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{z^2}{2\sigma^2}\right) dz + \int_{\gamma_1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{z^2}{2\sigma^2}\right) dz = 2Q\left(\frac{\gamma_1}{\sigma}\right)$$

We observe that we are able to determine the value $\gamma_1$ from the derived probability of false alarm without any knowledge of $m$. However, the probability of detection cannot be determined without $m$, but can be evaluated with $m$ as a parameter. Under hypothesis $H_1$, $Z_1 = \sum_{k=1}^{K} Y_1$ is Gaussian with mean $Km$ and variance $K\sigma^2$. Hence, the density function of $Z$ is Gaussian with mean $\sqrt{K}m$ and variance $\sigma^2$. That is,

$$f_{Z|H_1}(z \mid H_1) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left(z - \sqrt{K}m\right)^2}{2\sigma^2}\right]$$

The probability of detection for a given value of $m$, from Figure 6.3, is

$$P_D = P(\text{decide } H_1 \mid H_1 \text{ true})$$
$$= \int_{-\infty}^{-\gamma_1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left(z - \sqrt{K}m\right)^2}{2\sigma^2}\right] dz + \int_{\gamma_1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left(z - \sqrt{K}m\right)^2}{2\sigma^2}\right] dz$$
$$= 1 - Q\left(\frac{-\gamma_1 - \sqrt{K}m}{\sigma}\right) + Q\left(\frac{\gamma_1 - \sqrt{K}m}{\sigma}\right) = Q\left(\frac{\gamma_1 + \sqrt{K}m}{\sigma}\right) + Q\left(\frac{\gamma_1 - \sqrt{K}m}{\sigma}\right)$$

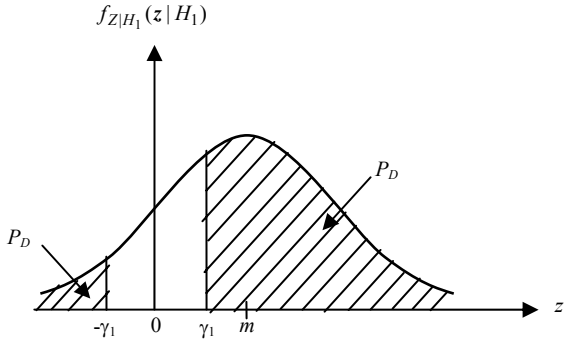


**Figure 6.3** Density function of $Z$ under $H_1$.

In Figure 3.31 of [1], it is shown that the generalized likelihood ratio test performs nearly as well as the Neyman-Pearson test.

## 6.4 SOME CRITERIA FOR GOOD ESTIMATORS

Since the estimator $\hat{\theta}$ is a random variable and may assume more than one value, some characteristics of a "good" estimate need to be determined.

*Unbiased Estimate* We say $\hat{\theta}$ is an unbiased estimator for $\theta$ if

$$E[\hat{\theta}] = \theta \quad \text{for all } \theta \tag{6.5}$$

*Bias of Estimator* Let

$$E[\hat{\theta}] = \theta + b(\theta) \tag{6.6}$$

1. If $b(\theta)$ does not depend on $\theta\,[b(\theta) = b]$, we say that the estimator $\hat{\theta}$ has a *known bias*. That is, $\left(\hat{\theta} - b\right)$ is an unbiased estimate.

2. When $b(\theta) \neq b$, an unbiased estimate cannot be obtained, since $\theta$ is unknown. In this case, we say that the estimator has an *unknown bias*.

When the parameter $\theta$ to be estimated satisfies (6.5) and is not random (i.e., there is no a priori probability distribution for $\theta$), it is sometimes referred to as *absolutely unbiased*.

The fact that the estimator is unbiased, which means that the average value of the estimate is close to the true value, does not necessarily guarantee that the estimator is "good." This is easily seen by the conditional density function of the estimator shown in Figure 6.4. We observe that even though the estimate is unbiased, sizable errors are likely to occur, since the variance of the estimate is large. However, if the variance is small, the variability of the estimator about its expected value is also small. Consequently, the variability of the estimator is close to the true value, since the estimate is unbiased, which is a desired feature. Hence, we say that the second measure of quality of the estimate is to have a small variance.

*Unbiased Minimum Variance* $\hat{\theta}$ is a minimum variance and unbiased (MVU) estimate of $\theta$ if, for all estimates $\theta'$ such that $E[\theta'] = \theta$, we have $\text{var}[\hat{\theta}] \le \text{var}[\theta']$ for all $\theta'$. That is, $\hat{\theta}$ has the smallest variance among all unbiased estimates of $\theta$.

*Consistent Estimate* $\hat{\theta}$ is a consistent estimate of the parameter $\theta$, based on $K$ observed samples, if

$$\lim_{K\to\infty} P\left(\left|\hat{\theta}-\theta\right| > \varepsilon\right) = 0 \qquad \text{for all}\, \varepsilon > 0 \tag{6.7}$$

where $P(\cdot)$ denotes probability.

Applying the above definition to verify the consistency of an estimate is not simple. The following theorem is used instead.

*Theorem*. Let $\hat{\theta}$ be an unbiased estimator of $\theta$ based on $K$ observed samples. If

$$\lim_{K\to\infty} E[\hat{\theta}] = \theta \tag{6.8}$$

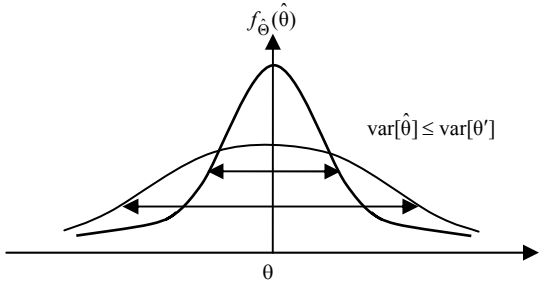


**Figure 6.4** Density function of the unbiased estimator $\hat{\theta}$.

and if

$$\lim_{K\to\infty} \text{var}\, E[\hat{\theta}] = 0 \tag{6.9}$$

then $\hat{\theta}$ is a consistent estimator of $\theta$.

**Example 6.3**

(a) Verify if the estimator $\hat{m}_{ml}$ of Example 6.1 is an unbiased estimate of $m$.

(b) Is the estimator $\hat{\sigma}^2_{ml}$ unbiased?

*Solution*

(a) The estimator $\hat{m}_{ml}$ is unbiased if $E[\hat{m}_{ml}] = m$. After substitution, we obtain

$$E[\hat{m}_{ml}] = E\left[\frac{1}{K}\sum_{k=1}^{K} Y_K\right] = \frac{1}{K}E\left[\sum_{k=1}^{K} Y_K\right] = \frac{1}{K}Km = m$$

Hence, $\hat{m}_{ml}$ is unbiased.

(b) The estimator $\hat{\sigma}^2_{ml}$ is unbiased if $E[\hat{\sigma}^2_{ml}] = \sigma^2$. That is,

$$E\left[\frac{1}{K}\sum_{k=1}^{K} (Y_k - m)^2\right] = \frac{1}{K}E\left[Km^2 + \sum_{k=1}^{K} Y_k^2 - 2m\sum_{k=1}^{K} Y_k\right] = \sigma^2$$

Hence, $\hat{\sigma}^2_{ml}$ is unbiased.

## 6.5 BAYES' ESTIMATION

In the Bayes' estimation, we assign a cost $C(\theta, \hat{\theta})$ to all pairs $(\theta, \hat{\theta})$. The cost is a nonnegative real value function of the two random variables $\theta$ and $\hat{\theta}$. As in the Bayes' detection, the risk function is defined to be the average value of the cost; that is,

$$\Re = E[C(\theta, \hat{\theta})] \tag{6.10}$$

The goal is to minimize the risk function in order to obtain $\hat{\theta}$, which is the optimum estimate. In many problems, only the error $\tilde{\theta}$ between the estimate and the true value is of interest; that is,

$$\tilde{\theta} = \theta - \hat{\theta} \tag{6.11}$$

Consequently, we will only consider costs which are a function of the error. Three cases will be studied, and their corresponding sketches are shown in Figure 6.5.

1. Squared error

$$C(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2 \tag{6.12}$$

2. Absolute value of error

$$C(\theta, \hat{\theta}) = \left| \theta - \hat{\theta} \right| \tag{6.13}$$

3. Uniform cost function

$$C(\theta, \hat{\theta}) = \begin{cases} 1, & \left| \theta - \hat{\theta} \right| \geq \dfrac{\Delta}{2} \\ 0, & \left| \theta - \hat{\theta} \right| < \dfrac{\Delta}{2} \end{cases} \tag{6.14}$$

The unknown parameter is assumed to be a continuous random variable with density function $f_{\Theta}(\theta)$. The risk function can then be expressed as

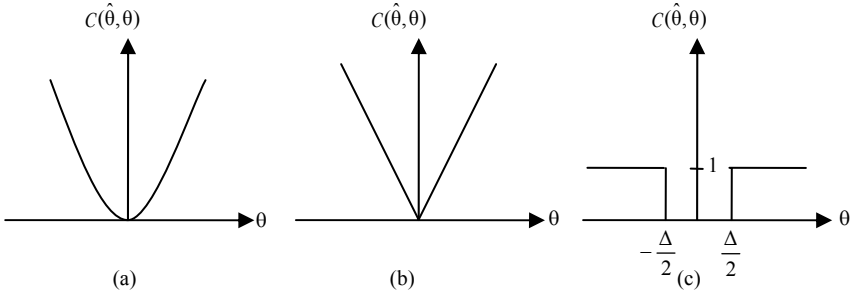


**Figure 6.5** Cost functions: (a) squared error, (b) absolute value of error, and (c) uniform.

$$\Re = E[C(\theta, \hat{\theta})] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(\theta, \hat{\theta}) f_{\Theta, \boldsymbol{Y}}(\theta, \boldsymbol{y}) d\theta d\boldsymbol{y} \tag{6.15}$$

Note that we take the cost average over all possible values of $\theta$ and $\boldsymbol{Y}$, where $\boldsymbol{Y}$ is the vector $[Y_1 \quad Y_2 \quad \ldots \quad Y_K]^T$. We now find the estimator for the three cost functions considered.

### 6.5.1 Minimum Mean-Square Error Estimate

The estimator that minimizes the risk function for the cost given in (6.12) is referred to as a minimum mean-square estimate (MMSE). The corresponding risk function is denoted by $\Re_{ms}$. We have

$$\Re_{ms} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\theta - \hat{\theta})^2 f_{\Theta, \boldsymbol{Y}}(\theta, \boldsymbol{y}) d\theta d\boldsymbol{y} = \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} (\theta - \hat{\theta})^2 f_{\Theta, \boldsymbol{Y}}(\theta, \boldsymbol{y}) d\theta d\boldsymbol{y} \tag{6.16}$$

Using (1.91), the risk function can be rewritten as

$$\Re_{ms} = \int_{-\infty}^{\infty} d\boldsymbol{y} f_{\boldsymbol{Y}}(\boldsymbol{y}) \left[ \int_{-\infty}^{\infty} (\theta - \hat{\theta})^2 f_{\Theta | \boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta \right] \tag{6.17}$$

Since the density function $f_{\boldsymbol{Y}}(\boldsymbol{y})$ is nonnegative, minimizing $\Re_{ms}$ is equivalent to minimizing the expression in brackets of the above equation. Hence, taking the derivative with respect to $\hat{\theta}$ and setting it equal to zero, we have

$$\frac{d}{d\hat{\theta}} \int_{-\infty}^{\infty} (\theta - \hat{\theta})^2 f_{\Theta | \boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta = 0 \tag{6.18}$$

Using Leibniz's rule given in (1.38), we obtain

$$\hat{\theta}_{ms} = \int_{-\infty}^{\infty} \theta \, f_{\Theta | \boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta = E[\theta \mid \boldsymbol{y}] \tag{6.19}$$

That is, the minimum mean-square estimate $\hat{\theta}_{ms}$ represents the conditional mean of $\theta$ given $\boldsymbol{Y}$. It can easily be shown that the second derivative with respect to $\hat{\theta}_{ms}$ is positive-definite, which corresponds to a unique minimum of $\Re_{ms}$, and is given by

$$\begin{aligned}\Re_{ms} &= \int_{-\infty}^{\infty} d\boldsymbol{y} f_{\boldsymbol{Y}}(\boldsymbol{y}) \int_{-\infty}^{\infty} (\theta - \hat{\theta}_{ms})^2 f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta \\ &= \int_{-\infty}^{\infty} d\boldsymbol{y} f_{\boldsymbol{Y}}(\boldsymbol{y}) \int_{-\infty}^{\infty} \{\theta - E[\theta \mid \boldsymbol{y}]\}^2 f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta \end{aligned} \tag{6.20}$$

The conditional variance of $\theta$ given $\boldsymbol{Y}$ is

$$\text{var}[\theta \mid \boldsymbol{y}] = \int_{-\infty}^{\infty} \{\theta - E[\theta \mid \boldsymbol{y}]\}^2 f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta \tag{6.21}$$

Hence, $\Re_{ms}$ is just the conditional variance of $\theta$ given $\boldsymbol{Y}$, averaged over all possible values of $\boldsymbol{Y}$. This estimation procedure using the squared error criterion is sometimes referred to as a minimum variance (MV) of error estimation.

### 6.5.2 Minimum Mean Absolute Value of Error Estimate

In this case, the cost function is given by (6.13), and the risk is

$$\Re_{abs} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\theta - \hat{\theta}| f_{\Theta,\boldsymbol{Y}}(\theta, \boldsymbol{y}) d\theta d\boldsymbol{y} = \int_{-\infty}^{\infty} f_{\boldsymbol{Y}}(\boldsymbol{y}) \left[ \int_{-\infty}^{\infty} |\theta - \hat{\theta}| f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta \right] d\boldsymbol{y} \tag{6.22}$$

Using the same arguments as in Section 6.5.1, the risk can be minimized by minimizing the integral in brackets, which is given by

$$\int_{-\infty}^{\hat{\theta}} (\hat{\theta} - \theta) f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta + \int_{\hat{\theta}}^{\infty} (\theta - \hat{\theta}) f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta \tag{6.23}$$

Differentiating (6.23) with respect to $\hat{\theta}$, and setting the result equal to zero, we obtain

$$\int_{-\infty}^{\hat{\theta}_{abs}} f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta = \int_{\hat{\theta}_{abs}}^{\infty} f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta \tag{6.24}$$

That is, the estimate $\hat{\theta}_{abs}$ is just the *median* of the conditional density function $f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y})$. This estimate is also known as the minimum mean absolute value of error (MAVE) estimate, and thus $\hat{\theta}_{abs} \equiv \hat{\theta}_{mave}$.

### 6.5.3 Maximum A Posteriori Estimate

For the uniform cost function given by (6.14), the Bayes' risk becomes

$$\Re_{unf} = \int_{-\infty}^{\infty} f_{\boldsymbol{Y}}(\boldsymbol{y}) \left[ \int_{-\infty}^{\hat{\theta}-\frac{\Delta}{2}} f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta + \int_{\hat{\theta}+\frac{\Delta}{2}}^{\infty} f_{\Theta|\boldsymbol{Y}}\theta \mid \boldsymbol{y}) d\theta \right] d\boldsymbol{y}$$

$$\Re_{unf} = \int_{-\infty}^{\infty} f_{\boldsymbol{Y}}(\boldsymbol{y}) \left[ 1 - \int_{\hat{\theta}-\frac{\Delta}{2}}^{\hat{\theta}+\frac{\Delta}{2}} f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta \right] d\boldsymbol{y} \tag{6.25}$$

where

$$\int_{\hat{\theta}-\frac{\Delta}{2}}^{\hat{\theta}+\frac{\Delta}{2}} f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) d\theta = P\left[ \hat{\theta} - \frac{\Delta}{2} \le \Theta \le \hat{\theta} + \frac{\Delta}{2} \mid \boldsymbol{y} \right] \tag{6.26}$$

$P[\cdot]$ denotes probability. Hence, the risk $\Re_{unf}$ is minimized by maximizing (6.26). Note that in maximizing (6.26) (minimizing $\Re_{unf}$), we are searching for the estimate $\hat{\theta}$, which minimizes $f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y})$. This is called the maximum a posteriori estimate (MAP), $\hat{\theta}_{map}$, which is defined as

$$\left. \frac{\partial f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y})}{\partial \theta} \right|_{\theta=\hat{\theta}_{map}} = 0 \tag{6.27}$$

Using the logarithm, which is a monotonically increasing function, (6.27) becomes

$$\frac{\partial \ln f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y})}{\partial \theta} = 0 \tag{6.28}$$

Equation (6.28) is called the *MAP equation*. This is a necessary but not sufficient condition, since $f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y})$ may have several local maxima. Using the Bayes' rule

$$f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) = \frac{f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \theta) f_{\Theta}(\theta)}{f_{\boldsymbol{Y}}(\boldsymbol{y})} \tag{6.29}$$

and the fact that

$$\ln f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y}) = \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \theta) + \ln f_{\Theta}(\theta) - \ln f_{\boldsymbol{Y}}(\boldsymbol{y}) \tag{6.30}$$

then the MAP equation may be rewritten as

$$\frac{\partial \ln f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y})}{\partial \theta} = \frac{\partial \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \theta)}{\partial \theta} + \frac{\partial \ln f_{\Theta}(\theta)}{\partial \theta} = 0 \tag{6.31}$$

We always assume that $\Delta$ is sufficiently small, so that the estimate $\hat{\theta}_{map}$ is given by the MAP equation. That is, the cost function shown in Figure 6.5 may be defined as

$$C(\hat{\theta}, \theta) = 1 - \delta(\theta, \hat{\theta}) \tag{6.32}$$

**Example 6.4**

Consider the problem where the observed samples are

$$Y_k = M + N_k, \quad k = 1, 2, \ldots, K$$

$M$ and $N_k$ are statistically independent Gaussian random variables with zero mean and variance $\sigma^2$. Find $\hat{m}_{ms}$, $\hat{m}_{map}$, and $\hat{m}_{mave}$.

*Solution*

From (6.19), the estimate $\hat{m}_{ms}$ is the conditional mean of $m$ given $\boldsymbol{Y}$. The density function $f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y})$ is expressed as

$$f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y}) = \frac{f_{\boldsymbol{Y}|M}(\boldsymbol{y} \mid m) f_M(m)}{f_{\boldsymbol{Y}}(\boldsymbol{y})}$$

where

$$f_M(m) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{m^2}{2\sigma^2}\right), \quad f_{\boldsymbol{Y}|M}(\boldsymbol{y} \mid m) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(y_k - m)^2}{2\sigma^2}\right]$$

and the marginal density function $f_{\boldsymbol{Y}}(\boldsymbol{y})$ is

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \int_{-\infty}^{\infty} f_{M,\boldsymbol{Y}}(m, \boldsymbol{y})dm = \int_{-\infty}^{\infty} f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y}) f_M(m)dm$$

Note that $f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y})$ is a function of *m*, but that $f_{\boldsymbol{Y}}(\boldsymbol{y})$ is a constant with $\boldsymbol{y}$ as a parameter needed to maintain the area under the conditional density function equal to one. That is,

$$f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y}) = \frac{1/(\sqrt{2\pi}\sigma)^{K+1}}{f_{\boldsymbol{Y}}(\boldsymbol{y})} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{k=1}^{K}(y_k - m)^2 + m^2\right]\right\}$$

Expanding the exponent, we have

$$\begin{aligned}\sum_{k=1}^{K}(y_k^2 - 2y_k m + m^2) + m^2 &= m^2(K+1) - 2m\sum_{k=1}^{K} y_k + \sum_{k=1}^{K} y_k^2 \\ &= (K+1)\left[m^2 - \frac{2m}{K+1}\sum_{k=1}^{K} y_k\right] + \sum_{k=1}^{K} y_k^2 \\ &= (K+1)\left(m - \frac{1}{K+1}\sum_{k=1}^{K} y_k\right)^2 - \left(\frac{1}{K+1}\sum_{k=1}^{K} y_k\right)^2 + \sum_{k=1}^{K} y_k^2\end{aligned}$$

The last two terms in the exponent do not involve *m*, and can be absorbed in the multiplicative constant to obtain

$$f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y}) = c(\boldsymbol{y}) \exp\left[-\frac{1}{2\sigma_m^2}\left(m - \frac{1}{K+1}\sum_{k=1}^{K} y_k^2\right)^2\right]$$

where $\sigma_m = \sigma/\sqrt{K+1}$. By inspection, the conditional mean is

$$\hat{m}_{ms} = E[M \mid \boldsymbol{y}] = \frac{1}{K+1}\sum_{k=1}^{K} y_k$$

According to (6.20), $\mathfrak{R}_{ms}$ is given by

$$\mathfrak{R}_{ms} = \int_{-\infty}^{\infty} \text{var}[M \mid \boldsymbol{y}] f_{\boldsymbol{Y}}(\boldsymbol{y})d\boldsymbol{y}$$

Hence, since $\int_{-\infty}^{\infty} f_{\boldsymbol{Y}}(\boldsymbol{y})d\boldsymbol{y} = 1$, then $\Re_{ms} = \sigma_m^2 \int_{-\infty}^{\infty} f_{\boldsymbol{Y}}(\boldsymbol{y})d\boldsymbol{y} = \sigma_m^2$.

The MAP estimate is obtained using (6.28) and (6.29). Taking the logarithm of $f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y})$, we have

$$\ln f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y}) = \ln c(\boldsymbol{y}) - \frac{1}{\sigma_m^2}\left(m - \frac{1}{K+1}\sum_{k=1}^{K} y_k\right)^2$$

Therefore,

$$\frac{\partial \ln f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y})}{\partial m} = -\frac{1}{\sigma_m^2}\left(m - \frac{1}{K+1}\sum_{k=1}^{K} y_k\right) = 0$$

$$\Rightarrow \hat{m}_{map} = \frac{1}{K+1}\sum_{k=1}^{K} y_k$$

That is, $\hat{m}_{map} = \hat{m}_{ms}$. We could have obtained this result directly by inspection, since we have shown that $f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y})$ is Gaussian. Consequently, the maximum of $f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y})$ occurs at its mean value.

Using the fact that the Gaussian density function is symmetric, and that $\hat{m}_{mave}$ is the median of the conditional density function $f_{M|\boldsymbol{Y}}(m \mid \boldsymbol{y})$, we conclude

$$\hat{m}_{mave} = \hat{m}_{ms} = \hat{m}_{map} = \frac{1}{K+1}\sum_{k=1}^{K} y_k$$

From (6.31), if $\theta$ is assumed to be random with $f_{\Theta}(\theta) = 0$ for $-\infty < \theta < \infty$, then the ML estimate can then be considered to be a special case of the MAP estimate. Such a density function for $\theta$ connotes zero a priori information about $\theta$. Furthermore, the MAP estimate of a Gaussian distributed parameter is equivalent to the ML estimate as the variance increases; that is, the distribution of the parameter to be estimated tends to be uniform. In general, for a symmetric distribution centered at the maximum, as shown Figure 6.6(a), the mean, mode, and median are identical. If the distribution of the parameter is uniform, then the MAP, the MMSE, and the MAVE estimates are identical. In Figure 6.6(b), we illustrate the different estimates when the density function is not symmetric. Recall that the *median* is the value of $y$ for which $P(Y \le y) = P(Y \ge y) = 1/2$, while the *mode* is the value that has the greatest probability of occurring.

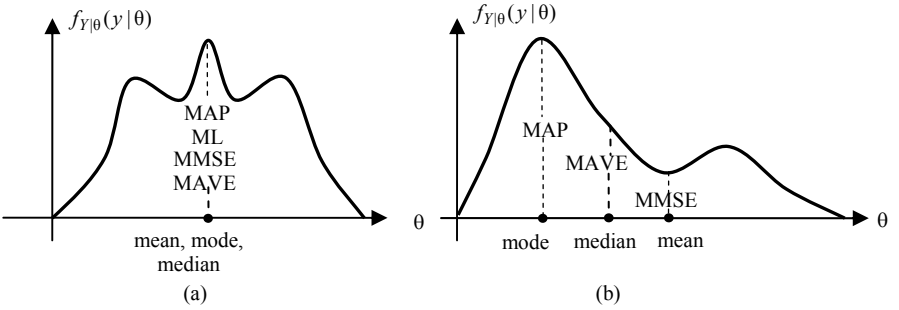


**Figure 6.6** Density functions showing relations to MAP, MAVE, and MMSE estimates: (a) symmetric pdf, and (b) nonsymmetric pdf. (*From*: [2]. © 2000 John Wiley and Sons, Inc. Reprinted with permission.)

## Example 6.5

Find $\hat{x}_{ms}$, the minimum mean-square error, and $\hat{x}_{map}$, the maximum a posteriori estimators, of $X$ from the observation $Y = X + N$. $X$ and $N$ are random variables with density functions

$$f_X(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-1) \quad \text{and} \quad f_N(n) = \frac{1}{2}e^{-|n|} = \begin{cases} \frac{1}{2}e^{n}, & n \le 0 \\ \frac{1}{2}e^{-n}, & n \ge 0 \end{cases}$$

*Solution*

The estimate $\hat{x}_{map}$ maximizes the density function $f_{X|Y}(x|y)$. Since the conditional probability density function is $f_{Y|X}(y|X) = (1/2)e^{-|n-x|}$, the probability density function of $Y$ is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)dx = \frac{1}{4}\int_{-\infty}^{\infty} e^{-|n-x|}[\delta(x) + \delta(x-1)]dx$$

$$= \frac{1}{4}\left\{e^{-|n|} + e^{-|n-1|}\right\} = \begin{cases} \frac{1}{4}\left(e^{n} + e^{n-1}\right), & y < 0 \\ \frac{1}{4}\left(e^{-n} + e^{n-1}\right), & 0 \le y < 1 \\ \frac{1}{4}\left(e^{-n} + e^{-n+1}\right), & y \ge 1 \end{cases}$$

The a posteriori density function is, from (6.29), given by

$$f_{X|Y}(x \mid y) = \frac{f_{Y|X}(y \mid x) f_X(x)}{f_Y(y)} = \frac{e^{-|n-x|}[\delta(x) + \delta(x-1)]}{e^{-|n|} + e^{-|n-1|}}$$

$f_{X|Y}(x \mid y)$ is zero except when $x = 0$ and $x = 1$. The above expression is maximized when $|n - x|$ is minimized. Since $x$ can take only two values, but must be close to $n$, we have

$$\hat{x}_{map} = \begin{cases} 1 & \text{for} \quad n \geq \dfrac{1}{2} \\ 0 & \text{for} \quad n < \dfrac{1}{2} \end{cases}$$

The mean-square error estimate is the mean of the a posteriori density function as given by (6.19). Hence,

$$\hat{x}_{ms} = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) dx = \int_{-\infty}^{\infty} x \frac{e^{-|n-x|}[\delta(x) + \delta(x-1)]}{e^{-|n|} + e^{-|n-1|}} dx$$

Since $\int_{-\infty}^{\infty} \delta(t - t_0) g(t) dt = g(t_0),$ the mean-square estimate is

$$\hat{x}_{ms} = \frac{e^{-|n-1|}}{e^{-|n|} + e^{-|n-1|}}$$

and we see that $\hat{x}_{map}$ is not identical to $\hat{x}_{ms}$.

## 6.6 CRAMER-RAO INEQUALITY

From the MAP equation of (6.31), if we set the density function of θ to zero, for all θ, we obtain the likelihood equation of (6.3). That is, the ML estimate can be considered as a special case of the MAP estimate. In this case, to check whether the estimate is "good," we need to compute its bias and error variance and determine its consistency. It may be very difficult to obtain an expression for the error variance. In this case, the "goodness" of the estimator is studied in terms of a lower bound on the error variance. This bound is known as the *Cramer-Rao*

*bound*. The Cramer-Rao bound of a constant parameter is given by the following theorem.

*Theorem*. Let the vector $\boldsymbol{Y} = [Y_1, Y_2, \ldots, Y_K]^T$ represent $K$ observations, and $\hat{\theta}$ be the unbiased estimator of $\theta$. Then

$$\text{var}[(\hat{\theta} - \theta) \mid \theta] \geq \frac{1}{E\left\{\left[\frac{\partial \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \theta)}{\partial \theta}\right]^2\right\}} \tag{6.33}$$

where

$$E\left\{\left[\frac{\partial \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \theta)}{\partial \theta}\right]^2\right\} = -E\left[\frac{\partial^2 \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \theta)}{\partial \theta^2}\right] \tag{6.34}$$

*Proof*. For an unbiased estimator $\hat{\theta}$, we have

$$E[\hat{\theta} \mid \theta] = \theta \tag{6.35}$$

Therefore,

$$E[(\hat{\theta} - \theta) \mid \theta] = \int_{-\infty}^{\infty} (\hat{\theta} - \theta) f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \theta) d\boldsymbol{y} = 0 \tag{6.36}$$

Differentiating (6.36) with respect to $\theta$, we obtain

$$\int_{-\infty}^{\infty} (\hat{\theta} - \theta) \frac{\partial f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \theta)}{\partial \theta} d\boldsymbol{y} = \int_{-\infty}^{\infty} f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \theta) d\boldsymbol{y} \tag{6.37}$$

The second integral is equal to one. Using the fact that

$$\frac{\partial \ln g(x)}{\partial x} = \frac{1}{g(x)} \frac{\partial g(x)}{\partial x} \tag{6.38}$$

where $g(x)$ is a function of $x$, we can express $\partial f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \theta) / \partial \theta$ as

$$\frac{\partial f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta} = f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)\frac{\partial \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta} \tag{6.39}$$

Substituting (6.39) into (6.37), we obtain

$$\int_{-\infty}^{\infty}(\hat{\theta}-\theta)f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)\frac{\partial \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta}\,d\boldsymbol{y} = 1 \tag{6.40}$$

The Schwarz inequality states that

$$\left[\int_{-\infty}^{\infty}x^2(t)dt\right]\left[\int_{-\infty}^{\infty}y^2(t)dt\right] \geq \left[\int_{-\infty}^{\infty}x(t)y(t)dt\right]^2 \tag{6.41}$$

where $x(t)$ and $y(t)$ are two functions of $t$. Equality holds if and only if $y(t) = cx(t)$, with $c$ a constant. Rewriting (6.39) in order to use the Schwarz inequality, we have

$$\int_{-\infty}^{\infty}\left[\frac{\partial \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta}\sqrt{f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}\right][\,(\hat{\theta}-\theta)\sqrt{f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}\,]\,d\boldsymbol{y} = 1 \tag{6.42}$$

or

$$\left[\int_{-\infty}^{\infty}(\hat{\theta}-\theta)^2\sqrt{f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}\,d\boldsymbol{y}\right]\left\{\int_{-\infty}^{\infty}\left[\frac{\partial \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta}\right]^2 f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)\,d\boldsymbol{y}\right\} \geq 1 \tag{6.43}$$

The first integral between brackets is actually $\mathrm{var}[(\hat{\theta}-\theta)\,|\,\theta]$. Hence, the inequality becomes

$$\mathrm{var}[(\hat{\theta}-\theta)\,|\,\theta] \geq \frac{1}{E\left\{\left[\dfrac{\partial \ln f_{Y|\Theta}(y\,|\,\theta)}{\partial\theta}\right]^2\right\}} \tag{6.44}$$

which proves (6.33).

We now prove (6.34), which says that the Cramer-Rao bound can be expressed in a different form. We know that

$$\int_{-\infty}^{\infty} f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)d\boldsymbol{y} = 1 \tag{6.45}$$

Differentiating both sides of the equation with respect to θ results in

$$\int_{-\infty}^{\infty} \frac{\partial f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta} d\boldsymbol{y} = 0 \tag{6.46}$$

Rewriting (6.46) and using (6.38), we have

$$\int_{-\infty}^{\infty} \frac{\partial \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta} f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)d\boldsymbol{y} = 0 \tag{6.47}$$

Differentiating again with respect to θ, we obtain

$$\int_{-\infty}^{\infty} \frac{\partial^2 \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta^2} f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)d\boldsymbol{y} + \int_{-\infty}^{\infty} \frac{\partial \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta} \frac{\partial f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta} = 0 \tag{6.48}$$

Substituting (6.47) for the second term of the second integral of (6.48), and rearranging terms yields

$$E\left[\frac{\partial^2 \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta^2}\right] = -E\left\{\left[\frac{\partial \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta}\right]^2\right\} \tag{6.49}$$

which is the same as (6.34), and the proof of the theorem is complete.

An important observation about (6.43) is that equality holds if and only if

$$\frac{\partial \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y}\,|\,\theta)}{\partial\theta} = c(\theta)[\hat{\theta} - \theta] \tag{6.50}$$

Any unbiased estimator that satisfies the equality in the Cramer-Rao inequality of (6.33) is said to be an *efficient estimator*.

If an efficient estimator exists, it can easily be shown that it equals the ML estimate. The ML equation is given by

$$\left.\frac{\partial \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{\partial\theta}\right|_{\theta=\hat{\theta}_{ml}} = 0 \tag{6.51}$$

Using (6.50), provided that an efficient estimate exists, we have

$$\left.\frac{\partial \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y}\,|\,\theta)}{\partial\theta}\right|_{\theta=\hat{\theta}_{ml}} = c(\theta)[\hat{\theta}-\theta]\Big|_{\theta=\hat{\theta}_{ml}} \tag{6.52}$$

which equals zero when $\hat{\theta}=\hat{\theta}_{ml}$.

**Example 6.6**

Consider $K$ observations, such that

$$Y_k = m + N_k, \quad k = 1, 2, \ldots, K$$

where $m$ is unknown and $N_k$s are statistically independent zero mean Gaussian random variables with unknown variance $\sigma^2$.

(a) Find the estimates $\hat{m}$ and $\hat{\sigma}^2$ for $m$ and $\sigma^2$, respectively.
(b) Is $\hat{m}$ an efficient estimator?
(c) Find the conditional variance of the error $\text{var}[(\hat{m}-m)\,|\,m]$.

*Solution*

(a) Using (6.2), we can determine $\hat{m}$ and $\hat{\sigma}^2$ simultaneously. The conditional density function of $\boldsymbol{Y}$ given $m$ and $\sigma^2$ is

$$f_{\boldsymbol{Y}}(\boldsymbol{y}\,|\,m,\sigma^2) = \prod_{k=1}^{K}\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(y_k-m)^2}{2\sigma^2}\right]$$

Taking the logarithm, we have

$$\ln f_{\boldsymbol{Y}}(\boldsymbol{y}\,|\,m,\sigma^2) = -\frac{K}{2}\ln\left(2\pi\sigma^2\right) - \sum_{k=1}^{K}\frac{(y_k-m)^2}{2\sigma^2}$$

We take the derivative of the above equation with respect to $m$ and $\sigma^2$ to obtain two equations in two unknowns. That is,

$$\frac{\partial \ln f_{\boldsymbol{Y}}(\boldsymbol{y}\,|\,m,\sigma^2)}{\partial m} = 2\sum_{k=1}^{K}\frac{y_k-m}{2\sigma^2} = 0$$

and

$$\frac{\partial \ln f_{\boldsymbol{Y}}(\boldsymbol{y} \mid m, \sigma^2)}{\partial \sigma^2} = -\frac{K}{2\sigma^2} + \sum_{k=1}^{K} \frac{(y_k - m)^2}{2\sigma^4} = 0$$

Solving for $\hat{m}_{ml}$ and $\hat{\sigma}^2_{ml}$ simultaneously, we obtain

$$\hat{m}_{ml} = \frac{1}{K} \sum_{k=1}^{K} y_k$$

and

$$\hat{\sigma}^2_{ml} = \frac{1}{K} \sum_{k=1}^{K} \left( y_k - \frac{1}{K} \sum_{k=1}^{K} y_k \right)^2 = \frac{1}{K} \sum_{k=1}^{K} (y_k - \hat{m}_{ml})^2$$

(b) $\hat{m}_{ml}$ is an unbiased estimator since

$$E[\hat{m}_{ml}] = \frac{1}{K} E\left[ \sum_{k=1}^{K} y_k \right] = m$$

To check if the estimator is efficient, we use (6.50) to obtain

$$\frac{\partial \ln f_{\boldsymbol{Y}}(\boldsymbol{y} \mid m, \sigma^2)}{\partial m} = \sum_{k=1}^{K} \frac{y_k - m}{\sigma^2} = \frac{K}{\sigma^2} \left( \frac{1}{K} \sum_{k=1}^{K} y_k - m \right)$$

where $c(m) = K / \sigma^2$ and $\hat{m} = (1/K) \sum_{k=1}^{K} y_k = \hat{m}_{ml}$. Hence, the estimator is efficient.

(c) To determine the conditional variance of error, we use (6.33) and (6.34). Taking the derivative of the likelihood equation with respect to *m*, we obtain

$$\frac{\partial^2 \ln f_{\boldsymbol{Y}}(\boldsymbol{y} \mid m, \sigma^2)}{\partial m^2} = -\frac{K}{\sigma^2}$$

Hence,

$$\text{var}[(\hat{m}-m)\,|\,m] = -\frac{1}{E\left[\dfrac{\partial^2 \ln f_{\boldsymbol{Y}}(\boldsymbol{y}\,|\,m,\sigma^2)}{\partial m^2}\right]} = \frac{\sigma^2}{K}$$

*Cramer-Rao Inequality for a Random Parameter*

We suppose that θ is a random parameter, such that the joint density function $f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)$ of the observation vector $\boldsymbol{Y}$ and the parameter θ are known. Then,

$$\text{var}[(\hat{\theta}-\theta)^2] \geq \frac{1}{E\left\{\left[\dfrac{\partial}{\partial\theta}\ln f_{\boldsymbol{Y},\Theta}(\boldsymbol{y},\theta)\right]^2\right\}} \tag{6.53}$$

where

$$E\left\{\left[\frac{\partial}{\partial\theta}\ln f_{\boldsymbol{Y},\Theta}(\boldsymbol{y},\theta)\right]^2\right\} = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f_{\boldsymbol{Y},\Theta}(\boldsymbol{y},\theta)\right] \tag{6.54}$$

Equality of (6.53) holds if and only if

$$\frac{\partial}{\partial\theta}\ln f_{\boldsymbol{Y},\boldsymbol{\Theta}}(\boldsymbol{y},\theta) = c(\hat{\theta}-\theta) \tag{6.55}$$

where $c$ is independent of $\boldsymbol{Y}$ and θ. Furthermore, the lower bound of (6.53) is achieved with equality if and if $f_{\theta|\boldsymbol{Y}}(\theta\,|\,\boldsymbol{y})$ is Gaussian.

It also can be shown that if the lower bound on the nonrandom parameter of (6.34) is denoted $J$ and if the lower bound on the random parameter of (6.54) is denoted $L$, then

$$L = J - E\left[\frac{\partial^2 \ln f_{\Theta}(\theta)}{\partial\theta^2}\right] \tag{6.56}$$

Next, we present the generalization of the Cramer-Rao bound for a vector parameter on multiple parameter estimation for both random and nonrandom parameters.

## 6.7 MULTIPLE PARAMETER ESTIMATION

In many radar and communication applications, it may be necessary to examine several parameters simultaneously. For example, in a radar application, a problem may be to estimate the range and velocity of a target; while in a communication application, the problem may be to estimate the amplitude, arrival time, and a carrier frequency of a received signal. Therefore, we can now extend the parameter estimation concepts to multiple parameters. The vector to be estimated may be random (in this case we use the Bayes' estimation) or nonrandom (in this case we use the maximum likelihood estimation).

### 6.7.1 $\boldsymbol{\theta}$ Nonrandom

In this case, the vector $\boldsymbol{\theta}$ is

$$\boldsymbol{\theta} = [\theta_1 \quad \theta_2 \quad \dots \quad \theta_K]^T \tag{6.57}$$

Then, (6.3) becomes the following set of simultaneous likelihood equations

$$\begin{aligned}
&\frac{\partial}{\partial\theta_1} \ln f_{\boldsymbol{Y}|\boldsymbol{\theta}}(y_1, y_2, \dots, y_K \mid \theta_1, \theta_2, \dots, \theta_K) = 0 \\
&\frac{\partial}{\partial\theta_2} \ln f_{\boldsymbol{Y}|\boldsymbol{\theta}}(y_1, y_2, \dots, y_K \mid \theta_1, \theta_2, \dots, \theta_K) = 0 \\
&\qquad\qquad \vdots \\
&\frac{\partial}{\partial\theta_K} \ln f_{\boldsymbol{Y}|\boldsymbol{\theta}}(y_1, y_2, \dots, y_K \mid \theta_1, \theta_2, \dots, \theta_K) = 0
\end{aligned} \tag{6.58}$$

In order to write (6.58) in a more compact form, we define the partial derivative column vector by

$$\nabla_{\boldsymbol{\theta}} = \left[\frac{\partial}{\partial\theta_1} \quad \frac{\partial}{\partial\theta_2} \quad \dots \quad \frac{\partial}{\partial\theta_K}\right]^T \tag{6.59}$$

This operation is generally applied to row vectors only. That is, if $\boldsymbol{X}^T = [X_1 \quad X_2 \quad \dots \quad X_n]$, then

$$\nabla_{\boldsymbol{\theta}} \boldsymbol{X}^T = \begin{bmatrix} \dfrac{\partial}{\partial \theta_1} \\ \dfrac{\partial}{\partial \theta_2} \\ \vdots \\ \dfrac{\partial}{\partial \theta_K} \end{bmatrix} [X_1 \quad X_2 \quad \cdots \quad X_n] = \begin{bmatrix} \dfrac{\partial X_1}{\partial \theta_1} & \dfrac{\partial X_2}{\partial \theta_1} & \cdots & \dfrac{\partial X_n}{\partial \theta_1} \\ \dfrac{\partial X_1}{\partial \theta_2} & \dfrac{\partial X_2}{\partial \theta_2} & \cdots & \dfrac{\partial X_n}{\partial \theta_2} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial X_1}{\partial \theta_K} & \dfrac{\partial X_2}{\partial \theta_K} & \cdots & \dfrac{\partial X_n}{\partial \theta_K} \end{bmatrix}$$

The ML equation is then

$$\nabla_{\boldsymbol{\theta}} [\ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \boldsymbol{\theta})] \Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{ml}(\boldsymbol{y})} = \mathbf{0} \tag{6.60}$$

We saw in Section 6.4 that a measure of quality of the estimate is the bias. The conditional mean of the estimate given by (6.6) becomes

$$E[\hat{\boldsymbol{\theta}}(\boldsymbol{y}) \mid \boldsymbol{\theta}] = \boldsymbol{\theta} + \boldsymbol{b}(\boldsymbol{\theta}) \tag{6.61}$$

If the bias vector $\boldsymbol{b}(\boldsymbol{\theta}) = \mathbf{0}$, that is, each component of the bias vector is zero for any $\boldsymbol{\theta}$, then the estimate is said to be unbiased. We note that

$$\boldsymbol{b}(\boldsymbol{\theta}) = E[(\tilde{\boldsymbol{\theta}}(\boldsymbol{y}) - \boldsymbol{\theta}) \mid \boldsymbol{\theta}] = E[\tilde{\boldsymbol{\theta}}(\boldsymbol{y})] = E[\hat{\boldsymbol{\theta}}(\boldsymbol{y})] - \boldsymbol{\theta} \tag{6.62}$$

A second measure of quality of the estimate is the conditional variance of the error. For multiple parameters, the corresponding conditional covariance matrix of the error is

$$\tilde{\boldsymbol{C}} = E[(\tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}_b)(\tilde{\boldsymbol{\theta}}^T - \tilde{\boldsymbol{\theta}}_b^T) \mid \boldsymbol{\theta}] \tag{6.63}$$

where $\tilde{\boldsymbol{\theta}}_b$ is the bias vector given by

$$\tilde{\boldsymbol{\theta}}_b = E[\tilde{\boldsymbol{\theta}}(\boldsymbol{y}) \mid \boldsymbol{\theta}] = \boldsymbol{b}(\boldsymbol{\theta}) \tag{6.64}$$

Note that $\tilde{\boldsymbol{C}}$ is a $K \times K$ matrix. The *ij*th element is

$$\tilde{\boldsymbol{C}}_{ij} = E[(\tilde{\theta}_i - \theta_{bi})(\tilde{\theta}_j - \theta_{bj}) \mid \boldsymbol{\theta}] \tag{6.65}$$

while the *i*th diagonal element is the conditional variance given by

$$\text{var}[\tilde{\sigma}_i] = \tilde{C}_{ii} = \text{var}[\tilde{\theta}_i \mid \boldsymbol{\theta}] = \text{var}[(\hat{\theta}_i(\boldsymbol{y}) - \theta_i) \mid \boldsymbol{\theta}] \tag{6.66}$$

*Cramer-Rao Bound*

The extension of the Cramer-Rao bound is given by the following theorem.

*Theorem*. If $\hat{\boldsymbol{\theta}}$ is any absolutely unbiased estimator of $\boldsymbol{\theta}$ based on the observation vector $\boldsymbol{Y}$, then the covariance of the error in the estimator is bounded by the inverse, assuming it exists, of the *Fisher information matrix* $\boldsymbol{J}$.

$$E[(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T \mid \boldsymbol{\theta}] \geq \boldsymbol{J}^{-1} \tag{6.67}$$

where

$$\boldsymbol{J} = E\left\{\left[\frac{\partial}{\partial \boldsymbol{\theta}} \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{Y} \mid \boldsymbol{\theta})\right]^T \left[\frac{\partial}{\partial \boldsymbol{\theta}} \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{Y} \mid \boldsymbol{\theta})\right] \middle| \boldsymbol{\theta}\right\} = -E\left\{\frac{\partial^2}{\partial \boldsymbol{\theta}^2} \ln f_{\boldsymbol{Y}|\Theta}(\boldsymbol{Y} \mid \boldsymbol{\theta}) \mid \boldsymbol{\theta}\right\} \tag{6.68}$$

$\boldsymbol{J}^{-1}$ is the inverse matrix of the Fisher information matrix. Equality holds only if

$$\left[\frac{\partial}{\partial \boldsymbol{\theta}} \ln f_{\mathrm{Y}|\boldsymbol{\Theta}}(\mathrm{y} \mid \boldsymbol{\theta})\right]^T = c(\boldsymbol{\theta})\left[\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right] \tag{6.69}$$

The derivatives $\partial f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \boldsymbol{\theta}) / \partial \boldsymbol{\theta}$ and $\partial^2 f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y} \mid \boldsymbol{\theta}) / \partial \boldsymbol{\theta}^2$ are assumed to exist and to be absolutely integrable. The Fisher information matrix is defined as

$$\mathrm{J} = E\left[\left\{\nabla_{\boldsymbol{\theta}}\left[\ln f_{\mathrm{Y}|\boldsymbol{\Theta}}(\mathrm{y} \mid \boldsymbol{\theta})\right]\right\}\left\{\nabla_{\boldsymbol{\theta}}\left[\ln f_{\mathrm{Y}|\boldsymbol{\Theta}}(\mathrm{y} \mid \boldsymbol{\theta})\right]\right\}^T \mid \boldsymbol{\theta}\right] \tag{6.70}$$

which can also be rewritten as

$$\mathrm{J} = -E\left[\nabla_{\boldsymbol{\theta}}\left\{\nabla_{\boldsymbol{\theta}}\left[\ln f_{\mathrm{Y}|\boldsymbol{\Theta}}(\mathrm{y} \mid \boldsymbol{\theta})\right]\right\}^T \mid \boldsymbol{\theta}\right] \tag{6.71}$$

For simplicity, we give the conditional variance on the error $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$, $i = 1, 2, \ldots, K$, which is bounded by the inequality

$$\sigma^2_{\tilde{\theta}_i} = \mathrm{var}\left[\tilde{\theta}_i \mid \boldsymbol{\theta}\right] = \mathrm{var}\left[(\hat{\theta}_i(\mathrm{y}) - \theta_i) \mid \boldsymbol{\theta}\right] \geq J^{ii} \tag{6.72}$$

$J^{ii}$ is the $i$th diagonal element in the $K \times K$ square matrix $\boldsymbol{J}^{-1}$. The $ij$th element of $\boldsymbol{J}$ in (6.70) is given by

$$J_{ij} = E\left[\frac{\partial \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(y \mid \boldsymbol{\theta})}{\partial \theta_i} \frac{\partial \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(y \mid \boldsymbol{\theta})}{\partial \theta_j} \mid \boldsymbol{\theta}\right] \tag{6.73}$$

whereas the $ij$th element of (6.71) is given by

$$J_{ij} = -E\left[\frac{\partial^2 \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(y \mid \boldsymbol{\theta})}{\partial \theta_i \, \partial \theta_j} \mid \boldsymbol{\theta}\right] \tag{6.74}$$

*Proof.* One way to prove the above theorem without resorting to excessive matrix operation is the following. Since the estimations are unbiased (the expected value of each estimator is the true value), we can write

$$E[\hat{\theta}_i(\boldsymbol{y}) \mid \boldsymbol{\theta}] = \int_{-\infty}^{\infty} \hat{\theta}_i(\boldsymbol{y}) f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y} \mid \boldsymbol{\theta}) d\boldsymbol{y} = \theta_i \tag{6.75}$$

or

$$\int_{-\infty}^{\infty} [\hat{\theta}_i(\boldsymbol{y}) - \theta_i] f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y} \mid \boldsymbol{\theta}) d\boldsymbol{y} = 0 \tag{6.76}$$

Differentiating both sides of (6.76) with respect to $\theta_j$, we have

$$\int_{-\infty}^{\infty} \hat{\theta}_i(\boldsymbol{y}) \frac{\partial \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y} \mid \boldsymbol{\theta})}{\partial \theta_j} d\boldsymbol{y} = \frac{\partial \theta_i}{\partial \theta_j} \tag{6.77}$$

Using (6.38) for the integral, and the fact that $\partial\theta_i / \partial\theta_j$ is the Kronecker $\delta_{ij}$ (unity for $i = j$, and zero otherwise), (6.77) can be rewritten as

$$\int_{-\infty}^{\infty} \hat{\theta}_i(\boldsymbol{y}) f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y} \mid \boldsymbol{\theta}) \frac{\partial \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y} \mid \boldsymbol{\theta})}{\partial \theta_j} d\boldsymbol{y} = \delta_{ij} \tag{6.78}$$

Consider the case when $j=1$, and define the $K+1$ dimensional vector $\boldsymbol{X}$ ($K$ is the number of parameters to be estimated) as

$$\boldsymbol{X}=\begin{bmatrix} \hat{\theta}_1(\boldsymbol{y})-\theta_1 \\ \dfrac{\partial \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y}\,|\,\boldsymbol{\theta})}{\partial\theta_1} \\ \dfrac{\partial \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y}\,|\,\boldsymbol{\theta})}{\partial\theta_2} \\ \vdots \\ \dfrac{\partial \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y}\,|\,\boldsymbol{\theta})}{\partial\theta_K} \end{bmatrix} \tag{6.79}$$

Note that the mean values of the components of $\boldsymbol{X}$ are all zero. The first term is zero because the estimate is unbiased, while the other terms are zero in light of (6.35), which can be written as

$$\int_{-\infty}^{\infty} \frac{\partial \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y}\,|\,\boldsymbol{\theta})}{\partial\theta} f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y}\,|\,\boldsymbol{\theta})d\boldsymbol{y} = E\left[\frac{\partial \ln f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y}\,|\,\boldsymbol{\theta})}{\partial\theta}\right]=0 \tag{6.80}$$

The covariance matrix of $\boldsymbol{X}$ is then

$$\boldsymbol{C}_{\boldsymbol{XX}}=E[\boldsymbol{X}\boldsymbol{X}^T]=\left[\begin{array}{c|cccc} \sigma^2_{\hat{\theta}_1} & 1 & 0 & \cdots & 0 \\ \hline 1 & J_{11} & J_{12} & \cdots & J_{1K} \\ 0 & J_{21} & J_{22} & \cdots & J_{2K} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & J_{K1} & J_{K2} & \cdots & J_{KK} \end{array}\right] \tag{6.81}$$

or in partitioned form,

$$\boldsymbol{C}_{\boldsymbol{XX}}=\left[\begin{array}{c|ccc} \sigma^2_{\hat{\theta}_1} & 1 & 0 & \cdots \quad 0 \\ \hline 1 & & & \\ 0 & & \boldsymbol{J} & \\ \vdots & & & \\ 0 & & & \end{array}\right] \tag{6.82}$$

Since the covariance matrix is nonnegative definite, and consequently its determinant is nonnegative definite, the determinant of (6.81) is given by

$$\det(\boldsymbol{C}_{\boldsymbol{XX}}) = \sigma^2_{\tilde{\theta}_i} |\boldsymbol{J}| - \begin{bmatrix} 1 & J_{12} & J_{13} & \cdots & J_{1K} \\ 0 & J_{22} & J_{23} & \cdots & J_{2K} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & J_{K2} & J_{K3} & \cdots & J_{KK} \end{bmatrix} \tag{6.83}$$

From (4.30), we observe that (6.83) can be written in terms of the cofactor $J_{11}$. Hence,

$$\det(\boldsymbol{C}_{\boldsymbol{XX}}) = \sigma^2_{\tilde{\theta}_i} |\boldsymbol{J}| - \begin{bmatrix} J_{22} & J_{23} & \cdots & J_{2K} \\ J_{32} & J_{33} & \cdots & J_{3K} \\ \vdots & \vdots & \vdots & \vdots \\ J_{K2} & J_{K3} & \cdots & J_{KK} \end{bmatrix} = \sigma^2_{\tilde{\theta}_i} |\boldsymbol{J}| - \text{cofactor } J_{11} \tag{6.84}$$

Assuming that the Fisher matrix $\boldsymbol{J}$ is nonsingular, we have

$$\boldsymbol{C}_{\boldsymbol{XX}} = E[\boldsymbol{X}\boldsymbol{X}^T] = \sigma^2_{\tilde{\theta}_i} |\boldsymbol{J}| - \text{cofactor } J_{11} \geq 0 \tag{6.85}$$

or

$$\sigma^2_{\tilde{\theta}_i} \geq \frac{\text{cofactor } J_{11}}{|\boldsymbol{J}|} = J^{ii} \tag{6.86}$$

which is the desired result given in (6.72).

### 6.7.2 θ Random Vector

In the Bayes' estimation, we minimize the cost function $C[\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}(\boldsymbol{y})]$. Consider now the extension of the mean-square error criterion and the MAP criterion for multiple parameters estimation.

*Mean-Square Estimation*

In this case, the cost function is the sum of the squares of the error samples given by

$$C[\tilde{\boldsymbol{\theta}}(\boldsymbol{y})] = c[\hat{\boldsymbol{\theta}}(\boldsymbol{y}) - \boldsymbol{\theta}] = \sum_{i=1}^{K} [\hat{\theta}_i(\boldsymbol{y}) - \theta_i(\boldsymbol{y})]^2 = \sum_{i=1}^{K} \tilde{\theta}_i(\boldsymbol{y})^2 = \tilde{\boldsymbol{\theta}}^T \tilde{\boldsymbol{\theta}}(\boldsymbol{y}) \tag{6.87}$$

The risk is

$$\Re_{ms} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C[\tilde{\boldsymbol{\theta}}(\boldsymbol{y})] f_{\boldsymbol{Y},\boldsymbol{\Theta}}(\boldsymbol{y}, \boldsymbol{\theta}) d\boldsymbol{y} d\boldsymbol{\theta} \tag{6.88}$$

Substituting (6.87) in (6.88) and using the Bayes' rule, the risk becomes

$$\Re_{ms} = \int_{-\infty}^{\infty} f_{\boldsymbol{Y}}(\boldsymbol{y}) d\boldsymbol{y} \int_{-\infty}^{\infty} \left\{ \sum_{i=1}^{K} [\hat{\theta}_i(\boldsymbol{y}) - \theta_i]^2 \right\} f_{\boldsymbol{\Theta}|\boldsymbol{Y}}(\boldsymbol{\theta}|\boldsymbol{y}) d\boldsymbol{\theta} \tag{6.89}$$

As before, minimizing the risk is equivalent to minimizing the expression in the brackets of (6.89). Each term between the brackets is positive, and thus the minimization is done term-by-term. From (6.19), the *i*th term $\hat{\theta}_i(\boldsymbol{y})$ is minimized for

$$\hat{\theta}_{msi}(\boldsymbol{y}) = \int_{-\infty}^{\infty} \theta_i \, f_{\boldsymbol{\Theta}|\mathbf{Y}}(\boldsymbol{\theta} \,|\, \mathbf{y}) d\boldsymbol{\theta} \tag{6.90}$$

In vector form, the MMSE is given by

$$\hat{\boldsymbol{\theta}}_{ms} = E[\boldsymbol{\theta} \,|\, \boldsymbol{y}] = \int_{-\infty}^{\infty} \boldsymbol{\theta} \, f_{\boldsymbol{\Theta}|\mathbf{Y}}(\boldsymbol{\theta} \,|\, \mathbf{y}) d\boldsymbol{\theta} \tag{6.91}$$

It can be shown that the mean-square estimation commutes over a *linear transformation* to yield

$$\hat{\boldsymbol{\phi}}_{ms}(\boldsymbol{y}) = \boldsymbol{D} \, \hat{\boldsymbol{\theta}}_{ms}(\boldsymbol{y}) \tag{6.92}$$

where $\boldsymbol{D}$ is an $L \times K$ matrix.

*MAP Estimation*

From (6.28), the MAP estimate $\hat{\boldsymbol{\theta}}_{map}$ is obtained by minimizing $f_{\boldsymbol{\Theta}|\boldsymbol{Y}}(\boldsymbol{\theta} \,|\, \boldsymbol{y})$. Generalizing the result to the estimation of multiple parameters estimation, we obtain the following set of MAP equations:

$$\left.\frac{\partial \ln f_{\boldsymbol{\Theta}|\boldsymbol{Y}}(\boldsymbol{\theta} \mid \boldsymbol{y})}{\partial \theta_i}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{map}(\boldsymbol{y})} = 0, \quad i = 1, 2, \ldots, K \tag{6.93}$$

Using (6.59), the MAP equation can be written in a single vector to be

$$\nabla_{\theta}[\ln f_{\Theta|\boldsymbol{Y}}(\theta \mid \boldsymbol{y})]\big|_{\theta=\hat{\theta}_{map}(\boldsymbol{y})} \tag{6.94}$$

*Cramer-Rao Bound*

The covariance matrix of the error of any unbiased estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is bounded below by the inverse of the Fisher information matrix, $\boldsymbol{L}$, and is given by

$$E[(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})^T] \geq \frac{1}{\boldsymbol{L}} \tag{6.95}$$

where

$$\boldsymbol{L} = -E\left[\frac{\partial^2}{\partial \boldsymbol{\theta}^2} \ln f_{\boldsymbol{Y},\boldsymbol{\Theta}}(\boldsymbol{y}, \boldsymbol{\theta})\right] \tag{6.96}$$

Note that the equality holds if and only if

$$\left[\frac{\partial}{\partial \boldsymbol{\theta}} \ln f_{\boldsymbol{Y},\boldsymbol{\Theta}}(\boldsymbol{y}, \boldsymbol{\theta})\right]^T = c(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}) \tag{6.97}$$

where $c$ is independent of $\boldsymbol{\theta}$ and $\boldsymbol{Y}$. If the conditional density function $f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y} \mid \boldsymbol{\theta})$ is Gaussian, the lower bound of (6.95) is achieved with equality.

The information matrix $\boldsymbol{L}$ can be written in terms of $\boldsymbol{J}$ as

$$\boldsymbol{L} = \boldsymbol{J} - E\left[\frac{\partial^2}{\partial \boldsymbol{\theta}^2} \ln f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\right] \tag{6.98}$$

## 6.8 BEST LINEAR UNBIASED ESTIMATOR

In many practical problems, it may be not possible to determine the MMSE estimators of a random or an unknown parameter, even if it exists. For example, we do not know the probability density function of the data, but we know the first-

order and second-order moments of it. In this case, the methods developed in estimating the parameters and determining the Cramer-Rao lower bound cannot be applied. However, we still would like to obtain a reasonable (suboptimum) or "best" estimator, in the sense that it is unbiased and has a minimum variance, usually called MVU estimator. To do so, we limit the estimator to be a *linear function* of the data, and thus it becomes possible to obtain an explicit expression for the best linear unbiased estimator (BLUE).

We first give the one parameter linear minimum mean-square estimation to present the fundamental concepts, and then generalize them to multiple parameters.

### 6.8.1 One Parameter Linear Mean-Square Estimation

The *linear minimum-square estimate* of a random parameter $\theta$ is given by

$$\hat{\theta}_{lms} = aY + b \tag{6.99}$$

The corresponding risk function is

$$\begin{aligned}\Re_{lms} &= E[C(\theta,\hat{\theta})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\theta-\hat{\theta})^2 f_{\Theta,Y}(\theta, y)d\theta dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\theta-ay-b)^2 f_{\Theta,Y}(\theta, y)d\theta dy \end{aligned} \tag{6.100}$$

Following the same procedure as we did in Section 6.5.1, we observe that minimizing the risk involves finding the constants $a$ and $b$, so that $\Re_{lms}$ is minimum. Hence, taking the derivatives of $\Re_{lms}$ with respect to $a$ and $b$ and setting them equal to zero, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\theta-ay-b)yf_{\Theta,Y}(\theta, y)d\theta dy = 0 \tag{6.101}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\theta-ay-b)f_{\Theta,\boldsymbol{Y}}(\theta, \boldsymbol{y})d\theta d\boldsymbol{y} = 0 \tag{6.102}$$

Using (1.45) and (1.108), (6.101) and (6.102) can be rewritten as

$$aE[Y^2]+bE[Y] = E[\theta\, Y] \tag{6.103}$$

and

$$aE[Y]+b=E[\theta] \tag{6.104}$$

We have two equations in two unknowns. Solving for $a$ and $b$, we obtain

$$a=\frac{E[\theta Y]-E[\theta]E[Y]}{E[Y^2]-E^2[Y]} \tag{6.105}$$

and

$$b=E[\theta]-E[Y]\frac{E[\theta Y]-E[\theta]E[Y]}{E[Y^2]-E^2[Y]} \tag{6.106}$$

Knowing that the correlation coefficient $\rho_{\theta Y}$ is given by

$$\rho_{\theta y}=\frac{E[(\hat{\theta}-m_{\hat{\theta}})(Y-m_y)]}{\sigma_\theta \sigma_y} \tag{6.107}$$

with $m_\theta=E[\theta],\ m_y=E[Y],\quad \sigma_\theta=\sqrt{E[(\theta-m_\theta)^2}$, and $\sigma_y=\sqrt{E[(Y-m_y)^2}$. Then,

$$a=\rho_{\theta y}\frac{\sigma_\theta}{\sigma_y} \tag{6.108}$$

and

$$b=m_\theta-\rho_{\theta y}m_y\frac{\sigma_\theta}{\sigma_y} \tag{6.109}$$

The optimal cost function can be obtained to be

$$\Re_{lms}=\sigma_\theta^2(1-\rho_{\theta y}^2) \tag{6.110}$$

It can be shown that if the joint density function $f_{Y,\Theta}(y,\theta)$ is Gaussian, then the conditional mean $E[\theta\,|\,y]$ is linear in the observation data, and thus the minimum mean-square estimate is linear. In addition, we usually assume for

convenience that the parameter θ and the observation $Y$ have zero means. In this case, $\hat{\theta}_{lms}$ is unbiased, and is given by

$$\theta_{blue} = C_{\theta} C_{yy}^{-1} y \tag{6.111}$$

where $C_{\theta y} = E[\theta Y]$ and $C_{yy}^{-1} = 1/E[Y^2]$. We now can generalize the result of (6.111) for multiple parameter estimation.

### 6.8.2 θ Random Vector

If now $\boldsymbol{\theta}$ is a random vector parameter and $\boldsymbol{\theta}$ and $\boldsymbol{Y}$ are assumed to have zero means, then it can be shown that the BLUE that minimizes the mean-square error (variance minimum) is given by

$$\hat{\boldsymbol{\theta}}_{blue} = \boldsymbol{C}_{\boldsymbol{\theta Y}} \boldsymbol{C}_{\boldsymbol{YY}}^{-1} \boldsymbol{Y} \tag{6.112}$$

and the mean-square error is

$$E[(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{blue})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{blue})^T] = \boldsymbol{C}_{\boldsymbol{\theta\theta}} - \boldsymbol{C}_{\boldsymbol{\theta Y}} \boldsymbol{C}_{\boldsymbol{YY}}^{-1} \boldsymbol{C}_{\boldsymbol{Y\theta}} \tag{6.113}$$

$\boldsymbol{C}_{\boldsymbol{YY}}$ is the covariance matrix of the observation vector $\boldsymbol{Y}$, $\boldsymbol{C}_{\boldsymbol{YY}}^{-1}$ is its inverse, and $\boldsymbol{C}_{\boldsymbol{\theta Y}}$ is the cross-covariance matrix between $\boldsymbol{Y}$ and $\boldsymbol{\theta}$. Note that the mean and covariance of the data are unknown, and the means of $\boldsymbol{Y}$ and $\boldsymbol{\theta}$ are assumed to be zero, and thus the linear mean-square estimator is unbiased.

*Proof.* We now give a derivation of the result given in (6.112). Since $\hat{\boldsymbol{\theta}}$ is restricted to be a linear estimator for $\boldsymbol{Y}$, that is a linear function of the data, then $\hat{\boldsymbol{\theta}}$ can be written as

$$\hat{\boldsymbol{\theta}} = \boldsymbol{D}\boldsymbol{Y} \tag{6.114}$$

The problem is to select the matrix $\boldsymbol{D}$ so that the mean-square given by (6.113) is minimized. Equation (6.113) is called the *matrix-valued squared error loss function*. Substituting (6.114) into (6.113), we have

$$E[(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T] = E[(\boldsymbol{\theta} - \boldsymbol{D}\boldsymbol{Y})(\boldsymbol{\theta} - \boldsymbol{D}\boldsymbol{Y})^T] \tag{6.115}$$

Using the fact that

$$E[(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T] = \text{tr}\, E[(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T] \tag{6.116}$$

then, (6.115) becomes

$$\begin{aligned} E[(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T] &= \text{tr}\, E[(\boldsymbol{\theta}-\boldsymbol{DY})(\boldsymbol{\theta}-\boldsymbol{DY})^T] \\ &= \text{tr}[\boldsymbol{C}_{\boldsymbol{\theta\theta}} - \boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{D}^T - \boldsymbol{D}\boldsymbol{C}_{\boldsymbol{Y\theta}} + \boldsymbol{D}\boldsymbol{C}_{\boldsymbol{YY}}\boldsymbol{D}^T] \end{aligned} \tag{6.117}$$

Note that

$$\begin{aligned} &(\boldsymbol{D}-\boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{C}_{\boldsymbol{YY}}^{-1})\,\boldsymbol{C}_{\boldsymbol{YY}}\,(\boldsymbol{D}-\boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{C}_{\boldsymbol{YY}}^{-1})^T \\ &\quad = \boldsymbol{D}\boldsymbol{C}_{\boldsymbol{YY}}\boldsymbol{D}^T - \boldsymbol{D}\boldsymbol{C}_{\boldsymbol{YY}}(\boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{C}_{\boldsymbol{YY}}^{-1})^T - \boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{D}^T + \boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{C}_{\boldsymbol{YY}}^{-1}\boldsymbol{C}_{\boldsymbol{Y\theta}} \\ &\quad = \boldsymbol{D}\boldsymbol{C}_{\boldsymbol{YY}}\boldsymbol{D}^T - \boldsymbol{D}\boldsymbol{C}_{\boldsymbol{Y\theta}} - \boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{D}^T + \boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{C}_{\boldsymbol{YY}}^{-1}\boldsymbol{C}_{\boldsymbol{Y\theta}} \end{aligned} \tag{6.118}$$

Using (6.118), we can write

$$E[(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T] = \text{tr}[\boldsymbol{C}_{\boldsymbol{YY}} + (\boldsymbol{D}-\boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{C}_{\boldsymbol{YY}}^{-1})\boldsymbol{C}_{\boldsymbol{YY}}(\boldsymbol{D}-\boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{C}_{\boldsymbol{YY}}^{-1})^{-1} - \boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{C}_{\boldsymbol{YY}}^{-1}\boldsymbol{C}_{\boldsymbol{Y\theta}}] \tag{6.119}$$

We observe that the gain matrix $\boldsymbol{D}$ appears only in the second term on the right-hand side of (6.119). Thus, each diagonal element in the matrix $E[(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T]$ is minimized when $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = \boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{C}_{\boldsymbol{YY}}^{-1} \tag{6.120}$$

Substituting (6.120) in (6.114), we have

$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}_{blue} = \boldsymbol{C}_{\boldsymbol{\theta Y}}\boldsymbol{C}_{\boldsymbol{YY}}^{-1}\boldsymbol{Y} \tag{6.121}$$

and the proof is complete.

Note that if $\boldsymbol{Y}$ and $\boldsymbol{\theta}$ are not zero mean, such that $E[\boldsymbol{Y}] = \boldsymbol{m}_{\boldsymbol{Y}}$ and $E[\boldsymbol{\theta}] = \boldsymbol{m}_{\boldsymbol{\theta}}$, then

$$\hat{\boldsymbol{\theta}}_{lms} = \boldsymbol{A}\boldsymbol{Y} + \boldsymbol{b} \tag{6.122}$$

where the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{b}$ are given by

$$\boldsymbol{A} = \left\{E[\boldsymbol{Y}\boldsymbol{Y}^T] - E[\boldsymbol{Y}]E[\boldsymbol{Y}^T]\right\}^{-1}\left\{E[\boldsymbol{\theta}\boldsymbol{Y}^T] - E[\boldsymbol{\theta}]E[\boldsymbol{Y}^T]\right\} = \boldsymbol{C}_{\boldsymbol{YY}}^{-1}\boldsymbol{C}_{\boldsymbol{\theta Y}} \tag{6.123}$$

and

$$\boldsymbol{b} = E[\boldsymbol{\theta}] - \boldsymbol{A}E[\boldsymbol{Y}] \tag{6.124}$$

By direct substitution, we obtain

$$\hat{\boldsymbol{\theta}}_{blue} = \boldsymbol{m}_{\boldsymbol{\theta}} + \boldsymbol{C}_{\boldsymbol{\theta Y}} \boldsymbol{C}_{\boldsymbol{YY}}^{-1} (\boldsymbol{Y} - \boldsymbol{m}_{\boldsymbol{Y}}) \tag{6.125}$$

The BLUE given in (6.121) has several properties of interest:

$$E[\hat{\boldsymbol{\theta}}_{blue} \boldsymbol{Y}^T] = \boldsymbol{C}_{\boldsymbol{\theta Y}} \tag{6.126}$$

$$E[\hat{\boldsymbol{\theta}}_{blue} \hat{\boldsymbol{\theta}}_{blue}^T] = \boldsymbol{C}_{\boldsymbol{\theta Y}} \boldsymbol{C}_{\boldsymbol{Y}}^{-1} \boldsymbol{C}_{\boldsymbol{Y\theta}} = \boldsymbol{C}_{\hat{\boldsymbol{\theta}}_{blue} \hat{\boldsymbol{\theta}}_{blue}} \tag{6.127}$$

$$E[(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{blue})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{blue})^T] = \boldsymbol{C}_{\boldsymbol{\theta\theta}} - \boldsymbol{C}_{\hat{\boldsymbol{\theta}}_{blue} \hat{\boldsymbol{\theta}}_{blue}} \tag{6.128}$$

$$E[(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{blue}) \boldsymbol{Y}^T] = 0 \tag{6.129}$$

$$E[(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{blue}) \hat{\boldsymbol{\theta}}_{blue}^T] = 0 \tag{6.130}$$

We observe that property (6.129) means that the error in the estimate is orthogonal to the data $\boldsymbol{Y}$, while property (6.130) means that the error in the estimate is orthogonal to the estimator $\hat{\boldsymbol{\theta}}_{blue}$. This concept of orthogonality is an important result, which will be developed and used extensively in the next chapter on filtering.

### 6.8.3 BLUE in White Gaussian Noise

Consider the general problem of estimating a random vector with $N$ parameters (denoted as the $N$-dimensional vectors $\boldsymbol{\theta}$), to be estimated from $K$ observations (denoted as the $K$-dimensional vector $\boldsymbol{Y}$), in white Gaussian noise. The parameters $\boldsymbol{\theta}$ and measurements $\boldsymbol{Y}$ are assumed to be related by the so-called *linear model*

$$\boldsymbol{Y} = \boldsymbol{H}\,\boldsymbol{\theta} + \boldsymbol{N} \tag{6.131}$$

$\boldsymbol{H}$ is a $K \times N$ known mapping matrix, $\boldsymbol{Y}$ is the $K \times 1$ observed random vector, $\boldsymbol{\theta}$ is an $N \times 1$ random vector to be estimated, and $\boldsymbol{N}$ is a $K \times 1$ vector representing errors in the measurement (noise). Assuming that $\boldsymbol{\theta}$ and $\boldsymbol{N}$ have zero means, then $\boldsymbol{Y}$ has zero mean. The covariance matrix of $\boldsymbol{Y}$ is

$$\boldsymbol{C}_{\boldsymbol{YY}} = [(\boldsymbol{H}\,\boldsymbol{\theta} + \boldsymbol{N})(\boldsymbol{H}\,\boldsymbol{\theta} + \boldsymbol{N})^T] = \boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\theta\theta}}\boldsymbol{H}^T + \boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\theta N}} + \boldsymbol{C}_{\boldsymbol{N\theta}}\boldsymbol{H}^T + \boldsymbol{C}_{\boldsymbol{NN}} \quad (6.132)$$

while the cross-covariance matrix of $\boldsymbol{Y}$ and $\boldsymbol{\theta}$ is

$$\boldsymbol{C}_{\boldsymbol{Y\theta}} = \boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\theta\theta}} + \boldsymbol{C}_{\boldsymbol{N\theta}} \quad (6.133)$$

Substituting (6.132) and (6.133) in (6.121), we obtain the BLUE estimate of $\boldsymbol{\theta}$ to be

$$\hat{\boldsymbol{\theta}}_{blue} = [\boldsymbol{C}_{\boldsymbol{\theta\theta}}\boldsymbol{H}^T + \boldsymbol{C}_{\boldsymbol{\theta N}}][\boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\theta\theta}}\boldsymbol{H}^T + \boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\theta N}} + \boldsymbol{C}_{\boldsymbol{N\theta}}\boldsymbol{H}^T + \boldsymbol{C}_{\boldsymbol{NN}}]^{-1}\boldsymbol{Y} \quad (6.134)$$

with error covariance matrix

$$\begin{aligned}\boldsymbol{C}_{\tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{\theta}}} &= \boldsymbol{C}_{\boldsymbol{\theta\theta}} - (\boldsymbol{C}_{\boldsymbol{\theta\theta}}\boldsymbol{H}^T + \boldsymbol{C}_{\boldsymbol{\theta N}}) \\ &\quad \cdot (\boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\theta\theta}}\boldsymbol{H}^T + \boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\theta N}} + \boldsymbol{C}_{\boldsymbol{N\theta}}\boldsymbol{H}^T + \boldsymbol{C}_{\boldsymbol{NN}})^{-1}(\boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\theta\theta}} + \boldsymbol{C}_{\boldsymbol{N\theta}}) \end{aligned} \quad (6.135)$$

When $\boldsymbol{\theta}$ and $\boldsymbol{N}$ are uncorrelated, which is the usual assumed case, $\mathrm{C}_{\theta\mathrm{N}} = \mathbf{0}$, and the BLUE of $\boldsymbol{\theta}$ reduces to

$$\hat{\boldsymbol{\theta}} = \boldsymbol{C}_{\boldsymbol{\theta\theta}}\boldsymbol{H}^T(\boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\theta\theta}}\boldsymbol{H}^T + \boldsymbol{C}_{\boldsymbol{NN}})^{-1}\boldsymbol{Y} \quad (6.136)$$

while the error matrix becomes

$$\boldsymbol{C}_{\tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{\theta}}} = \boldsymbol{C}_{\boldsymbol{\theta\theta}} - \boldsymbol{C}_{\boldsymbol{\theta\theta}}\boldsymbol{H}^T(\boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\theta\theta}}\boldsymbol{H}^T + \boldsymbol{C}_{\boldsymbol{NN}})^{-1}\boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\theta\theta}} \quad (6.137)$$

Using the *matrix inversion lemma* given in Chapter 4, and after some matrix operation, we have

$$\tilde{\boldsymbol{\theta}}_{blue} = \boldsymbol{C}_{\tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{\theta}}}\boldsymbol{H}^T\boldsymbol{C}_{\boldsymbol{NN}}^{-1}\boldsymbol{Y} \quad (6.138a)$$

where

$$\boldsymbol{C}_{\tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{\theta}}} = (\boldsymbol{C}_{\boldsymbol{\theta\theta}}^{-1} + \boldsymbol{H}^T\boldsymbol{C}_{\boldsymbol{NN}}^{-1}\boldsymbol{H})^{-1} \quad (6.138b)$$

If no a priori information about $\boldsymbol{\theta}$ is available, and thus if $\boldsymbol{C}_{\boldsymbol{\theta\theta}}^{-1}$ is assumed zero, the BLUE of $\hat{\boldsymbol{\theta}}$ is given by

$$\hat{\boldsymbol{\theta}} = (\boldsymbol{H}^T \boldsymbol{C}_{\boldsymbol{NN}}^{-1} \boldsymbol{H})^{-1} \boldsymbol{H}^T \boldsymbol{C}_{\boldsymbol{NN}}^{-1} \boldsymbol{Y} \tag{6.139}$$

Note that in these results, we only assumed that $\boldsymbol{\theta}$ is a random parameter. Consider now the problem of estimating the unknown vector $\boldsymbol{\theta}$**,** but which is constrained to be a linear function of the data (measurements).

*The Estimator as a Linear Function of Data*

In this case, we require

$$\hat{\boldsymbol{\theta}} = \sum_{k=1}^{K} a_{ik} Y_k + b_k, \quad i = 1, 2, \ldots, M \tag{6.140}$$

or, in matrix form

$$\hat{\boldsymbol{\theta}} = \boldsymbol{A}\boldsymbol{Y} + \boldsymbol{b} \tag{6.141}$$

where $\boldsymbol{A}$ is an $M \times K$ matrix, and $\boldsymbol{Y}$ and $\boldsymbol{b}$ are $K \times 1$ vectors. In order for $\hat{\boldsymbol{\theta}}$ to be unbiased, we must have

$$E[\hat{\boldsymbol{\theta}} \mid \boldsymbol{\theta}] = \boldsymbol{\theta} \tag{6.142}$$

Hence,

$$E[\boldsymbol{A}\boldsymbol{Y} + \boldsymbol{b} \mid \boldsymbol{\theta}] = \boldsymbol{A}E[\boldsymbol{Y} \mid \boldsymbol{\theta}] + \boldsymbol{b} = \boldsymbol{A}E[\boldsymbol{H}\,\boldsymbol{\theta} + \boldsymbol{N} \mid \boldsymbol{\theta}] + \boldsymbol{b} = \boldsymbol{A}\boldsymbol{H}\,\boldsymbol{\theta} + \boldsymbol{b} = \boldsymbol{\theta} \tag{6.143}$$

only if

$$\boldsymbol{A}\boldsymbol{H} = \boldsymbol{I} \tag{6.144a}$$

and

$$\boldsymbol{b} = \boldsymbol{0} \tag{6.144b}$$

The BLUE estimate is then given by

$$\hat{\boldsymbol{\theta}} = (\boldsymbol{H}^T \boldsymbol{C}_{\boldsymbol{NN}}^{-1} \boldsymbol{H})^{-1} \boldsymbol{H}^T \boldsymbol{C}_{\boldsymbol{NN}}^{-1} \boldsymbol{Y} \tag{6.145}$$

Therefore, with the noise Gaussian in the linear model, we can state the following result given by the *Gauss-Markov theorem*.

*Gauss-Markov Theorem*. If the data is of the general linear model form

$$\boldsymbol{Y} = \boldsymbol{H}\,\boldsymbol{\theta} + \boldsymbol{N} \tag{6.146}$$

where $\boldsymbol{H}$ is a known $K \times M$ matrix, $\boldsymbol{\theta}$ is an $M \times 1$ vector of parameters to be estimated, and $\boldsymbol{N}$ is a $K \times 1$ noise vector with mean zero and covariance matrix $\boldsymbol{C}_{\boldsymbol{NN}}$, then the BLUE of $\boldsymbol{\theta}$ that minimizes the mean-square error is

$$\hat{\boldsymbol{\theta}} = (\boldsymbol{H}^T \boldsymbol{C}_{\boldsymbol{NN}}^{-1} \boldsymbol{H})^{-1} \boldsymbol{H}^T \boldsymbol{C}_{\boldsymbol{NN}}^{-1} \boldsymbol{Y} \tag{6.147}$$

with error covariance matrix

$$\boldsymbol{C}_{\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}} = E[(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{blue})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{blue})^T \mid \boldsymbol{\theta}] = (\boldsymbol{H}^T \boldsymbol{C}_{\boldsymbol{NN}}^{-1} \boldsymbol{H})^{-1} \tag{6.148}$$

The minimum variance of $\hat{\boldsymbol{\theta}}_k$ is then

$$\text{var}[\hat{\boldsymbol{\theta}}_k] = [(\boldsymbol{H}^T \boldsymbol{C}_{\boldsymbol{NN}}^{-1} \boldsymbol{H})^{-1}]_{kk} \tag{6.149}$$

**Example 6.7**

Consider the problem of Example 6.2 where

$$Y_k = A + N_k, \quad k = 1, 2, \ldots, K$$

where $N_k$ is a zero mean white noise. Find the BLUE of $M$ if:

(a) The variance of $N_k$ , $k = 1, 2, \ldots, K$ is $\sigma^2$.

(b) The noise components are correlated with variance $\sigma_k^2$ , $k = 1, 2, \ldots, K$.

*Solution*

(a) The estimator is constrained to be a linear function of the data. Let

$$\hat{A}_k = \sum_{k=1}^{K} A_{jk} Y_k, \quad j = 1, 2, \ldots, M$$

where the $A_{jk}$s are the weighting coefficients to be determined. From (6.147), the BLUE is given by

$$\hat{A} = (\boldsymbol{H}^T \boldsymbol{C}_{\boldsymbol{NN}}{}^{-1} \boldsymbol{H})^{-1} \boldsymbol{H}^T \boldsymbol{C}_{\boldsymbol{NN}}{}^{-1} \boldsymbol{Y}$$

where

$$E[Y_k] = E[H_k A_k + N_k] = H_k E[A_k]$$

Since $A_k$ must be unbiased, then $E[A_k] = A_k$, $H_k = 1$, and thus $\boldsymbol{H} = \mathbf{1}$. Substituting, we have

$$\hat{A} = \left( \mathbf{1}^T \frac{1}{\sigma^2} \boldsymbol{I}\mathbf{1} \right)^{-1} \left( \mathbf{1}^T \frac{1}{\sigma^2} \boldsymbol{I}\boldsymbol{y} \right) = (\sigma^2 K)^{-1} \left( \sigma^2 \sum_{k=1}^{K} y_k \right) = \frac{1}{K} \sum_{k=1}^{K} y_k$$

Hence, we observe that the BLUE is the sample mean independently of the probability density function of the data, while the minimum variance is

$$\text{var}[\hat{A}] = \frac{1}{(\boldsymbol{H}^T \boldsymbol{C}_{\boldsymbol{NN}}^{-1} \boldsymbol{H})} = \frac{1}{\mathbf{1}^T \frac{1}{\sigma^2} \boldsymbol{I}\mathbf{1}} = \frac{\sigma^2}{K}$$

(b) In this case, the variance matrix is

$$\boldsymbol{C}_{\boldsymbol{NN}} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_K^2 \end{bmatrix}$$

After substitution, the BLUE is

$$\hat{A} = \frac{\sum_{k=1}^{K} \frac{1}{\sigma_k^2} y_k}{\sum_{k=0}^{K} \frac{1}{\sigma_k^2}}$$

while the minimum variance is

$$\text{var}(\hat{A}) = \frac{1}{\sum_{k=0}^{K} \frac{1}{\sigma_k^2}}$$

## 6.9 LEAST-SQUARE ESTIMATION

In studying parameter estimation in the previous sections, our criteria were to find a "good" estimator that was unbiased and had minimum variance. In the least-square estimation, the criterion is only to minimize the squared difference between the given data (signal plus noise) and the assumed signal data.

