A Geometric Approach to Differential Forms

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For the Instructor

The present work is not meant to contain any new material about differential forms. There are many good books out there which give nice, complete treatments of the subject. Rather, the goal here is to make the topic of differential forms accessible to the sophomore level undergraduate. The target audience for this material is primarily students who have completed three semesters of calculus, although the later sections will be of interest to advanced undergraduate and beginning graduate students. At many institutions a course in linear algebra is not a prerequisite for vector calculus. Consequently, these notes have been written so that the earlier chapters do not require many concepts from linear algebra.

What follows began as a set of lecture notes from an introductory course in differential forms, given at Portland State University, during the summer of 2000. The notes were then revised for subsequent courses on multivariable calculus and vector calculus at California Polytechnic State University. At some undetermined point in the future this may turn into a full scale textbook, so any feedback would be greatly appreciated!

I thank several people. First and foremost, I am grateful to all those students who survived the earlier versions of this book. I would also like to thank several of my colleagues for giving me helpful comments. Most notably, Don Hartig had several comments after using an earlier version of this text for a vector calculus course. John Etnyre and Danny Calegari gave me feedback regarding Chapter 6. Alvin Bachman had good suggestions regarding the format of this text. Finally, the idea to write this text came from conversations with Robert Ghrist while I was a graduate student at the University of Texas at Austin. He also deserves my gratitude.

Prerequisites. Most of the text is written for students who have completed three semesters of calculus. In particular, students are expected to be familiar with partial derivatives, multiple integrals, and parameterized curves and surfaces.

Concepts from linear algebra are kept to a minimum, although it will be important that students know how to compute the determinant of a matrix before delving into this material. Many will have learned this in secondary school. In practice they will only need to know how this works for $n \times n$ matrices with $n \leq 3$, although they should know that there is a way to compute it for higher values of n. It is crucial that they understand that the determinant of a matrix gives the volume of the parallelepiped spanned by its row vectors. If they have not seen this before the instructor should, at least, prove it for the 2×2 case.

The idea of a matrix as a linear transformation is only used in Section 2 of Chapter 5, when we define the pull-back of a differential form. Since at this point the students have already been computing pull-backs without realizing it, little will be lost by skipping this section.

The heart of this text is Chapters 2 through 5. Chapter 1 is purely motivational. Nothing from it is used in subsequent chapters. Chapter 7 is only intended for advanced undergraduate and beginning graduate students.

For the Student

It often seems like there are two types of students of mathematics: those who prefer to learn by studying equations and following derivations, and those who like pictures. If you are of the former type this book is not for you. However, it is the opinion of the author that the topic of differential forms is inherently geometric, and thus, should be learned in a very visual way. Of course, learning mathematics in this way has serious limitations: how can you visualize a 23 dimensional manifold? We take the approach that such ideas can usually be built up by analogy from simpler cases. So the first task of the student should be to really understand the simplest case, which CAN often be visualized.

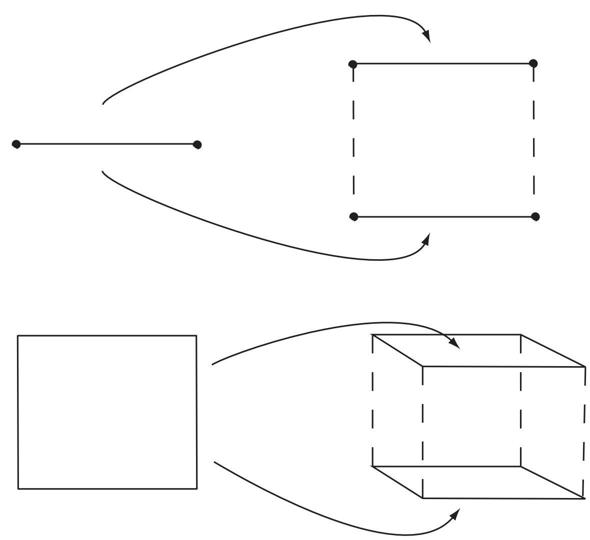


FIGURE 1. The faces of the n-dimensional cube come from connecting up the faces of two copies of an $(n-1)$-dimensional cube.

For example, suppose one wants to understand the combinatorics of the n- dimensional cube. We can visualize a 1-D cube (*i.e.* an interval), and see just from our mental picture that it has two boundary points. Next, we can visualize a 2-D cube (a square), and see from our picture that this has 4 intervals on its boundary. Furthermore, we see that we can construct this 2-D cube by taking two parallel copies of our original 1-D cube and connecting the endpoints. Since there are two endpoints, we get two new intervals, in addition to the two we started with (see Fig. 1). Now, to construct a 3-D cube, we place two squares parallel to each other, and connect up their edges. Each time we connect an edge of one to an edge of the other, we get a new square on the boundary of the 3-D cube. Hence, since there were 4 edges on the boundary of each square, we get 4 new squares, in addition to the 2 we started with, making 6 in all. Now, if the student understands this, then it should not be hard to convince him/her that every time we go up a dimension, the number of lower dimensional cubes on the boundary is the same as in the previous dimension, plus 2. Finally, from this we can conclude that there are 2n (n-1)-dimensional cubes on the boundary of the n-dimensional cube.

Note the strategy in the above example: we understand the “small” cases visually, and use them to generalize to the cases we cannot visualize. This will be our approach in studying differential forms.

Perhaps this goes against some trends in mathematics of the last several hundred years. After all, there were times when people took geometric intuition as proof, and later found that their intuition was wrong. This gave rise to the formalists, who accepted nothing as proof that was not a sequence of formally manipulated logical statements. We do not scoff at this point of view. We make no claim that the above derivation for the number of (n-1)-dimensional cubes on the boundary of an n-dimensional cube is actually a proof. It is only a convincing argument, that gives enough insight to actually produce a proof. Formally, a proof would still need to be given. Unfortunately, all too often the classical math book begins the subject with the proof, which hides all of the geometric intuition which the above argument leads to.

Contents

CHAPTER 1

Introduction

1. So what *is* a Differential Form?

A differential form is simply this: an integrand. In other words, it's a thing you can integrate over some (often complicated) domain. For example, consider the following integral: $\int\limits_0^1 x^2 dx$. This notation indicates that we are integrating x^2 over the interval $[0, 1]$. In this case, $x^2 dx$ is a differential form. If you have had no exposure to this subject this may make you a little uncomfortable. After all, in calculus we are taught that x^2 is the integrand. The symbol "dx" is only there to delineate when the integrand has ended and what variable we are integrating with respect to. However, as an object in itself, we are not taught any meaning for "dx". Is it a function? Is it an operator on functions? Some professors call it an "infinitesimal" quantity. This is very tempting... after all, $\int\limits_0^1 x^2 dx$ is defined to be the limit, as $n \to \infty$, of $\sum\limits_{i=1}^{n} x_i^2 \Delta x$, where $\{x_i\}$ are n evenly spaced points in the interval $[0, 1]$, and $\Delta x = 1/n$. When we take the limit, the symbol "$\sum$" becomes "$\int$", and the symbol "Δx" becomes "dx". This implies that $dx = \lim_{\Delta x \to 0} \Delta x$, which is absurd. $\lim_{\Delta x \to 0} \Delta x = 0$!! We are not trying to make the argument that the symbol "dx" should be done away with. It does have meaning. This is one of the many mysteries that this book will reveal.

One word of caution here: not all integrands are differential forms. In fact, in most calculus classes we learn how to calculate arc length, which involves an integrand which is not a differential form. Differential forms are just very natural objects to integrate, and also the first that one should study. As we shall see, this is much like beginning the study of all functions by understanding linear functions. The naive student may at first object to this, since linear functions are a very restrictive class. On the other hand, eventually we learn that any differentiable function (a much more general class) can be locally approximated by a linear function. Hence, in some sense, the linear functions are the most important ones. In the same way, one can make the argument that differential forms are the most important integrands.

2. Generalizing the Integral

Let's begin by studying a simple example, and trying to figure out how and what to integrate. The function $f(x, y) = y^2$ maps $\mathbb{R}^2$ to $\mathbb{R}$. Let M denote the top half of the circle of radius 1, centered at the origin. Let's restrict the function f to the domain, M, and try to integrate it. Here we encounter our first problem: I have given you a description of M which is not particularly useful. If M were something more complicated, it would have been much harder to describe it in words as I have just done. A parameterization is far easier to communicate, and far easier to use to determine which points of $\mathbb{R}^2$ are elements of M, and which aren't. But there are lots of parameterizations of M. Here are two which we shall use:

$\phi_1(a) = (a, \sqrt{1 - a^2})$, where $-1 \le a \le 1$, and

$\phi_2(t) = (\cos(t), \sin(t))$, where $0 \le t \le \pi$.

OK, now here's the trick: Integrating f over M is hard. It may not even be so clear as to what this means. But perhaps we can use ϕ_1 to translate this problem into an integral over the interval $[-1, 1]$. After all, an integral is a big sum. If we add up all the numbers $f(x, y)$ for all the points, (x, y), of M, shouldn't we get the same thing as if we added up all the numbers $f(\phi_1(a))$, for all the points, a, of $[-1, 1]$? (see Fig. 1)

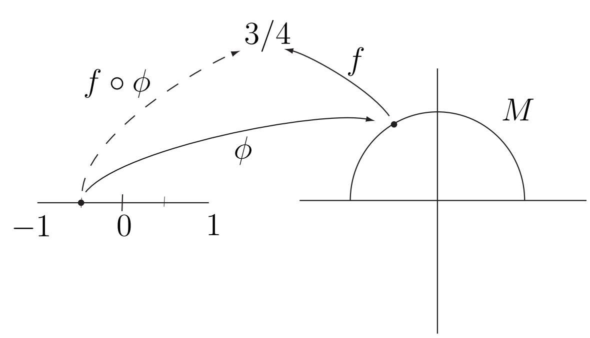


FIGURE 1. Shouldn't the integral of f over M be the same as the integral of $f \circ \phi$ over $[-1, 1]$?

Let's try it. $\phi_1(a) = (a, \sqrt{1-a^2})$, so $f(\phi_1(a)) = 1-a^2$. Hence, we are saying that the integral of f over M should be the same as $\int\limits_{-1}^{1} 1 - a^2 da$. Using a little calculus, we can determine that this evaluates to $4/3$.

Let's try this again, this time using ϕ_2. By the same argument, we have that the integral of f over M should be the same as $\int\limits_{0}^{\pi} f(\phi_2(t))dt = \int\limits_{0}^{\pi} \sin^2(t)dt = \pi/2$.

But hold on! The problem was stated *before* we chose any parameterizations. Shouldn't the answer be independent of which one we picked? It wouldn't be a very meaningful problem if two people could get different correct answers, depending on how they went about solving it. Something strange is going on!

3. Interlude: A review of single variable integration

In order to understand what happened, we must first review the definition of the Riemann integral. In the usual definition of the Riemann integral, the first step is to divide the interval up into n evenly spaced subintervals. Thus, $\int\limits_{a}^{b} f(x)dx$ is defined to be the limit, as $n \to \infty$, of $\sum\limits_{i=1}^{n} f(x_i)\Delta x$, where $\{x_i\}$ are n evenly spaced points in the interval $[a, b]$, and $\Delta x = (b-a)/n$. But what if the points $\{x_i\}$ are not evenly spaced? We can still write down a reasonable sum: $\sum\limits_{i=1}^{n} f(x_i)\Delta x_i$, where now $\Delta x_i = x_{i+1} - x_i$. In order to make the integral well defined, we can no longer take the limit as $n \to \infty$. Instead, we must let $\max\{\Delta x_i\} \to 0$. It is a basic result of analysis that if this limit converges, then it does not matter how we picked the points $\{x_i\}$; the limit will converge to the same number. It is this number that we define to be the value of $\int\limits_{a}^{b} f(x)dx$.

4. What went wrong?

We are now ready to figure out what happened in section 2. Obviously, $\int\limits_{-1}^{1} f(\phi_1(a))da$ was not what we wanted. But let's not give up on our general approach just yet: it would still be great if we could use ϕ_1 to find *some* function, that we can integrate on $[-1, 1]$, that will give us the same answer as the integral of f over M. For now, let's call this mystery function "$?(a)$". We'll figure out what it has to be in a moment.

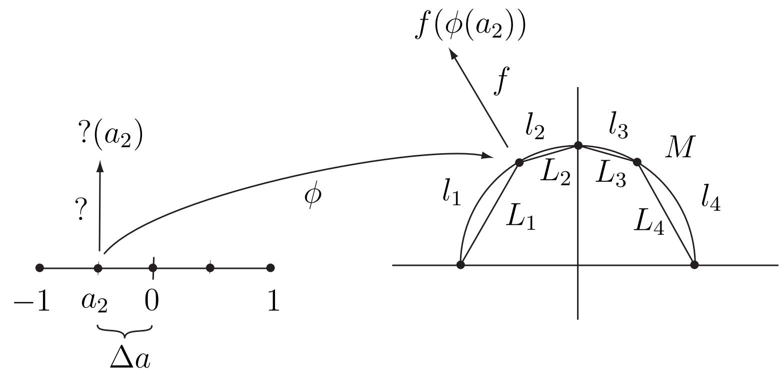


FIGURE 2. We want $?(a_1)\Delta a + ?(a_2)\Delta a + ?(a_3)\Delta a + ?(a_4)\Delta a = f(\phi(a_1))L_1 + f(\phi(a_2))L_2 + f(\phi(a_3))L_3 + f(\phi(a_4))L_4$.

Let's look at the Riemann sum that we get for $\int\limits_{-1}^{1} ?(a)da$, when we divide the interval up into n pieces, each of width $\Delta a = 2/n$. We get $\sum\limits_{i=1}^{n} ?(a_i)\Delta a$, where $a_i = -1+2/n$. Examine Figure 2 to see what happens to the points, a_i, under the function, ϕ_1, for $n = 4$. Notice that the points $\{\phi_1(a_i)\}$ are not evenly spaced along M. To use these points to estimate the integral of f over M, we would have to use the approach from the previous section. A Riemann sum for f over M would be:

$$\sum_{i=1}^{4} f(\phi_1(a_i))l_i$$
$$= f(-1,0)l_1 + f(-1/2, \sqrt{3/4})l_2 + f(0,1)l_3 + f(1/2, \sqrt{3/4})l_4$$
$$= (0)l_1 + (3/4)l_2 + (0)l_3 + (3/4)l_4$$

The l_i represent the arc length, along M, between $\phi_1(a_i)$ and $\phi_1(a_{i+1})$. This is a bit problematic, however, since arc-length is generally hard to calculate. Instead, we can approximate l_i by substituting the length of the line segment which connects $\phi_1(a_i)$ to $\phi_1(a_{i+1})$, which we shall denote as L_i. Note that this approximation gets better and better as we let $n \to \infty$. Hence, when we take the limit, it does not matter if we use l_i or L_i.

So our goal is to find a function, $?(a)$, on the interval $[-1, 1]$, so that the Riemann sum, $\sum\limits_{i=1}^{4} ?(a_i)\Delta a$ equals $(0)L_1 + (3/4)L_2 + (0)L_3 + (3/4)L_4$. In general, we want

$\sum\limits_{i=1}^{n} f(\phi_1(a_i))L_i = \sum\limits_{i=1}^{n} ?(a_i)\Delta a$. So, we must have $?(a_i)\Delta a = f(\phi_1(a_i))L_i$. Solving, we get $?(a_i) = \frac{f(\phi_1(a_i))L_i}{\Delta a}$.

What happens to this function as $\Delta a \to 0$? First, note that $L_i = |\phi_1(a_{i+1}) - \phi_1(a_i)|$. Hence,

$$\begin{aligned}
\lim_{\Delta a \to 0} ?(a_i) &= \lim_{\Delta a \to 0} \frac{f(\phi_1(a_i))L_i}{\Delta a} \\
&= \lim_{\Delta a \to 0} \frac{f(\phi_1(a_i))|\phi_1(a_{i+1}) - \phi_1(a_i)|}{\Delta a} \\
&= f(\phi_1(a_i)) \lim_{\Delta a \to 0} \frac{|\phi_1(a_{i+1}) - \phi_1(a_i)|}{\Delta a} \\
&= f(\phi_1(a_i)) \left| \lim_{\Delta a \to 0} \frac{\phi_1(a_{i+1}) - \phi_1(a_i)}{\Delta a} \right|
\end{aligned}$$

But $\lim_{\Delta a \to 0} \frac{\phi_1(a_{i+1}) - \phi_1(a_i)}{\Delta a}$ is precisely the definition of the derivative of ϕ_1 at a_i, $D_{a_i}\phi_1$. Hence, we have $\lim_{\Delta a \to 0} ?(a_i) = f(\phi_1(a_i))|D_{a_i}\phi_1|$. Finally, this means that the integral we want to compute is $\int\limits_{-1}^{1} f(\phi_1(a))|D_a\phi_1|da$, which should be a familiar integral from calculus.

Exercise 1.1. *Check that* $\int\limits_{-1}^{1} f(\phi_1(a))|D_a\phi_1|da = \int\limits_{0}^{\pi} f(\phi_2(t))|D_t\phi_2|dt$, *using the function,* f, *defined in section 2.*

Recall that $D_a\phi_1$ is a vector, based at the point $\phi(a)$, tangent to M. If we think of a as a time parameter, then the length of $D_a\phi_1$ tells us how fast $\phi_1(a)$ is moving along M. How can we generalize the integral, $\int\limits_{-1}^{1} f(\phi_1(a))|D_a\phi_1|da$? Note that the bars $|\cdot|$ are a function which "eats" vectors, and "spits out" real numbers. So we can generalize the integral by looking at other such functions. In other words, a more general integral would be $\int\limits_{-1}^{1} f(\phi_1(a))\omega(D_a\phi_1)da$, where f is a function of points and ω is a function of vectors.

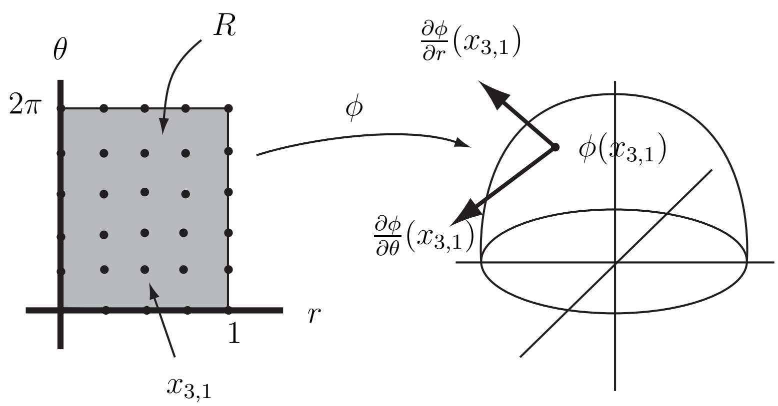
It is not the purpose of the present work to undertake a study of integrating with all possible functions, ω. However, as with the study of functions of real variables, a natural place to start is with *linear* functions. This is the study of differential forms. A differential form is precisely a linear function which eats vectors, spits out numbers, and is used in integration. The strength of differential forms lies in the fact that their integrals do not depend on a choice of parameterization.

5. What about surfaces?

Let's repeat the previous discussion (faster this time), bumping everything up a dimension. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = z^2$. Let M be the top half of the sphere of radius 1, centered at the origin. We can parameterize M by the function, ϕ, where $\phi(r, \theta) = (r\cos(\theta), r\sin(\theta), \sqrt{1 - r^2}), 0 \le r \le 1$, and $0 \le \theta \le 2\pi$. Again, our goal is not to figure out how to actually integrate f over M, but to use ϕ to set up an equivalent integral over the rectangle, $R = [0, 1] \times [0, 2\pi]$.

Let $\{x_{i,j}\}$ be a lattice of evenly spaced points in R. Let $\Delta r = x_{i+1,j} - x_{i,j}$, and $\Delta\theta = x_{i,j+1} - x_{i,j}$. By definition, the integral over R of a function, $?(x)$, is equal to $\lim_{\Delta r, \Delta\theta \to 0} \sum ?(x_{i,j})\Delta r \Delta\theta$.

To use the mesh of points, $\phi(x_{i,j})$, in M to set up a Riemann-Stiljes sum, we write down the following sum: $\sum f(\phi(x_{i,j}))Area(L_{i,j})$, where $L_{i,j}$ is the rectangle spanned by the vectors $\phi(x_{i+1,j}) - \phi(x_{i,j})$, and $\phi(x_{i,j+1}) - \phi(x_{i,j})$. If we want our Riemann sum over R to equal this sum, then we end up with $?(x_{i,j}) = \frac{f(\phi(x_{i,j}))Area(L_{i,j})}{\Delta r \Delta\theta}$.



We now leave it as an exercise to show that as Δr and $\Delta\theta$ get small, $\frac{Area(L_{i,j})}{\Delta r \Delta\theta}$ converges to the area of the parallelogram spanned by the vectors $\frac{\partial\phi}{\partial r}(x_{i,j})$, and $\frac{\partial\phi}{\partial\theta}(x_{i,j})$. The upshot of all this is that the integral we want to evaluate is the following:

$$\int\limits_R f(\phi(r, \theta))Area\left[\frac{\partial\phi}{\partial r}(r, \theta), \frac{\partial\phi}{\partial\theta}(r, \theta)\right] dr d\theta$$

EXERCISE 1.2. Compute the value of this integral for the function $f(x, y, z) = z^2$.

The point of all this is not the specific integral that we have arrived at, but the *form* of the integral. We are integrating $f \circ \phi$ (as in the previous section), times a function which takes *two* vectors and returns a real number. Once again, we can generalize this by using other such functions:

$$\int_R f(\phi(r,\theta))\omega\left[\frac{\partial\phi}{\partial r}(r,\theta), \frac{\partial\phi}{\partial\theta}(r,\theta)\right] dr d\theta$$

In particular, if we examine linear functions for ω, we arrive at a differential form. The moral is that if we want to perform an integral over a region parameterized by $\mathbb{R}$, as in the previous section, then we need to multiply by a function which takes a vector and returns a number. If we want to integrate over something parameterized by $\mathbb{R}^2$, then we need to multiply by a function which takes *two* vectors and returns a number. In general, an *n-form* is a linear function which takes n vectors, and returns a real number. One integrates n-forms over regions that can be parameterized by $\mathbb{R}^n$.

CHAPTER 2

Forms

1. Coordinates for vectors

Before we begin to discuss functions on vectors we first need to learn how to specify a vector. And before we can answer that we must first learn where vectors live. In Figure 1 we see a curve, C, and a tangent line to that curve. The line can be thought of as the set of all tangent vectors at the point, p. We denote that line as T_pC, the *tangent space* to C at the point p.

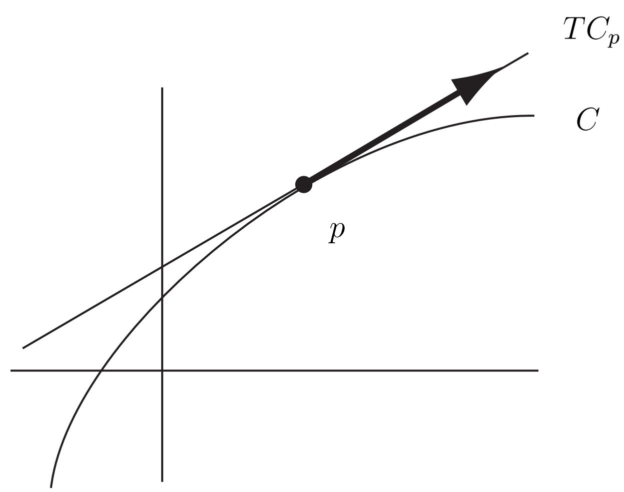


FIGURE 1. T_pC is the set of all vectors tangents to C at p.

What if C was actually a straight line? Would T_pC be the same line? To answer this, let's put down some coordinates. Suppose C were a straight line, with coordinates, and p is the point corresponding to the number 5. Now, suppose you were to draw a tangent vector to C, of length 2, which is tangent at p. Where would you draw it? Would you put it's base at 0 on C? Of course not...you'd put it's base at $p = 5$. So the origin for T_pC is in a different place as the origin for C. This is because we are thinking of C and T_pC as different lines, even though one may be right on top of the other.

Let's pause here for a moment to look at something a little more closely. What did we really do when we chose coordinates for C? What are "coordinates" anyway? They are a way of assigning a number (or, more generally, a set of numbers) to a point in our space. In other words, coordinates are functions which take points of a space and return (sets of) numbers. When we say that the x-coordinate of p is 5 we really mean that we have a function, $x : C \to \mathbb{R}$, such that $x(p) = 5$.

What about points in the plane? Of course we need two numbers to specify such a point, which means that we have two coordinate functions. Suppose we denote the plane by P and $x : P \to \mathbb{R}$ and $y : P \to \mathbb{R}$ are our coordinate functions. Then saying that the coordinates of a point, p, are $(2, 3)$ is the same thing as saying that $x(p) = 2$, and $y(p) = 3$. In other words, the coordinates of p are $(x(p), y(p))$.

So what do we use for coordinates in the tangent space? Well, first we need a *basis* for the tangent space of P at p. In other words, we need to pick two vectors which we can use to give the relative positions of all other points. Note that if the coordinates of p are (x, y) then $\frac{d(x+t,y)}{dt} = \langle 1, 0\rangle$, and $\frac{d(x,y+t)}{dt} = \langle 0, 1\rangle$. We have changed to the notation "$\langle\cdot,\cdot\rangle$" to indicate that we are not talking about points of P anymore, but rather vectors in T_pP. We take these two vectors to be a basis for T_pP. In other words, any point of T_pP can be written as $dx\langle 0, 1\rangle + dy\langle 1, 0\rangle$, where $dx, dy \in \mathbb{R}$. Hence, "dx" and "dy" are coordinate functions for T_pP. Saying that the coordinates of a vector V in T_pP are $\langle 2, 3\rangle$, for example, is the same thing as saying that $dx(V) = 2$ and $dy(V) = 3$. In general we may refer to the coordinates of an arbitrary vector in T_pP as $\langle dx, dy\rangle$, just as we may refer to the coordinates of an arbitrary point in P as (x, y).

It will be helpful in the future to be able to distinguish between the vector $\langle 2, 3\rangle$ in T_pP and the vector $\langle 2, 3\rangle$ in T_qP, where $p \neq q$. We will do this by writing $\langle 2, 3\rangle_p$ for the former and $\langle 2, 3\rangle_q$ for the latter.

Let's pause for a moment to address something that may have been bothering you since your first term of calculus. Let's look at the tangent line to the graph of $y = x^2$ at the point $(1, 1)$. We are no longer thinking of this tangent line as lying in the same plane that the graph does. Rather, it lies in $T_{(1,1)}\mathbb{R}^2$. The horizontal

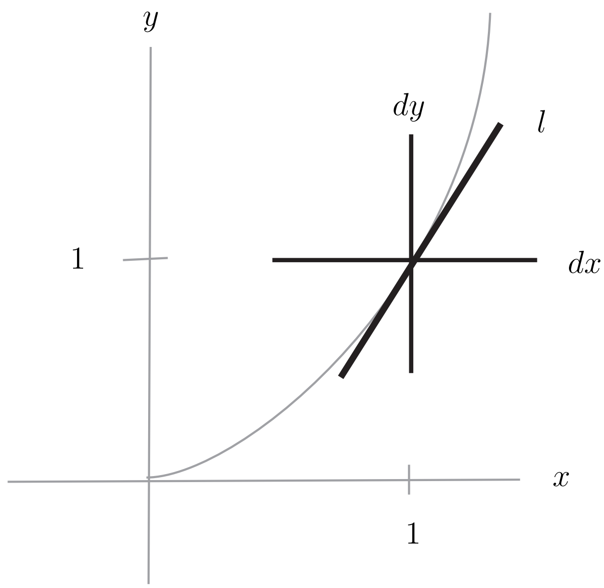


FIGURE 2. The line, l, lies in $T_{(1,1)}\mathbb{R}^2$. Its equation is $dy = 2dx$.

axis for $T_{(1,1)}\mathbb{R}^2$ is the "dx" axis and the vertical axis is the "dy" axis (see Fig. 2). Hence, we can write the equation of the tangent line as $dy = 2dx$. We can rewrite this as $\frac{dy}{dx} = 2$. Look familiar? This is one explanation of why we use the notation $\frac{dy}{dx}$ in calculus to denote the derivative.

