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PRINCIPLES AND APPLICATIONS OF TENSOR ANALYSIS

By MATTHEW S. SMITH

Principles and Applications of Tensor Analysis presents a detailed step-by-step development of tensor notation and theory, advanced concepts in tensor analysis, differential geometry, and analytical mechanics in tensor form.

CHAPTER 1. BASIC TENSOR THEORY

This chapter emphasizes the important concepts of relative tensors, covariant tensors, contravariant tensors, mixed tensors, metric tensors, base vectors, and Kronecker deltas.

CHARTER 2. CHRISTOFFEL SYMBOLS AND THE COVARIANT DERIVATIVES

Includes the Christoffel symbols, covariant derivatives, intrinsic derivatives, and Laplace's equation. Because of the many important applications, Laplace's equation is included in tensor form, using generalized curvilinear coordinates. Some of the more important applications in potential theory are included as illustrative examples.

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Principles & Applications of Tensor Analysis

by Matthew S. Smith

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Preface

Since the extensive use of tensors by Einstein, important applications in other fields, such as differential geometry, classical mechanics, and the theory of elasticity, have evolved. Therefore, the ability to understand and apply tensors is a definite advantage to engineers, physicists, and applied mathematicians. Tensor equations are powerful analysis tools that give added insight to the understanding of the fundamental laws of physics and engineering.

This text, using practical examples throughout, systematically explains and demonstrates basic tensor theory and its applications. The book is divided into four chapters. The first two chapters present basic tensor theory, including Christoffel symbols, covariant derivatives, and Laplace's equation. The third chapter includes the Riemann-Christoffel tensors, and the application of tensor analysis to several topics in differential geometry. The latter were selected because they have numerous important applications in the general theory of relativity, analytical dynamics, and the theory of elasticity.

The fourth chapter presents the application of tensor analysis to some of the most important concepts in classical and relativistic mechanics. The section on classical dynamics includes Lagrange's equations of motion and a solution of the two-body problems. The equations for the two-body problem are written as a geodesic to give the student a feeling for dynamical trajectories treated as geodesics prior to the study of the general theory of relativity.

The sections on relativistic mechanics include a discussion of space-time for the special theory of relativity, the Lorentz transformations, and the relativistic equations for momentum, energy, and force. The text ends with a short discussion of curved space and Einstein's gravitational equations for the general theory of relativity.

It is hoped that an understanding of the material in this text will provide the student with the ability to solve a variety of problems using tensor analysis, as well as giving him an additional understanding of the related sciences.

MATTHEW S. SMITH

February, 1963

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Chapter I

Basic Tensor Theory

Chapter I

Basic Tensor Theory

Tensor analysis is the study of invariant objects, whose properties must be independent of the co-ordinate systems used to describe the objects. A tensor is represented by a set of functions called components. For an object to be a tensor it must be an invariant that transforms from one acceptable co-ordinate system to another by the tensor laws. These laws are explained in detail, with examples, in Chapter 1.

Several examples of tensors are velocity vectors, base vectors, metric coefficients for the length of a line, Gaussian curvature, and the Newtonian gravitation potential.

Many of the important differential equations for physics, engineering, and applied mathematics can also be written as tensors. Examples of differential equations that can be written in tensor form are Lagrange's equations of motion and Laplace's equation. When an equation is written in tensor form it is in a general form that applies to all admissible co-ordinate systems.

SUMMATION NOTATION

The summation notation used throughout this text will be demonstrated explicitly by the examples in the first chapter. The general summation will be of the type:

$$S = a_1x_1 + a_2x_2 + \ldots + a_nx_n \tag{1.0}$$

This type of summation is generally expressed in the calculus:

$$S = \sum_{i=1}^{n} a_i x^i \tag{1.1}$$

The superscripts on x are not powers; they are used to distinguish between the various x's. In rectangular cartesian co-ordinates and vector notation, Equation 1.1 would be:

$$S = \sum_{i=1}^{3} a_i x^i$$

where,

$x^1 = x, x^2 = y, x^3 = z$

$a_1 = i, a_2 = j, a_3 = k$

With this interpretation of Equation 1.1 and the specific values for a_i and x^i as noted, sum S would be:

$$S = ix + jy + kz$$

For additional simplification, Einstein dropped the $\sum$ in Equation 1.1 and the summation is then expressed:

$$S = a_ix^i \tag{1.2}$$

In conclusion, it is to be remembered that Equations 1.0, 1.1, and 1.2 are equivalent, and x^i expresses variables, not powers of x.

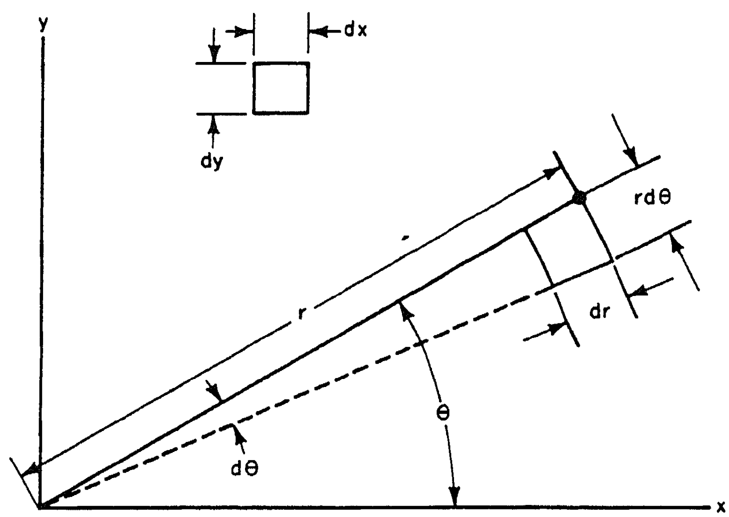
In the subsequent portions of the text, a superscript index will indicate a contravariant tensor, while a subscript index will indicate a covariant tensor.

The rank of a tensor is the sum of the covariant and contravariant indexes. This will be explained in detail in the section of this chapter on higher rank and mixed tensors.

RELATIVE TENSORS

The term relative tensor is used to describe scalars that are transformed from one co-ordinate system to another by means of the functional determinate known as the Jacobian. To illustrate this concept, the differential increment of area (dA) is indicated in Fig. 1-1.

In cartesian co-ordinates (x, y) it is:



$$dA = dx\,dy$$

In polar co-ordinates (r, Θ) it is:

$$dA = r\,d\Theta\,dr$$

Now, the connection between the x, y cartesian co-ordinates and the r, Θ polar co-ordinates is:

$$x = r\cos\Theta$$

$$y = r\sin\Theta$$

$$r = (x^2 + y^2)^{1/2}$$

$$\Theta = \tan^{-1}\frac{y}{x}$$

The Jacobian of the cartesian co-ordinates with respect to the polar co-ordinates is formed from the following partial derivatives:

$$\frac{\partial x}{\partial r} = \cos\Theta, \quad \frac{\partial x}{\partial \Theta} = -r\sin\Theta$$

$$\frac{\partial y}{\partial r} = \sin\Theta, \quad \frac{\partial y}{\partial \Theta} = r\cos\Theta$$

This set of partial derivatives are used to form the following Jacobian:

$$\begin{vmatrix} \cos\Theta & -r\sin\Theta \\ \sin\Theta & r\cos\Theta \end{vmatrix} = r\,(\cos^2\Theta + \sin^2\Theta) = r \qquad (1.3)$$

with respect to the cartesian co-ordinates is formed from the following partial derivatives:

$$\frac{\partial r}{\partial x} = \cos \Theta, \ \frac{\partial r}{\partial y} = \sin \Theta$$

$$\frac{\partial \Theta}{\partial x} = -\frac{1}{r}\sin \Theta, \ \frac{\partial \Theta}{\partial y} = \frac{1}{r}\cos \Theta$$

This set of partial derivatives are used to form the following Jacobian:

$$\begin{vmatrix} \cos \Theta & \sin \Theta \\ -\frac{1}{r}\sin \Theta & \frac{1}{r}\cos \Theta \end{vmatrix} = \frac{1}{r}\ (\cos^2\Theta + \sin^2\Theta) = \frac{1}{r} \qquad (1.4)$$

Now, returning to the expression for differential area in cartesian co-ordinates and polar co-ordinates, the following equation can be written:

$$S\ dx\ dy = \overline{S}\ dr\ d\Theta$$

where,

$S = 1$

$\overline{S} = r$

S and $\overline{S}$ are called relative tensors, as they are related by the equations:

$$\left| \partial x^i \right|^n$$

$$\bar{S} = \left| \frac{\partial x^i}{\partial y^j} \right|^n S \qquad (1.6)$$

Exponent n in Equations 1.5 and 1.6 is used to determine the weight of a relative scalar. The examples in this section are relative scalars having a weight equaling one; therefore, $n = 1$. An absolute scalar has a weight of zero; i.e., $n = 0$.

To illustrate Equation 1.5, we use the values:

$$\bar{S} = r$$

$$\left| \frac{\partial y^i}{\partial x^j} \right| = \begin{vmatrix} \cos \Theta & \sin \Theta \\ -\frac{1}{r} \sin \Theta & \frac{1}{r} \cos \Theta \end{vmatrix} = \frac{1}{r}$$

y^i ranges from $i = 1$ to $i = 2$

x^j ranges from $j = 1$ to $j = 2$

$$y^1 = r, \quad x^1 = x$$

$$y^2 = \Theta, \quad x^2 = y$$

$$S = \frac{1}{r}(r) = 1 \qquad (1.7)$$

Equation 1.7 is the desired result.

Now the notion of relative tensors can be extended to volumes and mass. To illustrate this concept, we start with the equation for an incremental mass in orthogonal cartesian coordinates.

$$dM = \rho \, dx \, dy \, dz \qquad (1.8)$$

Now the incremental mass in spherical co-ordinates is written in terms of relative tensor $\bar{S}$:

$$d\overline{M} = \overline{S}\,dr\,d\Phi\,d\Theta$$

$\overline{S}$ is evaluated by the relative tensor equation:

$$\overline{S} = \left|\frac{\partial x^i}{\partial y^j}\right| S \qquad (1.9)$$

In this example $S = \rho$, where ρ is called the scalar density.

$$x^1 = x,\ x^2 = y,\ x^3 = z$$

$$y^1 = r,\ y^2 = \Phi,\ y^3 = \Theta$$

The geometrical relationship between the cartesian co-ordinates and the spherical co-ordinates is indicated in Fig. 1-2. The corresponding mathematical relationship between the co-ordinates is:

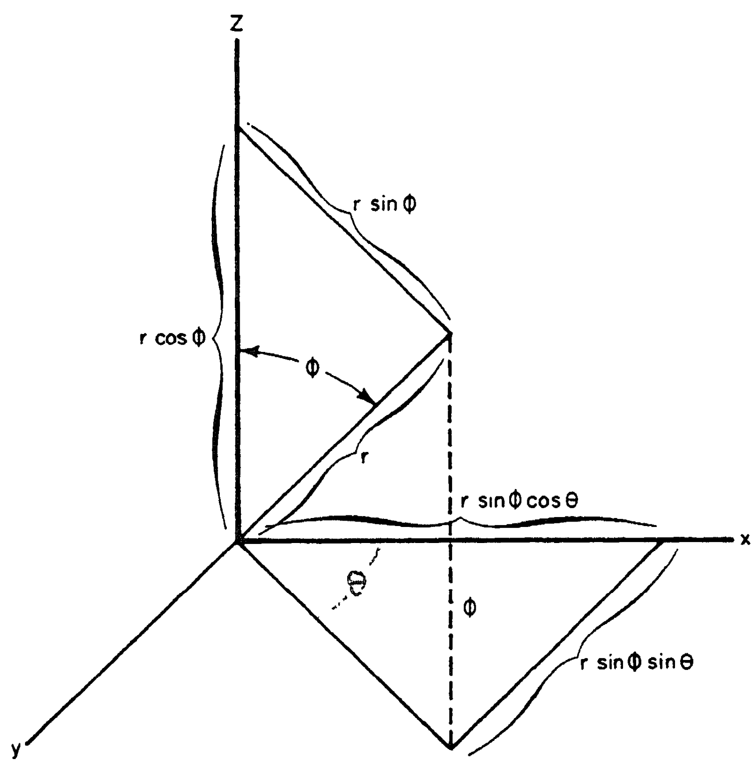


Fig. 1-2.

$$\left.\begin{aligned} x &= r \sin\Phi \cos\Theta \\ y &= r \sin\Phi \sin\Theta \\ z &= r \cos\Phi \end{aligned}\right\} \qquad (1.10)$$

The partial derivatives for the Jacobian $\left|\frac{\partial x^i}{\partial y^j}\right|$ are:

$$\frac{\partial x}{\partial r} = \sin\Phi \cos\Theta, \frac{\partial x}{\partial \Phi} = r \cos\Phi \cos\Theta, \frac{\partial x}{\partial \Theta} =$$

$$- r \sin\Phi \sin\Theta$$

$$\frac{\partial y}{\partial r} = \sin\Phi \sin\Theta, \frac{\partial y}{\partial \Phi} = r \cos\Phi \sin\Theta, \frac{\partial y}{\partial \Theta} =$$

$$r \sin\Phi \cos\Theta$$

$$\frac{\partial z}{\partial r} = \cos\Phi, \frac{\partial z}{\partial \Phi} = - r \sin\Phi, \frac{\partial z}{\partial \Theta} = 0$$

Using these values, the resultant determinate is:

$$\begin{vmatrix} \sin\Phi \cos\Theta & r \cos\Phi \cos\Theta & -r \sin\Phi \sin\Theta \\ \sin\Phi \sin\Theta & r \cos\Phi \sin\Theta & r \sin\Phi \cos\Theta \\ \cos\Phi & -r \sin\Phi & 0 \end{vmatrix} = r^2 \sin\Phi \qquad (1.11)$$

Now, Equation 1.6 can be evaluated:

$$\overline{S} = r^2 \sin\Phi\, \rho \qquad (1.12)$$

and the equation for $d\overline{M}$ is:

$$d\overline{M} = \rho\, r^2 \sin\Phi\, dr\, d\Phi\, d\Theta \qquad (1.13)$$

Equation 1.13 is the desired result for $d\overline{M}$. If the value for $d\overline{M}$ had been given initially in spherical co-ordinates, the corresponding value in cartesian co-ordinates could be found by the equation:

$$dM = S\, dx\, dy\, dz \qquad (1.14)$$

where,

$$S = \left|\frac{\partial y^i}{\partial x^j}\right| \overline{S} \qquad (1.5)$$

ADMISSIBLE TRANSFORMATIONS

From the previous examples, it has been demonstrated that relative tensors transform from one co-ordinate system to another by means of the functional determinate known as the Jacobian. Since a relative tensor is defined to be a function of the Jacobian, a necessary and sufficient condition for an admissible transformation of co-ordinates is that it is a member of a set in which the Jacobian does not vanish.

This condition is also necessary and sufficient for absolute tensors. Therefore, the set of all admissible transformations of co-ordinates form a group with nonvanishing Jacobians. If notation $|J|$ is used for the Jacobian, the definition for an admissible transformation of co-ordinates can be expressed:

$$|J| \neq 0 \qquad (1.15)$$

Another property of an admissible transformation is:

$$\left|\frac{\partial x^i}{\partial y^j}\right| \left|\frac{\partial y^i}{\partial x^j}\right| = 1 \qquad (1.16)$$

An example of Equation 1.16 can be found in Jacobian Equations 1.3 and 1.4.

$$(r)\left(\frac{1}{r}\right) = 1 \qquad (1.17)$$

N DIMENSIONAL SPACE

In general terms a co-ordinate system represents a one-to-one correspondence of a point or object with a set of numbers. To measure distance, we can use a rectangular cartesian co-ordinate system. This is called a metric manifold, or space of V_3.

Now, a space or manifold of N dimensions is expressed by the symbol V_N; and it is a co-ordinate system of N dimensions, if for each set of N numbers there is one corresponding point or object.

A sub space V_M where $M = N - 1$ is called a hypersurface. An example of hypersurface in Euclidean space is a plane. It is a hypersurface of V_2.

CONTRAVARIANT TENSORS

The prototype for contravariant tensors is the vector formed by taking the total differential of a variable in one co-ordinate system with respect to the variables in another admissible co-ordinate system. It is a contravariant tensor Rank 1. To present the concept, cartesian co-ordinates x and y; and polar co-ordinates r and Θ as indicated in Fig. 1-1 are used.

$$\left.\begin{aligned} x &= r\cos\Theta \\ y &= r\sin\Theta \end{aligned}\right\} \qquad (1.18)$$

Differential dx in terms of r and Θ is a contravariant tensor Rank 1.

$$dx = \frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial \Theta}\,d\Theta \qquad (1.19)$$

$$\frac{\partial x}{\partial r} = \cos\Theta$$

$$\frac{\partial x}{\partial \Theta} = -r\sin\Theta$$

If Equation 1.19 is divided by time differential dt, it becomes the familiar velocity vector.

$$\frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \Theta}\frac{d\Theta}{dt} \qquad (1.20)$$

Equation 1.20 is also a contravariant tensor Rank 1. The corresponding contravariant tensor equation for $\frac{dr}{dt}$ is:

$$\frac{dr}{dt} = \frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial y}\frac{dy}{dt} \qquad (1.21)$$

In tensor notation, a contravariant tensor, Rank 1, can be expressed:

$$B^i = \frac{\partial y^i}{\partial x^a}A^a \qquad (1.22)$$

Equation 1.22 is the contravariant law for tensors, Rank 1.

To illustrate Equation 1.22, we write the corresponding tensor notation for Equation 1.20.

$$B^1 = \frac{dx}{dt} \qquad (1.23)$$

$$\frac{\partial y^1}{\partial x^a} = \frac{\partial x}{\partial x^a}, \quad (a = 1, 2) \tag{1.24}$$

$$\left.\begin{aligned} x^1 &= r, \; A^1 = \frac{dr}{dt} \\ x^2 &= \Theta, \; A^2 = \frac{d\Theta}{dt} \end{aligned}\right\} \tag{1.25}$$

For $x = r \cos \Theta, \dfrac{\partial x}{\partial r} = \cos \Theta$, and $\dfrac{\partial x}{\partial \Theta} = -r \sin \Theta$

Equation 1.20 can be expressed:

$$\frac{dx}{dt} = \cos \Theta \frac{dr}{dt} - r \sin \Theta \frac{d\Theta}{dt} \tag{1.26}$$

The corresponding tensor equations for $\dfrac{dy}{dt}$ are:

$$\frac{dy}{dt} = \frac{\partial y}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial \Theta}\frac{d\Theta}{dt} \tag{1.27}$$

$$\frac{dy}{dt} = \sin \Theta \frac{dr}{dt} + r \cos \Theta \frac{d\Theta}{dt} \tag{1.28}$$

To find $\dfrac{dr}{dt}$ in terms of Equation 1.22, the terms in tensor notation are:

$$B^1 = \frac{dr}{dt}$$

$$\frac{\partial y^1}{\partial x^a} = \frac{\partial r}{\partial x^a}, \quad (a = 1, 2)$$

$$x^1 = x,\ A^1 = \frac{dx}{dt}$$

$$x^2 = y,\ A^2 = \frac{dy}{dt}$$

$$r = (x^2 + y^2)^{1/2} \tag{1.29}$$

$$\frac{\partial r}{\partial x} = \cos \Theta,\ \frac{\partial r}{\partial y} = \sin \Theta$$

With these values, tensor equation $B^i = \frac{\partial y^i}{\partial x^a} A^a$ gives the following result:

$$\frac{dr}{dt} = \cos \Theta \frac{dx}{dt} + \sin \Theta \frac{dy}{dt} \tag{1.30}$$

In summary, a velocity vector in Euclidean space has three components. These components form a contravariant tensor Rank 1. Each tensor component can be transformed to a new admissible co-ordinate system by using the corresponding partial derivative of the new co-ordinate system with respect to the old co-ordinate system.

COVARIANT TENSORS

The prototype for the covariant tensor is the covariant vector formed by taking the partial derivative of an absolute scalar. The chain rule for partial derivatives gives the correct pattern for this type of transformation. It is a covariant Tensor Rank 1.