Suppose we want to estimate $M$ parameters, denoting the $M$-dimensional vector $\boldsymbol{\theta}$, from the $K$ measurements, denoting the $K$-dimensional vector $\boldsymbol{Y}$ with $K \geq M$ . The relation between the parameters $\boldsymbol{\theta}$ and the observed data $\boldsymbol{Y}$ is given by the linear model

$$\boldsymbol{Y} = \boldsymbol{H}\,\boldsymbol{\theta} + \boldsymbol{N} \tag{6.150}$$

where $\boldsymbol{H}$ is a known $(K \times M)$ matrix, and $\boldsymbol{N}$ is the unknown $(K \times 1)$ error vector that occurs in the measurement of $\boldsymbol{\theta}$.

The least-square estimator (LSE) of $\boldsymbol{\theta}$ chooses the values that make $\boldsymbol{X} = \boldsymbol{H}\,\boldsymbol{\theta}$ closest to the observed data $\boldsymbol{Y}$. Hence, we minimize

$$\begin{aligned} J(\theta) = \sum_{k=1}^{K} (Y_k - X_k)^2 &= (\boldsymbol{Y} - \boldsymbol{H}\,\theta)^T (\boldsymbol{Y} - \boldsymbol{H}\,\theta) \\ &= \boldsymbol{Y}\boldsymbol{Y}^T - \boldsymbol{Y}^T \boldsymbol{H}\,\theta - \theta^T \boldsymbol{H}^T \boldsymbol{Y} + \theta^T \boldsymbol{H}^T \boldsymbol{H}\,\theta \\ &= \boldsymbol{Y}\boldsymbol{Y}^T - 2\boldsymbol{Y}^T \boldsymbol{H}\,\boldsymbol{\theta} + \boldsymbol{\theta}^T \boldsymbol{H}^T \boldsymbol{H}\,\boldsymbol{\theta} \end{aligned} \tag{6.151}$$

Note that $\boldsymbol{Y}^T \boldsymbol{H}\,\boldsymbol{\theta}$ is a scalar. Taking the first-order partial derivative of the cost function $J(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ (i.e., the gradient) and setting it equal to zero, we obtain the set of linear equations

$$\frac{\partial J(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -2\boldsymbol{H}^T \boldsymbol{Y} + 2\boldsymbol{H}^T \boldsymbol{H}\,\boldsymbol{\theta} = \boldsymbol{0} \tag{6.152}$$

and the LSE is found to be

$$\hat{\boldsymbol{\theta}}_{ls} = (\boldsymbol{H}^T \boldsymbol{H})^{-1} \boldsymbol{H}^T \boldsymbol{Y} \tag{6.153}$$

Note that the second-order partial derivative is

$$\frac{\partial^2 J(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2} = \boldsymbol{H}^T \boldsymbol{H} \tag{6.154}$$

This matrix is positive-definite as long as $\boldsymbol{H}$ is assumed to be of full rank to guarantee the inversion of $\boldsymbol{H}^T\boldsymbol{H}$. Thus, the solution (6.153) is unique and minimizes $J(\boldsymbol{\theta})$. The equations

$$\boldsymbol{H}^T\boldsymbol{H}\,\boldsymbol{\theta} = \boldsymbol{H}^T\boldsymbol{Y} \tag{6.155}$$

to be solved for $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}_{ls}$ are referred to as the *normal equations*.

We observe that the error in the estimator $\hat{\boldsymbol{\theta}}_{ls}$ is a linear function of the measurement errors $\boldsymbol{N}$, since

$$\begin{aligned}\tilde{\boldsymbol{\theta}}_{ls} \triangleq \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{ls} &= \boldsymbol{\theta} - \left(\boldsymbol{H}^T\boldsymbol{H}\right)^{-1}\boldsymbol{H}^T\boldsymbol{Y} = \boldsymbol{\theta} - \left(\boldsymbol{H}^T\boldsymbol{H}\right)^{-1}\boldsymbol{H}^T[\boldsymbol{H}\,\boldsymbol{\theta} + \boldsymbol{N}] \\ &= \theta - \left(\boldsymbol{H}^T\boldsymbol{H}\right)^{-1}\boldsymbol{H}^T\boldsymbol{H}\,\theta - \left(\boldsymbol{H}^T\boldsymbol{H}\right)^{-1}\boldsymbol{H}^T\boldsymbol{N} \\ &= -\left(\boldsymbol{H}^T\boldsymbol{H}\right)^{-1}\boldsymbol{H}^T\boldsymbol{N}\end{aligned} \tag{6.156}$$

The minimum least-square $J_{\min}$ can be shown, after some matrix operation, to be

$$\begin{aligned}J_{\min} = J(\hat{\theta}_{ls}) = (\boldsymbol{Y} - \boldsymbol{H}\,\hat{\theta})^T(\boldsymbol{Y} - \boldsymbol{H}\,\hat{\theta}) &= \boldsymbol{Y}\boldsymbol{Y}^T - \boldsymbol{Y}^T\boldsymbol{H}(\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{Y} \\ &= \boldsymbol{Y}^T(\boldsymbol{Y} - \boldsymbol{H}\,\hat{\boldsymbol{\theta}})\end{aligned} \tag{6.157}$$

*Generalization of the Least-Square Problem*

The least-square cost function can be generalized by introducing a $K \times K$ positive definite *weighting matrix* $\boldsymbol{W}$ to yield

$$J(\boldsymbol{\theta}) = (\boldsymbol{Y} - \boldsymbol{H}\,\boldsymbol{\theta})^T\boldsymbol{W}(\boldsymbol{Y} - \boldsymbol{H}\,\boldsymbol{\theta}) \tag{6.158}$$

The elements of the weighting can be chosen to emphasize specific values of the data that are more reliable for the estimate $\hat{\boldsymbol{\theta}}$.

The general form of the least-square estimator can be shown to be

$$\hat{\boldsymbol{\theta}} = (\boldsymbol{H}^T\boldsymbol{W}\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{W}\boldsymbol{Y} \tag{6.159}$$

while its minimum least-square error is

$$J_{\min} = \boldsymbol{Y}^T[\boldsymbol{W} - \boldsymbol{W}\boldsymbol{H}(\boldsymbol{H}^T\boldsymbol{W}\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{W}]\boldsymbol{Y} \tag{6.160}$$

The error covariance matrix becomes

$$\boldsymbol{C}_{\boldsymbol{NN}} = (\boldsymbol{H}^T\boldsymbol{W}\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{W}\boldsymbol{R}\boldsymbol{W}\boldsymbol{H}(\boldsymbol{H}^T\boldsymbol{W}\boldsymbol{H})^{-1} \tag{6.161}$$

where $\boldsymbol{R}_{\boldsymbol{NN}}$ is a known positive-definite covariance matrix given by

$$\boldsymbol{R}_{\boldsymbol{NN}} = E[\boldsymbol{N}\boldsymbol{N}^T] \tag{6.162}$$

since $E[\boldsymbol{N}] = \boldsymbol{0}$ (i.e., $\boldsymbol{R}_{\boldsymbol{NN}} = \boldsymbol{C}_{\boldsymbol{NN}}$).

If the measurement errors $\boldsymbol{N}$ are uncorrelated and have identical variance $\sigma^2$, then $\boldsymbol{R} = \sigma^2\boldsymbol{I}$; and if $\boldsymbol{W} = \sigma^2\boldsymbol{I}$, then (6.159) reduces to (6.153). That is, a constant scaling has no effect on the estimate.

It can also be shown that the least-square estimator and the linear minimum mean-square estimator are identical when the weighting matrix $\boldsymbol{W}$ is chosen as

$$\boldsymbol{W} = \boldsymbol{R}^{-1} \tag{6.163}$$

that is, the inverse of the measurement noise covariance matrix.

**Example 6.8**

Consider again the problem of Example 6.5 with $Y_k = A + N_k$, $k = 1, 2, \ldots, K$. From (6.153), the least-square estimate is $\hat{A} = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{Y}$. $\boldsymbol{H}$ is the $(K \times 1)$ column matrix denoted $\mathbf{1}^T = [1 \quad 1 \quad \ldots \quad 1]$. Hence,

$$\hat{A} = (\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T\boldsymbol{Y} = \frac{1}{K}\sum_{k=1}^{K} Y_k$$

which is the *sample mean*. Observe that for this simple operation, instead of applying a derived result, we could have started by writing the least-square cost function $J(A) = \sum_{k=1}^{K}(y_k - A)^2$, then differentiating $J(A)$ with respect to $A$, setting the result equal to zero, and solving for $\hat{A} = \hat{A}_{ls}$.

**Example 6.9**

Suppose that three measurements of signal $s_k = \theta \exp(k/2)$, where $\theta$ is the parameter to be estimated, are given by $y_1 = 1.5$, $y_2 = 3$, and $y_3 = 5$. Find the least-square estimate of $\theta$.

*Solution*

The data can be put in the form, $\boldsymbol{Y} = \boldsymbol{H}\,\theta + \boldsymbol{N}$ given by (6.150). Substituting for the values of *k*, we have

$$\begin{aligned} 1.5 &= 1.648\,\theta + N_1 \\ 3 &= 2.718\,\theta + N_2 \\ 5 &= 4.482\,\theta + N_3 \end{aligned}$$

where $\boldsymbol{y} = [1.5 \quad 3 \quad 5]^T$ is a realization of $\boldsymbol{Y}$, $\boldsymbol{H} = [1.648 \quad 2.718 \quad 4.482]^T$, and $\boldsymbol{N} = [N_1 \quad N_2 \quad N_3]$ a realization of $\boldsymbol{N}$. The least-square estimate is given by

$$\hat{\theta}_{ls} = (\boldsymbol{H}^T \boldsymbol{H})^{-1} \boldsymbol{H}^T \boldsymbol{y}$$

where $\boldsymbol{H}^T \boldsymbol{H} = \sum_{k=1}^{3} H_k^2 = 30.192$, and $\boldsymbol{H}^T \boldsymbol{y} = \sum_{k=1}^{3} H_k Y_k = 30.036$. Hence,

$$\hat{\theta}_{ls} = (\boldsymbol{H}^T \boldsymbol{H})^{-1} \boldsymbol{H}^T \boldsymbol{y} = \frac{\sum_{k=1}^{3} H_k Y_k}{\sum_{k=1}^{3} H_k^2} = 0.995$$

## 6.10 RECURSIVE LEAST-SQUARE ESTIMATOR

In real time estimation problems (filtering), it is necessary to write the estimator $\hat{\theta}$ in a recursive form for efficiency. For example, consider a situation where an estimate $\hat{\theta}$ is determined based on some data $\boldsymbol{Y}_K$. If new data $\boldsymbol{Y}_{K+1}$ is to be processed after having determined an estimate based on the data $\boldsymbol{Y}_K$, it is best to use the old solution along with the new data to determine the new least-square estimator. It is clear that discarding the estimate based on the data $\boldsymbol{Y}_K$ and restarting the computation for a solution is inefficient. This procedure of determining the least-square estimate from an estimate based on $\boldsymbol{Y}_K$ and the new data $\boldsymbol{Y}_{K+1}$ is referred to as *sequential least-square estimation*, or more commonly *recursive least-square* (RLS) *estimation*.

Consider the problem of estimating $\theta$ from the data vectors $\boldsymbol{Z}_M$ given by the linear model

$$\boldsymbol{Z}_M = \boldsymbol{H}_M \boldsymbol{\theta} + \boldsymbol{U}_M \tag{6.164a}$$

where

$$\boldsymbol{Z}_M = \begin{bmatrix} \boldsymbol{Y}_1 & \boldsymbol{Y}_2 & \dots & \boldsymbol{Y}_M \end{bmatrix}^T \tag{6.164b}$$

is an $(MK+1)$ collection of vectors $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \dots, \boldsymbol{Y}_M$, since each vector $\boldsymbol{Y}_k$, $k = 1, 2, \dots, M$, is a $(K+1)$ vector,

$$\boldsymbol{U}_M = \begin{bmatrix} \boldsymbol{N}_1 & \boldsymbol{N}_2 & \dots & \boldsymbol{N}_M \end{bmatrix}^T \tag{6.164c}$$

is an $(MK+1)$ error vector, and

$$\boldsymbol{H}_M = \begin{bmatrix} \boldsymbol{h}_1 & \boldsymbol{h}_2 & \dots & \boldsymbol{h}_M \end{bmatrix}^T \tag{6.164d}$$

is an $(MK \times n)$ mapping matrix relating $\boldsymbol{Z}_M$ to the $(n \times 1)$ parameter vector $\boldsymbol{\theta}$ to be estimated.

It can be shown that the RLS estimator is given by

$$\hat{\boldsymbol{\theta}}_M = \hat{\boldsymbol{\theta}}_{M-1} + \boldsymbol{V}_M [\boldsymbol{U}_M - \boldsymbol{H}_M \hat{\boldsymbol{\theta}}_{M-1}] \tag{6.165}$$

where

$$\boldsymbol{V}_M = \boldsymbol{C}_{\boldsymbol{UU}} \boldsymbol{H}_M^T \boldsymbol{R}_{MM}^{-1} \tag{6.166}$$

$\boldsymbol{C}$ is the error covariance matrix given by

$$\boldsymbol{C}_{\boldsymbol{UU}} = E[\boldsymbol{U}_M \boldsymbol{U}_M^T] = \begin{bmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} & \dots & \boldsymbol{R}_{11} \\ \boldsymbol{R}_{12}^T & \boldsymbol{R}_{22} & \dots & \boldsymbol{R}_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{R}_{1M}^T & \boldsymbol{R}_{2M}^T & \dots & \boldsymbol{R}_{MM} \end{bmatrix} \tag{6.167}$$

and

$$E[\boldsymbol{N}_i \boldsymbol{N}_j^T] = \boldsymbol{R}_i \, \delta_{ij} \tag{6.168}$$

The covariance matrix of the individual noise vector $\boldsymbol{N}$ is $\boldsymbol{R}_{ii} \triangleq \boldsymbol{R}_i$. Equation (6.170) indicates that the estimator $\hat{\boldsymbol{\theta}}_M$ based on $\boldsymbol{Z}_M$ is formed as a linear combination of $\hat{\boldsymbol{\theta}}_{M-1}$ and a correction term $\boldsymbol{V}_M[\boldsymbol{U}_M - \boldsymbol{H}_M \hat{\boldsymbol{\theta}}_{M-1}]$.

If $\boldsymbol{\theta}$ were a random variable, it can be shown that the generalization of the recursive least-square estimation leads to the Kalman filter [3]. In the next chapter on filtering, we present an introduction to Kalman filtering.

## 6.11 SUMMARY

In this chapter, we have developed the concept of parameter estimation. We used the maximum likelihood estimation to estimate nonrandom parameters. We first obtained the likelihood function in terms of the parameters to be estimated. Then, we maximized the likelihood function to obtain the estimator, which resulted from solving the likelihood equation. We linked this chapter to the previous one by presenting the generalized likelihood ratio test in Section 6.3. In the generalized likelihood ratio test, we used the maximum likelihood estimate of the unknown parameter in the composite hypothesis as its true value and then performed the likelihood ratio test. This was an alternative to the case where UMP tests did not exist. Measuring criteria for the estimator, such as bias and consistency, were presented to determine the quality of the estimator.

When the parameter to be estimated was a random variable, we used Bayes' estimation. In Bayes' estimation, we minimized the risk, which is a function of error between the estimate and the true value. Three cases were considered; the squared error, the absolute value error, and the uniform cost function. It was shown that the minimum mean-square error represents the conditional mean of the parameter (associated with the observation random variable) to be estimated. The resulting minimum risk was the conditional variance. In the absolute value error case, the estimate turned out to be the median of the conditional density function of the parameter to be estimated, given the observation random variable.

For the uniform Bayes' cost, the estimator was actually the solution of the MAP equation. In comparing the ML estimate and MAP estimate, it was observed that the ML estimate was a special case of the MAP estimate and is obtained by setting to zero the density function of the parameter to be estimated in the MAP equation. In order to measure the "goodness" of the estimator, the Cramer-Rao bound was given as an alternate way to measure the error variance, since an expression for the error variance was difficult to obtain. The above results were generalized to multiple parameter estimation in Section 6.7.

Then, we presented linear mean-square estimation for situations where it may have been difficult to find the MMSE, even if existed. We defined the BLUE in the sense that the mean-square value is minimized. We verified that for a joint Gaussian density function of the observation and the parameter to be estimated, the linear mean-square estimator is the optimum MMSE. An introduction to least-square estimation was presented. We noted that least-square estimation was not

based on the criteria of the unbiased and minimum variance estimator, but rather on minimizing the squared difference between the given data and the assumed signal data. We concluded the chapter with a brief section on recursive least-square estimation.

## PROBLEMS

**6.1** Let $Y_1, Y_2, \ldots, Y_K$ be the observed random variables, such that

$$Y_k = a + bx_k + Z_k, \quad k = 1, 2, \ldots, K$$

The constants $x_k$, $k = 1, 2, \ldots, K$, are known, while the constants $a$ and $b$ are not known. The random variables $Z_k$, $k = 1, 2, \ldots, K$, are statistically independent, each with zero mean and variance $\sigma^2$ known. Obtain the ML estimate of $(a, b)$.

**6.2** Let $Y$ be a Gaussian random variable with mean zero and variance $\sigma^2$.

(a) Obtain the ML estimates of $\sigma$ and $\sigma^2$.
(b) Are the estimates efficient?

**6.3** Let $Y_1$ and $Y_2$ be two statistically independent Gaussian random variables, such that $E[Y_1] = m$, $E[Y_2] = 3m$, and $\text{var}[Y_1] = \text{var}[Y_2] = 1$; $m$ is unknown.

(a) Obtain the ML estimates of $m$.
(b) If the estimator of $m$ is of the form $a_1 Y_1 + b_1 Y_2$, determine $a_1$ and $a_2$, so that the estimator is unbiased.

**6.4** The observation sample of the envelope of a received signal is given by the following exponential distribution

$$f_{Y_k}(y_k) = \frac{1}{\theta} \exp\left(-\frac{y_k}{\theta}\right), \quad k = 1, 2, \ldots, K$$

$\theta$ is an unknown parameter and the observations are statistically independent.

(a) Obtain the ML estimate of $\theta$.
(b) Is the estimator unbiased?
(c) Determine the lower bound on the estimator.
(d) Is the estimator consistent?

**6.5** Let the observation $Y$ satisfy the binomial law, such that the density function of $Y$ is

$$f_Y(y) = \binom{n}{k} p^n (1-p)^{n-k}$$

(a) Find an unbiased estimate for $p$.
(b) Is the estimate consistent?

**6.6** Obtain the ML estimates of the mean $m$ and variance $\sigma^2$ for the independent observations $Y_1, Y_2, \ldots, Y_K$, such that

$$f_{Y_k}(y_k) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[ -\frac{(y_k - m)^2}{2\sigma^2} \right], \quad k = 1, 2, \ldots, K$$

**6.7** Let $x$ be an unknown deterministic parameter that can have any value in the interval $[-1, 1]$. Suppose we take two observations of $x$ with independent samples of zero-mean Gaussian noise, and with variance $\sigma^2$ superimposed on each of the observations.
(a) Obtain the ML estimate of $x$.
(b) Is $\hat{x}_{ml}$ unbiased?

**6.8** Let $Y_1, Y_2, \ldots, Y_K$ be $K$ independent observed random variables, each having a Poisson distribution given by

$$f_{Y_k|\Theta}(y_k \mid \theta) = e^{-\theta} \frac{\theta^{y_k}}{y_k!}, \quad y_k \geq 0,\ k = 1, 2, \ldots, K.$$

The parameter $\theta$ is unknown.
(a) Obtain the ML estimate of $\theta$.
(b) Verify that the estimator is unbiased and determine the lower bound.

**6.9** Let $Y_1, Y_2, \ldots, Y_K$ be $K$ independent and identically distributed observations. The observations are uniformly distributed between $-\theta$ and $+\theta$, where $\theta$ is an unknown parameter to be estimated.
(a) Obtain the MLE of $\theta$.
(b) How is the estimator unbiased?

**6.10** Let $Y_1, Y_2, \ldots, Y_K$ be $K$ independent variables with $P(Y_k = 1) = p$ and $P(Y_k = 0) = 1 - p$, where $p,\ 0 \le p < 1$ is unknown.

(a) Obtain the ML estimate.

(b) Determine the lower bound on the variance of the estimator, assuming that the estimator is unbiased.

**6.11** Find $\hat{x}_{ms}$, the minimum mean-square error, and $\hat{x}_{map}$, the maximum a posteriori estimators, of $X$ from the observations

$$Y = X + N$$

$X$ and $N$ are random variables with density functions

$$f_X(x) = \frac{1}{2}[\delta(x-1) + \delta(x+1)] \quad \text{and} \quad f_N(n) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

**6.12** The conditional density function of the observed random variable $Y$ given a random parameter $X$ is given by

$$f_{Y|X}(y \mid x) = \begin{cases} xe^{-xy}, & y \ge 0 \text{ and } x > 0 \\ 0, & y < 0 \end{cases}$$

The a priori probability density function of $X$ is

$$f_X(x) = \begin{cases} \dfrac{\alpha^r}{\Gamma(r)} x^{r-1} e^{-\alpha x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

where $\alpha$ is a parameter, $r$ is a positive integer, and $\Gamma(r)$ is the gamma function.

(a) Obtain the a priori mean and variance of $X$.

(b) For $Y$ given,

1. Obtain the minimum mean-square error estimate of $X$.
2. What is the variance of this estimate?

(c) Suppose we take $K$ independent observations of $Y_k$, $k = 1, 2, \ldots, K$, such that

$$f_{Y_k|X}(y_k \mid x) = \begin{cases} xe^{-xy_k}, & y_k \ge 0 \text{ and } x > 0 \\ 0, & y_k < 0 \end{cases}$$

1. Determine the minimum mean-square error estimate of $X$.
2. What is the variance of this estimate?

(d) Verify if the MAP estimate equals the MMSE estimate.

**6.13** Consider the problem where the observation is given by $Y = \ln X + N$, where $X$ is the parameter to be estimated . $X$ is uniformly distributed over the interval $[0,1]$, and $N$ has an exponential distribution given by

$$f_N(n) = \begin{cases} e^{-n}, & n \geq 0 \\ 0 \quad , & \text{otherwise} \end{cases}$$

Obtain

(a) The mean-square estimate, $\hat{x}_{ms}$ .

(b) The MAP estimate, $\hat{x}_{map}$ .

(c) The MAVE estimate, $\hat{x}_{mave}$ .

**6.14** The observation $Y$ is given by $Y = X + N$, where $X$ and $N$ are two random variables. $N$ is normal with mean one and variance $\sigma^2$, and $X$ is uniformly distributed over the interval [0, 2]. Determine the MAP estimate of the parameter $X$.

**6.15** Show that the mean-square estimation $\hat{\boldsymbol{\theta}}_{ms} = E[\boldsymbol{\theta} \mid \boldsymbol{y}]$ commutes over a linear transformation.

**6.16** Suppose that the joint density function of the observation $Y$ and the parameter $\theta$ is Gaussian. The means $m_y$ and $m_\theta$ are assumed to be zero. $\theta$ can then be expressed as a linear form of the data. Determine an expression for the conditional density $f_{\Theta|Y}(\theta \mid y)$.

**6.17** Consider the problem of estimating a parameter $\theta$ from one observation $Y$. Then, $Y = \theta + N$, where $\theta$ and the noise $N$ are statistically independent with

$$f_\Theta(\theta) = \begin{cases} 1, & 0 \leq \theta \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_N(n) = \begin{cases} \dfrac{n}{2}, & 0 \leq n \leq 2 \\ 0 \ , & \text{otherwise} \end{cases}$$

Determine $\hat{\theta}_{blue}$, the best linear unbiased estimate of $\theta$.

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# Chapter 7

# Filtering

## 7.1 INTRODUCTION

In Chapter 6, we developed techniques for estimating random and nonrandom parameters. We also studied measures to determine the "goodness" of the estimates. In many applications, the goal was to estimate a signal waveform from a noisy version of the signal in an "optimal" manner.

In this chapter, we assume that the received signal is corrupted by an additive noise. We would like to extract the desired signal from the received signal based on the *linear minimum mean-square error* criterion. The received process signal, $Y(t)$, is observed over some interval of time $t \in [t_i, t_f]$, where $t_i$, denotes initial time and $t_f$ denotes final time. The problem is to determine $\hat{Y}(t)$, a linear estimate of $Y(t)$. When $t$ is outside the interval, we talk about *prediction*. If $t < t_i$, then $\hat{Y}(t)$ is a *backward predictor*. If $t > t_f$, then $\hat{Y}(t)$ is a *forward predictor*. When $t \in [t_i, t_f]$, the problem is referred to as *smoothing*. The process of extracting the information-carrying signal $S(t)$ from the observed signal $Y(t)$, where $Y(t) = S(t) + N(t)$ and $N(t)$ is a noise process, is called *filtering*. In Section 7.2, we define the linear transformation and present some related theorems in some detail. Recall that this concept was introduced and used in the previous chapter without any formal proof. The orthogonality principle theorem will be discussed in some detail, and we also show how it is used in different problems.

In Section 7.3, we discuss the problem of filtering by deriving the impulse response of the system for both realizable and unrealizable filters, continuous and discrete, using spectral factorization. Then, we derive a realizable discrete optimum Wiener filter of a transversal filter with an impulse response of finite duration using the "mean-square method." We conclude the chapter with a section on Kalman filtering.

## 7.2 LINEAR TRANSFORMATION AND ORTHOGONALITY PRINCIPLE

The estimate to be determined $\hat{Y}(t)$ is a *linear* transformation of the received signal $Y(t)$. In this section, we present some useful properties about linear transformations and discuss an important theorem known as "the orthogonality principle" before deriving the *estimation rule*. The estimate may be written as

$$\hat{Y}(t) = L[Y(t)] \tag{7.1}$$

where the operator $L[\cdot]$denotes linear transformation. The estimation rule is based on the minimum mean-square error. Hence, defining the error as

$$\varepsilon(t) = Y(t) - \hat{Y}(t) \tag{7.2}$$

we would like to derive the estimation rule $L[\cdot]$, so that the mean-square error

$$E\left[|\varepsilon(t)|^2\right] = E\left[\left|Y(t) - \hat{Y}(t)\right|^2\right] \tag{7.3}$$

is minimized.

By definition, a transformation $L[\cdot]$ is linear provided that

$$L[a_1 Y_1(t) + a_2 Y_2(t)] = a_1 L[Y_1(t)] + a_2 L[Y_2(t)] \tag{7.4}$$

for all constants $a_1$ and $a_2$ and processes $Y_1(t)$ and $Y_2(t)$. The difference transformation is also linear. That is, if $L_1[\cdot]$ and $L_2[\cdot]$ are two linear transformations, such that

$$L_1[a_1 Y_1(t) + a_2 Y_2(t)] = a_1 L_1[Y_1(t)] + a_2 L_1[Y_2(t)] \tag{7.5}$$

and

$$L_2[a_1 Y_1(t) + a_2 Y_2(t)] = a_1 L_2[Y_1(t)] + a_2 L_2[Y_2(t)] \tag{7.6}$$

then the difference transformation

$$L[\cdot] = L_2[\cdot] - L_1[\cdot] \tag{7.7}$$

is linear. The proof is straightforward by direct substitutions of (7.5) and (7.6) in (7.7). For a linear transformation, it can be shown that

$$E\{L[\cdot]\} = L\{E[\cdot]\} \tag{7.8}$$

where the operator $E[\cdot]$ denotes expectation.

If $Z(t)$ is a process orthogonal to $Y(\xi)$ for all $\xi$ in the interval $[t_i, t_f]$, then *any linear transformation* on $Y(\xi)$ is also orthogonal to $Z(t)$ in the interval $\xi \in [t_i, t_f]$. To prove this statement, we start from the definition that $Y(\xi)$ is orthogonal to $Z(t)$ in the given interval; that is,

$$E[Y(\xi)Z^*(t)] = 0 \quad \text{for} \quad \xi \in [t_i, t_f] \tag{7.9}$$

Let $L[Y(\xi)]$ be a linear transformation of $Y(\xi)$. Since linear operations and expectations are interchangeable as given in (7.8), we have

$$\begin{aligned} E\{L[Y(\xi)]Z^*(t)\} &= E\{L[Y(\xi)Z^*(t)]\} \\ &= L\{E[Y(\xi)Z^*(t)]\} = 0 \quad \text{for} \quad \xi \in [t_i, t_f] \end{aligned} \tag{7.10}$$

which proves that the linear transformation of $Y(\xi)$ is orthogonal to $Z(t)$.

*Theorem. Orthogonality Principle*

The linear transformation $L[\cdot]$ is the minimum mean-square error estimate if and only if the error $\varepsilon(t)$ is orthogonal to $Y(\xi)$ for $\xi \in [t_i, t_f]$.

*Proof.* Let all processes $Y(t), S(t),$ and $N(t)$, where $Y(t) = S(t) + N(t)$, be real and stationary. Consider the linear transformation $L_1[\cdot]$, $L_1[Y(\xi)] = \hat{S}(t)$ for all $\xi$, such that the mean-square error $E[\varepsilon_1^2(t)]$, $\varepsilon_1(t) = S(t) - \hat{S}(t)$ is minimum. That is,

$$L_1[Y(\xi)] = \hat{S}(t) \tag{7.11}$$

is the optimum estimator, and the mean-square error is

$$e_m = E\left[\varepsilon_1^2(t)\right] = E\left[\{S(t) - L_1[Y(\xi)]\}^2\right] = 0 \tag{7.12}$$

Consider the linear transformation $L_2[\cdot]$, $L_2[Y(\xi)] = \hat{S}(t)$ for all $\xi$, such that the error $\varepsilon_2(t)$, where $\varepsilon_2(t) = S(t) - \hat{S}(t)$ is orthogonal to the data $Y(\xi)$ for all $\xi$. That is,

$$E[\varepsilon_2(t)Y(t)] = E\left\{\left[S(t) - \hat{S}(t)\right]Y(t)\right\} = 0 \tag{7.13}$$

The error $\varepsilon_1(t)$ can then be expressed in terms of $\varepsilon_2(t)$ as

$$\begin{aligned}\varepsilon_1(t) &= S(t) - L_1[Y(\xi)] = S(t) + \{L_2[Y(\xi)] - L_2[Y(\xi)]\} - L_1[Y(\xi)] \\ &= \varepsilon_2(t) + L_2[Y(\xi)] - L_1[Y(\xi)] = \varepsilon_2(t) + L[Y(\xi)]\end{aligned} \tag{7.14}$$

where the difference transformation $L[\cdot] = L_2[\cdot] - L_1[\cdot]$ is linear as given by (7.7). Substituting (7.14) into (7.12), the linear mean-square error using the optimum estimator becomes

$$\begin{aligned}e_m &= E[\{S(t) - L_1[Y(\xi)]\}^2] = E\left[\{\varepsilon_2(t) + L[Y(\xi)]\}^2\right] \\ &= E\left[\varepsilon_2^2(t)\right] + 2E\{\varepsilon_2(t)L[Y(\xi)]\} + E\left[\{L[Y(\xi)]\}^2\right]\end{aligned} \tag{7.15}$$

Since $\varepsilon_2(t)$ is orthogonal to the data, $\varepsilon_2(t)$ is also orthogonal to $L[Y(\xi)]$ as shown in (7.9). Thus,

$$E\{\varepsilon_2(t)L[Y(\xi)]\} = 0 \tag{7.16}$$

and the minimum mean-square error reduces to

$$e_m = E\left[\varepsilon_2^2(t)\right] + E\left[\{L[Y(\xi)]\}^2\right] \tag{7.17}$$

where $E\left[\varepsilon_2^2(t)\right]$ is the mean-square error with $L_2[\cdot]$ as the estimator. Therefore,

$$e_m = E\left[\varepsilon_2^2(t)\right] + E\left[\left\{L\left[Y(\xi)\right]\right\}^2\right] = E\left[\varepsilon_2^2(t)\right] \tag{7.18}$$

if and only if the nonnegative quantity $E\left[\left\{L\left[Y(\xi)\right]\right\}^2\right]$ is zero. That is,

$$L[\cdot] = L_2[\cdot] - L_1[\cdot] = 0 \tag{7.19}$$

Hence, this proves our theorem, which says that the linear transformation for which the error is orthogonal to data results in the minimum mean-square error linear estimator and vice versa.

We now derive a simple expression for $e_m$, the minimum mean-square error, using the fact that the error is orthogonal to the data, which is given by

$$E\left\{\left[S(t) - \hat{S}(t)\right]\ Y(\xi)\right\} = 0 \tag{7.20}$$

Substituting (7.9) into (7.20), the above expression becomes

$$E\left[\left\{S(t) - L\left[Y(\xi)\right]\right\} L\left[Y(\xi)\right]\right] = 0 \tag{7.21}$$

Consequently, the minimum mean-square error reduces to

$$e_m = E\left[\left\{S(t) - L\left[Y(\xi)\right]\right\} S(t)\right] \tag{7.22}$$

**Example 7.1**

Let the observation process be $Y(t) = S(t) + N(t)$, where $S(t)$ and $N(t)$ are zero mean wide-sense stationary processes. Obtain an estimate of $S(t)$ in terms of the present value of $Y(t)$, and determine the minimum mean-square error.

*Solution*

In this case, the problem is to estimate the constant *a*, such that the estimate $\hat{S}(t)$ is given by

$$\hat{S}(t) = aY(t)$$

The linear minimum mean-square error estimator results in the requirement that the error $\varepsilon(t) = S(t) - \hat{S}(t)$ be orthogonal to the observed data $Y(t)$. That is,

$$E\left\{\left[S(t) - \hat{S}(t)\right]Y(t)\right\} = E\left\{\left[S(t) - aY(t)\right]Y(t)\right\} = E\left[S(t)Y(t)\right] - aE\left[Y^2(t)\right]$$
$$= R_{sy}(0) - aR_{yy}(0) = 0$$

Solving for $a$, we obtain

$$a = \frac{R_{sy}(0)}{R_{yy}(0)}$$

The minimum mean-square error is given by

$$e_m = E\left\{\left[S(t) - \hat{S}(t)\right]S(t)\right\} = E\left\{\left[S(t) - aY(t)\right]S(t)\right\} = E\left[S^2(t)\right] - aE\left[Y(t)S(t)\right]$$
$$= R_{ss}(0) - aR_{ys}(0)$$

Substituting for the value of $a$ and noting that $R_{sy}(0) = R_{ys}(0)$, the minimum mean-square error becomes

$$e_m = R_{ss}(0) - \frac{R_{sy}^2}{R_{yy}(0)} = \frac{R_{ss}(0)R_{yy}(0) - R_{sy}^2(0)}{R_{yy}(0)}$$

If, in addition, the signal and noise processes are statistically independent, then

$$R_{sy}(0) = E\left[S(t)Y(t)\right] = E\left\{S(t)\left[S(t) + N(t)\right]\right\} = R_{ss}(0)$$

since $E\left[S(t)N(t)\right] = E\left[S(t)\right]E\left[N(t)\right] = 0$. Also,

$$R_{yy}(0) = E\left\{\left[S(t) + N(t)\right]\left[S(t) + N(t)\right]\right\} = R_{ss}(0) + R_{nn}(0)$$

Therefore,

$$a = \frac{R_{ss}(0)}{R_{ss}(0) + R_{nn}(0)}$$

and

$$e_m = \frac{R_{ss}(0)R_{nn}(0)}{R_{ss}(0)+R_{nn}(0)} = aR_{nn}(0)$$

It should be noted that if the processes were not zero mean, then the estimator must solve for the constants *a* and *b*, such that

$$\hat{S}(t) = aY(t) + b$$

**Example 7.2** (*Interpolation*)

Estimate $Y(t)$ in the time interval $t \in [0, T]$ given $Y(0)$ and $Y(T)$. Determine the minimum mean-square error.

*Solution*

The problem of estimating a signal at any instant in an interval of time, given the values of the signal at the end of the interval, is known as *interpolation*. Using a linear estimator, the estimate $\hat{Y}(t)$ may be written as

$$\hat{Y}(t) = aY(0) + bY(T)$$

where *a* and *b* are constants to be determined. Since we require that the error be orthogonal to the data, we have

$$E\{[Y(t) - aY(0) - bY(T)]Y(0)\} = 0$$

$$E\{[Y(t) - aY(0) - bY(T)]Y(T)\} = 0$$

It follows that

$$R_{yy}(t) = aR_{yy}(0) + bR_{yy}(T)$$

$$R_{yy}(T-t) = aR_{yy}(T) + bR_{yy}(0)$$

We have two equations in two unknowns. Solving for *a* and *b*, we obtain

$$a = \frac{R_{yy}(0)R_{yy}(t) - R_{yy}(T)R_{yy}(T-t)}{R_{yy}^2(0) - R_{yy}^2(T)}$$

and

$$b = \frac{R_{yy}(0)R_{yy}(T-t) - R_{yy}(t)R_{yy}(T)}{R_{yy}^2(0) - R_{yy}^2(T)}$$

The minimum mean-square error is

$$e_m = E\{[Y(t) - aY(0) - bY(T)]Y(t)\} = R_{yy}(0) - aR_{yy}(t) - bR_{yy}(t-T)$$

where $a$ and $b$ are as given above.

We observe that if $t = T/2$, then

$$a = b = \frac{R_{yy}(T/2)}{R_{yy}(0) + R_{yy}(T)} \quad \text{and} \quad e_m = R_{yy}(0) - (a+b)R_{yy}(T/2)$$

**Example 7.3** (*The Yule-Walker Equations*)

We now consider the problem where we are given $K$ random variables, $Y_1, Y_2, \ldots, Y_K$, and we need to determine the linear minimum mean-square error estimator for the random variables $S$. Since the estimator is linear, the estimate $\hat{S}$ is given by

$$\hat{S} = a_1 Y_1 + a_2 Y_2 + \ldots + a_K Y_K \tag{7.23}$$

From the orthogonality principle, the mean-square error is minimum if and only if the constants $a_1, a_2, \ldots, a_K$ are chosen so that the error is orthogonal to the data. That is,

$$E[(S - a_1 Y_1 - a_2 Y_2 - \ldots - a_K Y_K)Y_k] = 0, \quad k = 1, 2, \ldots, K \tag{7.24a}$$

or

$$E[SY_k] - a_1 E[Y_1 Y_k] - a_2 E[Y_2 Y_k] - \ldots - a_K E[Y_K Y_k] = 0, \quad k = 1, 2, \ldots, K \tag{7.24b}$$

Defining

$$E[SY_k] = R_{0k} \tag{7.25a}$$

$$E[Y_j Y_k] = R_{jk} \tag{7.25b}$$

we obtain, from (7.24), the following set of $K$ equations in $K$ unknowns, known as the *Yule-Walker equations*, as discussed in Chapter 4.

$$\begin{aligned} a_1 R_{11} + a_2 R_{12} + \ldots + a_K R_{1K} &= R_{01} \\ a_1 R_{21} + a_2 R_{22} + \ldots + a_K R_{2K} &= R_{02} \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots & \\ a_1 R_{K1} + a_2 R_{K2} + \ldots + a_K R_{KK} &= R_{0K} \end{aligned} \tag{7.26}$$

The solution yields the constants $a_1, a_2, \ldots, a_K$. In matrix form, (7.26) can be written as

$$\boldsymbol{R}\boldsymbol{a} = \boldsymbol{R}_0 \tag{7.27a}$$

where

$$\boldsymbol{R} = \begin{bmatrix} R_{11} & R_{12} & \ldots & R_{1K} \\ R_{21} & R_{22} & \ldots & R_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ R_{K1} & R_{K2} & \ldots & R_{KK} \end{bmatrix} \tag{7.27b}$$

$$\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \ldots & a_K \end{bmatrix}^T \tag{7.27c}$$

and

$$\boldsymbol{R}_0 = \begin{bmatrix} R_{01} & R_{02} & \ldots & R_{0K} \end{bmatrix}^T \tag{7.27d}$$

Since $R_{jk} = R_{kj}$, $j, k = 1, 2, \ldots, K$, in the Yule-Walker equations, the coefficients are obtained from (7.27) to be

$$\boldsymbol{a} = \boldsymbol{R}^{-1}\boldsymbol{R}_0 \tag{7.28}$$

Note that the data correlation matrix is given by

$$\boldsymbol{R} = \boldsymbol{R}_{\boldsymbol{YY}} = E\left[\boldsymbol{Y}\boldsymbol{Y}^T\right] \tag{7.29a}$$

where $\boldsymbol{Y}^T$ denotes the transpose of $\boldsymbol{Y}$ and

$$\boldsymbol{Y} = \begin{bmatrix} Y_1 & Y_2 & \dots & Y_K \end{bmatrix}^T \tag{7.29b}$$

The estimator is simply the inner product of $\boldsymbol{a}$ and $\boldsymbol{Y}$; that is,

$$\hat{S} = \boldsymbol{a}^T \boldsymbol{Y} = \boldsymbol{Y}^T \boldsymbol{a} \tag{7.30}$$

Geometrically, $\hat{S}$ is the projection of $S$ onto the surface spanned by $Y_1, Y_2, \dots, Y_K$. This is illustrated in Figure 7.1 for $K = 2$.

The mean-square vector is

$$E\left[\hat{\boldsymbol{S}}^2\right] = E\left[\hat{\boldsymbol{S}}\boldsymbol{S}^T\right] = E\left[\boldsymbol{a}^T \boldsymbol{Y}\boldsymbol{Y}^T \boldsymbol{a}\right] = \boldsymbol{a}^T \boldsymbol{R}\boldsymbol{a} \tag{7.31}$$

Similarly, we can determine the minimum mean-square error to be

$$e_m = E\left[(S - \hat{S})S\right] = E\left[(S - a_1 Y_1 - a_2 Y_2 - \dots - a_K Y_K)S\right]$$

$$= R_{00} - a_1 R_{01} - a_2 R_{02} - \dots - a_K R_{0K} \tag{7.32}$$

where $R_{00} = E[S^2]$. In matrix form, the minimum mean-square error is expressed as

$$e_m = R_{00} - \boldsymbol{a}^T \boldsymbol{R}_0 \tag{7.33}$$

substituting for (7.28) in (7.33), we have

$$e_m = R_{00} - (\boldsymbol{R}^{-1}\boldsymbol{R}_0)^T \boldsymbol{R}_0 = R_{00} - \boldsymbol{R}_0^T (\boldsymbol{R}^{-1})^T \boldsymbol{R}_0 \tag{7.34}$$

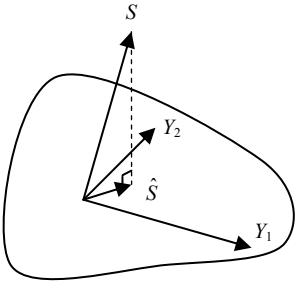


**Figure 7.1** Projection of $S$ onto plane spanned by $Y_1$ and $Y_2$.

When the data has a nonzero mean vector, we add a constant to the estimate given by (7.23), which becomes

$$\hat{S} = a_0 + a_1 Y_1 + a_2 Y_2 + \ \ldots \ + a_K Y_K \tag{7.35}$$

Following the above procedure, we obtain the set of $(K+1)$ equations

$$\begin{aligned} a_0 + m_1 a_1 + \ \ldots \ + m_K a_K &= m_S \\ m_1 a_0 + R_{11} a_1 + \ldots + R_{1K} a_K &= R_{00} \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad & \vdots \\ m_K a_0 + R_{K1} a_1 + \ \ldots \ + R_{KK} a_K &= R_{0K} \end{aligned} \tag{7.36}$$

where $m_s = E\left[\hat{S}\right]$ and $m_k = E\left[Y_k\right]$.

## 7.3 WIENER FILTERS

We now consider the case where the data is not a finite number of random variables as in the previous examples, but rather a random process, $X(\xi)$, observed over an interval of time $\xi \in [t_i, t_f]$. The goal is to estimate another process, $Y(t)$, by a *linear function* of $X(\xi)$, such that

$$\hat{Y}(t) = \int_{t_i}^{t_f} h(\xi) X(\xi) d\xi \tag{7.37}$$

The weighting function $h(\xi)$ is to be determined based on the minimum mean-square error criterion. The orthogonality principle requires that the error $\varepsilon(t) = Y(t) - \hat{Y}(t)$ be orthogonal to observed data $X(\xi)$ during the interval $\xi \in [t_i, t_f]$. Hence,

$$E\left\{\left[ Y(t) - \int_{t_i}^{t_f} h(\xi, \lambda) X(\xi) d\xi \right] X(\lambda)\right\} = 0, \quad \lambda \in [t_i, t_f] \tag{7.38}$$

Assuming that the processes are stationary, in which case the filter used is time-invariant, (7.38) becomes

$$R_{yx}(t-\lambda) = \int_{t_i}^{t_f} R_{xx}(\xi-\lambda)h(\xi)d\xi\,, \quad \lambda \in [t_i, t_f] \tag{7.39}$$

Solving the integral equation of (7.39) results in the desired weighting function $h(\xi)$.

The minimum mean-square error is

$$e_m = E\left\{\left[Y(t) - \int_{t_i}^{t_f} h(\xi)X(\xi)d\xi\right]Y(t)\right\} \tag{7.40}$$

or

$$e_m = R_{yy}(0) - \int_{t_i}^{t_f} h(\xi)R_{yx}(t-\xi)\,d\xi \tag{7.41}$$

### 7.3.1 The Optimum Unrealizable Filter

We now let $Y(t) = S(t) + N(t)$, where $S(t)$ is the desired signal to be estimated from the data $Y(t)$ for all time *t*. That is, we wish to obtain the optimum linear time-invariant filter, such that the mean-square error estimate of $S(t)$ is minimum. We assume all processes to be stationary, real, and zero mean. The system is required to be time-invariant, and consequently the desired estimate may be expressed as

$$\hat{S}(t) = \int_{-\infty}^{\infty} h(t-\xi)Y(\xi)d\xi \tag{7.42}$$

Note that in this case, the filter is not constrained to be realizable since we do not require a causal system. The impulse response $h(t-\xi)$ may be zero for $t < \xi$. This *filtering* problem can be represented by the block diagram of Figure 7.2.

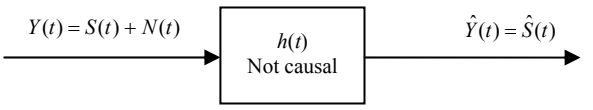


**Figure 7.2** Filtering *S*(*t*).

The orthogonality principle requires that

$$E\left\{\left[S(t)-\int_{-\infty}^{\infty}h(\xi)Y(t-\xi)d\xi\right]Y(\lambda)\right\}=0 \quad \text{for all } \lambda \tag{7.43}$$

That is,

$$R_{sy}(t-\lambda)=\int_{-\infty}^{\infty}R_{yy}(t-\xi-\lambda)h(\xi)d\xi \tag{7.44}$$

Let $t-\lambda=\tau$, then (7.44) becomes

$$R_{sy}(\tau)=\int_{-\infty}^{\infty}R_{yy}(\tau-\xi)h(\xi)d\xi=R_{yy}(\tau)*h(\tau) \tag{7.45}$$

where $*$ denotes convolution. It is easier to solve the above equation in frequency domain. Taking the Fourier transform of (7.45), we obtain

$$S_{sy}(f)=S_{yy}(f)H(f) \tag{7.46}$$

or

$$H(f)=\frac{S_{sy}(f)}{S_{yy}(f)} \tag{7.47}$$

The minimum mean-square error is given by

$$e_m=E\left\{\left[S(t)-\int_{-\infty}^{\infty}Y(t-\xi)h(\xi)d\xi\right]S(t)\right\}=R_{ss}(0)-\int_{-\infty}^{\infty}R_{sy}(\xi)h(\xi)d\xi \tag{7.48a}$$

where

$$\int_{-\infty}^{\infty}R_{sy}(\xi)h(\xi)d\xi=[R_{sy}(-\tau)*h(\tau)]\Big|_{\tau=0} \tag{7.48b}$$

Expressing the error of (7.48) in terms of the power spectral density, we obtain

$$e_m = \int_{-\infty}^{\infty} [S_{ss}(f) - S_{sy}(-f)H(f)]df \tag{7.49}$$

Substituting (7.47) in (7.49), the error becomes

$$e_m = \int_{-\infty}^{\infty} \left[ S_{ss}(f) - \frac{S_{sy}(-f)S_{sy}(f)}{S_{yy}(f)} \right] df \tag{7.50}$$

If $S(t)$ and $N(t)$ are statistically independent, then

$$E[S(t)N(t)] = E[S(t)]E[N(t)] = 0 \tag{7.51}$$

since they are zero-mean. It follows that

$$S_{yy}(f) = S_{ss}(f) + S_{nn}(f) \tag{7.52}$$

$$S_{sy}(f) = S_{ss}(f) \tag{7.53}$$

Therefore, the transfer function $H(f)$ is

$$H(f) = \frac{S_{sy}(f)}{S_{yy}(f)} = \frac{S_{ss}(f)}{S_{ss}(f) + S_{nn}(f)} \tag{7.54}$$

The resulting minimum mean-square error in this case is

$$e_m = \int_{-\infty}^{\infty} \left[ S_{ss}(f) - \frac{S_{ss}^2(f)}{S_{ss}(f) + S_{nn}(f)} \right] df = \int_{-\infty}^{\infty} \frac{S_{ss}(f)S_{nn}(f)}{S_{ss}(f) + S_{nn}(f)} df \tag{7.55}$$

We note that if the power spectral densities $S_{ss}(f)$ and $S_{nn}(f)$ do not overlap, then

$$\begin{aligned} S_{ss}(f) &= 0 \quad \text{when} \quad S_{nn}(f) \neq 0 \\ S_{nn}(f) &= 0 \quad \text{when} \quad S_{ss}(f) \neq 0 \end{aligned} \tag{7.56}$$

The transfer function $H(f)$ becomes

$$H(f)=\begin{cases}1 & \text{for } f \text{ such that } S_{ss}(f)\neq 0\\ 0 & \text{for } f \text{ such that } S_{nn}(f)\neq 0\end{cases} \tag{7.57}$$

For these power spectral densities of $S_{ss}(f)$ and $S_{nn}(f)$, which are nonoverlapping, the product $S_{ss}(f)S_{nn}(f)$ is zero, and thus the minimum mean-square error is zero.

**Example 7.4**

Let the observation process for all time $t$ be $Y(t)=S(t)+N(t)$.

(a) Obtain the linear mean-square error of $S'(t)$, the derivative of $S(t)$.

(b) Determine the impulse response $h(t)$ given that

$$R_{ss}=e^{-\alpha\tau^2},\quad R_{sn}(\tau)=0,\ \text{and}\quad R_{nn}(\tau)=k\delta(\tau)$$

*Solution*

(a) The linear mean-square error estimate of $S'(t)$ is given by

$$\hat{S}'(t)=\int_{-\infty}^{\infty}Y(t-\xi)h(\xi)d\xi$$

Since the error is orthogonal to the data, we have

$$E\left\{\left[\hat{S}'(t)-\int_{-\infty}^{\infty}Y(t-\alpha)h(\alpha)d\alpha\right]Y(\xi)\right\}=0\qquad\text{for all }\xi$$

or, $R_{s'y}(t-\xi)=\int_{-\infty}^{\infty}R_{yy}(t-\alpha-\xi)h(\alpha)d\alpha$. Let $\tau=t-\xi$, then

$$R_{s'y}(\tau)=\int_{-\infty}^{\infty}R_{yy}(\tau-\alpha)h(\alpha)d\alpha\qquad\text{for all }\tau$$

Taking the Fourier transform of the above expression, we obtain

$$jfS_{sy}(f)=S_{yy}(f)H(f)\quad\text{or}\quad H(f)=jf\frac{S_{sy}(f)}{S_{yy}(f)}$$

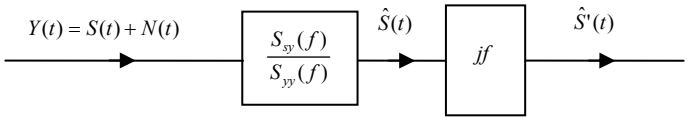


**Figure 7.3** Filter for estimating $\hat{S}'(t)$.

Hence, the optimum filter for estimating $S'(t)$ is a cascade of two systems; the first system has a transfer function $S_{sy}(f)/S_{yy}(f)$, while the second system is a differentiator as shown in Figure 7.3.

(b) Using (7.52) and (7.53), we have $R_{yy}(\tau) = R_{ss}(\tau) + R_{nn}(\tau)$ and $R_{sy}(\tau) = R_{ss}(\tau)$, since $R_{sn}(\tau) = 0$. Consequently, the transfer function becomes

$$H(f) = jf \frac{S_{ss}(f)}{S_{ss}(f) + S_{nn}(f)}$$

We need to determine $S_{ss}(f)$ from $R_{ss}(\tau)$. The Fourier transform of the autocorrelation function $R_{ss}(\tau)$ is

$$S_{ss}(f) = \int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-j2\pi f\tau} d\tau = \sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\pi^2 f^2}{\alpha}\right)$$

where we have used the fact that

$$\int_{-\infty}^{\infty} \exp[-(ax^2 + 2bx + c)]\, dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2 - ac}{a}\right) \quad \text{for} \quad a > 0$$

The transfer function becomes

$$H(f) = jf \frac{\sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\pi^2 f^2}{\alpha}\right)}{\sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\pi^2 f^2}{\alpha}\right) + k} = jf \frac{\frac{1}{k}\sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\pi^2 f^2}{\alpha}\right)}{1 + \frac{1}{k}\sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\pi^2 f^2}{\alpha}\right)} = jf\, G(f)$$

where

$$G(f) = \frac{\dfrac{1}{k}\sqrt{\dfrac{\pi}{\alpha}}\exp\left(-\dfrac{\pi^2 f^2}{\alpha}\right)}{1+\dfrac{1}{k}\sqrt{\dfrac{\pi}{\alpha}}\exp\left(-\dfrac{\pi^2 f^2}{\alpha}\right)}$$

Thus, $h(t) = dg(t)/dt$, where $g(t)$ is the inverse Fourier transform of $G(f)$. It can be shown that the impulse response $h(t)$ is given by

$$h(t) = \frac{2\alpha^{3/2}}{\sqrt{\pi}} t \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{3/2}} \left(\frac{1}{k}\sqrt{\frac{\pi}{\alpha}}\right)^n \exp\left(-\frac{t^2}{\alpha n}\right)$$

The above series converges, provided that $(1/k)\sqrt{\pi/\alpha} < 1$.

**Example 7.5**

(a) Determine the optimum unrealizable filter $h(t)$ of the observation process $Y(t) = S(t) + N(t)$. $S(t)$ and $N(t)$ are uncorrelated with autocorrelation functions

$$R_{ss}(\tau) = e^{-\alpha|\tau|} \qquad \text{and} \qquad R_{nn}(\tau) = \frac{N_0}{2}\delta(\tau)$$

(b) Calculate the minimum-mean-square error.

*Solution*

(a) The transfer function $H(f)$ is given by (7.54), where $S_{ss}(f)$ and $S_{nn}(f)$ are the Fourier transforms of $R_{ss}(\tau)$ and $R_{nn}(\tau)$, respectively. Hence,

$$H(f) = \frac{2\alpha/(\alpha^2 + 4\pi^2 f^2)}{[2\alpha/(\alpha^2 + 4\pi^2 f^2)] + N_0/2} = \frac{a}{b^2 + 4\pi^2 f^2}$$

where $a = 4\alpha/N_0$ and $b^2 = \alpha^2 + (4\alpha/N_0)$. It follows that

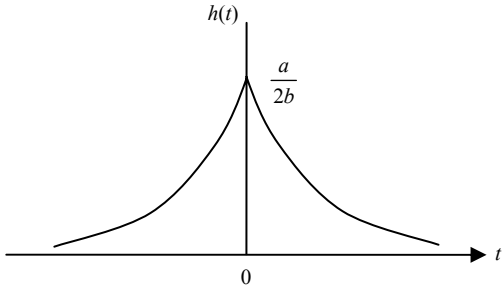


**Figure 7.4** Filter $h(t)$ for Example 7.5.

$$h(t) = \int_{-\infty}^{\infty} \frac{a}{b^2 + 4\pi^2 f^2} e^{j2\pi ft} df = \frac{a}{2b} e^{-b|t|} = \frac{2\alpha}{N_0 \sqrt{\alpha^2 + \frac{4\alpha}{N_0}}} \exp\left( -\sqrt{\alpha^2 + \frac{4\alpha}{N_0}} |t| \right)$$

From the plot of $h(t)$ shown in Figure 7.4, we observe that the impulse response $h(t)$ is noncausal and thus nonrealizable.

(b) The minimum mean-square error is given by (7.55). Substituting for $S_{ss}(f)$ and $S_{nn}(f)$ and solving the integral, we obtain the minimum mean-square error to be

$$e_m = \frac{\alpha}{b} = \frac{\alpha}{\sqrt{\alpha^2 + \frac{4\alpha}{N_0}}}$$

### 7.3.2 The Optimum Realizable Filter

In the previous section, we solved for the optimum unrealizable filter to extract the desired signal from the observation process $Y(t) = S(t) + N(t)$. We now consider the same problem with the constraint that the filter $h(t)$ is *realizable*; that is, $h(t) = 0$ for $t < 0$. The system representing the problem is shown in Figure 7.5.

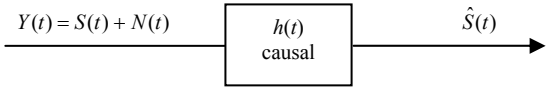


**Figure 7.5** Optimum realizable filter.

We assume that the signal process $Y(t)$ is known only up to the present moment $t$. Therefore, the estimate of $S(t)$ is

$$\hat{S}(t) = \int_{-\infty}^{t} h(t-\xi)Y(\xi)d\xi = \int_{0}^{\infty} Y(t-\xi)h(\xi)d\xi \tag{7.58}$$

The linear mean-square estimation requires that $h(t)$ be chosen so that the mean-square error is minimum. From the orthogonality principle, we have

$$E\left\{\left[S(t) - \int_{0}^{\infty} Y(t-\xi)h(\xi)d\xi\right]Y(t')\right\} = 0 \quad \text{for all } t' \leq t \tag{7.59}$$

The impulse response of the realizable filter satisfies the integral equation

$$R_{sy}(t-t') = \int_{0}^{\infty} R_{yy}(t-\xi-t')h(\xi)d\xi \tag{7.60}$$

Let $t - t' = \tau$; then the integral equation becomes

$$R_{sy}(\tau) = \int_{0}^{\infty} R_{yy}(\tau-\xi)h(\xi)d\xi \qquad \text{for all } \tau \geq 0 \tag{7.61}$$

Equation (7.61) is called the *Wiener-Hopf integral equation*. Furthermore, the mean-square error of the estimate reduces to

$$e_m = E\left\{\left[S(t) - \int_{0}^{\infty} Y(t-\xi)h(\xi)d\xi\right]S(t)\right\} = R_{ss}(0) - \int_{0}^{\infty} R_{sy}(\xi)h(\xi)d\xi \tag{7.62}$$

The solution to the Wiener-Hopf integral equation is not as easy as in the case of the unrealizable filter, since the integral is valid for positive $\tau$ only. Equation (7.61) can be written as

$$R_{sy}(\tau) = \int_{-\infty}^{\infty} R_{yy}(\tau-\xi)h(\xi)d\xi, \quad \tau \geq 0 \tag{7.63}$$

It can be shown that the choice of the above integral for negative $\tau$ is not arbitrary when the additional constraint $h(\xi)$ causal is imposed. Moreover,

$$\int_{-\infty}^{\infty} R_{yy}(\tau-\xi)h(\xi)d\xi \neq R_{sy}(\tau), \quad \tau < 0 \tag{7.64}$$

Consequently, we cannot obtain the impulse response of the optimum realizable filter by simply using the frequency domain approach, as we did with the unrealizable filter.

To obtain an integral equation valid for all $\tau$, $-\infty < \tau < \infty$, we combine (7.63) and (7.64), which results in

$$\int_{-\infty}^{\infty} R_{yy}(\tau-\xi)h(\xi)d\xi - R_{sy}(\tau) = a(\tau) \quad \text{for all } \tau \tag{7.65}$$

where $a(\tau)$ is zero for $\tau$ positive, and

$$a(\tau) = \int_{-\infty}^{\infty} A(f)e^{+j2\pi f\tau}df, \quad \tau < 0 \tag{7.66}$$

Assume that the power spectral density $S_{yy}(f)$ is a rational function and can be factored into

$$S_{yy}(f) = S_{yy}^{+}(f)S_{yy}^{-}(f) \tag{7.67}$$

$S_{yy}^{+}(f)$ and its conjugate $S_{yy}^{-}(f)$ are called the *spectral factorizations* of $S_{yy}(f)$. $S_{yy}^{+}(f)$ has all its poles and zeros in the left half-plane (LHP) of the *S*-plan ($s = j\omega$, $\omega = 2\pi f$), whereas $S_{yy}^{-}(f)$ has all its poles and zeros in the right half-plane (RHP). Taking the Fourier transform of (7.65), we have

$$S_{yy}(f)H(f) - S_{sy}(f) = A(f) \tag{7.68}$$

or

$$S_{yy}^{+}(f)S_{yy}^{-}(f)H(f) - S_{sy}(f) = A(f) \tag{7.69}$$

Dividing (7.69) by $S_{yy}^{-}(f)$, we obtain

$$S_{yy}^{+}(f)H(f)-\frac{S_{sy}(f)}{S_{yy}^{-}(f)}=\frac{A(f)}{S_{yy}^{-}(f)} \tag{7.70}$$

Note that since $S_{yy}^{+}(f)$ and $H(f)$ have all their poles in the LHP, the product $S_{yy}^{+}(f)H(f)$ has its poles in the LHP, and consequently the corresponding time function is zero for $\tau<0$. Since $S_{yy}^{-}(f)$ has its zeros in the RHP and $A(f)$ has all its poles in the RHP, the quotient $A(f)/S_{yy}^{-}(f)$ has all its poles in the RHP. Consequently, the corresponding time function is zero for $\tau>0$. The ratio $S_{sy}(f)/S_{yy}^{-}(f)$ has poles in the LHP and RHP. Thus, its corresponding time function is valid for all $\tau$. Splitting the poles and zeros, the ratio $S_{sy}(f)/S_{yy}^{-}(f)$ may be expressed as

$$\frac{S_{sy}(f)}{S_{sy}^{-}(f)}=\left[\frac{S_{sy}(f)}{S_{yy}^{-}(f)}\right]^{+}+\left[\frac{S_{sy}(f)}{S_{yy}^{-}(f)}\right]^{-} \tag{7.71}$$

$[S_{sy}(f)/S_{yy}^{-}(f)]^{+}$ has all its poles and zeros in the LHP, whereas $[S_{sy}(f)/S_{yy}^{-}(f)]^{-}$ has all its poles and zeros in the RHP. Substituting (7.71) into (7.70), we obtain

$$\underbrace{S_{yy}^{+}(f)H(f)}_{\text{LHP}}-\underbrace{\left[\frac{S_{sy}(f)}{S_{yy}^{-}(f)}\right]^{+}}_{\text{LHP}}-\underbrace{\left[\frac{S_{sy}(f)}{S_{yy}^{-}(f)}\right]^{-}}_{\text{RHP}}=\underbrace{\frac{A(f)}{S_{yy}^{-}(f)}}_{\text{RHP}} \tag{7.72}$$

Define

$$B^{+}(f)=\left[\frac{S_{sy}(f)}{S_{yy}^{-}(f)}\right]^{+},\quad B^{-}(f)=\left[\frac{S_{sy}(f)}{S_{yy}^{-}(f)}\right]^{-} \tag{7.73}$$

Equating the terms marked by LHP (all poles in LHP) in (7.72), we have

$$S_{yy}^{+}(f)H(f)-B^{+}(f)=0 \tag{7.74}$$

or, the transfer function $H(f)$ is

$$H(f) = \frac{B^+(f)}{S_{yy}^+(f)} \tag{7.75}$$

All poles of $H(f)$ are in LHP, and consequently the filter response $h(t)$ is zero for $t$ negative, and thus it is realizable. Therefore, the optimum filter is

$$h(t) = \int_{-\infty}^{\infty} \frac{B^+(f)}{S_{yy}^+(f)} e^{j2\pi ft} df \tag{7.76}$$

The corresponding minimum mean-square error is obtained by substituting for (7.75) into (7.62) and taking the Fourier transform, which results in

$$e_m = \int_{-\infty}^{\infty} \left[ S_{ss}(f) - S_{sy}(-f) \frac{B^+(f)}{S_{yy}^+(f)} \right] df \tag{7.77}$$

**Example 7.6**

Consider the problem where the signal $S(t)$ and the noise $N(t)$ are uncorrelated, and with autocorrelation functions $R_{ss}(\tau) = e^{-\alpha|\tau|}$ and $R_{nn}(\tau) = (N_0/2)\delta(\tau)$. For simplicity, assume $\alpha = N_0/2 = 1$.

(a) Obtain the optimum realizable filter.
(b) Calculate the minimum mean-square error.

*Solution*

(a) The optimum realizable filter is given by (7.76), where $B^+(f)$ and $S_{yy}^+(f)$ are defined in (7.73a) and (7.67), respectively. Since $S(t)$ and $N(t)$ are uncorrelated, the power spectral density of $Y(t)$ is as given by (7.52). Hence, $S_{yy}(f) = S_{ss}(f) + S_{nn}(f)$, where

$$S_{ss}(f) = \frac{1}{1 + 4\pi^2 f^2} \qquad \text{and} \qquad S_{nn}(f) = 1$$

Substituting for the expressions of $S_{ss}(f)$ and $S_{nn}(f)$, we obtain

$$S_{yy}(f) = \frac{1}{1+4\pi^2 f^2} + 1 = \frac{2+4\pi^2 f^2}{1+4\pi^2 f^2} = \frac{p^2-2}{p^2-1}$$

where $p = j2\pi f$. In order to differentiate between $S$ denoting the power spectral density and $P$ the Laplace transform, we denote the Laplace transform by $p = j2\pi f$. Consequently,

$$S_{yy}(p) = \frac{p^2-2}{p^2-1} = \frac{(p+\sqrt{2})(p-\sqrt{2})}{(p+1)(p-1)} = S_{yy}^{+}(p)S_{yy}^{-}(p)$$

where

$$S_{yy}^{+}(p) = \frac{p+\sqrt{2}}{p+1} \quad \text{and} \quad S_{yy}^{-}(p) = \frac{p-\sqrt{2}}{p-1}$$

We need to determine the cross-spectral density $S_{sy}(f)$. From (7.53), we have $S_{sy}(f) = S_{ss}(f) = 1/(1+4\pi^2 f^2)$, or in the Laplace domain $(p = 2j\pi f)$

$$S_{sy}(p) = S_{ss}(p) = \frac{-1}{p^2-1} = \frac{-1}{(p+1)(p-1)}$$

Hence, from (7.71),

$$\frac{S_{sy}(p)}{S_{sy}^{-}(p)} = B^{+}(p) + B^{-}(p) = \frac{-1}{(p+1)(p-\sqrt{2})} = \frac{1/(1+\sqrt{2})}{p+1} + \frac{-1/(1+\sqrt{2})}{p-\sqrt{2}}$$

Now that we have $B^{+}(p)$ and $S_{yy}^{+}(p)$, we substitute in (7.75) to obtain

$$H(p) = \frac{B^{+}(p)}{S_{yy}^{+}(p)} = \frac{1/(1+\sqrt{2})}{p+1}\frac{p+1}{p+\sqrt{2}} = \frac{1/(1+\sqrt{2})}{p+\sqrt{2}}$$

or, $H(f) = [1/(1+\sqrt{2})][1/(j2\pi f+\sqrt{2})]$. Taking the inverse Laplace transform of $H(p)$, we obtain the optimum realizable filter $h(t)$ to be

$$h(t) = \frac{1}{1+\sqrt{2}} e^{-\sqrt{2}t} u(t)$$

where $u(t)$ is the unit step function.

(b) The minimum mean-square error is given by (7.62) to be $e_m = R_{ss}(0) - \int_0^\infty R_{sy}(\xi) h(\xi) d\xi$. From part (a), we found that $S_{ss}(p) = S_{sy}(p) = -1/(p^2 - 1)$. Taking the inverse Laplace transform, we have

$$R_{ss}(\tau) = R_{sy}(\tau) = \frac{1}{2} e^{-|\tau|}$$

Substituting for the expression of $R_{ss}(0) = 1/2$ and $R_{sy}(\tau)$ into $e_m$, we obtain

$$e_m = \frac{1}{2} - \frac{1}{2(1+\sqrt{2})^2}$$

**Example 7.7**

Let $Y(t) = S(t) + N(t)$, where the signals $S(t)$ and $N(t)$ are statistically independent with zero mean and autocorrelation functions

$$R_{ss}(\tau) = e^{-|\tau|} \qquad \text{and} \qquad R_{nn}(\tau) = \delta(\tau) + 2e^{-|\tau|}$$

(a) Find the optimum unrealizable filter.
(b) Find the optimum realizable filter.

*Solution*

(a) The optimum unrealizable filter is given by (7.47) to be

$$H(f) = \frac{S_{sy}(f)}{S_{yy}(f)}$$

where $S_{sy}(f) = S_{ss}(f)$ and $S_{yy}(f) = S_{ss}(f) + S_{nn}(f)$, since the signal and noise are uncorrelated and zero mean. Hence, $S_{ss}(f) = 1/(1 + 4\pi^2 f^2) = S_{sy}(f)$,

$S_{nn}(f) = 1 + [2/(1+4\pi^2 f^2)]$, and $S_{yy}(f) = (4+4\pi^2 f^2)/(1+4\pi^2 f^2)$.
Substituting into the expression of $H(f)$, we obtain

$$H(f) = \frac{1}{4+4\pi^2 f^2}$$

Consequently,

$$h(t) = \frac{1}{4}e^{-2|t|} = \begin{cases} \frac{1}{4}e^{-2t}, & t \geq 0 \\ \frac{1}{4}e^{2t}, & t < 0 \end{cases}$$

(b) The optimum realizable filter is given by (7.75). First, we factor $S_{yy}(f)$ into $S_{yy}^{+}(f)$ and $S_{yy}^{-}(f)$. Then,

$$S_{yy}(p) = \frac{p^2-4}{p^2-1} = \frac{(p-2)(p+2)}{(p-1)(p+1)}$$

where

$$S_{yy}^{+}(p) = \frac{p+2}{p+1} \quad \text{and} \quad S_{yy}^{-}(p) = \frac{p-2}{p-1}$$

Also,

$$S_{sy}(p) = S_{ss}(p) = \frac{-1}{p^2-1} = \frac{-1}{(p+1)(p-1)}$$

Consequently,

$$\frac{S_{sy}(p)}{S_{yy}^{-}(p)} = \frac{-1}{(p+1)(p-2)} = \frac{1/3}{p+1} + \frac{-1/3}{p-2}$$

where $B^{+}(p) = 1/[3(p+1)]$ and $B^{-}(p) = -1/[3(p-2)]$. The transfer function of the realizable filter is

$$H(p) = \frac{B^+(p)}{S_{yy}^+(p)} = \frac{1/3}{p+2}$$

Taking the inverse Laplace transform of $H(p)$, we obtain the Wiener filter to be

$$h(t) = \begin{cases} \frac{1}{3}e^{-2t}, & t \geq 0 \\ 0 \quad , & t < 0 \end{cases}$$

## 7.4 DISCRETE WIENER FILTERS

We now consider the filtering problem where the observed signal is a discrete random sequence, and the goal is to estimate another random sequence. The incoming sequence composed of the signal sequence $S(n)$, $n = 0, 1, 2, \ldots$ and the additive noise sequence $N(n)$, $n = 0, 1, 2, \ldots$, enter a linear discrete-time filter with impulse response denoted by the sequences $h(n)$, $n = 0, 1, 2, \ldots$, as shown in Figure 7.6.

We assume that the sequences $S(n)$ and $N(n)$ are uncorrelated zero mean random variables. We wish to find $\hat{S}(n)$, the minimum linear mean-square error estimator. The estimate $\hat{S}(n)$ may be expressed as the correlation sum of the sequences $Y(n)$ and $h(n)$, and is given by

$$\hat{S}(n) = \sum_{k=-\infty}^{\infty} h(k)Y(n-k) \tag{7.78}$$

or

$$\hat{S}(n) = \sum_{k=0}^{\infty} h(k)Y(n-k) \tag{7.79}$$

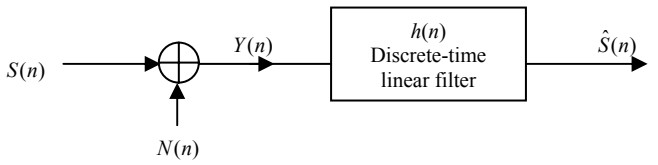


**Figure 7.6** Filtering the sequences $S(n)$.

Equation (7.78) indicates that all data $Y(n)$ is available for all $n$. The sequence is not finite, and thus the filter is *not realizable*. Equation (7.79) indicates that only present and past values of $Y$ are used in estimating $S(n)$. Thus, we have a finite sequence, and the filter is *causal* or *realizable*.

### 7.4.1 Unrealizable Filter

In this case, the estimator is given by (7.78). The criterion used to determine the filter sequence $h(n)$ is the mean-square error and is given by

$$E\left\{\left[S(n)-\hat{S}(n)\right]^2\right\} \tag{7.80}$$

The mean-square error is minimized by applying the orthogonality principle. Thus,

$$E\left\{\left[S(n)-\sum_{k=-\infty}^{\infty}h(k)Y(n-k)\right]Y(n-m)\right\} \quad \text{for all} \quad m \tag{7.81}$$

or

$$R_{sy}(m)=\sum_{k=-\infty}^{\infty}h(k)R_{yy}(m-k) \qquad \text{for all } m \tag{7.82a}$$

where

$$R_{sy}(m)=E\left[S(n)Y(n-m)\right] \tag{7.82b}$$

and

$$R_{yy}(n-m)=E\left[Y(n)Y(m)\right] \tag{7.82c}$$

We define the Fourier transform of a discrete sequence $f(k), k=0,\pm1,\pm2,\ldots$, as

$$F(e^{j\omega})=\sum_{k=-\infty}^{\infty}f(k)e^{-kj\omega} \tag{7.83}$$

where $\omega=2\pi f$. Making the change of variable $z=e^{j2\pi f}$, (7.82) becomes

$$F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k} \tag{7.84}$$

Note that the trajectory of $z = e^{j\omega}$ is the unit circle on the *Z*-plane. $F(e^{j\omega})$ is periodic with period $2\pi$, and hence the spectrum is usually plotted for $\omega \in [-\pi, \pi]$ or $f \in [-1/2, 1/2]$. Taking the *Z*-transform of (7.82), we obtain

$$S_{sy}(z) = H(z) S_{yy}(z) \tag{7.85}$$

or

$$H(z) = \frac{S_{sy}(z)}{S_{yy}(z)} \tag{7.86}$$

The resulting mean-square error is

$$e_m = R_{ss}(0) - \sum_{k=-\infty}^{\infty} h(k) R_{ys}(-k) \tag{7.87}$$

### 7.4.2 Realizable Filter

In this case, the filter is causal, and the estimator is given by (7.79) to be

$$\hat{S}(n) = \sum_{k=0}^{\infty} h(k) Y(n-k) \tag{7.88}$$

Assuming a linear filter with an impulse response of *finite duration*, single input, and single output, the filter can conceptually be realized as shown in Figure 7.7.

Applying the orthogonality principle, we obtain the discrete version of the Wiener-Hopf equation given by

$$R_{sy}(m) = \sum_{k=0}^{\infty} h(k) R_{yy}(m-k) \tag{7.89}$$

where $R_{sy}(m)$ and $R_{yy}(m-k)$ are as defined in (7.82b) and (7.82c), respectively. Let the spectral density $S_{yy}(z)$ be

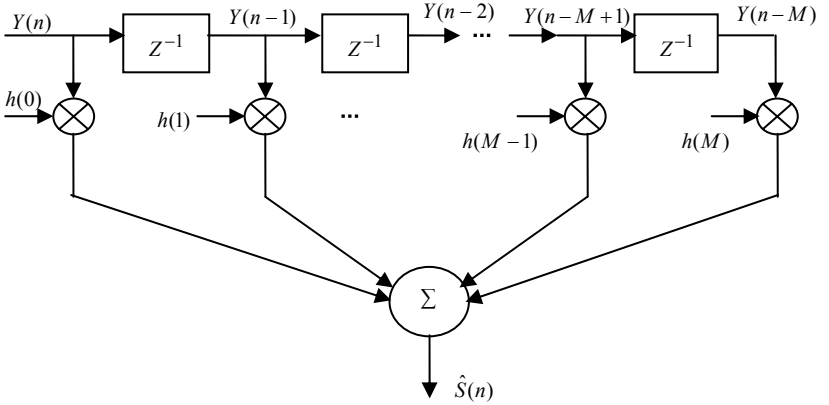


**Figure 7.7** Realization of a digital filter.

$$S_{yy}(z) = \sum_{k=-\infty}^{\infty} R_{yy}(k) z^{-k} \tag{7.90}$$

Following a similar procedure as in the continuous case, we first factor $S_{yy}(z)$ into

$$S_{yy}(z) = S_{yy}^{+}(z) S_{yy}^{-}(z) \tag{7.91}$$

such that the poles and zeros of $S_{yy}(z)$ inside the circle $|z| < 1$ are assigned to $S_{yy}^{+}(z)$, whereas the poles and zeros in the region $|z| > 1$ are assigned to $S_{yy}^{-}(z)$. Consequently, $S_{yy}^{+}(z)$ is analytic inside the unit circle, and $S_{yy}^{-}(z)$ is analytic in $|z| > 1$. Dividing (7.91) by $S_{yy}^{-}(z)$, and applying partial fraction expansion, we obtain

$$\frac{S_{sy}(z)}{S_{yy}^{-}(z)} = \left[\frac{S_{sy}(z)}{S_{yy}^{-}(z)}\right]^{+} + \left[\frac{S_{sy}(z)}{S_{yy}^{-}(z)}\right]^{-} \tag{7.92}$$

where $[\cdot]^{+}$ denotes poles and zeros inside $|z| < 1$, and $[\cdot]^{-}$ denotes poles and zeros in $|z| > 1$. Let

$$B^+(z) = \left[\frac{S_{sy}(z)}{S_{yy}^-(z)}\right]^+, \quad B^-(z) = \left[\frac{S_{sy}(z)}{S_{yy}^-(z)}\right]^- \tag{7.93}$$

The optimum causal filter is

$$H(z) = \frac{B^+(z)}{S_{yy}^-(z)} = \frac{1}{S_{yy}^+(z)}\left[\frac{S_{sy}(z)}{S_{yy}^-(z)}\right]^+ \tag{7.94}$$

We see that the optimum discrete realizable filter is a cascade of two filters as shown in Figure 7.8. The mean-square error is

$$e_m = R_{ss}(0) - \sum_{k=0}^{\infty} h(k) R_{sy}(k) \tag{7.95}$$

**Example 7.8**

Consider the problem where the received sequence is $Y(n) = S(n) + N(n)$. The signal sequence $S(n)$ is stationary and zero mean with power spectrum

$$S_{ss}(e^{j\omega}) = \frac{2}{5 - 4\cos\omega}$$

The noise sequence $N(n)$ is independent of the signal sequence $S(n)$, and has power spectrum $S_{nn}(e^{j\omega}) = 1$.

(a) Obtain the realizable filter.
(b) Find the unrealizable filter.

*Solution*

(a) Since the signal and noise sequences are independent, then

$$S_{sy}(e^{j\omega}) = S_{ss}(e^{j\omega})$$

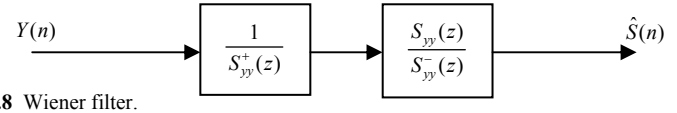


**Figure 7.8** Wiener filter.

Making the change of variables $z = e^{j\omega}$, we have $S_{nn}(z) = 1$

$$S_{sy}(z) = S_{ss}(z) = \frac{2z}{-2z^2 + 5z - 2}$$

Thus,

$$S_{yy}(z) = S_{ss}(z) + S_{nn}(z) = \frac{2z^2 - 7z + 2}{2z^2 - 5z + 2} = \frac{(z - 3.186)(z - 0.314)}{(z - 2)(z - 0.5)}$$

$$S_{yy}^{+}(z) = \frac{z - 0.314}{z - 0.5} \quad \text{and} \quad S_{yy}^{-}(z) = \frac{z - 3.186}{z - 2}$$

Dividing $S_{sy}(z)$ by $S_{yy}^{-}(z)$, we obtain

$$\frac{S_{sy}(z)}{S_{yy}^{-}(z)} = \frac{-z}{(z - 3.186)(z - 0.5)} = \frac{-1.186}{z - 3.186} + \frac{0.186}{z - 0.5}$$

where $B^{+}(z) = 0.186/(z - 0.5)$. Using (7.94), the optimum realizable filter is

$$H(z) = \frac{B^{+}(z)}{S_{yy}^{+}(z)} = \frac{0.186}{z - 0.5} \frac{z - 0.5}{z - 0.314} = \frac{0.186}{z - 0.314}$$

or

$$h(n) = 0.186(0.314)^n, \quad n = 0, 1, 2, \ldots$$

(b) The optimum unrealizable filter is given by (7.86) to be

$$H(z) = \frac{S_{sy}(z)}{S_{yy}(z)} = \frac{-z}{z^2 - 3.5z + 1} = \frac{-z}{(z - 3.186)(z - 0.314)}$$

Note that the pole at $z = 3.186$ outside the unit circle makes this filter unstable, and thus unrealizable in real time.