EXERCISE 2.1.

(1) Draw a vector with $dx = 1$, $dy = 2$, in the tangent space $T_{(1,-1)}\mathbb{R}^2$.
(2) Draw $\langle -3, 1\rangle_{(0,1)}$.

2. 1-forms

Recall from the previous chapter that a 1-form is a linear function which acts on vectors and returns numbers. For the moment let's just look at 1-forms on $T_p\mathbb{R}^2$ for some fixed point, p. Recall that a linear function, ω, is just one whose graph is a plane through the origin. Hence, we want to write down an equation of a plane though the origin in $T_p\mathbb{R}^2 \times \mathbb{R}$, where one axis is labelled dx, another dy, and the third, ω (see Fig. 3). This is easy: $\omega = a\ dx + b\ dy$. Hence, to specify a 1-form on $T_p\mathbb{R}^2$ we only need to know two numbers: a and b.

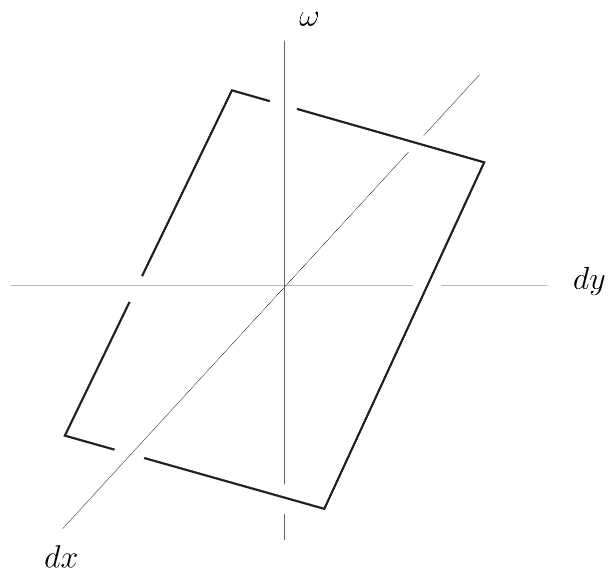


FIGURE 3. The graph of ω is a plane though the origin.

Here's a quick example: Suppose $\omega(\langle dx, dy\rangle) = 2dx + 3dy$ then

$$\omega(\langle -1, 2\rangle) = 2 \cdot -1 + 3 \cdot 2 = 4.$$

The alert reader may see something familiar here: the dot product. That is, $\omega(\langle -1, 2\rangle) = \langle 2, 3\rangle \cdot \langle -1, 2\rangle$. Recall the geometric interpretation of the dot product; you project $\langle -1, 2\rangle$ onto $\langle 2, 3\rangle$ and then multiply by $|\langle 2, 3\rangle| = \sqrt{13}$. In other words

Evaluating a 1-form on a vector is the same as projecting onto some line and then multiplying by some constant.

In fact, we can even interpret the act of multiplying by a constant geometrically. Suppose ω is given by $a\ dx + b\ dy$. Then the value of $\omega(V_1)$ is the length of the projection of V_1 onto the line, l, *where* $\frac{\langle a,b\rangle}{|\langle a,b\rangle|^2}$ *is a basis vector for* l.

This interpretation has a huge advantage... it's coordinate free. Recall from the previous section that we can think of the plane, P, as existing independent of our choice of coordinates. We only pick coordinates so that we can communicate to someone else the location of a point. Forms are similar. They are objects that exist independent of our choice of coordinates. This is one of the keys as to why they are so useful outside of mathematics.

There is still another geometric interpretation of 1-forms. Let's first look at the simple example $\omega(\langle dx, dy\rangle) = dx$. This 1-form simply returns the first coordinate of whatever vector you feed into it. This is also a projection; it's the projection of the input vector onto the dx-axis. This immediately gives us a new interpretation of the action of a general 1-form, $\omega = a\ dx + b\ dy$.

Evaluating a 1-form on a vector is the same as projecting onto each coordinate axis, scaling each by some constant, and adding the results.

Although this interpretation is a little more cumbersome it's the one that will generalize better when we get to n-forms.

Let's move on now to 1-forms in n dimensions. If $p \in \mathbb{R}^n$ then we can write p in coordinates as $(x_1, x_2, ..., x_n)$. The coordinates for a vector in $T_p\mathbb{R}^n$ are $\langle dx_1, dx_2, ..., dx_n\rangle$. A 1-form is a linear function, ω, whose graph (in $T_p\mathbb{R}^n \times \mathbb{R}$) is a plane through the origin. Hence, we can write it as $\omega = a_1 dx_1 + a_2 dx_2 + ... + a_n dx_n$. Again, this can be thought of as either projection onto the vector $\langle a_1, a_2, ..., a_n\rangle$ and then multiplying by $|\langle a_1, a_2, ..., a_n\rangle|$ or as projecting onto each coordinate axis, multiplying by a_i, and then adding.

EXERCISE 2.2. Let $\omega(\langle dx, dy\rangle) = -dx + 4dy$.

(1) Compute $\omega(\langle 1, 0\rangle)$, $\omega(\langle 0, 1\rangle)$, and $\omega(\langle 2, 3\rangle)$.
(2) What line does ω project vectors onto?

EXERCISE 2.3. Find a 1-form which

(1) projects vectors onto the line $dy = 2dx$ and scales by a factor of 2.
(2) projects vectors onto the line $dy = \frac{1}{3}dx$ and scales by a factor of $\frac{1}{5}$.
(3) projects vectors onto the dx-axis and scales by a factor of 3.
(4) projects vectors onto the dy-axis and scales by a factor of $\frac{1}{2}$.
(5) does both of the two preceding operations and adds the result.

3. Multiplying 1-forms

In this section we would like to explore a method of multiplying 1-forms. You may think, "What's the big deal? If ω and ν are 1-forms can't we just define $\omega \cdot \nu(V) = \omega(V) \cdot \nu(V)$?" Well, of course we *can*, but then $\omega \cdot \nu$ isn't a linear function, so we have left the world of forms.

The trick is to define the product of ω and ν to be a 2-form. So as not to confuse this with the product just mentioned we will use the symbol "$\wedge$" (pronounced "wedge") to denote multiplication. So how can we possibly define $\omega \wedge \nu$ to be a 2-form? To do this we have to say how it acts on a pair of vectors, (V_1, V_2).

Note first that there are four ways to combine all the ingredients:

$$\omega(V_1) \quad \nu(V_1) \quad \omega(V_2) \quad \nu(V_2)$$

The first two of these are associated with V_1 and the second two with V_2. In other words, ω and ν together give a way of taking each vector and returning a *pair* of numbers. And how do we visualize pairs of numbers? In the plane, of course! Let's define a new plane with one axis being the ω-axis and the other the ν-axis. So, the coordinates of V_1 in this plane are $[\omega(V_1), \nu(V_1)]$ and the coordinates of V_2 are $[\omega(V_2), \nu(V_2)]$. Note that we have switched to the notation "$[\cdot, \cdot]$" to indicate that we are describing points in a new plane. This may seem a little confusing at first. Just keep in mind that when we write something like $(1, 2)$ we are describing the location of a point in the x-y plane, whereas $\langle 1, 2 \rangle$ describes a vector in the dx-dy plane and $[1, 2]$ is a vector in the ω-ν plane.

Let's not forget our goal now. We wanted to use ω and ν to take the pair of vectors, (V_1, V_2), and return a number. So far all we have done is to take this pair of vectors and return another pair of vectors. But do we know of a way to take these vectors and get a number? Actually, we know several, but the most useful one turns out to be the area of the parallelogram that they span. This is precisely what we define to be the value of $\omega \wedge \nu(V_1, V_2)$ (see Fig. 4).

EXAMPLE 2.1. Let $\omega = 2dx - 3dy + dz$ and $\nu = dx + 2dy - dz$ be two 1-forms on $T_p\mathbb{R}^3$ for some fixed $p \in \mathbb{R}^3$. Let's evaluate $\omega \wedge \nu$ on the pair of

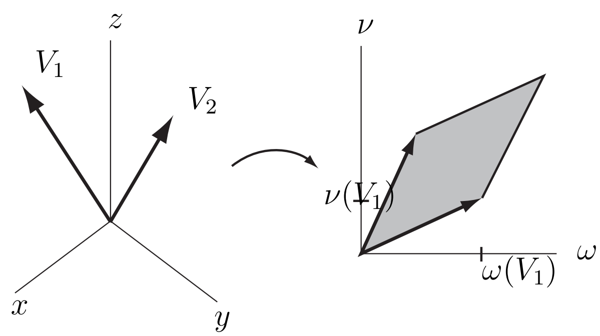


FIGURE 4. The product of ω and ν.

vectors, $(\langle 1,3,1\rangle, \langle (2,-1,3\rangle)$. First we compute the $[\omega, \nu]$ coordinates of the vector $\langle 1,3,1\rangle$.

$$\begin{aligned}[\omega(\langle 1,3,1\rangle), \nu(\langle 1,3,1\rangle)] &= [2\cdot 1 - 3\cdot 3 + 1\cdot 1, 1\cdot 1 + 2\cdot 3 - 1\cdot 1] \\ &= [-6, 6]\end{aligned}$$

Similarly we compute $[\omega(\langle 2,-1,3\rangle), \nu(\langle 2,-1,3\rangle)] = [10, -3]$. Finally, the area of the parallelogram spanned by $[-6,6]$ and $[10,-3]$ is

$$\begin{vmatrix} -6 & 6 \\ 10 & -3 \end{vmatrix} = 18 - 60 = -42$$

□

Should we have taken the absolute value? Not if we want to define a *linear* operator. The result of $\omega \wedge \nu$ isn't just an area, it's a *signed* area. It can either be positive or negative. We'll see a geometric interpretation of this soon. For now we define:

$$\omega \wedge \nu(V_1, V_2) = \begin{vmatrix} \omega(V_1) & \nu(V_1) \\ \omega(V_2) & \nu(V_2) \end{vmatrix}$$

EXERCISE 2.4. Let ω and ν be the following 1-forms:

$$\omega(\langle dx, dy\rangle) = 2dx - 3dy$$

$$\nu(\langle dx, dy\rangle) = dx + dy$$

(1) Let $V_1 = \langle -1, 2\rangle$ and $V_2 = \langle 1, 1\rangle$. Compute $\omega(V_1)$, $\nu(V_1)$, $\omega(V_2)$ and $\nu(V_2)$.
(2) Use your answers to the previous question to compute $\omega \wedge \nu(V_1, V_2)$.

(3) Find a constant c such that $\omega \wedge \nu = c\ dx \wedge dy$.

Exercise 2.5. $\omega \wedge \nu(V_1, V_2) = -\omega \wedge \nu(V_2, V_1)$ ($\omega \wedge \nu$ is *skew-symmetric*).

Exercise 2.6. $\omega \wedge \nu(V, V) = 0$. (This follows immediately from the previous exercise. It should also be clear from the geometric interpretation).

Exercise 2.7. $\omega \wedge \nu(V_1 + V_2, V_3) = \omega \wedge \nu(V_1, V_3) + \omega \wedge \nu(V_2, V_3)$ and $\omega \wedge \nu(cV_1, V_2) = \omega \wedge \nu(V_1, cV_2) = c\omega \wedge \nu(V_1, V_2)$, where c is any real number ($\omega \wedge \nu$ is *bilinear*).

Exercise 2.8. $\omega \wedge \nu(V_1, V_2) = -\nu \wedge \omega(V_1, V_2)$.

It's interesting to compare Exercises 2.5 and 2.8. Exercise 2.5 says that the 2-form, $\omega \wedge \nu$, is a skew-symmetric operator on pairs of vectors. Exercise 2.8 says that $\wedge$ can be thought of as a skew-symmetric operator on 1-forms.

Exercise 2.9. $\omega \wedge \omega(V_1, V_2) = 0$.

Exercise 2.10. $(\omega + \nu) \wedge \psi = \omega \wedge \psi + \nu \wedge \psi$ ($\wedge$ is distributive).

There is another way to interpret the action of $\omega \wedge \nu$ which is much more geometric, although it will take us some time to develop. Suppose $\omega = a\ dx + b\ dy + c\ dz$. Then we will denote the vector $\langle a, b, c\rangle$ as $\langle \omega \rangle$. From the previous section we know that if V is any vector then $\omega(V) = \langle \omega \rangle \cdot V$, and that this is just the projection of V onto the line containing $\langle \omega \rangle$, times $|\langle \omega \rangle|$.

Now suppose ν is some other 1-form. Choose a scalar x so that $\langle \nu - x\omega \rangle$ is perpendicular to $\langle \omega \rangle$. Let $\nu_\omega = \nu - x\omega$. Note that $\omega \wedge \nu_\omega = \omega \wedge (\nu - x\omega) = \omega \wedge \nu - x\omega \wedge \omega = \omega \wedge \nu$. Hence, any geometric interpretation we find for the action of $\omega \wedge \nu_\omega$ is also a geometric interpretation of the action of $\omega \wedge \nu$.

Finally, we let $\overline{\omega} = \frac{\omega}{|\langle \omega \rangle|}$ and $\overline{\nu_\omega} = \frac{\nu_\omega}{|\langle \nu_\omega \rangle|}$. Note that these are 1-forms such that $\langle \overline{\omega} \rangle$ and $\langle \overline{\nu_\omega} \rangle$ are perpendicular unit vectors. We will now present a geometric interpretation of the action of $\overline{\omega} \wedge \overline{\nu_\omega}$ on a pair of vectors, (V_1, V_2).

First, note that since $\langle \overline{\omega} \rangle$ is a unit vector then $\overline{\omega}(V_1)$ is just the projection of V_1 onto the line containing $\langle \overline{\omega} \rangle$. Similarly, $\overline{\nu_\omega}(V_1)$ is given by projecting V_1 onto the line containing $\langle \overline{\nu_\omega} \rangle$. As $\langle \overline{\omega} \rangle$ and $\langle \overline{\nu_\omega} \rangle$ are perpendicular, we can thus think of the quantity

$$\overline{\omega} \wedge \overline{\nu_\omega}(V_1, V_2) = \begin{vmatrix} \overline{\omega}(V_1) & \overline{\nu_\omega}(V_1) \\ \overline{\omega}(V_2) & \overline{\nu_\omega}(V_2) \end{vmatrix}$$

as being the area of parallelogram spanned by V_1 and V_2, projected onto the plane containing the vectors $\langle\overline{\omega}\rangle$ and $\langle\overline{\nu_\omega}\rangle$. This is the same plane as the one which contains the vectors $\langle\omega\rangle$ and $\langle\nu\rangle$.

Now observe the following:

$$\overline{\omega} \wedge \overline{\nu_\omega} = \frac{\omega}{|\langle\omega\rangle|} \wedge \frac{\nu_\omega}{|\langle\nu_\omega\rangle|} = \frac{1}{|\langle\omega\rangle||\langle\nu_\omega\rangle|}\omega \wedge \nu_\omega$$

Hence,

$$\omega \wedge \nu = \omega \wedge \nu_\omega = |\langle\omega\rangle||\langle\nu_\omega\rangle|\overline{\omega} \wedge \overline{\nu_\omega}$$

Finally, note that since $\langle\overline{\omega}\rangle$ and $\langle\overline{\nu_\omega}\rangle$ are perpendicular the quantity $|\langle\omega\rangle||\langle\nu_\omega\rangle|$ is just the area of the rectangle spanned by these two vectors. Furthermore, the parallelogram spanned by the vectors $\langle\omega\rangle$ and $\langle\nu\rangle$ is obtained from this rectangle by skewing. Hence, they have the same area. We conclude

Evaluating $\omega \wedge \nu$ on the pair of vectors (V_1, V_2) gives the area of parallelogram spanned by V_1 and V_2 projected onto the plane containing the vectors $\langle\omega\rangle$ and $\langle\nu\rangle$, and multiplied by the area of the parallelogram spanned by $\langle\omega\rangle$ and $\langle\nu\rangle$.

CAUTION: While every 1-form can be thought of as projected length *not* every 2-form can be thought of as projected area. The only 2-forms for which this interpretation is valid are those that are the product of 1-forms. See Exercise 2.15.

Let's pause for a moment to look at a particularly simple 2-form on $T_p\mathbb{R}^3$, $dx \wedge dy$. Suppose $V_1 = \langle a_1, a_2, a_3\rangle$ and $V_2 = \langle b_1, b_2, b_3\rangle$. Then

$$dx \wedge dy(V_1, V_2) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

This is precisely the (signed) area of the parallelogram spanned by V_1 and V_2 projected onto the dx-dy plane.

Exercise 2.11. $\omega \wedge \nu(\langle a_1, a_2, a_3\rangle, \langle b_1, b_2, b_3\rangle) = c_1 dx \wedge dy + c_2 dx \wedge dz + c_3 dy \wedge dz$, for some real numbers, c_1, c_2, and c_3.

The preceding comments, and this last exercise, give the following geometric interpretation of the action of a 2-form on the pair of vectors, (V_1, V_2):

> *Every 2-form projects the parallelogram spanned by V_1 and V_2 onto each of the (2-dimensional) coordinate planes, computes the resulting (signed) areas, multiplies each by some constant, and adds the results.*

This interpretation holds in all dimensions. Hence, to specify a 2-form we need to know as many constants as there are 2-dimensional coordinate planes. For example, to give a 2-form in 4-dimensional Euclidean space we need to specify 6 numbers:

$$c_1 dx \wedge dy + c_2 dx \wedge dz + c_3 dx \wedge dw + c_4 dy \wedge dz + c_5 dy \wedge dw + c_6 dz \wedge dw$$

The skeptic may argue here. Exercise 2.11 only shows that a 2-form which is a product of 1-forms can be thought of as a sum of projected, scaled areas. What about an arbitrary 2-form? Well, to address this we need to know what an arbitrary 2-form is! Up until now we have not given a complete definition. Henceforth, we shall define a 2-form to be a bi-linear, skew-symmetric, real-valued function on $T_p\mathbb{R}^n \times T_p\mathbb{R}^n$. That's a mouthful. This just means that it's an operator which eats pairs of vectors, spits out real numbers, and satisfies the conclusions of Exercises 2.5 and 2.7. Since these are the only ingredients necessary to do Exercise 2.11 our geometric interpretation is valid for all 2-forms.

EXERCISE 2.12. If $\omega(\langle dx, dy, dz\rangle) = dx+5dy-dz$, and $\nu(\langle dx, dy, dz\rangle) = 2dx-dy+dz$, compute

$$\omega \wedge \nu(\langle 1, 2, 3\rangle, \langle -1, 4, -2\rangle)$$

Answer: -127

EXERCISE 2.13. Let $\omega(\langle dx, dy, dz\rangle) = dx+5dy-dz$ and $\nu(\langle dx, dy, dz\rangle) = 2dx-dy+dz$. Find constants, c_1, c_2, and c_3, such that

$$\omega \wedge \nu = c_1 dx \wedge dy + c_2 dy \wedge dz + c_3 dx \wedge dz$$

Answer: $c_1 = -11$, $c_2 = 4$, and $c_3 = 3$

EXERCISE 2.14. Express each of the following as the product of two 1-forms:

(1) $3dx \wedge dy + dy \wedge dx$
(2) $dx \wedge dy + dx \wedge dz$
(3) $3dx \wedge dy + dy \wedge dx + dx \wedge dz$
(4) $dx \wedge dy + 3dz \wedge dy + 4dx \wedge dz$

4. 2-forms on $T_p\mathbb{R}^3$ (optional)

EXERCISE 2.15. Find a 2-form which is *not* the product of 1-forms.

In doing this exercise you may guess that in fact all 2-forms on $T_p\mathbb{R}^3$ *can* be written as a product of 1-forms. Let's see a proof of this fact that relies heavily on the geometric interpretations we have developed.

Recall the correspondence introduced above between vectors and 1-forms. If $\alpha = a_1 dx + a_2 dy + a_3 dz$ then we let $\langle \alpha \rangle = \langle a_1, a_2, a_3 \rangle$. If V is a vector then we let $\langle V \rangle^{-1}$ be the corresponding 1-form.

We now prove two lemmas:

LEMMA 2.1. *If α and β are 1-forms on $T_p\mathbb{R}^3$ and V is a vector in the plane spanned by $\langle \alpha \rangle$ and $\langle \beta \rangle$ then there is a vector, W, in this plane such that $\alpha \wedge \beta = \langle V \rangle^{-1} \wedge \langle W \rangle^{-1}$.*

PROOF. The proof of the above lemma relies heavily on the fact that 2-forms which are the product of 1-forms are very flexible. The 2-form $\alpha \wedge \beta$ takes pairs of vectors, projects them onto the plane spanned by the vectors $\langle \alpha \rangle$ and $\langle \beta \rangle$, and computes the area of the resulting parallelogram times the area of the parallelogram spanned by $\langle \alpha \rangle$ and $\langle \beta \rangle$. Note that for every non-zero scalar c the area of the parallelogram spanned by $\langle \alpha \rangle$ and $\langle \beta \rangle$ is the same as the area of the parallelogram spanned by $c\langle \alpha \rangle$ and $1/c\langle \beta \rangle$. (This is the same thing as saying that $\alpha \wedge \beta = c\alpha \wedge 1/c\beta$). The important point here is that we can scale one of the 1-forms as much as we want at the expense of the other and get the same 2-form as a product.

Another thing we can do is apply a rotation to the pair of vectors $\langle \alpha \rangle$ and $\langle \beta \rangle$ in the plane which they determine. As the area of the parallelogram spanned by these two vectors is unchanged by rotation, their product still determines the same 2-form. In particular, suppose V is any vector in the plane spanned by $\langle \alpha \rangle$ and $\langle \beta \rangle$. Then we can rotate $\langle \alpha \rangle$ and $\langle \beta \rangle$ to $\langle \alpha' \rangle$ and $\langle \beta' \rangle$ so that $c\langle \alpha' \rangle = V$, for some scalar c. We can then replace the pair $(\langle\alpha\rangle, \langle\beta\rangle)$ with the pair $(c\langle\alpha'\rangle, 1/c\langle\beta'\rangle) = (V, 1/c\langle\beta'\rangle)$. To complete the proof, let $W = 1/c\langle\beta'\rangle$. $\square$

LEMMA 2.2. *If $\omega_1 = \alpha_1 \wedge \beta_1$ and $\omega_2 = \alpha_2 \wedge \beta_2$ are 2-forms on $T_p\mathbb{R}^3$ then there exists 1-forms, α_3 and β_3, such that $\omega_1 + \omega_2 = \alpha_3 \wedge \beta_3$.*

PROOF. Let's examine the sum, $\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2$. Our first case is that the plane spanned by the pair $(\langle\alpha_1\rangle, \langle\beta_1\rangle)$ is the same as the plane spanned by the pair, $(\langle\alpha_2\rangle, \langle\beta_2\rangle)$. In this case it must be that $\alpha_1 \wedge \beta_1 = C\alpha_2 \wedge \beta_2$, and hence, $\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 = (1 + C)\alpha_1 \wedge \beta_1$.

If these two planes are not the same then they intersect in a line. Let V be a vector contained in this line. Then by the preceding lemma there are 1-forms γ and γ' such that $\alpha_1 \wedge \beta_1 = \langle V\rangle^{-1} \wedge \gamma$ and $\alpha_2 \wedge \beta_2 = \langle V\rangle^{-1} \wedge \gamma'$. Hence,

$$\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 = \langle V\rangle^{-1} \wedge \gamma + \langle V\rangle^{-1} \wedge \gamma' = \langle V\rangle^{-1} \wedge (\gamma + \gamma')$$

$\square$

Now note that *any* 2-form is the sum of products of 1-forms. Hence, this last lemma implies that any 2-form on $T_p\mathbb{R}^3$ is a product of 1-forms. In other words:

Every 2-form on $T_p\mathbb{R}^3$ projects pairs of vectors onto some plane and returns the area of the resulting parallelogram, scaled by some constant.

This fact is precisely why all of classical vector calculus works. We explore this in the next few exercises, and further in Section 4 of Chapter 5.

EXERCISE 2.16. Use the above geometric interpretation of the action of a 2-form on $T_p\mathbb{R}^3$ to justify the following statement: For every 2-form ω on $T_p\mathbb{R}^3$ there are non-zero vectors V_1 and V_2 such that V_1 is not a multiple of V_2, but $\omega(V_1, V_2) = 0$.

EXERCISE 2.17. Does Exercise 2.16 generalize to higher dimensions?

EXERCISE 2.18. Show that if ω is a 2-form on $T_p\mathbb{R}^3$ then there is a line l in $T_p\mathbb{R}^3$ such that if the plane spanned by V_1 and V_2 contains l then $\omega(V_1, V_2) = 0$.

Note that the conditions of Exercise 2.18 are satisfied when the vectors that are perpendicular to both V_1 and V_2 are also perpendicular to l.

EXERCISE 2.19. Show that if all you know about V_1 and V_2 is that they are vectors in $T_p\mathbb{R}^3$ that span a parallelogram of area A, then the value of $\omega(V_1, V_2)$ is maximized when V_1 and V_2 are perpendicular to the line l of Exercise 2.18.

Note that the conditions of this exercise are satisfied when the vectors perpendicular to V_1 and V_2 are parallel to l.

EXERCISE 2.20. Let N be a vector perpendicular to V_1 and V_2 in $T_p\mathbb{R}^3$ whose length is precisely the area of the parallelogram spanned by these two vectors. Show that there is a vector V_ω in the line l of Exercise 2.18 such that the value of $\omega(V_1, V_2)$ is precisely $V_\omega \cdot N$.

Remark. You may have learned that the vector N of the previous exercise is precisely the cross product of V_1 and V_2. Hence, the previous exercise implies that if ω is a 2-form on $T_p\mathbb{R}^3$ then there is a vector V_ω such that $\omega(V_1, V_2) = V_\omega \cdot (V_1 \times V_2)$

EXERCISE 2.21. Show that if $\omega = F_x\ dy \wedge dz - F_y\ dx \wedge dz + F_z\ dx \wedge dy$ then $V_\omega = \langle F_1, F_2, F_3 \rangle$.

5. n-forms

Let's think a little more about our multiplication, $\wedge$. If it's really going to be anything like multiplication we should be able to take three 1-forms, ω, ν, and ψ, and form the product $\omega \wedge \nu \wedge \psi$. How can we define this? A first guess might be to say that $\omega \wedge \nu \wedge \psi = \omega \wedge (\nu \wedge \psi)$, but $\nu \wedge \psi$ is a 2-form and we haven't defined the product of a 2-form and a 1-form. We're going to take a different approach and define $\omega \wedge \nu \wedge \psi$ directly.

