Contents

Chapter 1

Functions

In this chapter we review the basic concepts of functions, polynomial functions, rational functions, trigonometric functions, logarithmic functions, exponential functions, hyperbolic functions, algebra of functions, composition of functions and inverses of functions.

1.1 The Concept of a Function

Basically, a *function* f relates each element x of a set, say D_f, with exactly one element y of another set, say R_f. We say that D_f is the *domain* of f and R_f is the *range* of f and express the relationship by the equation $y = f(x)$. It is customary to say that the symbol x is an *independent variable* and the symbol y is the *dependent variable*.

Example 1.1.1 Let $D_f = \{a, b, c\}$, $R_f = \{1, 2, 3\}$ and $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$. Sketch the graph of f.

graph

Example 1.1.2 Sketch the graph of $f(x) = |x|$.

Let D_f be the set of all real numbers and R_f be the set of all non-negative real numbers. For each x in D_f, let $y = |x|$ in R_f. In this case, $f(x) = |x|$, the absolute value of x. Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We note that $f(0) = 0, f(1) = 1$ and $f(-1) = 1$.

If the domain D_f and the range R_f of a function f are both subsets of the set of all real numbers, then the *graph* of f is the set of all ordered pairs $(x, f(x))$ such that x is in D_f. This graph may be sketched in the xy-coordinate plane, using $y = f(x)$. The graph of the absolute value function in Example 2 is sketched as follows:

graph

Example 1.1.3 Sketch the graph of

$$f(x) = \sqrt{x - 4}.$$

In order that the range of f contain real numbers only, we must impose the restriction that $x \geq 4$. Thus, the domain D_f contains the set of all real numbers x such that $x \geq 4$. The range R_f will consist of all real numbers y such that $y \geq 0$. The graph of f is sketched below.

graph

Example 1.1.4 A useful function in engineering is the unit step function, u, defined as follows:

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

The graph of $u(x)$ has an upward *jump* at $x = 0$. Its graph is given below.

graph

Example 1.1.5 Sketch the graph of

$$f(x) = \frac{x}{x^2 - 4}.$$

It is clear that D_f consists of all real numbers $x \neq \pm 2$. The graph of f is given below.

graph

We observe several things about the graph of this function. First of all, the graph has three distinct pieces, separated by the dotted vertical lines $x = -2$ and $x = 2$. These vertical lines, $x = \pm 2$, are called the *vertical asymptotes*. Secondly, for large positive and negative values of x, $f(x)$ tends to zero. For this reason, the x-axis, with equation $y = 0$, is called a *horizontal asymptote*.

Let f be a function whose domain D_f and range R_f are sets of real numbers. Then f is said to be *even* if $f(x) = f(-x)$ for all x in D_f. And f is said to be *odd* if $f(-x) = -f(x)$ for all x in D_f. Also, f is said to be *one-to-one* if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

Example 1.1.6 Sketch the graph of $f(x) = x^4 - x^2$.

This function f is *even* because for all x we have

$$f(-x) = (-x)^4 - (-x)^2 = x^4 - x^2 = f(x).$$

The graph of f is symmetric to the y-axis because $(x, f(x))$ and $(-x, f(x))$ are on the graph for every x. *The graph of an even function is always symmetric to the y-axis.* The graph of f is given below.

graph

This function f is not one-to-one because $f(-1) = f(1)$.

Example 1.1.7 Sketch the graph of $g(x) = x^3 - 3x$.

The function g is an *odd* function because for each x,

$$g(-x) = (-x)^3 - 3(-x) = -x^3 + 3x = -(x^3 - 3x) = -g(x).$$

The graph of this function g is symmetric to the origin because $(x, g(x))$ and $(-x, -g(x))$ are on the graph for all x. *The graph of an odd function is always symmetric to the origin.* The graph of g is given below.

graph

This function g is not one-to-one because $g(0) = g(\sqrt{3}) = g(-\sqrt{3})$.

It can be shown that every function f can be written as the sum of an even function and an odd function. Let

$$g(x) = \frac{1}{2}(f(x) + f(-x)), h(x) = \frac{1}{2}(f(x) - f(-x)).$$

Then,

$$\begin{aligned} g(-x) &= \frac{1}{2}(f(-x) + f(x)) = g(x) \\ h(-x) &= \frac{1}{2}(f(-x) - f(x)) = -h(x). \end{aligned}$$

Furthermore

$$f(x) = g(x) + h(x).$$

Example 1.1.8 Express f as the sum of an even function and an odd function, where,

$$f(x) = x^4 - 2x^3 + x^2 - 5x + 7.$$

We define

$$\begin{aligned} g(x) &= \frac{1}{2}(f(x) + f(-x)) \\ &= \frac{1}{2}\{(x^4 - 2x^3 + x^2 - 5x + 7) + (x^4 + 2x^3 + x^2 + 5x + 7)\} \\ &= x^4 + x^2 + 7 \end{aligned}$$

and

$$\begin{aligned} h(x) &= \frac{1}{2}(f(x) - f(-x)) \\ &= \frac{1}{2}\{(x^4 - 2x^3 + x^2 - 5x + 7) - (x^4 + 2x^3 + x^2 + 5x + 7)\} \\ &= -2x^3 - 5x. \end{aligned}$$

Then clearly $g(x)$ is even and $h(x)$ is odd.

$$\begin{aligned} g(-x) &= (-x)^4 + (-x)^2 + 7 \\ &= x^4 + x^2 + 7 \\ &= g(x) \\ h(-x) &= -2(-x)^3 - 5(-x) \\ &= 2x^3 + 5x \\ &= -h(x). \end{aligned}$$

We note that

$$\begin{aligned} g(x) + h(x) &= (x^4 + x^2 + 7) + (-2x^3 - 5x) \\ &= x^4 - 2x^3 + x^2 - 5x + 7 \\ &= f(x). \end{aligned}$$

It is not always easy to tell whether a function is one-to-one. The graphical test is that if no horizontal line crosses the graph of f more than once, then f is one-to-one. To show that f is one-to-one mathematically, we need to show that $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Example 1.1.9 Show that $f(x) = x^3$ is a one-to-one function.

Suppose that $f(x_1) = f(x_2)$. Then

$$\begin{aligned} 0 &= x_1^3 - x_2^3 \\ &= (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) \qquad \text{(By factoring)} \end{aligned}$$

If $x_1 \neq x_2$, then $x_1^2 + x_1x_2 + x_2^2 = 0$ and

$$\begin{aligned} x_1 &= \frac{-x_2 \pm \sqrt{x_2^2 - 4x_2^2}}{2} \\ &= \frac{-x_2 \pm \sqrt{-3x_2^2}}{2}. \end{aligned}$$

This is only possible if x_1 is not a real number. This contradiction proves that $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$ and, hence, f is one-to-one. The graph of f is given below.

graph

If a function f with domain D_f and range R_f is one-to-one, then f has a unique *inverse* function g with domain R_f and range D_f such that for each x in D_f,

$$g(f(x)) = x$$

and for such y in R_f,

$$f(g(y)) = y.$$

This function g is also written as f^{-1}. It is not always easy to express g explicitly but the following algorithm helps in computing g.

Step 1 Solve the equation $y = f(x)$ for x in terms of y and make sure that there exists exactly one solution for x.

Step 2 Write $x = g(y)$, where $g(y)$ is the unique solution obtained in Step 1.

Step 3 If it is desirable to have x represent the independent variable and y represent the dependent variable, then exchange x and y in Step 2 and write

$$y = g(x).$$

Remark 1 If $y = f(x)$ and $y = g(x) = f^{-1}(x)$ are graphed on the same coordinate axes, then the graph of $y = g(x)$ is a mirror image of the graph of $y = f(x)$ through the line $y = x$.

Example 1.1.10 Determine the inverse of $f(x) = x^3$.

We already know from Example 9 that f is one-to-one and, hence, it has a unique inverse. We use the above algorithm to compute $g = f^{-1}$.

Step 1 We solve $y = x^3$ for x and get $x = y^{1/3}$, which is the unique solution.

Step 2 Then $g(y) = y^{1/3}$ and $g(x) = x^{1/3} = f^{-1}(x)$.

Step 3 We plot $y = x^3$ and $y = x^{1/3}$ on the same coordinate axis and compare their graphs.

graph

A polynomial function p of degree n has the general form

$$p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n, \; a_2 \neq 0.$$

The polynomial functions are some of the simplest functions to compute. For this reason, in calculus we approximate other functions with polynomial functions.

A rational function r has the form

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions. We will assume that $p(x)$ and $q(x)$ have no common non-constant factors. Then the domain of $r(x)$ is the set of all real numbers x such that $q(x) \neq 0$.

Exercises 1.1

1. Define each of the following in your own words.

   (a) f is a function with domain D_f and range R_f

   (b) f is an even function

   (c) f is an odd function

   (d) The graph of f is symmetric to the y-axis

   (e) The graph of f is symmetric to the origin.

   (f) The function f is one-to-one and has inverse g.

2. Determine the domains of the following functions

(a) $f(x) = \dfrac{|x|}{x}$ (b) $f(x) = \dfrac{x^2}{x^3 - 27}$

(c) $f(x) = \sqrt{x^2 - 9}$ (d) $f(x) = \dfrac{x^2 - 1}{x - 1}$

3. Sketch the graphs of the following functions and determine whether they are even, odd or one-to-one. If they are one-to-one, compute their inverses and plot their inverses on the same set of axes as the functions.

(a) $f(x) = x^2 - 1$ (b) $g(x) = x^3 - 1$

(c) $h(x) = \sqrt{9 - x}, x \geq 9$ (d) $k(x) = x^{2/3}$

4. If $\{(x_1, y_1), (x_2, y_2), \ldots, (x_{n+1}, y_{n+1})\}$ is a list of discrete data points in the plane, then there exists a unique nth degree polynomial that goes through all of them. Joseph Lagrange found a simple way to express this polynomial, called the Lagrange polynomial.

For $n = 2$, $P_2(x) = y_1 \left(\dfrac{x - x_2}{x_1 - x_2} \right) + y_2 \left(\dfrac{x - x_1}{x_2 - x_1} \right)$

For $n = 3$, $P_3(x) = y_1 \dfrac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \dfrac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + y_3 \dfrac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$

$$P_4(x) = y_1 \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} + y_2 \frac{(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} +$$

$$y_3 \frac{(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} + y_4 \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}$$

Consider the data $\{(-2, 1), (-1, -2), (0, 0), (1, 1), (2, 3)\}$. Compute $P_2(x)$, $P_3(x)$, and $P_4(x)$; plot them and determine which data points they go through. What can you say about $P_n(x)$?

5. A *linear function* has the form $y = mx + b$. The number m is called the slope and the number b is called the y-intercept. The graph of this function goes through the point $(0, b)$ on the y-axis. In each of the following determine the slope, y-intercept and sketch the graph of the given linear function:

a) $y = 3x - 5$ b) $y = -2x + 4$ c) $y = 4x - 3$

d) $y = 4$ e) $2y + 5x = 10$

6. A quadratic function has the form $y = ax^2 + bx + c$, where $a \neq 0$. On completing the square, this function can be expressed in the form

$$y = a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right\}.$$

The graph of this function is a *parabola* with *vertex* $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$ and line of symmetry axis being the vertical line with equation $x = \frac{-b}{2a}$. The graph opens upward if $a > 0$ and downwards if $a < 0$. In each of the following quadratic functions, determine the vertex, symmetry axis and sketch the graph.

a) $y = 4x^2 - 8$ b) $y = -4x^2 + 16$ c) $y = x^2 + 4x + 5$

d) $y = x^2 - 6x + 8$ e) $y = -x^2 + 2x + 5$ f) $y = 2x^2 - 6x + 12$

g) $y = -2x^2 - 6x + 5$ h) $y = -2x^2 + 6x + 10$ i) $3y + 6x^2 + 10 = 0$

j) $y = -x^2 + 4x + 6$ k) $y = -x^2 + 4x$ l) $y = 4x^2 - 16x$

7. Sketch the graph of the linear function defined by each linear equation and determine the x-intercept and y-intercept if any.

a) $3x - y = 3$ b) $2x - y = 10$ c) $x = 4 - 2y$

d) $4x - 3y = 12$ e) $3x + 4y = 12$ f) $4x + 6y = -12$

g) $2x - 3y = 6$ h) $2x + 3y = 12$ i) $3x + 5y = 15$

8. Sketch the graph of each of the following functions:

a) $y = 4|x|$ b) $y = -4|x|$

c) $y = 2|x| + |x - 1|$ d) $y = 3|x| + 2|x - 2| - 4|x + 3|$

e) $y = 2|x + 2| - 3|x + 1|$

9. Sketch the graph of each of the following piecewise functions.

a) $y = \begin{cases} 2 & \text{if } x \geq 0 \\ -2 & \text{if } x < 0 \end{cases}$ b) $y = \begin{cases} x^2 & \text{for } x \leq 0 \\ 2x + 4 & \text{for } x > 0 \end{cases}$

c) $y = \begin{cases} 4x^2 & \text{if } x \geq 0 \\ 3x^3 & x < 0 \end{cases}$ d) $y = \begin{cases} 3x^2 & \text{for } x \leq 1 \\ 4 & \text{for } x > 1 \end{cases}$

e) $y = n - 1$ for $n - 1 \leq x < n$, for each integer n.

f) $y = n$ for $n - 1 < x \leq n$ for each integer n.

10. The *reflection* of the graph of $y = f(x)$ is the graph of $y = -f(x)$. In each of the following, sketch the graph of f and the graph of its reflection on the same axis.

a) $y = x^3$ b) $y = x^2$ c) $y = |x|$

d) $y = x^3 - 4x$ e) $y = x^2 - 2x$ f) $y = |x| + |x - 1|$

g) $y = x^4 - 4x^2$ h) $y = 3x - 6$ i) $y = \begin{cases} x^2 + 1 & \text{for } x \leq 0 \\ x^3 + 1 & \text{if } x < 0 \end{cases}$

11. The graph of $y = f(x)$ is said to be

    (i) Symmetric with respect to the y-axis if (x, y) and $(-x, y)$ are both on the graph of f;

    (ii) Symmetric with respect to the origin if (x, y) and $(-x, -y)$ are both on the graph of f.

    For the functions in problems 10 a) – 10 i), determine the functions whose graphs are (i) Symmetric with respect to y-axis or (ii) Symmetric with respect to the origin.

12. Discuss the symmetry of the graph of each function and determine whether the function is even, odd, or neither.

    a) $f(x) = x^6 + 1$ b) $f(x) = x^4 - 3x^2 + 4$ c) $f(x) = x^3 - x^2$

    d) $f(x) = 2x^3 + 3x$ e) $f(x) = (x - 1)^3$ f) $f(x) = (x + 1)^4$

    g) $f(x) = \sqrt{x^2 + 4}$ h) $f(x) = 4|x| + 2$ i) $f(x) = (x^2 + 1)^3$

    j) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ k) $f(x) = \sqrt{4 - x^2}$ l) $f(x) = x^{1/3}$

1.2 Trigonometric Functions

The trigonometric functions are defined by the points (x, y) on the unit circle with the equation $x^2 + y^2 = 1$.

graph

Consider the points $A(0, 0), B(x, 0), C(x, y)$ where $C(x, y)$ is a point on the unit circle. Let θ, read theta, represent the length of the arc joining the points $D(1, 0)$ and $C(x, y)$. This length is the radian measure of the angle CAB. Then we define the following six trigonometric functions of θ as follows:

$$\sin\theta = \frac{y}{1}, \cos\theta = \frac{x}{1}, \tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta},$$

$$\csc\theta = \frac{1}{y} = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{x} = \frac{1}{\cos\theta}, \cot\theta = \frac{x}{y} = \frac{1}{\tan\theta}.$$

Since each revolution of the circle has arc length $2\pi, \sin\theta$ and $\cos\theta$ have period 2π. That is,

$$\sin(\theta + 2n\pi) = \sin\theta \text{ and } \cos(\theta + 2n\pi) = \cos\theta,\ n = 0, \pm 1, \pm 2, \ldots$$

The function values of some of the common arguments are given below:

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

θ	$7\pi/6$	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	$11\pi/6$	2π
$\sin\theta$	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0
$\cos\theta$	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1

A function f is said to have period p if p is the smallest positive number such that, for all x,

$$f(x + np) = f(x),\ n = 0, \pm 1, \pm 2, \ldots .$$

Since $\csc\theta$ is the reciprocal of $\sin\theta$ and $\sec\theta$ is the reciprocal of $\cos(\theta)$, their periods are also 2π. That is,

$$\csc(\theta + 2n\pi) = \csc(\theta) \text{ and } \sec(\theta + 2n\pi) = \sec\theta,\ n = 0, \pm 1, \pm 2, \ldots .$$

It turns out that $\tan\theta$ and $\cot\theta$ have period π. That is,

$$\tan(\theta + n\pi) = \tan\theta \text{ and } \cot(\theta + n\pi) = \cot\theta,\ n = 0, \pm 1, \pm 2, \ldots .$$

Geometrically, it is easy to see that $\cos\theta$ and $\sec\theta$ are the only even trigonometric functions. The functions $\sin\theta, \cos\theta, \tan\theta$ and $\cot\theta$ are all odd functions. The functions $\sin\theta$ and $\cos\theta$ are defined for all real numbers. The functions $\csc\theta$ and $\cot\theta$ are not defined for integer multiples of π, and $\sec\theta$ and $\tan\theta$ are not defined for odd integer multiples of $\pi/2$. The graphs of the six trigonometric functions are sketched as follows:

graph

The dotted vertical lines represent the vertical asymptotes.

There are many useful trigonometric identities and reduction formulas. For future reference, these are listed here.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sin^2\theta = 1 - \cos^2\theta \qquad \cos^2\theta = 1 - \sin^2\theta$$
$$\tan^2\theta + 1 = \sec^2\theta \qquad \tan^2\theta = \sec^2\theta - 1 \qquad \sec^2\theta - \tan^2\theta = 1$$
$$1 + \cot^2\theta = \csc^2\theta \qquad \cot^2\theta = \csc^2\theta - 1 \qquad \csc^2\theta - \cot^2\theta = 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = 2\cos^2\theta - 1 \qquad \cos 2\theta = 1 + 2\sin^2\theta$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y, \quad \cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\sin(x-y) = \sin x\cos y - \cos x\sin y, \quad \cos(x-y) = \cos x\cos y + \sin x\sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y} \qquad \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$

$$\cos x \sin y = \frac{1}{2}(\sin(x+y) - \sin(x-y))$$

$$\cos x \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$$

$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$

$$\sin(\pi \pm \theta) = \mp \sin\theta$$

$$\cos(\pi \pm \theta) = -\cos\theta$$

$$\tan(\pi \pm \theta) = \pm \tan\theta$$

$$\cot(\pi \pm \theta) = \pm \cot\theta$$

$$\sec(\pi \pm \theta) = -\sec\theta$$

$$\csc(\pi \pm \theta) = \mp \csc\theta$$

In applications of calculus to engineering problems, the graphs of $y = A\sin(bx+c)$ and $y = A\cos(bx+c)$ play a significant role. The first problem has to do with converting expressions of the form $A\sin bx + B\cos bx$ to one of the above forms. Let us begin first with an example.

Example 1.2.1 Express $y = 3\sin(2x) - 4\cos(2x)$ in the form $y = A\sin(2x \pm \theta)$ or $y = A\cos(2x \pm \theta)$.

First of all, we make a right triangle with sides of length 3 and 4 and compute the length of the hypotenuse, which is 5. We label one of the acute angles as θ and compute $\sin\theta, \cos\theta$ and $\tan\theta$. In our case,

$$\sin\theta = \frac{3}{5} \quad , \quad \cos\theta = \frac{4}{5} \quad , \text{and, } \tan\theta = \frac{3}{4}.$$

graph

Then,

$$\begin{aligned} y &= 3\sin 2x - 4\cos 2x \\ &= 5\left[(\sin(2x))\left(\frac{3}{5}\right) - (\cos(2x))\frac{4}{5}\right] \\ &= 5[\sin(2x)\sin\theta - \cos(2x)\cos\theta] \\ &= -5[\cos(2x)\cos\theta - \sin(2x)\sin\theta] \\ &= -5[\cos(2x+\theta)] \end{aligned}$$

Thus, the problem is reduced to sketching a cosine function, ???

$$y = -5\cos(2x+\theta).$$

We can compute the radian measure of θ from any of the equations

$$\sin\theta = \frac{3}{5},\ \cos\theta = \frac{4}{5} \text{ or } \tan\theta = \frac{3}{4}.$$

Example 1.2.2 Sketch the graph of $y = 5\cos(2x+1)$.

In order to sketch the graph, we first compute all of the zeros, relative maxima, and relative minima. We can see that the maximum values will be 5 and minimum values are -5. For this reason the number 5 is called the amplitude of the graph. We know that the cosine function has zeros at odd integer multiples of $\pi/2$. Let

$$2x_n + 1 = (2n+1)\frac{\pi}{2},\ x_n = (2n+1)\frac{\pi}{4} - \frac{1}{2},\ n = 0, \pm1, \pm2 \ldots .$$

The max and min values of a cosine function occur halfway between the consecutive zeros. With this information, we are able to sketch the graph of the given function. The period is π, phase shift is $\frac{1}{2}$ and frequency is $\frac{1}{\pi}$.

graph

For the functions of the form $y = A\sin(\omega t \pm d)$ or $y = A\cos(\omega t \pm d)$ we make the following definitions:

$$\textit{period} = \frac{2\pi}{\omega},\ \textit{frequency} = \frac{1}{\text{period}} = \frac{\omega}{2\pi},$$

$$\textit{amplitude} = |A|, \text{ and } \textit{phase shift} = \frac{d}{\omega}.$$

The motion of a particle that follows the curves $A\sin(\omega t \pm d)$ or $A\cos(\omega t \pm d)$ is called *simple harmonic motion.*

Exercises 1.2

1. Determine the amplitude, frequency, period and phase shift for each of the following functions. Sketch their graphs.

   (a) $y = 2\sin(3t - 2)$ (b) $y = -2\cos(2t - 1)$
   (c) $y = 3\sin 2t + 4\cos 2t$ (d) $y = 4\sin 2t - 3\cos 2t$
   (e) $y = \dfrac{\sin x}{x}$

2. Sketch the graphs of each of the following:

   (a) $y = \tan(3x)$ (b) $y = \cot(5x)$ (c) $y = x\sin x$
   (d) $y = \sin(1/x)$ (e) $y = x\sin(1/x)$

3. Express the following products as the sum or difference of functions.

   (a) $\sin(3x)\cos(5x)$ (b) $\cos(2x)\cos(4x)$ (c) $\cos(2x)\sin(4x)$
   (d) $\sin(3x)\sin(5x)$ (e) $\sin(4x)\cos(4x)$

4. Express each of the following as a product of functions:

   (a) $\sin(x + h) - \sin x$ (b) $\cos(x + h) - \cos x$ (c) $\sin(5x) - \sin(3x)$
   (d) $\cos(4x) - \cos(2x)$ (e) $\sin(4x) + \sin(2x)$ (f) $\cos(5x) + \cos(3x)$

5. Consider the graph of $y = \sin x$, $\dfrac{-\pi}{2} \le x \le \dfrac{\pi}{2}$. Take the sample points

$$\left\{\left(-\frac{\pi}{2}, -1\right),\ \left(-\frac{\pi}{6},\ -\frac{\pi}{2}\right), (0,0),\ \left(\frac{\pi}{6},\ \frac{1}{2}\right),\ \left(\frac{\pi}{2}, 1\right)\right\}.$$

Compute the fourth degree Lagrange Polynomial that approximates and agrees with $y = \sin x$ at these data points. This polynomial has the form

$$\begin{aligned} P_5(x) &= y_1 \frac{(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} + \\ & \quad y_2 \frac{(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} + \cdots \\ &+ y_5 \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}. \end{aligned}$$

6. Sketch the graphs of the following functions and compute the amplitude, period, frequency and phase shift, as applicable.

   a) $y = 3\sin t$ b) $y = 4\cos t$ c) $y = 2\sin(3t)$

   d) $y = -4\cos(2t)$ e) $y = -3\sin(4t)$ f) $y = 2\sin\left(t + \frac{\pi}{6}\right)$

   g) $y = -2\sin\left(t - \frac{\pi}{6}\right)$ h) $y = 3\cos(2t + \pi)$ i) $y = -3\cos(2t - \pi)$

   j) $y = 2\sin(4t + \pi)$ k) $y = -2\cos(6t - \pi)$ l) $y = 3\sin(6t + \pi)$

7. Sketch the graphs of the following functions over two periods.

   a) $y = 2\sec x$ b) $y = -3\tan x$ c) $y = 2\cot x$

   d) $y = 3\csc x$ e) $y = \tan(\pi x)$ f) $y = \tan\left(2x + \frac{\pi}{3}\right)$

   g) $y = 2\cot\left(3x + \frac{\pi}{2}\right)$ h) $y = 3\sec\left(2x + \frac{\pi}{3}\right)$ i) $y = 2\sin\left(\pi x + \frac{\pi}{6}\right)$

8. Prove each of the following identities:

   a) $\cos 3t = 3\cos t + 4\cos^3 t$ b) $\sin(3t) = 3\sin x - 4\sin^3 x$

   c) $\sin^4 t - \cos^4 t = -\cos 2t$ d) $\dfrac{\sin^3 t - \cos^3 t}{\sin t - \cos t} = 1 + \sin 2t$

   e) $\cos 4t \cos 7t - \sin 7t \sin 4t = \cos 11t$ f) $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$

9. If $f(x) = \cos x$, prove that

$$\frac{f(x+h)-f(x)}{h} = \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right).$$

10. If $f(x) = \sin x$, prove that

$$\frac{f(x+h)-f(x)}{h} = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right).$$

11. If $f(x) = \cos x$, prove that

$$\frac{f(x)-f(t)}{x-t} = \cos t\left(\frac{\cos(x-t) - 1}{x-t}\right) - \sin t\left(\frac{\sin(x-t)}{x-t}\right).$$

12. If $f(x) = \sin x$, prove that

$$\frac{f(x)-f(t)}{x-t} = \sin t\left(\frac{\cos(x-t) - 1}{x-t}\right) + \cos t\left(\frac{\sin(x-t)}{x-t}\right).$$

13. Prove that

$$\cos(2t) = \frac{1-\tan^2 t}{1+\tan^2 t}.$$

14. Prove that if $y = \tan\left(\frac{x}{2}\right)$, then

(a) $\cos x = \dfrac{1-u^2}{1+u^2}$ (b) $\sin x = \dfrac{2u}{1+u^2}$

1.3 Inverse Trigonometric Functions

None of the trigonometric functions are one-to-one since they are periodic. In order to define inverses, it is customary to restrict the domains in which the functions are one-to-one as follows.

1. $y = \sin x,\ -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$, is one-to-one and covers the range $-1 \le y \le 1$. Its inverse function is denoted $\arcsin x$, and we define $y = \arcsin x,\ -1 \le x \le 1$, if and only if, $x = \sin y,\ -\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$.

   graph

2. $y = \cos x,\ 0 \le x \le \pi$, is one-to-one and covers the range $-1 \le y \le 1$. Its inverse function is denoted $\arccos x$, and we define $y = \arccos x,\ -1 \le x \le 1$, if and only if, $x = \cos y,\ 0 \le y \le \pi$.

   graph

3. $y = \tan x,\ \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$, is one-to-one and covers the range $-\infty < y < \infty$ Its inverse function is denoted $\arctan x$, and we define $y = \arctan x,\ -\infty < x < \infty$, if and only if, $x = \tan y,\ \dfrac{-\pi}{2} < y < \dfrac{\pi}{2}$.

   graph

4. $y = \cot x,\ 0, x < \pi$, is one-to-one and covers the range $-\infty < y < \infty$. Its inverse function is denoted arccot x, and we define $y = \operatorname{arccot} x,\ -\infty < x < \infty$, if and only if $x = \cot y,\ 0 < y < \pi$.

   graph

5. $y = \sec x,\ 0 \le x \le \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} < x \le \pi$ is one-to-one and covers the range $-\infty < y \le -1$ or $1 \le y < \infty$. Its inverse function is denoted arcsec x, and we define $y = \text{arcsec } x,\ -\infty < x \le -1$ or $1 \le x < \infty$, if and only if, $x = \sec y,\ 0 \le y < \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} < y \le \pi$.

graph

6. $y = \csc x,\ \dfrac{-\pi}{2} \le x < 0$ or $0 < x \le \dfrac{\pi}{2}$, is one-to-one and covers the range $-\infty < y \le -1$ or $1 \le y < \infty$. Its inverse is denoted $\operatorname{arccsc} x$ and we define $y = \operatorname{arccsc} x,\ -\infty < x \le -1$ or $1 \le x < \infty$, if and only if, $x = \csc y,\ \dfrac{-\pi}{2} \le y < 0$ or $0 < y \le \dfrac{\pi}{2}$.

Example 1.3.1 Show that each of the following equations is valid.

(a) $\arcsin x + \arccos x = \dfrac{\pi}{2}$

(b) $\arctan x + \operatorname{arccot} x = \dfrac{\pi}{2}$

(c) $\operatorname{arcsec} x + \operatorname{arccsc} x = \dfrac{\pi}{2}$

To verify equation (a), we let $\arcsin x = \theta$.

graph

Then $x = \sin\theta$ and $\cos\left(\dfrac{\pi}{2} - \theta\right) = x$, as shown in the triangle. It follows that

$$\frac{\pi}{2} - \theta = \arccos x, \quad \frac{\pi}{2} = \theta + \arccos x = \arcsin x + \arccos x.$$

The equations in parts (b) and (c) are verified in a similar way.

Example 1.3.2 If $\theta = \arcsin x$, then compute $\cos\theta, \tan\theta, \cot\theta, \sec\theta$ and $\csc\theta$.

If θ is $-\dfrac{\pi}{2}$, 0, or $\dfrac{\pi}{2}$, then computations are easy.

graph

Suppose that $-\dfrac{\pi}{2} < x < 0$ or $0 < x < \dfrac{\pi}{2}$. Then, from the triangle, we get

$$\cos\theta = \sqrt{1-x^2}, \quad \tan\theta = \frac{x}{\sqrt{1-x^2}}, \quad \cot\theta = \frac{\sqrt{1-x^2}}{x},$$
$$\sec\theta = \frac{1}{\sqrt{1-x^2}} \text{ and } \csc\theta = \frac{1}{x}.$$

Example 1.3.3 Make the given substitutions to simplify the given radical expression and compute all trigonometric functions of θ.

(a) $\sqrt{4-x^2}$, $x = 2\sin\theta$ (b) $\sqrt{x^2-9}$, $x = 3\sec\theta$

(c) $(4+x^2)^{3/2}$, $x = 2\tan\theta$

(a) For part (a), $\sin\theta = \dfrac{x}{2}$ and we use the given triangle:

graph

Then

$$\cos\theta = \frac{\sqrt{4-x^2}}{2}, \quad \tan\theta = \frac{x}{\sqrt{4-x^2}}, \quad \cot\theta = \frac{\sqrt{4-x^2}}{x},$$
$$\sec\theta = \frac{2}{\sqrt{4-x^2}}, \quad \csc\theta = \frac{2}{x}.$$

Furthermore, $\sqrt{4-x^2} = 2\cos\theta$ and the radical sign is eliminated.

(b) For part (b), $\sec\theta = \dfrac{x}{3}$ and we use the given triangle:

graph

Then,

$$\sin\theta = \frac{\sqrt{x^2-4}}{x}, \quad \cos\theta = \frac{3}{x}, \quad \tan\theta = \frac{\sqrt{x^2-4}}{3}$$

$$\cot\theta = \frac{3}{\sqrt{x^2-9}}, \quad \csc\theta = \frac{x}{\sqrt{x^2-9}}.$$

Furthermore, $\sqrt{x^2-9} = 3\tan\theta$ and the radical sign is eliminated.

(c) For part (c), $\tan\theta = \dfrac{x}{2}$ and we use the given triangle:

graph

Then,

$$\sin\theta = \frac{x}{\sqrt{x^2+4}}, \quad \cos\theta = \frac{2}{\sqrt{x^2+4}}, \quad \cot\theta = \frac{2}{x},$$

$$\sec\theta = \frac{\sqrt{x^2+4}}{2}, \quad \csc\theta = \frac{\sqrt{x^2+4}}{x}.$$

Furthermore, $\sqrt{x^2+4} = 2\sec\theta$ and hence

$$(4+x)^{3/2} = (2\sec\theta)^3 = 8\sec^3\theta.$$

Remark 2 The three substitutions given in Example 15 are very useful in calculus. In general, we use the following substitutions for the given radicals:

(a) $\sqrt{a^2 - x^2},\ x = a\sin\theta$ (b) $\sqrt{x^2 - a^2},\ x = a\sec\theta$

(c) $\sqrt{a^2 + x^2},\ x = a\tan\theta$.

Exercises 1.3

1. Evaluate each of the following:

   (a) $3\arcsin\left(\dfrac{1}{2}\right) + 2\arccos\left(\dfrac{\sqrt{3}}{2}\right)$

   (b) $4\arctan\left(\dfrac{1}{\sqrt{3}}\right) + 5\text{arccot}\left(\dfrac{1}{\sqrt{3}}\right)$

   (c) $2\text{arcsec}\,(-2) + 3\arccos\left(-\dfrac{2}{\sqrt{3}}\right)$

   (d) $\cos(2\arccos(x))$

   (e) $\sin(2\arccos(x))$

2. Simplify each of the following expressions by eliminating the radical by using an appropriate trigonometric substitution.

   (a) $\dfrac{x}{\sqrt{9-x^2}}$ (b) $\dfrac{3+x}{\sqrt{16+x^2}}$ (c) $\dfrac{x-2}{x\sqrt{x^2-25}}$

   (d) $\dfrac{1+x}{\sqrt{x^2+2x+2}}$ (e) $\dfrac{2-2x}{\sqrt{x^2-2x-3}}$

   (Hint: In parts (d) and (e), complete squares first.)

3. Some famous polynomials are the so-called Chebyshev polynomials, defined by

$$T_n(x) = \cos(n\arccos x),\ -1 \le x \le 1,\ n = 0, 1, 2, \ldots .$$

(a) Prove the recurrence relation for Chebyshev polynomials:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ for each } n \geq 1.$$

(b) Show that $T_0(x) = 1,\ T_1(x) = x$ and generate $T_2(x), T_3(x), T_4(x)$ and $T_5(x)$ using the recurrence relation in part (a).

(c) Determine the zeros of $T_n(x)$ and determine where $T_n(x)$ has its absolute maximum or minimum values, $n = 1, 2, 3, 4, ?.$
(Hint: Let $\theta = \arccos x, x = \cos\theta$. Then $T_n(x) = \cos(n\theta), T_{n+1}(x) = \cos(n\theta + \theta), T_{n-1}(x) = \cos(n\theta - \theta)$. Use the expansion formulas and then make substitutions in part (a)).

4. Show that for all integers m and n,

$$T_n(x)T_m(x) = \frac{1}{2}\left[T_{m+n}(x) + T_{|m-n|}(x)\right]$$

(Hint: use the expansion formulas as in problem 3.)

5. Find the exact value of y in each of the following

a) $y = \arccos\left(-\frac{1}{2}\right)$ b) $y = \arcsin\left(\frac{\sqrt{3}}{2}\right)$ c) $y = \arctan(-\sqrt{3})$

d) $y = \operatorname{arccot}\left(-\frac{\sqrt{3}}{3}\right)$ e) $y = \operatorname{arcsec}(-\sqrt{2})$ f) $y = \operatorname{arccsc}(-\sqrt{2})$

g) $y = \operatorname{arcsec}\left(-\frac{2}{\sqrt{3}}\right)$ h) $y = \operatorname{arccsc}\left(-\frac{2}{\sqrt{3}}\right)$ i) $y = \operatorname{arcsec}(-2)$

j) $y = \operatorname{arccsc}(-2)$ k) $y = \arctan\left(\frac{-1}{\sqrt{3}}\right)$ l) $y = \operatorname{arccot}(-\sqrt{3})$

6. Solve the following equations for x in radians (all possible answers).

a) $2\sin^4 x = \sin^2 x$ b) $2\cos^2 x - \cos x - 1 = 0$

c) $\sin^2 x + 2\sin x + 1 = 0$ d) $4\sin^2 x + 4\sin x + 1 = 0$

e) $2\sin^2 x + 5\sin x + 2 = 0$ f) $\cot^3 x - 3\cot x = 0$

g) $\sin 2x = \cos x$ h) $\cos 2x = \cos x$

i) $\cos^2\left(\frac{x}{2}\right) = \cos x$ j) $\tan x + \cot x = 1$

7. If $\arctan t = x$, compute $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\csc x$ in terms of t.

8. If $\arcsin t = x$, compute $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\csc x$ in terms of t.

9. If $\operatorname{arcsec} t = x$, compute $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\csc x$ in terms of t.

10. If $\arccos t = x$, compute $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\csc x$ in terms of t.

Remark 3 Chebyshev polynomials are used extensively in approximating functions due to their properties that minimize errors. These polynomials are called equal ripple polynomials, since their maxima and minima alternate between 1 and -1.

1.4 Logarithmic, Exponential and Hyperbolic Functions

Most logarithmic tables have tables for $\log_{10} x, \log_e x, e^x$ and e^{-x} because of their universal applications to scientific problems. The key relationship between logarithmic functions and exponential functions, using the same base, is that each one is an inverse of the other. For example, for base 10, we have

$$N = 10^x \text{ if and only if } x = \log_{10} N.$$

We get two very interesting relations, namely

$$x = \log_{10}(10^x) \text{ and } N = 10^{(\log_{10} N)}.$$

For base e, we get

$$x = \log_e(e^x) \text{ and } y = e^{(\log_e y)}.$$

If $b > 0$ and $b \neq 1$, then b is an admissible base for a logarithm. For such an admissible base b, we get

$$x = \log_b(b^x) \text{ and } y = b^{(\log_b y)}.$$

The Logarithmic function with base $b,\ b > 0,\ b \neq 1$, satisfies the following important properties:

1. $\log_b(b) = 1,\ \log_b(1) = 0$, and $\log_b(b^x) = x$ for all real x.

2. $\log_b(xy) = \log_b x + \log_b y,\ x > 0, y > 0$.

3. $\log_b(x/y) = \log_b x - \log_b y,\ x > 0, y > 0$.

4. $\log_b(x^y) = y \log_b x,\ x > 0, x \neq 1$, for all real y.

5. $(\log_b x)(\log_a b) = \log_a xb > 0, a > 0, b \neq 1, a \neq 1$. Note that $\log_b x = \dfrac{\log_a x}{\log_a b}$.

This last equation (5) allows us to compute logarithms with respect to any base b in terms of logarithms in a given base a.

The corresponding laws of exponents with respect to an admissible base $b, b > 0, b \neq 1$ are as follows:

1. $b^0 = 1,\ b^1 = b$, and $b^{(\log_b x)} = x$ for $x > 0$.

2. $b^x \times b^y = b^{x+y}$

3. $\dfrac{b^x}{b^y} = b^{x-y}$

4. $(b^x)^y = b^{(xy)}$

Notation: If $b = e$, then we will express

$$\log_b(x) \text{ as } \ln(x) \text{ or } \log(x).$$

The notation $\exp(x) = e^x$ can be used when confusion may arise.

The graph of $y = \log x$ and $y = e^x$ are reflections of each other through the line $y = x$.

graph

In applications of calculus to science and engineering, the following six functions, called *hyperbolic functions*, are very useful.

1. $\sinh(x) = \frac{1}{2}\,(e^x - e^{-x})$ for all real x, read as *hyperbolic sine* of x.

2. $\cosh(x) = \frac{1}{2}\,(e^x + e^{-x})$, for all real x, read as *hyperbolic cosine* of x.

3. $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, for all real x, read as *hyperbolic tangent* of x.

4. $\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, $x \neq 0$, read as *hyperbolic cotangent* of x.

5. $\operatorname{sech}(x) = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$, for all real x, read as *hyperbolic secant* of x.

6. $\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$, $x \neq 0$, read as *hyperbolic cosecant* of x.

The graphs of these functions are sketched as follows:

graph

Example 1.4.1 Eliminate quotients and exponents in the following equation by taking the natural logarithm of both sides.

$$y = \frac{(x+1)^3(2x-3)^{3/4}}{(1+7x)^{1/3}(2x+3)^{3/2}}$$

$$\begin{aligned}
\ln(y) &= \ln\left[\frac{(x+1)^3(2x-3)^{3/4}}{(1+7x)^{1/3}(2x+3)^{3/2]}}\right] \\
&= \ln[(x+1)^3(2x-3)^{3/4}] - \ln[(1+7x)^{1/3}(2x+3)^{3/2}] \\
&= \ln(x+1)^3 + \ln(2x-3)^{3/4} - \{\ln(1+7x)^{1/3} + \ln(2x+3)^{3/2}\} \\
&= 3\ln(x+1) + \frac{3}{4}\ \ln(2x-3) - \frac{1}{3}\ \ln(1+7x) - \frac{3}{2}\ \ln(2x+3)
\end{aligned}$$

Example 1.4.2 Solve the following equation for x:

$$\log_3(x^4) + \log_3 x^3 - 2\log_3 x^{1/2} = 5.$$

Using logarithm properties, we get

$$\begin{aligned}
4\log_3 x + 3\log_3 x - \log_3 x &= 5 \\
6\log_3 x &= 5 \\
\log_3 x &= \frac{5}{6} \\
x &= (3)^{5/6}.
\end{aligned}$$

Example 1.4.3 Solve the following equation for x:

$$\frac{e^x}{1+e^x} = \frac{1}{3}.$$

On multiplying through, we get

$$\begin{aligned}
3e^x = 1 + e^x \text{ or } 2e^x = 1, e^x\ &= \frac{1}{2} \\
x =\ &\ln(1/2) = -\ln(2).
\end{aligned}$$

Example 1.4.4 Prove that for all real x, $cosh^2 x - \sinh^2 x = 1$.

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2 \\
&= \frac{1}{4}[e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})] \\
&= \frac{1}{4}[4] \\
&= 1
\end{aligned}$$

Example 1.4.5 Prove that

(a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

(b) $\sinh 2x = 2 \sinh x \cosh y$.

Equation (b) follows from equation (a) by letting $x = y$. So, we work with equation (a).

$$
\begin{aligned}
\text{(a)}\quad \sinh x \cosh y + \cosh x \sinh y &= \frac{1}{2}(e^x - e^{-x}) \cdot \frac{1}{2}(e^y + e^{-y}) \\
&\quad + \frac{1}{2}(e^x + e^{-x}) \cdot \frac{1}{2}(e^y - e^{-y}) \\
&= \frac{1}{4}[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) \\
&\quad + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})] \\
&= \frac{1}{4}[2(e^{x+y} - e^{-(x+y)}] \\
&= \frac{1}{2}(e^{(x+y)} - e^{-(x+y)}) \\
&= \sinh(x+y).
\end{aligned}
$$

Example 1.4.6 Find the inverses of the following functions:

(a) $\sinh x$ (b) $\cosh x$ (c) $\tanh x$

(a) Let $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$. Then

$$
\begin{aligned}
2e^x y = 2e^x \left(\frac{1}{2}(e^x - e^{-x})\right) &= e^{2x} - 1 \\
e^{2x} - 2ye^x - 1 &= 0 \\
(e^x)^2 - (2y)e^x - 1 &= 0 \\
e^x = \frac{2y \pm \sqrt{4y^2+4}}{2} &= y \pm \sqrt{y^2+1}
\end{aligned}
$$

Since $e^x > 0$ for all x, $e^x = y + \sqrt{1+y^2}$.

On taking natural logarithms of both sides, we get

$$x = \ln(y + \sqrt{1 + y^2}).$$

The inverse function of $\sinh x$, denoted arcsinh x, is defined by

$$\boxed{\operatorname{arcsinh} x = \ln(x + \sqrt{1 + x^2})}$$

(b) As in part (a), we let $y = \cosh x$ and

$$\begin{aligned} 2e^x y &= 2e^x \cdot \frac{1}{2}(e^x + e^{-x}) = e^{2x} + 1 \\ e^{2x} &- (2y)e^x + 1 = 0 \\ e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\ e^x &= y \pm \sqrt{y^2 - 1}. \end{aligned}$$

We observe that $\cosh x$ is an even function and hence it is not one-to-one. Since $\cosh(-x) = \cosh(x)$, we will solve for the larger x. On taking natural logarithms of both sides, we get

$$x_1 = \ln(y + \sqrt{y^2 - 1}) \text{ or } x_2 = \ln(y - \sqrt{y^2 - 1}).$$

We observe that

$$\begin{aligned} x_2 = \ln(y - \sqrt{y^2 - 1}) &= \ln\left[\frac{(y - \sqrt{y^2 - 1})(y + \sqrt{y^2 - 1})}{y + \sqrt{y^2 - 1}}\right] \\ &= \ln\left(\frac{1}{y + \sqrt{y^2 - 1}}\right) \\ &= -\ln(y + \sqrt{y^2 - 1}) = -x_1. \end{aligned}$$

Thus, we can define, as the principal branch,

$$\boxed{\operatorname{arccosh} x = \ln(x + \sqrt{x^2 - 1}), \ x \geq 1}$$

(c) We begin with $y = \tanh x$ and clear denominators to get

$$\begin{aligned}
y &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad &, \quad |y| < 1 \\
e^x[(e^x + e^{-x})y] &= e^x[(e^x - e^{-x})] \quad &, \quad |y| < 1 \\
(e^{2x} + 1)y &= e^{2x} - 1 \quad &, \quad |y| < 1 \\
e^{2x}(y - 1) &= -(1 + y) \quad &, \quad |y| < 1 \\
e^{2x} &= -\frac{(1 + y)}{y - 1} \quad &, \quad |y| < 1 \\
e^{2x} &= \frac{1 + y}{1 - y} \quad &, \quad |y| < 1 \\
2x &= \ln\left(\frac{1 + y}{1 - y}\right) \quad &, \quad |y| < 1 \\
x &= \frac{1}{2}\ln\left(\frac{1 + y}{1 - y}\right) \quad &, |y| < 1.
\end{aligned}$$

Therefore, the inverse of the function $\tanh x$, denoted $\text{arctanh}x$, is defined by

$$\boxed{\text{arctanh}\,, x = \frac{1}{2}\ \ln\left(\frac{1 + x}{1 - x}\right), \quad |x| < 1.}$$

Exercises 1.4

1. Evaluate each of the following

(a) $\log_{10}(0.001)$ (b) $\log_2(1/64)$ (c) $\ln(e^{0.001})$

(d) $\log_{10}\left(\frac{(100)^{1/3}(0.01)^2}{(.0001)^{2/3}}\right)^{0.1}$ (e) $e^{\ln(e^{-2})}$

2. Prove each of the following identities

(a) $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

(b) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

(c) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

(d) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

3. Simplify the radical expression by using the given substitution.

(a) $\sqrt{a^2+x^2},\ x = a\sinh t$ (b) $\sqrt{x^2-a^2},\ x = a\cosh t$

(c) $\sqrt{a^2-x^2},\ x = a\tanh t$

4. Find the inverses of the following functions:

(a) $\coth x$ (b) sech x (c) csch x

5. If $\cosh x = \dfrac{3}{2}$, find $\sinh x$ and $\tanh x$.

6. Prove that $\sinh(3t) = 3\sinh t + 4\sinh^3 t$ (Hint: Expand $\sinh(2t+t)$.)

7. Sketch the graph of each of the following functions.

a) $y = 10^x$ b) $y = 2^x$ c) $y = 10^{-x}$ d) $y = 2^{-x}$

e) $y = e^x$ f) $y = e^{-x^2}$ g) $y = xe^{-x^2}$ i) $y = e^{-x}$

j) $y = \sinh x$ k) $y = \cosh x$ l) $y = \tanh x$ m) $y = \coth x$

n) $y = \operatorname{sech} x$ o) $y = \operatorname{csch} x$

8. Sketch the graph of each of the following functions.

a) $y = \log_{10} x$ b) $y = \log_2 x$ c) $y = \ln x$ d) $y = \log_3 x$

e) $y = \operatorname{arcsinh} x$ f) $y = \operatorname{arccosh} x$ g) $y = \operatorname{arctanh} x$

9. Compute the given logarithms in terms $\log_{10} 2$ and $\log_{10} 3$.

a) $\log_{10} 36$ b) $\log_{10}\left(\frac{27}{16}\right)$ c) $\log_{10}\left(\frac{20}{9}\right)$

d) $\log_{10}(600)$ e) $\log_{10}\left(\frac{30}{16}\right)$ f) $\log_{10}\left(\frac{6^{10}}{(20)^5}\right)$

10. Solve each of the following equations for the independent variable.

a) $\ln x - \ln(x+1) = \ln(4)$ b) $2\log_{10}(x-3) = \log_{10}(x+5) + \log_{10} 4$

c) $\log_{10} t^2 = (\log_{10} t)^2$ d) $e^{2x} - 4e^x + 3 = 0$

e) $e^x + 6e^{-x} = 5$ f) $2\sinh x + \cosh x = 4$

Chapter 2

Limits and Continuity

2.1 Intuitive treatment and definitions

2.1.1 Introductory Examples

The concepts of limit and continuity are very closely related. An intuitive understanding of these concepts can be obtained through the following examples.