To present the concept, Newtonian gravitational potential V in cartesian co-ordinates and polar co-ordinates is used. Using Fig. 1-3 as a guide, the equation for gravitational potential V is:

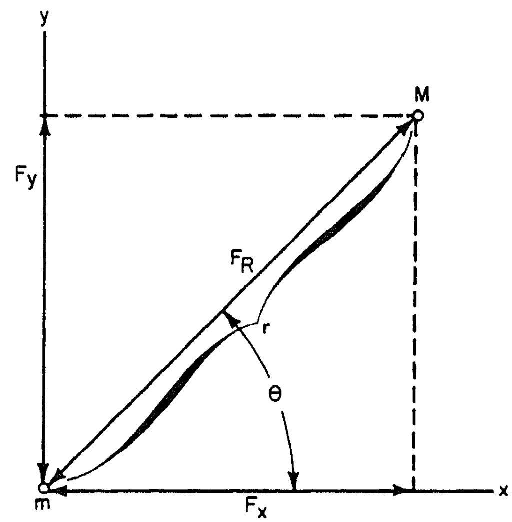


Fig. 1-3.

$$V = -\frac{\mu Mm}{r}, \mu = \frac{(\text{Length})^4}{(\text{Force})\ (\text{Time})^4} \tag{1.31}$$

$$\left.\begin{aligned} r &= (x^2 + y^2)^{1/2} \\ \Theta &= \tan^{-1}\frac{y}{x} \end{aligned}\right\} \tag{1.32}$$

Force F_x in the direction of the x axis is:

$$F_x = \frac{\partial V}{\partial x} \tag{1.33}$$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \Theta}\frac{\partial \Theta}{\partial x} \tag{1.34}$$

For a particle M in orbit about a central body m, $\frac{\partial V}{\partial \Theta} =$ o. Now, using the values $\frac{\partial V}{\partial r} = \frac{\mu Mm}{r^2}$, and $\frac{\partial r}{\partial x} = \cos\Theta$ in Equation 1.34 yields the result:

$$\frac{\partial V}{\partial x} = \frac{\mu Mm}{r^2} \cos \Theta = F_x \qquad (1.35)$$

In tensor notation a covariant tensor, Rank 1 can be expressed:

$$B_i = \frac{\partial x^a}{\partial y^i} A_a \qquad (1.36)$$

Equation 1.36 is the covariant law for tensors, Rank 1. Using the tensor notation in Equation 1.36, the force F_y in the direction of the y axis is calculated as follows:

$$B_2 = \frac{\partial V}{\partial y}, \; y^2 = y \qquad (1.37)$$

$$\left.\begin{aligned} x^1 = r, \quad A_1 &= \frac{\partial V}{\partial r} \\ x^2 = \Theta, \quad A_2 &= \frac{\partial V}{\partial \Theta} \\ \frac{\partial r}{\partial y} = \sin \Theta, \quad \frac{\partial \Theta}{\partial y} &= \frac{1}{r} \cos \Theta \end{aligned}\right\} \qquad (1.38)$$

As $\dfrac{\partial V}{\partial \Theta} = 0$, F_y has the value:

$$\frac{\partial V}{\partial y} = \frac{\mu Mm}{r^2} \sin \Theta \qquad (1.39)$$

HIGHER RANK AND MIXED TENSORS

The equation for a covariant tensor of Rank 2 is:

$$B_{ij} = \frac{\partial x^{\alpha}}{\partial y^{i}} \frac{\partial x^{\beta}}{\partial y^{j}} A_{\alpha\beta} \tag{1.40}$$

The equation for a contravariant tensor, Rank 2, is:

$$B^{ij} = \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial y^{j}}{\partial x^{\beta}} A^{\alpha\beta} \tag{1.41}$$

The equation for mixed tensor, Rank 2, is:

$$B^{i}{}_{j} = \frac{\partial x^{\alpha}}{\partial y^{i}} \frac{\partial y^{j}}{\partial x^{\beta}} A^{\alpha}_{\beta} \tag{1.42}$$

Equation 1.42 is covariant, Rank 1 and contravariant, Rank 1.

For a tensor of Rank N, where N is any positive number, components A would be multiplied by N partial derivatives.

METRIC TENSORS AND THE LINE ELEMENT

Metric tensors assume a position of prime importance in tensor analysis. They are used in the Christoffel symbols, covariant derivatives, intrinsic derivatives, and all of the differential equations in tensor form that are used in this text.

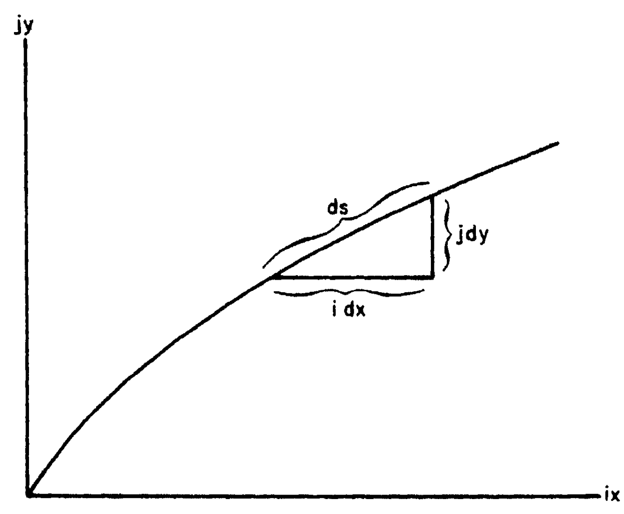


Fig. 1-4.

Metric tensors g_{ij} of the line element ds^2 are covariant tensors, Rank 2. To present the concept of the metric tensors, line element ds is written as a vector in cartesian and polar co-ordinates. Using Fig. 1-4 as a reference, line element ds in cartesian co-ordinates is:

$$ds = i\,dx + j\,dy \tag{1.43}$$

The length of the line element squared is found by squaring both sides of Equation 1.43 and taking the dot product of the right-hand side.

$$ds^2 = i \cdot i\,dx^2 + 2i \cdot j\,dx\,dy + j \cdot j\,dy^2 \tag{1.44}$$

Using an orthogonal co-ordinate system and elementary vector analysis, the dot products are:

$$\left.\begin{aligned} i \cdot i &= 1 \\ i \cdot j &= 0 \\ j \cdot j &= 1 \end{aligned}\right\} \tag{1.45}$$

Therefore, the square of the line element is:

$$ds^2 = dx^2 + dy^2 \tag{1.46}$$

Now, if the square of the line element is computed in polar co-ordinates:

$$\left.\begin{aligned} ix &= i\,r\cos\Theta \\ jy &= j\,r\sin\Theta \end{aligned}\right\} \tag{1.47}$$

The result is:

$$ds^2 = r^2\,d\Theta^2 + dr^2 \tag{1.48}$$

Equation 1.48 is computed using the relation:

$$ds = (jr \cos \Theta - ir \sin \Theta)\, d\Theta + (i \cos \Theta + j \sin \Theta)\, dr \tag{1.49}$$

The procedure is to square both sides of Equation 1.49 and take the dot product of the right-hand side, using the results in Equation 1.45.

Now, using the metric tensors g_{ij} in Equation 1.46 yields the results:

$$\left.\begin{aligned} g_{11} &= 1, \quad g_{22} = 1 \\ ds^2 &= g_{11} dx^2 + g_{22}\, dy^2 \end{aligned}\right\} \tag{1.50}$$

In a similar fashion, metric tensors $\bar{g}_{ij}$ in Equation 1.48 yields the results:

$$\left.\begin{aligned} \bar{g}_{11} &= 1, \bar{g}_{22} = r^2 \\ ds^2 &= \bar{g}_{11}\, dr^2 + \bar{g}_{22}\, d\Theta^2 \end{aligned}\right\} \tag{1.51}$$

Coefficients $\bar{g}_{11}$, $\bar{g}_{22}$ and g_{11}, g_{22} are covariant tensors, Rank 2.

To illustrate the transformation of the metric tensors, we will use Equation 1.40 for the transformation of covariant tensors, Rank 2.

$$B_{ij} = \frac{\partial x^{\alpha}}{\partial y^{i}} \frac{\partial x^{\beta}}{\partial y^{j}} A_{\alpha\beta} \tag{1.40}$$

Now, to determine $\bar{g}_{22}$ and $\bar{g}_{11}$ in Equations 1.51 in terms of g_{11} and g_{22} in Equations 1.50, the corresponding terms in Equation 1.40 are:

$$B_{ij} = B_{22} = \bar{g}_{22} \tag{1.52}$$

$$\bar{g}_{22} = \frac{\partial x}{\partial \Theta}\frac{\partial x}{\partial \Theta} g_{11} + \frac{\partial y}{\partial \Theta}\frac{\partial y}{\partial \Theta} g_{22} \tag{1.53}$$

$$g_{11} = 1, g_{22} = 1$$

$$\frac{\partial x}{\partial \Theta} = -r \sin \Theta, \quad \frac{\partial y}{\partial \Theta} = r \cos \Theta$$

$$\bar{g}_{22} = r^2 \ (\sin^2 \Theta + \cos^2 \Theta) = r^2 \tag{1.54}$$

To calculate $\bar{g}_{11}$ in terms of g_{11} and g_{22} we repeat the procedure with the values:

$$B_{ij} = B_{11} = \bar{g}_{11} \tag{1.55}$$

$$\bar{g}_{11} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} g_{11} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} g_{22} \tag{1.56}$$

$$\frac{\partial x}{\partial r} = \cos \Theta, \quad \frac{\partial y}{\partial r} = \sin \Theta$$

$$\bar{g}_{11} = \cos^2 \Theta + \sin^2 \Theta = 1 \tag{1.57}$$

Using the computed values for $\bar{g}_{11}$ and $\bar{g}_{22}$, as indicated in Equations 1.57 and 1.54, we get the familiar result for the line element in polar co-ordinates r and Θ:

$$ds^2 = dr^2 + r^2 d\Theta^2 \tag{1.58}$$

The covariant metric tensor Rank 2 is called the covariant fundamental tensor. The corresponding contravariant metric tensor is called the contravariant fundamental tensor. For orthogonal co-ordinate systems, the contravariant metric tensor is found by the equation:

$$g^{ij} = \frac{1}{g_{ij}} \tag{1.59}$$

The contravariant metric tensor g^{ij} transforms by the equation:

$$B^{ij} = \frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\beta} A^{\alpha\beta} \tag{1.41}$$

To illustrate the procedure, the metric tensors from the following equation are used:

$$ds^2 = g_{11}\, dx^2 + g_{22}\, dy^2$$

$$= \bar{g}_{11}\, dr^2 + \bar{g}_{22}\, d\Theta^2$$

Using Equation 1.59, the corresponding contravariant metric tensors are calculated:

$$g^{11} = \frac{1}{g_{11}} = 1, \; g^{22} = \frac{1}{g_{22}} = 1$$

$$\bar{g}^{11} = \frac{1}{\bar{g}_{11}} = 1, \; \bar{g}^{22} = \frac{1}{\bar{g}_{22}} = \frac{1}{r^2}$$

To calculate $\bar{g}^{22}$ from g^{11} and g^{22}, Equation 1.41 is used.

$$\bar{g}^{22} = g^{11} \frac{\partial \Theta}{\partial x} \frac{\partial \Theta}{\partial x} + g^{22} \frac{\partial \Theta}{\partial y} \frac{\partial \Theta}{\partial y} \tag{1.60}$$

$$\bar{g}^{22} = \frac{1}{r^2} (\sin^2 \Theta + \cos^2 \Theta)$$

$$\bar{g}^{22} = \frac{1}{r^2} \tag{1.61}$$

This is the desired result.

The determinate of the metric tensors is denoted by symbol g. For Euclidian space, it is:

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} \tag{1.62}$$

If the co-ordinate system is orthogonal, the determinate reduces to:

$$g = \begin{vmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{vmatrix} \tag{1.63}$$

The general formula to find the contravariant metric tensors is:

$$g^{ij} = \frac{G^{ij}}{g}$$

G^{ij} is a cofactor of elements g^{ij} in the determinate g.

For all of the examples in this text, orthogonal co-ordinate systems are used. This simplifies the evaluation of the metric tensors and the tensor equations.

BASE VECTORS

A base vector for a given co-ordinate system is used to determine a differential change in a position vector with respect to a variable. To clarify and illustrate this statement, a set of orthogonal cartesian co-ordinates x, y, and z, with unit vectors i, j, and k, and position vector P are indicated in Fig. 1-5.

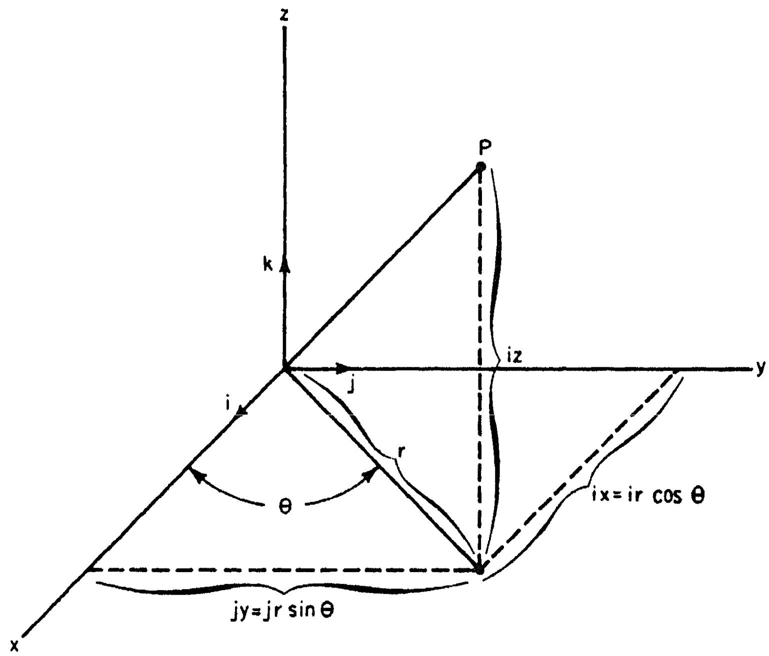


Fig. 1-5.

$$P = ix + jy + kz \tag{1.65}$$

The corresponding base vectors are:

$$\left.\begin{aligned} \frac{\partial P}{\partial x} &= e_1 = i \\ \frac{\partial P}{\partial y} &= e_2 = j \\ \frac{\partial P}{\partial z} &= e_3 = k \end{aligned}\right\} \tag{1.66}$$

Now, the total differential of position vector P is:

$$\left.\begin{aligned} dP &= \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \\ dP &= e_1 dx + e_2 dy + e_3 dz \\ dP &= i\,dx + j\,dy + k\,dz \end{aligned}\right\} \tag{1.67}$$

Now, the same position vector P expressed in cylindrical co-ordinates r, Θ, and z is:

$$P = i\, r \cos\Theta + j\, r \sin\Theta + k\, z \qquad (1.68)$$

The corresponding base vectors in cylindrical co-ordinates are:

$$\left.\begin{aligned} \frac{\partial P}{\partial r} &= \bar{e}_1 = i \cos\Theta + j \sin\Theta \\ \frac{\partial P}{\partial \Theta} &= \bar{e}_2 = -i\, r \sin\Theta + j\, r \cos\Theta \\ \frac{\partial P}{\partial z} &= \bar{e}_3 = k \end{aligned}\right\} \qquad (1.69)$$

Now, the total differential of position vector P in cylindrical co-ordinates is:

$$\left.\begin{aligned} dP &= \frac{\partial P}{\partial r}\, dr + \frac{\partial P}{\partial \Theta}\, d\Theta + \frac{\partial P}{\partial z}\, dz \\ dP &= \bar{e}_1\, dr + \bar{e}_2\, d\Theta + \bar{e}_3\, dz \\ dP &= (i \cos\Theta + j \sin\Theta)\, dr + \\ &\quad (-i\, r \sin\Theta + j\, r \cos\Theta)\, d\Theta + k\, dz \end{aligned}\right\} \qquad (1.70)$$

Base vectors are covariant tensors, Rank 1. To illustrate this relation e_1 in Equations 1.66 will be found in terms of $\bar{e}_1$ and $\bar{e}_2$ in Equations 1.69.

$$e_1 = \bar{e}_1 \frac{\partial r}{\partial x} + \bar{e}_2 \frac{\partial \Theta}{\partial x} \qquad (1.71)$$

$$\begin{aligned} e_1 = {} & (i \cos\Theta + j \sin\Theta) \cos\Theta \\ & + (-i\, r \sin\Theta + j\, r \cos\Theta) \left(-\frac{1}{r} \sin\Theta\right) \end{aligned}$$

$$e_1 = i$$

The metric tensors can be expressed as the dot product of the base vectors. To illustrate the procedure, the following examples are presented:

$$\left.\begin{aligned} g_{11} &= e_1 \bullet e_1 = i \bullet i = 1 \\ g_{22} &= e_2 \bullet e_2 = j \bullet j = 1 \end{aligned}\right\} \qquad (1.72)$$

$$\left.\begin{aligned} \bar{g}_{11} = \bar{e}_1 \bullet \bar{e}_1 &= (i \cos \Theta + j \sin \Theta) \bullet \\ &\quad (i \cos \Theta + j \sin \Theta) = 1 \\ \bar{g}_{22} = \bar{e}_2 \bullet \bar{e}_2 &= (-i\, r \sin \Theta + j\, r \cos \Theta) \bullet \\ &\quad (-i\, r \sin \Theta + j\, r \cos \Theta) = r^2 \end{aligned}\right\} \quad (1.73)$$

Therefore, in any admissible orthogonal co-ordinate system, the metric tensors are:

$$g_{ii} = e_i \bullet e_i \qquad (1.74)$$

ASSOCIATED TENSORS AND THE INNER PRODUCT

Either the product of a covariant metric tensor and a contravariant tensor, or a contravariant metric tensor and a covariant tensor are called associated tensors. An example of an associated tensor is the product of a contravariant metric tensor and a covariant base vector. The result is a contravariant base vector. To illustrate this fact, polar co-ordinates r and Θ, and cartesian co-ordinates x and y are used. In the polar co-ordinates, the metric tensors and covariant base vectors are:

$$\left.\begin{aligned} \bar{g}_{11} &= 1, \quad \bar{g}^{11} = 1 \\ \bar{g}_{22} &= r^2, \quad \bar{g}^{22} = \frac{1}{r^2} \\ \bar{e}_1 &= i \cos \Theta + j \sin \Theta \\ \bar{e}_2 &= -i\, r \sin \Theta + j\, r \cos \Theta \end{aligned}\right\} \tag{1.75}$$

Now, to find the contravariant base vectors, the following equation is used:

$$\bar{e}^i = \bar{g}^{ii}\, \bar{e}_i \tag{1.76}$$

Using Equation 1.76 and the values in Equations 1.75, base vectors $\bar{e}^1$ and $\bar{e}^2$ are:

$$\left.\begin{aligned} \bar{e}^1 &= \bar{g}^{11}\ (i \cos \Theta + j \sin \Theta) \\ &= i \cos \Theta + j \sin \Theta \\ \bar{e}^2 &= \bar{g}^{22}\ (-i\, r \sin \Theta + j\, r \cos \Theta) \\ &= -\frac{i}{r} \sin \Theta + \frac{j}{r} \cos \Theta \end{aligned}\right\} \tag{1.77}$$

To show that $\bar{e}^1$ and $\bar{e}^2$ are indeed contravariant base vectors, Rank 1, they will be used to compute $e^1 = i$.

$$\left.\begin{aligned} e^1 &= \left(i \cos \Theta + j \sin \Theta\right) \frac{\partial x}{\partial r} \\ &\quad + \left(-\frac{i}{r} \sin \Theta + \frac{j}{r} \cos \Theta\right) \frac{\partial x}{\partial \Theta} \\ e^1 &= i\ (\cos^2 \Theta + \sin^2 \Theta) \\ &\quad + j \sin \Theta \cos \Theta - j \cos \Theta \sin \Theta \\ e^1 &= i \end{aligned}\right\} \tag{1.78}$$

From Equation 1.76 and the following calculations it can be seen that the contravariant metric tensor, Rank 2, and the covariant base vector, Rank 1, form a contravariant base vector or tensor, Rank 1. Now, the corresponding equation for a covariant base vector in terms of a contravariant base vector and a covariant metric tensor is:

$$\bar{e}_i = \bar{g}_{ii}\,\bar{e}^i \tag{1.79}$$

In repetition, associated tensors are formed by taking the inner product of any covariant and contravariant tensors. The inner product is the product of a covariant component, Rank P and the corresponding contravariant component, Rank Q. This product forms a new tensor covariant, Rank P-1, and contravariant, Rank Q-1.