This is completely analogous to the previous section. ω, ν, and ψ each act on a vector, V, to give three numbers. In other words, they can be thought of as coordinate functions. We say the coordinates of V are $[\omega(V), \nu(V), \psi(V)]$. Hence, if we have three vectors, V_1, V_2, and V_3, we can compute the $[\omega, \nu, \psi]$ coordinates of each. This gives us three new vectors. The signed volume of the parallelepiped which they span is what we define to be the value of $\omega \wedge \nu \wedge \psi(V_1, V_2, V_3)$.

There is no reason to stop at 3-dimensions. Suppose $\omega_1, \omega_2, ..., \omega_n$ are 1-forms and $V_1, V_2, ..., V_n$ are vectors. Then we define the value of $\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, V_2, ..., V_n)$

to be the signed (n-dimensional) volume of the parallelepiped spanned by the vectors $[\omega_1(V_i), \omega_2(V_i), ..., \omega_n(V_i)]$. Algebraically,

$$\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, V_2, ..., V_n) = \begin{vmatrix} \omega_1(V_1) & \omega_2(V_1) & ... & \omega_n(V_1) \\ \omega_1(V_2) & \omega_2(V_2) & ... & \omega_n(V_2) \\ \vdots & \vdots & & \vdots \\ \omega_1(V_n) & \omega_2(V_n) & ... & \omega_n(V_n) \end{vmatrix}$$

It follows from linear algebra that if we swap any two rows or columns of this matrix the sign of the result flips. Hence, if the n-tuple, $\mathbf{V}' = (V_{i_1}, V_{i_2}, ..., V_{i_n})$ is obtained from $\mathbf{V} = (V_1, V_2, ..., V_n)$ by an even number of exchanges then the sign of $\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(\mathbf{V}')$ will be the same as the sign of $\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(\mathbf{V})$. If the number of exchanges were odd then the sign would be opposite. We sum this up by saying that the n-form, $\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n$ is *alternating*.

The wedge product of 1-forms is also *multilinear*, in the following sense:

$$\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, ..., V_i + V_i', ..., V_n)$$
$$= \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, ..., V_i, ..., V_n) + \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, ..., V_i', ..., V_n),$$

and

$$\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, ..., cV_i, ..., V_n) = c\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, ..., V_i, ..., V_n),$$

for all i and any real number, c.

In general, we define an n-form to be any alternating, multilinear real-valued function which acts on n-tuples of vectors.

EXERCISE 2.22. Prove the following geometric interpretation: *Hint: All of the steps are completely analogous to those in the last section.*

> *An m-form on $T_p\mathbb{R}^n$ can be thought of as a function which takes the parallelepiped spanned by m vectors, projects it onto each of the m-dimensional coordinate planes, computes the resulting areas, multiplies each by some constant, and adds the results.*

EXERCISE 2.23. How many numbers do you need to give to specify a 5-form on $T_p\mathbb{R}^{10}$?

We turn now to the simple case of an n-form on $T_p\mathbb{R}^n$. Notice that there is only one n-dimensional coordinate plane in this space, namely, the space itself. Such a form, evaluated on an n-tuple of vectors, must therefore give the n-dimensional volume of the parallelepiped which it spans, multiplied by some constant. For this reason such a form is called a *volume form* (in 2-dimensions, an *area form*).

EXAMPLE 2.2. Consider the forms, $\omega = dx + 2dy - dz, \nu = 3dx - dy + dz$, and $\psi = -dx - 3dy + dz$, on $T_p\mathbb{R}^3$. By the above argument $\omega \wedge \nu \wedge \psi$ must be a volume form. But which volume form is it? One way to tell is to compute its value on a set of vectors which we *know* span a parallelepiped of volume 1, namely $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$, and $\langle 0, 0, 1\rangle$. This will tell us how much the form scales volume.

$$\omega \wedge \nu \wedge \psi(\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle) = \begin{vmatrix} 1 & 3 & -1 \\ 2 & -1 & -3 \\ -1 & 1 & 1 \end{vmatrix} = 4$$

So, $\omega \wedge \nu \wedge \psi$ must be the same as the form $4dx \wedge dy \wedge dz$. □

EXERCISE 2.24. Let $\omega(\langle dx, dy, dz\rangle) = dx + 5dy - dz$, $\nu(\langle dx, dy, dz\rangle) = 2dx - dy + dz$, and $\gamma(\langle dx, dy, dz) = -dx + dy + 2dz$.

(1) If $V_1 = \langle 1, 0, 2\rangle, V_2 = \langle 1, 1, 2\rangle$, and $V_3 = \langle 0, 2, 3\rangle$, compute $\omega \wedge \nu \wedge \gamma(V_1, V_2, V_3)$.
Answer: -87
(2) Find a constant, c, such that $\omega \wedge \nu \wedge \gamma = c\ dx \wedge dy \wedge dz$.
Answer: $c = -29$
(3) Let $\alpha = 3dx \wedge dy + 2dy \wedge dz - dx \wedge dz$. Find a constant, c, such that $\alpha \wedge \gamma = c\ dx \wedge dy \wedge dz$.
Answer: $c = 5$

EXERCISE 2.25. Simplify:

$$dx \wedge dy \wedge dz + dx \wedge dz \wedge dy + dy \wedge dz \wedge dx + dy \wedge dx \wedge dy$$

EXERCISE 2.26. Let ω be an n-form and ν an m-form. Show that

$$\omega \wedge \nu = (-1)^{nm} \nu \wedge \omega$$

CHAPTER 3

Differential **Forms**

1. Families of forms

Let's now go back to the example in Chapter 1. In the last section of that chapter we showed that the integral of a function, $f : \mathbb{R}^3 \to \mathbb{R}$, over a surface parameterized by $\phi : R \subset \mathbb{R}^2 \to \mathbb{R}^3$ is

$$\int\limits_R f(\phi(r,\theta))Area\left[\frac{\partial\phi}{\partial r}(r,\theta), \frac{\partial\phi}{\partial\theta}(r,\theta)\right] drd\theta$$

This was one of the motivations for studying differential forms. We wanted to generalize this integral by considering functions other than "$Area(\cdot,\cdot)$" which eat pairs of vectors and return numbers. But in this integral the point at which such a pair of vectors is based changes. In other words, $Area(\cdot,\cdot)$ does *not* act on $T_p\mathbb{R}^3 \times T_p\mathbb{R}^3$ *for a fixed* p. We can make this point a little clearer by re-examining the above integrand. Note that it is of the form $f(\star)Area(\cdot,\cdot)$. For a fixed point, $\star$, of $\mathbb{R}^3$ this is an operator on $T_\star\mathbb{R}^3 \times T_\star\mathbb{R}^3$, much like a 2-form is.

But so far all we have done is to define 2-forms at fixed points of $\mathbb{R}^3$. To really generalize the above integral we have to start considering entire families of 2-forms, $\omega_p : T_p\mathbb{R}^3 \times T_p\mathbb{R}^3 \to \mathbb{R}$, where p ranges over all of $\mathbb{R}^3$. Of course, for this to be useful we'd like such a family to have some "niceness" properties. For one thing, we would like it to be *continuous*. That is, if p and q are close then ω_p and ω_q should be similar.

An even stronger property that we will insist on is that the family, ω_p, is *differentiable*. To see what this means recall that for a fixed p, a 2-form ω_p can always be written as $a_p dx \wedge dy + b_p dy \wedge dz + c_p dx \wedge dz$, where a_p, b_p, and c_p are constants. But if we let our choice of p vary over all of $\mathbb{R}^3$ then so will these constants. In other words, a_p, b_p and c_p are all functions from $\mathbb{R}^3$ to $\mathbb{R}$. To say that the family, ω_p, is differentiable we mean that each of these functions is differentiable. If ω_p is differentiable then we will refer to it as a *differential* form. When there can be no confusion we will suppress the subscript, p.

EXAMPLE 3.1. $\omega = x^2y\, dx \wedge dy - xz\, dy \wedge dz$ is a differential 2-form on $\mathbb{R}^3$. On the space $T_{(1,2,3)}\mathbb{R}^3$ it is just the 2-form $2dx \wedge dy - 3dy \wedge dz$. We will denote vectors in $T_{(1,2,3)}\mathbb{R}^3$ as $\langle dx, dy, dz\rangle_{(1,2,3)}$. Hence, the value of $\omega(\langle 4, 0, -1\rangle_{(1,2,3)}, \langle 3, 1, 2\rangle_{(1,2,3)})$ is the same as the 2-form, $2dx \wedge dy + dy \wedge dz$, evaluated on the vectors $\langle 4, 0, -1\rangle$ and $\langle 3, 1, 2\rangle$, which we compute:

$$\begin{aligned} \omega(\langle 4, 0, -1\rangle_{(1,2,3)} \quad &, \quad \langle 3, 1, 2\rangle_{(1,2,3)}) \\ &= 2dx \wedge dy - 3dy \wedge dz\ (\langle 4, 0, -1\rangle, \langle 3, 1, 2\rangle) \\ &= 2\begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 5 \end{aligned}$$

□

Suppose ω is a differential 2-form on $\mathbb{R}^3$. What does ω act on? It takes a pair of vectors at each point of $\mathbb{R}^3$ and returns a number. In other words, it takes two vector *fields* and returns a function from $\mathbb{R}^3$ to $\mathbb{R}$. A vector field is simply a choice of vector in $T_p\mathbb{R}^3$, for each $p \in \mathbb{R}^3$. In general, a differential n-form on $\mathbb{R}^m$ acts on n vector fields to produce a function from $\mathbb{R}^m$ to $\mathbb{R}$ (see Fig. 1).

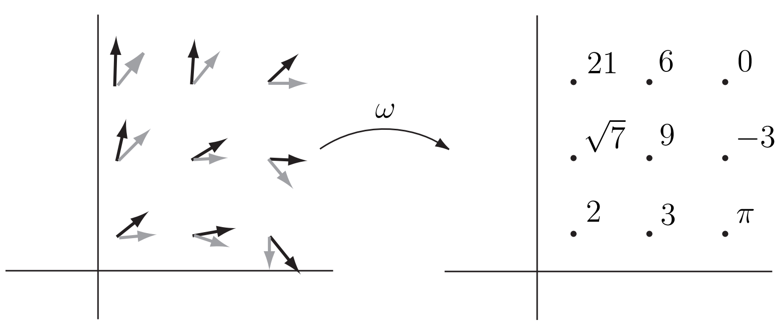


FIGURE 1. A differential 2-form, ω, acts on a pair of vector fields, and returns a function from $\mathbb{R}^n$ to $\mathbb{R}$.

EXAMPLE 3.2. $V_1 = \langle 2y, 0, -x\rangle_{(x,y,z)}$ is a vector field on $\mathbb{R}^3$. For example, it contains the vector $\langle 4, 0, -1\rangle \in T_{(1,2,3)}\mathbb{R}^3$. If $V_2 = \langle z, 1, xy\rangle_{(x,y,z)}$ and ω is the differential 2-form, $x^2y\ dx \wedge dy - xz\ dy \wedge dz$, then

$$\begin{aligned} \omega(V_1, V_2) &= x^2y\ dx \wedge dy - xz\ dy \wedge dz(\langle 2y, 0, x\rangle_{(x,y,z)}, \langle z, 1, xy\rangle_{(x,y,z)}) \\ &= x^2y\begin{vmatrix} 2y & 0 \\ z & 1 \end{vmatrix} - xz\begin{vmatrix} 0 & -x \\ 1 & xy \end{vmatrix} = 2x^2y^2 - x^2z, \end{aligned}$$

which is a function from $\mathbb{R}^3$ to $\mathbb{R}$.

Notice that V_2 contains the vector $\langle 3, 1, 2\rangle_{(1,2,3)}$. So, from the previous example we would expect that $2x^2y^2 - x^2z$ equals 5 at the point $(1, 2, 3)$, which is indeed the case. □

EXERCISE 3.1. Let ω be the differential 2-form on $\mathbb{R}^3$ given by

$$\omega = xyz\ dx \wedge dy + x^2z\ dy \wedge dz - y\ dx \wedge dz$$

Let V_1 and V_2 be the following vector fields:

$$V_1 = \langle y, z, x^2\rangle_{(x,y,z)}, V_2 = \langle xy, xz, y\rangle_{(x,y,z)}$$

(1) What vectors do V_1 and V_2 contain at the point $(1, 2, 3)$?
(2) Which 2-form is ω on $T_{(1,2,3)}\mathbb{R}^3$?
(3) Use your answers to the previous two questions to compute $\omega(V_1, V_2)$ at the point $(1, 2, 3)$.
(4) Compute $\omega(V_1, V_2)$ at the point (x, y, z). Then plug in $x = 1$, $y = 2$, and $z = 3$ to check your answer against the previous question.

2. Integrating Differential 2-Forms

Let us now examine more closely integration of functions on subsets of $\mathbb{R}^2$, which you learned in calculus. Suppose $R \subset \mathbb{R}^2$ and $f : R \to \mathbb{R}$. How did we learn to define the integral of f over R? We summarize the procedure in the following steps:

(1) Choose a lattice of points in R, $\{(x_i, y_j)\}$.
(2) For each i, j define $V^1_{i,j} = (x_{i+1}, y_j) - (x_i, y_j)$ and $V^2_{i,j} = (x_i, y_{j+1}) - (x_i, y_j)$ (See Fig. 2). Notice that $V^1_{i,j}$ and $V^2_{i,j}$ are both vectors in $T_{(x_i,y_j)}\mathbb{R}^2$.

(3) For each i, j compute $f(x_i, y_j)Area(V^1_{i,j}, V^2_{i,j})$, where $Area(V, W)$ is the function which returns the area of the parallelogram spanned by the vectors V and W.
(4) Sum over all i and j.
(5) Take the limit as the maximal distance between adjacent lattice points goes to 0. This is the number that we define to be the value of $\int\limits_R f\ dx\ dy$.

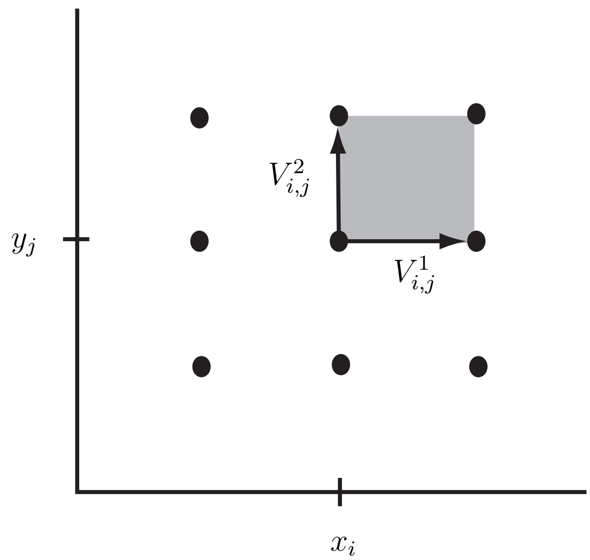


FIGURE 2. The steps toward integration.

Let's focus on Step 3. Here we compute $f(x_i, y_j)Area(V^1_{i,j}, V^2_{i,j})$. Notice that this is exactly the value of the differential 2-form $\omega = f(x, y)dx \wedge dy$, evaluated on the vectors $V^1_{i,j}$ and $V^2_{i,j}$ at the point (x_i, y_j). Hence, in step 4 we can write this sum as $\sum\limits_i \sum\limits_j f(x_i, y_j)Area(V^1_{i,j}, V^2_{i,j}) = \sum\limits_i \sum\limits_j \omega_{(x_i,y_j)}(V^1_{i,j}, V^2_{i,j})$. It is reasonable, then, to adopt the shorthand "$\int\limits_R \omega$" to denote the limit in Step 5. The upshot of all this is the following:

$$\boxed{\textit{If } \omega = f(x, y)dx \wedge dy \textit{ then } \int\limits_R \omega = \int\limits_R f\ dx\ dy.}$$

Since all differential 2-forms on $\mathbb{R}^2$ are of the form $f(x, y)dx \wedge dy$ we now know how to integrate them.

CAUTION! When integrating 2-forms on $\mathbb{R}^2$ it is tempting to always drop the "$\wedge$" and forget you have a differential form. This is only valid with $dx \wedge dy$. It is NOT valid with $dy \wedge dx$. This may seem a bit curious since

$$\int f\ dx \wedge dy = \int f\ dx\ dy = \int f\ dy\ dx$$

All of these are equal to $-\int f\ dy \wedge dx$.

EXERCISE 3.2. Let $\omega = xy^2\ dx \wedge dy$ be a differential 2-form on $\mathbb{R}^2$. Let D be the region of $\mathbb{R}^2$ bounded by the graphs of $x = y^2$ and $y = x - 6$. Calculate $\int\limits_D \omega$. *Answer:* 189.

What about integration of differential 2-forms on $\mathbb{R}^3$? As remarked at the end of Section 5 we do this only over those subsets of $\mathbb{R}^3$ which can be parameterized by subsets of $\mathbb{R}^2$. Suppose M is such a subset, like the top half of the unit sphere. To define what we mean by $\int\limits_M \omega$ we just follow the steps above:

(1) Choose a lattice of points in M, $\{p_{i,j}\}$.
(2) For each i, j define $V^1_{i,j} = p_{i+1,j} - p_{i,j}$ and $V^2_{i,j} = p_{i,j+1} - p_{i,j}$. Notice that $V^1_{i,j}$ and $V^2_{i,j}$ are both vectors in $T_{p_{i,j}}\mathbb{R}^3$ (see Fig. 3).
(3) For each i, j compute $\omega_{p_{i,j}}(V^1_{i,j}, V^2_{i,j})$.
(4) Sum over all i and j.
(5) Take the limit as the maximal distance between adjacent lattice points goes to 0. This is the number that we define to be the value of $\int\limits_M \omega$.

Unfortunately these steps aren't so easy to follow. For one thing, it's not always clear how to pick the lattice in Step 1. In fact there is an even worse problem. In Step 3 why did we compute $\omega_{p_{i,j}}(V^1_{i,j}, V^2_{i,j})$ instead of $\omega_{p_{i,j}}(V^2_{i,j}, V^1_{i,j})$? After all, $V^1_{i,j}$ and $V^2_{i,j}$ are two randomly oriented vectors in $T\mathbb{R}^3_{p_{i,j}}$. There is no reasonable way to decide which should be first and which second. There is nothing to be done about this. At some point we just have to make a choice and make it clear which choice we have made. Such a decision is called an *orientation*. We will have much more to say about this later. For now, we simply note that a different choice will only change our answer by changing its sign.

While we are on this topic we also note that we would end up with the same number in Step 5 if we had calculated $\omega_{p_{i,j}}(-V^1_{i,j}, -V^2_{i,j})$ in Step 4, instead. Similarly, if it turns out later that we should have calculated $\omega_{p_{i,j}}(V^2_{i,j}, V^1_{i,j})$ then we could have

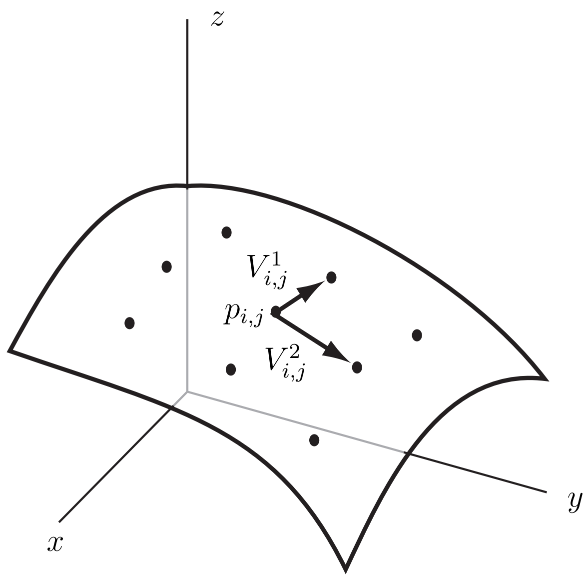


FIGURE 3. The steps toward integrating a 2-form.

also gotten the right answer by computing $\omega_{p_{i,j}}(-V^1_{i,j}, V^2_{i,j})$. In other words, there are really only two possibilities: either $\omega_{p_{i,j}}(V^1_{i,j}, V^2_{i,j})$ gives the correct answer or $\omega_{p_{i,j}}(-V^1_{i,j}, V^2_{i,j})$ does. Which one will depend on our choice of orientation.

Despite all the difficulties with using the above definition of $\int\limits_M \omega$, all hope is not lost. Remember that we are only integrating over regions which can be parameterized by subsets of $\mathbb{R}^2$. The trick is to use such a parameterization to translate the problem into an integral of a 2-form over a region in $\mathbb{R}^2$. The steps are analogous to those in Section 5 of Chapter 1.

Suppose $\phi : R \subset \mathbb{R}^2 \to M$ is a parameterization. We want to find a 2-form, $f(x, y)\ dx \wedge dy$, such that a Riemann sum for this 2-form over R gives the same result as a Riemann sum for ω over M. Let's begin:

(1) Choose a rectangular lattice of points in R, $\{(x_i, y_j)\}$. This also gives a lattice, $\{\phi(x_i, y_j)\}$, in M.
(2) For each i, j, define $V^1_{i,j} = (x_{i+1}, y_j) - (x_i, y_j)$, $V^2_{i,j} = (x_i, y_{j+1}) - (x_i, y_j)$, $\mathcal{V}^1_{i,j} = \phi(x_{i+1}, y_j) - \phi(x_i, y_j)$, and $\mathcal{V}^2_{i,j} = \phi(x_i, y_{j+1}) - \phi(x_i, y_j)$ (see Fig. 4). Notice that $V^1_{i,j}$ and $V^2_{i,j}$ are vectors in $T_{(x_i,y_j)}\mathbb{R}^2$ and $\mathcal{V}^1_{i,j}$ and $\mathcal{V}^2_{i,j}$ are vectors in $T_{\phi(x_i,y_j)}\mathbb{R}^3$.

(3) For each i, j compute $f(x_i, y_j)\ dx \wedge dy(V^1_{i,j}, V^2_{i,j})$ and $\omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})$.
(4) Sum over all i and j.

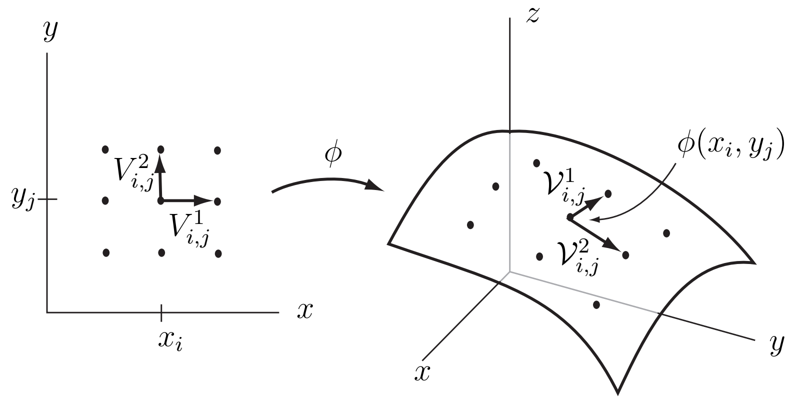


FIGURE 4. Using ϕ to integrate a 2-form.

At the conclusion of Step 4 we have two sums, $\sum\limits_i \sum\limits_j f(x_i, y_j)\ dx \wedge dy(V^1_{i,j}, V^2_{i,j})$, and $\sum\limits_i \sum\limits_j \omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})$. In order for these to be equal we must have:

$$f(x_i, y_j)\ dx \wedge dy(V^1_{i,j}, V^2_{i,j}) = \omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})$$

And so,

$$f(x_i, y_j) = \frac{\omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})}{dx \wedge dy(V^1_{i,j}, V^2_{i,j})}$$

But, since we are using a rectangular lattice in R we know $dx \wedge dy(V^1_{i,j}, V^2_{i,j}) = Area(V^1_{i,j}, V^2_{i,j}) = |V^1_{i,j}| \cdot |V^2_{i,j}|$. We now have

$$f(x_i, y_j) = \frac{\omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})}{|V^1_{i,j}| \cdot |V^2_{i,j}|}$$

Using the bilinearity of ω this reduces to

$$f(x_i, y_j) = \omega_{\phi(x_i,y_j)}\left(\frac{\mathcal{V}^1_{i,j}}{|V^1_{i,j}|}, \frac{\mathcal{V}^2_{i,j}}{|V^2_{i,j}|}\right)$$

But, as the distance between adjacent points of our partition tends toward 0,

$$\frac{\mathcal{V}^1_{i,j}}{|V^1_{i,j}|} = \frac{\phi(x_{i+1}, y_j) - \phi(x_i, y_j)}{|(x_{i+1}, y_j) - (x_i, y_j)|} = \frac{\phi(x_{i+1}, y_j) - \phi(x_i, y_j)}{|x_{i+1} - x_i|} \to \frac{\partial \phi}{\partial x}(x_i, y_j)$$

Similarly, $\frac{\mathcal{V}^2_{i,j}}{|V^2_{i,j}|}$ converges to $\frac{\partial \phi}{\partial y}(x_i, y_j)$.

Let's summarize what we have so far. We have defined $f(x,y)$ so that

$$\sum_i \sum_j \omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j}) = \sum_i \sum_j f(x_i, y_j)\ dx \wedge dy(V^1_{i,j}, V^2_{i,j})$$

$$= \sum_i \sum_j \omega_{\phi(x_i,y_j)}\Big(\frac{\mathcal{V}^1_{i,j}}{|V^1_{i,j}|}, \frac{\mathcal{V}^2_{i,j}}{|V^2_{i,j}|}\Big)\ dx \wedge dy(V^1_{i,j}, V^2_{i,j})$$

We have also shown that when we take the limit as the distance between adjacent partition point tends toward 0 this sum converges to the sum

$$\sum_i \sum_j \omega_{\phi(x,y)}\Big(\frac{\partial \phi}{\partial x}(x,y), \frac{\partial \phi}{\partial y}(x,y)\Big) dx \wedge dy(V^1_{i,j}, V^2_{i,j})$$

Hence, it must be that

$$\boxed{(1) \qquad \int\limits_M \omega = \int\limits_R \omega_{\phi(x,y)}\Big(\frac{\partial \phi}{\partial x}(x,y), \frac{\partial \phi}{\partial y}(x,y)\Big) dx \wedge dy}$$

At first glance, this seems like a very complicated formula. Let's break it down by examining the integrand on the right. The most important thing to notice is that this is just a differential 2-form on R, even though ω is a 2-form on $\mathbb{R}^3$. For each pair of numbers, (x,y), the function $\omega_{\phi(x,y)}\Big(\frac{\partial \phi}{\partial x}(x,y), \frac{\partial \phi}{\partial y}(x,y)\Big)$ just returns some real number. Hence, the entire integrand is of the form $g\ dx \wedge dy$, where $g : R \to \mathbb{R}$.

The only way to *really* convince oneself of the usefulness of this formula is to actually use it.