The method described above in solving the Wiener-Hopf equation is called *spectral factorization*. Another approach to obtain the Wiener filter is based on the

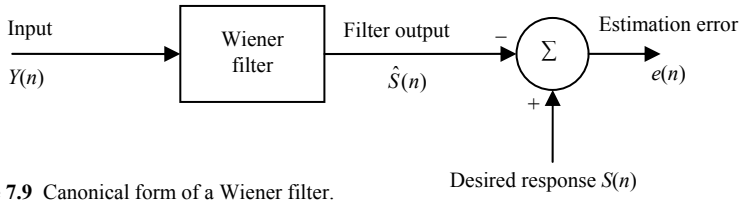


**Figure 7.9** Canonical form of a Wiener filter.

least-square principle discussed in the previous chapter. We minimize the error $e(n)$ between the actual output and the desired output, as shown in Figure 7.9.

*Mean-Square Method*

Consider the linear transversal filter with $M-1$ *delays* as shown in Figure 7.10. Note that the *tap weights* $h(0), h(1), \ldots, h(M-1)$ of Figure 7.7 are now denoted $\omega_0^*, \omega_1^*, \ldots, \omega_{M-1}^*$, respectively. Since in many practical situations such as communications, radar, and sonar, the *information carrying signal* may be complex, we assume the general case that the time series $Y(n)$, $Y(n-1)$, …, $Y(n-M+1)$ is *complex valued*. Following the approach developed by Haykin [1], the filter output is then given by the convolution sum

$$\hat{S}(n) = \sum_{k=0}^{M-1} \omega_k^* Y(n-k) \tag{7.96}$$

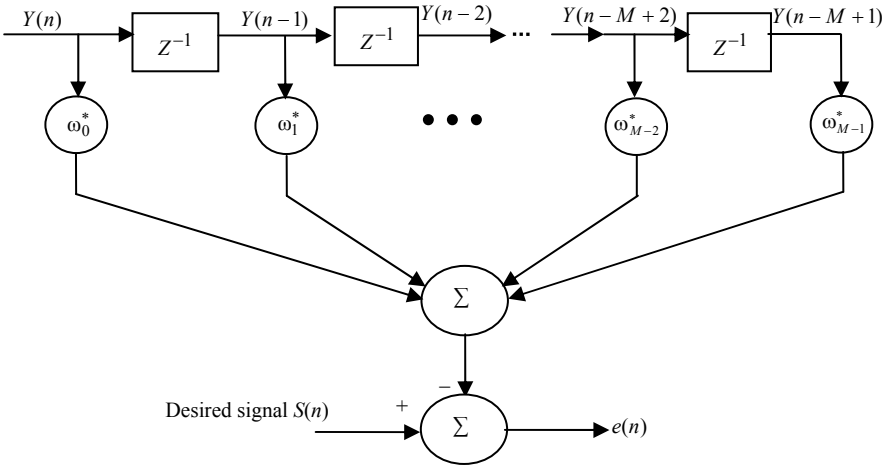


**Figure 7.10** Transversal filter.

Since the estimation error is

$$e(n) = S(n) - \hat{S}(n) \tag{7.97}$$

the goal is to minimize the cost function

$$C(\boldsymbol{\omega}) = E\left[e(n)e^*(n)\right] \tag{7.98}$$

which yields the *optimum linear filter in the mean-square sense*. Let the weight vector $\boldsymbol{\omega}$ be

$$\boldsymbol{\omega} = [\omega_1 \quad \omega_2 \quad \ldots \quad \omega_{M-1}]^T \tag{7.99}$$

and the input vector $\boldsymbol{Y}(n)$ be

$$\boldsymbol{Y}(n) = [Y(n) \quad Y(n-1) \quad \ldots \quad Y(n-M)]^T \tag{7.100}$$

Then, in matrix form,

$$\hat{S}^*(n) = \boldsymbol{\omega}^H \boldsymbol{Y}(n) \tag{7.101}$$

where *H* denotes the Hermitian transpose and

$$\hat{S}^*(n) = \boldsymbol{Y}^H(n)\boldsymbol{\omega} \tag{7.102}$$

Substituting (7.101) and (7.102) into (7.98), the cost function becomes

$$\begin{aligned} C(\boldsymbol{\omega}) &= E\left\{\left[S(n) - \boldsymbol{\omega}^H \mathrm{Y}(n)\right]\left[S^*(n) - \mathrm{Y}^H(n)\boldsymbol{\omega}\right]\right\} \\ &= E\left[S(n)S^*(n)\right] - \boldsymbol{\omega}^H E\left[\boldsymbol{Y}(n)S^*(n)\right] - E\left[S(n)\boldsymbol{Y}^H(n)\right]\boldsymbol{\omega} + \boldsymbol{\omega}^H E\left[\boldsymbol{Y}(n)\mathrm{Y}^H(n)\right]\boldsymbol{\omega} \end{aligned} \tag{7.103}$$

Assuming $S(n)$ has zero mean and variance $\boldsymbol{\sigma}_s^2$, then

$$\sigma_s^2 = E\left[\hat{S}(n)S^*(n)\right] \tag{7.104}$$

The cross-correlation vector between the input sequence and the desired signal is

$$\begin{aligned}\boldsymbol{R}_{\boldsymbol{YS}} &= E[\boldsymbol{Y}(n)S^*(n)] = E\{[Y(n) \quad Y(n-1) \quad \ldots \quad Y(n-M+1)]S^*(n)\} \\ &= [r_{ys}(0) \quad r_{ys}(1) \quad \ldots \quad r_{ys}(M-1)]\end{aligned} \tag{7.105}$$

where $r_{ys}(\cdot)$ is the cross-correlation between $y(\cdot)$ and $S(\cdot)$. The autocorrelation matrix of the input sequence is given by

$$\begin{aligned}\mathrm{R}_{\mathrm{YY}} &= E\left[\boldsymbol{Y}(n)\boldsymbol{Y}^H(n)\right] \\ &= \begin{bmatrix} r_{yy}(0) & r_{yy}(1) & \ldots & r_{yy}(M-1) \\ r_{yy}(-1) & r_{yy}(0) & \ldots & r_{yy}(M-2) \\ \vdots & \vdots & \vdots & \vdots \\ r_{yy}(-M+1) & r_{yy}(-M+2) & \ldots & r_{yy}(0) \end{bmatrix}\end{aligned} \tag{7.106}$$

Note again that we use the lowercase letter *r* to represent the correlation elements of a matrix or a vector and the subscript capital to denote matrix.

After substitution of (7.104), (7.105), and (7.106) in (7.103), the cost function can be written as

$$C(\boldsymbol{\omega}) = \sigma_s^2 - \boldsymbol{R}_{\boldsymbol{Ys}}^H\boldsymbol{\omega} - \boldsymbol{\omega}^H\boldsymbol{R}_{\boldsymbol{Ys}} + \boldsymbol{\omega}^H\boldsymbol{R}_{\boldsymbol{YY}}\boldsymbol{\omega} \tag{7.107}$$

The cost function is a second-order function of a weight vector, $\boldsymbol{\omega}$ and thus the dependence of the cost function on the weights $\omega_0, \omega_1, \ldots, \omega_{M-1}$, can be visualized as a bowl-shaped surface with a unique minimum. This surface is referred to as the *error performance surface* of the filter. The minimum-mean-square error values for which the filter operates at the *minimum point* of the error performance surface yields the *optimum weight vector* $\boldsymbol{\omega}_0$. Hence, we need to take the derivative of (7.107) with respect to the vector $\boldsymbol{\omega}$. Before giving the optimum weight vector, we need to show the differentiation with respect to a vector.

*Differentiation with Respect to a Vector*

Let *g* be a scalar-value function of a $K \times 1$ vector $\boldsymbol{\omega}$ with elements

$$\omega_k = a_k + jb_k, \quad k = 1, 2, \ldots, K \tag{7.108}$$

The derivative of *g* with respect to the vector $\boldsymbol{\omega}$ is defined as

$$\frac{dg}{d\boldsymbol{\omega}} = \begin{bmatrix} \dfrac{\partial g}{\partial a_1} + j\dfrac{\partial g}{\partial b_1} \\ \dfrac{\partial g}{\partial a_2} + j\dfrac{\partial g}{\partial b_2} \\ \vdots \\ \dfrac{\partial g}{\partial a_K} + j\dfrac{\partial g}{\partial b_K} \end{bmatrix} \tag{7.109}$$

**Example 7.9**

This example has been reprinted from [1] with permission by Pearson Education. Given the scalar $g$, a $K \times 1$ vector $\boldsymbol{c}$, and a $K \times K$ matrix $\boldsymbol{Q}$, determine the derivative $\partial g / \partial \boldsymbol{\omega}$ for

(a) $g = \boldsymbol{c}^H \boldsymbol{\omega}$

(b) $g = \boldsymbol{\omega}^H \boldsymbol{c}$

(c) $g = \boldsymbol{\omega}^H \boldsymbol{Q} \boldsymbol{\omega}$.

*Solution*

(a) $g = \boldsymbol{c}^H \boldsymbol{\omega}$ can be written in expanded form as

$$g = \sum_{k=1}^{K} c_k^* \omega_k = \sum_{k=1}^{K} c_k^* (a_k + jb_k)$$

Taking the derivative with respect to $a_k$ and $b_k$, respectively, we have

$$\frac{\partial g}{\partial a_k} = c_k^* \qquad , \quad k = 1, 2, \ldots, K$$

and

$$\frac{\partial g}{\partial b_k} = jc_k^*, \quad k = 1, 2, \ldots, K$$

Substituting in (7.109), we obtain

$$\frac{\partial g}{\partial \boldsymbol{\omega}}(\boldsymbol{c}^H \boldsymbol{\omega}) = \mathbf{0} \tag{7.110}$$

(b) Similarly, $g = \boldsymbol{\omega}^H \boldsymbol{c}$ can be written as

$$g = \sum_{k=1}^{K} c_k \, \omega_k^* = \sum_{k=1}^{K} c_k (a_k - jb_k)$$

Hence,

$$\frac{\partial g}{\partial a_k} = c_k, \quad k = 1, 2, \ldots, K$$

$$\frac{\partial g}{\partial b_k} = -jc_k, \quad k = 1, 2, \ldots, K$$

After substitution, we have

$$\frac{d}{d\boldsymbol{\omega}} = (\boldsymbol{\omega}^H \boldsymbol{c}) = 2\boldsymbol{c} \tag{7.111}$$

(c) In this case $g = \boldsymbol{\omega}^H \boldsymbol{Q} \boldsymbol{\omega}$. Let $\boldsymbol{c}_1 = \boldsymbol{Q}^H \boldsymbol{\omega}$ be a constant; then $\boldsymbol{c}_1^H = \boldsymbol{\omega}^H \boldsymbol{Q}$. Therefore,

$$\frac{dg}{d\boldsymbol{\omega}} = \frac{d}{d\boldsymbol{\omega}}(\boldsymbol{c}_1 \boldsymbol{\omega}) = \mathbf{0}$$

$$\frac{dg}{d\boldsymbol{\omega}} = \frac{d}{d\boldsymbol{\omega}}(\boldsymbol{\omega}^H \boldsymbol{c}_1) = 2\boldsymbol{c}_1$$

Summing both results, we obtain

$$\frac{dg}{d\boldsymbol{\omega}} = \frac{d}{d\boldsymbol{\omega}}(\boldsymbol{\omega}^H \boldsymbol{Q} \boldsymbol{\omega}) = 2\boldsymbol{Q} \boldsymbol{\omega} \tag{7.112}$$

Now, taking the derivative of the cost function given in (7.103) with respect to $\boldsymbol{\omega}$, and using (7.110), (7.111), and (7.112), we obtain

$$\frac{\partial C(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}} = -2\boldsymbol{R}_{\boldsymbol{Y}s} + 2\boldsymbol{R}_{\boldsymbol{YY}}\boldsymbol{\omega} = \mathbf{0} \tag{7.113}$$

or

$$\boldsymbol{R}_{\boldsymbol{YY}}\boldsymbol{\omega} = \boldsymbol{R}_{\boldsymbol{Y}s} \tag{7.114}$$

Equation (7.114) is the Wiener-Hopf equation in the discrete form, and is called the *normal equation*. Solving (7.114), we obtain the optimum weight vector to be

$$\boldsymbol{\omega}_0 = \boldsymbol{R}_{\boldsymbol{YY}}^{-1}\boldsymbol{R}_{\boldsymbol{Y}s} \tag{7.115}$$

Note that from the principle of orthogonality (i.e., the error is orthogonal to the data), we have

$$E\left[\boldsymbol{Y}(n)e_0^*(n)\right] = \mathbf{0} \tag{7.116}$$

where $e_0^*(n)$ is the estimate error resulting from the use of the optimum filter and is given by

$$e_0^*(n) = S^*(n) - \boldsymbol{Y}^H(n)\boldsymbol{\omega}_0 \tag{7.117}$$

It can be shown that

$$E\left[\hat{S}(n)e_0^*(n)\right] = 0 \tag{7.118}$$

which means that the estimate at the output of the optimum filter and the estimation error $e_0(n)$ are also orthogonal as shown in Figure 7.11. This is why the Wiener-Hopf equations in discrete form are also referred to as normal equations.

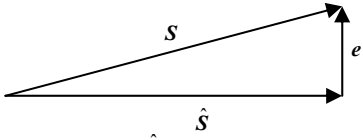


**Figure 7.11** Error orthogonal to filter output $\hat{\boldsymbol{S}}$.

The minimum mean-square error is given by

$$e_m = \sigma_s^2 - \boldsymbol{R}_{\boldsymbol{Y}s}^H \boldsymbol{\omega}_0 = \sigma_s^2 - \boldsymbol{R}_{\boldsymbol{Y}s}^H \boldsymbol{R}_{\boldsymbol{YY}}^{-1} \boldsymbol{R}_{\boldsymbol{Y}s} \tag{7.119}$$

or

$$e_m = \sigma_s^2 - \sum_{k=0}^{M-1} \omega_{0k} r_{ys}^*(k) \tag{7.120}$$

Assuming that the desired response $S(n)$ and the input sequence $\boldsymbol{Y}(n)$ have zero means, the minimum mean-square error is

$$e_m = \sigma_s^2 - \sigma_{\hat{s}}^2 \tag{7.121}$$

## 7.5 KALMAN FILTER

In this section, we present the optimum Kalman filter. We consider the *state model* approach. In this case, filtering means estimating the state vector at the present time based upon past observed data. Prediction is estimating the state vector at a future time. Since it can be shown that the filtered estimate of the state vector is related to the one-step prediction of the state, we first develop the concept of prediction, and then derive the equations for the filtered state.

We shall state the problem for the scalar case and then generalize it to the vector case. We follow this approach for all necessary steps in order to understand the resulting general equations. We assume the *state model* or *signal model*

$$S(n) = \Phi(n)S(n-1) + W(n) \tag{7.122}$$

where $S(n)$ is a zero mean Gaussian sequence and $\Phi(n)$ is a series of known constants. The additive random noise disturbance is also Gaussian and white with variance $Q(n)$ [or $\sigma_w^2(n)$ ].

The observation $Y(n)$ is modeled as

$$Y(n) = H(n)S(n) + N(n) \tag{7.123}$$

where $H(n)$ is a measurement relating the state $S(n)$ to the observation $Y(n)$, and $N(n)$ is a zero mean white Gaussian noise with variance $R(n)$ [or $\sigma_n^2(n)$ ].

The corresponding state vector model is of the form

$$\boldsymbol{S}(n) = \boldsymbol{\Phi}(n)\boldsymbol{S}(n-1) + \boldsymbol{W}(n) \tag{7.124}$$

where $\boldsymbol{S}(n)$ is the $m \times 1$ state vector, $\boldsymbol{\Phi}(n)$ is an $m \times m$ known state transition matrix, and $\boldsymbol{W}(n)$ is an $m \times 1$ noise vector. We assume that the vector random sequence $\boldsymbol{S}(n)$ is zero mean Gaussian, and the noise vector process $\boldsymbol{W}(n)$ is also zero mean and white with autocorrelation

$$E[\boldsymbol{W}(n)\boldsymbol{W}^T(k)] = \begin{cases} \boldsymbol{Q}(n), & n = k \\ 0 \quad , & n \neq k \end{cases} \tag{7.125}$$

Let $\boldsymbol{Y}(n)$ be the $p \times 1$ observation vector consisting of a Gaussian random sequence. The observation can be modeled as

$$\boldsymbol{Y}(n) = \boldsymbol{H}(n)\boldsymbol{S}(n) + \boldsymbol{N}(n) \tag{7.126}$$

where $\boldsymbol{H}(n)$ is a $p \times m$ measurement matrix relating the state vector to the observation vector, and $\boldsymbol{N}(n)$ is a known $p \times 1$ measurement error. $\boldsymbol{N}(n)$ is a Gaussian zero mean white noise sequence with autocorrelation

$$E\left[\boldsymbol{N}(n)\boldsymbol{N}^T(k)\right] = \begin{cases} \mathrm{R}(n), & n = k \\ 0 \quad , & n \neq k \end{cases} \tag{7.127}$$

In order to obtain the Kalman filter state, $\hat{\boldsymbol{S}}(n)$, we first solve for $\hat{\boldsymbol{S}}(n+1)$, the one-step linear predictor, using the concept of innovations.

### 7.5.1 Innovations

In this section, we first present the concept of innovations for random variables and give some important properties. The results, which will then be generalized to random vectors, will be used to solve for Kalman filter. Let $Y(1), Y(2), \ldots, Y(n)$ be a sequence of zero mean Gaussian random variables. The innovation process $V(n)$ represents the new information, which is not carried from the observed data $Y(1), Y(2), \ldots, Y(n-1)$, to obtain the predicted estimate $\hat{Y}(n)$ of the observed random variables. Specifically, let $\hat{S}(n-1)$ be the linear minimum mean-square estimate of a random variable $S(n-1)$ based on the observation data $Y(1), Y(2), \ldots, Y(n-1)$. Suppose that we take an additional observation $Y(n)$ and desire to obtain $\hat{S}(n)$ the estimate of $S(n)$. In order to avoid redoing the

computations from the beginning for $\hat{S}(n-1)$, it is more efficient to use the previous estimate $\hat{S}(n-1)$ based on the $(n-1)$ observation random variables $Y(1), Y(2), \ldots, Y(n-1)$, and compute $\hat{S}(n)$ *recursively* based on the $n$ random variables; $Y(1), Y(2), \ldots, Y(n-1)$ and the additional new observation variable $Y(n)$. We define

$$V(n) = Y(n) - \hat{Y}[n \mid Y(1), Y(2), \ldots, Y(n-1)], \quad n = 1, 2, \ldots \tag{7.128}$$

where $V(n)$ denotes the innovation process and $\hat{Y}[n \mid Y(1), Y(2), \ldots, Y(n-1)]$ is the estimate of $Y(n)$ based on the $(n-1)$ observations, $Y(1), Y(2), \ldots, Y(n-1)$. We see form (7.128) that because $V(n)$ represents a new information *measure* in the observation variable $Y(n)$, it is referred to as "*innovation*."

The innovation $V(n)$ has several important properties as follows.

1. The innovation $V(n)$ associated with the observation $Y(n)$ is orthogonal to the past variables, $Y(1), Y(2), \ldots, Y(n-1)$; that is,

$$E[V(n)Y(k)] = 0, \quad k = 1, 2, \ldots, n-1 \tag{7.129}$$

This is simply the principle of orthogonality.

2. The innovations $V(k)$, $k = 1, 2, \ldots, n$, are orthogonal to each other; that is,

$$E[V(n)V(k)] = 0, \quad k \neq n \tag{7.130}$$

3. There is a one-to-one correspondence between the observed data $\{Y(1), Y(2), \ldots, Y(n)\}$ and innovations $\{V(1), V(2), \ldots, V(n)\}$, in the sense that one sequence may be obtained from the other without any loss of information. That is,

$$\{Y(1), Y(2), \ldots, Y(n)\} \leftrightarrow \{V(1), V(2), \cdots, V(n)\} \tag{7.131}$$

Using property (3), $\hat{S}(n)$ is the minimum mean-square estimate of $S(n)$ based on the observations $Y(1), Y(2), \ldots, Y(n)$. Equivalently, $\hat{S}(n)$ is the minimum mean-square estimate of $S(n)$ given the innovations $V(1), V(2), \ldots, V(n)$. Hence, defining the estimate $\hat{S}(n)$ as a linear combination of the innovations $V(1), V(2), \ldots, V(n)$, we have

$$\hat{S}(n) = \sum_{k=1}^{n} b_k V(k) \tag{7.132a}$$

$$= \sum_{k=1}^{n-1} b_k V(k) + b_n V(n) \tag{7.132b}$$

Using property (2) and the fact that $b_k$ is chosen so that the minimum mean-square value of the error $S(n) - \hat{S}(n)$ is minimized, we obtain

$$b_k = \frac{E\left[S(n)V(k)\right]}{E\left[V^2(k)\right]}, \quad k = 1, 2, \ldots, n \tag{7.133}$$

Recognizing that the estimate $\hat{S}(n-1) = \sum_{k=1}^{n-1} b_k V(k)$, we observe that the estimate $\hat{S}(n)$ based on the $n$ observations, $Y(1), Y(2), \ldots, Y(n)$, is related to the estimate $\hat{S}(n-1)$ based on the $(n-1)$ observations, $Y(1), Y(2), \ldots, Y(n-1)$, by the following recursive rule

$$\hat{S}(n) = \hat{S}(n-1) + b_n V(n) \tag{7.134}$$

where the constant $b_n$ is given by

$$b_n = \frac{E\left[S(n)V(n)\right]}{E\left[V^2(n)\right]} \tag{7.135}$$

Generalizing the results given in (7.128), (7.129), and (7.130) to random vectors, we obtain

$$\boldsymbol{V}(n) = \boldsymbol{Y}(n) - \hat{\boldsymbol{Y}}[n \mid \boldsymbol{Y}(1), \boldsymbol{Y}(2), \ldots, \boldsymbol{Y}(n-1)], \quad n = 1, 2, \ldots \tag{7.136}$$

$$E\left[\boldsymbol{V}(n)\boldsymbol{Y}^T(k)\right] = \boldsymbol{0}, \quad k = 1, 2, \ldots, n-1 \tag{7.137}$$

and

$$E\left[\boldsymbol{V}(n)\boldsymbol{V}^T(k)\right] = \boldsymbol{0}, \quad k \neq n \tag{7.138}$$

### 7.5.2 Prediction and Filtering

The optimum linear mean-square error one-step predictor based on the Gaussian assumptions is given by

$$\hat{S}(n+1) = E\left[S(n+1) \mid Y(1), Y(2), \ldots, Y(n)\right] \tag{7.139}$$

The goal is to write $\hat{S}(n+1)$ in a recursive form. Since there is a one-to-one correspondence between the set of observation vectors and the set representing the innovations [property (3)], then

$$\hat{S}(n+1) = E\left[S(n+1) \mid V(1), V(2), \ldots, V(n)\right] \tag{7.140}$$

$\hat{S}(n+1)$ is also a linear combination of the innovations, and thus

$$\hat{S}(n+1) = \sum_{k=1}^{n} a_k V(k) \tag{7.141}$$

where $a_k$ is a constant to be determined. Since the error is orthogonal to the observations (innovations), we have

$$E\left\{\left[S(n+1) - \hat{S}(n+1)\right] \; V(k)\right\} = 0, \quad k = 1, 2, \ldots, n \tag{7.142}$$

Substituting (7.141) in (7.142), we obtain

$$E\left[S(n+1)V(k)\right] = \frac{a_k}{E\left[V^2(k)\right]} \tag{7.143}$$

or

$$a_k = \frac{E\left[S(n+1)V(k)\right]}{E\left[V^2(k)\right]} \tag{7.144}$$

Substituting for the value of $a_k$ in (7.141) and using the state model $S(n+1) = \Phi(n+1)S(n) + W(n)$, we obtain

$$\hat{S}(n+1) = \sum_{k=1}^{n} \frac{E\{[\boldsymbol{\Phi}(n+1)S(n) + W(n+1)]V(k)\}}{E[V^2(k)]} V(k)$$

$$= \sum_{k=1}^{n} \boldsymbol{\Phi}(n+1) \frac{E[S(n)V(k)]}{E[V^2(k)]} V(k) + \sum_{k=1}^{n} \frac{E[W(n+1)V(k)]}{E[V^2(k)]} V(k) \quad (7.145)$$

Note that the second term of (7.145) is zero because $W(n+1)$ is zero mean and statistically independent of $S(k)$ and $N(k)$, and thus independent of $Y(k)$ and $V(k)$, since $V(k)$ is a linear combination of the observations $Y(k)$, $k = 1, 2, \ldots, n$. Hence, (7.145) becomes

$$\hat{S}(n+1) = \boldsymbol{\Phi}(n+1) \sum_{k=1}^{n} \frac{E[S(n)V(k)]}{E[V^2(k)]} V(k) \quad (7.146a)$$

$$= \boldsymbol{\Phi}(n+1) \left\{ \sum_{k=1}^{n-1} \frac{E[S(n)V(k)]}{E[V^2(k)]} V(k) + \frac{E[S(n)V(n)]}{E[V^2(n)]} V(n) \right\} \quad (7.146b)$$

Using (7.132a) and (7.133), (7.146b) becomes

$$\hat{S}(n+1) = \Phi(n+1)[\hat{S}(n) + b_n V(n)] \quad (7.147)$$

Note that using properties (1) and (3), we observe that

$$E[Y(n) \mid V(1), V(2), \ldots, V(n)] = \hat{S}(n) \quad (7.148)$$

where $\hat{S}(n)$ is the linear minimum MSE estimator of $S(n)$, and thus

$$V(n) = Y(n) - \hat{S}(n) \quad (7.149)$$

Defining

$$k(n) = \frac{d(n)}{\Phi(n+1)} \quad (7.150)$$

and using (7.149) and (7.150) in (7.146), after some mathematical manipulation, we obtain

$$\hat{S}(n+1) = \mathbf{\Phi}(n+1)\left[\hat{S}(n) + k(n)V(n)\right] \tag{7.151a}$$

$$= \mathbf{\Phi}(n+1)\left\{\left[1-k(n)\right]\hat{S}(n) + k(n)Y(n)\right\} \tag{7.151b}$$

Equation (7.151) indicates that the optimum prediction is a linear combination of the previous best estimator $\hat{S}(n)$ and the innovation $V(n) = Y(n) - \hat{S}(n)$ of $Y(n)$.

We now need to determine $k(n)$, which is unknown. To do so, we use the estimation error

$$\tilde{S}(n) = S(n) - \hat{S}(n) \tag{7.152}$$

and define

$$P(n) = E\left[\tilde{S}^2(n)\right] \tag{7.153}$$

Substituting (7.152) in (7.153), and then using (7.149), (7.122), and the orthogonality principle, after some back-and-forth substitutions we obtain

$$P(n+1) = \mathbf{\Phi}^2(n+1)\left\{\left[1+k(n)\right]^2 P(n) + k^2(n)R(n)\right\} + Q(n+1) \tag{7.154}$$

That is, $P(n+1)$ is the error at stage $n+1$ using all previous observations until stage *n*. Minimizing (7.154) with respect to $k(n)$, we obtain

$$k(n) = \frac{P(n)}{P(n)+R(n)} \tag{7.155}$$

which is referred to as *Kalman filter gain*. Again, by back substitutions, it can be shown that [2]

$$P(n+1) = \Phi^2(n+1)[1-k(n)]P(n) + Q(n+1) \tag{7.156}$$

In summary, to start the algorithm at $n=1$, we need the observation $Y(1)$, and to assume some initial values for $P(1)$ and $\hat{S}(1)$. The usual practical assumptions are $\hat{S}(1) = 0$, $P(1) = \sigma_w^2(1)$ [ $P(1) = \sigma_n^2(1)$ ]. We first calculate $k(n)$

using (7.155). Then, we revise $\hat{S}(n)$ based on the innovation (new information) due to the measurement $Y(n)$, so that we can project to the next stage using $\Phi(n+1)$. Then we apply (7.156).

We can now generalize the above concepts to the vector Kalman filter by giving the main results only. The optimum linear mean-square error one-step predictor is

$$\hat{\boldsymbol{S}}(n+1) = E\left[\boldsymbol{S}(n+1) \mid \boldsymbol{Y}(1), \boldsymbol{Y}(2), \ldots, \boldsymbol{Y}(n)\right] \tag{7.157}$$

Using (7.136), and the fact that there is a one-to-one correspondence between the set of the observation vectors and the set representing the innovations process, we can write that

$$\hat{\boldsymbol{S}}(n+1) = \sum_{k=1}^{n} \boldsymbol{A}(n,k)\boldsymbol{V}(k) \tag{7.158}$$

where $\boldsymbol{A}(n,k)$ is an $m \times p$ matrix to be determined.

In accordance with the orthogonality principle, we have,

$$E\left\{\left[\boldsymbol{S}(n+1) - \hat{\boldsymbol{S}}(n+1)\right]\boldsymbol{V}(n)\right\} = \boldsymbol{0} \qquad k = 1, 2, \ldots, n \tag{7.159}$$

Substituting (7.158) in (7.159) and simplifying, we obtain

$$E\left[\boldsymbol{S}(n+1)\boldsymbol{V}^T(\ell)\right] = \boldsymbol{A}(n,\ell)E\left[\boldsymbol{V}(\ell)\boldsymbol{V}^T(\ell)\right] = \boldsymbol{A}(n,\ell)\boldsymbol{C}_{\boldsymbol{VV}}(\ell) \tag{7.160}$$

where $\boldsymbol{C}_{\boldsymbol{VV}}(\ell)$ is the correlation matrix of the innovations process. Solving for $\boldsymbol{A}(n,\ell)$ and substituting in (7.159), the predictor state becomes

$$\hat{\boldsymbol{S}}(n+1) = \sum_{k=1}^{\infty} E\left[\boldsymbol{S}(n+1)\boldsymbol{V}^T(k)\right]\boldsymbol{C}_{\boldsymbol{VV}}^{-1}(k)\boldsymbol{V}(k) \tag{7.161}$$

Upgrading (7.124) to $(n+1)$ and substituting into (7.161), we have

$$E\left[\boldsymbol{S}(n+1)\boldsymbol{V}^T(k)\right] = \boldsymbol{\Phi}(n+1)E\left[\boldsymbol{S}(n)\boldsymbol{V}^T(k)\right] \quad , \quad k = 0, 1, 2, \ldots, n \tag{7.162}$$

where we have used the fact that

$$E\left[\boldsymbol{Y}(k)\boldsymbol{W}^T(n)\right]=\mathbf{0} \tag{7.163}$$

and the fact that the innovations depend on the observation vectors. Substituting (7.162) into (7.161), and after some manipulations, the predictor state becomes

$$\hat{\boldsymbol{S}}(n+1)=\boldsymbol{\Phi}(n+1)\hat{\boldsymbol{S}}(n)+\boldsymbol{K}(n)\boldsymbol{V}(n) \tag{7.164}$$

where $\boldsymbol{K}(n)$ is an $m\times p$ matrix called the *predictor gain matrix*, and defined as

$$\boldsymbol{K}(n)=\boldsymbol{\Phi}(n+1)E\left[\mathrm{S}(n)\boldsymbol{V}^T(n)\right]\boldsymbol{C}_{\boldsymbol{VV}}^{-1}(n) \tag{7.165}$$

Equations (7.164) and (7.165) can be simplified further for computational purposes. If we define

$$\tilde{\boldsymbol{S}}(n)=\boldsymbol{S}(n)-\hat{\boldsymbol{S}}(n) \tag{7.166}$$

and

$$\boldsymbol{P}(n)=E\left[\tilde{\boldsymbol{S}}(n)\tilde{\boldsymbol{S}}^T(n)\right] \tag{7.167}$$

where $\tilde{\boldsymbol{S}}(n)$ is called the *predicted state-error vector* and $\boldsymbol{P}(n)$ is the *predicted state-error correlation matrix*, then it can be shown that [3]

$$\boldsymbol{K}(n)=\boldsymbol{\Phi}(n+1)\boldsymbol{P}(n)\boldsymbol{H}^T(n)\boldsymbol{C}_{\boldsymbol{VV}}^{-1}(n) \tag{7.168}$$

It can also be shown that $\boldsymbol{P}(n)$ can be updated recursively as

$$\begin{aligned}\boldsymbol{P}(n+1)=&\left[\boldsymbol{\Phi}(n+1)-\boldsymbol{K}(n)\boldsymbol{H}(n)\right]\boldsymbol{P}(n)\left[\boldsymbol{\Phi}(n+1)-\boldsymbol{K}(n)\boldsymbol{H}(n)\right]^T\\&+\boldsymbol{Q}(n)+\boldsymbol{K}(n)\boldsymbol{R}(n)\boldsymbol{K}^T(n)\end{aligned} \tag{7.169}$$

and that the filter state is

$$\hat{\boldsymbol{S}}(n)=\boldsymbol{\Phi}(n)\hat{\boldsymbol{S}}(n-1)+\boldsymbol{K}(n)\boldsymbol{C}_{\boldsymbol{VV}}(n) \tag{7.170}$$

where $\boldsymbol{K}(n)$ is an $m \times m$ matrix called the *filter gain matrix*, and is given by

$$\boldsymbol{K}(n) = \boldsymbol{\Phi}(n)\boldsymbol{P}(n) \tag{7.171}$$

Equation (7.169) can be decomposed into a pair of coupled equations to constitute the *Ricatti difference equations*.

*Relationship Between Kalman and Wiener Filters*

The Kalman filter can also be derived for continuous time. If all signal processes considered are stationary, the measurement noise is white and uncorrelated with the signal, and the observation interval is semi-infinite, then the Kalman filter reduces to the Wiener filter. That is, both Kalman and Wiener filters lead to the same result in estimating a stationary process.

In discrete time, the Kalman filter, which is an optimum recursive filter based on the concept of innovations, has the ability to consider nonstationary processes; whereas the Wiener filter, which is an optimum nonrecursive filter, does not.

## 7.6 SUMMARY

In this chapter, we have covered the concept of filtering. We first presented the orthogonality principle theorem, the definition of linear transformations, and related theorems. Realizable and unrealizable Wiener filters for continuous-time were presented in Section 7.4. To obtain the linear mean-square error realizable filter, we needed to solve the Wiener-Hopf integral equation. An approach called spectral factorization using Laplace transform to solve the Wiener-Hopf equation was shown. Then, we extended the concept of the Wiener filter to discrete-time. For a realizable discrete Wiener filter, we considered a transversal filter with an impulse response of finite duration. We used the "mean-square approach" and solved for the optimum weights. We concluded this chapter with a section about Kalman filtering. Since vector Kalman filter development can be "heavy," we gave more details for the scalar case only.

## PROBLEMS

**7.1** Let the observation process be $Y(t) = S(t) + N(t)$. The signal process $S(t)$ and the zero mean white noise process $N(t)$ are uncorrelated with power spectral densities

$$S_{ss}(f) = \frac{2\alpha}{\alpha^2 + 2\pi^2 f^2} \qquad \text{and} \qquad S_{nn}(f) = \frac{N_0}{2}$$

(a) Obtain the optimum unrealizable linear filter for estimating the delayed signal $S(t-t_0)$.
(b) Compute the minimum mean-square error.

**7.2** Let the observation process be $Y(t)=S(t)+N(t)$. The signal process $S(t)$ and the zero mean noise process $N(t)$ are uncorrelated with autocorrelations $R_{ss}(\tau)=e^{-0.5|\tau|}$ and $R_{nn}(\tau)=\delta(\tau)$.
(a) Find the optimum unrealizable filter.
(b) Obtain the optimum realizable filter.
(c) Compute the minimum mean-square error for both filters and compare the results.

**7.3** Let the observation process be $Y(t)=S(t)+N_1(t)$.The signal process $S(t)$ and the zero mean noise process $N_1(t)$ are uncorrelated. The autocorrelation function of $N_1(t)$ is $R_{n_1n_1}(\tau)=e^{-|\tau|}$. Assume that the signal $S(t)$ is given by the expression $S'(t)+S(t)=N_2(t)$ for *t* positive. $S'(t)$ denotes the derivative of $S(t)$ with respect to *t*. $N_2(t)$ is a white Gaussian noise with power spectral density 2. Determine the Wiener filter if the processes $N_1(t)$ and $N_2(t)$ are independent.

**7.4** Let the observation process be $Y(t)=S(t)+N(t)$, for $-\infty<t\le\xi$. The signal process $S(t)$ and the noise process $N(t)$ are uncorrelated with power spectral densities

$$S_{ss}(f)=\frac{1}{1+4\pi^2 f} \qquad \text{and} \qquad S_{nn}(f)=\frac{1}{2}$$

Obtain the optimum linear filter to estimate $S'(t)$; $S'(t)$ is the derivative of the signal $S(t)$ with respect to *t*.

**7.5** Let the observation process be $Y(t)=S(t)+N(t)$. The signal process $S(t)$ and the zero mean noise process $N(t)$ are uncorrelated with autocorrelation functions

$$R_{ss}(\tau)=\frac{5}{3}e^{-\frac{1}{2}|\tau|} \qquad \text{and} \qquad R_{nn}(\tau)=\frac{7}{6}e^{-|\tau|}$$

Obtain the optimum linear filter to estimate $S(t+\alpha)$, $\alpha > 0$.

**7.6** Let $Y(n) = S(n) + N(n)$ be the received sequence. The signal sequence $S(n)$ and the noise sequence $N(n)$ are zero mean and independent with autocorrelation functions

$$R_{ss}(n) = \frac{1/2^{|n|}}{1-(1/4)} \qquad \text{and} \qquad R_{nn}(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

(a) Obtain the optimum realizable filter.
(b) Compute the mean-square error.

**7.7** Let $Y(n) = S(n) + N(n)$ represent the received sequence. The signal sequence $S(n)$ and the noise sequence $N(n)$ are zero mean and independent with autocorrelation functions

$$R_{ss}(n) = \frac{1}{2^{|n|}} \qquad \text{and} \qquad R_{nn}(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

(a) Obtain the optimum realizable filter.
(b) Compute the mean-square error.

**7.8** Consider the Wiener filter consisting of a transversal filter with two delays, as shown in Figure P7.8, with $\omega_0 = 1$, correlations matrix $\boldsymbol{R_{YY}} = \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$

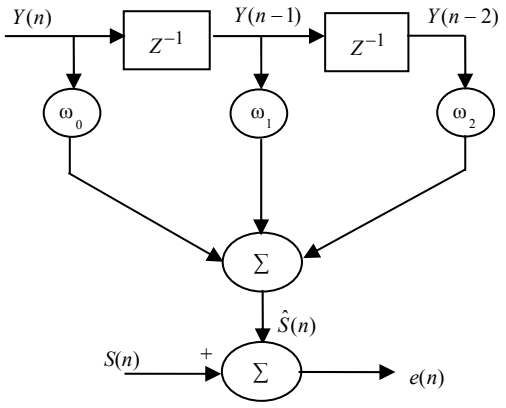


**Figure P7.8** Wiener filter.

and $\boldsymbol{R}_{YS} = \begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix}$

(a) Determine the optimum weights.

(b) Determine the minimum mean-square error $e_m$ if the signal variance is 0.9486.

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# Chapter 8

# Representation of Signals

## 8.1 INTRODUCTION

In this chapter, we study some mathematical principles that will be very useful to us in order to understand the next two chapters. First, we define the meaning of orthogonal functions, which are used to represent deterministic signals in a series expansion known as the generalized Fourier series. We use the Gram-Schmidt procedure to transform a set of $M$ linear dependent or independent functions into a set of $K$, $K \le M$, orthogonal functions. We also discuss geometric representation of signals in the signal space, which can be used to determine decision regions in $M$-ary detection of signals in noise, as be will be seen later. Then, integral equations are studied. The relation between integral equations and their corresponding linear differential equations are established through Green's function or the kernel. In solving integral equations, we present an approach by which we obtain the eigenfunctions and eigenvalues from the linear differential equation. In Section 8.4, we discuss the series representation of random processes by orthogonal functions known as Karhunen-Loève expansion. Specifically, we consider processes with rational power spectral densities, the Wiener process, and the white Gaussian noise process.

## 8.2 ORTHOGONAL FUNCTIONS

From vector analysis, we say that two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are orthogonal (perpendicular) if their dot or inner product is zero. That is,

$$\boldsymbol{X} \cdot \boldsymbol{Y} = 0 \tag{8.1}$$

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two vectors in $\Re^K$, such that $\boldsymbol{X} = [x_1 \quad x_2 \quad \dots \quad x_K]^T$ and $\boldsymbol{Y} = [y_1 \quad y_2 \quad \dots \quad y_K]^T$. Then

$$\boldsymbol{X} \cdot \boldsymbol{Y} = x_1 y_1 + x_2 y_2 + \ \ldots \ + x_K y_K \tag{8.2}$$

The distance $d(x, y)$ between the points $x$ and $y$ is given by

$$d(x, y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ \ldots \ + (y_K - x_K)^2} \tag{8.3}$$

The length or norm of the vector $\boldsymbol{X}$, denoted $|\boldsymbol{X}|$, is defined by

$$|\boldsymbol{X}| = \sqrt{\boldsymbol{X} \cdot \boldsymbol{X}} = \sqrt{x_1^2 + x_2^2 + \ \ldots \ + x_K^2} \tag{8.4}$$

If the length $|\boldsymbol{X}| = 1$, we say that $\boldsymbol{X}$ is a *normalized* vector. Geometrically, (8.1) says that the angle θ between the vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ is $90^\circ$. For an arbitrary angle θ between the two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, θ is defined by

$$\cos\theta = \frac{\boldsymbol{X} \cdot \boldsymbol{Y}}{|\boldsymbol{X}||\boldsymbol{Y}|} \tag{8.5}$$

We now generalize the above concepts to continuous functions of time. Let $\{s_k(t)\}, k = 1, 2, \ldots$, be a set of deterministic functions with finite energies defined over the interval $t \in [0, T]$. Let $E_k$ denote the energy of $s_k(t)$. Then,

$$E_k = \int_0^T |s_k(t)|^2 \, dt < \infty \tag{8.6}$$

The norm of $s_k(t)$, $k = 1, 2, \ldots$, can be written as

$$|s_k(t)| = \left[ \int_0^T s_k^2(t) dt \right]^{\frac{1}{2}} \tag{8.7}$$

Geometrically, (8.7) represents the square root of the area under the curve $s_k^2(t)$. The "distance" between the two signals $s_k(t)$ and $s_j(t)$ is

$$|s_k(t) - s_j(t)| = \left\{ \int_0^T [s_k(t) - s_j(t)]^2 \, dt \right\}^{\frac{1}{2}} \tag{8.8}$$

We say that the set of functions (signals), $\{s_k(t)\}, k = 1, 2, \ldots$, are orthogonal when

$$\int_0^T s_k(t) s_j(t)\, dt = 0, \quad k \neq j \tag{8.9}$$

A set of functions $\{\phi_k(t)\}, k = 1, 2, \ldots$, are *orthonormal* if

$$\int_0^T \phi_k(t)\phi_j(t) dt = \delta_{kj} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \tag{8.10}$$

where $\delta_{kj}$ is the Kronecker's delta function. Note that the set of functions $\{\phi_k(t)\}, k = 1, 2, \ldots$, is normalized.

### 8.2.1 Generalized Fourier Series

Let $s(t)$ be a deterministic signal with finite energy $E$ and observed over the interval $t \in [0, T]$. Given an orthonormal set of functions $\{\phi_k(t)\}, k = 1, 2, \ldots$, for the specified time $t \in [0, T]$, it may be possible to represent the signal $s(t)$ as a linear combination of functions $\phi_k(t), k = 1, 2, \ldots$, as

$$s(t) = s_1\phi_1(t) + s_2\phi_2(t) + \ \ldots\ + s_k\phi_k(t) + \ \ldots = \sum_{k=1}^{\infty} s_k\phi_k(t) \tag{8.11}$$

Assuming the series of (8.11) converges to $s(t)$, then

$$s_k = \int_0^T s(t)\phi_k(t) dt \tag{8.12}$$

where we have used the fact that $\int_0^T \phi_k(t)\phi_j(t) dt = \delta_{kj}$. In this case, the coefficients $s_k, k = 1, 2, \ldots$, are called the *generalized Fourier coefficients*. The series in (8.11) with the coefficients as given by (8.12) is called the *generalized Fourier series*.

If there exists a set of orthonormal functions $\{\phi_k(t)\}, k = 1, 2, \ldots, K,$ such that the signal $s(t)$ may be expressed as

$$s(t) = \sum_{k=1}^{K} s_k \phi_k(t) \tag{8.13}$$

where $s_k$ is as given by (8.12), then the set of orthonormal functions $\{\phi_k(t)\}, k = 1, 2, \ldots, K,$ is said to be *complete*.

Consider the finite sum $s_K(t)$, such that

$$s_K(t) = \sum_{k=1}^{K} s_k \phi_k(t) \tag{8.14}$$

where $s_K(t)$ is an approximation to the signal $s(t)$ observed over the interval $t \in [0, T]$. In general, it is practical to only use a finite number of terms $K$. The goal is to select the coefficients $s_k$ such that the mean-square error is minimum. We define the error $\varepsilon_K(t)$ as

$$\varepsilon_K(t) = s(t) - s_K(t) \tag{8.15}$$

and its corresponding energy as

$$E_{\varepsilon K} = \int_0^T \varepsilon_K^2(t) dt \tag{8.16}$$

The mean-square error is

$$< E_{\varepsilon K}^2(t) > = \frac{1}{T} \int_0^T \varepsilon_K^2(t) dt \tag{8.17}$$

where $< \cdot >$ denotes time average. We observe from (8.16) and (8.17) that minimizing the mean-square error is equivalent to minimizing the energy. Hence,

$$E_{\varepsilon K} = \int_0^T \left[ s(t) - \sum_{k=1}^{K} s_k \phi_k(t) \right]^2 dt \tag{8.18}$$

Differentiating (8.18) with respect to $s_k$, we obtain

$$\frac{dE_{\varepsilon K}}{ds_k} = -2\int_0^T \left[ s(t) - \sum_{k=1}^{K} s_k \phi_k(t) \right] \phi_j(t)dt = -2\int_0^T s(t)\phi_j(t)dt + 2\sum_{k=1}^{K} s_k \int_0^T \phi_k(t)\phi_j(t)dt \tag{8.19}$$

Setting (8.19) equal to zero and using (8.10), the coefficients of $s_k$ are given by

$$s_k = \int_0^T s(t)\phi_k(t)dt \tag{8.20}$$

Note that the second derivative $d^2E_{\varepsilon k} / ds_k^2 = 2$ is positive, and thus the coefficients $s_k$, $k = 1, 2, \ldots, K$, minimize the energy or the mean-square error. The set $\{\phi_k(t)\}$ forms a complete orthonormal set in the interval $[0, T]$. That is,

$$\lim_{K\to\infty} \int_0^T [s(t) - s_K(t)]^2 dt = 0 \tag{8.21}$$

or

$$\underset{K\to\infty}{\text{l.i.m.}}\ s_K(t) = s(t) \tag{8.22}$$

Equation (8.22) is read as the *limit in the mean of* $s_K(t)$ *as* $K \to \infty$ *equals* $s(t)$, or $s_K(t)$ *converges in the mean to* $s(t)$ *as* $K \to \infty$. Substituting the result of (8.20) in (8.18) and solving for $E_{\varepsilon K}$, we obtain

$$E_{\varepsilon K} = \int_0^T s^2(t)dt - \sum_{k=1}^{K} s_k^2 = E - \sum_{k=1}^{K} s_k^2 \tag{8.23}$$

We observe that $E_{\varepsilon K}$ is minimum when the set of orthonormal signals $\{\phi_k\}$ is complete. That is,

$$E_{\varepsilon K} = \int_0^T s^2(t)dt = \sum_{k=1}^{\infty} s_k^2 \tag{8.24}$$

$s_k^2$ may be interpreted as the energy of the signal in the *k*th component. Equation (8.24) is referred to as *Parseval's identity for orthonormal series of functions*. The set of orthonormal functions $\{\phi_k(t)\}$ over the interval $[0, T]$ can be obtained by

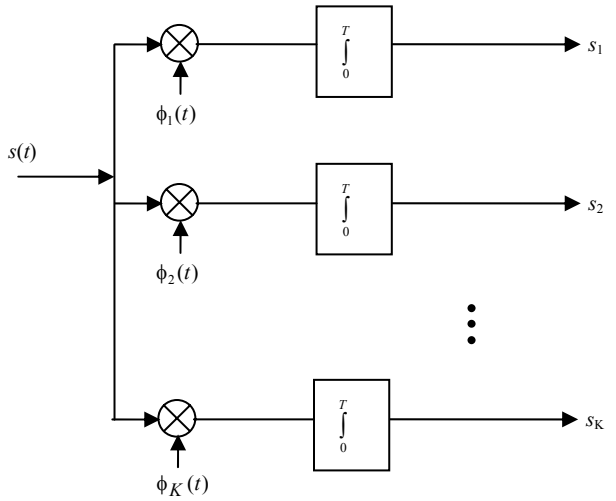


**Figure 8.1** Correlation operation for generating the set of coefficients $\{s_k\}$.

using the Gram-Schmidt orthogonalization procedure, which will be given in Section 8.2.2. The coefficients $s_k, k = 1, 2, \ldots, K,$ may be determined by a correlation operation as shown in Figure 8.1. An equivalent operation is filtering. The signal $s(t)$ is passed through a set of linear filters, *matched filters*, with impulse response $h_k(\tau) = \phi_k(T - \tau)$, and the outputs of the matched filters are then observed at time $t = T$. This is shown in Figure 8.2. Due to the importance of matched filters, we will study them in some detail in Chapter 10.

Let the output of the *k*th channel be $y_k(T)$. The output of the *k*th filter is

$$y_k(t) = \int_0^T s(\tau) h_k(t - \tau) d\tau = \int_0^T s(\tau) \phi_k(T - t + \tau) d\tau \tag{8.25}$$

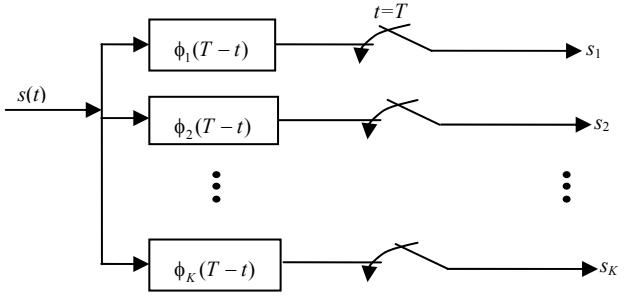


**Figure 8.2** Filtering operation for generating the set of coefficients $\{s_k\}$.

Sampling $y_k(t)$ at time $t = T$, we obtain

$$y_k(t) = \int_0^T s(\tau)\phi_k(T - T + \tau)d\tau = \int_0^T s(\tau)\phi_k(\tau)d\tau = s_k \tag{8.26}$$

### 8.2.2 Gram-Schmidt Orthogonalization Procedure

Given a set of *M* signals $s_k(t), k = 1, 2, \ldots, M$, we would like to represent these signals as a linear combination of *K orthonormal basis functions*, $K \le M$. The signals $s_1(t), s_2(t), \ldots, s_M(t)$ are real-valued, and each is of duration *T*. From (8.13), we may represent these energy signals in the form

$$s_m(t) = \sum_{j=1}^{K} s_{kj}\phi_j(t) \qquad \begin{matrix} 0 \le t \le T \\ k = 1, 2, \ldots, K \\ m = 1, 2, \ldots, M \end{matrix} \tag{8.27}$$

where the coefficients $s_{kj}, j = 1, 2, \ldots, K$, of the signal $s_k(t)$ are defined by

$$s_{kj} = \int_0^T s_k(t)\phi_j(t)dt \qquad k, j = 1, 2, \ldots, K \tag{8.28}$$

The orthonormal functions $\phi_j(t), j = 1, 2, \ldots, K$, are as defined in (8.10). That is, $\int_0^T \phi_k(t)\phi_j(t)dt = \delta_{kj}$. The orthogonalization procedure is as follows.

1. Normalize the first signal $s_1(t)$ to obtain $\phi_1(t)$. That is,

$$\phi_1(t) = \frac{s_1}{\sqrt{\int_0^T s_1^2(t)dt}} = \frac{s_1}{\sqrt{E_1}} \tag{8.29}$$

where $E_1$ is the energy of $s_1(t)$. Thus,

$$s_1(t) = \sqrt{E_1}\phi_1(t) = s_{11}\phi_1(t) \tag{8.30}$$

where the coefficient $s_{11} = \sqrt{E_1}$.

2. Using the signal $s_2(t)$, we compute the projection of $\phi_1(t)$ onto $s_2(t)$, which is

$$s_{21} = \int_0^T s_2(t)\phi_1(t)dt \tag{8.31}$$

We then subtract $s_{21}\phi_1(t)$ from $s_2(t)$ to yield

$$f_2(t) = s_2(t) - s_{21}\phi_1(t) \tag{8.32}$$

which is orthogonal to $\phi_1(t)$ over the interval $0 \le t \le T$. $\phi_2(t)$ is obtained by normalizing $f_2(t)$; that is,

$$\phi_2(t) = \frac{f_2(t)}{\sqrt{\int_0^T f_2^2(t)dt}} = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}} \tag{8.33}$$

where $E_2$ is the energy of the signal $s_2(t)$. Note that from (8.33), $\int_0^T \phi_2^2(t)dt = 1$ and $\int_0^T \phi_2(t)\phi_1(t)dt = 0$. That is, $\phi_1(t)$ and $\phi_2(t)$ are orthonormal.

3. Continuing in this manner, we can determine all $K$ $(K \le M)$ orthonormal functions to be

$$\phi_k(t) = \frac{f_k(t)}{\sqrt{\int_0^T f_k^2(t)dt}} \tag{8.34}$$

where

$$f_k(t) = s_k - \sum_{j=1}^{k-1} s_{kj}\phi_j(t) \tag{8.35}$$

and the coefficients $s_{kj}$, $j = 1, 2, \ldots, k-1,$ are defined by

$$s_{kj} = \int_0^T s_k(t)\phi_j(t)dt \tag{8.36}$$

If all *M* signals $s_1(t), s_2(t), \ldots, s_M(t)$ are independent, (i.e., no signal is a linear combination of the other), then the dimensionality *K* of the signal space is equal to *M*.

*Modified Gram-Schmidt*

The proposed Gram-Schmidt procedure defined in (8.34), (8.35), and (8.36) is referred to as the classical Gram-Schmidt (CGS) procedure. The concept of subtracting away the components in the direction of $\phi_1(t), \phi_2(t), \ldots, \phi_{k-1}(t)$ is sometimes numerically unstable. A slight modification in the algorithm makes it stable and efficient. This modification yields the modified Gram-Schmidt (MGS) procedure. For simplicity, we show only the first two steps. We compute the projection of $s_k(t)$ onto $\phi_1(t), \phi_2(t), \ldots, \phi_{k-1}(t)$. We start with $s_{k1}\phi_1(t)$ and subtract it *immediately*. That is, we are left with a new function $s_k^1(t)$, such that

$$s_k^1(t) = s_k(t) - s_{k1}(t)\phi_1(t) \tag{8.37}$$

where $s_{k1}$ is as defined in (8.36). Then, we project $s_k^1(t)$ instead of the original signal $s_k(t)$ onto $\phi_2(t)$ and subtract that projection. That is,

$$s_k^2(t) = s_k^1(t) - s_{21}(t)\phi_2(t) \tag{8.38}$$

where

$$s_{21}^1 = \int_0^T s_k^1(t)\phi_2(t)dt \tag{8.39}$$

and the power 2 on $s_k(t)$ denotes a *superscript*. Observe that this is identical in principle to the classical Gram-Schmidt procedure, which projects $s_k(t)$ onto both $\phi_1(t)$ and $\phi_2(t)$ to yield $f_k(t)$. Substituting (8.37) and (8.39) into (8.38), we obtain

$$\begin{aligned} s_k^2(t) &= \left[s_k(t) - s_{k1}\phi_1(t)\right] - \left\{\phi_2(t)\left[s_k(t) - s_{k1}\phi_1(t)\right]\phi_2(t)\right\} \\ &= s_k(t) - s_{k1}\phi_1(t) - s_{k2}\phi_2(t) = f_k(t) \end{aligned} \tag{8.40}$$

since $\int_0^T \phi_1(t)\phi_2(t)dt = 0.$

### 8.2.3 Geometric Representation

In order to have a geometric interpretation of the signals, we write the *M* signals by their corresponding *vectors* of coefficients. That is, the *M signal vectors* are

$$\boldsymbol{s}_k = \begin{bmatrix} s_{k1} & s_{k2} & \dots & s_{kK} \end{bmatrix}^T \quad k = 1, 2, \dots, M \tag{8.41}$$

The vectors $s_k, k = 1, 2, \dots, M,$ may be visualized as *M* points in a *K-dimensional Euclidean space*. The *K* mutually perpendicular axes are labeled $\phi_1(t), \phi_2(t), \dots, \phi_K(t).$ This *K*-dimensional Euclidean space is referred to as the *signal space*.

Using (8.4), we say that the inner product of the vector $\boldsymbol{s}_k$ with itself, which is the norm of $\boldsymbol{s}_k$, is

$$\left|\boldsymbol{s}_k\right|^2 = (\boldsymbol{s}_k, \boldsymbol{s}_k) = \sum_{j=1}^{K} s_{kj}^2 \tag{8.42}$$

Since the *K* orthonormal functions form a complete set, (8.42) also represents the energy of signal $s_k(t)$ as shown in the previous section. Thus,

$$E_k = \sum_{j=1}^{K} s_{kj}^2 \tag{8.43}$$

From (8.3), (8.41), and (8.43), the Euclidean distance between the points represented by the signal vectors $\boldsymbol{s}_k$ and $\boldsymbol{s}_j$ can be written as

$$\left|\boldsymbol{s}_k - \boldsymbol{s}_j\right|^2 = \sum_{i=1}^{K} (s_{ki}^2 - s_{ji}^2) = \int_0^T [s_k(t) - s_j(t)]^2 dt \tag{8.44}$$

The correlation coefficient between the signals $s_k(t)$ and $s_j(t)$ is defined by

$$s_{kj} = \frac{\int_0^T s_k(t)s_j(t)dt}{\sqrt{E_k E_j}} = \frac{\int_0^T \left[\sum_{i=1}^{K} s_{ki}\phi_i\right]\left[\sum_{i=1}^{K} s_{ji}\phi_i\right] dt}{\sqrt{E_k E_j}} = \frac{\sum_{i=1}^{K} s_{ki}s_{ji}}{\sqrt{E_k E_j}} = \frac{\boldsymbol{s}_k^T \boldsymbol{s}_j}{\left|\boldsymbol{s}_k\right|\left|\boldsymbol{s}_j\right|} \tag{8.45}$$

where $\boldsymbol{s}_k$ is given in (8.41), and $\boldsymbol{s}_j$ is

$$\boldsymbol{s}_j = \begin{bmatrix} s_{j1} & s_{j2} & \dots & s_{jK} \end{bmatrix}^T \tag{8.46}$$

**Example 8.1**

Consider the signals $s_1(t), s_2(t), s_3(t)$, and $s_4(t)$ as shown in Figure 8.3. Use the Gram-Schmidt procedure to determine the orthonormal basis functions for $s_k(t), k = 1,\ 2,\ 3,\ 4$.

*Solution*

From (8.29), the first function $\phi_1(t)$ is

$$\phi_1 = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} \sqrt{\dfrac{3}{T}}, & 0 \le t \le \dfrac{T}{3} \\ 0 \quad , \text{ otherwise} \end{cases}$$

where $E_1 = \int_0^{T/3} (1)^2\, dt = T/3$. To find $\phi_2(t)$, we first use (8.31) to determine $s_{21}$; that is, $s_{21} = \int_0^T s_2(t)\phi_1(t)dt = \sqrt{T/3}$. From (8.32), $f_2(t)$ is given by

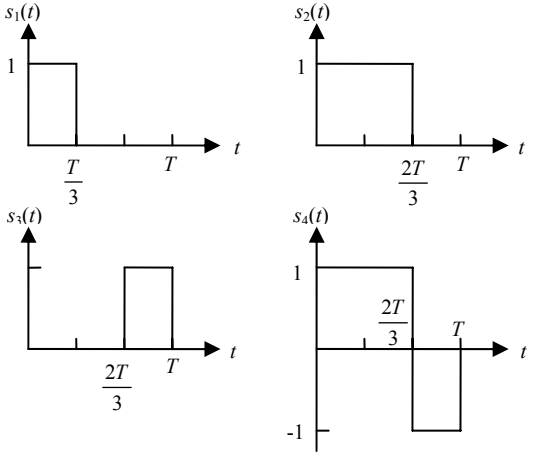


**Figure 8.3** Set of signals $\{s_k(t)\}$.

$$f_2(t) = s_2(t) - s_{21}\phi_1(t) = \begin{cases} 1, & \frac{T}{3} \le t \le \frac{2T}{3} \\ 0, & \text{otherwise} \end{cases}$$

Normalizing $f_2(t)$, we have

$$\phi_2(t) = \frac{f_2(t)}{\sqrt{\int_0^T f_2^2(t)dt}} = \begin{cases} \sqrt{\frac{3}{T}}, & \frac{T}{3} \le t \le \frac{2T}{3} \\ 0, & \text{otherwise} \end{cases}$$

We use (8.35) and (8.36) to find the coefficients $s_{31}$ and $s_{32}$; that is, $s_{31} = \int_0^T s_3(t)\phi_1(t)dt = 0$ and $s_{32} = \int_0^T s_3(t)\phi_2(t)dt = 0$. Thus, $f_3(t) = s_3(t)$, and the normalized signal $\phi_3(t)$ is

$$\phi_3(t) = \frac{s_3(t)}{\sqrt{E_3}} = \begin{cases} \sqrt{\frac{3}{T}}, & \frac{2T}{3} \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

We observe that $s_4(t) = s_2(t) - s_3(t)$ is a linear combination of $s_2(t)$ and $s_3(t)$. The complete set of orthonormal functions is $\phi_1(t), \phi_2(t)$, and $\phi_3(t)$; that is, the dimensionality is $K = 3$. The basis functions are shown in Figure 8.4.

**Example 8.2**

(a) Find a set of orthonormal basis functions that can be used to represent the signals shown in Figure 8.5.

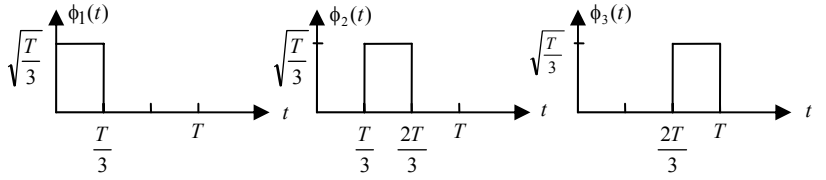


**Figure 8.4** Orthonormal basis functions $\{\phi_k(t)\}$.

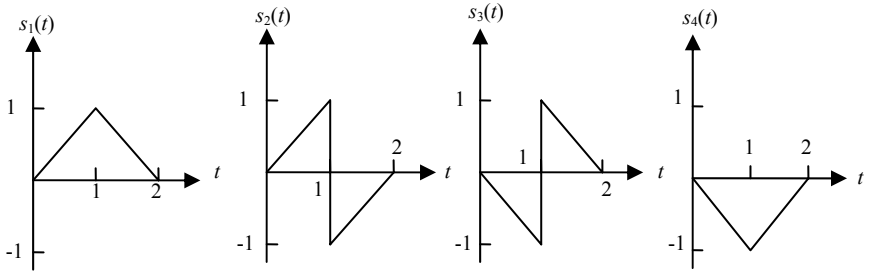


**Figure 8.5** Signal set for Example 8.2.

(b) Find the vector corresponding to each signal for the orthonormal basis set found in (a), and sketch the location of each signal in the signal space.

*Solution*

(a) In this example, we are not going to do a formal mathematical derivation as we did in the previous one, but instead we solve it by inspection. We see that the given waveforms can be decomposed into two basis functions $\phi_1(t)$ and $\phi_2(t)$, as shown in Figure 8.6.

Since $\phi_1(t)$ and $\phi_2(t)$ must have unit energy, we have $E = \int_0^1 (At)^2\, dt = 1$ or $A = \sqrt{3}$.

(b) The signal vectors are

$$\boldsymbol{s}_1 = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right], \quad \boldsymbol{s}_2 = \left[\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right], \quad \boldsymbol{s}_3 = \left[\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right], \text{ and } \quad \boldsymbol{s}_4 = \left[\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right]$$

Thus, the signal space is as shown in Figure 8.7.

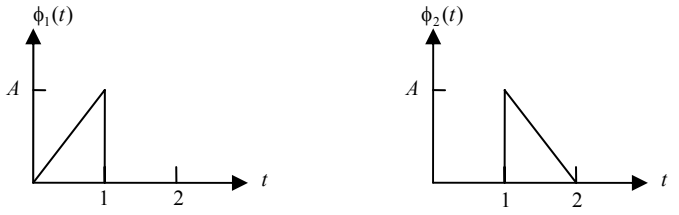


**Figure 8.6** Basis functions for Example 8.2

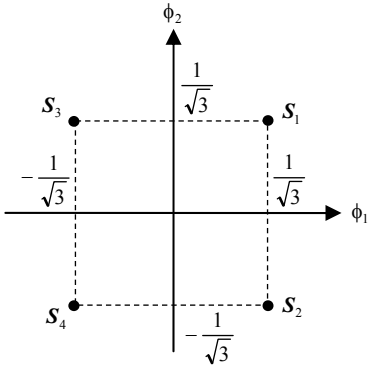


**Figure 8.7** Signal space for Example 8.2.

### Example 8.3

Consider the three possible functions

$$\phi_k(t) = E\cos\frac{2k\pi t}{T}, \quad \begin{matrix} k = 1, 2, 3 \\ 0 \le t \le T \end{matrix}$$

(a) Does $\phi_k$ constitute an orthonormal set?
(b) What geometric figure does the vector $\boldsymbol{s}_k$, $k = 1, 2, 3,$ form in the signal space?

*Solution*

(a) To check for orthogonality,

$$\begin{aligned}(\phi_k, \phi_j) &= \int_0^T \phi_k(t)\phi_j(t)dt = E^2 \int_0^T \cos\frac{2k\pi t}{T}\cos\frac{2j\pi t}{T}\,dt \\ &= \frac{E^2}{2}\left[\int_0^T \cos\frac{2\pi t(k-j)}{T}\,dt + \int_0^T \cos\frac{2\pi t(k+j)}{T}\,dt\right] \\ &= \frac{E^2}{2}\left\{\left[\frac{T}{2\pi(k-j)}\sin\frac{2\pi t(k+j)}{T}\right]_0^T + \left[\frac{T}{2\pi(k+j)}\sin\frac{2\pi t(k+j)}{T}\right]_0^T\right\} \\ &= 0, \quad \text{for } k \ne j\end{aligned}$$

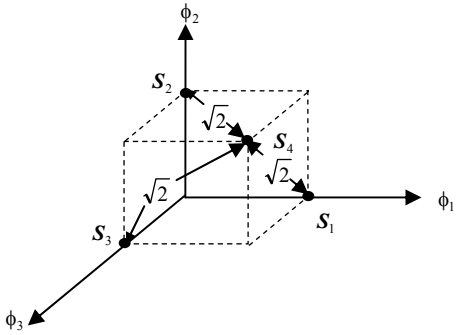


**Figure 8.8** Signal space for Example 8.3.

If $k = j$, we have $(\phi_k, \phi_k) = E^2 \int_0^T \cos(2k\pi t / T)\, dt = E^2 T / 2.$ Hence, $\phi$ constitutes an orthonormal set.

(b) The signal vectors for the set of signals $\{s_k(t)\}$ are

$$\boldsymbol{s}_1 = [1 \quad 0 \quad 0],\ \boldsymbol{s}_2 = [0 \quad 1 \quad 0],\ \boldsymbol{s}_3 = [0 \quad 0 \quad 1],\ \text{and}\ \boldsymbol{s}_4 = [1 \quad 1 \quad 1]$$

as shown in Figure 8.8.

### 8.2.4 Fourier Series

If the signal is periodic with period $T_0$, such that

$$s(t) = s(t + kT_0), \quad k = \pm 1, \pm 2, \ldots \tag{8.47}$$

it can be represented by an infinite set of orthonormal functions made of sines and cosines. This is the most common representation of a signal by a set of orthonormal functions and is known as the *Fourier series*. The *trigonometric form* of the series is

$$s(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + \sum_{k=1}^{\infty} b_k \sin k\omega_0 t \tag{8.48}$$

where $\omega_0 = 2\pi / T = 2\pi f_0$, $f_0 = 1/T_0$, and $a_0$, $a_k$, and $b_k$ are the Fourier coefficients given by

$$a_0 = \frac{1}{T_0}\int_{T_0} s(t)dt \tag{8.49}$$

$$a_k = \frac{2}{T_0}\int_{T_0} s(t)\cos k\omega_0 t\, dt \tag{8.50}$$

and

$$b_k = \frac{2}{T_0}\int_{T_0} s(t)\sin k\omega_0 t\, dt \tag{8.51}$$

$\int_{T_0}(\cdot)\,dt$ denotes integration over any period.

Let the normalized set of functions of sines and cosines and a constant term be

$$\left\{\frac{1}{\sqrt{T_0}}, \sqrt{\frac{2}{T_0}}\cos k\omega_0 t, \sqrt{\frac{2}{T_0}}\sin j\omega_0 t\right\}, \quad k, j = 1, 2, \ldots \tag{8.52}$$

From (8.11) and (8.12), the generalized Fourier series of (8.52) corresponding to the signal $s(t)$ with respect to the orthonormal set is

$$s(t) = \frac{1}{\sqrt{T_0}}\int_0^{T_0} s(t)\frac{1}{\sqrt{T_0}}dt + \sum_{k=1}^{\infty}\left[\sqrt{\frac{2}{T_0}}\cos k\omega_0 t\int_0^{T_0} s(t)\sqrt{\frac{2}{T_0}}\cos k\omega_0 t dt + \sqrt{\frac{2}{T_0}}\sin k\omega_0 t\int_0^{T_0} s(t)\sqrt{\frac{2}{T_0}}\sin k\omega_0 t dt\right] \tag{8.53}$$

Hence, the generalized Fourier series (8.53) is the series (8.48), and the generalized Fourier coefficients $s_k$ of (8.12) are the coefficients (8.49), (8.50), and (8.51). This correspondence can be rewritten as

$$s(t) = \frac{1}{T_0}\int_0^{T_0} s(t)dt + \frac{2}{T_0}\sum_{k=1}^{\infty}\left[\cos k\omega_0 t\int_0^{T_0} s(t)\cos k\omega_0 t dt + \sin k\omega_0 t\int_0^{T_0} s(t)\sin k\omega_0 t dt\right] \tag{8.54}$$

which confirms that the Fourier series, which consists of sines and cosines as the orthonormal set of functions, represents a periodic signal with period $T_0$ for all $t$. Note also that the constant term $a_0$ in the series is the average value of $s(t)$ over the period $T_0$.

Since the Fourier series is well known, we give only a brief discussion. Another useful form of the Fourier series is the *exponential* or *complex form*, which is given by

$$s(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \tag{8.55}$$

where

$$c_k = \frac{1}{T_0} \int_{T_0} s(t) e^{-jk\omega_0 t} dt, \quad k = 1, 2, \ldots \tag{8.56}$$

$c_k$ is the complex number, which is written in *polar form* as

$$c_k = |c_k| e^{j\theta_n} \tag{8.57}$$

$|c_k|$, $k = 0, \pm 1, \pm 2, \ldots$, is the *amplitude spectrum* of $s(t)$. When $s(t)$ is a *real signal*,

$$c_{-k} = c_k^* = |c_k| e^{-\theta_k} \tag{8.58}$$

and

$$|c_{-k}| = |c_k| \tag{8.59}$$

That is, the amplitude spectrum of real signals is even. *The phase spectrum* is the set of numbers of $\theta_k$, $k = 0, \pm 1, \pm 2, \ldots$. For $s(t)$ real,

$$\theta_{-k} = \theta_k \tag{8.60}$$

and thus the phase spectrum is odd. The relationship between the trigonometric and complex form depends on the different ways we write the trigonometric Fourier series. If

$$s(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \tag{8.61a}$$

then

$$A_0 = c_0, \quad A_k = 2|c_k| \tag{8.61b}$$

If

$$s(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + \sum_{k=1}^{\infty} b_k \cos k\omega_0 t \tag{8.62}$$

then

$$a_0 = c_0, \quad a_k = 2\,\Re e\{c_k\}, \; b_k = -2\,\Im m\{c_k\} \tag{8.63}$$

where $\Re e\{\cdot\}$ and $\Im m\{\cdot\}$ denote the real part and imaginary part, respectively. Note that

$$\theta_k = -\tan^{-1}\left(\frac{b_k}{a_k}\right) \tag{8.64}$$

All three forms of the Fourier series are then equivalent.

## 8.3 LINEAR DIFFERENTIAL OPERATORS AND INTEGRAL EQUATIONS

In the representation of signals, we frequently encounter integral equations, as will be seen in upcoming sections and chapters. In this section, we give the mathematical foundations for their solutions by using their inverse: the linear differential equations. We establish the relationship and the approach to solve them through the use of the kernel (Green's function), and the use of eigenfunctions and eigenvalues. A brief analogy to matrix operation also will be given.

From spectral analysis of differential systems, let $f$ be a function in the space $C^2(0,T)$ of twice continuously differentiable functions in the interval $[0,T]$. Then $-f''$ will be in the space of continuous functions $C(0,T)$ [or $C^0(0,T)$]. Consider the following linear differential equation

$$-f'' = \phi \tag{8.65a}$$

with boundary conditions

$$f(0) = \alpha_1, \quad f(T) = \alpha_2 \tag{8.65b}$$

The ordinary solution to differential equations will be to solve the equation $-f'' = \phi$ while ignoring the boundary conditions. Then, we apply the boundary conditions to eliminate the arbitrary constants in the solution. However, if we consider the operator $-D^2$ as being restricted throughout the entire solution process to act only on functions that satisfy the boundary conditions, then the computed constants in the solution of the differential equation are not arbitrary anymore. Rather, they are unknown specific functions of the boundary values $\alpha_1$ and $\alpha_2$. We define the differential operator in modeling systems as $\boldsymbol{T} : C^2(0,T) \to C(0,T) \times \Re^2$, where $\Re^2$ is the set of a couple of real numbers such as

$$\boldsymbol{T} f \triangleq [-f'', f(0), f(T)] \tag{8.66}$$

and where $\triangleq$ denotes definition. The system of equations in (8.65) can be written as

$$\boldsymbol{T} f = (\phi, \alpha_1, \alpha_2) \tag{8.67}$$

The goal is to find an explicit expression for the inverse operator $\boldsymbol{T}^{-1}$, such that $f = \boldsymbol{T}^{-1}(\phi, \alpha_1, \alpha_2)$. To do so, we decompose the differential system (8.65) into two functions, one function $f_d$ involving the distributed input, and the other function $f_b$ involving only the boundary conditions. Hence, we have

$$f_d'' = \phi \quad \text{with} \quad f_d(0) = f_d(T) = 0 \tag{8.68}$$

and

$$f_b'' = 0 \quad \text{with} \quad f_b(0) = \alpha_1, f_b(T) = \alpha_2 \tag{8.69}$$

The superposition of the solutions of (8.68) and (8.69) yields a unique solution $f$ to (8.65). For the purpose of performing *inverse operations*, we define the modified differential operator $\boldsymbol{T}_d : v \to C(0,b)$ by $\boldsymbol{T}_d f \triangleq -D^2 f$ for all $f$ in $v$, where $v$ is

the subspace of functions in $C^2(0,b)$ satisfying the homogeneous boundary conditions $f(0) = f(T) = 0$. Then,

$$\boldsymbol{T}_d f_d = \phi \tag{8.70}$$

includes the boundary conditions in the definition of the operator. Similarly, the differential system (8.69) can be expressed in terms of the modified operator $\boldsymbol{T}_b : \mathcal{P}^2 \to \Re^2$ by $\boldsymbol{T}_b f \triangleq [f(0), f(T)]$ for all $f$ in $\mathcal{P}^2$, where $\mathcal{P}^2$ is the space of functions of the form $f(t) = c_1 t + c_2$. Note that $f''(t) = 0 \Rightarrow f'(t) = c_1 \Rightarrow f(t) = c_1 t + c_2$. Hence, (8.69) can be expressed in terms of the operator $\boldsymbol{T}_b$ as a two-dimensional equation including the differential equation and the boundary conditions to yield

$$\boldsymbol{T}_b f_b = (\alpha_1, \alpha_2) \tag{8.71}$$

Hence, the solution of (8.65) is

$$f = f_d + f_b = \boldsymbol{T}_d^{-1}\phi + \boldsymbol{T}_b^{-1}(\alpha_1 \alpha_2) = \boldsymbol{T}^{-1}(\phi, \alpha_1, \alpha_2) \tag{8.72}$$

Since $\boldsymbol{T}_d$ is a differential operator, its inverse $\boldsymbol{T}_d^{-1}$ is then an integrator, and $f_d$ is given by

$$f_d(t) = (\boldsymbol{T}_d^{-1}\phi)(t) = \int_0^T k(u,t)\phi(t)dt, \quad 0 \le u, t \le T \tag{8.73}$$

The *kernel function* $k(u,t)$ is also referred to as *Green's function* for the differential system (8.65). Note that $f_d(t)$ must satisfy the differential system (8.68). Hence, substituting (8.73) in (8.68) yields

$$-f_d''(t) = -\frac{d^2}{dt^2} = \int_0^T k(u,t)\phi(u)du = \int_0^T -\frac{d^2 k(u,t)}{dt^2}\phi(u)du = \phi(t) \tag{8.74}$$

with

$$f_d(0) = \int_0^T k(0,u)\phi(u)du = 0 \tag{8.75a}$$

and

$$f_d(T) = \int_0^T k(T,u)\phi(u)du = 0 \tag{8.75b}$$

(8.74) and (8.75) are satisfied for all $\phi$ continuous if and only if

$$-\frac{d^2k(u,t)}{dt^2} = \delta(t-u), \quad k(0,u) = k(T,u) = 0 \tag{8.76}$$

We can do the same operations to obtain the solution of (8.69) to be

$$f_b = \boldsymbol{T}_b^{-1}(\alpha_1, \alpha_2) = \alpha_1\rho_1 + \alpha_2\rho_2 \tag{8.77}$$

where $\rho_1$ and $\rho_2$ are functions in $\mathcal{P}^2$, known as the *boundary kernel* for the differential system (8.65). It can be shown that

$$f_b''(t) = \alpha_1\rho_1''(t) \quad \alpha_2\rho_2''(t) = 0 \tag{8.78}$$

with

$$f_b(0) = \alpha_1\rho_1(0) + \alpha_2\rho_2(0) = \alpha_1 \tag{8.79a}$$

and

$$f_b(T) = \alpha_1\rho_1(T) + \alpha_2\rho_2(T) = \alpha_2 \tag{8.79b}$$

for all $\alpha_1$ and $\alpha_2$, and thus, the boundary kernel $\rho(t)$ must obey

$$\rho_1''(t) = 0, \quad \rho_1(0) = 1 \ , \ \rho_1(T) = 0 \tag{8.80a}$$

and

$$\rho_2''(t) = 0, \quad \rho_2(0) = 0 \ , \ \rho_2(T) = 1 \tag{8.80b}$$

Having $\boldsymbol{T}_d^{-1}$ and $\boldsymbol{T}_b^{-1}$, we combine the two inverses to obtain the complete solution of $\phi(t)$ to be

$$\phi(t) = \int_0^T k(u,t)\phi(u)du + \alpha_1\rho_1(t) + \alpha_2\rho_2(t) \tag{8.81}$$

### 8.3.1 Green's Function

Green's function of a differential system is also known as the *kernel function*. Green's function associated with (8.65) for $t \in [0, T]$ must satisfy the differential equation and the boundary conditions, and thus it must satisfy

$$-\frac{d^2 k(u,t)}{dt^2} = \delta(t-u), \qquad \begin{matrix} 0 < u, t < T \\ k(0,u) = k(1,t) = 0 \end{matrix} \tag{8.82}$$

as was shown following the procedure developed by Dorny [1]. We now show how we solve (8.82) in Figure 8.9. Integrating (8.82) and permitting the value of $c_1$ to depend upon the point *u* at which the unit impulse is applied, we have

$$-\frac{dk(u,t)}{dt} = \begin{cases} c_1(u), & 0 < t < u \\ c_1(u)+1, & u < t < T \end{cases} \tag{8.83}$$

which is shown in Figure 8.10. The integration of $-dk(u,t)/dt$ yields a continuity of $-k(u,t)$ at *u*, such that

$$-k(u,t) = \begin{cases} c_1(u)t + c_2(u), & 0 \le t \le u \\ c_1(u)u + c_2(u) + [c_1(u)+1](t-u), & u \le t \le T \end{cases} \tag{8.84}$$

and is shown in Figure 8.11. Thus, the function $k(u,t)$ exists and is unique. It is explicitly given by

$$k(u,t) = \begin{cases} \dfrac{t(T-u)}{T}, & 0 \le t < u \\ \dfrac{u(T-t)}{T}, & u < t \le T \end{cases} \tag{8.85}$$

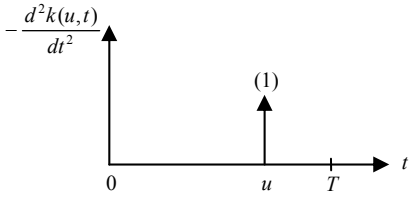


**Figure 8.9** Graph representing (8.82).

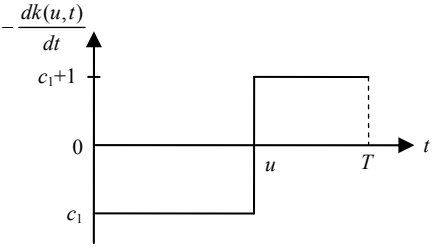


**Figure 8.10** Graph representing (8.83).

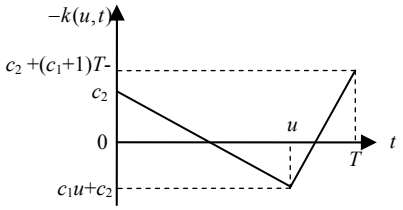


**Figure 8.11** Graph representing (8.84).

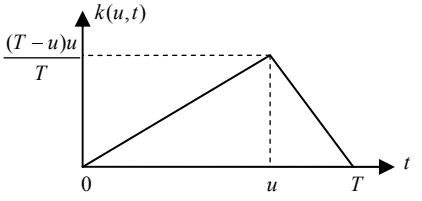


**Figure 8.12** Green's function.

and is shown in Figure 8.12. If $T=1$, then $k(u,t)$ reduces to

$$k(u,t)=\begin{cases} t(1-u), & 0\le t<u \\ u(1-t), & u<t\le 1 \end{cases} \tag{8.86}$$

Similarly, solving for the boundary kernel $\rho(t)$ associated with (8.65), we obtain

$$\rho_1(t)=\frac{T-t}{T},\ \rho_2(t)=\frac{t}{T} \tag{8.87}$$

Substituting (8.85) and (8.87) in (8.81), we obtain the complete solution of $\phi(t)$ to be

$$\begin{aligned}\phi(t) &= \int_0^T k(u,t)\phi(u)du+\alpha_1\rho_1(t)+\alpha_2\rho_2(t) \\ &= \int_0^T \frac{(T-t)u}{T}\phi(u)du+\int_t^T \frac{(T-u)t}{T}\phi(u)du+\alpha_1\frac{T-t}{T}+\alpha_2\frac{t}{T}\end{aligned} \tag{8.88}$$

In general, Green's function cannot be determined by direct integration techniques when the system is not as simple as the second order we just treated. A different approach would be to use the *eigenfunction expansion*, which is to be developed in the next section on integral equations.