Another example of an inner product is the product of covariant and contravariant metric tensors. Using the metric tensors in Equations 1.75, the product $g_{11}\, g^{11} = 1$ will be calculated to demonstrate that $g_{11}\, g^{11}$ is a mixed tensor, covariant Rank 1, and contravariant Rank 1.

$$g_{11}\, g^{11} = \bar{g}_{11}\, \bar{g}^{11} \frac{\partial r}{\partial x} \frac{\partial x}{\partial r} + \bar{g}_{22}\, \bar{g}^{22} \frac{\partial \Theta}{\partial x} \frac{\partial x}{\partial \Theta} \tag{1.80}$$

$$\bar{g}_{11}\, \bar{g}^{11} = (1)\ (1) = 1$$

$$\bar{g}_{22}\, \bar{g}^{22} = (r^2) \left(\frac{1}{r^2}\right) = 1$$

$$\frac{\partial r}{\partial x} = \cos \Theta, \frac{\partial x}{\partial r} = \cos \Theta$$

$$\frac{\partial \Theta}{\partial x} = -\frac{1}{r} \sin \Theta,\ \frac{\partial x}{\partial \Theta} = -r \sin \Theta$$

$$g_{11}\, g^{11} = 1\ (\cos^2 \Theta + \sin^2 \Theta) = 1 \tag{1.81}$$

KRONECKER DELTAS

The Kronecker Delta is defined by the equations:

$$\left.\begin{aligned} \delta^{k}_{j} &= 1, \text{ if } j = k \\ \delta^{k}_{j} &= 0, \text{ if } j \neq k \end{aligned}\right\} \qquad (1.82)$$

An example of a Kronecker Delta is the equation:

$$\frac{\partial x^k}{\partial y^i}\frac{\partial y^i}{\partial x^j} = \delta^{k}_{j} \qquad (1.83)$$

To illustrate Equations 1.82 and 1.83, the relation between cartesian co-ordinates x and y, and polar co-ordinates r and Θ is used.

First, we let $k = j = 1$, and $i = 1, 2$

$$\delta^{1}_{1} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial x}{\partial \Theta}\frac{\partial \Theta}{\partial x}$$

$$\delta^{1}_{1} = \cos^2 \Theta + \sin^2 \Theta = 1$$

Next, we let $k = j = 2$, and $i = 1, 2$

$$\delta^{2}_{2} = \frac{\partial y}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial y}{\partial \Theta}\frac{\partial \Theta}{\partial y}$$

$$\delta^{2}_{2} = \sin^2 \Theta + \cos^2 \Theta = 1$$

The Kronecker Delta is a mixed tensor, Rank 2, whose components in any other admissible co-ordinate system again form a Kronecker Delta.

Chapter II

Christoffel Symbols and the Covariant Derivatives

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Christoffel Symbols and the Covariant Derivatives

CHRISTOFFEL SYMBOLS

Christoffel symbols are certain combinations of the partial derivatives of fundamental tensors g_{ij}, and g^{ij}. The Christoffel symbols are not tensors, but they are used to form tensor equations, called covariant derivatives, and intrinsic derivatives. The derivation of the Christoffel symbols and these important resulting tensor equations will be presented in this chapter.

In the following section, Christoffel symbols of the first the second kind will be derived. The Christoffel symbols of the first kind will be derived for a three dimensional co-ordinate system. Three indices, i, j, and k (each ranging from one to three), will be used.

We start with the equation for covariant metric tensor g_{ij} in terms of base vectors ε_i, ε_j, ε_k.

$$g_{ij} = \varepsilon_i \cdot \varepsilon_j \tag{2.0}$$

Now, the partial derivative of the fundamental metric tensor can be expressed:

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial \varepsilon_i}{\partial x^k} \bullet \varepsilon_j + \frac{\partial \varepsilon_j}{\partial x^k} \bullet \varepsilon_i \qquad (2.1)$$

Permutating indices i, j, and k in Equation 2.1 we get the following two equations:

$$\frac{\partial g_{ik}}{\partial x^j} = \frac{\partial \varepsilon_i}{\partial x^j} \bullet \varepsilon_k + \frac{\partial \varepsilon_k}{\partial x^j} \bullet \varepsilon_i \qquad (2.2)$$

$$\frac{\partial g_{jk}}{\partial x^i} = \frac{\partial \varepsilon_j}{\partial x^i} \bullet \varepsilon_k + \frac{\partial \varepsilon_k}{\partial x^i} \bullet \varepsilon_j \qquad (2.3)$$

The next step is to add Equations 2.2 and 2.3; then subtract Equation 2.1 from the sum. For this summation we will make use of the following relations:

$$\frac{\partial \varepsilon_i}{\partial x^j} = \frac{\partial \varepsilon_j}{\partial x^i} \qquad (2.4)$$

Equation 2.4 is true for the following conditions:

$$\varepsilon_i = \frac{\partial \bar{r}}{\partial x^i}, \text{ and } \varepsilon_j = \frac{\partial \bar{r}}{\partial x^j} \qquad (2.5)$$

The term $\bar{r}$ in Equation 2.5 is a position vector.

Now, the corresponding expressions for $\frac{\partial \varepsilon_i}{\partial x^j}$ and $\frac{\partial \varepsilon_j}{\partial x^i}$

written in terms of position vector $\bar{r}$ are:

$$\frac{\partial \varepsilon_i}{\partial x^j} = \frac{\partial^2 \bar{r}}{\partial x^j\, x^i}, \text{ and } \frac{\partial \varepsilon_j}{\partial x^i} = \frac{\partial^2 \bar{r}}{\partial x^i\, x^j}$$

To complete the proof, we need condition:

$$\frac{\partial^2 \bar{r}}{\partial x^j x^i} = \frac{\partial^2 \bar{r}}{\partial x^i x^j} \qquad (2.6)$$

Making use of Equation 2.4, the sum of the Equations (2.2 + 2.3 – 2.1) can be expressed:

$$\frac{\partial \varepsilon_i}{\partial x^j} \bullet \varepsilon_k = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right] \qquad (2.7)$$

Equation 2.7 is a Christoffel symbol of the first kind. A notation used in tensor analysis for Christoffel symbols of the first kind is [i j, k].

Using this notation, the equation is:

$$[i\,j, k] = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right] \qquad (2.8)$$

In Euclidean cartesion co-ordinates, all Christoffel symbols are zero as metric tensors g_{ij} are constants. To illustrate Christoffel symbols of the first kind that are not zero, polar co-ordinates r and Θ are used. The corresponding metric tensors are:

$$g_{11} = 1, \; g_{22} = r^2$$

The variables x^1, x^2, and x^3 are:

$$x^1 = r, \; x^2 = \Theta, \; x^3 = 0$$

For these values, the Christoffel symbols of the first kind are:

$$[1\,1, 1] = 0, \; [2\,2, 2] = 0$$

$$[1\,2, 1] = o, \quad [2\,1, 2] = r$$

$$[1\,1, 2] = o, \quad [2\,2, 1] = -r$$

$$[1\,2, 2] = r, \quad [2\,1, 1] = o$$

Now, to derive the formula for Christoffel symbols of the second kind in a three dimensional co-ordinate system, we start with Equation 2.7. From using Equation 2.7 and the notation in Equation 2.8, the following equation can be written, where the three indices i, j, and k range from one to three as in the previous derivations.

$$\frac{\partial \varepsilon_i}{\partial x^j} = [ij, k]\ \varepsilon^k \tag{2.9}$$

If the dot product of ε_k is made with both sides of Equation 2.9, and $\varepsilon^k \bullet \varepsilon_k = 1$, Equation 2.9 will reduce to Equation 2.8. Now, if the dot product of ε^a is made with both sides of Equation 2.9, it will give the formula for the Christoffel symbols of the second kind.

$$\frac{\partial \varepsilon_i}{\partial x^j} \bullet \varepsilon^a = [ij, k]\ \varepsilon^k \bullet \varepsilon^a \tag{2.10}$$

The dot products $\varepsilon^k \bullet \varepsilon^a$ are the contravariant metric tensors. A notation used in tensor analysis for Christoffel symbols of the second kind is: $\left\{ \begin{matrix} \alpha \\ i\,j \end{matrix} \right\}$.

Using this notation and the result $g^{ka} = \varepsilon^k \bullet \varepsilon^a$, Christoffel symbols of the second kind can be expressed:

$$\left\{ \begin{matrix} a \\ i\,j \end{matrix} \right\} = g^{ka}\,[ij, k] = \frac{g^{ka}}{2}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{g_{ij}}{\partial x^k}\right] \tag{2.11}$$

The Christoffel symbols of the second kind for polar co-ordinates r and Θ in terms of Equation 2.11 are:

$$\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} = 0, \quad \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} = 0$$

$$\left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\} = 0, \quad \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} = \frac{1}{r}$$

$$\left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} = 0, \quad \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} = \frac{1}{r}$$

$$\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = -r, \quad \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} = 0$$

TRANSFORMATION OF CHRISTOFFEL SYMBOLS

As initially noted in the preceding section, the Christoffel symbols are not tensors. To demonstrate this fact, the following transformations are included. The transformation equations of the metric tensors in a y and an x co-ordinate system can be expressed:

$$\bar{g}_{ij} = \frac{\partial x^{\alpha}}{\partial y^{i}} \frac{\partial x^{\beta}}{\partial y^{j}} g_{\alpha\beta} \tag{2.12}$$

g_{ij} are the metric tensors in the y co-ordinate system and $g_{\alpha\beta}$ are the metric tensors in the x co-ordinate system. The Christoffel symbols of the first kind in the y co-ordinate system will be expressed:

$$[i\,j, k]_y = \frac{1}{2}\left[\frac{\partial \bar{g}_{ik}}{\partial y^{j}} + \frac{\partial \bar{g}_{jk}}{\partial y^{i}} - \frac{\partial \bar{g}_{ij}}{\partial y^{k}} \right] \tag{2.13}$$

Now the partial derivatives of Equation 2.12 are:

$$\frac{\partial \bar{g}_{ij}}{\partial y^k} = g_{\alpha\beta}\left[\frac{\partial^2 x^\alpha}{\partial y^k \partial y^i}\frac{\partial x^\beta}{\partial y^j} + \frac{\partial^2 x^\beta}{\partial y^k \partial y^j}\frac{\partial x^\alpha}{\partial y^i}\right] + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \qquad (2.14)$$

For a three dimensional orthogonal co-ordinate system, Equation 2.14 represents nine partial derivatives as i, j, and k each range from one to three. These partial derivatives are:

$$\frac{\partial \bar{g}_{11}}{\partial y^1}, \quad \frac{\partial \bar{g}_{22}}{\partial y^1}, \quad \frac{\partial \bar{g}_{33}}{\partial y^1}$$

$$\frac{\partial \bar{g}_{11}}{\partial y^2}, \quad \frac{\partial \bar{g}_{22}}{\partial y^2}, \quad \frac{\partial \bar{g}_{33}}{\partial y^2}$$

$$\frac{\partial \bar{g}_{11}}{\partial y^3}, \quad \frac{\partial \bar{g}_{22}}{\partial y^3}, \quad \frac{\partial \bar{g}_{33}}{\partial y^3}$$

Of course, any or all of the partial derivatives could be zero. If all three of the metric tensors were constants, all of the partial derivatives would be zero. As previously stated, this is the case in rectangular cartesian co-ordinates.

In Equation 2.14, the dummy indices α and β of the second term in the parenthesis can be interchanged because $g_{\alpha\beta} = g_{\beta\alpha}$. With the interchange of α and β, Equation 2.14 can now be expressed:

$$\frac{\partial \bar{g}_{ij}}{\partial y^k} = g_{\alpha\beta}\left[\frac{\partial^2 x^\alpha}{\partial y^k \partial y^i}\frac{\partial x^\beta}{\partial y^j} + \frac{\partial^2 x^\alpha}{\partial y^k \partial y^j}\frac{\partial x^\beta}{\partial y^i}\right] + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \qquad (2.15)$$

Equation 2.15 is the last term in the brackets on the right hand side of Equation 2.13. The first two terms in the brackets $\frac{\partial \bar{g}_{ik}}{\partial y^j}$ and $\frac{\partial \bar{g}_{jk}}{\partial y^i}$ can be found in terms of Equation 2.15 by permutation of indices. Referring back to Equation 2.12, changing indices and taking the partial derivatives as indicated in Equation 2.14, we get the following two equations:

$$\frac{\partial \bar{g}_{ik}}{\partial y^j} = g_{\alpha\gamma}\left[\frac{\partial^2 x^\alpha}{\partial y^j \partial y^i}\frac{\partial x^\gamma}{\partial y^k} + \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k}\frac{\partial x^\gamma}{\partial y^i}\right] + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\gamma}{\partial y^k}\frac{\partial x^\beta}{\partial y^j}\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} \tag{2.16}$$

$$\frac{\partial \bar{g}_{jk}}{\partial y^i} = g_{\beta\gamma}\left[\frac{\partial^2 x^\beta}{\partial y^i \partial y^j}\frac{\partial x^\gamma}{\partial y^k} + \frac{\partial^2 x^\beta}{\partial y^i \partial y^k}\frac{\partial x^\gamma}{\partial y^j}\right] + \frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \tag{2.17}$$

If Equations 2.16 and 2.17 are added and Equation 2.15 is subtracted from the sum, the following equation will result:

$$\Big[\, i\, j, k \,\Big]_y = \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k}\Big[\, \alpha\beta, \gamma \,\Big]_x + \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j}\frac{\partial x^\beta}{\partial y^k}\, g_{\alpha\beta} \tag{2.18}$$

The second term on the right-hand side of Equation 2.18 is one-half the sum of the first terms on the right-hand side of sum (2.16 + 2.17 − 2.15). If this term were zero, the Christoffel symbols of the first kind would be covariant tensors, Rank 3. But this is not the case and nonvanishing Christoffel symbols of the first kind are not tensors.

Now it will be demonstrated that Christoffel symbols of the second kind are not tensors. First, the Christoffel symbols of the second kind in a y co-ordinate system are derived from Equation 2.13 as follows:

$$\left\{ \begin{matrix} k \\ i\,j \end{matrix} \right\}_y = \bar{g}^{k\mu} \left[ij, \mu \right]_y \tag{2.19}$$

where,

$$g^{k\mu} = \frac{\partial y^k}{\partial x^\rho} \frac{\partial y^\mu}{\partial x^\delta} g^{\rho\delta} \tag{2.20}$$

Now multiply Equation 2.18 by the corresponding sides of Equation 2.20 and substitute $\mu = k$ in Equation 2.18 to obtain the following equations:

$$\left\{ \begin{matrix} k \\ i\,j \end{matrix} \right\}_y = \frac{\partial y^k}{\partial x^\rho} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} g^{\rho\gamma} \left[\alpha\beta, \gamma \right]_x$$

$$+ \frac{\partial y^k}{\partial x^\rho} \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} g^{\rho\beta}\, g_{\alpha\beta}$$

$$\left\{ \begin{matrix} k \\ i\,j \end{matrix} \right\}_y = \frac{\partial y^k}{\partial x^\rho} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \left\{ \begin{matrix} \rho \\ \alpha\,\beta \end{matrix} \right\}_x + \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} \frac{\partial y^k}{\partial x^\alpha} \tag{2.21}$$

Equation 2.21 can be solved for $\dfrac{\partial^2 x^\alpha}{\partial y^i \partial y^j}$. The first step is to multiply Equation 2.21 by $\dfrac{\partial x^m}{\partial y^k}$; then substitute $\gamma = k$ and sum:

$$\left\{ \begin{matrix} \gamma \\ i\,j \end{matrix} \right\}_y \frac{\partial x^m}{\partial y^\gamma} = \frac{\partial x^m}{\partial y^\gamma} \frac{\partial y^\gamma}{\partial x^\rho} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \left\{ \begin{matrix} \rho \\ \alpha\,\beta \end{matrix} \right\}_x + \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} \frac{\partial x^m}{\partial y^\gamma} \frac{\partial y^\gamma}{\partial x^\alpha} \tag{2.22}$$

As explained in Chapter 1, the products listed below are Kronecker deltas:

$$\frac{\partial x^m}{\partial y^\gamma} \frac{\partial y^\gamma}{\partial x^\rho} = \delta^m_\rho$$

$$\frac{\partial x^m}{\partial y^\gamma} \frac{\partial y^\gamma}{\partial x^\alpha} = \delta^m_\alpha$$

When the summation is made for $^m = {}^\rho$, and $^m = {}^\alpha$, both terms have a value of one; but all other values $^m = {}^\rho$, and $^m \neq {}^\alpha$ are zero. With this summation, Equation 2.22 takes the form:

$$\frac{\partial^2 x^m}{\partial y^i \partial y^j} = \left\{ \begin{matrix} \gamma \\ i\,j \end{matrix} \right\}_y \frac{\partial x^m}{\partial y^\gamma} - \left\{ \begin{matrix} m \\ \alpha\,\beta \end{matrix} \right\}_x \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \tag{2.23}$$

Equation 2.23 is the second partial derivative of one co-ordinate system with respect to another admissible co-ordinate system. It is not a tensor, but it is used to develop the tensor equations for covariant derivatives.

COVARIANT DERIVATIVES

If we write the tensor equation $B_i = \frac{\partial x^\alpha}{\partial y^i} A_\alpha$ and take the partial derivative using the operator $\frac{\partial}{\partial y^j}$, the result is not a tensor.

$$\frac{\partial B_i}{\partial y^j} = \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial A_\alpha}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^j \partial y^i} A_\alpha \qquad (2.24)$$

Using Equation 2.23 to evaluate the term $\frac{\partial^2 x^\alpha}{\partial y^j \partial y^i}$ in Equation 2.24, and substituting α = m, and γ = α yields the result:

$$\frac{\partial B_i}{\partial y^j} = \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial A_\alpha}{\partial x^\beta} + \left\{ \begin{matrix} \gamma \\ i\,j \end{matrix} \right\}_y \frac{\partial x^\alpha}{\partial y^\gamma} A_\alpha$$

$$- \left\{ \begin{matrix} \alpha \\ \gamma\,\beta \end{matrix} \right\}_x \frac{\partial x^\gamma}{\partial y^i}\frac{\partial x^\beta}{\partial y^j} A_\alpha \qquad (2.25)$$

Now substituting the value $B_\gamma = \frac{\partial x^\alpha}{\partial y^\gamma} A_\alpha$ in Equation 2.25 yields the result:

$$\frac{\partial B_i}{\partial y^j} - \left\{ \begin{matrix} \gamma \\ i\,j \end{matrix} \right\}_y B_\gamma = \left[\frac{\partial A_\alpha}{\partial x^\beta} - \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\}_x A_\gamma \right] \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j} \qquad (2.26)$$

The term in brackets of Equation 2.26 is called the covariant derivative of a covariant tensor, Rank 1. It is a covariant tensor, Rank 2. Now, the covariant derivative of a covariant tensor, Rank 1, can be written:

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \left\{ \begin{matrix} \alpha \\ i\,j \end{matrix} \right\} A_\alpha \qquad (2.27)$$

In repetition, Equation 2.27 is a covariant tensor, Rank 2.

In a similar fashion, we will find the covariant derivative of a contravariant tensor, Rank 1. The partial derivative of the equation $B^i = \frac{\partial y^i}{\partial x^\alpha} A^\alpha$ with respect to the operator $\frac{\partial}{\partial y^j}$ is:

$$\frac{\partial B^i}{\partial y^j} = \frac{\partial A^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^j}\frac{\partial y^i}{\partial x^\alpha} + A^\alpha \frac{\partial^2 y^i}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\beta}{\partial y^j} \tag{2.28}$$

By making use of Equation 2.23, Equation 2.28 can be revised to yield the result:

$$\frac{\partial B^i}{\partial y^j} + \left\{ \begin{matrix} i \\ \gamma\, j \end{matrix} \right\}_y B^\gamma = \left[\frac{\partial A^\alpha}{\partial x^\beta} + \left\{ \begin{matrix} \alpha \\ \gamma\, \beta \end{matrix} \right\}_x A^\gamma \right] \frac{\partial x^\beta}{\partial y^j}\frac{\partial y^i}{\partial x^\alpha} \tag{2.29}$$

The term in the brackets on the right-hand side of Equation 2.29 is the covariant derivative of a contravariant tensor, Rank 1. It is a mixed tensor, Rank 2.

In tensor notation, the covariant derivative of a contravariant tensor, Rank 1, can be written:

$$A,^i{}_j = \frac{\partial A^i}{\partial x^j} + \left\{ \begin{matrix} i \\ \alpha\, j \end{matrix} \right\} A^\alpha \tag{2.30}$$

In repetition, Equation 2.30 is a mixed tensor, Rank 2.

HIGHER RANK AND MIXED COVARIANT DERIVATIVES

The covariant derivative of a mixed tensor, $A^i{}_j$, Rank 2, can be expressed in tensor notation by the equation:

$$A^i{}_{j,k} = \frac{\partial A^i{}_j}{\partial x^k} - \left\{ \begin{matrix} \alpha \\ j\,k \end{matrix} \right\} A^i{}_\alpha + \left\{ \begin{matrix} i \\ \alpha\,k \end{matrix} \right\} A^\alpha{}_j \tag{2.31}$$

Equation 2.31 is a mixed tensor, Rank 3. It is covariant, Rank 2, and contravariant, Rank 1. The covariant derivative of a covariant tensor, A_{ij}, Rank 2, can be expressed in tensor notation by the equation:

$$A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - \left\{ \begin{matrix} \alpha \\ i\,k \end{matrix} \right\} A_{\alpha j} - \left\{ \begin{matrix} \alpha \\ j\,k \end{matrix} \right\} A_{i\alpha} \tag{2.32}$$

Equation 2.32 is covariant tensor, Rank 3.