Example 2.1.1 Consider the function $f(x) = x^2$ as x tends to 2.

As x tends to 2 from the right or from the left, $f(x)$ tends to 4. The value of f at 2 is 4. The graph of f is in one piece and there are no holes or jumps in the graph. We say that f is continuous at 2 because $f(x)$ tends to $f(2)$ as x tends to 2.

graph

The statement that $f(x)$ tends to 4 as x tends to 2 *from the right* is expressed in symbols as

$$\lim_{x \to 2^+} f(x) = 4$$

and is read, "the limit of $f(x)$, as x goes to 2 *from the right*, equals 4."

The statement that $f(x)$ tends to 4 as x tends to 2 *from the left* is written

$$\lim_{x \to 2^-} f(x) = 4$$

and is read, "the limit of $f(x)$, as x goes to 2 from the left, equals 4."

The statement that $f(x)$ tends to 4 as x tends to 2 either from the right or from the left, is written

$$\lim_{x \to 2} f(x) = 4$$

and is read, "the limit of $f(x)$, as x goes to 2, equals 4."

The statement that $f(x)$ is continuous at $x = 2$ is expressed by the equation

$$\lim_{x \to 2} f(x) = f(2).$$

Example 2.1.2 Consider the unit step function as x tends to 0.

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

graph

The function, $u(x)$ tends to 1 as x tends to 0 from the right side. So, we write

$$\lim_{x \to 0^+} u(x) = 1 = u(0).$$

The limit of $u(x)$ as x tends to 0 from the left equals 0. Hence,

$$\lim_{x \to 0^-} u(x) = 0 \neq u(0).$$

Since

$$\lim_{x \to 0^+} u(x) = u(0),$$

we say that $u(x)$ is continuous at 0 from the right. Since

$$\lim_{x \to 0^-} u(x) \neq u(0),$$

we say that $u(x)$ is not continuous at 0 from the left. In this case the jump at 0 is 1 and is defined by

$$\begin{aligned}\text{jump }(u(x),0) &= \lim_{x\to 0^+} u(x) - \lim_{x\to 0^-} u(x)\\ &= 1.\end{aligned}$$

Observe that the graph of $u(x)$ has two pieces that are not joined together. Every horizontal line with equation $y = c,\ 0 < c < 1$, separates the two pieces of the graph without intersecting the graph of $u(x)$. This kind of jump discontinuity at a point is called "finite jump" discontinuity.

Example 2.1.3 Consider the signum function, sign(x), defined by

$$\text{sign }(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

If $x > 0$, then sign$(x) = 1$. If $x < 0$, then sign$(x) = -1$. In this case,

$$\begin{aligned}\lim_{x\to 0^+} \text{sign}(x) &= 1\\ \lim_{x\to 0^-} \text{sign}(x) &= -1\\ \text{jump }(\text{sign}(x),0) &= 2.\end{aligned}$$

Since sign(x) is not defined at $x = 0$, it is not continuous at 0.

Example 2.1.4 Consider $f(\theta) = \dfrac{\sin\theta}{\theta}$ as θ tends to 0.

graph

The point $C(\cos\theta, \sin\theta)$ on the unit circle defines $\sin\theta$ as the vertical length BC. The radian measure of the angle θ is the arc length DC. It is clear that the vertical length BC and arc length DC get closer to each other as θ tends to 0 from above. Thus,

graph

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1.$$

For negative θ, $\sin\theta$ and θ are both negative.

$$\lim_{\theta \to 0^+} \frac{\sin(-\theta)}{-\theta} = \lim_{\theta \to 0^+} \frac{-\sin\theta}{-\theta} = 1.$$

Hence,

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

This limit can be verified by numerical computation for small θ.

Example 2.1.5 Consider $f(x) = \dfrac{1}{x}$ as x tends to 0 and as x tends to $\pm\infty$.

graph

It is intuitively clear that

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$
$$\lim_{x \to +\infty} \frac{1}{x} = 0$$
$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$
$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

The function f is not continuous at $x = 0$ because it is not defined for $x = 0$. This discontinuity is not removable because the limits from the left and from the right, at $x = 0$, are not equal. The horizontal and vertical axes divide the graph of f in two separate pieces. The vertical axis is called the *vertical asymptote* of the graph of f. The horizontal axis is called the *horizontal asymptote* of the graph of f. We say that f has an essential discontinuity at $x = 0$.

Example 2.1.6 Consider $f(x) = \sin(1/x)$ as x tends to 0.

graph

The period of the sine function is 2π. As observed in Example 5, $1/x$ becomes very large as x becomes small. For this reason, many cycles of the sine wave pass from the value -1 to the value $+1$ and a rapid oscillation occurs near zero. None of the following limits exist:

$$\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right), \qquad \lim_{x\to 0^-} \sin\left(\frac{1}{x}\right), \qquad \lim_{x\to 0} \sin\left(\frac{1}{x}\right).$$

It is not possible to define the function f at 0 to make it continuous. This kind of discontinuity is called an "oscillation" type of discontinuity.

Example 2.1.7 Consider $f(x) = x\sin\left(\dfrac{1}{x}\right)$ as x tends to 0.

graph

In this example, $\sin\left(\dfrac{1}{x}\right)$, oscillates as in Example 6, but the amplitude

$|x|$ tends to zero as x tends to 0. In this case,

$$\lim_{x \to 0^+} x \sin\left(\frac{1}{x}\right) = 0$$

$$\lim_{x \to 0^-} x \sin\left(\frac{1}{x}\right) = 0$$

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

The discontinuity at $x = 0$ is removable. We define $f(0) = 0$ to make f continuous at $x = 0$.

Example 2.1.8 Consider $f(x) = \dfrac{x-2}{x^2-4}$ as x tends to ± 2.

This is an example of a rational function that yields the *indeterminate* form 0/0 when x is replaced by 2. When this kind of situation occurs in rational functions, it is necessary to cancel the common factors of the numerator and the denominator to determine the appropriate limit if it exists. In this example, $x-2$ is the common factor and the reduced form is obtained through cancellation.

graph

$$\begin{aligned} f(x) &= \frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)} \\ &= \frac{1}{x+2}. \end{aligned}$$

In order to get the limits as x tends to 2, we used the reduced form to get 1/4. The discontinuity at $x = 2$ is removed if we define $f(2) = 1/4$. This function still has the essential discontinuity at $x = -2$.

Example 2.1.9 Consider $f(x) = \dfrac{\sqrt{x} - \sqrt{3}}{x^2 - 9}$ as x tends to 3.

In this case f is not a rational function; still, the problem at $x = 3$ is caused by the common factor $(\sqrt{x} - \sqrt{3})$.

graph

$$\begin{aligned} f(x) &= \frac{\sqrt{x} - \sqrt{3}}{x^2 - 9} \\ &= \frac{(\sqrt{x} - \sqrt{3})}{(x+3)(\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3})} \\ &= \frac{1}{(x+3)(\sqrt{x} + \sqrt{3})}. \end{aligned}$$

As x tends to 3, the reduced form of f tends to $1/(12\sqrt{3})$. Thus,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = \lim_{x \to 3} f(x) = \frac{1}{12\sqrt{3}}.$$

The discontinuity of f at $x = 3$ is removed by defining $f(3) = \dfrac{1}{12\sqrt{3}}$. The other discontinuities of f at $x = -3$ and $x = -\sqrt{3}$ are essential discontinuities and cannot be removed.

Even though calculus began intuitively, formal and precise definitions of limit and continuity became necessary. These precise definitions have become the foundations of calculus and its applications to the sciences. Let us assume that a function f is defined in some open interval, (a, b), except possibly at one point c, such that $a < c < b$. Then we make the following definitions using the Greek symbols: ϵ, read "epsilon" and δ, read, "delta."

2.1.2 Limit: Formal Definitions

Definition 2.1.1 The limit of $f(x)$ as x goes to c from the right is L, if and only if, for each $\epsilon > 0$, there exists some $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \quad \text{whenever, } c < x < c + \delta.$$

The statement that the limit of $f(x)$ as x goes to c from the right is L, is expressed by the equation

$$\lim_{x \to c^+} f(x) = L.$$

graph

Definition 2.1.2 The limit of $f(x)$ as x goes to c from the left is L, if and only if, for each $\epsilon > 0$, there exists some $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \quad \text{whenever, } c - \delta < x < c.$$

The statement that the limit of $f(x)$ as x goes to c from the left is L, is written as

$$\lim_{x \to c^-} f(x) = L.$$

graph

Definition 2.1.3 The (two-sided) limit of $f(x)$ as x goes to c is L, if and only if, for each $\epsilon > 0$, there exists some $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \quad \text{whenever } 0 < |x - c| < \delta.$$

graph

The equation

$$\lim_{x\to c} f(x) = L$$

is read "the (two-sided) limit of $f(x)$ as x goes to c equals L."

2.1.3 Continuity: Formal Definitions

Definition 2.1.4 The function f is said to be continuous at c from the right if $f(c)$ is defined, and

$$\lim_{x\to c^+} f(x) = f(c).$$

Definition 2.1.5 The function f is said to be continuous at c from the left if $f(c)$ is defined, and

$$\lim_{x\to c^-} f(x) = f(c).$$

Definition 2.1.6 The function f is said to be (two-sided) continuous at c if $f(c)$ is defined, and

$$\lim_{x\to c} f(x) = f(c).$$

Remark 4 The continuity definition requires that the following conditions be met if f is to be continuous at c:

(i) $f(c)$ is defined as a finite real number,

(ii) $\lim_{x\to c^-} f(x)$ exists and equals $f(c)$,

(iii) $\lim_{x\to c^+} f(x)$ exists and equals $f(c)$,

(iv) $\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c^+} f(x)$.

When a function f is not continuous at c, one, or more, of these conditions are not met.

Remark 5 All polynomials, $\sin x, \cos x, e^x, \sinh x, \cosh x, b^x, b \neq 1$ are continuous for all real values of x. All logarithmic functions, $\log_b x, b > 0, b \neq 1$ are continuous for all $x > 0$. Each rational function, $p(x)/q(x)$, is continuous where $q(x) \neq 0$. Each of the functions $\tan x, \cot x, \sec x, \csc x, \tanh x, \coth x$, sech x, and csch x is continuous at each point of its domain.

Definition 2.1.7 (Algebra of functions) Let f and g be two functions that have a common domain, say D. Then we define the following for all x in D:

1. $(f + g)(x) = f(x) + g(x)$ (sum of f and g)

2. $(f - g)(x) = f(x) - g(x)$ (difference of f and g)

3. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$, if $g(x) \neq 0$ (quotient of f and g)

4. $(gf)(x) = g(x)f(x)$ (product of f and g)

   If the range of f is a subset of the domain of g, then we define the composition, $g \circ f$, of f followed by g, as follows:

5. $(g \circ f)(x) = g(f(x))$

Remark 6 The following theorems on limits and continuity follow from the definitions of limit and continuity.

Theorem 2.1.1 *Suppose that for some real numbers L and M, $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then*

(i) $\lim_{x\to c} k = k$, *where k is a constant function.*

(ii) $\lim_{x\to c} (f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

(iii) $\lim_{x\to c} (f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$

(iv) $\lim_{x\to c} (f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

(v) $\lim_{x\to c} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, *if* $\lim_{x\to c} g(x) \neq 0$

Proof.
Part (i) Let $f(x) = k$ for all x and $\epsilon > 0$ be given. Then

$$|f(x) - k| = |k - k| = 0 < \epsilon$$

for all x. This completes the proof of Part (i).

For Parts (ii)–(v) let $\epsilon > 0$ be given and let

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M.$$

By definition there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - L| < \frac{\epsilon}{3} \quad \text{whenever} \quad 0 < |x - c| < \delta_1 \tag{1}$$

$$|g(x) - M| < \frac{\epsilon}{3} \quad \text{whenever} \quad 0 < |x - c| < \delta_2 \tag{2}$$

Part (ii) Let $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x - c| < \delta$ implies that

$$0 < |x - c| < \delta_1 \quad \text{and} \quad |f(x) - L| < \frac{\epsilon}{3} \quad \text{(by (1))} \tag{3}$$

$$0 < |x - c| < \delta_2 \quad \text{and} \quad |g(x) - M| < \frac{\epsilon}{3} \quad \text{(by (2))} \tag{4}$$

Hence, if $0 < |x - c| < \delta$, then

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} \qquad \text{(by (3) and (4))} \\ &< \epsilon. \end{aligned}$$

This completes the proof of Part (ii).

Part (iii) Let δ be defined as in Part (ii). Then $0 < |x - c| < \delta$ implies that

$$\begin{aligned} |(f(x) - g(x)) - (L - M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &< \epsilon. \end{aligned}$$

This completes the proof of Part (iii).

Part (iv) Let $\epsilon > 0$ be given. Let

$$\epsilon_1 = \min\left(1, \frac{\epsilon}{1 + |L| + |M|}\right).$$

Then $\epsilon_1 > 0$ and, by definition, there exist δ_1 and δ_2 such that

$$|f(x) - L| < \epsilon_1 \quad \text{whenever} \quad 0 < |x - c| < \delta_1 \tag{5}$$
$$|g(x) - M| < \epsilon_1 \quad \text{whenever} \quad 0 < |x - c| < \delta_2 \tag{6}$$

Let $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x - c| < \delta$ implies that

$$0 < |x - c| < \delta_1 \quad \text{and} \quad |f(x) - L| < \epsilon_1 \qquad \text{(by (5))} \tag{7}$$
$$0 < |x - c| < \delta_2 \quad \text{and} \quad |g(x) - M| < \epsilon_1 \qquad \text{(by (6))} \tag{8}$$

Also,

$$\begin{aligned} |f(x)g(x) - LM| &= |(f(x) - L + L)(g(x) - M + M) - LM| \\ &= |(f(x) - L)(g(x) - M) + (f(x) - L)M + L(g(x) - M)| \\ &\leq |f(x) - L|\ |g(x) - M| + |f(x) + L|\ |M| + |L|\ |g(x) - M| \\ &< \epsilon_1^2 + |M|\epsilon_1 + |L|\epsilon_1 \\ &\leq \epsilon_1 + |M|\epsilon_1 + |L|\epsilon_1 \\ &= (1 + |M| + |N|)\epsilon_1 \\ &\leq \epsilon. \end{aligned}$$

This completes the proof of Part (iv).

Part (v) Suppose that $M > 0$ and $\lim_{x\to c} g(x) = M$. Then we show that

$$\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}.$$

Since $M/2 > 0$, there exists some $\delta_1 > 0$ such that

$$\begin{aligned} |g(x) - M| &< \frac{M}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1, \\ -\frac{M}{2} + M < g(x) &< \frac{3M}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1, \\ 0 < \frac{M}{2} < g(x) &< \frac{3M}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1, \\ \frac{1}{|g(x)|} &< \frac{2}{M} \quad \text{whenever} \quad 0 < |x - c| < \delta_1. \end{aligned}$$

Let $\epsilon > 0$ be given. Let $\epsilon_1 = M^2\epsilon/2$. Then $\epsilon_1 > 0$ and there exists some $\delta > 0$ such that $\delta < \delta_1$ and

$$\begin{aligned} |g(x) - M| < \epsilon_1 \quad &\text{whenever} \quad 0 < |x - c| < \delta < \delta_1, \\ \left|\frac{1}{g(x)} - \frac{1}{M}\right| &= \left|\frac{M - g(x)}{g(x)M}\right| = \frac{|g(x) - M|}{|g(x)|M} \\ &= \frac{1}{M} \cdot \frac{1}{|g(x)|} \; |g(x) - M| \\ &< \frac{1}{M} \cdot \frac{2}{M} \cdot \epsilon_1 \\ &= \frac{2\epsilon_1}{M^2} \\ &= \epsilon \qquad \text{whenever} \quad 0 < |x - c| < \delta. \end{aligned}$$

This completes the proof of the statement

$$\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M} \quad \text{whenever} \quad M > 0.$$

The case for $M < 0$ can be proven in a similar manner. Now, we can use Part (iv) to prove Part (v) as follows:

$$\begin{aligned} \lim_{x \to c} \frac{f(x)}{g(x)} &= \lim_{x \to c} \left(f(x) \cdot \frac{1}{g(x)} \right) \\ &= \lim_{x \to c} f(x) \cdot \lim_{x \to c} \left(\frac{1}{g(x)} \right) \\ &= L \cdot \frac{1}{M} \\ &= \frac{L}{M}. \end{aligned}$$

This completes the proof of Theorem 2.1.1.

Theorem 2.1.2 *If f and g are two functions that are continuous on a common domain D, then the sum, $f+g$, the difference, $f-g$ and the product, fg, are continuous on D. Also, f/g is continuous at each point x in D such that $g(x) \neq 0$.*

Proof. If f and g are continuous at c, then $f(c)$ and $g(c)$ are real numbers and

$$\lim_{x\to c} f(x) = f(c), \ \lim_{x\to c} g(x) = g(c).$$

By Theorem 2.1.1, we get

$$\begin{aligned}
\lim_{x\to c}(f(x)+g(x)) &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c)+g(c)\\
\lim_{x\to c}(f(x)-g(x)) &= \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = f(c)-g(c)\\
\lim_{x\to c}(f(x)g(x)) &= \left(\lim_{x\to c} f(x)\right)\lim_{x\to c}(g(x)) = f(c)g(c)\\
\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) &= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{f(c)}{g(c)}, \text{ if } g(c)\neq 0.
\end{aligned}$$

This completes the proof of Theorem 2.1.2.

2.1.4 Continuity Examples

Example 2.1.10 Show that the constant function $f(x) = 4$ is continuous at every real number c. Show that for every constant k, $f(x) = k$ is continuous at every real number c.

First of all, if $f(x) = 4$, then $f(c) = 4$. We need to show that

$$\lim_{x\to c} 4 = 4.$$

graph

For each $\epsilon > 0$, let $\delta = 1$. Then

$$|f(x) - f(c)| = |4-4| = 0 < \epsilon$$

for all x such that $|x - c| < 1$. Secondly, for each $\epsilon > 0$, let $\delta = 1$. Then

$$|f(x) - f(c)| = |k - k| = 0 < \epsilon$$

for all x such that $|x - c| < 1$. This completes the required proof.

Example 2.1.11 Show that $f(x) = 3x - 4$ is continuous at $x = 3$.

Let $\epsilon > 0$ be given. Then

$$\begin{aligned} |f(x) - f(3)| &= |(3x - 4) - (5)| \\ &= |3x - 9| \\ &= 3|x - 3| \\ &< \epsilon \end{aligned}$$

whenever $|x - 3| < \dfrac{\epsilon}{3}$.

We define $\delta = \dfrac{\epsilon}{3}$. Then, it follows that

$$\lim_{x \to 3} f(x) = f(3)$$

and, hence, f is continuous at $x = 3$.

Example 2.1.12 Show that $f(x) = x^3$ is continuous at $x = 2$.

Since $f(2) = 8$, we need to prove that

$$\lim_{x \to 2} x^3 = 8 = 2^3.$$

graph

Let $\epsilon > 0$ be given. Let us concentrate our attention on the open interval

$(1, 3)$ that contains $x = 2$ at its mid-point. Then

$$\begin{aligned}|f(x) - f(2)| &= |x^3 - 8| = |(x-2)(x^2 + 2x + 4)| \\ &= |x-2|\ |x^2 + 2x + 4| \\ &\le |x-2|(|x|^2 + 2|x| + 4) \quad \text{(Triangle Inequality } |u+v| \le |u| + |v|) \\ &\le |x-2|(9 + 18 + 4) \\ &= 31|x-2| \\ &< \epsilon\end{aligned}$$

Provided

$$|x - 2| < \frac{\epsilon}{31}.$$

Since we are concentrating on the interval $(1, 3)$ for which $|x - 2| < 1$, we need to define δ to be the minimum of 1 and $\frac{\epsilon}{31}$. Thus, if we define $\delta = \min\{1, \epsilon/31\}$, then

$$|f(x) - f(2)| < \epsilon$$

whenever $|x - 2| < \delta$. By definition, $f(x)$ is continuous at $x = 2$.

Example 2.1.13 Show that every polynomial $P(x)$ is continuous at every c.

From algebra, we recall that, by the Remainder Theorem,

$$P(x) = (x - c)Q(x) + P(c).$$

Thus,

$$|P(x) - P(c)| = |x - c||Q(x)|$$

where $Q(x)$ is a polynomial of degree one less than the degree of $P(x)$. As in Example 12, $|Q(x)|$ is bounded on the closed interval $[c - 1, c + 1]$. For example, if

$$Q(x) = q_0 x^{n-1} + q_1 x^{n-2} + \cdots + q_{n-2} x + q_{n-1}$$

$|Q(x)| \le |q_0|\ |x|^{n-1} + |q_1|\ |x|^{n-2} + \cdots + |q_{n-2}|\ |x| + |q_{n-1}|$.

Let $m = \max\{|x| : c - 1 \le x \le c + 1\}$. Then

$$|Q(x)| \le |q_0| m^{n-1} + |q_1| m^{n-2} + \cdots + q_{n-2} m + |q_{n-1}| = M,$$

for some M. Then

$$|P(x) - P(c)| = |x - c|\ |Q(x)| \leq M|x - c| < \epsilon$$

whenever $|x - c| < \dfrac{\epsilon}{M}$. As in Example 12, we define $\delta = \min\left\{1, \dfrac{\epsilon}{M}\right\}$. Then $|P(x) - P(c)| < \epsilon$, whenever $|x - c| < \delta$. Hence,

$$\lim_{x \to c} P(x) = P(c)$$

and by definition $P(x)$ is continuous at each number c.

Example 2.1.14 Show that $f(x) = \dfrac{1}{x}$ is continuous at every real number $c > 0$.

We need to show that

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{c}.$$

Let $\epsilon > 0$ be given. Let us concentrate on the interval $|x - c| \leq \dfrac{c}{2}$; that is, $\dfrac{c}{2} \leq x \leq \dfrac{3c}{2}$. Clearly, $x \neq 0$ in this interval. Then

$$\begin{aligned}
|f(x) - f(c)| &= \left|\frac{1}{x} - \frac{1}{c}\right| \\
&= \left|\frac{c - x}{cx}\right| \\
&= |x - c| \cdot \frac{1}{c} \cdot \frac{1}{|x|} \\
&< |x - c| \cdot \frac{1}{c} \cdot \frac{2}{c} \\
&= \frac{2}{c^2}\ |x - c| \\
&< \epsilon
\end{aligned}$$

whenever $|x - c| < \dfrac{c^2\epsilon}{2}$.

We define $\delta = \min\left\{\dfrac{c}{2}, \dfrac{c^2\epsilon}{2}\right\}$. Then for all x such that $|x - c| < \delta$,

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \epsilon.$$

Hence,

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$

and the function $f(x) = \dfrac{1}{x}$ is continuous at each $c > 0$.

A similar argument can be used for $c < 0$. The function $f(x) = \dfrac{1}{x}$ is continuous for all $x \neq 0$.

Example 2.1.15 Suppose that the domain of a function g contains an open interval containing c, and the range of g contains an open interval containing $g(c)$. Suppose further that the domain of f contains the range of g. Show that if g is continuous at c and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

We need to show that

$$\lim_{x \to c} f(g(x)) = f(g(c)).$$

Let $\epsilon > 0$ be given. Since f is continuous at $g(c)$, there exists $\delta_1 > 0$ such that

1. $|f(y) - f(g(c))| < \epsilon$, whenever, $|y - g(c)| < \delta_1$.

   Since g is continuous at c, and $\delta_1 > 0$, there exists $\delta > 0$ such that

2. $|g(x) - g(c)| < \delta_1$, whenever, $|x - c| < \delta$.

On replacing y by $g(x)$ in equation (1), we get

$$|f(g(x)) - f(g(c))| < \epsilon, \text{ whenever, } |x - c| < \delta.$$

By definition, it follows that

$$\lim_{x \to c} f(g(x)) = f(g(c))$$

and the composition $f \circ g$ is continuous at c.

Example 2.1.16 Suppose that two functions f and g have a common domain that contains one open interval containing c. Suppose further that f and g are continuous at c. Then show that

(i) $f + g$ is continuous at c,

(ii) $f - g$ is continuous at c,

(iii) kf is continuous at c for every constant $k \neq 0$,

(iv) $f \cdot g$ is continuous at c.

Part (i) We need to prove that

$$\lim_{x \to c} [f(x) + g(x)] = f(c) + g(c).$$

Let $\epsilon > 0$ be given. Then $\frac{\epsilon}{2} > 0$. Since f is continuous at c and $\frac{\epsilon}{2} > 0$, there exists some $\delta_1 > 0$ such that

(1) $$|f(x) - f(c)| < \frac{\epsilon}{2}, \text{ whenever, } |x - c| \leq \delta_1.$$

Also, since g is continuous at c and $\frac{\epsilon}{2} > 0$, there exists some $\delta_2 > 0$ such that

(2) $$|g(x) - g(c)| < \frac{\epsilon}{2}, \text{ whenever, } |x - c| < \frac{\delta}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$. Let $|x - c| < \delta$. Then $|x - c| < \delta_1$ and $|x - c| < \delta_2$. For this choice of x, we get

$$\begin{aligned}
&|\{f(x) + g(x)\} - \{f(c) + g(c)\}| \\
&\quad = |\{f(x) - f(c)\} + \{g(x) - g(c)\}| \\
&\quad \leq |f(x) - f(c)| + |g(x) - g(c)| \qquad \text{(by triangle inequality)} \\
&\quad < \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&\quad = \epsilon.
\end{aligned}$$

It follows that

$$\lim_{x \to 0} (f(x) + g(x)) = f(c) + g(c)$$

and $f + g$ is continuous at c. This proves part (i).

Part (ii) For Part (ii) we chose $\epsilon, \epsilon/2, \delta_1, \delta_2$ and δ exactly as in Part (i). Suppose $|x-c| < \delta$. Then $|x-c| < \delta_1$ and $|x-c| < \delta_2$. For these choices of x we get

$$\begin{aligned}
&|\{f(x)-g(x)\} - \{f(c)-g(c)\}| \\
&\qquad = |\{f(x)-f(c)\} - \{g(x)-g(c)\}| \\
&\qquad \leq |f(x)-f(c)| + |g(x)-g(c)| \qquad \text{(by triangle inequality)} \\
&\qquad < \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&\qquad = \epsilon.
\end{aligned}$$

It follows that

$$\lim_{x \to c} (f(x) - g(x)) = f(c) - g(c)$$

and, hence, $f - g$ is continuous at c.

Part (iii) For Part (iii) let $\epsilon > 0$ be given. Since $k \neq 0$, $\dfrac{\epsilon}{|k|} > 0$. Since f is continuous at c, there exists some $\delta > 0$ such that

$$|f(x) - f(c)| < \frac{\epsilon}{|k|}, \quad \text{whenever, } |x - c| < \delta.$$

If $|x - c| < \delta$, then

$$\begin{aligned}
|kf(x) - kf(c)| &= |k(f(x) - f(c))| \\
&= |k| \; |(f(x) - f(c)| \\
&< |k| \cdot \frac{\epsilon}{|k|} \\
&= \epsilon.
\end{aligned}$$

It follows that

$$\lim_{x \to c} kf(x) = kf(c)$$

and, hence, kf is continuous at c.

Part (iv) We need to show that

$$\lim_{x \to c} (f(x)g(x)) = f(c)g(c).$$

Let $\epsilon > 0$ be given. Without loss of generality we may assume that $\epsilon < 1$. Let $\epsilon_1 = \dfrac{\epsilon}{2(1+|f(c)|+|g(c)|)}$. Then $\epsilon_1 > 0, \epsilon_1 < 1$ and $\epsilon_1(1+|f|+|g(c)|) = \dfrac{\epsilon}{2} < \epsilon$. Since f is continuous at c and $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$|f(x) - f(c)| < \epsilon_1 \qquad \text{whenever, } |x - c| < \delta_1.$$

Also, since g is continuous at c and $\epsilon_1 > 0$, there exists $\delta_2 > 0$ such that

$$|g(x) - g(c)| < \epsilon_1 \quad \text{whenever, } |x - c| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and $|x - c| < \delta$. For these choices of x, we get

$$\begin{aligned}
&|f(x)g(x) - f(c)g(c)| \\
&= |(\overline{f(x) - f(c)} + f(c))(\overline{g(x) - g(c)} + g(c)) - f(c)g(c)| \\
&= |(\overline{f(x) - f(c)})(g(x) - g(c)) + (\overline{f(x) - f(c)})g(c) + f(c)(\overline{g(x) - g(c)})| \\
&\leq |f(x) - f(c)|\ |g(x) - g(c)| + |f(x) - f(c)|\ |g(c)| + |f(c)|\ |g(x) - g(c)| \\
&< \epsilon_1 \cdot \epsilon_1 + \epsilon_1|g(c)| + \epsilon_1|f(c)| \\
&< \epsilon_1(1 + |g(c)| + |f(c)|) \quad , \text{ (since } \epsilon_1 < 1) \\
&< \epsilon.
\end{aligned}$$

It follows that

$$\lim_{x\to c} f(x)g(x) = f(c)g(c)$$

and, hence, the product $f \cdot g$ is continuous at c.

Example 2.1.17 Show that the quotient f/g is continuous at c if f and g are continuous at c and $g(c) \neq 0$.

First of all, let us observe that the function $1/g$ is a composition of $g(x)$ and $1/x$ and hence $1/g$ is continuous at c by virtue of the arguments in Examples 14 and 15. By the argument in Example 16, the product $f(1/g) = f/g$ is continuous at c, as required in Example 17.

Example 2.1.18 Show that a rational function of the form $p(x)/q(x)$ is continuous for all c such that $g(c) \neq 0$.

In Example 13, we showed that each polynomial function is continuous at every real number c. Therefore, $p(x)$ is continuous at every c and $q(x)$ is continuous at every c. By virtue of the argument in Example 17, the quotient $p(x)/q(x)$ is continuous for all c such that $q(c) \neq 0$.

Example 2.1.19 Suppose that $f(x) \leq g(x) \leq h(x)$ for all x in an open interval containing c and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L.$$

Then, show that,

$$\lim_{x \to c} g(x) = L.$$

Let $\epsilon > 0$ be given. Then there exist $\delta_1 > 0$, $\delta_2 > 0$, and $\delta = \min\{\delta_1, \delta_2\}$ such that

$$|f(x) - L| < \frac{\epsilon}{2} \text{ whenever } 0 < |x - c| < \delta_1$$
$$|h(x) - L) < \frac{\epsilon}{2} \text{ whenever } 0 < |x - c| < \delta_2.$$

If $0 < |x - c| < \delta_1$, then $0 < |x - c| < \delta_1,\ 0 < |x - c| < \delta_2$ and, hence,

$$-\frac{\epsilon}{2} < f(x) - L < g(x) - L < h(x) - L < \frac{\epsilon}{2}.$$

It follows that

$$|g(x) - L| < \frac{\epsilon}{2} < \epsilon \text{ whenever } 0 < |x - c| < \delta,$$

and

$$\lim_{x \to c} g(x) = L.$$

Example 2.1.20 Show that $f(x) = |x|$ is continuous at 0.

We need to show that

$$\lim_{x \to 0} |x| = 0.$$

Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. Then $|x - 0| < \epsilon$ implies that $|x| < \epsilon$ Hence,

$$\lim_{x \to 0} |x| = 0$$

Example 2.1.21 Show that

(i) $\lim_{\theta\to 0} \sin\theta = 0$ (ii) $\lim_{\theta\to 0} \cos\theta = 1$

(iii) $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ (iv) $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$

graph

Part (i) By definition, the point $C(\cos\theta, \sin\theta)$, where θ is the length of the arc CD, lies on the unit circle. It is clear that the length $BC = \sin\theta$ is less than θ, the arclength of the arc CD, for small positive θ. Hence,

$$-\theta \leq \sin\theta \leq \theta$$

and

$$\lim_{\theta\to 0^+} \sin\theta = 0.$$

For small negative θ, we get

$$\theta \leq \sin\theta \leq -\theta$$

and

$$\lim_{\theta\to 0^-} \sin\theta = 0.$$

Therefore,

$$\lim_{\theta\to 0} \sin\theta = 0.$$

Part (ii) It is clear that the point B approaches D as θ tends to zero. Therefore,

$$\lim_{\theta\to 0} \cos\theta = 1.$$

Part (iii) Consider the inequality

$$\text{Area of triangle } ABC \leq \text{ Area of sector } ADC \leq \text{ Area of triangle } ADE$$

$$\frac{1}{2}\cos\theta\sin\theta \leq \frac{1}{2}\,\theta \leq \frac{1}{2}\,\frac{\sin\theta}{\cos\theta}.$$

Assume that θ is small but positive. Multiply each part of the inequality by $2/\sin\theta$ to get

$$\cos\theta \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}.$$

On taking limits and using the squeeze theorem, we get

$$\lim_{\theta\to 0^+} \frac{\theta}{\sin\theta} = 1.$$

By taking reciprocals, we get

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

Since

$$\frac{\sin(-\theta)}{-\theta} = \frac{\sin\theta}{\theta},$$

$$\lim_{\theta - 0^-} \frac{\sin\theta}{\theta} = 1.$$

Therefore,

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

Part (iv)

$$\begin{aligned}
\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0} \frac{(1-\cos\theta)(1+\cos\theta)}{\theta(1+\cos\theta)} \\
&= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta} \cdot \frac{1}{(1+\cos\theta)} \\
&= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1+\cos\theta} \\
&= 1 \cdot \frac{0}{2} \\
&= 0.
\end{aligned}$$

Example 2.1.22 Show that

(i) $\sin\theta$ and $\cos\theta$ are continuous for all real θ.

(ii) $\tan\theta$ and $\sec\theta$ are continuous for all $\theta \neq 2n\pi \pm \frac{\pi}{2}, n$ integer.

(iii) $\cot\theta$ and $\csc\theta$ are continuous for all $\theta \neq n\pi, n$ integer.

Part (i) First, we show that for all real c,

$$\lim_{\theta\to c} \sin\theta = \sin c \text{ or equivalently } \lim_{\theta\to c} |\sin\theta - \sin c| = 0.$$

We observe that

$$\begin{aligned} 0 \leq |\sin\theta - \sin c| &= \left|2\cos\frac{\theta+c}{2}\ \sin\frac{\theta-c}{2}\right| \\ &\leq \left|2\sin\frac{(\theta-c)}{2}\right| \\ &= |(\theta-c)|\ \left|\frac{\sin\frac{(\theta-c)}{2}}{\frac{(\theta-c)}{2}}\right| \end{aligned}$$

Therefore, by squeeze theorem,

$$0 \leq \lim_{\theta - c} |\sin\theta - \sin c| \leq 0 \cdot 1 = 0.$$

It follows that for all real $c, \sin\theta$ is continuous at c.
Next, we show that

$$\lim_{x\to c} \cos x = \cos c \text{ or equivalently } \lim_{x\to c} |\cos x - \cos c| = 0.$$

We observe that

$$\begin{aligned} 0 \leq |\cos x - \cos c| &= \left|-2\sin\frac{x+c}{2}\ \sin\frac{(x-c)}{2}\right| \\ \leq |\theta - c|\ \left|\frac{\sin\left(\frac{x-c}{2}\right)}{\left(\frac{x-c}{2}\right)}\right|\ ; &\qquad \left(\left|\sin\ \frac{x+c}{2}\right| \leq 1\right) \end{aligned}$$

Therefore,

$$0 \leq \lim_{x\to c} |\cos x - \cos c| \leq 0 \cdot 1 = 0$$

and $\cos x$ is continuous at c.

Part (ii) Since for all $\theta \neq 2n\pi \pm \frac{\pi}{2}, n$ integer,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \ \sec\theta = \frac{1}{\cos\theta}$$

it follows that $\tan\theta$ and $\sec\theta$ are continuous functions.

Part (iii) Both $\cot\theta$ and $\csc\theta$ are continuous as quotients of two continuous functions where the denominators are not zero for $n \neq n\pi, n$ integer.

Exercises 2.1 Evaluate each of the following limits.

1. $\lim\limits_{x\to1} \dfrac{x^2-1}{x^3-1}$
2. $\lim\limits_{x\to0} \dfrac{\sin(2x)}{x}$
3. $\lim\limits_{x\to0} \dfrac{\sin 5x}{\sin 7x}$
4. $\lim\limits_{x\to2^+} \dfrac{1}{x^2-4}$
5. $\lim\limits_{x\to2^-} \dfrac{1}{x^2-4}$
6. $\lim\limits_{x\to2} \dfrac{x-2}{x^2-4}$
7. $\lim\limits_{x\to2^+} \dfrac{x-2}{|x-2|}$
8. $\lim\limits_{x\to2^-} \dfrac{x-2}{|x-2|}$
9. $\lim\limits_{x\to2} \dfrac{x-2}{|x-2|}$
10. $\lim\limits_{x\to3} \dfrac{x^2-9}{x-3}$
11. $\lim\limits_{x\to3} \dfrac{x^2-9}{x+3}$
12. $\lim\limits_{x\to\frac{\pi}{2}} \tan x$
13. $\lim\limits_{x\to\frac{\pi}{2}^+} \tan x$
14. $\lim\limits_{x\to0^-} \csc x$
15. $\lim\limits_{x\to0^+} \csc x$
16. $\lim\limits_{x\to0^+} \cot x$
17. $\lim\limits_{x\to0^-} \cot x$
18. $\lim\limits_{x\to\frac{\pi}{2}^+} \sec x$
19. $\lim\limits_{x\to\frac{\pi}{2}} \sec x$
20. $\lim\limits_{x\to0} \dfrac{\sin 2x + \sin 3x}{x}$
21. $\lim\limits_{x\to4^-} \dfrac{\sqrt{x}-2}{x-4}$
22. $\lim\limits_{x\to4^+} \dfrac{\sqrt{x}-2}{x-4}$

23 $\lim\limits_{x\to4} \dfrac{\sqrt{x}-2}{x-4}$

24. $\lim\limits_{x\to3} \dfrac{x^4-81}{x^2-9}$

Sketch the graph of each of the following functions. Determine all the discontinuities of these functions and classify them as (a) removable type, (b) finite jump type, (c) essential type, (d) oscillation type, or other types.

25. $f(x) = 2\frac{x-1}{|x-1|} - \frac{x-2}{|x-2|}$

26. $f(x) = \frac{x}{x^2-9}$

27. $f(x) = \begin{cases} 2x & \text{for } x \leq 0 \\ x^2+1 & \text{for } x > 0 \end{cases}$

28. $f(x) = \begin{cases} \sin x & \text{if } x \leq 0 \\ \sin\left(\frac{2}{x}\right) & \text{if } x > 0 \end{cases}$

29. $f(x) = \frac{x-1}{(x-2)(x-3)}$

30. $f(x) = \begin{cases} |x-1| & \text{if } x \leq 1 \\ |x-2| & \text{if } x > 1 \end{cases}$

Recall the unit step function $u(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$

Sketch the graph of each of the following functions and determine the left hand limit and the right hand limit at each point of discontinuity of f and g.

31. $f(x) = 2u(x-3) - u(x-4)$

32. $f(x) = -2u(x-1) + 4u(x-5)$

33. $f(x) = u(x-1) + 2u(x+1) - 3u(x-2)$

34. $f(x) = \sin x\left[u\left(x+\frac{\pi}{2}\right) - u\left(x-\frac{\pi}{2}\right)\right]$

35. $g(x) = (\tan x)\left[u\left(x+\frac{\pi}{2}\right) - u\left(x-\frac{\pi}{2}\right)\right]$

36. $f(x) = [u(x) - u(x-\pi)]\cos x$

2.2 Linear Function Approximations

One simple application of limits is to approximate a function $f(x)$, in a small neighborhood of a point c, by a line. The approximating line is called the tangent line. We begin with a review of the equations of a line.

A vertical line has an equation of the form $x = c$. A vertical line has no slope. A horizontal line has an equation of the form $y = c$. A horizontal line has slope zero. A line that is neither horizontal nor vertical is called an oblique line.

Suppose that an oblique line passes through two points, say (x_1, y_1) and (x_2, y_2). Then the slope of this line is define as

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

If (x, y) is any arbitrary point on the above oblique line, then

$$m = \frac{y - y_1}{x - x_1} = \frac{y - y_2}{x - x_2}.$$

By equating the two forms of the slope m we get an equation of the line:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad \frac{y - y_2}{x - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.$$

On multiplying through, we get the "two point" form of the equation of the line, namely,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}\,(x - x_1) \quad \text{or} \quad y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}\,(x - x_2).$$

Example 2.2.1 Find the equations of the lines passing through the following pairs of points:

(i) $(4, 2)$ and $(6, 2)$
(ii) $(1, 3)$ and $(1, 5)$
(iii) $(3, 4)$ and $(5, -2)$
(iv) $(0, 2)$ and $(4, 0)$.

Part (i) Since the y-coordinates of both points are the same, the line is horizontal and has the equation $y = 2$. This line has slope 0.

Part (ii) Since the x-coordinates of both points are equal, the line is vertical and has the equation $x = 1$.

Part (iii) The slope of the line is given by

$$m = \frac{-2 - 4}{5 - 3} = -3.$$

The equation of this line is

$$y - 4 = -3(x - 3) \quad \text{or} \quad y + 2 = -3(x - 5).$$

On solving for y, we get the equation of the line as

$$y = -3x + 13.$$

This line goes through the point $(0, 13)$. The number 13 is called the y-intercept. The above equation is called the *slope-intercept form* of the line.

Example 2.2.2 Determine the equations of the lines satisfying the given conditions:

(i) slope $= 3$, passes through $(2, 4)$

(ii) slope $= -2$, passes through $(1, -3)$

(iii) slope $= m$, passes through (x_1, y_1)

(iv) passes through $(3, 0)$ and $(0, 4)$

(v) passes through $(a, 0)$ and $(0, b)$

Part (i) If (x, y) is on the line, then we equate the slopes and simplify:

$$3 = \frac{y-4}{x-2} \quad \text{or} \quad y - 4 = 3(x-2).$$

Part (ii) If (x, y) is on the line, then we equate slopes and simplify:

$$-2 = \frac{y+3}{x-1} \quad \text{or} \quad y + 3 = -2(x-1).$$

Part (iii) On equating slopes and clearing fractions, we get

$$m = \frac{y-y_1}{x-x_1} \quad \text{or} \quad y - y_1 = m(x - x_1).$$

This form of the line is called the "point-slope" form of the line.

Part (iv) Using the two forms of the line we get

$$\frac{y-0}{x-3} = \frac{4-0}{0-3} \quad \text{or} \quad y = -\frac{4}{3}(x-3).$$

If we divide by 4 we get

$$\frac{x}{3} + \frac{y}{4} = 1.$$

The number 3 is called the x-intercept and the number 4 is called the y-intercept of the line. This form of the equation is called the "two-intercept" form of the line.

Part (v) As in Part (iv), the "two-intercept" form of the line has the equation

$$\frac{x}{a} + \frac{y}{b} = 1.$$

In order to approximate a function f at the point c, we first define the slope m of the line that is tangent to the graph of f at the point $(c, f(c))$.

graph

$$m = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Then the equation of the tangent line is

$$y - f(c) = m(x - c),$$

written in the point-slope form. The point $(c, f(c))$ is called the point of tangency. This tangent line is called the linear approximation of f about $x = c$.

Example 2.2.3 Find the equation of the line tangent to the graph of $f(x) = x^2$ at the point $(2, 4)$.

The slope m of the tangent line at $(3, 9)$ is

$$\begin{aligned} m &= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \to 3} (x + 3) \\ &= 6. \end{aligned}$$

The equation of the tangent line at $(3, 9)$ is

$$y - 9 = 6(x - 3).$$

Example 2.2.4 Obtain the equation of the line tangent to the graph of $f(x) = \sqrt{x}$ at the point $(9, 3)$.

The slope m of the tangent line is given by

$$\begin{aligned} m &= \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \\ &= \lim_{x \to 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(x - 9)(\sqrt{x} + 3)} \\ &= \lim_{x \to 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} \\ &= \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} \\ &= \frac{1}{6}. \end{aligned}$$

The equation of the tangent line is

$$y - 3 = \frac{1}{6}(x - 9).$$

Example 2.2.5 Derive the equation of the line tangent to the graph of $f(x) = \sin x$ at $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

The slope m of the tangent line is given by

$$\begin{aligned}
m &= \lim_{x\to\frac{\pi}{6}} \frac{\sin x - \sin\left(\frac{\pi}{6}\right)}{x-\frac{\pi}{6}} \\
&= \lim_{x\to\frac{\pi}{6}} \frac{2\cos\left(\frac{x+\pi/6}{2}\right)\sin\left(\frac{x-\pi/6}{2}\right)}{(x-\pi/6)} \\
&= \cos(\pi/6)\cdot \lim_{x\to\frac{\pi}{6}} \frac{\sin\left(\frac{x-\pi/6}{2}\right)}{\left(\frac{x-\pi/6}{2}\right)} \\
&= \cos(\pi/6) \\
&= \frac{\sqrt{3}}{2}.
\end{aligned}$$

The equation of the tangent line is

$$y - \frac{1}{2} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right).$$

Example 2.2.6 Derive the formulas for the slope and the equation of the line tangent to the graph of $f(x) = \sin x$ at $(c, \sin c)$.