### 8.3.2 Integral Equations

Consider the nonhomogeneous linear equation defined over the interval $t\in[0,T]$ and given by

$$\frac{d^2\phi(t)}{dt^2} + f(t) = 0, \quad \phi(0) = \alpha, \phi(T) = \beta \tag{8.89}$$

The *forcing function*, $f(t)$, and the *boundary values*, $\phi(0)$ and $\phi(T)$, are known. The general solution to (8.89) is of the form

$$\phi(t) = \int_0^T k(u,t) f(u) du + (1-t)\alpha + \beta \tag{8.90}$$

where the kernel $k(u,t)$ is Green's function. Note that in comparing (8.90) with (8.89), we observe that

$$\alpha = \frac{\alpha_1 - \alpha_2}{T} \tag{8.91}$$

and

$$\beta = \alpha_1 \tag{8.92}$$

If the boundary values are zero, $\phi(0) = \phi(T) = 0$, then (8.90) becomes

$$\phi(t) = \int_0^T k(u,t) f(u) du \tag{8.93}$$

An *integral equation* is an equation having the form

$$\int_0^T k(u,t)\phi(u) du - \lambda\phi(t) = f(t), \quad 0 \le t \le T \tag{8.94}$$

Given the kernel $k(u,t)$, the nonhomogeneous term $f(t)$, and the eigenvalue $\lambda$, we would like to determine the unknown function $\phi(t)$, and study its dependence on the function $f(t)$ and the eigenvalue $\lambda$.

The integral equation in (8.94) is known as the *Fredholm equation of the second kind* when $\lambda \neq 0$, and as the *Fredholm equation of the first kind* when $\lambda = 0$. When the function $f(t)$ is zero, we have the *eigenvalue problem*

$$\int_0^T k(u,t)\phi(u) du = \lambda\phi(t), \quad 0 \le t \le T \tag{8.95}$$

For simplicity and without loss of generality, if $T = 1$, then the nonhomogeneous linear differential equation becomes

$$\frac{d^2\phi(t)}{dt^2} + f(t) = 0, \qquad \begin{matrix} 0 < t < 1 \\ \phi(0) = \phi(1) = 0 \end{matrix} \tag{8.96}$$

Using *eigenfunction* expansion, the associated *eigenproblem* is given by

$$\frac{d^2\phi(t)}{dt^2} + \lambda\phi(t) = 0, \qquad \begin{matrix} 0 < t < 1 \\ \phi(0) = \phi(1) = 0 \end{matrix} \tag{8.97}$$

The nontrivial solutions $[\phi(t) \neq 0]$ to the boundary value problem of (8.97) are called *eigenfunctions*. The corresponding values of $\lambda$ are known as *eigenvalues*. Before developing the solution, we first give some important properties of integral equations that will be used in solving the eigenproblem.

*Properties*

1. There exists at *least one* square integrable function $\phi(t)$ corresponding to a definite eigenvalue $\lambda$ $(\lambda \neq 0)$ that satisfies (8.95).

2. If the eigenfunction $\phi_k(t)$ is a solution, then $c\phi_k(t)$, with $c$ a constant, is also a solution. This means that we can normalize the eigenfunction.

3. An eigenvalue may be associated with more than one independent eigenfunction.

4. If $\phi_1(t)$ and $\phi_2(t)$ are two eigenfunctions corresponding to the same eigenvalue $\lambda$, then $c_1\phi_1(t)$ and $c_2\phi_2(t)$, where $c_1$ and $c_2$ are constants, are also eigenfunctions corresponding to the same eigenvalue $\lambda$.

5. Let $\lambda_k$ and $\lambda_j$ be any two distinct eigenvalues with corresponding eigenfunctions $\phi_k(t)$ and $\phi_j(t)$. Then, $\phi_k(t)$ and $\phi_j(t)$ are orthogonal. Since $\phi_k(t)$ and $\phi_j(t)$ are the eigenfunctions corresponding to $\lambda_k$ and $\lambda_j$ $(\lambda_k \neq \lambda_j)$, then

$$\lambda_k\phi_k(t) = \int_0^T k(t,u)\phi_k(u)du \tag{8.98}$$

and

$$\lambda_j \phi_j(t) = \int_0^T k(t,u)\phi_j(u)du \tag{8.99}$$

It follows that

$$\int_0^T \phi_k(t)\phi_j(t)dt = \int_0^T \phi_k(t)\left[\frac{1}{\lambda_j}\int_0^T k(t,u)\phi_j(t)du\right]dt \tag{8.100}$$

Interchanging the integrations and noting that $k(t,u) = k(u,t)$, we have

$$\int_0^T \phi_k(t)\,\phi_j(t)dt = \frac{1}{\lambda_j}\int_0^T \phi_j(u)du\left[\int_0^T \phi_k(t)\,k(u,t)dt\right] \tag{8.101}$$

From the definition of the integral equation, the integral in brackets is $\lambda_k \phi_k(t)$. Hence,

$$\int_0^T \phi_k(t)\,\phi_j(t)dt = \frac{\lambda_k}{\lambda_j}\int_0^T \phi_k(u)\phi_j(u)du \tag{8.102a}$$

or

$$(\lambda_k - \lambda_j)\int_0^T \phi_k(t)\phi_j(t)dt = 0 \tag{8.102b}$$

Since $\lambda_k \neq \lambda_j$, then $\int_0^T \phi_k(t)\phi_j(t)dt = 0$, and we conclude that the eigenfunctions $\phi_k(t)$ and $\phi_j(t)$ are orthogonal.

6. Because the kernel $k(t,u)$ is nonnegative-definite, it can be expressed in the Fourier series expansion to yield

$$k(u,t) = \sum_{k=1}^{\infty} \lambda_k \phi_k(t)\phi_j(u) \quad 0 \le t,u \le T \tag{8.103}$$

This is known as *Mercer's formula*.

Now, solving (8.97) and imposing the boundary conditions, we obtain the nontrivial solutions

$$\phi_k(t) = \sin k\pi t, \quad k = 1, 2, \ldots \tag{8.104}$$

corresponding to the eigenvalues

$$\lambda_k = k^2\pi^2, \quad k = 1, 2, \ldots \tag{8.105}$$

The eigenfunctions corresponding to different eigenvalues are orthogonal; that is,

$$\int_0^1 \sin(k\pi t)\sin(j\pi t)dt = 0 \quad \text{for} \quad k \neq j \tag{8.106}$$

Note that (8.97) is a problem of the type of (8.96), with the forcing function $f(t) = \lambda\phi(t)$. Since the "solution" of (8.96) is given by (8.93), it becomes

$$\phi(t) = \lambda\int_0^1 k(u,t)\phi(u)du, \quad 0 < t < 1 \tag{8.107}$$

The function $\phi(t)$ appears in both sides of (8.107); that is, we really have not solved for $\phi(t)$, but we have shown that the nonhomogeneous linear equation (8.97) is equivalent to the *integral equation* (8.95). The boundary conditions are incorporated in the integral equation through its Green's function, known as *kernel* $k(u,t)$. Hence, we write

$$\phi(t) = \lambda\int_0^1 k(u,t)\phi(u)du\,, \quad 0 < t < 1 \tag{8.108}$$

**Example 8.4**

Consider a differential system of the form

$$\frac{d\phi(t)}{dt} + \phi(t) = f(t), \quad 0 \le t \le 1$$

$$\phi(0) = \phi(1) = \alpha$$

Determine the kernel $k(u,t)$.

*Solution*

A solution to the homogeneous equation

$$\frac{d\phi(t)}{dt} + \phi(t) = 0$$

is $\phi_h(t) = ce^{-t}$. We guess a solution for $k(u,t)$ to be

$$k(u,t) = \begin{cases} c_1 e^{-t}, & 0 \le t \le u \\ c_2 e^{-t}, & u \le t \le 1 \end{cases}$$

In this case, the kernel satisfies $dk(u,t)/dt = \delta(t-u)$, and hence at $t = u$ we have $-c_1 e^{-u} + c_2 e^{-u} = 1$. From the boundary conditions, we have $c_1 + c_2 e^{-1} = 0$. Solving for $c_1$ and $c_2$, we obtain

$$c_1 = \frac{e^{u-1}}{1+e^{-1}} \quad \text{and} \quad c_2 = \frac{e^{-u}}{1+e^{-1}}$$

Consequently, the kernel is

$$k(u,t) = \begin{cases} -\dfrac{e^{-u-t-1}}{1+e^{-1}}, & 0 \le t \le u \\ \dfrac{e^{u-t}}{1+e^{-1}}, & u \le t \le 1 \end{cases}$$

**Example 8.5**

Consider the homogeneous eigenvalue problem $K\phi - \lambda\phi = 0$. Let the associated eigenproblem be given by the following homogeneous differential equation

$$\frac{d^2\phi(t)}{dt^2} + \lambda\phi(t) = 0 \qquad 0 < t < 1$$
$$\phi'(0) = \phi(1) = 0$$

(a) Determine the kernel and write the corresponding integral equation.
(b) Find the eigenvalues and eigenfunctions.

*Solution*

(a) Let $\phi''(t)$ denote $d^2\phi(t)/dt^2$, and $\phi'(t)$ denote $d\phi(t)/dt$. Integrating the differential equation with respect to *t*, we have

$$\phi'(t) - \phi'(0) + \lambda \int_0^t \phi(u)du = 0$$

Integrating again results in

$$\phi(t) - \phi(0) - t\phi'(0) + \lambda \int_0^t (t-u)\phi(u)du = 0$$

Applying the boundary conditions, $\phi'(0) = 0$ yields

$$\phi(t) - \phi(0) + \lambda \int_0^t (t-u)\phi(u)du = 0$$

To determine $\phi(0)$ in the above equation, we apply the boundary condition $\phi(1) = 0$ at $t = 1$; that is,

$$\phi(1) - \phi(0) + \lambda \int_0^1 (1-u)\phi(u)du = 0 \quad \text{or} \quad \phi(0) = \lambda \int_0^1 (1-u)\phi(u)du$$

Substituting for $\phi(0)$ into $\phi(t)$, we have

$$\begin{aligned}
\phi(t) &= \lambda \int_0^1 (1-u)\phi(u)du - \lambda \int_0^t (t-u)\phi(u)du \\
&= \lambda \left[ \int_0^t (1-u)\phi(u)du + \int_t^1 (1-u)\phi(u)du \right] - \lambda \int_0^t (t-u)\phi(u)du \\
&= \lambda \int_0^t (1-u)\phi(u)du + \lambda \int_0^1 (1-u)\phi(u)du = \lambda \int_0^1 k(u,t)\phi(u)du
\end{aligned}$$

where the kernel $k(u,t)$ is

$$k(u,t)=\begin{cases}1-t, & 0\le t\le u\\ 1-u, & t\le u\le 1\end{cases}$$

(b) From the homogeneous differential equation $\phi''(t)+\lambda\phi'(t)=0$, we have the general solution

$$\phi(t)=A\cos\sqrt{\lambda}t+B\sin\sqrt{\lambda}t$$

The derivative of $\phi(t)$ is

$$\phi'(t)=\ -\sqrt{\lambda}A\sin\sqrt{\lambda}t+\sqrt{\lambda}B\cos\sqrt{\lambda}t$$

Applying the boundary condition, $\phi'(t)=0\ \Rightarrow\ \phi'(0)=-\sqrt{\lambda}B=0$. Since $\lambda\neq 0$ $\Rightarrow\ B=0$, then $\phi(t)=A\cos\sqrt{\lambda}t$.

Applying the other boundary condition, $\phi(1)=0$ yields

$$\phi(1)=A\cos\sqrt{\lambda}=0\Rightarrow\sqrt{\lambda}=(2k-1)\frac{\pi}{2},\quad k=1,2,\ldots$$

Hence, the eigenvalues are

$$\lambda_k=\frac{(2k-1)^2\pi^2}{4},\quad k=1,2,\ldots$$

and the corresponding eigenfunctions are

$$\phi(t)=A\cos(2k-1)\frac{\pi}{2}t,\quad 0\le t\le 1$$

but $\int_0^1\phi^2(t)dt=1\ \Rightarrow\ A=\sqrt{2}$. Therefore, the nonzero eigenvalues and the corresponding normalized eigenfunctions are

$$\lambda_k=\frac{(2k-1)^2\pi^2}{4}\ \text{ and }\ \phi_k(t)=\sqrt{2}\cos(2k-1)\frac{\pi}{2}t,\qquad \begin{matrix}0\le t\le 1\\ k=1,2,\ldots\end{matrix}$$

### 8.3.3 Matrix Analogy

A differential equation with an appropriate set of boundary conditions is analogous to a square matrix equation. In order to remove any abstractness from the differential operators and their inverses, we explore this analogy by considering a simple example. Let $\boldsymbol{A}$ be a $2\times 2$ square matrix, and $\boldsymbol{x}$ and $\boldsymbol{y}$ two column vectors, such that $\boldsymbol{Ax} = \boldsymbol{y}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{y}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\det(\boldsymbol{A})} \begin{bmatrix} a_{22} & -a_{12} \\ a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\det(\boldsymbol{A}) = a_{11}a_{22} - a_{12}a_{21}$. The element of $\boldsymbol{x}$ can be expressed as

$$x_1 = \frac{1}{\det(\boldsymbol{A})}(a_{22}y_1 - a_{12}y_2) \text{ and } x_2 = \frac{1}{\det(\boldsymbol{A})}(a_{21}y_1 + a_{11}y_2)$$

In general form, for an $n\times n$ matrix, we have

$$x_i = \sum_{j=1}^{n} (\boldsymbol{A}^{-1})_{ij}\, y_j, \quad i = 1, 2, \ldots, n \tag{8.109}$$

We observe that the inverse matrix is analogous to the integral equation (inverse differential equation). The symbol $\left(\boldsymbol{A}^{-1}\right)_{ij}$ represents the elements in row $i$ and column $j$ of the $n\times n$ matrix $\boldsymbol{A}^{-1}$, as clearly shown by the example of the $2\times 2$ matrix. Hence, the form (8.109) (of $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{y}$) is the discrete analog for the integral equation $\phi_d(t) = \int_0^T k(u,t)\phi(u)du$ of (8.73). The kernel $k(u,t)$ is the analogue of the inverse matrix $\boldsymbol{A}^{-1}$. If we compare (8.103) to (8.81), the analogy is not very clear, because the true analog of $\boldsymbol{A}^{-1}$ is the *pair* of kernel functions $k(u,t)$ and $\rho(t)$.

Based on the above discussion, to determine the eigenvalues for which the integral equation of (8.95) has no trivial solution, we must solve the eigenvalue problem

$$(\boldsymbol{K} - \boldsymbol{I}\lambda)\boldsymbol{\Phi} = \boldsymbol{0} \tag{8.110}$$

where $\boldsymbol{K}$ is a symmetric nonnegative definite matrix representing the transformation operator.

In summary, from Section 4.3, if an operator $\boldsymbol{T}$ is invertible and $\boldsymbol{T}\boldsymbol{x} = \lambda\boldsymbol{x}$, then $\boldsymbol{T}^{-1}\boldsymbol{x} = (1/\lambda)\boldsymbol{x}$. That is, the eigenvectors of $\boldsymbol{T}$ and $\boldsymbol{T}^{-1}$ are identical and correspond to the reciprocal eigenvalues. We have seen that a differential system $\boldsymbol{T}$ is invertible if and only if the trivial solution $\lambda = 0$ is not an eigenvalue of $\boldsymbol{T}$, and thus the kernel of $\boldsymbol{T}$ does not exist. Invertible differential equations come in *pairs*, the integral equation and the inverse. Hence, we use the integral form to obtain any information about the eigenfunctions and the solutions of equations. However, integral equations are difficult to solve. We thus return to the differential form and standard differential equation techniques to determine the eigenfunctions, as we have shown in Example 8.5.

## 8.4 REPRESENTATION OF RANDOM PROCESSES

In Section 8.2, we represented deterministic finite energy signals in terms of an orthogonal series expansion. We now extend this concept to random processes.

Let $X(t)$ be a random process to be represented by a complete set of orthonormal functions $\{\phi_k(t)\}$ specified over the interval [0,$T$]. That is, we write

$$X(t) = \lim_{k\to\infty} \sum_{k=1}^{K} X_k \phi_k(t) \tag{8.111}$$

where the random variable $X_k$ is given by

$$X_k = \int_0^T X(t)\phi_k(t)dt \tag{8.112}$$

The above ordinary limit is not practical, since it requires that all sample functions of the random process satisfy (8.111), which is not possible. Instead, we use a little more relaxed type of the convergence, which is the *mean-square convergence*. That is, we require

$$\lim_{K\to\infty} E\left\{\left[X(t)-\sum_{k=1}^{K} X_k \phi_k(t)\right]^2\right\}=0 \tag{8.113}$$

Equivalently, we say

$$X(t)=\lim_{K\to\infty}\sum_{k=1}^{K} X_k \phi_k(t) \tag{8.114}$$

Since it is generally easier to solve problems in which the random variables are uncorrelated, we would select the set $\{\phi_k(t)\}$, such that the coefficients $X_k, k=1,2,\ldots,K,$ are uncorrelated provided that

$$E[X_k X_j]=\lambda_k \delta_{kj} \tag{8.115}$$

Substituting (8.112) into (8.115), we obtain

$$\begin{aligned} E[X_k X_j] &= E\left[\int_0^T X(t)\phi_k(t)dt \int_0^T X(u)\phi_j(u)du\right] \\ &= \int_0^T \phi_k(t)dt \int_0^T K_{xx}(t,u)\phi_j(u)du = \lambda_k \delta_{kj} \end{aligned} \tag{8.116}$$

where $K_{xx}(t,u)=E[X(t)X(u)]$ is the autocovariance function of $X(t)$. Equation (8.116) is satisfied if

$$\int_0^T K_{xx}(t,u)\phi_j(u)du=\lambda_j \phi_j(t) \tag{8.117}$$

Equation (8.117) is the homogenous linear integral equation as defined in the previous section, and the autocovariance function represents the kernel. The kernel $K_{xx}(t,u)$ can always be expanded in the series

$$K_{xx}(t,u)=\sum_{k=1}^{\infty}\lambda_k \phi_k(t)\,\phi_k(u), \quad 0\le t,u\le T \tag{8.118}$$

where the convergence is uniform for $0\le t,u\le T$. This is *Mercer's theorem*.

To compute the discussion on the properties of the integral equations given in the previous section, we add the following properties.

1. If $K_{xx}(t,u)$ is positive-definite, the eigenfunctions form a complete orthonormal set. If $K_{xx}(t,u)$ is not positive-definite, the eigenfunctions cannot form a complete orthonormal set. In such a situation, the eigenfunctions are augmented with additional orthonormal functions to complete the rest. These additional functions are referred to as eigenfunctions with zero eigenvalues.

2. The mean energy of the random process $X(t)$ in the interval $(0,T)$ is the infinite sum of the eigenvalues; that is,

$$E\left[\int_0^T X^2(t)dt\right] = \int_0^T K_{xx}(t,t)dt = \sum_{k=1}^{\infty} \lambda_k \tag{8.119}$$

We have

$$\begin{aligned} E\left[\int_0^T X^2(t)dt\right] &= E\left[\int_0^T \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} X_j X_k \phi_j(t)\phi_k(t)dt\right] \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} E[X_j X_k]\int_0^T \phi_j(t)\phi_k(t)dt = \sum_{k=1}^{\infty} E[X_k^2] \end{aligned} \tag{8.120}$$

Assuming $X(t)$ is zero mean, we can use (8.115), and thus (8.120) reduces to

$$E\left[\int_0^T X^2(t)dt\right] = \int_0^T K_{xx}(t,t)dt = \sum_{k=1}^{\infty} \lambda_k \tag{8.121}$$

*Karhunen-Loève Expansion*

The series expansion of $X(t)$

$$X(t) = \sum_{k=1}^{K} X_k \phi_k(t) \tag{8.122}$$

is known as the *Karhunen-Loève* expansion. We now show the mean-square convergence for this series representation. We define the error $\varepsilon_K(t)$ as

$$\varepsilon_K(t) = E\left\{\left[X(t) - \sum_{k=1}^{K} X_k \phi_k(t)\right]^2\right\} \tag{8.123}$$

Expanding $\varepsilon_K(t)$, we have

$$\begin{aligned}\varepsilon_K(t) &= E\left[X^2(t) - 2X(t)\sum_{k=1}^{K} X_k\phi_k(t) + \sum_{j=1}^{K}\sum_{k=1}^{K} X_j X_k \phi_j(t)\phi_k(t)\right] \\ &= K_{xx}(t) - 2E\left\{X(t)\sum_{k=1}^{K}\left[\int_0^T X(u)\phi_k(u)du\right]\right\}\phi_k(t) \\ &\quad + \sum_{j=1}^{K}\sum_{k=1}^{K} E[X_j X_k]\phi_j(t)\phi_k(t) \end{aligned} \tag{8.124}$$

Using the fact that $E[X_j X_k] = \lambda_k \delta_{jk}$ and $\int_0^T K_{xx}(t,u)\phi_k(u)du = \lambda_k\phi_k(t)$, results in

$$\varepsilon_K(t) = K_{xx}(t,t) - \sum_{k=1}^{K}\lambda_k\phi_k^2(t) \tag{8.125}$$

From Mercer's theorem, we have

$$K_{xx}(t,u) = \sum_{k=1}^{\infty}\lambda_k\phi_k(t)\,\phi_k(u) \tag{8.126}$$

or

$$K_{xx}(t,t) = \sum_{k=1}^{\infty}\lambda_k\phi_k^2(t) \tag{8.127}$$

Hence,

$$\lim_{K\to\infty}\varepsilon_K(t) = 0 \tag{8.128}$$

and the series converges in the mean-square sense.

### 8.4.1 The Gaussian Process

In Chapters 3 and 4, we presented the Gaussian random variable and Gaussian random process. The definition of the Gaussian process was deduced from the property of jointly Gaussian random variables. We now give a formal definition of the Gaussian process and some of the properties that follow from the definition,

while using the new concept of orthogonal functions and the Karhunen-Loève expansion that we have just developed in this chapter.

Recall that *K* random variables $X_1, X_2, \ldots, X_K$ are jointly Gaussian, if

$$Y = \sum_{k=1}^{K} g_k Y_k \tag{8.129}$$

is a Gaussian random variable for all possible finite $g_k$. If the number of the random variables *K* is infinite, then the mean-square value of *Y*, $E[Y^2]$, must be finite.

*Definition*. A random process $X(t)$ defined over some interval $[T_i, T_f]$ with mean value function $m_x(t)$ and covariance function $K_{xx}(t,u)$ is said to be *Gaussian* if and only if the random variable

$$Y = \int_{T_i}^{T_f} g(u)x(u)du \tag{8.130}$$

is Gaussian for all possible functions $g(u)$, such that $E[Y^2] < \infty$.

*Properties*

1. For any set of times $t_1, t_2, \ldots, t_n$ in the interval $[T_i, T_f]$, the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly Gaussian random variables.

*Proof*. Let $X(t)$ be a Gaussian random process, and let

$$g(u) = \sum_{k=1}^{n} g_k \delta(u - t_k) \tag{8.131}$$

Then, from (8.127), the random variable

$$Y = \int_{T_i}^{T_f} g(u)x(u)du = \sum_{k=1}^{n} g_k \int_{T_i}^{T_f} x(u)\delta(u - t_k)du = \sum_{k=1}^{n} g_k x(t_k) \tag{8.132}$$

is a Gaussian random variable for any set $g_k$, and $X(t_k)$ is a random variable corresponding to the sampling instants $t_k, k = 1, 2, \ldots, n$. Hence, the random

variables $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly Gaussian variables.

2. If $X(t)$ is a Gaussian random process applied to a linear system with impulse response $h(t,u)$, then the output $Y(t)$ is also a Gaussian process.

*Proof.* The output $Y(t)$ is given by

$$Y(t) = \int_{T_i}^{T_f} h(t,u)x(u)du, \quad T_1 < t < T_2 \tag{8.133}$$

where the interval $[T_1, T_2]$ is the range over which $Y(t)$ is defined. From (8.130), $Y(t)$ is a Gaussian random variable. The goal is to show that any linear function of $Y(t)$ is Gaussian. Hence, let

$$Z = \int_{T_1}^{T} g_y(t)y(t)dt \tag{8.134}$$

where $g_y(t)$ is any arbitrary function, such that $E[Z^2] < \infty$. Substituting (8.133) in (8.134), we obtain

$$Z = \int_{T_1}^{T_2} g_y(t) \int_{T_i}^{T_f} h(t,u)x(u)dudt \tag{8.135}$$

Integrating first with respect to *t*, we have

$$Z = \int_{T_i}^{T_f} x(u) \left[ \int_{T_1}^{T_f} g_y(t)h(t,u)dt \right] du = \int_{T_i}^{T_f} x(u)g(u)du \tag{8.136}$$

since the integral between brackets is $g(u)$. Since $X(t)$ is a Gaussian random process in the interval $[T_i, T_f]$, then $Z$ is a Gaussian random variable for every choice of $g(u)$, and $Y(t)$ is a Gaussian random process for every $g_y(t)$.

3. If

$$Y_1 = \int_{T_i}^{T_f} g_1(u)x(u)du \tag{8.137}$$

and

$$Y_2 = \int_{T_i}^{T_f} g_2(u)x(u)du \tag{8.138}$$

where $X(u)$ is a Gaussian random process, then $Y_1$ and $Y_2$ are jointly Gaussian.

*Proof*. Let $Y$ be a linear combination of $Y_1$ and $Y_2$. Then,

$$Y = c_1Y_1 + c_2Y_2 \tag{8.139}$$

where $c_1$ and $c_2$ are constants. Substituting (8.137) and (8.138) in (8.139), we have

$$Y = \int_{T_i}^{T_f} [c_1g_1(u) + c_2g_2(u)]x(u)du \tag{8.140}$$

which is, from (8.129) and (8.130), a Gaussian random variable for all possible $c_1$ and $c_2$. Thus, $Y_1$ and $Y_2$ are jointly Gaussian.

4. Let $\phi_j(t)$ and $\phi_k(t)$ be two orthonormalizing eigenfunctions in the interval $t \in [T_i, T_f]$ of

$$\lambda_j\phi_j(t) = \int_{T_i}^{T_f} K_{xx}(t,u)\phi_j(u)du \tag{8.141}$$

where $K_{xx}(t,u)$ is the kernel. If

$$X_j = \int_{T_i}^{T_f} \phi_j(u)x(u)du \tag{8.142}$$

and

$$X_k = \int_{T_i}^{T_f} \phi_k(u)x(u)du \tag{8.143}$$

then $X_j$ and $X_k$, $j \neq k$, are statistically independent Gaussian random variables.

*Proof.* From property (3), $X_j$ and $X_k$ are jointly Gaussian random variables. From (8.115), $E[X_j X_k] = \lambda_k \delta_{kj}$, which means that $X_j$ and $X_k$ are uncorrelated random variables. Since the random variables are Gaussian, they are also statistically independent. In this case, the density function $X_k$ is given by

$$f_{X_k}(x_k) = \frac{1}{\sqrt{2\pi\lambda_k}} \exp\left[-\frac{(x_k - m_k)^2}{2\lambda_k}\right] \tag{8.144}$$

where $\lambda_k$ is the corresponding eigenvalue and $m_k$ is the mean given by

$$m_k = E[X_k] = \int_{T_i}^{T_f} m_x(t)\phi_k(t)dt \tag{8.145}$$

Observe that property (4) is the base of the characterization of a Gaussian random process into a Karhunen-Loève expansion; that is, in a series of countably infinite sets of the statistically independent Gaussian random variables, as was shown in the previous section.

### 8.4.2 Rational Power Spectral Densities

Let $X(t)$ be a zero mean wide-sense stationary process with a rational power spectrum density of the form

$$S_{xx}(\omega) = \frac{N(\omega^2)}{D(\omega^2)} \tag{8.146}$$

where $\omega = 2\pi f$. $S_{xx}(\omega)$ is an even function of ω, and forms a Fourier transform pair with autocorrelation function $R_{xx}(\tau)$, which is equal to the autocovariance function, since $E[X(t)] = 0$. Thus,

$$R_{xx}(\tau) = E[X^2(t)] = K_{xx}(\tau) \tag{8.147}$$

and

$$E[X^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega \tag{8.148}$$

Let $N(\omega^2)$ be a polynomial of degree $q$ in $\omega^2$, and $D(\omega^2)$ be a polynomial of degree $r$ in $\omega^2$, where $q < r$, since the mean-square value $E[X^2(t)]$ is assumed finite.

For a rational function of the form given in (8.146), the solution to the integral equation can always be obtained from the corresponding linear differential equation with constant coefficients. The integral equation is

$$\lambda\phi(t) = \int_0^T K_{xx}(t-u)\phi(u)du, \quad 0 \le t \le T \tag{8.149}$$

Since $\phi$ is zero outside the interval $[0, T]$, (8.149) can be written as

$$\lambda\phi(t) = \int_{-\infty}^{\infty} K_{xx}(t-u)\phi(u)du, \quad -\infty < t < \infty \tag{8.150}$$

Taking the Fourier transform, we have

$$\lambda\Phi(j\omega) = S_{xx}(\omega)\Phi(j\omega) = \frac{N(\omega^2)}{D(\omega^2)}\Phi(j\omega) \tag{8.151}$$

or

$$[\lambda D(\omega^2) - N(\omega^2)]\Phi(j\omega) = 0 \tag{8.152}$$

Let $p = j\omega$, then $p^2 = -\omega^2$. Substituting in (8.152), we have

$$[\lambda D(-P^2) - N(-P^2)]\Phi(p) = 0 \tag{8.153}$$

where $p$ can be implemented as an operator, and thus (8.153) can be transformed into a homogeneous linear differential equation, such that $p$ denotes $d/dt$. Since the polynomial in (8.153) is of degree $2r$, there are $2r$ homogeneous solutions

denoted as $\phi_{h_k}(t)$, $k = 1, 2, \ldots, 2r$, for every eigenvalue $\lambda$. Once we obtain $\phi_{h_k}(t)$, $k = 1, 2, \cdots, 2r$, we form

$$\phi(t) = \sum_{k=1}^{2r} c_k \phi_{h_k}(t) \tag{8.154}$$

where $c_k$ is a constant, and then substitute into the original integral equation to determine $\lambda$ and $c_k$, as shown in Example 8.4.

**Example 8.6**

Let $X(t)$ be a zero mean wide-sense stationary process with power spectrum density

$$S_{xx}(\omega) = \frac{2\alpha\sigma^2}{\omega^2 + \alpha^2} \quad \text{for all } \omega$$

$\alpha$ and $\sigma^2$ are constants.

(a) Obtain the differential equation.
(b) Determine the eigenvalues $\lambda$.

*Solution*

(a) The autocorrelation function is given by $R_{xx}(\tau) = \sigma^2 e^{-\alpha|\tau|}$. Observe that $R_{xx}(0) = \sigma^2$, which is the variance of $X(t)$, since it is zero mean. Assuming a symmetric observation interval, the integral equation is

$$\lambda\phi(t) = \int_{-T}^{T} K_{xx}(t,u)\phi(u)du = \int_{-T}^{T} \sigma^2 e^{-|t-u|}\phi(u)du$$

since $R_{xx}(\tau) = K_{xx}(\tau)$, where $\tau = t - u$. The above expression can be rewritten as

$$\lambda\phi(t) = \int_{-T}^{t} \sigma^2 e^{-\alpha(t-u)}\phi(u)du + \int_{t}^{T} \sigma^2 e^{-\alpha(u-t)}\phi(u)du$$

Differentiating with respect to *t*, we obtain

$$\lambda \frac{d\phi(t)}{dt} = -\alpha \int_{-T}^{t} \sigma^2 e^{-\alpha(t-u)} \phi(u) du + \alpha \int_{t}^{T} \sigma^2 e^{-\alpha(u-t)} \phi(u) du$$

Differentiating again results in

$$\lambda \frac{d^2\phi(t)}{dt^2} = \alpha^2 \int_{-T}^{t} \sigma^2 e^{-\alpha(t-u)} \phi(u) du - 2\alpha\sigma^2 \phi(t) + \alpha^2 \int_{t}^{T} \sigma^2 e^{-\alpha(u-t)} \phi(u) du$$

$$= -2\alpha\sigma^2 \phi(t) + \alpha^2 \int_{-T}^{T} \sigma^2 e^{-\alpha|t-u|} \phi(u) du = -2\alpha\sigma^2 \phi(t) + \alpha^2 \lambda \phi(t)$$

or

$$\frac{d^2\phi(t)}{dt^2} - \frac{\alpha^2}{\lambda}\left(\lambda - \frac{2\sigma^2}{\alpha}\right)\phi(t) = 0$$

(b) Since the eigenvalues must be nonnegative, we have four cases:

$$(1)\ \lambda = 0 \qquad (2)\ 0 < \lambda < \frac{2\sigma^2}{\alpha} \qquad (3)\ \lambda = \frac{2\sigma^2}{\alpha} \qquad (4)\ \lambda > \frac{2\sigma^2}{\alpha}$$

*Case* (*1*). $\lambda = 0$. The differential equation is $-2\alpha\sigma^2\phi(t) = 0$. Since $\alpha\sigma^2 \neq 0$, we only have the trivial solution $\phi(t) = 0$, and thus $\lambda = 0$ is not an eigenvalue.

*Case* (*2*). $0 < \lambda < 2\sigma^2/\alpha$. Let $\beta^2 = (\alpha^2/\lambda)/[\lambda - (2\sigma^2/\alpha)] \Rightarrow 0 < \beta^2 < \infty$, and the differential equation becomes $[d^2\phi(t)/dt^2] + \beta^2\phi(t) = 0$. This has a general solution of the form $\phi(t) = c_1 e^{j\beta t} + c_2 e^{-j\beta t}$, where $c_1$ and $c_2$ are constants to be determined. Substituting for $\phi(t)$ in the integral equation and integrating, we obtain

$$\lambda(c_1 e^{j\beta t} + c_2 e^{-j\beta t}) = c_1\sigma^2 e^{j\beta t}\left(\frac{1}{\alpha + j\beta} + \frac{1}{\alpha - j\beta}\right) + c_2\sigma^2 e^{-j\beta t}\left(\frac{1}{\alpha - j\beta} + \frac{1}{\alpha + j\beta}\right)$$

$$-\sigma^2 e^{\alpha t}\left(\frac{c_1 e^{-(\alpha+j\beta)T}}{\alpha + j\beta} + \frac{c_2 e^{-(\alpha-j\beta)T}}{\alpha - j\beta}\right) - \sigma^2 e^{\alpha t}\left(\frac{c_1 e^{-(\alpha-j\beta)T}}{\alpha - j\beta} + \frac{c_2 e^{-(\alpha+j\beta)T}}{\alpha + j\beta}\right)$$

By inspection of the above expression, we have

$$\frac{c_1 e^{-(\alpha+j\beta)T}}{\alpha+j\beta}+\frac{c_2 e^{-(\alpha-j\beta)}}{\alpha-j\beta}=0, \quad \frac{c_1 e^{-(\alpha-j\beta)T}}{\alpha-j\beta}+\frac{c_2 e^{-(\alpha+j\beta)}}{\alpha+j\beta}=0,$$

$$\lambda c_1 = c_1\sigma^2\left(\frac{1}{\alpha+j\beta}+\frac{1}{\alpha-j\beta}\right), \text{ and } \lambda c_2 = c_2\sigma^2\left(\frac{1}{\alpha-j\beta}+\frac{1}{\alpha+j\beta}\right)$$

Solving for $\lambda, c_1, \text{and } c_2$, we obtain the eigenvalue $\lambda = 2\alpha\sigma^2(\alpha^2+\beta^2)$ and $c_1^2 = c_2^2 \Rightarrow c_1 = c_2$ or $c_1 = -c_2$.

When $c_1 = c_2$, the $k$th eigenfunction is $\phi_k(t) = c_1(e^{j\beta_k t}+e^{-j\beta_k t})$ $= A_k \cos\beta_k t$. Since $\int_{-T}^{T}\phi_k^2(t)=1$, we can solve for $A_k$ as $A_k = 1/\sqrt{T[1+(\sin 2\beta_k T/2\beta_k)]}$. It should be noted that when $c_1 = c_2$, we have $(\alpha+j\beta)e^{j\beta T} = (-\alpha+j\beta)e^{-j\beta T}$ or $\tan(\beta/T)=\alpha/\beta$. The values of β satisfying the above equation can be determined graphically, as shown in Figure 3.8 of Van Trees [2].

When $c_1 = -c_2$, we follow the same procedure as before to obtain $\phi_k(t) = B_k \sin\beta_k t$, where $B_k = 1/\sqrt{T[1-(\sin 2\beta_k T/2\beta_k)]}$ and $\tan\beta T = -\alpha/\beta$.

*Case* (*3*). $\lambda = 2\sigma^2/\alpha$. In this case, the differential equation becomes

$$\frac{d^2\phi(t)}{dt^2}=0 \Rightarrow \phi(t)=c_1 t + c_2$$

Substituting for $\phi(t)$ into the integral equation to determine the constants $c_1$ and $c_2$, we obtain $c_1 = c_2[T-(1/\alpha)]$ and $c_1 = -c_2[T-(1/\alpha)]$. For $T \neq 1/\alpha$ $\Rightarrow c_1 = c_2 = 0$ we only have the trivial solution $\phi(t)=0$, and $\lambda = 2\sigma^2/\alpha$ is not an eigenvalue.

*Case* (*4*). $\lambda > 2\sigma^2/\alpha$. Let $\gamma = (\alpha^2/\lambda)[\lambda-(2\sigma^2/\alpha)] \Rightarrow 0<\gamma^2<\alpha^2$. For $|\gamma|<\alpha$, the differential equation is $[d^2\phi(t)/dt^2]-\gamma\phi(t)=0$, which has the solution $\phi(t)=c_1e^{\gamma t}+c_2e^{-\gamma t}$. As in Case (2), we obtain $c_2/c_1 = -[(\alpha+\gamma)/(\alpha-\gamma)]e^{2\gamma T}$ and $c_2/c_1 = -[(\alpha-\gamma)/(\alpha+\gamma)]e^{2\gamma T}$. No solution satisfies this equation, and hence $\lambda > 2\sigma^2/\alpha$ is not an eigenvalue.

### 8.4.3 The Wiener Process

In Chapter 3, we showed how the Wiener process (a nonstationary process) is obtained from the random walk. In this section, we derive the eigenvalues and the corresponding eigenfunctions of the Wiener process in order to write the Karhunen-Loève expansion.

Let $X(t)$ be the Wiener process. To determine the covariance function $K_{xx}(t,u) = E[X(t)X(u)]$, consider the increments $X(t) - X(u)$ and $X(t) - X(0)$, where $t \geq u > 0$. Since $X(u) - X(0) = X(u)$, then $X(t) - X(u)$ and $X(u)$ are statistically independent. Consequently,

$$E\{[X(t) - X(u)]X(u)\} = E[X(t)X(u)] - E[X^2(u)] = K_{xx}(t,u) - \alpha u = 0 \quad (8.155)$$

since $E[X(t)] = 0$. Hence,

$$K_{xx}(t,u) = \alpha u, \quad t \geq u \quad (8.156)$$

Similarly, for $u \geq t > 0$, we obtain

$$K_{xx}(t,u) = \alpha t, \quad u \geq t \quad (8.157)$$

The covariance function of the Wiener process is

$$K_{xx}(t,u) = \alpha \min(u,t) = \begin{cases} \alpha u, & u \leq t \\ \alpha t, & t \leq u \end{cases} \quad (8.158)$$

To solve for the eigenfunctions, we use the integral equation

$$\lambda\phi(t) = \int_0^T K_{xx}(t,u)\phi(u)du = \alpha\int_0^t u\phi(u)du + \alpha t\int_t^T \phi(u)du \quad (8.159)$$

Differentiating twice with respect to *t*, and using Leibniz's rule, we obtain the differential equation

$$\lambda\frac{d^2\phi(t)}{dt^2} + \alpha\phi(t) = 0 \quad (8.160)$$

We have two cases: (1) $\lambda = 0$ and (2) $\lambda > 0$.

*Case (1).* $\lambda = 0$. In this case $\phi(t) = 0$, which is the trivial solution.

*Case* (*2*). $\lambda > 0$. Let $\beta^2 = \alpha / \lambda$. Then the differential equation is

$$\frac{d^2\phi(t)}{dt^2} + \beta^2\phi(t) = 0 \tag{8.161}$$

where

$$\phi(t) = c_1 e^{j\beta t} + c_2 e^{-j\beta t} \tag{8.162}$$

Substituting into the integral equation and solving for $\lambda_k$ and $\phi_k(t)$, we obtain

$$\lambda_k = \frac{\alpha}{\beta_k} = \frac{\alpha T^2}{\pi^2[k-(1/2)]^2} \tag{8.163}$$

where $\beta_k = (\pi / T)[k-(1/2)]$, and the normalized $\phi_k(t)$ is

$$\phi_k(t) = \sqrt{\frac{2}{T}} \sin\left[\frac{\pi}{T}\left(k - \frac{1}{2}\right)t\right], \quad 0 \le t \le T \tag{8.164}$$

Therefore, the Karhunen-Loève expansion of the Wiener process is

$$X(t) = \sum_{k=1}^{\infty} X_k \phi_k(t) = \sum_{k=1}^{\infty} X_k \sqrt{\frac{2}{T}} \sin\left[\frac{\pi}{T}\left(k - \frac{1}{2}\right)t\right] \tag{8.165}$$

where the mean-square value of the coefficient $X_k$ is

$$E[X_k^2] = \lambda_k = \frac{\alpha T^2}{[k-(1/2)]\pi^2} \tag{8.166}$$

### 8.4.4 The White Noise Process

The white noise process can be derived from the Wiener process. Let $\alpha = \sigma^2$ and the *K*-term approximation of the Wiener process $X(t)$ be $X_K(t)$. That is,

$$X_K(t) = \sum_{k=1}^{K} X_k \sqrt{\frac{2}{T}} \sin\left[\left(k - \frac{1}{2}\right)\frac{\pi}{T}t\right] \tag{8.167}$$

Taking the derivative $X_K(t)$ with respect to $t$, we obtain

$$\frac{dX_K(t)}{dt} = \sum_{k=1}^{K} X_k\left(k-\frac{1}{2}\right)\frac{\pi}{T}\sqrt{\frac{2}{T}}\cos\left[\left(k-\frac{1}{2}\right)\frac{\pi}{T}t\right] = \sum_{k=1}^{K} W_k\sqrt{\frac{2}{T}}\cos\left[\left(k-\frac{1}{2}\right)\frac{\pi}{T}t\right] \tag{8.168}$$

where

$$W_k = X_k\left(k-\frac{1}{2}\right)\frac{\pi}{T} \tag{8.169}$$

and

$$E[W_k^2] = \sigma^2 \quad \text{for all } k \tag{8.170}$$

Note also that the functions $\phi_k(t) = \sqrt{2/T}\cos\{[k-(1/2)](\pi/T)t\}$ are orthonormal in the observation interval $[0,T]$ and are possible eigenfunctions to the derivative process.

To show that the set of functions $\{\phi_k(t)\}$ are eigenfunctions for the approximate integral equation corresponding to the white noise process, we need to define the white Gaussian noise process.

*Definition*. A white Gaussian process is a Gaussian process with covariance function given by

$$\sigma^2\delta(t-u) \tag{8.171}$$

where $\delta$ is the data function. The coefficients along each of the coordinate functions are statistically independent Gaussian random variables with variance $\sigma^2$.

Now considering the derivative of the covariance function of the Wiener process, we have

$$K_{x'x'}(t,u) = E\left[\frac{dX(t)}{dt}\frac{dX(u)}{du}\right] = \frac{\partial^2}{\partial t\partial u}E[X(t)X(u)] = \frac{\partial^2}{\partial t\partial u}K_{xx}(t,u)$$
$$= \frac{\partial^2}{\partial t\partial u}[\sigma^2\min(u,t)] = \sigma^2\delta(t-u) \tag{8.172}$$

The corresponding integral equation is

$$\lambda\phi(t) = \int_0^T \sigma^2 \delta(t-u)\phi(u)du = \sigma^2\phi(t) \tag{8.173}$$

and hence the integral equation is satisfied for *any* set of orthonormal functions $\{\phi_k(t)\}$. In addition, we observe that

$$\lambda_k = \sigma^2 \quad \text{for all } k \tag{8.174}$$

Note that the energy over the interval $[0,T]$ is not finite as $K \to \infty$, since

$$\sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} \sigma^2 \to \infty \tag{8.175}$$

Therefore, this derivative process is not realizable. Nevertheless, one possible representation, which is not unique, is

$$W(t) = \frac{dX(t)}{dt} = \sum_{k=1}^{\infty} W_k \sqrt{\frac{2}{T}} \cos\left[\left(k - \frac{1}{2}\right)\frac{\pi}{T}t\right] \tag{8.176}$$

## 8.5 SUMMARY

In this chapter, we have shown how a deterministic signal can be represented in a series expansion of orthonormal functions. In doing this, we needed to cover the fundamental mathematical concepts of orthogonal functions and generalized Fourier series. Then, we used the Gram-Schmidt orthogonalization procedure to show how a set of dependent or independent functions can be decomposed into another set of orthonormal and independent functions.

We also showed how a random process can be represented by an orthonormal series expansion, known as the Karhunen-Loève expansion. Specific processes such as the rational power spectral densities, the Wiener process, and the white noise process were considered. We showed how the white Gaussian noise process can be derived from the Wiener process. This required solving for the eigenvalues and eigenvectors of linear transformations. We discussed Green's function and showed how integral equations can be reduced to linear differential equations in order to solve for the eigenvalues and their corresponding eigenfunctions.

The mathematical concepts covered, such as solving for eigenvalues and eigenvectors/eigenfunctions, matrix diagonalization, and series representation of

signals will be useful to us in the next two chapters, which deal with the general Gaussian problem and detection in noise.

## PROBLEMS

**8.1** (a) Is the set of functions

$$\left\{\frac{1}{\sqrt{T}}, \sqrt{\frac{2}{T}}\cos\left(\frac{k\pi}{T}\right)t\right\}, \quad k = 1, 2, \ldots$$

orthonormal in the interval $[0, T]$?

(b) Using the fact that the functions in (a) are orthonormal on the interval $[0, T]$, show that the set

$$\left\{\frac{1}{\sqrt{2T}}, \frac{1}{\sqrt{T}}\cos\left(\frac{k\pi}{T}\right)t\right\}, \quad k = 1, 2, \ldots$$

is orthonormal on the interval $[-T, T]$.

**8.2** Let $s_1(t) = 1$ and $s_2(t) = t$ be defined on the interval $[-1, 1]$.

(a) Are $s_1(t)$ and $s_2(t)$ orthogonal in the given interval?

(b) Determine the constants $\alpha$ and $\beta$, such that $s_3(t) = 1 + \alpha t + \beta t^2$ is orthogonal to both $s_1(t)$ and $s_2(t)$ in the given interval.

**8.3** (a) Find a set of orthonormal basis functions for the set of signals shown in Figure P8.3.

(b) Find the vector corresponding to each signal for the orthonormal basis set found in (a), and sketch the signal constellation.

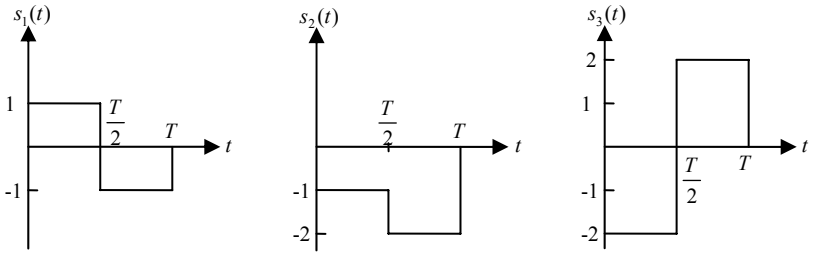


**Figure P8.3** Set of signals.

**8.4** Show by substitution that

$$\phi(t) = \int_{-\pi}^{\pi} \exp[j(n\theta - t\sin\theta)]d\theta$$

is a solution of

$$\frac{d}{dt}\left(t\frac{d\phi(t)}{dt}\right) + \left(t - \frac{n^2}{t}\right)\phi(t) = 0$$

**8.5** Find the kernel for the differential system

$$\frac{d\phi(t)}{dt} + \phi(t) = u(t), \quad 0 \le t \le 1$$
$$\phi'(0) = 0 = \phi(1)$$

**8.6** Consider the integral equation $\lambda\phi(t) = \int_0^{\pi/2} k(u,t)\phi(u)du, \quad 0 \le t \le \pi/2$, where $k(u,t) = \begin{cases} u, & u < t \\ t, & u > t \end{cases}$. Find all eigenvalues and eigenfunctions in the interval $[0, \pi/2]$.

**8.7** Consider the integral equation $\lambda\phi(t) = \int_0^{T} k(u,t)\phi(u)du, \quad 0 \le t \le T$ ,where $k(u,t) = \begin{cases} T-t, & u < t \\ T-u, & u > t \end{cases}$. Determine the eigenvalues and eigenfunctions in the interval $[0, T]$.

**8.8** Determine the eigenvalues and eigenfunctions for the linear differential equation

$$\frac{d^2\phi(t)}{dt^2} + n\omega\frac{d\phi(t)}{dt} = 0 \qquad 0 \le t \le T$$
$$\phi(0) = \phi(T) = 0$$

Assume $\phi(t)$ is continuous, such that $\phi'(u-0) - \phi'(u+0) = 1$ and $\sin n\omega T \neq 0$.

**8.9** As in Problem 8.8, find the solution $k(t,u)$ of $d^2k(t)/dt^2 = 0$ that has the properties $k(0,u) = k(T,u) = 0$, where $k(t,u)$ is continuous, and $k_t(u-0,u) - k_t(u+0,u) = 1$ for $0 \le t,u \le T$.

**8.10** If $k(t,u)$ is the solution of Problem 8.9 and $\phi(t)$ is any twice continuously differentiable function, then show that

$$\frac{d^2}{dt^2}\int_0^T k(t,u)\phi(u)du = -\phi(t)$$

and $\int_0^T k(t,u)\phi(u)du = 0$ at $t = 0$ and $T$. Thus, the solution of the differential equation (if it exists)

$$\frac{d^2\phi(t)}{dt^2} + \lambda\phi(t) = 0, \quad \phi(0) = \phi(T) = 0$$

is also a solution to the integral equation $\phi(t) = \lambda\int_0^T k(t,u)\phi(t)dt$.

**8.11** Verify that the kernel $k(t,u) = k(u,t)$ for both Problems 8.8 and 8.9.

**8.12** The differential equation of Problem 8.9 has twice continuously differentiable solutions only when $\lambda \in \{\lambda_n = (n\pi/T)^2\}$. The corresponding orthonormal set of solutions is $[\phi_n(t) = \sqrt{2/T}\sin(n\pi t/T)]$. Calculate the coefficients in the expansion

$$k(t,u) = \sum_{n=1}^{\infty} c_n(u)\phi_n(t)$$

Show from the solution of Problem 8.11, represented as $h(t,u)$, that

$$\phi(t) = \lambda\int_0^T h(t,u)\phi(u)du$$

is a solution of

$$\phi''(t) + [(n\omega^2) + \lambda]\phi(t) = 0, \quad \phi(0) = \phi(T) = 0$$

**8.13** Show from the solution of Problem 8.8, represented as $h(t,u)$, that

$$\phi(t) = \lambda \int_0^T h(t,u)\phi(u)du$$

is a solution of

$$\frac{d^2\phi(t)}{dt} + [(n\omega^2) + \lambda]\phi(t) = 0, \quad \phi(0) = \phi(T) = 0$$

Use the integral equation to obtain $c_n(u)$ in $h(t,u) = \sum_{n=1}^{\infty} c_n(u)\phi_n(t)$. Note that $\{\phi_n(t)\}$ is the set of functions of Problem 8.12, and $\lambda \in \left[\lambda_n = (n\pi/T)^2 - (m\omega)^2\right]$.

**8.14** Consider the integral equation

$$\lambda\phi(t) = \int_{-\infty}^{\infty} k(t,u)\phi(u)du \qquad \text{for all } t$$

where

$$k(t,u) = \frac{1}{\sqrt{1-s^2}} \exp\left(\frac{t^2+u^2}{2}\right) \exp\left(-\frac{t^2+u^2-2stu}{1-s^2}\right)$$

and $s$, $0 < s < 1$, is fixed. Show that $\phi(t) = e^{-t^2/2}$ is an eigenfunction corresponding to the eigenvalue $\lambda = \sqrt{\pi}$.

**8.15** Determine the integral equation corresponding to the following second-order linear differential equation

$$\frac{d^2\phi(t)}{dt^2} + \lambda\phi(t) = 0$$

where $\lambda$ is constant, $d\phi(t)/dt\big|_{t=0} = 0$, and $\phi(t)\big|_{t=1} = 0$.

**8.16** Consider the integral equation with the following corresponding linear differential equation

$$\frac{d^2\phi(t)}{dt^2} + \lambda\phi(t) = 0\ , \qquad 0 \le t \le T$$
$$\alpha\phi(1) + \phi'(1) = 0\ ,\ \phi(0) = 0$$

where $\alpha$ is a positive constant. Determine all eigenvalues and eigenfunctions.

**8.17** Determine all eigenvalues and eigenfunctions for the integral equation with the corresponding linear differential equation

$$\frac{d^2\phi(t)}{dt^2} + \lambda\phi(t) = 0\ , \qquad 0 \le t \le T$$
$$\phi'(0) = \phi'(T) = 0$$

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# Chapter 9

# The General Gaussian Problem

## 9.1 INTRODUCTION

In Chapter 2, we discussed the Gaussian random variable. In Sections 3.4.4 and 8.4.1, we discussed Gaussian random processes. Due to the wide use of the Gaussian process, we formulate the general Gaussian problem in Section 9.2. In Section 9.3, we cover the general Gaussian problem with equal covariance matrix under either hypothesis $H_1$ or $H_0$. For nondiagonal covariance matrices, we use an orthogonal transformation into a new coordinate system so that the matrix is diagonalized. In Section 9.4, we also solve the general Gaussian binary hypothesis problems but with mean vectors equal under both hypotheses. In Section 9.5, we consider symmetric hypotheses and obtain the likelihood ratio test (LRT).

## 9.2 BINARY DETECTION

In this section, we formulate the general Gaussian problem for binary hypothesis testing. Consider the hypotheses

$$\begin{aligned} H_1 &: \boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{N} \\ H_0 &: \boldsymbol{Y} = \quad + \boldsymbol{N} \end{aligned} \tag{9.1}$$

where the vector observation $\boldsymbol{Y}$, the signal vector $\boldsymbol{X}$, and the noise vector $\boldsymbol{N}$ are given by

$$\boldsymbol{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_K \end{bmatrix}, \qquad \boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_K \end{bmatrix}, \qquad \boldsymbol{N} = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_K \end{bmatrix} \tag{9.2}$$

The noise components are Gaussian random variables. By definition, a hypothesis testing problem is called a general Gaussian problem if the conditional density function $f_{\boldsymbol{Y}|H_j}(\boldsymbol{y}\,|\,H_j)$ for all $j$ is a Gaussian density function. Similarly, an estimation problem is called a general Gaussian problem if the conditional density function $f_{\boldsymbol{Y}|\boldsymbol{\Theta}}(\boldsymbol{y}\,|\,\boldsymbol{\theta})$ has a Gaussian density for all $\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is the parameter to be estimated.

Consider the binary hypothesis testing problem given in (9.1). Let the mean vectors $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ under hypotheses $H_1$ and $H_2$, respectively, be

$$\boldsymbol{m}_1 = E[\boldsymbol{Y}\,|\,H_1] \tag{9.3a}$$

and

$$\boldsymbol{m}_0 = E[\boldsymbol{Y}\,|\,H_0] \tag{9.3b}$$

The covariance matrices under each hypothesis are given by

$$\boldsymbol{C}_1 = E[(\boldsymbol{Y}-\boldsymbol{m}_1)(\boldsymbol{Y}-\boldsymbol{m}_1)^T\,|\,H_1] \tag{9.4a}$$

and

$$\boldsymbol{C}_0 = E[(\boldsymbol{Y}-\boldsymbol{m}_0)(\boldsymbol{Y}-\boldsymbol{m}_0)^T\,|\,H_0] \tag{9.4b}$$

In Chapter 5, we have seen that applying the Bayes' criterion to the binary hypothesis problem resulted in the likelihood ratio test; that is,

$$\Lambda(\boldsymbol{y}) = \frac{f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\,|\,H_1)}{f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\,|\,H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \tag{9.5}$$

where

$$f_{\mathrm{Y}|H_j}(\mathrm{y}\,|\,H_j) = \frac{1}{(2\pi)^{K/2}\left|\boldsymbol{C}_j\right|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{m}_j)^T\boldsymbol{C}^{-1}(\boldsymbol{y}-\boldsymbol{m}_j)\right], \quad j=0,1 \tag{9.6}$$

Substituting (9.6) into (9.5) yields

$$\Lambda(\boldsymbol{y}) = \frac{|\boldsymbol{C}_0|^{1/2} \exp\left[-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{m}_1)^T \boldsymbol{C}_1^{-1}(\boldsymbol{y}-\boldsymbol{m}_1)\right]}{|\boldsymbol{C}_1|^{1/2} \exp\left[-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{m}_0)^T \boldsymbol{C}_0^{-1}(\boldsymbol{y}-\boldsymbol{m}_0)\right]} \begin{matrix} H_1 \\ > \\ < \\ H_0 \end{matrix} \eta \tag{9.7}$$

Taking the logarithm on both sides of the above equation, an equivalent test is

$$\frac{1}{2}(\boldsymbol{y}-\boldsymbol{m}_0)^T \boldsymbol{C}_0^{-1}(\boldsymbol{y}-\boldsymbol{m}_0) - \frac{1}{2}(\boldsymbol{y}-\boldsymbol{m}_1)^T \boldsymbol{C}_1^{-1}(\boldsymbol{y}-\boldsymbol{m}_1) \begin{matrix} H_1 \\ > \\ < \\ H_0 \end{matrix} \gamma \tag{9.8a}$$

where

$$\gamma = \ln \eta + \frac{1}{2}(\ln|\boldsymbol{C}_1| - \ln|\boldsymbol{C}_0|) \tag{9.8b}$$

Thus, the likelihood ratio test reduces to the difference of two quadratic forms. The evaluation of such difference depends on several constraints on the mean vectors and covariance matrices under each hypothesis.

## 9.3 SAME COVARIANCE

In this case, we assume the covariance matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_0$ under both hypotheses $H_1$ and $H_0$ are the same; that is,

$$\boldsymbol{C}_1 = \boldsymbol{C}_0 = \boldsymbol{C} \tag{9.9}$$

Substituting (9.9) into (9.8a), the LRT can be written as

$$\frac{1}{2}(\boldsymbol{y}-\boldsymbol{m}_0)^T \boldsymbol{C}^{-1}(\boldsymbol{y}-\boldsymbol{m}_0) - \frac{1}{2}(\boldsymbol{y}-\boldsymbol{m}_1)^T \boldsymbol{C}^{-1}(\boldsymbol{y}-\boldsymbol{m}_1) \begin{matrix} H_1 \\ > \\ < \\ H_0 \end{matrix} \gamma \tag{9.10}$$

Expanding the above expression, we obtain

$$-\frac{1}{2}\boldsymbol{m}_0^T \boldsymbol{C}^{-1}\boldsymbol{y} - \frac{1}{2}\boldsymbol{y}^T \boldsymbol{C}^{-1}\boldsymbol{m}_0 + \frac{1}{2}\boldsymbol{m}_0^T \boldsymbol{C}^{-1}\boldsymbol{m}_0 + \frac{1}{2}\boldsymbol{m}_1^T \boldsymbol{C}^{-1}\boldsymbol{y}$$

$$+\frac{1}{2}\boldsymbol{y}^T\boldsymbol{C}^{-1}\boldsymbol{m}_1-\frac{1}{2}\boldsymbol{m}_1^T\boldsymbol{C}^{-1}\boldsymbol{m}_1 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \tag{9.11}$$

Using the fact that the inverse covariance matrix $\boldsymbol{C}^{-1}$ is symmetric, that is $\boldsymbol{C}^{-1}=(\boldsymbol{C}^{-1})^T$, and the fact that the transpose of the scalar is equal to itself, that is

$$\boldsymbol{y}^T\boldsymbol{C}^{-1}\boldsymbol{m}_j=(\boldsymbol{y}^T\boldsymbol{C}^{-1}\boldsymbol{m}_j)^T=\boldsymbol{m}_j^T(\boldsymbol{C}^{-1})^T\boldsymbol{y}=\boldsymbol{m}_j^T\boldsymbol{C}^{-1}\boldsymbol{y},\quad j=0,1 \tag{9.12}$$

Equation (9.11) reduces to the following test

$$\boldsymbol{m}_1^T\boldsymbol{C}^{-1}\boldsymbol{y}-\boldsymbol{m}_0^T\boldsymbol{C}^{-1}\boldsymbol{y}+\frac{1}{2}\boldsymbol{m}_0^T\boldsymbol{C}^{-1}\boldsymbol{m}_0-\frac{1}{2}\boldsymbol{m}_1^T\boldsymbol{C}^{-1}\boldsymbol{m}_1 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \tag{9.13}$$

Rearranging terms, an equivalent test is

$$(\boldsymbol{m}_1^T-\boldsymbol{m}_0^T)\boldsymbol{C}^{-1}\boldsymbol{y} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_1 \tag{9.14a}$$

where

$$\gamma_1=\gamma+\frac{1}{2}(\boldsymbol{m}_1^T\boldsymbol{C}^{-1}\boldsymbol{m}_1-\boldsymbol{m}_0^T\boldsymbol{C}^{-1}\boldsymbol{m}_0) \tag{9.14b}$$

Note that all terms in $\boldsymbol{y}$ are on one side, and the others are on the other side. Hence, the sufficient statistic $T(\boldsymbol{y})$ is

$$T(\boldsymbol{y})=(\boldsymbol{m}_1^T-\boldsymbol{m}_0^T)\boldsymbol{C}^{-1}\boldsymbol{y} \tag{9.15}$$

Let the difference mean vector be

$$\Delta\boldsymbol{m}=\boldsymbol{m}_1-\boldsymbol{m}_0 \tag{9.16}$$

Substituting (9.16) into (9.14a), the LRT becomes

$$T(\boldsymbol{y}) = \Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} \boldsymbol{y} = \boldsymbol{y}^T \boldsymbol{C}^{-1} \Delta \boldsymbol{m} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_1 \tag{9.17}$$

We observe that $T(\boldsymbol{y})$ is a linear combination of jointly Gaussian random variables, and hence, by definition, it is a Gaussian random variable. Therefore, we only need to find the mean and variance of the sufficient statistic under each hypothesis, and perform the test in (9.17) against the threshold $\gamma_1$ to determine the performance of this test. The mean and variance of $T(\boldsymbol{y})$ are given by

$$\begin{aligned} E[T(\boldsymbol{Y}) \mid H_j] &= E[\Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} \boldsymbol{Y} \mid H_j] = \Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} E[\boldsymbol{Y} \mid H_j] \\ &= \Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} \boldsymbol{m}_j, \quad j = 0,1 \end{aligned} \tag{9.18}$$

and

$$\begin{aligned} \text{var}[T(\boldsymbol{Y}) \mid H_j] &= E\{(T(\boldsymbol{Y}) - E[T(\boldsymbol{Y}) \mid H_j])^2 \mid H_j\} \\ &= E\{[\Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} \boldsymbol{Y} - \Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} \boldsymbol{m}_j]^2 \mid H_j\} \\ &= E\{[\Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} \boldsymbol{Y} - \Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} \boldsymbol{m}_j][\boldsymbol{Y}^T \boldsymbol{C}^{-1} \Delta \boldsymbol{m} - \boldsymbol{m}_j^T \boldsymbol{C}^{-1} \Delta \boldsymbol{m}] \mid H_j\} \\ &= E\{[\Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} (\boldsymbol{Y} - \boldsymbol{m}_j)][(\boldsymbol{Y}^T - \boldsymbol{m}_j^T) \boldsymbol{C}^{-1} \Delta \boldsymbol{m}] \mid H_j\} \\ &= \Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} E[(\boldsymbol{Y} - \boldsymbol{m}_j)(\boldsymbol{Y}^T - \boldsymbol{m}_j^T) \mid H_j] \boldsymbol{C}^{-1} \Delta \boldsymbol{m} \end{aligned} \tag{9.19}$$

Using (9.4) and (9.9), the conditional variance of the sufficient statistic becomes

$$\text{var}[T(\boldsymbol{Y}) \mid H_j] = \Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} \boldsymbol{C} \boldsymbol{C}^{-1} \Delta \boldsymbol{m} = \Delta \boldsymbol{m}^T \boldsymbol{C}^{-1} \Delta \boldsymbol{m} \tag{9.20}$$

since $\boldsymbol{C}\boldsymbol{C}^{-1} = \boldsymbol{I}$ is the identity matrix. Note that the variance is independent of any hypothesis. The performance of this test is affected by the choice $\boldsymbol{C}$, which we will study next.

In (5.75), we defined the detection parameter when the variance was normalized to one. When the variance is not normalized to one, the equivalent definition of the detection parameter $d$ is

$$d^2 \triangleq \frac{\{E[T(\boldsymbol{Y}) \mid H_1 - E[T(\boldsymbol{Y}) \mid H_0]\}^2}{\text{var}[T(\boldsymbol{Y}) \mid H_0]} \tag{9.21}$$

Substituting (9.18) and (9.20) in (9.21), we obtain

$$d^2 = \frac{(\Delta \boldsymbol{m}^T \boldsymbol{C} \boldsymbol{m}_1 - \Delta \boldsymbol{m}^T \boldsymbol{C} \boldsymbol{m}_0)^2}{\Delta \boldsymbol{m}^T \boldsymbol{C} \Delta \boldsymbol{m}} = \Delta \boldsymbol{m}^T \boldsymbol{C} \Delta \boldsymbol{m} \tag{9.22}$$

Hence, for this case of an equal variance matrix, the performance of the system is determined by the quadratic form of $d^2$. We now study the different cases for the covariance matrix.

### 9.3.1 Diagonal Covariance Matrix

Let the covariance matrix $\boldsymbol{C}$ be diagonal and given by

$$\boldsymbol{C} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_K^2 \end{bmatrix} \tag{9.23}$$

This means that the components $Y_k, k = 1, 2, \dots, K$, are statistically independent. Two possible cases may arise. The variances of the components are either (1) equal and in this case $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_K^2 = \sigma^2$, or (2) unequal and in this case $\sigma_1^2 \neq \sigma_2^2 \neq \dots \neq \sigma_K^2$.

*Equal Variance* In this case, the covariance matrix is given by

$$\boldsymbol{C} = \begin{bmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sigma^2 \end{bmatrix} = \sigma^2 \boldsymbol{I} \tag{9.24}$$

that is,

$$E[(Y_j - m_j)(Y_k - m_k)] = \begin{cases} \sigma^2, & j = k \\ 0, & j \neq k \end{cases} \tag{9.25}$$

The inverse covariance matrix is $\boldsymbol{C}^{-1} = (1/\sigma^2)\boldsymbol{I}$. Substituting in (9.15), the sufficient statistic is

$$T(\boldsymbol{y}) = \frac{1}{\sigma^2}\Delta\boldsymbol{m}^T\boldsymbol{y} \tag{9.26}$$

which is simply the dot product between the mean difference vector $\Delta\boldsymbol{m}$ and the observation vector $\boldsymbol{Y}$. The corresponding detection parameter is simply

$$d^2 = \Delta\boldsymbol{m}^T\boldsymbol{C}\Delta\boldsymbol{m} = \frac{1}{\sigma^2}(\Delta\boldsymbol{m}^T\Delta\boldsymbol{m}) = \frac{1}{\sigma^2}\sum_{k=1}^{K}(\Delta m_k)^2 = \frac{|\Delta\boldsymbol{m}|^2}{\sigma^2} \tag{9.27}$$

where $|\Delta\boldsymbol{m}|$ is the magnitude of the vector $\Delta\boldsymbol{m}$ . Hence,

$$d = \frac{|\Delta\boldsymbol{m}|}{\sigma} = \frac{|\boldsymbol{m}_1 - \boldsymbol{m}_0|}{\sigma} \tag{9.28}$$

is the *distance* between the two mean value vectors divided by the standard deviation of the observation $Y_k, k = 1, 2, \ldots, K$ as shown in Figure 9.1 [1].

*Unequal Variance* In this case, the covariance matrix is as given in (9.23), where $\sigma_1 \neq \sigma_2 \neq \ldots \neq \sigma_K$. The inverse covariance matrix is given by

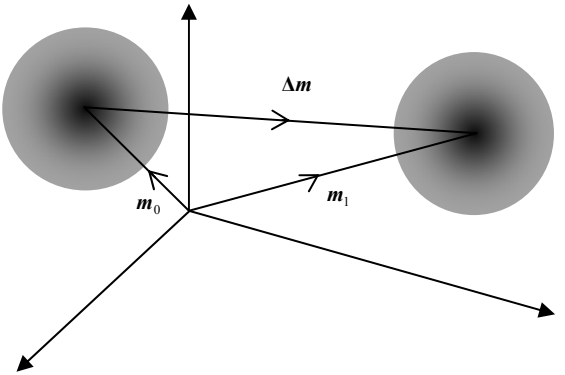


**Figure 9.1** Mean value vectors. (*From*: [1]. © 1968 John Wiley and Sons, Inc. Reprinted with permission.)

$$\boldsymbol{C}^{-1} = \begin{bmatrix} 1/\sigma_1^2 & 0 & \dots & 0 \\ 0 & 1/\sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1/\sigma_K^2 \end{bmatrix} \tag{9.29}$$

Consequently, after substitution into (9.15), the sufficient statistic becomes

$$T(\boldsymbol{y}) = \boldsymbol{\Delta m}^T \boldsymbol{C}^{-1} \boldsymbol{y} = \sum_{k=1}^{K} \frac{\Delta m_k y_k}{\sigma_k^2} \tag{9.30}$$

It follows that

$$d^2 = \boldsymbol{\Delta m}^T \boldsymbol{C}^{-1} \boldsymbol{\Delta m} = \boldsymbol{\Delta m}^T \begin{bmatrix} \frac{\Delta m_1}{\sigma_1} \\ \frac{\Delta m_2}{\sigma_2} \\ \vdots \\ \frac{\Delta m_K}{\sigma_K} \end{bmatrix} = \sum_{k=1}^{K} \frac{(\Delta m_k)^2}{\sigma_k^2} \tag{9.31}$$

Making a change of variables, let

$$y_k' = \frac{y_k}{\sigma_k} \tag{9.32}$$

Then, $Y_k, k = 1, 2, \dots, K$ is Gaussian with mean $m_k / \sigma_k$ and variance one. The corresponding mean difference vector is

$$\boldsymbol{\Delta m}' = \begin{bmatrix} \frac{\Delta m_1}{\sigma_1} & \frac{\Delta m_2}{\sigma_2} & \dots & \frac{\Delta m_K}{\sigma_K} \end{bmatrix}^T \tag{9.33}$$

The detection parameter becomes

$$d^2 = \sum_{k=1}^{K} \left( \frac{\Delta m_k}{\sigma_k} \right)^2 = \sum_{k=1}^{K} (\Delta m_k')^2 = \left| \boldsymbol{\Delta m}' \right|^2 \tag{9.34}$$

or

$$d = \left|\Delta \boldsymbol{m}'\right| = \left|\boldsymbol{m}_1' - \boldsymbol{m}_0'\right| \tag{9.35}$$

That is, it can be interpreted as the distance between the mean value vectors in the new coordinate system. The sufficient statistic is

$$T(\boldsymbol{y}) = \sum_{k=1}^{K} \frac{\Delta m_k}{\sigma_k} \frac{y_k}{\sigma_k} = \sum_{k=1}^{K} \Delta m_k' y_k' = (\Delta \boldsymbol{m}')^T \boldsymbol{y}' = (\boldsymbol{y}')^T \Delta \boldsymbol{m}' \tag{9.36}$$

### 9.3.2 Nondiagonal Covariance Matrix

In general, the covariance matrix $\boldsymbol{C}$ will not be a diagonal matrix, and thus the components of the received random vector $\boldsymbol{Y}$ are not statistically independent. In order to make the components independent, we need to find a new coordinate system in which the transformed components are independent. That is, the covariance matrix in the new coordinate system must be diagonal.

The concept of diagonalizing a matrix, which can be done by the similarity transformation, was presented in Chapter 4 and now can be used. Let the new coordinate system have coordinate axes denoted by the set of orthonormal vectors $\{\boldsymbol{\Phi}_k\}, k = 1, 2, \ldots, K$. Let $\boldsymbol{Y}$ be the original observation vector and $\boldsymbol{Y}'$ its transformed vector in the new coordinate system. The vector $\boldsymbol{Y}'$ also has $K$ components, where the $k$th component, denoted $Y_k'$, is just the projection of the observation vector $\boldsymbol{Y}$ onto the coordinate $\boldsymbol{\Phi}_k$ of the new system. This geometric interpretation mathematically represents the dot product between the vector $\boldsymbol{Y}$ and the vector $\boldsymbol{\Phi}_k$. That is,

$$Y_k' = \boldsymbol{\Phi}_k^T \boldsymbol{Y} = \boldsymbol{Y}^T \boldsymbol{\Phi}_k \tag{9.37}$$

Assuming we have a three-dimentional vector, the transformation of the vector $\boldsymbol{Y}$ into $\boldsymbol{Y}'$ in the new coordinate system may be as shown in Figure 9.2. The mean of $\boldsymbol{Y}'$ in the new coordinate system is

$$\boldsymbol{m}' = E[\boldsymbol{Y}'] = E[\boldsymbol{\Phi}^T \boldsymbol{Y}] = \boldsymbol{\Phi}^T E[\boldsymbol{Y}] = \boldsymbol{\Phi}^T \boldsymbol{m} = \boldsymbol{m}^T \boldsymbol{\Phi} \tag{9.38}$$

The covariance matrix of $\boldsymbol{Y}'$ is diagonal since the components in this new coordinate system are now statistically independent; that is,

$$E[(Y_j' - m_j)(Y_k' - m_k)] = \lambda_k \delta_{jk} \tag{9.39}$$

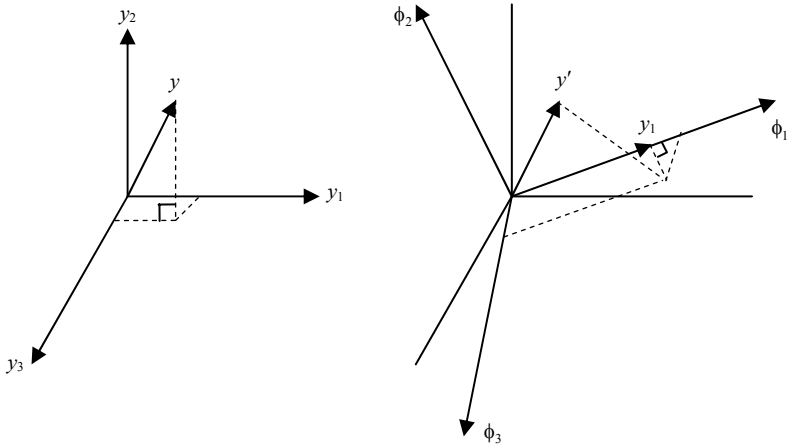


**Figure 9.2** New coordinate system representing transformation of vector $\boldsymbol{Y}$ into $\boldsymbol{Y}'$.

where $m_k = E[Y_k]$, and $\delta_{jk}$ is the Kronecker delta function. Using the fact that $Y_k' = \boldsymbol{\Phi}_k^T \boldsymbol{Y} = \boldsymbol{Y}^T \boldsymbol{\Phi}_k$ and $m_k = \boldsymbol{\Phi}_k^T \boldsymbol{m} = \boldsymbol{m}^T \boldsymbol{\Phi}_k$, (9.39) can be written as

$$E[\boldsymbol{\Phi}_j^T (\boldsymbol{Y} - \boldsymbol{m})(\boldsymbol{Y}^T - \boldsymbol{m}^T)\boldsymbol{\Phi}_k] = \lambda_k \delta_{jk} \tag{9.40}$$

or

$$\boldsymbol{\Phi}_j^T \boldsymbol{C}\, \boldsymbol{\Phi}_k = \lambda_k \delta_{jk} \tag{9.41}$$

Hence, (9.41) is only true when

$$\boldsymbol{C}\boldsymbol{\Phi}_k = \lambda_k \boldsymbol{\Phi}_k \tag{9.42}$$

since

$$\boldsymbol{\Phi}_j^T \boldsymbol{\Phi}_k = \delta_{jk} \tag{9.43}$$

Consequently, the solution reduces to solving the eigenproblem

$$\boldsymbol{C}\boldsymbol{\Phi} = \lambda \boldsymbol{\Phi} \tag{9.44}$$

or

$$(\boldsymbol{C} - \boldsymbol{I}\lambda)\boldsymbol{\Phi} = 0 \tag{9.45}$$

The solution to the homogeneous equations of (9.45) was studied in detail in Chapter 8. That is, we first obtain the nonzero eigenvalues from the equation $|\boldsymbol{C} - \boldsymbol{I}\lambda| = 0$. Then, using (9.44), we solve the set of $K$ equations in $K$ unknowns to obtain the eigenvectors $\Phi_k$, $k = 1, 2, \ldots, K$, corresponding to the eigenvalues $\lambda_k$, $k = 1, 2, \ldots, K$. The eigenvectors are linearly independent. We form the modal matrix $\boldsymbol{M}$ given by

$$\boldsymbol{M} = [\boldsymbol{\Phi}_1 \vdots \boldsymbol{\Phi}_2 \vdots \cdots \vdots \boldsymbol{\Phi}_K] \tag{9.46}$$

and then use the similarity transformation to diagonalize the covariance matrix $\boldsymbol{C}$. We obtain

$$\boldsymbol{\lambda} = \boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{M} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_K \end{bmatrix} \tag{9.47}$$

It should also be noted that since the covariance matrix $\boldsymbol{C}$ is real and symmetric, the inverse of the modal matrix $\boldsymbol{M}$ equals its transpose $\left(\boldsymbol{M}^{-1} = \boldsymbol{M}^T\right)$. Thus, the orthogonal transformation can be used to diagonalize $\boldsymbol{C}$. The vector $\boldsymbol{y}'$ in the new coordinate system is given by

$$\boldsymbol{y}' = \boldsymbol{M}^T \boldsymbol{y} \tag{9.48}$$

or

$$\boldsymbol{y} = \boldsymbol{M}\,\boldsymbol{y}' \tag{9.49}$$

The above transformation corresponds to a rotation, and hence the norm of $\boldsymbol{y}'$ in the new system is equal to the norm of $\boldsymbol{y}$ in the original system.

Now, we can apply the LRT to the binary hypothesis problem in the new coordinate system. The sufficient statistic is still of the form given in (9.15). Let $\boldsymbol{m}_1$ and $\boldsymbol{m}_0$ be the mean vectors in the original coordinate system under $H_1$ and $H_0$, respectively, such that

$$\boldsymbol{m}_1 = \begin{bmatrix} m_{11} \\ m_{12} \\ \vdots \\ m_{1K} \end{bmatrix}, \quad \boldsymbol{m}_0 = \begin{bmatrix} m_{01} \\ m_{02} \\ \vdots \\ m_{0K} \end{bmatrix} \tag{9.50}$$

The transformed mean vectors are given by (9.38). Hence, the transformed mean difference vector $\Delta \boldsymbol{m}' = \boldsymbol{m}_1' - \boldsymbol{m}_0'$ is

$$\Delta \boldsymbol{m}' = \begin{bmatrix} \boldsymbol{\Phi}_1^T \Delta \boldsymbol{m} \\ \boldsymbol{\Phi}_2^T \Delta \boldsymbol{m} \\ \vdots \\ \boldsymbol{\Phi}_K^T \Delta \boldsymbol{m} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_1^T \\ \boldsymbol{\Phi}_2^T \\ \vdots \\ \boldsymbol{\Phi}_K^T \end{bmatrix} \Delta \boldsymbol{m} = \boldsymbol{W} \Delta \boldsymbol{m} \tag{9.51}$$

where $\boldsymbol{W}$ is a $K \times K$ matrix with the vectors $\boldsymbol{\Phi}_k^T$, $k = 1, 2, \ldots, K$. That is, $\boldsymbol{W} = \boldsymbol{M}^T = \boldsymbol{M}^{-1}$ and hence,

$$\Delta \boldsymbol{m} = \boldsymbol{W}^{-1} \Delta \boldsymbol{m}' = \boldsymbol{M} \Delta \boldsymbol{m}' \tag{9.52}$$

Substituting (9.48) and (9.52) into (9.15), the sufficient statistic in the new coordinate system becomes

$$\begin{aligned} T(\boldsymbol{y}') = \Delta \boldsymbol{m} \boldsymbol{C}^{-1} \boldsymbol{y} &= (\boldsymbol{M} \Delta \boldsymbol{m}')^T \boldsymbol{C}^{-1} (\boldsymbol{M}\, \boldsymbol{y}') = (\Delta \boldsymbol{m}')^T \boldsymbol{M}^T \boldsymbol{C}^{-1} \boldsymbol{M}\, \boldsymbol{y}' \\ &= (\Delta \boldsymbol{m}')^T \boldsymbol{M}^{-1} \boldsymbol{C}^{-1} \boldsymbol{M} \boldsymbol{y}' \end{aligned} \tag{9.53}$$

Using (9.47), the sufficient statistic reduces to

$$T(\boldsymbol{y}') = (\Delta \boldsymbol{m}')^T \boldsymbol{\lambda}^{-1} \boldsymbol{y}' = \sum_{k=1}^{K} \frac{\Delta m_k' y_k'}{\lambda_k} \tag{9.54}$$

where

$$\boldsymbol{\lambda}^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & \ldots & 0 \\ 0 & 1/\lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1/\lambda_K \end{bmatrix} \tag{9.55}$$

**Example 9.1**

Consider the binary hypothesis problem with specifications

$$\boldsymbol{m}_0 = \mathbf{0}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix}, \quad \boldsymbol{m}_1 = \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \end{bmatrix}$$

Obtain the sufficient statistic in the new coordinate system for which the components of the observation vector are independent.