The covariant derivative of a contravariant tensor, A^{ij}, Rank 2, can be expressed in tensor notation by the equation:

$$A^{ij}{}_{,k} = \frac{\partial A^{ij}}{\partial x^k} + \left\{ \begin{matrix} i \\ \alpha\,k \end{matrix} \right\} A^{\alpha j} + \left\{ \begin{matrix} j \\ \alpha\,k \end{matrix} \right\} A^{i\alpha} \tag{2.33}$$

Equation 2.32 is a mixed tensor, Rank 3, contravariant, Rank 2, and covariant, Rank 1.

INTRINSIC DERIVATIVES

In Chapter 1 it was shown that the velocity vector is a contravariant tensor, Rank 1. Now, if we take the derivative of a velocity vector with respect to time, the result is not a tensor. The following example illustrates the case:

$$\frac{dr}{dt} = \frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial y}\frac{dy}{dt} \tag{2.34}$$

$$\frac{d^2r}{dt^2} = \frac{\partial^2 r}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + \frac{\partial r}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial^2 r}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial r}{\partial y}\frac{d^2y}{dt^2} \tag{2.35}$$

It is obvious that Equation 2.34 is a contravariant tensor, Rank 1, while Equation 2.35 is not a tensor.

Now, to put the acceleration term in the form of a tensor, we use the intrinsic derivative. The equation for the intrinsic derivative will be derived using an orthogonal cartesian coordinate system and the vector:

$$A = A^i \varepsilon_i \tag{2.36}$$

The total differential of Equation 2.36 is:

$$dA = dA^i \varepsilon_i + A^i d\varepsilon_i \tag{2.37}$$

If Equation 2.37 is divided by dx^j, we have the partial derivative:

$$\frac{\partial A}{\partial x^j} = \frac{\partial A^i}{\partial x^j}\varepsilon_i + A^i \frac{\partial \varepsilon_i}{\partial x^j} \tag{2.38}$$

Now, using Equation 2.10,

$$\frac{\partial \varepsilon_i}{\partial x^j} \cdot \varepsilon^\alpha = [ij,k]\, \varepsilon^k \cdot \varepsilon^\alpha$$

we can establish the results:

$$\frac{\partial \varepsilon_i}{\partial x^j} \cdot \varepsilon^\alpha = \left\{ \begin{matrix} \alpha \\ i\,j \end{matrix} \right\} \tag{2.39}$$

$$\frac{\partial \varepsilon_i}{\partial x^j} = \left\{ {\alpha \atop i\,j} \right\} \varepsilon_\alpha \tag{2.40}$$

The next step is to substitute Equation 2.40 in Equation 2.38,

$$\frac{\partial A}{\partial x^j} = \frac{\partial A^i}{\partial x^j} \varepsilon_i + \left\{ {\alpha \atop i\,j} \right\} A^i \varepsilon_\alpha \tag{2.41}$$

$$\frac{\partial A}{\partial x^j} = \left[\frac{\partial A^\alpha}{\partial x^j} + \left\{ {\alpha \atop i\,j} \right\} A^i \right] \varepsilon_\alpha \tag{2.42}$$

$$\frac{\partial A}{\partial x^j} = A,^\alpha{}_j \; \varepsilon_\alpha \tag{2.43}$$

The term in the brackets of Equation 2.42 is the covariant derivative $A,^\alpha{}_j$ of a contravariant tensor. Therefore, the covariant derivative $A,^\alpha{}_j$ of the vector A^α is a vector whose components are the components of $\frac{\partial A}{\partial x^j}$ referred to a base vector system ε_i. An equivalent form for $\frac{\partial A}{\partial x^j}$ is:

$$\frac{\partial A}{\partial x^j} = A_{\alpha,j} \; \varepsilon^\alpha \tag{2.44}$$

Now, to find the intrinsic derivative of a tensor, we differentiate a vector with respect to parameter t.

$$\frac{dA}{dt} = \frac{\partial A}{\partial x^j} \frac{dx^j}{dt} \tag{2.45}$$

The term $\frac{\partial A}{\partial x^j}$ in Equation 2.45 is a covariant derivative. Therefore, we can use Equation 2.42 for $\frac{\partial A}{\partial x^j}$ and substitute the result in Equation 2.45, and use the relation:

$$\frac{dA^\alpha}{dt} = \frac{\partial A^\alpha}{\partial x^j} \frac{dx^j}{dt}$$

to obtain:

$$\frac{dA}{dt} = \left[\frac{dA^\alpha}{dt} + \left\{ \begin{matrix} \alpha \\ i\,j \end{matrix} \right\} A^i \frac{dx^j}{dt} \right] \varepsilon_\alpha \qquad (2.46)$$

Now the vector $\frac{\delta A^\alpha}{\delta t}$ is defined by the equation:

$$\frac{\delta A^\alpha}{\delta t} = \frac{dA^\alpha}{dt} + \left\{ \begin{matrix} \alpha \\ i\,j \end{matrix} \right\} A^i \frac{dx^j}{dt}, \quad (\alpha = 1, 2, 3) \qquad (2.47)$$

Equation 2.47 is called the absolute or intrinsic derivative of A^α with respect to parameter t.

Now, if we return to the problem of finding the total accelerations in terms of polar co-ordinates r and Θ, the results are readily obtained with Equation 2.47.

First, to find the total radial acceleration, we let $\frac{dr}{dt} =$ A^α

$$\frac{\delta A^r}{\delta t} = \frac{d^2r}{dt^2} + \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} \frac{d\Theta}{dt} \frac{d\Theta}{dt} \qquad (2.48)$$

$\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = -r$, and all other Christoffel symbols of the second kind with a (1) in the upper position of the symbol are zero. Therefore, Equation 2.48 gives the familiar result for the total radial acceleration that is the sum of linear and centrifugal accelerations.

$$\frac{\delta A^r}{\delta t} = \frac{d^2r}{dt^2} - r\left(\frac{d\Theta}{dt}\right)^2 \tag{2.49}$$

Now, to find the total angular acceleration we let $\frac{d\Theta}{dt} = A^\alpha$

$$\frac{\delta A^\theta}{\delta t} = \frac{d^2\Theta}{dt^2} + \left\{ \begin{matrix} 2 \\ 1\,2 \end{matrix} \right\} \frac{dr}{dt}\frac{d\Theta}{dt} + \left\{ \begin{matrix} 2 \\ 2\,1 \end{matrix} \right\} \frac{d\Theta}{dt}\frac{dr}{dt}$$

$$\left\{ \begin{matrix} 2 \\ 2\,1 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 1\,2 \end{matrix} \right\} = \frac{1}{r}, \tag{2.50}$$

and all other Christoffel symbols of the second kind with a (2) in the upper position of the symbol are zero. This gives the familiar result for the total angular acceleration.

$$\frac{\delta A^\theta}{\delta t} = \frac{d^2\Theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\Theta}{dt} \tag{2.51}$$

Equation 2.51 can be converted into the familiar equation for the total tangential acceleration by multiplying it with r.

$$\frac{r\delta A^\theta}{\delta t} = r\frac{d^2\Theta}{dt^2} + 2\frac{dr}{dt}\frac{d\Theta}{dt} \tag{2.52}$$

In conclusion, the intrinsic derivative of a tensor is a tensor of the same type as the initial tensor. Therefore, the intrinsic derivative of the velocity vector with respect to parameter t is a contravariant tensor, Rank 1. To illustrate this fact, we will find $\frac{d^2x}{dt^2}$ from Equations 2.49 and 2.51.

$$\frac{d^2x}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\Theta}{dt}\right)^2\right]\frac{\partial x}{\partial r} + \left[\frac{d^2\Theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\Theta}{dt}\right]\frac{\partial x}{\partial \Theta}$$

$$\frac{d^2x}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\Theta}{dt}\right)^2\right]\cos\Theta$$

$$- \left[\frac{d^2\Theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\Theta}{dt}\right] r \sin\Theta \qquad (2.53)$$

In repetition, the intrinsic derivative of a tensor is a tensor of the same type and rank as the initial tensor. The corresponding equation for this intrinsic derivative of a covariant tensor Rank 1 is:

$$\frac{\delta A_i}{\delta t} = \frac{dA_i}{dt} - \left\{ {k \atop i\,j} \right\} A_k \frac{dx^j}{dt} \qquad (2.53A)$$

LAPLACE'S EQUATION

Laplace's equation $\nabla^2 \Phi$ has many applications in engineering and physics. It can be written using the contravariant metric tensors and the covariant derivative of a covariant vector.

The divergence of the covariant vector $\Phi_{,i}$, in curvilinear co-ordinates is a Laplacian. In tensor notation, the equation is:

$$\nabla^2\Phi = g^{ij}\,(\Phi_{,i})_j = g^{ij}\left[\frac{\partial^2\Phi}{\partial x^i x^j} - \left\{ {k \atop i\,j} \right\}\frac{\partial\Phi}{\partial x^k}\right] \qquad (2.54)$$

To illustrate Equation 2.54, the Laplacian will be found for curvilinear co-ordinates r and Θ. Now, the corresponding notation for the co-ordinates r and Θ in Equation 2.54 is:

$$x^1 = r,\ x^2 = \Theta$$

$$\bar{g}_{11} = 1,\ \bar{g}_{22} = r^2$$

$$\bar{g}^{11} = 1,\ \bar{g}^{22} = \frac{1}{r^2}$$

$$\nabla^2\Phi = 1\left[\frac{\partial^2\Phi}{\partial r^2} - \left\{ {1 \atop 1\,1} \right\}\frac{\partial\Phi}{\partial r} - \left\{ {2 \atop 1\,1} \right\}\frac{\partial\Phi}{\partial\Theta}\right]$$

$$+ \frac{1}{r^2}\left[\frac{\partial^2\Phi}{\partial\Theta^2} - \left\{ {1 \atop 2\,2} \right\}\frac{\partial\Phi}{\partial r} - \left\{ {2 \atop 2\,2} \right\}\frac{\partial\Phi}{\partial\Theta}\right] \tag{2.55}$$

The only Christoffel symbol in Equation 2.55 that is not zero is:

$$\left\{ {1 \atop 2\,2} \right\} = -r$$

Now, Equation 2.55 reduces to the familiar result:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\Theta^2} \tag{2.56}$$

If cartesian co-ordinates x and y are used, Equation 2.54 yields the familiar result:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} \tag{2.57}$$

The two terms in Equation 2.57 are a result of the fact that all Christoffel symbols in cartesian linear co-ordinates are zero.

As initially noted in this section, Laplace's equation has many applications in engineering and physics. Two of the most important are:

1. The Laplacian of a potential

$$\left| \Phi \right| = \left| \frac{M}{r} \right|$$

2. The wave equation.

Therefore, these two forms will be used to illustrate the use of Laplace's equation.

The first example will be for the Laplacian of potential $\Phi = -\frac{M}{r}$. The absolute value of this function is the form for the Newtonian Gravitational Potential, Einstein's Gravitational Potential for the General Theory of Relativity, the Electrostatic Potential, and the Potential for Magnetism.

If two bodies, m_1 and m_2, are separated by a distance r, in a Newtonian field, the potential is:

$$\Phi = -\frac{M}{r} = -\frac{\mu\, m_1\, m_2}{r} \tag{2.58}$$

If the same two bodies are separated by a relativistic distance r, in an Einstein field for the General Theory of Relativity, the potential is:

$$\Phi = 1 - \frac{M}{r} \tag{2.59}$$

where,

$$M \backsim \frac{2\,\mu\, m_1\, m_2}{C^2}$$

C is the speed of light.

If two electrostatic charges, or two magnetic poles are separated by a distance r, the potential is:

$$\Phi = \pm \frac{M}{r} = \pm \frac{\varepsilon_1 \, \varepsilon_2}{r} \qquad (2.60)$$

The plus or minus is used because like charges or poles repel, and opposite charges or poles attract.

Now using Equations 2.58 and 2.54 it will be proved that the Laplacian of a potential function that satisfies the inverse square law is zero.

$$g^{ij}\left[\frac{\partial^2 \Phi}{\partial x^i \, \partial x^j} - \left\{ \begin{matrix} k \\ i \; j \end{matrix} \right\} \frac{\partial \Phi}{\partial x^k}\right] = 0 \qquad (2.61)$$

If an orthogonal cartesian co-ordinate system is used, all Christoffel symbols are zero and Equation 2.61 assumes the form:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad (2.62)$$

The next step in the proof is to operate on $\left(- \frac{M}{r} \right)$ with $\frac{\partial^2}{\partial x^2}$.

$$\frac{\partial}{\partial x}\left(- \frac{M}{r} \right) = \frac{M}{r^2} \frac{\partial r}{\partial x} \qquad (2.63)$$

Because it is an Orthogonal Co-ordinate System, $r = (x^2 + y^2 + z^2)^{1/2}$ and $\frac{\partial r}{\partial x} = \frac{x}{r}$. Using these results, Equation 2.63 is modified and operated on a second time by $\frac{\partial}{\partial x}$.

$$\frac{\partial}{\partial x}\left(\frac{Mx}{r^3} \right) = \frac{M}{r^3} - \frac{3Mx}{r^4} \frac{\partial r}{\partial x} \qquad (2.64)$$

$$\frac{\partial^2}{\partial x^2}\left(-\frac{M}{r}\right) = \frac{M}{r^3} - \frac{3Mx^2}{r^5} \qquad (2.65)$$

In a similar fashion the next two terms in the Laplacian can be evaluated. The results are:

$$\frac{\partial^2}{\partial y^2}\left(-\frac{M}{r}\right) = \frac{M}{r^3} - \frac{3My^2}{r^5} \qquad (2.66)$$

$$\frac{\partial^2}{\partial z^2}\left(-\frac{M}{r}\right) = \frac{M}{r^3} - \frac{3Mz^2}{r^5} \qquad (2.67)$$

Now the three terms are added and set equal to zero:

$$\frac{3M}{r^3} - \frac{3M}{r^3}\left(\frac{x^2 + y^2 + z^2}{r^2}\right) = 0 \qquad (2.68)$$

Equation 2.68 is equal to zero because $r^2 = x^2 + y^2 + z^2$. In conclusion, the Laplacian of the function, $\Phi = -\frac{M}{r}$, was proved to be zero in cartesian co-ordinates. But Equation 2.61 is a tensor. Therefore, if the Laplacian vanishes in one co-ordinate system, it must vanish in all acceptable co-ordinate systems.

The final examples are for the wave equation. It applies to sound waves, electromagnetic waves, gravitational waves for the General Theory of Relativity, and the mechanical vibratons of strings and membranes.

If the tensor form of the Laplacian is set equal to the function $\frac{1}{C^2}\frac{\partial^2\Phi}{\partial t^2}$, the result is the complete wave equation for a three dimensional Euclidean space.

$$\frac{1}{C^2}\frac{\partial^2\Phi}{\partial t^2} = g^{ij}\left[\frac{\partial^2\Phi}{\partial x^j\,\partial x^i} - \left\{ {k \atop i\,j} \right\}\frac{\partial\Phi}{\partial x^i}\right] \qquad (2.69)$$

Term C is called "characteristic velocity," and it is the speed of propagation. For sound waves, it is the speed of sound. For electromagnetic or gravitational waves, it is the speed of light.

If the wave equation is written in orthogonal cartesian co-ordinates, again because all Christoffel symbols are zero, it assumes the form:

$$\frac{1}{C^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \qquad (2.69)$$

A solution of Equation 2.69 for spherical waves propagating from a point source is:

$$\Phi = A \sin \frac{2\,\pi}{\lambda}(lx + my + nz - Ct) \qquad (2.70)$$

l, m, and n are direction cosines that satisfy the condition:

$$l^2 + m^2 + n^2 = 1 \qquad (2.71)$$

λ is the wave length and it can be computed using the equation

$$\lambda = \frac{C}{f} \qquad (2.72)$$

f is the frequency of the source. For sound waves it could be the vibrations of a metal bar. For an electromagnetic radio wave, it could be an oscillating circuit.

To write the equation for plane waves, or the mechanical vibrations of a string, only one co-ordinate is used:

$$\frac{1}{C^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{\partial^2\Phi}{\partial x^2} \qquad (2.73)$$

A solution for plane waves is:

$$\Phi = A \sin \frac{2\pi}{\lambda}(x - Ct) \qquad (2.74)$$

Solutions in the form of Equations 2.70 and 2.74 are called retarded potentials. Therefore, it takes a wave a finite time t to travel a distance D.

$$t = \frac{D}{C} \qquad (2.75)$$

If Equation 2.73 is used for the mechanical vibrations of a string, $\frac{1}{C^2}$ is replaced by $\frac{m}{T}$. m is the mass per unit length of the string, and T is the tension force applied to the string.

A solution for the mechanical vibrations of a string is:

$$\Phi = \sum_{n=1}^{n=\infty} A_n \sin \omega_n t \sin \frac{n\pi x}{L} \qquad (2.76)$$

This is a Fourier trigonometric series where L is the length of the string and ω_n are the natural frequencies for each mode length.

In summary, there are many other important applications of Laplace's equation. Some of the better known are the Heat equation, Poisson's equation, the Continuity equation for incompressible fluids expressed in terms of a velocity potential, and the Compatibility equations for stresses in the Theory of Elasticity.

Chapter III

Riemann-Christoffel Tensors & Differential Geometry

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Riemann - Christoffel Tensors & Differential Geometry

RIEMANN - CHRISTOFFEL TENSORS

A Riemann - Christoffel tensor of the second kind is a mixed tensor, Rank 4. It is contravariant, Rank 1, and covariant, Rank 3. To derive this tensor, we take the second covariant derivative of a covariant tensor, Rank 1.

A first covariant derivative of the tensor A_i is:

$$A_{i,\,j} = \frac{\partial A_i}{\partial x^j} - \left\{ \begin{matrix} \alpha \\ i\,j \end{matrix} \right\} A_\alpha \qquad (3.1)$$

Now, if we use Equation 2.32 and take the second covariant derivative of Equation 3.1 with respect to x^k, the result is:

$$A_{i,\,jk} = \frac{\partial^2 A_i}{\partial x^k \partial x^j} - \frac{\partial \left\{ \begin{matrix} \alpha \\ i\,j \end{matrix} \right\}}{\partial x^k} A_\alpha - \left\{ \begin{matrix} \alpha \\ i\,j \end{matrix} \right\} \frac{\partial A_\alpha}{\partial x^k}$$

(Equation 3.2 continued on page 70.)

$$- \left\{ \begin{matrix} \alpha \\ i\,k \end{matrix} \right\} \frac{\partial A_\alpha}{\partial x^j} + \left\{ \begin{matrix} \alpha \\ i\,k \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \alpha\, j \end{matrix} \right\} A_\beta$$

$$- \left\{ \begin{matrix} \alpha \\ j\,k \end{matrix} \right\} \frac{\partial A_i}{\partial x^\alpha} + \left\{ \begin{matrix} \alpha \\ j\,k \end{matrix} \right\} \left\{ \begin{matrix} \gamma \\ i\,\alpha \end{matrix} \right\} A_\gamma \qquad (3.2)$$

The next step is to form another second covariant derivative of A_i, $A_{i,\,kj}$, and subtract it from $A_{i,\,jk}$. The result is:

$$A_{i,\,jk} - A_{i,\,kj} = \left\{ \begin{matrix} \alpha \\ i\,k \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \alpha\, j \end{matrix} \right\} A_\beta - \frac{\partial \left\{ \begin{matrix} \alpha \\ i\,j \end{matrix} \right\}}{\partial x^k} A_\alpha$$

$$- \left\{ \begin{matrix} \alpha \\ i\,j \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \alpha\, k \end{matrix} \right\} A_\beta + \frac{\partial \left\{ \begin{matrix} \alpha \\ i\,k \end{matrix} \right\}}{\partial x^j} A_\alpha \qquad (3.3)$$

An interchange of the indices α and β in the two terms of Equation 3.3 that have the product of the Christoffel symbols yields the result:

$$A_{i,\,jk} - A_{i,\,kj} = \left[\frac{\partial \left\{ \begin{matrix} \alpha \\ i\,k \end{matrix} \right\}}{\partial x^j} - \frac{\partial \left\{ \begin{matrix} \alpha \\ i\,j \end{matrix} \right\}}{\partial x^k} + \right.$$

$$\left. \left\{ \begin{matrix} \beta \\ i\,k \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \beta\, j \end{matrix} \right\} - \left\{ \begin{matrix} \beta \\ i\,j \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \beta\, k \end{matrix} \right\} \right] A_\alpha \qquad (3.4)$$

The term in the brackets of Equation 3.4 is the mixed Riemann-Christoffel tensor, covariant, Rank 3, and contravariant, Rank 1. Using tensor notation, it can be expressed:

$$R^{\alpha}{}_{ijk} = \frac{\partial \left\{ {\alpha \atop ik} \right\}}{\partial x^j} - \frac{\partial \left\{ {\alpha \atop ij} \right\}}{\partial x^k} +$$

$$\left\{ {\beta \atop ik} \right\}\left\{ {\alpha \atop \beta j} \right\} - \left\{ {\beta \atop ij} \right\}\left\{ {\alpha \atop \beta k} \right\} \qquad (3.5)$$

Equation 3.5 is called a Riemann - Christoffel tensor of the second kind. A Riemann - Christoffel tensor of the first kind is the associated tensor:

$$R_{ijkl} = g_{i\alpha} \; R^{\alpha}{}_{jkl}$$

This result can be written:

$$R_{ijkl} = \frac{\partial [jl, i]}{\partial x^k} - \frac{\partial [jk, i]}{\partial x^l} +$$

$$\left\{ {\alpha \atop jk} \right\} [i\,l, \alpha] - \left\{ {\alpha \atop jl} \right\} [i\,k, \alpha] \qquad (3.6)$$

Equation 3.6 is the covariant Riemann - Christoffel tensor of the first kind, Rank 4. If the Christoffel symbols in Equation 3.6 are written out in terms of the metric tensors, and the indicated partial derivatives, the following identities result:

$$\left.\begin{aligned} R_{jikl} &= -R_{ijkl} \\ R_{ijlk} &= -R_{ijkl} \\ R_{klij} &= R_{ijkl} \\ R_{ijkl} &+ R_{iklj} + R_{iljk} = 0 \end{aligned}\right\} \qquad (3.7)$$

In two dimensions there is only one distinct nonvanishing component, R_{1212}. It is called the curvature tensor of a surface.