As in Example 27, replacing $\pi/6$ by c, we get

$$\begin{aligned}
m &= \lim_{x\to c} \frac{\sin x - \sin c}{x-c} \\
&= \lim_{x\to c} \frac{2\cos\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right)}{x-c} \\
&= \lim_{x\to c} \cos\left(\frac{x+c}{2}\right)\cdot \lim_{x\to c} \frac{\sin\left(\frac{x-c}{2}\right)}{\left(\frac{x-c}{2}\right)} \\
&= \cos c.
\end{aligned}$$

Therefore the slope of the line tangent to the graph of $f(x) = \sin x$ at $(c, \sin c)$ is $\cos c$.

The equation of the tangent line is

$$y - \sin c = (\cos c)(x - c).$$

Example 2.2.7 Derive the formulas for the slope, m, and the equation of the line tangent to the graph of $f(x) = \cos x$ at $(c, \cos c)$. Then determine the slope and the equation of the tangent line at $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.

As in Example 28, we replace the sine function with the cosine function,

$$\begin{aligned}
m &= \lim_{x\to c} \frac{\cos x - \cos c}{x - c} \\
&= \lim_{x\to c} \frac{-2\sin\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right)}{x-c} \\
&= \lim_{x\to c} \sin\left(\frac{x+c}{2}\right) \lim_{x\to c} \frac{\sin\left(\frac{x-c}{2}\right)}{\left(\frac{x-c}{2}\right)} \\
&= -\sin(c).
\end{aligned}$$

The equation of the tangent line is

$$y - \cos c = -\sin c(x - c).$$

For $c = \frac{\pi}{3}$, slope $= -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and the equation of the tangent line

$$y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right).$$

Example 2.2.8 Derive the formulas for the slope, m, and the equation of the line tangent to the graph of $f(x) = x^n$ at the point (c, c^n), where n is a natural number. Then get the slope and the equation of the tangent line for $c = 2, n = 4$.

By definition, the slope m is given by

$$m = \lim_{x\to c} \frac{x^n - c^n}{x - c}.$$

To compute this limit for the general natural number n, it is convenient to let $x = c + h$. Then

$$\begin{aligned} m &= \lim_{h \to 0} \frac{(c+h)^n - c^n}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \left[\left(c^n + nc^{n-1}h + \frac{n(n-1)}{2!}\, c^{n-2}h^2 + \cdots + h^n \right) - c^n \right] \\ &= \lim_{h \to 0} \frac{1}{h} \left[nc^{n-1}h + \frac{n(n-1)}{2!}\, c^{n-2}h^2 + \cdots + h^n \right] \\ &= \lim_{h \to 0} \left[nc^{n-1} + \frac{n(n-1)}{2!}\, c^{n-2}h + \cdots + h^{n-1} \right] \\ &= nc^{n-1}. \end{aligned}$$

Therefore, the equation of the tangent line through (c, c^n) is

$$y - c^n = nc^{n-1}(x - c).$$

For $n = 4$ and $c = 2$, we find the slope, m, and equation for the tangent line to the graph of $f(x) = x^4$ at $c = 2$:

$$\begin{aligned} &m = 4c^3 = 32 \\ &y - 2^4 = 32(x-2) \quad \text{or} \quad y - 16 = 32(x-2). \end{aligned}$$

Definition 2.2.1 Suppose that a function f is defined on a closed interval $[a, b]$ and $a < c < b$. Then c is called a *critical point* of f if the slope of the line tangent to the graph of f at $(c, f(c))$ is zero or undefined. The *slope function* of f at c is defined by

$$\begin{aligned} \text{slope } (f(x), c) &= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}. \end{aligned}$$

Example 2.2.9 Determine the slope functions and critical points of the following functions:

(i) $f(x) = \sin x,\ 0 \le x \le 2\pi$ (ii) $f(x) = \cos x,\ 0 \le x \le 2\pi$
(iii) $f(x) = |x|,\ -1 \le x \le 1$ (iv) $f(x) = x^3 - 4x,\ -2 \le x \le 2$

Part (i) In Example 28, we derived the slope function formula for $\sin x$, namely

$$\text{slope }(\sin x, c) = \cos c.$$

Since $\cos c$ is defined for all c, the non-end point critical points on $[0, 2\pi]$ are $\pi/2$ and $3\pi/2$ where the cosine has a zero value. These critical points correspond to the maximum and minimum values of $\sin x$.

Part (ii) In Example 29, we derived the slope function formula for $\cos x$, namely

$$\text{slope }(\cos x, c) = -\sin c.$$

The critical points are obtained by solving the following equation for c:

$$\begin{aligned} -\sin c &= 0, \quad 0 \le c \le 2\pi \\ c &= 0, \pi, 2\pi. \end{aligned}$$

These values of c correspond to the maximum value of $\cos x$ at $c = 0$ and 2π, and the minimum value of $\cos x$ at $c = \pi$.

Part (iii)

$$\begin{aligned} \text{slope }(|x|, c) &= \lim_{x \to c} \frac{|x| - |c|}{x - c} \\ &= \lim_{x \to c} \frac{|x| - |c|}{x - c} \cdot \frac{|x| + |c|}{|x| + |c|} \\ &= \lim_{x \to c} \frac{x^2 - c^2}{(x - c)(|x| + |c|)} \\ &= \lim_{x \to c} \frac{x + c}{|x| + |c|} \\ &= \frac{2c}{2|c|} \\ &= \frac{c}{|c|} \\ &= \begin{cases} 1 & \text{if } c > 0 \\ -1 & \text{if } c < 0 \\ \text{undefined} & \text{if } c = 0 \end{cases} \end{aligned}$$

The only critical point is $c = 0$, where the slope function is undefined. This critical point corresponds to the minimum value of $|x|$ at $c = 0$. The slope function is undefined because the tangent line does not exist at $c = 0$. There is a sharp corner at $c = 0$.

Part (iv) The slope function for $f(x) = x^3 - 4x$ is obtained as follows:

$$\begin{aligned}
\text{slope } (f(x), c) &= \lim_{h \to 0} \frac{1}{h} \left[((c+h)^3 - 4(c+h)) - (c^3 - 4c)\right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[c^3 + 3c^2h + 3ch^2 + h^3 - 4c - 4h - c^3 + 4c\right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[3c^2h + 3ch^2 + h^3 - 4h\right] \\
&= \lim_{h \to 0} \left[3c^2 + 3ch + h^2 - 4\right] \\
&= 3c^2 - 4
\end{aligned}$$

graph

The critical points are obtained by solving the following equation for c:

$$\begin{aligned}
3c^2 - 4 &= 0 \\
c &= \pm \frac{2}{\sqrt{3}}
\end{aligned}$$

At $c = \frac{-2}{\sqrt{3}}$, f has a local maximum value of $\frac{16}{3\sqrt{3}}$ and at $c = \frac{2}{\sqrt{3}}$, f has a local minimum value of $\frac{-16}{3\sqrt{3}}$. The end point $(-2, 0)$ has a local end-point minimum and the end point $(2, 0)$ has a local end-point maximum.

Remark 7 The zeros and the critical points of a function are helpful in sketching the graph of a function.

Exercises 2.2

1. Express the equations of the lines satisfying the given information in the form $y = mx + b$.

   (a) Line passing through $(2, 4)$ and $(5, -2)$

   (b) Line passing through $(1, 1)$ and $(3, 4)$

   (c) Line with slope 3 which passes through $(2, 1)$

   (d) Line with slope 3 and y-intercept 4

   (e) Line with slope 2 and x-intercept 3

   (f) Line with x-intercept 2 and y-intercept 4.

2. Two oblique lines are parallel if they have the same slope. Two oblique lines are perpendicular if the product of their slopes is -1. Using this information, solve the following problems:

   (a) Find the equation of a line that is parallel to the line with equation $y = 3x - 2$ which passes through $(1, 4)$.

   (b) Solve problem (a) when "parallel" is changed to "perpendicular."

   (c) Find the equation of a line with y-intercept 4 which is parallel to $y = -3x + 1$.

   (d) Solve problem (c) when "parallel" is changed to "perpendicular."

   (e) Find the equation of a line that passes through $(1, 1)$ and is

      (i) parallel to the line with equation $2x - 3y = 6$.

      (ii) perpendicular to the line with equation $3x + 2y = 6$

3. For each of the following functions $f(x)$ and values c,

   (i) derive the slope function, slope $(f(x), c)$ for arbitrary c;

   (ii) determine the equations of the tangent line and normal line (perpendicular to tangent line) at the point $(c, f(c))$ for the given c;

   (iii) determine all of the critical points $(c, f(c))$.

      (a) $f(x) = x^2 - 2x, \quad c = 3$

      (b) $f(x) = x^3 \quad , \quad c = 1$

(c) $f(x) = \sin(2x), \quad c = \dfrac{\pi}{12}$

(d) $f(x) = \cos(3x), \quad c = \dfrac{\pi}{9}$

(e) $f(x) = x^4 - 4x^2, \quad c = -2, 0, 2, -\sqrt{2}, \sqrt{2}.$

2.3 Limits and Sequences

We begin with the definitions of sets, sequences, and the completeness property, and state some important results. If x is an element of a set S, we write $x \in S$, read "x is in S." If x is not an element of S, then we write $x \notin S$, read "x is not in S."

Definition 2.3.1 If A and B are two sets of real numbers, then we define

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

and

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both}\}.$$

We read "$A \cap B$" as the "intersection of A and B." We read "$A \cup B$" as the "union of A and B." If $A \cap B$ is the empty set, $\emptyset$, then we write $A \cap B = \emptyset$.

Definition 2.3.2 Let A be a set of real numbers. Then a number m is said to be an *upper bound* of A if $x \leq m$ for all $x \in A$. The number m is said to be a *least upper bound* of A, written lub(A) if and only if,

(i) m is an upper bound of A, and,

(ii) if $q < m$, then there is some $x \in A$ such that $q < x \leq m$.

Definition 2.3.3 Let B be a set of real numbers. Then a number ℓ is said to be a *lower bound* of B if $\ell \leq y$ for each $y \in B$. This number ℓ is said to be the *greatest lower bound* of B, written, glb(b), if and only if,

(i) ℓ is a lower bound of B, and,

(ii) if $\ell < p$, then there is some element $y \in B$ such that $\ell \leq y < p$.

Definition 2.3.4 A real number p is said to be a *limit point* of a set S if and only if every open interval that contains p also contains an element q of S such that $q \neq p$.

Example 2.3.1 Suppose $A = [1, 10]$ and $B = [5, 15]$.

Then $A \cap B = [5, 10]$, $A \cup B = [1, 15]$, $\text{glb}(A) = 1$, $\text{lub}(A) = 10$, $\text{glb}(B) = 5$ and $\text{lub}(B) = 15$. Each element of A is a limit point of A and each element of B is a limit point of B.

Example 2.3.2 Let $S = \left\{ \frac{1}{n} : n \text{ is a natural number} \right\}$.

Then no element of S is a limit point of S. The number 0 is the only limit point of S. Also, $\text{glb}(S) = 0$ and $\text{lub}(S) = 1$.

Completeness Property: The *completeness property* of the set R of all real numbers states that if A is a non-empty set of real numbers and A has an upper bound, then A has a least upper bound which is a real number.

Theorem 2.3.1 *If B is a non-empty set of real numbers and B has a lower bound, then B has a greatest lower bound which is a real number.*

Proof. Let m denote a lower bound for B. Then $m \leq x$ for every $x \in B$. Let $A = \{-x : x \in B\}$. then $-x \leq -m$ for every $x \in B$. Hence, A is a non-empty set that has an upper bound $-m$. By the completeness property, A has a least upper bound $\text{lub}(A)$. Then, -lub$(A) = \text{glb}(B)$ and the proof is complete.

Theorem 2.3.2 *If x_1 and x_2 are real numbers such that $x_1 < x_2$, then $x_1 < \frac{1}{2}(x_1 + x_2) < x_2$.*

Proof. We observe that

$$\begin{aligned} x_1 \leq \frac{1}{2}(x_1 + x_2) < x_2 &\leftrightarrow 2x_1 < x_1 + x_2 < 2x_2 \\ &\leftrightarrow x_1 < x_2 < x_2 + (x_2 - x_1). \end{aligned}$$

This completes the proof.

Theorem 2.3.3 *Suppose that A is a non-empty set of real numbers and $m = \text{lub}(A)$. If $m \notin A$, then m is a limit point of A.*

Proof. Let an open interval (a, b) contain m. That is, $a < m < b$. By the definition of a least upper bound, **a** is not an upper bound for A. Therefore, there exists some element q of A such that $a < q < m < b$. Thus, every open interval (a, b) that contains m must contain a point of A other than m. It follows that m is a limit point of A.

Theorem 2.3.4 (Dedekind-Cut Property). *The set R of all real numbers is not the union of two non-empty sets A and B such that*

(i) if $x \in A$ and $y \in B$, then $x < y$,

(ii) A contains no limit point of B, and,

(iii) B contains no limit point of A.

Proof. Suppose that $R = A \cup B$ where A and B are non-empty sets that satisfy conditions (i), (ii) and (iii). Since A and B are non-empty, there exist real numbers a and b such that $a \in A$ and $b \in B$. By property (i), **a** is a lower bound for B and b is an upper bound for A. By the completeness property and theorem 2.3.1, A has a least upper bound, say m, and B has a greatest lower bound, say M. If $m \notin A$, then m is a limit point of A. Since B contains no limit point of A, $m \in A$. Similarly, $M \in B$. It follows that $m < M$ by condition (i). However, by Theorem 2.3.2,

$$m < \frac{1}{2}(m + M) < M.$$

The number $\frac{1}{2}(m + M)$ is neither in A nor in B. This is a contradiction, because $R = A \cup B$. This completes the proof.

Definition 2.3.5 An *empty set* is considered to be a *finite set.* A *non-empty set S* is said to be *finite* if there exists a natural number n and a one-to-one function that maps S onto the set $\{1, 2, 3, \ldots, n\}$. Then we say that S has n elements. If S is not a finite set, then S is said to be an *infinite set.* We say that an infinite set has an infinite number of elements. Two sets are said to have the same number of elements if there exists a one-to-one correspondence between them.

Example 2.3.3 Let $A = \{a, b, c\}, B = \{1, 2, 3\}, C = \{1, 2, 3, \dots\}$, and $D = \{0, 1, -1, 2, -2, \dots\}$.

In this example, A and B are finite sets and contain three elements each. The sets C and D are infinite sets and have the same number of elements. A one-to-one correspondence f between n, C and D can be defined as $f : C \to D$ such that

$$f(1) = 0,\ f(2n) = n \text{ and } f(2n+1) = -n \text{ for } n = 1, 2, 3, \dots .$$

Definition 2.3.6 A set that has the same number of elements as $C = \{1, 2, 3, \dots\}$ is said to be *countable*. An infinite set that is not countable is said to be *uncountable*.

Remark 8 The set of all rational numbers is *countable* but the set of all real numbers is *uncountable*.

Definition 2.3.7 A *sequence* is a function, say f, whose domain is the set of all natural numbers. It is customary to use the notation $f(n) = a_n, n = 1, 2, 3, \dots$. We express the sequence as a list without braces to avoid confusion with the set notation:

$$a_1, a_2, a_3, \dots, a_n, \dots \quad \text{or, simply,} \quad \{a_n\}_{n=1}^{\infty}.$$

The number a_n is called the nth term of the sequence. The sequence is said to *converge* to the limit **a** if for every $\epsilon > 0$, there exists some natural number, say N, such that $|a_m - a| < \epsilon$ for all $m \geq N$. We express this convergence by writing

$$\lim_{n \to \infty} a_n = a.$$

If a sequence does not converge to a limit, it is said to diverge or be divergent.

Example 2.3.4 For each natural number n, let

$$a_n = (-1)^n,\ b_n = 2^{-n},\ c_n = 2^n,\ d_n = \frac{(-1)^n}{n}.$$

The sequence $\{a_n\}$ does not converge because its terms oscillate between -1 and 1. The sequence $\{b_n\}$ converges to 0. The sequence $\{c_n\}$ diverges to ∞. The sequence $\{d_n\}$ converges to 0.

Definition 2.3.8 A sequence $\{a_n\}_{n=1}^{\infty}$ diverges to ∞ if, for every natural number N, there exists some m such that

$$a_{m+j} \geq N \text{ for all } j = 1, 2, 3, \cdots .$$

The sequence $\{a_n\}_{n=1}^{\infty}$ is said to diverge to $-\infty$ if, for every natural number N, there exists some m such that

$$a_{m+j} \leq -N \ , \text{ for all } j = 1, 2, 3, \ldots .$$

Theorem 2.3.5 *If p is a limit point of a non-empty set A, then every open interval that contains p must contain an infinite subset of A.*

Proof. Let some open interval (a, b) contain p. Suppose that there are only two finite subsets $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_m\}$ of distinct elements of A such that

$$a < a_1 < a_2 < \cdots < a_n < p < b_m < b_{m-1} < \cdots < b_1 < b.$$

Then the open interval (a_n, b_m) contains p but no other points of A distinct from p. Hence p is not a limit point of A. The contradiction proves the theorem.

Theorem 2.3.6 *If p is a limit point of a non-empty set A, then there exists a sequence $\{p_n\}_{n=1}^{\infty}$, of distinct points p_n of A, that converges to p.*

Proof. Let $a_1 = p - \dfrac{1}{2}$, $b_1 = p + \dfrac{1}{2}$. Choose a point p_1 of A such that $p_1 \neq p$ and $a_1 < p_1 < p < b_1$ or $a_1 < p < p_1 < b_1$. If $a_1 < p_1 < p < b_1$, then define $a_2 = \max\left\{p_1, p - \dfrac{1}{2^2}\right\}$ and $b_2 = p + \dfrac{1}{2^2}$. Otherwise, define $a_2 = p - \dfrac{1}{2^2}$ and $b_2 = \min\left\{p_1, p + \dfrac{1}{2^2}\right\}$. Then the open interval (a_2, b_2) contains p but not p_1 and $b_2 - a_2 \leq \dfrac{1}{2}$. We repeat this process indefinitely to select the sequence $\{p_n\}$, of distinct points p_n of A, that converges to p. The fact that $\{p_n\}$ is an infinite sequence is guaranteed by Theorem 2.3.5. This completes the proof.

Theorem 2.3.7 *Every bounded infinite set A has at least one limit point p and there exists a sequence $\{p_n\}_{n=1}^{\infty}$, of distinct points of A, that converges to p.*

Proof. We will show that A has a limit point. Since A is bounded, there exists an open interval (a, b) that contains all points of A. Then either $\left(a, \frac{1}{2}(a+b)\right)$ contains an infinite subset of A or $\left(\frac{1}{2}(a+b), b\right)$ contains an infinite subset of A. Pick one of the two intervals that contains an infinite subset of A. Let this interval be denoted (a_1, b_1). We continue this process repeatedly to get an open interval (a_n, b_n) that contains an infinite subset of A and $|b_n - a_n| = \frac{|b-a|}{2^n}$. Then the lub of the set $\{a_n, a_2, \ldots\}$ and glb of the set $\{b_1, b_2, \ldots\}$ are equal to some real number p. It follows that p is a limit point of A. By Theorem 2.3.6, there exists a sequence $\{p_n\}$, of distinct points of A, that converges to p. This completes the proof.

Definition 2.3.9 A set is said to be a *closed set* if it contains all of its limit points. The complement of a closed set is said to be an *open set.* (Recall that the complement of A is $\{x \in R : x \notin A\}$.)

Theorem 2.3.8 *The interval $[a, b]$ is a closed and bounded set. Its complement $(-\infty, a) \cup (b, \infty)$ is an open set.*

Proof. Let $p \in (-\infty, a) \cup (b, \infty)$. Then $-\infty < p < a$ or $b < p < \infty$. The intervals $\left(p - \frac{1}{2}, \frac{1}{2}(a+p)\right)$ or $\left(\frac{1}{2}(b+p), p + \frac{1}{2}\right)$ contain no limit point of $[a, b]$. Thus $[a, b]$ must contain its limit points, because they are not in the complement.

Theorem 2.3.9 *If a non-empty set A has no upper bound, then there exists a sequence $\{p_n\}_{n=1}^{\infty}$, of distinct points of A, that diverges to ∞. Furthermore, every subsequence of $\{p_n\}_{n=1}^{\infty}$ diverges to ∞*

Proof. Since 1 is not an upper bound of A, there exists an element p_1 of A such that $1 < p_1$. Let $a_1 = \max\{2, p_1\}$. Choose a point, say p_2, of A such that $a_1 < p_2$. By repeating this process indefinitely, we get the sequence $\{p_n\}$ such that $p_n > n$ and $p_1 < p_2 < p_3 < \ldots$. Clearly, the sequence $\{p_n\}_{n=1}^{\infty}$ diverges to ∞. It is easy to see that every subsequence of $\{p_n\}_{n=1}^{\infty}$ also diverges to ∞.

Theorem 2.3.10 *If a non-empty set B has no lower bound, then there exists a sequence $\{q_n\}_{n=1}^{\infty}$, of distinct points of B, that diverges to $-\infty$. Furthermore, every subsequence of $\{q_n\}_{n=1}^{\infty}$ diverges to $-\infty$.*

Proof. Let $A = \{-x : x \in B\}$. Then A has no upper bound. By Theorem 2.3.9, there exists a sequence $\{p_n\}_{n=1}^{\infty}$, of distinct points of A, that diverges to ∞. Let $q_n = -p_n$. Then $\{q_n\}_{n=1}^{\infty}$ is a sequence that meets the requirements of the Theorem 2.3.10. Also, every subsequence of $\{q_n\}_{n=1}^{\infty}$ diverges to $-\infty$.

Theorem 2.3.11 *Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points of a closed set S that converges to a point p of S. If f is a function that is continuous on S, then the sequence $\{f(p_n)\}_{n=1}^{\infty}$ converges to $f(p)$. That is, continuous functions preserve convergence of sequences on closed sets.*

Proof. Let $\epsilon > 0$ be given. Since f is continuous at p, there exists a $\delta > 0$ such that

$$|f(x) - f(p)| < \epsilon \quad \text{whenever} \quad |x - p| < \delta, \quad \text{and } x \in S.$$

The open interval $(p - \delta, p + \delta)$ contains the limit point p of S. The sequence $\{p_n\}_{n=1}^{\infty}$ converges to p. There exists some natural numbers N such that for all $n \geq N$,

$$p - \delta < p_n < p + \delta.$$

Then

$$|f(p_n) - f(p)| < \epsilon \quad \text{whenever } n \geq N.$$

By definition, $\{f(p_n)\}_{n=1}^{\infty}$ converges to $f(p)$. We write this statement in the following notation:

$$\lim_{n\to\infty} f(p_n) = f\left(\lim_{n\to\infty} p_n\right).$$

That is, continuous functions allow the interchange of taking the limit and applying the function. This completes the proof of the theorem.

Corollary 1 *If S is a closed and bounded interval $[a, b]$, then Theorem 2.3.11 is valid for $[a, b]$.*

Theorem 2.3.12 *Let a function f be defined and continuous on a closed and bounded set S. Let $R_f = \{f(x) : x \in S\}$. Then R_f is bounded.*

Proof. Suppose that R_f has no upper bound. Then there exists a sequence $\{f(x_n)\}_{n=1}^{\infty}$, of distinct points of R_f, that diverges to ∞. The set $A = \{x_1, x_2, \dots\}$ is an infinite subset of S. By Theorem 2.3.7, the set A has some limit point, say p. Since S is closed, $p \in S$. There exists a sequence

$\{p_n\}_{n=1}^{\infty}$, of distinct points of A that converges to p. By the continuity of f, $\{f(p_n)\}_{n=1}^{\infty}$ converges to $f(p)$. Without loss of generality, we may assume that $\{f(p_n)\}_{n=1}^{\infty}$ is a subsequence of $\{f(x_n)\}_{n=1}^{\infty}$. Hence $\{f(p_n)\}_{n=1}^{\infty}$ diverges to ∞, and $f(p) = \infty$. This is a contradiction, because $f(p)$ is a real number. This completes the proof of the theorem.

Theorem 2.3.13 *Let a function f be defined and continuous on a closed and bounded set S. Let $R_f = \{f(x) : x \in S\}$. Then R_f is a closed set.*

Proof. Let q be a limit point of R_f. Then there exists a sequence $\{f(x_n)\}_{n=1}^{\infty}$, of distinct points of R_f, that converges to q. As in Theorem 2.3.12, the set $A = \{x_1, x_2, \dots\}$ has a limit point $p, p \in S$, and there exists a subsequence $\{p_n\}_{n=1}^{\infty}$, of $\{x_n\}_{n=1}^{\infty}$ that converges to p. Since f is defined and continuous on S,

$$q = \lim_{n\to\infty} f(p_n) = f\left(\lim_{n\to\infty} p_n\right) = f(p).$$

Therefore, $q \in R_f$ and R_f is a closed set. This completes the proof of the theorem.

Theorem 2.3.14 *Let a function f be defined and continuous on a closed and bounded set S. Then there exist two numbers c_1 and c_2 in S such that for all $x \in S$,*

$$f(c_1) \le f(x) \le f(c_2).$$

Proof. By Theorems 2.3.12 and 2.3.13, the range, R_f, of f is a closed and bounded set. Let

$$m = \text{glb}(R_f) \text{ and } M = \text{lub}(R_f).$$

Since R_f is a closed set, m and M are in R_f. Hence, there exist two numbers, say c_1 and c_2, in S such that

$$m = f(c_1) \text{ and } M = f(c_2).$$

This completes the proof of the theorem.

Definition 2.3.10 A set S of real numbers is said to be compact, if and only if S is closed and bounded.

Theorem 2.3.15 *A continuous function maps compact subsets of its domain onto compact subsets of its range.*

Proof. Theorems 2.3.13 and 2.3.14 together prove Theorem 2.1.15.

Definition 2.3.11 Suppose that a function f is defined and continuous on a compact set S. A number m is said to be an *absolute minimum* of f on S if $m \leq f(x)$ for all $x \in S$ and $m = f(c)$ for some c in S.

A number M is said to be an *absolute maximum* of f on S if $M \geq f(x)$ for all $x \in S$ and $M = f(d)$ for some d in S.

Theorem 2.3.16 *Suppose that a function f is continuous on a compact set S. Then there exist two points c_1 and c_2 in S such that $f(c_1)$ is the absolute minimum and $f(c_2)$ is the absolute maximum of f on S.*

Proof. Theorem 2.3.14 proves Theorem 2.3.16.

Exercises 2.3

1. Find lub(A), glb(A) and determine all of the limit points of A.

   (a) $A = \{x : 1 \leq x^2 \leq 2\}$
   (b) $A = \{x : x\sin(1/x),\ x > 0\}$
   (c) $A = \{x^{2/3} : -8 < x < 8\}$
   (d) $A = \{x : 2 < x^3 < 5\}$
   (e) $A = \{x : x \text{ is a rational number and } 2 < x^3 < 5\}$

2. Determine whether or not the following sequences converge. Find the limit of the convergent sequences.

   (a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$
   (b) $\left\{\dfrac{n}{n^2}\right\}_{n=1}^{\infty}$
   (c) $\left\{(-1)^n\,\dfrac{n}{3n+1}\right\}_{n=1}^{\infty}$
   (d) $\left\{\dfrac{n^2}{n+1}\right\}_{n=1}^{\infty}$
   (e) $\{1 + (-1)^n\}_{n=1}^{\infty}$

3. Show that the Dedekind-Cut Property is equivalent to the completeness property.

4. Show that a convergent sequence cannot have more than one limit point.

5. Show that the following principle of mathematical induction is valid: If $1 \in S$, and $k+1 \in S$ whenever $k \in S$, then S contains the set of all natural numbers. (Hint: Let $A = \{n : n \notin S\}$. A is bounded from below by 2. Let $m = \text{glb}(A)$. Then $k = m - 1 \in S$ but $k + 1 = m \notin S$. This is a contradiction.)

6. Prove that every rational number is a limit point of the set of all rational numbers.

7. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

   (i) $\{a_n\}_{n=1}^{\infty}$ is said to be *increasing* if $a_n < a_{n+1}$, for all n.

   (ii) $\{a_n\}_{n=1}^{\infty}$ is said to be *non-decreasing* if $a_n \leq a_{n+1}$ for all n.

   (iii) $\{a_n\}_{n=1}^{\infty}$ is said to be *non-increasing* if $a_n \geq a_{n+1}$ for all n.

   (iv) $\{a_n\}_{n=1}^{\infty}$ is said to be *decreasing* if $a_n > a_{n+1}$ for all n.

   (v) $\{a_n\}_{n=1}^{\infty}$ is said to be *monotone* if it is increasing, non-decreasing, non-increasing or decreasing.

   (a) Determine which sequences in Exercise 2 are monotone.

   (b) Show that every bounded monotone sequence converges to some point.

   (c) A sequence $\{b_m\}_{m=1}^{\infty}$ is said to be a *subsequence* of the $\{a_n\}_{n=1}^{\infty}$ if and only if every b_m is equal to some a_n, and if

   $$b_{m_1} = a_{n_1} \quad \text{and} \quad b_{m_2} = a_{n_2} \quad \text{and} \quad n_1 < n_2, \text{ then } m_1 < m_2.$$

   That is, a subsequence preserves the order of the parent sequence. Show that if $\{a_n\}_{n=1}^{\infty}$ converges to p, then every subsequence of $\{a_n\}_{n=1}^{\infty}$ also converges to p

   (d) Show that a divergent sequence may contain one or more convergent sequences.

(e) In problems 2(c) and 2(e), find two convergent subsequences of each. Do the parent sequences also converge?

8. (Cauchy Criterion) A sequence $\{a_n\}_{n=1}^{\infty}$ is said to satisfy a *Cauchy Criterion*, or be a *Cauchy sequence*, if and only if for every $\epsilon > 0$, there exists some natural number N such that $(a_n - a_m) < \epsilon$ whenever $n \geq N$ and $m \geq N$. Show that a sequence $\{a_n\}_{n=1}^{\infty}$ converges if and only if it is a Cauchy sequence. (Hint: (i) If $\{a_n\}$ converges to p, then for every $\epsilon > 0$ there exists some N such that if $n \geq N$, then $|a_n - p| < \epsilon/2$. If $m \geq N$ and $n \geq N$, then

$$\begin{aligned} |a_n - a_m| &= |(a_n - p) + (p - a_m)| \\ &\leq |a_n - p| + |a_m - p| \qquad \text{(why?)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So, if $\{a_n\}$ converges, then it is Cauchy.

(ii) Suppose $\{a_n\}$ is Cauchy. Let $\epsilon > 0$. Then there exists $N > 0$ such that

$$|a_n - a_m| < \epsilon \quad \text{whenever} \quad n \geq N \text{ and } m \geq N.$$

In particular,

$$|a_n - a_N| < \epsilon \quad \text{whenever} \quad n \geq N.$$

Argue that the sequence $\{a_n\}$ is bounded. Unless an element is repeated infinitely many times, the set consisting of elements of the sequence has a limit point. Either way, it has a convergent subsequence that converges, say to p. Then show that the Cauchy Criterion forces the parent sequence $\{a_n\}$ to converge to p also.)

9. Show that the set of all rational numbers is countable. (Hint: First show that the positive rationals are countable. List them in reduced form without repeating according to denominators, as follows:

$$\begin{aligned} &\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \ldots . \\ &\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{2}{2}, \ldots . \\ &\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \frac{8}{3}, \frac{10}{3}, \ldots . \end{aligned}$$

Count them as shown, one-by-one. That is, list them as follows:

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, 2, 3, \frac{5}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}.$$

Next, insert the negative rational right after its absolute value, as follows:

$$\left\{0, 1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \cdots\right\}.$$

Now assign the even natural numbers to the positive rationals and the odd natural numbers to the remaining rationals.)

10. A non-empty set S has the property that if $x \in S$, then there is some open interval (a, b) such that $x \in (a, b) \subset S$. Show that the complement of S is closed and hence S is open.

11. Consider the sequence $\left\{a_n = \frac{\pi}{2} + \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$. Determine the convergent or divergent properties of the following sequences:

    (a) $\{\sin(a_n)\}_{n=1}^{\infty}$
    (b) $\{\cos(a_n)\}_{n=1}^{\infty}$
    (c) $\{\tan(a_n)\}_{n=1}^{\infty}$
    (d) $\{\cot(a_n)\}_{n=1}^{\infty}$
    (e) $\{\sec(a_n)\}_{n=1}^{\infty}$
    (f) $\{\csc(a_n)\}_{n=1}^{\infty}$

12. Let

    (a) $f(x) = x^2, -2 \le x \le 2$
    (b) $g(x) = x^3, -2 \le x \le 2$
    (c) $h(x) = \sqrt{x}, 0 \le x \le 4$
    (d) $p(x) = x^{1/3}, -8 \le x \le 8$

    Find the absolute maximum and absolute minimum of each of the functions f, g, h, and p. Determine the points at which the absolute maximum and absolute minimum are reached.

13. A function f is said to have a *fixed point* p if $f(p) = p$. Determine all of the fixed points of the functions f, g, h, and p in Exercise 12.

14. Determine the range of each of the functions in Exercise 12, and show that it is a closed and bounded set.

2.4 Properties of Continuous Functions

We recall that if two functions f and g are defined and continuous on a common domain D, then $f + g,\ f - g,\ af + bg,\ g \cdot f$ are all continuous on D, for all real numbers a and b. Also, the quotient f/g is continuous for all x in D where $g(x) \neq 0$. In section 2.3 we proved the following:

(i) Continuous functions preserve convergence of sequences.

(ii) Continuous functions map compact sets onto compact sets.

(iii) If a function f is continuous on a closed and bounded interval $[a, b]$, then $\{f(x) : x \in [a, b]\} \subseteq [m, M]$, where m and M are absolute minimum and absolute maximum of f, on $[a, b]$, respectively.

Theorem 2.4.1 *Suppose that a function f is defined and continuous on some open interval (a, b) and $a < c < b$.*

(i) If $f(c) > 0$, then there exists some $\delta > 0$ such that $f(x) > 0$ whenever $c - \delta < x < c + \delta$.

(ii) If $f(c) < 0$, then there exists some $\delta > 0$ such that $f(x) < 0$ whenever $c - \delta < x < c + \delta$.

Proof. Let $\epsilon = \dfrac{1}{2}\ |f(c)|$. For both cases (i) and (ii), $\epsilon > 0$. Since f is continuous at c and $\epsilon > 0$, there exists some $\delta > 0$ such that $a < (c - \delta) < c < (c + \delta) < b$ and

$$|f(x) - f(c)| < \epsilon \quad \text{whenever} \quad |x - c| < \delta.$$

We observe that

$$\begin{aligned}|f(x)-f(c)|<\epsilon \leftrightarrow |f(x)-f(c)| &< \frac{1}{2}\,|f(c)| \\ &\leftrightarrow -\frac{1}{2}\,|f(c)| < f(x)-f(c) < \frac{1}{2}\,|f(c)| \\ &\leftrightarrow f(c)-\frac{1}{2}\,|f(c)| < f(x) < f(c)+\frac{1}{2}\,|f(c)|.\end{aligned}$$

We note also that the numbers $f(c)-\frac{1}{2}\,|f(c)|$ and $f(c)+\frac{1}{2}\,|f(c)|$ have the same sign as $f(c)$. Therefore, for all x such that $|x-c|<\delta$, we have $f(x)>0$ in part (i) and $f(x)<0$ in part (ii) as required. This completes the proof.

Theorem 2.4.2 *Suppose that a function f is defined and continuous on some closed and bounded interval $[a,b]$ such that either*

(i) $f(a)<0<f(b)$ *or* *(ii)* $f(b)<0<f(a)$.

Then there exists some c such that $a<c<b$ and $f(c)=0$.

Proof. *Part (i)* Let $A\ \{x : x\in[a,b] \text{ and } f(x)<0\}$. Then A is nonempty because it contains a. Since A is a subset of $[a,b]$, A is bounded. Let $c_1=\text{lub}(A)$. We claim that $f(c_1)=0$. Suppose $f(c_1)\neq 0$. Then $f(c_1)>0$ or $f(c_1)<0$. By Theorem 2.4.1, there exists $\delta>0$ such that $f(x)$ has the same sign as $f(c_1)$ for all x such that $c_1-\delta<x<c_1+\delta$.

If $f(c_1)<0$, then $f(x)<0$ for all x such that $c_1<x<c_1+\delta$ and hence $c_1\neq \text{lub}(A)$. If $f(c_1)>0$, then $f(x)>0$ for all x such that $c_1-\delta<x<c_1$ and hence $c_1\neq\text{lub}(A)$. This contradiction proves that $f(c_1)=0$.

Part (ii) is proved by a similar argument.

Example 2.4.1 Show that Theorem 2.4.2 guarantees the validity of the following method of bisection for finding zeros of a continuous function f:

Bisection Method: We wish to solve $f(x)=0$ for x.

Step 1. Locate two points such that $f(a)f(b)<0$.

Step 2. Determine the sign of $f\left(\frac{1}{2}(a+b)\right)$.

(i) If $f\left(\frac{1}{2}(a+b)\right) = 0$, stop the procedure; $\frac{1}{2}(a+b)$ is a zero of f.

(ii) If $f\left(\frac{1}{2}(a+b)\right) \cdot f(a) < 0$, then let $a_1 = a, b_1 = \frac{1}{2}(a+b)$.

(iii) If $f\left(\frac{1}{2}(a+b)\right) \cdot f(b) < 0$, then let $a_1 = \frac{1}{2}(a+b),\ b_1 = b$.

Then $f(a_1) \cdot f(b_1) < 0$, and $|b_1 - a_1| = \frac{1}{2}(b-a)$.

Step 3. Repeat Step 2 and continue the loop between Step 2 and Step 3 until

$$|b_n - a_n|/2^n < \text{ Tolerance Error.}$$

Then stop.

This method is slow but it approximates the number c guaranteed by Theorem 2.4.2. This method is used to get close enough to the zero. The switchover to the faster Newton's Method that will be discussed in the next section.

Theorem 2.4.3 (Intermediate Value Theorem). *Suppose that a function is defined and continuous on a closed and bounded interval* $[a, b]$. *Suppose further that there exists some real number* k *such that either (i)* $f(a) < k < f(b)$ *or (ii)* $f(b) < k < f(a)$. *Then there exists some* c *such that* $a < c < b$ *and* $f(c) = k$.

Proof. Let $g(x) = f(x) - k$. Then g is continuous on $[a, b]$ and either (i) $g(a) < 0 < g(b)$ or (ii) $g(b) < 0 < g(a)$. By Theorem 2.4.2, there exists some c such that $a < c < b$ and $g(c) = 0$. Then

$$0 = g(c) = f(c) - k$$

and

$$f(c) = k$$

as required. This completes the proof.

Theorem 2.4.4 *Suppose that a function f is defined and continuous on a closed and bounded interval $[a, b]$. Then there exist real numbers m and M such that*

$$[m, M] = \{f(x) : a \leq x \leq b\}.$$

That is, a continuous function f maps a closed and bounded interval $[a, b]$ onto a closed and bounded interval $[m, M]$.

Proof. By Theorem 2.3.14, there exist two numbers c_1 and c_2 in $[a, b]$ such that for all $x \in [a, b]$,

$$m = f(c_1) \leq f(x) \leq f(c_2) = M.$$

By the Intermediate Value Theorem (2.4.3), every real value between m and M is in the range of f contained in the interval with end points c_1 and c_2. Therefore,

$$[m, M] = \{f(x) : a \leq x \leq b\}.$$

Recall that m = absolute minimum and M = absolute maximum of f on $[a, b]$. This completes the proof of the theorem.

Theorem 2.4.5 *Suppose that a function f is continuous on an interval $[a, b]$ and f has an inverse on $[a, b]$. Then f is either strictly increasing on $[a, b]$ or strictly decreasing on $[a, b]$.*

Proof. Since f has in inverse on $[a, b]$, f is a one-to-one function on $[a, b]$. So, $f(a) \neq f(b)$. Suppose that $f(a) < f(b)$. Let

$$A = \{x : f \text{ is strictly increasing on } [a, x] \text{ and } a \leq x \leq b\}.$$

Let c be the least upper bound of A. If $c = b$, then f is strictly increasing on $[a, b]$ and the proof is complete. If $c = a$, then there exists some d such that $a < d < b$ and $f(d) < f(a) < f(b)$. By the intermediate value theorem there must exist some x such that $d < x < b$ and $f(x) = f(a)$. This contradicts the fact that f is one-to-one. Then $a < c < b$ and there exists some d such that $c < d < b$ and $f(a) < f(d) < f(c)$. By the intermediate value theorem there exists some x such that $a < x < c$ and $f(x) = f(c)$ and f is not one-to-one. It follows that c must equal b and f is strictly increasing on $[a, b]$. Similarly, if $f(a) > f(b)$, f will be strictly decreasing on $[a, b]$. This completes the proof of the theorem.

Theorem 2.4.6 *Suppose that a function f is continuous on $[a, b]$ and f is one-to-one on $[a, b]$. Then the inverse of f exists and is continuous on $J = \{f(x) : a \le x \le b\}$.*

Proof. By Theorem 2.4.4, $J = [m, M]$ where m and M are the absolute minimum and the absolute maximum of f on $[a, b]$. Also, there exist numbers c_1 and c_2 on $[a, b]$ such that $f(c_1) = m$ and $f(c_2) = M$. Since f is either strictly increasing or strictly decreasing on $[a, b]$, either $a = c_1$ and $b = c_2$ or $a = c_2$ and $b = c_1$. Consider the case where f is strictly increasing and $a = c_1, b = c_2$. Let $m < d < M$ and $d = f(c)$. Then $a < c < b$. We show that f^{-1} is continuous at d. Let $\epsilon > 0$ be such that $a < c - \epsilon < c < c + \epsilon 2b$. Let $d_1 = f(c - \epsilon), d_2 = f(c + \epsilon)$. Since f is strictly increasing, $d_1 < d < d_2$. Let $\delta = \min(d - d_1, d_2 - d)$. It follows that if $0 < |y - d| < \delta$, then $|f^{-1}(y) - f^{-1}(d)| < \epsilon$ and f^{-1} is continuous at d. Similarly, we can prove the one-sided continuity of f^{-1} at m and M. A similar argument will prove the continuity of f^{-1} if f is strictly decreasing on $[a, b]$.

Theorem 2.4.7 *Suppose that a function f is continuous on an interval I and f is one-to-one on I. Then the inverse of f exists and is continuous on I.*

Proof. Let $J = \{f(x) : x \text{ is in } I\}$. By the intermediate value theorem J is also an interval. Let d be an interior point of J. Then there exists a closed interval $[m, M]$ contained in I and $m < d < M$. Let $c_1 = f^{-1}(m)$, $c_2 = f^{-1}(b)$, $a = \min\{c_1, c_2\}$ Since the theorem is valid on $[a, b]$, f^{-1} is continuous at d. The end points can be treated in a similar way. This completes the proof of the theorem. (See the proof of Theorem 2.4.6).

Theorem 2.4.8 (Fixed Point Theorem). *Let f satisfy the conditions of Theorem 2.4.4. Suppose further that $a \le m \le M \le b$, where m and M are the absolute minimum and absolute maximum, respectively, of f on $[a, b]$. Then there exists some $p \in [a, b]$ such that $f(p) = p$. That is, f has a fixed point p on $[a, b]$.*

Proof. If $f(a) = a$, then a is a fixed point. If $f(b) = b$, then b is a fixed point. Suppose that neither a nor b is a fixed point of f. Then we define

$$g(x) = f(x) - x$$

for all $x \in [a, b]$.

We observe that $g(b) < 0 < g(a)$. By the Intermediate Value Theorem (2.4.3) there exists some p such that $a < p < b$ and $g(p) = 0$. Then

$$0 = g(p) = f(p) - p$$

and hence,

$$f(p) = p$$

and p is a fixed point of f on $[a, b]$. This completes the proof.

Remark 9 The Fixed Point Theorem (2.4.5) is the basis of the fixed point iteration methods that are used to locate zeros of continuous functions. We illustrate this concept by using Newton's Method as an example.

Example 2.4.2 Consider $f(x) = x^3 + 4x - 10$.

Since $f(1) = -5$ and $f(2) = 6$, by the Intermediate Value Theorem (2.4.3) there is some c such that $1 < c < 2$ and $f(c) = 0$. We construct a function g whose fixed points agree with the zeros of f. In Newton's Method we used the following general formula:

$$g(x) = x - \frac{f(x)}{\text{slope}(f(x), x)}.$$

Note that if $f(x) = 0$, then $g(x) = x$, provided slope $(f(x), x) \neq 0$. We first compute

$$\begin{aligned}
\text{Slope}(f(x), x) &= \lim_{h \to 0} \frac{1}{h} \left[f(x+h) - f(x)\right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\{(x+h)^3 + 4(x+h) - 10\} - \{x^3 + 4x - 10\}\right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[3x^2h + 3xh^2 + h^3 + 4h\right] \\
&= \lim_{h \to 0} \left[3x^2 + 3xh + h^2 + 4\right] \\
&= 3x^2 + 4.
\end{aligned}$$

We note that $3x^2 + 4$ is never zero. So, Newton's Method is defined.

The fixed point iteration is defined by the equation

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{\text{slope}(f(x), x_n)}$$

or

$$\boxed{x_{n+1} = x_n - \frac{x_n^3 + 4x_n - 10}{3x_n^2 + 4}.}$$

Geometrically, we draw a tangent line at the point $(x_n, f(x_n))$ and label the x-coordinate of its point of intersection with the x-axis as x_{n+1}.

graph

Tangent line: $y - f(x_n) = m(x - x_n)$

$$0 - f(x_n) = m(x_{n+1} - x_n)$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{m}} \quad ,$$

where $m =$ slope $(f(x), x_n) = 3x_n^2 + 4$.

To begin the iteration we required a guess x_0. This guess is generally obtained by using a few steps of the Bisection Method described in Example 36. Let $x_0 = 1.5$. Next, we need a stopping rule. Let us say that we will stop when a few digits of x_n do not change anymore. Let us stop when

$$|x_{n+1} - x_n| < 10^{-4}.$$

We will leave the computation of $x_1, x_2, x_3, \ldots$ as an exercise.

Remark 10 Newton's Method is fast and quite robust as long as the initial guess is chosen close enough to the intended zeros.

Example 2.4.3 Consider the same equation ($x^3 + 4x - 10 = 0$) as in the preceding example.

We solve for x in some way, such as,

$$x = \left(\frac{10}{4 + x}\right)^{1/2} = g(x).$$

In this case the new equation is good enough for positive roots. We then define

$$x_{n+1} = g(x_n), x_0 = 1.5$$

and stop when

$$|x_{n+1} - x_n| < 10^{-4}.$$

We leave the computations of $x_1, x_2, x_3 \ldots$ as an exercise. Try to compare the number of iterations needed to get the same accuracy as Newton's Method in the previous example.

Exercises 2.4

1. Perform the required iterations in the last two examples to approximate the roots of the equation $x^3 + 4x - 10 = 0$.