Example 3.3. Let M denote the top half of the unit sphere in $\mathbb{R}^3$. Let $\omega = z^2 dx \wedge dy$ be a differential 2-form on $\mathbb{R}^3$. Calculating $\int\limits_M \omega$ directly by setting up a Riemann sum would be next to impossible. So we employ the parameterization $\phi(r,t) = (r\cos t, r\sin t, \sqrt{1-r^2})$, where $0 \le t \le 2\pi$ and $0 \le r \le 1$.

$$\begin{aligned}
\int\limits_M \omega &= \int\limits_R \omega_{\phi(r,t)}\Big(\frac{\partial \phi}{\partial r}(r,t), \frac{\partial \phi}{\partial t}(r,t)\Big) dr \wedge dt \\
&= \int\limits_R \omega_{\phi(r,t)}\Big(\langle \cos t, \sin t, \frac{-r}{\sqrt{1-r^2}}\rangle, \langle -r\sin t, r\cos t, 0\rangle\Big) dr \wedge dt \\
&= \int\limits_R (1-r^2) \begin{vmatrix} \cos t & \sin t \\ -r\sin t & r\cos t \end{vmatrix} dr \wedge dt \\
&= \int\limits_R (1-r^2)(r) dr \wedge dt \\
&= \int\limits_0^{2\pi} \int\limits_0^1 r - r^3 dr\ dt = \frac{\pi}{2}
\end{aligned}$$

Notice that as promised, the term $\omega_{\phi(r,t)}\Big(\frac{\partial \phi}{\partial r}(r,t), \frac{\partial \phi}{\partial t}(r,t)\Big)$ in the second integral above simplified to a function from R to $\mathbb{R}$, $r - r^3$. $\square$

EXERCISE 3.3. Integrate the 2-form

$$\omega = \frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz$$

over the top half of the unit sphere using the following parameterizations from rectangular, cylindrical, and spherical coordinates:

(1) $(x, y) \to (x, y, \sqrt{1 - x^2 - y^2})$, where $\sqrt{x^2 + y^2} \leq 1$.
(2) $(r, \theta) \to (r\cos\theta, r\sin\theta, \sqrt{1-r^2})$, where $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$.
(3) $(\theta, \phi) \to (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{2}$.

Answer: 4π.

EXERCISE 3.4. Let S be the surface in $\mathbb{R}^3$ parameterized by

$$\Psi(\theta, z) = (\cos\theta, \sin\theta, z)$$

where $0 \leq \theta \leq \pi$ and $0 \leq z \leq 1$. Let $\omega = xyz\ dy \wedge dz$. Calculate $\int\limits_S \omega$. *Answer:* $\frac{1}{3}$

EXERCISE 3.5. Let ω be the differential 2-form on $\mathbb{R}^3$ given by

$$\omega = xyz\ dx \wedge dy + x^2 z\ dy \wedge dz - y\ dx \wedge dz$$

(1) Let P be the portion of the plane $3 = 2x + 3y - z$ in $\mathbb{R}^3$ which lies above the square $\{(x, y)|0 \le x \le 1, 0 \le y \le 1\}$. Calculate $\int\limits_P \omega$. *Answer:* $-\frac{17}{12}$.

(2) Let M be the portion of the graph of $z = x^2 + y^2$ in $\mathbb{R}^3$ which lies above the rectangle $\{(x, y)|0 \le x \le 1, 0 \le y \le 2\}$. Calculate $\int\limits_M \omega$. *Answer:* $-\frac{29}{6}$.

EXERCISE 3.6. Let $\omega = f(x, y, z)\ dx \wedge dy$ be a differential 2-form on $\mathbb{R}^3$. Let D be some region in the xy-plane. Let M denote the portion of the graph of $z = g(x, y)$ that lies above D. Show that

$$\int\limits_M \omega = \int\limits_D f(x, y, g(x, y))\ dx\ dy$$

EXERCISE 3.7. Let S be the surface obtained from the graph of $z = f(x) = x^3$, where $0 \le x \le 1$, by rotating around the z-axis. Integrate the 2-form $\omega = y\ dx \wedge dz$ over S. (Hint: use cylindrical coordinates to parameterize S.) *Answer:* $\frac{3\pi}{5}$.

3. Orientations

What would have happened in Example 3.3 if we had used the parameterization $\phi'(r, t) = (-r \cos t, r \sin t, \sqrt{1 - r^2})$ instead? We leave it to the reader to check that we end up with the answer $-\pi/2$ rather than $\pi/2$. This is a problem. We defined $\int\limits_M \omega$ before we started talking about parameterizations. Hence, the value which we calculate for this integral should not depend on our choice of parameterization. So what happened?

To analyze this completely, we need to go back to the definition of $\int\limits_M \omega$ from the previous section. We noted at the time that a choice was made to calculate $\omega_{p_{i,j}}(V^1_{i,j}, V^2_{i,j})$ instead of $\omega_{p_{i,j}}(-V^1_{i,j}, V^2_{i,j})$. But was this choice correct? The answer is a resounding *maybe!* We are actually missing enough information to tell. An *orientation* is precisely some piece of information about M which we can use to make the right choice. This way we can tell a friend what M is, what ω is, and what the orientation on M is, and they are sure to get the same answer. Recall Equation 1:

$$\int\limits_M \omega = \int\limits_R \omega_{\phi(x,y)}\left(\frac{\partial \phi}{\partial x}(x, y), \frac{\partial \phi}{\partial y}(x, y)\right) dx \wedge dy$$

Depending on the specified orientation of M, it may be incorrect to use Equation 1. Sometimes we may want to use:

$$\int\limits_M \omega = \int\limits_R \omega_{\phi(x,y)} \left(-\frac{\partial \phi}{\partial x}(x,y), \frac{\partial \phi}{\partial y}(x,y) \right) dx \wedge dy$$

Both ω and $\int$ are linear. This just means the negative sign in the integrand on the right can come all the way outside. Hence, we can write both this equation and Equation 1 as

$$\int\limits_M \omega = \pm \int\limits_R \omega_{\phi(x,y)} \Big(\frac{\partial \phi}{\partial x}(x,y), \frac{\partial \phi}{\partial y}(x,y) \Big) dx \wedge dy \tag{2}$$

We now define:

Definition. *An orientation of M is any piece of information that can be used to decide, for each choice of parameterization ϕ, whether to use the "+" or "−" sign in Equation 2, so that the integral will always yield the same answer.*

We will see several ways to specify an orientation on M. The first way is to simply pick a point p of M and choose any 2-form ν on $T_p\mathbb{R}^3$ such that $\nu(V_p^1, V_p^2) \neq 0$ whenever V_p^1 and V_p^2 are vectors tangent to M, and V_1 is not a multiple of V_2. Don't confuse this 2-form with the *differential* 2-form, ω, of Equation 2. The 2-form ν is only defined at the single tangent space $T_p\mathbb{R}^3$, whereas ω is defined everywhere.

Let's see now how we can use ν to decide whether to use the "+" or "−" sign in Equation 2. All we must do is calculate $\nu\left(\frac{\partial \phi}{\partial x}(x_p, y_p), \frac{\partial \phi}{\partial y}(x_p, y_p)\right)$, where $\phi(x_p, y_p) = p$. If the result is positive then we will use the "+" sign to calculate the integral in Equation 2. If it's negative then we use the "−" sign. Let's see how this works with an example.

Example 3.4. Let's revisit Example 3.3. The problem was to integrate the form $z^2 dx \wedge dy$ over M, the top half of the unit sphere. But no orientation was ever given for M, so the problem wasn't very well stated. Let's pick an easy point, p, on M: $(0, \sqrt{2}/2, \sqrt{2}/2)$. The vectors $\langle 1, 0, 0\rangle_p$ and $\langle 0, 1, -1\rangle_p$ in $T_p\mathbb{R}^3$ are both tangent to M. To give an orientation on M all we have to do is specify a

2-form ν on $T_p\mathbb{R}^3$ such that $\nu(\langle 1,0,0\rangle, \langle 0,1,-1\rangle) \neq 0$. Let's pick an easy one: $\nu = dx \wedge dy$.

Now, let's see what happens when we try to evaluate the integral by using the parameterization $\phi'(r,t) = (-r\cos t, r\sin t, \sqrt{1-r^2})$. First note that $\phi'(\sqrt{2}/2, \pi/2) = (0, \sqrt{2}/2, \sqrt{2}/2)$ and

$$\left(\frac{\partial \phi'}{\partial r}(\frac{\sqrt{2}}{2}, \frac{\pi}{2}), \frac{\partial \phi'}{\partial t}(\frac{\sqrt{2}}{2}, \frac{\pi}{2})\right) = (\langle 0,1,-1\rangle, \langle \frac{\sqrt{2}}{2}, 0, 0\rangle)$$

Now we check the value of ν when this pair is plugged in:

$$dx \wedge dy(\langle 0,1,-1\rangle, \langle \frac{\sqrt{2}}{2}, 0, 0\rangle) = \begin{vmatrix} 0 & 1 \\ \frac{\sqrt{2}}{2} & 0 \end{vmatrix} = -\frac{\sqrt{2}}{2}$$

The sign of this result is "$-$" so we need to use the negative sign in Equation 2 in order to use ϕ' to evaluate the integral of ω over M.

$$\begin{aligned} \int\limits_M \omega &= -\int\limits_R \omega_{\phi'(r,t)}\left(\frac{\partial \phi'}{\partial r}(r,t), \frac{\partial \phi'}{\partial t}(r,t)\right) dr \wedge dt \\ &= -\int\limits_R (1-r^2) \begin{vmatrix} -\cos t & \sin t \\ r\sin t & r\cos t \end{vmatrix} dr\ dt = \frac{\pi}{2} \end{aligned}$$

$\square$

Very often the surface that we are going to integrate over is given to us by a parameterization. In this case there is a very natural choice of orientation. Just use the "+" sign in Equation 2! We will call this the orientation of M *induced* by the parameterization. In other words, if you see a problem phrased like this, "*Calculate the integral of the form ω over the manifold M given by parameterization ϕ with the induced orientation,*" then you should just go back to using Equation 1 and don't worry about anything else.

EXERCISE 3.8. Let M be the image of the parameterization, $\phi(a,b) = (a, a+b, ab)$, where $0 \leq a \leq 1$, and $0 \leq b \leq 1$. Integrate the form $\omega = 2z\ dx \wedge dz + y\ dy \wedge dz - x\ dx \wedge dz$ over M using the orientation induced by ϕ. *Answer:* $-5/6$

There is one subtle technical point here that should be addressed. The novice reader may want to skip this for now. Suppose someone gives you a surface defined by a parameterization and tells you to integrate some 2-form over it, using the induced orientation. But you are clever, and you realize that if you change parameterizations you can make the integral easier. Which orientation do you use? The problem is that the orientation induced by your new parameterization may not be the same as the one induced by the original parameterization.

To fix this we need to see how we can define a 2-form on some tangent space $T_p\mathbb{R}^3$, where p is a point of M, that yields an orientation of M that is consistent with the one induced by a parameterization ϕ. This is not so hard. If $dx \wedge dy\left(\frac{\partial \phi}{\partial x}(x_p, y_p), \frac{\partial \phi}{\partial y}(x_p, y_p)\right)$ is positive then we simply let $\nu = dx \wedge dy$. If it is negative then we let $\nu = -dx \wedge dy$. In the unlikely event that $dx \wedge dy\left(\frac{\partial \phi}{\partial x}(x_p, y_p), \frac{\partial \phi}{\partial y}(x_p, y_p)\right) = 0$ we can remedy things by either changing the point p or by using $dx \wedge dz$ instead of $dx \wedge dy$. Once we have defined ν we know how to integrate M using any other parameterization.

4. Integrating n-forms on $\mathbb{R}^m$

In the previous sections we saw how to integrate a 2-form over a region in $\mathbb{R}^2$, or over a subset of $\mathbb{R}^3$ parameterized by a region in $\mathbb{R}^2$. The reason that these dimensions were chosen was because there is nothing new that needs to be introduced to move to the general case. In fact, if the reader were to go back and look at what we did he/she would find that almost nothing would change if we had been talking about n-forms instead.

Before we jump to the general case, we will work one example showing how to integrate a 1-form over a parameterized curve.

Example 3.5. Let C be the curve in $\mathbb{R}^2$ parameterized by

$$\phi(t) = (t^2, t^3)$$

where $0 \leq t \leq 2$. Let ν be the 1-form $y\ dx + x\ dy$. We calculate $\int\limits_C \nu$.

The first step is to calculate

$$\frac{d\phi}{dt} = \langle 2t, 3t^2 \rangle$$

So, $dx = 2t$ and $dy = 3t^2$. From the parameterization we also know $x = t^2$ and $y = t^3$. Hence, since $\nu = y\ dx + x\ dy$, we have

$$\nu_{\phi(t)}\left(\frac{d\phi}{dt}\right) = (t^3)(2t) + (t^2)(3t^2) = 5t^4$$

Finally, we integrate:

$$\begin{aligned}\int_C \nu &= \int_0^2 \nu_{\phi(t)}\left(\frac{d\phi}{dt}\right) dt \\ &= \int_0^2 5t^4 \ dt \\ &= t^5\Big|_0^2 \\ &= 32\end{aligned}$$

$\square$

EXERCISE 3.9. Let C be the curve in $\mathbb{R}^3$ parameterized by $\phi(t) = (t, t^2, 1+t)$, where $0 \le t \le 2$. Integrate the 1-form $\omega = y \ dx + z \ dy + xy \ dz$ over C using the induced orientation. *Answer:* 16.

EXERCISE 3.10. Let M be the line segment in $\mathbb{R}^2$ which connects $(0,0)$ to $(4,6)$. An orientation on M is specified by the 1-form $-dx$ on $T_{(2,3)}\mathbb{R}^2$. Integrate the form $\omega = \sin y \ dx + \cos x \ dy$ over M. *Answer:* $\frac{2}{3}\cos 6 - \frac{3}{2}\sin 4 - \frac{2}{3}$

To proceed to the general case, we need to know what the integral of an n-form over a region of $\mathbb{R}^n$ is. The steps to define such an object are precisely the same as before, and the results are similar. If our coordinates in $\mathbb{R}^n$ are $(x_1, x_2, ..., x_n)$ then an n-form is always given by $f(x_1, ..., x_n)dx_1 \wedge dx_2 \wedge ... \wedge dx_n$. Going through the steps we find that the definition of $\int\limits_{\mathbb{R}^n} \omega$ is exactly the same as the definition we learned in calculus for $\int\limits_{\mathbb{R}^n} f \ dx_1 dx_2 ... dx_n$.

EXERCISE 3.11. Let Ω be the cube in $\mathbb{R}^3$

$$\{(x,y,z) | \ 0 \le x, y, z \le 1\}$$

Let γ be the 3-form $x^2 z \ dx \wedge dy \wedge dz$. Calculate $\int\limits_\Omega \gamma$. *Answer:* $\frac{1}{6}$

Moving on to integrals of n-forms over subsets of $\mathbb{R}^m$ parameterized by a region in $\mathbb{R}^n$ we again find nothing surprising. Suppose we denote such a subset as M. Let $\phi : R \subset \mathbb{R}^n \to M \subset \mathbb{R}^m$ be a parameterization. Then we find that the following generalization of Equation 2 must hold:

$$\int\limits_M \omega = \pm \int\limits_R \omega_{\phi(x_1,\dots,x_n)} \left(\frac{\partial \phi}{\partial x_1}(x_1, \dots x_n), \dots, \frac{\partial \phi}{\partial x_n}(x_1, \dots x_n) \right) dx_1 \wedge \dots \wedge dx_n \tag{3}$$

To decide whether or not to use the negative sign in this equation we must specify an orientation. Again, one way to do this is to give an n-form ν on $T_p\mathbb{R}^m$, where p is some point of M. We use the negative sign when the value of

$$\nu\Big(\frac{\partial \phi}{\partial x_1}(x_1, \dots x_n), \dots, \frac{\partial \phi}{\partial x_n}(x_1, \dots x_n) \Big)$$

is negative, where $\phi(x_1, \dots x_n) = p$. If M was originally given by a parameterization and we are instructed to use the induced orientation then we can ignore the negative sign.

Example 3.6. Suppose $\phi(a, b, c) = (a + b, a + c, bc, a^2)$, where $0 \le a, b, c \le 1$. Let M be the image of ϕ with the induced orientation. Suppose $\omega = dy \wedge dz \wedge dw - dx \wedge dz \wedge dw - 2y\ dx \wedge dy \wedge dz$. Then,

$$\begin{aligned}
\int\limits_M \omega &= \int\limits_R \omega_{\phi(a,b,c)} \left(\frac{\partial \phi}{\partial a}(a, b, c), \frac{\partial \phi}{\partial b}(a, b, c), \frac{\partial \phi}{\partial c}(a, b, c) \right) da \wedge db \wedge dc \\
&= \int\limits_R \omega_{\phi(a,b,c)} \left(\langle 1, 1, 0, 2a \rangle, \langle 1, 0, c, 0 \rangle, \langle 0, 1, b, 0 \rangle \right) da \wedge db \wedge dc \\
&= \int\limits_R \begin{vmatrix} 1 & 0 & 2a \\ 0 & c & 0 \\ 1 & b & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 2a \\ 1 & c & 0 \\ 0 & b & 0 \end{vmatrix} - 2(a + c) \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & c \\ 0 & 1 & b \end{vmatrix} da \wedge db \wedge dc \\
&= \int_0^1 \int_0^1 \int_0^1 2bc + 2c^2\ da\ db\ dc = \frac{7}{6}
\end{aligned}$$

$\square$

5. Integrating n-forms on parameterized subsets of $\mathbb{R}^n$

There is a special case of Equation 3 which is worth noting. Suppose ϕ is a parameterization that takes some subregion, R, of $\mathbb{R}^n$ into some other subregion, M, of $\mathbb{R}^n$ and ω is an n-form on $\mathbb{R}^n$. At each point ω is just a volume form, so it can be written as $f(x_1, ..., x_n)\ dx_1 \wedge ... \wedge dx_n$. If we let $\bar{x} = (x_1, ...x_n)$ then Equation 3 can be written as:

$$\int\limits_M f(\bar{x})dx_1...dx_n = \pm \int\limits_R f(\phi(\bar{x})) \left| \frac{\partial \phi}{\partial x_1}(\bar{x}), ..., \frac{\partial \phi}{\partial x_n}(\bar{x}) \right| dx_1...dx_n$$

When $n = 1$ this is just the substitution rule for integration from Calculus. For other n this is the general change of variables formula.

Example 3.7. We will use the parameterization $\Psi(u, v) = (u, u^2 + v^2)$ to evaluate

$$\int\int\limits_R (x^2 + y)\ dA$$

where R is the region of the xy-plane bounded by the parabolas $y = x^2$ and $y = x^2 + 4$, and the lines $x = 0$ and $x = 1$.

The first step is to find out what the limits of integration will be when we change coordinates.

$$y = x^2 \Rightarrow u^2 + v^2 = u^2 \Rightarrow v = 0$$
$$y = x^2 + 4 \Rightarrow u^2 + v^2 = u^2 + 4 \Rightarrow v = 2$$
$$x = 0 \Rightarrow u = 0$$
$$x = 1 \Rightarrow u = 1$$

Next, we will need the partial derivatives.

$$\frac{\partial \Psi}{\partial u} =< 1, 2u >$$
$$\frac{\partial \Psi}{\partial v} =< 0, 2v >$$

Finally, we can integrate.

$$\begin{aligned}
\int\int_R (x^2+y)\, dA &= \int_R (x^2+y)\, dx \wedge dy \\
&= \int_0^2 \int_0^1 u^2 + (u^2+v^2) \begin{vmatrix} 1 & 2u \\ 0 & 2v \end{vmatrix} du\, dv \\
&= \int_0^2 \int_0^1 4vu^2 + 2v^3\, du\, dv \\
&= \int_0^2 \frac{4}{3}v + 2v^3\, dv \\
&= \frac{8}{3} + 8 = \frac{32}{3}
\end{aligned}$$

□

EXERCISE 3.12. Let E be the region in $\mathbb{R}^2$ parameterized by $\Psi(u,v) = (u^2+v^2, 2uv)$, where $0 \leq u \leq 1$ and $0 \leq v \leq 1$. Evaluate

$$\iint_E \frac{1}{\sqrt{x-y}}\, dA$$

Answer: 4

EXERCISE 3.13. Let R be the region of the xy-plane bounded by the ellipse $9x^2 + 4y^2 = 36$. Integrate the 2-form $\omega = x^2 dx \wedge dy$ over R. (Hint: Use the parameterization $\phi(u,v) = (2u, 3v)$.) *Answer:* 6π.

EXAMPLE 3.8. Often in multivariable calculus classes we integrate functions $f(x,y)$ over regions R bounded by the graphs of equations $y = g_1(x)$ and $y = g_2(x)$, and by the lines $x = a$ and $x = b$, where $g_1(x) < g_2(x)$ for all $x \in [a,b]$. We show here that such problems can always be translated into integrals over rectangular regions.

The region R described above is parameterized by

$$\Psi(u,v) = (u, (1-v)g_1(u) + vg_2(u))$$

where $a \le u \le b$ and $0 \le v \le 1$. The partials of this parameterization are

$$\frac{\partial \Psi}{\partial u} = \left\langle 1, (1-v)\frac{dg_1(u)}{du} + v\frac{dg_2(u)}{du} \right\rangle$$

$$\frac{\partial \Psi}{\partial v} = \langle 0, -g_1(u) + g_2(u) \rangle$$

Hence,

$$dx \wedge dy = \begin{vmatrix} 1 & (1-v)\frac{dg_1(u)}{du} + v\frac{dg_2(u)}{du} \\ 0 & -g_1(u) + g_2(u) \end{vmatrix} = -g_1(u) + g_2(u)$$

We conclude with the identity

$$\begin{aligned} \int_a^b \int_{g_1(u)}^{g_2(u)} f(x,y)\ dy\ dx &= \int_a^b \int_0^1 f(u, (1-v)g_1(u) + vg_2(u))(g_2(u) - g_1(u))\ dv\ du \\ &= \int_a^b \int_0^1 g_2(u) f(u, (1-v)g_1(u) + vg_2(u))\ dv\ du \\ &\quad - \int_a^b \int_0^1 g_1(u) f(u, (1-v)g_1(u) + vg_2(u))\ dv\ du \end{aligned}$$

This may be of more theoretical importance than practical (see Example 5.1). □

EXAMPLE 3.9. Let $V = \{(r, \theta, z) | 1 \le r \le 2,\ 0 \le z \le 1\}$. ($V$ is the region between the cylinders of radii 1 and 2 and between the planes $z = 0$ and $z = 1$.) We will calculate

$$\int_V z(x^2 + y^2)\ dx \wedge dy \wedge dz$$

The region V is best parameterized using cylindrical coordinates:

$$\Psi(r, \theta, z) = (r\cos\theta, r\sin\theta, z),$$

where $1 \le r \le 2$, $1 \le \theta \le 2\pi$, and $0 \le z \le 1$.
Computing the partials:

$$\begin{aligned}
\frac{\partial \Psi}{\partial r} &= \langle \cos\theta, \sin\theta, 0\rangle \\
\frac{\partial \Psi}{\partial \theta} &= \langle -r\sin\theta, r\cos\theta, 0\rangle \\
\frac{\partial \Psi}{\partial z} &= \langle 0, 0, 1\rangle
\end{aligned}$$

Hence,

$$dx \wedge dy \wedge dz = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Also,

$$z(x^2+y^2) = z(r^2\cos^2\theta + r^2\sin^2\theta) = zr^2$$

So we have

$$\begin{aligned}
\int\limits_V z(x^2+y^2)\ dx \wedge dy \wedge dz &= \int\limits_0^1 \int\limits_0^{2\pi} \int\limits_1^2 (zr^2)(r)\ dr\ d\theta\ dz \\
&= \int\limits_0^1 \int\limits_0^{2\pi} \int\limits_1^2 zr^3\ dr\ d\theta\ dz \\
&= \frac{15}{4} \int\limits_0^1 \int\limits_0^{2\pi} z\ d\theta\ dz \\
&= \frac{15\pi}{2} \int\limits_0^1 z\ dz \\
&= \frac{15\pi}{4}
\end{aligned}$$

□

EXERCISE 3.14. Integrate the 3-form $\omega = x\ dx \wedge dy \wedge dz$ over the region of $\mathbb{R}^3$ in the first octant bounded by the cylinders $x^2+y^2=1$ and $x^2+y^2=4$, and the plane $z=2$. *Answer:* $\frac{14}{3}$

EXERCISE 3.15. Let R be the region in the first octant of $\mathbb{R}^3$ bounded by the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Integrate the 3-form $\omega = dx \wedge dy \wedge dz$ over R. *Answer:* $\frac{-7\pi}{6}$

6. Summary: How to Integrate a Differential Form

6.1. The Steps. To compute the integral of a differential n-form, ω, over a region, S, the steps are as follows:

(1) Choose a parameterization, $\Psi : R \to S$, where R is a subset of $\mathbb{R}^n$ (see Figure 5).

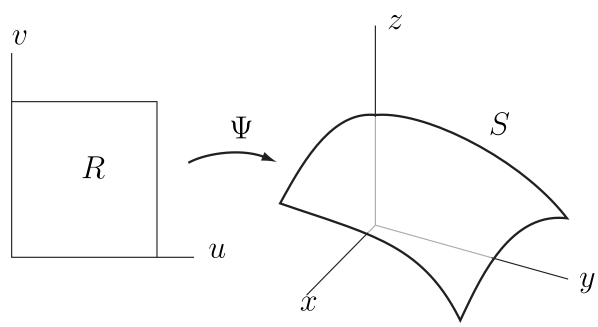


FIGURE 5.

(2) Find all n vectors given by the partial derivatives of Ψ. Geometrically, these are tangent vectors to S which span its tangent space (see Figure 6).

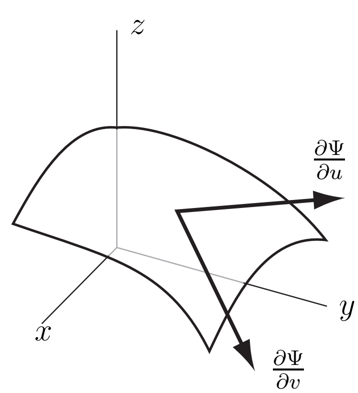


FIGURE 6.

(3) Plug the tangent vectors into ω at the point $\Psi(u_1, u_2, ..., u_n)$.

(4) Integrate the resulting function over R.

6.2. Integrating 2-forms. The best way to see the above steps in action is to look at the integral of a 2-form over a surface in $\mathbb{R}^3$. In general, such a 2-form is given by

$$\omega = f_1(x,y,z)\ dx \wedge dy + f_2(x,y,z)\ dy \wedge dz + f_3(x,y,z)\ dx \wedge dz$$

To integrate ω over S we now follow the steps:

(1) Choose a parameterization, $\Psi : R \to S$, where R is a subset of $\mathbb{R}^2$.

$$\Psi(u,v) = (g_1(u,v), g_2(u,v), g_3(u,v))$$

(2) Find both vectors given by the partial derivatives of Ψ.

$$\begin{aligned} \frac{\partial \Psi}{\partial u} &= \left\langle \frac{\partial g_1}{\partial u}, \frac{\partial g_2}{\partial u}, \frac{\partial g_3}{\partial u} \right\rangle \\ \frac{\partial \Psi}{\partial v} &= \left\langle \frac{\partial g_1}{\partial v}, \frac{\partial g_2}{\partial v}, \frac{\partial g_3}{\partial v} \right\rangle \end{aligned}$$

(3) Plug the tangent vectors into ω at the point $\Psi(u,v)$.

To do this, x, y, and z will come from the coordinates of Ψ. That is, $x = g_1(u,v)$, $y = g_2(u,v)$, and $z = g_3(u,v)$. Terms like $dx \wedge dy$ will be determinants of 2×2 matrices, whose entries come from the vectors computed in the previous step. Geometrically, the value of $dx \wedge dy$ will be the area of the parallelogram spanned by the vectors $\frac{\partial \Psi}{\partial u}$ and $\frac{\partial \Psi}{\partial v}$ (tangent vectors to S), projected onto the dx-dy plane (see Figure 7).