*Solution*

For the components of $\boldsymbol{Y}$ to be independent, the covariance matrix $\boldsymbol{C}$ in the new coordinate system must be diagonal. This can be achieved using the orthogonal transformation. First, we solve for the eigenvalues of $\boldsymbol{C}$ using

$$|\boldsymbol{C} - \boldsymbol{I}\lambda| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0.5 & 0.25 \\ 0.5 & 1-\lambda & 0.5 \\ 0.25 & 0.5 & 1-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 2.4375\lambda + 0.5625$$

Therefore, $\lambda_1 = 0.4069$, $\lambda_2 = 0.75$, and $\lambda_3 = 1.8431$. To obtain the first eigenvector $\boldsymbol{\Phi}_1$, we solve

$$\boldsymbol{C}\boldsymbol{\Phi}_1 = \lambda_1 \boldsymbol{\Phi}_1 \Rightarrow \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \\ \phi_{13} \end{bmatrix} = 0.4069 \begin{bmatrix} \phi_{11} \\ \phi_{12} \\ \phi_{13} \end{bmatrix}$$

Solving for $\phi_{11}$, $\phi_{12}$, and $\phi_{13}$, we obtain

$$\boldsymbol{\Phi}_1 = \begin{bmatrix} 0.4544 \\ -0.7662 \\ 0.4544 \end{bmatrix}$$

Similarly, we solve for $\boldsymbol{\Phi}_2$ and $\boldsymbol{\Phi}_3$, using $\boldsymbol{C}\boldsymbol{\Phi}_2 = \lambda_2 \boldsymbol{\Phi}_2$ and $\boldsymbol{C}\boldsymbol{\Phi}_3 = \lambda_3 \boldsymbol{\Phi}_3$, to obtain

$$\boldsymbol{\Phi}_2 = \begin{bmatrix} -0.7071 \\ 0.0000 \\ 0.7071 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Phi}_3 = \begin{bmatrix} 0.5418 \\ 0.6426 \\ 0.5418 \end{bmatrix}$$

Hence, the modal matrix $\boldsymbol{M}$ is

$$\boldsymbol{M} = [\boldsymbol{\Phi}_1 \,\vdots\, \boldsymbol{\Phi}_2 \,\vdots\, \boldsymbol{\Phi}_3] = \begin{bmatrix} 0.4544 & -0.7071 & 0.5418 \\ -0.7662 & 0.0000 & 0.6426 \\ 0.4544 & 0.7071 & 0.5418 \end{bmatrix}$$

Since $\boldsymbol{y} = \boldsymbol{M}\,\boldsymbol{y}' \Rightarrow \boldsymbol{y}' = \boldsymbol{M}^{-1}\boldsymbol{y} = \boldsymbol{M}^T\boldsymbol{y}$, we have

$$\boldsymbol{y}' = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.4544 & -0.7662 & 0.4544 \\ -0.7071 & 0.0000 & 0.7071 \\ 0.5418 & 0.6426 & 0.5418 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \end{bmatrix}$$

Since $\boldsymbol{m}_0 = \boldsymbol{0} \Rightarrow \boldsymbol{m}_0' = \boldsymbol{M}^T\boldsymbol{m}_0 = \boldsymbol{0}$ and $\Delta\boldsymbol{m}' = \boldsymbol{m}_1'$ where

$$\boldsymbol{m}_1' = \begin{bmatrix} m_1' \\ m_2' \\ m_3' \end{bmatrix} = \begin{bmatrix} 0.4544 & -0.7662 & 0.4544 \\ -0.7071 & 0.0000 & 0.7071 \\ 0.5418 & 0.6426 & 0.5418 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \end{bmatrix}$$

Therefore, the sufficient statistic is

$$T(\boldsymbol{y}') = \sum_{k=1}^{3} \frac{\Delta\, m_k' y_k'}{\lambda_k} = \sum_{k=1}^{3} \frac{m_k' y_k'}{\lambda_k}$$

**Example 9.2**

Consider the problem of Example 9.1, but $K = 2$, $\boldsymbol{m}_0 = \boldsymbol{0}$, and $\boldsymbol{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.

*Solution*

Following the same procedure as in Example 9.1, we solve for the eigenvalues using $|\boldsymbol{C} - \boldsymbol{I}\lambda| = 0$. That is,

$$\begin{vmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 - \rho^2 = 0$$

Thus, $\lambda_1 = 1+\rho$ and $\lambda_2 = 1-\rho$. To obtain the eigenvector $\boldsymbol{\Phi}_1$, we have $\boldsymbol{C}\boldsymbol{\Phi}_1 = \lambda_1 \boldsymbol{\Phi}_1$, or

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = (1+\rho) \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix}$$

Solving for $\phi_{11}$ and $\phi_{12}$, such that $\boldsymbol{\Phi}_1^T \boldsymbol{\Phi}_1 = \phi_{11}^2 + \phi_{12}^2 = 1$, we obtain the normalized eigenvector $\Phi_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$. Similarly, we obtain $\Phi_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$. The modal matrix is

$$\boldsymbol{M} = [\boldsymbol{\Phi}_1 \vdots \boldsymbol{\Phi}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Note that

$$\boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{M} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \boldsymbol{\lambda}$$

The observation vector $\boldsymbol{y}'$ in the new coordinate system is

$$\boldsymbol{y}' = \boldsymbol{M}^T \boldsymbol{y} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow y_1' = \frac{y_1 + y_2}{\sqrt{2}} \quad \text{and} \quad y_2' = \frac{y_1 - y_2}{\sqrt{2}}$$

Similarly, the mean vector $\boldsymbol{m}_1'$ is

$$\boldsymbol{m}_1' = \boldsymbol{M}^T \boldsymbol{m}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \end{bmatrix} \Rightarrow m_{11}' = \frac{m_{11} + m_{12}}{\sqrt{2}} \quad \text{and} \quad m_{12}' = \frac{m_{11} - m_{12}}{\sqrt{2}}$$

The difference mean vector is $\Delta \boldsymbol{m}' = \boldsymbol{m}_1' - \boldsymbol{m}_0' = \boldsymbol{m}_1'$. Therefore, the sufficient statistic is given by

$$T(\boldsymbol{y}') = \sum_{k=1}^{2} \frac{\Delta m_k' y_k'}{\lambda_k} = \frac{m_{11}' y_1'}{1+\rho} + \frac{m_{12}' y_2'}{1-\rho}$$
$$= \frac{(m_{11}+m_{12})(y_1+y_2)}{2(1+\rho)} + \frac{(m_{11}-m_{12})(y_1-y_2)}{2(1-\rho)}$$

## 9.4 SAME MEAN

In the previous section, the constraint was that the covariance matrices under both hypotheses were the same. Now, we consider the case with the constraint that the mean vectors under both hypotheses are equal. That is,

$$\boldsymbol{m}_1 = \boldsymbol{m}_0 = \boldsymbol{m} \tag{9.56}$$

Substituting (9.56) into the LRT given in (9.8), we obtain

$$\frac{1}{2}(\boldsymbol{y}-\boldsymbol{m})^T (\boldsymbol{C}_0^{-1} - \boldsymbol{C}_1^{-1})(\boldsymbol{y}-\boldsymbol{m}) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \tag{9.57}$$

Note that the mean vector $\boldsymbol{m}$ of the test in (9.57) does not affect the decision as to which hypothesis is true. Consequently, for simplicity and without loss of generality, let $\boldsymbol{m} = \boldsymbol{0}$. The LRT reduces to

$$T(\boldsymbol{y}) = \boldsymbol{y}^T (\boldsymbol{C}_0^{-1} - \boldsymbol{C}_1^{-1})\boldsymbol{y} \underset{H_0}{\overset{H_1}{\gtrless}} 2\gamma = \gamma_2 \tag{9.58}$$

Furthermore, assume that this binary hypothesis problem can be characterized by

$$\begin{aligned} H_1 &: Y_k = S_k + N_k, \quad k = 1, 2, \ldots, K \\ H_0 &: Y_k = \quad N_k, \quad k = 1, 2, \ldots, K \end{aligned} \tag{9.59}$$

That is, we only have noise under hypothesis $H_0$, while under hypothesis $H_1$ we have signal plus noise. The signal and noise components are assumed to be statistically independent. In addition, the noise components are uncorrelated with equal variances $\sigma_n^2$, $k = 1, 2, \ldots, K$. Thus, the noise components under hypothesis $H_0$ are a multivariate Gaussian with variance matrix

$$\boldsymbol{C}_n = \sigma_n^2 \boldsymbol{I} = \boldsymbol{C}_0 \tag{9.60}$$

If the signal components are assumed to be independent of each other, then the covariance matrix $\boldsymbol{C}_s$ is diagonal. The signal components are also a multivariate Gaussian with covariance matrix $\boldsymbol{C}_s$. Since the signal and noise components are independent, the covariance matrix $\boldsymbol{C}_1$ under hypothesis $H_1$ is

$$\boldsymbol{C}_1 = \boldsymbol{C}_s + \boldsymbol{C}_n = \boldsymbol{C}_s + \sigma_n^2 \boldsymbol{I} \tag{9.61}$$

Substituting (9.60) and (9.61) into (9.58), the LRT becomes

$$T(\boldsymbol{y}) = \boldsymbol{y}^T \left[ \frac{1}{\sigma_n^2} \boldsymbol{I} - (\boldsymbol{C}_s - \sigma_n^2 \boldsymbol{I})^{-1} \right] \boldsymbol{y} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_2 \tag{9.62}$$

We note that the LRT can be further reduced depending on the structure of the signal covariance matrix, which we consider next.

### 9.4.1 Uncorrelated Signal Components and Equal Variances

In this case, we assume that the signal components are uncorrelated and identically distributed. Thus, the covariance matrix is diagonal with equal diagonal elements $\sigma_s^2$; that is,

$$\boldsymbol{C}_s = \sigma_s^2 \boldsymbol{I} \tag{9.63}$$

Consequently, the LRT reduces to

$$T(\boldsymbol{y}) = \frac{\sigma_s^2}{\sigma_n^2 (\sigma_s^2 + \sigma_n^2)} \boldsymbol{y}^T \boldsymbol{y} = \frac{\sigma_s^2}{\sigma_n^2 (\sigma_s^2 + \sigma_n^2)} \sum_{k=1}^{K} y_k^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_2 \tag{9.64}$$

where $\gamma_2 = 2\gamma$, and $\gamma$ is given in (9.8b). Simplifying (9.64) further, we obtain the equivalent test

$$T(\boldsymbol{y}) = \sum_{k=1}^{K} y_k^2 \underset{H_0}{\overset{H_1}{>}} \gamma_3 \tag{9.65}$$

where $\gamma_3 = [\sigma_n^2(\sigma_s^2 + \sigma_n^2)/\sigma_s^2]\gamma_2$. Hence, the sufficient statistic is $T(\boldsymbol{y}) = \sum_{k=1}^{K} y_k^2$. Since $Y_k$ is independent and an identically distributed Gaussian random variable, $T(\boldsymbol{Y}) = Y_1^2 + Y_1^2 + \ldots + Y_K^2$ is a chi-square random variable with $K$ degrees of freedom, as was shown in Chapter 2. Consequently, we can carry the test further, and obtain an expression for $P_D$, the probability of detection, and $P_F$, the probability of false alarm. Note that once we obtain $P_D$ and $P_F$, we can plot the receiver operating characteristics.

Using the concept of transformation of random variables developed in Chapter 1, the density function of the random variable $Y = X^2$, where $X$ is Gaussian with mean zero and variance $\sigma^2$, is

$$f_Y(y) = \begin{cases} \dfrac{1/2\sigma^2}{\Gamma(1/2)}\left(\dfrac{1}{2\sigma^2}y\right)^{\frac{1}{2}} e^{-\frac{y}{2\sigma^2}}, & y > 0 \\ 0, & \text{otherwise} \end{cases} \tag{9.66}$$

where, from (2.102), $\alpha = 1/2$ and $\beta = 2\sigma^2$. Hence, the mean and the variance of $Y$ are $E[Y] = \alpha\beta = \sigma^2$ and $\text{var}[Y] = \alpha\beta^2 = 2\sigma^4$. From (2.106), the characteristic function of $Y = X^2$ is

$$\Phi_x(j\omega) = E[e^{j\omega X}] = \frac{1}{(1 - j\beta\omega)^{\alpha}} \tag{9.67}$$

Generalizing the result in (9.67) to $Y = Y_1 + Y_2 + \ldots + Y_K$, the sum of $K$ independent random variables, we obtain

$$\Phi_y(j\omega) = E[e^{j\omega Y}] = E[e^{j\omega(Y_1+Y_2+\ldots+Y_K)}] = E[e^{j\omega Y_1}]E[e^{j\omega Y_2}]\ldots E[e^{j\omega Y_K}]$$

$$= \Phi_{y_1}(j\omega)\,\Phi_{y_2}(j\omega)\ldots\Phi_{y_K}(j\omega) = \frac{1}{(1-j\beta\omega)^{\alpha_1+\alpha_2+\ldots+\alpha_K}} \tag{9.68}$$

and hence the density function of $Y$ is

$$f_Y(y) = \begin{cases} \dfrac{1/\beta}{\Gamma(\alpha_1+\alpha_2+\ldots+\alpha_K)}\left(\dfrac{y}{\beta}\right)^{\alpha_1+\alpha_2+\ldots+\alpha_K} e^{-\frac{y}{\beta}}, & y>0 \\ 0, & \text{otherwise} \end{cases} \tag{9.69}$$

Using $\alpha = 1/2$ and $\beta = 2\sigma^2$, we obtain the density function of the sufficient statistic to be

$$f_Y(y) = \begin{cases} \dfrac{1}{2^{K/2}\sigma^K\,\Gamma(K/2)} y^{\frac{K}{2}-1} e^{-\frac{y}{2\sigma^2}}, & y>0 \\ 0, & \text{otherwise} \end{cases} \tag{9.70}$$

Note that the variance $\sigma^2$ of $Y_k$, $k = 1, 2, \ldots, K$, denotes $\sigma_n^2$ under hypothesis $H_0$ and $(\sigma_s^2+\sigma_n^2)$ under hypothesis $H_1$. That is, the density function of the sufficient statistic $T(\boldsymbol{y})$ under each hypothesis is

$$f_{T|H_0}(t\,|\,H_0) = \begin{cases} \dfrac{1}{2^{K/2}\sigma_0^K\,\Gamma(K/2)} t^{\frac{K}{2}-1} e^{-\frac{t}{2\sigma_0^2}}, & t>0 \\ 0, & \text{otherwise} \end{cases} \tag{9.71}$$

$$f_{T|H_1}(t\,|\,H_1) = \begin{cases} \dfrac{1}{2^{K/2}\sigma_1^K\,\Gamma(K/2)} t^{\frac{K}{2}-1} e^{-\frac{t}{2\sigma_1^2}}, & t>0 \\ 0, & \text{otherwise} \end{cases} \tag{9.72}$$

where $\sigma_0^2 = \sigma_n^2$ and $\sigma_1^2 = \sigma_s^2+\sigma_n^2$. Knowing the conditional density functions $f_{T|H_1}(t\,|\,H_1)$ and $f_{T|H_0}(t\,|\,H_0)$, we can obtain expressions for $P_D$ and $P_F$. From (9.65), the probabilities of detection and false alarm are

$$P_F = \int_{\gamma_3}^{\infty} f_{T|H_0}(t \mid H_0)dt = \frac{1}{2^{K/2}\sigma_0^K\,\Gamma(K/2)} \int_{\gamma_3}^{\infty} t^{\frac{K}{2}-1} e^{-\frac{t}{2\sigma_0^2}} dt \tag{9.73}$$

and

$$P_D = \int_{\gamma_3}^{\infty} f_{T|H_1}(t \mid H_1)dt = \frac{1}{2^{K/2}\sigma_1^K\,\Gamma(K/2)} \int_{\gamma_3}^{\infty} t^{\frac{K}{2}-1} e^{-\frac{t}{2\sigma_1^2}} dt \tag{9.74}$$

### 9.4.2 Uncorrelated Signal Components and Unequal Variances

In this case, we assume that the signal components are uncorrelated, and thus the covariance matrix $\boldsymbol{C}_s$ is diagonal. We also assume that the variances of the different components are not equal; that is,

$$\boldsymbol{C}_s = \begin{bmatrix} \sigma_{s_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{s_2}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{s_K}^2 \end{bmatrix} \tag{9.75}$$

From the LRT in (9.62), let the term in brackets be denoted $\boldsymbol{H}$; that is,

$$\boldsymbol{H} = \frac{1}{\sigma_n^2}\boldsymbol{I} - (\boldsymbol{C}_s + \sigma_n^2\boldsymbol{I})^{-1} \tag{9.76}$$

Substituting (9.75) into (9.76) and rearranging terms, the $\boldsymbol{H}$ matrix reduces to

$$\boldsymbol{H} = \begin{bmatrix} \dfrac{\sigma_{s_1}^2}{\sigma_n^2(\sigma_{s_1}^2+\sigma_n^2)} & 0 & \cdots & 0 \\ 0 & \dfrac{\sigma_{s_2}^2}{\sigma_n^2(\sigma_{s_2}^2+\sigma_n^2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \dfrac{\sigma_{s_K}^2}{\sigma_n^2(\sigma_{s_K}^2+\sigma_n^2)} \end{bmatrix} \tag{9.77}$$

and consequently, the LRT becomes

$$T(\boldsymbol{y}) = \boldsymbol{y}^T \boldsymbol{H}\, \boldsymbol{y} = \frac{1}{\sigma_n^2} \sum_{k=1}^{K} \frac{\sigma_{s_k}^2}{(\sigma_{s_k}^2 + \sigma_n^2)} y_k^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_2 \tag{9.78}$$

We observe that the above expression is not as simple as the one in the previous section, and consequently it may not be easy to obtain expressions for $P_D$ and $P_F$.

*Remark.* If the signal components are not independent, and thus the signal covariance matrix is not diagonal, we can diagonalize the matrix using an orthogonal transformation, following the procedure given in Section 9.3.2.

**Example 9.3**

Consider the binary hypothesis problem

$$\begin{aligned} H_1 &: Y_k = S_k + N_k, \quad k = 1, 2 \\ H_0 &: Y_k = \quad\; N_k, \quad k = 1, 2 \end{aligned}$$

where the noise components are zero mean and uncorrelated Gaussian random variables with variance $\sigma_n^2$, $k = 1, 2$. The signal components are also independent and zero mean with variance $\sigma_s^2$, $k = 1, 2$. The signal and noise components are independent. Obtain:

(a) the optimum decision rule.
(b) expressions for the probabilities of detection and false alarm.

*Solution*

(a) This is the case where the noise components are independent and identically distributed, and the signal components are also independent and identically distributed. Both covariance matrices $\boldsymbol{C}_s$ and $\boldsymbol{C}_n$ of the signal and noise are diagonal. The optimum decision rule is given by (9.65) to be

$$T(\boldsymbol{y}) = y_1^2 + y_2^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_3$$

where $\gamma_3 = \left\{[\sigma_n^2(\sigma_s^2+\sigma_n^2)]/\sigma_n^2\right\}\gamma_2$, $\gamma_2 = 2\gamma$, and $\gamma = \ln\eta + (1/2)(\ln|\boldsymbol{C}_1| - \ln|\boldsymbol{C}_0|)$. The covariance matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_0$ under hypotheses $H_1$ and $H_0$ are

$$\boldsymbol{C}_1 = \boldsymbol{C}_s + \boldsymbol{C}_n = \begin{bmatrix} \sigma_s^2+\sigma_n^2 & 0 \\ 0 & \sigma_s^2+\sigma_n^2 \end{bmatrix} \text{ and } \boldsymbol{C}_0 = \boldsymbol{C}_n = \begin{bmatrix} \sigma_n^2 & 0 \\ 0 & \sigma_n^2 \end{bmatrix}$$

Rearranging terms, the decision rule becomes

$$T(\boldsymbol{y}) = y_1^2 + y_2^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\frac{\sigma_n^2(\sigma_s^2+\sigma_n^2)}{\sigma_s^2}\left(\ln\eta + \ln\frac{\sigma_s^2+\sigma_n^2}{\sigma_s^2}\right) = \gamma_3$$

Consequently, the sufficient statistic is $T(\boldsymbol{y}) = y_1^2 + y_2^2$.

(b) Using the results derived in (9.71) and (9.72), the conditional probability density function of the sufficient statistic under each hypothesis is

$$f_{T|H_0}(t\,|\,H_0) = \begin{cases} \dfrac{1}{2\sigma_0^2}\exp\left(-\dfrac{t}{2\sigma_0^2}\right), & t>0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{T|H_1}(t\,|\,H_1) = \begin{cases} \dfrac{1}{2\sigma_1^2}\exp\left(-\dfrac{t}{2\sigma_1^2}\right), & t>0 \\ 0, & \text{otherwise} \end{cases}$$

where $\sigma_0^2 = \sigma_n^2$ and $\sigma_1^2 = \sigma_s^2 + \sigma_n^2$. Consequently, the probability of detection and probability of false alarm are

$$P_D = \frac{1}{2\sigma_1^2}\int_{\gamma_3}^{\infty} e^{-\frac{t}{2\sigma_1^2}}dt = e^{-\frac{\gamma_3}{2\sigma_1^2}} \text{ and } P_F = \frac{1}{2\sigma_0^2}\int_{\gamma_3}^{\infty} e^{-\frac{t}{2\sigma_0^2}}dt = e^{-\frac{\gamma_3}{2\sigma_0^2}}$$

## 9.5 SAME MEAN AND SYMMETRIC HYPOTHESES

Consider the binary symmetric hypothesis problem given by

$$H_1: \begin{array}{ll} Y_k = S_k + N_k, & k = 1, 2, \ldots, K \\ Y_k = N_k, & k = (K+1), (K+2), \ldots, 2K \end{array}$$

$$H_0: \begin{array}{ll} Y_k = N_k, & k = 1, 2, \ldots, K \\ Y_k = S_k + N_k, & k = (K+1), (K+2), \ldots, 2K \end{array} \tag{9.79}$$

We assume, as before, that the mean vectors $\boldsymbol{m}_1 = \boldsymbol{m}_0 = \boldsymbol{0}$ and that the noise components are uncorrelated with variance $\sigma_n^2$. Thus, the noise covariance matrix is $\boldsymbol{C}_n = \sigma_n^2 \boldsymbol{I}$. Let $\boldsymbol{C}_s$ denote the signal covariance matrix. Then, the $2K \times 2K$ covariance matrices $\boldsymbol{C}_0$ and $\boldsymbol{C}_1$ under hypotheses $H_0$ and $H_1$, respectively, can be partitioned into $K \times K$ submatrices. That is,

$$\boldsymbol{C}_1 = \left[\begin{array}{c|c} \boldsymbol{C}_s + \boldsymbol{C}_n & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{C}_n \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{C}_s + \sigma_n^2 \boldsymbol{I} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \sigma_n^2 \boldsymbol{I} \end{array}\right] \tag{9.80}$$

and

$$\boldsymbol{C}_0 = \left[\begin{array}{c|c} \boldsymbol{C}_n & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{C}_s + \boldsymbol{C}_n \end{array}\right] = \left[\begin{array}{c|c} \sigma_n^2 \boldsymbol{I} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{C}_s + \sigma_n^2 \boldsymbol{I} \end{array}\right] \tag{9.81}$$

Let the difference of the inverse covariance matrices of $\boldsymbol{C}_0$ and $\boldsymbol{C}_1$ be denoted by

$$\Delta \boldsymbol{C}^{-1} = \boldsymbol{C}_0^{-1} - \boldsymbol{C}_1^{-1} \tag{9.82}$$

Thus,

$$\Delta \boldsymbol{C}^{-1} = \left[\begin{array}{c|c} \dfrac{1}{\sigma_n^2} \boldsymbol{I} & \boldsymbol{0} \\ \hline \boldsymbol{0} & (\boldsymbol{C}_s + \sigma_n^2 \boldsymbol{I})^{-1} \end{array}\right] - \left[\begin{array}{c|c} (\boldsymbol{C}_s + \sigma_n^2 \boldsymbol{I})^{-1} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \dfrac{1}{\sigma_n^2} \boldsymbol{I} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \dfrac{1}{\sigma_n^2}\boldsymbol{I} - (\boldsymbol{C}_s + \sigma_n^2 \boldsymbol{I})^{-1} & \mathbf{0} \\ \hline \mathbf{0} & (\boldsymbol{C}_s + \sigma_n^2 \boldsymbol{I})^{-1} - \dfrac{1}{\sigma_n^2}\boldsymbol{I} \end{array}\right] \tag{9.83}$$

Partitioning the $2K \times 1$ vector $\boldsymbol{Y}$ into two $K \times 1$ vectors such that

$$\boldsymbol{Y} = \left[\begin{array}{c} \boldsymbol{Y}_1 \\ \hline \boldsymbol{Y}_2 \end{array}\right] \tag{9.84}$$

and substituting (9.84) and (9.83) into (9.58), the LRT becomes

$$\begin{aligned} T(\boldsymbol{y}) &= \boldsymbol{y}^T \Delta \boldsymbol{C}^{-1} \boldsymbol{y} \\ &= \left[\begin{array}{c} \boldsymbol{y}_1^T \\ \hline \boldsymbol{y}_2^T \end{array}\right] \left[\begin{array}{c|c} \dfrac{1}{\sigma_n^2}\boldsymbol{I} - (\boldsymbol{C}_s + \sigma_n^2 \boldsymbol{I})^{-1} & \mathbf{0} \\ \hline \mathbf{0} & (\boldsymbol{C}_s + \sigma_n^2 \boldsymbol{I})^{-1} - \dfrac{1}{\sigma_n^2}\boldsymbol{I} \end{array}\right] \left[\begin{array}{c} \boldsymbol{y}_1 \\ \hline \boldsymbol{y}_2 \end{array}\right] \\ &= \boldsymbol{y}_1^T \left[\frac{1}{\sigma_n^2}\boldsymbol{I} - (\boldsymbol{C}_s + \sigma_n^2 \boldsymbol{I})^{-1}\right] \boldsymbol{y}_1 + \boldsymbol{y}_2^T \left[(\boldsymbol{C}_s + \sigma_n^2 \boldsymbol{I})^{-1} - \frac{1}{\sigma_n^2}\boldsymbol{I}\right] \boldsymbol{y}_2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_2 \end{aligned} \tag{9.85}$$

Again, depending on the structure of the signal covariance matrix $\boldsymbol{C}_s$, the above expression may be reduced as in the previous section.

### 9.5.1 Uncorrelated Signal Components and Equal Variances

In order to carry the test in (9.85) further, let the signal components be uncorrelated and identically distributed. That is,

$$\boldsymbol{C}_s = \sigma_s^2 \boldsymbol{I} \tag{9.86}$$

Substituting the above value of the signal covariance matrix into (9.85), the LRT test is obtained to be

$$T(\boldsymbol{y}) = \boldsymbol{y}_1^T \left[ \frac{1}{\sigma_n^2} \boldsymbol{I} - (\sigma_s^2 \boldsymbol{I} + \sigma_n^2 \boldsymbol{I})^{-1} \right] \boldsymbol{y}_1 + \boldsymbol{y}_2^T \left[ (\sigma_s^2 \boldsymbol{I} + \sigma_n^2 \boldsymbol{I})^{-1} - \frac{1}{\sigma_n^2} \boldsymbol{I} \right] \boldsymbol{y}_2$$

$$= \frac{\sigma_s^2}{\sigma_n^2(\sigma_s^2 + \sigma_n^2)} \boldsymbol{I} (\boldsymbol{y}_1^T \boldsymbol{y}_1 - \boldsymbol{y}_2^T \boldsymbol{y}_2) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_2 \tag{9.87}$$

or

$$T(\boldsymbol{y}) = \sum_{k=1}^{K} y_k^2 - \sum_{k=K+1}^{2K} y_k^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_3 \tag{9.88}$$

where $\gamma_3$ is defined in (9.65).

We can have more insight into this problem by assuming that we have a minimum probability of error criterion, and that both hypotheses are equally likely. Thus, the threshold $\eta$ equals one, and $\gamma_2$ and $\gamma_3$ become zero. We observe that the determinants of both covariance matrices are equal $\left( |\boldsymbol{C}_1| = |\boldsymbol{C}_0| \right)$, since the hypotheses are symmetrical. Consequently, the LRT reduces to

$$T_1(\boldsymbol{y}) = \sum_{k=1}^{K} y_k^2 \underset{H_0}{\overset{H_1}{\gtrless}} \sum_{k=K+1}^{2K} y_k^2 = T_0(\boldsymbol{y}) \tag{9.89}$$

The probability of error is defined as

$$P(\varepsilon) = P(\varepsilon \mid H_0) P(H_0) + P(\varepsilon \mid H_1) P(H_1) = \frac{1}{2} [P(\varepsilon \mid H_0) + P(\varepsilon \mid H_1)] \tag{9.90}$$

Since the test is symmetrical with respect to both hypotheses, we have

$$P(\varepsilon \mid H_0) = P(\varepsilon \mid H_1) \tag{9.91}$$

Thus, the probability of error is just

$$P(\varepsilon) = P(\varepsilon \mid H_0) = P(T_0 < T_1 \mid H_0) = P(\varepsilon \mid H_1) = P(T_1 < T_0 \mid H_1) \tag{9.92}$$

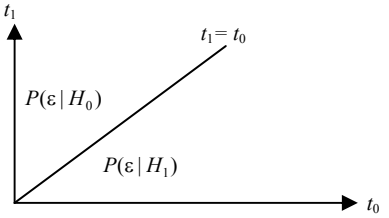


**Figure 9.3** Regions of integration for $P(\varepsilon)$.

From Figure 9.3, we see that the probability of error is given by

$$P(\varepsilon) = P(\varepsilon \mid H_0) = \int_0^{\infty} \int_0^{t_1} f_{T_1 T_0}(t_1, t_0 \mid H_0) dt_0 dt_1$$
$$= P(\varepsilon \mid H_1) = \int_0^{\infty} \int_{t_1}^{\infty} f_{T_1 T_0}(t_1, t_0 \mid H_1) dt_0 dt_1 \tag{9.93}$$

From (9.70), $T_1(\boldsymbol{Y})$ and $T_2(\boldsymbol{Y})$ are statistically independent and chi-square distributed, and thus

$$f_{T_1}(t_1) = \frac{1}{2^{K/2} \sigma_1^K \, \Gamma(K/2)} t_1^{\frac{n}{2}-1} e^{-\frac{t_1}{2\sigma_1^2}} \tag{9.94}$$

$$f_{T_0}(t_0) = \frac{1}{2^{K/2} \sigma_0^K \, \Gamma(K/2)} t_0^{\frac{n}{2}-1} e^{-\frac{t_0}{2\sigma_0^2}} \tag{9.95}$$

where $\sigma_0^2 = \sigma_n^2$ and $\sigma_1^2 = \sigma_s^2 + \sigma_n^2$. Substituting (9.94) and (9.95) into (9.93) and solving the integral, it can be shown that the probability of error reduces to

$$P(\varepsilon) = \left( \frac{\sigma_n^2}{\sigma_1^2 + \sigma_n^2} \right)^{\frac{K}{2}} \sum_{k=0}^{\frac{K}{2}-1} \binom{\frac{K}{2} + k - 1}{k} \left( 1 - \frac{\sigma_n^2}{\sigma_1^2 + \sigma_n^2} \right)^k \tag{9.96}$$

### 9.5.2 Uncorrelated Signal Components and Unequal Variances

In this case, we assume that the signal components are uncorrelated but their corresponding variances are not equal. That is, the signal covariance matrix is still diagonal but with unequal elements. Thus, we have

$$\boldsymbol{C}_s = \begin{bmatrix} \sigma_{s_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{s_2}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{s_K}^2 \end{bmatrix} \tag{9.97}$$

Substituting (9.97) into (9.82) and rearranging terms, we obtain

$$\Delta \boldsymbol{C}^{-1} = \left[\begin{array}{ccc:ccc} \dfrac{\sigma_{s_1}^2}{\sigma_n^2(\sigma_{s_1}^2+\sigma_n^2)} & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \dfrac{\sigma_{s_K}^2}{\sigma_n^2(\sigma_{s_K}^2+\sigma_n^2)} & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -\dfrac{\sigma_{s_1}^2}{\sigma_{s_1}^2+\sigma_n^2} & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\dfrac{\sigma_{s_K}^2}{\sigma_{s_K}^2+\sigma_n^2} \end{array}\right] \tag{9.98}$$

It follows that the test (9.85) becomes

$$T(\boldsymbol{y}) = \frac{\sigma_1^2}{\sigma_n^2}\left(\sum_{k=1}^{K}\frac{\sigma_{s_k}^2}{\sigma_{s_k}^2+\sigma_n^2}y_k^2 - \sum_{k=K+1}^{2K}\frac{\sigma_{s_{(k-K)}}^2}{\sigma_{s_{(k-K)}}^2+\sigma_n^2}y_k^2\right) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_2 \tag{9.99}$$

This expression is too complicated to proceed any further with the test.

## 9.6 SUMMARY

In this chapter we have discussed the general Gaussian problem. We considered the binary hypothesis problem. Due to the characteristics of the Gaussian process and Gaussian random variables, the general Gaussian problem was considered in

terms of the covariance matrices and mean vectors under each hypothesis. First, we considered the case of an equal covariance matrix for both hypotheses. The noise samples were always assumed uncorrelated and thus statistically independent with equal variances. The signal components considered, however, were either independent or not independent. When the signal components were independent and of equal variance, the problem was relatively simple, since the covariance matrix is diagonal with equal value elements. When the signal component variances were not equal, the expressions were more difficult, and in this case we were able to solve for the sufficient statistic only.

In the case when the covariance matrices are general, we transformed the problem from one coordinate system into another coordinate system, such that the covariance matrix is diagonal. We solved for the eigenvalues and eigenvectors, and then used an orthogonal transformation to diagonalize the covariance matrix. In Sections 9.4 and 9.5, we considered the case of equal mean vectors and obtained the LRT.

## PROBLEMS

**9.1** For the binary hypothesis problem with $\boldsymbol{m}_0 = \mathbf{0}$, let the covariance matrix $\boldsymbol{C}$ be

(a) $\boldsymbol{C} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$ (b) $\boldsymbol{C} = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}$ (c) $\boldsymbol{C} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$

Determine the LRT for the three cases above.

**9.2** Repeat Problem 9.1, assuming that the covariance matrix $\boldsymbol{C}$ is

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 2 \end{bmatrix}$$

**9.3** Consider the binary hypothesis problem

$$H_1 : Y_k = S_k + N_k, \quad k = 1, 2$$
$$H_1 : Y_k = \quad N_k, \quad k = 1, 2$$

where the noise components are zero mean and uncorrelated Gaussian random variables with variances $\sigma_n^2 = 1$, $k = 1, 2$. The signal components are also independent and zero mean with variances $\sigma_s^2 = 2$, $k = 1, 2$. The signal and noise components are independent.

(a) Obtain the optimum decision rule.

(b) Determine the minimum probability of error for $P(H_0) = P(H_1) = 1/2$.

**9.4** Repeat Problem 9.3 with $k = 1, 2, 3, 4$.

**9.5** Plot the receiver operating characteristics of Problem 9.3 with the ratio $\sigma_s^2 / \sigma_n^2$ as a parameter.

**9.6** Consider Problem 9.3 with signal covariance matrix

$$\boldsymbol{C}_s = \begin{bmatrix} \sigma_s^2 & 0 \\ 0 & \sigma_s^2 \end{bmatrix}$$

Design an optimum test.

**9.7** Consider the symmetrical binary hypothesis problem

$$H_1 : \begin{array}{ll} Y_k = S_k + N_k, & k = 1, 2 \\ Y_k = N_k, & k = 3, 4 \end{array}$$

$$H_0 : \begin{array}{ll} Y_k = N_k, & k = 1, 2 \\ Y_k = S_k + N_k, & k = 3, 4 \end{array}$$

Let the mean vectors under each hypothesis be zero for both hypotheses $H_0$ and $H_1$. The noise components are identically distributed Gaussian random variance with variance 1. The signal components are also independent and identically distributed with variance 2. The signal and noise components are independent.

(a) Design an optimum test.
(b) Determine the probability of error assuming minimum probability of error criterion and $P_0 = P_1 = 1/2$.

**9.8** Repeat Problem 9.1 if the covariance matrix is given by

(a) $\boldsymbol{C} = \begin{bmatrix} 1 & 0.9 & 0.5 \\ 0.9 & 1 & 0.1 \\ 0.5 & 0.1 & 1 \end{bmatrix}$

(b) $\boldsymbol{C} = \begin{bmatrix} 1 & 0.8 & 0.6 & 0.2 \\ 0.8 & 1 & 0.8 & 0.6 \\ 0.6 & 0.8 & 1 & 0.8 \\ 0.2 & 0.6 & 0.8 & 1 \end{bmatrix}$

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# Chapter 10

# Detection and Parameter Estimation

## 10.1 INTRODUCTION

In Chapters 1 and 3, we presented the fundamentals of probability theory and stochastic processes. In Chapters 5 and 6, we developed the basic principles needed for solving decision and estimation problems. The observations considered were represented by random variables. In Chapter 7, we presented the orthogonality principle and its application in the optimum linear mean-square estimation. In Chapter 8, we presented some mathematical principles, such as Gram-Schmidt orthogonalization procedure, diagonalization of a matrix and similarity transformation, integral equations, and generalized Fourier series. The concept of generalized Fourier series was then used to represent random processes by an orthogonal series expansion, referred to as the Karhunen-Loève expansion. Chapter 8 gave us the basic mathematical background for Chapters 9 and 10. In Chapter 9, we covered the general detection Gaussian problem.

In this chapter, we extend the concepts of decision and estimation problems to time varying waveforms. If a signal is transmitted, then the received waveform is composed of the transmitted signal and an additive noise process. If no signal is transmitted, then the received waveform is noise only. The goal is to design an optimum receiver (detector) according to some criterion. In Section 10.2, we discuss the general and simple binary detection of known signals corrupted by an additive white Gaussian noise process with mean zero and power spectral density $N_0/2$. The received waveforms are observed over the interval of time $t \in [0, T]$. In Section 10.3, we extend the concepts of binary detection to *M*-ary detection. In Section 10.4, we assume that the received signals in the presence of the additive white Gaussian noise process have some unknown parameters, which need to be estimated. Some linear estimation techniques are used to estimate these unknown parameters, which may be either random or nonrandom. Nonlinear estimation is presented in Section 10.5. In Section 10.6, we consider the general binary detection with unwanted parameters in additive white Gaussian noise. In this case the received waveform is not completely known a priori, as in the previous

sections. The unknown parameters of the signal are referred to as *unwanted parameters*. We consider signals with random phase. We obtain the sufficient statistic and solve for the probabilities of detection and false alarm through an example showing all steps. We show how the incoherent matched filter is used for this type of application. Then, we consider signals with two random parameters, the phase and amplitude. Other cases, such as signals with random frequency, signals with different random phases, frequency shift keying signals with Rayleigh fading, and signal with random time of arrival that may arise in radar and communication applications are also discussed.

We conclude this chapter with a section on detection in colored noise. Specifically, we consider the general binary detection in nonwhite Gaussian noise. Two different approaches, using Karhunen-Loève expansion and whitening, are suggested to solve this problem.

## 10.2 BINARY DETECTION

In a binary communication problem, the transmitter may send a deterministic signal $s_0(t)$ under the null hypothesis $H_0$, or a deterministic signal $s_1(t)$ under the alternate hypothesis $H_1$. At the receiver, the signal is corrupted by $W(t)$, which is an additive white Gaussian noise process. Assume that the additive noise is zero mean and has a double-sided power spectral density of $N_0/2$. The goal is to design an optimum receiver that observes the received signal $Y(t)$ over the interval $t \in [0, T]$, and then decides whether hypothesis $H_0$ or hypothesis $H_1$ is true.

### 10.2.1 Simple Binary Detection

In a simple binary detection problem, the transmitted signal under hypothesis $H_1$ is $s(t)$, and no signal is transmitted under the null hypothesis $H_0$. At the receiver, we have

$$\begin{aligned} H_1 &: Y(t) = s(t) + W(t), \quad 0 \le t \le T \\ H_0 &: Y(t) = \qquad W(t), \quad 0 \le t \le T \end{aligned} \tag{10.1}$$

Note that the signal is a continuous time function. In order to obtain a set of countable random variables so that we may apply the concepts developed in Chapter 5, we need to take $K$ samples, where $K$ may be infinite. However, in Chapter 8, we saw that a continuous time signal may be represented by Karhunen-Loève expansion using a set of $K$ complete orthonormal functions. The coefficients in the series expansion are the desired set of random variables.

The energy of the known deterministic signal is

$$E = \int_0^T s^2(t)dt \tag{10.2}$$

Thus, let the first normalized function $\phi_1(t)$ be

$$\phi_1(t) = \frac{s(t)}{\sqrt{E}} \tag{10.3}$$

or

$$s(t) = \sqrt{E}\phi_1(t) \tag{10.4}$$

Consequently, the first coefficient in the Karhunen-Loève expansion of $Y(t)$ is

$$Y_1 = \int_0^T Y(t)\phi_1(t)dt = \begin{cases} H_1 : \int_0^T [s(t)+W(t)]\phi_1(t)dt = \sqrt{E} + W_1 \\ H_0 : \int_0^T W(t)\phi_1(t)dt = W_1 \end{cases} \tag{10.5}$$

where $W_1$ is the first coefficient in the series expansion of $W(t)$. T $s(t)+W(t)$ he rest of the coefficients $Y_k$ , $k = 2,3,\ldots$ , are obtained by using arbitrary orthogonal functions $\phi_k$ , $k = 2,3,\ldots$ . $\phi_k$ orthogonal to $\phi_1(t)$ $\left(\int_0^T \phi_k(t)\phi_1(t)dt = 0\right)$. Thus,

$$Y_k = \begin{cases} H_1 : \int_0^T [s(t)+W(t)]\phi_k(t)dt = W_k \\ H_0 : \int_0^T W(t)\phi_k(t)dt = W_k \end{cases} \tag{10.6}$$

Since $W(t)$ is a Gaussian process, the random variables $W_k$ , $k = 2,3,\ldots$ , are Gaussian. We observe from (10.6) that the coefficients $Y_k$ , $k = 2,3,\ldots$ , are coefficients of a white Gaussian process ($W_k$), and do not depend on which hypothesis is true. Only the coefficient $Y_1$ depends on the hypotheses $H_1$ and $H_0$.

We need to find a sufficient statistic for this infinite number of random variables in order to make a decision as to which hypothesis is true. Since the

coefficients $W_j$ and $W_k$, $j \neq k$, of Karhunen-Loève expansion are uncorrelated, that is,

$$E[W_j W_k \mid H_0] = E[W_j W_k \mid H_1] = E[W_j W_k] = 0, \quad j \neq k \tag{10.7}$$

and are jointly Gaussian, they are statistically independent. Thus, all $Y_k$, $k = 2, 3, \ldots$, are statistically independent of $Y_1$ and have no effect on the decision. Hence, the sufficient statistic is only $Y_1$; that is,

$$T(\boldsymbol{Y}) = Y_1 \tag{10.8}$$

We learn from (10.8) that the infinite observation space has been reduced to a one-dimensional decision space. Thus, the equivalent problem to (10.1) is

$$\begin{aligned} H_1 &: Y_1 = \sqrt{E} + W_1 \\ H_0 &: Y_1 = \quad\; W_1 \end{aligned} \tag{10.9}$$

where $W_1$ is Gaussian, with means

$$E[W_1 \mid H_1] = E[W_1 \mid H_0] = E\left[\int_0^T \phi_1(t) W(t)\, dt\right] = \int_0^T \phi_1(t) E[W(t)]\, dt = 0 \tag{10.10}$$

and variances

$$\begin{aligned} E[W_1^2 \mid H_1] = E[W_1^2 \mid H_0] &= E\left[\int_0^T \int_0^T \phi_1(t)\phi_1(u) W(t) W(u)\, dt\, du\right] \\ &= \int_0^T \int_0^T \phi_1(t)\phi_1(u) E[W(t)W(u)]\, dt\, du \end{aligned} \tag{10.11}$$

The power spectral density of $W(t)$ is $N_0/2$ for all frequency *f*, and thus its autocorrelation function $R_{ww}(t,u)$ is

$$E[W(t)W(u)] = R_{ww}(t,u) = \frac{N_0}{2}\delta(t-u) = C_{ww}(t,u) \tag{10.12}$$

where $C_{ww}(t,u)$ is the covariance function. Substituting (10.12) into (10.11), we obtain the variance of $W_1$ to be

$$E\left[W_1^2\right] = \frac{N_0}{2}\int_0^T\int_0^T \phi_1(t)\phi_1(u)\delta(t-u)dtdu = \frac{N_0}{2}\int_0^T \phi_1^2(t)dt = \frac{N_0}{2} \tag{10.13}$$

We observe that the problem given by (10.9) is the same as the one solved in Example 5.1, with $m = \sqrt{E}$ and $\sigma^2 = N_0/2$. Consequently, the optimum decision rule is

$$T(y) = y_1 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2\sqrt{E}}\ln\eta + \frac{\sqrt{E}}{2} = \gamma \tag{10.14}$$

The detection parameter is given by

$$d^2 \triangleq \frac{\{E[T(Y)\,|\,H_1] - E[T(Y)\,|\,H_0]\}^2}{\text{var}[T(Y)\,|\,H_0]} = \frac{2E}{N_0} \tag{10.15}$$

The probabilities of detection and false alarm are then

$$P_D = Q\left(2\frac{\gamma - \sqrt{E}}{N_0}\right) \tag{10.16}$$

and

$$P_F = Q\left(\frac{2\gamma}{N_0}\right) \tag{10.17}$$

where $Q(\cdot)$ is the $Q$-function, also denoted $\text{erfc}_*(\cdot)$ in many other books. Thus, the only factors affecting the performance of such a receiver are the signal energy $E$ and the noise power spectral density $N_0/2$. From Chapter 8, we note that the optimum receiver is either a *correlation receiver* or a *matched filter receiver*. The receivers are illustrated in Figures 10.1 and 10.2.

Note that the impulse response $h(t)$ of the matched filter is

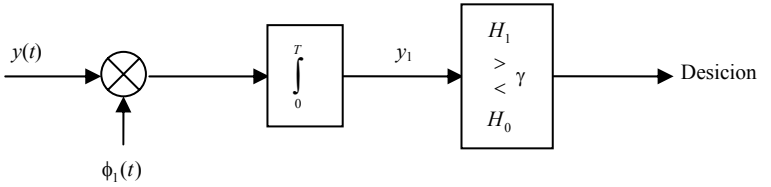


**Figure 10.1** Correlation receiver.

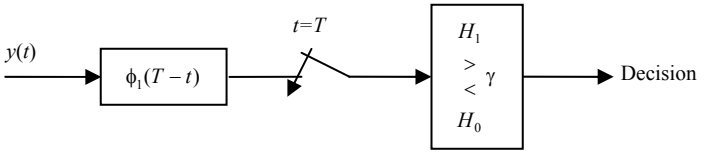


**Figure 10.2** Matched filter receiver .

$$h(t) = \begin{cases} \phi_1(T-t), & 0 \le t \le T \\ 0 \quad , & \text{otherwise} \end{cases} \tag{10.18}$$

We now derive the optimum receiver without resorting to the concept of sufficient statistics. Given a complete set $\{\phi_k(t)\}$ of $K$ orthonormal functions, the Karhunen-Loève expansion of the received process $Y(t)$ is

$$Y(t) = \sum_{k=1}^{K} Y_k \phi_k(t), \quad 0 \le t \le T \tag{10.19}$$

where

$$Y_k = \int_0^T Y(t)\phi_k(t)dt, \quad k = 1, 2, \ldots, K \tag{10.20}$$

The observation vector is $\boldsymbol{Y} = [Y_1 \quad Y_2 \quad \ldots \quad Y_k]^T$ . Under hypothesis $H_0$, $Y_k$ is expressed as

$$Y_k = \int_0^T W(t)\phi_k(t)dt = W_k \tag{10.21}$$

while under hypothesis $H_1$, $Y_k$ is

$$Y_k = \int_0^T [s(t) + W(t)]\phi_k(t)dt = \int_0^T s(t)\phi_k(t)dt + \int_0^T W(t)\phi_k(t)dt = s_k + W_k \quad (10.22)$$

$Y_k$ indicates Gaussian random variables, and thus we only need to find the means and variances under each hypothesis to have a complete description of the conditional density functions. The means and variances of $Y_k$ are

$$E[Y_k \mid H_0] = E[W_k] = 0 \quad (10.23)$$

$$E[Y_k \mid H_1] = E[s_k + W_k] = s_k \quad (10.24)$$

$$\text{var}[Y_k \mid H_0] = E[Y_k^2 \mid H_0] = E[W_k^2 \mid H_0] = R_{ww}(0) = \frac{N_0}{2} \quad (10.25)$$

and

$$\text{var}[Y_k \mid H_1] = E[(Y_k - s_k)^2 \mid H_1] = E[W_k^2 \mid H_1] = R_{ww}(0) = \frac{N_0}{2} \quad (10.26)$$

Since uncorrelated Gaussian random variables are statistically independent, the conditional density functions are

$$f_{\boldsymbol{Y}|H_1}(\boldsymbol{y} \mid H_1) = \prod_{k=1}^{K} \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(y_k - s_k)^2}{N_0}\right] \quad (10.27)$$

and

$$f_{\boldsymbol{Y}|H_0}(\boldsymbol{y} \mid H_0) = \prod_{k=1}^{K} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{y_k^2}{N_0}\right) \quad (10.28)$$

From (8.11), (8.22), and (8.24), we have

$$s(t) = \lim_{K \to \infty} s_K(t) \quad (10.29)$$

where

$$s_K(t) = \sum_{k=1}^{K} s_k \phi_k(t) \quad (10.30)$$

Consequently, the likelihood ratio is

$$\Lambda[y(t)] = \lim_{K\to\infty} \Lambda[y_K(t)] = \frac{f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\,|\,H_1)}{f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\,|\,H_0)} = \frac{\prod_{k=1}^{K} \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(y_k - s_k)^2}{N_0}\right]}{\prod_{k=1}^{K} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{y_k^2}{N_0}\right)} \tag{10.31}$$

where $\Lambda[y_K(t)]$ is the $K$-term likelihood ratio. Taking the logarithm and simplifying, (10.31) may be rewritten as

$$\lim_{K\to\infty} \ln\Lambda\left[Y_K(t)\right] = \lim_{K\to\infty}\left(\frac{2}{N_0}\sum_{k=1}^{K} Y_k s_k - \frac{1}{N_0}\sum_{k=1}^{K} s_k^2\right) \tag{10.32}$$

where

$$\lim_{K\to\infty} \sum_{k=1}^{K} Y_k s_k = \int_0^T Y_K(t) s_K(t)\,dt \tag{10.33}$$

and

$$\lim_{K\to\infty} \sum_{k=1}^{K} s_k^2 = \int_0^T s_K^2(t)\,dt \tag{10.34}$$

The likelihood ratio, letting $K \to \infty$, is

$$\ln \Lambda\left[Y(t)\right] = \frac{2}{N_0}\int_0^T Y(t)s(t)\,dt - \frac{1}{N_0}\int_0^T s^2(t)\,dt \tag{10.35}$$

and the decision rule is given by

$$\ln \Lambda\left[y(t)\right] \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta \tag{10.36}$$

Substituting (10.2) into (10.4), and then into (10.36), we obtain

$$\ln \Lambda[y(t)] = \frac{2}{N_0}\int_0^T y(t)s(t)dt - \frac{E}{N_0} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta \tag{10.37}$$

Since $s(t) = \sqrt{E}\phi_1(t)$, the test reduces to

$$\frac{2\sqrt{E}}{N_0}\int_0^T y(t)\phi_1(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta + \frac{E}{N_0} \tag{10.38}$$

or

$$\int_0^T y(t)\phi_1(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \tag{10.39}$$

which is the optimum receiver derived earlier in (10.14) using the sufficient statistic.

**Example 10.1**

Consider the digital communication system shown in Figure 10.3. The information source is binary, and produces zeros and ones with equal probability. The communication system uses amplitude shift keying (ASK) so that the received signals under hypotheses $H_1$ and $H_0$ are

$$H_1 : Y(t) = As(t) + W(t), \quad 0 \le t \le T$$
$$H_0 : Y(t) = \quad W(t), \quad 0 \le t \le T$$

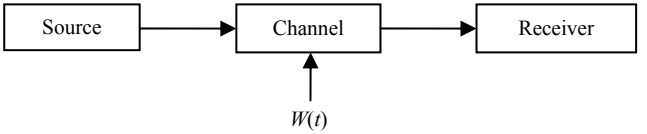


**Figure 10.3** Digital communication system.

The attenuation $A$ produced by the communication channel is a Gaussian random variable with mean zero and variance $\sigma_a^2$. The signal $s(t)$ is deterministic with energy $E$, and $W(t)$ is an additive white Gaussian noise with mean zero and power spectral density $N_0/2$. Determine the optimum receiver assuming minimum probability of error criterion.

*Solution*

From (10.3), the first normalized function $\phi_1(t)$ is $\phi_1(t) = s(t)/\sqrt{E}$. Following the same procedure described from (10.3) to (10.9), the problem reduces to

$$\begin{aligned} H_1 : Y_1 &= A\sqrt{E} + W_1 \\ H_0 : Y_1 &= \qquad W_1 \end{aligned}$$

The conditional density functions are

$$f_{Y_1|H_0}(y_1 \mid H_0) = f_{W_1}(w_1) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{y_1^2}{N_0}\right)$$

and $f_{Y_1|H_1}(y_1 \mid H_1) = f_A(a) * f_{W_1}(w_1)$, where * denotes convolution. The convolution of two Gaussian density functions is Gaussian with mean

$$E\left[Y_1 \mid H_1\right] = E\left[A\sqrt{E} + W_1 \mid H_1\right] = 0$$

and variance

$$\begin{aligned} \operatorname{var}\left[Y_1 \mid H_1\right] = \operatorname{var}\left[A\sqrt{E} + W_1 \mid H_1\right] = E\left[\left(A\sqrt{E} + W_1\right)^2 \mid H_1\right] &= E\operatorname{var}[A] + \operatorname{var}[W_1] \\ &= E\sigma_a^2 + \frac{N_0}{2} \end{aligned}$$

since the random variables $A$ and $W_1$ are independent and each with mean zero. Thus, the conditional density function under hypothesis $H_1$ is

$$f_{Y_1|H_1}(y_1 \mid H_1) = \frac{1}{\sqrt{2\pi}\sqrt{E\sigma_a^2 + (N_0/2)}} \exp\left\{-\frac{y_1^2}{2[E\sigma_a^2 + (N_0/2)]}\right\}$$

Applying the likelihood ratio test, we have

$$\Lambda(y_1)=\frac{f_{Y_1|H_1}(y_1\mid H_1)}{f_{Y_1|H_0}(y_1\mid H_0)}=\frac{\dfrac{1}{\sqrt{\pi(2E\sigma_a^2+N_0)}}\exp\left(-\dfrac{y_1^2}{2E\sigma_a^2+N_0}\right)}{\dfrac{1}{\sqrt{\pi N_0}}\exp\left(-\dfrac{y_1^2}{N_0}\right)}\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

Taking the logarithm and rearranging terms, an equivalent test is

$$y_1^2\left[\frac{2E\sigma_a^2}{N_0(2E\sigma_a^2+N_0)}\right]\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta-\frac{1}{2}\ln\frac{N_0}{2E\sigma_a^2+N_0}$$

or

$$y_1^2\underset{H_0}{\overset{H_1}{\gtrless}}\frac{N_0(2E\sigma_a^2+N_0)}{2E\sigma_a^2}\left(\ln\eta-\frac{1}{2}\ln\frac{N_0}{2E\sigma_a^2+N_0}\right)$$

For minimum probability of error, $C_{00}=C_{11}=0$ and $C_{01}=C_{10}=1$, we have $\eta=P_0(C_{10}-C_{00})/P_1(C_{01}-C_{11})=P_0/P_1=1$, since the hypotheses are equally likely. Thus, $\ln\eta=0$, and the optimum decision rule becomes

$$y_1^2\underset{H_0}{\overset{H_1}{\gtrless}}\frac{N_0(2E\sigma_a^2+N_0)}{4E\sigma_a^2}\ln\frac{2E\sigma_a^2+N_0}{N_0}=\gamma$$

The sufficient statistic is $T(\boldsymbol{y})=y_1^2$, and the optimum receiver is as shown in Figure 10.4.

### 10.2.2 General Binary Detection

In this case, the transmitter sends the signal $s_1(t)$ under hypothesis $H_1$ and $s_0(t)$ under hypothesis $H_0$. At the receiver, we have

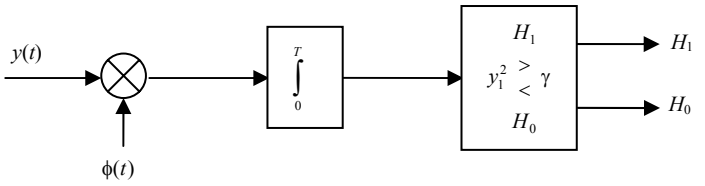


**Figure 10.4** Optimum receiver for Example 10.1.

$$\begin{aligned} H_1 &: Y(t) = s_1(t) + W(t), \quad 0 \le t \le T \\ H_0 &: Y(t) = s_0(t) + W(t), \quad 0 \le t \le T \end{aligned} \tag{10.40}$$

Let the signal $s_0(t)$ and $s_1(t)$ have energies

$$E_0 = \int_0^T s_0^2(t)dt \tag{10.41}$$

and

$$E_1 = \int_0^T s_1^2(t)dt \tag{10.42}$$

and correlation coefficient $\rho, -1 \le \rho \le 1$, such that

$$\rho = \frac{1}{\sqrt{E_0 E_1}} \int_0^T s_0(t) s_1(t)dt \tag{10.43}$$

Following the same procedure as in the previous section, we use the Gram-Schmidt orthogonalization procedure to obtain a complete set of orthonormal functions. The first basis function is

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\int_0^T s_1^2(t)dt}} = \frac{s_1(t)}{\sqrt{E_1}} \tag{10.44}$$

The second basis function $\phi_2(t)$ orthogonal to $\phi_1(t)$ is

$$\phi_2(t) = \frac{f_2(t)}{\sqrt{\int_0^T f_2^2(t)dt}} \tag{10.45}$$

where

$$f_2(t) = s_0(t) - s_{01}\phi_1(t) \tag{10.46}$$

and

$$s_{01} = \int_0^T s_0(t)\phi_1(t)dt \tag{10.47}$$

Substituting (10.44) into (10.47), we obtain

$$s_{01}\frac{1}{\sqrt{E_1}}\int_0^T s_0(t)s_1(t)dt = \rho\sqrt{E_0} \tag{10.48}$$

Thus,

$$f_2(t) = s_0(t) - \rho\sqrt{E_0}\phi_1(t) \tag{10.49}$$

and

$$\phi_2(t) = \frac{1}{\sqrt{E_0(1-\rho^2)}}\left[s_0(t) - \rho\sqrt{E_0}\phi_1(t)\right] \tag{10.50}$$

The remaining $\phi_k(t), k = 3, 4, \ldots,$ needed to complete the orthonormal set can be selected from any set orthogonal to $\phi_1(t)$ and $\phi_2(t)$. In terms of the basis functions, $s_1(t)$ and $s_0(t)$ are

$$s_1(t) = \sqrt{E_1}\phi_1(t) \tag{10.51}$$

$$s_0(t) = \left[\rho\sqrt{E_0}\phi_1(t) + \sqrt{E_0(1-\rho^2)}\phi_2(t)\right] \tag{10.52}$$

The general binary hypothesis problem is now given by

$$\begin{aligned} H_1 &: Y(t) = \sqrt{E_1}\phi_1(t) + W(t) \qquad\qquad , \quad 0 \le t \le T \\ H_0 &: Y(t) = \left[\rho\sqrt{E_0}\phi_1(t) + \sqrt{E_0(1-\rho^2)}\phi_2(t)\right] + W(t), \quad 0 \le t \le T \end{aligned} \tag{10.53}$$

To obtain the random variables $Y_k$, $k = 1, 2, \ldots$, we need to determine Karhunen-Loève coefficients of $Y(t)$. Thus,

$$Y_1 = \begin{cases} H_1 : \int_0^T [Y(t)]\phi_1(t)dt = \int_0^T \sqrt{E_1}\phi_1^2(t)dt + \int_0^T W(t)\phi_1(t)dt \\ H_0 : \int_0^T \left\{\left[\rho\sqrt{E_0}\phi_1(t) + \sqrt{E_0(1-\rho^2)}\phi_2(t)\right] + W(t)\right\}\phi_1(t)dt \end{cases} \tag{10.54}$$

or

$$Y_1 = \begin{cases} H_1 : \sqrt{E_1} \quad + W_1 \\ H_0 : \rho\sqrt{E_0} + W_1 \end{cases} \tag{10.55}$$

since $\int_0^T \phi_1(t)\phi_2(t)dt = 0$ and $\int_0^T \phi_1^2(t)dt = 1$. Also,

$$Y_2 = \begin{cases} H_1 : \int_0^T [Y(t)]\phi_2(t)dt = \int_0^T \sqrt{E_1}\phi_1(t)\phi_2(t)dt + \int_0^T W(t)\phi_2(t)dt \\ H_0 : \int_0^T \left\{\left[\rho\sqrt{E_0}\phi_1(t) + \sqrt{E_0(1-\rho^2)}\phi_2(t)\right] + W(t)\right\}\phi_2(t)dt \end{cases} \tag{10.56}$$

or

$$Y_2 = \begin{cases} H_1 : W_2 \\ H_0 : \sqrt{E_0(1-\rho^2)} + W_2 \end{cases} \tag{10.57}$$

The random variable $Y_k$ for $k > 2$ is not dependent on the choice of the hypotheses, and thus it has no effect on the decision. It is

$$Y_k = \begin{cases} H_1 : \int_0^T \left[\sqrt{E_1}\phi_1(t) + W(t)\right]\phi_k(t)dt = W_k \\ H_0 : \int_0^T \left\{\left[\rho\sqrt{E_0}\phi_1(t) + \sqrt{E_0(1-\rho^2)}\phi_2(t)\right] + W(t)\right\}\phi_k(t)dt = W_k \end{cases} \tag{10.58}$$

Since $W_k$, $k = 1, 2, \ldots$, is a coefficient of Karhunen-Loève expansion of the white Gaussian process with mean zero and power spectral density $N_0/2$, it is a statistically independent Gaussian random variable with mean zero and variance $N_0/2$.

The equivalent problem to (10.40) is now two dimensional, and is given by

$$H_1 : \begin{cases} Y_1 = \sqrt{E_1} + W_1 \\ Y_2 = \quad\quad W_2 \end{cases} \tag{10.59a}$$

$$H_0 : \begin{cases} Y_1 = \rho\sqrt{E_0} \quad\quad + W_1 \\ Y_2 = \sqrt{E_0(1-\rho^2)} + W_2 \end{cases} \tag{10.59b}$$

In vector form, the received vector $\boldsymbol{Y}$ and the signal vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_0$ are

$$\boldsymbol{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \qquad \boldsymbol{s}_1 = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}, \qquad \boldsymbol{s}_0 = \begin{bmatrix} s_{01} \\ s_{02} \end{bmatrix} \tag{10.60}$$

$Y_1$ and $Y_2$ are statistically independent Gaussian random variables with mean vector $\boldsymbol{m}_1$ under hypothesis $H_1$, and mean vector $\boldsymbol{m}_0$ under hypothesis $H_0$, given by

$$\boldsymbol{m}_1 = \begin{bmatrix} m_{11} \\ m_{12} \end{bmatrix} = E[\boldsymbol{Y} \mid H_1] = \begin{bmatrix} \sqrt{E_1} \\ 0 \end{bmatrix} = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix} = \boldsymbol{s}_1 \tag{10.61}$$

and

$$\boldsymbol{m}_0 = \begin{bmatrix} m_{01} \\ m_{02} \end{bmatrix} = E[\boldsymbol{Y} \mid H_0] = \begin{bmatrix} \rho\sqrt{E_0} \\ \sqrt{E_0(1-\rho^2)} \end{bmatrix} = \begin{bmatrix} s_{01} \\ s_{02} \end{bmatrix} = \boldsymbol{s}_0 \tag{10.62}$$

Since the components of $\boldsymbol{Y}$ are uncorrelated, the covariance matrix of $\boldsymbol{Y}$ under each

hypothesis is diagonal and is given by

$$\boldsymbol{C}_1 = \begin{bmatrix} N_0/2 & 0 \\ 0 & N_0/2 \end{bmatrix} = \boldsymbol{C}_0 = \boldsymbol{C} \tag{10.63}$$

Thus, using the results in (9.14a) for diagonal equal covariances, the decision rule is

$$T(\boldsymbol{y}) = \left(\boldsymbol{m}_1^T - \boldsymbol{m}_0^T\right)\boldsymbol{C}^{-1}\boldsymbol{y} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \tag{10.64}$$

where

$$\gamma = \ln\eta + \frac{1}{2}\left(\boldsymbol{m}_1^T\boldsymbol{C}^{-1}\boldsymbol{m}_1 - \boldsymbol{m}_0^T\boldsymbol{C}^{-1}\boldsymbol{m}_0\right) \tag{10.65}$$

and

$$\boldsymbol{C}^{-1} = \begin{bmatrix} 2/N_0 & 0 \\ 0 & 2/N_0 \end{bmatrix} \tag{10.66}$$

Since $\boldsymbol{C}^{-1}$ is also diagonal, the decision rule reduces to

$$T(\boldsymbol{y}) = \boldsymbol{y}^T\left(\boldsymbol{m}_1 - \boldsymbol{m}_0\right) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\eta + \frac{1}{2}\left(\left|\boldsymbol{m}_1\right|^2 - \left|\boldsymbol{m}_0\right|^2\right) = \gamma_1 \tag{10.67}$$

The sufficient statistic is

$$T(\boldsymbol{y}) = \boldsymbol{y}^T\left(\boldsymbol{m}_1 - \boldsymbol{m}_0\right) \tag{10.68}$$

Substituting (10.59) to (10.61) in (10.68), the sufficient statistic can be written as

$$\begin{aligned} T(\mathrm{y}) &= y_1\left(m_{11} - m_{01}\right) - y_2\left(m_{12} - m_{02}\right) = y_1\left(\sqrt{E_1} - \rho\sqrt{E_0}\right) - y_2\sqrt{E_0\left(1-\rho^2\right)} \\ &= \left(\sqrt{E_1} - \rho\sqrt{E_0}\right)\int_0^T y(t)\phi_1(t)dt - \sqrt{E_0\left(1-\rho^2\right)}\int_0^T y(t)\phi_2(t)dt \end{aligned} \tag{10.69}$$

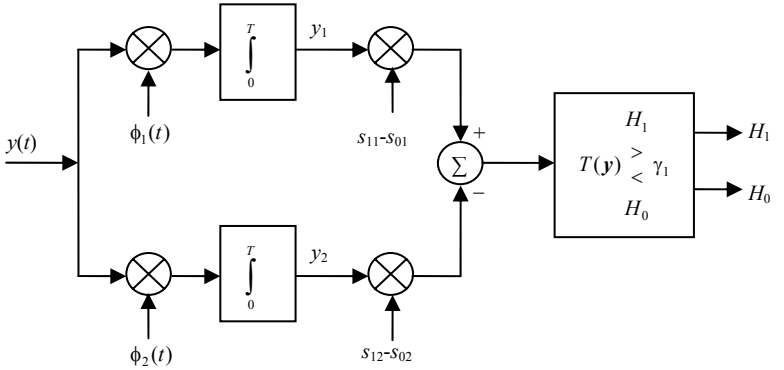


**Figure 10.5** Optimum receiver for general binary detection.

The optimum correlation receiver is shown in Figure 10.5.

This optimum receiver can be implemented in terms of a single correlator. Substituting for the values of $\phi_1(t)$ and $\phi_2(t)$ in (10.69), we have

$$T(\mathrm{y}) = (\sqrt{E_1} - \rho\sqrt{E_0})\int_0^T y(t)\frac{s_1(t)}{\sqrt{E_1}}dt - \sqrt{E_0(1-\rho^2)}\int_0^T y(t)\frac{s_0(t)-\rho\sqrt{E_0}\,\phi_1(t)}{\sqrt{E_0(1-\rho^2)}}dt$$

$$= \int_0^T y(t)[s_1(t)-s_0(t)]dt = \int_0^T y(t)s_\Delta(t)dt \tag{10.70}$$

where

$$s_\Delta(t) = s_1(t) - s_0(t) \tag{10.71}$$

The decision in this case is

$$\int_0^T y(t)\big[s_1(t)-s_0(t)\big]dt \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_1 \tag{10.72}$$

where

$$\gamma_1 = \frac{N_0}{2}\ln\eta + \frac{1}{2}\int_0^T \big[s_1^2(t) - s_0^2(t)\big]dt \tag{10.73}$$

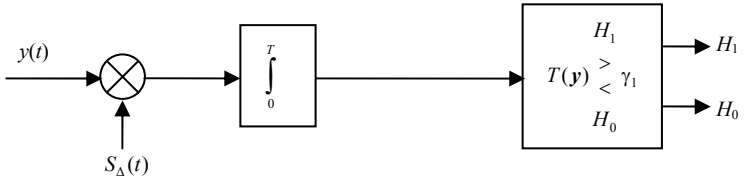


**Figure 10.6** Optimum receiver for general binary detection problem with one correlator.

The corresponding optimum receiver is shown in Figure 10.6.

We now study the performance of this detector. Since the sufficient statistic is Gaussian, we only need to solve for the means and variances under each hypothesis to have a complete description of the conditional density functions. Solving for the means, we have

$$\begin{aligned} E[T(\mathrm{Y}) \mid H_1] &= \left(\sqrt{E_1} - \rho\sqrt{E_0}\right) E[Y_1 \mid H_1] - \sqrt{E_0(1-\rho^2)}\, E[Y_2 \mid H_1] \\ &= \left(\sqrt{E_1} - \rho\sqrt{E_0}\right)\sqrt{E_1} = E_1 - \rho\sqrt{E_0 E_1} \end{aligned} \tag{10.74}$$

and

$$\begin{aligned} E[T(\mathrm{Y}) \mid H_1] &= \left(\sqrt{E_1} - \rho\sqrt{E_0}\right) E[Y_1 \mid H_0] - \sqrt{E_0(1-\rho^2)}\, E[Y_2 \mid H_0] \\ &= \left(\sqrt{E_1} - \rho\sqrt{E_0}\right)\rho\sqrt{E_0} - \sqrt{E_0(1-\rho^2)}\sqrt{E_0(1-\rho^2)} \\ &= \rho\sqrt{E_0 E_1} - E_0 \end{aligned} \tag{10.75}$$

The variances are

$$\begin{aligned} \mathrm{var}[T(\mathrm{Y}) \mid H_1] &= \mathrm{var}[T(\mathrm{Y}) \mid H_0] \\ &= \left(\sqrt{E_1} - \rho\sqrt{E_0}\right)^2 \mathrm{var}[Y_1 \mid H_1] + \left(\sqrt{E_0(1-\rho^2)}\right)^2 \mathrm{var}[Y_2 \mid H_1] \\ &= \left[\left(\sqrt{E_1} - \rho\sqrt{E_0}\right)^2 + \left(\sqrt{E_0(1-\rho^2)}\right)^2\right]\frac{N_0}{2} \\ &= \left(E_1 + E_0 - 2\rho\sqrt{E_0 E_1}\right)\frac{N_0}{2} = \sigma^2 \end{aligned} \tag{10.76}$$

The performance index, after substitution of (10.75) and (10.76), is given by

$$d^2 \triangleq \frac{\{E[T(\mathrm{Y}) \mid H_1] - E[T(\mathrm{Y}) \mid H_0]\}^2}{\mathrm{var}[T(\mathrm{Y}) \mid H_0]} = \frac{2}{N_0}\left(E_1 + E_0 - 2\rho\sqrt{E_1 E_0}\right) \tag{10.77}$$

Therefore, the probability of detection is

$$P_D = \int_{\gamma_1}^{\infty} f_{T|H_1}(t \mid H_1)dt = \int_{\gamma_1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[ -\frac{1}{2} \frac{(t - E_1 + \rho\sqrt{E_0 E_1})^2}{\sigma^2} \right] dt$$

$$= Q\left( \frac{\gamma_1 - E_1 + \rho\sqrt{E_0 E_1}}{\sigma} \right) \tag{10.78}$$

where

$$\gamma_1 = \frac{1}{2}\left( N_0 \ln \eta + E_1 - E_0 \right) \tag{10.79}$$

The probability of false alarm is

$$P_F = \int_{\gamma_1}^{\infty} f_{T|H_0}(t \mid H_0)dt = \int_{\gamma_1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[ -\frac{1}{2} \frac{\left(t - \rho\sqrt{E_0 E_1} + E_0\right)^2}{\sigma^2} \right] dt$$

$$= Q\left( \frac{\gamma_1 + E_0 - \rho\sqrt{E_0 E_1}}{\sigma} \right) \tag{10.80}$$

We get more insight into the performance of this system by assuming that the hypotheses are equally likely, and by using minimum probability of error criterion. In this case,

$$\gamma_1 = \frac{1}{2}\left( E_1 - E_0 \right) \tag{10.81}$$

Define the constant

$$\alpha = E_1 + E_0 - 2\rho\sqrt{E_1 E_0} \tag{10.82}$$

Substituting (10.81) and (10.82) into (10.79) and (10.80), and rearranging terms, we obtain

$$P_F = Q\left( \frac{1}{2}\sqrt{\frac{2\alpha}{N_0}} \right) \tag{10.83}$$

and

$$P_D = Q\left(-\frac{1}{2}\sqrt{\frac{2\alpha}{N_0}}\right) = 1 - Q\left(\frac{1}{2}\sqrt{\frac{2\alpha}{N_0}}\right) \tag{10.84}$$

Since the probability of miss $P_M = 1 - P_D$, then the probability of error is

$$P(\varepsilon) = P_F = P_M = Q\left(\frac{1}{2}\sqrt{\frac{2\alpha}{N_0}}\right) \tag{10.85}$$

We observe that the probability of error decreases as $\alpha$ increases, while $N_0$ is fixed. Thus, from (10.82), the optimum system is obtained when the correlation coefficient $\rho = -1$. In this case, $s_1(t) = -s_0(t)$, and we say that the signals are *antipodal*. If, in addition, the signal energies are equal, $E_0 = E_1 = E$, then the likelihood ratio test is

$$T(\boldsymbol{y}) = \boldsymbol{y}^T(\boldsymbol{m}_1 - \boldsymbol{m}_0) \underset{H_0}{\overset{H_1}{\gtrless}} 0 \tag{10.86}$$

or

$$T(\mathrm{y}) = \sqrt{E}\,(1-\rho)\int_0^T y(t)\phi_1(t)dt - \sqrt{E(1-\rho)^2}\int_0^T y(t)\phi_2(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} 0 \tag{10.87}$$

Substituting for the values of $\phi_1(t)$ and $\phi_2(t)$ in terms of $s_1(t)$ and $s_0(t)$ into (10.85) and simplifying, an equivalent test is

$$\int_0^T y(t)s_1(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \int_0^T y(t)s_0(t)dt \tag{10.88}$$

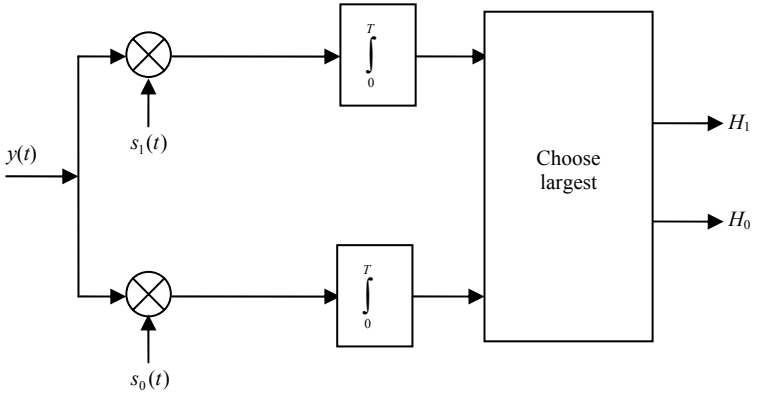


**Figure 10.7** Optimum receiver representing (10.88).

The corresponding receiver is shown in Figure 10.7. The decision rule of (10.88) means that the receiver chooses the signal that has the largest correlation coefficient with the received one.

**Example 10.2**

Consider a communication system with binary transmission during each duration $T_b = 2\pi / \omega_b$ seconds. The transmitted signal under each hypothesis is

$$H_1 : s_1(t) = A \sin \omega_b t \ , \quad 0 \le t \le T_b$$
$$H_0 : s_0(t) = A \sin 2\omega_b t, \quad 0 \le t \le T_b$$

The hypotheses are equally likely. During transmission, the channel superimposes on the signals a white Gaussian noise process with mean zero and power spectral density $N_0 / 2$. Determine the optimum receiver and calculate the probability of error. Assume minimum probability of error criterion.

*Solution*

The received signal is characterized by

$$H_1 : Y(t) = s_1(t) + W(t), \quad 0 \le t \le T_b$$
$$H_0 : Y(t) = s_0(t) + W(t), \quad 0 \le t \le T_b$$

We observe that the signals $s_1(t)$ and $s_0(t)$ are orthogonal with energies

$$E_1 = \frac{A^2 T_b}{2} = \frac{A^2 \pi}{\omega_b} = E_0 = E$$

Thus, the orthonormal basis functions are

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E}} = \sqrt{\frac{2}{T_b}} \sin \omega_b t \quad \text{and} \quad \phi_2(t) = \frac{s_0(t)}{\sqrt{E}} = \sqrt{\frac{2}{T_b}} \sin 2\omega_b t$$

Using (10.54) and (10.56), we obtain the equivalent decision problem

$$Y_1 = \begin{cases} H_1 : \int_0^T [s_1(t) + W(t)]\phi_1(t)dt = \sqrt{E} + W_1 \\ H_0 : \int_0^T [s_0(t) + W(t)]\phi_1(t)dt = W_1 \end{cases}$$

and

$$Y_2 = \begin{cases} H_1 : \int_0^T [s_1(t) + W(t)]\phi_2(t)dt = W_2 \\ H_0 : \int_0^T [s_0(t) + W(t)]\phi_2(t)dt = \sqrt{E} + W_2 \end{cases}$$

Correspondingly, the coefficients of the signal vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_0$ are

$$\boldsymbol{s}_1 = \begin{bmatrix} \sqrt{E} \\ 0 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{s}_0 = \begin{bmatrix} 0 \\ \sqrt{E} \end{bmatrix}$$

Applying the decision rule of (10.67), we have

$$T(\boldsymbol{y}) = \boldsymbol{y}^T(\boldsymbol{s}_1 - \boldsymbol{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}\left(|\boldsymbol{s}_1|^2 - |\boldsymbol{s}_0|^2\right)$$

where $\ln \eta$ is zero, since we are using minimum probability of error criterion and $P_0 = P_1$. Substituting for the values of $\boldsymbol{y}$, $\boldsymbol{s}_1$, and $\boldsymbol{s}_0$, the test reduces to

$$T(\boldsymbol{y}) = y_1 - y_2 \underset{H_0}{\overset{H_1}{\gtrless}} 0 \quad \text{or} \quad y_1 \underset{H_0}{\overset{H_1}{\gtrless}} y_2$$

To determine the probability of error, we need to solve for the mean and variance of the sufficient statistic $T(\boldsymbol{Y}) = Y_1 - Y_2$. Since $Y_1$ and $Y_2$ are uncorrelated Gaussian random variables, $T(\boldsymbol{Y}) = Y_1 - Y_2$ is also Gaussian with means

$$E\left[T(\mathrm{Y}) \mid H_1\right] = E\left[Y_1 - Y_2 \mid H_1\right] = \sqrt{E}$$

$$E\left[T(\mathrm{Y}) \mid H_0\right] = E\left[Y_1 - Y_2 \mid H_0\right] = -\sqrt{E}$$

and variances

$$\mathrm{var}\left[T(\mathrm{Y}) \mid H_1\right] = \mathrm{var}\left[T(\mathrm{Y}) \mid H_0\right] = \mathrm{var}\left[Y_1 \mid H_j\right] + \mathrm{var}\left[Y_2 \mid H_j\right], \quad j = 0,1$$

The variance of $Y_1$ under hypothesis $H_0$ is

$$\begin{aligned}\mathrm{var}\left[Y_1 \mid H_0\right] = E\left[Y_1^2 \mid H_0\right] &= E\left[\int_0^T \int_0^T W(t)\phi_1(t)W(u)\phi_1(u)dtdu\right] \\ &= \int_0^T \int_0^T \phi_1(t)\phi_1(u)E\left[W(t)W(u)\right]dtdu\end{aligned}$$

where

$$E\left[W(t)W(u)\right] = R_{ww}(t,u) = C_{ww}(t,u) = \frac{N_0}{2}\delta(t-u)$$

Thus,

$$\mathrm{var}\left[Y_1 \mid H_0\right] = \frac{N_0}{2}\int_0^T \int_0^T \phi_1(t)\phi_1(u)\delta(t-u)dtdu = \frac{N_0}{2}\int_0^T \phi_1^2(t)dt = \frac{N_0}{2}$$

and

$$\text{var}\left[T(\text{Y}) \mid H_1\right] = \text{var}\left[T(\text{Y}) \mid H_0\right] = N_0$$

The conditional density functions of the sufficient statistic are

$$f_{T|H_1}(t \mid H_1) = \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{1}{2}\frac{\left(t-\sqrt{E}\right)^2}{N_0}\right]$$

$$f_{T|H_0}(t \mid H_0) = \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{1}{2}\frac{\left(t+\sqrt{E}\right)^2}{N_0}\right]$$

The probability of error in this case is

$$P(\varepsilon) = P(\varepsilon \mid H_1)P(H_1) + P(\varepsilon \mid H_0)P(H_0) = P(\varepsilon \mid H_1) = P(\varepsilon \mid H_0)$$
$$= \frac{1}{\sqrt{2\pi N_0}} \int_0^\infty \exp\left[-\frac{1}{2}\frac{\left(t-\sqrt{E}\right)^2}{N_0}\right] dt = Q\left(\sqrt{\frac{E}{N_0}}\right) = Q\left(\sqrt{\frac{A^2 T_b}{2N_0}}\right)$$

The optimum receiver is shown in Figure 10.8.

## 10.3 *M*-ARY DETECTION

We now generalize the concepts developed for binary hypothesis to $M$ hypotheses. In this case, the decision space consists of, at most, $(M-1)$ dimensions.