2. Let $f(x) = x - \cos x$. Then slope $(f(x), x) = 1 + \sin x > 0$ on $\left[0, \frac{\pi}{2}\right]$. Approximate the zeros of $f(x)$ on $\left[0, \frac{\pi}{2}\right]$ by Newton's Method:

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n}, x_0 = 0.8$$

and stop when

$$|x_{n+1} - x_n| < 10^{-4}.$$

3. Let $f(x) = x - 0.8 - 0.4 \sin x$ on $\left[0, \frac{\pi}{2}\right]$. then slope $(f(x), x) = 1 - 0.4 \cos x > 0$ on $\left[0, \frac{\pi}{2}\right]$. Approximate the zero of f using Newton's Iteration

$$x_{n+1} = x_n - \frac{x_n - 0.8 - 0.4\sin(x_n)}{1 - 0.4\cos(x_n)}, x_0 = 0.5$$

4. To avoid computing the slope function f, the Secant Method of iteration uses the slope of the line going through the previous two points

$(x_n, f(x_n))$ and $(x_{n+1}, f(x_{n+1}))$ to define x_{n+2} as follows: Given x_0 and x_1, we define

$$x_{n+2} = x_{n+1} - \frac{f(x_{n+1})}{\left(\frac{f(x_{n+1})-f(x_n)}{x_{n+1}-x_n}\right)}$$

$$\boxed{x_{n+2} = x_{n+1} - \frac{f(x_{n+1})(x_{n+1} - x_n)}{f(x_{n+1} - f(x_n)}}$$

This method is slower than Newton's Method, but faster than the Bisection. The big advantage is that we do not need to compute the slope function for f. The stopping rule can be the same as in Newton's Method. Use the secant Method for Exercises 2 and 3 with $x_0 = 0.5, x_1 = 0.7$ and $|x_{n+1} - x_n| < 10^{-4}$. Compare the number of iterations needed with Newton's Method.

5. Use the Bisection Method to compute the zero of $x^3+4x-10$ on $[1, 2]$ and compare the number of iterations needed for the stopping rule $|x_{n+1} - x_n| < 10^{-4}$.

6. A set S is said to be *connected* if S is not the union of two non-empty sets A and B such that A contains no limit point of B and B contains no limit point of A. Show that every closed and bounded interval $[a, b]$ is connected.

   (Hint: Assume that $[a, b]$ is not connected and $[a, b] = A \cup B$, $a \in A, B \neq \emptyset$ as described in the problem. Let $m = \text{lub}(A)$, $M = \text{glb}(B)$. Argue that $m \in A$ and $m \in B$. Then $\frac{1}{2}(m + M) \notin (A \cup B)$. The contradiction proves the result.

7. Show that the Intermediate Value Theorem (2.4.3) guarantees that continuous functions map connected sets onto connected sets. (Hint: Let S be connected and f be continuous on S. Let $R_f = \{f(x) : x \in S\}$. Suppose $R_f = A \cup B, A \neq \emptyset, B \neq \emptyset$, such that A contains no limit point of B and B contains no limit point of A. Let $U = \{x \in S : f(x) \in A\}$, $V = \{x \in S : f(x) \in B\}$. Then $S = U \cup V, U \neq \emptyset$ and $V \neq \emptyset$. Since S is connected, either U contains a limit point of V or V contains a limit point of U. Suppose $p \in V$ and p is a limit point of U. Then choose a

sequence $\{u_n\}$ that converges to $p, u_n \in U$. By continuity, $\{f(u_n)\}$ converges to $f(p)$. But $f(u_n) \in A$ and $f(p) \in B$. This is a contradiction.)

8. Find all of the fixed points of the following:

   (a) $f(x) = x^2, \quad -4 \leq x \leq 4$

   (b) $f(x) = x^3, \quad -2 \leq x \leq 2$

   (c) $f(x) = x^2 + 3x + 1$

   (d) $f(x) = x^3 - 3x, \quad -4 \leq x \leq 4$

   (e) $f(x) = \sin x$

9. Determine which of the following sets are
   (i) bounded, (ii) closed, (iii) connected.

   (a) $N = \{1, 2, 3, \ldots,\}$

   (b) $Q = \{x : x \text{ is rational number}\}$

   (c) $R = \{x : x \text{ is a real number}\}$

   (d) $B_1 = \{\sin x : -\pi \leq x \leq \pi\}$

   (e) $B_2 = \{\sin x : -\pi < x < \pi\}$

   (f) $B_3 = \left\{\sin x : \frac{-\pi}{2} < x < \frac{\pi}{2}\right\}$

   (g) $B_4 = \left\{\tan x : \frac{-\pi}{2} < x < \frac{\pi}{2}\right\}$

   (h) $C_1 = [(-1, 0) \cup (0, 1]$

   (i) $C_2 = \left\{f(x) : -\pi \leq x \leq \pi, f(x) = \frac{\sin x}{x}, x \neq 0; f(0) = 2\right\}$

   (j) $C_3 = \left\{g(x) : -\pi \leq x \leq \pi, g(x) = \frac{1 - \cos x}{x}, g(0) = 1\right\}$

10. Suppose f is continuous on the set of all real numbers. Let the open interval (c, d) be contained in the range of f. Let

$$A = \{x : c < f(x) < d\}.$$

Show that A is an open set.

(Hint: Let $p \in A$. Then $f(p) \in (c, d)$. Choose $\epsilon > 0$ such that $c < p - \epsilon < p + \epsilon < d$. Since f is continuous at p, there is $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ whenever $|x - p| < \delta$. This means that the open interval $(p - \delta, p + \delta)$ is contained in A. By definition, A is open. This proves that the inverse of a continuous function maps an open set onto an open set.)

2.5 Limits and Infinity

The convergence of a sequence $\{a_n\}_{n=1}^{\infty}$ depends on the limit of a_n as n tends to ∞.

Definition 2.5.1 Suppose that a function f is defined on an open interval (a, b) and $a < c < b$. Then we define the following limits:

(i) $\lim\limits_{x \to c^-} f(x) = +\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) > M$ whenever $c - \delta < x < c$.

(ii) $\lim\limits_{x \to c^+} f(x) = +\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) > M$ whenever $c < x < c + \delta$.

(iii) $\lim\limits_{x \to c} f(x) = +\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$.

(iv) $\lim\limits_{x \to c} f(x) = -\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) < -M$ whenever $0 < |x - c| < \delta$.

(v) $\lim\limits_{x \to c^+} f(x) = -\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) < -M$ whenever $c < x < c + \delta$.

(vi) $\lim_{x \to c^-} f(x) = -\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) < -M$ whenever $c - \delta < x < c$.

Definition 2.5.2 Suppose that a function f is defined for all real numbers.

(i) $\lim_{x \to +\infty} f(x) = L$

if and only if for every $\epsilon > 0$ there exists some $M > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > M$.

(ii) $\lim_{x \to -\infty} f(x) = L$

if and only if for every $\epsilon > 0$ there exists some $M > 0$ such that $|f(x) - L| < \epsilon$ whenever $x < -M$.

(iii) $\lim_{x \to +\infty} f(x) = \infty$

if and only if for every $M > 0$ there exists some $N > 0$ such that $f(x) > M$ whenever $x > N$.

(iv) $\lim_{x \to +\infty} f(x) = -\infty$

if and only if for every $M > 0$ there exists some $N > 0$ such that $f(x) < -M$ whenever $x > M$.

(v) $\lim_{x \to -\infty} f(x) = \infty$

if and only if for every $M > 0$ there exists some $N > 0$ such that $f(x) > M$ whenever $x < -N$.

(vi) $\lim_{x \to -\infty} f(x) = -\infty$

if and only if for every $M > 0$ there exists some $N > 0$ such that $f(x) < -M$ whenever $x < -N$.

Definition 2.5.3 The vertical line $x = c$ is called a *vertical asymptote* to the graph of f if and only if either

(i) $\lim_{x \to c} f(x) = \infty$ or $-\infty$; or

(ii) $\lim_{x \to c^-} f(x) = \infty$ or $-\infty$; or both.

Definition 2.5.4 The horizontal line $y = L$ is a *horizontal asymptote* to the graph of f if and only if

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L, \text{ or both.}$$

Example 2.5.1 Compute the following limits:

(i) $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}$

(ii) $\displaystyle\lim_{x\to\infty} \frac{\cos x}{x}$

(iii) $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{3x^3+10}$

(iv) $\displaystyle\lim_{x\to-\infty} \frac{x^3-2}{3x^3+2x-3}$

(v) $\displaystyle\lim_{x\to-\infty} \frac{3x^3+4x-7}{2x^2+5x+2}$

(vi) $\displaystyle\lim_{x\to-\infty} \frac{-x^4+3x-10}{2x^2+3x-5}$

(i) We observe that $-1 \le \sin x \le 1$ and hence

$$0 = \lim_{x\to\infty} \frac{-1}{x} \le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x} = 0.$$

Hence, $y = 0$ is the horizontal asymptote and

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

(ii) $-1 \le \cos x \le 1$ and, by a similar argument as in part (i),

$$\lim_{x\to\infty} \frac{\cos x}{x} = 0.$$

(iii) We divide the numerator and denominator by x^2 and then take the limit as follows:

$$\lim_{x\to\infty} \frac{x^2+1}{3x^3+10} = \lim_{x\to\infty} \frac{1+1/x^2}{3x+10/x^2} = 0.$$

(iv) We divide the numerator and denominator by x^3 and then take the limit as follows:

$$\lim_{x\to-\infty} \frac{x^3-2}{3x^3+2x-3} = \lim_{x\to-\infty} \frac{1-2/x^3}{3+2/x^2-3/x^3} = \frac{1}{3}.$$

(v) We divide the numerator and denominator by x^2 and then take the limit as follows:

$$\lim_{x\to-\infty} \frac{3x^3+4x-7}{2x^2+5x-2} = \lim_{x\to-\infty} \frac{3x+4/x-7/x^2}{2+5/x+2/x^2} = -\infty.$$

(vi) We divide the numerator and denominator by x^2 and then take the limit as follows:

$$\lim_{x\to-\infty} \frac{-x^4+3x-10}{2x^2+3x-5} = \lim_{x\to-\infty} \frac{-x^2+3/x-10/x^2}{2+3/x-5/x^2} = -\infty.$$

Example 2.5.2

(i) $\lim\limits_{n\to\infty} \dfrac{(-1)^n+1}{n} = 0$

(ii)
$$\begin{aligned}\lim_{n\to\infty} \left\{\frac{n^2}{n+3} - \frac{n^2}{n+4}\right\} &= \lim_{n\to\infty} \frac{n^3+4n^2-n^3-3n^2}{n^2+7n+12} \\ &= \lim_{n\to\infty} \frac{n^2}{n^2+7n+12} \\ &= \lim_{n\to\infty} \frac{1}{1+7/n+12/n^2} \\ &= 1\end{aligned}$$

(iii)
$$\begin{aligned}\lim_{n\to\infty} (\sqrt{n+4}-n) &= \lim_{n\to\infty} \frac{(\sqrt{n+4}-\sqrt{n})(\sqrt{n+4}+\sqrt{n})}{(\sqrt{n+4}+\sqrt{n})} \\ &= \lim_{n\to\infty} \frac{4}{(\sqrt{n+4}+\sqrt{n})} \\ &= 0\end{aligned}$$

(vi) $\lim_{n\to\infty} \left\{ \frac{n^2}{1+n^2} \sin\left(\frac{n\pi}{2}\right) \right\}$ does not exist because it oscillates:

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n = 2m \\ 1 & \text{if } n = 2m+1 \\ -1 & \text{if } n = 2m+3 \end{cases}$$

(v) $\lim_{n\to\infty} \frac{3^n}{4+3^n} = \lim_{h\to\infty} \frac{1}{4\cdot e^{-n}+1} = 1$

(vi) $\lim_{n\to\infty} \{\cos(n\pi)\} = \lim_{n\to\infty} (-1)^n$ does not exist.

Exercises 2.5 Evaluate the following limits:

1. $\lim_{x\to 2} \frac{x}{x^2-4}$
2. $\lim_{x\to 2^+} \frac{x}{x^2-4}$
3. $\lim_{x\to 1^-} \frac{x}{x^2-1}$
4. $\lim_{x\to \frac{\pi}{2}^-} \tan(x)$
5. $\lim_{x\to \frac{\pi}{2}^+} \sec x$
6. $\lim_{x\to 0^+} \cot x$
7. $\lim_{x\to 0^-} \csc x$
8. $\lim_{x\to\infty} \frac{3x^2-7x+5}{4x^2+5x-7}$
9. $\lim_{x\to -\infty} \frac{x^2+4}{4x^3+3x-5}$
10. $\lim_{x\to\infty} \frac{-x^4+2x-1}{x^2+3x+2}$
11. $\lim_{x\to\infty} \frac{\cos(n\pi)}{n^2}$
12. $\lim_{x\to\infty} \frac{1+(-1)^n}{n^3}$
13. $\lim_{x\to\infty} \frac{\sin(n)}{n}$
14. $\lim_{x\to\infty} \frac{1-\cos n}{n}$
15. $\lim_{x\to\infty} \frac{\cos\left(\frac{n\pi}{2}\right)}{n}$
16. $\lim_{x\to\infty} \tan\left(\frac{n\pi}{n}\right)$

Chapter 3

Differentiation

In Definition 2.2.2, we defined the slope function of a function f at c by

$$\begin{aligned} \text{slope}(f(x), c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}. \end{aligned}$$

The slope $(f(x), c)$ is called the *derivative* of f at c and is denoted $f'(c)$. Thus,

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}.$$

Link to another file.

3.1 The Derivative

Definition 3.1.1 Let f be defined on a closed interval $[a, b]$ and $a < x < b$. Then the derivative of f at x, denoted $f'(x)$, is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

whenever the limit exists. When $f'(x)$ exists, we say that f is differentiable at x. At the end points a and b, we define one-sided derivatives as follows:

(i) $f'(a^+) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}.$

We call $f'(a+)$ the right-hand derivative of f at a.

(ii) $f'(b^-) = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b} = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$.

We call $f'(b)$ the left-hand derivative of f at b.

Example 3.1.1 In Example 28 of Section 2.2, we proved that if $f(x) = \sin x$, then $f'(c) = \text{slope } (\sin x, c) = \cos c$. Thus, $f'(x) = \cos x$ if $f(x) = \sin x$.

Example 3.1.2 In Example 29 of Section 2.2, we proved that if $f(x) = \cos x$, then $f'(c) = -\sin c$. Thus, $f'(x) = -\sin x$ if $f(x) = \cos x$.

Example 3.1.3 In Example 30 of Section 2.2, we proved that if $f(x) = x^n$ for a natural number n, then $f'(c) = nc^{n-1}$. Thus $f'(x) = nx^{n-1}$, when $f(x) = x^n$, for any natural number n.

In order to find derivatives of functions obtained from the basic elementary functions using the operations of addition, subtraction, multiplication and division, we state and prove the following theorem.

Theorem 3.1.1 *If f is differentiable at c, then f is continuous at c. The converse is false.*

Proof. Suppose that f is differentiable at c. Then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

and $f'(c)$ is a real number. So,

$$\begin{aligned}
\lim_{x \to c} f(x) &= \lim_{x \to c} \left[\left(\frac{f(x) - f(c)}{x - c} \right) (x - c) + f(c) \right] \\
&= \lim_{x \to 0} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c) + f(c) \\
&= f'(c) \cdot 0 + f(c) \\
&= f(c).
\end{aligned}$$

Therefore, if f is differentiable at c, then f is continuous at c.

To prove that the converse is false we consider the function $f(x) = |x|$. This function is continuous at $x = 0$. But

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \left[\frac{|x+h| - |x|}{h}\right] \\ &= \lim_{h\to 0} \frac{(|x+h| - |x|)(|x+h| + |x|)}{h(|x+h| + |x|)} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h(|x+h| + |x|)} \\ &= \lim_{h\to 0} \frac{2x+h}{|x+h|\ |x|} \\ &= \frac{x}{|x|} \\ &= \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \\ \text{undefined for } x = 0. \end{cases} \end{aligned}$$

Thus, $|x|$ is continuous at 0 but not differentiable at 0. This completes the proof of Theorem 3.1.1.

Theorem 3.1.2 *Suppose that functions f and g are defined on some open interval (a, b) and $f'(x)$ and $g'(x)$ exist at each point x in (a, b). Then*

(i) $(f + g)'(x) = f'(x) + g'(x)$ (The Sum Rule)

(ii) $(f - g)'(x) = f'(x) - g'(x)$ (The Difference Rule)

(iii) $(kf)'(x) = kf'(x)$, *for each constant k.* (The Multiple Rule)

(iv) $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ (The Product Rule)

(v) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$, *if $g(x) \neq 0$.* (The Quotient Rule)

Proof.

Part (i) $(f+g)'(x) = \lim_{h\to 0} \dfrac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x).$$

Part (ii) $(f-g)'(x) = \lim_{h\to 0} \dfrac{[f(x+h)-g(x+h)]-[f(x)-g(x)]}{h}$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$
$$= f'(x) - g'(x).$$

Part (iii) $(kf)'(x) = \lim_{h\to 0} \dfrac{kf(x+h)-kf(x)}{h}$

$$= k \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= kf'(x).$$

Part (iv)

$$\begin{aligned}(f\cdot g)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0} \frac{1}{h}\,[(f(x+h)-f(x))g(x+h)+f(x)(g(x+h)-g(x))]\\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0}\, g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\\ &= f'(x)g(x) + f(x)g'(x).\end{aligned}$$

Part (v)
$$\left(\frac{f}{g}\right)'(x) = \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)\cdot g(x) - g(x+h)f(x)}{g(x+h)g(x)}\right]$$
$$= \frac{1}{(g(x))^2} \lim_{h\to 0} \left[\frac{(f(x+h)-f(x))}{h}\, g(x) - f(x)\, \frac{(g(x+h)-g(x))}{h}\right]$$
$$= \frac{1}{(g(x))^2}\cdot [f'(x)g(x) - f(x)g'(x)]$$
$$= \frac{g(x)f'(x) - g(x)g'(x)}{(g(x))^2}\text{ , if } g(x)\neq 0.$$

To emphasize the fact that the derivatives are taken with respect to the independent variable x, we use the following notation, as is customary:

$$f'(x) = \frac{d}{dx}\,(f(x)).$$

Based on Theorem 3.1.2 and the definition of the derivative, we get the following theorem.

Theorem 3.1.3

(i) $\dfrac{d(k)}{dx} = 0$, *where k is a real constant.*

(ii) $\dfrac{d}{dx}\,(x^n) = nx^{n-1}$, *for each real number x and natural number n.*

(iii) $\dfrac{d}{dx}\,(\sin x) = \cos x$, *for all real numbers (radian measure) x.*

(iv) $\dfrac{d}{dx}\,(\cos x) = -\sin x$, *for all real numbers (radian measure) x.*

(v) $\dfrac{d}{dx}\,(\tan x) = \sec^2 x$, *for all real numbers $x \neq (2n+1)\,\dfrac{\pi}{2}$, n = integer.*

(vi) $\dfrac{d}{dx}(\cot x) = -\csc^2 x$, *for all real numbers* $x \neq n\pi$, n = *integer.*

(vii) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$, *for all real numbers* $x \neq (2n+1)\dfrac{\pi}{2}$, n = *integer.*

(viii) $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$, *for all real numbers* $x \neq n\pi$, n = *integer.*

Proof.

Part(i)

$$\frac{d(k)}{dx}(k) = \lim_{h\to 0} \frac{k-k}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Part (ii) For each natural n, we get

$$\begin{aligned}\frac{d}{dx}(x^n) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \qquad \text{(Binomial Expansion)}\\ &= \lim_{h\to 0} \frac{1}{h}\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2!}\, x^{n-2}h^2 + \cdots + h^n - x^n\right]\\ &= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2!}\, x^{n-2}h + \cdots + h^{n-1}\right]\\ &= nx^{n-1}.\end{aligned}$$

Part (iii) By definition, we get

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}\\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}\\ &= \lim_{h\to 0}\left[\cos x\, \frac{\sin h}{h} - \sin x\, \left(\frac{1-\cos h}{h}\right)\right]\\ &= \cos x \cdot 1 - \sin x \cdot 0\\ &= \cos x\end{aligned}$$

since

$$\lim_{h\to 0} \frac{\sin h}{h} = 1,\ \lim_{h\to 0} \frac{1-\cos h}{h} = 0. \quad \text{(Why?)}$$

Part (iv) By definition, we get

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\cos x \cos h - \sin x \sin h - \cos x\right] \\
&= \lim_{h\to 0} \left[-\sin x \cdot \frac{\sin h}{h} - \cos x \left(\frac{1-\cos h}{h}\right)\right] \\
&= -\sin x \cdot 1 - \cos x \cdot 0 \qquad \text{(Why?)} \\
&= -\sin x.
\end{aligned}$$

Part (v) Using the quotient rule and parts (iii) and (iv), we get

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x(\sin x)' - \sin x(\cos x)'}{(\cos x)^2} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \qquad \text{(Why?)} \\
&= \sec^2 x, \quad x \neq (2n+1)\frac{\pi}{2}, n = \text{ integer.}
\end{aligned}$$

Part (vi) Using the quotient rule and Parts (iii) and (iv), we get

$$\begin{aligned}
\frac{d}{dx}\,(\cot x) &= \frac{d}{dx}\,\left(\frac{\cos x}{\sin x}\right) \\
&= \frac{(\sin x)(\cos x)' - (\cos x)(\sin x)'}{(\sin x)^2} \\
&= \frac{-\sin^2 x - \cos^2 x}{(\sin x)^2} \qquad \text{(Why?)} \\
&= \frac{-1}{(\sin x)^2} \qquad \text{(why?)} \\
&= -\csc^2 x, \quad x \neq n\pi,\ n = \text{ integer.}
\end{aligned}$$

Part (vii) Using the quotient rule and Parts (iii) and (iv), we get

$$\begin{aligned}
\frac{d}{dx}\,(\sec x) &= \frac{d}{dx}\,\left(\frac{1}{\cos x}\right) \\
&= \frac{(\cos x)\cdot 0 - 1\cdot(\cos x)'}{(\cos x)^2} \\
&= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} \qquad \text{(Why?)} \\
&= \sec x \tan x, \quad x \neq (2n+1)\frac{\pi}{2},\ n = \text{ integer.}
\end{aligned}$$

Part (viii) Using the quotient rule and Parts (iii) and (iv), we get

$$\begin{aligned}
\frac{d}{dx}\,(\csc x) &= \frac{d}{dx}\,\left(\frac{1}{\sin x}\right) \\
&= \frac{\sin x\cdot 0 - 1\cdot(\sin x)'}{(\sin x)^2} \\
&= \frac{1}{\sin x}\cdot\frac{-\cos x}{\sin x} \qquad \text{(Why?)} \\
&= -\csc x \cot x,\ x \neq n\pi, n = \text{ integer.}
\end{aligned}$$

This concludes the proof of Theorem 3.1.3.

Example 3.1.4 Compute the following derivatives:

(i) $\dfrac{d}{dx}\,(4x^3 - 3x^2 + 2x + 10)$ (ii) $\dfrac{d}{dx}\,(4\sin x - 3\cos x)$

(iii) $\dfrac{d}{dx}\,(x\sin x + x^2\cos x)$ (iv) $\dfrac{d}{dx}\left(\dfrac{x^3+1}{x^2+4}\right)$

Part (i) Using the sum, difference and constant multiple rules, we get

$$\begin{aligned}\frac{d}{dx}\,(4x^3 - 3x^2 + 2x + 10) &= 4\,\frac{d}{dx}(x^3) - 3\,\frac{d}{dx}\,(x^2) + 2\,\frac{d}{dx} + 0\\ &= 12x^2 - 6x + 2.\end{aligned}$$

Part (ii) $$\begin{aligned}\frac{d}{dx}\,(4\sin x - 3\cos x) &= 4\,\frac{d}{dx}\,(\sin x) - 3\,\frac{d}{dx}\,(\cos x)\\ &= 4\cos x - 3(-\sin x)\\ &= 4\cos x + 3\sin x.\end{aligned}$$

Part (iii) Using the sum and product rules, we get

$$\begin{aligned}\frac{d}{dx}\,(x\sin x + x^2\cos x) &= \frac{d}{dx}\,(x\sin x) + \frac{d}{dx}\,(x^2\cos x) \quad \text{(Sum Rule)}\\ &= \left[\frac{d}{dx}\,\sin x + x\,\frac{d}{dx}\,(\sin x)\right]\\ &\quad + \left[\frac{d}{dx}\,(x^2)\cos x + x^2\,\frac{d}{dx}\,(\cos x)\right]\\ &= 1\cdot\sin x + x\cos x + 2x\cos x + x^2(-\sin x)\\ &= \sin x + 3x\cos x - x^2\sin x.\end{aligned}$$

Part (iv). Using the sum and quotient rules, we get

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^3+1}{x^2+4}\right) &= \frac{(x^2+4)\frac{d}{dx}(x^3+1) - (x^3+1)\frac{d}{dx}(x^2+4)}{(x^2+4)^2} && \text{(Why?)}\\
&= \frac{(x^2+4)(3x^2) - (x^3+1) = x}{(x^2+4)^2} && \text{(Why?)}\\
&= \frac{3x^4 + 12x^2 - 2x^3 - 2x}{(x^2+4)^2} && \text{(Why?)}\\
&= \frac{3x^4 - 2x^3 + 12x^2 - 2x}{(x^2+4)^2}.
\end{aligned}$$

Exercises 3.1

1. From the definition, prove that $\frac{d}{dx}(x^3) = 3x^2$.

2. From the definition, prove that $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$.

Compute the following derivatives:

3. $\frac{d}{dx}(x^5 - 4x^2 + 7x - 2)$

4. $\frac{d}{dx}(4\sin x + 2\cos x - 3\tan x)$

5. $\frac{d}{dx}\left(\frac{2x+1}{x^2+1}\right)$

6. $\frac{d}{dx}\left(\frac{x^4+2}{3x+1}\right)$

7. $\frac{d}{dx}(3x\sin x + 4x^2\cos x)$

8. $\frac{d}{dx}(4\tan x - 3\sec x)$

9. $\frac{d}{dx}(3\cot x + 5\csc x)$

10. $\frac{d}{dx}(x^2\tan x + x\cot x)$

Recall that the equation of the line tangent to the graph of f at $(c, f(c))$ has slope $f'(c)$ and equations.

Tangent Line: $\quad y - f(c) = f'(c)(x - c)$

The normal line has slope $-1/f'(c)$, if $f'(c) \neq 0$ and has the equation:

Normal Line: $$y - f(c) = \frac{-1}{f'(c)}\ (x - c).$$

In each of the following, find the equation of the tangent line and the equation of the normal line for the graph of f at the given c.

11. $f(x) = x^3 + 4x - 12,\ c = 1$
12. $f(x) = \sin x,\ c = \pi/6$
13. $f(x) = \cos x,\ c = \pi/3$
14. $f(x) = \tan x,\ c = \pi/4$
15. $f(x) = \cot x,\ c = \pi/4$
16. $f(x) = \sec x,\ c = \pi/3$
17. $f(x) = \csc x, c = \pi/6$
18. $f(x) = 3\sin x + 4\cos x,\ c = 0.$

Recall that Newton's Method solves $f(x) = 0$ for x by using the fixed point iteration algorithm:

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}\ ,\ x_0 = \text{ given},$$

with the stopping rule, for a given natural number n,

$$|x_{n+1} - x_n| < 10^{-n}.$$

In each of the following, set up Newton's Iteration and perform 3 calculations for a given x_0.

19. $f(x) = 2x - \cos x\ ,\ x_0 = 0.5$
20. $f(x) = x^3 + 2x + 1\ ,\ x_0 = -0.5$
21. $f(x) = x^3 + 3x^2 - 1 = 0,\ x_0 = 0.5$
22. Suppose that $f'(c)$ exists. Compute each of the following limits in terms of $f'(c)$

    (a) $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c}$
    (b) $\lim\limits_{h\to 0} \dfrac{f(c+h) - f(c)}{h}$
    (c) $\lim\limits_{h\to 0} \dfrac{f(c-h) - f(c)}{h}$
    (d) $\lim\limits_{t\to c} \dfrac{f(c) - f(t)}{t - c}$
    (e) $\lim\limits_{h\to 0} \dfrac{f(c+h) - f(c-h)}{2h}$
    (f) $\lim\limits_{h\to 0} \dfrac{f(c+2h) - f(c-2h)}{h}$

23. Suppose that g is differentiable at c and

$$f(t) = \begin{cases} \frac{g(t)-g(c)}{t-c} & \text{if } t \neq c \\ g'(c) & \text{if } t = c. \end{cases}$$

Show that f is continuous at c.

Suppose that a business produces and markets x units of a commercial item. Let

$$\begin{aligned} C(x) = & \text{ The total cost of producing } x\text{-units.} \\ p(x) = & \text{ The sale price per item when } x\text{-units are on the market.} \\ R(x) = xp(x) = & \text{ The revenue for selling } x\text{-units.} \\ P(x) = R(x) - C(x) = & \text{ The gross profit for selling } x\text{-items.} \\ C'(x) = & \text{ The marginal cost.} \\ R'(x) = & \text{ The marginal revenue.} \\ P'(x) = & \text{ The marginal profit.} \end{aligned}$$

In each of the problems 24–26, use the given functions $C(x)$ and $p(x)$ and compute the revenue, profit, marginal cost, marginal revenue and marginal profit.

24. $C(x) = 100x - (0.2)x^2,\ 0 \leq x \leq 5000,\ p(x) = 10 - x$

25. $C(x) = 5000 + \dfrac{2}{x},\ 1 \leq x \leq 5000,\ p(x) = 20 + \dfrac{1}{x}$

26. $C(x) = 1000 + 4x - 0.1x^2,\ 1 \leq x \leq 2000,\ p(x) = 10 - \dfrac{1}{x}$

In exercises 27–60, compute the derivative of the given function.

27. $f(x) = 4x^3 - 2x^2 + 3x - 10$

28. $f(x) = 2\sin x - 3\cos x + 4$

29. $f(x) = 3\tan x - 4\sec x$

30. $f(x) = 2\cot x + 3\csc x$

31. $f(x) = 2x^2 + 4x + 5$

32. $f(x) = x^{2/3} - 4x^{1/3} + 5$

33. $f(x) = 3x^{-4/3} + 3x^{-2/3} + 10$

34. $f(x) = 2\sqrt{x} + 4$

35. $f(x) = \frac{2}{x^2}$

36. $f(x) = \frac{4}{x^3} - \frac{3}{x^2} + \frac{2}{x} + 1$

37. $f(x) = x^4 - 4x^2$

38. $f(x) = (x^2 + 2)(x^2 + 1)$

39. $f(x) = (x + 2)(x - 4)$

40. $f(x) = (x^3 + 1)(x^3 - 1)$

41. $y = (x^2 + 1) \sin x$

42. $y = x^2 \cos x$

43. $y = (x^2 + 1)(x^{10} - 5)$

44. $y = x^2 \tan x$

45. $y = (x^{1/2} + 4)(x^{1/3} - 5)$

46. $y = (2x + \sin x)(x^2 + 4)$

47. $y = x^5 \sin x$

48. $y = x^4(2 \sin x - 3 \cos x)$

49. $y = x^2 \cot x - 2x + 5$

50. $y = (x + \sin x)(4 + \csc x)$

51. $y = (\sec x + \tan x)(\sin x + \cos x)$

52. $y = x^2(2 \cot x - 3 \csc x)$

53. $y = \frac{x^2 + 1}{x^2 + 4}$

54. $y = \frac{1 + \sin x}{1 + \cos x}$

55. $y = \frac{x^{1/2} + 1}{3x^{3/2} + 2}$

56. $y = \frac{\sin x - \cos x}{\sin x + \cos x}$

57. $y = \frac{t^2 + 3t + 2}{t^3 + 1}$

58. $y = \frac{x^2 e^x}{1 + e^x}$

59. $y = \frac{3 + \sin t \cos t}{4 + \sec t \tan t}$

60. $y = \frac{t^2 \sin t}{4 + t^2}$

3.2 The Chain Rule

Suppose we have two functions, u and y, related by the equations:

$$u = g(x) \quad \text{and} \quad y = f(u).$$

Then $y = (f \circ g)(x) = f(g(x))$.

The chain rule deals with the derivative of the composition and may be stated as the following theorem:

Theorem 3.2.1 (The Chain Rule). *Suppose that g is defined in an open interval I containing c, and f is defined in an open interval J containing $g(c)$, such that $g(x)$ is in J for all x in I. If g is differentiable at c, and f is differentiable at $g(c)$, then the composition $(f \circ g)$ is differentiable at c and*

$$(f \circ g)'(c) = f'(g(c)) \cdot g'(c).$$

In general, if $u = g(x)$ and $y = f(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx.}$$

Proof. Let F be defined on J such that

$$F(u) = \begin{cases} \frac{f(u)-f(g(c))}{u-g(c)} & \text{if } u \neq g(c) \\ f'(g(c)) & \text{if } u = g(c) \end{cases}$$

since f is differentiable at $g(c)$,

$$\begin{aligned} \lim_{u \to g(c)} F(u) &= \lim_{u \to g(c)} \frac{f(u) - f(g(c))}{u - g(c)} \\ &= f'(g(c)) \\ &= F(g(c)). \end{aligned}$$

Therefore, F is continuous at $g(c)$. By the definition of F,

$$f(u) - f(g(c)) = F(u)(u - g(c))$$

for all u in J. For each x in I, we let $y = g(x)$ on I. Then

$$\begin{aligned} (f \circ g)'(c) &= \lim_{x \to c} \frac{(f \circ g)(x) - (f \circ g)(c)}{x - c} \\ &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} \\ &= \lim_{u \to g(c)} F(u) \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= f'(g(c)) \cdot g'(c). \end{aligned}$$

It follows that $f \circ g$ is differentiable at c. The general result follows by replacing c by the independent variable x. This completes the proof of Theorem 3.2.1.

Example 3.2.1 Let $y = u^2 + 1$ and $u = x^3 + 4$. Then

$$\frac{dy}{du} = 2u \text{ and } \frac{du}{dx} = 3x^2.$$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 2u \cdot 3x^2 \\ &= \boxed{6x^2(x^3+4)}\ . \end{aligned}$$

Using the composition notation, we get

$$y = (x^3+4)^2 + 1 = x^6 + 8x^3 + 17$$

and

$$\begin{aligned} \frac{dy}{dx} &= 6x^5 + 24x^2 \\ &= \boxed{6x^2(x^3+4)}\ . \end{aligned}$$

Using

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$

we see that

$$(f \circ g)(x) = (x^3+4)^2 + 1$$

and

$$\begin{aligned} (f \circ g)'(x) &= f'(g(x)) \cdot g'(x) \\ &= 2(x^3+4)^1 \cdot (3x^2) \\ &= \boxed{6x^2(x^3+4)}\ . \end{aligned}$$

Example 3.2.2 Suppose that $y = \sin(x^2 + 3)$.

We let $u = x^2 + 3$, and $y = \sin u$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (\cos u)(2x) \\ &= (\cos(x^2 + 3)) \cdot (2x).\end{aligned}$$

Example 3.2.3 Suppose that $y = w^2$, $w = \sin u + 3$, and $u = (4x + 1)$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx} \\ &= (2w) \cdot (\cos u) \cdot 4 \\ &= 8w \cos u \\ &= 8[\sin(4x + 1) + 3] \cdot \cos(4x + 1) \cdot 4 \\ &= 8(\sin(4x + 1) + 3) \cdot \cos(4x + 1).\end{aligned}$$

If we express y in terms of x explicitly, then we get

$$y = (\sin(4x + 1) + 3)^2$$

and

$$\begin{aligned}\frac{dy}{dx} &= 2(\sin(4x + 1) + 3)^1 \cdot ((\cos(4x + 1)) \cdot 4 + 0) \\ &= 8(\sin(4x + 1) + 3)\ \cos(4x + 1).\end{aligned}$$

Example 3.2.4 Suppose that $y = (\cos(3x + 1))^5$. Then

$$\begin{aligned}\frac{dy}{dx} &= 5(\cos(3x + 1))^4 \cdot (-\sin(3x + 1)) \cdot 3 \\ &= -15(\cos(3x + 1))^4\ \sin(3x + 1).\end{aligned}$$

Example 3.2.5 Suppose that $y = \tan^3(2x^2+1)$. Then

$$\begin{aligned}\frac{dy}{dx} &= 3(\tan^2(2x^2+1)) \cdot (\sec^2(2x^2+1)) \cdot 4x \\ &= 12x \cdot \tan^2(2x^2+1) \cdot \sec^2(2x^2+1).\end{aligned}$$

Example 3.2.6 Suppose that $y = \cot\left(\frac{x+1}{x^2+1}\right)$. Then

$$\begin{aligned}\frac{dy}{dx} &= \left[-\csc^2\left(\frac{x+1}{x^2+1}\right)\right]\ \left[\frac{(x^2+1)\cdot 1-(x+1)^{2x}}{(x^2+1)\cdot 2x}\right] \\ &= \frac{x^2+2x-1}{(x^2+1)^2}\ \csc^2\left(\frac{x+1}{x^2+1}\right).\end{aligned}$$

Example 3.2.7 Suppose that $y = \sec\left(\frac{x^2+1}{x^4+2}\right)^3$.

Since the function y has a composition of several functions, let us define some intermediate functions. Let

$$y = \sec w,\ \ w = u^3, \quad \text{and} \quad u = \frac{x^2+1}{x^4+2}.$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dw}\cdot\frac{dw}{du}\cdot\frac{du}{dx} \\ &= [\sec(w)\tan(w)]\cdot[3u^2]\cdot\ \frac{(x^4+2)\cdot 2x-(x^2+1)\cdot 4x^3}{(x^4+2)^2} \\ &= 3u^2(\sec w\tan w)\cdot\ \frac{4x-4x^3-2x^5}{(x^4+2)^2} \\ &= 3\left(\frac{x^2+1}{x^4+2}\right)^2\ \sec\left(\frac{x^2+1}{x^4+2}\right)^3\ \tan\left(\frac{x^2+1}{x^4+2}\right)^3\cdot\frac{4x-4x^3-2x^5}{(x^4+2)^5}.\end{aligned}$$

Example 3.2.8 Suppose that $y = \csc(2x+5)^4$. Then

$$\frac{dy}{dx} = [-\csc(2x+5)^4 \cot(2x+5)^4] \cdot 4(2x+5)^3 \cdot 2$$

$$= -8(2x+5)^3 \csc(2x+5)^4 \cot(2x+5)^4.$$

Exercises 3.2 Evaluate $\dfrac{dy}{dx}$ for each of the following:

1. $y = (2x-5)^{10}$
2. $y = \left(\dfrac{x^2+2}{x^5+4}\right)^3$
3. $y = \sin(3x+5)$
4. $y = \cos(x^3+1)$
5. $y = \tan^5(3x+1)$
6. $y = \sec^2(x^2+1)$
7. $y = \cot^4(2x-4)$
8. $y = \csc^3(3x^2+2)$
9. $y = \left(\dfrac{3x+1}{x^2+2}\right)^5$
10. $y = \left(\dfrac{x^2+1}{x^3+2}\right)^4$
11. $y = \sin(w),\ w = u^3,\ u = (2x-1)$
12. $y = \cos(w),\ w = u^2+1,\ u = (3x+5)$
13. $y = \tan(w),\ w = v^2,\ v = u^3+1,\ u = \left(\dfrac{1}{x}\right)$
14. $y = \sec w,\ w = v^3,\ v = 2u^2-1,\ u = \dfrac{x}{x^2+1}$
15. $y = \csc w,\ w = 3v+2,\ v = (u+1)^3,\ u = (x^2+3)^2$

In exercises 16–30, compute the derivative of the given function.

16. $y = \left(\dfrac{x^3+1}{x^2+4}\right)^3$
17. $y = (x^2-1)^{10}$

18. $y = (x^2 + x + 2)^{100}$

19. $y = (2 \sin t - 3 \cos t)^3$

20. $y = (x^{2/3} + x^{4/3})^2$

21. $y = (x^{1/2} + 1)^{50}$

22. $y = \sin(3x + 2)$

23. $y = \cos(3x^2 + 1)$

24. $y = \sin(2x) \cos(3x)$

25. $y = \sec 2x + \tan 3x$

26. $y = \sec 2x \tan 3x$

27. $y = (x^2 + 1)^2 \sin 2x$

28. $y = x \sin(1/x^2)$

29. $y = \sin^2(3x) + \sec^2(5x)$

30. $y = \cot(x^2) + \csc(3x)$

In exercises 31–60, assume that

(a) $\dfrac{d}{dx}(e^x) = e^x$ (b) $\dfrac{d}{dx}(e^{-x}) = -e^{-x}$ (c) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

(d) $\dfrac{d}{dx}(b^x) = b^x \ln b$ (e) $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x \ln b}$ for $b > 0$ and $b \neq 1$.

Compute the derivative of the given function.

31. $y = \sinh x$

32. $y = \cosh x$

33. $y = \tanh x$

34. $y = \coth x$

35. $y = \operatorname{sech} x$

36. $y = \operatorname{csch} x$

37. $y = \ln(1 + x)$

38. $y = \ln(1 - x)$

39. $y = \dfrac{1}{2} \ln\left(\dfrac{1 - x}{1 + x}\right)$

40. $y = \ln\left(x + \sqrt{x^2 + 1}\right)$

41. $y = \ln\left(x + \sqrt{x^2 - 1}\right)$

42. $y = xe^{-x^2}$

43. $y = e^{\sin 3x}$

44. $y = e^{2x} \sin 4x$

45. $y = e^{x^2}(2\sin 3x - 4\cos 5x)$

46. $y = xe^{-x^2} + 4e^{-x}$

47. $y = 4^{x^2}$

48. $y = 10^{(x^2+4)}$

49. $y = 10^{\sin 2x}$

50. $y = 3^{\cos 3x}$

51. $y = \log_{10}(x^2 + 10)$

52. $y = \log_3(x^2 \sin x + x)$

53. $y = \ln(\sin(e^{2x}))$

54. $y = \ln(1 + e^{-x})$

55. $y = \ln(\cos x + 2)$

56. $y = \ln(\ln(x^2 + 4))$

57. $y = \left\{\ln\left(\dfrac{x^4+3}{x^2+10}\right)\right\}^3$

58. $y = (1 + \sin^2 x)^{3/2}$

59. $y = \ln(\sec 2x + \tan 2x)$

60. $y = \ln(\csc 3x - \cot 3x)$

3.3 Differentiation of Inverse Functions

One of the applications of the chain rule is to compute the derivatives of inverse functions. We state the exact result as the following theorem:

Theorem 3.3.1 *Suppose that a function f has an inverse, f^{-1}, on an open interval I. If $u = f^{-1}(x)$, then*

(i) $\dfrac{du}{dx} = \dfrac{1}{\left(\frac{dx}{du}\right)}$

(ii) $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{f'(u)}$

Proof. By comparison, $x = f(f^{-1}(x)) = x$. Hence, by the chain rule

$$1 = \frac{dx}{dx} = f'(f^{-1}(x)) \cdot (f^{-1})'(x)$$

and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

In the $u = f^{-1}(x)$ notation, we have

$$\frac{du}{dx} = \frac{1}{\left(\frac{dx}{du}\right)}.$$

Remark 11 In Examples 76–81, we assume that the inverse trigonometric functions are differentiable.

Example 3.3.1 Let $u = \arcsin x$, $-1 \leq x \leq 1$, and $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$. Then $x = \sin u$ and by the chain rule, we get

$$\begin{aligned} 1 = \frac{dx}{dx} &= \frac{d(\sin u)}{du} \cdot \frac{du}{dx} \\ &= \cos u \cdot \frac{du}{dx} \\ \frac{du}{dx} &= \frac{1}{\cos u}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{dx}(\arcsin x) &= \frac{1}{\cos u}\ , \ -\frac{\pi}{2} < u < \frac{\pi}{2}, \\ &= \frac{1}{\sqrt{1 - \sin^2 u}} \qquad \text{(Why?)} \\ &= \frac{1}{\sqrt{1 - x^2}}\ , \ -1 < x < 1. \qquad \text{(Why?)} \end{aligned}$$

Thus,

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}\ , \ -1 < x < 1.$$

We note that $x = \pm 1$ are excluded.

Example 3.3.2 Let $u = \arccos x$, $-1 \leq x \leq 1$, and $0 \leq u \leq \pi$. Then $x = \cos u$ and

$$\begin{aligned} 1 = \frac{dx}{dx} &= -\sin u \, \frac{du}{dx} \\ \frac{du}{dx} &= -\frac{1}{\sin u}, \; 0 < u < \pi \\ &= -\frac{1}{\sqrt{1-\cos^2 u}}, \; 0 < u < \pi \qquad \text{(Why?)} \\ &= -\frac{1}{\sqrt{1-x^2}}, \; -1 < x < 1. \qquad \text{(Why?)} \end{aligned}$$

We note again that $x = \pm 1$ are excluded.