RICCI TENSOR

The Ricci tensor is defined:

$$R_{ij} = g^{lk} R_{kijl}$$

It can be written out in tensor notation as:

$$R_{ij} = \frac{\partial \left\{ {\alpha \atop i\,\alpha} \right\}}{\partial x^j} - \frac{\partial \left\{ {\alpha \atop i\,j} \right\}}{\partial x^\alpha} + \left\{ {\alpha \atop \beta\,j} \right\} \left\{ {\beta \atop i\,\alpha} \right\} - \left\{ {\alpha \atop \beta\,\alpha} \right\} \left\{ {\beta \atop i\,j} \right\} \quad (3.8)$$

The curvature invariant R is defined:

$$R = g^{ij} R_{ij} \quad (3.9)$$

The Ricci tensors can be used to find the Gaussian curvature of a surface, and the curvature invariant is twice the Gaussian curvature. These important facts will be explained in the following section.

GAUSSIAN CURVATURE

The Gaussian curvature or the total curvature of a surface in tensor notation is:

$$K = \frac{R_{1212}}{g} \quad (3.10)$$

If a curved surface is expressed in orthogonal curvilinear co-ordinates, Equation 3.10 can be evaluated in terms of Equation 3.6, and the determinate of the metric tensors. For othogonal curvilinear co-ordinates u^1, u^2, and metric tensors g_{11}, g_{22}, the equation for K is:

$$K = -\frac{1}{2\sqrt{g}}\left[\frac{\partial}{\partial u^1}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial u^2}\right)\right] \tag{3.11}$$

Using the same co-ordinates u^1 and u^2, total curvature K can also be expressed in terms of the Ricci tensors by the equations:

$$\left.\begin{aligned} -K &= \frac{R_{11}}{g_{11}} \\ -K &= \frac{R_{22}}{g_{22}} \end{aligned}\right\} \tag{3.12}$$

Equations 3.12 are the components of the invariant:

$$R = -2K \tag{3.13}$$

R is called the Einstein curvature.

To illustrate Equations 3.11 and 3.12, the case for the surface of a sphere with a radius r will be used. The line element for the surface is:

$$ds^2 = r^2 d\Phi^2 + r^2 \sin^2 \Phi \, d\Theta^2$$

Variables u^1 and u^2, metric tensors g_{11}, g_{22}, and determinate g are:

$$u^1 = \Phi, \quad g_{11} = r^2$$

$$u^2 = \Theta, \quad g_{22} = r^2 \sin^2\Phi$$

$$g = r^4 \sin^2\Phi,$$

$$\sqrt{g} = r^2 \sin \Phi$$

With these values and Equation 3.11, we get the following value for K:

$$K = \frac{-1}{2\sqrt{r^4 \sin^2\Phi}}\left[\frac{\partial}{\partial\Phi}\left(\frac{2\, r^2 \sin \Phi \cos \Phi}{\sqrt{r^4 \quad \sin^2 \quad \Phi}}\right) + 0\right]$$

$$K = \frac{1}{r^2}$$

For Equations 3.12, the corresponding Ricci tensors are:

$$R_{11} = -1$$

$$R_{22} = -\sin^2 \Phi$$

With these results, Equations 3.12 give the identical result for the curvature:

$$K = \frac{1}{r^2}$$

From this example, it should be clear that the total curvature of a surface can be found from the Riemann - Christoffel tensors or the Ricci tensors.

In the general theory of relativity, the Ricci tensors are used to extend the concept of curvature to the study of curved space. This will be explained in the section of Chapter 4 on Einstein's gravitational equations.

A space of V_N is said to be flat if the Riemann - Christoffel tensor is identically zero. The condition $R_{ijkl} = 0$ is necessary and sufficient for a flat space. An example of a flat space is the Euclidean plane with the familiar line element $ds^2 = dx^2$

$+ \, dy^2$ in rectangular Cartesian co-ordinates or $ds^2 = dr^2 + r^2 d\Theta^2$ in polar co-ordinates.

SERRET - FRENET FORMULAS

To lead into the important Serret - Frenet formulas for space curves, a brief introduction using vector analysis is included. In Fig. 3-1 we have a space curve $r = r(s)$. The length of the arc PP^1 is ds, and as P^1 approaches P, as a limit $ds = dr$. With this equality, the unit tangent vector T is:

$$\frac{dr}{ds} = T \tag{3.14}$$

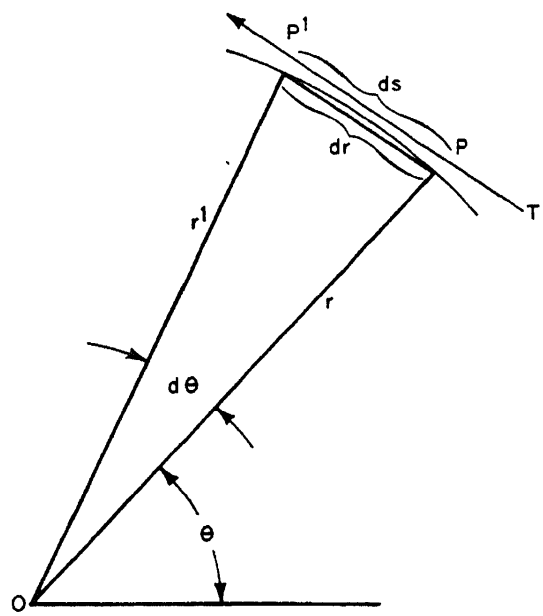

Fig. 3-1.

Since T is a unit tangent vector, it changes as r moves through an angle $d\Theta$ by an amount $dT = Td\Theta$, that is always normal to T. Therefore, the first Frenet formula in terms of T is:

$$\frac{dT}{ds} = kN \tag{3.15}$$

N is the unit normal and $k = \frac{1}{r}$ is called the curvature.

Since T is normal to N, a third vector, B, the unit binormal, indicated in Fig. 3-2 can be written:

$$B = T \times N \tag{3.16}$$

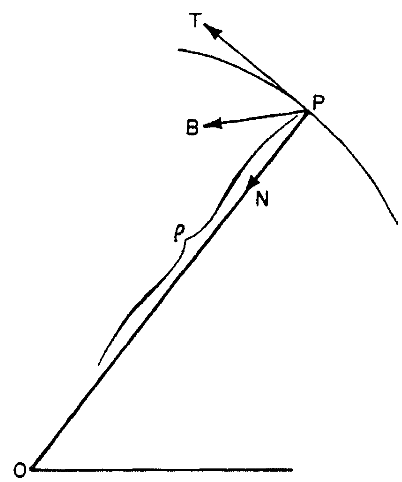

Fig. 3-2.

Now, the second Frenet formula expressed in terms of the unit binormal B is:

$$\frac{dB}{ds} = -\tau N \tag{3.17}$$

$\tau = \frac{1}{\rho}$ is called the torsion of the curve at P.

The next step is to compute $\frac{dN}{ds}$, using the relation $N = B \times T$.

$$\frac{dN}{ds} = \frac{dB}{ds} \times T + B \times \frac{dT}{ds} \tag{3.18}$$

If we substitute Equations 3.15 and 3.17 in Equation 3.18, the third Frenet formula in terms of the normal N is:

$$\frac{dN}{ds} = -kT + \tau B \qquad (3.19)$$

Equations 3.15, 3.17, and 3.19 are the three Serret - Frenet formulas. They are fundamental in the theory of space curves. The three Frenet formulas can be expressed in terms of the Christoffel symbols. In the first Frenet formula, Equation 3.15, the unit tangent T can be expressed as the sum of components, T^i. Therefore, the unit normal $N = \frac{\delta T}{\delta s}$ has the corresponding components $N^i = \frac{\delta T^i}{\delta s}$ and they are intrinsic derivatives of the contravariant vectors:

$$T^i = \frac{dx^i}{ds}$$

Using Equation 2.47 and substituting ds for dt, the Frenet equation in tensor notation for the curvature k is:

$$kN^i = \frac{dT^i}{ds} + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} T^j \frac{dx^k}{ds} \qquad (3.20)$$

To illustrate Equation 3.20, we use a circle on the surface of a cylinder. The line element in cylindrical co-ordinates is:

$$ds^2 = dr^2 + r^2\, d\Theta^2 + dz^2$$

The corresponding value for the line element on the surface of a cylinder is:

$$ds^2 = r^2\, d\Theta^2 + dz^2$$

Now the components of tangent T to the circle are $T^i = \dfrac{dx^i}{ds}$,

where $T^1 = \dfrac{dr}{ds} = 0$, $T^2 = \dfrac{d\Theta}{ds}$, and $T^3 = \dfrac{dz}{ds} = 0$.

To find the unit tangent vector, $T = T^2$, we use the length of the line element for the circle:

$$ds^2 = r^2 \, d\Theta^2$$

From the length of the line element, it is obvious that:

$$\left(\frac{d\Theta}{ds}\right)^2 = \frac{1}{r^2}$$

$$T^2 = \frac{d\Theta}{ds} = \frac{1}{r}$$

Using the following results and Formula 3.20, we will now compute the curvature kN.

$$T^2 = \frac{1}{r}, \qquad \frac{dT^2}{ds} = 0$$

$$\frac{dx^1}{ds} = T^1 = 0, \text{ as } x^1 = r = \text{constant}$$

$$\frac{dx^3}{ds} = T^3 = 0, \text{ as } x^3 = z = 0 \quad \text{(for the circle)}$$

For this example, the components of Formula 3.20 are:

$$kN^1 = \frac{dT^1}{ds} + \left\{ \begin{matrix} 1 \\ 2\,2 \end{matrix} \right\} T^2 \frac{dx^2}{ds}$$

$$\left\{ \begin{matrix} 1 \\ 2\,2 \end{matrix} \right\} = -r$$

$$T^2 = \frac{dx^2}{ds} = \frac{1}{r}$$

$$kN^1 = -\frac{1}{r}$$

$$kN^2 = \frac{dT^2}{ds} + \left\{ \begin{matrix} 2 \\ 2\,1 \end{matrix} \right\} T^2 \frac{dx^1}{ds} = 0$$

$$kN^3 = \frac{dT^3}{ds} + \left\{ \begin{matrix} 3 \\ j\,k \end{matrix} \right\} T^j \frac{dx^k}{ds} = 0$$

This gives the familiar result for a circle, $kN = \frac{1}{r}$, where $N = -1$. Now, the second Frenet formula 3.17, expressed in terms of the Christoffel symbols, is:

$$-\tau N^i = \frac{dB^i}{ds} + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} B^j \frac{dx^k}{ds} \qquad (3.21)$$

Where B^i are the components of the unit binormal B.
The third Frenet Formula 3.19, expressed in terms of the Christoffel symbols is:

$$-kT^i + \tau B^i = \frac{dN^i}{ds} + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} N^j \frac{dx^k}{ds} \qquad (3.22)$$

T^i are the components of the unit tangent vector T, and B^i are the components of the unit binormal B.

In conclusion, the three Frenet Formulas, 3.20, 3.21, and 3.22, in tensor notation are the intrinsic derivatives of contravariant vectors. Using the symbol for the intrinsic derivative δ, these formulas are:

$$\frac{\delta T^i}{\delta s} = kN^i \qquad (3.23)$$

$$\frac{\delta B^i}{\delta s} = -\tau N^i \qquad (3.24)$$

$$\frac{\delta N^i}{\delta s} = -kT^i + \tau B^i \qquad (3.25)$$

A simple example of a dynamics problem using the Serret-Frenet formulas is a mass M, traveling along a curved path with a velocity $\dot{T}$, that is tangent to the trajectory. For this case centrifugal force F_N is

$$F^i{}_N = M\,K\,N^i\,(\dot{T}^i)^2$$

GEODESICS

A geodesic is the shortest line between two points. On a Euclidean plane it is a straight line. On the surface of a sphere, it is the arc of a great circle. The equations for geodesics can be derived by the calculus of variations, from the metric line element:

$$ds = \sqrt{g_{ij}\,dx^i\,dx^j} \qquad (3.26)$$

Equation 3.26 can be expressed in terms of the velocity components,

$$\dot{x}^i = \frac{dx^i}{dt}$$

$$ds = -\sqrt{g_{ij}\,\dot{x}^i\,\dot{x}^j}\,dt \qquad (3.27)$$

The length of the line element for a time interval t_1 to t_2 is:

$$s = \int_{t_1}^{t_2} \sqrt{g_{ij}\,\dot{x}^i\,\dot{x}^j}\,dt \qquad (3.28)$$

Now, to find the extremals or minimum length for Equation 3.28, Euler's equation for a functional can be used. To derive Euler's equation, the calculus of variations is applied to the functional J:

$$J = \int_{t_1}^{t_2} F\,(t, x, \dot{x})\,dt \qquad (3.29)$$

Now we write a second functional $\overline{J}$ that differs from J by a small variation. It is the sum $J + \delta J$, where δJ is the small variation. For this variation, the boundary conditions at t_1 and t_2 are $\delta J = O$.

Now, for the functional J to be an extremal, the integral:

$$\delta J = \int_{t_1}^{t_2} \delta F\,dt$$

must be zero.

$$\delta J = \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x}\,\delta x + \frac{\partial F}{\partial \dot{x}}\,\delta \dot{x} \right) dt = O \qquad (3.30)$$

$$\delta F = \frac{\partial F}{\partial x}\,\delta x + \frac{\partial F}{\partial \dot{x}}\,\delta \dot{x} \qquad (3.31)$$

The second term under the integral in Equation 3.30 can be integrated by parts.

$$\int_{t_1}^{t_2} \frac{\partial F}{\partial \dot{x}}\,\delta \dot{x} = \frac{\partial F}{\partial \dot{x}}\,\delta x \,\Big]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) \delta x\,dt \qquad (3.32)$$

The first term on the right side of Equation 3.32 is zero, as $\delta x = 0$ at boundaries t_1 and t_2. Equation 3.30 then becomes:

$$\delta J = \int_{t_1}^{t_2} \left[\frac{\partial F}{\partial x} \delta x - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) \delta x \right] dt = 0 \tag{3.33}$$

Since the integral from t_1 to t_2 must be zero, the term in the brackets can be divided by δx and set equal to zero. This result is Euler's equation for the extremal of a functional.

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \tag{3.34}$$

Now, Euler's Equation 3.34 can be expressed in generalized co-ordinates x^k, and $\frac{\partial F}{\partial x^k} = F x^k$ The result is:

$$F x^k - \frac{dF \dot{x}^k}{dt} = 0 \tag{3.35}$$

The next step is to evaluate Equation 3.35 for the line element:

$$ds = \sqrt{g_{ji} \dot{x}^i \dot{x}^j}\, dt \tag{3.27}$$

The corresponding values for F, Fx^k, and $F\dot{x}^k$ are:

$$F = \sqrt{g_{ij} \dot{x}^i \dot{x}^j} \tag{3.36}$$

$$Fx^k = \tfrac{1}{2}\left(g_{ij}\dot{x}^i\dot{x}^j \right) \frac{-\frac{1}{2}\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j$$

$$F\dot{x}^k = \left(g_{ij}\dot{x}^i\dot{x}^j \right)^{-\frac{1}{2}} g_{ik} \dot{x}^i$$

Using these values, Euler's equation for the line is:

$$\frac{d}{dt}\left[\frac{g_{ik}\,\dot{x}^i}{\sqrt{g_{ij}\,\dot{x}^i\,\dot{x}^j}}\right] - \frac{\frac{\partial g_{ij}}{\partial x^k}\;\dot{x}^i\dot{x}^j}{2\sqrt{g_{ij}\,\dot{x}^i\,\dot{x}^j}} = 0 \qquad (3.37)$$

From Equation 3.27 we can get the result:

$$\frac{ds}{dt} = \dot{s} = \sqrt{g_{ij}\,\dot{x}^i\,\dot{x}^j} \qquad (3.38)$$

Now, substitute Equation 3.38 into Equation 3.37:

$$\frac{d}{dt}\left(\frac{g_{ik}\,\dot{x}^i}{\dot{s}}\right) - \frac{\frac{\partial g_{ij}}{\partial x^k}\,\dot{x}^i\,\dot{x}^j}{2\quad \dot{s}} = 0 \qquad (3.39)$$

If we perform the indicated differentiation in Equation 3.39, the result is:

$$g_{ik}\,\ddot{x}^i + \frac{\partial g_{ik}}{\partial x^j}\,\dot{x}^i\,\dot{x}^j - \tfrac{1}{2}\,\frac{\partial g_{ij}}{\partial x^k}\,\dot{x}^i\,\dot{x}^j = \frac{g_{ik}\,\dot{x}^i\,\ddot{s}}{\dot{s}} \qquad (3.40)$$

The next step is to let parameter t become arc length s, and set:

$$\dot{s} = \frac{ds}{ds} = \sqrt{g_{ij}\,\dot{x}^i\,\dot{x}^j} = 1 \qquad (3.41)$$

Then, the left-hand term in Equation 3.40 becomes zero, and the dots denote differentiation with respect to arc parameter s. The second term on the right-hand side can be expressed as the sum of two terms. With these changes in Equation 3.40, we obtain:

$$g_{ik}\frac{d^2x^i}{ds^2}+\frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^k}\right]\frac{dx^i}{ds}\frac{dx^j}{ds}=0 \tag{3.42}$$

The second term in Equation 3.42 is a Christoffel symbol of the first kind. If we form the inner product of Equation 3.42 and the contravariant metric tensor $g^{\alpha k}$, the result is the equation for a geodesic in terms of the Christoffel symbols of the second kind.

$$\frac{d^2x^\alpha}{ds^2}+\left\{ {\alpha \atop i\,j} \right\}\frac{dx^i}{ds}\frac{dx^j}{ds}=0 \tag{3.43}$$

Now the equation for a geodesic on a two-dimensional manifold, or surface V_2 with the co-ordinate u^i is:

$$\frac{d^2u^i}{ds^2}+\left\{ {i \atop j\,k} \right\}\frac{du^j}{ds}\frac{du^k}{ds}=0 \tag{3.44}$$

$$(i, j, k = 1, 2)$$

The concept of the geodesic is important in the study of differential geometry, Newtonian dynamics, and the general theory of relativity. To illustrate the application of Equation 3.44, two examples will be used. The first example is the geodesic line on a Euclidean plane. If the space is Euclidean, the metric tensors are constants and the Christoffel symbols vanish. Then, Equation 3.44 reduces to:

$$\frac{d^2\,u^i}{ds^2}=0 \qquad (i = 1, 2)$$

$$\frac{d\,u^i}{ds}=A_i$$

$$u^i = A_i s + B_i$$

This result, where A_i and B_i are the constants of integration, gives the two components of a straight line on a plane. Therefore, the geodesics on a Euclidean plane are straight lines.