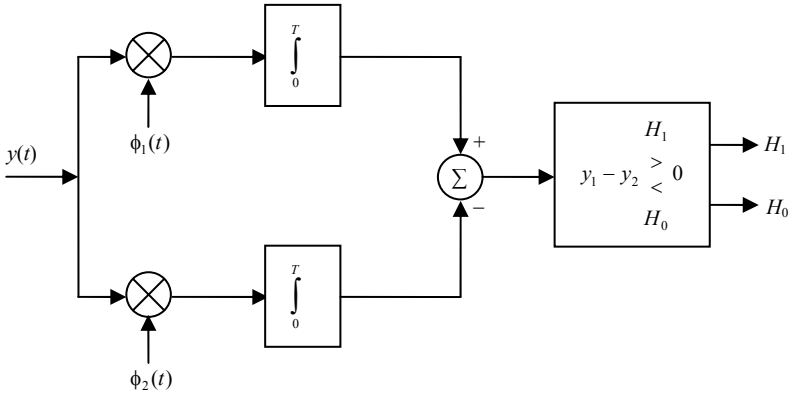


**Figure 10.8** Optimum receiver for Example 10.2.

### 10.3.1 Correlation Receiver

The problem may be characterized as follows

$$H_k : Y(t) = s_k(t) + W(t), \qquad \begin{matrix} 0 \le t \le T \\ k = 1, 2, \ldots, M \end{matrix} \tag{10.89}$$

where $s_k(t)$ is a known deterministic signal with energy $E_k$, such that

$$E_k = \int_0^T s_k^2(t)dt, \quad k = 1, 2, \ldots, M \tag{10.90}$$

and $W(t)$ is an additive white Gaussian noise process with mean zero and power spectral density $N_0 / 2$, or of covariance (autocorrelation) function

$$C_{ww}(t,u) = R_{ww}(t,u) = \frac{N_0}{2}\delta(t-u) \tag{10.91}$$

The *M* signals may be dependent and correlated with autocorrelation coefficients

$$\rho_{jk} = \frac{1}{\sqrt{E_j E_k}} \int_0^T s_j(t) s_k(t)dt, \quad j,k = 1, 2, \ldots, M \tag{10.92}$$

As before, we need to find a set of orthonormal basis functions in order to expand the received process $Y(t)$; that is $W(t)$ into the Karhunen-Loève expansion, since $C_{yy}(t,u) = C_{ww}(t,u)$.

Using the Gram-Schmidt orthogonalization procedure, we can find a set of *K* basis functions, $K \le M$, if only *K* signals $\{s_k(t)\}$ are linearly independent out of the original *M* signals. Once the complete set of *K* orthonormal functions $\{\phi_j(t)\}$, $j = 1, 2, \ldots, K$, are obtained, we generalize the corresponding coefficients by

$$Y_j = \int_0^T Y(t)\phi_j(t)dt, \quad j = 1, 2, \ldots, K \tag{10.93}$$

From (10.29), the signals $s_k(t), k = 1, 2, \ldots, M$, may be written as

$$s_k(t) = \sum_{j=1}^{K} s_{kj}\phi_j(t), \qquad \begin{matrix} 0 \le t \le T \\ k = 1, 2, \ldots, M \end{matrix} \tag{10.94}$$

where $s_{kj}$ is as defined in (8.36). Substituting (10.94) into (10.23), the equivalent $M$-ary decision problem becomes

$$\begin{aligned} H_k : Y_k &= \int_0^T [s(t) + W(t)]\phi_k(t)dt = \int_0^T \left[\sum_{j=1}^{K} s_{kj} + W(t)\right]\phi_k(t)dt \\ &= \begin{cases} s_{kj} + W_k, & k = 1, 2, \ldots, K \\ W_k, & k = K+1, K+2, \ldots \end{cases} \end{aligned} \tag{10.95}$$

We observe that $Y_k$ is a statistically independent Gaussian random variable with variance $N_0/2$, and that only the first $K$ terms affect the decision, since for $k > K$ the coefficients are $W_k$, irrespective of the hypothesis considered. That is, we have reduced the decision space to $K, K \le M$. The mean of the first $K$ coefficients under each hypothesis is

$$E[Y_k \mid H_j] = m_{kj} = s_{kj}, \qquad \begin{matrix} j = 1, 2, \ldots, M \\ k = 1, 2, \ldots, K \end{matrix} \tag{10.96}$$

whereas, for $k > K$, the mean is

$$E[Y_k \mid H_k] = E[W_k] = 0 \tag{10.97}$$

From (5.56), we have seen that the optimum decision is based on the computation of the a posteriori probability $P(H_j \mid \boldsymbol{Y})$. A decision is made in favor of the hypothesis corresponding to the largest a posteriori probability. Since the set of $K$ statistically independent random variables is described by the joint density function

$$\begin{aligned} f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j) &= \prod_{k=1}^{K} \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(\boldsymbol{y} - \boldsymbol{m}_k)^2}{N_0}\right] \\ &= \frac{1}{(\pi N_0)^{K/2}} \exp\left[-\frac{1}{N_0}\sum_{k=1}^{K}(y_k - m_{kj})^2\right] \end{aligned} \tag{10.98}$$

and the a posteriori probability on which the decision is based is given by

$$P(H_j \mid \boldsymbol{Y}) = \frac{P(H_j) f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j)}{f_{\boldsymbol{Y}}(\boldsymbol{y})} \tag{10.99}$$

the sufficient statistic can be expressed as

$$T_j(\boldsymbol{y}) = P_j f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j), \quad j = 1, 2, \ldots, M \tag{10.100}$$

Note that $f_{\boldsymbol{Y}}(\boldsymbol{y})$, which is the denominator of (10.99), is common to all signals, and hence it does not affect the decision and need not be included in the computation. Substituting (10.98) into (10.100) and taking the logarithm, an equivalent sufficient statistic is

$$T_j^1(\boldsymbol{y}) = \ln P_j - \frac{1}{N_0} \sum_{k=1}^{K} \left( y_k - m_{kj} \right)^2, \quad j = 1,2, \ldots, M \tag{10.101}$$

where

$$T_j^1(\boldsymbol{y}) = T_j(\boldsymbol{y}) + \ln\left(\pi N_0\right)^{K/2} \tag{10.102}$$

The 1 of $T_j^1(\boldsymbol{y})$ is a superscript. From (10.99) and (10.97), the signal vector is equal to the mean vector. That is,

$$\boldsymbol{s}_j = \begin{bmatrix} s_{1j} \\ s_{2j} \\ \vdots \\ s_{Kj} \end{bmatrix} = E[\boldsymbol{Y} \mid H_j] = \boldsymbol{m}_j = \begin{bmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{Kj} \end{bmatrix}, \quad j = 1, 2, \ldots, M \tag{10.103}$$

We observe that if the hypotheses are equally likely, $P_j = 1/M$ for all $j$, then (10.100) means to compute $f_{\boldsymbol{Y}|H_j}(\boldsymbol{y} \mid H_j)$ and select the maximum. That is, the MAP criterion is reduced to the ML criterion. The sufficient statistic reduces to

$$T_j^2(\boldsymbol{y}) = -\sum_{k=1}^{K} \left( y_k - m_{kj} \right)^2 = -\sum_{k=1}^{K} \left( y_k - s_{kj} \right)^2 = \left| \boldsymbol{y} - \boldsymbol{s}_j \right|^2, \quad j = 1, 2, \ldots M \tag{10.104}$$

where

$$T_j^2(\boldsymbol{y}) = N_0\left[T_j^1(\boldsymbol{y}) + \ln M\right] \tag{10.105}$$

and 2 of $T_j^2(\boldsymbol{y})$ is a superscript. In other words, the receiver decides in favor of the signal that maximizes the metric. Dropping the minus sign in (10.104) means that the receiver computes $\sum_{k=1}^{K}(y_k - s_{kj})^2$ and decides in favor of the signal with the smallest distance.

The computation of the decision random variables given by the sufficient statistic in (10.105) can be simplified if the signals transmitted have equal energy. The equivalent sufficient statistic is (see Problem 10.11)

$$T_j^3(\boldsymbol{y}) = \boldsymbol{s}_j^T \boldsymbol{Y} = \int_0^T s_j(t)Y(t)dt, \quad j = 1,2,\ldots,M \tag{10.106}$$

where the 3 of $T_j^3(\boldsymbol{y})$ is a superscript. The optimum receiver computes the decision variables from (10.106) and decides in favor of one. This receiver is referred to as the "largest of" receiver and is shown in Figure 10.9.

*Probability of Error of M-Orthogonal Signals*

We have seen that when all hypotheses are equally likely and when all signals have equal energy *E*, the optimum receiver is the "largest of" receiver, as shown in

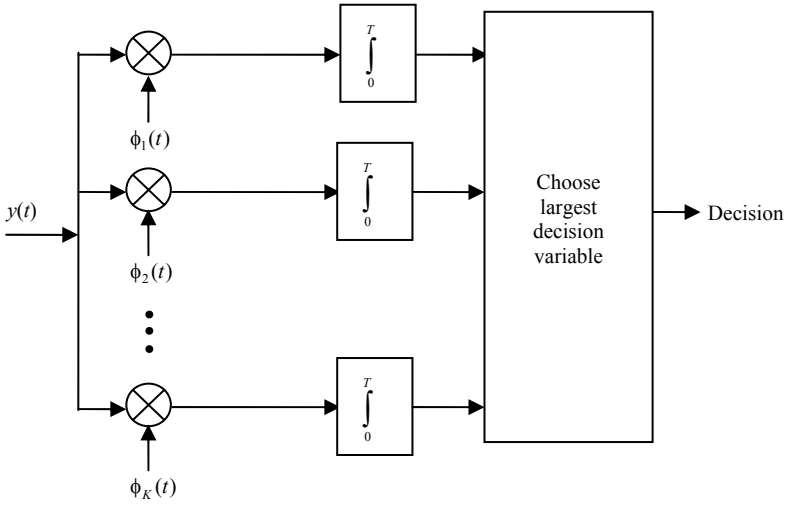


**Figure 10.9** "Largest of" receiver.

Figure 10.9, which computes the sufficient statistics given in (10.106) and decides in favor of the hypothesis with the largest $T_j$. The probability of error is given by

$$P(\varepsilon) = P_1 P(\varepsilon \mid H_1) + P_2 P(\varepsilon \mid H_2) + \ \ldots \ + P_M P(\varepsilon \mid H_M) \tag{10.107}$$

Assuming $H_1$ is true, it is easier to calculate $P(\varepsilon)$ using the complement. Thus,

$$P(\varepsilon) = 1 - P_c = 1 - P(\text{all } T_k < T_1 \ , k = 2, 3, \ldots, M \mid H_1) \tag{10.108}$$

where $P_c$ is the probability of a correct decision. A correct decision for $H_1$ means that the receiver decides $H_1$ ($T_1 > T_k$ for all $k \neq 1$) when $H_1$ is transmitted.

Since the variables $Y_k$ , $k = 1, 2, \ldots, M$ , are Gaussian and uncorrelated, the sufficient statistics are also Gaussian and uncorrelated, and thus statistically independent. They are given by

$$T_k = \int_0^T s_k(t)\left[s_1(t) + W(t)\right]dt = \begin{cases} E + W_1, & k = 1 \\ W_k, & k = 2, 3, \ldots, M \end{cases} \tag{10.109}$$

The mean and variance for $T_k$ , $k = 1, 2, \ldots, M$ , under hypothesis $H_1$ are

$$E\left[T_k \mid H_1\right] = \begin{cases} E, & k = 1 \\ 0, & k = 2, 3, \ldots, M \end{cases} \tag{10.110}$$

and

$$\text{var}\left[T_k \mid H_1\right] = \frac{N_0}{2} \quad \text{for all } k \tag{10.111}$$

Hence, the conditional density functions of the sufficient statistics are

$$f_{T_1|H_1}(t_1 \mid H_1) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(t_1 - E)^2}{N_0}\right] \tag{10.112}$$

and

$$f_{T_k|H_1}(t_k \mid H_1) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{t_k^2}{N_0}\right), \quad k = 2, 3, \ldots, M \tag{10.113}$$

The probability of error is given by

$$P(\varepsilon) = 1 - P_c \tag{10.114}$$

where $P_c$ is given by

$$\begin{aligned} P_c &= P\left(T_2 < T_1, T_3 < T_1, \ldots, T_M < T_1 \mid H_1\right) \\ &= P\left(T_2 < T_1 \mid H_1\right) P\left(T_3 < T_1 \mid H_1\right) \ldots P\left(T_M < T_1 \mid H_1\right) \end{aligned} \tag{10.115}$$

Given a value of the random variable $T_1$, we have

$$P\left(T_k < t_1, k = 2, 3, \ldots, M \mid H_1\right) = \left[ \int_{-\infty}^{t_1} f_{T_k \mid H_1}(t_k \mid H_1) dt_k \right]^{M-1} \tag{10.116}$$

Averaging all possible values of $T_1$, the probability of a correct decision is

$$P_c = \int_{-\infty}^{\infty} f_{T_1 \mid H_1}(t_1 \mid H_1) \left[ \int_{-\infty}^{t_1} f_{T_k \mid H_1}(t_k \mid H_1) dt_k \right]^{M-1} dt_1 \tag{10.117}$$

Thus, $P(\varepsilon)$ is obtained to be

$$P_c = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{\infty} \left[ 1 - Q\left( t_1 \sqrt{\frac{2}{N_0}} \right) \right]^{M-1} \exp\left[ -\frac{(t_1 - E)^2}{N_0} \right] dt_1 \tag{10.118}$$

**Example 10.3**

A signal source generates the following waveforms

$$\begin{aligned} s_1(t) &= \cos \omega_c t \quad , \quad 0 \le t \le T \\ s_2(t) &= \cos\left( \omega_c t + \frac{2\pi}{3} \right), \quad 0 \le t \le T \\ s_3(t) &= \cos\left( \omega_c t - \frac{2\pi}{3} \right), \quad 0 \le t \le T \end{aligned}$$

where $\omega_c = 2\pi / T$. During transmission, the channel superimposes on the signal a Gaussian noise with mean zero and power spectral density $N_0 / 2$. Determine the

optimum receiver, and show the decision regions on the signal space. Assume that the signals are equally likely and minimum probability of error criterion.

*Solution*

We observe that the three signals $s_1(t)$, $s_2(t)$, and $s_3(t)$ have equal energy $E = T/2$. Let the first basis function be

$$\phi_1 = \frac{s_1(t)}{\sqrt{E}} = \sqrt{\frac{2}{T}} \cos \omega_c t, \quad 0 \le t \le T$$

Using trigonometric identities, $s_2(t)$ and $s_3(t)$ can be written as

$$s_2(t) = \cos\left(\omega_c t + \frac{2\pi}{3}\right) = \cos(\omega_c t)\cos\left(\frac{2\pi}{3}\right) - \sin(\omega_c t)\sin\left(\frac{2\pi}{3}\right)$$

$$s_3(t) = \cos\left(\omega_c t - \frac{2\pi}{3}\right) = \cos(\omega_c t)\cos\left(\frac{2\pi}{3}\right) + \sin(\omega_c t)\sin\left(\frac{2\pi}{3}\right)$$

where $\cos(2\pi/3) = -1/2$ and $\sin(2\pi/3) = \sqrt{3/2}$. By inspection, $k = 2$ orthonormal functions are needed to span the signal set. Hence,

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos \omega_c t, \quad 0 \le t \le T$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin \omega_c t, \quad 0 \le t \le T$$

The optimum receiver is the "largest of " receiver, as shown in Figure 10.10. In terms of the basis functions, the signal set $\{s_k(t)\}$ may be expressed as

$$s_1(t) = \sqrt{\frac{T}{2}}\phi_1(t)$$

$$s_2(t) = -\frac{1}{2}\sqrt{\frac{T}{2}}\phi_1(t) - \frac{1}{2}\sqrt{\frac{3T}{2}}\phi_2(t)$$

$$s_3(t) = -\frac{1}{2}\sqrt{\frac{T}{2}}\phi_1(t) + \frac{1}{2}\sqrt{\frac{3T}{2}}\phi_2(t)$$

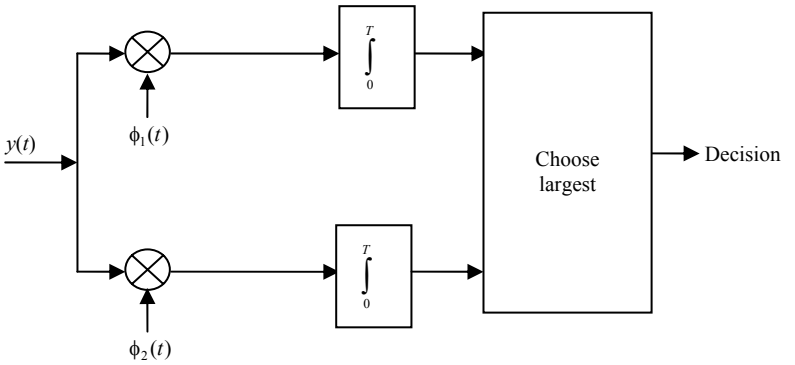


**Figure 10.10** Optimum receiver for Example 10.3.

The signal constellation and the decision regions are shown in Figure 10.11.

**Example 10.4**

Consider the problem given in Example 10.3, assuming the signal set

$$s_k = A\sin\left[\omega_c t + (k-1)\frac{\pi}{2}\right], \quad \begin{matrix} 0 \le t \le T \\ k = 1, 2, 3, 4 \end{matrix}$$

*Solution*

Using trigonometric identities, $s_k(t)$ can be written as

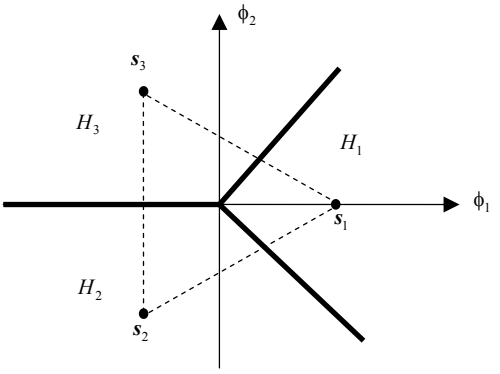


**Figure 10.11** Decision regions for Example 10.3.

$$s_k(t) = A\sin(\omega_c t)\cos\left[(k-1)\frac{\pi}{2}\right] + A\cos(\omega_c t)\sin\left[(k-1)\frac{\pi}{2}\right], \quad k = 1, 2, 3, 4$$

or $s_1(t) = A\sin\omega_c t$, $s_2(t) = A\cos\omega_c t$, $s_3(t) = -A\sin\omega_c t$, and $s_4(t) = -A\cos\omega_c t$

The signals have equal energy $E = A^2T/2$. By inspection, $K = 2$ orthonormal functions are needed to span the signal set $\{s_k(t)\}$, $k = 1, 2, 3, 4$. Thus, we have

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos\omega_c t \quad \text{and} \quad \phi_2(t) = \sqrt{\frac{2}{T}}\sin\omega_c t \quad \text{for} \quad 0 \le t \le T$$

Again, since the signals have equal energy and are equally likely, the optimum receiver is the "largest of " receiver, and the decision regions, which are based on the "nearest neighbor" rule, are shown in Figure 10.12.

Note that a rotation of the signal set does not affect the probability of error. For convenience, let the new signal set be as shown in Figure 10.13. Assuming that the signal $s_1(t)$ is transmitted, the probability of error is

$$P(\varepsilon \mid H_1) = P(\boldsymbol{Y} \text{ falls outside first quadrant} \mid H_1)$$

Due to symmetry and the fact that $P_j = 1/4$ for $j = 1, 2, 3, 4$,

$$P(\varepsilon \mid H_1) = P(\varepsilon \mid H_2) = P(\varepsilon \mid H_3) = P(\varepsilon \mid H_4) = P(\varepsilon)$$

$Y_1$ and $Y_2$ are statistically independent Gaussian random variables with means

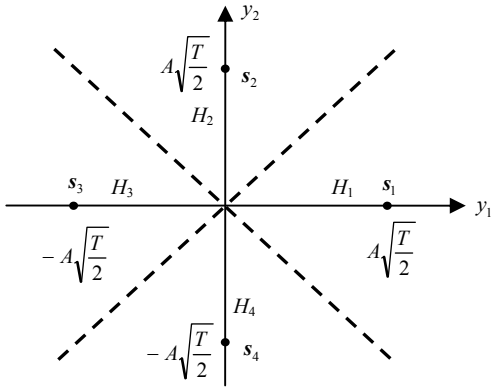


**Figure 10.12** Decision space for Example 10.4.

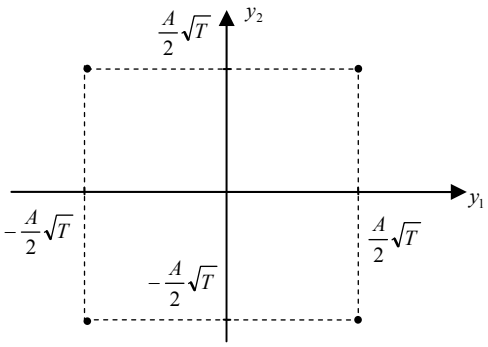


**Figure 10.13** Signal set for Example 10.4 after rotation.

$$E[Y_1 \mid H_1] = E[Y_2 \mid H_1] = \frac{A}{2}\sqrt{T}$$

and variances

$$\text{var}[Y_1 \mid H_1] = \text{var}[Y_2 \mid H_1] = \frac{N_0}{2}$$

Therefore,

$$P_c = P(0 \le Y_1 \le \infty)P(0 \le Y_2 \le \infty) = \left\{ \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left[ -\frac{\left( y - \frac{A}{2}\sqrt{T} \right)^2}{N_0} \right] dy \right\}^2$$

$$= \left[ Q\left( -\frac{A}{2}\sqrt{\frac{2T}{N_0}} \right) \right]^2 = \left[ 1 - Q\left( \frac{A}{2}\sqrt{\frac{2T}{N_0}} \right) \right]^2$$

and the probability of error is

$$P(\varepsilon) = 1 - \left[ 1 - Q\left( \frac{A}{2}\sqrt{\frac{2T}{N_0}} \right) \right]^2$$

### 10.3.2 Matched Filter Receiver

The sufficient statistic given by (10.101) in the previous section using a correlation receiver can also be obtained using a matched filter. The matched filter is a particularly important topic in detection theory either for communication or radar applications. The output signal-to-noise ratio (SNR) is an efficient measure of the system. Instead of using a bank of $K$ correlators, as shown in Figure 10.9, we use $K$ matched filters, as shown in Figure 10.14. The impulse responses of the $K$ filters are

$$h_k(t) = \phi_k(T-t), \qquad \begin{matrix} 0 \le t \le T \\ k = 1, 2, \ldots, K \end{matrix} \tag{10.119}$$

where $\{\phi_k(t)\}$ form the set of basis functions.

If $s(t)$ is the input to a linear filter with impulse response $h(t)$, as shown in Figure 10.15, the output $y(t)$ is just the convolution of $s(t)$ and $h(t)$ to yield

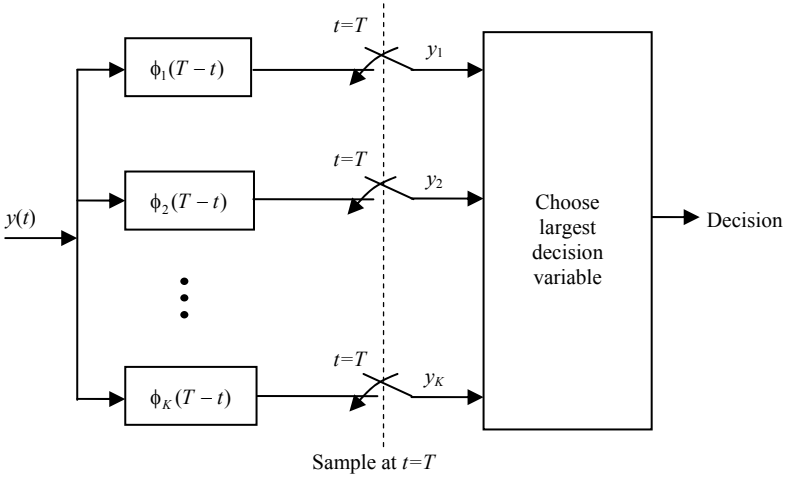


**Figure 10.14** Matched filter receiver.

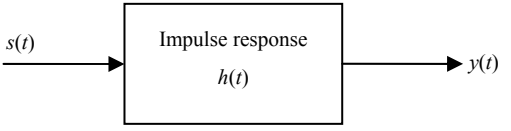


**Figure 10.15** Linear filter.

$$y(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau \tag{10.120}$$

If $h(t)$ is as given by (10.119), the resulting filter output is

$$y(t) = \int_{-\infty}^{\infty} s(\tau)\phi(T-t+\tau)d\tau \tag{10.121}$$

Sampling at $t = T$, we obtain

$$y(T) = \int_{-\infty}^{\infty} s(\tau)\phi(\tau)d\tau = \int_{0}^{T} s(\tau)\phi(\tau)d\tau \tag{10.122}$$

since $\phi(t)$ is zero outside the interval $0 \le t \le T$. A filter whose impulse response $h(t) = s(T-t)$ is a time-reversed and delayed version of a signal $s(t)$, $0 \le t \le T$, is said to be *matched* to the signal $s(t)$. Correspondingly, the optimum receiver shown in Figure 10.14 is referred to as the *matched filter receiver,* since the $K$ matched filters are matched to the basis functions $\{\phi_k(t)\}$ and generate the observation variables $Y_1, Y_2, \ldots, Y_K$.

*Maximization of Output Signal-to-Noise Ratio*

Consider the system shown in Figure 10.16 with a known input $s(t)$, impulse response $h(t)$, and an additive white Gaussian noise $W(t)$ of mean zero and power spectral density $N_0/2$. The input is

$$X(t) = s(t) + W(t), \quad 0 \le t \le T \tag{10.123}$$

The resulting output $Y(t)$ of the linear filter may be expressed as

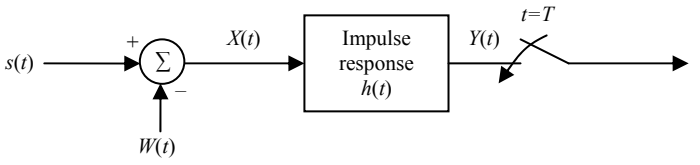


**Figure 10.16** System for derivation of matched filter.

$$Y(t) = s_0(t) + W(t) \tag{10.124}$$

where $s_0(t)$ and $W(t)$ are produced by the signal and noise components of the input $X(t)$, respectively. The largest *output signal-to-noise ratio* is defined at the sampling time $t = T$ as

$$d_{out}^2 = \mathrm{SNR}_0 = \frac{[s_0(T)]^2}{E\left[W^2(t)\right]} \tag{10.125}$$

Note that the denominator of (10.125) is actually the variance of the noise. We now show that maximization of the SNR occurs when the filter is matched to the input known signal $s(t)$.

Let $S(f)$ and $H(f)$ denote the Fourier transforms of $s(t)$ and $h(t)$, respectively. Then, $s_0(t)$ can be written in terms of the inverse Fourier transform to be

$$s_0(t) = \int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi ft}\,df \tag{10.126}$$

At sampling time $t = T$, we may write

$$\left|s_0(T)\right|^2 = \left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}\,df\right|^2 \tag{10.127}$$

Evaluating the output average power of noise, we have

$$E\left[W^2(t)\right] = \int_{-\infty}^{\infty} S_{w_0 w_0}(f)df = \frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df \tag{10.128}$$

Substituting (10.127) and (10.128) into (10.125), we obtain

$$d_{out}^2 = \mathrm{SNR}_0 \frac{\left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}\,df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df} \tag{10.129}$$

Using the Schwarz inequality for the numerator of (10.129), we have

$$\left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}\,df\right|^2 \le \int_{-\infty}^{\infty}|S(f)|^2\,df\int_{-\infty}^{\infty}|H(f)|^2\,df \tag{10.130}$$

The output SNR becomes

$$d_{out}^2 = \mathrm{SNR}_0 \le \frac{2}{N_0}\int_{-\infty}^{\infty}|S(f)|^2\,df \tag{10.131}$$

We observe that the right-hand side of (10.131) does not depend on the transfer function $H(f)$ of the filter, but depends only on the signal energy and noise power spectral density. Hence, the signal-to-noise ratio in (10.131) is maximum when equality holds; that is, we choose $H(f) = H_{opt}(f)$ so that

$$[\mathrm{SNR}_0]_{\max} = \frac{2}{N_0}\int_{-\infty}^{\infty}|S(f)|^2\,df \tag{10.132}$$

Again, using the Schwarz inequality, the optimum value of the transfer function is defined as

$$H_{opt}(f) = S^*(f)e^{-j2\pi fT} \tag{10.133}$$

where $S^*(f)$ is the complex conjugate of the Fourier transform of the input signal $s(t)$. For a real valued signal, $S^*(f) = S(-f)$ and the impulse response of the optimum filter (10.133) is then

$$h_{opt}(t) = \int_{-\infty}^{\infty} S(-f)e^{-j2\pi f(T-t)}\,dt = s(T-t) \tag{10.134}$$

which is a time-reversed and delayed version of the input signal $s(t)$, and thus matched to the input signal.

**Example 10.5**

Let $s_1(t)$ and $s_2(t)$ be two signals as shown in Figure 10.17, which are used to transmit a binary sequence.

(a) Sketch the matched filters.

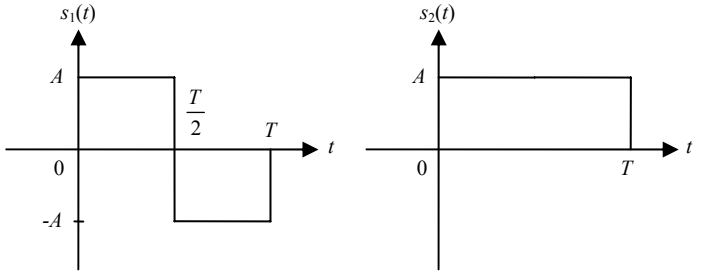


**Figure 10.17** Signals $s_1(t)$ and $s_2(t)$ for Example 10.5.

(b) Determine and sketch the response to $s_2(t)$ of the matched filter.

*Solution*

(a) The matched filters to the signals $s_1(t)$ and $s_2(t)$ are $h_1(t) = s_1(T-t)$ and $h_2(t) = s_2(T-t)$, as shown in Figure 10.18.

(b) The output to the input $s_2(t)$ is $y_2(t) = s_2(t) * h_2(t)$. Solving the convolution, we obtain

$$y_2(t) = \begin{cases} \dfrac{A^2}{2}t, & 0 \le t \le T \\ A^2\left(T - \dfrac{t}{2}\right), & T \le t \le 2T \\ 0, & \text{otherwise} \end{cases}$$

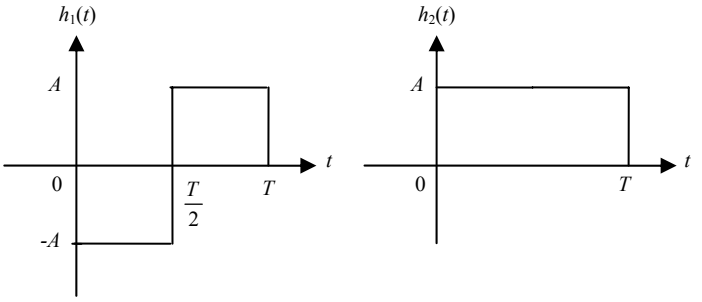


**Figure 10.18** Matched filters to $s_1(t)$ and $s_2(t)$.

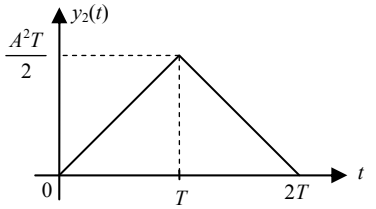


**Figure 10.19** Response $y_2(t)$ of matched filter.

which is shown in Figure 10.19. We observe that the maximum of the response is at the sampling time $t = T$ .

## 10.4 LINEAR ESTIMATION

In Chapter 6, we studied some techniques for parameter estimation in some optimum way, based on a finite number of samples of the signal. In this section, we consider parameter estimation of the signal, but in the presence of an additive white Gaussian noise process with mean zero and power spectral density $N_0 / 2$. The received waveform is of the form

$$Y(t) = s(t, \theta) + W(t), \quad 0 \le t \le T \tag{10.135}$$

where $\theta$ is the unknown parameter to be estimated and $s(t)$ is a deterministic signal with energy $E$. The parameter $\theta$ may be either random or nonrandom. If it is random, we use Bayes estimation; otherwise, we use the maximum likelihood estimation. We assume that $s(t, \theta)$, which is a mapping of the parameter $\theta$ into a time function, is linear. That is, the superposition principle holds, such that

$$s(t, \theta_1 + \theta_2) = s(t, \theta_1) + s(t, \theta_2) \tag{10.136}$$

The estimator of the above-mentioned problem is *linear*, as will be shown later, and thus we refer to the problem as a *linear estimation* problem.

Systems that use linear mappings are known as *linear signaling* or *linear modulation systems*. For such signaling, the received waveform may be expressed as

$$Y(t) = \theta s(t) + W(t), \quad 0 \le t \le T \tag{10.137}$$

We now consider the cases where the parameter is nonrandom and random.

### 10.4.1 ML Estimation

In this case, $\theta$ is a nonrandom parameter. $Y(t)$ may be expressed in a series of orthonormal functions, such that

$$Y(t) = \lim_{K\to\infty} \sum_{k=1}^{K} Y_k \phi_k(t) \tag{10.138}$$

where

$$Y_k = \int_0^T Y(t)\phi_k(t)dt \tag{10.139}$$

and the function $\phi_k$ forms a complete set of orthonormal functions. Thus, the first basis function is

$$\phi_1(t) = \frac{s(t)}{\sqrt{E}} \tag{10.140}$$

Substituting (10.140) into (10.139), with $k > 1$, we obtain

$$Y_k = \int_0^T \left[\theta\sqrt{E}\phi_1(t) + W(t)\right]\phi_k(t)dt = W_k \tag{10.141}$$

which does not depend on the parameter to be estimated, whereas

$$Y_1 = \int_0^T \left[\theta\sqrt{E}\phi_1(t) + W(t)\right]\phi_1(t)dt = \theta\sqrt{E} + W_1 \tag{10.142}$$

depends on $\theta$. Consequently, $Y_1$ is a sufficient statistic. $Y_1$ is a Gaussian random variable with mean $\theta\sqrt{E}$ and variance $N_0/2$.

The likelihood function is

$$L(\theta) = f_{Y_1|\Theta}(y_1 \mid \theta) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{\left(y_1 - \theta\sqrt{E}\right)^2}{N_0}\right] \tag{10.143}$$

We know, from (6.3), that the ML estimate $\hat{\theta}$ is obtained by solving the likelihood

equation. That is,

$$\frac{\partial}{\partial\theta}\ln L(\theta)=\frac{\partial}{\partial\theta}\left[-\frac{1}{2}\ln\pi N_0-\frac{\left(y_1-\theta\sqrt{E}\right)^2}{N_0}\right]=\frac{2\sqrt{E}}{N_0}\left(y_1-\theta\sqrt{E}\right)=0 \qquad (10.144)$$

or

$$\hat{\theta}_{ml}=\frac{Y_1}{\sqrt{E}} \qquad (10.145)$$

Therefore, this optimum estimator is a correlation of the received signal with the signal $s(t)$ normalized as shown in Figure 10.20.

To check if an estimate is "good," we need to compute its bias, error variance or Cramer-Rao bound, and determine its consistency. We observe that $\hat{\theta}_{ml}$ is unbiased since $E[Y_1]=\theta\sqrt{E}$, and thus from (10.145)

$$E\left[\hat{\theta}_{ml}(Y_1)\right]=\frac{1}{\sqrt{E}}E[Y_1]=\theta \qquad (10.146)$$

For an unbiased estimate, the variance of the error is equal to the lower bound of the Cramer-Rao inequality, provided it is efficient. Using (6.50) and (10.144), we have

$$\frac{\partial\ln f_{Y_1|\Theta}(y_1\mid\theta)}{\partial\theta}=\frac{2\sqrt{E}}{N_0}\left(y_1-\theta\sqrt{E}\right)=\frac{2E}{N_0}\left(\frac{y_1}{\sqrt{E}}-\theta\right)$$
$$=c(\theta)\left[\hat{\theta}(y_1)-\theta\right]_{\theta=\hat{\theta}_{ml}} \qquad (10.147)$$

which means that $\text{var}[\hat{\theta}_{ml}-\theta]$ equals the lower bound of the Cramer-Rao inequality given in (6.33).

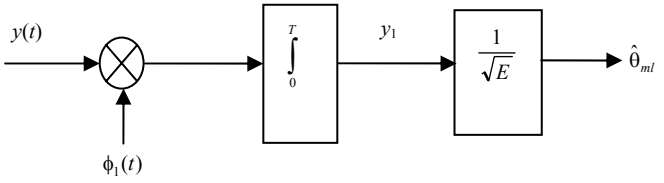


**Figure 10.20** Optimum ML estimator.

### 10.4.2 MAP Estimation

Following the same procedure as in Section 10.4.1, we obtain the sufficient statistic $Y_1$. However, since $\theta$ is now assumed to be a random variable, the MAP estimate is obtained by solving the MAP equation in (6.31). Assume that $\theta$ is Gaussian with mean zero and variance $\sigma_\theta^2$; that is,

$$f_\Theta(\theta) = \frac{1}{\sqrt{2\pi}\sigma_\theta} \exp\left(-\frac{\theta^2}{2\sigma_\theta^2}\right) \tag{10.148}$$

The MAP equation is

$$\frac{\partial \ln f_{\Theta|Y_1}(\theta \mid y_1)}{\partial\theta} = \frac{\partial \ln f_{Y_1|\Theta}(y_1 \mid \theta)}{\partial\theta} + \frac{\partial \ln f_\Theta(\theta)}{\partial\theta} = \frac{2\sqrt{E}}{N_0}\left(y_1 - \theta\sqrt{E}\right) + \frac{\theta}{\sigma_\theta^2} = 0 \tag{10.149}$$

Solving for $\theta$, we obtain the MAP estimate to be

$$\theta_{map}(Y_1) = \frac{2\sqrt{E}/N_0}{\left(2E/N_0\right) + \left(1/\sigma_\theta^2\right)} Y_1 = \alpha\hat{\theta}_{ml} \tag{10.150}$$

where

$$\alpha = \frac{2\sqrt{E}/N_0}{\left(2E/N_0\right) + \left(1/\sigma_\theta^2\right)} \tag{10.151}$$

It is easily shown that the mean-square error of the MAP estimate is equal to the lower bound of the Cramer-Rao inequality; that is,

$$\mathrm{var}\left\{\left[\theta_{map}(Y_1) - \theta\right]^2\right\} = -\frac{1}{E\left[\dfrac{\partial^2 \ln f_{Y_1|\Theta}(y_1 \mid \theta)}{\partial\theta^2}\right]} = \frac{\sigma_\theta^2 N_0}{2\sigma_\theta^2 E + N_0} \tag{10.152}$$

The optimum MAP estimator is shown in Figure 10.21.

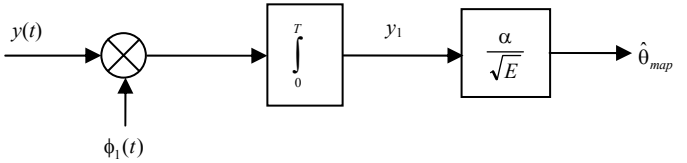


**Figure 10.21** Optimum MAP estimator.

## 10.5 NONLINEAR ESTIMATION

The function $s(t,\theta)$ is now a nonlinear function in $\theta$. Again, $\theta$ may be random or nonrandom.

### 10.5.1 ML Estimation

Let $\{\phi_k(t)\}$ be a set of $K$ orthonormal basis functions. Since we require an infinite number of basis functions to represent $Y(t)$, we approximate the received signal $Y(t)$ as

$$Y(t) = \sum_{k=1}^{K} Y_k \phi_k(t) \tag{10.153}$$

where

$$Y_k = \int_0^T Y(t)\phi_k(t)dt \tag{10.154}$$

Substituting (10.135) into (10.154), we have

$$\begin{aligned} Y_k &= \int_0^T [s(t,\theta) + W(t)]\phi_k(t)dt = \int_0^T s(t,\theta)\phi_k(t)dt + \int_0^T W(t)\phi_k(t)dt \\ &= s_k(\theta) + W_k, \quad k = 1, 2, \ldots, K \end{aligned} \tag{10.155}$$

where

$$s_k(\theta) = \int_0^T s(t,\theta)\phi_k(t)dt \tag{10.156}$$

The $Y_k$ is a statistically independent Gaussian random variable with mean $s_k(\theta)$ and variance $N_0/2$. Thus, the likelihood function, from (6.2), is

$$L(\theta) = f_{\mathrm{Y}|\Theta}(\mathrm{y}\,|\,\theta) = \frac{1}{(\pi N_0)^{K/2}} \prod_{k=1}^{K} \exp\left\{ -\frac{[y_k - s_k(\theta)]^2}{N_0} \right\} \tag{10.157}$$

As $K \to \infty$, (10.157) is not well defined. In fact,

$$\lim_{K\to\infty} = f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta) = \begin{cases} 0 & \text{for } \pi N_0 > 1 \\ \infty & \text{for } \pi N_0 < 1 \end{cases} \tag{10.158}$$

Since the likelihood function is not affected if it is divided by any function that does not depend on $\theta$, we avoid the convergence difficulty of (10.156) by dividing $L(\theta)$ by

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \prod_{k=1}^{K} \frac{1}{\sqrt{\pi N_0}} \exp\left[ -\frac{y_k^2}{N_0} \right] \tag{10.159}$$

Consequently, we define $\Lambda'[\boldsymbol{y}, \theta]$ as

$$\Lambda'[\boldsymbol{y}, \theta] \triangleq \frac{f_{\boldsymbol{Y}|\Theta}(\boldsymbol{y}\,|\,\theta)}{f_{\boldsymbol{Y}}(\boldsymbol{y})} = \exp\left[ \frac{2}{N_0} \sum_{k=1}^{K} y_k s_k(\theta) - \frac{1}{N_0} \sum_{k=1}^{K} s_k^2(\theta) \right] \tag{10.160}$$

The ML estimate is the value of $\theta$ for which $\Lambda_k[\boldsymbol{Y}, \theta]$ is maximum. Using Parseval's theorem and the fact that $\lim_{K\to\infty} y_k(t) = y(t)$ and $\lim_{K\to\infty} s_k(t, \theta) = s(t, \theta)$, we obtain

$$\lim_{K\to\infty} \sum_{k=1}^{K} y_k s_k(\theta) = \int_0^T y(t)\, s(t, \theta)\, dt \tag{10.161}$$

and

$$\lim_{K\to\infty} \sum_{k=1}^{K} s_k^2(\theta) = \int_0^T s^2(t, \theta)\, dt \tag{10.162}$$

Using (10.161) and (10.162), and taking the logarithm as $K \to \infty$, the likelihood function is

$$\ln \Lambda'[Y(t), \theta] = \frac{2}{N_0} \int_0^T Y(t) s(t, \theta) dt - \frac{1}{N_0} \int_0^T s^2(t, \theta) dt \tag{10.163}$$

To obtain the ML estimate $\hat{\theta}_{ml}$, which maximizes the likelihood function, we differentiate (10.163) with respect to $\theta$ and set the result equal to zero. We find that the ML estimate $\hat{\theta}$ is the solution to the equation

$$\int_0^T \left[ Y(t) - s(t, \hat{\theta}) \right] \frac{\partial s(t, \hat{\theta})}{\partial \hat{\theta}} dt = 0 \tag{10.164}$$

Since $\hat{\theta}_{ml}$ is an unbiased estimate, it can be shown that the error variance from the inequality

$$\text{var}\{\hat{\theta}[Y(t)] - \theta\} \geq \frac{N_0}{2 \int_0^T \left[ \frac{\partial s(t, \theta)}{\partial \theta} \right]^2 dt} \tag{10.165}$$

equals the lower bound of the Cramer-Rao if and only if, as $K \to \infty$,

$$\frac{\partial \ln \Lambda' [Y(t), \theta]}{\partial \theta} = c(\theta) \left\{ \hat{\theta}[Y(t)] - \theta \right\} \tag{10.166}$$

**Example 10.6**

Consider a known signal of the form

$$s(t, \theta) = A_c \sin(2\pi f_c t + \theta), \quad 0 \leq t \leq T$$

where the amplitude $A_c$, the frequency $f_c$, $\omega_c = k\pi / T$, and the integer $k$ are known. We wish to estimate the unknown phase $\theta$.

*Solution*

The ML estimate $\hat{\theta}_{ml}$ is the solution to (10.164). That is,

$$\int_0^T \left[Y(t) - A_c \sin(2\pi f_c t + \hat{\theta})\right] \cos(2\pi f_c t + \hat{\theta}) dt = 0$$

or

$$\int_0^T Y(t) \cos(2\pi f_c t + \hat{\theta}) dt = 0$$

since $\int_0^T A_c \sin(2\pi f_c t + \hat{\theta}) \cos(2\pi f_c t + \hat{\theta}) dt = 0$. Using trigonometric identities, we can express the above integral as

$$\cos\hat{\theta} \int_0^T Y(t) \cos(2\pi f_c t) dt - \sin\hat{\theta} \int_0^T Y(t) \sin(2\pi f_c t) dt = 0$$

Solving for $\hat{\theta}$, we obtain the ML estimate to be

$$\hat{\theta} = \tan^{-1} \frac{\int_0^T Y(t) \cos(2\pi f_c t) dt}{\int_0^T Y(t) \sin(2\pi f_c t) dt}$$

### 10.5.2 MAP Estimation

Now $\Theta$ is a random variable with density function $f_\Theta(\theta)$. Following the same approach as in Section 10.5.1, and using the fact that $\hat{\theta}_{map}$ is that value of $\theta$ for which the conditional density function $f_{\Theta|Y}(\theta \mid \boldsymbol{y})$ is maximum,

$$\begin{aligned} \hat{\theta}_{map} &= \frac{\partial \ln \Lambda'[Y(t), \theta]}{\partial \theta} \\ &= \frac{2}{N_0} \int_0^T [Y(t) - s(t, \theta)] \frac{\partial s(t, \theta)}{\partial \theta} dt + \frac{d}{d\theta} \ln f_\Theta(\theta) \end{aligned} \qquad (10.167)$$

If $\theta$ is Gaussian with mean zero and variance $\sigma_\theta^2$, then $d \ln f_\Theta(\theta) / d\theta = -\theta / \sigma_\theta^2$, and the MAP estimate becomes

$$\hat{\theta}_{map} = \frac{2\sigma_\theta^2}{N_0}\int_0^T [Y(t) - s(t,\theta)]\frac{\partial s(t,\theta)}{\partial \theta}\,dt \tag{10.168}$$

## 10.6 GENERAL BINARY DETECTION WITH UNWANTED PARAMETERS

In this section, we consider the general binary detection of signals in an additive white Gaussian noise process with mean zero and power spectral density $N_0/2$. However, the received waveform is not completely known in advance as in the previous section, where we assumed that the only uncertainties were due to additive white Gaussian noise. These signals, which are not completely known in advance, arise in many applications due to factors such as fading, random phase in an echo pulse, and so on. The unknown parameters of the signal are known as *unwanted parameters*.

Consider the general binary detection problem where the received signal under hypotheses $H_1$ and $H_0$ is given by

$$\begin{aligned} H_1 &: Y(t) = s_1(t,\boldsymbol{\theta}_1) + W(t), \quad 0 \le t \le T \\ H_0 &: Y(t) = s_0(t,\boldsymbol{\theta}_0) + W(t), \quad 0 \le t \le T \end{aligned} \tag{10.169}$$

where $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_0$ are the unknown random vectors. Note that if $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_0$ are known, the signals $s_1(t,\boldsymbol{\theta}_1)$ and $s_0(t,\boldsymbol{\theta}_0)$ are deterministics, and thus they are completely specified.

The unknown parameter $\boldsymbol{\theta}_j$, $j = 0,1$, may be either random or nonrandom. In our case, we assume that $\boldsymbol{\theta}_j$, $j = 0,1$, is a random vector with a known a priori density function. That is, the joint density function of the components of $\boldsymbol{\theta}_j$, $j = 0,1$, is known. The approach to solve this problem is to obtain a set of $K$ orthonormal functions $\{\phi_k(t)\}$, approximate $Y(t)$ with the $K$-term series expansion, and let $K \to \infty$. We form the $K$-term approximate to the likelihood ratio, and let $K \to \infty$ to obtain

$$\Lambda[Y(t)] = \lim_{K\to\infty} \Lambda[Y_K(t)] = \frac{f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\,|\,H_1)}{f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\,|\,H_0)} \tag{10.170}$$

where

$$\begin{aligned} f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\,|\,H_1) &= \int_{\chi_{\boldsymbol{\theta}_1}} f_{\boldsymbol{Y},\boldsymbol{\Theta}_1|H_1}(\boldsymbol{y},\boldsymbol{\theta}_1\,|\,H_1)\,d\boldsymbol{\theta}_1 \\ &= \int_{\chi_{\boldsymbol{\theta}_1}} f_{\boldsymbol{Y}|\boldsymbol{\Theta}_1,H_1}(\boldsymbol{y}\,|\,\boldsymbol{\theta}_1,H_1) f_{\boldsymbol{\Theta}_1|H_1}(\boldsymbol{\theta}_1\,|\,H_1)\,d\boldsymbol{\theta}_1 \end{aligned} \tag{10.171}$$

and

$$f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\mid H_0)=\int_{\chi_{\boldsymbol{\theta}_0}} f_{\boldsymbol{Y},\boldsymbol{\Theta}_0|H_0}(\boldsymbol{y},\boldsymbol{\theta}_0\mid H_0)\,d\boldsymbol{\theta}_0$$
$$=\int_{\chi_{\boldsymbol{\theta}_0}} f_{\boldsymbol{Y}|\boldsymbol{\Theta}_0,H_0}(\boldsymbol{y}\mid\boldsymbol{\theta}_0,H_0)f_{\boldsymbol{\Theta}_0|H_0}(\boldsymbol{\theta}_0\mid H_0)d\boldsymbol{\theta}_0 \quad (10.172)$$

where $\chi_{\boldsymbol{\theta}_j}$, $j=0,1$, denotes the space of the parameter $\boldsymbol{\theta}_j$.

We now solve for $f_{\boldsymbol{Y}|\boldsymbol{\Theta}_j,H_j}(\boldsymbol{y}\mid\boldsymbol{\theta}_j,H_j)$, $j=0,1$, under the given conditions. Let

$$Y_K(t)=\sum_{k=1}^{K} Y_k\phi_k(t) \quad (10.173)$$

where

$$Y_k=\int_0^T Y_k\phi_k(t)dt \quad (10.174)$$

The observation vector is

$$\mathrm{Y}_K=\begin{bmatrix}Y_1 & Y_2 & \dots & Y_K\end{bmatrix}^T \quad (10.175)$$

Substituting (10.169) into (10.174), we obtain that $Y_k$ under hypothesis $H_1$ is

$$Y_k=\int_0^T s_1(t,\boldsymbol{\theta}_1)\phi_k(t)dt+\int_0^T W(t)\phi_k(t)dt=s_{k1}+W_k \quad (10.176)$$

while under hypothesis $H_0$ is

$$Y_k=\int_0^T s_0(t,\boldsymbol{\theta}_0)\phi_k(t)dt+\int_0^T W(t)\phi_k(t)dt=s_{k0}+W_k \quad (10.177)$$

Given $\boldsymbol{\theta}_j$, $j=0,1$, $Y_k$ is a statistically independent Gaussian random variable with means

$$E\left[Y_k\mid\boldsymbol{\theta}_1,H_1\right]=s_{k1} \quad (10.178)$$

$$E[Y_k \mid \boldsymbol{\theta}_0, H_0] = s_{k0} \tag{10.179}$$

and variances

$$\text{var}[Y_k \mid \boldsymbol{\theta}_1, H_1] = \text{var}[Y_k \mid \boldsymbol{\theta}_0, H_0] = \frac{N_0}{2} \tag{10.180}$$

Thus, the conditional density functions are

$$f_{\boldsymbol{Y}|\boldsymbol{\Theta}_1, H_1}(\boldsymbol{y} \mid \boldsymbol{\theta}_1, H_1) = \prod_{k=1}^{K} \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(y_k - s_{k1})^2}{N_0}\right] \tag{10.181}$$

$$f_{\boldsymbol{Y}|\boldsymbol{\Theta}_0, H_1}(\boldsymbol{y} \mid \boldsymbol{\theta}_0, H_1) = \prod_{k=1}^{K} \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(y_k - s_{k0})^2}{N_0}\right] \tag{10.182}$$

We observe that $\boldsymbol{\Lambda}[Y_k(t)]$ is the ratio of (10.181) and (10.182). In the limit as $K \to \infty$, the terms in the exponent of (10.181) and (10.182), which can be approximated as summations, become

$$\lim_{K\to\infty} \sum_{k=1}^{K} (y_k - s_{k1})^2 = \int_0^T [y(t) - s_1(t, \boldsymbol{\theta}_1)]^2 \, dt \tag{10.183}$$

and

$$\lim_{K\to\infty} \sum_{k=1}^{K} (y_k - s_{k0})^2 = \int_0^T [y(t) - s_0(t, \boldsymbol{\theta}_0)]^2 \, dt \tag{10.184}$$

Substituting (10.183) and (10.184) into (10.171) and (10.172), respectively, we obtain

$$f_{\boldsymbol{Y}|H_1}(\boldsymbol{y} \mid H_1) = \int_{\chi_{\boldsymbol{\theta}_1}} f_{\boldsymbol{\Theta}_1|H_1}(\boldsymbol{\theta}_1 \mid H_1) \exp\left\{-\frac{1}{N_0} \int_0^T [y(t) - s_1(t, \boldsymbol{\theta}_1)]^2 \, dt\right\} d\boldsymbol{\theta}_1 \tag{10.185}$$

and

$$f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\mid H_0)=\int_{\chi_{\boldsymbol{\theta}_0}} f_{\boldsymbol{\Theta}_0|H_0}(\boldsymbol{\theta}_0\mid H_0)\exp\left\{-\frac{1}{N_0}\int_0^T\left[y(t)-s_0(t,\boldsymbol{\theta}_0)\right]^2 dt\right\}d\boldsymbol{\theta}_0 \tag{10.186}$$

Hence, the likelihood ratio is the ratio of (10.185) and (10.186) to yield

$$\Lambda[Y(t)]=\frac{\int_{\chi_{\boldsymbol{\theta}_1}} f_{\boldsymbol{\Theta}_1|H_1}(\boldsymbol{\theta}_1\mid H_1)\exp\left\{-\frac{1}{N_0}\int_0^T\left[y(t)-s_1(t,\boldsymbol{\theta}_1)\right]^2 dt\right\}d\boldsymbol{\theta}_1}{\int_{\chi_{\boldsymbol{\theta}_0}} f_{\boldsymbol{\Theta}_0|H_0}(\boldsymbol{\theta}_0\mid H_0)\exp\left\{-\frac{1}{N_0}\int_0^T\left[y(t)-s_0(t,\boldsymbol{\theta}_0)\right]^2 dt\right\}d\boldsymbol{\theta}_0} \tag{10.187}$$

### 10.6.1 Signals with Random Phase

We assume that the uncertainty in the received signal is due to a random phase angle, which is probably the most common random signal parameter. Let the two hypotheses be characterized by

$$\begin{aligned} H_1 &: Y(t)=A\cos(\omega_c t+\Theta)+W(t), \quad 0\le t\le T \\ H_0 &: Y(t)= \qquad\qquad\qquad W(t), \quad 0\le t\le T \end{aligned} \tag{10.188}$$

where the amplitude $A$ and frequency $\omega=2\pi f$ are assumed to be known. The phase $\Theta$ is a random variable having an a priori density function

$$f_\Theta(\theta)=\begin{cases}\dfrac{1}{2\pi}, & -\pi\le\theta\le\pi \\ 0\ , & \text{otherwise}\end{cases} \tag{10.189}$$

We observe that $s_1(t,\theta)=A\cos(\omega t+\Theta)$ and $s_0(t)=0$. The goal is to design a receiver that chooses between the two signals $s_1(t)$ or $s_0(t)$. Since $s_0(t,\theta)=0$, the denominator of the likelihood ratio given by (10.186) becomes

$$\int_{\chi_{\boldsymbol{\theta}}} f_{\boldsymbol{\Theta}|H_0}(\boldsymbol{\theta}\mid H_0)\exp\left[-\frac{1}{N_0}\int_0^T y^2(t)dt\right]d\boldsymbol{\theta}$$
$$=\exp\left[-\frac{1}{N_0}\int_0^T y^2(t)dt\right]\int_{\chi_{\boldsymbol{\theta}}} f_{\boldsymbol{\Theta}|H_0}(\boldsymbol{\theta}\mid H_0)\,d\boldsymbol{\theta}$$

$$= \exp\left[ -\frac{1}{N_0} \int_0^T y^2(t)dt \right] \tag{10.190}$$

because $\int_{\chi_{\boldsymbol{\theta}}} f_{\boldsymbol{\Theta}|H_0}(\boldsymbol{\theta} \mid H_0)\, d\boldsymbol{\theta} = 1$. Substituting (10.190) into (10.187) and simplifying, the resulting likelihood ratio is

$$\Lambda[y(t)] = \int_{-\pi}^{\pi} f_{\Theta}(\theta) d\theta \exp\left[ \frac{2}{N_0} \int_0^T y(t) s_1(t,\theta) dt - \frac{1}{N_0} \int_0^T s_1^2(t,\theta) dt \right] \tag{10.191}$$

Solving for the integral between brackets, we obtain

$$\int_0^T A^2 \cos^2(\omega_c t + \theta) dt = \frac{A^2 T}{2} \tag{10.192a}$$

and

$$\begin{aligned} \int_0^T y(t) s_1(t) dt &= A \int_0^T y(t) \cos(\omega_c t + \theta) dt \\ &= A \cos\theta \int_0^T y(t) \cos(\omega_c t) dt - A \sin\theta \int_0^T y(t) \sin(\omega_c t) dt \end{aligned} \tag{10.192b}$$

where we have used

$$\cos(\omega_c t + \theta) = \cos(\omega_c t)\cos\theta - \sin(\omega_c t)\sin\theta \tag{10.192c}$$

For convenience, define the quantities

$$y_c = \int_0^T y(t) \cos\omega_c t \, dt \tag{10.193a}$$

and

$$y_s = \int_0^T y(t) \sin\omega_c t \, dt \tag{10.193b}$$

Substituting (10.192) and (10.193) into (10.191), we have

$$\Lambda[y(t)] = \exp\left(-\frac{A^2T}{2N_0}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[\frac{2A}{N_0}\left(y_c\cos\theta - y_s\sin\theta\right)\right]d\theta$$
$$= \exp\left(-\frac{A^2T}{2N_0}\right)I_0\left(\frac{2A}{N_0}\sqrt{y_c^2+y_s^2}\right) \tag{10.194}$$

where $I_0(\cdot)$ is the modified Bessel function given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[a\cos x + b\sin x]dx = I_0\left(\sqrt{a^2+b^2}\right) \tag{10.195}$$

The likelihood ratio test is then

$$\Lambda[y(t)] = \exp\left(-\frac{A^2T}{2N_0}\right)I_0\left(\frac{2A}{N_0}\sqrt{y_c^2+y_s^2}\right)\underset{H_0}{\overset{H_1}{\gtrless}}\eta \tag{10.196}$$

Taking the natural logarithm, an equivalent test is

$$T_1(y) = \ln I_0\left(\frac{2A}{N_0}\sqrt{y_c^2+y_s^2}\right)\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta + \frac{2N_0}{A^2T} = \gamma_1 \tag{10.197}$$

The optimum receiver computes only the sufficient statistic $T_1(y)$. A possible realization is shown in Figure 10.22.

The Bessel function $I_0(x)$ is a monotonically increasing function of $x$. Recognizing that the plus sign is associated with the square root, the decision may be taken on $x$ or $x^2$. Removing the square root blocks, an equivalent sufficient statistic is

$$T_2(y) = \left(y_c^2+y_s^2\right)\underset{H_0}{\overset{H_1}{\gtrless}}\gamma_2 \tag{10.198}$$

and the alternate realization of the optimum receiver is shown in Figure 10.23.

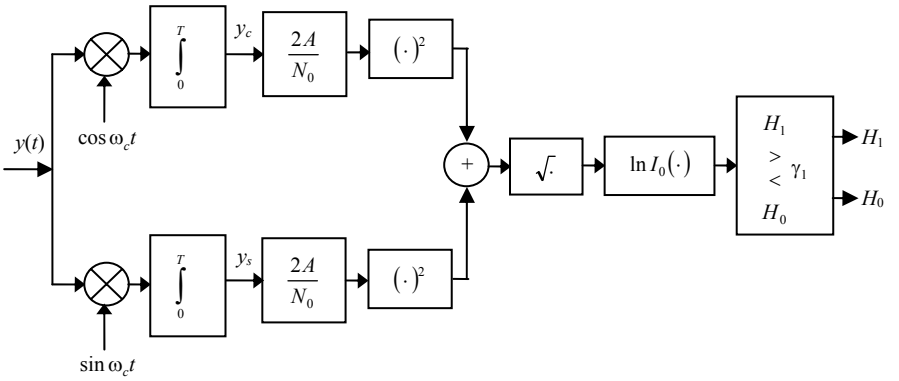


**Figure 10.22** An optimum receiver for the problem stated in (10.188).

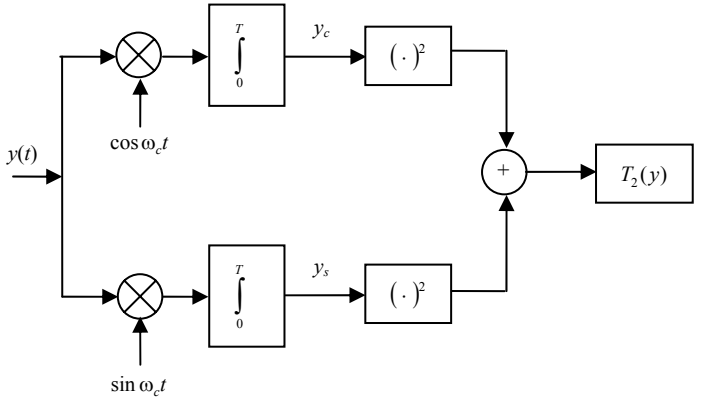


**Figure 10.23** Simplified form of the optimum receiver.

Note that in deriving the decision rule, we kept $y_c$ and $y_s$ as defined in (10.193a) and (10.193b) to show how the quadrature components are used in the decision process. If we now use polar transformations

$$\begin{aligned} y_c &= r\cos\theta \\ y_s &= r\sin\theta \end{aligned} \tag{10.199}$$

such that $\theta = \tan^{-1}(y_c / y_s)$, (10.194) becomes

$$\Lambda[Y(t)] = \exp\left(-\frac{A^2T}{2N_0}\right) I_0\left(\frac{2A}{N_0}r\right) \tag{10.200}$$

This is a "nice" result [1], that will be used in the next section on signals with random phase and amplitude.

*Incoherent Matched Filter*

We now show that the optimum receiver can be implemented in terms of a bandpass filter followed by an envelope detector and a sampler. The combination of the matched filter and envelop detector is often called an *incoherent matched filter*. We observe that by substituting (10.193) in (10.198), the decision rule can be written explicitly as

$$T_2(y) = \left[\int_0^T y(t)\cos\omega_c(t)\right]^2 + \left[\int_0^T y(t)\sin\omega_c(t)\right]^2 \underset{H_2}{\overset{H_1}{\gtrless}} \gamma_2 \tag{10.201}$$

which is the known *quadrature receiver* shown in Figure 10.24.

We have shown in the previous section that a correlator filter is equivalent to a matched filter having an impulse response $h(t) = s(T-t)$, $0 \le t \le T$, followed by a sampler at $t = T$. The incoming signals are in this case $\cos\omega_c t$ and $\sin\omega_c t$, $0 \le t \le T$. Hence, the equivalent quadrature receiver is as shown in Figure 10.25.

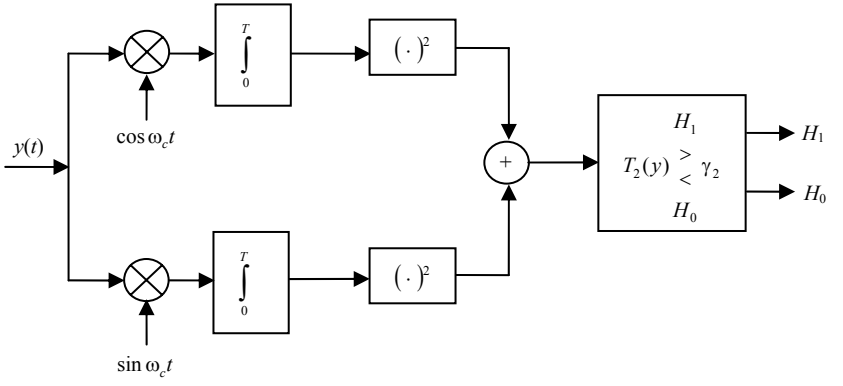


**Figure 10.24** Quadrature receiver.

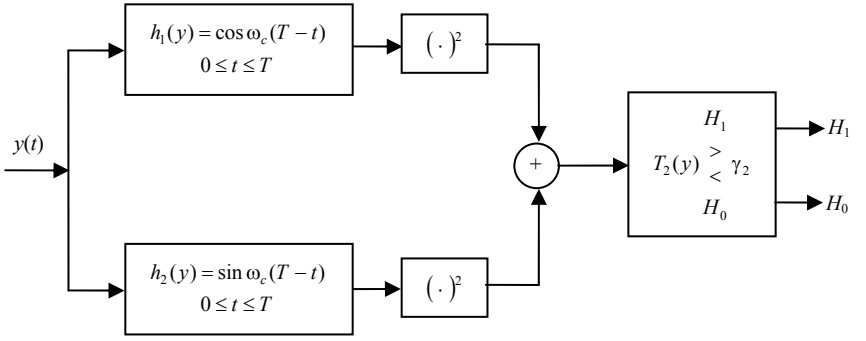


**Figure 10.25** Quadrature receiver using matched filters.

The impulse response of a bandpass filter can be written as

$$h(t) = h_L(t) \cos[\omega_c t + \phi_L(t)] \tag{10.202}$$

where $h_L(t)$ is the lowpass representation of $h(t)$. That is, it is very slowly varying compared to $\cos \omega_c t$. The phase $\phi_L(t)$ is also very slowly varying compared to $\cos \omega_c t$, and it can be shown that setting it equal to zero will not make any difference. Hence, the bandpass filter becomes

$$h(t) = h_L(t) \cos \omega_c t = \Re e\left\{ h_L(t) e^{j\omega_c t} \right\} \tag{10.203}$$

If the input to the bandpass filter is $s(t)$, then the output at time $T$ is

$$y(T) = \int_0^T h(T-t) s(t) dt = \int_0^T s(t) h_L(T-t) \cos(\omega_c T - \omega_c t)\, dt \tag{10.204}$$

Expanding the cosine, we obtain

$$\begin{aligned} y(T) &= \cos \omega_c T \int_0^T s(t) h_L(T-t) \cos \omega_c t\, dt + \sin \omega_c T \int_0^T s(t) h_L(T-t) \sin \omega_c t\, dt \\ &= \cos \omega_c T\, y_c(T) + \sin \omega_c T\, y_s(T) \end{aligned} \tag{10.205}$$

where

$$y_c(T) = \int_0^T s(t)h_L(T-t)\cos\omega_c t\,dt = \Re e\left\{\int_0^T s(t)h_L(T-t)e^{j\omega_c t}\,dt\right\} \quad (10.206a)$$

and

$$y_s(T) = \int_0^T s(t)h_L(T-t)\sin\omega_c t\,dt = \Im m\left\{\int_0^T s(t)h_L(T-t)e^{j\omega_c t}\,dt\right\} \quad (10.206b)$$

Equation (10.203) can be written in terms of the amplitude and phase as

$$y(T) = \sqrt{y_c^2(T) + y_s^2(T)}\cos[\omega_c t + \phi(t)] \quad (10.207a)$$

where

$$\phi(t) = -\tan^{-1}\frac{y_s(T)}{y_c(T)} \quad (10.207b)$$

Let $Z$ be given by

$$Z = \int_0^T s(t)h_L(T-t)e^{j\omega_c t}\,dt \quad (10.208)$$

Then, $|Z| = \sqrt{\Re e^2\{Z\} + \Im m^2\{Z\}}$. We conclude that

$$\sqrt{y_s^2(T) + y_c^2(T)} = |Z| = \left|\int_0^T s(t)h_L(T-t)e^{j\omega_c t}\,dt\right| \quad (10.209)$$

That is, $\sqrt{y_s^2(T) + y_c^2(T)}$ is the envelope of the response at time $T$ and can be obtained by the incoherent matched filter shown in Figure 10.26.

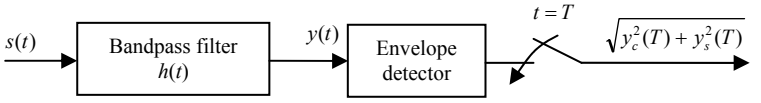


**Figure 10.26** Incoherent matched filter.

Suppose now that our signal is $s_1(t) = A(t)\cos(\omega_c t + \theta)$, $0 \le t \le T$, and the amplitude variation $A(t)$ is slow compared to $\cos \omega_c t$. By definition,

$$y_c = \int_0^T y(t)A(t)\cos \omega_c t \, dt = \Re e\left\{\int_0^T y(t)A(t)e^{j\omega_c t}\, dt\right\} \qquad (10.210a)$$

and

$$y_s = \int_0^T y(t)A(t)\sin \omega_c t \, dt = \Im m\left\{\int_0^T y(t)A(t)e^{j\omega_c t}\, dt\right\} \qquad (10.210b)$$

It follows that

$$\sqrt{y_c^2 + y_s^2} = \left|\int_0^T y(t)A(t)e^{j\omega_c t}\, dt\right| \qquad (10.211)$$

In comparing (10.211) and (10.209), we observe that $\sqrt{y_c^2(T) + y_s^2(T)}$ is identical to $\sqrt{y_c^2 + y_s^2}$ when $A(t) = h_L(T-t)$ or $h_L(t) = A(T-t)$. That is, when the impulse response of the bandpass filter has the envelope matched to the amplitude of the signal, the output of the bandpass filter at time $T$ is

$$T(y)\big|_{t=T} = \sqrt{y_c^2 + y_s^2}\,\cos\left(\omega_c t - \tan^{-1}\frac{y_s}{y_c}\right) \qquad (10.212)$$

Hence, the sufficient statistic is the output of the envelope detector at time $t = T$.

**Example 10.7**

Given the problem in (10.188), and assuming $\Theta$ uniformly distributed over the interval $[-\pi, \pi]$, determine

(a) the probability of false alarm.
(b) the probability of detection.

*Solution*

(a) We found that the optimum receiver is the quadrature receiver with sufficient statistic $T(y) = y_c^2 + y_s^2$, with $y_c$ and $y_s$ as defined in (10.193). $W(t)$ is a white

Gaussian noise with mean zero and power spectral density $N_0/2$. Under $H_0$, $Y(t) = W(t)$, and the probability of false alarm is given by

$$P_F = P[T(y) > \gamma \mid H_0] = P\left[Y_c^2 + Y_s^2 > \gamma \mid H_0\right]$$

From (8.130), $y_c$ and $y_s$ are Gaussian random variables, and thus we need only determine the means and variances. We observe that

$$E[Y_c \mid H_0] = E[Y_s \mid H_0] = 0$$

Also,

$$\begin{aligned}
\text{var}[Y_c \mid H_0] = E\left[Y_c^2 \mid H_0\right] &= \int_0^T \int_0^T \cos \omega_c t E[W(t)W(u)] \cos \omega_c u \, dt \, du \\
&= \frac{N_0}{2} \int_0^T \int_0^T \cos \omega_c t \, \delta(t-u) \cos \omega_c u \, dt \, du \\
&= \frac{N_0}{2} \int_0^T \cos^2 \omega_c t \, dt = \frac{N_0 T}{4} + \frac{N_0}{4} \int_0^T \cos 2\omega_c t \, dt \approx \frac{N_0 T}{4}
\end{aligned}$$

since the integral of the double frequency term is negligible. In a similar manner,

$$\text{var}[Y_s \mid H_0] \approx \frac{N_0 T}{4}$$

We now show that $Y_c$ and $Y_s$ are approximately uncorrelated

$$\begin{aligned}
E[Y_c Y_s \mid H_0] &= \int_0^T \int_0^T \cos \omega_c t E[W(t)W(u)] \sin \omega_c u \, dt \, du \\
&= \frac{N_0}{2} \int_0^T \cos \omega_c t \sin \omega_c t \, dt = \frac{N_0}{4} \int_0^T \sin 2\omega_c t \, dt \approx 0
\end{aligned}$$

since, again, the integral of the double frequency term is negligible. Hence, the joint density function of $Y_c$ and $Y_s$ is

$$f_{Y_c Y_s | H_0}(y_c, y_s \mid H_0) = \frac{1}{2\pi\sigma^2} \exp\left( -\frac{y_c^2 + y_s^2}{2\sigma^2} \right)$$

with $\sigma^2 = N_0 T / 4$. The probability of false alarm is given by

$$P_F = P[T(y) > \gamma \mid H_0] = P\left[Y_c^2 + Y_s^2 > \gamma \mid H_0\right] = \iint_D \frac{1}{2\pi\sigma^2} \exp\left(-\frac{y_c^2 + y_s^2}{2\sigma^2}\right) dy_c\, dy_s$$

where $D$ is the region in the $y_c$-$y_s$ plane outside the circle of radius $\sqrt{\gamma}$, as shown in Figure 10.27.

Using polar coordinates, we have $y_c = r\cos\alpha$, $y_s = r\sin\alpha$, $r^2 = y_c^2 + y_s^2$, $\alpha^2 = \tan^{-1}(y_s / y_c)$, and $dy_c dy_s = r dr d\alpha$. Hence,

$$P_F = \int_{\sqrt{\gamma}}^{\infty} \int_{-\pi}^{\pi} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\, r d\alpha\, dr = \int_{\sqrt{\gamma}}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = e^{-\frac{\gamma}{2\sigma^2}} \tag{10.213}$$

(b) Assuming $\theta$ is known, the probability of detection is given by

$$P_{D|\theta} = P_D(\theta) = P[T(y) > \gamma \mid \theta, H_1] = P\left[Y_c^2 + Y_s^2 > \gamma \mid H_1\right]$$

Under hypothesis $H_1$, $Y(t) = A\cos(\omega_c t + \Theta) + W(t)$. Thus,

$$Y_c = \int_0^T A\cos(\omega_c t + \theta)\cos\omega_c t\, dt + \int_0^T AW(t)\cos(\omega_c t + \theta)\, dt$$

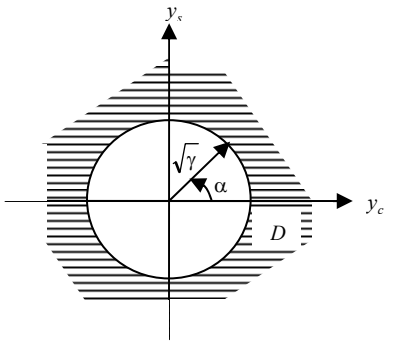


**Figure 10.27** Region $D$ in the $y_c$-$y_s$ plane.

Since $E[W(t)]=0$ , then

$$E[Y_c \mid \theta, H_1]=A\int_0^T \cos(\omega_c t+\theta)\cos\omega_c t\,dt=\frac{AT}{2}\cos\theta+\frac{A}{2}\int_0^T E[\cos(2\omega_c t+\theta)]dt$$

Once again, the double frequency term is negligible, and thus

$$E[Y_c \mid \theta, H_1]=\frac{AT}{2}\cos\theta$$

Similarly, it can be shown that

$$E[Y_s \mid \theta, H_1]=\frac{AT}{2}\sin\theta \quad \text{and} \quad \text{var}[Y_c \mid \theta, H_1]=\text{var}[Y_s \mid \theta, H_1]=\frac{N_0 T}{4}$$

In addition, $Y_c$ and $Y_s$ are jointly Gaussian and statistically independent under the assumption that $\theta$ is known. Hence,

$$f_{Y_c Y_s|\boldsymbol{\Theta},H_1}(y_c, y_s \mid \theta, H_1)=\frac{1}{2\pi\sigma^2}\exp\left\{-\frac{[y_c-(AT/2)\cos\theta]^2+[y_s-(AT)\sin\theta]^2}{2\sigma^2}\right\}$$

The probability of detection is then

$$P_D(\theta)=\iint_{D_1}\frac{1}{2\pi\sigma^2}\exp\left\{-\frac{[y_c-(AT/2)\cos\theta]^2+[y_s-(AT)\sin\theta]^2}{2\sigma^2}\right\}dy_c\,dy_s$$

Using polar coordinates, as in (a) with $y_c=r\cos\alpha$ and $y_s=r\sin\alpha$ , then

$$P_D(\theta)=\int_{\sqrt{\gamma}}^{\infty}\int_{-\pi}^{\pi}\frac{1}{2\pi\sigma^2}\exp\left[-\frac{\left(r\cos\alpha-\frac{AT}{2}\cos\alpha\right)^2+\left(r\sin\alpha-\frac{AT}{2}\sin\alpha\right)^2}{2\sigma^2}\right]r\,dr\,d\alpha$$

Expanding the exponent, $P_D(\theta)$ becomes

$$P_D(\theta) = \int_{\sqrt{\gamma}}^{\infty} \frac{r}{2\pi\sigma^2} \exp\left[ -\left(\frac{AT}{2}\right)^2 \frac{1}{2\sigma^2} \right] e^{-\frac{r^2}{2\sigma^2}} dr \int_{-\pi}^{\pi} \exp\left[ \frac{AT}{2\sigma^2} r\cos(\alpha - \theta) \right] d\theta$$

$$= \int_{\sqrt{\gamma}}^{\infty} \frac{r}{\sigma^2} \exp\left[ -\frac{\left(\frac{AT}{2}\right)^2 + r^2}{2\sigma^2} \right] I_0\left( \frac{AT}{2\sigma^2} r \right) dr \tag{10.214}$$

where

$$\int_{-\pi}^{\pi} \exp\left[ \frac{AT}{2\sigma^2} r\cos(\alpha - \theta) \right] d\theta = 2\pi I_0\left( \frac{AT}{2\sigma^2} r \right)$$

We observe that $P_D(\theta)$ is no longer a function of $\theta$, and thus it does not matter whether or not $\theta$ is known. It follows that $P_D(\theta) = P_D$. Defining

$$d^2 = \frac{(AT/2)^2}{\sigma^2} \tag{10.215a}$$

and

$$z = \frac{r}{\sigma} \tag{10.215b}$$

then

$$P_D = \int_{\sqrt{\gamma}/\sigma}^{\infty} z \exp\left( -\frac{z^2 + d^2}{2} \right) I_0(d\,z)\, dz \tag{10.216}$$

This is Marcum's $Q$-function, where

$$Q(a,b) = \int_{b}^{\infty} z \exp\left( -\frac{z^2 + a^2}{2} \right) I_0(az)\, dz$$

It does not have a closed form, but the integral is evaluated numerically and tabulated. Thus,

$$P_D = Q\left(d, \frac{\sqrt{\gamma}}{\sigma}\right) \tag{10.217}$$

Since $P_F = \exp(-\gamma / 2\sigma^2)$, $\ln P_F = -\gamma / 2\sigma^2$, and the threshold becomes $\gamma = -2\sigma^2 \ln P_F$. Hence, the probability of detection also can be written as $P_D = Q(d, \sqrt{-2 \ln P_F})$, and the ROC can be plotted as shown in Figure 10.28.

### 10.6.2 Signals with Random Phase and Amplitude

In many applications, both the signal amplitude and phase may be random. The received hypotheses are

$$\begin{aligned} H_1 &: Y(t) = S(t) + W(t), \quad 0 \le t \le T \\ H_0 &: Y(t) = W(t), \quad 0 \le t \le T \end{aligned} \tag{10.218}$$

where $S(t) = A\cos(\omega_c t + \Theta)$. The amplitude $A$ and the phase $\Theta$ are random variables, even through they are assumed constant during an observation time interval $[0, T]$. The a priori distributions of the amplitude and phase are assumed to be known. $W(t)$ is the white Gaussian noise with mean zero and power spectral density $N_0 / 2$. Using (10.170) and (10.171), the decision rule is

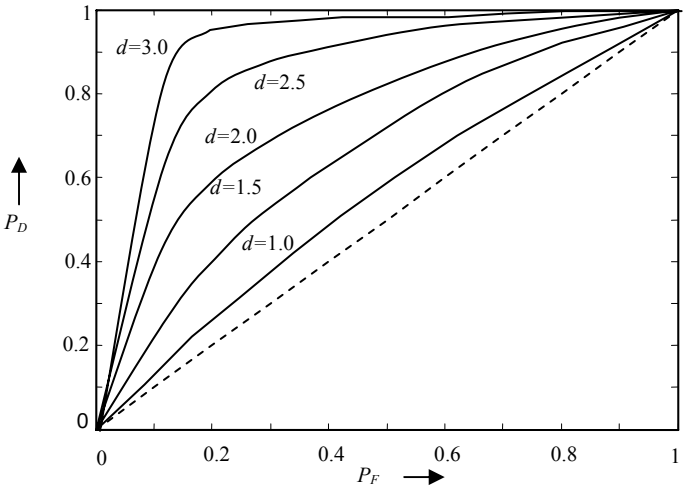


**Figure 10.28** ROC for problem (10.188) with random phase.

$$\Lambda[y(t)] = \frac{\int_{\chi_{\theta_1}} f_{\mathbf{Y}|\boldsymbol{\Theta}_1,H_1}(\boldsymbol{y} \mid \boldsymbol{\theta}_1, H_1) f_{\boldsymbol{\Theta}_1|H_1}(\boldsymbol{\theta}_1 \mid H_1) d\boldsymbol{\theta}_1}{f_{\mathbf{Y}|H_0}(\boldsymbol{y} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \quad (10.219)$$

where the vector $\boldsymbol{\theta}_1$ represents the unknown parameters $a$ and $\theta_1$; that is, $\boldsymbol{\theta}_1 = (a, \theta)$. Since the random variables A and Θ are assumed independent, the likelihood ratio becomes

$$\Lambda[y(t)] = \frac{\int_A \int_{\Theta} f_{\mathbf{Y}|A,\Theta_1,H_1}(\boldsymbol{y} \mid a, \theta, H_1) f_A(a) f_{\Theta}(\theta) da d\theta}{f_{\mathbf{Y}|H_0}(\boldsymbol{y} \mid H_0)} \quad (10.220)$$

Defining the *conditional likelihood ratio* as

$$\Lambda[\mathbf{y} \mid A] = \frac{\int_{\Theta} f_{\mathbf{Y}|A,\Theta_1,H_1}(\mathbf{y} \mid a, \theta, H_1) f_{\Theta}(\theta) da d\theta}{f_{\mathbf{Y}|H_0}(\mathbf{y} \mid H_0)} \quad (10.221)$$

the likelihood ratio is

$$\Lambda[\boldsymbol{y}] = \int_A \Lambda[\boldsymbol{y}|a] f_A(a) da \quad (10.222)$$

Using the result obtained in (10.200), the conditional likelihood ratio is then

$$\Lambda[\boldsymbol{y} \mid a] = \exp\left(-\frac{a^2 T}{2N_0}\right) I_0\left(\frac{2a}{N_0} r\right) \quad (10.223)$$

Assume Θ is uniformly distributed over the interval $[0, 2\pi]$ and the amplitude $A$ has a Rayleigh density function given by

$$f_A(a) = \begin{cases} \dfrac{a}{\sigma_a^2} \exp\left(-\dfrac{a^2}{2\sigma_a^2}\right), & a \geq 0 \\ 0 \qquad\qquad\qquad , & \text{otherwise} \end{cases} \quad (10.224)$$

This is called a *slow Rayleigh fading*, since the channel is considered constant over the signaling interval $T$. Substituting (10.223) and (10.224) in (10.222) and solving the integral, the likelihood ratio is then

$$\Lambda[\boldsymbol{y}] = \int_0^\infty \frac{a}{\sigma_a^2} \exp\left[ -\frac{a^2}{2}\left( \frac{T}{N_0} + \frac{1}{\sigma_a^2} \right) \right] I_0\left( \frac{2a}{N_0} r \right) da$$

$$= \frac{N_0}{N_0 + T\sigma_a^2} \exp\left[ \frac{2\sigma_a^2}{N_0(N_0 + T\sigma_a^2)} r^2 \right] \tag{10.225}$$

Taking the natural logarithm on both sides of (10.225) and rearranging terms, the decision rule is

$$T(y) = r \underset{H_0}{\overset{H_1}{>}} \gamma \tag{10.226}$$

where

$$\gamma = \left\{ \frac{N_0 + (N_0 + T\sigma_a^2)}{2\sigma_a^2} \ln\left[ \frac{\eta\left(N_0 + T\sigma_a^2\right)}{N_0} \right] \right\}^{\frac{1}{2}} \tag{10.227a}$$

and

$$r = \sqrt{y_c^2 + y_s^2} \tag{10.227b}$$

Hence, the optimum receiver is the matched (or correlation) filter followed by an envelope detector, as shown in Figure 10.29.