Thus,

$$\frac{d}{dx}\,(\arccos x) = \frac{-1}{\sqrt{1-x^2}}, \; -1 < x < 1.$$

Example 3.3.3 Let $u = \arctan x$, $-\infty < x < \infty$, and $-\frac{\pi}{2} < u < \frac{\pi}{2}$. Then,

$$\begin{aligned} x &= \tan u, \; -\frac{\pi}{2} < u < \frac{\pi}{2} \\ 1 = \frac{dx}{dx} &= (\sec^2 u), \; \frac{du}{dx} \;, \; -\frac{\pi}{2} < u < \frac{\pi}{2} \\ \frac{du}{dx} &= \frac{1}{\sec^2 u} \\ &= \frac{1}{1+\tan^2 u}, \; -\frac{\pi}{2} < u < \frac{\pi}{2} \\ &= \frac{1}{1+x^2}, \; -\infty < x < \infty \end{aligned}$$

Therefore,

$$\frac{d}{dx}\,(\arctan x) = \frac{1}{1+x^2}, \; -\infty < x < \infty.$$

Example 3.3.4 Let $u = \operatorname{arcsec} x,\ x \in (-\infty, -1] \cup [1, \infty)$ and $u \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Then,

$$\begin{aligned}
x &= \sec u \\
1 = \frac{dx}{dx} &= \sec u \tan u \cdot \frac{du}{dx}\ , \qquad u \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \\
\frac{du}{dx} &= \frac{1}{\sec u \tan u}\ , \qquad u \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \\
&= \frac{1}{|\sec u|\sqrt{\sec^2 u - 1}} \qquad \text{(Why the absolute value?)} \\
&= \frac{1}{|x|\sqrt{x^2 - 1}}, \qquad x \in (-\infty, -1) \cup (1, \infty).
\end{aligned}$$

Thus,

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2 - 1}},\ x \in (-\infty, -1) \cup (1, \infty).$$

Example 3.3.5 Let $u = \operatorname{arccsc} x,\ x \in (-\infty, -1] \cup [1, \infty)$, and $u \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. Then,

$$\begin{aligned}
x &= \csc u\ , \quad u \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \\
1 = \frac{dx}{dx} &= -\csc u \cot u \cdot \frac{du}{dx}, \quad u \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \\
\frac{du}{dx} &= \frac{-1}{\csc u \cot u}, \quad u \in \left(\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right), \quad \text{(Why?)} \\
&= \frac{1}{|\csc u|\sqrt{\csc^2 u - 1}} \qquad \text{(Why?)} \\
&= \frac{1}{|x|\sqrt{x^2 - 1}}, \quad x \in (-\infty, -1) \cup (1, \infty).
\end{aligned}$$

Note that $x = \pm 1$ are excluded.
Thus,

$$\frac{d}{dx}\ (\operatorname{arccsc} x) = \frac{-1}{x\sqrt{x^2 - 1}},\ x \in (-\infty, 1] \cup (1, \infty).$$

Example 3.3.6 Let $u = \text{arccot}\ x,\ x \in (-\infty, 0] \cup [0, \infty)$ and $u \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right)$. Then

$$x = \cot u, \quad u \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right)$$

and

$$\begin{aligned} 1 = \frac{dx}{dx} &= -\csc^2(u) \cdot \frac{du}{dx}, u \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right) \\ \frac{du}{dx} &= \frac{-1}{\csc^2 u}, \quad u \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right) \\ &= \frac{-1}{1 + \cot^2 u}, \quad u \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right) \\ &= \frac{-1}{1 + x^2}, \quad x \in (-\infty, 0] \cup [0, \infty). \end{aligned}$$

Therefore,

$$\frac{d}{dx}\,(\text{arccot} x) = \frac{-1}{1 + x^2}, \quad x \in (-\infty, 0] \cup [0, \infty).$$

The results of these examples are summarized in the following theorem:

Theorem 3.3.2 (The Inverse Trigonometric Functions) *The following differentiation formulas are valid for the inverse trigonometric functions:*

(i) $\frac{d}{dx}\,(\arcsin x) = \frac{1}{\sqrt{1 - x^2}},\ -1 < x < 1.$

(ii) $\frac{d}{dx}\,(\arccos x) = \frac{-1}{\sqrt{1 - x^2}},\ -1 < x < 1.$

(iii) $\frac{d}{dx}\,(\arctan x) = \frac{1}{1 + x^2},\ -\infty < x < \infty.$

(iv) $\frac{d}{dx}\,(\text{arccot}\ x) = \frac{-1}{1 + x^2},\ -\infty < x < \infty.$

(v) $\frac{d}{dx}\,(\text{arcsec}\ x) = \frac{1}{|x|\sqrt{x^2 - 1}},\ -\infty < x < -1 \ \ \textit{or}\ \ 1 < x < \infty.$

(vi) $\dfrac{d}{dx}\,(\text{arccsc } x) = \dfrac{-1}{|x|\sqrt{x^2-1}}, \quad -\infty < x < -1 \quad or \quad 1 < x < \infty.$

Proof. Proof of Theorem 3.3.2 is outlined in Examples 76–80.

Theorem 3.3.3 (Logarithmic and Exponential Functions)

(i) $\dfrac{d}{dx}\,(\ln x) = \dfrac{1}{x}$ *for all* $x > 0$.

(ii) $\dfrac{d}{dx}\,(e^x) = e^x$ *for all real* x.

(iii) $\dfrac{d}{dx}\,(\log_b x) = \dfrac{1}{x \ln b}$ *for all* $x > 0$ *and* $b \neq 1$.

(iv) $\dfrac{d}{dx}\,(b^x) = b^x(\ln b)$ *for all real* x, $b > 0$ *and* $b \neq 1$.

(v) $\dfrac{d}{dx}\,(u(x)^{v(x)} = (u(x))^{v(x)}\left[v'(x)\ln(u(x)) + v(x)\,\dfrac{u'(x)}{u(x)}\right].$

Proof. Proof of Theorem 3.3.3 is outlined in the proofs of Theorems 5.5.1–5.5.5. We illustrate the proofs of parts (iii), (iv) and (v) here.

Part (iii) By definition for all $x > 0$, $b > 0$ and $b \neq 1$,

$$\log_b x = \frac{\ln x}{\ln b}.$$

Then,

$$\begin{aligned}
\frac{d}{dx}\,(\log_b x) &= \frac{d}{dx}\,\left(\left(\frac{1}{\ln b}\right)\ln x\right) \\
&= \left(\frac{1}{\ln b}\right)\cdot\frac{1}{x} \\
&= \frac{1}{x\ln b}.
\end{aligned}$$

Part (iv) By definition, for real x, $b > 0$ and $b \neq 1$,

$$b^x = e^{x\ln b}.$$

Therefore,

$$\begin{aligned}
\frac{d}{dx}\,(b^x) &= \frac{d}{dx}\,(e^{x\ln b}) \\
&= e^{x\ln b},\frac{d}{dx}\,(x\ln b) \qquad \text{(by the chain rule)} \\
&= b^x \ln b. \qquad \text{(Why?)}
\end{aligned}$$

Part (v)

$$\begin{aligned}
\frac{d}{dx}\,(u(x))^{v(x)} &= \frac{d}{dx}\left\{e^{v(x)\ln(u(x))}\right\} \\
&= e^{v(x)\ln(u(x))}\left\{v'(x)\ln(u(x)) + v(x)\,\frac{u'(x)}{u(x)}\right\} \\
&= (u(x))^{v(x)}\left\{v'(x)\ln u(x) + v(x)\,\frac{u'(x)}{u(x)}\right\}
\end{aligned}$$

Example 3.3.7 Let $y = \log_{10}(x^2+1)$. Then

$$\begin{aligned}
\frac{d}{dx}\,(\log_{10}(x^2+1)) &= \frac{d}{dx}\,\left(\frac{\ln(x^2+1)}{\ln 10}\right) \\
&= \frac{1}{\ln 10}\,\left(\frac{1}{x^2+1}\cdot 2x\right) \qquad \text{(by the chain rule)} \\
&= \frac{2x}{(x^2+1)\ln 10}.
\end{aligned}$$

Example 3.3.8 Let $y = e^{x^2+1}$. Then, by the chain rule, we get

$$\begin{aligned}
\frac{dy}{dx} &= e^{x^2+1}\cdot 2x \\
&= 2xe^{x^2+1}.
\end{aligned}$$

Example 3.3.9 Let $y = 10^{(x^3+2x+1)}$. By definition and the chain rule, we get

$$\frac{dy}{dx} = 10^{(x^3+2x+1)} \cdot (\ln 10) \cdot (3x^2 + 2).$$

Example 3.3.10

$$\frac{dx}{dx}\,(x^2+1)^{\sin x} = (x^2+1)^{\sin x}\left\{\cos x \ln(x^2+1) + \sin x \cdot \frac{2x}{x^2+1}\right\}$$

$$\frac{d}{dx}\,(x^2+1)^{\sin x} = \frac{d}{dx}\,\left[e^{\sin x \ln(x^2+1)}\right] = 3^{\sin x \ln(x^2+1)} \cdot \left[\cos x \ln(x^2+1) + \sin x \cdot \frac{2x}{x^2+1}\right]$$

$$= (x^2+1)^{\sin x}\left[\cos x \ln(x^2+1) + \frac{2x \sin x}{x^2+1}\right].$$

Theorem 3.3.4 (Differentiation of Hyperbolic Functions)

(i) $\frac{d}{dx}\,(\sinh x) = \cosh x$

(ii) $\frac{d}{dx}\,(\cosh x) = \sinh x$

(iii) $\frac{d}{dx}\,(\tanh x) = \text{sech}^2 x$

(iv) $\frac{d}{dx}\,(\text{coth} x) = -\text{csch}^2 x$

(v) $\frac{d}{dx}\,(\text{sech } x) = -\text{sech } x \tanh x$

(vi) $\frac{d}{dx}\,(\text{csch } x) = -\text{csch } x \text{ coth } x.$

Proof.

Part (i)

$$\begin{aligned}\frac{d}{dx}\,(\sinh x) &= \frac{d}{dx}\,\left(\frac{1}{2}\,(e^x - e^{-x})\right)\\ &= \frac{1}{2}\,(e^x - e^{-x}(-1)) \qquad \text{(by the chain rule)}\\ &= \frac{1}{2}\,(e^x + e^{-x})\\ &= \cosh x.\end{aligned}$$

Part (ii)

$$\begin{aligned}
\frac{d}{dx}\,(\cosh x) &= \frac{d}{dx}\,\left(\frac{1}{2}\,(e^x + e^{-x})\right) \\
&= \frac{1}{2}\,(e^x + e^{-x}(-1)) \qquad \text{(by the chain rule)} \\
&= \frac{1}{2}\,(e^x - e^{-x}) \\
&= \sinh x.
\end{aligned}$$

Part (iii)

$$\begin{aligned}
\frac{d}{dx}\,(\tanh x) &= \frac{d}{dx}\,\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) \\
&= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
&= \frac{4}{(e^x + e^{-x})^2} \\
&= \left(\frac{2}{e^x + e^{-x}}\right)^2 \\
&= \operatorname{sech}^2 x.
\end{aligned}$$

Part (iv)

$$\begin{aligned}
\frac{d}{dx}\,(\operatorname{sech}\ x) &= \frac{d}{dx}\,\left(\frac{2}{e^x + e^{-x}}\right) \\
&= \frac{(e^x + e^{-x}) \cdot 0 - 2(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
&= -\frac{2}{e^x + e^{-x}} \cdot \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
&= -\operatorname{sech} x \tanh x.
\end{aligned}$$

Part (v)

$$
\begin{aligned}
\frac{d}{dx}\,(\coth x) &= \frac{d}{dx}\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right), \quad x \neq 0 \\
&= \frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2} \quad x \neq 0 \\
&= \frac{-4}{(e^x-e^{-x})^2}, \quad x \neq 0 \\
&= -\left(\frac{2}{e^x-e^{-x}}\right)^2, \quad x \neq 0 \\
&= -\operatorname{csch}^2 x, \quad x \neq 0.
\end{aligned}
$$

Part (vi)

$$
\begin{aligned}
\frac{d}{dx}\,(\operatorname{csch} x) &= \frac{d}{dx}\left(\frac{2}{e^x-e^{-x}}\right), \quad x \neq 0 \\
&= \frac{(e^x-e^{-x})\cdot 0-2(e^x+e^{-x})}{(e^x-e^{-x})^2}, \quad x \neq 0 \\
&= -\frac{2}{e^x-e^{-x}}\cdot\frac{e^x+e^{-x}}{e^x-e^{-x}}, \quad x \neq 0 \\
&= -\operatorname{csch} x\,\coth x, \quad x \neq 0.
\end{aligned}
$$

Theorem 3.3.5 (Inverse Hyperbolic Functions)

(i) $\frac{d}{dx}\,(\operatorname{arcsinh} x) = \frac{1}{\sqrt{1+x^2}}$

(ii) $\frac{d}{dx}\,(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$

(iii) $\frac{d}{dx}\,(\operatorname{arctanh} x) = \frac{1}{1-x^2}, \quad |x| < 1$

Proof.

Part (i)

$$\begin{aligned}
\frac{d}{dx}\,(\operatorname{arcsinh} x) &= \frac{d}{dx}\,\ln(x+\sqrt{1+x^2}) \\
&= \frac{1}{x+\sqrt{1+x^2}}\cdot\left[1+\frac{x}{\sqrt{1+x^2}}\right] \qquad \text{(by chain rule)} \\
&= \frac{1}{x+\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} \\
&= \frac{1}{\sqrt{1+x^2}}.
\end{aligned}$$

Part (ii)

$$\begin{aligned}
\frac{d}{dx}\,(\operatorname{arccosh} x) &= \frac{d}{dx}\,\ln(x+\sqrt{x^2-1})\ , \quad x\ge 1 \\
&= \frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{x}{\sqrt{x^2-1}}\right)\ , \quad x>0 \\
&= \frac{1}{x+\sqrt{x^2-1}}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\ , x>0 \\
&= \frac{1}{\sqrt{x^2-1}}\ , x>0.
\end{aligned}$$

Part (iii)

$$\begin{aligned}
\frac{d}{dx}\,(\operatorname{arctanh} x) &= \frac{d}{dx}\,\left[\frac{1}{2}\,\ln\left(\frac{1+x}{1-x}\right)\right]\ , \quad |x|<1 \\
&= \frac{d}{dx}\,\left[\frac{1}{2}\,\ln(1+x)-\ln(1-x)\right]\ , \quad |x|<1 \\
&= \frac{1}{2}\,\left[\frac{1}{1+x}-\frac{-1}{1-x}\right]\ , \quad |x|<1 \\
&= \frac{1}{2}\,\left[\frac{1}{1+x}+\frac{1}{1-x}\right]\ , \quad |x|<1 \\
&= \frac{1}{2}\,\left[\frac{1-x+1+x}{1-x^2}\right]\ , \quad |x|<1 \\
&= \frac{1}{1-x^2}\ , \quad |x|<1.
\end{aligned}$$

Exercises 3.3 Compute $\frac{dy}{dx}$ for each of the following:

1. $y = \ln(x^2+1)$
2. $y = \ln\left(\frac{1-x}{1+x}\right)$, $-1 < x < 1$
3. $y = \log_2(x)$
4. $y = \log_5(x^3+1)$
5. $y = \log_{10}(3x+1)$
6. $y = \log_{10}(x^2+4)$
7. $y = 2e^{-x}$
8. $y = e^{x^2}$
9. $y = \frac{1}{2}\left(e^{x^2} - e^{-x^2}\right)$
10. $y = \frac{1}{2}\left(e^{x^2} + e^{-x^2}\right)$
11. $y = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$
12. $y = \frac{2}{e^{x^2} + e^{-x^2}}$
13. $y = \frac{2}{e^{x^3} - e^{-x^3}}$
14. $y = \frac{2}{e^{x^4} + e^{-x^4}}$
15. $y = \arcsin\left(\frac{x}{2}\right)$
16. $y = \arccos\left(\frac{x}{3}\right)$
17. $y = \arctan\left(\frac{x}{5}\right)$
18. $y = \operatorname{arccot}\left(\frac{x}{7}\right)$
19. $y = \operatorname{arcsec}\left(\frac{x}{2}\right)$
20. $y = \operatorname{arccsc}\left(\frac{x}{3}\right)$
21. $y = 3\sinh(2x) + 4\cosh 3x$
22. $y = e^x(3\sin 2x + 4\cos 2x)$
23. $y = e^{-x}(4\sin 3x - 3\cos 3x)$
24. $y = 4\sinh 2x + 3\cosh 2x$
25. $y = 3\tanh(2x) - 7\coth(2x)$
26. $y = 3\operatorname{sech}(5x) + 4\operatorname{csch}(3x)$
27. $y = 10^{x^2}$
28. $y = 2^{(x^3+1)}$
29. $y = 5^{(x^4+x^2)}$
30. $y = 3^{\sin x}$

31. $y = 4^{\cos(x^2)}$

32. $y = 10^{\tan(x^3)}$

33. $y = 2^{\cot x}$

34. $y = 10^{\sec(2x)}$

35. $y = 4^{\csc(x^2)}$

36. $y = e^{-x}(2\sin(x^2) + 3\cos(x^3))$

37. $y = \text{arcsinh}\left(\dfrac{x}{2}\right)$

38. $y = \text{arccosh}\left(\dfrac{x}{3}\right)$

39. $y = \arctan\left(\dfrac{x}{4}\right)$

40. $y = x\ \text{arcsinh}\left(\dfrac{x}{3}\right)$

In exercises 41–50, use the following procedure to compute the derivative of the given functions:

$$\begin{aligned}\frac{d}{dx}\left[(f(x)^{g(x)}\right] &= \frac{d}{dx}\left[e^{g(x)\ln(f(x))}\right] \\ &= e^{g(x)\ln(f(x))}\cdot\left[g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right] \\ &= (f(x))^{g(x)}\cdot\left[g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right].\end{aligned}$$

41. $y = (x^2+4)^{3x}$

42. $y = (2+\sin x)^{\cos x}$

43. $y = (3+\cos x)^{\sin 2x}$

44. $y = (x^2+4)^{x^2+1}$

45. $y = (1+x)^{1/x}$

46. $y = (1+x^2)^{\cos 3x}$

47. $y = (2\sin x + 3\cos x)^{x^3}$

48. $y = (1+\ln x)^{1/x^2}$

49. $y = (1+\sinh x)^{\cosh x}$

50. $y = (\sinh^2 x + \cosh^2 x)^{x^2+3}$

3.4 Implicit Differentiation

So far we have dealt with explicit functions such as x^2, $\sin x$, $\cos x$, $\ln x$, e^x, $\sinh x$ and $\cosh x$ etc. In applications, two variables can be related by an equation such as

(i) $x^2+y^2=16$ (ii) $x^3+y^3=4xy$ (iii) $x\sin y+\cos 3y=\sin 2y$.

In such cases, it is not always practical or desirable to solve for one variable explicitly in terms of the other to compute derivatives. Instead, we may implicitly assume that y is some function of x and differentiate each term of the equation with respect to x. Then we solve for y', noting any conditions under which the derivative may or may not exist. This process is called implicit differentiation. We illustrate it by examples.

Example 3.4.1 Find $\dfrac{dy}{dx}$ if $x^2+y^2=16$.

Assuming that y is to be considered as a function of x, we differentiate each term of the equation with respect to x.

graph

$$\begin{aligned}\frac{d}{dx}\,(x^2)+\frac{d}{dx}\,(y^2) &= \frac{d}{dx}\,(16)\\ 2x+2y\left(\frac{dy}{dx}\right) &= 0 \qquad\qquad \text{(Why?)}\\ 2y\,\frac{dy}{dx} &= -2x\\ \frac{dy}{dx} &= -\frac{x}{y}\ ,\ \text{provided } y\neq 0.\end{aligned}$$

We observe that there are two points, namely $(4,0)$ and $(-4,0)$ that satisfy the equation. At each of these points, the tangent line is vertical and hence, has no slope.

If we solve for y in terms of x, we get two solutions, each representing a function of x:

$$y=(16-x^2)^{1/2} \quad \text{or} \quad y=-(16-x^2)^{1/2}.$$

On differentiating each function with respect to x, we get, respectively,

$$\frac{dy}{dx} = \frac{1}{2}\,(16 - x^2)^{-1/2}(-2x)\ ; \quad \text{or} \quad \frac{dy}{dx} = -\frac{1}{2}\,(16 - x^2)^{-1/2}(-2x)$$
$$\frac{dy}{dx} = -\frac{x}{(16 - x^2)^{1/2}}\ ; \quad \text{or} \quad \frac{x}{-(16 - x^2)^{1/2}}$$
$$\frac{dy}{dx} = -\frac{x}{y}\ , y \neq 0; \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}\ , y \neq 0.$$

In each case, the final form is the same as obtained by implicit differentiation.

Example 3.4.2 Compute $\dfrac{dy}{dx}$ for the equation $x^3 + y^3 = 4xy$.

As in Example 2.4.1, we differentiate each term with respect to x, assuming that y is a function of x.

$$\frac{dy}{dx}\,(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}\,(4xy)$$
$$3x^2 + 3y^2\left(\frac{dy}{dx}\right) = 4\left[\frac{dx}{dx}\,y + x\,\frac{dy}{dx}\right] \qquad \text{(Why?)}$$
$$(3y^2)\,\frac{dy}{dx} - 4x\frac{dy}{dx} = 4y - 3x^2 \qquad \text{(Why?)}$$
$$(3y^2 - 4x)\,\frac{dy}{dx} = 4y - 3x^2 \qquad \text{(Why?)}$$
$$\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}\ , \text{if } 3y^2 - 4x \neq 0. \qquad \text{(Why?)}$$

This differentiation formula is valid for all points (x, y) on the given curve, where $3y^2 - 4x \neq 0$.

Example 3.4.3 Compute $\dfrac{dy}{dx}$ for the equation $x \sin y + \cos 3y = \sin 2y$. In this example, it certainly is not desirable to solve for y explicitly in terms of

x. We consider y to be a function of x, differentiate each term of the equation with respect to x and then algebraically solve for y in terms of x and y.

$$\begin{aligned} \frac{d}{dx}(x\sin y) + \frac{d}{dx}(\cos 3y) &= \frac{d}{dx}(\sin 2y) \\ \left[\left(\frac{dx}{dx}\right)(\sin y) + x\,\frac{d}{dx}(\sin y)\right] + (-3\sin 3y)\,\frac{dy}{dx} &= (\cos 2y)\left(2\frac{dy}{dx}\right) \\ \sin y + x(\cos y)\,\frac{dy}{dx} - 3\sin(3y)\,\frac{dy}{dx} &= (2\cos 2y)\,\frac{dy}{dx}. \end{aligned}$$

Upon collecting all terms containing $\frac{dy}{dx}$ on the left-side, we get

$$\begin{aligned} [x\cos y - 3\sin 3y - 2\cos 2y]\,\frac{dy}{dx} &= -\sin y \\ \frac{dy}{dx} &= -\frac{\sin y}{x\cos y - 3\sin 3y - 2\cos 2y} \end{aligned}$$

whenever

$$x\cos y - 3\sin 3y - 2\cos 2y \neq 0.$$

Example 3.4.4 Find $\frac{dy}{dx}$ for $\frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1$.

On differentiating each term with respect to x, we get

graph

$$\begin{aligned} \frac{d}{dx}\left(\frac{(x-2)^2}{9}\right) + \frac{d}{dx}\left(\frac{(y-3)^2}{16}\right) &= \frac{d}{dx}(1) \\ \frac{2}{9}(x-2) + \frac{2}{16}(y-3)\,\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{2(x-2)/9}{2(y-3)/16}\ , \text{ if } y \neq 3 \\ &= -\frac{16(x-2)}{9(y-3)}\ , \text{ if } y = 3. \end{aligned}$$

The tangent lines are vertical at $(-1,3)$ and $(5,3)$. The graph of this equation is an ellipse.

Example 3.4.5 Find $\dfrac{dy}{dx}$ for the *astroid* $x^{2/3}+y^{2/3}=16$.

graph

$$\begin{aligned}
\frac{d}{dx}\,(x^{2/3})+\frac{d}{dx}\,(y^{2/3}) &= 0\\
\frac{2}{3}\,x^{-1/3}+\frac{2}{3}\,y^{-1/3}\,\frac{dy}{dx} &= 0,\ \text{if } x\neq 0 \text{ and } y\neq 0\\
\frac{dy}{dx} = -\frac{y^{-1/3}}{x^{-1/3}} &= -\left(\frac{x}{y}\right)^{1/3},\ \text{if } x\neq 0 \text{ and } y\neq 0.
\end{aligned}$$

Example 3.4.6 Find $\dfrac{dy}{dx}$ for the lemniscate with equation $(x^2+y^2)^2 = 4(x^2-y^2)$.

graph

$$\begin{aligned}
\frac{d}{dx}\ ((x^2+y^2)^2) &= 4\ \frac{d}{dx}\ (x^2-y^2)\\
2(x^2+y^2)\left(2x+2y\ \frac{dy}{dx}\right) &= 4\left[2x-2y\ \frac{dy}{dx}\right]\\
[4y(x^2+y^2)+8y]\ \frac{dy}{dx} &= 8x-4x(x^2+y^2) \qquad \text{(Why?)}\\
\frac{dy}{dx} &= \frac{8x-4x(x^2+y^2)}{4y(x^2+y^2)+8y}\ ,\ \text{if } 4y(x^2+y^2)+8y\neq 0, y\neq 0.
\end{aligned}$$

Example 3.4.7 Find the equations of the tangent and normal lines at (x_0, y_0) to the graph of an ellipse of the form

$$\frac{(x-k)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

First, we find $\frac{dy}{dx}$ by implicit differentiation as follows:

$$\begin{aligned}
\frac{d}{dx}\ \left(\frac{(x-h)^2}{a^2}\right)+\frac{d}{dx}\ \left(\frac{(y-k)^2}{b^2}\right) &= \frac{d}{dx}\ (1)\\
\frac{2}{a^2}\ (x-h)+\frac{2}{b^2}\ (y-k)\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= -\frac{2}{a^2}\ (x-h)\cdot\frac{b^2}{2(y-k)}\ ,\ \text{if } y\neq k\\
&= \frac{-b^2}{a^2}\ \left(\frac{x-h}{y-k}\right)\ , y\neq k.
\end{aligned}$$

It is clear that at $(a+h, k)$ and $(-a+h, k)$, the tangent lines are vertical and have the equations

$$x=a+h \quad \text{and} \quad x=-a+h.$$

Let (x_0, y_0) be a point on the ellipse such that $y_0\neq k$. Then the equation of the line tangent to the ellipse at (x_0, y_0) is

$$y-y_0=\frac{-b^2}{a^2}\ \left(\frac{x_0-h}{y_0-k}\right)(x-x_0).$$

We may express this in the form

$$\frac{(y-y_0)(y_0-k)}{b^2}+\frac{(x-x_0)(x_0-h)}{a^2}=0.$$

By rearranging some terms, we can simplify the equation in the following traditional form:

$$\frac{(y-k)+(k-y_0)}{b^2}\cdot(y_0-k)+\frac{(x-h)+(h-x_0)}{a^2}\,(x_0-h)=0$$

$$\frac{(y-k)(y_0-k)}{b^2}+\frac{(x-h)(x_0-h)}{a^2}=\frac{(x_0-h)^2}{a^2}+\frac{(y_0-k)^2}{b^2}=1.$$

$$\boxed{\frac{(y-k)(y_0-k)}{b^2}+\frac{(x-h)(x_0-h)}{a^2}=1}\,.$$

Exercises 3.4 In each of the following, find $\frac{dy}{dx}$ by implicit differentiation.

1. $y^2+3xy+2x^2=16$
2. $x^{3/4}+y^{3/4}=10^{3/4}$
3. $x^5+4x^3y^2+3y^4=8$
4. $\sin(x-y)=x^2y\cos x$
5. $\frac{x^2}{4}-\frac{y^2}{9}=1$
6. $\frac{x^2}{16}+\frac{y^2}{9}=1$

Find the equation of the line tangent to the graph of the given equation at the given point.

7. $\frac{x^2}{9}+\frac{y^2}{4}=1$ at $\left(2,\frac{2\sqrt{5}}{3}\right)$
8. $\frac{x^2}{9}-\frac{y^2}{4}=1$ at $\left(\frac{3}{2}\sqrt{5},1\right)$
9. $x^2y^2=(y+1)^2(9-y^2)$ at $\left(\frac{3}{2}\sqrt{5},2\right)$
10. $y^2=x^3(4-x)$ at $(2,4)$

Two curves are said to be *orthogonal* at each point (x_0, y_0) of their intersection if their tangent lines are perpendicular. Show that the following families of curves are orthogonal.

11. $x^2 + y^2 = r^2,\ y + mx = 0$

12. $(x-h)^2 + y^2 = h^2,\ x^2 + (y-k)^2 = k^2$

Compute y' and y'' in exercises 13–20.

13. $4x^2 + 9y^2 = 36$

14. $4x^2 - 9y^2 = 36$

15. $x^{2/3} + y^{2/3} = 16$

16. $x^3 + y^3 = a^3$

17. $x^2 + 4xy + y^2 = 6$

18. $\sin(xy) = x^2 + y^2$

19. $x^4 + 2x^2y^2 + 4y^4 = 26$

20. $(x^2 + y^2)^2 = x^2 - y^2$

3.5 Higher Order Derivatives

If the vertical height y of an object is a function f of time t, then $y'(t)$ is called its velocity, denoted $v(t)$. The derivative $v'(t)$ is called the acceleration of the object and is denoted $a(t)$. That is,

$$y(t) = f(t),\ y'(t) = v(t),\ v'(t) = a(t).$$

We say that $a(t)$ is the second derivative of y, with respect to t, and write

$$y''(t) = a(t) \quad \text{or} \quad \frac{d^2y}{dt^2} = a(t).$$

Derivatives of order two or more are called higher derivatives and are represented by the following notation:

$$y'(x) = \frac{dy}{dx}\ ,\ y''(x) = \frac{d^2y}{dx^2}\ ,\ y'''(x) = \frac{d^3y}{dx^3}\ ,\ldots,\ y^{(n)}(x) = \frac{d^ny}{dx^n}.$$

The definition is given as follows by induction:

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) \quad \text{and} \quad \frac{d^nf}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}f}{dx^{n-1}}\right), n = 2, 3, 4, \cdots .$$

A convenient notation is

$$f^{(n)}(x) = \frac{d^n f}{dx^n}$$

which is read as "the nth derivative of f with respect to x."

Example 3.5.1 Compute the second derivative y'' for each of the following functions:

(i) $y = \sin(3x)$ (ii) $y = \cos(4x^2)$ (iii) $y = \tan(3x)$

(iv) $y = \cot(5x)$ (v) $y = \sec(2x)$ (vi) $y = \csc(x^2)$

Part (i) $y' = 3\cos(3x),\ y'' = -9\sin(3x)$

Part (ii) $y' = -8x\sin(4x^2),\ y'' = -8[\sin(4x^2) + x \cdot (8x) \cdot \cos(4x^2)]$

Part(iii) $y' = 3\sec^2(3x),\ y'' = 3[2\sec(3x) \cdot \sec(3x)\tan(3x) \cdot 3]$

$$y'' = 18\sec^2(3x)\tan(3x)$$

Part(iv) $y' = -5\csc^2(5x),\ y'' = -10\csc(5x)[(-\csc 5x \cot 5x) \cdot 5]$

$$y'' = 50\csc^2(5x)\cot(5x)$$

Part(v) $y' = 2\sec(2x)\tan(2x)$

$$y'' = 2[(2\sec(2x)\tan(2x)) \cdot \tan(2x) + \sec(2x) \cdot (2\sec^2(2x))]$$

$$y'' = 4\sec(2x)\tan^2(2x) + 4\sec^3(2x)$$

Part(vi) $y' = -2x\csc(x^2)\cot(x^2)$

$$y'' = -2[1 \cdot \csc(x^2)\cot(x^2) + x(-2x\csc(x^2)\cot(x^2)) \cdot \cot(x^2)$$

$$+ \ x\csc(x^2)\cdot(-2x\csc^2(x^2))]$$

$$= -2\csc(x^2)\cot(x^2) + 4x^2\csc(x^2)\cot^2(x^2) + 4x^2\csc^3(x^2)$$

Example 3.5.2 Compute the second order derivative of each of the following functions:

(i) $y = \sinh(3x)$ (ii) $y = \cosh(x^2)$ (iii) $y = \tanh(2x)$

(iv) $y = \coth(4x)$ (v) $y = \text{sech}(5x)$ (vi) $y = \text{csch}(10x)$

Part (i) $y' = 3\cosh(3x),\ y'' = 9\sinh(3x)$

Part (ii) $y' = 2x\sinh(x^2),\ y'' = 2\sinh(x^2) + 2x(2x\cosh x^2)$ or

$y'' = 2\sinh(x^2) + 4x^2\cosh(x^2)$

Part (iii) $y' = 2\ \text{sech}^2(2x),\ y'' = 2\cdot(2\ \text{sech}(2x)\cdot(-\text{sech}(2x)\tanh(2x)\cdot 2)),$

$y'' = -8\ \text{sech}^2(2x)\tanh(2x)$

Part (iv) $y' = -4\ \text{csch}^2(4x),\ y'' = -4(2(\text{csch}(4x))\cdot(-\text{csch}(4x)\ \coth(4x)\cdot 4))$

$y'' = 32\ \text{csch}^2(4x)\ \coth(4x)$

Part (v) $y' = -5\ \text{sech}\ (5x)\tanh(5x)$

$y' = -5[-5\ \text{sech}(5x)\tanh(5x)\cdot\tanh(5x) + \ \text{sech}(5x)\cdot\ \text{sech}^2(5x)\cdot 5]$

$y' = 25\ \text{sech}(5x)\tanh^2(5x) - 25\ \text{sech}^3(5x).$

Part (vi) $y' = -10\ \text{csch}(10x)\ \coth(10x)$

$y'' = -10[-10\ \text{csch}(10x)\ \coth(10x)\cdot\coth(10x)$

$$+\ \text{csch}(10x)(-10\ \text{csch}^2(10x))]$$

$$y'' = 100\ \text{csch}(10x)\ \coth^2(10x) + 100\ \text{csch}^3(10x)$$

Example 3.5.3 Compute the second order derivatives for the following functions:

(i) $y = \ln(x^2)$ (ii) $y = e^{x^2}$ (iii) $\log_{10}(x^2+1)$

(iv) $y = 10^{x^2}$ (v) $y = \arcsin x$ (vi) $y = \arctan x$

Part (i) $y' = \dfrac{2x}{x2} = \dfrac{2}{x} = 2x^{-1}$

$$y'' = -2x^{-2} = \frac{-2}{x^2}.$$

Part (ii) $y' = 2xe^{x^2},\ y'' = 2e^{x^2} + 4x^2e^{x^2} = (2+4x^2)e^{x^2}.$

Part (iii) $y' = \dfrac{1}{\ln 10} \cdot \dfrac{2x}{x^2+1}\ , y'' = \dfrac{2}{\ln 10}\ \left[\dfrac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2}\right],$

$$y'' = \frac{2}{\ln 10} \cdot \left[\frac{1-x^2}{(x^2+1)^2}\right]$$

Part (iv) $y' = 10^{x^2} \cdot (\ln 10) \cdot 2x$

$$y'' = 2\ln 10[10^{x^2} + x \cdot 10^{x^2} \ln 10 \cdot 2x]$$

$$y'' = 10^{x^2}[2\ln 10 + (2\ln 10)^2 x^2]$$

Part (v) $y' = \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$

$$y'' = \frac{-1}{2}\,(1-x^2)^{-3/2}(-2x)$$

$$y'' = \frac{x}{(1-x^2)^{3/2}}.$$

Part (vi) $y' = \dfrac{1}{1+x^2} = (1+x^2)^{-1}$

$$y'' = -1(1+x^2)^{-2}\cdot 2x = \frac{-2x}{(1+x^2)^2}$$

Example 3.5.4 Compute the second derivatives of the following functions:

(i) $y = \operatorname{arcsinh} x$ (ii) $y = \operatorname{arccosh} x$ (iii) $y = \operatorname{arctanh} x$

From Section 1.4, we recall that

arcsinh $x = \ln(x + \sqrt{1+x^2})$

arccosh $x = \ln(x + \sqrt{x^2-1})$, $x \geq 1$

arctanh $x = \dfrac{1}{2}\,\ln\left(\dfrac{1+x}{1-x}\right) = \dfrac{1}{2}\,[\ln(1+x) - \ln(1-x)],\ |x| < 1.$

Then

Part (i)

$$\begin{aligned} y' &= \frac{1}{\sqrt{1+x^2}} \\ \frac{d^2}{dx^2}\,(\operatorname{arcsinh} x) &= \frac{d}{dx}\,(1+x^2)^{-1/2} \\ &= \frac{-1}{2}\,(2x)(1+x^2)^{-3/2} \\ &= -\frac{x}{(1+x^2)^{3/2}}. \end{aligned}$$

Part (ii)

$$\begin{aligned}
y' &= \frac{1}{\sqrt{x^2-1}}\ ,\ x>1 \\
\frac{d^2}{dx^2}\,(\operatorname{arccosh} x) &= \frac{d}{dx}\,(x^2-1)^{-1/2} \\
&= \frac{-1}{2}\,(2x)(x^2-1)^{-3/2} \\
&= -\frac{x}{(x^2-1)^{3/2}}\ ,\ x>1
\end{aligned}$$

Part (iii)

$$\begin{aligned}
y' &= \frac{1}{1-x^2}\ ,\ |x|<1. \\
\frac{d^2}{dx}\,(\operatorname{arctanh} x) &= \frac{d}{dx}\,(1-x^2)^{-1} \\
&= (-1)(1-x^2)^{-2}(-2x) \\
&= \frac{x}{(1-x^2)^2}\ ,\ |x|<1.
\end{aligned}$$

Example 3.5.5 Find y'' for the equation $x^2+y^2=4$.

First, we find y' by implicit differentiation.

$$2x+2yy'=0 \to y', \frac{x}{y}.$$

Now, we differentiate again with respect to x.

$$\begin{aligned}
y'' &= \frac{y\cdot 1 - xy'}{y^2} \\
&= -\frac{y-x(-x/y)}{y^2} && \text{(replace } y' \text{ by } -x/y\text{)} \\
&= -\frac{y^2+x^2}{y^3} && \text{(Why?)} \\
&= -\frac{4}{y^3} && \text{(since } x^2+y^2=4\text{)}
\end{aligned}$$

Example 3.5.6 Compute y'' for $x^3 + y^3 = 4xy$.

From Example 25 in the last section we found that

$$y' = \frac{4y - 3x^2}{3y^2 - 4x} \quad \text{if} \quad 3y^2 - 4x \neq 0.$$

To find y'', we differentiate y' with respect to x to get

$$y'' = \frac{(3y^2 - 4x)(4y' - 3x^2) - (4y - 3x^2)(6yy' - 4)}{3y^2 - 4x}\ ,\ 3y^2 - 4x \neq 0.$$

In order to simplify any further, we must first replace y' by its computed value. We leave this as an exercise.

Example 3.5.7 Compute $f^{(n)}(c)$ for the given f and c and all natural numbers n:

(i) $f(x) = \sin x,\ c = 0$ (ii) $f(x) = \cos x,\ x = 0$ (iii) $f(x) = \ln(x),\ c = 1$

(iv) $f(x) = e^x,\ c = 0$ (v) $f(x) = \sinh x,\ x = 0$ (vi) $f(x) = \cosh x,\ x = 0$

To compute the general nth derivative formula we must discover a pattern and then generalize the pattern.

Part (i) $f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = \cos x, f^4(x) = \sin x$. Then the next four derivatives are repeated and so on. We get

$$f^{(4n)}(n) = \sin x,\ f^{(4n+1)}(x) = \cos x,\ f^{(4n+2)}(x) = -\sin x,\ f^{(4n+3)}(x) = -\cos x.$$

By evaluating these at $c = 0$, we get

$$f^{(4n)}(0) = 0,\ f^{(4n+2)}(0);\ f^{(4n+1)}(0) = 1 \ \text{ and } \ f^{(4n+3)}(0) = -1,$$

for $n = 0, 1, 2, \cdots$

Part (ii) This part is similar to Part (i) and is left as an exercise.

Part (iii) $f(x) = \ln x,\ f'(x) = x^{-1},\ f''(x) = (-1)x^{-2},\ f^{(3)}(x) = (-1)(-2)x^{-3}, \ldots .,$ $f^{(n)}(x) = (-1)(-2)\ldots(-(n-1))x^{-n} = (-1)^{n-1}(n-1)!x^{-n},\ f^{(n)}(1) = (-1)^{n-1}(n-1)!,\ n = 1, 2, \ldots$

Part (iv) $f(x) = e^x,\ f'(x) = e^x,\ f''(x) = e^x, \dots,\ f^{(n)}(x) = e^x,\ f^{(n)}(0) = 1,\ n = 0, 1, 2, \dots$

Part (v) $f(x) = \sinh x, f'(x) = \cosh x, f''(x) = \sinh x, \dots f^{(2n)}(x) = \sinh x, f^{(2n+1)}(x) = \cosh x, f^{(2n)}(0) = 0,\ f^{(2n+1)}(0) = 1,\ n = 0, 1, 2, \dots$

Part (vi) $f(x) = \cosh x,\ f'(x) = \sinh x,\ f''(x) = \cosh x, \dots, f^{(2n)}(x) = \cosh x,\ f^{(2n+1)}(x) = \sinh x,\ f^{(2n)}(0) = 1, f^{(2n+1)}(0) = 0, n = 0, 1, 2, \dots$

Exercises 3.5 Find the first two derivatives of each of the following functions f.

1. $f(t) = 4t^3 - 3t^2 + 10$
2. $f(x) = 4\sin(3x) + 3\cos(4x)$
3. $f(x) = (x^2 + 1)^3$
4. $f(x) = x^2 \sin(3x)$
5. $f(x) = e^{3x} \sin 4x$
6. $f(x) = e^{2x} \cos 4x$
7. $f(x) = \dfrac{x^2}{2x+1}$
8. $f(x) = (x^2 + 1)^{10}$
9. $f(x) = \ln(x^2 + 1)$
10. $f(x) = \log_{10}(x^4 + 1)$
11. $f(x) = 3\sinh(4x) + 5\cosh(4x)$
12. $f(x) = \tanh(3x)$
13. $f(x) = x \tan x$
14. $f(x) = x^2 e^x$
15. $f(x) = \arctan(3x)$
16. $f(x) = \operatorname{arcsinh}(2x)$
17. $f(x) = \cos(nx)$
18. $f(x) = (x^2 + 1)^{100}$

Show that the given $y(x)$ satisfies the given equation:

19. $y = A\sin(4x) + B\cos(4x)$ satisfies $y'' + 16y = 0$

20. $y = A\sinh(4x) + B\cosh(4x)$ satisfies $y'' - 16y = 0$

21. $y = e^{-x}(a\sin(2x) + b\cos(2x))$ satisfies $y'' - 2y' + 2y = 0$

22. $y = e^{x}(a\sin(3x) + b\cos(3x))$ satisfies $y'' - 2y' + 10y = 0$

Compute the general nth derivative for each of the following:

23. $f(x) = e^{2x}$

24. $f(x) = \sin 3x$

25. $f(x) = \cos 4x$

26. $f(x) = \ln(x+1)$

27. $f(x) = \sinh(2x)$

28. $f(x) = \cosh(3x)$

29. $f(x) = (x+1)^{100}$

30. $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

Find y' and y'' for the following equations:

31. $x^4 + y^4 = 20$

32. $x^2 + xy + y^2 = 16$

Chapter 4

Applications of Differentiation

One of the important problems in the real world is *optimization*. This is the problem of maximizing or minimizing a given function. Differentiation plays a key role in solving such real world problems.

4.1 Mathematical Applications

Definition 4.1.1 A function f with domain D is said to have an *absolute maximum* at c if $f(x) \leq f(c)$ for all $x \in D$. The number $f(c)$ is called the absolute maximum of f on D. The function f is said to have a *local maximum* (or *relative maximum*) at c if there is some open interval (a, b) containing c and $f(c)$ is the absolute maximum of f on (a, b).

Definition 4.1.2 A function f with domain D is said to have an *absolute minimum* at c if $f(c) \leq f(x)$ for all x in D. The number $f(c)$ is called the *absolute minimum* of f on D. The number $f(c)$ is called a *local minimum* (or *relative minimum*) of f if there is some open interval (a, b) containing c and $f(c)$ is the absolute minimum of f on (a, b).

Definition 4.1.3 An absolute maximum or absolute minimum of f is called an *absolute extremum* of f. A local maximum or minimum of f is called a *local extremum* of f.

Theorem 4.1.1 (Extreme Value Theorem) *If a function f is continuous on a closed and bounded interval $[a, b]$, then there exist two points, c_1 and c_2, in $[a, b]$ such that $f(c_1)$ is the absolute minimum of f on $[a, b]$ and $f(c_2)$ is the absolute maximum of f on $[a, b]$.*

Proof. Since $[a, b]$ is a closed and bounded set and f is continuous on $[a, b]$, Theorem 4.1.1 follows from Theorem 2.3.14.

Definition 4.1.4 A function f is said to be *increasing* on an open interval (a, b) if $f(x_1) < f(x_2)$ for all x_1 and x_2 in (a, b) such that $x_1 < x_2$. The function f is said to be *decreasing* on (a, b) if $f(x_1) > f(x_2)$ for all x_1 and x_2 in (a, b) such that $x_1 < x_2$. The function f is said to be *non-decreasing* on (a, b) if $f(x_1) \leq f(x_2)$ for all x_1 and x_2 in (a, b) such that $x_1 < x_2$. The function f is said to be *non-increasing* on (a, b) if $f(x_1) \geq f(x_2)$ for all x_1 and x_2 in (a, b) such that $x_1 < x_2$.

Theorem 4.1.2 *Suppose that a function f is defined on some open interval (a, b) containing a number c such that $f'(c)$ exists and $f'(c) \neq 0$. Then $f(c)$ is not a local extremum of f.*

Proof. Suppose that $f'(c) \neq 0$. Let $\epsilon = \dfrac{1}{2}\,|f'(c)|$. Then $\epsilon > 0$.

Since $\epsilon > 0$ and

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

there exists some $\delta > 0$ such that if $0 < |x - c| < \delta$, then

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \frac{1}{2}\,|f'(c)|$$

$$-\frac{1}{2}\,|f'(c)| < \frac{f(x) - f(c)}{x - c} - f'(c) < \frac{1}{2}\,|f'(c)|$$

$$f'(c) - \frac{1}{2}\,|f'(c)| < \frac{f(x) - f(c)}{x - c} < f'(c) + \frac{1}{2}\,|f'(c)|.$$

The following three numbers have the same sign, namely,

$$f'(c), \ f'(c) - \frac{1}{2}\,|f'(c)| \quad \text{and} \quad f'(c) + \frac{1}{2}\,|f'(c)|.$$

Since $f'(c) > 0$ or $f'(c) < 0$, we conclude that

$$0 < \frac{f(x) - f(c)}{x - c} \quad \text{or} \quad \frac{f(x) - f(c)}{x - c} < 0$$

for all x such that $0 < |x - c| < \delta$. Thus, if $c - \delta < x_1 < c < x_2 < c + \delta$, then either $f(x_1) < f(c) < f(x_2)$ or $f(x_1) > f(c) > f(x_2)$. It follows that $f(c)$ is not a local extremum.

Theorem 4.1.3 *If f is defined on an open interval (a, b) containing c, $f(c)$ is a local extremum of f and $f'(c)$ exists, then $f'(c) = 0$.*

Proof. This theorem follows immediately from Theorem 4.1.2.

Theorem 4.1.4 (Rolle's Theorem) *Suppose that a function f is continuous on a closed and bounded interval $[a, b]$, differentiable on the open interval (a, b) and $f(a) = f(b)$. Then there exists some c such that $a < c < b$ and $f'(c) = 0$.*

Proof. Since f is continuous on $[a, b]$, there exist two numbers c_1 and c_2 on $[a, b]$ such that $f(c_1) \leq f(x) \leq f(c_2)$ for all x in $[a, b]$. (Extreme Value Theorem.) If $f(c_1) = f(c_2)$, then the function f has a constant value on $[a, b]$ and $f'(c) = 0$ for $c = \frac{1}{2}\,(a + b)$. If $f(c_1) \neq f(c_2)$, then either $f(c_1) \neq f(a)$ or $f(c_2) \neq f(a)$. But $f'(c_1) = 0$ and $f'(c_2) = 0$. It follows that $f'(c_1) = 0$ or $f'(c_2) = 0$ and either c_1 or c_2 is between a and b. This completes the proof of Rolle's Theorem.

Theorem 4.1.5 (The Mean Value Theorem) *Suppose that a function f is continuous on a closed and bounded interval $[a, b]$ and f is differentiable on the open interval (a, b). Then there exists some number c such that $a < c < b$ and*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. We define a function $g(x)$ that is obtained by subtracting the line joining $(a, f, (a))$ and $(b, f(b))$ from the function f:

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\,(x - a) + f(a)\right].$$

The g is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, $g(a) = g(b) = 0$. By Rolle's Theorem, there exists some number c such that $a < c < b$ and

$$\begin{aligned} 0 &= g'(c) \\ &= f'(c) - \frac{f(b) - f(a)}{b - a} \ . \end{aligned}$$

Hence,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

as required.