The result of all this will be:

$$f_1(g_1,g_2,g_3)\begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_2}{\partial u} \\ \frac{\partial g_1}{\partial u} & \frac{\partial g_2}{\partial v} \end{vmatrix} + f_2(g_1,g_2,g_3)\begin{vmatrix} \frac{\partial g_2}{\partial u} & \frac{\partial g_3}{\partial u} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_3}{\partial v} \end{vmatrix}$$
$$+ f_3(g_1,g_2,g_3)\begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_3}{\partial u} \\ \frac{\partial g_1}{\partial u} & \frac{\partial g_3}{\partial v} \end{vmatrix}$$

Note that when you simplify this you get a function of u and v.

(4) Integrate the resulting function over R. In other words, if $h(u,v)$ is the function you ended up with in the previous step, then compute

$$\int\int_R h(u,v)\ du\ dv$$

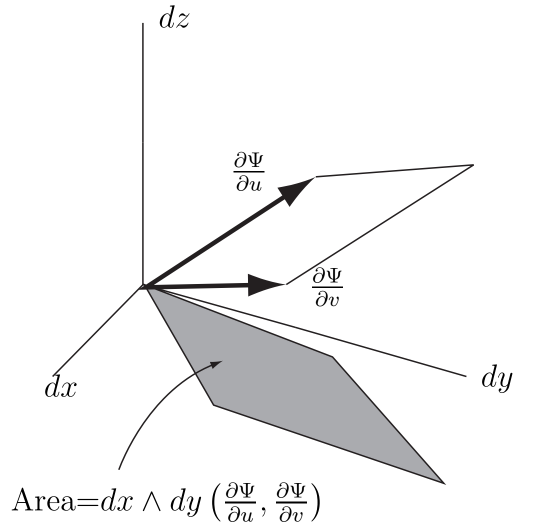


FIGURE 7. Evaluating $dx \wedge dy$ geometrically

In practice, the limits of integration will come from the shape of R, determined in Step 1. They will all be constants only if R was a rectangle.

6.3. A sample 2-form. Let $\omega = (x^2 + y^2)\ dx \wedge dy + z\ dy \wedge dz$. Let S denote the subset of the cylinder $x^2 + y^2 = 1$ that lies between the planes $z = 0$ and $z = 1$.

(1) Choose a parameterization, $\Psi : R \to S$.

$$\Psi(u, v) = (\cos\theta, \sin\theta, z)$$

Where $R = \{(\theta, z) | 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1\}$.

(2) Find both vectors given by the partial derivatives of Ψ.

$$\begin{aligned} \frac{\partial \Psi}{\partial \theta} &= \langle -\sin\theta, \cos\theta, 0\rangle \\ \frac{\partial \Psi}{\partial z} &= \langle 0, 0, 1\rangle \end{aligned}$$

(3) Plug the tangent vectors into ω at the point $\Psi(\theta, z)$. We get

$$(\cos^2\theta + \sin^2\theta)\begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} + z\begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix}$$

This simplifies to the function $z\cos\theta$.

(4) Integrate the resulting function over R.

$$\int_0^1 \int_0^{2\pi} z \cos\theta \ d\theta \ dz$$

Note that the integrand comes from Step 3 and the limits of integration come from Step 1.

CHAPTER 4

Differentiation of Forms.

1. The derivative of a differential 1-form

The goal of this section is to figure out what we mean by the derivative of a differential form. One way to think about a derivative is as a function which measures the variation of some other function. Suppose ω is a 1-form on $\mathbb{R}^2$. What do we mean by the "variation" of ω? One thing we can try is to plug in V_p, a particular vector in the tangent space at the point, p. We can then look at how $\omega(V_p)$ changes as we vary p. But p can vary in lots of ways, so we need to pick one. In calculus we learn how to take another vector, W_p, and use it to vary p. Hence, the derivative of ω, which we shall denote "$d\omega$", is a function that acts on both V_p and W_p. In other words, it must be a 2-form!

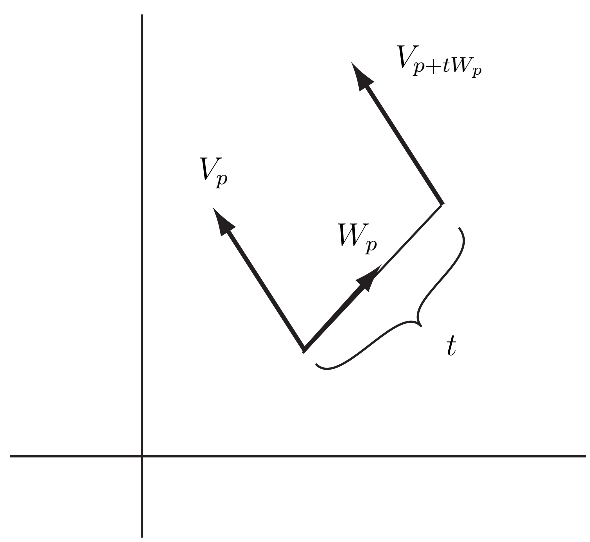


FIGURE 1. Using W_p to vary V_p.

Let's see how to use W_p to calculate the variation in $\omega(V_p)$ in a specific example. Suppose $\omega = y\ dx - x^2 dy$, $p = (1,1)$, $V_p = \langle 1, 2\rangle_{(1,1)}$, and $W_p = \langle 2, 3\rangle_{(1,1)}$. Notice that

$V_{p+tW} = \langle 1,2\rangle_{(1+2t,1+3t)}$ is a vector similar to V_p, but pushed away by t in the direction of W_p. Hence, the variation of $\omega(V_p)$, in the direction of W_p, can be calculated as follows:

$$\begin{aligned}
&\lim_{t\to 0}\frac{\omega(\langle 1,2\rangle_{(1+2t,1+3t)})-\omega(\langle 1,2\rangle_{(1,1)})}{t}\\
&= \lim_{t\to 0}\frac{[(1+3t)(1)-(1+2t)^2(2)]-[(1)(1)-(1)^2(2)]}{t}\\
&= \lim_{t\to 0}\frac{-5t-8t^2}{t} = -5
\end{aligned}$$

What about the variation of $\omega(W_p)$ in the direction of V_p? We calculate:

$$\begin{aligned}
&\lim_{t\to 0}\frac{\omega(\langle 2,3\rangle_{(1+t,1+2t)})-\omega(\langle 2,3\rangle_{(1,1)})}{t}\\
&= \lim_{t\to 0}\frac{[(1+2t)(2)-(1+t)^2(3)]-[(1)(2)-(1)^2(3)]}{t}\\
&= \lim_{t\to 0}\frac{-2t-3t^2}{t} = -2
\end{aligned}$$

This is a small problem. We want $d\omega$ to be a 2-form. Hence, $d\omega(V_p, W_p)$ should equal $-d\omega(W_p, V_p)$. How can we use the variations above to define $d\omega$ so this is true? Simple. We just define it to be the difference in these variations:

$$d\omega(V_p, W_p) = \lim_{t\to 0}\frac{\omega(W_{p+tV})-\omega(W_p)}{t} - \lim_{t\to 0}\frac{\omega(V_{p+tW})-\omega(V_p)}{t} \tag{4}$$

Hence, in the above example, $d\omega(\langle 1,2\rangle_{(1,1)}, \langle 2,3\rangle_{(1,1)}) = -2-(-5) = 3$.

Before going further we introduce some notation from calculus to make Equation 4 a little more readable. Suppose f is a function from $\mathbb{R}^n$ to $\mathbb{R}$ and $V \in T_p\mathbb{R}^n$. Then the derivative of f at p, in the direction of V, can be written as "$\nabla_V f$". That is, we define

$$\nabla_V f = \lim_{t\to 0}\frac{f(p+tV)-f(p)}{t}$$

One may also recall from calculus that $\nabla_V f = \nabla f(p)\cdot V$, where $\nabla f(p)$ denotes the gradient of f evaluated at p. Using this notation, we can rewrite Equation 4 as

$$d\omega(V_p, W_p) = \nabla_{V_p}\omega(W_p) - \nabla_{W_p}\omega(V_p)$$

There are other ways to determine what $d\omega$ is than by using Equation 4. Recall that a 2-form acts on a pair of vectors by projecting them onto each coordinate plane, calculating the area they span, multiplying by some constant, and adding. So the 2-form is completely determined by the constants that you multiply by after projecting. In order to figure out what these constants are we are free to examine the action of the 2-form on any pair of vectors. For example, suppose we have two vectors that lie in the x-y plane and span a parallelogram with area 1. If we run these through some 2-form and end up with the number 5 then we know that the multiplicative constant for that 2-form, associated with the x-y plane is 5. This, in turn, tells us that the 2-form equals 5 $dx \wedge dy+?$. To figure out what "?" is, we can examine the action of the 2-form on other pairs of vectors.

Let's try this with a general differential 2-form on $\mathbb{R}^3$. Such a form always looks like $d\omega = a(x,y,z)dx \wedge dy + b(x,y,z)dy \wedge dz + c(x,y,z)dx \wedge dz$. To figure out what $a(x,y,z)$ is, for example, all we need to do is determine what $d\omega$ does to the vectors $\langle 1,0,0\rangle_{(x,y,z)}$ and $\langle 0,1,0\rangle_{(x,y,z)}$. Let's compute this using Equation 4, assuming $\omega = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz$.

$$
\begin{aligned}
& d\omega(\langle 1,0,0\rangle_{(x,y,z)}, \langle 0,1,0\rangle_{(x,y,z)}) \\
= \;& \lim_{t\to 0} \frac{\omega(\langle 0,1,0\rangle_{(x+t,y,z)}) - \omega(\langle 0,1,0\rangle_{(x,y,z)})}{t} \\
& -\lim_{t\to 0} \frac{\omega(\langle 1,0,0\rangle_{(x,y+t,z)}) - \omega(\langle 1,0,0\rangle_{(x,y,z)})}{t} \\
= \;& \lim_{t\to 0} \frac{g(x+t,y,z)}{t} - \lim_{t\to 0} \frac{f(x,y+t,z)}{t} \\
= \;& \frac{\partial g}{\partial x}(x,y,z) - \frac{\partial f}{\partial y}(x,y,z)
\end{aligned}
$$

Similarly, direct computation shows:

$$d\omega(\langle 0,1,0\rangle_{(x,y,z)}, \langle 0,0,1\rangle_{(x,y,z)}) = \frac{\partial h}{\partial y}(x,y,z) - \frac{\partial g}{\partial z}(x,y,z)$$

and,

$$d\omega(\langle 1,0,0\rangle_{(x,y,z)}, \langle 0,0,1\rangle_{(x,y,z)}) = \frac{\partial h}{\partial x}(x,y,z) - \frac{\partial f}{\partial z}(x,y,z)$$

Hence, we conclude that

$$d\omega = (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})dx \wedge dy + (\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z})dy \wedge dz + (\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z})dx \wedge dz$$

EXERCISE 4.1. Suppose $\omega = f(x,y)dx + g(x,y)dy$ is a 1-form on $\mathbb{R}^2$. Show that $d\omega = (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})dx \wedge dy$.

EXERCISE 4.2. Suppose $\omega = xy^2\ dx + x^3 z\ dy - (y + z^9)\ dz$, $V = \langle 1, 2, 3\rangle_{(2,3,-1)}$, and $W = \langle -1, 0, 1\rangle_{(2,3,-1)}$

(1) Compute $\nabla_V \omega(W)$ and $\nabla_W \omega(V)$, at the point $(2, 3, -1)$.
(2) Use your answer to the previous question to compute $d\omega(V, W)$.

EXERCISE 4.3. If $\omega = y\ dx - x^2\ dy$, find $d\omega$. Verify that $d\omega(\langle 1, 2\rangle_{(1,1)}, \langle 2, 3\rangle_{(1,1)}) = 3$.

2. Derivatives of n-forms

Before jumping to the general case let's look at the derivative of a 2-form. A 2-form, ω, acts on a pair of vectors, V_p and W_p, to return some number. To find some sort of variation of ω we can vary the vectors V_p and W_p and examine how $\omega(V_p, W_p)$ varies. As in the last section one way to vary a vector is to push it in the direction of some other vector, U_p. Hence, whatever $d\omega$ turns out to be, it will be a function of the vectors U_p, V_p, and W_p. So, we would like to define it to be a 3-form.

Let's start by looking at the variation of $\omega(V_p, W_p)$ in the direction of U_p. We write this as $\nabla_{U_p}\omega(V_p, W_p)$. If we were to define this as the value of $d\omega(U_p, V_p, W_p)$ we would find that in general it would not be alternating. That is, usually $\nabla_{U_p}\omega(V_p, W_p) \neq -\nabla_{V_p}\omega(U_p, W_p)$. To remedy this, we simply define $d\omega$ to be the alternating sum of *all* the variations:

$$d\omega(U_p, V_p, W_p) = \nabla_{U_p}\omega(V_p, W_p) - \nabla_{V_p}\omega(U_p, W_p) + \nabla_{W_p}\omega(U_p, V_p)$$

We leave it to the reader to check that $d\omega$ is alternating and multilinear.

It shouldn't be hard for the reader to now jump to the general case. Suppose ω is an n-form and $V_p^1, ..., V_p^{n+1}$ are $n+1$ vectors. Then we define

$$d\omega(V_p^1, ..., V_p^{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \nabla_{V_p^i}\ \omega(V_p^1, ..., V_p^{i-1}, V_p^{i+1}, ..., V_p^{n+1})$$

In other words, $d\omega$, applied to $n+1$ vectors, is the alternating sum of the variation of ω applied to n of those vectors in the direction of the remaining one. Note that we can think of "d" as an operator which takes n-forms to $(n+1)$-forms.

EXERCISE 4.4. $d\omega$ is alternating and multilinear.

EXERCISE 4.5. Suppose $\omega = f(x, y, z)\ dx \wedge dy + g(x, y, z)\ dy \wedge dz + h(x, y, z)\ dx \wedge dz$. Find $d\omega$ (*Hint*: Compute $d\omega(\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle)$). Compute $d(x^2y\ dx \wedge dy + y^2z\ dy \wedge dz)$.

3. Interlude: 0-forms

Let's go back to Section 1, when we introduced coordinates for vectors. At that time we noted that if C was the graph of the function $y = f(x)$ and p was a point of C then the tangent line to C at p lies in $T_p\mathbb{R}^2$ and has equation $dy = m\ dx$, for some constant, m. Of course, if $p = (x_0, y_0)$ then m is just the derivative of f evaluated at x_0.

Now, suppose we had looked at the graph of a function of 2-variables, $z = f(x, y)$, instead. At some point, $p = (x_0, y_0, z_0)$, on the graph we could look at the tangent plane, which lies in $T_p\mathbb{R}^3$. It's equation is $dz = m_1dx + m_2dy$. Since $z = f(x, y)$, $m_1 = \frac{\partial f}{\partial x}(x_0, y_0)$, and $m_2 = \frac{\partial f}{\partial y}(x_0, y_0)$, we can rewrite this as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Notice that the right-hand side of this equation is a differential 1-form. This is a bit strange; we applied the "d" operator to something and the result was a 1-form. However, we know that when we apply the "d" operator to a differential n-form we get a differential $(n+1)$-form. So, it must be that $f(x, y)$ is a *differential 0-form* on $\mathbb{R}^2$!

In retrospect, this should not be so surprising. After all, the input to a differential n-form on $\mathbb{R}^m$ is a point, and n vectors based at that point. So, the input to a differential 0-form should be a point of $\mathbb{R}^m$, and *no* vectors. In other words, a 0-form on $\mathbb{R}^m$ is just another word for a real-valued function on $\mathbb{R}^m$.

Let's extend some of the things we can do with forms to 0-forms. Suppose f is a 0-form, and ω is an n-form (where n may also be 0). What should we mean by $f \wedge \omega$? Since the wedge product of an n-form and an m-form is an $(n+m)$-form, it must be that $f \wedge \omega$ is an n form. It's hard to think of any other way to define this as just the product, $f\omega$.

What about integration? Remember that we integrate n-forms over subsets of $\mathbb{R}^m$ that can be parameterized by a subset of $\mathbb{R}^n$. So 0-forms get integrated over things parameterized by $\mathbb{R}^0$. In other words, we integrate a 0-form over a *point*. How do we do this? We do the simplest possible thing; define the value of a 0-form, f, integrated over the point, p, to be $\pm f(p)$. To specify an orientation we just need to say whether or not to use the $-$ sign. We do this just by writing "$-p$" instead of "p" when we want the integral of f over p to be $-f(p)$.

One word of caution here...beware of orientations! If $p \in \mathbb{R}^n$ then we use the notation "$-p$" to denote p with the negative orientation. So if $p = -3 \in \mathbb{R}^1$ then $-p$ is not the same as the point, 3. $-p$ is just the point, -3, with a negative orientation. So, if $f(x) = x^2$ then $\int_{-p} f = -f(p) = -9$.

Exercise 4.6. If f is the 0-form x^2y^3, p is the point $(-1, 1)$, q is the point $(1, -1)$, and r is the point $(-1, -1)$, then compute the integral of f over the points $-p$, $-q$, and $-r$, with the indicated orientations.

Let's go back to our exploration of derivatives of n-forms. Suppose $f(x, y)\ dx$ is a 1-form on $\mathbb{R}^2$. Then we have already shown that $d(f\ dx) = \frac{\partial f}{\partial y}\ dy \wedge dx$. We now compute:

$$\begin{aligned} df \wedge dx &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \wedge dx \\ &= \frac{\partial f}{\partial x}dx \wedge dx + \frac{\partial f}{\partial y}dy \wedge dx \\ &= \frac{\partial f}{\partial y}dy \wedge dx \\ &= d(f\ dx) \end{aligned}$$

Exercise 4.7. If ω is an n-form, and f is a 0-form, then $d(f\ d\omega) = df \wedge d\omega$.

Exercise 4.8. $d(d\omega) = 0$.

Exercise 4.9. If ω is an n-form, and μ is an m-form, then $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^n \omega \wedge d\mu$.

4. Algebraic computation of derivatives

In this section we break with the spirit of the text briefly. At this point we have amassed enough algebraic identities that computing derivatives of forms can become quite routine. In this section we quickly summarize these identities and work a few examples.

4.1. Identities involving $\wedge$ only. Let ω be an n-form and ν be an m-form.

$$\begin{aligned}
\omega \wedge \omega &= 0\\
\omega \wedge \nu &= (-1)^{nm}\nu \wedge \omega\\
\omega \wedge (\nu + \psi) &= \omega \wedge \nu + \omega \wedge \psi\\
(\nu + \psi) \wedge \omega &= \nu \wedge \omega + \psi \wedge \omega
\end{aligned}$$

4.2. Identities involving "d". Let ω be an n-form, μ an m-form, and f a 0-form.

$$\begin{aligned}
d(d\omega) &= 0\\
d(\omega + \mu) &= d\omega + d\mu\\
d(\omega \wedge \mu) &= d\omega \wedge \mu + (-1)^n \omega \wedge d\mu\\
d(f\ d\omega) &= df \wedge d\omega\\
df &= \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + ... + \frac{\partial f}{\partial x_n} dx_n
\end{aligned}$$

4.3. Some examples.

EXAMPLE 4.1.

$$d\left(xy\ dx - xy\ dy + xy^2 z^3\ dz\right)$$

$$\begin{aligned}
&= d(xy) \wedge dx - d(xy) \wedge dy + d(xy^2z^3) \wedge dz \\
&= (y\ dx + x\ dy) \wedge dx - (y\ dx + x\ dy) \wedge dy \\
&\quad +(y^2z^3\ dx + 2xyz^3\ dy + 3xy^2z^2\ dz) \wedge dz \\
&= \cancel{y\ dx \wedge dx} + x\ dy \wedge dx - y\ dx \wedge dy - \cancel{x\ dy \wedge dy} \\
&\quad +y^2z^3\ dx \wedge dz + 2xyz^3\ dy \wedge dz + \cancel{3xy^2z^2\ dz \wedge dz} \\
&= x\ dy \wedge dx - y\ dx \wedge dy + y^2z^3\ dx \wedge dz + 2xyz^3\ dy \wedge dz \\
&= -x\ dx \wedge dy - y\ dx \wedge dy + y^2z^3\ dx \wedge dz + 2xyz^3\ dy \wedge dz \\
&= (-x - y)\ dx \wedge dy + y^2z^3\ dx \wedge dz + 2xyz^3\ dy \wedge dz
\end{aligned}$$

□

EXAMPLE 4.2.

$$\begin{aligned}
&d\left(x^2(y+z^2)\ dx \wedge dy + z(x^3+y)\ dy \wedge dz\right) \\
&\qquad = d(x^2(y+z^2)) \wedge dx \wedge dy + d(z(x^3+y)) \wedge dy \wedge dz \\
&\qquad = 2x^2z\ dz \wedge dx \wedge dy + 3x^2z\ dx \wedge dy \wedge dz \\
&\qquad = 5x^2z\ dx \wedge dy \wedge dz
\end{aligned}$$

□

EXERCISE 4.10. For each differential n-form, ω, find $d\omega$.

(1) $\sin y\ dx + \cos x\ dy$
(2) $xy^2\ dx + x^3z\ dy - (y + z^9)\ dz$
(3) $xy^2\ dy \wedge dz + x^3z\ dx \wedge dz - (y + z^9)\ dx \wedge dy$
(4) $x^2y^3z^4\ dx \wedge dy \wedge dz$

EXERCISE 4.11. If f is the 0-form x^2y^3 and ω is the 1-form $x^2z\ dx + y^3z^2\ dy$ (on $\mathbb{R}^3$) then use the identity $d(f\ d\omega) = df \wedge d\omega$ to compute $d(f\ d\omega)$. *Answer:* $(3x^4y^2 - 2xy^6z)dx \wedge dy \wedge dz$.

EXERCISE 4.12. Let f, g, and h be functions from $\mathbb{R}^3$ to $\mathbb{R}$. If $\omega = f\ dy \wedge dz - g\ dx \wedge dz + h\ dx \wedge dy$ then compute $d\omega$.

CHAPTER 5

Stokes' Theorem

1. Cells and Chains

Up until now we have not been very specific as to the types of subsets of $\mathbb{R}^m$ on which one integrates a differential n-form. All we have needed is a subset that can be parameterized by a region in $\mathbb{R}^n$. To go further we need to specify what types of regions.

DEFINITION. *Let $I = [0, 1]$. An n-cell, σ, is the image of differentiable map, $\phi : I^n \to \mathbb{R}^m$, with a specified orientation. We denote the same cell with opposite orientation as $-\sigma$. We define a 0-cell to be a point of $\mathbb{R}^m$.*

EXAMPLE 5.1. Suppose $g_1(x)$ and $g_2(x)$ are functions such that $g_1(x) < g_2(x)$ for all $x \in [a, b]$. Let R denote the subset of $\mathbb{R}^2$ bounded by the graphs of the equations $y = g_1(x)$ and $y = g_2(x)$, and by the lines $x = a$ and $x = b$. In Example 3.8 we show that R is a 2-cell (assuming the induced orientation). $\square$

We would like to treat cells as algebraic objects which can be added and subtracted. But if σ is a cell it may not at all be clear what "2σ" represents. One way to think about it is as two copies of σ, placed right on top of each other.

DEFINITION. *An n-chain is a formal linear combination of n-cells.*

As one would expect, we assume the following relations hold:

$$\sigma - \sigma = \emptyset$$

$$n\sigma + m\sigma = (n + m)\sigma$$

$$\sigma + \tau = \tau + \sigma$$

You can probably guess what the integral of an n-form, ω, over an n-chain is. Suppose $C = \sum n_i \sigma_i$. Then we define

$$\int\limits_C \omega = \sum_i n_i \int\limits_{\sigma_i} \omega$$

EXERCISE 5.1. If f is the 0-form x^2y^3, p is the point $(-1,1)$, q is the point $(1,-1)$, and r is the point $(-1,-1)$, then compute the integral of f over the following 0-chains:

(1) $p - q$; $r - p$
(2) $p + q - r$

Another concept that will be useful for us is the *boundary* of an n-chain. As a warm-up, we define the boundary of a 1-cell. Suppose σ is the 1-cell which is the image of $\phi : [0,1] \to \mathbb{R}^m$ with the induced orientation. Then we define the boundary of σ (which we shall denote "$\partial\sigma$") as the 0-chain, $\phi(1) - \phi(0)$. In general, if the n-cell σ is the image of the parameterization $\phi : I^n \to \mathbb{R}^m$ with the induced orientation then

$$\partial\sigma = \sum_{i=1}^{n} (-1)^{i+1} \left(\phi|_{(x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_n)} - \phi|_{(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)} \right)$$

So, if σ is a 2-cell then

$$\begin{aligned} \partial\sigma &= (\phi(1,x_2) - \phi(0,x_2)) - (\phi(x_1,1) - \phi(x_1,0)) \\ &= \phi(1,x_2) - \phi(0,x_2) - \phi(x_1,1) + \phi(x_1,0) \end{aligned}$$

If σ is a 3-cell then

$$\begin{aligned} \partial\sigma &= (\phi(1,x_2,x_3) - \phi(0,x_2,x_3)) - (\phi(x_1,1,x_3) - \phi(x_1,0,x_3)) \\ &\quad + (\phi(x_1,x_2,1) - \phi(x_1,x_2,0)) \\ &= \phi(1,x_2,x_3) - \phi(0,x_2,x_3) - \phi(x_1,1,x_3) + \phi(x_1,0,x_3) \\ &\quad + \phi(x_1,x_2,1) - \phi(x_1,x_2,0) \end{aligned}$$

An example will hopefully clear up the confusion this all was sure to generate:

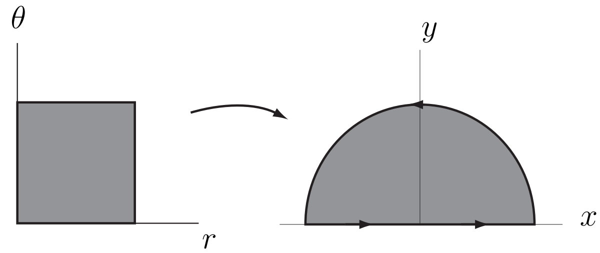


FIGURE 1. Orienting the boundary of a 2-cell.

EXAMPLE 5.2. Suppose $\phi(r,\theta) = (r\cos\pi\theta, r\sin\pi\theta)$. The image of ϕ is the 2-cell, σ, depicted in Figure 1. By the above definition,

$$\begin{aligned}\partial\sigma &= (\phi(1,\theta) - \phi(0,\theta)) - (\phi(r,1) - \phi(r,0))\\ &= (\cos\pi\theta, \sin\pi\theta) - (0,0) + (r,0) - (-r,0)\end{aligned}$$

This is the 1-chain depicted in Figure 1. □

Finally, we are ready to define what we mean by the boundary of an n-chain. If $C = \sum n_i\sigma_i$, then we define $\partial C = \sum n_i \partial\sigma_i$.

EXAMPLE 5.3. Suppose

$$\begin{aligned}\phi_1(r,\theta) &= (r\cos 2\pi\theta, r\sin 2\pi\theta, \sqrt{1-r^2}),\\ \phi_2(r,\theta) &= (-r\cos 2\pi\theta, r\sin 2\pi\theta, -\sqrt{1-r^2}),\end{aligned}$$

$\sigma_1 = Im(\phi_1)$ and $\sigma_2 = Im(\phi_2)$. Then $\sigma_1 + \sigma_2$ is a sphere in $\mathbb{R}^3$. One can check that $\partial(\sigma_1 + \sigma_2) = \emptyset$. □

EXERCISE 5.2. If σ is an n-cell then $\partial\partial\sigma = \emptyset$.