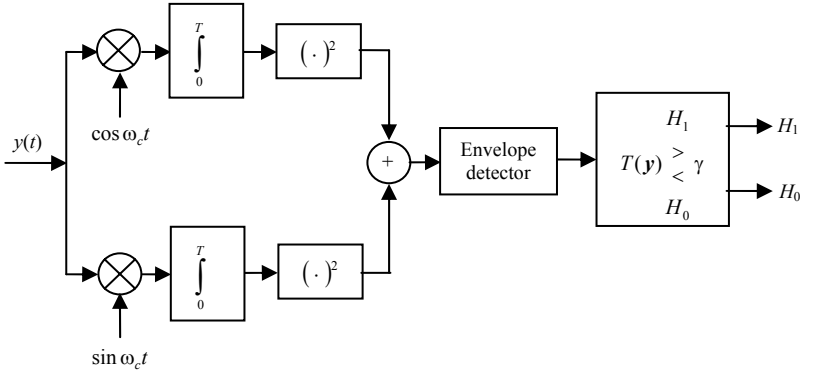


**Figure 10.29** Optimum receiver for a slow Rayleigh fading channel.

The probability of false alarm is given in (10.213) to be

$$P_F = \exp\left(-\frac{\gamma}{2\sigma^2}\right) = \exp\left(-\frac{2\gamma}{N_0 T}\right) \tag{10.228}$$

since $\sigma^2 = N_0 T / 4$. For a given value of $A = a$, the probability of detection is

$$P_D = \int_A P_D(a) f_A(a) da \tag{10.229}$$

where $P_D(a)$ is given by (10.214). Substituting (10.214) and (10.224) in (10.229) and solving the integral, we obtain

$$P_D = \exp\left(-\frac{2\gamma}{N_0 T + \sigma_0^2 T^2}\right) = \exp\left\{-\frac{2\gamma}{N_0 T}\left[\frac{1}{1+(\sigma_a^2 T / N_0)}\right]\right\} = (P_F)^{\beta} \tag{10.230}$$

where $\beta = N_0 / (N_0 + \sigma_a^2 T)$. The average signal energy, for a given signal level $A = a$, is

$$E_{av} = \int_0^{\infty} \frac{Ea}{\sigma_a^2} \exp\left(-\frac{a^2}{2\sigma_a^2}\right) da = \sigma_a^2 T \tag{10.231}$$

where $E = A^2 T / 2$. The probability of detection then becomes

$$P_D = (P_F)^{\frac{N_0}{N_0 + E_{av}}} \tag{10.232}$$

### 10.6.3 Signals with Random Parameters

Once the fundamental concepts of binary detection with unwanted parameters are understood, we can apply them to the many radar or communication situations that may arise. After we treated in detail the detection of signals with random phase, and random amplitude and phase, we can now give the procedure of some other applications.

*Signals with Random Frequency*

This is a typical radar application where the frequency of the echo signal of a moving target may differ from the frequency of the transmitted signal by a

frequency $f_d$ known as *Doppler frequency*. The problem may be defined as follows

$$\begin{aligned} H_1 &: Y(t) = A\cos(\omega t + \theta) + W(t), \quad 0 \le t \le T \\ H_0 &: Y(t) = W(t), \quad 0 \le t \le T \end{aligned} \tag{10.233}$$

where the phase is uniformly distributed. The signal amplitude and time of arrival are assumed to be known, while the frequency is a random variable uniformly distributed over the interval $[\omega_\ell, \omega_h]$ with density function $f_W(w)$. $\omega = 2\pi f$ and $\omega_\ell$ and $\omega_h$ denote the lowest and the highest possible frequencies. $W(t)$ is the additive Gaussian noise with mean zero and power spectral density $N_0/2$.

Using the approach developed by Whalen [1], from (10.200), the conditional likelihood ratio is

$$\Lambda[\boldsymbol{y} \mid \omega] = \exp\left(-\frac{A^2 T}{2N_0}\right) I_0\left(\frac{2A}{N_0} r\right) \tag{10.234}$$

while the average likelihood function is

$$\Lambda[\boldsymbol{y}] = \int_{\omega_\ell}^{\omega_h} \Lambda[\boldsymbol{y} \mid \omega] f_W(\omega) d\omega \tag{10.235}$$

To solve the integral (10.235), the density function is approximated by a discrete density function, such that

$$f_W(\omega) \approx \sum_{k=1}^{M} p(\omega_k)\delta(\omega - \omega_k) \tag{10.236}$$

where

$$p(\omega_k) = f_W(\omega_k)\,\Delta\omega \tag{10.237a}$$

$$\omega_i = \omega_\ell + k\,\Delta\omega, \quad k = 1, 2, \ldots, \frac{\omega_h - \omega_\ell}{\Delta\omega} \tag{10.237b}$$

and

$$\frac{\omega_h - \omega_\ell}{\Delta\omega} \triangleq K \tag{10.237c}$$

Hence, the likelihood function is

$$\Lambda[\boldsymbol{y}] = \sum_{k=1}^{K} \Lambda[\boldsymbol{y} \mid \omega_k] p(\omega_k) \tag{10.238}$$

and the optimum receiver may be as shown in Figure 10.30.

*Signals with Random Frequency and Rayleigh Fading Amplitude*

This is the same problem as the one defined in (10.231), while the amplitude has the density function as given by (10.224). The conditional likelihood ratio is then

$$\Lambda[\boldsymbol{y} \mid \omega] = \frac{N_0}{N_0 + E_{av}} \exp\left[ \frac{2\sigma^2}{N_0 (N_0 + E_{av})} r^2 \right] \tag{10.239}$$

Using the discrete approximation (10.226), the likelihood ratio is then

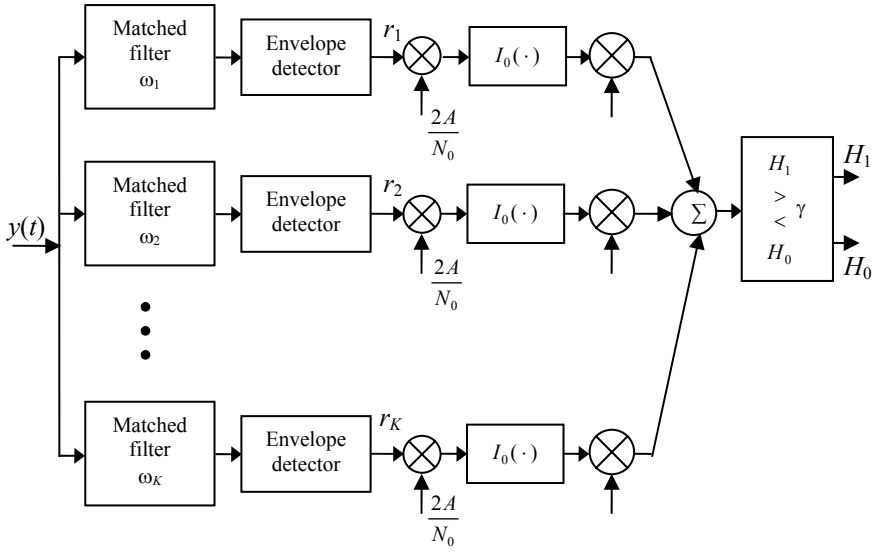


**Figure 10.30** Optimum receiver for signals with random frequency. (*From*: [1]. © 1971 Elsevier. Reprinted with permission.)

$$\Lambda[\boldsymbol{y}] = \int \Lambda[\boldsymbol{y} \mid \omega] f_W(\omega) d\omega = \frac{N_0}{N_0 + E_{av}} \sum_{k=1}^{K} p(\omega_k) \exp\left[\frac{2\sigma^2}{N_0(N_0 + E_{av})} r_k^2\right] \tag{10.240}$$

The optimum receiver is shown in Figure 10.31.

*Signals with Different Random Phases*

This is known as the binary frequency shift keying (FSK) in communications. One of two frequencies is transmitted with an equal probability. At the receiver, we have

$$\begin{aligned} H_1 &: Y(t) = S_1(t) + W(t) \\ H_0 &: Y(t) = S_0(t) + W(t) \end{aligned} \tag{10.241}$$

Where

$$S_1(t) = A\cos(\omega_1 t + \Theta) \tag{10.242a}$$

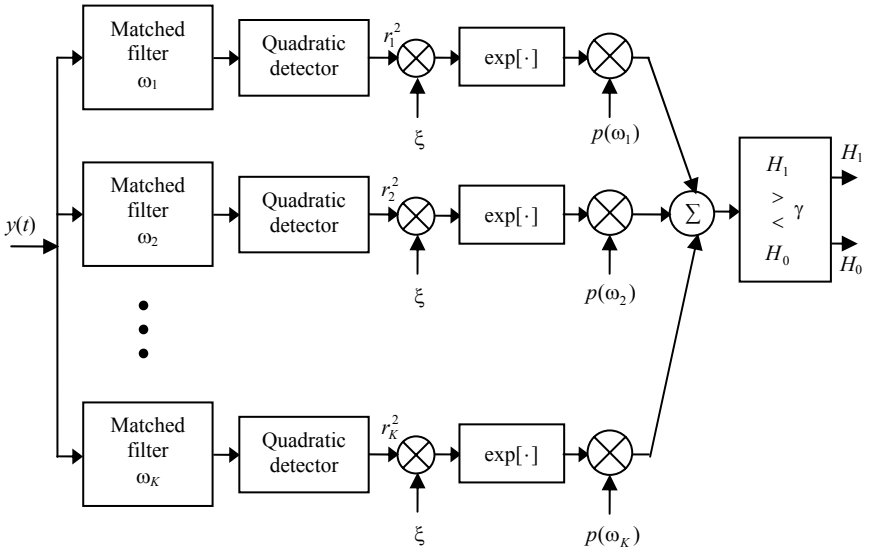


**Figure 10.31** Optimum receiver for signals with random frequency and Rayleigh fading amplitude; $\xi = 2\sigma_a^2 / [N_0^2(1 + \sigma_a^2 T)]$ . (*From*: [1]. © 1971 Elsevier. Reprinted with permission.)

$$S_0(t) = A\cos(\omega_0 t + \Phi) \tag{10.242b}$$

The random phases $\Theta$ and $\Phi$ are statistically independent and uniformly distributed over the interval $[0, 2\pi]$. From (10.187), the likelihood ratio is

$$\Lambda[y] = \frac{\frac{1}{2\pi}\int_0^{2\pi} \exp\left\{-\frac{1}{N_0}\int_0^T [y(t) - A\cos(\omega_1 t + \theta)]^2\, dt\right\} d\theta}{\frac{1}{2\pi}\int_0^{2\pi} \exp\left\{-\frac{1}{N_0}\int_0^T [y(t) - A\cos(\omega_0 t + \phi)]^2\, dt\right\} d\phi} \tag{10.243}$$

We follow the same approach as we did for signals with random phase in Section 10.6.1, but in this case, we develop both the numerator and denominator of (10.243) to obtain

$$\Lambda[\boldsymbol{y}] = \frac{f_{\boldsymbol{Y}|H_1}(\boldsymbol{y}\,|\,H_1)}{f_{\boldsymbol{Y}|H_0}(\boldsymbol{y}\,|\,H_0)} = \frac{\exp\left(-\frac{A^2T}{2N_0}\right) I_0\left(\frac{2A}{N_0} r_1\right)}{\exp\left(-\frac{A^2T}{2N_0}\right) I_0\left(\frac{2A}{N_0} r_0\right)} \tag{10.244}$$

where $r_1 = \sqrt{y_{1c}^2 + y_{1s}^2}$, $y_{1c} = r_1\cos\theta$, $y_{1s} = r_1\sin\theta$, $r_0 = \sqrt{y_{0c}^2 + y_{0s}^2}$, $y_{0c} = r_0\cos\phi$, and $y_{0s} = r_0\sin\phi$. The likelihood ratio test is then

$$\Lambda[\boldsymbol{y}] = \frac{I_0\left(\frac{2A}{N_0} r_1\right)}{I_0\left(\frac{2A}{N_0} r_0\right)} \underset{H_0}{\overset{H_1}{>\atop <}} \eta \tag{10.245}$$

In communications problems, we are interested in the minimum probability of error, and thus $C_{00} = C_{11} = 0$ and $C_{10} = C_{01} = 1$. If $P_0 = P_1 = 1/2$, then $\eta = 1$, and the decision rule is

$$I_0\left(\frac{2A}{N_0} r_1\right) \underset{H_0}{\overset{H_1}{>\atop <}} I_0\left(\frac{2A}{N_0} r_0\right) \tag{10.246}$$

or equivalently

$$r_1 \underset{H_0}{\overset{H_1}{\gtrless}} r_0 \tag{10.247}$$

The corresponding optimum receiver is shown in Figure 10.32.

The probability of error can be shown to be

$$P(\varepsilon) = \frac{1}{2}\exp\left(-\frac{E}{2N_0}\right) \tag{10.248}$$

where $E = A^2T/2$ is the signal energy.

*FSK Signals with Rayleigh Fading*

Due to multipath, the Rayleigh amplitude is often assumed in communication systems. Applying the Rayleigh fading model to FSK signals, the received signals are modeled as given by (10.241), with $S_1(t) = A\cos(\omega_1 t + \theta)$ and $S_2(t) = B\cos(\omega_0 t + \phi)$. $S_1(t)$ and $S_2(t)$ are transmitted with equal probabilities. Assuming *slowly fading channel*, the density functions of $A$ and $B$ are given by

$$f_A(a) = \frac{a}{\sigma^2}\exp\left(-\frac{a^2}{2\sigma^2}\right) \tag{10.249a}$$

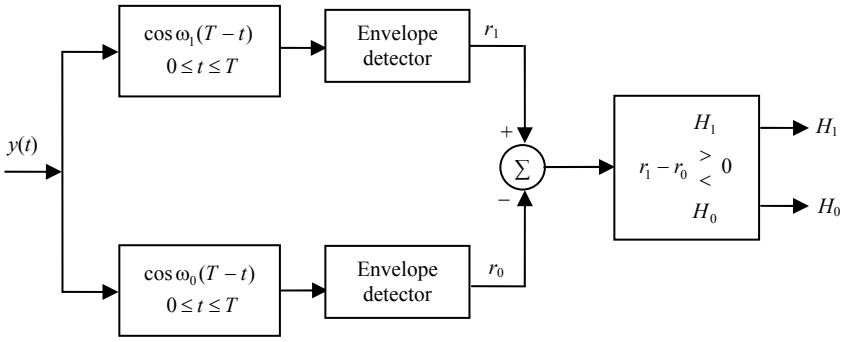


**Figure 10.32** Noncoherent receiver for binary FSK signals.

and

$$f_B(b) = \frac{b}{\sigma^2} \exp\left( -\frac{b^2}{2\sigma^2} \right) \tag{10.249b}$$

The random phases $\Theta$ and $\Phi$ are statistically independent and uniformly distributed over the interval $[0, 2\pi]$. $W(t)$ is the white Gaussian noise with mean zero and power spectral density $N_0 / 2$. In this case, the likelihood ratio is

$$\Lambda[\boldsymbol{y}] = \frac{\int_A \int_\Theta f_{\boldsymbol{Y}|A,\Theta,H_1}(\boldsymbol{y} \mid a, \theta, H_1) f_A(a) f_\Theta(\theta) da\, d\theta}{\int_B \int_\Phi f_{\boldsymbol{Y}|B,\Phi,H_0}(\boldsymbol{y} \mid b, \phi, H_0) f_B(b) f_\Phi(\phi) db\, d\phi} \tag{10.250}$$

Solving for the decision rule after substitution, we obtain

$$\Lambda[\boldsymbol{y}] = \frac{\exp\left[ \dfrac{2\sigma^2}{N_0(N_0 + \sigma^2 T)} r_1^2 \right]}{\exp\left[ \dfrac{2\sigma^2}{N_0(N_0 + \sigma^2 T)} r_0^2 \right]} \begin{matrix} H_1 \\ > \\ < \\ H_0 \end{matrix} \eta \tag{10.251}$$

Taking the natural logarithm, the decision rule becomes

$$r_1^2 - r_0^2 \begin{matrix} H_1 \\ > \\ < \\ H_0 \end{matrix} \gamma \tag{10.252a}$$

where

$$\gamma = \frac{N_0(N_0 + \sigma^2 T)}{2\sigma^2} \ln \eta \tag{10.252b}$$

Assuming minimum probability of error, the decision rule becomes

$$r_1 \begin{matrix} H_1 \\ > \\ < \\ H_0 \end{matrix} r_0 \tag{10.253}$$

The optimum receiver is the same as the one shown in Figure 10.32. The probability of error can be shown to be

$$P(\varepsilon) = \frac{N_0}{2N_0 + \sigma^2 T} = \frac{N_0}{2 + E_{av}} \tag{10.254}$$

where $E_{av}$ is the average signal energy given by (10.231), and $E = A^2 T / 2$ is the signal energy over the interval $T$ for a given signal level.

*Signals with Random Time of Arrival*

In this case, the received hypotheses are

$$\begin{aligned} H_1 &: Y(t) = S(t-\tau) + W(t) \\ H_0 &: Y(t) = \qquad\quad W(t) \end{aligned} \tag{10.255}$$

where $s(t) = A\cos(\omega_c t + \theta), 0 \le t \le T$, and the arrival time $\tau$ has a density function $f_{\mathrm{T}}(\tau)$ for $\tau$ defined in $0 \le \tau \le \tau_K$. The conditional likelihood is then

$$\Lambda[\boldsymbol{y} \mid \tau] = \exp\left(-\frac{E}{N_0}\right) I_0\left[\frac{2A}{N_0} r(\tau + T)\right] \tag{10.256}$$

where

$$r(\tau + T) = \sqrt{y_c^2 + y_s^2} \tag{10.257a}$$

$$y_c = \int_{\tau}^{\tau+T} y(t)\cos\omega_c(t-\tau)dt \tag{10.257b}$$

$$y_s = \int_{\tau}^{\tau+T} y(t)\sin\omega_c(t-\tau)dt \tag{10.257c}$$

The likelihood ratio test is

$$\Lambda[\boldsymbol{y}] = \int_0^{\tau_K} \Lambda[\boldsymbol{y} \mid \tau] f_{\mathrm{T}}(\tau) d\tau = \int_T^{T+\tau_K} \exp\left(-\frac{A^2 T}{2N_0}\right) I_0\left[\frac{2A}{N_0} r(u)\right] p(u-T)du \tag{10.258}$$

and the optimum receiver is as shown in Figure 10.33.

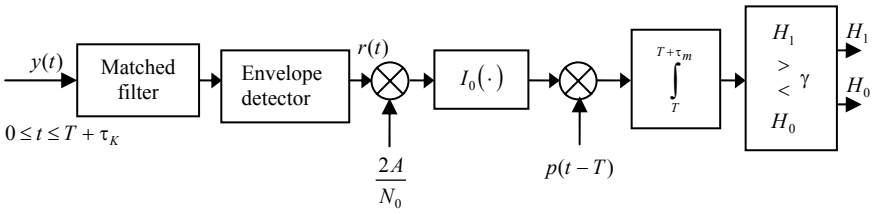


**Figure 10.33** Optimum receiver for signals with random time of arrival.

If the arrival time $0 \le \tau \le \tau_K$ is divided in $K$ discrete times $\tau_k$, $k = 1, 2, \ldots, K$, then, by analogy to the random frequency case, the optimum receiver is as shown in Figure 10.34.

## 10.7 BINARY DETECTION IN COLORED NOISE

In the previous sections, we assumed that the additive Gaussian noise is zero mean and white. However, in many applications this assumption is not valid. We now consider detection of signals in nonwhite Gaussian noise. Consequently, the power

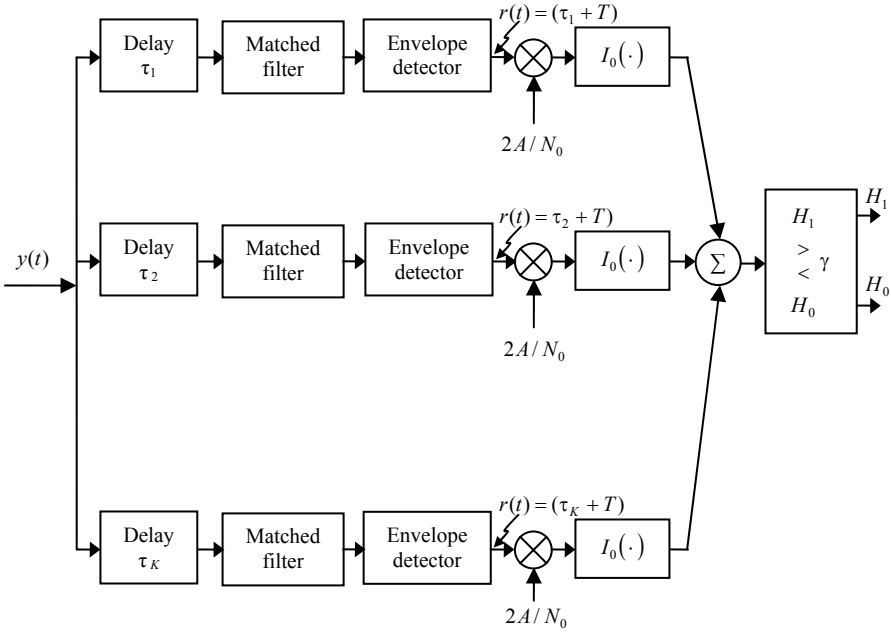


**Figure 10.34** Optimum receiver for signals with discrete random time of arrival. (*From*: [1]. © 1971 Elsevier. Reprinted with permission.)

spectral density is not constant in the filter bandwidth. The noise samples are no longer uncorrelated, and thus they are statistically dependent. One way to deal with this problem is to extend to colored Gaussian noise the concepts using Karhunen-Loève expansion for white Gaussian noise. Another way may be to use some preliminary processing for the noise (referred to as whitening) to make the colored noise white, and then use the Karhunen-Loève expansion.

The problem under consideration is to design a receiver to test for the general binary detection given by

$$\begin{aligned} H_1 &: Y(t) = s_1(t) + N(t), \quad 0 \le t \le T \\ H_0 &: Y(t) = s_0(t) + N(t), \quad 0 \le t \le T \end{aligned} \tag{10.259}$$

where $Y(t)$ is the received waveform, $s_1(t)$ and $s_0(t)$ are known deterministic signals, and $N(t)$ is the additive colored Gaussian with mean zero and covariance function $C_{nn}(t,u)$.

### 10.7.1 Karhunen-Loève Expansion Approach

The solution to the binary detection problem with Gaussian noise was relatively simple, since the coefficients of the Karhunen-Loève expansion generated by any set of orthonormal basis function resulted in independent samples. The coefficients $Y_1, Y_2, \ldots, Y_K$ were statistically independent Gaussian random variables, and thus the likelihood function was the joint probability density function of these coefficients in the limit as $K \to \infty$. The goal is still to generate uncorrelated coefficients from the likelihood ratio and obtain the decision rule. That is, the corresponding orthonormal functions and eigenfunctions satisfy the integral equation

$$\int_0^T C_{nn}(t,u) f_k(u) du = \lambda_k f_k, \quad 0 \le t \le T \tag{10.260}$$

where $\lambda_k$ is the eigenvalue. This means that the coefficients

$$Y_k = \int_0^T Y(t) f_k(t) dt = \int_0^T \left[ s(t) + N(t) \right] f_k(t) dt = s_k + N_k \tag{10.261}$$

where

$$s_k = \int_0^T s(t) f_k(t) dt \tag{10.262}$$

and

$$N_k = \int_0^T N(t) f_k(t) dt \tag{10.263}$$

are obtained by the correlation operation and are uncorrelated. The noise components are uncorrelated Gaussian random variables with zero mean, such that

$$E\left[N_k\right] = \int_0^T f_k(t) E\left[N(t)\right] dt = 0 \tag{10.264}$$

and

$$E\left[N_j N_k\right] = \lambda_k \delta_{jk} \tag{10.265}$$

as shown in Section 8.5. The series expansion of noise is

$$N(t) = \lim_{K \to \infty} \sum_{k=1}^{K} N_k f_k(t) \tag{10.266}$$

Karhunen-Loève coefficients under hypotheses $H_1$ and $H_0$ are

$$H_1 : Y_k = s_{1k} + N_k \tag{10.267}$$

and

$$H_0 : Y_k = s_{0k} + N_k \tag{10.268}$$

with means

$$E\left[Y_k \mid H_1\right] = s_{1k} \tag{10.269}$$

and

$$E\left[Y_k \mid H_0\right] = s_{0k} \tag{10.270}$$

and variances

$$\text{var}\left[Y_k \mid H_1\right] = \text{var}\left[Y_k \mid H_0\right] = \lambda_k \tag{10.271}$$

Since the coefficients $Y_k, k = 1, 2, \ldots, K,$ are statistically independent under each hypothesis, the conditional density functions are given by

$$f_{\boldsymbol{Y}|H_1}(\boldsymbol{y} \mid H_1) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi\lambda_k}} \exp\left[-\frac{(y_k - s_{1k})^2}{2\lambda_k}\right] \tag{10.272}$$

and

$$f_{\boldsymbol{Y}|H_0}(\boldsymbol{y} \mid H_0) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi\lambda_k}} \exp\left[-\frac{(y_k - s_{0k})^2}{2\lambda_k}\right] \tag{10.273}$$

Consequently, the *K*-term approximation of the likelihood ratio is

$$\begin{aligned}\Lambda[y_K(t)] &= \frac{\displaystyle\prod_{k=1}^{K} \frac{1}{\sqrt{2\pi\lambda_k}} \exp\left[-\frac{(y_k - s_{1k})^2}{2\lambda_k}\right]}{\displaystyle\prod_{k=1}^{K} \frac{1}{\sqrt{2\pi\lambda_k}} \exp\left[-\frac{(y_k - s_{0k})^2}{2\lambda_k}\right]} \\ &= \exp\left\{\left[\sum_{k=1}^{K} \frac{1}{2\lambda_k} s_{1k}(2y_k - s_{1k})\right] - \left[\sum_{k=1}^{K} \frac{1}{2\lambda_k} s_{0k}(2y_k - s_{0k})\right]\right\}\end{aligned} \tag{10.274}$$

Taking the logarithm, we obtain

$$\ln \Lambda[y_K(t)] = \sum_{k=1}^{K} \frac{1}{2\lambda_k} s_{1k}(2y_k - s_{1k}) - \sum_{k=1}^{K} \frac{1}{2\lambda_k} s_{0k}(2y_k - s_{0k}) \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta \tag{10.275}$$

Letting $K \to \infty$, the log-likelihood ratio is

$$\ln \Lambda[y_K(t)] = \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} s_{1k}(2y_k - s_{1k}) - \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} s_{0k}(2y_k - s_{0k}) \tag{10.276}$$

Substituting (10.259) and (10.260) in (10.274), we obtain the likelihood ratio in terms of the correlation to be

$$\ln \Lambda[y(t)] = \frac{1}{2}\int_0^T\int_0^T s_1(t)[2y(u) - s_1(u)]\sum_{k=1}^{\infty}\frac{f_k(t)f_k(u)}{\lambda_k}\,dt\,du - \frac{1}{2}\int_0^T\int_0^T s_0(t)[2y(u) - s_0(u)]\sum_{k=1}^{\infty}\frac{f_k(t)f_k(u)}{\lambda_k}\,dt\,du \quad (10.277)$$

Define

$$h_1(t) = \int_0^T s_1(t)\sum_{k=1}^{\infty}\frac{f_k(t)f_k(u)}{\lambda_k}\,dt \quad (10.278)$$

and

$$h_0(t) = \int_0^T s_0(t)\sum_{k=1}^{\infty}\frac{f_k(t)f_k(u)}{\lambda_k}\,dt \quad (10.279)$$

Substituting (10.278) and (10.279) into (10.277), the log-likelihood ratio test becomes

$$\frac{1}{2}\int_0^T h_1(t)[2y(t) - s_1(t)]dt - \frac{1}{2}\int_0^T h_0(t)[2y(t) - s_0(t)]dt \underset{H_0}{\overset{H_1}{\gtrless}} \ln\eta \quad (10.280)$$

or

$$\int_0^T y(t)h_1(t)dt - \int_0^T y(t)h_0(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \quad (10.281)$$

where

$$\gamma = \ln\eta + \frac{1}{2}\int_0^T s_1(t)h_1(t)\,dt - \frac{1}{2}\int_0^T s_0(t)h_0(t)dt \quad (10.282)$$

Hence, from (10.280), we see that the receiver for detection of signals in colored noise can be interpreted as a correlation detector, as shown in Figure 10.35. To build such a receiver, we need to determine $h_1(t)$ and $h_0(t)$. Substituting (10.278)

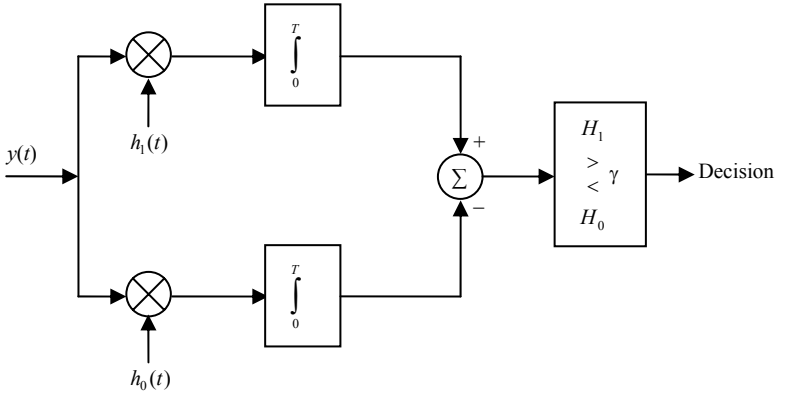


**Figure 10.35** Correlation receiver for signals in colored noise.

and (10.279) into (10.260), we obtain

$$\int_0^T C_{nn}(t,u)h_1(u)du = \int_0^T C_{nn}(t,u)\int_0^T \sum_{k=1}^{\infty}\frac{f_k(u)}{\lambda_k}s_1(t)f_k(t)dtdu$$

$$= \sum_{k=1}^{\infty}\frac{s_{k1}}{\lambda_k}\int_0^T C_{nn}(t,u)f_k(u)du = \sum_{k=1}^{\infty}s_{k1}f_k(t) = s_1(t) \qquad (10.283)$$

and similarly,

$$\int_0^T C_{nn}(t,u)h_0(u)du = s_0(t) \qquad (10.284)$$

That is, $h_1(t)$ and $h_0(t)$ are solutions to the integral equations in (10.283) and (10.284), respectively.

### 10.7.2 Whitening Approach

Another approach to detect signals in colored noise is to do a preliminary processing of the colored noise. The received signal is passed through a linear time-invariant filter, such that the noise at the output of the filter is white, as shown in Figure 10.36.

The process of converting colored noise to white noise is referred to as whitening. Once the noise is white, the problem becomes a detection of a known signal in additive white Gaussian noise, which we have covered in the previous sections.

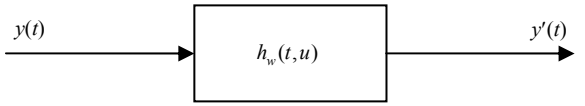


**Figure 10.36** Whitening filter.

We now solve the binary detection problem in colored noise. The output of the whitening filter under hypothesis $H_1$ is given by

$$Y'(t) = \int_0^T h_w(t,u)Y(u)du \tag{10.285a}$$

$$= \int_0^T h_w(t,u)[s_1(u) + N(u)]du \tag{10.285b}$$

$$= s_1'(t) + N'(t) \tag{10.285c}$$

where

$$s_1'(t) = \int_0^T h_w(t,u)s_1(u)du, \quad 0 \le t \le T \tag{10.286}$$

and

$$N'(t) = \int_0^T h_w(t,u)N(u)du, \quad 0 \le t \le T \tag{10.287}$$

under hypothesis $H_0$, $Y'(t)$ is

$$Y'(t) = \int_0^T h_w(t,u)[s_0(u) + N(u)]du = s_0'(t) + N'(t), \qquad 0 \le t \le T \tag{10.288}$$

where

$$s_0'(t) = \int_0^T h_w(t)s_0(u)du, \quad 0 \le t \le T \tag{10.289}$$

Since $N'(t)$ is the white Gaussian noise, its covariance function is given by

$$C_{n'n'}(t,u) = E[N'(t)N'(u)] = \delta(t-u), \quad 0 \le t,u \le T \tag{10.290}$$

where we have assumed $N_0 = 2$. Thus, we have reduced the problem to general binary detection in white Gaussian noise. The equivalent problem is summarized as follows:

$$\begin{aligned} H_1 &: Y'(t) = s_1'(t) + N'(t) \\ H_0 &: Y'(t) = s_0'(t) + N'(t) \end{aligned} \tag{10.291}$$

This problem was solved in Section 10.2.2. Thus, by analogy to (10.71) and (10.72), the decision rule can be written as

$$\int_0^T y'(t)\left[s_1'(t) - s_0'(t)\right]dt \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \tag{10.292}$$

where

$$\gamma = \frac{1}{2}\ln\eta - \frac{1}{2}\int_0^T \left\{\left[s_0'(t)\right]^2 - \left[s_1'(t)\right]^2\right\}dt \tag{10.293}$$

Note that $y'(t)$, $s_1'(t)$, and $s_0'(t)$ are given in (10.285a), (10.286), and (10.289), respectively, in terms of the original signals $y(t)$, $s_1(t)$, and $s_0(t)$. Rewriting (10.292) and (10.293) in terms of the original signals, we obtain

$$\int_0^T \int_0^T h_w(t,u)Y(u)\int_0^T h_w(t,v)\left[s_1(v) - s_0(v)\right]dv\,du\,dt \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \tag{10.294}$$

which can be implemented as shown in Figure 10.37.

Again, construction of the receiver requires knowledge of the impulse response $h_w(t,u)$, which can be obtained by substituting (10.287) into (10.290). We have

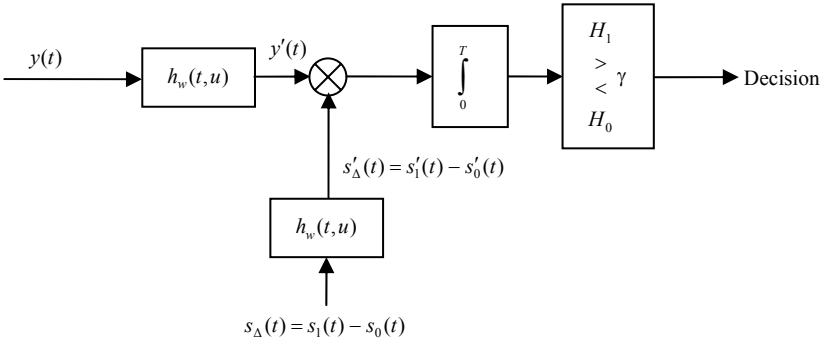


**Figure 10.37** Receiver for colored noise using whitening.

$$E[N'(t)N'(u)] = E\left[\int_0^T\int_0^T h_w(t,\alpha)h_w(u,\beta)N(\alpha)N(\beta)d\alpha d\beta\right]$$
$$= \int_0^T\int_0^T h_w(t,\alpha)h_w(u,\beta)\,C_{nn}(\alpha,\beta)d\alpha d\beta = \delta(t-u) \tag{10.295}$$

The solution to the integral equation in (10.295) yields $h_w(t,w)$.

Another way to define the integral equation is in terms of the function $Q(a,b)$. In some applications, we may need to express the colored noise as the sum of two components, such as

$$N(t) = N_c(t) + N'(t) \tag{10.296}$$

where $N_c(t)$ is not known. In this case, the function $Q(a,b)$ is useful in obtaining the minimum mean-square error, $\hat{N}_c(t)$ of $N_c(t)$. We define

$$Q_{nn}(a,b) = \int_0^T h_w(t,a)h_w(t,b)dt = Q_{nn}(b,a) \tag{10.297}$$

In order to write the integral equation in terms of $Q(a,b)$, we multiply both sides of (10.295) by $h_w(t,v)$, and integrate with respect to *t* to obtain

$$\int_0^T h_w(t,u)\delta(t,u)dt = h_w(u,v) \tag{10.298a}$$

$$= \int_0^T \int_0^T \int_0^T h_w(t,v) h_w(t,a) h_w(u,b) C_{nn}(a,b) da\, db\, dt \quad (10.298b)$$

$$= \int_0^T h_w(u,b) \int_0^T C_{nn}(a,b) \int_0^T h_w(t,v) h_w(t,a) da\, db\, dt \quad (10.298c)$$

Substituting (10.297) into (10.298c) results in

$$h_w(u,v) = \int_0^T h_w(u,b) \int_0^T C_{nn}(a,b) Q_{nn}(v,a) da\, db \quad (10.299)$$

From (10.298a) and (10.299), we deduce that

$$\delta(b-v) = \int_0^T C_{nn}(a,b) Q_{nn}(v,a) da \quad (10.300)$$

which means that given the covariance function $C_{nn}(a,b)$, we solve (10.300) to yield $Q_{nn}(v,a)$.

### 10.7.3 Detection Performance

In this section, we study how the colored noise affects the performance. Recall that for binary detection in white noise, the decision rule, from (10.71), (10.72), and (10.73), was

$$T(y) = \int_0^T y(t)\left[s_1(t) - s_0(t)\right] dt - \frac{1}{2} \int_0^T \left[s_1^2(t) - s_0^2(t)\right] dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2} \ln \eta \quad (10.301)$$

Using the whitening approach, the nonwhite noise $N(t)$ is transformed into white noise $N'(t)$ with $N_0 = 2$. The received waveform $Y(t)$ is transformed into $Y'(t)$, and the transmitted signals $s_1(t)$ and $s_0(t)$ are transformed into $s_1'(t)$ and $s_0'(t)$, respectively. Assuming minimum probability of error criterion and that the hypotheses are equally likely, the test may be expressed as

$$T(y') = \int_0^T y'(t)s_1'(t)dt - \int_0^T y'(t)s_0'(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}\int_0^T \left\{[s_1'(t)]^2 - [s_0'(t)]^2\right\}dt \qquad (10.302)$$

The sufficient statistic $T(Y')$ is Gaussian with means

$$E[T \mid H_1] = \int_0^T [s_1'(t)]^2\,dt - \int_0^T s_1'(t)s_0'(t)dt = \int_0^T dt \int_0^T\int_0^T h_w(t,u)s_1(u)h_w(t,v)s_1(v)dudv$$
$$- \int_0^T dt \int_0^T\int_0^T h_w(t,u)s_1(u)h_w(t,v)s_0(v)dudv \qquad (10.303)$$

and

$$E[T \mid H_0] = \int_0^T s_0'(t)s_1'(t)dt - \int_0^T [s_0'(t)]^2\,dt = \int_0^T dt \int_0^T\int_0^T h_w(t,u)s_0(u)h_w(t,v)s_1(v)dudv$$
$$- \int_0^T dt \int_0^T\int_0^T h_w(t,u)h_w(t,v)s_0(u)s_0(v)dudv \qquad (10.304)$$

The variances under hypotheses $H_1$ and $H_0$ are the same. The expression is cumbersome. However, it can be shown to have a value of twice the mean of $T$ under $H_1$. Denote this variance by $\sigma^2$, and then the probability of error is

$$P(\varepsilon) = \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{[t+(1/2)\sigma^2]^2}{2\sigma^2}\right\}dt = \int_{\sigma/2}^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)du \qquad (10.305)$$

where $\sigma$ is given by (10.303). The calculation of (10.305) is involved. However, we observe that the probability of error is a function of the signal's shape, unlike the case of detection in white noise, where the performance was a function of the signal-to-noise ratio only. Consequently, to minimize the probability of error, we need to find the signals shape. We also see from (10.305) that the probability of error is minimized if $\sigma$ is maximized, subject to the constraint that the energy is fixed to a value $E$. Hence, we form the objective function $J$ and solve the equation

$$J = \sigma^2 - \lambda E \qquad (10.306)$$

where $\lambda$ is the Lagrange multiplier and $E$ is given by

$$E = \frac{1}{2}\int_0^T \left[s_1^2(t) - s_0^2(t)\right]dt \tag{10.307}$$

The solution of (10.306) results in the optimum signal's shape, which is obtained to be

$$s_1(t) = -s_0(t), \quad 0 \le t \le T \tag{10.308}$$

That is, we have optimum performance when the correlation coefficient $\rho = -1$, which is the same result obtained for binary detection in white Gaussian noise.

## 10.8 SUMMARY

In this chapter, we have discussed the problem of detection of signal waveforms and parameter estimation of signals in the presence of additive noise. We first covered binary and *M*-ary detection. The approach adopted was to decompose the signal waveform into a set of *K* independent random variables, and write the signal in Karhunen-Loève expansion. The coefficients of Karhunen-Loève expansion are in a sense samples of the received signal. Since the additive noise was white and Gaussian, the coefficients of the Karhunen-Loève expansion were uncorrelated and jointly Gaussian. Consequently, the problem was reduced to an equivalent decision problem, as developed in Chapter 5.

In Sections 10.4 and 10.5, we assumed that the received signals may contain some unknown parameters that needed to be estimated. Linear and nonlinear estimation were considered. When the parameter to be estimated was nonrandom, we used maximum likelihood estimation. The maximum a posteriori estimation was used for a random parameter. The "goodness" of the estimation techniques was studied as well.

The general binary detection with unknown parameters was presented in Section 10.6. Again using Karhunen-Loève coefficients, we obtained the aproximated *K*-term likelihood ratio, and then we let $K \to \infty$ to obtain the likelihood ratio. This approach of obtaining a *K*-term approximation of Karhunen-Loève coefficients and letting $K \to \infty$ was also used in solving for the parameter-estimates discussed in Sections 10.4 and 10.5. Specifically, we considered signals with random phase, and derived the incoherent matched filter. Then, we considered signals with random phase and amplitude. In Section 10.6.3, we treated examples of signals with random parameters, such as signals with random frequency, signals with random frequency and Rayleigh fading amplitude, signals with different random phases, FSK signals with Rayleigh fading, and signals with random time of arrival.

We concluded the chapter with a section on binary detection in colored noise. Since the noise was not white anymore, the generated Karhunen-Loève

coefficients were no longer uncorrelated. In solving this problem, we first used the *K*-term approximation from Karhunen-Loève coefficients. However, due to the nature of noise, some integral equations needed to be solved in order to design the optimum receiver. The second approach used to solve this problem was whitening. That is, we did a preliminary processing by passing the received signal through a linear time-invariant system, such that the noise at the output of the filter was white. Once the noise became white, the techniques developed earlier for binary detection were then used to obtain the optimum receiver. A brief study on the performance of detection of signals in colored noise was also presented.

## PROBLEMS

**10.1** A signal source generates signals as shown in Figure P10.1. The signals are expressed as $s_1(t) = \cos(2\pi t)\,\text{rect}(t)$, $s_2(t) = \cos[2\pi t + (2\pi/3)]\,\text{rect}(t)$, and $s_3(t) = \cos[2\pi t - (2\pi/3)]\,\text{rect}(t)$.

(a) Describe a correlation receiver for these signals.
(b) Draw the corresponding decision regions on a signal space.

**10.2** A rectangular pulse of known amplitude $A$ is transmitted starting at time instant $t_0$ with probability $1/2$. The duration $T$ of the pulse is a random variable uniformly distributed over the interval $[T_1, T_2]$. The additive noise to the pulse is white Gaussian with mean zero and variance $N_0/2$.

(a) Determine the likelihood ratio.
(b) Describe the likelihood ratio receiver.

**10.3** Consider the general binary detection problem

$$H_1 : Y(t) = s_1(t) + W(t), \quad 0 \le t \le T$$
$$H_0 : Y(t) = s_0(t) + W(t), \quad 0 \le t \le T$$

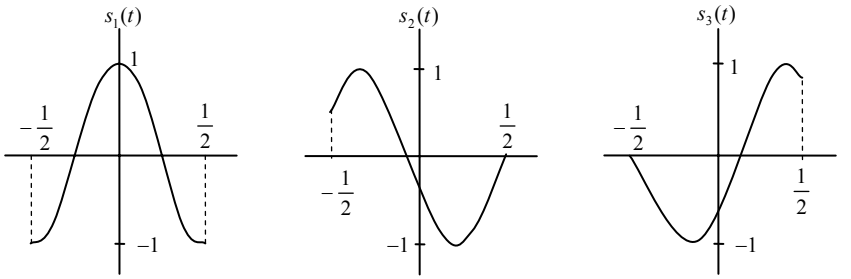


**Figure P10.1** Signal set.

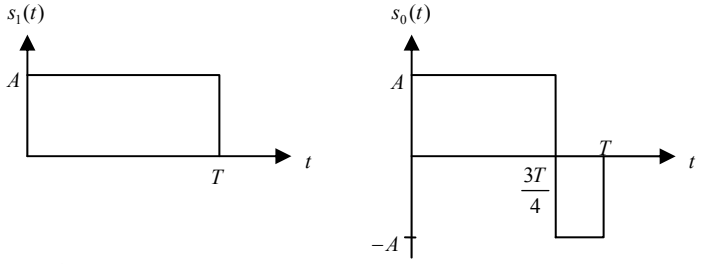


**Figure P10.3** Signal set.

where $s_1(t)$ and $s_0(t)$ are as shown in Figure P10.3, and $W(t)$ is a white Gaussian noise with mean zero and power spectral density $N_0/2$.

(a) Determine the probability of error, assuming minimum probability of error criterion and $P(H_0) = P(H_1) = 1/2$.

(b) Draw a block diagram of the optimum receiver

**10.4** In a binary detection problem, the transmitted signal under hypothesis $H_1$ is either $s_1(t)$ or $s_2(t)$, with respective probabilities $P_1$ and $P_2$. Assume $P_1 = P_2 = 1/2$, and $s_1(t)$ and $s_2(t)$ orthogonal over the observation time $t \in [0, T]$. No signal is transmitted under hypothesis $H_0$. The additive noise is white Gaussian with mean zero and power spectral density $N_0/2$.

(a) Obtain the optimum decision rule, assuming minimum probability of error criterion and $P(H_0) = P(H_1) = 1/2$.

(b) Draw a block diagram of the optimum receiver.

**10.5** Consider the binary detection problem

$$H_1 : Y(t) = s_1(t) + W(t), \quad 0 \le t \le 2$$
$$H_0 : Y(t) = s_0(t) + W(t), \quad 0 \le t \le 2$$

where $s_1(t) = -s_0(t) = e^{-t}$, and $W(t)$ is an additive white Gaussian noise with mean zero and covariance function $C_{ww}(t,u) = (N_0/2)\delta(t-u)$.

(a) Determine the probability of error, assuming minimum probability of error criterion.

(b) Draw a block diagram of the optimum receiver.

**10.6** A binary transmission uses two signaling waveforms $s_1(t)$ and $s_2(t)$, such that

$$s_1(t)=\begin{cases}\sin\dfrac{\pi}{T}t, & 0\le t\le T\\ 0, & \text{otherwise}\end{cases} \quad \text{and} \quad s_2(t)=\begin{cases}\sin\dfrac{2\pi}{T}t, & 0\le t\le T\\ 0, & \text{otherwise}\end{cases}$$

$s_1(t)$ and $s_2(t)$ are transmitted with equal probability. The additive noise during transmission is white Gaussian with mean zero and power spectral density $N_0/2$. Determine the minimum probability of error at the receiver.

**10.7** A binary transmission is constructed from two orthogonal signals $s_1(t)$ and $s_2(t)$, $0\le t\le T$, with energies $E_1=1$ and $E_2=0.5$, respectively. The additive noise is white Gaussian with mean zero and power spectral density 0.5. $s_1(t)$ and $s_2(t)$ are transmitted with equal a priori probabilities.

(a) Determine the achievable probability of error.

(b) Determine the minimum signal energy to achieve the same error performance.

**10.8** Consider the following binary detection problem. At the receiver, we have

$$\begin{aligned}H_1 &: Y(t)=\sqrt{E}s(t)+W(t), \quad 0\le t\le T\\ H_0 &: Y(t)=\qquad\quad W(t), \quad 0\le t\le T\end{aligned}$$

The additive noise is Gaussian with mean zero and power spectral density $N_0/2$. However, when a signal is transmitted, it can be either $s_1(t)$ or $s_2(t)$, which occur with probabilities $P_1$ and $P_2$, respectively. $s_1(t)$ and $s_2(t)$ are orthogonal over the observation interval, and have energies $E_1$ and $E_2$, respectively. Determine the decision rule that minimizes the probability of error.

**10.9** Let $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$ be three orthonormal functions over the interval $[0,T]$. Define $s_k(t), k=0,1,2,\ldots,7$, as

$$\begin{aligned}s_0(t)&=A[\phi_1(t)+\phi_2(t)+\phi_3(t)] & s_4(t)&=A[-\phi_1(t)+\phi_2(t)+\phi_3(t)]\\ s_1(t)&=A[\phi_1(t)+\phi_2(t)-\phi_3(t)] & s_5(t)&=A[-\phi_1(t)+\phi_2(t)-\phi_3(t)]\\ s_2(t)&=A[\phi_1(t)-\phi_2(t)+\phi_3(t)] & s_6(t)&=A[-\phi_1(t)-\phi_2(t)+\phi_3(t)]\\ s_3(t)&=A[\phi_1(t)-\phi_2(t)-\phi_3(t)] & s_7(t)&=A[-\phi_1(t)-\phi_2(t)-\phi_3(t)]\end{aligned}$$

The signals $s_k(t), k=0,1,2,\ldots,7$, are transmitted with equal a priori probabilities, and the received signal is given by

$$Y(t) = s_i(t) + W(t), \quad 0 \le t \le T$$

where $W(t)$ is the white Gaussian noise with mean zero and power spectral density $N_0/2$.

(a) Determine $A$, such that the energy of $s_k(t)$ is equal to $E$.
(b) Determine the receiver for minimum probability of error criterion.
(c) Show the decision regions.
(d) Find the minimum probability of error.

**10.10** During transmission of 16 quadrature amplitude modulated signals, an additive white Gaussian noise with mean zero and power spectral density $N_0/2$ is superimposed on the signals. The signal space is shown in Figure P10.10. The signal points are spaced $d$ units apart. They are given by

$$s_k(t) = a_k \phi_1(t) + b_k \phi_2(t), \quad \frac{-T}{2} \le t \le \frac{T}{2}, \quad k = 1, 2, \ldots, 16$$

where $\phi_1(t) = \sqrt{2/T} \cos 2\pi f_0 t$ and $\phi_2(t) = \sqrt{2/T} \sin 2\pi f_0 t$

Assume minimum probability of error criterion.

(a) Draw a block diagram of the optimum receiver.
(b) Show the decision regions in the signal space.
(c) Determine the probability of error.

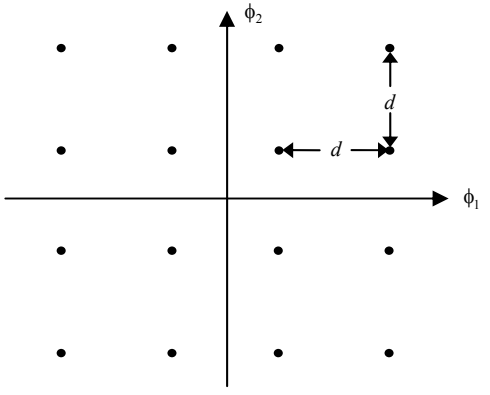


**Figure P10.10** Signal set.

**10.11** Starting from (10.104), derive the expression in (10.106).

**10.12** Consider the situation where the received signal is given by

$$H_1 : Y(t) = As(t) + W(t), \quad 0 \le t \le T$$
$$H_0 : Y(t) = \qquad W(t), \quad 0 \le t \le T$$

Let $A$ be an unknown constant and $W(t)$ be a white Gaussian noise process with mean zero and power spectral density $N_0/2$. Design the optimum receiver, assuming minimum probability of error criterion.

**10.13** Consider the estimation problem

$$Y(t) = s(t, \theta) + W(t), \quad 0 \le t \le T$$

where $s(t, \theta) = (1/\theta)s(t)$. $\theta$ is an unknown constant, whereas $s(t)$ is a known signal with energy $E$. $W(t)$ is a white Gaussian noise with mean zero and covariance function $C_{ww}(t, u) = (N_0/2)\delta(t-u)$. Determine $\hat{\theta}_{ml}$, the maximum likelihood estimate of $\theta$.

**10.14** Consider Problem 10.13, where $\theta$ is now a Gaussian random variable with mean zero and variance $\sigma_\theta^2$. Determine the equation for which a solution is $\hat{\theta}_{map}$, the maximum a posteriori estimate of $\theta$, and show that this equation also gives $\hat{\theta}_{ml}$ as the variance $\sigma_\theta^2 \to \infty$.

**10.15** Assume the received signal is given by

$$Y(t) = A\cos(\omega_c t + \theta) + W(t)$$

where $\theta$ is an unknown constant, and $W(t)$ is the white Gaussian noise with mean zero and power spectral density $N_0/2$.

(a) Determine the likelihood equation satisfied by the maximum likelihood estimate for $\hat{\theta}$. Assume the integral involving the double frequency terms is zero.

(b) Assuming $\hat{\theta}_{ml}$ unbiased, and apply the Cramer-Rao inequality to obtain a bound for $\text{var}[\hat{\theta}_{ml}]$ when $(A^2T/N_0) << 1$; that is, when the signal-to-noise ratio is very small.

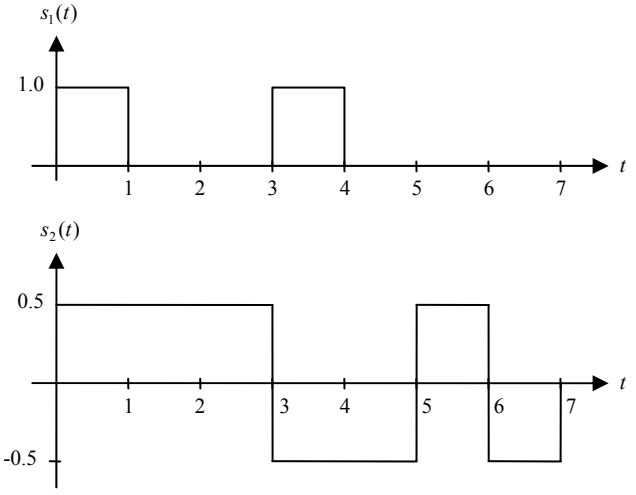


**Figure P10.16** Signals $s_1(t)$ and $s_2(t)$.

**10.16** Let $s_1(t)$ and $s_2(t)$ be the signals shown in Figure P10.16.

(a) Specify the matched filter for each of the signals shown.
(b) Sketch the filter output as a function of time when the signal matched to it is the input.
(c) Sketch the output of the filter matched to $s_2(t)$ when the input is $s_1(t)$.

**10.17** Consider the general binary detection with

$$s_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \le t \le \frac{T}{2} \\ 0 \quad , & \text{otherwise} \end{cases} \quad \text{and} \quad s_2(t) = \begin{cases} \sqrt{\frac{2}{T}}, & \frac{T}{2} \le t \le T \\ 0 \quad , & \text{otherwise} \end{cases}$$

over an additive white Gaussian noise channel. The noise is assumed to have zero mean with power spectral density $N_0 / 2$.

(a) Determine the matched filters.
(b) Draw the responses (noise-free) of the matched filters.
(c) Compute the SNR at the output of the matched filters.

**10.18** Consider a digital communications system with a source using On-Off signaling. The channel superimposes on the transmitted signal an additive

white Gaussian noise process with mean zero and power spectral density $N_0/2$. The received waveforms are given by

$$\begin{aligned} H_1 &: Y(t) = A\cos(\omega_c t + \Theta) + W(t), \quad 0 \le t \le T \\ H_0 &: Y(t) = W(t), \quad 0 \le t \le T \end{aligned}$$

where the amplitude $A$ and the phase $\Theta$ are independent random variables with known density functions. Assume that $\omega_c = 2n\pi/T$ where $n$ is an integer, $\Theta$ is uniformly distributed over the interval $[0, 2\pi]$, and $A$ is Rayleigh distributed with density function

$$f_A(a) = \begin{cases} \dfrac{a}{\sigma_a}\exp\left(-\dfrac{a^2}{2\sigma_a^2}\right), & a \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

The signal $s_1(t) = \cos\omega_c t$, $0 \le t \le T$, has energy $E$.

(a) Determine the optimum decision rule.
(b) Draw a block diagram of the optimum receiver.

**10.19** Consider the problem of signals with random phase and amplitude given in Section 10.6.2. Using the Neyman-Pearson test, determine the probability of detection for a specified probability of false alarm. The random variables $\Theta$ and $A$ are assumed to be statistically independent, with $\Theta$ uniformly distributed over the interval $[0, 2\pi]$ and $A$ Rayleigh distributed.

**10.20** Consider the estimation problem

$$Y(t) = s(t, \theta) + N(t), \quad 0 \le t \le T$$

where $N(t)$ is a nonwhite Gaussian noise with mean zero and covariance function

$$C_{nn}(t, u) = C_{n'n'}(t, u) + C_{n_c n_c}(t, u)$$

where $C_{n'n'}(t, u) = (N_0/2)\delta(t - u)$. The received waveform is passed through a correlation operation to yield

$$Y_k = s_k(\theta) + N_k, \quad k = 1, 2, \ldots, K$$

such that

$$s_k(\theta) = \int_0^T s(t,\theta)\phi_k(t)dt$$

and $N_k, k = 1, 2, \ldots, K$, are random variables. $\phi_k(t), k = 1, 2, \ldots, K$, are eigenfunctions corresponding to $\lambda_k, k = 1, 2, \ldots, K$, associated with the covariance function $C_{n_c n_c}(t,u)$. In the limit as $K \to \infty$, we have

$$Y(t) = \lim_{K\to\infty} \sum_{k=1}^{K} s_k(\theta)\phi_k(t) + \lim_{K\to\infty} \sum_{k=1}^{K} N_k \phi_k(t)$$

(a) Determine the mean and variance of the Karhunen-Loève coefficients $N_k$.
(b) Are the noise components $N_k, k = 1, 2, \ldots$, statistically independent?
(c) If $Y(t)$ is passed through a whitening filter to obtain

$$\begin{aligned} Y'(t) &= s'(t,\theta) + N'(t), \qquad 0 \le t \le T \\ &= \lim_{K\to\infty} \sum_{k=1}^{K} s'_k(\theta)\phi_k(t) + \lim_{K\to\infty} \sum_{k=1}^{K} N'_k(\theta)\phi_k(t) \end{aligned}$$

determine the mean and variance of the white noise component $N_k$.

**10.21** Consider a noise process, such that $N_1(t) = W$ in the interval $t \in [0, T]$. $W$ is Gaussian with mean zero and variance $\sigma_w^2$.
(a) Can $N_1(t)$ be whitened in the given interval? Explain.
(b) Repeat (a) assuming that another independent noise process $N_2(t)$ is superimposed on $N_1(t)$, such that the new noise process is

$$N(t) = N_1(t) + N_2(t), \quad 0 \le t \le T$$

and the covariance function of $N_2(t)$ is

$$C_{n_2 n_2}(t,u) = \frac{N_0}{2}\delta(t-u), \quad 0 \le t, u \le T$$

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# Chapter 11

# Adaptive Thresholding CFAR Detection

## 11.1 INTRODUCTION

In signal detection, the basic goal is to derive the optimum receiver structure based on some criterion that is determined by the application. Such optimal (ideal) detectors require an essentially complete statistical description of the input signals and noise. In practice, this information may not be available a priori, and the statistics of the input data may also vary with time. These constraints lead to the consideration of other (nonoptimal) detectors.

In practical radar signal detection systems, the problem is to automatically detect a target in thermal noise plus clutter. *Clutter* is the term applied to any unwanted radar signal from scatterers that are not of interest to the radar user [1]. Examples of unwanted echoes, or clutter, in radar signal detection are reflections from terrain, sea, rain, birds, insects, chaff, and so on. *Chaff* consisting of dipole reflectors, usually metallic, is dropped from aircrafts to mask the real target from the radar. Due to the development of Doppler processing, it is now mainly used for slow moving targets [2]. Since the environment in which a radar operates depends on factors such as weather conditions and the physical location of operation, the returned signals are statistically nonstationary, with unknown variance at the receiver input. Thus, the ideal detector using a fixed threshold is extremely sensitive to the total noise (thermal noise plus clutter) variance. In fact, a small increase in the total noise power results in a corresponding increase of several orders of magnitude in the probability of false alarm. For a single pulse detection, it can be shown that the probability of false alarm is given by [3]

$$P_F = \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) \tag{11.1}$$

where $\gamma$ is the threshold level and $\sigma^2$ is the total noise variance. Let $P_{F_d}$ be the design probability of false alarm based on a known variance $\sigma_d^2$. For a fixed

threshold γ, the probability of false alarm in terms of the noise level and design probability of false alarm is obtained from (11.1) as

$$P_F = \left(P_{F_d}\right)^{\sigma_d^2/\sigma^2} \tag{11.2}$$

The subscript *d* denotes design value. As illustrated in Figure 11.1, for a design probability of false alarm of $10^{-6}$, an increase of only 3 dB in the noise power causes the actual probability of false alarm to increase by more than 1,000, which is intolerable for data processing, either by a computer or by a human operator. Therefore, adaptive threshold techniques are needed to maintain a CFAR. The receiver is desired to achieve CFAR and maximum probability of detection of the target.

In order to appreciate the practical aspects of adaptive thresholding detection, we first give a brief description of radar principles. It should be noted that radar concepts can be very involved, and many books are written in that sense. Thus, we only give the necessary fundamentals to be able to understand adaptive thresholding CFAR detection. Then, we discuss some of the adaptive CFAR techniques. We also introduce the concept of adaptive CFAR detection in mobile communications. In particular, we consider applications of adaptive CFAR detection in code division multiple access (CDMA), and thus we must also give a brief description of spread spectrum communication systems. Again, we present only the necessary fundamentals to be able to appreciate the applications and possible future developments.

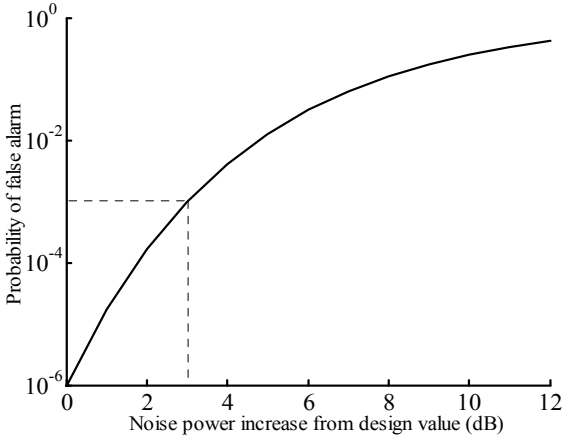


**Figure 11.1** Effect of the noise power increase on the probability of false alarm for a fixed threshold; design $P_F = 10^{-6}$.

## 11.2 RADAR ELEMENTARY CONCEPTS

Radar is derived from the initials of the phrase RAdio Detection And Ranging. Radar is an electromagnetic system used for the detection and location of objects (or targets). It achieves these two purposes by transmitting an electromagnetic energy and then extracting the necessary information about the target from the returned echo signal, as shown in Figure 11.2. This information is drawn from the changes observed in the signal parameters. The range, or distance, is determined from the measurements of the time taken for the radar signal to travel to the target and back (time delay). The direction, or angular location, of the target relative to the radar is found with a directive antenna. The shift in the carrier frequency of the received echo signal caused by a moving target (Doppler effect) yields information on the range rate or velocity, and also may be used to distinguish moving targets from stationary objects. Thus, from the measurements of these parameters with time, the radar can derive the track, or trajectory, of a moving target, and predict its future location.

In general, the transmitter and the receiver are in the same location. This is called a *monostatic* radar. This particular system shares a single antenna between the transmitter and the receiver ($R_b = 0, R_1 = R_2$) via a waveguide called the *duplexer* [4]. A simplified block diagram of a modern monostatic radar is shown in Figure 11.3.

The major blocks—the modulator, the transmitter, the receiver, the signal processor, the data processor, and the display—and their functions are now briefly described [5].

*The Modulator* Upon reception of each timing pulse, the modulator produces a high-pulse direct current and supplies it to the transmitter.

*The Transmitter* The transmitter is a high-power oscillator. It generates a high-peak power coherent train of pulses to illuminate the target.

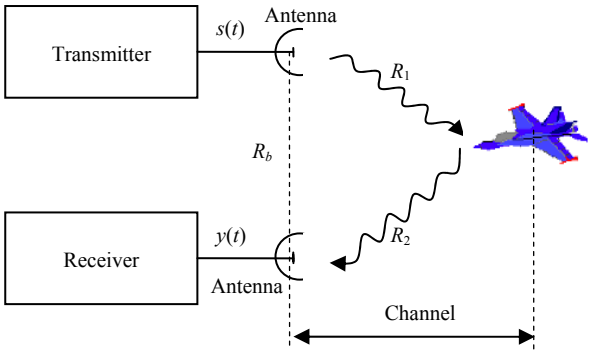


**Figure 11.2** Basic radar scene.

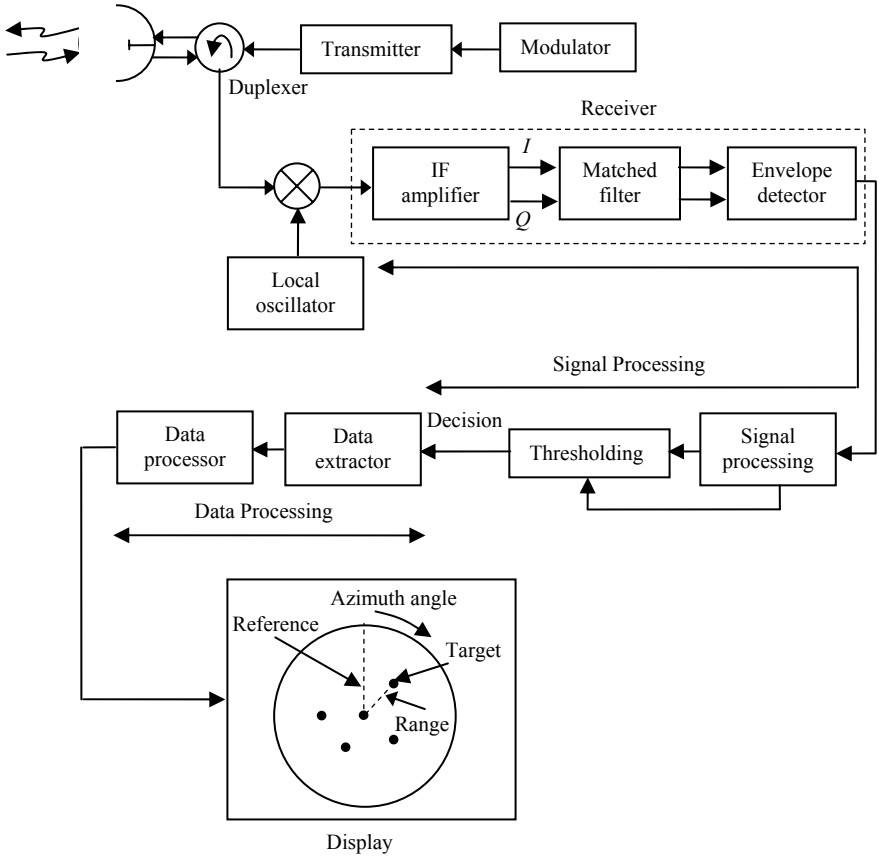


**Figure 11.3** Signal and data processing in a modern pulse radar system.

*The Receiver* Typically, the receiver is of a superheterodyne type. It provides frequency conversion (to a lower frequency called intermediate frequency, IF), interference rejection, and low-noise amplification. Noise reduction is an important consideration in radar receiver design, and it is accomplished by a matched filter that maximizes the SNR at the output.

*Signal Processor* This device processes the target echoes and interfering signals to increase the signal-to-interference ratio. The operations may be pulse compression, Doppler range clutter suppression techniques, and CFAR processing. This is the part that will be developed in detail. However, we say briefly that the CFAR circuit keeps the rate of occurrence of false decisions (alarms) due to background noise and clutter at a constant and relatively low rate. This prevents saturation of the system and/or user. It estimates the noise and clutter level from a

number of range, Doppler, and/or azimuth cells to allow the threshold to be set correctly.

*Data Processor* It provides the target measurements in range, angle (azimuth and elevation), radial velocity, and possibly the target signature.

*Display* The output is generally conveyed to a display to visualize the information contained in the target echo signal in a form suitable for operator action and interpretation. *The plan position indicator* (PPI) is the usual display employed in the radar receiver, and it indicates the range and azimuth of a detected target.

When the transmitter and the receiver are in separate locations ( $R_b \neq 0$ ), this is called a *bistatic* radar. In this case, the ranges $R_1$ and $R_2$ may not be the same. A *multistatic* radar is a radar with one transmitting antenna, but many receiving stations, all in a network. Most radars nowadays are *active* and of *pulse* type; that is, the radars have a transmitter, and the signal transmitted is a pulse.

### 11.2.1 Range, Range Resolution, and Unambiguous Range

When a pulse is transmitted, the radar clock begins counting the time. The time taken for the pulse to travel to the target and return is called the *time delay*, $\tau_d$ . The range of the target is given by

$$R = \frac{c\,\tau_d}{2} \tag{11.3}$$

where $c$ is velocity of light $c = 3\times10^8$ m/s. Since radar signals propagate in real atmosphere and not a vacuum, $c$ is actually the velocity of propagation given by $c = 2.9979\times10^8$ m/s. The factor 2 appears in the denominator of (11.3) because of the two-way travel of the pulse to the target and back.

If the time delay between the echoes from two targets is equal to or greater than the pulse duration, then two separate echoes are observed, as shown in Figure 11.4(a, b), respectively. In this case, the targets are resolvable. However, if the time delay is less than a pulse duration, the targets are not resolvable, as shown in Figure 11.4(c). Instead of seeing two targets, we see one large one. Hence, the range resolution between the two targets is [2]

$$\Delta R = \frac{c\tau}{2} \tag{11.4}$$

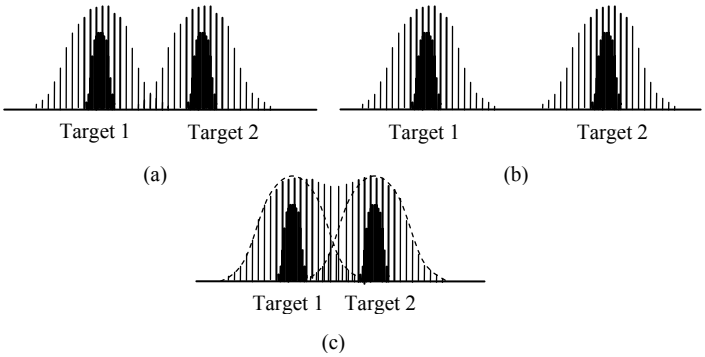


**Figure 11.4** Returned target echoes: (a) $\tau_d = \tau$ , (b) $\tau_d > \tau$, and (c) $\tau_d < \tau$.

The radar receiver samples the output every $\tau$ seconds, and thus each sample represents a distance $\Delta R$ , called *range gate* or *range bin*. For example, if the radar pulse duration is $\tau = 1\,\mu s$ and we desire a receiver output every 150m in range, we would use a 1-MHz *A*/*D* sampler.

The rate at which pulses are transmitted is called pulse repetition frequency (PRF), $f_p$, and it is determined by the maximum at which the targets are expected, such that

$$f_p \le \frac{c}{2R_{\max}} \tag{11.5}$$

That is, in transmitting multiple pulses, the limit occurs when the second pulse is transmitted before the first one has completed its two-way trip to the target and back. This maximum range is called the *unambiguous range*, also denoted $R_u$. For example, if we use a pulse repetition frequency of $f_p = 1$ kHz, the maximum range is $R_{\max} \le c/2f_p = 150$ km. If we now want to survey this range with a higher PRF of 1.5 kHz, which has an $R_{\max}$ of 100 km, the echo of the first pulse may be confused with the echo of the second one, as shown in Figure 11.5. We observe that target *A* at 30 km is within the unambiguous range. However, target *B* at 130 km could be the echo of the first pulse at 130 km, or the echo of the second pulse at 30 km.

A typical radar transmits a series of *N* pulses. The *pulse width* $\tau$, the *interpulse period* *T*, and the transmission duty cycle $\tau/T$, as shown in Figure 11.6, are constant throughout the transmission of all *N* pulses [6]. *T* is called the pulse repetition interval (PRI) and is the inverse of the pulse repetition frequency, $T = 1/f_p$. The *N* transmitted pulses are coherent; that is, they are in-phase, and

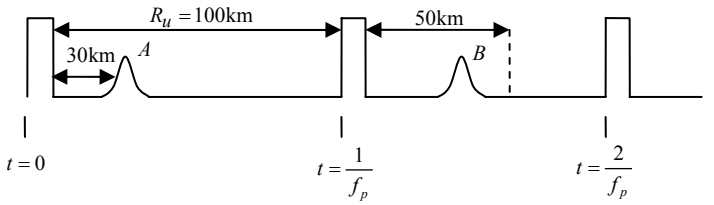


**Figure 11.5** Illustration of ambiguous range.

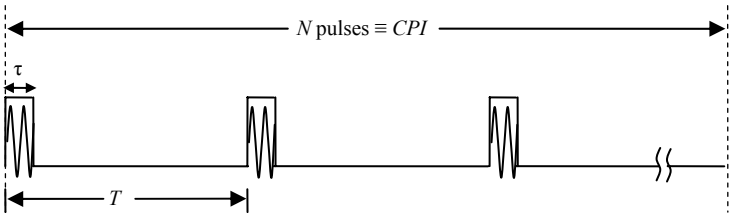


**Figure 11.6** Coherent pulse train.

the set of $N$ coherent pulses transmitted during the time interval $T$ is called a *coherent pulse-train*. The time spanned by the coherent pulse-train is called a coherent processing interval (CPI).

### 11.2.2 Doppler Shift

An accurate way of measuring the speed of a target is the use of *Doppler frequency* shift, which is the difference between the received frequency and the transmitted frequency caused by the motion of the target. In this case, a *coherent* system is needed, in the sense that the transmitter and the receiver oscillators are phase locked, in order to detect any difference in the echo signal. Thus,

$$f_d = f_r - f_t \tag{11.6}$$

where $f_d$ is Doppler frequency, $f_r$ is the receiver frequency, and $f_t$ is the transmitter frequency. Doppler frequency is given in terms of $v_r$, the radial component of the target speed toward the radar, by

$$f_d = \frac{2v_r}{\lambda} \text{ Hz} \tag{11.7}$$

where $v_r << c$, $c$ is the speed of light, and the *wavelength* $\lambda$ is given by

$$\lambda = \frac{c}{f_t} \tag{11.8}$$

For fixed objects, $f_d$ equals zero.

## 11.3 PRINCIPLES OF ADAPTIVE CFAR DETECTION

The input signal at the radar receiver, when a target is present, is an attenuated randomly phase-shifted version of the transmitted pulse in noise. A typical radar processor for a single-range cell sums the $K$ samples of the matched filter output and compares the sum to a fixed threshold, as shown in Figure 11.7. When the transmitted pulse is embedded in white Gaussian, the clutter return signal envelope is Rayleigh distributed [7]. The optimum Neyman-Pearson detector, for this case, is shown in Figure 11.8, where $y(t)$ denotes the received signal and $\omega_c$ the carrier angular frequency. From Figure 11.1, we saw that a small increase in noise power causes the probability of false alarm to increase intolerably. Hence, when the noise variance is not known, and in order to regulate the false alarm probability, Finn and Johnson [8] proposed the use of a reference channel, from which an estimate of the noise environment can be obtained, and upon which the decision threshold is adapted. The radar uses the range cells surrounding the cell

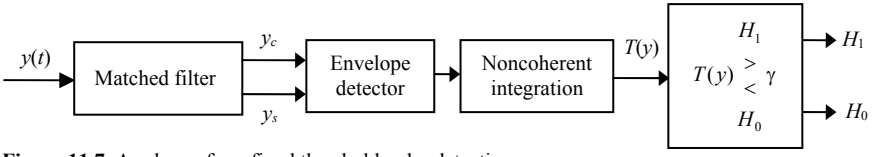


**Figure 11.7** A scheme for a fixed threshold radar detection.

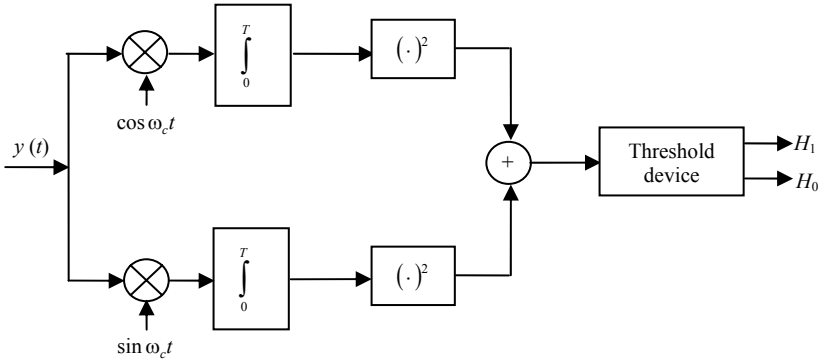


**Figure 11.8** Optimum receiver, square realization.

under test as reference cells, as shown in Figure 11.9. The detector proposed in [8] is the cell-averaging constant false alarm (CA-CFAR), where the adaptive threshold is obtained from the arithmetic mean of the reference cells. For a homogeneous background noise, and independent and identically distributed reference cells outputs, the arithmetic is the maximum likelihood estimate. This means that the detection threshold is designed to adapt to changes in the environment. These noise observations are obtained by sampling in range and Doppler, as shown in Figure 11.10. The bandwidth of each Doppler, (bandpass) filter is equal to the bandwidth of the transmitted rectangular pulse $B$, where

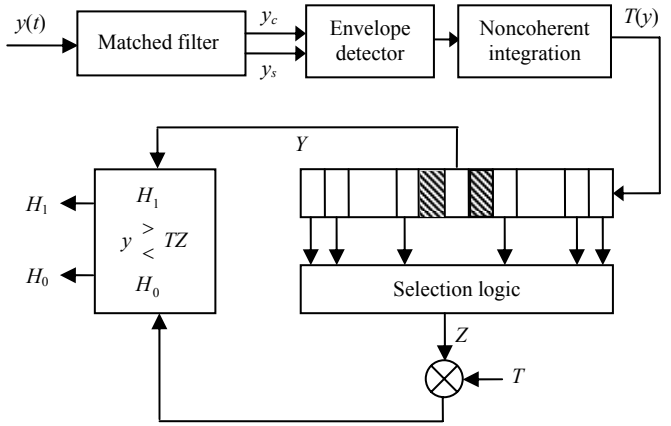


**Figure 11.9** A scheme for an adaptive threshold radar detection.

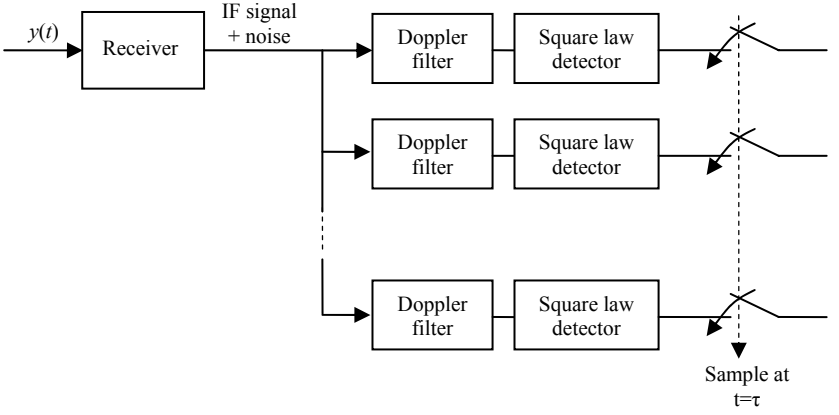


**Figure 11.10** Range and Doppler sampling process.

$B = 1/\tau$ and $\tau$ is the transmitted pulse width. The output of each square-law detector is sampled every $\tau$ seconds, which corresponds to a range interval of $c\tau/2$. Hence, each sample can be considered as the output of a range-Doppler resolution cell with dimensions $\tau$ in time and $1/\tau$ in frequency [9]. Therefore, we obtain a matrix of range and Doppler resolution cells, as shown in Figure 11.11. For simplicity and without loss of generality, we show the CA-CFAR detector in Figure 11.12 for range cells only and for a specific Doppler frequency.