Theorem 4.1.6 (Cauchy-Mean Value Theorem) *Suppose that two functions f and g are continuous on a closed and bounded interval $[a, b]$, differentiable on the open interval (a, b) and $g'(x) \neq 0$ for all x in (a, b). Then there exists some number c in (a, b) such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \ .$$

Proof. We define a new function h on $[a, b]$ as follows:

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} \, (g(x) - g(a)).$$

Then h is continuous on $[a, b]$ and differentiable on (a, b). Furthermore,

$$h(a) = 0 \quad \text{and} \quad h(b) = 0.$$

By Rolle's Theorem, there exist some c in (a, b) such that $h'(c) = 0$. Then

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} \, g'(c)$$

and, hence,

$$\frac{f(b) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}$$

as required. This completes the proof of Theorem 4.1.6.

Theorem 4.1.7 (L'Hospital's Rule, $\frac{0}{0}$ Form) *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval (a, b) containing c (except possibly at c). Suppose that*

$$\lim_{x \to c} f(x) = 0 \ , \quad \lim_{x \to c} g(x) = 0 \quad \text{and} \quad \lim_{x \to c} \frac{f'(x)}{g'(x)} = L,$$

where L is a real number, ∞, or $-\infty$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} = L.$$

Proof. We define $f(c) = 0$ and $g(c) = 0$. Let $x \in (c, b)$. Then f and g are continuous on $[c, x]$, differentiable on (c, x) and $g'(y) \neq 0$ on (c, x). By the Cauchy Mean Value Theorem, there exists some point $y \in (c, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(y)}{g'(y)} \ .$$

Then

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{y \to c^+} \frac{f'(y)}{g'(y)} = L.$$

Similarly, we can prove that

$$\lim_{x \to c^-} \frac{f(x)}{g(x)} = L.$$

Therefore,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} = L.$$

Remark 12 Theorem 4.1.7 is valid for one-sided limits as well as the two-sided limit. This theorem is also true if $c = \infty$ or $c = -\infty$.

Theorem 4.1.8 *Theorem 4.1.7 is valid for the case when*

$$\lim_{x \to c} f(x) = \infty \quad \text{or} \quad -\infty \quad \text{and} \quad \lim_{x \to c} g(x) = \infty \quad \text{or} \quad -\infty.$$

Proof of Theorem 4.1.8 is omitted.

Example 4.1.1 Find each of the following limits using L'Hospital's Rule.

(i) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 5x}$ (ii) $\lim\limits_{x\to 0} \dfrac{\tan 2x}{\tan 3x}$ (iii) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

(iv) $\lim\limits_{x\to 0} \dfrac{x}{\sin x}$ (v) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$ (vi) $\lim\limits_{x\to 0} x\ln x$

We compute these limits as follows:

(i) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 5x} = \lim\limits_{x\to 0} \dfrac{3\cos 3x}{5\cos 5x} = \dfrac{3}{5}$

(ii) $\lim\limits_{x\to 0} \dfrac{\tan 2x}{\tan 3x} = \lim\limits_{x\to 0} \dfrac{2\sec^2 x}{3\sec^2 3x} = \dfrac{2}{3}$

(iii) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{x\to 0} \dfrac{\cos x}{1} = 1$

(iv) $\lim\limits_{x\to 0} \dfrac{x}{\sin x} = \lim\limits_{x\to 0} \dfrac{1}{\cos x} = 1$

(v) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = \lim\limits_{x\to 0} \dfrac{\sin x}{1} = 0$

(vi) $\lim\limits_{x\to 0} x\ln x = \lim\limits_{x\to 0} \dfrac{\ln x}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to 0} \dfrac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = \lim\limits_{x\to 0} (-x) = 0.$

Theorem 4.1.9 *Suppose that two functions f and g are continuous on a closed and bounded interval $[a, b]$ and are differentiable on the open interval (a, b). Then the following statements are true:*

(i) If $f'(x) > 0$ for each x in (a, b), then f is increasing *on (a, b).*

(ii) If $f'(x) < 0$ for each x in (a, b), then f is decreasing *on (a, b).*

(iii) If $f'(x) \geq 0$ for each x in (a, b), then f is non-decreasing *on (a, b).*

(iv) If $f'(x) \leq 0$ for each x in (a, b), then f is non-increasing *on (a, b).*

(v) If $f'(x) = 0$ for each x in (a, b), then f is constant *on (a, b).*

(vi) If $f'(x) = g'(x)$ on (a, b), then $f(x) = g(x)+C$, for constant C, on (a, b).

Proof.
Part (i) Suppose $a < x_1 < x_2 < b$. Then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Mean Value Theorem, there exists some c such that $a < x_1 < c < x_2 < b$ and

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0.$$

Since $x_2 - x_1 > 0$, it follows that $f(x_2) - f(x_1) > 0$ and $f(x_2) > f(x_1)$. By definition, f is increasing on (a, b). The proof of Parts (ii)–(v) are similar and are left as an exercise.

Part (vi) Let $F(x) = f(x) - g(x)$ for all x in $[a, b]$. Then F is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, $F'(x) = 0$ on (a, b). Hence, by Part (v), there exists some constant C such that for each x in (a, b),

$$F(x) = C, \ f(x) - g(x) = c, \ f(x) = g(x) + C.$$

This completes the proof of the theorem.

Theorem 4.1.10 (First Derivative Test for Extremum) *Let f be continuous on an open interval (a, b) and $a < c < b$.*

(i) If $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b), then $f(c)$ is a local maximum *of f on (a, b).*

(ii) If $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b), then $f(c)$ is a local minimum *of f on (a, b).*

Proof. This theorem follows immediately from Theorem 4.1.9 and its proof is left as an exercise.

Theorem 4.1.11 (Second Derivative Test for Extremum) *Suppose that f, f' and f'' exist on an open interval (a, b) and $a < c < b$. Then the following statements are true:*

(i) If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local minimum *of f.*

(ii) If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum *of f.*

(iii) *If* $f'(c) = 0$ *and* $f''(c) = 0$, *then* $f(c)$ *may or may not be a local extremum.*

Proof.
Part (i) If $f''(c) > 0$, then by Theorem 4.1.2, there exists some $\delta > 0$ such that for all x in $(c - \delta, c + \delta)$,

$$\frac{f'(c)}{x - c} = \frac{f'(x) - f'(c)}{x - c} > 0.$$

Hence, $f'(x) > 0$ on $(c, c + \delta)$ and $f'(x) < 0$ on $(c - \delta, c)$. By the first derivative test, $f(c)$ is a local minimum of f.

Part (ii) The proof of Part (ii) is similar to Part (i) and is left as an exercise.

Part (iii) Let $f(x) = x^3$ and $g(x) = x^4$. Then

$$f'(0) = g'(0) = f''(0) = g''(0).$$

However, f has no local extremum at 0 but g has a local maximum at 0. This completes the proof of this theorem.

Definition 4.1.5 (Concavity) Suppose that f is defined in some open interval (a, b) containing c and $f'(c)$ exists. Let

$$y = g(x) = f'(c)(x - c) + f(c)$$

be the equation of the line tangent to the graph of f at c.

(i) If there exists $\delta > 0$ such that $f(x) > g(x)$ for all x in $(c-\delta, c+\delta)$, $x \neq c$, then the graph of f is said to be *concave upward at* c. If the graph of f is concave upward at every c in (a, b), then it is said to be *concave upward on* (a, b).

(ii) If there exists $\delta > 0$ such that $f(x) < g(x)$ for all x in $(c-\delta, c+\delta)$, $x \neq c$, then the graph of f is said to be *concave downward at* c. If the graph of f is concave downward at every c in (a, b), then it is said to be *concave downward on* (a, b).

(iii) The point $(c, f(c))$ is said to be a *point of inflection* if there exists some $\delta > 0$ such that either

(i) the graph of f is concave upward on $(c-\delta, c)$ and concave downward on $(c, c+\delta)$, or

(ii) the graph of f is concave downward on $(c-\delta, c)$ and concave upward on $(c, c+\delta)$.

Remark 13 The first derivative test, second derivative test and concavity test are very useful in graphing functions.

Example 4.1.2 Let $f(x) = x^4 - 4x^2, \ -3 \leq x \leq 3$

(a) Locate the local extrema, and point extrema and points of inflections.

(b) Locate the intervals where the graph of f is increasing, decreasing, concave up and concave down.

(c) Sketch the graph of f. Determine the absolute maximum and the absolute minimum of the graph of f on $[-3, 3]$.

Part (a)

(i) $f(x) = x^4 - 4x^2 = x^2(x^2 - 4) = 0 \to x = 0, x = -2, x = 2$ are zeros of f.

(ii) $f'(x) = 4x^3 - 8x = 0 = 4x(x^2 - 2) = 0 \to x = 0, x = -\sqrt{2}$ and $x = \sqrt{2}$ are the critical points of f.

(iii) $f''(x) = 12x^2 - 4 = 12\left(x^2 - \frac{1}{3}\right) = 0 \to x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$ are the x-coordinates of the points of inflections of the graph of f, since f'' changes sign at these points.

(iv) $f'(0) = 0, f''(0) = -4 \to f(0) = 0$ is a local minimum of f.
$f'(-\sqrt{2}) = 0, f''(-\sqrt{2}) > 0 \to f(-\sqrt{2}) = -8$ is a local minimum of f.
$f'(\sqrt{2}) = 0, f''(\sqrt{2}) > 0 \to f(\sqrt{2}) = -8$ is a local minimum of f.

(v) $f''(x)$ changes sign at $x = \pm\frac{1}{\sqrt{3}}$ and hence $\left(\pm\frac{1}{\sqrt{3}}, \frac{-11}{9}\right)$ are the points of inflection of the graph of f.

Part (b) The function f is decreasing on $(-\infty, -\sqrt{2}) \cup (0, \sqrt{2})$ and is increasing on $(-\sqrt{2}, 0) \cup (\sqrt{2}, \infty)$. The graph of f is concave up on $\left(-\infty, \dfrac{-1}{\sqrt{3}}\right) \cup \left(\dfrac{1}{\sqrt{3}}, \infty\right)$ and is concave down on $\left(\dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$.

(c) $f(-3) = f(3) = 45$ is the absolute maximum of f and is obtained at the end points of the interval.

Also, $f(-\sqrt{2}) = f(\sqrt{2}) = -8$ is the absolute minimum of f on $[-3, 3]$. We note that $f(0) = 0$ is a local maximum of f. The graph is sketched with the above information.

graph

Example 4.1.3 Consider $g(x) = x^2 - x^{2/3}, -2 \leq x \leq 3$. Sketch the graph of g, locating extrema, zeros, points of inflection, intervals where f is increasing or decreasing, and intervals where the graph of f is concave up or concave down.

Let us compute the zeros and critical points of g.

(i) $g(x) = x^{2/3}(x^{4/3} - 1) = 0 \to x = 0, -1, 1.$

$$g'(x) = 2x - \frac{2}{3}\, x^{-1/3} = 2x^{-1/3}\left(x^{4/3} - \frac{1}{3}\right) = 0 \to x = \pm\left(\frac{1}{3}\right)^{3/4}.$$

We note that $g'(0)$ is undefined. The critical points are, $0, \pm\left(\dfrac{1}{3}\right)^{3/4}$.

(ii) $g''(x) = 2 + \dfrac{2}{9}\, x^{-4/3} > 0$ for all x, except $x = 0$, where $g''(x)$ does not exist.

The function g is decreasing on $\left(-\infty, -\left(\dfrac{1}{3}\right)^{3/4}\right)$ and $\left(0, \left(\dfrac{1}{3}\right)^{3/4}\right)$.

The function g is increasing on $\left(-\left(\dfrac{1}{3}\right)^{3/4}, 0\right) \cup \left(\left(\dfrac{1}{3}\right)^{3/4}, \infty\right)$.

(iii) The point $(0, 0)$ is not an inflection point, since the graph is concave up everywhere on $(-\infty, 0) \cup (0, \infty)$.

Exercises 4.1 Verify that each of the following Exercises 1–2 satisfies the hypotheses and the conclusion of the Mean Value Theorem. Determine the value of the admissible c.

1. $f(x) = x^2 - 4x, \ -2 \leq x \leq 2$

2. $g(x) = x^3 - x^2$ on $[-2, 2]$

3. Does the Mean Value Theorem apply to $y = x^{2/3}$ on $[-8, 8]$? If not, why not?

4. Show that $f(x) = x^2 - x^3$ cannot have more than two zeros by using Rolle's Theorem.

5. Show that $f(x) = \ln x$ is an increasing function. (Use Mean Value Theorem.)

6. Show that $f(x) = e^{-x}$ is a decreasing function.

7. How many real roots does $f(x) = 12x^4 - 14x^2 + 2$ have?

8. Show that if a polynomial has four zeros, then there exists some c such that $f'''(c) = 0$.

   A function f is said to satisfy a Lipschitz condition with constant M if

$$|f(x) - f(y)| \leq M|x - y|$$

   for all x and y. The number M is called a Lipschitz constant for f.

9. Show that $f(x) = \sin x$ satisfies a Lipschitz condition. Find a Lipschitz constant.

10. Show that $g(x) = \cos x$ satisfies a Lipschitz condition. Find a Lipschitz constant for g.

In each of the following exercises, sketch the graph of the given function over the given interval. Locate local extrema, absolute extrema, intervals where the function is increasing, decreasing, concave up or concave down. Locate the points of inflection and determine whether the points of inflection are oblique or not.

11. $f(x) = \dfrac{x^2}{2x^2+1}, [-1, 1]$

12. $f(x) = x^2(1-x)^2, [-2, 2]$

13. $f(x) = |x-1| + 2|x+2|, [-4, 4]$

14. $f(x) = 2x^2 + \dfrac{1}{x^2}, [-1, 1]$

15. $f(x) = \sin x - \cos x, [0, 2\pi]$

16. $f(x) = x - \cos x, [0, 2\pi]$

17. $f(x) = \dfrac{2x}{x^2-9}, [-4, 4]$

18. $f(x) = 2x^{3/5} - x^{6/5}, [-2, 2]$

19. $f(x) = (x^2-1)e^{-x^2}, [-2, 2]$

20. $f(x) = 3\sin 2x + 4\cos 2x, [0, 2\pi]$

Evaluate each of the following limits by using the L'Hospital's Rule.

21. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\tan 5x}$

22. $\lim\limits_{x\to 0} \dfrac{x+\sin \pi x}{x-\sin \pi x}$

23. $\lim\limits_{x\to 1} \dfrac{x \ln x}{1-x}$

24. $\lim\limits_{x\to 0} \dfrac{e^x-1}{\ln(x+1)}$

25. $\lim\limits_{x\to 0} \dfrac{e^x-1}{x}$

26. $\lim\limits_{x\to 0} \dfrac{10^x-1}{x}$

27. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sinh(5x)}$

28. $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \csc x\right)$

29. $\lim\limits_{x\to 0} \dfrac{x+\tan x}{x+\sin x}$

30. $\lim\limits_{x\to 1} \dfrac{(1-x^2)}{(1-x^3)}$

4.2 Antidifferentiation

The process of finding a function $g(x)$ such that $g(x) = f(x)$, for a given $f(x)$, is called antidifferentiation.

Definition 4.2.1 Let f and g be two continuous functions defined on an open interval (a, b). If $g'(x) = f(x)$ for each x in (a, b), then g is called an *antiderivative* of f on (a, b).

Theorem 4.2.1 *If $g_1(x)$ and $g_2(x)$ are any two antiderivatives of $f(x)$ on (a, b), then there exists some constant C such that*

$$g_1(x) = g_2(x) + C.$$

Proof. If $h(x) = g_1(x) - g_2(x)$, then

$$\begin{aligned} h'(x) &= g_1'(x) - g_2'(x) \\ &= f(x) - f(x) \\ &= 0 \end{aligned}$$

for all x in (a, b). By Theorem 4.1.9, Part (iv), there exists some constant c such that for all x in (a, b),

$$\begin{aligned} C = h(x) &= g_1(x) - g_2(x) \\ g_2(x) &= g_1(x) + C. \end{aligned}$$

Definition 4.2.2 If $g(x)$ is an antiderivative of f on (a, b), then the set $\{g(x)+C : C \text{ is a constant}\}$ is called a one-parameter family of antiderivatives of f. We called this one-parameter family of antiderivatives the *indefinite integral* of $f(x)$ on (a, b) and write

$$\int f(x)dx = g(x) + C.$$

The expression "$\int f(x)dx$" is read as "the indefinite integral of $f(x)$ with respect to x." The function "$f(x)$" is called the integrand, "$\int$" is called the integral sign and "x" is called the variable of integration. When dealing with indefinite integrals, we often use the terms antidifferentiation and integration interchangeably. By definition, we observe that

$$\frac{d}{dx}\left(\int f(x)dx\right) = g'(x) = f(x).$$

Example 4.2.1 The following statements are true:

1. $\displaystyle\int x^3 dx = \frac{1}{4}\, x^4 + c$

2. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

3. $\displaystyle\int \frac{1}{x}\ dx = \ln|x| + c$

4. $\displaystyle\int \sin x\ dx = -\cos x + c$

5. $\displaystyle\int \sin(ax)dx = \frac{-1}{a}\ \cos(ax) + c$

6. $\displaystyle\int \cos x\ dx = \sin x + c$

7. $\displaystyle\int \cos(ax)dx = \frac{1}{a}\ \sin(ax) + c$

8. $\displaystyle\int \tan x\ dx = \ln|\sec x| + c$

9. $\displaystyle\int \tan(ax)dx = \frac{1}{a}\ \ln|\sec(ax)| + c$

10. $\displaystyle\int \cot x\ dx = \ln|\sin x| + c$

11. $\displaystyle\int \cot(ax)dx = \frac{1}{a}\ln|\sin(ax)| + c$

12. $\displaystyle\int e^x dx = e^x + c$

13. $\displaystyle\int e^{-x}dx = -e^{-x} + c$

14. $\displaystyle\int e^{ax}dx = \frac{1}{a}\ e^{ax} + c$

15. $\displaystyle\int \sinh x dx = \cosh x + c$

16. $\displaystyle\int \cosh x\ dx = \sinh x + c$

17. $\displaystyle\int \tanh x\ dx = \ln|\cosh x| + c$

18. $\displaystyle\int\ \coth x\ dx = \ln|\sinh x| + c$

19. $\displaystyle\int \sinh(ax) = \frac{1}{a}\ \cosh(ax) + c$

20. $\displaystyle\int \cosh(ax)dx = \frac{1}{a}\ \sinh(ax) + c$

21. $\displaystyle\int \tanh(ax)dx = \frac{1}{a}\ \ln|\cosh ax| + c$

22. $\displaystyle\int\ \coth\,(ax)dx = \frac{1}{a}\ \ln|\sinh(ax)| + c$

23. $\displaystyle\int \sec x\ dx = \ln|\sec x + \tan x| + c$

24. $\displaystyle\int \csc x\ dx = -\ln|\csc x + \cot x| + c$

25. $\displaystyle\int \sec(ax)dx = \frac{1}{a}\ \ln|\sec(ax) + \tan(ax)| + c$

26. $\displaystyle\int \csc(ax)dx = \frac{-1}{a}\ \ln|\csc(ax) + \cot(ax)| + c$

27. $\displaystyle\int \sec^2 x dx = \tan x + c$

28. $\displaystyle\int \sec^2(ax)dx = \frac{1}{a}\ \tan(ax) + c$

29. $\displaystyle\int \csc^2 x\ dx = -\cot x + c$

30. $\displaystyle\int \csc^2(ax)dx = \frac{-1}{a}\ \cot(ax) + c$

31. $\displaystyle\int \tan^2 x\ dx = \tan x - x + c$

32. $\displaystyle\int \cot^2 x\ dx = -\cot x - x + c$

33. $\displaystyle\int \sin^2 x\ dx = \frac{1}{2}\ (x - \sin x \cos x) + c = \frac{1}{2}\ \left(x - \frac{\sin 2x}{2}\right) + c$

34. $\displaystyle\int \cos^3 x dx = \frac{1}{2}\ (x + \sin x \cos x) + c = \frac{1}{2}\ \left(x + \frac{\sin 2x}{2}\right) + c$

35. $\displaystyle\int \sec x \tan x\ dx = \sec x + c$

36. $\displaystyle\int \csc x \cot x \; dx = -\csc x + c$

Each of these indefinite integral formulas can be proved by differentiating the right sides of the equation. We show some details in selected cases.

Part 3. Recall that

$$\frac{d}{dx}\left(|x|\right) = \frac{x}{|x|} = \frac{|x|}{x}\;, x \neq 0.$$

Hence,

$$\frac{d}{dx}\left(\ln|x| + c\right) = \frac{1}{|x|} \cdot \left(\frac{|x|}{x} + 0\right) = \frac{1}{x}.$$

The absolute values are necessary because $\ln(x)$ is defined for positive numbers only.

Part 23.
$$\begin{aligned}\frac{d}{dx}\left(\ln|\sec x + \tan x|\right) &= \frac{1}{\sec x + \tan x} \cdot \left(\sec x \tan x + \sec^2 x\right)\\ &= \frac{\sec x(\tan x + \sec x)}{(\sec x + \tan x)}\\ &= \sec x.\end{aligned}$$

Part 31. $\displaystyle\frac{d}{dx}\left(\tan x - x + c\right) = \sec^2 x - 1 = \tan^2 x.$

Part 33. $\displaystyle\frac{d}{dx}\left(\frac{1}{2}\left(x - \sin x \cos x\right) + c\right)$

$$= \frac{d}{dx}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) \qquad \text{(Trigonometric Identity)}$$

$$= \frac{1}{2} - \frac{2\cos 2x}{4}$$

$$= \frac{1}{2}\left(1 - \cos x\right)$$

$$= \sin^2 x \qquad \text{(Trigonometric Identity)}$$

Part 34. $\dfrac{d}{dx}\left(\dfrac{1}{2}\,(x+\sin x\cos x)+c\right)$

$$= \frac{d}{dx}\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)$$

$$= \frac{1}{2}+\frac{1}{2}\cos 2x$$

$$= \frac{1}{2}\,(1+\cos 2x)$$

$$= \cos^2 x \qquad \text{(Trigonometric Identity)}$$

Example 4.2.2 The following statements are true:

1. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$

2. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}+c$

3. $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx = \operatorname{arcsinh} x + c = \ln(x+\sqrt{1+x^2})+c$

4. $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2}+c$

5. $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx = \operatorname{arccosh} x + c = \ln|x+\sqrt{x^2-1}|+c$

6. $\displaystyle\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}+c$

7. $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + c$

8. $\displaystyle\int \frac{1}{1-x^2}\,dx = \operatorname{arctanh} x + c = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+c$

9. $\displaystyle\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x + c$

10. $\displaystyle\int b^x dx = \frac{b^x}{\ln b}+c, b>0, b\neq 1$

All of these integration formulas can be verified by differentiating the right sides of the equations.

Remark 14 In the following exercises, use the substitution to reduce the integral to a familiar form and then use the integral tables if necessary.

Exercises 4.2 In each of the following, evaluate the indefinite integral by using the given substitution. Use the formula:

$$\int f(g(t))g'(t)dt = \int f(u)du, \quad \text{where} \quad u = g(t), \; du = g'(t)dt.$$

1. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx, x = 2\sin t$

2. $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx, x = 2\cosh t$

3. $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx, x = 3\tan t$

4. $\displaystyle\int \frac{1}{x\sqrt{x^2-9}}\,dx, x = 3\sec t$

5. $\displaystyle\int xe^{-x^2}dx, u = -x^2$

6. $\displaystyle\int \sin(7x+1)dx; u = 7x+1$

7. $\displaystyle\int \sec^2(3x+1)dx, u = 3x+1$

8. $\displaystyle\int \cos^2(2x+1)dx, u = 2x+1$

9. $\displaystyle\int x\sin^2(x^2)dx, u = x^2$

10. $\displaystyle\int \tan^2(5x+7)dx, u = 5x+7$

11. $\displaystyle\int \sec(2x-3)\tan(2x-3)dx, u = 2x-3$

12. $\displaystyle\int \cot(5x+2)dx, u = 5x+2$

13. $\displaystyle\int x(x^2+1)^{10}dx, u = x^2+1$

14. $\displaystyle\int \frac{x}{(x^2+1)^{1/3}}\,dx, u = x^2+1$

15. $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx, u = e^x$

16. $\displaystyle\int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\,dx, u = e^{2x}+e^{-2x}$

17. $\displaystyle\int \sin^3(2x)\cos 2x\ dx, u = \sin 2x$

18. $\displaystyle\int e^{\sin 3x}\cos 3x\ dx, u = \sin 3x$

19. $\displaystyle\int \sec^2 x\tan x\ dx, u = \sec x$

20. $\displaystyle\int \tan^{10} x\sec^2 x\ dx, u = \tan x$

21. $\displaystyle\int \frac{x\ln(x^2+1)}{x^2+1}\,dx, u = \ln(x^2+1)$

22. $\displaystyle\int \frac{x}{\sqrt{4+x^2}}\,dx, u = 4+x^2$

23. $\displaystyle\int \frac{x\;dx}{\sqrt{4-x^2}}, u = 4-x^2$

24. $\displaystyle\int \frac{x}{9+x^2}\,dx, u = 9+x^2$

25. $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx,\; u = 2\sinh x$

26. $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx,\; u = 2\cosh x$

4.3 Linear First Order Differential Equations

Definition 4.3.1 If $p(x)$ and $q(x)$ are defined on some open interval, then an equation of the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

is called a *linear first order differential equation* in the variable y.

Example 4.3.1 (Exponential Growth). A model for exponential growth is the first order differential equation

$$\frac{dy}{dx} = ky,\; k > 0,\; y(0) = y_0.$$

To solve this equation we divide by y, integrate both sides with respect to x, replacing $\left(\dfrac{dy}{dx}\right)\,dx$ by dy as follows:

$$\begin{aligned}
\int \frac{1}{y}\left(\frac{dy}{dx}\right)\,dx &= \int k\;dx \\
\int \frac{1}{y}\,dy &= kx + c \\
\ln|y| &= kx + c \\
|y| &= e^{kx+c} = e^c e^{kx} \\
y &= \pm e^c e^{kx}.
\end{aligned}$$

Next, we impose the condition $y(0) = y_0$ to get

$$y(0) = \pm e^c = y_0$$
$$\boxed{y = y_0 e^{kx}}\ .$$

The number y_0 is the value of y at $x = 0$. If the variable x is replaced by the time variable t, we get

$$y(t) = y(0)e^{kt}.$$

If $k > 0$, this is an exponential growth model. If $k < 0$, this is an example of an exponential decay model.

Theorem 4.3.1 (Linear First Order Differential Equations) *If $p(x)$ and $q(x)$ are continuous, then the differential equation*

$$\frac{dy}{dx} + p(x)y = q(x) \tag{1}$$

has the one-parameter family of solutions

$$y(x) = e^{-\int p(x)dx}\left[\int q(x)e^{\int p(x)dx}dx + c\right].$$

Proof. We multiply the given differential equation (1) by $e^{\int p(x)dx}$, which is called the integrating factor.

$$e^{\int p(x)dx}\frac{dy}{dx} + p(x)e^{\int p(x)dx}y = q(x)e^{\int p(x)dx}. \tag{2}$$

Since the integrating factor is never zero, the equation (2) has exactly the same solutions as equation (1). Next, we observe that the left side of the equation is the derivative of the product the integrating factor and y:

$$\frac{d}{dx}\left(e^{\int p(x)dx}y\right) = q(x)e^{\int p(x)dx}. \tag{3}$$

By the definition of the indefinite integral, we express equation (3) as follows:

$$e^{\int p(x)dx}y = \int \left(q(x)e^{\int p(x)dx}dx\right) + c. \tag{4}$$

Next, we multiply both sides of equation (4) by $e^{-\int p(x)dx}$:

$$\boxed{y = e^{-\int p(x)dx}\left[\int q(x)e^{\int p(x)dx}dx + c\right]}. \tag{5}$$

Equation (5) gives a one-parameter family of solutions to the equation. To pick a particular member of the family, we specify either a point on the curve, or the slope at a point of the curve. That is,

$$y(0) = y_0 \quad \text{or} \quad y'(0) = y_0'.$$

Then c is uniquely determined. This completes the proof.

Example 4.3.2 Solve the differential equation

$$y' + 4y = 10\ ,\ y(0) = 200.$$

Step 1. We multiply both sides by the integrating factor

$$e^{\int 4dx} = e^{4x}$$

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = 10e^{4x}. \tag{6}$$

Step 2. We observe that the left side is the derivative of the integrating factor and y.

$$\frac{d}{dx}\left(e^{4x}y\right) = 10e^{4x}. \tag{7}$$

Step 3. Using the definition of the indefinite integral, we antidifferentiate:

$$e^{4x}y = \int (10e^{4x})dx + c.$$

Step 4. We multiply both sides by e^{-4x}.

$$y = e^{-4x}\left[\int (10e^{4x})dx + c\right]$$

$$y = e^{-4x}\left[10 \cdot \frac{e^{4x}}{4} + c\right]$$

$$\boxed{y(x) = \frac{10}{4} + ce^{-4x}}. \tag{8}$$

Step 5. We impose the condition $y(0) = 200$ to solve for c.

$$y(0) = 200 = \frac{5}{2} + c, \quad c = 200 - \frac{5}{2}.$$

Step 6. We replace c by its value in solution (8)

$$\boxed{y(x) = \frac{5}{2} + \left(200 - \frac{5}{2}\right) e^{-4x}}\,.$$

Exercises 4.3 Find $y(t)$ in each of the following:

1. $y' = 4y,\ y(0) = 100$
2. $y' = -2y,\ y(0) = 1200$
3. $y' = -4(y - c),\ y(0) = y_0$
4. $L\dfrac{dy}{dt} + Ry = E,\ y(0) = y_0$
5. $y' + 3y(t) = 32,\ y(0) = 0$
6. $y' = ty,\ y(0) = y_0$

$$\left(\text{Hint: } y' = ty, \ \int \frac{1}{y}\frac{dy}{dt}\,dt = \int t dt; \ \int \frac{1}{y}\,dy = \frac{t^2}{2} + c.\right)$$

7. The population $P(t)$ of a certain country is given by the equation:

$$P'(t) = 0.02P(t), \quad P(0) = 2 \text{ million}.$$

   (i) Find the time when the population will double.
   (ii) Find the time when the population will be 3 million.

8. Money grows at the rate of $r\%$ compounded continuously if

$$A'(t) = \left(\frac{r}{100}\right) A(t), \quad A(0) = A_0,$$

   where $A(t)$ is the amount of money at time t.

   (i) Determine the time when the money will double.
   (ii) If $A = \$5000$, determine the time for which $A(t) = \$15{,}000$.

9. A radioactive substance satisfies the equation

$$A'(t) = -0.002A(t), \quad A(0) = A_0,$$

where t is measured in years.

(i) Determine the time when $A(t) = \frac{1}{2} A_0$. This time is called the half-life of the substance.

(ii) If $A_0 = 20$ grams, find the time t for which $A(t)$ equals 5 grams.

10. The number of bacteria in a test culture increases according to the equation

$$N'(t) = rN(t), \quad n(0) = N_0,$$

where t is measured in hours. Determine the doubling period. If $N_0 = 100$, $r = 0.01$, find t such that $N(t) = 300$.

11. Newton's law of cooling states that the time rate of change of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium. Suppose that K stands for the constant of proportionality. Then this law may be expressed as

$$T'(t) = K(A - T(t)).$$

Solve for $T(t)$ in terms of time t and $T_0 = T(0)$.

12. In a draining cylindrical tank, the level y of the water in the tank drops according to Torricelli's law

$$y'(t) = -Ky^{1/2}$$

for some constant K. Solve for y in terms of t and K.

13. The rate of change $P'(t)$ of a population $P(t)$ is proportional to the square root of $P(t)$. Solve for $P(t)$.

14. The rate of change $v'(t)$ of the velocity $v(t)$ of a coasting car is proportional to the square of v. Solve for $v(t)$.

In exercises 15–30, solve for y.

15. $y' = x - y,\ y(0) = 5$
16. $y' + 3x^2y = 0,\ y(0) = 6$
17. $xy'(x) + 3y(x) = 2x^5,\ y(2) = 1$
18. $xy' + y = 3x^2,\ y(1) = 4$
19. $y' + y = e^x,\ y(0) = 100$
20. $y' = -6xy,\ y(0) = 9$
21. $y' = (\sin x)y,\ y(0) = 5$
22. $y' = xy^3,\ y(0) = 2$
23. $y' = \dfrac{1+\sqrt{x}}{1+\sqrt{y}},\ y(0) = 10$
24. $y' - 2y = 1,\ y(1) = 3$
25. $y' = ry - c,\ y(0) = A$
26. $y' - 3y = 2\sin x,\ y(0) = 12$
27. $y' - 2y = 4e^{2x},\ y(0) = 4$
28. $y' - 3x^2y = e^{x^3},\ y(0) = 7$
29. $y' - \dfrac{1}{2x}y = \sin x,\ y(1) = 3$
30. $y' - 3y = e^{2x},\ y(0) = 1$

4.4 Linear Second Order Homogeneous Differential Equations

Definition 4.4.1 A linear second order differential equation in the variable y is an equation of the form

$$y'' + p(x)y' + q(x)y = r(x).$$

If $r(x) = 0$, we say that the equation is homogeneous; otherwise it is called non-homogeneous. If $p(x)$ and $q(x)$ are constants, we say that the equation has constant coefficients.

Definition 4.4.2 If f and g are differentiable functions, then the Wronskian of f and g is denoted $W(f, g)$ and defined by

$$W(f, g) = f(x)g'(x) - f'(x)g(x).$$

Example 4.4.1 Compute the following Wronskians:

(i) $W(\sin(mx), \cos(mx))$ (ii) $W(e^{px}\sin(qx), e^{px}\cos(qx))$

(iii) $W(x^n, x^m)$ (iv) $W(x\sin(mx), x\cos(mx))$

Part (i) $W(\sin mx, \cos mx) = \sin(mx)\frac{d}{dx}(\cos(mx)) - \frac{d}{dx}(\sin(mx))\cos(mx)$
$= -m\sin^2(mx) - m\cos^2(mx)$
$= -m(\sin^2(mx) + \cos^2(mx))$
$= -m$

Part (iii) $W(e^{px}\sin qx, e^{px}\cos qx)$
$= e^{px}\sin qx(pe^{px}\cos qx - qe^{px}\sin qx)$
$-e^{px}\cos qx(pe^{px}\sin qx + qe^{px}\cos qx)$
$= -qe^{2px}(\sin^2 qx + \cos^2 qx)$
$= -qe^{2px}.$

Part (iii) $W(x^n, x^m) = x^n \cdot mx^{m-1} - x^m \cdot nx^{n-1}$
$= (m-n)x^{n+m-1}.$

Part (iv) $W(x\sin mx, x\cos mx) = (x\sin mx)(\cos mx - mx\sin mx)$
$-(x\cos mx)(\sin mx + mx\cos mx)$
$= -mx^2(\sin^2 mx + \cos^2 mx)$
$= -mx^2.$

Definition 4.4.3 Two differentiable functions f and g are said to be *linearly independent* if their Wronskian, $W(f(x), g(x))$, is not zero for all x in the domains of both f and g.

Example 4.4.2 Which pairs of functions in Example 8 are *linearly independent*?

(i) In Part (i), $W(\sin mx, \cos mx) = -m \neq 0$ unless $m = 0$. Therefore, $\sin mx$ and $\cos mx$ are linearly independent if $m \neq 0$.

(ii) In Part (ii),

$$W(e^{px}\sin qx, e^{px}\cos qx) = -qe^{2px} \not\equiv \text{ if } q \neq 0.$$

Therefore, $e^{px}\sin(qx)$ and $e^{px}\cos qx$ are linearly independent if $q \neq 0$.

(iii) In Part (iii), $W(x^n, x^m) = (m-n)x^{n+m-1} \not\equiv 0$ if $m \neq n$. Therefore, if m and n are not equal, then x^n and x^m are linearly independent.

(iv) In Part (iv),
$$W(x \sin mx, x \cos mx) = -mx^2 \not\equiv 0$$
if $m \neq 0$. Therefore, $x \sin mx$ and $x \cos mx$ are linearly independent if $m \neq 0$.

Theorem 4.4.1 *Consider the linear homogeneous second order differential equation*
$$y'' + p(x)y' + q(x)y = 0. \tag{1}$$

(i) If $y_1(x)$ and $y_2(x)$ are any two solutions of (1), then every linear combination $y(x)$, with constants A and B,
$$y(x) = Ay_1(x) + By_2(x)$$
is also a solution of (1).

(ii) If $y_1(x)$ and $y_2(x)$ are any two linearly independent *solutions of (1), then every solution $y(x)$ of (1) has the form*
$$y(x) = Ay_1(x) + By_2(x)$$
for some constants A and B.

Proof.
Part (i) Suppose that y_1 and y_2 are solutions of (1), A and B are any constants. Then

$$\begin{aligned}
&(Ay_1 + By_2)'' + p(Ay_1 + By_2)' + q(Ay_1 + By_2) \\
&\quad = Ay_1'' + By_2'' + Apy_1' + ABy_2' + Aqy_1 + BqBy_2 \\
&\quad = A(y_1'' + py_1' + qy_1) + B(y_2'' + py_2' + qy_2) \\
&\quad = A(0) + B(0) \qquad \text{(Because } y_1 \text{ and } y_2 \text{ are solutions of (1))} \\
&\quad = 0.
\end{aligned}$$

Hence, $y = Ay_1 + By_2$ are solutions of (1) whenever y_1 and y_2 are solutions of (1).

Part (ii) Let y be any solution of (1) and suppose that

$$y = Ay_1 + By_2 \quad (2)$$
$$y' = Ay_1' + By_2' \quad (3)$$

We solve for A and B from equations (2) and (3) to get

$$A = \frac{yy_2' - y_2y'}{y_1y_2' - y_2y_1'} = \frac{W(y, y_2)}{W(y_1, y_2)}$$
$$B = \frac{y_1y' - y_1'y}{y_1y_2' - y_2y_1'} = \frac{W(y_1, y)}{(y_1, y_2)}\ .$$

Since y_1 and y_2 are linearly independent, $W(y_1, y_2) \neq 0$, and hence, A and B are uniquely determined.

Remark 15 It turns out that the Wronskian of two solutions of (1) is either identically zero or never zero for any value of x.

Theorem 4.4.2 *Let y_1 and y_2 be any two solutions of the homogeneous equation*

$$y'' + py' + qy = 0. \quad (1)$$

Let

$$W(x) = W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Then

$$W'(x) = -pW(x)$$
$$W(x) = ce^{-\int p(x)dx}$$

for some constant c. If $c = 0$, then $W(x) = 0$ for every x. If $c \neq 0$, then $W(x) \neq 0$ for every x.

Proof. Since y_1 and y_2 are solutions of (1),

$$y_1'' = -py_1' - qy_1 \quad (2)$$
$$y_2'' = -py_2' - qy_2 \quad (3)$$

Then,

$$\begin{aligned}
W'(x) &= (y_1y_2' - y_1'y_2)' \\
&= y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' \\
&= y_1y_2'' - y_2y_1'' \\
&= y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1) \qquad \text{(from (2) and (3))} \\
&= -p[y_1y_2' - y_2y_1'] \\
&= -pW(x).
\end{aligned}$$

Thus,

$$W'(x) + pW(x) = 0.$$

By Theorem 4.3.1

$$W(x) = e^{-\int p dx}\left[\int 0 \; dx + c\right] = ce^{-\int p dx}.$$

If $c = 0$, $W(x) \equiv 0$; otherwise $W(x)$ is never zero.

Theorem 4.4.3 (Homogeneous Second Order) *Consider the linear second order homogeneous differential equation with constant coefficients:*

$$ay'' + by' + cy = 0, \quad a \neq 0. \tag{1}$$

(i) If $y = e^{mx}$ is a solution of (1), then

$$am^2 + bm + c = 0. \tag{2}$$

Equation (2) is called the characteristic equation of (1).

(ii) Let $m_1 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $m_2 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$. Then the following three cases arise:

Case 1. *The discriminant $b^2 - 4ac > 0$. Then m_1 and m_2 are real and distinct. The two linearly independent solutions of (1) are e^{m_1x} and e^{m_2x} and its general solution has the form*

$$y(x) = Ae^{m_1x} + Be^{m_2x}.$$

Case 2. *The discriminant* $b^2 - 4ac = 0$. *Then* $m_1 = m_2 = m$, *and only one real solution exists for equation (2). The roots are repeated. In this case,* e^{mx} *and* xe^{mx} *are two linearly independent solutions of (1) and the general solution of (1) has the form*

$$y(x) = Ae^{mx} + Bxe^{mx} = e^{mx}(A + Bx).$$

Case 3 $b^2 - 4ac < 0$. *Then* $m_1 = p - iq$, *and* $m_2 = p + iq$ *where* $p = -b/2a$, *and* $q = \sqrt{4ac - b^2}/2a$. *In this case, the functions* $e^{px}\sin qx$ *and* $e^{px}\cos qx$ *are two linearly independent solutions of (1) and the most general solution of (1) has the form*

$$y(x) = e^{px}(A\sin qx + B\sin qx).$$

Proof. Let $y = e^{mx}$. Then $y' = me^{mx}, y'' = m^2e^{mx}$ and

$$\begin{aligned} ay'' + by' + cy &= (am^2 + bm + c)e^{mx} = 0, a \neq 0 \leftrightarrow \\ am^2 + bm + c &= 0, a \neq 0 \leftrightarrow \\ m &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

This proves Part (i).

Case 1. For Case 1, e^{m_1x} and e^{m_2x} are solutions of (1). We show that these are linearly independent by showing that their Wronskian is not zero.

$$\begin{aligned} W(e^{m_1x}, e^{m_2x}) &= e^{m_1x} \cdot m_2e^{m_2x} - m_1e^{m_1x} \cdot e^{m_2x} \\ &= (m_2 - m_1)e^{(m_1+m_2)x}. \end{aligned}$$

Since $m_1 \neq m_2, W(e^{m_1x}, e^{m_2x}) \neq 0$.

Case 2. We already know that $e^{mx} \neq 0$, and $m = -b/2a$. Let us try $y = xe^{mx}$. Then

$$\begin{aligned} ay'' + by' + cy &= a(2m + m^2x)e^{mx} + b(1 + mx)e^{mx} + xe^{mx} \\ &= (b + 2am)e^{mx} + (am^2 + bm + c)xe^{mx} \\ &= (b + 2a(-b/2a))e^{mx} \\ &= 0. \end{aligned}$$

Therefore, e^{mx} and xe^{mx} are both solutions. We only need to show that they are linearly independent.

$$\begin{aligned} W(e^{mx}, xe^{mx}) &= e^{mx}(e^{mx} + mxe^{mx}) - me^{mx}(xe^{mx}) \\ &= e^{2mx} + mxe^{2mx} - mxe^{2mx} \\ &= e^{2mx} \\ &\neq 0. \end{aligned}$$

Hence, e^{mx} and xe^{mx} are linearly independent and the general solution of (1) has the form

$$y(x) = Ae^{mx} + Bxe^{mx} = e^{mx}(A + Bx).$$

Case 3. In Example 8, we showed that

$$W(e^{px}\sin qx, e^{px}\cos qx) = -qe^{2px} \neq 0$$

since $q \neq 0$.

We only need to show that $e^{px}\sin qx$ and $e^{px}\cos qx$ are solutions of (1).

Let $y_1 = e^{px}\sin qx$ and $y_2 = e^{px}\cos qx$. Then,

$$\begin{aligned} y_1' &= pe^{px}\sin qx + qe^{px}\cos qx \\ y_1'' &= p^2e^{px}\sin qx + pqe^{px}\cos qx + pqe^{px}\cos qx - q^2\sin qx \end{aligned}$$

$$\begin{aligned} ay_1'' + by_1' + cy_1 &= ae^{px}(p^2\sin qx + 2pq\cos qx - q^2\sin qx) \\ &\quad + be^{px}(p\sin qx + q\cos qx) + ce^{px}\sin qx \\ &= e^{px}\sin qx[a(p^2 - q^2) + (bp + c)] + e^{px}\cos qx[2apq + bq] \\ &= e^{px}\sin(qx)\left[a\left(\frac{b^2}{4a^2} - \frac{4ac - b^2}{4a^2}\right) + b\left(\frac{-b}{2a}\right) + c\right] \\ &\quad + e^{px}\cos qx\left[\left\{2a\left(\frac{-b}{2a}\right) + b\right\}\sqrt{4ac - b^2}\right] \\ &= e^{px}\sin(qx)\left[\frac{b^2 - 2ac + b^2 + 2ac}{2a}\right] \\ &\quad + e^{px}\cos(qx)[0] \\ &= 0. \end{aligned}$$

Therefore $y_1 = e^{px}\sin(qx)$ is a solution of (1). Similarly, we can show that $y_2 = e^{px}\cos qx$ is a solution of (1). We leave this as an exercise.

Remark 16 Use Integral Tables or computer algebra in evaluating the indefinite integrals as needed.

Example 4.4.3 Solve the differential equations for $y(t)$.

(i) $y'' - 5y' + 14y = 0$

(ii) $y'' - 6y' + 9y = 0$

(iii) $y'' - 4y' + 5y = 0$

We let $y = e^{mt}$. We then solve for m and determine the solution. We observe that $y' = me^{mt}, y'' = m^2e^{mt}$.