The second example is for the geodesics on the surface of a sphere. The first step is to use the spherical co-ordinates, $u^1 = \Phi$, $u^2 = \Theta$, a radius r, and the metric line element.

$$ds^2 = r^2\, d\Phi^2 + r^2 \sin^2 \Phi\, d\Theta^2 \qquad (3.45)$$

The metric tensors g_{ij} and g^{ij} for Equation 3.45 are:

$$g_{11} = r^2, \quad g^{11} = \frac{1}{r^2}$$

$$g_{22} = r^2 \sin^2 \Phi, \quad g^{22} = \frac{1}{r^2 \sin^2 \Phi}$$

and the only nonvanishing Christoffel symbols have the following values:

$$\left\{ \begin{matrix} 1 \\ 2\,2 \end{matrix} \right\} = -\sin \Phi \cos \Phi$$

$$\left\{ \begin{matrix} 2 \\ 1\,2 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 2\,1 \end{matrix} \right\} = \cot \Phi$$

Using these Christoffel symbols and Equation 3.44, the two equations for the components of the geodesics on the surface of the sphere are:

$$\frac{d^2\Phi}{ds^2} - \sin \Phi \cos \Phi \left(\frac{d\Theta}{ds} \right)^2 = 0$$

$$\frac{d^2\Theta}{ds^2} + 2 \cot \Phi \frac{d\Theta}{ds} \frac{d\Phi}{ds} = 0$$

Now, if we let Θ be a constant,

$$\frac{d\Theta}{ds} = 0, \text{ and } \frac{d\Phi}{ds} = \frac{1}{r},$$

as the line element in Equation 3.45 reduces to $ds^2 = r^2 d\Phi^2$, $\frac{d^2\Phi}{ds^2}$ is also equal to zero, as $\frac{1}{r}$ is constant. Now, it can be concluded that the arcs of great circles are geodesics.

The concept of a geodesic can be extended to the dynamics of bodies in gravitational fields. In Chapter 4 the geodesic concept will be used for orbital bodies in a Newtonian gravitational field, and for orbital bodies in Einstein's relativistic gravitational field.

PARALLEL DISPLACEMENTS

Equation 3.44 can be written in terms of the components of the unit tangent vectors to a geodesic. First, we let A^i represent the components of the tangent vectors:

$$A^i = \frac{du^i}{ds} = T^i \qquad (3.46)$$

The corresponding line element is:

$$ds^2 = g_{ij}\, u^i\, u^j \qquad (3.47)$$

Now, Equation 3.44 in terms of the tangent vectors A^i, and the condition stipulated by Equation 3.47 is:

$$\frac{dA^i}{ds} + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} A^j \frac{du^k}{ds} = 0 \qquad (3.48)$$

$$(i, j, k = 1, 2)$$

In conclusion, Equation 3.48 defines the parallel displacements of the tangent vectors to a geodesic. The parallel propagation of the tangent vectors to a given curve is a geodesic. But, it must be emphasized that the parallel displacement of tangent vectors do not form a straight line unless the surface is a Euclidean plane.

SURFACES, FIRST FUNDAMENTAL FORM

The distance (ds) from a point p to a point p^1 on a surface can be measured by the change in the position vector (dr). If the position vector to the surface is expressed in curvilinear co-ordinates u^1 and u^2, distance ds^2 is:

$$ds^2 = dr \cdot dr = \left[\frac{\partial r}{\partial u^1} du^1 + \frac{\partial r}{\partial u^2} du^2 \right] \cdot \left[\frac{\partial r}{\partial u^1} du^1 + \frac{\partial r}{\partial u^2} du^2 \right] \tag{3.49}$$

$$ds^2 = \frac{\partial r}{\partial u^1} \cdot \frac{\partial r}{\partial u^1} \left(du^1 \right)^2 + 2 \frac{\partial r}{\partial u^1} \cdot \frac{\partial r}{\partial u^2} du^1 du^2 + \frac{\partial r}{\partial u^2} \cdot \frac{\partial r}{\partial u^2} \left(du^2 \right)^2 \tag{3.50}$$

The metric tensors for the surface in curvilinear co-ordinates u^1 and u^2, are:

$$\bar{g}_{11} = \frac{\partial r}{\partial u^1} \cdot \frac{\partial r}{\partial u^1} \tag{3.51}$$

$$\bar{g}_{12} = \frac{\partial r}{\partial u^1} \cdot \frac{\partial r}{\partial u^2} \tag{3.52}$$

$$\bar{g}_{22} = \frac{\partial r}{\partial u^2} \cdot \frac{\partial r}{\partial u^2} \tag{3.53}$$

With these results, the first fundamental form for the surface in terms of the curvilinear co-ordinates u^1 and u^2 is:

$$ds^2 = \bar{g}_{11}\,(du^1)^2 + \bar{g}_{12}\,du^1\,du^2 + \bar{g}_{22}\,(du^2)^2 \qquad (3.54)$$

If curvilinear co-ordinates u^1 and u^2 are orthogonal, g_{12} is zero, and Equation 3.54 reduces to the familiar form for the line element of a surface.

$$ds^2 = \bar{g}_{11}\,(du^1)^2 + \bar{g}_{22}\,(du^2)^2 \qquad (3.55)$$

SURFACES, SECOND FUNDAMENTAL FORM

The tangent vector to a surface in terms of co-ordinates u^1 and u^2 is:

$$T = \frac{dr}{ds} = \frac{\partial r}{\partial u^1}\frac{du^1}{ds} + \frac{\partial r}{\partial u^2}\frac{du^2}{ds}$$

The corresponding term for the curvature is:

$$\frac{dT}{ds} = \frac{d^2r}{ds^2} = \frac{\partial^2 r}{\partial (u^1)^2}\left[\frac{du^1}{ds}\right]^2 + 2\,\frac{\partial^2 r}{\partial u^1\,\partial u^2}\frac{du^1}{ds}\frac{du^2}{ds}$$

$$+ \frac{\partial^2 r}{\partial (u^2)^2}\left[\frac{du^2}{ds}\right]^2 + \frac{\partial r}{\partial u^1}\frac{d^2u^1}{ds^2} + \frac{\partial r}{\partial u^2}\frac{d^2u^2}{ds^2} \qquad (3.56)$$

Since $\frac{dT}{ds} = k_nN$, curvature k_n can be found by the equation:

$$k_n = N \cdot \frac{d^2r}{ds^2} \qquad (3.57)$$

$$k_n = \left[N \cdot \frac{\partial^2 r}{\partial (u^1)^2} \right] \left[\frac{du^1}{ds} \right]^2 + 2 \left[N \cdot \frac{\partial^2 r}{\partial u^1 \, \partial u^2} \right] \frac{du^1}{ds} \frac{du^2}{ds}$$

$$+ \left[N \cdot \frac{\partial^2 r}{\partial (u^2)^2} \right] \left[\frac{du^2}{ds} \right]^2 \qquad (3.58)$$

The terms with $\frac{\partial r}{\partial u^1}$ and $\frac{\partial r}{\partial u^2}$ drop out as:

$$N \cdot \frac{\partial r}{\partial u^1} = 0, \quad N \cdot \frac{\partial r}{\partial u^2} = 0$$

Now, we make the following substitutions in Equation 3.58:

$$\bar{b}_{11} = N \cdot \frac{\partial^2 r}{\partial (u^1)^2} \qquad (3.59)$$

$$\bar{b}_{12} = N \cdot \frac{\partial^2 r}{\partial u^1 \, \partial u^2} \qquad (3.60)$$

$$\bar{b}_{22} = N \cdot \frac{\partial^2 r}{\partial (u^2)^2} \qquad (3.61)$$

$$k_n = \frac{\bar{b}_{11} \, (du^1)^2 + 2 \, \bar{b}_{12} \, du^1 du^2 + \bar{b}_{22} \, (du^2)^2}{ds^2} \qquad (3.62)$$

The numerator of Equation 3.62 is called the second fundamental form for a surface. In tensor notation, it can be expressed:

$$\beta = \bar{b}_{11} \, (du^1)^2 + 2 \, \bar{b}_{12} \, du^1 du^2 + \bar{b}_{22} \, (du^2)^2 \qquad (3.63)$$

Terms $\bar{b}_{11}$, $\bar{b}_{12}$, and $\bar{b}_{22}$ are covariant tensors, Rank 2.
The normal curvatures in the directions of co-ordinate curves u^1 and u^2 are:

$$k_1 = \frac{\bar{b}_{11}}{g_{11}} \tag{3.64}$$

$$k_2 = \frac{\bar{b}_{22}}{g_{22}} \tag{3.65}$$

The total curvature K can be expressed in terms of the determinate $\bar{b}$ of covariant tensors $\bar{b}_{11}$, $\bar{b}_{12}$, and $\bar{b}_{22}$ and determinate $\bar{g}$ of the corresponding metric tensors $\bar{g}_{11}$, $\bar{g}_{12}$, and $\bar{g}_{22}$ for the first fundamental form.

$$K = \frac{\bar{b}}{\bar{g}} \tag{3.66}$$

The equations of Gauss in terms of $\bar{b}$ are:

$$\bar{b} = \bar{b}_{11}\,\bar{b}_{22} - \bar{b}^2_{12} = R_{1212} \tag{3.67}$$

$$K = \frac{\bar{b}}{\bar{g}} = \frac{R_{1212}}{\bar{g}} \tag{3.68}$$

To demonstrate that $\bar{b} = R_{1212}$ the following analysis is included. Let the position vector to the surface be described by orthogonal cartesian co-ordinates x^i, x^j, and x^k; and the surface be described by curvilinear co-ordinates U^α and U^β. Using this notation, the surface vectors can be expressed:

$$\frac{\partial x^i}{\partial u^\alpha} = x^i_{\alpha} \tag{3.69}$$

The first covariant derivative of Equation 3.69 is:

$$x^i_{\alpha.\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \; \partial u^\beta} + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\}_x x^i_{\alpha} \; x^k_{\beta} - \left\{ \begin{matrix} \gamma \\ \alpha\,\beta \end{matrix} \right\}_u x^i_{\gamma} \tag{3.70}$$

As noted, x^i are cartesian co-ordinates, and the Christoffel symbols $\left\{ {i \atop j\,k} \right\}_x$ are zero. Therefore, the covariant derivative of Equation 3.70 at the pole is:

$$x^i_{\alpha,\beta\gamma} = \frac{\partial^3 x^i}{\partial u^\alpha\, \partial u^\beta\, \partial u^\gamma} - \frac{\partial}{\partial u^\gamma}\left\{ {\delta \atop \alpha\,\beta} \right\}_u x^i_\delta \qquad (3.71)$$

Permutation of indices in Equation 3.71 and subtracting yields the result:

$$x^i_{\alpha,\beta\gamma} - x^i_{\alpha,\gamma\beta} = \left[\frac{\partial}{\partial u^\beta}\left\{ {\delta \atop \alpha\,\gamma} \right\}_u - \frac{\partial}{\partial u^\gamma}\left\{ {\delta \atop \alpha\,\beta} \right\}_u \right] x^i_\delta \qquad (3.72)$$

The term in the brackets is a Riemann-Christoffel tensor of the second kind. It is of the same form as Equation 3.5 with the last two terms equal to zero. Now using the symbol for Riemann-Christoffel tensors, Equation 3.72 assumes the form:

$$x^i_{\alpha,\beta\gamma} - x^i_{\alpha,\gamma\beta} = R^\delta_{\alpha\beta\gamma} x^i_\delta \qquad (3.73)$$

If N^i is a unit normal to the surface, and referring to Equations 3.59 through 3.61 $\bar{b}_{\alpha\beta}$ has the values:

$$\bar{b}_{\alpha\beta} = N^i \cdot x^i_{\alpha,\beta}, \text{ where } x^i_\alpha = \frac{\partial r}{\partial u^\alpha} \qquad (3.74)$$

in terms of $\bar{b}_{\alpha\beta}$ and N^i, $x^i_{\alpha,\beta}$ is:

$$x^i_{\alpha.\beta} = \bar{b}_{\alpha\beta} N^i \qquad (3.75)$$

The covariant derivative of Equation 3.75 is:

$$x^i_{\alpha,\beta\gamma} = N^i_{,\gamma}\ \bar{b}_{\alpha\beta} + \bar{b}_{\alpha\beta,\gamma} N^i \qquad (3.76)$$

To evaluate $N,^{i}_{\gamma}$ Weingarten's formula is used.

$$N,^{i}_{\gamma} = -\bar{g}^{\gamma\beta}\bar{b}_{\alpha\gamma}\, x^{i}_{\beta} \tag{3.77}$$

$$x^{i}_{\alpha,\beta\gamma} = b_{\alpha\beta,\gamma}\, N^{i} - g^{\epsilon\delta} b_{\alpha\beta}\, b_{\gamma\epsilon}\, x^{i}_{\beta} \tag{3.78}$$

Substituting Equation 3.78 in Equation 3.73 and the permutation of symbols yields the result:

$$(\bar{b}_{\alpha\beta,\gamma} - \bar{b}_{\alpha\gamma,\beta})N^{i} - g^{\epsilon\delta}\,(\bar{b}_{\alpha\beta}\bar{b}_{\gamma\epsilon} - \bar{b}_{\alpha\gamma}\bar{b}_{\beta\epsilon})x^{i}_{\delta} = R^{\delta}{}_{\alpha\beta\gamma}x^{i}_{\delta} \tag{3.79}$$

The inner product of Equation 3.79 by N^{i} and the orthogonal condition, $N^{i} \cdot x^{i}_{\delta} = 0$, yields the result:

$$\bar{b}_{\alpha\beta,\gamma} - \bar{b}_{\alpha\gamma,\beta} = 0 \tag{3.80}$$

The inner product of Equation 3.79 by $g_{ij}x^{i}_{\rho}$, yields the results:

$$\left.\begin{aligned} \bar{b}_{\alpha\gamma}\,\bar{b}_{\rho\beta} - \bar{b}_{\alpha\beta}\,\bar{b}_{\rho\gamma} &= R_{\rho\alpha\beta\gamma} \\ \bar{b} &= R_{\rho\alpha\beta\gamma} \end{aligned}\right\} \tag{3.81}$$

Equations 3.80 and 3.81 are called the Codazzi equations. Therefore, as initially noted in Equation 3.68 for two dimensional surfaces:

$$\bar{b} = R_{1212} \tag{3.82}$$

Chapter IV

Classical and Relativistic Mechanics

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Classical and Relativistic Mechanics

THE DYNAMICS OF A PARTICLE FOR CLASSICAL MECHANICS

In the Newtonian physics the law for the force on a free particle is $F^i = Ma^i$. From this law we can derive the concepts of momentum, Mv^i, and kinetic energy, $\frac{M}{2}(v^i)^2$.

Velocity vector v^i, is a contravariant tensor, Rank 1. Acceleration vector a^i, can be expressed as a tensor by writing it in terms of the intrinsic derivative of the velocity vector.

$$v^i = \frac{dx^i}{dt}$$

$$a^i = \frac{d^2x^i}{dt^2} + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \frac{dx^j}{dt} \frac{dx^k}{dt} \qquad (4.1)$$

Equation 4.1 is the equivalent of Equation 2.47 with $\frac{dx^i}{dt} = A^i$. It is a contravariant tensor, Rank 1. Examples of the acceleration tensor are included in the section for intrinsic derivatives in Chapter 2.

The product of the mass of a particle and its intrinsic derivative, a^i, can be thought of as the contravariant tensor components, $\overline{F}^i$, of a force, F.

$$\overline{F}^i = Ma^i \qquad (4.2)$$

The actual components of the force, F^i, are found by the equation:

$$F^i = \sqrt{g_{ii}}\ \overline{F}^i \qquad (4.3)$$

The corresponding term for an increment of work, dW^i, is:

$$dW^i = g_{ii}\ \overline{F}^i\, dx^i \qquad (4.4)$$

Now, kinetic energy T of a particle can also be expressed as a tensor equation:

$$T = \frac{M}{2}\, g_{ij}\, \dot{x}^i\, \dot{x}^j \qquad (4.5)$$

An example of the kinetic energy in polar co-ordinates, r and Θ, using Equation 4.5 is:

$$\dot{x}^1 = \dot{r}, \quad g_{11} = 1$$

$$\dot{x}^2 = \dot{\Theta}, \quad g_{22} = r^2$$

$$T = \frac{M}{2}(\dot{r}^2 + r^2\, \dot{\Theta}^2)$$

LAGRANGE'S EQUATIONS OF MOTION FOR CLASSICAL MECHANICS

Newton's Law, for a force F_i, in the form used by Lagrange can be expressed:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}^i}\right] - \frac{\partial T}{\partial x^i} = F_i \qquad (4.6)$$

The term on the right-hand side of Equation 4.6 for a conservative system expressed as a function of the Newtonian gravitational potential is:

$$F_i = -\frac{\partial V}{\partial x^i} \qquad (4.7)$$

As explained in Chapter 1, Equation 4.7 is a covariant tensor, Rank 1.

Now, referring back to Equation 4.6, the equivalent form in tensor notation for the terms on the left-hand side is determined as follows:

$$\frac{\partial T}{\partial \dot{x}^i} = Mg_{ij}\,\dot{x}^j \qquad (4.8)$$

$$\frac{d}{dt}\left(Mg_{ij}\,\dot{x}^j \right) = M\left(g_{ij}\,\ddot{x}^j + \frac{\partial g_{ij}}{\partial x^k}\dot{x}^j\,\dot{x}^k \right) \qquad (4.9)$$

$$\frac{\partial T}{\partial x^i} = \frac{M}{2}\,\frac{\partial g_{jk}}{\partial x^i}\,\dot{x}^j\,\dot{x}^k \qquad (4.10)$$

If we subtract Equation 4.10 from Equation 4.9, and use the result:

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{1}{2}\left[\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j}\right]$$

we get the following equations:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}^i}\right] - \frac{\partial T}{\partial x^i} = M\left(g_{ij}\,\ddot{x}^j + [jk, i]\;\dot{x}^j\,\dot{x}^k \right) \qquad (4.11)$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}^i}\right] - \frac{\partial T}{x^i} = Mg_{il}\left[\ddot{x}^l + \left\{\frac{l}{j\,k}\right\}\dot{x}^j\,\dot{x}^k\right] \tag{4.12}$$

Using Equations 4.12 and 4.7, we have Lagrange's equations of motion for a particle in a Newtonian gravitational field.

$$Mg_{il}\left[\ddot{x}^l + \left\{\begin{matrix} l \\ j\,k \end{matrix}\right\}\dot{x}^j\,\dot{x}^k\right] = -\frac{\partial V}{\partial x^i} \tag{4.13}$$

To illustrate Equation 4.13, we use the motion of a particle, M, described by polar co-ordinates r and Θ.

$$x^1 = r, \quad g_{11} = 1$$

$$x^2 = \Theta, \quad g_{22} = r^2$$

$$Mg_{11}\left[\ddot{x}^1 + \left\{\begin{matrix} 1 \\ 2\,2 \end{matrix}\right\}(\dot{x}^2)^2\right] = -\frac{\partial V}{\partial x^1} \tag{4.14}$$

$$Mg_{22}\left[\ddot{x}^2 + \left\{\begin{matrix} 2 \\ 1\,2 \end{matrix}\right\}\dot{x}^1\,\dot{x}^2 + \left\{\begin{matrix} 2 \\ 2\,1 \end{matrix}\right\}\dot{x}^2\,\dot{x}^1\right] = 0 \tag{4.15}$$

The right-hand side of Equation 4.15 is zero because the gravitational potential is a function of distance r. Using the values for g_{11} and g_{22}, the only nonvanishing Christoffel symbols are:

$$\left\{\begin{matrix} 1 \\ 2\,2 \end{matrix}\right\} = -r, \quad \left\{\begin{matrix} 2 \\ 1\,2 \end{matrix}\right\} = \left\{\begin{matrix} 2 \\ 2\,1 \end{matrix}\right\} = \frac{1}{r}$$

Now, Equations 4.14 and 4.15 assume the familiar form for the orbit of a particle, M, about a central body, m $\gg$ M.

$$M\left(\ddot{r} - r\dot{\Theta}^2\right) = -\frac{\partial V}{\partial r} \qquad (4.16)$$

$$M\left(r^2\ddot{\Theta} + 2r\dot{\Theta}\dot{r}\right) = 0 \qquad (4.17)$$

CLASSICAL SOLUTION OF THE TWO-BODY PROBLEM

The solution of Equations 4.16 and 4.17 will give the orbit of a particle, M, about a central body, m»M, (Fig. 4-1). These equations can be set equal to zero and expressed:

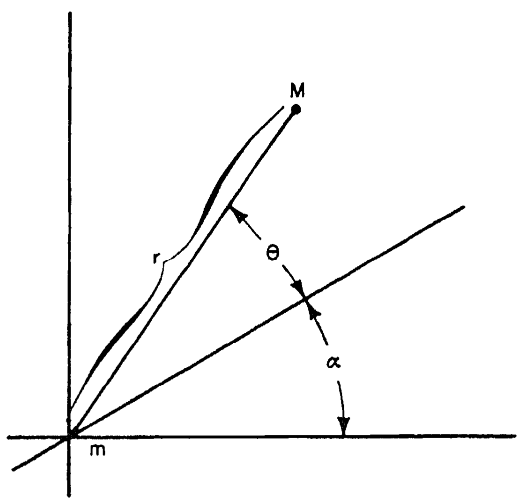

Fig. 4-1.

$$\ddot{r} - r\dot{\Theta}^2 + \frac{\mu m}{r^2} = 0 \qquad (4.18)$$

$$\frac{d}{dt}\left(r^2\,\dot{\Theta}\right) = 0 \qquad (4.19)$$

If Equation 4.19 is integrated, we get the result:

$$r^2\dot{\Theta} = h \qquad (4.20)$$

Letting $\frac{d\Theta}{dt} = \dot{\Theta}$, Equation 4.20 can be revised.

$$\frac{r^2}{2}\,d\Theta = \frac{h}{2}\,dt \tag{4.21}$$

This is Kepler's law of areas; the area swept out by the radius vector in any given time interval is the same for all parts of the path. The term $\frac{r^2}{2}\,d\Theta$ is the differential area of a triangle, and h is a constant of integration. Now, the following relations can be derived from Equation 4.21, and used to eliminate parameter t from Equation 4.18.

$$\left(\frac{d\Theta}{dt}\right)^2 = \frac{h^2}{r^4} \tag{4.22}$$

$$\frac{dr}{dt} = \frac{dr}{d\Theta}\,\frac{h}{r^2} \tag{4.23}$$

$$\frac{d^2 r}{dt^2} = \frac{h}{r^2}\,\frac{d}{d\Theta}\left(\frac{h}{r^2}\,\frac{dr}{d\Theta}\right) \tag{4.24}$$

Substituting Equations 4.22 and 4.24 in Equation 4.18 yields the result:

$$h\,\frac{d}{d\Theta}\left(\frac{h}{r^2}\,\frac{dr}{d\Theta}\right) - \frac{h^2}{r} + \mu m = 0 \tag{4.25}$$

Now, if we substitute $u = \frac{1}{r}$ in Equation 4.25, we get the following equation:

$$\frac{d^2u}{d\Theta^2} + u = \frac{\mu m}{h^2} \tag{4.26}$$

The solution of this equation is:

$$u = \frac{\mu m}{h^2}\left[1 - \varepsilon \cos\,(\Theta - \alpha)\right]$$

$$r = \frac{h^2}{\mu m\ [1 - \varepsilon \cos (\Theta - \alpha)]}$$

is the eccentricity and α is called the perihelion constant.