We now describe the system in more detail. The output from the square-law detector is fed into a tapped delay line forming the reference cells. To avoid any signal energy spill from the test into directly adjacent range cells, which may

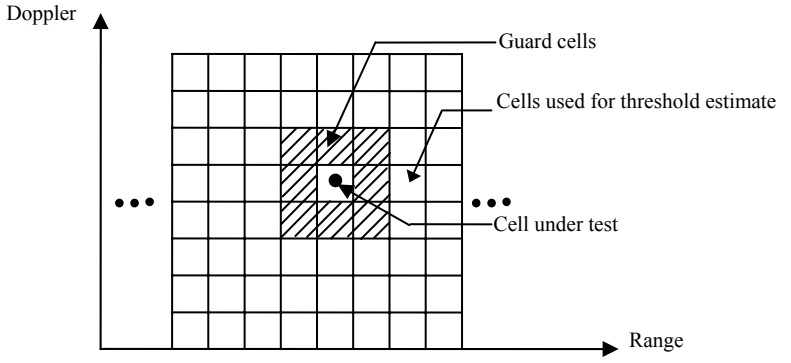


**Figure 11.11** Matrix of range and Doppler cells.

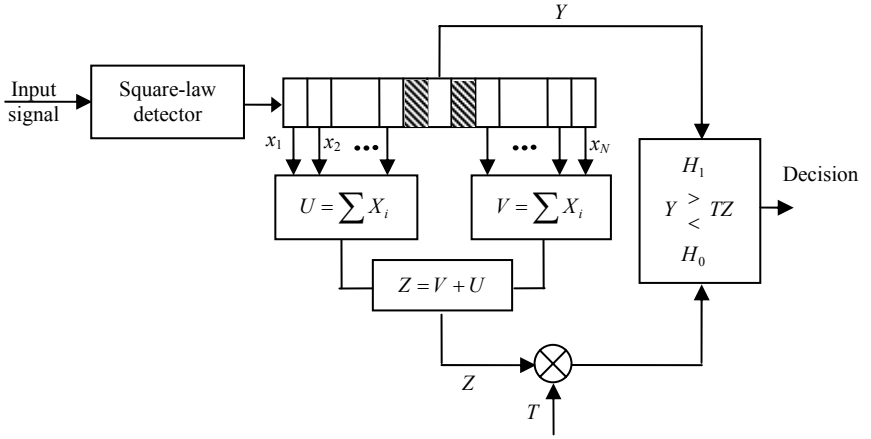


**Figure 11.12** Cell averaging CFAR detector.

affect the clutter power estimate, the adjacent cells, called *guard cells*, are completely ignored. Each resolution cell is tested separately in order to make a decision for the whole range of the radar. We assume that the cell under test is the one in the middle. The statistics of the reference windows *U* and *V* are obtained from the sum of the $N/2$ leading cells and $N/2$ lagging cells, respectively. Thus, a total of *N* noise samples are used to estimate the background environment. The reference windows *U* and *V* are combined to obtain the estimate of the clutter power level *Z*. To maintain the probability of false alarm, $P_F$, at the desired value, the adaptive threshold is multiplied by a scaling factor called the *threshold multiplier T*. The product $TZ$ is the resulting adaptive threshold. The output $Y$ from the test cell (center tap) is then compared with the threshold in order to make a decision.

We assume that the target model at the test cell, called the *primary target*, is a slowly fluctuating target of *Swerling Case 1*. The signal-to-noise ratio of the target is denoted *S*. We further assume that the total background noise is white Gaussian. Since both the noise and Rayleigh targets have Gaussian quadrature components, the output of the square-law detector has an exponential probability density function [2]. If the noise variance is $\sigma^2$, then the conditional density function of the output of the test cell is given by

$$f_{Y|H_i}(y \mid H_i) = \begin{cases} \dfrac{1}{2\sigma^2(1+S)} \exp\left[-\dfrac{y}{2\sigma^2(1+S)}\right], & \text{for } H_1 \\ \dfrac{1}{2\sigma^2} \exp\left(-\dfrac{y}{2\sigma^2}\right), & \text{for } H_0 \end{cases} \tag{11.9}$$

The hypothesis $H_0$ represents the case of noise alone, while hypothesis $H_1$ represents the noise plus target signal case.

The probability of detection is given by

$$P_D = \int_0^\infty P(Y > TZ \mid Z, H_1) f_{Y|H_1}(y \mid H_1) dy = E_Z[P(Y > TZ \mid Z, H_1)] \tag{11.10}$$

where *Z* is the estimated homogeneous background noise power level, $f_Z(z)$ is the density function of *Z*, and $E_Z[\cdot]$ is the expected value over all values of *z*. Substituting (11.9) into (11.10) and solving the integral, we obtain

$$P_D = E_Z\left\{\int_{Tz}^\infty \frac{1}{2\sigma^2(1+S)} \exp\left[-\frac{y}{2\sigma^2(1+S)}\right] dy\right\} = E_Z\left\{\exp\left[-\frac{TZ}{2\sigma^2(1+S)}\right]\right\}$$

$$= M_Z\left[\frac{T}{2\sigma^2(1+S)}\right] \tag{11.11}$$

where $M_Z(\cdot)$ denotes the MGF of the random variable $Z$. We can obtain the probability of false alarm in a similar way, or by setting the target SNR, $S$, to zero to obtain

$$P_F = M_Z\left(\frac{T}{2\sigma^2}\right) \tag{11.12}$$

Hence, for a design probability $P_F$, the threshold multiplier $T$ can be computed from (11.12). For the CA-CFAR detector, the reference window is

$$Z = \sum_{i=1}^{N} X_i \tag{11.13}$$

with $X_i, i = 1, 2, \ldots, N,$ independent and identically distributed random variables.

From Chapter 2, the gamma density function $G(\alpha, \beta)$ given in (2.98) is

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \tag{11.14}$$

with MGF

$$M_x(t) = \frac{1}{(1-\beta t)^\alpha} \tag{11.15}$$

If we set $\alpha = 1,$ we obtain the exponential distribution $G(1, \beta)$ with density function

$$f_X(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}} \tag{11.16}$$

which is equivalent to $f_{Y|H_i}(y \mid H_i)$ given in (11.9), with $\beta = 2\sigma^2$ under hypothesis $H_0$, and $\beta = 2\sigma^2(1+S)$ under hypothesis $H_1$. Thus, using (11.15), the probability of false alarm of the distribution $G(N, 2\sigma^2)$ is

$$P_F = M_z\left(\frac{T}{2\sigma^2}\right) = \left[1 - 2\sigma^2\left(\frac{T}{2\sigma^2}\right)\right]^{-N} = \frac{1}{(1+T)^N} \tag{11.17}$$

The threshold multiplier is then

$$T = -1 + P_F^{-\frac{1}{N}} \tag{11.18}$$

Replacing $T/2\sigma^2$ by $T/[2\sigma^2(1+S)]$, the probability of detection is [10, 11]

$$P_D = \left(1 + \frac{T}{1+S}\right)^{-N} = \left(\frac{1+S}{1+S+T}\right)^N \tag{11.19}$$

For this homogeneous clutter background, the detection performance of the CA-CFAR detector is optimum in the sense that its probability of detection approaches, that of the (ideal) Neyman-Pearson detector as the number of reference cells becomes infinite. Hence, there is an inherent loss in the probability of detection of the adaptive CFAR detector when compared to the Neyman-Pearson detector.

In general, the CFAR loss in the design, while computing the scale factor *T*, is a function of the background noise assumed, the design probability of false alarm, and the reference window size *N* [12, 13]. It is also a function of the CFAR algorithm adopted, as we will see. This gives an idea about the many CFAR processors we can have for different applications. In fact, hundreds of papers were published to deal with the different applications. Thus, we can only give a rough sketch showing the evolution and variety of classes of CFAR problems.

Note also in deriving expressions for the probabilities of detection and false alarm, we assumed a target model of Swerling Case 1, which we did not define. This means that other targets may be considered depending on the application. Before giving the definitions of target models in the next section, it should be noted that there are different types of radar targets. The simplest target that we are considering is the *point target*, but there are other types of targets. A *point target* is one whose largest physical dimension is small relative to the range cell $(c\tau/2)$ of the transmitted pulse [4]. Such targets may be many aircrafts, satellites, small boats, people and animals, and land vehicles. These targets are small enough so that no significant "smearing" or spreading in time occurs in the received pulses. Larger targets that can cause spreading in the received pulses, such as large buildings, ships, and some aircraft, are called *extended targets*. Still larger targets are called *distributed targets*. In this latter case, there is a class of targets called *area targets*, which represents targets such as forests, oceans, and mountains. Another class of targets, representing targets such as rain, snow, fog, smoke, and chaff, is called *volume targets*.

### 11.3.1 Target Models

When a target is present, the amplitude of the signal at the receiver depends on the target radar cross section (RCS), which is the effective scattering area of a target as seen by the radar. In general, the target RCS fluctuates because targets consist of many scattering elements, and returns from each scattering element vary. The effect of the fluctuation is to require a higher signal-to-noise ratio for high probability of detection, and lower values for low probability of detection than those required with nonfluctuating signals. In addition, returns from the same scattering element are functions of the illumination angle, the frequency and polarization of the transmitted wave, the target motion and vibration, and the kinematics associated with the radar itself [1, 2, 4, 12–15]. Target RCS fluctuations are often modeled according to the four Swerling target cases, Swerling case 1 to 4. These fluctuating models assume that the target RCS fluctuation follows either a Rayleigh or one-dominant-plus Rayleigh distribution with *scan-to-scan* or *pulse-to pulse* statistical independence.

A *scan* is when the antenna main beam of the radar makes one complete search of a surveillance region, as shown in Figure 11.13. When the antenna's main beam crosses a target, the radar receives a group of $N$ pulses within a resolution angle of the surveillance region. If the reflected target amplitude is constant over the entire time it takes to observe a resolution angle, as the antenna returns to again search the area containing the target, the RCS may have changed. This *slow* fluctuation of the radar reflected target amplitude from a pulse-group to a pulse-group, but not within a group, is called *scan-to-scan* fluctuation. However, when the radar-reflected target amplitude is *fast* enough so that it can be considered independent for each pulse within the group of $N$-pulses, this fluctuation is called *pulse-to-pulse*. The four Swerling cases are defined as follows.

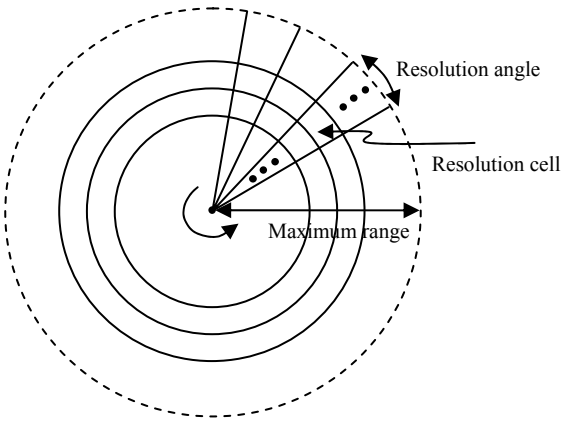


**Figure 11.13** A radar scan.

*Swerling Case 1* In this case, the returned signal power per pulse on any one scan is assumed to be constant, but these echo pulses are independent (uncorrelated) from scan to scan. A returned signal of this type is then a scan-to-scan fluctuation. The envelope of the entire pulse-train is a single Rayleigh-distributed independent random variable given by

$$P(S) = \frac{1}{m_s} \exp\left(-\frac{S}{m_s}\right), \quad S \geq 0 \tag{11.20}$$

where $m_s$ is the average cross section (average of RCS or signal-to-noise power ratio *S*) over all target fluctuations.

*Swerling Case 2* In this case, the fluctuations are more rapid than in Case 1, and are assumed to be independent from pulse-to-pulse instead of from scan-to-scan. This is pulse-to-pulse fluctuation. The probability density function for the target cross section is the same as given in (11.20).

*Swerling Case 3* In this case, the fluctuations are scan-to-scan as in Case 1, but the probability density function is given by

$$P(S) = \frac{4S}{m_s^2} \exp\left(-\frac{2S}{m_s}\right), \quad S \geq 0 \tag{11.21}$$

*Swerling Case 4* In this case, the fluctuations are pulse-to-pulse as in Case 2, but the probability density function is given by (11.21).

Note that in Cases 1 and 2, the targets are assumed to be composed of a large number of independent scatterers, none of which dominates (e.g., large aircraft). Cases 3 and 4 represent targets that have a single dominant nonfluctuating scatterer, together with other smaller independent scatterers (e.g., missiles). Observe that Cases 1 and 2 targets produce signals whose envelopes are Rayleigh distributed, while Cases 3 and 4 targets produce signals whose envelopes are chi-squared distributed.

*Swerling Case 5* Often, nonfluctuating targets are said to have Swerling Case 5 or Swerling Case 0. In this case, the received signal amplitude is assumed unknown, and there is no amplitude (or RCS) fluctuation.

Swerling Cases 1 to 4 are the models most commonly used, even though other models have been developed. They are summarized in the chi-square target models family by [13]

$$P_k(S) = \frac{1}{\Gamma(k)} \frac{k}{m_s} \left( \frac{kS}{m_s} \right)^{k-1} \exp\left( -\frac{kS}{m_s} \right), \quad S \geq 0 \tag{11.22}$$

where $\Gamma(k) = (k-1)!$, $S = A^2 / 2\sigma^2$ is the target signal-to-noise power ratio (radar cross section), $m_s$ is the average signal-to-noise ratio (mean cross section), $k = m_s^2 / \text{var}[S]$, $\sigma^2$ is the noise variance, and $A$ is the signal amplitude. Table 11.1 shows the different Swerling target models for different values of $k$.

### 11.3.2 Review of Some CFAR Detectors

There are three main approaches to the CFAR problem: the adaptive threshold processor, the nonparametric processor, and the nonlinear receiver approach. The adaptive threshold processor is the one most commonly used, because it provides the lowest CFAR loss when the actual environment closely matches the design environment. Of the hundreds of papers published in this field, we shall mention only a few to give a sketch of the advance of this rich field up to the actual interest when using high-resolution radars.

A real environment in which a radar operates cannot be described by a single clutter model. We refer to homogeneous clutter in situations where the outputs of the range cells are identically distributed and statistically independent. In a nonhomogeneous background, the adaptive threshold setting is seriously affected, resulting in a degradation of the performance.

*Clutter Edge*

This model is defined to describe situations where there is a transition in the clutter power distribution. The transition is not relatively smooth, and it is assumed that

**Table 11.1**
Different Cases to Which Swerling Models Apply

| *Model* | $k$ | *Fluctuations* | | *Scatterers* |
|---|---|---|---|---|
| | | *Scan-to-Scan* | *Pulse-to-Pulse* | |
| Swerling Case 1 | 1 | ✓ | | Many independent |
| Swerling Case 2 | 1 | | ✓ | |
| Swerling Case 3 | 2 | ✓ | | One dominant |
| Swerling Case 4 | 2 | | ✓ | |
| Swerling Case 5 | ∞ | Nonfluctuating | | |

the total noise power as a function of range can be represented by the step function, as shown in Figure 11.14. This may represent the boundary of a precipitation area. Two cases may be encountered in this severe clutter environment. In the first case, the cell under test is in the clear, but a group of reference cells are immersed in the clutter. This results in a higher adaptive threshold, and the probabilities of detection and false alarm are reduced. This is also known as the masking effect. In the second case, if the cell under test is immersed in the clutter but some of the reference cells are in the clear region, the threshold is relatively low, and the probability of false alarm increases intolerably. Hansen and Sawyers [16] proposed the greatest-of-selection logic in cell averaging constant false-alarm rate detector (GO-CFAR) to control the increase in the probability of false alarm. In the GO-CFAR detector, the estimate of the noise level in the cell under test is selected to be the maximum of $U$ and $V$, $X = \max(U, V)$, where $U$ and $V$ are the sums of the outputs of the leading and lagging cells, respectively. If one or more interfering targets are present, Weiss [17] has shown that the GO-CFAR detector performs poorly, and suggested the use of the smallest-of-selection logic in cell averaging constant false-alarm rate detector (SO-CFAR). In the SO-CFAR detector, the minimum of $U$ and $V$, $X = \min(U, V)$, is selected to represent the noise level estimate in the cell under test. The SO-CFAR detector was first proposed by Trunk [18] while studying the target resolution of some adaptive threshold detectors. We can intuitively see that the SO-CFAR detector performs well for the case shown in Figure 11.14(a).

*Homogeneous Background Plus Interfering Targets*

This model is defined to describe situations where the clutter background is composed of homogeneous white Gaussian noise plus interfering targets. The targets appear as spikes in individual range cells. These interfering targets may fall in either the leading or lagging reference cells, or in both leading and lagging range cells at the same time [19].

When interfering targets lie in the reference cells of the target under consideration, the primary target, the threshold is raised and the detection of

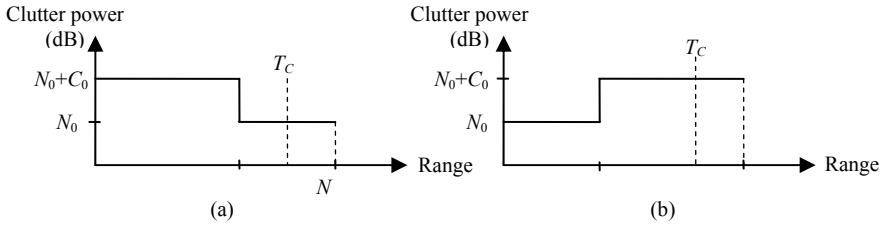


**Figure 11.14** Model of a clutter edge, test cell in (a) clear and (b) clutter. $N_0$ = noise power, $C_0$ = clutter power.

the primary target is seriously degraded. This is known as the capture effect. With the threshold too high, some targets may be undetected, as illustrated in Figure 11.15. On the other hand, if the threshold is not high enough, as illustrated in Figure 11.16, the number of false alarms due to noise spikes increases. To alleviate such problems, much research work has been proposed in the literature. Rickard and Dillard [20] proposed the censored mean level detector (CMLD), in which target samples are censored and the noise level estimate is obtained from the remaining noise samples. Ritcey [21] studied the performance of the CMLD for a fixed number of interfering Swerling Case 2 targets. Gandhi and Kassam [22] proposed the trimmed mean level CFAR (TM-CFAR) detector, that implements trimmed averaging after ordering the samples in the window. When the number of interfering targets is not known a priori, Barkat et al. [23] proposed the generalized censored mean level detector (GCMLD), in which the number of interfering targets is determined and their corresponding samples are then sampled. In the censored mean level detector, the outputs of the range cells are ranked in *ascending* order according to their magnitude to yield the *N*-ordered samples

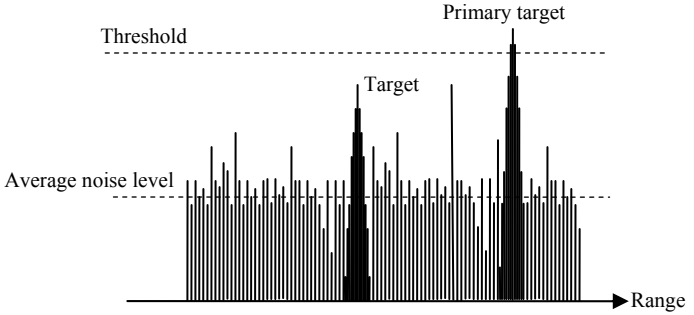


**Figure 11.15** Threshold too high.

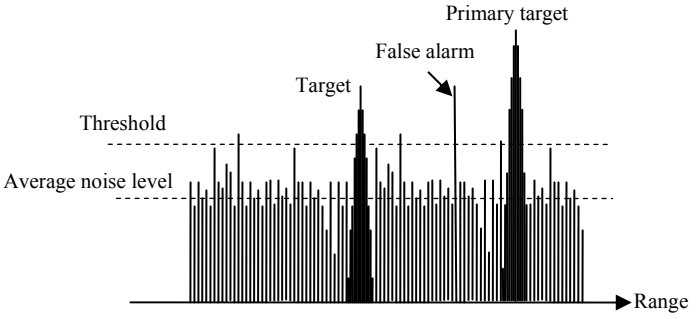


**Figure 11.16** Threshold not high enough.

$$X_{(1)} \le X_{(2)} \le \ldots \le X_{(k)} \le \ldots \le X_{(N-1)} \le X_{(N)} \tag{11.23}$$

Then, a censoring algorithm is applied according to the application. Rohling [24] proposed the order-statistic CFAR (OS-CFAR) detector which chooses *one* ordered sample to represent the noise level estimate in the cell under test. The *k*th ordered sample value, $X_{(k)}$, selected as the test statistic *Z*, is multiplied by the scale factor *T* to achieve the desired probability of false alarm, and then a decision is made by comparing the output of the cell under test *Y* with the adaptive threshold *TZ*. The value suggested in [24] to represent a good background estimate for typical radar applications in Gaussian noise is $k = 3N/4$. The calculations of the probabilities of detection and false alarm are relatively simple, and that makes the OS-CFAR detector a relatively more popular detector. The probability density function of the *k*th ranked sample in a Gaussian homogeneous background is given by [11, 24, 25]

$$f_{X_{(k)}}(z) = k\binom{N}{k}[1-F(z)]^{N-k}[F(z)]^{k-1} f(z) \tag{11.24}$$

where the noise density function is

$$f(z) = \frac{1}{2\sigma}\exp\left(-\frac{z}{2\sigma}\right), \quad z \ge 0 \tag{11.25}$$

and $F(z)$ is the corresponding distribution function given by

$$F(z) = 1 - e^{-z} \tag{11.26}$$

Substituting (11.25) and (11.26) in (11.24), we obtain

$$f_{X_{(k)}}(z) = \frac{k}{2\sigma}\binom{N}{k}\left[\exp\left(-\frac{z}{2\sigma}\right)\right]^{N-k+1}\left[1-\exp\left(-\frac{z}{2\sigma}\right)\right]^{k-1} \tag{11.27}$$

Using (11.12), the probability of false alarm is then

$$\begin{aligned} P_F &= \frac{k}{2\sigma}\binom{N}{k}\int_0^\infty \left[1-\exp\left(-\frac{z}{2\sigma}\right)\right]^{k-1}\left[\exp\left(-\frac{z}{2\sigma}\right)\right]^{T+N-k+1} dz \\ &= k\binom{N}{k}\int_0^\infty \left(1-e^{-z}\right)^{k-1}\left(e^{-z}\right)^{T+N-k+1} dz = \prod_{i=0}^{k-1}\frac{N-i}{N-i+T} \end{aligned} \tag{11.28}$$

Replacing $T$ by $T/(1+S)$ in (11.28), we obtain the probability of detection to be

$$P_D = \prod_{i=0}^{k-1} \frac{N-i}{N-i+\dfrac{T}{1+S}} \tag{11.29}$$

*Clutter Edge and Spikes*

This model describes the most general case in which there is not only a transition in the clutter power distribution, but also interfering targets, as illustrated in Figure 11.17. Himonas and Barkat [26] proposed the generalized two-level censored mean level detector (GTL-CMLD), which uses an automatic censoring algorithm of the unwanted samples when both interfering targets and extended clutter are present in the reference window of the cell under test. Khalighi and Bastani [27] presented another variation called the AEXGO-LOG processor.

Many papers were published using different variations of the above detectors for specific environments. For example, El-Mashade [28] studied the performance of the mean-level CFAR processor in multiple target environments when using $M$-correlated sweeps. In [29], an intelligent CFAR processor based on data variability was proposed. In [30], they considered an automatic censoring approach based also on ordered data variability, and proposed an automatic censoring CFAR detector for nonhomogeneous environments.

*Non-Gaussian Noise*

Non-Gaussian distributions have been considered since the beginning of adaptive thresholding techniques to represent certain types of clutter, such as sea clutter, land clutter, and weather clutter. The log-normal, Weibull, and gamma distributions have been used to represent envelope-detected non-Gaussian clutter distributions. In recent years, the $K$-distribution has been used mostly to model the

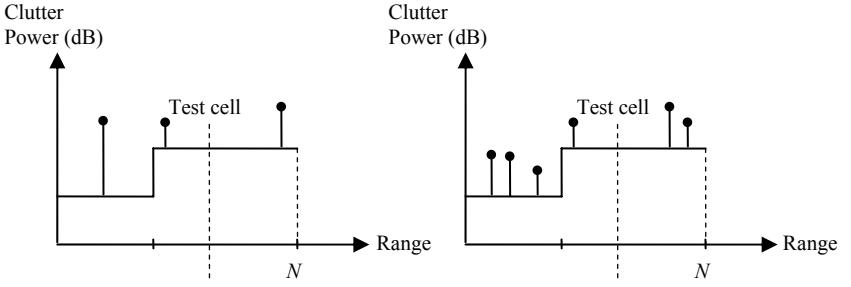


**Figure 11.17** Sample clutter power distribution when clutter edge and spikes appear in the reference range cells; $N_0$ = thermal noise power, $C_0$ = clutter power.

sea clutter [31–41]. The most important characteristic of the *K*-distribution is its ability to take into account the correlation properties of the sea echo. This ability is a result of the fact that the *K*-distribution is a compound distribution made up of a Rayleigh distributed component termed "speckle," whose mean level component varies slowly in time according to a chi-distribution, as discussed in Chapter 2. This is equivalent to modulating the square law detected speckle $S$ with a gamma distributed power modulation process $\tau$, referred to as "texture." A characteristic of all non-Gaussian distributions used in radar detection is their having a much longer "tail" than the Gaussian distribution. Thus, the optimum detectors used assuming a Gaussian background are no longer optimum, resulting in a significant increase in the probability of false alarm. If the threshold is raised to maintain a constant false alarm rate, then the probability of detection is seriously reduced. Thus, better signal processors are needed to obtain a high performance.

*High-Resolution Radars*

In early studies, the resolution capabilities of radars were relatively low, and the Gaussian representation of the background noise (that is, the amplitude is Rayleigh distributed) was a good statistical representation. Optimal detection approaches as discussed in the previous chapter were considered. As the resolution capabilities of radar systems improved, it was believed that the radar would intercept less clutter, and thus improve the detection performance. However, the detection performance did not improve, but rather the radar system was plagued by target-like "spikes" that gave rise to an intolerable increase in the false alarm rate [42]. It was then observed that the noise statistic was no longer Gaussian, as it was assumed. Hence, new clutter models were needed to reduce the effects of spikes to improve the detection performance. Studies showed that "good" distributions to represent spiky non-Gaussian clutter possess "longer tails," such as the Weibull distribution, log-normal distribution, and *K*-distribution, which are two parameter distributions. Anastassopoulos et al. [43] showed that these distribution models are special cases of the compound-Gaussian model. In Chapter 2, we discussed the different cases obtained from the compound-Gaussian model. There is a lot of research ongoing to improve detection performances while controlling the false alarm rate. Gini et al. [44] published a list of almost 700 references on radar signal processing, which comprises more than 120 papers on CFAR detection. In [45], a review of some CFAR detection techniques in radar systems was presented. Another reference is the paper published by Shnidman [46] on a generalized radar clutter model. Recently, Conte et al. [47] presented a statistical compatibility of real clutter data with the compound Gaussian model.

The literature on CFAR detection is very rich. I apologize to the many authors who contributed in this field but were not cited explicitly.

## 11.4 ADAPTIVE THRESHOLDING IN CODE ACQUISITION OF DIRECT-SEQUENCE SPREAD SPECTRUM SIGNALS

The concept of adaptive thresholding CFAR in digital communication systems started to appear in the literature in the last seven years. The basic CFAR operation is the same but the philosophy and approach are completely different. In the previous section, we introduced the concept of adaptive thresholding. We needed to give a very brief description of some radar principles, so that we can understand the application and its philosophy. Similarly in this section, we first present a brief description of spread spectrum signals in digital communication systems and then show how adaptive thresholding techniques are applied.

Spread spectrum communication signals have been used in military systems for decades because of their ability to reject interference. The interference can be unintentional when another transmitter tries to transmit simultaneously through the channel, or intentional when a hostile transmitter attempts to jam the transmission. By definition, for a communication system to be considered spread spectrum, it must satisfy two conditions. First, the bandwidth of the transmitted data must be much greater than the message bandwidth. Second, the system spreading is accomplished before transmission by some function (e.g., code or a PN sequence) that is independent of the message but known to the receiver. This same code is then used at the receiver to despread the signal so that the original data may be recovered. Thus, synchronization between the PN sequence generated at the receiver and the PN sequence used in the transmitted signal is necessary for demodulation. This may be achieved by sending a fixed PN sequence that the receiver will recognize in the presence of interference. After the time synchronization is established, transmission of information may commence.

The two main modulating techniques in spread spectrum communication systems are direct-sequence (DS) or pseudonoise (PN) spread spectrum, and frequency-hop (FH) spread spectrum. Direct-sequence and pseudonoise are used interchangeably, with no distinction between them. In direct-sequence spread spectrum technique, a *pseudorandom* or a *pseudonoise sequence*, which is a noise-like spreading code, is used to transform the narrowband data sequence into a wideband sequence. Then, the resulting wideband signal undergoes a second modulation using phase shift keying (PSK) techniques. In frequency-hopping spread spectrum, the information sequence bandwidth is still widened by a pseudonoise sequence but with a changing carrier frequency. A typical spread spectrum digital communication system is shown in Figure 11.18.

Spread spectrum signals appear like random noise, which makes them difficult to demodulate by receivers other than the intended ones, or even difficult to detect in the presence of background noise. Thus, spread spectrum systems are not useful in combating white noise, but have important applications such as antijam capabilities and interference rejection.

Interference arises also in multiple access communication, in which a number of independent users share a common channel. The conventional way to provide

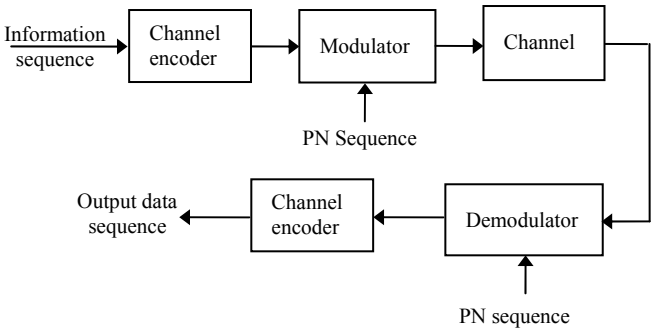


**Figure 11.18** Typical spread spectrum system.

multiple access communication uses frequency division multiple access (FDMA) or time division multiple access (TDMA) communication. In FDMA, each user is assigned a particular frequency channel, which presents a fraction of the channel bandwidth until system capacity is reached, when the whole bandwidth is used. In TDMA, the channel time-bandwidth is apportioned into fixed time slots. Each user is assigned a particular time slot until capacity is reached, when all time slots are used. A more efficient way to accomplish multiple access communications is code division multiple access (CDMA). In CDMA, each user is assigned a particular code, which is either a PN sequence or a frequency-hopping pattern, to perform the spread spectrum modulation. Since each user has its own code, the receiver can recover the transmitted signal by knowing the code used by the transmitter. However, each code used must be approximately *orthogonal* to all other codes; that is, it must have low cross-correlation.

CDMA offers secure communication privacy, due to the fact that the messages intended for one user may not be decodable by other users because they may not know the proper codes. In addition, as the number of users increases beyond a certain threshold, a gradual degradation in the performance is tolerated, and thus CDMA can accommodate more users. Because of its low power level, the spread spectrum signal may be hidden in the background noise, and in this case it is called "*covert*." It has a low probability of being detected and is called a low-probability of intercept (LPI) signal. Because of the above advantages, DS-CDMA became in the late 1980s increasingly of interest in cellular type communications for commercial purposes [48]. Next, we present the pseudonoise sequence.

### 11.4.1 Pseudonoise or Direct Sequences

The most widely used PN sequences are the *maximum length sequences*, which are coded sequences of 1s and 0s with certain autocorrelation properties. They have long periods, and are simply generated by a *linear feedback shift register*. An

*m*-sequence is periodic with period (length) $N = 2^m - 1$ bits, and is generated by a shift register of length *m*, which uses *m* flip-flops, as shown in Figure 11.19. Some properties of the maximum length sequences are as follows [49].

1. *Balance Property* Each period of the sequences contains $2^{m-1}$ ones and $2^{m-1} - 1$ zeros; that is, the number of ones is always one more than the number of zeros.

2. *Run Property* Among the runs (subsequences of identical symbols) of ones or zeros in each period of a maximum-length sequence, one-half of runs of each kind are of length one, one-fourth are of length two, one-eighth are of length three, and so forth, as long as these fractions have a meaningful number of runs. The total number of runs is $(m+1)/2$.

3. *Correlation Property* The autocorrelation function of a maximum-length sequence is periodic and binary valued.

**Example 11.1**

Consider the $m = 3$-stage feedback shift register shown in Figure 11.20. The systematic code generated is of length $N = 2^3 - 1 = 7$, as shown in Table 11.2. Assuming that the initial state of the shift register is 100, the successive states will be 100, 110, 111, 011, 101, 010, 001, 100, ….

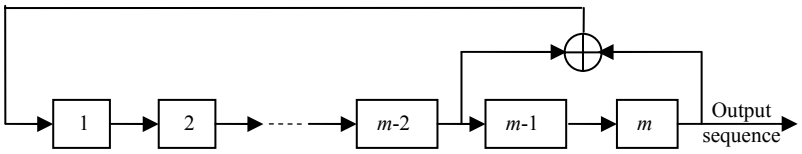


**Figure 11.19** Maximum-length PN code generator.

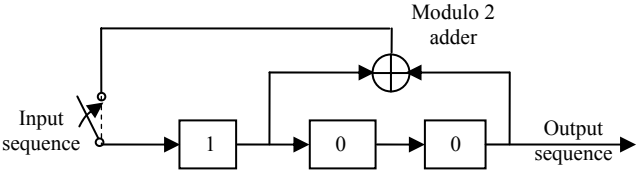


**Figure 11.20** Three-stage (*m* = 3) feedback shift register.

Note that the choice of 100 as an initial state is arbitrary. Any other choice from the six possible states would result in a shifted version of this cyclic code, as shown in Table 11.2. The state 000 results in the *catastrophic cyclic code*.

The output sequence is the code $\{c_n\} = \underbrace{00111010}_{N=7}\ldots$. Note that we have four runs: 00, 111, 0, and 1. Two of the runs (one-half of the total) are of length one, and one run (one-quarter of the total) is of length two.

In terms of the levels $-1$ and $+1$, let zero represent $-1$, and thus the output sequence is as shown in Figure 11.21. The small time increments representing the duration of binary symbols 0 or 1 in the sequence are commonly referred to as *chips*, and denoted $T_c$, and $N$ is the length of one period of the sequence. The autocorrelation function is given by

**Table 11.2**
Maximum-Length Shift Register Codes for $m = 3$

| *Information Bits* | *Code Words* |
|---|---|
| 000 | 0000000 |
| 001 | 1001110 |
| 010 | 0100111 |
| 011 | 1101001 |
| 100 | 0011101 |
| 101 | 1010011 |
| 110 | 0111010 |
| 111 | 1110100 |

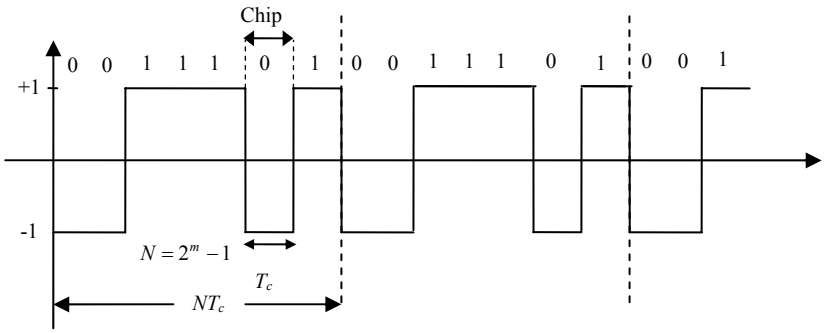


**Figure 11.21** Periodic binary PN sequence.

$$R_c(k) = \frac{1}{N}\sum_{n=1}^{N} c_n c_{n-k} = \begin{cases} 1, & k = \ell N \\ -\dfrac{1}{N}, & k \neq \ell N \end{cases} \tag{11.30}$$

where $\ell$ is any integer. The autocorrelation function is shown in Figure 11.22. Note that the autocorrelation function is periodic and binary valued.

### 11.4.2 Direct-Sequence Spread Spectrum Modulation

One way of widening the bandwidth of the information-bearing signal is by *modulation* of the PN sequence on the spread spectrum carrier, which can be binary phase-shift keying (BPSK), as shown in Figure 11.23. First, the binary message $m(t)$ and the PN sequence $p(t)$ are applied to a product modulator, as shown in Figure 11.24(a). The assumed sequences $m(t)$ and $p(t)$ are represented in their *polar* forms, as shown in Figures 11.24(b, c). Note the duration of a rectangular pulse $T_b = MT_c$, where $M$ is an integer representing the number of chips per information bit. Therefore, it also represents the number of phase shifts

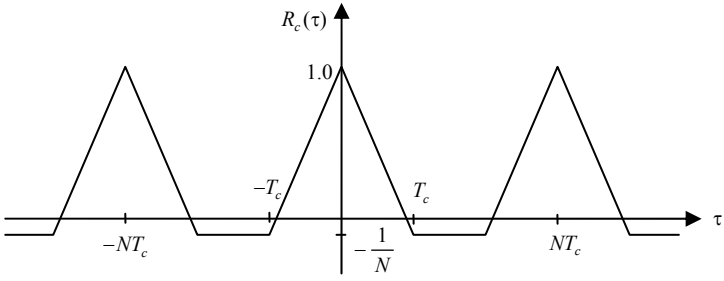


**Figure 11.22** Autocorrelation function of PN sequence.

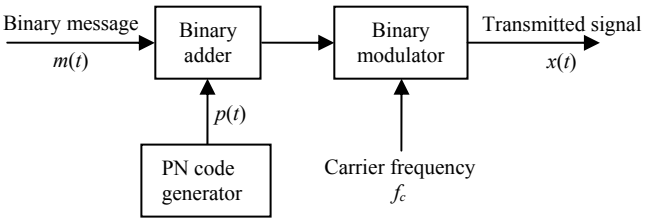


**Figure 11.23** Direct-sequence transmitter.

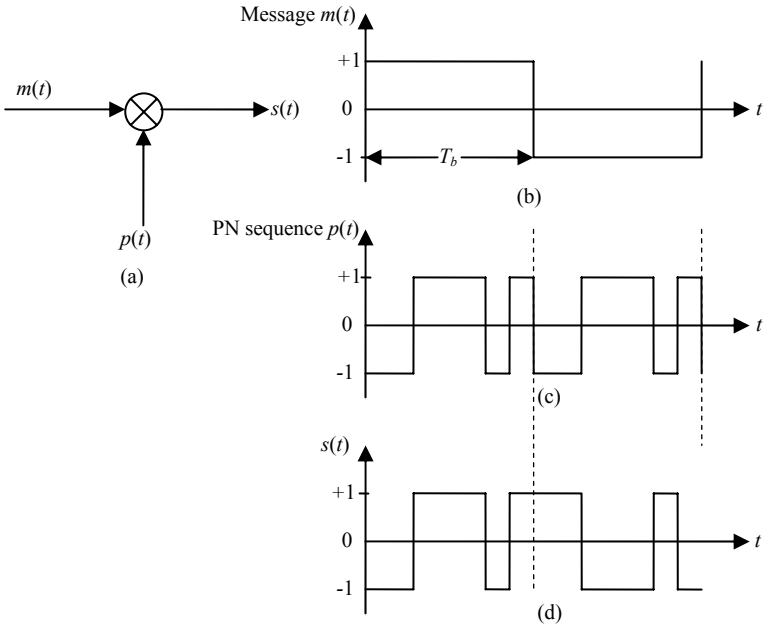


**Figure 11.24** Simplified spread spectrum transmitter and waveforms.

that occur in the transmitted signal during the bit duration $T_b$. Since the information sequence $m(t)$ is narrowband and the PN sequence is wideband, the product signal $s(t)$ will have a spectrum nearly the same as the PN sequence. That is, the spectrum of the transmitted signal is widened by the PN sequence, which is a *spreading code*. Thus, the transmitted signal is

$$s(t) = m(t)\,p(t) \tag{11.31}$$

The transmitted signal is corrupted by some additive *interference* $i(t)$, as shown in Figure 11.25(a). The received signal $y(t)$ is

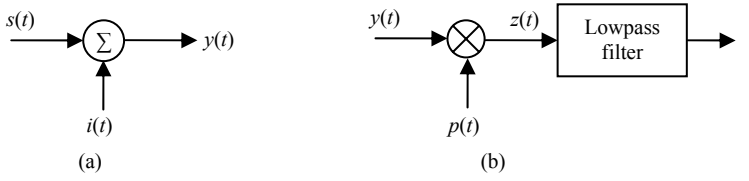


**Figure 11.25** Spread spectrum model: (a) channel and (b) receiver.

$$y(t) = s(t) + i(t) = m(t)\,p(t) + i(t) \tag{11.32}$$

To recover the original information sequence $m(t)$, the receiver signal is applied to a synchronous demodulator, which is a multiplier followed by a lowpass filter, as shown in Figure 11.25(b). The resulting demodulated signal is

$$z(t) = y(t)\,p(t) = m(t)\,p^2(t) + p(t)i(t) = m(t) + p(t)i(t) \tag{11.33}$$

since $p^2(t) = 1$ for all $t$. Thus, we obtain the original narrowband message $m(t)$ plus a wideband term $p(t)i(t)$. The filter reduces significantly the power of the interference. This is just to illustrate the baseband transmission and reception. In reality, the message is transmitted over a bandpass channel with a carrier frequency $f_c$, as illustrated in Figure 11.23. Thus, for direct-sequence binary phase-shift keying (DS/BPSK) transmission, the transmitted signal is

$$s(t) = A\cos[\omega_c t + \theta(t)] \tag{11.34}$$

where $\omega_c = 2\pi f_c$ is the carrier frequency, and the phase $\theta(t)$ is given by the truth table in Table 11.3. The general model of a direct-sequence spread spectrum phase-shift keying system is shown in Figure 11.26.

**Table 11.3**
Truth Table for Phase $\theta(t)$

| | *Message* $m(t)$ | |
|---|---|---|
| Polarity of PN Sequence | $+1$ | $-1$ |
| $+1$ | 0 | $\pi$ |
| $-1$ | $\pi$ | 0 |

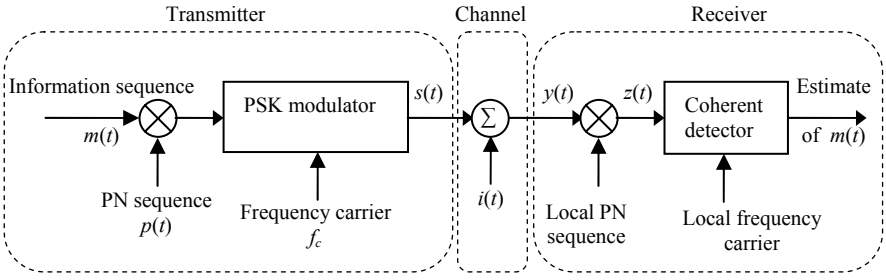


**Figure 11.26** Conceptual model of DS/BPSK system.

### 11.4.3 Frequency-Hopped Spread Spectrum Modulation

In an FH spread spectrum communications system, the frequency is constant during each time chip but changes from chip to chip, as illustrated in Figure 11.27. The bandwidth is thus subdivided into a large number of contiguous frequency slots. The modulation of FH systems is commonly binary or *M*-ary frequency shift keying (FH/FSK or FH/MFSK) [50, 51]. A block diagram of an FH/MFSK transmitter and noncoherent receiver is shown in Figure 11.28.

### 11.4.4 Synchronization of Spread Spectrum Systems

For both DS and FH spread spectrum systems, time synchronization of the local code generated at the receiver and the code embedded in the receiving signal is done in two phases. The initial synchronization, called *acquisition*, consists of bringing the two spreading signals into *coarse* alignment with one another within one chip interval $T_c$. Hence, the problem of acquisition is one of searching through a region of time and frequency in order to synchronize the received spread spectrum signal with the locally generated spreading signal. Once the received spectrum signal is acquired in the acquisition phase, then the second phase, called *tracking*, performs a *fine* synchronization within a small fraction of a chip, and maintains the PN code generator at the receiver in synchronism with the incoming signal while the demodulator is in progress. The usual way for establishing initial synchronization is for the transmitter to send a known pseudorandom data sequence to the receiver, and thus the initial synchronization may be viewed as establishing a time synchronization between the transmitter clock and the receiver clock. There is an initial timing uncertainty between the transmitter and the receiver for the following reasons [52].

1. Uncertainty in the range between the transmitter and the receiver, which translates into uncertainty in the amount of propagation delay.

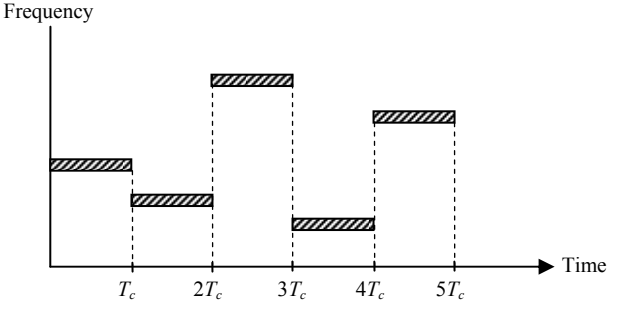


**Figure 11.27** Frequency-hopping signal.

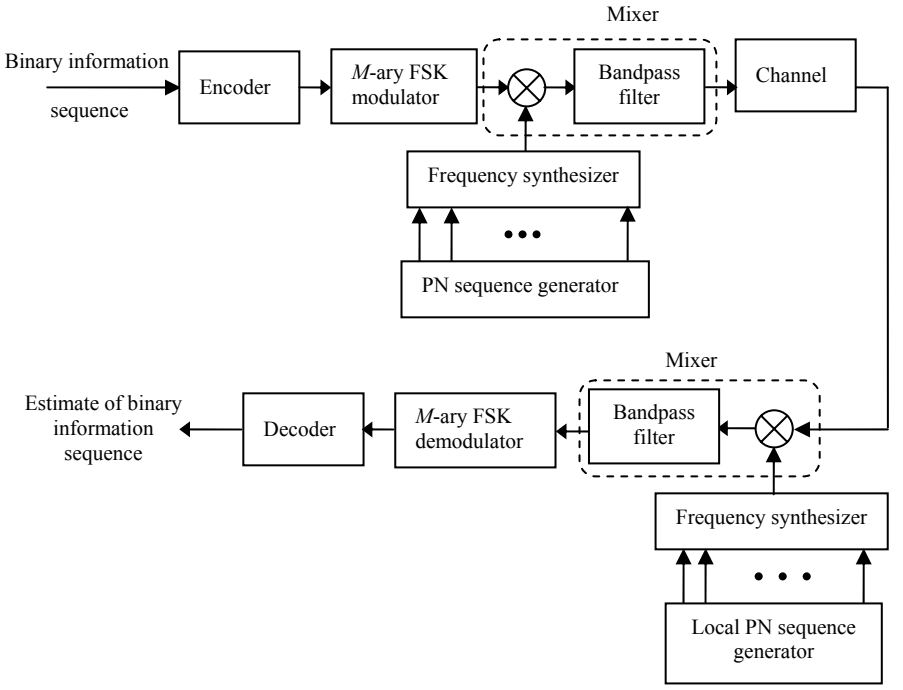


**Figure 11.28** Block diagram of an FH/MFSK spread spectrum system.

2. Relative clock instabilities between the transmitter and the receiver, which results in phase differences between the transmitter and the receiver spreading signals.

3. Uncertainty of the receiver's relative velocity with respect to the transmitter, which translates into uncertainty in a Doppler frequency offset value of the incoming signal.

4. Relative oscillator instabilities between the transmitter and the receiver, which results in frequency offset between the incoming signal and the locally generated signal.

Note that most acquisition schemes utilize noncoherent detection because the spreading process typically takes place before carrier synchronization, and thus the carrier phase is unknown at this point. Acquisition can be realized in principle by a filter matched to the spreading code or cross-correlation, which are optimum methods.

*Serial Search*

A popular strategy for the acquisition of direct-sequence spread spectrum signals is the use of a *sliding correlator*, as shown in Figure 11.29. This single correlator searches serially for the correct phase of the DS code signal.

The incoming PN signal is correlated with the locally generated PN signal in discrete time instants, usually in time intervals of $T_c/2$. In order to test synchronism at each time instant, the cross-correlation is performed over fixed intervals of $NT_c$, called *search dwell time*. The correlator output signal is compared to a preset threshold. If the output is below the threshold, the phase of the locally generated reference code signal is advanced in time by a fraction (usually one-half) of a chip, and the correlation process is repeated. These operations are performed until a signal is detected; that is, when the threshold is exceeded. In this case, the PN code is assumed to have been acquired, the phase-incrementing process of the local reference code is inhibited, and the tracking phase is initiated.

If $N$ chips are examined during each correlation, the maximum time required—$(T_{acq})_{\max}$—for a fully serial DS search, assuming increments of $T_c/2$, is

$$(T_{acq})_{\max} = 2NN_cT_c \tag{11.35}$$

where $N_c$ chips is the time uncertainty between the local reference code and the receiver code (searched region). The mean acquisition time can be shown, for $N_c >> T_c/2$, to be [52]

$$\overline{T}_{acq} = \frac{(2-P_D)(1+KP_F)}{P_D}(NN_cT_c) \tag{11.36}$$

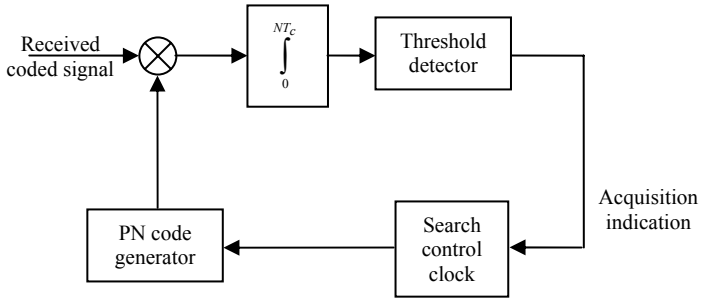


**Figure 11.29** A sliding correlator for DS serial search acquisition.

where $P_D$ is the probability of detection, $P_F$ is the probability of false alarm, and $KNT_c\,(K >> 1)$ the time interval needed to verify a detection.

A similar process may also be used for frequency-hopping signals. In this case, the problem is to search for the correct hopping pattern of the FH signal.

*Parallel Search*

Consider the direct-sequence *parallel search* acquisition shown in Figure 11.30. We observe that the incoming signal is correlated with the locally generated code and its delayed versions with one-half chip $(T_c / 2)$ apart. If the time uncertainty between the local code and the received code is $N_c$ chips, then we need $2N_c$ correlators to make a complete parallel search in a single search time. The locally generated code corresponding to the correlator with the largest output is chosen. As the number of chips $N$ increases, the probability of choosing the incorrect code alignment (synchronization error) decreases, and the maximum acquisition time given by

$$\left(T_{acq}\right)_{\max} = NT_c \tag{11.37}$$

increases. Thus, $N$ is chosen as a compromise between the acquisition time and the error probability of synchronization. The mean acquisition time is [52]

$$\overline{T}_{acq} = \frac{NT_c}{P_D} \tag{11.38}$$

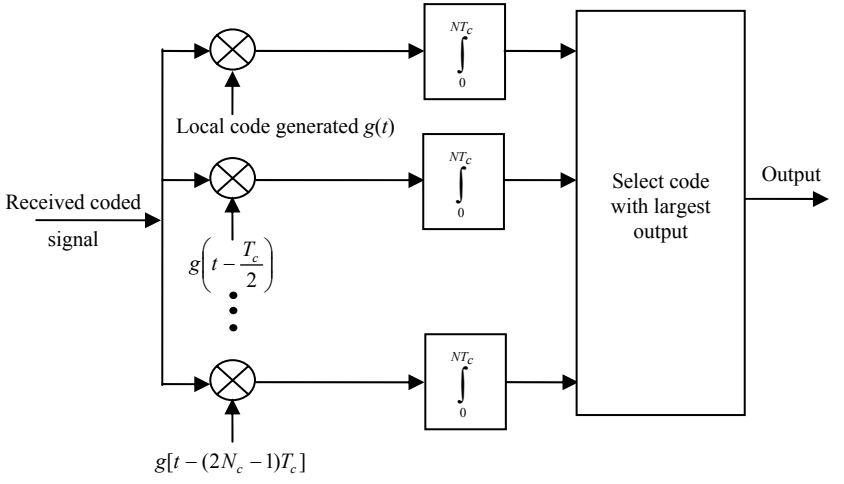


**Figure 11.30** Correlator for DS parallel search acquisition.

The number of correlators can be large, which makes this parallel acquisition less attractive. Other approaches or combinations have been proposed in the literature.

### 11.4.5 Adaptive Thresholding with False Alarm Constraint

Threshold setting plays an important role in the performance of the system, since it is the base for the decision of synchronization. Several methods for setting the threshold have been published in the literature. In the last seven years, the concept of adaptive CFAR thresholding has been introduced. Consider a single dwell serial search scheme with a noncoherent detection, as shown in Figure 11.31. This system consists of a single adaptive detector with a correlation tap size $N$. The adaptive detector consists of two blocks. The first block is the conventional noncoherent matched filter (MF) detector, as shown in Figure 11.32. The second block illustrates the adaptive CFAR operation for the decision process. Figure 11.33 illustrates the overall operation in some detail. The received PN signal plus noise and any interference are arriving at the input of the adaptive detector. If the adaptive detector declares that the present cell is the correct one, the tracking loop is activated, and the relative time delay of the local PN signal is retarded by $\Delta T_c$, where $T_c$ is the chip time, to examine the next cell. The whole testing procedure is repeated. Usually, the value of $\Delta$ is 0.25, 0.5, or 1. On the other hand, if the

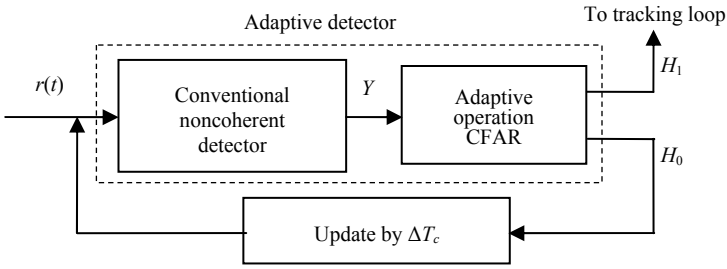


**Figure 11.31** Adaptive serial search acquisition scheme.

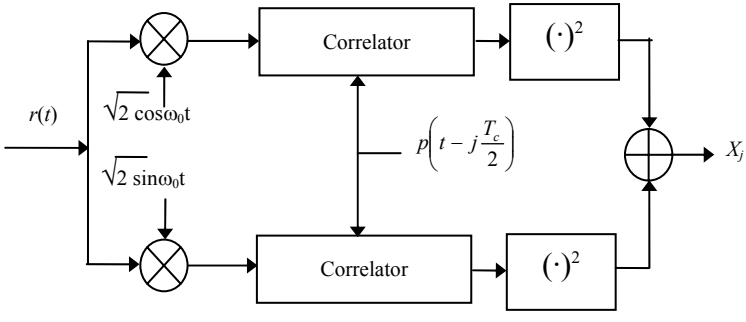


**Figure 11.32** I-Q noncoherent matched filter.

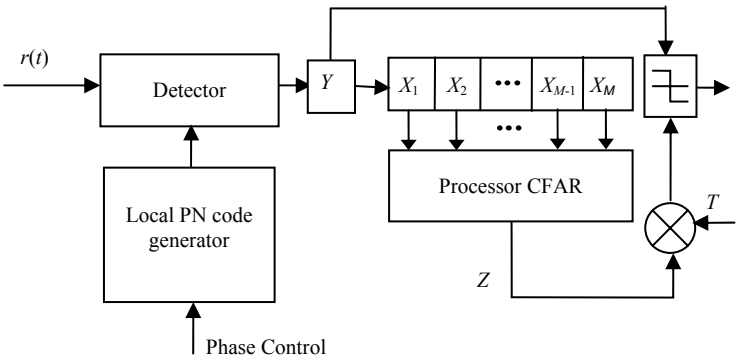


**Figure 11.33** Block diagram of adaptive detector.

adaptive detector declares $H_0$, the phases of the two codes (incoming and local) are automatically adjusted to the next offset position, and the test is repeated.

For the adaptive operation of the decision processor, the threshold value of the comparator in the adaptive detector is adapted in accordance with the magnitude of the incoming signals. Accordingly, the outputs of the correlator are serially fed into a shift register of length $M + 1$. The first register, denoted as $Y$, stores the output of the multiplication of the power of the incoming signal with the value of the partial correlation between the local and incoming PN sequences. The following $M$ registers, denoted by $X_j$, $j = 1, 2, \ldots, M$, and called reference windows, store the output of the previous $M$ phases. Note that the data stored in the register forming the reference window is like the radar reference window in CFAR adaptive thresholding. A selection logic is then used to set the threshold based on a fixed probability of false alarm.

Note that the first register stores the output of the test phase. This is a *fundamental* difference from radar CFAR detection. However, the operations of thresholding are the same, and thus much research can be pursued in this field. Linatti [53], while studying threshold principles in code acquisition of direct sequence spread spectrum signals, showed that better performances may be obtained using CFAR criterion under certain conditions. Different CFAR algorithms have been suggested in the literature [54–58], and the results look promising.

## 11.5 SUMMARY

In this chapter, we considered applications of adaptive CFAR thresholding in radar systems and code division multiple access communication systems. We first showed the need of adaptive thresholding CFAR in radar automatic detection due to the nonstationary nature of signals. Then, we presented briefly the simplified

basic concepts of radar systems. The theory of radar systems can be very involved, but we presented only the necessary steps that lead us to understand the principles of automatic detection. The cell-averaging CFAR detector was then presented in some detail, since it is the first detector presented in adaptive thresholding CFAR detection. Different detectors were then discussed to show the evolution of adaptive CFAR detection in different environments. The OS-CFAR detector was also presented in some detail. The literature in this field is very rich, and thus we had to limit ourselves to only a few papers.

In Section 11.3, we briefly described spread spectrum communication systems. Then, we presented the concepts of adaptive thresholding CFAR applied to spread spectrum communication systems, which started to appear in the literature in the last few years. We showed how the philosophy of radar adaptive thresholding is different from spread spectrum communications adaptive thresholding, but the operations of computing the adaptive threshold and the scale parameter for a CFAR are the same.

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# Chapter 12

# Distributed CFAR Detection

## 12.1 INTRODUCTION

The concept of employing multiple sensors with data fusion is widely used in surveillance systems. For a large area of coverage and/or a large number of targets under consideration, a number of geographically separated receivers may be used to monitor the same volume in space, as shown in Figure 12.1. In such space diversity systems, complete observations can be transmitted by the sensors to a central processor for data processing. Diversity systems are more robust and more reliable than single sensor systems. However, the enhanced performance of these systems is essentially derived from the diversity of the system configuration, at the expense of a required large communication bandwidth between the local receivers

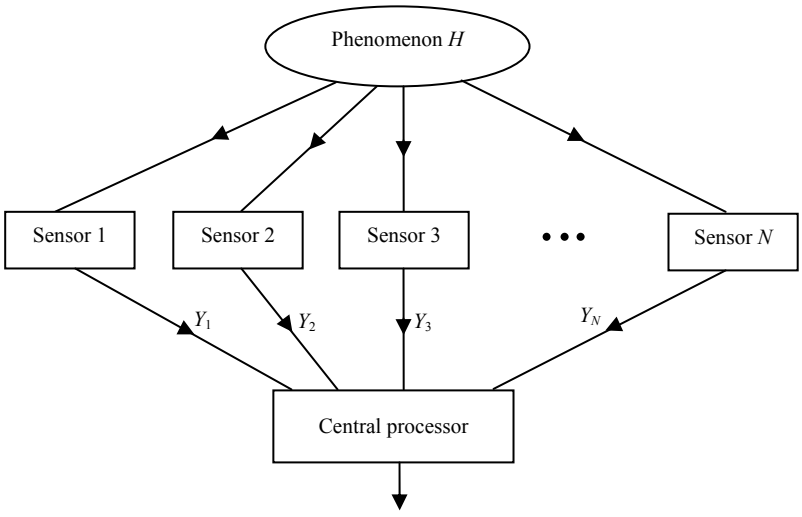


**Figure 12.1** Distributed sensor system with central computation.

and the central processor. Thus, due to the constraints on the bandwidth of the communication channels, distributed signal processing with a data fusion center is preferred in many situations. In such distributed detection systems, some processing of the signal is done at each sensor, which then sends partial results (compressed data) to the central processor, or in the context of distributed detection, to the data fusion center, as shown in Figure 12.2. These partial results are combined according to a suitable data fusion rule to yield the desired global result. In our case, the partial results are decisions from the individual detectors, $D_i, i = 1, 2, \ldots, N$, where $D_i \in \{0, 1\}$. The values of $D_i$ are combined to yield a final decision, $D_0$, which may again be zero or one.

A lot of work on distributed detection using a *fixed* threshold has been reported in the literature, for example [1–11]. When the target is embedded in nonstationary clutter and noise, adaptive thresholding techniques are used.

## 12.2 DISTRIBUTED CA-CFAR DETECTION

The theory of distributed CA-CFAR detection was first developed by Barkat and Varshney [12, 13]. They considered the problem of detecting a Swerling target model I, embedded in a white Gaussian noise of unknown level. For a given target SNR common to all local detectors and a known fusion rule at the data fusion center, they obtained the optimum threshold multipliers of the individual detectors and derived an expression for the probability of detection at the data fusion center.

The probability of detection, $P_{D_i}$, for detector $i$, $i = 1, 2, \ldots, N$, is given by

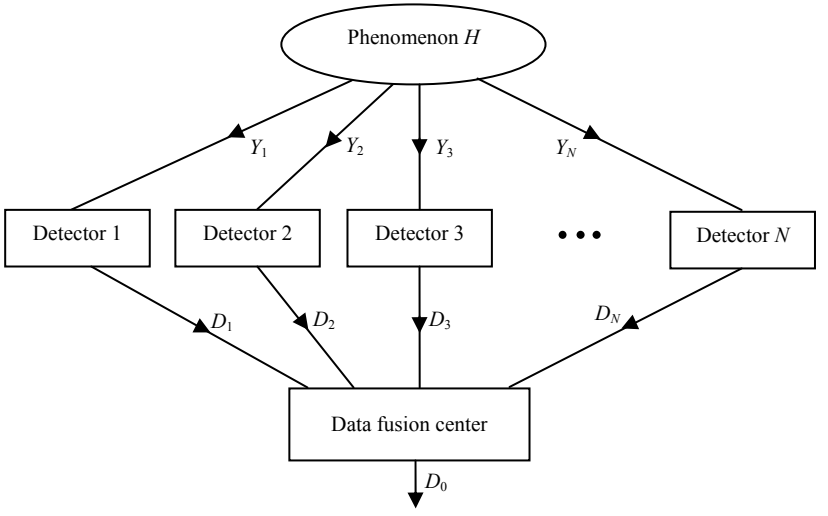


**Figure 12.2** Distributed sensor system with data fusion.

$$P_{D_i} = \int_0^\infty P\left(Y_0^i > T_i Z^i \mid Z^i, H_1\right) P_{Z^i}\left(z^i\right) dz^i \tag{12.1}$$

where $T_i$ is the threshold multiplier at the CA-CFAR detector $i$, $i = 1, 2, \ldots, N$, and $P_{Z^i}\left(z^i\right)$ denotes the probability density function of the adaptive threshold at the $i$th CA-CFAR detector. Also,

$$P\left(Y_0^i > T_i Z^i \mid z^i, H_1\right) = \int_{T_i z^i}^\infty P_{Y^i|H_1}\left(y^i \mid H_1\right) dy^i = \exp\left(-\frac{T_i y^i}{1+S}\right) \tag{12.2}$$

Since the noise samples for each CA-CFAR detector are identically distributed, the probability of detection of the individual detectors can be written, from the previous chapter, as

$$P_{D_i} = \frac{(1+S)^{N_i}}{(1+S+T_i)^{N_i}}, \quad i = 1, 2, \ldots, N \tag{12.3}$$

The goal is to maximize the overall probability of detection while keeping the overall probability of false alarm constant. To do this, we use the calculus of extrema and form the objective function

$$J(T_1, T_2, \ldots, T_N) = P_D(S, T_1, T_2, \ldots, T_N) + \lambda\left[P_F(T_1, T_2, \ldots, T_N) - \alpha\right] \tag{12.4}$$

where α is the desired false alarm probability at the data fusion center, λ is the *Lagrange multiplier*, and $T_i$, $i = 1, 2, \ldots, N$, is the threshold multiplier at each detector. To maximize $P_D(S, T_1, T_2, \ldots, T_N)$, subject to the constraint that $P_F(T_1, T_2, \ldots, T_N)$ is a constant, we must maximize the objective function $J(T_1, T_2, \ldots, T_N)$. We set the derivative of $J(T_1, T_2, \ldots, T_N)$ with respect to $T_i$, $i = 1, 2, \ldots, N$, equal to zero, and solve the following system of $N$ nonlinear equations in $N$ unknowns.

$$\frac{\partial J(T_1, T_2, \ldots, T_N)}{\partial T_j} = 0, \quad j = 1, 2, \ldots, N \tag{12.5}$$

Once the threshold multipliers, $T_i$ , $i = 1, 2, \ldots, N,$ are obtained, all the values of $P_{F_i}$ are fixed and the optimum $P_D$ results. Now, we give specific results for the "AND" and "OR" fusion rules. We also find the optimum threshold multipliers so as to maximize $P_D$ while $P_F$ is maintained at the desired value.

*AND Fusion Rule*

In this case, the global probabilities of detection and false alarm, in terms of the local ones, are

$$P_D = \prod_{i=1}^{N} P_{D_i} \tag{12.6}$$

and

$$P_F = \prod_{i=1}^{N} P_{F_i} \tag{12.7}$$

That is,

$$P_D = \prod_{i=1}^{N} \frac{(1+S)^{N_i}}{(1+S+T_i)^{N_i}} \tag{12.8}$$

and

$$P_F = \prod_{i=1}^{N} \frac{1}{(1+T_i)^{N_i}} \tag{12.9}$$

Substituting (12.8) and (12.9) into (12.4), the objective function is

$$J(T_1, T_2, \ldots, T_N) = \prod_{i=1}^{N} \frac{(1+S)^{N_i}}{(1+S+T_i)^{N_i}} + \lambda \left[ \prod_{i=1}^{N} \frac{1}{(1+T_i)^{N_i}} - \alpha \right] \tag{12.10}$$

Taking the derivative of $J(T_1, T_2, \ldots, T_N)$ with respect to $T_i$ , $i = 1, 2, \ldots, N,$ and setting it equal to zero, we obtain

$$\frac{\partial J(T_1, T_2, \ldots, T_N)}{\partial T_j} = \prod_{\substack{i=1 \\ i \neq j}}^{N} \frac{(1+S)^{N_i+N_j}}{(1+S+T_j)^{N_j+1}(1+S+T_i)^{N_i}} + \prod_{\substack{i=1 \\ i \neq j}}^{N} \frac{1}{(1+T_j)^{N_j+1}(1+T_i)^{N_i}} = 0, \quad j = 1, 2, \ldots, N \tag{12.11}$$

The threshold multiplier, $T$, can be obtained by solving the above set of coupled nonlinear equations along with the constraint

$$P_F = \prod_{i=1}^{N} \frac{1}{(1+T_i)^{N_i}} = \alpha \tag{12.12}$$

*OR Fusion Rule*

In this case, we have

$$P_M = \prod_{i=1}^{N} P_{M_i} \tag{12.13}$$

and

$$P_F = 1 - \prod_{i=1}^{N} (1 - P_{F_i}) \tag{12.14}$$

where $P_M$ is the probability of miss, and recall that $P_M = 1 - P_D$. The objective function then becomes

$$\begin{aligned} J(T_1, T_2, \ldots, T_N) &= P_M + \lambda [P_F(T_1, T_2, \ldots, T_N) - \alpha] \\ &= 1 - \prod_{i=1}^{N} \left[ 1 - \frac{(1+S)^{N_i}}{(1+S+T_i)^{N_i}} \right] + \lambda \left\{ 1 - \prod_{i=1}^{N} \left[ 1 - \frac{1}{(1+T_i)^{N_i}} - \alpha \right] \right\} \end{aligned} \tag{12.15}$$

Note that in this case we have to minimize $J(T_1, T_2, \ldots, T_N)$, since we are minimizing the overall probability of a miss, which is equivalent to maximizing $P_D$ at the data fusion center as defined by (12.4). Taking the derivative of the objective function with respect to $T_j$, $j = 1, 2, \ldots, N$, and setting it equal to zero, we obtain

$$\frac{\partial J(T_1, T_2, \ldots, T_N)}{\partial T_j} = \frac{(1+S)^{N_j}}{(1+S+T_j)^{N_j+1}} \prod_{\substack{i=1 \\ j \neq i}}^{N} \left[ 1 - \frac{(1+S)^{N_i}}{(1+S+T_i)^{N_i}} \right]$$

$$+ \frac{\lambda}{(1+T_j)^{N_j+1}} \prod_{\substack{i=1 \\ j \neq i}}^{N} \left[ 1 - \frac{1}{(1+T_i)^{N_i}} \right] = 0, \quad j = 0, 1, 2, \ldots, N \qquad (12.16)$$

Hence, we obtain a system of coupled equations. Then, we use the following constraint

$$1 - \prod_{i=1}^{N} \left[ 1 - \frac{1}{(1+T_i)^{N_i}} \right] = \alpha \qquad (12.17)$$

to solve for the unknown threshold multipliers recursively.

## 12.3 FURTHER RESULTS

In [14], Elias-Fusté et al. extended the work in [12] to $N$ receivers using cell-averaging and order statistic CFAR. They considered a "$k$ out of $N$" fusion rule at the data fusion center, and solved for the optimum thresholds of the local receivers by maximizing the overall probability of detection, while the global probability of false alarm is maintained constant. Then, they assumed that the local receivers are based on identical ordered statistics CFAR for a multiple target situation. Recall in OS-CFAR detection, an order number of the estimating cell is used to represent the background level. The problem of nonidentical OS-CFAR local detectors was considered in [15]. For a given set of ordered number cells, $k_i, i = 1, 2, \ldots, N,$ they form the objective function at the data fusion center, which is given by

$$J[(T_1, k_1), (T_2, k_2), \ldots, (T_N, k_N)] = P_D[(T_1, k_1), (T_2, k_2), \ldots, (T_N, k_N)]$$
$$+ \lambda \{ P_F[(T_1, k_1), (T_2, k_2), \ldots, (T_N, k_N)] - \alpha \} \qquad (12.18)$$

Subject to the constraint that the overall desired probability of false alarm at the data fusion center is α, λ is again a Lagrange multiplier. Then, they obtain the optimum threshold multipliers $T_1, T_2, \ldots, T_N$, by solving the set of nonlinear equations

$$\frac{\partial J[(T_1, k_1), (T_2, k_2), \ldots, (T_N, k_N)]}{\partial T_j} = 0, \quad j = 1, 2, \cdots, N \qquad (12.19)$$

for the constraint

$$P_F\left[(T_1,k_1),(T_2,k_2),\ldots,(T_N,k_N)\right]=\alpha \tag{12.20}$$

The corresponding objective functions for the AND and OR fusion rules are, respectively, given by

$$J\left[(T_1,k_1),(T_2,k_2),\ldots,(T_N,k_N)\right]=\prod_{i=1}^{N}\left(\prod_{j=0}^{k_i-1}\frac{N_i-j}{N_i-j+\dfrac{T_i}{1+s}}\right)+\lambda\left[\prod_{i=1}^{N}\left(\prod_{j=0}^{k_i-1}\frac{N_i-j}{N_i-j+T_i}\right)-\alpha\right] \tag{12.21}$$

and

$$J\left[(T_1,k_1),(T_2,k_2),\ldots,(T_N,k_N)\right]=1-\prod_{i=1}^{N}\left(1-\prod_{j=0}^{k_i-1}\frac{N_i-j}{N_i-j+\dfrac{T_i}{1+s}}\right)+\lambda\left[1-\prod_{i=1}^{N}\left(1-\prod_{j=0}^{k_i-1}\frac{N_i-j}{N_i-j+T_i}\right)-\alpha\right] \tag{12.22}$$

Further results based on decentralized cell-averaging CFAR detection and decentralized order statistic CFAR detection were developed by Blum et al. [16, 17]. In [18], different target models were considered.

Non-Gaussian clutter such as the Weibull distribution or the distribution *K* were considered in [19–21]. The literature is very rich, and further developments can be found in [22–32].

Again, I apologize to the many authors who contributed in this field and were not cited explicitly. As discussed in the previous chapters, high-resolution radars and different topologies with embedded systems may be considered for this quest of better detection performances.

## 12.4 SUMMARY

In this chapter, we introduced the concept of adaptive thresholding CFAR using multiple sensors and data fusion. We showed how the problem is formulated and gave the necessary steps to obtain the optimum scale factors using the AND and

OR fusion rules at the data fusion center. Other approaches using OS-CFAR detection were also discussed. Then, we presented some papers that enriched this concept of adaptive CFAR detection with multiple sensors and data fusion, for non-Gaussian clutter environments, and under different constraints.

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# Appendix

The density function of the Gaussian, also called normal, distribution is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[ -\frac{(x-m)^2}{2\sigma^2} \right] \quad \text{for all } x \tag{A.1}$$

where $m$ and $\sigma$ are the mean and standard deviation of $X$, respectively, and satisfy the conditions $-\infty < m < \infty$ and $\sigma > 0$. The corresponding distribution function is given by

$$F_X(x) = P(X \le x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} \exp\left[ -\frac{(u-m)^2}{2\sigma^2} \right] du \tag{A.2}$$

The distribution function can be determined in terms of the *error function* as

$$F_X(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left( \frac{x}{\sqrt{2}} \right) \tag{A.3}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^2} du \tag{A.4}$$

Letting $u = (x-m)/\sigma$ in (A.1), then

$$I(x) \triangleq F_X(x) = P(X \le x) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \tag{A.5}$$

where

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{A.6}$$

is the standard normal distribution with mean $m = 0$ and variance $\sigma^2 = 1$, and also denoted $\mathcal{N}(0,1)$. Tabulated values of $I(x)$ and $\text{erf}(x)$ are given in Tables A.1 [1] and A.2 [2], respectively.

Other important results are the *complementary error function* and the *Q-function* given by

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du \tag{A.7}$$

such that

$$\text{erfc}(x) = 1 - \text{erfc}(x) \tag{A.8}$$

and

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}}\, du \tag{A.9}$$

where

$$Q(0) = \frac{1}{2} \tag{A.10}$$

and

$$Q(-x) = 1 - Q(x) \quad \text{for } x \geq 0 \tag{A.11}$$

The $Q$-function can be written in terms of the error function to be

$$Q(x) = \frac{1}{2}\left[1 - \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right] = \frac{1}{2}\,\text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{A.12}$$

Also note that

$$I(x) + Q(x) = 1 \tag{A.13}$$

and

$$Q(x) \cong \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \text{for } x > 4 \tag{A.14}$$

In some books, $Q(x)$ defined in (A.9) is denoted $\text{erfc}_*(x)$, while $I(x)$ in (A.5) is denoted $\text{erfc}_*(x)$, and thus $\text{erf}_*(x) + \text{erfc}_*(x) = 1$, as in (A.13).

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**Table A.1** Values of the Standard Normal Distribution Function

$$I(x) \triangleq \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = P(X \le x)$$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| − 3.0 | 0.0013 | 0.0010 | 0.0007 | 0.0005 | 0.0003 | 0.0002 | 0.0002 | 0.0001 | 0.0001 | 0.0000 |
| − 2.9 | 0.0019 | 0.0018 | 0.0017 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| − 2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| − 2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| − 2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| − 2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| − 2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| − 2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| − 2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0126 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| − 2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| − 2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| − 1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0238 | 0.0233 |
| − 1.8 | 0.0359 | 0.0352 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0300 | 0.0294 |
| − 1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| − 1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| − 1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0570 | 0.0559 |
| − 1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0722 | 0.0708 | 0.0694 | 0.0681 |
| − 1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| − 1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| − 1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| − 1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| − 0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| − 0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| − 0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2297 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| − 0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| − 0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| − 0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| − 0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| − 0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| − 0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| − 0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |

**Table A.1** (continued) Values of the Standard Normal Distribution Function

$$I(x) \triangleq \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = P(X \le x)$$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7703 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9278 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9430 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9648 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9700 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9762 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9874 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9990 | 0.9993 | 0.9995 | 0.9997 | 0.9998 | 0.9998 | 0.9999 | 0.9999 | 1.0000 |

**Table A.2** Error Function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

| $x$ | erf($x$) | $x$ | erf($x$) | $x$ | erf($x$) | $x$ | erf($x$) |
|---|---|---|---|---|---|---|---|
| 0.00 | 0.00000 | 0.25 | 0.27633 | 0.50 | 0.52050 | 0.75 | 0.71116 |
| 0.01 | 0.01128 | 0.26 | 0.28690 | 0.51 | 0.52924 | 0.76 | 0.71754 |
| 0.02 | 0.02256 | 0.27 | 0.29742 | 0.52 | 0.53790 | 0.77 | 0.72382 |
| 0.03 | 0.03384 | 0.28 | 0.30788 | 0.53 | 0.54646 | 0.78 | 0.73001 |
| 0.04 | 0.04511 | 0.29 | 0.31828 | 0.54 | 0.55494 | 0.79 | 0.73610 |
| 0.05 | 0.05637 | 0.30 | 0.32863 | 0.55 | 0.56332 | 0.80 | 0.74210 |
| 0.06 | 0.06762 | 0.31 | 0.33891 | 0.56 | 0.57162 | 0.81 | 0.74800 |
| 0.07 | 0.07885 | 0.32 | 0.34913 | 0.57 | 0.57982 | 0.82 | 0.75381 |
| 0.08 | 0.09007 | 0.33 | 0.35928 | 0.58 | 0.58792 | 0.83 | 0.75952 |
| 0.09 | 0.10128 | 0.34 | 0.36836 | 0.59 | 0.59594 | 0.84 | 0.76514 |
| 0.10 | 0.11246 | 0.35 | 0.37938 | 0.60 | 0.60386 | 0.85 | 0.77067 |
| 0.11 | 0.12362 | 0.36 | 0.38933 | 0.61 | 0.61168 | 0.86 | 0.77610 |
| 0.12 | 0.13476 | 0.37 | 0.39921 | 0.62 | 0.61941 | 0.87 | 0.78144 |
| 0.13 | 0.14587 | 0.38 | 0.40901 | 0.63 | 0.62705 | 0.88 | 0.78669 |
| 0.14 | 0.15695 | 0.39 | 0.41874 | 0.64 | 0.63459 | 0.89 | 0.79184 |
| 0.15 | 0.16800 | 0.40 | 0.42839 | 0.65 | 0.64203 | 0.90 | 0.79691 |
| 0.16 | 0.17901 | 0.41 | 0.43797 | 0.66 | 0.64938 | 0.91 | 0.80188 |
| 0.17 | 0.18999 | 0.42 | 0.44747 | 0.67 | 0.65663 | 0.92 | 0.80677 |
| 0.18 | 0.20094 | 0.43 | 0.45689 | 0.68 | 0.66378 | 0.93 | 0.81156 |
| 0.19 | 0.21184 | 0.44 | 0.46623 | 0.69 | 0.67084 | 0.94 | 0.81627 |
| 0.20 | 0.22270 | 0.45 | 0.47548 | 0.70 | 0.67780 | 0.95 | 0.82089 |
| 0.21 | 0.23352 | 0.46 | 0.48466 | 0.71 | 0.68467 | 0.96 | 0.82542 |
| 0.22 | 0.24430 | 0.47 | 0.49375 | 0.72 | 0.69143 | 0.97 | 0.82987 |
| 0.23 | 0.25502 | 0.48 | 0.50275 | 0.73 | 0.69810 | 0.98 | 0.83423 |
| 0.24 | 0.26570 | 0.49 | 0.51167 | 0.74 | 0.70468 | 0.99 | 0.83851 |
| 0.25 | 0.27633 | 0.50 | 0.52050 | 0.75 | 0.71116 | 1.00 | 0.84270 |

**Table A.2** (continued) Error Function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

| $x$ | erf($x$) | $x$ | erf($x$) | $x$ | erf($x$) | $x$ | erf($x$) |
|---|---|---|---|---|---|---|---|
| 1.00 | 0.84270 | 1.25 | 0.92290 | 1.50 | 0.96611 | 1.75 | 0.98667 |
| 1.01 | 0.84681 | 1.26 | 0.92524 | 1.51 | 0.96728 | 1.76 | 0.98719 |
| 1.02 | 0.85084 | 1.27 | 0.92751 | 1.52 | 0.96841 | 1.77 | 0.98769 |
| 1.03 | 0.85478 | 1.28 | 0.92973 | 1.53 | 0.96952 | 1.78 | 0.98817 |
| 1.04 | 0.85685 | 1.29 | 0.93190 | 1.54 | 0.97059 | 1.79 | 0.98864 |
| 1.05 | 0.86244 | 1.30 | 0.93401 | 1.55 | 0.97162 | 1.80 | 0.98909 |
| 1.06 | 0.86614 | 1.31 | 0.93606 | 1.56 | 0.97263 | 1.81 | 0.98952 |
| 1.07 | 0.86977 | 1.32 | 0.93807 | 1.57 | 0.97360 | 1.82 | 0.98994 |
| 1.08 | 0.87333 | 1.33 | 0.94002 | 1.58 | 0.97455 | 1.83 | 0.99035 |
| 1.09 | 0.87680 | 1.34 | 0.94191 | 1.59 | 0.97546 | 1.84 | 0.99074 |
| 1.10 | 0.88021 | 1.35 | 0.94376 | 1.60 | 0.97635 | 1.85 | 0.99111 |
| 1.11 | 0.88353 | 1.36 | 0.94556 | 1.61 | 0.97721 | 1.86 | 0.99147 |
| 1.12 | 0.88679 | 1.37 | 0.94731 | 1.62 | 0.97804 | 1.87 | 0.99182 |
| 1.13 | 0.88997 | 1.38 | 0.94902 | 1.63 | 0.97884 | 1.88 | 0.99216 |
| 1.14 | 0.89308 | 1.39 | 0.95067 | 1.64 | 0.97962 | 1.89 | 0.99247 |
| 1.15 | 0.89612 | 1.40 | 0.95229 | 1.65 | 0.98038 | 1.90 | 0.99279 |
| 1.16 | 0.98910 | 1.41 | 0.95385 | 1.66 | 0.98110 | 1.91 | 0.99308 |
| 1.17 | 0.90200 | 1.42 | 0.95530 | 1.67 | 0.98181 | 1.92 | 0.99338 |
| 1.18 | 0.90484 | 1.43 | 0.95686 | 1.68 | 0.98249 | 1.93 | 0.99366 |
| 1.19 | 0.90761 | 1.44 | 0.95830 | 1.69 | 0.98315 | 1.94 | 0.99392 |
| 1.20 | 0.91031 | 1.45 | 0.95970 | 1.70 | 0.98379 | 1.95 | 0.99418 |
| 1.21 | 0.91296 | 1.46 | 0.96105 | 1.71 | 0.98441 | 1.96 | 0.99442 |
| 1.22 | 0.91553 | 1.47 | 0.96237 | 1.72 | 0.98500 | 1.97 | 0.99466 |
| 1.23 | 0.91805 | 1.48 | 0.96365 | 1.73 | 0.98558 | 1.98 | 0.99489 |
| 1.24 | 0.92051 | 1.49 | 0.96490 | 1.74 | 0.98613 | 1.99 | 0.99511 |
| 1.25 | 0.92290 | 1.50 | 0.96611 | 1.75 | 0.98667 | 2.00 | 0.99532 |

# About the Author

Mourad Barkat received a B.S. with high honors, magna cum laude, and an M.S. and a Ph.D. in electrical engineering from Syracuse University in 1981, 1983, and 1987, respectively.

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