Part (i) By substituting $y = e^{mt}$ in the equation we get

$$\begin{aligned}&m^2e^{mt} - 5me^{mt} + 14e^{mt} = e^{mt}(m^2 - 5m + 14) = 0 \to\\&m^2 - 5m + 14 = 0 = (m-7)(m+2) \to m = 7, -2.\end{aligned}$$

Therefore,

$$y(t) = Ae^{-2t} + Be^{7t}.$$

Part (ii) Again, by substituting $y = e^{mt}$, we get

$$\begin{aligned}m^2e^{mt} - 6me^{mt} + 9e^{mt} &= 0\\m^2 - 6m + 9 = (m-3)^2 &= 0\\m &= 3, 3.\end{aligned}$$

The solution is

$$y(t) = Ae^{3t} + Bte^{3t}.$$

Part (iii) By substituting $y = e^{mt}$, we get

$$\begin{aligned}m^2e^{mt} - 4me^{mt} + 5e^{mt} &= e^{mt}(m^2 - 4m + 5) = 0 \to\\m^2 - 4m + 5 &= 0\\m &= \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm 1i.\end{aligned}$$

The general solution is

$$y(t) = e^{2t}(A\cos t + B\sin t).$$

Example 4.4.4 Solve the differential equation for $y(t)$ where $y(t)$ satisfies the conditions

$$y''(t) - 2y'(t) - 15y(t) = 0; \ y(0) = 1, \ y'(0) = -1.$$

We assume that $y(t) = e^{mt}$. By substitution we get the characteristic equation:

$$m^2 - 2m - 15 = 0, \ m = 5, -3.$$

The general solution is

$$y(t) = Ae^{-3t} + Be^{5t}.$$

We now impose the additional conditions $y(0) = 1, y'(0) = -1$.

$$\begin{aligned} y(t) &= Ae^{-3t} + Be^{5t} \\ y'(t) &= -3Ae^{-3t} + 5Be^{5t} \\ y(0) &= A + B = 1 \\ y'(0) &= -3A + 5B = -1 \end{aligned}$$

On solving these two equations simultaneously, we get

$$A = \frac{3}{4}, \ B = \frac{1}{4} \ .$$

Then the exact solution is

$$y(t) = \frac{3}{4} \ e^{-3t} + \frac{1}{4} \ e^{5t}.$$

Exercises 4.4 Solve for $y(t)$ from each of the following:

1. $y'' - y' - 20y = 0$

2. $y'' - 8y' + 16y = 0$

3. $y'' + 9y' + 20y = 0$

4. $y'' + 4y' + 4 = 0$

5. $y'' - 8y' + 12y = 0$

6. $y'' - 6y' + 10y = 0$

7. $y' - y' - 6y = 0,\ y(0) = 10,\ y'(0) = 15$

8. $y'' - 4y' + 4y - 0,\ y(0) = 4,\ y'(0) = 8$

9. $y'' + 8y' + 12y = 0,\ y(0) = 1,\ y'(0) = 3$

10. $y'' + 6y' + 10y = 0,\ y(0) = 5,\ y'(0) = 7$

11. $y'' - 4y = 0,\ y(0) = 1,\ y'(0) = -1$

12. $y'' - 9y = 0,\ y(0) = -1,\ y'(0) = 1$

13. $y'' + 9y = 0,\ y(0) = 2,\ y'(0) = 3$

14. $y'' + 4y = 0,\ y(0) = -1,\ y'(0) = 2$

15. $y'' - 3y' + 2y = 0,\ y(0) = 2,\ y'(0) = -2$

16. $y'' - y' - 6y = 0,\ y(0) = 6, y'(0) = 5$

17. $y'' + 4y' + 4y = 0,\ y(0) = 1,\ y'(0) = 4$

18. $y'' - 6y' + 9y = 0,\ y(0) = 1,\ y'(0) = -1$

19. $y'' + 6y' + 13y = 0,\ y(0) = 1, y'(0) = 2$

20. $y'' - 3y' + 2y = 0$

21. $y'' + 3y' + 2y = 0$

22. $y'' + m^2y = 0$

23. $y'' - m^2y = 0$

24. $y'' + 2my' + m^2y^2 = 0$

25. $y'' + 2my' + (m^2 + 1)y = 0$

26. $y'' - 2my' + (m^2 + 1)y = 0$

27. $y'' + 2my' + (m^2 - 1)y = 0$

28. $y'' - 2my' + (m^2 - 1)y = 0$

29. $9y'' - 12y' + 4y = 0$

30. $4y'' + 4y' + y = 0$

4.5 Linear Non-Homogeneous Second Order Differential Equations

Theorem 4.5.1 (Variation of Parameters) *Consider the equations*

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$
$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

Suppose that y_1 and y_2 are any two linearly independent solutions of (2). Then the general solution of (1) is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \left[y_2(x)\int \frac{y_1(x)r(x)}{W(y_1,y_2)}\,dx - y_1(x)\int \frac{y_2(x)r(x)}{W(y_1,y_2)}\,dx\right].$$

Proof. It is already shown that $c_1y_1(x)+c_2y_2(x)$ is the most general solution of the homogeneous equation (2), where c_1 and c_2 are arbitrary constants. We observe that the difference of any two solutions of (1) is a solution of (2).

Suppose that $y^*(x)$ is any solution of (1). We wish to find two functions, u_1 and u_2, such that

$$y^*(x) = u_1(x)y_1(x) + u_2(x)y_2(x). \tag{3}$$

By differentiation of (3), we get

$$y^{*\prime}(x) = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2'). \tag{4}$$

We impose the following condition (5) on u_1 and u_2:

$$u_1'y_1 + u_2'y_2 = 0. \tag{5}$$

Then

$$\begin{aligned} y^{*\prime}(x) &= u_1y_1' + u_2y_2' \\ y^{*\prime\prime}(x) &= (u_1y_1'' + u_2y_2'') + u_1'y_1' + u_2'y_2'. \end{aligned}$$

Since $y^*(x)$ is a solution of (1), we get

$$\begin{aligned} r(x) &= y^{*\prime\prime} + p(x)y^{*\prime} + q(x)y^* \\ &= (u_1y_1'' + u_2y_2'') + (u_1'y_1' + u_2'y_2') + p(x)[u_1y_1' + u_2y_2'] \\ &\quad + q(x)(u_1y_1 + u_2y_2) \\ &= u_1[y_1'' + p(x)y_1' + q(x)y_1] + u_2[y_2'' + p(x)y_2' + q(x)y_2] \\ &\quad + (u_1'y_1 + u_2'y_2') \\ &= u_1'y_1' + u_2'y_2'. \end{aligned}$$

Hence, another condition on u_1 and u_2 is

$$u_1'y_1' + u_2'y_2' = r(x). \tag{6}$$

By solving equations (5) and (6) simultaneously for u_1' and u_2', we get

$$u_1' = \frac{-y_2 r(x)}{y_1 y_2' - y_2 y_1'} \quad \text{and} \quad u_2'(x) = \frac{y_1 r(x)}{y_1 y_2' - y_2 y_1'} \ . \tag{7}$$

The denominator of the solution (7) is the Wronskian of y_1 and y_2, which is not zero for any x since y_1 and y_2 are linearly independent by assumption. By taking the indefinite integrals in equation (7), we obtain u_1 and u_2.

$$u_1(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)}\, dx \quad \text{and} \quad u_2(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}\, dx.$$

By substituting these values in (3), we get a particular solution

$$\begin{aligned} y^*(x) &= y_2 u_2 + y_1 u_1 \\ &= y_2(x) \int \frac{y_1(x)r(x)}{W(y_1, y_2)}\, dx - y_1(x) \int \frac{y_2(x)r(x)}{W(y_1, y_2)}\, dx. \end{aligned}$$

This solution $y^*(x)$ is called a particular solution of (1). To get the general solution of (1), we add the general solution $c_1 y_1(x) + c_2\, y_2(x)$ of (2) to the particular solution of $y^*(x)$ and get

$$\boxed{y(x) = (c_1 y_1(x) + c_2 y_2(x)) + \left\{ y_2(x) \int \frac{y_1(x)r(x)}{W(y_1, y_2)}\, dx - y_1(x) \int \frac{y_2(x)r(x)}{W(y_1, y_2)}\, dx \right\}.}$$

This completes the proof of this theorem.

Remark 17 The general solution of (2) is called the complementary solution of (1) and is denoted $y_c(x)$.

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x).$$

The particular solution y^* of (1) is generally written as y_p.

$$y_p = y_2 \int \frac{r(x)y_1(x)}{W(y_1, y_2)}\, dx - y_1 \int \frac{r(x)y_2(x)}{W(y_1, y_2)}\, dx.$$

The general solution $y(x)$ of (1) is the sum of y_c and y_p,

$$y = y_c(x) + y_p(x).$$

Example 4.5.1 Solve the differential equation

$$y'' + 8y' + 12y = e^{-3x}.$$

We find the general solution of the homogeneous equation

$$y'' + 8y' + 12y = 0.$$

We let $y = e^{mx}$ be a solution. Then $y' = me^{mx}, y'' = m^2 e^{mx}$ and

$$\begin{aligned} m^2 e^{mx} + 8me^{mx} + 12e^{mx} &= 0 \\ m^2 + 8m + 12 = 0 \quad m &= -6, -2. \end{aligned}$$

So,

$$y_c(x) = Ae^{-6x} + Be^{-2x}$$

is the complementary solution. We compute the Wronskian

$$\begin{aligned} W(e^{-6x}, e^{-2x}) &= e^{-6x}(-2)e^{-2x} - e^{-2x}(-6)e^{-6x} \\ &= e^{-8x}(-2+6) \\ &= 4e^{-8x} \\ &\neq 0. \end{aligned}$$

By Theorem 4.4.1, the particular solution is given by

$$\begin{aligned} y_p &= e^{-2x} \int \frac{e^{-6x} \cdot e^{-3x}}{4e^{-8x}}\, dx - e^{-6x} \int \frac{e^{-2x} \cdot e^{-3x}}{4e^{-8x}}\, dx \\ &= e^{-2x} \int \frac{1}{4}\, e^{-x} dx - e^{-6x} \int \frac{1}{4}\, e^{3x} dx \\ &= -e^{-2x} \left(\frac{1}{4}\, e^{-x} \right) - e^{-6x} \left(\frac{1}{4}\, e^{3x} \right) \\ &= -\frac{1}{4}\, e^{-3x} - \frac{1}{12}\, e^{-3x} \\ &= -\frac{1}{3}\, e^{-3x}. \end{aligned}$$

The complete solution is the sum of the complementary solution y_c and the particular solution y_p.

$$y(t) = Ae^{-6x} + Be^{-2x} - \frac{1}{3}\, e^{-3x}.$$

Exercises 4.5 Find the complementary, particular and the complete solution for each of the following. Use tables of integrals or computer algebra to do the integrations, if necessary.

1. $y'' + 4y = \sin(3x)$
2. $y'' - 9y = e^{2x}$
3. $y'' + 9y = \cos 2x$
4. $y'' - 4y = e^{-x}$
5. $y'' - y = xe^{x}$
6. $y'' - 5y' + 6y = 3e^{4x}$
7. $y'' - 4y' + 4y = e^{-x}$
8. $y'' + 5y' + 4y = 2e^{x}$
9. $my'' - py' = mg$
10. $y'' + 5y' + 6y = x^2e^{2x}$

In exercises 11–20, compute the complete solution for y.

11. $y'' + y = 4x,\ y(0) = 2, y'(0) = 1$
12. $y'' - 9y = e^{x},\ y(0) = 1,\ y'(0) = 5$
13. $y'' - 2y' - 3y = 4,\ y(0) = 2,\ y'(0) = -1$
14. $y'' - 3y' + 2y = 4x$
15. $y'' + 4y = \sin 2x$
16. $y'' - 4y = e^{2x}$
17. $y'' - 4y = e^{-2x}$
18. $y'' + 4y = \cos 2x$
19. $y'' + 9y = 2\sin 3x + 4\cos 3x$
20. $y'' + 4y' + 5y = \sin x - 2\cos x$

Chapter 5

The Definite Integral

5.1 Area Approximation

In Chapter 4, we have seen the role played by the indefinite integral in finding antiderivatives and in solving first order and second order differential equations. The definite integral is very closely related to the indefinite integral. We begin the discussion with finding areas under the graphs of positive functions.

Example 5.1.1 Find the area bounded by the graph of the function $y = 4,\ y = 0,\ x = 0,\ x = 3$.

graph

From geometry, we know that the area is the height 4 times the width 3 of the rectangle.

Area = 12.

Example 5.1.2 Find the area bounded by the graphs of $y = 4x,\ y = 0,\ x = 0,\ x = 3$.

graph

From geometry, the area of the triangle is $\frac{1}{2}$ times the base, 3, times the height, 12.

Area = 18.

Example 5.1.3 Find the area bounded by the graphs of $y = 2x, \ y = 0, \ x = 1, \ x = 4$.

graph

The required area is covered by a trapezoid. The area of a trapezoid is $\frac{1}{2}$ times the sum of the parallel sides times the distance between the parallel sides.

Area $= \frac{1}{2}\,(2 + 8)(3) = 15$.

Example 5.1.4 Find the area bounded by the curves $y = \sqrt{4 - x^2}, \ y = 0, \ x = -2, \ x = 2$.

graph

By inspection, we recognize that this is the area bounded by the upper half of the circle with center at $(0, 0)$ and radius 2. Its equation is

$$x^2 + y^2 = 4 \quad \text{or} \quad y = \sqrt{4 - x^2}, \ -2 \le x \le 2.$$

Again from geometry, we know that the area of a circle with radius 2 is $\pi r^2 = 4\pi$. The upper half of the circle will have one half of the total area. Therefore, the required area is 2π.

Example 5.1.5 Approximate the area bounded by $y = x^2$, $y = 0$, $x = 0$, and $x = 3$. Given that the exact area is 9, compute the error of your approximation.

Method 1. We divide the interval $[0, 3]$ into six equal subdivisions at the points $0, \frac{1}{2}, 1, \frac{3}{2}, \frac{5}{2}$ and 2. Such a subdivision is called a *partition* of $[0, 3]$. We draw vertical segments joining these points of division to the curve. On each subinterval $[x_1, x_2]$, the minimum value of the function x^2 is at x_1^2. The maximum value x_2^2 of the function is at the right hand end point x_2. Therefore,

graph

The lower approximation, denoted L, is given by

$$\begin{aligned}
L &= 0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2} + \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} + (2)^2 \cdot \frac{1}{2} + \left(\frac{5}{2}\right)^2 \cdot \frac{1}{2} \\
&= \frac{1}{2} \cdot \left[0 + 1 + \frac{9}{4} + 4 + \frac{25}{4}\right] \\
&= \frac{27}{4} \approx 8 \cdot 75.
\end{aligned}$$

This approximation is called the *left-hand* approximation of the area. The error of approximation is -0.25.

The Upper approximation, denoted U, is given by

$$\begin{aligned}
U &= \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2} + \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} + (2)^2 \cdot \frac{1}{2} + \left(\frac{5}{2}\right)^2 \cdot \frac{1}{2} + (3)^2 \cdot \frac{1}{2} \\
&= \frac{1}{2}\left[\frac{1}{4} + 1 + \frac{9}{4} + 4 + \frac{25}{4} + 9\right] \\
&= \frac{1}{2}\left[\frac{91}{4}\right] \\
&= \frac{91}{8} \approx 11 \cdot 38.
\end{aligned}$$

The error of approximation is +2.28.
This approximation is called the *right-hand* approximation.

Method 2. (Trapezoidal Rule) In this method, for each subinterval $[x_1, x_2]$, we join the point (x_1, x_1^2) with the point (x_2, x_2^2) by a straight line and find the area under this line to be a trapezoid with area $\frac{1}{2}\,(x_2 - x_1)(x_1^2 + x_2^2)$. We add up these areas as the Trapezoidal Rule approximation, T, that is given by

$$\begin{aligned} T &= \frac{1}{2}\left(\frac{1}{2}-0\right)\left(0^2+\left(\frac{1}{2}\right)^2\right)+\frac{1}{2}\left(1-\frac{1}{2}\right)\left(1^2+\left(\frac{1}{2}\right)^2\right) \\ &\quad +\frac{1}{2}\left(\frac{3}{2}-1\right)\left(\left(\frac{3}{2}\right)^2+1^2\right)+\frac{1}{2}\left(2-\frac{3}{2}\right)\left(2^2+\left(\frac{3}{2}\right)^2\right) \\ &\quad +\frac{1}{2}\left(\frac{5}{2}-2\right)\left(\left(\frac{5}{2}\right)^2+2^2\right)+\frac{1}{2}\left(3-\frac{5}{2}\right)\left(3^2+\left(\frac{5}{2}\right)^2\right) \\ &= \frac{1}{4}\left[0^2+2\cdot\left(\frac{1}{2}\right)^2+2(1^2)+2\cdot\left(\frac{3}{2}\right)^2+2(2)^2+2\cdot\left(\frac{5}{2}\right)^2+3^2\right] \\ &= \frac{1}{4}\left[1+2+\frac{9}{2}+8+\frac{25}{2}+9\right] \\ &= \frac{37}{4} = 9\cdot 25. \end{aligned}$$

The error of this Trapezoidal approximation is +0.25.

Method 3. (Simpson's Rule) In this case we take two intervals, say $[x_1, x_2] \cup [x_2, x_3]$, and approximate the area over this interval by

$$\frac{1}{6}\,[f(x_1) + 4f(x_2) + f(x_3)] \cdot (x_3 - x_1)$$

and then add them up. In our case, let $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$, $x_4 = 2$, $x_5 = \frac{5}{2}$ and $x_6 = 3$. Then the Simpson's rule approximation, S, is given by

$$
\begin{aligned}
S &= \frac{1}{6}\left[0^2 + 4\cdot\left(\frac{1}{2}\right)^2 + (1)^2\right]\cdot(1) + \frac{1}{6}\left[(1)^2 + 4\cdot\left(\frac{3}{2}\right)^2 + 2^2\right](1) \\
&\quad + \frac{1}{6}\left[2^2 + 4\cdot\left(\frac{5}{2}\right)^2 + 3^2\right]\cdot(1) \\
&= \frac{1}{6}\left[0^2 + 4\left(\frac{1}{2}\right)^2 + 2\cdot 1^2 + 4\cdot\left(\frac{3}{2}\right)^2 + 2\cdot 2^2 + 4\cdot\left(\frac{5}{2}\right)^2 + 3^2\right] \\
&= \frac{54}{6} = 9 = \text{ Exact Value!}
\end{aligned}
$$

For positive functions, $y = f(x)$, defined over a closed and bounded interval $[a, b]$, we define the following methods for approximating the area A, bounded by the curves $y = f(x),\ y = 0,\ x = a$ and $x = b$. We begin with a common equally-spaced partition,

$$P = \{a = x_0 < x_1 < x_2 < x_3 < \ldots < x_n = b\},$$

such that $x_i = a + \dfrac{b-a}{n}\, i$, for $i = 0, 1, 2, \ldots, n$.

Definition 5.1.1 (Left-hand Rule) The left-hand rule approximation for A, denoted L, is defined by

$$L = \frac{b-a}{n}\cdot[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})].$$

Definition 5.1.2 (Right-hand Rule) The right-hand rule approximation for A, denoted R, is defined by

$$R = \frac{b-a}{n}\cdot[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)].$$

Definition 5.1.3 (Mid-point Rule) The mid-point rule approximation for A, denoted M, is defined by

$$M = \frac{b-a}{n}\left[f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right)\right].$$

Definition 5.1.4 (Trapezoidal Rule) The trapezoidal rule approximation for A, denoted T, is defined by

$$\begin{aligned} T &= \frac{b-a}{n}\left[\frac{1}{2}\left(f(x_0)+f(x_1)\right)+\frac{1}{2}\left(f(x_1)+f(x_2)\right)+\cdots+\frac{1}{2}\left(f(x_{n-1})+f(x_n)\right)\right] \\ &= \frac{b-a}{n}\left[\frac{1}{2}\,f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\frac{1}{2}f(x_n)\right]. \end{aligned}$$

Definition 5.1.5 (Simpson's Rule) The Simpson's rule approximation for A, denoted S, is defined by

$$\begin{aligned} S = \frac{b-a}{n}&\left[\frac{1}{6}\left\{f(x_0)+4\ f\left(\frac{x_0+x_1}{2}\right)+f(x_1)\right\}\right. \\ &+\frac{1}{6}\left\{f(x_1)+4\ f\left(\frac{x_1+x_2}{2}\right)+f(x_2)\right\} \\ &\left.+\cdots+\frac{1}{6}\left\{f(x_{n-1}+4\ f\left(\frac{x_{n-1}+x_n}{2}\right)+f(x_n)\right\}\right] \\ = \left(\frac{b-a}{n}\right)\cdot\frac{1}{6}\cdot&\left[f(x_0)+4\ f\left(\frac{x_0+x_1}{2}\right)+2\ f(x_1)+4\ f\left(\frac{x_1+x_2}{2}\right)\right. \\ &\left.+\cdots 2\ f(x_{n-1})+4\ f\left(\frac{x_{n-1}+x_n}{2}\right)+f(x_n)\right]. \end{aligned}$$

Examples

Exercises 5.1

1. The sum of n terms $a_1, a_2, \cdots, a_n$ is written in compact form in the so called sigma notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The variable k is called the index, the number 1 is called the lower limit and the number n is called the upper limit. The symbol $\displaystyle\sum_{k=1}^{n} a_k$ is read "the sum of a_k from $k = 1$ to $k = n$."

Verify the following sums for $n = 5$:

(a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

(d) $\displaystyle\sum_{k=1}^{n} 2^r = 2^{n+1} = 1$

2. Prove the following statements by using mathematical induction:

(a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

(d) $\displaystyle\sum_{k=1}^{n} 2^r = 2^{n+1} - 1$

3. Prove the following statements:

(a) $\displaystyle\sum_{k=1}^{n}(c\ a_k) = c\ \sum_{k=1}^{n} a_k$

(b) $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(c) $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(d) $\displaystyle\sum_{k=1}^{n}(a\ a_k + b\ b_k) = a\sum_{k=1}^{n} a_k + b\sum_{k=1}^{n} b_k$

4. Evaluate the following sums:

(a) $\displaystyle\sum_{i=0}^{6}(2i)$

(b) $\displaystyle\sum_{j=1}^{5}\left(\frac{1}{j}\right)$

(c) $\displaystyle\sum_{k=0}^{4}(1+(-1)^k)^2$

(d) $\displaystyle\sum_{m=2}^{5}(3m-2)$

5. Let $P=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$ be a partition of $[a,b]$ such that $x_k=a+\left(\dfrac{b-a}{n}\right)k,\ k=0,1,2,\cdots,n$. Let $f(x)=x^2$. Let A denote the area bounded by $y=f(x),\ y=0,\ x=0$ and $x=2$. Show that

(a) Left-hand Rule approximation of A is $\displaystyle\frac{2}{n}\sum_{k=1}^{n-1}x_{k-1}^2$.

(b) Right-hand Rule approximation of A is $\displaystyle\frac{2}{n}\sum_{k=1}^{n-1}x_k^2$.

(c) Mid-point Rule approximation of A is $\displaystyle\frac{2}{n}\sum_{k=1}^{n}\left(\frac{x_{k-1}+x_k}{2}\right)^2$.

(d) Trapezoidal Rule approximation of A is $\displaystyle\frac{2}{n}\left\{2+\sum_{k=1}^{n-1}x_k^2\right\}$.

(e) Simpson's Rule approximation of A

$$\frac{1}{3n}\left\{4+4\sum_{k=1}^{n}\left(\frac{x_{k-1}+x_k}{2}\right)^2+2\sum_{k=1}^{n-1}x_k^2\right\}.$$

In problems 6–20, use the function f, numbers a, b and n, and compute the approximations LH, RH, MP, T, S for the area bounded by $y = f(x),\ y = 0,\ x = a,\ x = b$ using the partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}, \text{ where } x_k = a + k\left(\frac{b-a}{n}\right), \text{ and}$$

(a) $LH = \dfrac{b-a}{n}\displaystyle\sum_{k=1}^{n} f(x_{k-1})$

(b) $RH = \dfrac{b-a}{n}\displaystyle\sum_{k=1}^{n} f(x_k)$

(c) $MP = \dfrac{b-a}{n}\displaystyle\sum_{k=1}^{n} f\left(\frac{x_{n-1}+x_k}{2}\right)$

(d) $T = \dfrac{b-a}{n}\left\{\displaystyle\sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}(f(x_0)+f(x_n))\right\}$

(e) $S = \dfrac{b-a}{6n}\left\{(f(x_0)+f(x_n)) + 2\displaystyle\sum_{k=1}^{n-1} f(x_n) + 4\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\right\}$
$= \dfrac{1}{6}\{LH + 4MP + RH\}$

6. $f(x) = 2x,\ a = 0,\ b = 2,\ n = 6$

7. $f(x) = \dfrac{1}{x},\ a = 1,\ b = 3,\ n = 6$

8. $f(x) = x^2,\ a = 0,\ b = 3,\ n = 6$

9. $f(x) = x^3,\ a = 0,\ b = 2,\ n = 4$

10. $f(x) = \dfrac{1}{1+x}, a = 0,\ b = 3,\ n = 6$

11. $f(x) = \dfrac{1}{1+x^2},\ a = 0,\ b = 1,\ n = 4$

12. $f(x) = \dfrac{1}{\sqrt{4-x^2}},\ a = 0,\ b = 1,\ n = 4$

13. $f(x) = \dfrac{1}{4-x^2},\ a=0,\ b=1,\ n=4$

14. $f(x) = \dfrac{1}{4+x^2},\ a=0,\ b=2,\ n=4$

15. $f(x) = \dfrac{1}{\sqrt{4+x^2}},\ a=0,\ b=2,\ n=4$

16. $f(x) = \sqrt{4+x^2},\ a=0,\ b=2,\ n=4$

17. $f(x) = \sqrt{4-x^2},\ a=0,\ b=2,\ n=4$

18. $f(x) = \sin x,\ a=0,\ b=\pi,\ n=4$

19. $f(x) = \cos x,\ a=-\dfrac{\pi}{2},\ b=\dfrac{\pi}{2}, n=4$

20. $f(x) = \sin^2 x,\ a=0,\ b=\pi,\ n=4$

5.2 The Definite Integral

Let f be a function that is continuous on a bounded and closed interval $[a,b]$. Let $p = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ be a partition of $[a,b]$, not necessarily equally spaced. Let

$$\begin{aligned} m_i &= \min\{f(x) : x_{i-1} \leq x \leq x_i\},\ i=1,2,\ldots,n; \\ M_i &= \max\{f(x) : x_{i-1} \leq x \leq x_i\},\ i=1,2,\ldots,n; \\ \Delta x_i &= x_i - x_{i-1},\ i=1,2,\ldots,n; \\ \Delta &= \max\{\Delta x_i : i=1,2,\ldots,n\}; \end{aligned}$$

$L(p) = m_1\Delta x_1 + m_2\Delta x_2 + \ldots + m_n\Delta x_n$
$U(p) = M_1\Delta x_i + M_2\Delta x_2 + \ldots + M_n\Delta x_n.$

We call $L(p)$ the lower Riemann sum. We call $U(p)$ the upper Riemann sum. Clearly $L(p) \leq U(p)$, for every partition. Let

$$\begin{aligned} L_f &= \text{lub}\{L(p) : p \text{ is a partition of } [a,b]\} \\ U_f &= \text{glb}\{U(p) : p \text{ is a partition of } [a,b]\}. \end{aligned}$$

Definition 5.2.1 If f is continuous on $[a, b]$ and $L_f = U_f = I$, then we say that:

(i) f is integrable on $[a, b]$;

(ii) the definite integral of $f(x)$ from $x = a$ to $x = b$ is I;

(iii) I is expressed, in symbols, by the equation

$$I = \int_a^b f(x)dx;$$

(iv) the symbol "$\int$" is called the "*integral sign*"; the number "a" is called the "*lower limit*"; the number "b" is called the "*upper limit*"; the function "$f(x)$" is called the "*integrand*"; and the variable "x" is called the (dummy) "*variable of integration*."

(v) If $f(x) \geq 0$ for each x in $[a, b]$, then the area, A, bounded by the curves $y = f(x)$, $y = 0$, $x = a$ and $x = b$, is defined to be the definite integral of $f(x)$ from $x = a$ to $x = b$. That is,

$$A = \int_a^b f(x)dx.$$

(vi) For convenience, we define

$$\int_a^a f(x)dx = 0, \quad \int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Theorem 5.2.1 *If a function f is continuous on a closed and bounded interval $[a, b]$, then f is* integrable *on $[a, b]$.*

Proof. See the proof of Theorem 5.6.3.

Theorem 5.2.2 (Linearity) *Suppose that f and g are continuous on $[a, b]$ and c_1 and c_2 are two arbitrary constants. Then* *(i)* $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(ii) $\int_a^b (f(x)-g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

(iii) $\int_a^b c_1 f(x)dx = c_1 \int_a^b f(x)dx, \int_a^b c_2 g(x)dx = c_2 \int_a^b g(x)dx$ *and*

$\int_a^b (c_1 f(x) + c_2 g(x))dx = c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx$

Proof.
Part (i) Since f and g are continuous, $f+g$ is continuous and hence by Theorem 5.2.1 each of the following integrals exist:

$$\int_a^b f(x)dx, \int_a^b g(x)dx, \quad \text{and} \quad \int_a^b (f(x)+g(x))dx.$$

Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$. For each i, there exist number c_1, c_2, c_3, d_1, d_2, and d_3 on $[x_{i-1}, x_i]$ such that

$f(c_1)$ = absolute minimum of f on $[x_{i-1}, x_i]$,
$g(c_2)$ = absolute minimum of f on $[x_{i-1}, x_i]$,
$f(c_3) + g(c_3)$ = absolute minimum of $f+g$ on $[x_{i-1}, x_i]$,
$f(d_1)$ = absolute maximum of f on $[x_{i-1}, x_i]$,
$g(d_2)$ = absolute maximum of g on $[x_{i-1}, x_i]$,
$f(d_3) + g(d_3)$ = absolute maximum of $f+g$ on $[x_{i-1}, x_i]$.

It follows that

$$f(c_1) + g(c_2) \le f(c_3) + g(c_3) \le f(d_3) + g(d_3) \le f(d_1) + g(d_2)$$

Consequently,

$$L_f + L_g \le L_{(f+g)} \le U_{(f+g)} \le U_f + U_g \qquad \text{(Why?)}$$

Since f and g are integrable,

$$L_f = U_f = \int_a^b f(x)dx; \quad L_g = U_g = \int_a^b g(x)dx.$$

By the squeeze principle,

$$L_{(f+g)} = U_{(f+g)} = \int_a^b (f(x)+g(x))dx$$

and

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

This completes the proof of Part (i) of this theorem.

Part (iii) Let k be a positive constant and let F be a function that is continuous on $[a, b]$. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$ be any partition of $[a, b]$. Then for each i there exist numbers c_i and d_i such that $F(c_i)$ is the absolute minimum of F on $[x_{i-1}, x_i]$ and $F(d_i)$ is absolute maximum of F on $[x_{i-1}, x_i]$. Since k is a positive constant,

$kF(c_i) =$ absolute minimum of kF on $[x_{i-1}, x_i]$,

$kF(d_i) =$ absolute maximum of kF on $[x_{i-1}, x_i]$,

$-kF(d_i) =$ absolute minimum of $(-k)F$ on $[x_{i-1}, x_i]$,

$-kF(c_i) =$ absolute maximum of $(-k)F$ on $[x_{i-1}, x_i]$.

Then

$$\begin{aligned}
L(P) &= F(c_1)\Delta x_1 + F(c_2)\Delta x_2 + \cdots + F(c_n)\Delta x_n,\\
U(P) &= F(d_1)\Delta x_1 + F(d_2)\Delta x_2 + \cdots + F(d_n)\Delta x_n,\\
kL(P) &= (kF)(c_1)\Delta x_1 + (kF)(c_2)\Delta x_2 + \cdots + (kF)(c_n)\Delta x_n,\\
kU(P) &= (kF)(d_1)\Delta x_1 + (kF)(d_2)\Delta x_2 + \cdots + (kF)(d_n)\Delta x_n,\\
-kU(P) &= (-kF)(d_1)\Delta x_1 + (-kF)(d_2)\Delta x_2 + \cdots + (-kF)(d_n)\Delta x_n,\\
-kL(P) &= (-kF)(c_1)\Delta x_1 + (-kF)(c_2)\Delta x_2 + \cdots + (-kF)(c_n)\Delta x_n.
\end{aligned}$$

Since F is continuous, kF and $(-k)F$ are both continuous and

$$\begin{aligned}
L_f = U_p &= \int_a^b F(x)dx,\\
L_{(kF)} = U_{(kF)} &= k(L_F) = k(U_F) = k\int_a^b F(x)dx\\
L_{(-kF)} &= (-k)U_F, U_{(-kF)} = -kL_F,
\end{aligned}$$

and hence

$$L_{(-kF)} = U_{(-kF)} = (-k)\int_a^b F(x)dx.$$

Therefore,

$$\int_a^b (c_1 f(x) + c_2 g(x)) = \int_a^b c_1 f(x)dx + \int_a^b c_2 g(x)dx \qquad \text{(Part (i))}$$
$$= c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx \qquad \text{(Why?)}$$

This completes the proof of Part (iii) of this theorem.
Part (ii) is a special case of Part (iii) where $c_1 = 1$ and $c_2 = -1$. This completes the proof of the theorem.

Theorem 5.2.3 (Additivity) *If f is continuous on $[a, b]$ and $a < c < b$, then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Proof. Suppose that f is continuous on $[a, b]$ and $a < c < b$. Then f is continuous on $[a, c]$ and on $[c, b]$ and, hence, f is integrable on $[a, b], [a, c]$ and $[c, b]$. Let $P = \{a = x_0 < x_1 < x_2 < \cdots x_n = b\}$. Suppose that $x_{i-1} \leq c \leq x_i$ for some i. Let $P_1 = \{a = x_0 < x_1 < x_2 < \cdots < x_{i-1} \leq c\}$ and $P_2 = \{c \leq x_i < x_{i+1} < \cdots < x_n = b\}$. Then there exist numbers c_1, c_2, c_3, d_1, d_2, and d_3 such that

$f(c_1) =$ absolute minimum of f on $[x_{i-1}, c]$,
$f(d_1) =$ absolute maximum of f on $[x_{i-1}, c]$,
$f(c_2) =$ absolute minimum of f on $[c, x_i]$,
$f(d_2) =$ absolute maximum of f on $[c, x_i]$,
$f(c_3) =$ absolute minimum of f on $[x_{i-1}, x_i]$,
$f(d_3) =$ absolute maximum of f on $[x_{i-1}, x_i]$,

Also,

$$f(c_3) \leq f(c_1),\ f(c_3) \leq f(c_2),\ f(d_1) \leq f(d_3) \quad \text{and} \quad f(d_2) \leq f(d_3).$$

It follows that

$$L(P) \leq L(P_1) + L(P_2) \leq U(P_1) + U(P_2) \leq U(P).$$

It follows that

$$\int_a^b f(x) = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

This completes the proof of the theorem.

Theorem 5.2.4 (Order Property) *If f and g are continuous on $[a, b]$ and $f(x) \leq g(x)$ for all x in $[a, b]$, then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Proof. Suppose that f and g are continuous on $[a, b]$ and $f(x) \leq g(x)$ for all x in $[a, b]$. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of $[a, b]$. For each i there exists numbers c_i, c_i^*, d_i and d_i^* such that

$f(c_i) =$ absolute minimum of f on $[x_{i-1}, x_i]$,
$f(d_i) =$ absolute maximum of f on $[x_{i-1}, x_i]$,
$g(c_i^*) =$ absolute minimum of g on $[x_{i-1}, x_i]$,
$g(d_i^*) =$ absolute maximum of g on $[x_{i-1}, x_i]$.

By the assumption that $f(x) \leq g(x)$ on $[a, b]$, we get

$$f(c_i) \leq g(c_i^*) \quad \text{and} \quad f(d_i) \leq g(d_i^*).$$

Hence

$$L_f \leq L_g \quad \text{and} \quad U_f \leq U_g.$$

It follows that

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

This completes the proof of this theorem.

Theorem 5.2.5 (Mean Value Theorem for Integrals) *If f is continuous on $[a, b]$, then there exists some point c in $[a, b]$ such that*

$$\int_a^b f(x)dx = f(c)(b-a).$$

Proof. Suppose that f is continuous on $[a, b]$, and $a < b$. Let

$m =$ absolute minimum of f on $[a, b]$, and
$M =$ absolute maximum of f on $[a, b]$.

Then, by Theorem 5.2.4,

$$m(b-a) \leq \int_a^b m\ dx \leq \int_a^b f(x)dx \leq \int_a^b M\ dx = M(b-a)$$

and

$$m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M.$$

By the intermediate value theorem for continuous functions, there exists some c such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx$$

and

$$\int_a^b f(x)dx = f(c)(b-a).$$

For $a = b$, take $c = a$. This completes the proof of this theorem.

Definition 5.2.2 The number $f(c)$ given in Theorem 5.2.6 is called the *average* value of f on $[a, b]$, denoted $f_{av}[a, b]$. That is

$$f_{av}[a, b] = \frac{1}{b-a} \int_a^b f(x)dx.$$

Theorem 5.2.6 (Fundamental Theorem of Calculus, First Form) *Suppose that f is continuous on some closed and bounded interval $[a, b]$ and*

$$g(x) = \int_a^x f(t)dt$$

for each x in $[a, b]$. Then $g(x)$ is continuous on $[a, b]$, differentiable on (a, b) and for all x in $(a, b), g'(x) = f(x)$. That is

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x).$$

Proof. Suppose that f is continuous on $[a, b]$ and $a < x < b$. Then

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{1}{h}[g(x+h) - g(x)] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x} f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt\right] \qquad \text{(Why?)} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\int_x^{x+h} f(t)dt\right] \\
&= \lim_{h\to 0} \frac{1}{h}[f(c)(x+h-x)] \qquad \text{by Theorem 5.2.5)} \\
&= \lim_{h\to 0} f(c)
\end{aligned}$$

for some c between x and $x + h$.

Since f is continuous on $[a, b]$ and c is between x and $x + h$, it follows that

$$g'(x) = \lim_{h\to 0} f(c) = f(x)$$

for all x such that $a < x < b$.

At the end points a and b, a similar argument can be used for one sided derivatives, namely,

$$\begin{aligned}
g'(a^+) &= \lim_{h\to 0^+} \frac{g(x+h) - g(x)}{h} \\
g'(b^-) &= \lim_{h\to 0^-} \frac{g(x+h) - g(x)}{h}.
\end{aligned}$$

We leave the end points as an exercise. This completes the proof of this theorem.

Theorem 5.2.7 (Fundamental Theorem of Calculus, Second Form) *If f and g are continuous on a closed and bounded interval $[a, b]$ and $g'(x) = f(x)$ on $[a, b]$, then*

$$\int_a^b f(x)dx = g(b) - g(a).$$

We use the notation: $[g(x)]_a^b = g(b) - g(a)$.

Proof. Let f and g be continuous on the closed and bounded interval $[a, b]$ and for each x in $[a, b]$, let

$$G(x) = \int_a^x f(t)dt.$$

Then, by Theorem 5.2.6, $G'(x) = f(x)$ on $[a, b]$. Since $G'(x) = g(x)$ for all x on $[a, b]$, there exists some constant C such that

$$G(x) = g(x) + C$$

for all x on $[a, b]$. Since $G(a) = 0$, we get $C = -g(a)$. Then

$$\begin{aligned}\int_a^b f(x)dx &= G(b) \\ &= g(b) + C \\ &= g(b) - g(a).\end{aligned}$$

This completes the proof of Theorem 5.2.7.

Theorem 5.2.8 (Leibniz Rule) *If $\alpha(x)$ and $\beta(x)$ are differentiable for all x and f is continuous for all x, then*

$$\frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} f(t)dt\right] = f(\beta(x)) \cdot \beta'(x) - f(\alpha(x)) \cdot \alpha'(x).$$

Proof. Suppose that f is continuous for all x and $\alpha(x)$ and $\beta(x)$ are differentiable for all x. Then

$$\begin{aligned}&\frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} f(t)dt\right] = \frac{d}{dx}\left[\int_{\alpha(x)}^{0} f(t)dt + \int_0^{\beta(x)} f(t)dt\right] \\ &= \frac{d}{dx}\left[\int_0^{\beta(x)} f(t)dt - \int_0^{\alpha(x)} f(t)dt\right] \\ &= \frac{d}{d(\beta(x))}\left(\int_0^{\beta(x)} f(t)dt\right) \cdot \frac{d(\beta(x))}{dx} - \frac{d}{d(\alpha(x))}\left(\int_0^{\alpha(x)} f(x)dt\right) \frac{d(\alpha(x))}{dx} \\ &= f(\beta(x))\ \beta'(x) - f(\alpha(x))\alpha'(x) \quad \text{(by Theorem 5.2.6)}\end{aligned}$$

This completes the proof of Theorem 5.2.8.

Example 5.2.1 Compute each of the following definite integrals and sketch the area represented by each integral:

(i) $\int_0^4 x^2 dx$

(ii) $\int_0^{\pi} \sin x \; dx$

(iii) $\int_{-\pi/2}^{\pi/2} \cos x \; dx$

(iv) $\int_0^{10} e^x dx$

(v) $\int_0^{\pi/3} \tan x \; dx$

(vi) $\int_{\pi/6}^{\pi/2} \cot x \; dx$

(vii) $\int_{-\pi/4}^{\pi/4} \sec x \; dx$

(viii) $\int_{\pi/4}^{3\pi/4} \csc x \; dx$

(xi) $\int_0^1 \sinh x \; dx$

(x) $\int_0^1 \cosh x \; dx$

We note that each of the functions in the integrand is positive on the respective interval of integration, and hence, represents an area. In order to compute these definite integrals, we use the Fundamental Theorem of Calculus, Theorem 5.2.2. As in Chapter 4, we first determine an anti-derivative $g(x)$ of the integrand $f(x)$ and then use

$$\int_a^b f(x)dx = g(b) - g(a) = [g(x)]_a^b \, .$$

graph

(i) $\int_0^4 x^2 dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{64}{3}$

graph

(ii) $\displaystyle\int_0^{\pi} \sin x \; dx = [-\cos x]_0^{\pi} = 1-(-1) = 2$

graph

(iii) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \; dx = [\sin x]_{-\pi/2}^{\pi/2} = 1-(-1) = 2$

graph

(iv) $\displaystyle\int_0^{10} e^x \; dx = [e^x]_0^{10} = e^{10} - e^0 = e^{10} - 1$

graph

(v) $\displaystyle\int_0^{\pi/3} \tan x \; dx = [\ln|\sec x|]_0^{\pi/3} = \ln\left|\sec\left(\frac{\pi}{3}\right)\right| = \ln 2$

graph

(vi) $\displaystyle\int_{\pi/6}^{\pi/2} \cot x \; dx = [\ln|\sin x|]_{\pi/6}^{\pi/2} = \ln(1) - \ln\left(\frac{1}{2}\right) = \ln 2$

graph

(vii) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec x \; dx = [\ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4} = \ln|\sqrt{2}+1| - \ln|\sqrt{2}-1|$

graph

(viii) $$\int_{\pi/4}^{3\pi/4} \csc x \; dx = [-\ln|\csc x + \cot x|]_{\pi/4}^{3\pi/4}$$
$$= -\ln|\sqrt{2}-1| + \ln|\sqrt{2}+1|$$

graph

(ix) $\displaystyle\int_{0}^{1} \sinh x \; dx = [\cosh x]_0^1 = \cosh 1 - \cosh 0 = \cosh 1 - 1$

graph

(x) $\displaystyle\int_0^1 \cosh x \; dx = [\sinh x]_0^1 = \sinh 1$

graph

Example 5.2.2 Evaluate each of the following integrals:

(i) $\displaystyle\int_1^{10} \frac{1}{x} \; dx$

(ii) $\displaystyle\int_0^{\pi/2} \sin(2x) dx$

(iii) $\displaystyle\int_0^{\pi/6} \cos(3x) dx$

(iv) $\displaystyle\int_0^2 (x^4 - 3x^2 + 2x - 1) dx$

(v) $\displaystyle\int_0^3 \sinh(4x) dx$

(vi) $\displaystyle\int_0^4 \cosh(2x) dx$

(i) Since $\dfrac{d}{dx} (\ln|x|) = \dfrac{1}{x}$,

$$\int_1^{10} \frac{1}{x} \; dx = [\ln|x|]_1^{10} = \ln(10)$$

(ii) Since $\dfrac{d}{dx} \left(\dfrac{-1}{2} \cos(2x) \right) = \sin(2x)$,

$$\int_0^{\pi/2} \sin 2x \; dx = \left[\frac{-1}{2} \cos(2x) \right]_0^{\pi/2} = \frac{1}{2} + \frac{1}{2} = 1.$$

(iii) $\displaystyle\int_0^{\pi/6} \cos(3x) = \left[\frac{1}{3} \sin(3x) \right]_0^{\pi/6} = \frac{1}{3} \sin\left(\frac{\pi}{2}\right) = \frac{1}{3}.$

(iv) $$\int_0^2 (x^4 - 3x^2 + 2x - 1)dx = \left[\frac{1}{5}x^5 - x^3 + x^2 - x\right]_0^2 = \left(\frac{32}{5} - 8 + 4 - 2\right) - 0 = \frac{2}{5}.$$

(v) $$\int_0^3 \sinh(4x)dx = \left[\frac{1}{4}\cosh(4x)\right]_0^3 = \frac{1}{4}\ \cosh(12) - \frac{1}{4}\ \cosh(0) = \frac{1}{4}\ (\cosh(12) - 1)$$

(vi) $$\int_0^4 \cosh(2x)dx = \left[\frac{1}{2}\ \sinh(2x)\right]_0^4 = \frac{1}{2}\ \cosh(8)$$

Example 5.2.3 Verify each of the following:

(i) $$\int_0^4 x^2dx = \int_0^3 x^2dx + \int_3^4 x^2dx$$

(ii) $$\int_1^4 x^2dx < \int_1^4 x^3dx$$

(iii) $$\frac{d}{dx}\ \left[\int_0^x (t^2 + 3t + 1)dt\right] = x^2 + 3x + 1$$

(iv) $$\frac{d}{dx}\ \left[\int_{x^2}^{x^3} \cos(t)dt\right] = 3x^2\cos(x^3) - 2x\cos(x^2).$$

(v) If $f(x) = \sin x$, then $f_{av}[0, \pi] = \dfrac{2}{\pi}$.

(i) $$\int_0^4 x^2dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{64}{3}$$

$$\int_0^3 x^2dx + \int_3^4 x^2dx = \left[\frac{x^3}{3}\right]_0^3 + \left[\frac{x^3}{3}\right]_3^4$$

$$= \left(\frac{27}{3} - 0\right) + \left(\frac{64}{3} - \frac{27}{3}\right) = \frac{64}{3}.$$

Therefore,

$$\int_1^4 x^2 dx = \int_0^3 x^2 dx + \int_3^4 x^2 dx.$$

(ii) $\displaystyle\int_1^4 x^2 dx = \left[\frac{x^3}{3}\right]_1^4 = \frac{64}{3} - \frac{1}{3} = 21$

$\displaystyle\int_1^4 x^3 dx = \left[\frac{x^4}{4}\right]_1^4 = \left(64 - \frac{1}{4}\right)$

Therefore, $\displaystyle\int_1^4 x^2 dx < \int_1^4 x^3 dx$. We observe that $x^2 < x^3$ on $(1, 4]$.