2. Pull-backs

Before getting to the central theorem of the text we need to introduce one more concept. Let's reexamine Equation 3:

$$\int\limits_M \omega = \pm \int\limits_R \omega_{\phi(x_1,\ldots,x_n)}\Big(\frac{\partial\phi}{\partial x_1}(x_1,\ldots x_n), \ldots, \frac{\partial\phi}{\partial x_n}(x_1,\ldots x_n)\Big) dx_1 \wedge \ldots \wedge dx_n$$

The form in the integrand on the right was defined so as to integrate to give the same answer as the form on the left. This is what we would like to generalize. Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is a parameterization, and ω is a k-form on $\mathbb{R}^m$. We define the *pull-back of* ω *under* ϕ to be the form on $\mathbb{R}^n$ which gives the same integral over any k-cell, σ, as ω does when integrated over $\phi(\sigma)$. Following convention, we denote the pullback of ω under ϕ as "$\phi^*\omega$".

So how do we decide how $\phi^*\omega$ acts on a k-tuple of vectors in $T_p\mathbb{R}^n$? The trick is to use ϕ to translate the vectors to a k-tuple in $T_{\phi(p)}\mathbb{R}^m$, and then plug them into ω. The matrix $D\phi$, whose columns are the partial derivatives of ϕ, is an $n \times m$ matrix. This matrix acts on vectors in $T_p\mathbb{R}^n$, and returns vectors in $T_{\phi(p)}\mathbb{R}^m$. So, we define (see Figure 2):

$$\phi^*\omega(V_p^1, ..., V_p^k) = \omega(D\phi(V_p^1), ..., D\phi(V_p^k))$$

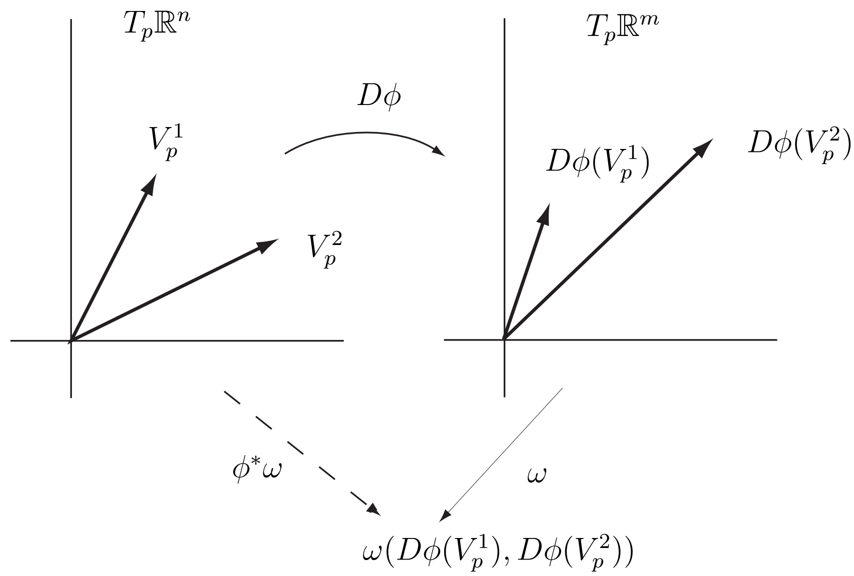


FIGURE 2. Defining $\phi^*\omega$.

EXAMPLE 5.4. Suppose $\omega = y\ dx + z\ dy + x\ dz$ is a 1-form on $\mathbb{R}^3$, and $\phi(a, b) = (a+b, a-b, ab)$ is a map from $\mathbb{R}^2$ to $\mathbb{R}^3$. Then $\phi^*\omega$ will be a 1-form on

$\mathbb{R}^2$. To determine which one, we can examine how it acts on the vectors $\langle 1,0\rangle_{(a,b)}$ and $\langle 0,1\rangle_{(a,b)}$.

$$\begin{aligned}
\phi^*\omega(\langle 1,0\rangle_{(a,b)}) &= \omega(D\phi(\langle 1,0\rangle_{(a,b)})) \\
&= \omega\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ b & a \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(a,b)}\right) \\
&= \omega(\langle 1,1,b\rangle_{(a+b,a-b,ab)}) \\
&= (a-b)+ab+(a+b)b \\
&= a-b+2ab+b^2
\end{aligned}$$

Similarly,

$$\begin{aligned}
\phi^*\omega(\langle 0,1\rangle_{(a,b)}) &= \omega(\langle 1,-1,a\rangle_{(a+b,a-b,ab)}) \\
&= (a-b)-ab+(a+b)a \\
&= a-b+a^2
\end{aligned}$$

Hence,

$$\phi^*\omega = (a-b+2ab+b^2)\ da + (a-b+a^2)\ db$$

$\square$

EXERCISE 5.3. If $\omega = x^2 dy \wedge dz + y^2 dz \wedge dw$ is a 2-form on $\mathbb{R}^4$, and $\phi(a,b,c) = (a,b,c,abc)$, then what is $\phi^*\omega$?

EXERCISE 5.4. If ω is an n-form on $\mathbb{R}^m$ and $\phi : \mathbb{R}^n \to \mathbb{R}^m$, then

$$\phi^*\omega = \omega_{\phi(x_1,\ldots,x_n)}\left(\frac{\partial\phi}{\partial x_1}(x_1,\ldots x_n),\ldots,\frac{\partial\phi}{\partial x_n}(x_1,\ldots x_n)\right) dx_1\wedge\ldots\wedge dx_n$$

In light of the preceding exercise Equation 3 can be re-written as

$$\int\limits_M \omega = \int\limits_R \phi^*\omega$$

EXERCISE 5.5. If σ is a k-cell in $\mathbb{R}^n$, $\phi : \mathbb{R}^n \to \mathbb{R}^m$, and ω is a k-form on $\mathbb{R}^m$ then

$$\int_{\sigma} \phi^* \omega = \int_{\phi(\sigma)} \omega$$

EXERCISE 5.6. If $\phi : \mathbb{R}^n \to \mathbb{R}^m$ and ω is a k-form on $\mathbb{R}^m$ then $d(\phi^*\omega) = \phi^*(d\omega)$.

3. Stokes' Theorem

In calculus we learn that when you take a function, differentiate it, and then integrate the result, something special happens. In this section we explore what happens when we take a form, differentiate it, and then integrate the resulting form over some chain. The general argument is quite complicated, so we start by looking at forms of a particular type integrated over very special regions.

Suppose $\omega = a\, dx_2 \wedge dx_3$ is a 2-form on $\mathbb{R}^3$, where $a : \mathbb{R}^3 \to \mathbb{R}$. Let R be the unit cube, $I^3 \subset \mathbb{R}^3$. We would like to explore what happens when we integrate $d\omega$ over R. Note first that Exercise 4.7 implies that $d\omega = \frac{\partial a}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3$.

Recall the steps used to define $\int\limits_R d\omega$:

(1) Choose a lattice of points in R, $\{p_{i,j,k}\}$. Since R is a cube, we can choose this lattice to be rectangular.
(2) Define $V^1_{i,j,k} = p_{i+1,j,k} - p_{i,j,k}$. Define $V^2_{i,j,k}$ and $V^3_{i,j,k}$ similarly.
(3) Compute $d\omega_{p_{i,j,k}}(V^1_{i,j,k}, V^2_{i,j,k}, V^2_{i,j,k})$.
(4) Sum over all i, j and k.
(5) Take the limit as the maximal distance between adjacent lattice points goes to 0.

Let's focus on Step 3 for a moment. Let t be the distance between $p_{i+1,j,k}$ and $p_{i,j,k}$, and assume t is small. Then $\frac{\partial a}{\partial x_1}(p_{i,j,k})$ is approximately equal to $\frac{a(p_{i+1,j,k}) - a(p_{i,j,k})}{t}$. This approximation gets better and better when we let $t \to 0$, in Step 5.

The vectors, $V^1_{i,j,k}$ through $V^3_{i,j,k}$, form a little cube. If we say the vector $V^1_{i,j,k}$ is "vertical", and the other two are horizontal, then the "height" of this cube is t, and the area of its base is $dx_2 \wedge dx_3(V^2_{i,j,k}, V^3_{i,j,k})$, which makes its volume $t\ dx_2 \wedge dx_3(V^2_{i,j,k}, V^3_{i,j,k})$. Putting all this together, we find that

$$\begin{aligned} d\omega_{p_{i,j,k}}(V^1_{i,j,k}, V^2_{i,j,k}, V^2_{i,j,k}) &= \frac{\partial a}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3(V^1_{i,j,k}, V^2_{i,j,k}, V^2_{i,j,k}) \\ &\approx \frac{a(p_{i+1,j,k}) - a(p_{i,j,k})}{t} t\ dx_2 \wedge dx_3(V^2_{i,j,k}, V^3_{i,j,k}) \\ &= \omega(V^2_{i+1,j,k}, V^3_{i+1,j,k}) - \omega(V^2_{i,j,k}, V^3_{i,j,k}) \end{aligned}$$

Let's move on to Step 4. Here we sum over all i, j and k. Suppose for the moment that i ranges between 1 and N. First, we fix j and k, and sum over all i. The result is $\omega(V^2_{N,j,k}, V^3_{N,j,k}) - \omega(V^2_{1,j,k}, V^3_{1,j,k})$. Now notice that $\sum\limits_{j,k} \omega(V^2_{N,j,k}, V^3_{N,j,k})$ is a Riemann sum for the integral of ω over the "top" of R, and $\sum\limits_{j,k} \omega(V^2_{1,j,k}, V^3_{1,j,k})$ is a Riemann sum for ω over the "bottom" of R. Lastly, note that ω, evaluated on any pair of vectors which lie in the sides of the cube, gives 0. Hence, the integral of ω over a side of R is 0. Putting all this together, we conclude:

$$\int\limits_R d\omega = \int\limits_{\partial R} \omega \tag{5}$$

EXERCISE 5.7. Prove that Equation 5 holds if $\omega = b\ dx_1 \wedge dx_3$, or if $\omega = c\ dx_1 \wedge dx_2$. **Caution!** Beware of signs and orientations.

EXERCISE 5.8. Use the previous problem to conclude that if $\omega = a\ dx_2 \wedge dx_3 + b\ dx_1 \wedge dx_3 + c\ dx_1 \wedge dx_2$ is an arbitrary 2-form on $\mathbb{R}^3$ then Equation 5 holds.

EXERCISE 5.9. If ω is an arbitrary $(n-1)$-form on $\mathbb{R}^n$ and R is the unit cube in $\mathbb{R}^n$ then show that Equation 5 still holds.

This exercise prepares us to move on to the general case. Suppose σ is an n-cell in $\mathbb{R}^m$, $\phi : I^n \subset \mathbb{R}^n \to \mathbb{R}^m$ is a parameterization of σ, and ω is an $(n-1)$-form on $\mathbb{R}^m$. Then we can combine Exercises 5.5, 5.6, and 5.9 to give us

$$\int\limits_{\partial\sigma} \omega = \int\limits_{\phi(\partial I^n)} \omega = \int\limits_{\partial I^n} \phi^*\omega = \int\limits_{I^n} d(\phi^*\omega) = \int\limits_{I^n} \phi^*(d\omega) = \int\limits_{\phi(I^n)} d\omega = \int\limits_{\sigma} d\omega$$

In general, this implies that if $C = \sum n_i \sigma_i$ is an n-chain, then

$$\boxed{\int_{\partial C} \omega = \int_C d\omega}$$

This equation is called the *Generalized Stokes' Theorem*. It is unquestionably the most crucial result of this text. In fact, everything we have done up to this point has been geared toward developing this equation and everything that follows will be applications of this equation.

EXAMPLE 5.5. Let $\omega = x\ dy$ be a 1-form on $\mathbb{R}^2$. Let σ be the 2-cell which is the image of the parameterization $\phi(r,\theta) = (r\cos\theta, r\sin\theta)$, where $0 \le r \le R$ and $0 \le \theta \le 2\pi$. By the Generalized Stokes' Theorem,

$$\int_{\partial\sigma} \omega = \int_\sigma d\omega = \int_\sigma dx \wedge dy = \int_\sigma dx\ dy = Area(\sigma) = \pi R^2$$

□

EXERCISE 5.10. Verify directly that $\int_{\partial\sigma} \omega = \pi R^2$

EXAMPLE 5.6. Let $\omega = x\ dy + y\ dx$ be a 1-form on $\mathbb{R}^2$, and let σ be any 2-cell. Then $\int_{\partial\sigma} \omega = \int_\sigma d\omega = 0$ □

EXERCISE 5.11. Find a 1-chain in $\mathbb{R}^2$ which bounds a 2-cell and integrate the form $x\ dy + y\ dx$ over this curve.

EXAMPLE 5.7. Let C be the curve in $\mathbb{R}^2$ parameterized by $\phi(t) = (t^2, t^3)$, where $-1 \le t \le 1$. Let f be the 0-form x^2y. We use the Generalized Stokes Theorem to calculate $\int_C df$.

The curve C goes from the point (1,-1) , when $t = -1$, to the point (1,1), when $t = 1$. Hence, ∂C is the 0-chain $(1,1) - (1,-1)$. Now we use Stokes:

$$\int_C df = \int_{\partial C} f = \int_{(1,1)-(1,-1)} x^2y = 1 - (-1) = 2$$

□

EXERCISE 5.12. Calculate $\int\limits_C df$ directly.

EXAMPLE 5.8. Let $\omega = (x^2 + y)dx + (x - y^2)dy$ be a 1-form on $\mathbb{R}^2$. We wish to integrate ω over σ, the top half of the unit circle. First, note that $d\omega = 0$, so that if we integrate ω over the boundary of any 2-cell, we would get 0. Let τ denote the line segment connecting (-1,0) to (1,0). Then the 1-chain $\sigma - \tau$ bounds a 2-cell. So $\int\limits_{\sigma-\tau} \omega = 0$, which implies that $\int\limits_\sigma \omega = \int\limits_\tau \omega$. This latter integral is a bit easier to compute. Let $\phi(t) = (t, 0)$ be a parameterization of τ, where $-1 \le t \le 1$. Then

$$\int\limits_\sigma \omega = \int\limits_\tau \omega = \int\limits_{[-1,1]} \omega_{(t,0)}(\langle 1, 0\rangle)\ dt = \int\limits_{-1}^{1} t^2\ dt = \frac{2}{3}$$

□

EXERCISE 5.13. Let $\omega = (x + y^3)\ dx + 3xy^2\ dy$ be a differential 1-form on $\mathbb{R}^2$. Let Q be the rectangle $\{(x, y)|0 \le x \le 3, 0 \le y \le 2\}$.

(1) Compute $d\omega$.
(2) Use the generalized Stokes Theorem to compute $\int\limits_{\partial Q} \omega$.
(3) Compute $\int\limits_{\partial Q} \omega$ directly, by integrating ω over each each edge of the boundary of the rectangle, and then adding in the appropriate manner. *Answer: If L, R, T, and B represent the 1-cells that are the left, right, top, and bottom of Q then*

$$\int\limits_{\partial Q} \omega = \int\limits_{(R-L)-(T-B)} \omega = \int\limits_R \omega - \int\limits_L \omega - \int\limits_T \omega + \int\limits_B \omega = 24 - 0 - 28\frac{1}{2} + 4\frac{1}{2} = 0$$

(4) How does $\int\limits_{R-T-L} \omega$ compare to $\int\limits_B \omega$?
(5) Let S be any curve in the upper half plane ($i.e.$, the set $\{(x, y)|y \ge 0\}$) that connects the point $(0, 0)$ to the point $(3, 0)$. What is $\int\limits_S \omega$? Why?

(6) Let S be *any* curve that connects the point $(0, 0)$ to the point $(3, 0)$. What is $\int\limits_S \omega$? WHY???

EXERCISE 5.14. Calculate the volume of a ball of radius 1, $\{(\rho, \theta, \phi)|\rho \leq 1\}$, by integrating some 2-form over the sphere of radius 1, $\{(\rho, \theta, \phi)|\rho = 1\}$.

EXERCISE 5.15. Calculate

$$\int\limits_C x^3 \ dx + \left(\frac{1}{3}x^3 + xy^2\right) \ dy$$

where C is the circle of radius 2, centered about the origin. *Answer:* 8π

EXERCISE 5.16. Suppose $\omega = x \ dx + x \ dy$ is a 1-form on $\mathbb{R}^2$. Let C be the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Determine the value of $\int\limits_C \omega$ by integrating some 2-form over the region bounded by the ellipse. (Hint: the region bounded by the ellipse can be parameterized by $\phi(r, \theta) = (2r\cos(\theta), 3r\sin(\theta))$, where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.) *Answer:* 6π

4. Vector calculus and the many faces of Stokes' Theorem

Although the language and notation may be new, you have already seen Stokes' Theorem in many guises. For example, let $f(x)$ be a 0-form on $\mathbb{R}$. Then $df = f'(x)dx$. Let $[a, b]$ be a 1-cell in $\mathbb{R}$. Then Stokes' Theorem tells us

$$\int\limits_a^b f'(x)dx = \int\limits_{[a,b]} f'(x)dx = \int\limits_{\partial[a,b]} f(x) = \int\limits_{b-a} f(x) = f(b) - f(a)$$

Which is, of course, the Fundamental Theorem of Calculus. If we let R be some 2-chain in $\mathbb{R}^2$ then Stokes' Theorem implies

$$\int\limits_{\partial R} P \ dx + Q \ dy = \int\limits_R d(P \ dx + Q \ dy) = \int\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \ dx \ dy$$

This is what we call "Green's Theorem" in Calculus. To proceed further, we restrict ourselves to $\mathbb{R}^3$. In this dimension there is a nice correspondence between vector fields and both 1- and 2-forms.

$$\begin{aligned}\mathbf{F} = \langle F_x, F_y, F_z\rangle \quad &\leftrightarrow \quad \omega^1_{\mathbf{F}} = F_x dx + F_y dy + F_z dz \\ &\leftrightarrow \quad \omega^2_{\mathbf{F}} = F_x dy \wedge dz - F_y dx \wedge dz + F_z dx \wedge dy\end{aligned}$$

On $\mathbb{R}^3$ there is also a useful correspondence between 0-forms (functions) and 3-forms.

$$f(x,y,z) \leftrightarrow \omega^3_f = f\ dx \wedge dy \wedge dz$$

We can use these correspondences to define various operations involving functions and vector fields. For example, suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a 0-form. Then df is the 1-form, $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$. The vector field associated to this 1-form is then $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$. In calculus we call this vector field **grad** f, or ∇f. In other words, ∇f is the vector field associated with the 1-form, df. This can be summarized by the equation

$$df = \omega^1_{\nabla f}$$

It will be useful to think of this diagrammatically as well.

$$\begin{tikzcd} f \arrow[r, "\mathbf{grad}"] \arrow[d, equal] & \nabla f \\ f \arrow[r, "d"'] & df \arrow[u] \end{tikzcd}$$

EXAMPLE 5.9. Suppose $f = x^2y^3z$. Then $df = 2xy^3z\ dx + 3x^2y^2z\ dy + x^3y^3\ dz$. The associated vector field, **grad** f, is then $\nabla f = \langle 2xy^3z, 3x^2y^2z, x^3y^3\rangle$. □

Similarly, if we start with a vector field, $\mathbf{F}$, form the associated 1-form, $\omega^1_{\mathbf{F}}$, differentiate it, and look at the corresponding vector field, then the result is called **curl F**, or $\nabla\times\mathbf{F}$. So, $\nabla\times\mathbf{F}$ is the vector field associated with the 2-form, $d\omega^1_{\mathbf{F}}$. This can be summarized by the equation

$$d\omega^1_{\mathbf{F}} = \omega^2_{\nabla\times\mathbf{F}}$$

This can also be illustrated by the following diagram.

$$
\begin{tikzcd}
\mathbf{F} \arrow[r, "\mathbf{curl}"] \arrow[d] & \nabla \times \mathbf{F} \\
\omega^1_{\mathbf{F}} \arrow[r, "d"'] & d\omega^1_{\mathbf{F}} \arrow[u]
\end{tikzcd}
$$

EXAMPLE 5.10. Let $\mathbf{F} = \langle xy, yz, x^2 \rangle$. The associated 1-form is then

$$\omega^1_{\mathbf{F}} = xy\ dx + yz\ dy + x^2\ dz.$$

The derivative of this 1-form is the 2-form

$$d\omega^1_{\mathbf{F}} = -y\ dy \wedge dz + 2x\ dx \wedge dz - x\ dx \wedge dy.$$

The vector field associated to this 2-form is **curl F**, which is

$$\nabla \times \mathbf{F} = \langle -y, -2x, -x \rangle.$$

$\square$

Lastly, we can start with a vector field, $\mathbf{F} = \langle F_x, F_y, F_z \rangle$, and then look at the 3-form, $d\omega^2_{\mathbf{F}} = (\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}) dx \wedge dy \wedge dz$ (See Exercise 4.12). The function, $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$ is called **div F**, or $\nabla \cdot \mathbf{F}$. This is summarized in the following equation and diagram.

$$d\omega^2_{\mathbf{F}} = \omega^3_{\nabla \cdot \mathbf{F}}$$

$$
\begin{tikzcd}
\mathbf{F} \arrow[r, "\mathbf{div}"] \arrow[d] & \nabla \cdot \mathbf{F} \\
\omega^2_{\mathbf{F}} \arrow[r, "d"'] & d\omega^2_{\mathbf{F}} \arrow[u]
\end{tikzcd}
$$

EXAMPLE 5.11. Let $\mathbf{F} = \langle xy, yz, x^2 \rangle$. The associated 2-form is then

$$\omega^2_{\mathbf{F}} = xy\ dy \wedge dz - yz\ dx \wedge dz + x^2\ dx \wedge dy.$$

The derivative is the 3-form

$$d\omega^2_{\mathbf{F}} = (y + z)\ dx \wedge dy \wedge dz.$$

So **div F** is the function $\nabla \cdot \mathbf{F} = y + z$. $\square$

Two important vector identities follow from the fact that for a differential form, ω, calculating $d(d\omega)$ always yields zero (see Exercise 4.8 of Chapter 4). For the first, consider the following diagram.

$$\begin{tikzcd}
f \arrow[r, "\textbf{grad}"] \arrow[d, equal] & \nabla f \arrow[r, "\textbf{curl}"] & \nabla \times (\nabla f) \\
f \arrow[r, "d"'] & df \arrow[u] \arrow[r, "d"'] & ddf \arrow[u]
\end{tikzcd}$$

This shows that if f is a 0-form then the vector field corresponding to ddf is $\nabla \times (\nabla f)$. But $ddf = 0$, so we conclude

$$\nabla \times (\nabla f) = 0$$

For the second identity, consider this diagram.

$$\begin{tikzcd}
\mathbf{F} \arrow[r, "\textbf{curl}"] \arrow[d] & \nabla \times \mathbf{F} \arrow[r, "\textbf{div}"] \arrow[d] & \nabla \cdot (\nabla \times \mathbf{F}) \\
\omega^1_{\mathbf{F}} \arrow[r, "d"'] & d\omega^1_{\mathbf{F}} \arrow[r, "d"'] & dd\omega^1_{\mathbf{F}} \arrow[u]
\end{tikzcd}$$

This shows that if $dd\omega^1_{\mathbf{F}}$ is written as $g\ dx \wedge dy \wedge dz$ then the function g is equal to $\nabla \cdot (\nabla \times \mathbf{F})$. But $dd\omega^1_{\mathbf{F}} = 0$, so we conclude

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

In vector calculus we also learn how to integrate vector fields over parameterized curves (1-chains) and surfaces (2-chains). Suppose first that σ is some parameterized curve. Then we can integrate the component of $\mathbf{F}$ which points in the direction of the tangent vectors to σ. This integral is usually denoted $\int\limits_\sigma \mathbf{F} \cdot d\mathbf{s}$, and its definition is precisely the same as the definition we learned here for $\int\limits_\sigma \omega^1_{\mathbf{F}}$. A special case of this integral arises when $\mathbf{F} = \nabla f$, for some function, f. In this case, $\omega^1_{\mathbf{F}}$ is just df, so the definition of $\int\limits_\sigma \nabla f \cdot d\mathbf{s}$ is the same as $\int\limits_\sigma df$.

We also learn to integrate vector fields over parameterized surfaces. In this case, the quantity we integrate is the component of the vector field which is normal to the surface. This integral is often denoted $\int\limits_S \mathbf{F} \cdot d\mathbf{S}$. Its definition is precisely the same as that of $\int\limits_S \omega^2_{\mathbf{F}}$ (see Exercises 2.20 and 2.21). A special case of this is when $\mathbf{F} = \nabla \times \mathbf{G}$, for some vector field, $\mathbf{G}$. Then $\omega^2_{\mathbf{G}}$ is just $d\omega^1_{\mathbf{G}}$, so we see that $\int_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}$ must be the same as $\int_S d\omega^1_{\mathbf{G}}$.

The most basic thing to integrate over a 3-dimensional region (*i.e.* a 3-chain), Ω, in $\mathbb{R}^3$ is a function $f(x, y, x)$. In calculus we denote this integral as $\int_\Omega f\ dV$. Note that this is precisely the same as $\int_\Omega \omega^3_f$. A special case is when $f = \nabla \cdot \mathbf{F}$, for some vector field $\mathbf{F}$. In this case $\int_\Omega f\ dV = \int_\Omega (\nabla \cdot \mathbf{F}) dV$. But we can write this integral with differential forms as $\int_\Omega d\omega^2_{\mathbf{F}}$.

We summarize the equivalence between the integrals developed in vector calculus and various integrals of differential forms in the following table:

Vector Calculus	Differential Forms
$\int_\sigma \mathbf{F} \cdot d\mathbf{s}$	$\int_\sigma \omega^1_{\mathbf{F}}$
$\int_\sigma \nabla f \cdot d\mathbf{s}$	$\int_\sigma df$
$\int_S \mathbf{F} \cdot d\mathbf{S}$	$\int_S \omega^2_{\mathbf{F}}$
$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$	$\int_S d\omega^1_{\mathbf{F}}$
$\int_\Omega f\ dV$	$\int_\Omega \omega^3_f$
$\int_\Omega (\nabla \cdot \mathbf{F}) dV$	$\int_\Omega d\omega^2_{\mathbf{F}}$

Let us now apply the Generalized Stokes' Theorem to various situations. First, we start with a parameterization, $\phi : [a, b] \to \sigma \subset \mathbb{R}^3$, of a curve in $\mathbb{R}^3$, and a function, $f : \mathbb{R}^3 \to \mathbb{R}$. Then we have

$$\int_\sigma \nabla f \cdot d\mathbf{s} \equiv \int_\sigma df = \int_{\partial\sigma} f = f(\phi(b)) - f(\phi(a))$$

This shows the independence of path of line integrals of gradient fields. We can use this to prove that a line integral of a gradient field over any simple closed curve is 0, but for us there is an easier, direct proof, which again uses the Generalized Stokes' Theorem. Suppose σ is a simple closed loop in $\mathbb{R}^3$ (*i.e.* $\partial\sigma = \emptyset$). Then $\sigma = \partial D$, for some 2-chain, D. We now have

$$\int_\sigma \nabla f \cdot d\mathbf{s} \equiv \int_\sigma df = \int_D ddf = 0$$

Now, suppose we have a vector field, $\mathbf{F}$, and a parameterized surface, S. Yet another application of the Generalized Stokes' Theorem yields

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} \equiv \int_{\partial S} \omega^1_{\mathbf{F}} = \int_S d\omega^1_{\mathbf{F}} \equiv \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

In vector calculus we call this equality "Stokes' Theorem". In some sense, $\nabla \times \mathbf{F}$ measures the "twisting" of $\mathbf{F}$ at points of S. So Stokes' Theorem says that the net twisting of $\mathbf{F}$ over all of S is the same as the amount $\mathbf{F}$ circulates around ∂S.