Equation 4.26 will be compared with Einstein's equation for the two-body problem, using the general theory of relativity. Einstein's solution is presented in the last section of this chapter.

THE GEODESIC EQUATIONS FOR THE TWO-BODY PROBLEM IN CLASSICAL MECHANICS

The two-body problem can be treated in Newtonian physics as a geodesic. The geodesic equations for the dynamical curves of an orbital trajectory in a Newtonian gravitational field are the same as Lagrange's equations of motion. This fact will be demonstrated to explain the concept of treating the two-body problem as a geodesic. The geodesic concept is used by Einstein to develop his equations of gravity in the general theory of relativity.

To present the equations of dynamical curves as geodesics that correspond to any given energy constant, c, we use a modified line element in Riemannian space.

$$ds^2 = 2(c - v)\, \bar{g}_{ij}\, dx^i\, dx^j \tag{4.27}$$

The $\bar{g}_{ij}$ are functions of kinetic energy $T = (c - v)$

$$\bar{g}_{ij} = M g_{ij} \tag{4.28}$$

$$T = \tfrac{1}{2} \bar{g}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \tag{4.29}$$

In equation 4.27 line element ds^2 has the dimensions $(\text{length})^2$, $(\text{force})^2$, and $(\text{time})^2$; and v is the Newtonian gravitational potential. If we divide Equation 4.27 by dt^2, we get the velocity along the trajectory:

$$\frac{ds}{dt} = \left[2\ (c - v)\ \bar{g}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right]^{\frac{1}{2}} \qquad (4.30)$$

Using the term for the kinetic energy in Equation 4.29, and the relation T = (c − v), we can simplify Equation 4.30 and get the result:

$$\frac{ds}{dt} = 2\ (c - v) \qquad (4.31)$$

The next step is to find the velocities and accelerations in terms of curvilinear co-ordinates x^i. First, we let $K = 2(c - v)$ and take the inverse of Equation 4.31.

$$\frac{dt}{ds} = K^{-1} \qquad (4.32)$$

Now, we can operate on Equation 4.32 and find the velocities and accelerations with respect to x^i.

$$\frac{dx^i}{ds} = \frac{dx^i}{dt} K^{-1} \qquad (4.33)$$

$$\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{dt^2} K^{-2} - \frac{dx^i}{dt} \frac{dK}{dt} K^{-3} \qquad (4.34)$$

The geodesic differential equations of the dynamical trajectory corresponding to the line element in Equation 4.27 are:

$$\frac{d^2x^i}{ds^2} + \overset{*}{\left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\}} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \qquad (4.35)$$

The term $\overset{*}{\left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\}}$ is the Christoffel symbols with respect to the dynamic metric coefficients, h_{ij}.

$$\left.\begin{aligned} h_{ij} &= K\bar{g}_{ij},\ h^{ij} = \frac{\bar{g}^{ij}}{K} \\ K &= 2\,(c - v) \\ \bar{g}_{ij} &= Mg_{ij} \end{aligned}\right\} \qquad (4.36)$$

Using Equations 4.33 and 4.34 in Equation 4.35 yields the result:

$$\frac{1}{K^2}\frac{d^2x^i}{dt^2} - \frac{1}{K^3}\frac{dK}{dt}\frac{dx^i}{dt} + {}^{*}\left\{ {i \atop j\,k} \right\} \frac{1}{K^2}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \qquad (4.37)$$

The Christoffel symbols ${}^{*}\left\{ {i \atop j\,k} \right\}$ are evaluated using the relations in Equations 4.36, as indicated in Equation 4.38.

$${}^{*}\left\{ {i \atop j\,k} \right\} = \frac{\bar{g}^{\beta i}}{2K}\left[\frac{K\partial\bar{g}_{j\beta}}{\partial x^k} + \frac{K\partial\bar{g}_{k\beta}}{\partial x^j} - \frac{K\partial\bar{g}_{jk}}{\partial x^\beta}\right] +$$

$$\frac{\bar{g}^{\beta i}}{2K}\left[\bar{g}_{j\beta}\frac{\partial K}{\partial x^k} + \bar{g}_{k\beta}\frac{\partial K}{\partial x^j} - \bar{g}_{jk}\frac{\partial K}{\partial x^\beta}\right] \qquad (4.38)$$

The first term in Equation 4.38 reduces to Christoffel symbols of the second kind with respect to $\bar{g}_{ij}$ and it will be denoted by the symbol $\left\{ {i \atop j\,k} \right\}$. The second term is reduced by taking the inner product of $\frac{g^{\beta i}}{2K}$ and the first quantities in the brackets and a substitution for $\frac{\partial K}{\partial x^\beta}$ in the last quantity. To perfrom these operations the following relations are used:

$$\bar{g}^{\beta i}\,\bar{g}_{j\beta} = 0,\ i = j$$

$$\bar{g}^{\beta i}\,\bar{g}_{k\beta} = 0,\ i = k$$

$$\frac{\partial K}{\partial x^\beta} = -\frac{2\partial v}{\partial x^\beta}$$

Using these relations and multiplying Equation 4.37 by K^2 and letting $\frac{dK}{dt} = \frac{\partial K}{\partial x^i}\frac{dx^i}{dt}$ we get the following equation:

$$\frac{d^2x^i}{dt^2} + \left\{ {i \atop j\,k} \right\}\frac{dx^j}{dt}\frac{dx^k}{dt} = -\frac{1}{K}\bar{g}^{\beta i}\frac{\partial v}{\partial x^\beta}\bar{g}_{jk}\frac{\partial x^j}{dt}\frac{\partial x^k}{dt} \tag{4.39}$$

Now, using the value $K = \bar{g}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}$ in Equation 4.39, we have Lagrange's equations of motion for the conservative system consisting of a particle in orbit about a central force system.

$$\frac{d^2x^i}{dt^2} + \left\{ {i \atop j\,k} \right\}\frac{dx^j}{dt}\frac{dx^k}{dt} = -\bar{g}^{\beta i}\frac{\partial v}{\partial x^\beta} \tag{4.40}$$

Equation 4.40 is the equivalent of Equation 4.13. The metric coefficients $\bar{g}_{ij}$ in Equation 4.40 are related to the metric coefficients g_{ij} in Equation 4.13 by:

$$\bar{g}_{ij} = Mg_{ij}$$

MINKOWSKI SPACE-TIME AND THE LORENTZ TRANSFORMATIONS

In the special theory of relativity, Newton's laws are revised in accordance with Einstein's concept of space-time. Einstein was led to his conclusions by the results of experiments that proved:

1. The velocity of light in a vacuum is a constant with respect to any inertial system. That is, it does not depend on the relative velocity of its source and the observer. An inertial system is defined as a system at rest, or having a constant velocity in a straight line and free from rotation.
2. The measured length of a line in an inertial system varies with the relative velocity of the observer. If the observer is traveling with an inertial system that has a velocity, V, the measured length of a line in this inertial system will be $\bar{L}$. If the observer is at rest with respect to the inertial system containing the line, and the line is in the direction of the velocity vector, V; the measured length of the line will be smaller. The length measured by the observer at rest will be:

$$L = \sqrt{1 - \frac{V^2}{C^2}}\,\bar{L} \tag{4.41}$$

This phenomenon is called the Fitzgerald contraction.

In conjunction with these experimental results, Einstein postulated that physical laws must have the same form with respect to all inertial systems.

Now, to write the laws of physics for the special theory of relativity, we must find an invariant with respect to all inertial systems. It is obvious that the Euclidean line element $dS^2 = dx^2 + dy^2 + dz^2$ is not an invariant with respect to all inertial systems, because of the Fitzgerald contraction. To meet this need for an invariant that is consistent with the constant velocity of light and the Fitzgerald contraction, we have the four dimensional space-time of Minkowski and the

Lorentz transformations. The space-time line element in Minkowski space using orthogonal cartesian co-ordinates is defined:

$$dS^2 = dx^2 + dy^2 + dz^2 - C^2dt^2 = 0 \qquad (4.42)$$

To explain Equation 4.42, we refer to Fig. 4-2. The square of the distance from a light source at point P to the origin, in any inertial system, must be the sum of the squares of the projected lengths of $\overline{PO}$ along the co-ordinate axis, x, y, z. It is also true that the distance from P to the origin must be equal to the product of the velocity of light, C, and the time, dt, required for light to travel from P to the origin.

Now, if we have another inertial system, $\bar{x}$, $\bar{y}$, $\bar{z}$, that has a velocity, V, with respect to the initial system, x, y, z, the space - time line element must also be zero.

$$d\bar{S}^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 - C^2\, d\bar{t}^2 = 0 \qquad (4.43)$$

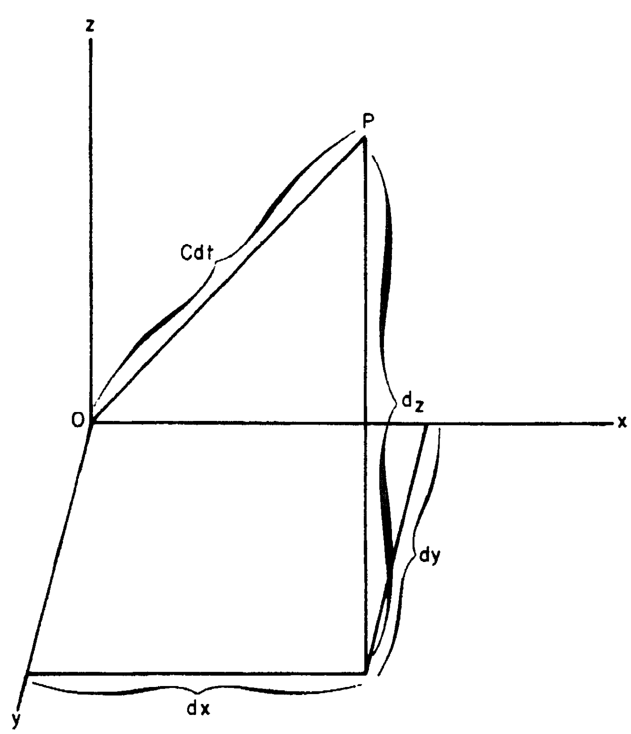

Fig. 4-2.

If the orientation of the axis in the two inertial systems is the same, and relative velocity vector V is parallel to the x and $\bar{x}$ axis; measurements taken along $\bar{x}$ by an observer in the x, y, z system will be shortened in accordance with the Fitzgerald contraction.

$$dx^2 = \left[1 - \frac{V^2}{C^2} \right] d\bar{x}^2 \qquad (4.44)$$

But the velocity of light is a constant. Therefore, the increment of time, $d\bar{t}$, measured by the observer in the x, y, z system must be modified for Equation 4.43 to maintain its invariant value of zero. The complete set of transformations for the invariant four-dimensional space-time line elements are the Lorentz transformations.

$$\left.\begin{aligned} \bar{x} &= \frac{1}{\sqrt{1 - \frac{V^2}{C^2}}} (x - Vt) \\ \bar{y} &= y \\ \bar{z} &= z \\ \bar{t} &= \frac{1}{\sqrt{1 - \frac{V^2}{C^2}}} \left(t - \frac{Vx}{C^2} \right) \end{aligned}\right\} \qquad (4.45)$$

Now, to prove that the Minkowski space - time line element is invariant with respect to a Lorentz transformation, we check the equation for line element $d\bar{S}^2$ in the $\bar{x}, \bar{y}, \bar{z}$ inertial system that is moving with a constant velocity, V, with respect to the x, y, z inertial system. Using Equations 4.45, $\bar{y} = y$, and $\bar{z} = z$. Therefore, $d\bar{x}^2 - C^2 d\bar{t}^2$ must be identical to $dx^2 - C^2 dt^2$.

$$d\bar{x} = \frac{dx - V\,dt}{\left[1 - \frac{V^2}{C^2}\right]^{\frac{1}{2}}}$$

$$d\bar{x}^2 = \frac{C^2\,dx^2 - 2\,C^2\,V\,dx\,dt + C^2\,V^2\,dt^2}{C^2 - V^2} \qquad (4.45)$$

$$d\bar{t} = \frac{dt - \frac{V}{C^2}\,dx}{\left[1 - \frac{V^2}{C^2}\right]^{\frac{1}{2}}}$$

$$d\bar{t}^2 = \frac{C^2\,dt^2 - 2V\,dx\,dt + \frac{V^2}{C^2}\,dx^2}{C^2 - V^2} \qquad (4.46)$$

Now, multiply Equation 4.46 by C^2 and subtract the result from Equation 4.45.

$$d\bar{x}^2 - C^2\,d\bar{t}^2 = \frac{C^2\,dx^2 - V^2\,dx^2 + C^2\,V^2\,dt^2 - C^4\,dt^2}{(C^2 - V^2)}$$

$$d\bar{x}^2 - C^2\,d\bar{t}^2 = dx^2 - C^2\,dt \qquad (4.47)$$

Thus, it can be concluded that Minkowski space - time line element dS^2 is invariant with respect to a Lorentz transformation. Using the concept of the metric tensors, the space - time line element can be written in the following two forms:

$$dS^2 = C^2\,dt^2 - g_{ij}\,dx^i\,dx^j, \ (i, j = 1, 2, 3) \qquad (4.48)$$

$$dS^2 = a_{\alpha\beta}\,dx^\alpha\,dx^\beta, \ (\alpha, \beta = 1, 2, 3, 4) \qquad (4.49)$$

$$a_{11} = g_{11}, \ a_{22} = g_{22}, \ a_{33} = g_{33}, \ a_{44} = C^2$$

The Minkowski space - time line element is used as the invariant for Einstein's special theory of relativity. In the following sections on the special theory of relativity, Newton's laws of motion are rewritten to comply with the Lorentz transformations and the Minkowski space - time line element.

THE FOUR-DIMENSIONAL MINKOWSKI MOMENTUM VECTOR AND EINSTEIN'S ENERGY EQUATION

Because of the Lorentz transformation, the Newtonian velocity vector,

$$\frac{dx^i}{dt}$$

and the corresponding acceleration vector,

$$\frac{d^2 x^i}{dt^2}$$

do not comply with Einstein's postulate that physical laws must have the same form with respect to all inertial systems. But, with the differential, dS, in terms of the Minkowski space - time line element, we do have the required invariant to write the laws of analytical mechanics for the special theory of relativity. Using invariant dS, the Minkowski four-dimensional momentum vector is defined.

$$P^i = m_o C \frac{dx^i}{dS}, \quad (i = 1, 2, 3, 4) \qquad (4.50)$$

To evaluate the components of velocity vector $\frac{dx^i}{dS}$ we use a set of orthogonal co-ordinates, $x = x^1$, $y = x^2$, $z = x^3$, and $t = x^4$.

$$dS = (C^2\,dt^2 - dx^2 - dy^2 - dz^2)^{1/2}$$

$$dS = \left[C^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2 \right]^{1/2} dt \qquad (4.51)$$

In Equation 4.51, the total velocity squared is:

$$V^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \qquad (4.52)$$

If we substitute corresponding value V^2 in Equation 4.51, we will have line element dS in terms of velocity of light C and relative velocity V.

$$dS = (C^2 - V^2)^{1/2}\ dt = C\sqrt{1 - \frac{V^2}{C^2}}\ dt \qquad (4.53)$$

Substituting dS as determined by Equation 4.53 into Equation 4.50 yields the result:

$$P^i = \frac{M_o\,C}{C\sqrt{1 - \frac{V^2}{C^2}}}\ \frac{dx^i}{dt} = \frac{M_o}{\sqrt{1 - \frac{V^2}{C^2}}}\ \frac{dx^i}{dt} \qquad (4.54)$$

Component $P^4 = m_o\ C\ \frac{dt}{dS}$ is called relativistic mass m of a particle.

$$P^4 = m = \frac{m_o}{\sqrt{1 - \frac{V^2}{C^2}}} \qquad (4.55)$$

Quantity m_o is called the rest mass of the particle. The rest mass is the value calculated by an observer moving with the same velocity as the particle.

From Equation 4.55, Einstein's energy equation can be derived. The first step is to multiply by C^2.

$$mC^2 = \frac{m_o C^2}{\sqrt{1 - \frac{V^2}{C^2}}} \tag{4.56}$$

If $V << C$, a Maclaurin expansion yields the approximate result:

$$mC^2 = m_o C^2 + \frac{m_o}{2} V^2 + \quad . . . \tag{4.57}$$

The first term on the right, $m_o C^2$, is called the intrinsic energy of matter. The second term on the right, $\frac{m_o}{2} V^2$, is the familiar term in classical mechanics for kinetic energy.

Returning to Equation 4.55, relativistic mass m becomes infinite as the velocity of the particle approaches the speed of light. Therefore, the relativistic mass must be thought of as the variable resistance to acceleration, and not as the measure of ponderable matter in the Newtonian sense.

THE MINKOWSKI ACCELERATION AND FORCE VECTORS

Prior to presenting the Minkowski acceleration and force vectors, the Newtonian force vector will be explained in terms of the time rate of change of the relativistic momentum. This procedure is used to illustrate the important fact that in the special theory of relativity, the total acceleration and resulting force vector are not always in the same direction. The equation for the Newtonian force vector is:

$$F^i = \frac{d}{dt}\left[\frac{m_o V^i}{\sqrt{1 - \frac{V^2}{C^2}}}\right] \tag{4.58}$$

To illustrate the fact that the total acceleration vector and resulting force vector are not always in the same direction, we let $V_x = V^1$ be the total velocity of a particle with respect to an inertial frame, and compute the component of force, F_x, along the x axis. Using Equation 4.58, we get the following results:

$$F_x = \frac{m_o}{\sqrt{1 - \frac{(V_x)^2}{C^2}}}\dot{V}_x + \frac{m_o \frac{(V_x)^2}{C^2}}{\left[\sqrt{1 - \frac{(V_x)^2}{C^2}}\right]^3}\dot{V}_x \tag{4.59}$$

Letting $a_x = \dot{V}_x$, and combining terms in Equation 4.59, F_x has the value:

$$F_x = \frac{m_o}{\left[\sqrt{1 - \frac{(V_x)^2}{C^2}}\right]^3} a_x \tag{4.60}$$

In a similar fashion, the component of force, F_y, for an acceleration vector, a_y, is:

$$F_y = \frac{m_o}{\sqrt{1 - \frac{(V_x)^2}{C^2}}} a_y \tag{4.61}$$

Now, if we take the vector sum, $a_x + a_y$, it will be at an angle with initial velocity vector V_x. But, the vector sum, $F_x + F_y$, will make a smaller angle with initial velocity vector V_x, because inertial mass m_x in the x direction is larger than inertial mass m_y in the y direction.

$$m_x = \frac{m_o}{\left[\sqrt{1 - \frac{(V_x)^2}{C^2}}\right]^3} \tag{4.62}$$

$$m_y = \frac{m_o}{\sqrt{1 - \frac{(V_x)^2}{C^2}}} \tag{4.63}$$

The foregoing analysis indicates that we must think of the inertial mass as the variable resistance of ponderable matter to acceleration. Now, to find a force vector, $\overline{P}^i$, in terms of Minkowski space-time invariant dS, it is necessary to find a new equation for the acceleration. The equation that complies with this condition is the Minkowski acceleration vector, f^i.

$$f^i = \frac{\delta u^i}{\delta S} = \frac{d^2x^i}{dS^2} + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \frac{dx^j}{dS}\frac{dx^k}{dS} \tag{4.64}$$

$$(i, j = 1, 2, 3, 4)$$

Equation 4.64 is essentially Equation 2.47 as applied to the four-dimensional space - time manifold. The term $u^i = \frac{dx^i}{dS}$ is a component of the Minkowski velocity vector. The equation for Minkowski force vector $\overline{P}^i$, in terms of rest mass m_o and Minkowski acceleration vector f^i, is:

$$\overline{P}^i = m_oC^2f^i \quad (i = 1, 2, 3) \tag{4.65}$$

The term $f^4 = 0$, as $x^4 = t$.