EXAMPLE 5.12. Suppose we are faced with a problem phrased thusly: "Use Stokes' Theorem to calculate $\int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = x + 1$, and $\mathbf{F}$ is the vector field $\langle -x^2y, xy^2, z^3 \rangle$."

We will solve this problem by translating to the language of differential forms, and using the Generalized Stokes' Theorem instead. To begin, note that $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \omega^1_{\mathbf{F}}$, and $\omega^1_{\mathbf{F}} = -x^2y \; dx + xy^2 \; dy + z^3 \; dz$.

Now, to use the Generalized Stokes' Theorem we will need to calculate

$$d\omega^1_{\mathbf{F}} = (x^2 + y^2) \; dx \wedge dy.$$

Let D denote the subset of the plane $z = x + 1$ bounded by C. Then $\partial D = C$. Hence, by the Generalized Stokes' Theorem we have

$$\int_C \omega^1_{\mathbf{F}} = \int_D d\omega^1_{\mathbf{F}} = \int_D (x^2 + y^2) \; dx \wedge dy$$

The region D is parameterized by $\Psi(r, \theta) = (r\cos\theta, r\sin\theta, r\cos\theta + 1)$, where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Using this one can (and should!) show that $\int_D (x^2 + y^2) \; dx \wedge dy = 8\pi$. $\square$

EXERCISE 5.17. Let C be the square with sides $(x, \pm 1, 1)$, where $-1 \leq x \leq 1$ and $(\pm 1, y, 1)$, where $-1 \leq y \leq 1$, with the indicated orientation (see Figure 3). Let $\mathbf{F}$ be the vector field $\langle xy, x^2, y^2 z\rangle$. Compute $\int\limits_C \mathbf{F} \cdot d\mathbf{s}$. *Answer:* 0

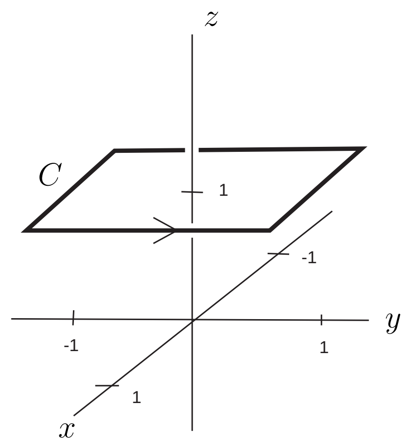


FIGURE 3.

Suppose now that Ω is some volume in $\mathbb{R}^3$. Then we have

$$\int\limits_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S} \equiv \int\limits_{\partial\Omega} \omega^2_{\mathbf{F}} = \int\limits_{\Omega} d\omega^2_{\mathbf{F}} \equiv \int\limits_{\Omega} (\nabla \cdot \mathbf{F}) dV$$

This last equality is called "Gauss' Divergence Theorem". $\nabla \cdot \mathbf{F}$ is a measure of how much $\mathbf{F}$ "spreads out" at a point. So Gauss' Theorem says that the total spreading out of $\mathbf{F}$ inside Ω is the same as the net amount of $\mathbf{F}$ "escaping" through $\partial\Omega$.

EXERCISE 5.18. Let Ω be the cube $\{(x, y, z) | 0 \leq x, y, z \leq 1\}$. Let $\mathbf{F}$ be the vector field $\langle xy^2, y^3, x^2y^2\rangle$. Compute $\int\limits_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S}$. *Answer:* $\frac{4}{3}$

CHAPTER 6

Applications

1. Maxwell's Equations

As a brief application we show how the language of differential forms can greatly simplify the classical vector equations of Maxwell. These equations describe the relationship between electric and magnetic fields. Classically both electricity and magnetism are described as a 3-dimensional vector field which varies with time:

$$\mathbf{E} = \langle E_x, E_y, E_z \rangle$$

$$\mathbf{B} = \langle B_x, B_y, B_z \rangle$$

Where E_x, E_z, E_z, B_x, B_y, and B_z are all functions of x, y, z and t.

Maxwell's equations are then:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} &= -4\pi\mathbf{J} \end{aligned}$$

The quantity ρ is called the *charge density* and the vector $\mathbf{J} = \langle J_x, J_y, J_z \rangle$ is called the *current density*.

We can make all of this look much simpler by making the following definitions. First we define a 2-form called the *Faraday*, which simultaneously describes both the electric and magnetic fields:

$$\begin{aligned} \mathbf{F} = \; & E_x \; dx \wedge dt + E_y \; dy \wedge dt + E_z \; dz \wedge dt \\ & + B_x \; dy \wedge dz + B_y \; dz \wedge dx + B_z \; dx \wedge dy \end{aligned}$$

Next we define the "dual" 2-form, called the *Maxwell*:

$$\begin{aligned} {}^*\mathbf{F} &= E_x\ dy \wedge dz + E_y\ dz \wedge dx + E_z\ dx \wedge dy \\ &\quad + B_x\ dt \wedge dx + B_y\ dt \wedge dy + B_z\ dt \wedge dz \end{aligned}$$

We also define the *4-current*, $\mathbf{J}$, and it's "dual", ${}^*\mathbf{J}$:

$$\begin{aligned} \mathbf{J} &= \langle \rho, J_x, J_y, J_z \rangle \\ {}^*\mathbf{J} &= \rho\ dx \wedge dy \wedge dz \\ &\quad - J_x\ dt \wedge dy \wedge dz \\ &\quad - J_y\ dt \wedge dz \wedge dx \\ &\quad - J_z\ dt \wedge dx \wedge dy \end{aligned}$$

Maxwell's four vector equations now reduce to:

$$\begin{aligned} d\mathbf{F} &= 0 \\ d^*\mathbf{F} &= 4\pi^*\mathbf{J} \end{aligned}$$

EXERCISE 6.1. Show that the equation $d\mathbf{F} = 0$ implies the first two of Maxwell's equations.

EXERCISE 6.2. Show that the equation $d^*\mathbf{F} = 4\pi^*\mathbf{J}$ implies the second two of Maxwell's equations.

The differential form version of Maxwell's equation has a huge advantage over the vector formulation: *it is coordinate free!* A 2-form such as $\mathbf{F}$ is an operator that "eats" pairs of vectors and "spits out" numbers. The way it acts is completely geometric... that is, it can be defined without any reference to the coordinate system (t, x, y, z). This is especially poignant when one realizes that Maxwell's equations are laws of nature that should not depend on a man-made construction such as coordinates.

2. Foliations and Contact Structures

Everyone has seen tree rings and layers in sedimentary rock. These are examples of *foliations*. Intuitively, a foliation is when some region of space has been "filled up" with lower dimensional surfaces. A full treatment of foliations is a topic for a much larger textbook than this one. Here we will only be discussing foliations of $\mathbb{R}^3$.

Let U be an open subset of $\mathbb{R}^3$. We say U has been *foliated* if there is a family $\phi^t : R_t \to U$ of parameterizations (where for each t the domain $R_t \subset \mathbb{R}^2$) such that every point of U is in the image of exactly one such parameterization. In other words, the images of the parameterizations ϕ^t are surfaces that fill up U, and no two overlap.

Suppose p is a point of U and U has been foliated as above. Then there is a unique value of t such that p is a point in $\phi^t(R_t)$. The partial derivatives, $\frac{\partial \phi^t}{\partial x}(p)$ and $\frac{\partial \phi^t}{\partial y}(p)$ are then two vectors that span a plane in $T_p\mathbb{R}^3$. Let's call this plane Π_p. In other words, if U is foliated then at every point p of U we get a plane Π_p in $T_p\mathbb{R}^3$.

The family $\{\Pi_p\}$ is an example of a *plane field*. In general a plane field is just a choice of a plane in each tangent space which varies smoothly from point to point in $\mathbb{R}^3$. We say a plane field is *integrable* if it consists of the tangent planes to a foliation.

This should remind you a little of first-term calculus. If $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a differentiable function then at every point p on its graph we get a line in $T_p\mathbb{R}^2$ (see Figure 2). If we just know the lines and want the original function then we are integrating.

There is a theorem that says that every *line field* on $\mathbb{R}^2$ is integrable. The question we would like to answer in this section is whether or not this is true of plane fields on $\mathbb{R}^3$. The first step is to figure out how to specify a plane field in some reasonably nice way. This is where differential forms come in. Suppose $\{\Pi_p\}$ is a plane field. At each point p we can define a line in $T_p\mathbb{R}^3$ (*i.e.* a *line field*) by looking at the set of all vectors that are perpendicular to Π_p. We can then define a 1-form ω by projecting vectors onto these lines. So, in particular, if V_p is a vector in Π_p then $\omega(V_p) = 0$. Another way to say this is that the plane Π_p is the set of all vectors which yield zero when plugged into ω. As shorthand we write this set as Ker ω ("Ker" comes from the word "Kernel", a term from linear algebra). So all we are saying is that ω is a 1-form such that $\Pi_p = \text{Ker } \omega$. This is very convenient. To specify a plane field all we have to do now is write down a 1-form!

EXAMPLE 6.1. Suppose $\omega = dx$. Then at each point p of $\mathbb{R}^3$ the vectors of $T_p\mathbb{R}^3$ that yield zero when plugged into ω are all those in the dy-dz plane. Hence, Ker ω is the plane field consisting of all of the dy-dz planes (one for every point of $\mathbb{R}^3$). It is obvious that this plane field is integrable; at each point p we just have the tangent plane to the plane parallel to the y-z plane through p. $\square$

In the above example note that any 1-form that looks like $f(x, y, z)dx$ defines the same plane field, as long as f is non-zero everywhere. So, knowing something about a plane field (like the assumption that it is integrable) seems like it might not say much about the 1-form ω, since so many different 1-forms give the same plane field. Let's investigate this further.

First, let's see if there's anything special about the derivative of a 1-form that looks like $\omega = f(x, y, z)dx$. This is easy: $d\omega = \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial f}{\partial z} dz \wedge dx$. Nothing too special so far. How about combining this with ω? Let's compute:

$$\omega \wedge d\omega = f(x, y, z)dx \wedge \left(\frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial f}{\partial z} dz \wedge dx \right) = 0$$

Now that's special! In fact, recall our emphasis earlier that forms are *coordinate free*. In other words, any computation one can perform with forms will give the same answer regardless of what coordinates are chosen. The wonderful thing about foliations is that near every point you can always choose coordinates so that your foliation looks like planes parallel to the y-z plane. In other words, the above computation is not as special as you might think:

THEOREM 6.1. *If Ker ω is an integrable plane field then $\omega \wedge d\omega = 0$ at every point of $\mathbb{R}^3$.*

It should be noted that we have only chosen to work in $\mathbb{R}^3$ for ease of visualization. There are higher dimensional definitions of foliations and plane fields. In general, if the kernel of a 1-form ω defines an integrable plane field then $\omega \wedge d\omega^n = 0$.

Our search for a plane field that is *not* integrable (*i.e.* not the tangent planes to a foliation) has now been reduced to the search for a 1-form ω for which $\omega \wedge d\omega \neq 0$ *somewhere*. There are many such forms. An easy one is $x\ dy + dz$. We compute:

$$(x\ dy + dz) \wedge d(x\ dy + dz) = (x\ dy + dz) \wedge (dx \wedge dy) = dz \wedge dx \wedge dy$$

Our answer is quite special. All we needed was a 1-form such that $\omega \wedge d\omega \neq 0$ somewhere. What we found was a 1-form for which $\omega \wedge d\omega \neq 0$ *everywhere*. This means that there is not a single point of $\mathbb{R}^3$ which has a neighborhood in which the planes given by Ker $x\ dy + dz$ are tangent to a foliation. Such a plane field is called a *contact structure*.

At this point you're probably wondering, "What could Ker $x\ dy + dz$ possibly look like?!" It's not so easy to visualize this, but we have tried to give you some indication in Figure 1[1]. A good exercise is to stare at this picture long enough to convince yourself that the planes pictured cannot be the tangent planes to a foliation.

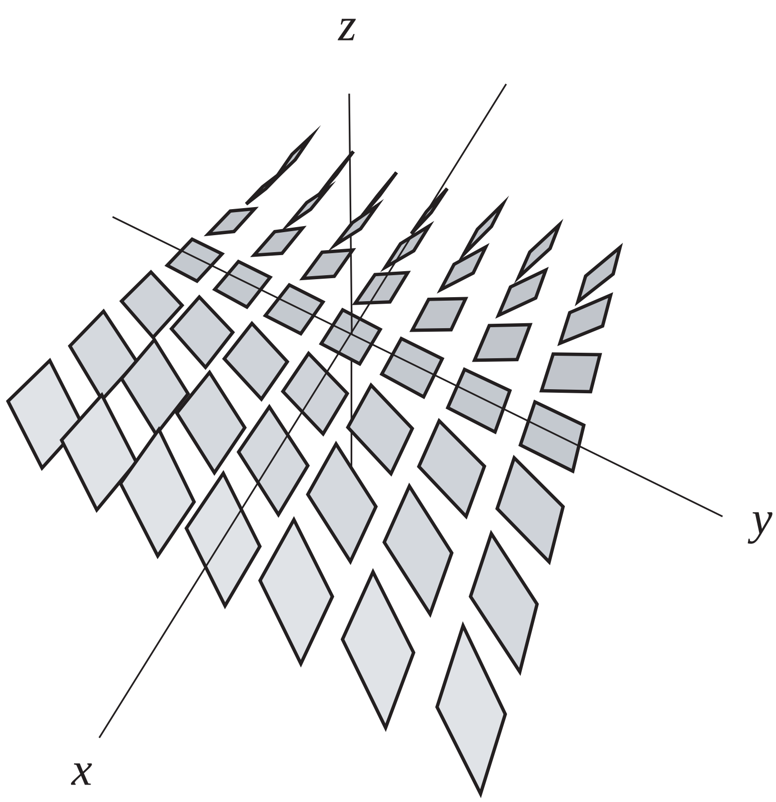


FIGURE 1. The plane field Ker $x\ dy + dz$.

We have just seen how we can use differential forms to tell if a plane field is integrable. But one may still wonder if there is more we can say about a 1-form, assuming its kernel is integrable. Let's go back to the expression $\omega \wedge d\omega$. Recall that ω is a 1-form, which makes $d\omega$ a 2-form, and hence $\omega \wedge d\omega$ a 3-form.

[1]Figure drawn by Stephan Schoenenberger. Taken from *Introductory Lectures on Contact Geometry* by John B. Etnyre

A 3-form on $T_p\mathbb{R}^3$ measures the volume of the parallelepiped spanned by three vectors, multiplied by a constant. For example, if $\psi = \alpha \wedge \beta \wedge \gamma$ is a 3-form then the constant it scales volume by is given by the volume of the parallelepiped spanned by the vectors $\langle \alpha \rangle$, $\langle \beta \rangle$, and $\langle \gamma \rangle$ (where "$\langle \alpha \rangle$" refers to the vector dual to the 1-form α introduced in Section 3 of Chapter 3). If it turns out that ψ is the zero 3-form then the vector $\langle \alpha \rangle$ must be in the plane spanned by the vectors $\langle \beta \rangle$ and $\langle \gamma \rangle$.

On $\mathbb{R}^3$ the results of Section 3 of Chapter 3 tell us that a 2-form such as $d\omega$ can always be written as $\alpha \wedge \beta$, for some 1-forms α and β. If ω is a 1-form with integrable kernel then we have already seen that $\omega \wedge d\omega = \omega \wedge \alpha \wedge \beta = 0$. But this tells us that $\langle \omega \rangle$ must be in the plane spanned by the vectors $\langle \alpha \rangle$ and $\langle \beta \rangle$. Now we can invoke Lemma 2.1 of Chapter 3, which says that we can rewrite $d\omega$ as $\omega \wedge \nu$, for some 1-form ν.

If we start with a foliation and choose a 1-form ω whose kernel consists of planes tangent to the foliation then the 1-form ν that we have just found is in no way canonical. We made lots of choices to get to ν, and different choices will end up with different 1-forms. But here's the amazing fact: the integral of the 3-form $\nu \wedge d\nu$ does *not* depend on any of our choices! It is completely determined by the original foliation. Whenever a mathematician runs into a situation like this they usually throw up their hands and say, "Eureka! I've discovered an *invariant*." The quantity $\int \nu \wedge d\nu$ is referred to as the *Gobillion-Vey invariant* of the foliation. It is a topic of current research to identify exactly what information this number tells us about the foliation.

Two special cases are worth noting. First, it may turn out that $\nu \wedge d\nu = 0$ everywhere. This tells us that the plane field given by Ker ν is integrable, so we get another foliation. The other interesting case is when $\nu \wedge d\nu$ is nowhere zero. Then we get a contact structure.

3. How *not* to visualize a differential 1-form

There are several contemporary physics texts that attempt to give a visual interpretation of differential forms that seems quite different from the one presented here. As this alternate interpretation is much simpler than anything described in these notes, one may wonder why we have not taken this approach.

Let's look again at the 1-form dx on $\mathbb{R}^3$. Given a vector V_p at a point p the value of $dx(V_p)$ is just the projection of V_p onto the dx axis in $T_p\mathbb{R}^3$. Now, let C be some parameterized curve in $\mathbb{R}^3$ for which the x-coordinate is always increasing. Then $\int\limits_C dx$ is just the length of the projection of C onto the x-axis. To the nearest integer, this is just the number of evenly spaced planes that C punctures that are parallel to the y-z plane. So one way that you might visualize the form dx is by picturing these planes.

This view is very appealing. After all, every 1-form ω, at every point p, projects vectors onto some line l_p. So can't we integrate ω along a curve C (at least to the nearest integer) by counting the number of surfaces punctured by C whose tangent planes are perpendicular to the lines l_p (see Figure 2)? If you've read the previous section you might guess that the answer is a categorical *NO!*

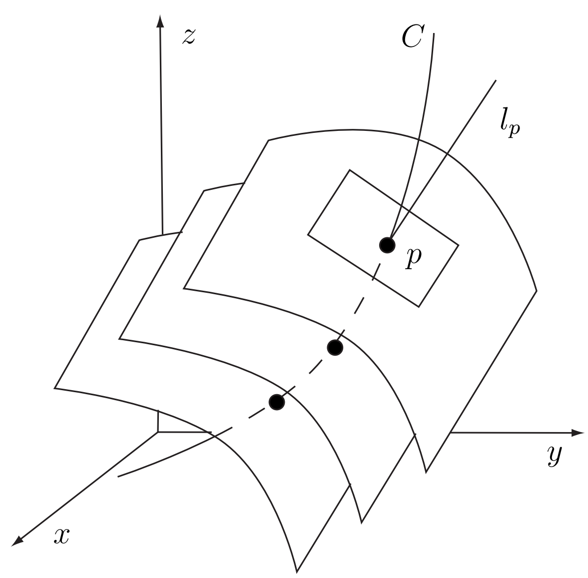


FIGURE 2. "Surfaces" of ω?

Recall that the planes perpendicular to the lines l_p are precisely Ker ω. To say that there are surfaces whose tangent planes are perpendicular to the lines l_p is the same thing as saying that Ker ω is an integrable plane field. But we have seen in the previous section that there are 1-forms as simple as $x\ dy + dz$ whose kernels are *nowhere* integrable.

Can we at least use this interpretation for a 1-form whose kernel *is* integrable? Unfortunately, the answer is still no. Let ω be the 1-form on the solid torus whose kernel consists of the planes tangent to the foliation pictured in Figure 3 (This is called the *Reeb foliation* of the solid torus). The surfaces of this foliation spiral continually outward. So if we try to pick some number of "sample" surfaces then they will "bunch up" near the boundary torus. This would seem to indicate that if we wanted to integrate ω over any path that cut through the solid torus then we should get an infinite answer, since such a path would intersect our "sample" surfaces an infinite number of times. However, we can certainly find a 1-form ω for which this is not the case.

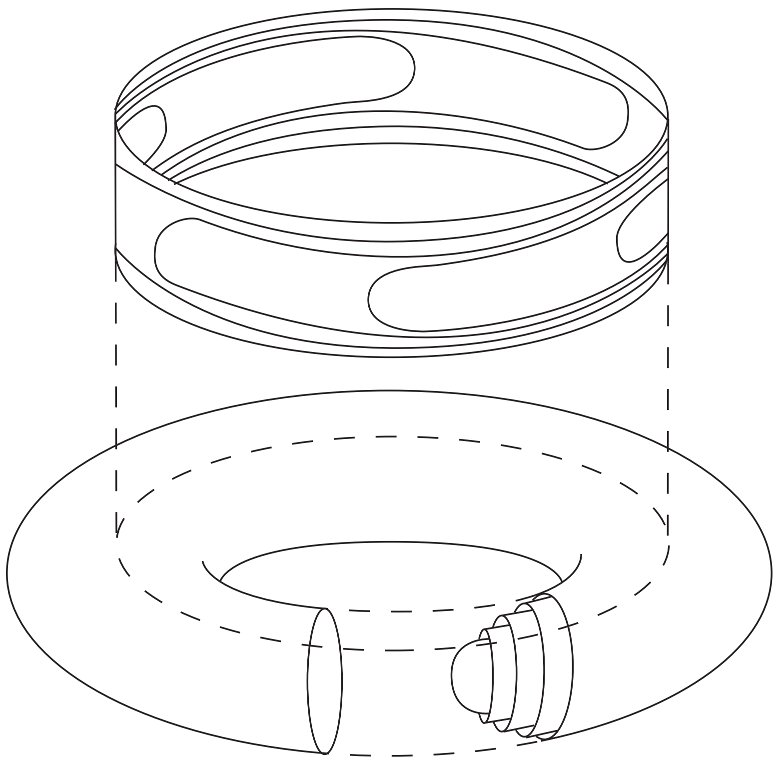


FIGURE 3. The Reeb foliation of the solid torus.

We don't want to end this section on such a down note. Although it is not in general valid to visualize a 1-form as a sample collection of surfaces from a foliation, we *can* visualize it as a plane field. For example, Figure 1 is a pretty good depiction of the 1-form $x\ dy + dz$. All that we have pictured there is a few evenly spaced elements of it's kernel, but this is enough. To get a rough idea of the value of $\int\limits_C x\ dy + dz$ we can just count the number of (transverse) intersections of the planes pictured with C. So, for example, if C is a curve whose tangents are always contained in one of these planes (a so called *Legendrian* curve) then $\int\limits_C x\ dy + dz$ will be zero. Inspection of the picture reveals that examples of such curves are the lines parallel to the x-axis.

EXERCISE 6.3. Show that if C is a line parallel to the x-axis then $\int\limits_C x\ dy + dz = 0$.

CHAPTER 7

Manifolds

1. Forms on subsets of $\mathbb{R}^n$

The goal of this chapter is to slowly work up to defining forms in a much more general setting than just on $\mathbb{R}^n$. One reason for this is because Stokes' Theorem actually tells us that forms on $\mathbb{R}^n$ just aren't very interesting. For example, let's examine how a 1-form, ω, on $\mathbb{R}^2$, for which $d\omega = 0$ (*i.e.* ω is *closed*), integrates over an 1-chain, C, such that $\partial C = \emptyset$ (*i.e.* C is *closed*). It is a basic result of Topology that any such 1-chain bounds a 2-chain, D. Hence, $\int\limits_C \omega = \int\limits_D d\omega = 0$!!

Fortunately, there is no reason to restrict ourselves to differential forms which are defined on all of $\mathbb{R}^n$. Instead, we can simply consider forms which are defined on subsets, U, of $\mathbb{R}^n$. For technical reasons, we will always assume such subsets are *open* (*i.e.* for each $p \in U$, there is an ϵ such that $\{q \in \mathbb{R}^n | d(p,q) < \epsilon\} \subset U$). In this case, $TU_p = T\mathbb{R}^n_p$. Since a differential n-form is nothing more than a choice of n-form on $T\mathbb{R}^n_p$, for each p (with some condition about differentiability), it makes sense to talk about a differential form on U.

Example 7.1.

$$\omega_0 = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

is a differential 1-form on $\mathbb{R}^2 - (0,0)$. □

Exercise 7.1. Show that $d\omega_0 = 0$.

Exercise 7.2. Let C be the unit circle, oriented counter-clockwise. Show that $\int\limits_C \omega_0 = 2\pi$. *Hint:* Let $\omega' = -y\ dx + x\ dy$. Note that on C, $\omega_0 = \omega'$.

If C is any closed 1-chain in $\mathbb{R}^2 - (0,0)$, then the quantity $\frac{1}{2\pi}\int\limits_C \omega_0$ is called the *winding number* of C, since it computes the number of times C winds around the origin.

EXERCISE 7.3. Let x^+ denote the positive x-axis in $\mathbb{R}^2 - (0,0)$, and let C be any closed 1-chain. Suppose V_p is a basis vector of TC_p which agrees with the orientation of C at p. A *positive* (resp. *negative*) intersection of C with x^+ is one where V_p has a component which points "up" (resp. "down"). Assume all intersections of C with x^+ are either positive or negative. Let P denote the number of positive ones, and N the number of negative ones. Show that $\frac{1}{2\pi}\int\limits_C \omega_0 = P - N$. *Hint:* Use the Generalized Stokes' Theorem.

2. Forms on Parameterized Subsets

Recall that at each point a differential from is simply an alternating, multilinear map on a tangent plane. So all we need to define a differential form on a more general space is a well defined tangent space. One case in which this happens is when we have a parameterized subset of $\mathbb{R}^m$. Let $\phi : U \subset \mathbb{R}^n \to M \subset \mathbb{R}^m$ be a (one-to-one) parameterization of M. Then recall that TM_p is defined to be the span of the partial derivatives of ϕ at $\phi^{-1}(p)$, and is an n-dimensional Euclidean space, regardless of the point, p. Hence, we say the *dimension* of M is n.

A differential k-form on M is simply an alternating, multilinear, real-valued function on TM_p, for each $p \in M$, *which varies differentiably with* p. In other words, a differential k-form on M is a whole family of k-forms, each one acting on TM_p, for different points, p. It is not so easy to say precisely what we mean when we say the form varies in a differentiable way with p. Fortunately, we have already introduced the tools necessary to do this. Let's say that ω is a family of k-forms, defined on TM_p, for each $p \in M$. Then $\phi^*\omega$ is a family of k-forms, defined on $T\mathbb{R}^n_{\phi^{-1}(p)}$, for each $p \in M$. We say that ω is a *differentiable* k-form on M, if $\phi^*\omega$ is a differentiable family on U.

This definition illustrates an important technique which is used often when dealing with differential forms on manifolds. Rather than working in M directly we use the map ϕ^* to translate problems about forms on M into problems about forms on U. These are nice because we already know how to work with forms which are defined on open subsets of R^n. We will have much more to say about this later.

EXAMPLE 7.2. The infinitely long cylinder, L, of radius 1, centered along the z-axis, is given by the parameterization, $\phi(a,b) = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, \ln\sqrt{a^2+b^2}\right)$, whose domain is $\mathbb{R}^2 - (0,0)$. We can use ϕ^* to solve any problem about forms on L, by translating it back to a problem about forms on U. $\square$

EXERCISE 7.4. Consider the 1-form, $\tau' = -y\ dx + x\ dy$, on $\mathbb{R}^3$. In particular, this form acts on vectors in TL_p, where L is the cylinder of the previous example, and p is any point in L. Let τ be the restriction of τ' to vectors in TL_p. So, τ is a 1-form on L. Compute $\phi^*\tau$. What does this tell you that τ measures?

If ω is a k-form on M, then what do we mean by $d\omega$? Whatever the definition, we clearly want $d\phi^*\omega = \phi^* d\omega$. So why don't we use this to define $d\omega$? After all, we know what $d\phi^*\omega$ is, since $\phi^*\omega$ is a form on $\mathbb{R}^n$. Recall that $D\phi_p$ is a map from $T\mathbb{R}^n_p$ to $T\mathbb{R}^m_p$. However, if we restrict the range to TM_p, then $D\phi_p$ is 1-1, so it makes sense to refer to $D\phi_p^{-1}$. We now define

$$d\omega(V_p^1, ..., V_p^{k+1}) = d\phi^*\omega(D\phi_p^{-1}(V_p^1), ..., D\phi_p^{-1}(V_p^{k+1}))$$

EXERCISE 7.5. If τ' and τ are the 1-forms on $\mathbb{R}^3$ and L, respectively, defined in the previous section, compute $d\tau'$ and $d\tau$. *Answer:* $d\tau' = 2\ dx \wedge dy$ and $d\tau = 0$.

3. Forms on quotients of $\mathbb{R}^n$ (optional)

This section requires some knowledge of Topology and Algebra. It is not essential for the flow of the text.

While we are on the subject of differential forms on subsets of $\mathbb{R}^n$, there is a very common construction of a topological space for which it is very easy to define what we mean by a differential form. Let's look again at the cylinder, L, of the previous section. One way to construct L is to start with the plane, $\mathbb{R}^2$, and "roll it up." More technically, we can consider the map, $\mu(\theta, z) = (\cos\theta, \sin\theta, z)$. In general this is a many-to-one map, so it is not a parameterization, in the strict sense. To remedy this, one might try and restrict the domain of μ to $\{(\theta, z) \in \mathbb{R}^2 | 0 \le \theta < 2\pi\}$, however this set is not open.

Note that for each point, $(\theta, z) \in \mathbb{R}^2$, $D\mu$ is a 1-1 map from $T\mathbb{R}^2_{(\theta,z)}$ to $TL_{\mu(\theta,z)}$. This is all we need in order for $\mu^*\tau$ to make sense, where τ is the form on L defined in the previous section.

EXERCISE 7.6. Show that $\mu^*\tau = d\theta$.

In this case, we say that μ is a *covering map*, $\mathbb{R}^2$ is a *cover* of L, and $d\theta$ is the *lift* of τ to $\mathbb{R}^2$.

EXERCISE 7.7. Suppose ω_0 is the 1-form on $\mathbb{R}^2$ which we used to define the winding number. Let $\mu(r, \theta) = (r\cos\theta, r\sin\theta)$. Let $U = \{(r, \theta) | r > 0\}$. Then $\mu : U \to \{\mathbb{R}^2 - (0, 0)\}$ is a covering map. Hence, there is a 1-1 correspondence between a quotient of U and $\mathbb{R}^2 - (0, 0)$. Compute the lift of ω_0 to U.

Let's go back to the cylinder, L. Another way to look at things is to ask: How can we recover L from the θ-z plane? The answer is to view L as a quotient space. Let's put an equivalence relation, R, on the points of $\mathbb{R}^2$: $(\theta_1, z_1) \sim (\theta_2, z_2)$ if and only if $z_1 = z_2$, and $\theta_1 - \theta_2 = 2n\pi$, for some $n \in \mathbb{Z}$. We will denote the quotient of $\mathbb{R}^2$ under this relation as $\mathbb{R}^2/R$. μ now induces a 1-1 map, $\bar{\mu}$, from $\mathbb{R}^2/R$ onto L. Hence, these two spaces are homeomorphic.

Let's suppose now that we have a form on U, an open subset of $\mathbb{R}^n$, and we would like to know when it *descends* to a form on a quotient of U. Clearly, if we begin with the lift of a form, then it will descend. Let's try and see why. In general, if $\mu : U \subset \mathbb{R}^n \to M \subset \mathbb{R}^m$ is a many-to-one map, differentiable at each point of U, then the sets, $\{\mu^{-1}(p)\}$, partition U. Hence, we can form the quotient space, U/μ^{-1}, under this partition. For each $x \in \mu^{-1}(p)$, $D\mu_x$ is a 1-1 map from TU_x to TM_p, and hence, $D\mu_x^{-1}$ is well defined. If x and y are both in $\mu^{-1}(p)$, then $D\mu_y^{-1} \circ D\mu_x$ is a 1-1 map from TU_x to TU_y. We will denote this map as $D\mu_{xy}$. We say a k-form, ω, on $\mathbb{R}^n$ *descends* to a k-form on U/μ^{-1} if and only if $\omega(V_x^1, ..., V_x^k) = \omega(D\mu_{xy}(V_x^1), ..., D\mu_{xy}(V_x^1))$, for all $x, y \in U$ such that $\mu(x) = \mu(y)$.

EXERCISE 7.8. If τ is a differential k-form on M, then $\mu^*\tau$ (the lift of τ) is a differential k-form on U which descends to a differential k-form on U/μ^{-1}.

Now suppose that we have a k-form, $\tilde{\omega}$, on U which descends to a k-form on U/μ^{-1}, where $\mu : U \subset \mathbb{R}^n \to M \subset \mathbb{R}^m$ is a covering map. How can we get a k-form on M? As we have already remarked, $\bar{\mu} : U/\mu^{-1} \to M$ is a 1-1 map. Hence, we can use it to *push forward* the form, ω. In other words, we can define a k-form on M as follows: Given k vectors in TM_p, we first choose a point, $x \in \mu^{-1}(p)$. We then define

$$\mu_* \omega(V_p^1, ..., V_p^k) = \tilde{\omega}(D\mu_x^{-1}(V_p^1), ..., D\mu_x^{-1}(V_p^k))$$

It follows from the fact that $\tilde{\omega}$ descends to a form on U/μ^{-1} that it did not matter which point, x, we chose in $\mu^{-1}(p)$. Note that although μ is not 1-1, $D\mu_x$ is, so $D\mu_x^{-1}$ makes sense.

If we begin with a form on U, there is a slightly more general construction of a form on a quotient of U, which does not require the use of a covering map. Let Γ be a group of transformations of U. We say Γ acts *discretely* if for each $p \in U$, there exists an $\epsilon > 0$ such that $N_\epsilon(p)$ does not contain $\gamma(p)$, for any non-identity element, $\gamma \in \Gamma$. If Γ acts discretely, then we can form the quotient of U by Γ, denoted U/Γ, as follows: $p \sim q$ if there exists $\gamma \in \Gamma$ such that $\gamma(p) = q$ (The fact that Γ acts discretely is what guarantees a "nice" topology on U/Γ).

Now, suppose $\tilde{\omega}$ is a k-form on U. We say $\tilde{\omega}$ descends to a k-form, ω, on U/Γ, if and only if $\tilde{\omega}(V_p^1, ..., V_p^k) = \tilde{\omega}(D\gamma(V_p^1), ..., D\gamma(V_p^1))$, for all $\gamma \in \Gamma$.

Now that we have decided what a form on a quotient of U is, we still have to define n-chains, and what we mean by integration of n-forms over n-chains. We say an n-chain, $\tilde{C} \subset U$, descends to an n-chain, $C \subset U/\Gamma$, if $\gamma(\tilde{C}) = \tilde{C}$, for all $\gamma \in \Gamma$. The n-chains of U/Γ are simply those which are descendants of n-chains in U.

Integration is a little more subtle. For this we need the concept of a *fundamental domain for* Γ. This is nothing more than a closed subset of U, whose interior does not contain two equivalent points. Furthermore, for each equivalence class, there is at least one representative in a fundamental domain. Here is one way to construct a fundamental domain: First, choose a point, $p \in U$. Now, let $D = \{q \in U | d(p, q)) \leq d(\gamma(p), q)$, for all $\gamma \in \Gamma\}$.

Now, let $\tilde{C}$ be an n-chain on U which descends to an n-chain, C, on U/Γ, and let $\tilde{\omega}$ be an n-form that descends to an n-form, ω. Let D be a fundamental domain for Γ in U. Then we define

$$\int\limits_C \omega \equiv \int\limits_{\tilde{C} \cap D} \tilde{\omega}$$

Technical note: In general, this definition is invariant of which point was chosen in the construction of the fundamental domain, D. However, a VERY unlucky choice will result in $\tilde{C} \cap D \subset \partial D$, which could give a different answer for the above integral. Fortunately, it can be shown that the set of such "unlucky" points has *measure zero*. That is, if we were to choose the point at random, then the odds of picking an "unlucky" point are 0%. Very unlucky indeed!

EXAMPLE 7.3. Suppose Γ is the group of transformations of the plane generated by $(x, y) \to (x+1, y)$, and $(x, y) \to (x, y+1)$. The space $\mathbb{R}^2/\Gamma$ is often denoted T^2, and referred to as a *torus*. Topologists often visualize the torus as the surface of a donut. A fundamental domain for Γ is the unit square in $\mathbb{R}^2$. The 1-form, dx, on $\mathbb{R}^2$ descends to a 1-form on T^2. Integration of this form over a closed 1-chain, $C \subset T^2$, counts the number of times C wraps around the "hole" of the donut. □

4. Defining Manifolds

As we have already remarked, a differential n-form on $\mathbb{R}^m$ is just an n-form on $T_p\mathbb{R}^m$, for each point $p \in \mathbb{R}^m$, along with some condition about how the form varies in a differentiable way as p varies. All we need to define a form on a space other than $\mathbb{R}^m$ is some notion of a tangent space at every point. We call such a space a *manifold*. In addition, we insist that at each point of a manifold the tangent space has the same dimension, n, which we then say is the dimension of the manifold.

How do we guarantee that a given subset of $\mathbb{R}^m$ is a manifold? Recall that we defined the tangent space to be the span of some partial derivatives of a parameterization. However, insisting that the whole manifold is capable of being parameterized is very restrictive. Instead, we only insist that every point of a manifold lies in a subset that can be parameterized. Hence, if M is an n-manifold in $\mathbb{R}^m$ then there is a set of open subsets, $\{U_i\} \subset \mathbb{R}^n$, and a set of differentiable maps, $\{\phi_i : U_i \to M\}$, such that $\{\phi_i(U_i)\}$ is a *cover* of M. (That is, for each point, $p \in M$, there is an i, and a point, $q \in U_i$, such that $\phi_i(q) = p$).

EXAMPLE 7.4. S^1, the unit circle in R^2, is a 1-manifold. Let $U_i = (-1, 1)$, for $i = 1, 2, 3, 4$, $\phi_1(t) = (t, \sqrt{1-t^2})$, $\phi_2(t) = (t, -\sqrt{1-t^2})$, $\phi_3(t) = (\sqrt{1-t^2}, t)$, and $\phi_4(t) = (-\sqrt{1-t^2}, t)$. Then $\{\phi_i(U_i)\}$ is certainly a cover of S^1 with the desired properties. □

EXERCISE 7.9. Show that S^2, the unit sphere in $\mathbb{R}^3$, is a 2-manifold.

5. Differential Forms on Manifolds

Basically, the definition of a differential n-form on an m-manifold is the same as the definition of an n-form on a subset of $\mathbb{R}^m$ which was given by a single parameterization. First and foremost it is just an n-form on T_pM, for each $p \in M$.

Let's say M is an m-manifold. Then we know there is a set of open sets, $\{U_i\} \subset \mathbb{R}^m$, and a set of differentiable maps, $\{\phi_i : U_i \to M\}$, such that $\{\phi_i(U_i)\}$ covers M. Now, let's say that ω is a family of n-forms, defined on T_pM, for each $p \in M$. Then we say that the family, ω, is a *differentiable* n-form on M if $\phi_i^*\omega$ is a differentiable n-form on U_i, for each i.

EXAMPLE 7.5. In the previous section we saw how S^1, the unit circle in $\mathbb{R}^2$, is a 1-manifold. If (x, y) is a point of S^1, then $TS^1_{(x,y)}$ is given by the equation $dy = -\frac{x}{y}dx$, in $T\mathbb{R}^2_{(x,y)}$, as long as $y \neq 0$. If $y = 0$, then $TS^1_{(x,y)}$ is given by $dx = 0$. We define a 1-form on S^1, $\omega = -y\ dx + x\ dy$. (Actually, ω is a 1-form on all of $\mathbb{R}^2$. To get a 1-form on just S^1, we restrict the domain of ω to the tangent lines to S^1.) To check that this is really a differential form, we must compute all pull-backs:

$$\phi_1^*\omega = \frac{-1}{\sqrt{1-t^2}}\ dt \quad , \quad \phi_2^*\omega = \frac{1}{\sqrt{1-t^2}}\ dt$$
$$\phi_3^*\omega = \frac{1}{\sqrt{1-t^2}}\ dt \quad , \quad \phi_4^*\omega = \frac{-1}{\sqrt{1-t^2}}\ dt$$

Since all of these are differentiable on $U_i = (-1, 1)$, we can say that ω is a differential form on S^1. □

We now move on to integration of n-chains on manifolds. The definition of an n-chain is no different than before; it is just a formal linear combination of n-cells in

M. Let's suppose that C is an n-chain in M, and ω is an n-form. Then how do we define $\int\limits_C \omega$? If C lies entirely in $\phi_i(U_i)$, for some i, then we could define the value of this integral to be the value of $\int\limits_{\phi_i^{-1}(C)} \phi_i^*\omega$. But it may be that part of C lies in both $\phi_i(U_i)$ and $\phi_j(U_j)$. If we define $\int\limits_C \omega$ to be the sum of the two integrals we get when we pull-back ω under ϕ_i and ϕ_j, then we end up "double counting" the integral of ω on $C \cap \phi_i(U_i) \cap \phi_j(U_j)$. Somehow, as we move from $\phi_i(U_i)$ into $\phi_j(U_j)$, we want the effect of the pull-back of ω under ϕ_i to "fade out", and the effect of the pull back under ϕ_j to "fade in". This is accomplished by a *partition of unity.*

The technical definition of a *partition of unity subordinate to the cover,* $\{\phi_i(U_i)\}$ is a set of differentiable functions, $f_i : M \to [0,1]$, such that $f_i(p) = 0$ if $p \notin \phi_i(U_i)$, and $\sum\limits_i f_i(p) = 1$, for all $p \in M$. We refer the reader to any book on differential topology for a proof of the existence of partitions of unity.

We are now ready to give the full definition of the integral of an n-form on an n-chain in an m-manifold.

$$\int\limits_C \omega \equiv \sum_i \int\limits_{\phi_i^{-1}(C)} \phi_i^*(f_i\omega)$$

We start with a very simple example to illustrate the use of a partition of unity.

EXAMPLE 7.6. Let M be the manifold which is the interval $(1,10) \subset \mathbb{R}$. Let $U_i = (i, i+2)$, for $i = 1, ..., 8$. Let $\phi_i : U_i \to M$ be the identity map. Let $\{f_i\}$ be a partition of unity, subordinate to the cover, $\{\phi_i(U_i)\}$. Let ω be a 1-form on M. Finally, let C be the 1-chain which consists of the single 1-cell, $[2,8]$. Then we have

$$\int\limits_C \omega \equiv \sum_{i=1}^{8} \int\limits_{\phi_i^{-1}(C)} \phi_i^*(f_i\omega) = \sum_{i=1}^{8} \int\limits_C f_i\omega = \int\limits_C \sum_{i=1}^{8}(f_i\omega) = \int\limits_C \left(\sum_{i=1}^{8} f_i\right)\omega = \int\limits_C \omega$$

as one would expect! □

Example 7.7. Let S^1, U_i, ϕ_i, and ω be defined as in Examples 7.4 and 7.5. A partition of unity subordinate to the cover $\{\phi_i(U_i)\}$ is as follows:

$$f_1(x,y) = \begin{cases} y^2 & y \geq 0 \\ 0 & y < 0 \end{cases}, \quad f_2(x,y) = \begin{cases} 0 & y > 0 \\ y^2 & y \leq 0 \end{cases}$$

$$f_3(x,y) = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}, \quad f_4(x,y) = \begin{cases} 0 & x > 0 \\ x^2 & x \leq 0 \end{cases}$$

(Check this!) Let $\mu : [0,\pi] \to S^1$ be defined by $\mu(\theta) = (\cos\theta, \sin\theta)$. Then the image of μ is a 1-cell, σ, in S^1. Let's integrate ω over σ:

$$\begin{aligned}
\int\limits_{\sigma} \omega &\equiv \sum_{i=1}^{4} \int\limits_{\phi_i^{-1}(\sigma)} \phi_i^*(f_i\omega) \\
&= \int\limits_{-(-1,1)} -\sqrt{1-t^2}\, dt + \ 0 \ + \int\limits_{[0,1)} \sqrt{1-t^2}\, dt + \int\limits_{-[0,1)} -\sqrt{1-t^2}\, dt \\
&= \int\limits_{-1}^{1} \sqrt{1-t^2}\, dt + 2\int\limits_{0}^{1} \sqrt{1-t^2}\, dt \\
&= \pi
\end{aligned}$$

CAUTION: Beware of orientations! □

6. Application: DeRham cohomology

One of the predominant uses of differential forms is to give *global* information about manifolds. Consider the space $\mathbb{R}^2 - (0,0)$, as in Example 7.1. Near every point of this space we can find an open set which is identical to an open set around a point of $\mathbb{R}^2$. This means that all of the *local* information in $\mathbb{R}^2 - (0,0)$ is the same as the local information in $\mathbb{R}^2$. The fact that the origin is missing is a global property.

For the purposes of detecting global properties certain forms are interesting, and certain forms are completely uninteresting. We will spend some time discussing both. The interesting forms are the ones whose derivative is zero. Such forms are said to be *closed*. An example of a closed 1-form was ω_0, from Example 7.1 of the previous chapter. For now let's just focus on closed 1-forms so that you can keep this example in mind.

Let's look at what happens when we integrate a closed 1-form ω_0 over a 1-chain C such that $\partial C = 0$ (*i.e.* C is a *closed* 1-chain). If C bounds a disk D then Stokes' theorem says

$$\int_C \omega_0 = \int_D d\omega_0 = \int_D 0 = 0$$

In a sufficiently small region of every manifold every closed 1-chain bounds a disk. So integrating closed 1-forms on "small" 1-chains gives us no information. In other words, closed 1-forms give no local information.

Suppose now that we have a closed 1-form ω_0 and a closed 1-chain C such that $\int_C \omega_0 \neq 0$. Then we know C does not bound a disk. The fact that there exists such a 1-chain is global information. This is why we say that the closed forms are the ones that are interesting, from the point of view of detecting only global information.

Now let's suppose that we have a 1-form ω_1 that is the derivative of a 0-form f (*i.e.* $\omega_1 = df$). We say such a form is *exact*. Again, let C be a closed 1-chain. Let's pick two points, p and q, on C. Then $C = C_1 + C_2$, where C_1 goes from p to q and C_2 goes from q back to p. Now let's do a quick computation:

$$\begin{aligned}
\int_C \omega_1 &= \int_{C_1+C_2} \omega_1 \\
&= \int_{C_1} \omega_1 + \int_{C_2} \omega_1 \\
&= \int_{C_1} df + \int_{C_2} df \\
&= \int_{p-q} f + \int_{q-p} f \\
&= 0
\end{aligned}$$

So integrating an exact form over a closed 1-chain always gives zero. This is why we say the exact forms are completely uninteresting. Unfortunately, in Exercise 4.8 we learned that every exact form is also closed. This is a problem, since this would say that all of the completely uninteresting forms are also interesting! To remedy this we define an *equivalence relation*.

We pause here for a moment to explain what this means. An equivalence relation is just a way of taking one set and creating a new set by declaring certain objects in the original set to be "the same". You do this kind of thing every time you tell time. To construct the clock numbers you start with the integers and declare two to be "the same" if they differ by a multiple of 12. So $10 + 3 = 13$, but 13 is the same as 1, so if it's now 10 o'clock then in 3 hours it will 1 o'clock.

We play the same trick for differential forms. We will restrict ourselves to the closed forms, but we will consider two of them to be "the same" if their difference is an exact form. The set which we end up with is called the *cohomology* of the manifold in question. For example, if we start with the closed 1-forms then, after our equivalence relation, we end up with the set which we will call H^1, or the *first* cohomology (see Figure 1).

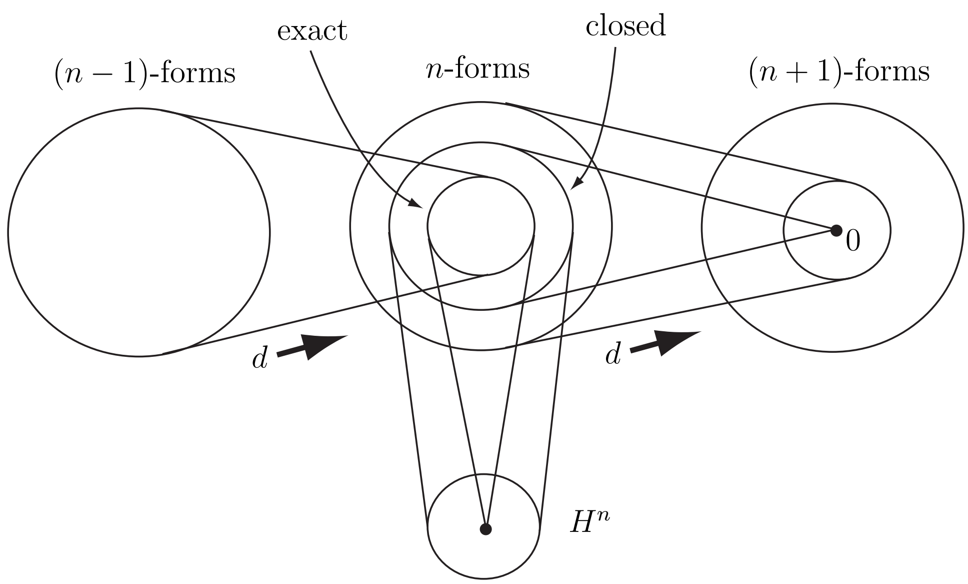


FIGURE 1. Defining H^n.

Note that the difference between an exact form and the form which always returns the number zero is an exact form. Hence, every exact form is equivalent to 0 in H^n, as in the figure.

For each n the set H^n contains a lot of information about the manifold in question. For example, if $H^1 \cong \mathbb{R}^1$ (as it turns out is the case for $\mathbb{R}^2 - (0,0)$) then this tells us that the manifold has one “hole” in it. Studying manifolds via cohomology is the topic of a field of mathematics called *Algebraic Topology*.

APPENDIX A

Non-linear forms

1. Surface area and arc length

Now that we have developed some proficiency with differential forms, let's see what *else* we can integrate. A basic assumption that we used to come up with the definition of an n-form was the fact that at every point it is a *linear* function which "eats" n vectors and returns a number. But what about the non-linear functions?

Let's go all the way back to Section 5 of Chapter 1. There we decided that the integral of a function f over a surface R in $\mathbb{R}^3$ should look something like:

$$\int_R f(\phi(r,\theta))Area\left[\frac{\partial \phi}{\partial r}(r,\theta), \frac{\partial \phi}{\partial \theta}(r,\theta)\right] dr d\theta \tag{6}$$

At the heart of the integrand is the *Area* function, which takes two vectors and returns the area of the parallelogram that it spans. The 2-form $dx \wedge dy$ does this for two vectors in $T_p\mathbb{R}^2$. In $T_p\mathbb{R}^3$ the right function is the following:

$$Area(V_p^1, V_p^2) = \sqrt{(dy \wedge dz)^2 + (dx \wedge dz)^2 + (dx \wedge dy)^2}$$

(The reader may recognize this as the magnitude of the cross product between V_p^1 and V_p^2.) This is clearly non-linear!

Example A.1. The area of the parallelogram spanned by $\langle 1,1,0\rangle$ and $\langle 1,2,3\rangle$ can be computed as follows:

$$\begin{aligned} Area(\langle 1,1,0\rangle,\langle 1,2,3\rangle) &= \sqrt{\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}^2 + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix}^2 + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}^2} \\ &= \sqrt{3^2+3^2+1^2} \\ &= \sqrt{19} \end{aligned}$$

□

The thing that makes (linear) differential forms so useful is the Generalized Stokes theorem. We don't have anything like this for non-linear forms, but that's not to say that they don't have their uses. For example, there is no differential 2-form on $\mathbb{R}^3$ that one can integrate over arbitrary surfaces to find their surface area. For that we would need to compute the following:

$$Area(R) = \int_S \sqrt{(dy\wedge dz)^2 + (dx\wedge dz)^2 + (dx\wedge dy)^2}$$

For relatively simple surfaces this integrand can be evaluated by hand. Integrals such as this play a particularly important role in certain applied problems. For example, if one were to dip a loop of bent wire into a soap film, the resulting surface would be the one of minimal area. Before one can even begin to figure out what surface this is for a given piece of wire, one must be able to know how to compute the area of an arbitrary surface, as above.

Example A.2. We compute the surface area of a sphere of radius r in $\mathbb{R}^3$. A parameterization is given by

$$\Phi(\theta,\phi) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

Now we compute:

$Area\left(\frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial\phi}\right)$

$= Area\left(\langle -r\sin\phi\sin\theta, r\sin\phi\cos\theta, 0\rangle, \langle r\cos\phi\cos\theta, r\cos\phi\sin\theta, -r\sin\phi\rangle\right)$

$= \sqrt{(-r^2\sin^2\phi\cos\theta)^2 + (r^2\sin^2\phi\sin\theta)^2 + (-r^2\sin\phi\cos\phi)^2}$

$= r\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}$

$= r\sin\phi$

And so the desired area is given by

$$\begin{aligned} & \int\limits_S Area\left(\frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial\phi}\right) d\theta\ d\phi \\ = & \int\limits_0^{\pi}\int\limits_0^{2\pi} r\sin\phi\ d\theta\ d\phi \\ = & \ 4\pi r \end{aligned}$$

$\square$

EXERCISE A.1. Compute the surface area of a sphere of radius r in $\mathbb{R}^3$ using the parameterizations

$$\Phi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, \pm\sqrt{r^2-\rho^2})$$

for the top and bottom halves, where $0 \le \rho \le r$ and $0 \le \theta \le 2\pi$.

Let's now go back to Equation 6. Classically this is called a *surface integral*. It might be a little clearer how to compute such an integral if we write it as follows:

$$\int\limits_R f(x,y,z)\ dS = \int\limits_R f(x,y,z)\sqrt{(dy\wedge dz)^2 + (dx\wedge dz)^2 + (dx\wedge dy)^2}$$

Lengths are very similar to areas. In calculus you learn that if you have a curve C in the plane, for example, parameterized by the function $\phi(t) = (x(t), y(t))$, where $a \le t \le b$, then its length is given by

$$Length(C) = \int\limits_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

We can write this without making reference to the parameterization by employing a non-linear 1-form:

$$Length(C) = \int\limits_C \sqrt{dx^2 + dy^2}$$

Finally, we can define what is classically called a *line integral* as follows:

$$\oint_C f(x,y)\ ds = \int_C f(x,y)\sqrt{dx^2 + dy^2}$$