The Minkowski force is related to Newtonian relativistic force F^i by the equation:

$$F^i = \overline{P}^i \sqrt{1 - \frac{V^2}{C^2}} \qquad (4.66)$$

To illustrate Equations 4.65 and 4.66, Minkowski force $\overline{P}^x$ along the x axis for a total relative velocity V_x will be computed, and the results compared to Equation 4.60. Using Equation 4.53, the Minkowski velocity vector, $u^x = \frac{dx}{dS}$ is:

$$u^x = \frac{1}{C\sqrt{1 - \frac{(V_x)^2}{C^2}}} \frac{dx}{dt} \qquad (4.67)$$

Now, if the reference frame has orthogonal cartesian co-ordinates, the Christoffel symbols are zero and Equation 4.64 reduces to:

$$f^x = \frac{\delta u^x}{\delta S} = \frac{d^2 x}{dS^2} \qquad (4.68)$$

To find δu^x, Equation 4.67 is differentiated with respect to time:

$$\delta u^x = \frac{\delta u^x}{\delta t} dt$$

$$\delta u^x = \frac{1}{C\sqrt{1 - \frac{(V_x)^2}{C^2}}} \frac{d^2x}{dt^2} dt + \frac{\frac{(V_x)^2}{C^2}}{C\left[\sqrt{1 - \frac{(V_x)^2}{C^2}}\right]^3} \frac{d^2x}{dt^2} dt$$

$$\delta u^x = \frac{1}{C\left[\sqrt{1 - \frac{(V_x)^2}{C^2}}\right]^3} \frac{d^2x}{dt^2} dt \qquad (4.69)$$

Now, if Equation 4.69 is divided by the line element:

$$dS = C\sqrt{1 - \frac{(V_x)^2}{C^2}}\, dt$$

Minkowski acceleration vector f^x along the x axis is:

$$f^x = \frac{1}{C^2\left[\sqrt{1 - \frac{(V_x)^2}{C^2}}\right]^4} \frac{d^2x}{dt^2} \qquad (4.70)$$

Substituting Equation 4.70 into Equation 4.65 gives Minkowski force vector $\overline{P}^x$.

$$\overline{P}^x = \frac{m_0}{\left[\sqrt{1 - \frac{(V_x)^2}{C^2}}\right]^4} \frac{d^2x}{dt^2} \qquad (4.71)$$

In Equation 4.71 $\frac{d^2x}{dt^2} = a_x$, and Minkowski force $\overline{P}^x$ is indeed related to Newtonian relativistic force F^x, as calculated in Equation 4.60 by the factor:

$$F^x = \overline{P}^x \sqrt{1 - \frac{(V_x)^2}{C^2}} \qquad (4.72)$$

In conclusion, the total acceleration vector and the total force vector are not always in the same direction because of the relativistic mass; and the force required to accelerate a particle becomes infinite as the relative velocity of the particle approaches the speed of light.

EINSTEIN'S GRAVITATIONAL EQUATIONS FOR THE GENERAL THEORY OF RELATIVITY

In the special theory of relativity, all measurements are made with respect to inertial systems. Thus, the distance (r) and the inertial mass (M) of a particle in the Newtonian gravitational potential must vary in accordance with the Lorentz transformations. But in the real universe, the concept of an inertial system at best can only be an approximation. We know that all bodies of ponderable matter are moving through fields of force. These fields of force were described, to a high degree of approximation, by Newton's gravitational potential. This means that all bodies are subjected to gravitational forces and, as a result, their motion must be accelerated. Therefore, Einstein concluded that the laws of motion should be independent of co-ordinate systems, and that the gravitational field influences and even determines the metrical laws of space-time. In an attempt to correct for the inherent errors in using the Lorentz transformations for accelerated particles in a gravitational field he wrote his general theory of relativity including new equations for space-time and gravity.

In the special theory, the line element in polar co-ordinates is:

$$dS^2 = -dr^2 - r^2\, d\Theta^2 - r^2 \sin^2 \Theta d\Phi^2 + C^2\, dt^2 \quad (4.73)$$

This is a flat space (Euclidean), and curvature tensor $R_{ijkl} = 0$. Now, if the line element of a particle, M, is written with metric tensors g_{ij} as functions of the force field, the g_{ij} become functions of the gravitational potential.

The spherically symmetrical metric form for the line element, due to Schwartzchild, with the g_{ij} as functions of the gravitational potential is:

$$dS^2 = -\varepsilon^{\lambda} dr^2 - r^2\, d\Theta^2 - r^2 \sin^2 \Theta d\Phi^2 + C^2\, \varepsilon^{\gamma} dt^2 \quad (4.74)$$

λ and γ are functions of distance r. The line element in Equation 4.74 does not correspond to flat space. The space is curved, as differential distance dr, and differential time dt are functions of total distance r. But, as distance r becomes infinite, the space must become flat as the gravitational force approaches zero as a limit. And, the line element in Equation 4.74 will approach the line element in Equation 4.73 as a limit.

To evaluate λ and v, Ricci tensors R_{ij} are evaluated to comply with the boundary condition, $R_{ij} = 0$, when $r = \infty$. The g_{ij} for Equation 4.74 are:

$$g_{11} = -\varepsilon^{\gamma}, \; g_{22} = -r^2, \; g_{33} = -r^2 \sin^2\Theta, \; g_{44} = \varepsilon^{\gamma}$$

Ricci tensors R_{ij} can be evaluated by Equation 3.8.

$$R_{ij} = \frac{\partial \begin{Bmatrix} \alpha \\ i\,\alpha \end{Bmatrix}}{\partial x^j} - \frac{\partial \begin{Bmatrix} \alpha \\ i\,j \end{Bmatrix}}{\partial x^a} +$$

$$\begin{Bmatrix} \alpha \\ \beta\, j \end{Bmatrix}\begin{Bmatrix} \beta \\ i\,\alpha \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \beta\,\alpha \end{Bmatrix}\begin{Bmatrix} \beta \\ i\,j \end{Bmatrix} \qquad (3.8)$$

There are four definite equations for the Ricci tensors R_{11}, R_{22}, R_{33}, and R_{44}. Using metric tensors g_{ij} for the line element as expressed by Equation 4.74, the nonvanishing Christoffel symbols are:

$$\left.\begin{aligned}
\begin{Bmatrix} 1 \\ 1\,1 \end{Bmatrix} &= \frac{g^{11}}{2}\frac{\partial g_{11}}{\partial r} = \frac{d\lambda}{2dr} \\
\begin{Bmatrix} 1 \\ 2\,2 \end{Bmatrix} &= -\frac{g^{11}}{2}\frac{\partial g_{22}}{\partial r} = -r\varepsilon^{-\lambda} \\
\begin{Bmatrix} 1 \\ 3\,3 \end{Bmatrix} &= -\frac{g^{11}}{2}\frac{\partial g_{33}}{\partial r} = -r \sin^2\Theta\, \varepsilon^{-\lambda} \\
\begin{Bmatrix} 1 \\ 4\,4 \end{Bmatrix} &= -\frac{g^{11}}{2}\frac{\partial g_{44}}{\partial r} = \frac{\varepsilon^{\gamma-\lambda}}{2}\frac{d\gamma}{dr}
\end{aligned}\right\} \qquad (4.75)$$

$$\left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\} = \frac{g^{22}}{2} \frac{\partial g_{22}}{\partial r} = \frac{1}{r}$$

$$\left\{ \begin{matrix} 2 \\ 3\ 3 \end{matrix} \right\} = - \frac{g^{22}}{2} \frac{\partial g_{33}}{\partial \Theta} = -\sin\Theta \cos\Theta$$

$$\left\{ \begin{matrix} 3 \\ 1\ 3 \end{matrix} \right\} = \frac{g^{33}}{2} \frac{\partial g_{33}}{\partial r} = \frac{1}{r} \qquad \text{(4.75 cont.)}$$

$$\left\{ \begin{matrix} 3 \\ 2\ 3 \end{matrix} \right\} = \frac{g^{33}}{2} \frac{\partial g_{33}}{\partial \Theta} = \cot\Theta$$

$$\left\{ \begin{matrix} 4 \\ 1\ 4 \end{matrix} \right\} = \frac{g^{44}}{2} \frac{\partial g_{44}}{\partial r} = \frac{d\gamma}{2dr}$$

The values for the Christoffel symbols in Equations 4.75 yield the following values for the Ricci tensors when they are substituted in Equation 3.8. These tensors are set equal to zero and evaluated for the boundary condition, $R_{ij} = 0$ as $r \rightarrow \infty$

$$R_{11} = \frac{1}{2}\frac{d^2\gamma}{dr^2} + \frac{1}{4}\left(\frac{d\gamma}{dr}\right)^2 - \frac{1}{4}\frac{d\lambda}{dr}\frac{d\gamma}{dr} - \frac{1}{r}\frac{d\lambda}{dr} = 0$$

$$R_{22} = \varepsilon^{-\lambda} \left[1 + \frac{r}{2}\left(\frac{d\gamma}{dr} - \frac{d\lambda}{dr}\right)\right] -1 = 0$$

$$R_{33} = \sin^2\Theta\,\varepsilon^{-\lambda} \left[1 + \frac{r}{2}\left(\frac{d\gamma}{dr} - \frac{d\lambda}{dr}\right)\right] -\sin^2\Theta = 0$$

$$R_{44} = \varepsilon^{\gamma-\lambda} \left[-\frac{1}{2}\frac{d^2\gamma}{dr^2} - \frac{1}{4}\left(\frac{d\gamma}{dr}\right)^2 + \frac{1}{4}\frac{d\lambda}{dr}\frac{d\gamma}{dr} - \frac{1}{r}\frac{d\gamma}{dr}\right] = 0$$

(4.76)

Now, if we divide R_{44} by $\varepsilon^{\gamma-\lambda}$ and add the result to R_{11}, we get

$$\left.\begin{aligned} \frac{d\lambda}{dr} + \frac{d\gamma}{dr} &= 0 \\ \lambda + \gamma &= \text{Constant} \end{aligned}\right\} \tag{4.77}$$

But, we must select the values for λ and γ that will let Equation 4.74 approach Equation 4.73 as a limit for $r \to \infty$. This requires that λ and γ approach zero as r approaches infinity. To meet these conditions:

$$\lambda + \gamma = 0, \quad \lambda = -\gamma \tag{4.78}$$

Using the relation $\lambda = -\gamma$ and $R_{22} = 0$, we can obtain the result:

$$\varepsilon^{\gamma}\left(1 + r\frac{d\gamma}{dr}\right) = 1 \tag{4.79}$$

Now, we let $K_0 = \varepsilon^{\gamma}$, and $\dfrac{d\gamma}{dr} = \dfrac{1}{K_0}\,\dfrac{K_0}{dr}$

Then, substituting these relations in Equation 4.79 yields the following results:

$$K_0\left(1 + \frac{r}{K_0}\,\frac{dK_0}{dr}\right) = 1 \tag{4.80}$$

$$\frac{dK_0}{1 - K_0} = \frac{dr}{r} \tag{4.81}$$

We integrate Equation 4.81 and obtain:

$$K_o = 1 - \frac{2m}{C^2 r} \tag{4.82}$$

The term $\frac{2m}{C^2}$ is the constant of integration and it is associated with the mass of a central body, m » M, where M is the mass of the orbital body. Using Equation 4.82 and the relations $K_o = \varepsilon^{\gamma}$, $\lambda = -\gamma$, the line element in Equation 4.74 assumes the form:

$$dS^2 = -\frac{dr^2}{1 - \frac{2m}{C^2 r}} - r^2 d\Theta^2 - r^2 \sin^2\Theta d\Phi^2 +$$

$$C^2 \left(1 - \frac{2m}{C^2 r}\right) dt^2 \tag{4.83}$$

RELATIVISTIC SOLUTION OF THE TWO-BODY PROBLEM FOR THE GENERAL THEORY OF RELATIVITY

To solve for the orbit of a particle, M, about a central body, m » M, in terms of Einstein's gravitational equations for the general theory of relativity, we will use the Schwartzchild line element; and the dynamical trajectory will be treated as a geodesic.

$$\frac{d^2x^i}{dS^2} + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \frac{dx^j}{dS} \frac{dx^k}{dS} = 0 \tag{4.84}$$

Equation 4.84 is of the same form as Equations 3.44 and 4.35. The first term is analogous to the inertia term in Newton's equations of motion and the second term is analogous to the Newtonian gravitational force.

The first step in the solution of the orbital problem is to write and evaluate the four equations for the geodesics in terms of variables Θ, r, Φ, and t. The first geodesic in terms of Θ is:

$$\frac{d^2\Theta}{dS^2} + 2\left\{ {2 \atop 1\ 2} \right\} \frac{dr}{dS}\frac{d\Theta}{dS} + \left\{ {2 \atop 3\ 3} \right\} \left(\frac{d\Phi}{dS}\right)^2 = 0 \tag{4.85}$$

$$\frac{d^2\Theta}{dS^2} + \frac{2}{r}\frac{dr}{dS}\frac{d\Theta}{dS} - \sin\Theta\cos\Theta\left(\frac{d\Phi}{dS}\right)^2 = 0 \tag{4.86}$$

If the orbit is in the plane, $\Theta = \frac{\pi}{2}$, $\frac{d\Theta}{dS} = 0$, and Equation 4.86 is equal to zero.

The next geodesic equation in terms of variable r is:

$$\frac{d^2r}{dS^2} + \left\{ {1 \atop 1\ 1} \right\} \left(\frac{dr}{dS}\right)^2 + \left\{ {1 \atop 3\ 3} \right\} \left(\frac{d\Phi}{dS}\right)^2 + \left\{ {1 \atop 4\ 4} \right\} \left(\frac{dt}{dS}\right)^2 = 0 \tag{4.87}$$

$$\frac{d^2r}{dS^2} + \frac{1}{2}\frac{d\lambda}{dr}\left(\frac{dr}{dS}\right)^2 - r\varepsilon^{-\lambda}\left(\frac{d\Phi}{dS}\right)^2 + \tfrac{1}{2}\varepsilon^{\gamma-\lambda}\frac{d\gamma}{dr}\left(\frac{dt}{dS}\right)^2 = 0 \tag{4.88}$$

Two more equations can be written; one in terms of Φ, and the other in terms of t. Making use of the condition

$\Theta = \frac{\pi}{2}$, these equations are:

$$\frac{d^2\Phi}{dS^2} + 2 \left\{ \begin{matrix} 3 \\ 1\ 3 \end{matrix} \right\} \frac{dr}{dS} \frac{d\Phi}{dS} = 0 \tag{4.89}$$

$$\frac{d^2\Phi}{dS^2} + \frac{2}{r} \frac{dr}{dS} \frac{d\Phi}{dS} = 0 \tag{4.90}$$

$$\frac{d^2t}{dS^2} + 2 \left\{ \begin{matrix} 4 \\ 1\ 4 \end{matrix} \right\} \frac{dt}{dS} \frac{dr}{dS} = 0 \tag{4.91}$$

$$\frac{d^2t}{dS^2} + \frac{d\gamma}{dr} \frac{dt}{dS} \frac{dr}{dS} = 0 \tag{4.92}$$

Integrating Equations 4.90 and 4.92 respectively yields the results:

$$r^2 \frac{d\Phi}{dS} = h \tag{4.93}$$

$$\mathrm{Log} \frac{dt}{dS} + \gamma = \log C \tag{4.94}$$

But, $K_0 = \varepsilon^{\gamma}$, and $\log K_0 = \gamma$.

Now, if we substitute the relation, $\log K_0 = \gamma$, in Equation 4.94 we get the relation:

$$\frac{dt}{dS} = \frac{C}{K_0} \tag{4.95}$$

h and C in Equations 4.93 and 4.95 are constants of integration.

Because $\Theta = \frac{\pi}{2}$, Equation 4.83 for the line element reduces to:

$$dS^2 = -\frac{1}{K_0} dr^2 - r^2 d\Theta^2 + C^2 K_0\, dt^2 \qquad (4.96)$$

$$K_0 = 1 - \frac{2m}{C^2 r}$$

If we divide Equation 4.96 by dS^2, we get the equation:

$$1 = -\frac{1}{K_0}\left(\frac{dr}{dS}\right)^2 - r^2\left(\frac{d\Phi}{dS}\right)^2 + C^2K_0\left(\frac{dt}{dS}\right)^2 \qquad (4.97)$$

From Equation 4.93 we can get the additional relations:

$$\frac{dr}{dS} = \frac{h}{r^2}\frac{dr}{d\Phi}$$

$$\left(\frac{dr}{dS}\right)^2 = \left(\frac{h}{r^2}\frac{dr}{d\Phi}\right)^2 \qquad (4.98)$$

$$\frac{d\Phi}{dS} = \frac{h}{r^2}$$

$$\left(\frac{d\Phi}{dS}\right)^2 = \frac{h^2}{r^4} \qquad (4.99)$$

Now, if we substitute Equations 4.99, 4.98, and 4.95 in Equation 4.97 we get the result:

$$1 = -\frac{1}{K_0}\left(\frac{h}{r^2}\frac{dr}{d\Phi}\right)^2 - r^2\frac{h^2}{r^4} + \frac{C^4}{K_0} \qquad (4.100)$$

Now, if Equation 4.100 is multiplied by K_0, for the value

$$K_0 = \left(1 - \frac{2m}{C^2 r}\right),$$

the resulting equation is:

$$\left(\frac{h}{r^2}\frac{dr}{d\Phi}\right)^2 + \frac{h^2}{r^2} = C^4 - 1 + \frac{2m}{C^2 r} + \frac{2mh^2}{C^2 r^3} \tag{4.101}$$

To simplify Equation 4.101, it is divided by h^2 and the substitution $u = \frac{1}{r}$ is made.

$$\left(\frac{du}{d\Phi}\right)^2 + u^2 = \frac{C^4 - 1}{h^2} + \frac{2m}{C^2 h^2} u + \frac{2mu^3}{C^2} \tag{4.102}$$

Now, if Equation 4.102 is differentiated with respect to Φ, the result is the equation for an orbital body in terms of Einstein's gravitational theory using the Schwartzchild line element. The resulting equation is:

$$\frac{d^2u}{d\Phi^2} + u = \frac{m}{C^2 h^2} + \frac{3mu^2}{C^2} \tag{4.103}$$

Equation 4.103 is of the same form as Equation 4.26 in classical mechanics, if the Newtonian gravitational constant is modified to accommodate the term $\frac{1}{C^2}$. This solution differs from the classical solution by the small value associated with the term $\frac{3mu^2}{C^2}$.

In conclusion, $\frac{m}{C^2 h^2} \gg \frac{3mu^2}{C^2}$; therefore, neglecting $\frac{3mu^2}{C^2}$ gives Newton's solution for the two-body problem. The difference in the solutions has been used to account for the advance of the Perihelion of Mercury.

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PRINCIPLES AND APPLICATIONS OF TENSOR ANALYSIS

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CHAPTER 4. CLASSICAL AND RELATIVISTIC MECHANICS

Discusses the dynamics of a particle for classical and relativistic mechanics. Lagrange's equations of motion are developed as tensors for conservative systems, then written for the two-body problem and solved. The geodesic concept is introduced by writing the equations of motion for the two-body problem in classical mechanics, using dynamic curves as modified line elements in a Riemannian space. The geodesic concept is used in conjunction with the Schwartzchild line element for curved space to present Einstein's solution of the two-body problem for the general theory of relativity.

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