(iii) $\displaystyle\int_0^x (t^2 + 3t + 1) dt = \left[\frac{t^3}{3} + 3\frac{t^2}{2} + t\right]_0^x$

$$= \frac{x^3}{3} + \frac{3}{2}x^2 + x$$

$$\frac{d}{dx}\left(\frac{x^3}{3} + \frac{3}{2}x^2 + x\right) = x^2 + 3x + 1.$$

(iv) $\displaystyle\frac{d}{dx}\left[\int_{x^2}^{x^3} \cos t dt\right] = \frac{d}{dx}\left[[\sin t]_{x^2}^{x^3}\right]$

$$= \frac{d}{dx}[\sin(x^3) - \sin(x^2)]$$

$$= \cos(x^3) \cdot 3x^2 - \cos(x^2) \cdot 2x$$

$$= 3x^2 \cos(x^3) - 2x \cos(x^2).$$

Using the Leibniz Rule, we get

$$\frac{d}{dx}\left(\int_{x^2}^{x^3} \cos t dt\right) = \cos(x^3) \cdot 3x^2 - \cos(x^2) \cdot 2x$$

$$= 3x^2 \cos x^3 - 2x \cos x^2.$$

(v) The average value of $\sin x$ on $[0,\pi]$ is given by

$$\begin{aligned}\frac{1}{\pi-0}\left(\int_0^\pi \sin x\ dx\right) &= \frac{1}{\pi}\left[-\cos x\right]_0^\pi \\ &= \frac{1}{\pi}[-(-1)+1] \\ &= \frac{2}{\pi}.\end{aligned}$$

Basic List of Indefinite Integrals:

1. $\displaystyle\int x^3 dx = \frac{1}{4}x^4 + c$

2. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq 1$

3. $\displaystyle\int \frac{1}{x}\ dx = \ln|x| + c$

4. $\displaystyle\int \sin x\ dx = -\cos x + c$

5. $\displaystyle\int \sin(ax) dx = \frac{-1}{a}\ \cos(ax) + c$

6. $\displaystyle\int \cos x\ dx = \sin x + c$

7. $\displaystyle\int \cos(ax)\ dx = \frac{1}{a}\ \sin(ax) + c$

8. $\displaystyle\int \tan x dx = \ln|\sec x| + c$

9. $\displaystyle\int \tan(ax)\ dx = \frac{1}{a}\ \ln|\sec(ax)| + c$

10. $\displaystyle\int \cot x\ dx = \ln|\sin x| + c$

11. $\displaystyle\int \cot(ax)\ dx = \frac{1}{a}\ \ln|\sin(ax)| + c$

12. $\displaystyle\int e^x\ dx = e^x + c$

13. $\displaystyle\int e^{-x}\ dx = -e^{-x} + c$

14. $\displaystyle\int e^{ax}\ dx = \frac{1}{a}\ e^{ax} + c$

15. $\displaystyle\int \sinh x\ dx = \cosh x + c$

16. $\displaystyle\int \cosh x\ dx = \sinh x + c$

17. $\displaystyle\int \tanh x\ dx = \ln|\cosh x| + c$

18. $\displaystyle\int \coth x\ dx = \ln|\sinh x| + c$

19. $\displaystyle\int \sinh(ax)\ dx = \frac{1}{a}\ \cosh(ax) + c$

20. $\displaystyle\int \cosh(ax)\ dx = \frac{1}{a}\ \sinh(ax) + c$

21. $\displaystyle\int \tanh(ax)\ dx = \frac{1}{a}\ \ln|\cosh ax| + c$

22. $\displaystyle\int \coth(ax)\ dx = \frac{1}{a}\ \ln|\sinh(ax)| + c$

23. $\displaystyle\int \sec x\ dx = \ln|\sec x + \tan x| + c$

24. $\displaystyle\int \csc x\ dx = -\ln|\csc x + \cot x| + c$

25. $\displaystyle\int \sec(ax)\ dx = \frac{1}{a}\ \ln|\sec(ax) + \tan(ax)| + c$

26. $\displaystyle\int \csc(ax)\ dx = \frac{-1}{a}\ \ln|\csc(ax) + \cot(ax)| + c$

27. $\displaystyle\int \sec^2 x\ dx = \tan x + c$

28. $\displaystyle\int \sec^2(ax)\ dx = \frac{1}{a}\ \tan(ax) + c$

29. $\displaystyle\int \csc^2 x\ dx = -\cot x + c$

30. $\displaystyle\int \csc^2(ax)\ dx = \frac{-1}{a}\ \cot(ax) + c$

31. $\displaystyle\int \tan^2 x\ dx = \tan x - x + c$

32. $\displaystyle\int \cot^2 x\ dx = -\cot x - x + c$

33. $\displaystyle\int \sin^2 x\ dx = \frac{1}{2}\ (x - \sin x \cos x) + c$

34. $\displaystyle\int \cos^2 x\ dx = \frac{1}{2}\ (x + \sin x \cos x) + c$

35. $\displaystyle\int \sec x \tan x\ dx = \sec x + c$

36. $\displaystyle\int \csc x\ dx = -\csc x + c$

Exercises 5.2 Using the preceding list of indefinite integrals, evaluate the following:

1. $\displaystyle\int_1^5 \frac{1}{t}\ dt$

2. $\displaystyle\int_0^{3\pi/2} \sin x\ dx$

3. $\displaystyle\int_0^{3\pi/2} \cos x\ dx$

4. $\displaystyle\int_0^{10} e^x\ dx$

5. $\displaystyle\int_0^{\pi/10} \sin(5x)\ dx$

6. $\displaystyle\int_0^{\pi/6} \cos(5x)\ dx$

7. $\int_{\pi/12}^{\pi/6} \cot(3x)\ dx$

8. $\int_{-1}^{1} e^{-x} dx$

9. $\int_{0}^{2} e^{3x}\ dx$

10. $\int_{0}^{2} \sinh(2x)\ dx$

11. $\int_{0}^{4} \cosh(3x)\ dx$

12. $\int_{0}^{1} \tanh(2x)\ dx$

13. $\int_{1}^{2} \coth(3x)\ dx$

14. $\int_{\pi/12}^{\pi/6} \sec(2x) dx$

15. $\int_{\pi/12}^{\pi/6} \csc(2x)\ dx$

16. $\int_{0}^{\pi/8} \sec^2(2x)\ dx$

17. $\int_{\pi/12}^{\pi/6} \csc^2(2x)$

18. $\int_{0}^{\pi/4} \tan^2 x\ dx$

19. $\int_{\pi/6}^{\pi/4} \cot^2 x\ dx$

20. $\int_{0}^{\pi} \sin^2 x dx$

21. $\int_{-\pi/2}^{\pi/2} \cos^2 x\ dx$

22. $\int_{\pi/6}^{\pi/4} \sec x \tan x\ dx$

23. $\int_{\pi/6}^{\pi/4} \csc x \cot x\ dx$

24. $\int_{0}^{2} e^{-3x}\ dx$

Compute the average value of each given f on the given interval.

25. $f(x) = \sin x,\ \left[\dfrac{-\pi}{2}, \pi\right]$

26. $f(x) = x^{1/3}, [0, 8]$

27. $f(x) = \cos x,\ \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

28. $f(x) = \sin^2 x, [0, \pi]$

29. $f(x) = \cos^2 x, [0, \pi]$

30. $f(x) = e^{-x}, [-2, 2]$

Compute $g'(x)$ without computing the integrals explicitly.

31. $g(x) = \int_{0}^{x} (1 + t^2)^{2/3} dt$

32. $g(x) = \int_{x^2}^{4x^3} \arctan(x)\ dx$

33. $g(x) = \int_{x^3}^{x^2} (1 + t^3)^{1/3} dt$

34. $g(x) = \int_{\arcsin x}^{\operatorname{arcsinh} x} (1 + t^2)^{3/2} dt$

35. $g(x) = \int_1^x \left(\frac{1}{t}\right) dt$

36. $g(x) = \int_{\sin 2x}^{\sin 3x} (1+t^2)^{1/2} dt$

37. $g(x) = \int_{\sin(x^2)}^{\sin(x^3)} (1+t^3)^{1/3} dt$

38. $\int_x^{4x} \frac{1}{1+t^2}\, dt$

39. $\int_{x^2}^{x^3} \arcsin(x)\ dx$

40. $\int_{\ln x}^{e^x} 2^t dt$

5.3 Integration by Substitution

Many functions are formed by using compositions. In dealing with a composite function it is useful to change variables of integration. It is convenient to use the following differential notation:

If $u = g(x)$, then $du = g'(x)\ dx$.

The symbol "du" represents the "differential of u," namely, $g'(x)dx$.

Theorem 5.3.1 (Change of Variable) *If f, g and g' are continuous on an open interval containing $[a, b]$, then*

(i) $\int_a^b f(g(x)) \cdot g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)du$

(ii) $\int f(g(x))g'(x)\ dx = \int f(u)du,$

where $u = g(x)$ and $du = g'(x)\ dx$.

Proof. Let f, g, and g' be continuous on an open interval containing $[a, b]$. For each x in $[a, b]$, let

$$F(x) = \int_a^x f(g(x))g'(x)dx$$

and

$$G(x) = \int_{g(a)}^{g(x)} f(u)du.$$

Then, by Leibniz Rule, we have

$$F'(x) = f(g(x))g'(x),$$

and

$$G'(x) = f(g(x))g'(x)$$

for all x on $[a, b]$.

It follows that there exists some constant C such that

$$F(x) = G(x) + C$$

for all x on $[a, b]$. For $x = a$ we get

$$0 = F(a) = G(a) + C = 0 + C$$

and, hence,

$$C = 0.$$

Therefore, $F(x) = G(x)$ for all x on $[a, b]$, and hence

$$\begin{aligned}\int_a^b f(g(x))g'(x)dx &= F(b)\\ &= G(b)\\ &= \int_{g(a)}^{g(b)} f(u)du.\end{aligned}$$

This completes the proof of this theorem.

Remark 18 We say that we have changed the variable from x to u through the substitution $u = g(x)$.

Example 5.3.1

(i) $\displaystyle\int_0^2 \sin(3x)\ dx = \int_0^6 \frac{1}{3}\ \sin u du = \frac{1}{3}\ [-\cos u]_0^6 = \frac{1}{3}\ (1 - \cos 6),$

where $u = 3x,\ du = 3\ dx,\ dx = \dfrac{1}{3}\ du.$

(ii) $$\int_0^2 3x\cos(x^2)\ dx = \int_0^4 \cos u \left(\frac{3}{2}\ du\right)$$
$$= \frac{3}{2}\ [\sin u]_0^4$$
$$= \frac{3}{2}\ \sin 4,$$

where $u = x^2,\ du = 2x\ dx,\ 3x\ dx = \frac{3}{2}\ du.$

(iii) $$\int_0^3 e^{x^2} x\ dx = \int_0^9 e^u \frac{1}{2}\ du = \frac{1}{2}\ [e^u]_0^9 = \frac{1}{2}\ (e^9 - 1),$$

where $u = x^2,\ du = 2x\ dx,\ x\ dx = \frac{1}{2}\ dx.$

Definition 5.3.1 Suppose that f and g are continuous on $[a, b]$. Then the area bounded by the curves $y = f(x),\ y = g(x),\ y = a$ and $x = b$ is defined to be A, where

$$A = \int_a^b |f(x) - g(x)|\ dx.$$

If $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$A = \int_a^b (f(x) - g(x))\ dx.$$

If $g(x) \geq f(x)$ for all x in $[a, b]$, then

$$A = \int_a^b (g(x) - f(x))\ dx.$$

Example 5.3.2 Find the area, A, bounded by the curves $y = \sin x,\ y = \cos x,\ x = 0$ and $x = \pi$.

graph

We observe that $\cos x \geq \sin x$ on $\left[0, \frac{\pi}{4}\right]$ and $\sin x \geq \cos x$ on $\left[\frac{\pi}{4}, \pi\right]$. Therefore, the area is given by

$$\begin{aligned}
A &= \int_0^{\pi} |\sin x - \cos x| \; dx \\
&= \int_0^{\pi/4} (\cos x - \sin x) \; dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\
&= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi} \\
&= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1\right) + \left[1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] \\
&= 2\sqrt{2}.
\end{aligned}$$

Example 5.3.3 Find the area, A, bounded by $y = x^2, \; y = x^3, \; x = 0$ and $x = 2$.

graph

We note that $x^3 \leq x^2$ on $[0, 1]$ and $x^3 \geq x^2$ on $[1, 2]$. Therefore, by definition,

$$\begin{aligned}
A &= \int_0^1 (x^2 - x^3) \; dx + \int_1^2 (x^3 - x^2) \; dx \\
&= \left[\frac{1}{3} \, x^3 - \frac{1}{4} \, x^4\right]_0^1 + \left[\frac{1}{4} \, x^4 - \frac{1}{3} \, x^3\right]_1^2 \\
&= \left(\frac{1}{3} - \frac{1}{4}\right) + \left[\left(4 - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right] \\
&= \frac{1}{12} + \frac{4}{3} + \frac{1}{12} \\
&= \frac{3}{2}.
\end{aligned}$$

Example 5.3.4 Find the area bounded by $y = x^3$ and $y = x$. To find the interval over which the area is bounded by these curves, we find the points of intersection.

graph

$$x^3 = x \leftrightarrow x^3 - x = 0 \leftrightarrow x(x^2-1) = 0$$
$$\leftrightarrow x = 0,\ x = 1,\ x = -1.$$

The curve $y = x$ is below $y = x^3$ on $[-1, 0]$ and the curve $y = x^3$ is below the curve $y = x$ on $[0, 1]$. The required area is A, where

$$\begin{aligned} A &= \int_{-1}^{0} (x^3 - x)\ dx + \int_{0}^{1} (x - x^3)\ dx \\ &= \left[\frac{1}{4}\ x^4 - \frac{1}{2}\ x^2\right]_{-1}^{0} + \left[\frac{1}{2}\ x^2 - \frac{x^4}{4}\right]_{0}^{1} \\ &= \left[\frac{1}{2} - \frac{1}{4}\right] + \left[\frac{1}{2} - \frac{1}{4}\right] \\ &= \frac{1}{2} \end{aligned}$$

Exercises 5.3 Find the area bounded by the given curves.

1. $y = x^2,\ y = x^3$
2. $y = x^4,\ y = x^3$
3. $y = x^2,\ y = \sqrt{x}$
4. $y = 8 - x^2,\ y = x^2$
5. $y = 3 - x^2,\ y = 2x$
6. $y = \sin x,\ y = \cos x,\ x = \frac{-\pi}{2}, x = \frac{\pi}{2}$
7. $y = x^2 + 4x,\ y = x$
8. $y = \sin 2x,\ y = x,\ x = \frac{\pi}{2}$
9. $y^2 = 4x,\ x - y = 0$
10. $y = x + 3,\ y = \cos x,\ x = 0,\ x = \frac{\pi}{2}$

Evaluate each of the following integrals:

11. $\int \sin 3x \; dx$

12. $\int \cos 5x \; dx$

13. $\int e^{x^2} x \; dx$

14. $\int x \sin(x^2) \; dx$

15. $\int x^2 \tan(x^3+1) \; dx$

16. $\int \sec^2(3x+1) \; dx$

17. $\int \csc^2(2x-1) \; dx$

18. $\int x \sinh(x^2) \; dx$

19. $\int x^2 \cosh(x^3+1) \; dx$

20. $\int \sec(3x+5) \; dx$

21. $\int \csc(5x-7) \; dx$

22. $\int x \tanh(x^2+1) \; dx$

23. $\int x^2 \coth(x^3) \; dx$

24. $\int \sin^3 x \cos x \; dx$

25. $\int \tan^5 x \sec^2 x \; dx$

26. $\int \cot^3 x \csc^2 x \; dx$

27. $\int \sec^3 x \tan x \; dx$

28. $\int \csc^3 x \cot x \; dx$

29. $\int \frac{(\arcsin x)^4}{\sqrt{1-x^2}} \; dx$

30. $\int \frac{(\arctan x)^3}{1+x^2} \; dx$

31. $\int_0^1 x e^{x^2} \; dx$

32. $\int_0^{\pi/6} \sin(3x) dx$

33. $\int_0^{\pi/4} \cos(4x) \; dx$

34. $\int_0^3 \frac{1}{(3x+1)} \; dx$

35. $\int_0^{\pi/2} \sin^3 x \cos x \; dx$

36. $\int_0^{\pi/6} \cos^3(3x) \sin 3x \; dx$

5.4 Integration by Parts

The product rule of differentiation yields an integration technique known as integration by parts. Let us begin with the product rule:

$$\frac{d}{dx}\,(u(x)v(x)) = \frac{du(x)}{dx}\,v(x) + u(x)\frac{dv(x)}{dx}\ .$$

On integrating each term with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b \frac{d}{dx}\,(u(x)v(x))\ dx = \int_a^b v(x)\left(\frac{du(x)}{dx}\right)\ dx + \int_a^b u(x)\left(\frac{dv(x)}{dx}\right)\ dx.$$

By using the differential notation and the fundamental theorem of calculus, we get

$$[u(x)v(x)]_a^b = \int_a^b v(x)u'(x)\ dx + \int_a^b u(x)v'(x)\ dx.$$

The standard form of this integration by parts formula is written as

(i) $\displaystyle\int_a^b u(x)v'(x)\ dx = [u(x)v(x)]_a^b - \int_a^b v(x)u'(x)\ dx$

and

(ii) $\displaystyle\int udv = uv - \int vdu$

We state this result as the following theorem:

Theorem 5.4.1 (Integration by Parts) *If $u(x)$ and $v(x)$ are two functions that are differentiable on some open interval containing $[a, b]$, then*

(i) $\displaystyle\int_a^b u(x)v'(x)\ dx = [u(x)v(x)]_a^b - \int_a^b v(x)u'(x)\ dx$

for definite integrals and

(ii) $\displaystyle\int udv = uv - \int vdu$

for indefinite integrals.

Proof. Suppose that u and v are differentiable on some open interval containing $[a, b]$. For each x on $[a, b]$, let

$$F(x) = \int_a^x u(x)v'(x)dx + \int_a^x v(x)u'(x)dx.$$

Then, for each x on $[a, b]$,

$$\begin{aligned} F'(x) &= u(x)v'(x) + v(x)u'(x) \\ &= \frac{d}{dx}\,(u(x)v(x)). \end{aligned}$$

Hence, there exists some constant C such that for each x on $[a, b]$,

$$F(x) = u(x)v(x) + C.$$

For $x = a$, we get

$$F(a) = 0 = u(a)v(a) + C$$

and, hence,

$$C = -u(a)v(a).$$

Then,

$$\begin{aligned} \int_a^b u(x)v'(x)dx + \int_a^b v(x)u'(x)dx &= F(b) \\ &= u(b)v(b) + C \\ &= u(b)v(b) - u(a)v(a). \end{aligned}$$

Consequently,

$$\int_a^b u(x)v'(x)dx = [u(b)v(b) - u(a)v(a)] - \int_a^b v(x)u'(x)dx.$$

This completes the proof of Theorem 5.4.1.

Remark 19 The "two parts" of the integrand are "$u(x)$" and "$v'(x)dx$" or "u" and "dv". It becomes necessary to compute $u'(x)$ and $v(x)$ to make the integration by parts step.

Example 5.4.1 Evaluate the following integrals:

(i) $\int x \sin x \ dx$ (ii) $\int xe^{-x} \ dx$ (iii) $\int (\ln x) \ dx$

(iv) $\int \arcsin x \ dx$ (v) $\int \arccos x \ dx$ (vi) $\int x^2 e^x \ dx$

(i) We let $u = x$ and $dv = \sin x \ dx$. Then $du = dx$ and

$$\begin{aligned} v(x) &= \int \sin x \ dx \\ &= -\cos x + c. \end{aligned}$$

We drop the constant c, since we just need one $v(x)$. Then, by the integration by parts theorem, we get

$$\begin{aligned} \int x \sin x \ dx &= \int udv \\ &= uv - \int vdu \\ &= x(-\cos x) - \int (-\cos x) \ dx \\ &= -x \cos x + \sin x + c. \end{aligned}$$

(ii) We let $u = x, \ du = \ dx, \ dv = e^{-x} dx, \ v = \int e^{-x} \ dx = -e^{-x}$. Then,

$$\begin{aligned} \int xe^{-x} \ dx &= x(-e^{-x}) - \int (-e^{-x}) \ dx \\ &= -xe^{-x} - e^{-x} + c. \end{aligned}$$

(iii) We let $u = (\ln x), \ du = \frac{1}{x} \ dx, \ dv = \ dx, \ v = x$. Then,

$$\begin{aligned} \int \ln x \ dx &= x \ln x - \int x \cdot \frac{1}{x} \ dx \\ &= x \ln x - x + c. \end{aligned}$$

(iv) We let $u = \arcsin x,\ du = \dfrac{1}{\sqrt{1-x^2}}\ dx,\ dv = \ dx,\ v = x$. Then,

$$\int \arcsin x\ dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\ dx.$$

To evaluate the last integral, we make the substitution $y = 1 - x^2$. Then, $dy = -2xdx$ and $x\ dx = (-1/2)du$ and hence

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\ dx &= \int \frac{(-1/2)du}{u^{1/2}} \\ &= -\frac{1}{2}\int u^{-1/2}du \\ &= -u^{1/2} + c \\ &= -\sqrt{1-x^2} + c.\end{aligned}$$

Therefore,

$$\int \arcsin x\ dx = x \arcsin x - \sqrt{1-x^2} + c.$$

(v) Part (v) is similar to part (iv) and is left as an exercise.

(vi) First we let $u = x^2,\ du = 2x\ dx,\ dv = e^x\ dx,\ v = \int e^x dx = e^x$. Then,

$$\begin{aligned}\int x^2 e^x\ dx &= x^2 e^x - \int 2xe^x\ dx \\ &= x^2 e^x - 2\int xe^x\ dx.\end{aligned}$$

To evaluate the last integral, we let $u = x,\ du = \ dx,\ dv = e^x dx,\ v = e^x$. Then

$$\begin{aligned}\int xe^x\ dx &= xe^x - \int e^x\ dx \\ &= xe^x - e^x + c.\end{aligned}$$

Therefore,

$$\begin{aligned}\int x^2 e^x\ dx &= x^2 e^x - 2(xe^x - e^x + c) \\ &= x^2 e^x - 2xe^x + 2e^x - 2c \\ &= e^x(x^2 - 2x + 2) + D.\end{aligned}$$

Example 5.4.2 Evaluate the given integrals in terms of integrals of the same kind but with a lower power of the integrand. Such formulas are called the reduction formulas. Apply the reduction formulas for $n = 3$ and $n = 4$.

(i) $\int \sin^n x \;\; dx$ (ii) $\int \csc^{m+2} x \quad dx$ (iii) $\int \cos^n x \;\; dx$ (iv) $\int \sec^{m+2} x \quad dx$

(i) We let

$$u = (\sin x)^{n-1}, \;\; du = (n-1)(\sin x)^{n-2} \cos x \;\; dx$$
$$dv = \sin x \;\; dx, \;\; v = \int \sin x \;\; dx = -\cos x.$$

Then

$$\begin{aligned}
\int \sin^n x \;\; dx &= \int (\sin x)^{n-1} (\sin x \;\; dx) \\
&= (\sin x)^{n-1}(-\cos x) - \int (-\cos x)(n-1)(\sin x)^{n-2} \cos x \;\; dx \\
&= -(\sin x)^{n-1} \cos x + (n-1) \int (\sin x)^{n-2} (1 - \sin^2 x) \;\; dx \\
&= -(\sin x)^{n-1} \cos x + (n-1) \int (\sin x)^{n-2} dx \\
&\qquad - (n-1) \int \sin^n x \;\; dx.
\end{aligned}$$

We now use algebra to solve the integral as follows:

$$\int \sin^n x \;\; dx + (n-1) \int \sin^n x \;\; dx = -(\sin x)^{n-1} \cos x + (n-1) \int \sin^{n-2} x \;\; dx$$

$$n \int \sin^n x \;\; dx = -(\sin x)^{n-1} \cos x + (n-1) \int \sin^{n-2} x \;\; dx$$

$$\boxed{\int \sin^n x \;\; dx = \frac{-1}{n} \;\; (\sin x)^{n-1} \cos x + \frac{n-1}{n} \int \sin^{n-2} x \;\; dx}\,. \qquad (1)$$

We have reduced the exponent of the integrand by 2. For $n = 3$, we get

$$\begin{aligned}
\int \sin^3 x \;\; dx &= \frac{-1}{3} \;\; (\sin x)^2 \cos x + \frac{2}{3} \int \sin x \;\; dx \\
&= \frac{-1}{3} \;\; (\sin x)^2 \cos x \frac{-2}{3} \;\; \cos x + c.
\end{aligned}$$

For $n = 2$, we get

$$\begin{aligned}
\int \sin^2 x \, dx &= \frac{-1}{2} \, (\sin x) \cos x + \frac{1}{2} \int 1 \, dx \\
&= \frac{-1}{2} \, \sin x \cos x + \frac{x}{2} + c \\
&= \frac{1}{2} \, (x - \sin x \cos x) + c.
\end{aligned}$$

For $n = 4$, we get

$$\begin{aligned}
\int \sin^4 x \, dx &= \frac{-1}{4} \, (\sin x)^3 \cos x + \frac{3}{4} \int \sin^2 x \, dx \\
&= \frac{-1}{4} \, (\sin x)^3 \cos x + \frac{3}{4} \cdot \frac{1}{2} \, (x - \sin x \cos x) + c.
\end{aligned}$$

In this way, we have a reduction formula by which we can compute the integral of any positive integral power of $\sin x$. If n is a negative integer, then it is useful to go in the direction as follows:

Suppose $n = -m$, where m is a positive integer. Then, from equation (1) we get

$$\begin{aligned}
\frac{n-1}{n} \int \sin^{n-2} x \, dx &= \frac{1}{n} \, (\sin x)^{n-1} \cos x + \int (\sin x)^n \, dx \\
\int \sin^{n-2} x \, dx &= \frac{1}{n-1} \, (\sin x)^{n-1} \cos x + \frac{n}{n-1} \int (\sin x)^n \, dx \\
\int \sin^{-m-2} x \, dx &= \frac{1}{-m-1} \, (\sin x)^{-m-1} \cos x \\
&\quad + \frac{-m}{-m-1} \int (\sin x)^{-m} \, dx
\end{aligned}$$

$$\boxed{\int \csc^{m+2} x \, dx = \frac{-1}{m+1} \, (\csc x)^m \cot x + \frac{m}{m+1} \int (\csc x)^m \, dx}. \quad (2)$$

This gives us the reduction formula for part (iii). Also,

$$\int \csc^n x \, dx = \frac{-1}{n-1} (\csc x^{n-2}) \cot x + \frac{n-2}{n-1} \int (\csc x^{n-2}) \, dx.$$

(iii) We can derive a formula by a method similar to part (i). However, let us make use of a trigonometric reduction formula to get it. Recall that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ and $\cos\left(\frac{\pi}{2} - x\right) = \sin x$. Then

$$\begin{aligned}
\int \cos^n x\ dx &= \int \sin^n\left(\frac{\pi}{2} - x\right)\ dx \qquad \left(\text{let } u = \frac{\pi}{2} - x,\ du = -dx\right)\\
&= \int \sin^n(u)(-du)\\
&= -\int \sin^n u du\\
&= -\left[\frac{-1}{n}\ (\sin u)^{n-1}\cos u + \frac{n-1}{n}\int \sin^{n-2} u du\right] \qquad \text{(by (1))}\\
&= \frac{1}{n}\ \left(\sin\left(\frac{\pi}{2} - x\right)\right)^{n-1}\cos\left(\frac{\pi}{2} - x\right)\\
&\quad - \frac{n-1}{n}\int \left(\sin\left(\frac{\pi}{2} - x\right)\right)^{n-2} d\left(\frac{\pi}{2} - x\right)
\end{aligned}$$

$$\boxed{\int \cos^n x\ dx = \frac{1}{n}\ (\cos x)^{n-1}\sin x + \frac{n-1}{n}\int \cos^{n-2} x\ dx}\ . \qquad (3)$$

To get part (iv) we replace n by $-m$ and get

$$\begin{aligned}
\int \cos^{-m} x\ dx &= \frac{1}{-m}\ (\cos x)^{-m-1}\sin x + \frac{-m-1}{-m}\int \cos^{-m-2} x\ dx\\
\int \sec^m x\ dx &= \frac{-1}{m}\ (\sec x)^m \tan x + \frac{m+1}{m}\int \sec^{m+2} x\ dx.
\end{aligned}$$

On solving for the last integral, we get

$$\boxed{\int \sec^{m+2} x\ dx = \frac{1}{m+1}\ (\sec x)^m \tan x + \frac{m}{m+1}\int \sec^m x\ dx}\ . \qquad (4)$$

Also, $\displaystyle\int \sec^n x\ dx = \frac{1}{n-1}\ \sec^{n-2} x\ \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx.$

In parts (ii), (iii) and (vi) we leave the cases for $n = 3$ and 4 as an exercise. These are handled as in part (i).

Example 5.4.3 Develop the reduction formulas for the following integrals:

(i) $\int \tan^n x\,dx$ (ii) $\int \cot^n x\,dx$ (iii) $\int \sinh^n x\,dx$ (iv) $\int \cosh^n x\,dx$

(i) First, we break $\tan^2 x = \sec^2 x - 1$ away from the integrand:

$$\begin{aligned}\int \tan^n x\ dx &= \int \tan^{n-2} x \cdot \tan^2 x\ dx \\ &= \int \tan^{n-2} x(\sec^2 x - 1)\ dx \\ \int \tan^n x\ dx &= \int \tan^{n-2} x \sec^2 x\ dx - \int \tan^{n-2} x\ dx.\end{aligned}$$

For the middle integral, we let $u = \tan x$ as a substitution.

$$\begin{aligned}\int \tan^n x\ dx &= \int u^{n-2} du - \int \tan^{n-2} x\ dx \\ &= \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x\ dx \\ &= \frac{(\tan x)^{n-1}}{n-1} - \int \tan^{n-2} x\ dx.\end{aligned}$$

Therefore,

$$\boxed{\int \tan^n x\ dx = \frac{(\tan x)^{n-1}}{n-1} - \int \tan^{n-2} x\ dx \quad n \neq 1}. \tag{5}$$

$\int \tan x\ dx = \ln|\sec x| + c$ for $n = 1$.

(ii) We use the reduction formula $\tan\left(\frac{\pi}{2}-x\right)=\cot x$ in (5).

$$\begin{aligned}
\int \cot^n x\ dx &= \int \tan^n\left(\frac{\pi}{2}-x\right)\ dx; \qquad \left(\text{let } u=\frac{\pi}{2}-x,\ du=-dx\right)\\
&= -\int \tan^n u(-du)\\
&= -\int \tan^n u\ du\\
&= -\left[\frac{\tan^{n-1}(u)}{n-1}-\int \tan^{n-2} u\ du\right], n\neq 1\\
&= -\frac{\cot^{n-1} x}{n-1}-\int \cot^{n-2} x(-dx),\ n\neq 1\\
&= -\frac{\cot^{n-1} x}{n-1}+\int \cot^{n-2} x\ dx,\ n\neq 1
\end{aligned}$$

$$\int \cot x\ dx = \ln|\sin x|+c,\ \text{for } n=1.$$

Therefore,

$$\boxed{\int \cot^n(x)\ dx = -\frac{\cot^{n-1} x}{n-1}+\int \cot^{n-2} x\ dx, n\neq 1} \tag{6}$$

$$\int \cot x\ dx = \ln|\sin x|+c.$$

(iii)
$$\begin{aligned}
\int \sinh^n x\ dx &= \int (\sinh^{n-1} x)(\sinh x\ dx); u=\sinh^{n-1} x,\ dv=\sinh x\ dx\\
&= \sinh^{n-1} x\cosh x-\int \cosh x\cdot(n-1)\sinh^{n-2} x\cosh x dx\\
&= \sinh^{n-1} x\cosh x-(n-1)\int \sinh^{n-2} x(\cosh^2 x)\ dx\\
&= \sinh^{n-1} x\cosh x-(n-1)\int \sinh^{n-2} x(1+\sinh^2 x)\ dx\\
&= \sinh^{n-1} x\cosh x-(n-1)\int \sinh^{n-2} x\ dx-(n-1)\int \sinh^n x\ dx.
\end{aligned}$$

On bringing the last integral to the left, we get

$$n\int \sinh^n x\ dx = \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x\ dx$$

$$\boxed{\int \sinh^n x\ dx = \frac{1}{n}\ \sinh^{n-1}\cosh x - \frac{n-1}{n}\ \int \sinh^{n-2} x\ dx}. \qquad (7)$$

(iv) $\int \cosh^n x\ dx = \int (\cosh^{n-1} x)(\cosh x\ dx); \quad u = \cosh^{n-1} x,\ dv = \cosh x\ dx, v = \sinh x$

$$= \cosh^{n-1}(x)\sinh x - \int \sinh x (n-1)\cosh^{n-2} x \sinh x dx$$

$$= \cosh^{n-1} x \sinh x - (n-1)\int \cosh^{n-2} x \sinh^2 x\ dx$$

$$= \cosh^{n-1} x \sinh x - (n-1)\int \cosh^{n-2} x(\cosh^2 x - 1)\ dx$$

$$= \cosh^{n-1} x \sinh x - (n-1)\int \cosh^n x\ dx$$

$$+(n-1)\int \cosh^{n-2} x\ dx$$

$$\int \cosh^n x\ dx \ +(n-1)\int \cosh^n x\ dx = \cosh^{n-1} x \sinh x$$

$$+(n-1)\int \cosh^{n-2} x\ dx$$

$$n\int \cosh^n x\ dx \ = \cosh^{n-1} x \sinh x + (n-1)\int \cosh^{n-2} x\ dx$$

$$\boxed{\int \cosh^n x\ dx = \frac{1}{n}\ \cosh^{n-1} x \sinh x + \frac{n-1}{n}\ \int \cosh^{n-2} x\ dx} \qquad (8)$$

Example 5.4.4 Develop reduction formulas for the following:

(i) $\int x^n e^x \, dx$ (ii) $\int x^n \ln x \, dx$ (iii) $\int (\ln x)^n \, dx$

(iv) $\int x^n \sin x$ (v) $\int x^n \cos x \, dx$ (vi) $\int e^{ax} \sin(\ln x) \, dx$

(vii) $\int e^{ax} \cos(\ln x) \, dx$

(i) We let $u = x^n,\ dv = e^x \ dx,\ du = nx^{n-1}dx,\ v = e^x$. Then

$$\begin{aligned}\int x^n e^x \ dx &= x^n e^x - \int e^x (nx^{n-1}) \ dx \\ &= x^n e^x - n \int x^{n-1} e^x \ dx.\end{aligned}$$

Therefore,

$$\boxed{\int x^n e^x \ dx = x^n e^x - n \int x^{n-1} e^x \ dx}\ . \tag{9}$$

(ii) We let $u = \ln x,\ du = (1/x) \ dx,\ dv = x^n \ dx,\ v = x^{n+1}/(n+1)$. Then,

$$\begin{aligned}\int x^n \ln x \ dx &= (\ln x)\frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} \ dx \\ &= \frac{x^{n+1}(\ln x)}{n+1} - \frac{1}{n+1} \int x^n \ dx \\ &= \frac{x^{n+1}(\ln x)}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c.\end{aligned}$$

Therefore,

$$\boxed{\int x^n \ln x \ dx = \frac{x^{n+1}}{(n+1)^2} \ [(n+1)\ln(x) - 1] + c}\ . \tag{10}$$

(iii) We let $u = (\ln x)^n,\ du = n(\ln x)^{n-1}\frac{1}{x} \ dx,\ dv = \ dx,\ v = x$. Then,

$$\begin{aligned}\int (\ln x)^n \ dx &= x(\ln x)^n - \int x \cdot n(\ln x)^{n-1} \cdot \frac{1}{x} \ dx \\ &= x(\ln x)^n - n \int (\ln x)^{n-1} \ dx\end{aligned}$$

Therefore,

$$\boxed{\int (\ln x)^n \ dx = x(\ln x)^n - n \int (\ln x)^{n-1} \ dx}. \tag{11}$$

(iv) We let $u = x^n,\ du = nx^{n-1}dx,\ dv = \sin x\ dx,\ v = -\cos x$. Then,

$$\boxed{\begin{aligned}\int x^n \sin x\ dx &= x^n(-\cos x) - \int (-\cos x)nx^{n-1}\ dx \\ &= -x^n \cos x = n \int x^{n-1} \cos x\ dx.\end{aligned}} \tag{$*$}$$

Again in the last integral we let $u = x^{n-1},\ du = (n-1)x^{n-2}dx,\ dv = \cos x\ dx,\ v = \sin x$. Then

$$\boxed{\begin{aligned}\int x^{n-1} \cos x\ dx &= x^{n-1} \sin x - \int \sin x(n-1)x^{n-2}dx \\ &= x^{n-1} \sin x - (n-1) \int x^{n-2} \sin x\ dx.\end{aligned}} \tag{$**$}$$

By substitution, we get the reduction formula

$$\int x^n \sin x\ dx = -x^n \cos x + n \left[x^{n-1} \sin x - (n-1) \int x^{n-2} \sin x\ dx\right]$$

$$\boxed{\int x^n \sin x\ dx = -x^n \cos x + nx^{n-1} \sin x - n(n-1) \int x^{n-2} \sin x\ dx} \tag{12}$$

(v) We can use ($**$) and ($*$) in part (iv) to get the following:

$$\int x^{n-1} \cos x\ dx = x^{n-1} \sin x - (n-1) \int x^{n-2} \sin x\ dx \qquad \text{by } (**)$$

$$= x^{n-1} \sin x - (n-1) \left[-x^{n-2} \cos x + (n-2) \int x^{n-3} \cos x\ dx\right] \qquad \text{by } (*)$$

$$\int x^{n-1}\cos x\,dx = x^{n-1}x+(n-1)x^{n-2}\cos x-(n-1)(n-2)\int x^{n-3}\cos x\,dx.$$

If we replace n by $n+1$ throughout the last equation, we get

$$\boxed{\int x^n \cos x\ dx = x^n \sin x + nx^{n-1}\cos x - n(n-1)\int x^{n-2}\cos x\ dx} \tag{13}$$

(vi) We let $dv = e^{ax}\ dx,\ v = \dfrac{1}{a}\ e^{ax},\ u = \sin(bx),\ du = b\cos(bx)\ dx$. Then

$$\int e^{ax}\sin(bx)\ dx = \frac{1}{a}\ e^{ax}\sin(bx) - \frac{b}{a}\int e^{ax}\cos(bx)\ dx. \qquad (***)$$

In the last integral, we let $dv = e^{ax}dx,\ v = \dfrac{1}{a}\ e^{ax},\ u = \cos bx$. Then

$$\int e^{ax}\cos(bx)\ dx = \frac{1}{a}\ e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\ dx \qquad (****)$$

First we substitute $(****)$ into $(***)$ and then solve for

$$\int e^{ax}\sin bx\ dx.$$

$$\begin{aligned}
\int e^{ax}\sin bx\ dx &= \frac{1}{a}\ e^{ax}\sin bx - \frac{b}{a}\ \left[\frac{1}{a}\ e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\ dx\right] \\
&= \frac{e^{ax}}{a^2}\ (a\sin bx - b\cos bx) - \frac{b^2}{a^2}\int ax\sin bx\ dx \\
\left(1+\frac{b^2}{a^2}\right)\int e^{ax}\sin bx\ dx &= \frac{e^{ax}}{a^2}\ (a\sin bx - b\cos bx\ dx)
\end{aligned}$$

$$\boxed{\int e^{ax}\sin bx\ dx = \frac{e^{ax}}{a^2+b^2}\ (a\sin bx - b\cos bx) + c}\,. \tag{14}$$

(vii) We start with $(****)$ and substitute in (14) without the constant c and get

$$\begin{aligned}\int e^{ax}\cos bx\ dx &= \frac{1}{a}\ e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\ dx\\ &= \frac{1}{a}\ e^{ax}\cos bx + \frac{b}{a}\left[\frac{e^{ax}}{a^2b^2}\ (a\sin bx - b\cos bx)\right] + c\\ &= e^{ax}\left[\frac{1}{a}\ \cos bx + \frac{1}{a^2+b^2}\ \left(b\sin bx - \frac{b^2}{a}\ \cos bx\right)\right] + c\\ &= \frac{e^{ax}}{a^2+b^2}\ [b\sin b + a\cos bx] + c.\end{aligned}$$

Therefore,

$$\boxed{\int e^{ax}\cos bx\ dx = \frac{e^{ax}}{a^2+b^2}\ [b\sin bx + a\cos bx] + c}\ . \tag{15}$$

Exercises 5.4 Evaluate the following integrals and check your answers by differentiation. You may use the reduction formulas given in the examples.

1. $\displaystyle\int xe^{-2x}\ dx$
2. $\displaystyle\int x^3\ln x$
3. $\displaystyle\int \frac{dx}{x(\ln x)4}$
4. $\displaystyle\int (\ln x)^3\ dx$
5. $\displaystyle\int e^{2x}\sin 3x\ dx$
6. $\displaystyle\int e^{3x}\cos 2x\ dx$
7. $\displaystyle\int x^2\sin 2x\ dx$
8. $\displaystyle\int x^2\cos 3x\ dx$
9. $\displaystyle\int x\ln(x+1)\ dx$
10. $\displaystyle\int \arcsin(2x)\ dx$
11. $\displaystyle\int \arccos(2x)\ dx$
12. $\displaystyle\int \arctan(2x)\ dx$
13. $\displaystyle\int \sec^3 x\ dx$
14. $\displaystyle\int \sec^5 x\ dx$
15. $\displaystyle\int \tan^5 x\ dx$
16. $\displaystyle\int x^2\ln x\ dx$
17. $\displaystyle\int x^3\sin x\ dx$
18. $\displaystyle\int x^3\cos x\ dx$

19. $\int x \sinh x \, dx$

20. $\int x \cosh x \, dx$

21. $\int x(\ln x)^3 dx$

22. $\int x \arctan x \, dx$

23. $\int x \operatorname{arccot} x \, dx$

24. $\int \sin^3 x \, dx$

25. $\int \cos^3 x \, dx$

26. $\int \sin^4 x \, dx$

27. $\int \cos^4 x \, dx$

28. $\int \sinh^2 x \, dx$

29. $\int \cosh^2 x \, dx$

30. $\int \sinh^3 x \, dx$

31. $\int x^2 \sinh x \, dx$

32. $\int x^2 \cosh x \, dx$

33. $\int x^3 \sinh x \, dx$

34. $\int x^3 \cosh x \, dx$

35. $\int x^2 e^{2x} dx$

36. $\int x^3 e^{-x} dx$

37. $\int x \sin(3x) \, dx$

38. $\int x \cos(x+1) dx$

39. $\int x \ln(x+1) dx$

40. $\int x \; 2^x dx$

41. $\int x \; 10^{2x} dx$

42. $\int x^2 \; 10^{3x} dx$

43. $\int x^2 (\ln x)^3 dx$

44. $\int \operatorname{arcsinh}(3x) dx$

45. $\int \operatorname{arccosh}(2x) dx$

46. $\int \operatorname{arctanh}(2x) dx$

47. $\int \operatorname{arccoth}(3x) dx$

48. $\int x \operatorname{arcsec} x \, dx$

50. $\int x \operatorname{arccsc} x \, dx$

5.5 Logarithmic, Exponential and Hyperbolic Functions

With the Fundamental Theorems of Calculus it is possible to rigorously develop the logarithmic, exponential and hyperbolic functions.

Definition 5.5.1 For each $x > 0$ we define the *natural logarithm of* x, denoted $\ln x$, by the equation

$$\ln(x) = \int_1^x \frac{1}{t}\,dt \quad , \quad x > 0.$$

Theorem 5.5.1 (Natural Logarithm) *The natural logarithm,* $\ln x$, *has the following properties:*

(i) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x} > 0$ *for all* $x > 0$.
The natural logarithm is an increasing, continuous and differentiable function on $(0,\infty)$.

(ii) If $a > 0$ *and* $b > 0$, *then* $\ln(ab) = \ln(a) + \ln(b)$.

(iii) If $a > 0$ *and* $b > 0$, *then* $\ln(a/b) = \ln(a) + \ln(b)$.

(iv) If $a > 0$ *and* n *is a natural number, then* $\ln(a^n) = n\ln a$.

(v) The range of $\ln x$ *is* $(-\infty,\infty)$.

(vi) $\ln x$ *is one-to-one and has a unique inverse, denoted* e^x.

Proof.

(i) Since $1/t$ is continuous on $(0,\infty)$, (i) follows from the Fundamental Theorem of Calculus, Second Form.

(ii) Suppose that $a > 0$ and $b > 0$. Then

$$\begin{aligned}
\ln(ab) &= \int_1^{ab} \frac{1}{t}\,dt \\
&= \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt \\
&= \ln a + \int_1^{b} \frac{1}{au}\,adu \ ; \ \left(u = \frac{1}{a}\,t,\ du = \frac{1}{a}\,dt\right) \\
&= \ln a + \ln b.
\end{aligned}$$

(iii) If $a > 0$ and $b > 0$, then

$$\begin{aligned}
\ln\left(\frac{a}{b}\right) &= \int_1^{\left(\frac{a}{b}\right)} \frac{1}{t}\ dt \\
&= \int_1^a \frac{1}{t}\ dt + \int_a^{\frac{a}{b}} \frac{1}{t}\ dt;\ \left(u = \frac{b}{a}\ t,\ du = \frac{b}{a}\ dt\right) \\
&= \int_1^a \frac{1}{t}\ dt + \int_b^1 \frac{1}{\left(\frac{au}{b}\right)}\ \left(\frac{a}{b}\ dt\right) \\
&= \int_1^a \frac{1}{t}\ dt - \int_1^b \frac{1}{u}\ du \\
&= \ln a - \ln b.
\end{aligned}$$

(iv) If $a > 0$ and n is a natural number, then

$$\begin{aligned}
\ln(a^n) &= \int_1^{a^n} \frac{1}{t}\ dt\ ;\ t = u^n,\ dt = nu^{n-1}du \\
&= \int_1^a \frac{1}{u^n} \cdot nu^{n-1}du \\
&= n \int_1^a \frac{1}{u}\ du \\
&= n \ln a
\end{aligned}$$

as required.

(v) From the partition $\{1, 2, 3, 4, \cdots\}$, we get the following inequality using upper and lower sum approximations:

graph

$$\frac{13}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} < \ln 4 < 1 + \frac{1}{2} + \frac{1}{3}.$$

Hence, $\ln 4 > 1$. $\ln(4^n) = n \ln 4 > n$ and $\ln 4^{-n} = -n \ln 4 < -n$. By the intermediate value theorem, every interval $(-n, n)$ is contained in the range of $\ln x$. Therefore, the range of $\ln x$ is $(-\infty, \infty)$, since the derivative of $\ln x$ is always positive, $\ln x$ is increasing and hence one-to-one. The inverse of $\ln x$ exists.

(vi) Let e denote the number such that $\ln(e) = 1$. Then we define $y = e^x$ if and only if $x = \ln(y)$ for $x \in (-\infty, \infty), y > 0$.

This completes the proof.

Definition 5.5.2 If x is any real number, we define $y = e^x$ if and only if $x = \ln y$.

Theorem 5.5.2 (Exponential Function) *The function $y = e^x$ has the following properties:*

(i) $e^0 = 1$, $\ln(e^x) = x$ *for every real* x *and* $\frac{d}{dx}(e^x) = e^x$.

(ii) $e^a \cdot e^b = e^{a+b}$ *for all real numbers* a *and* b.

(iii) $\frac{e^a}{e^b} = e^{a-b}$ *for all real numbers* a *and* b.

(iv) $(e^a)^n = e^{na}$ *for all real numbers* a *and natural numbers* n.

Proof.

(i) Since $\ln(1) = 0$, $e^0 = 1$. By definition $y = e^x$ if and only if $x = \ln(y) = \ln(e^x)$. Suppose $y = e^x$. Then $x = \ln y$. By implicit differentiation, we get

$$1 = \frac{1}{y}\frac{dy}{dx}, \quad \frac{dy}{dx} = y = e^x.$$

Therefore,

$$\frac{d}{dx}(e^x) = e^x.$$

(ii) Since $\ln x$ is increasing and, hence, one-to-one,

$$\begin{aligned} e^a \cdot e^b = e^{a+b} &\leftrightarrow \\ \ln(e^a \cdot e^b) = \ln(e^{a+b}) &\leftrightarrow \\ \ln(e^a) + \ln(e^b) = a + b &\leftrightarrow \\ a + b = a + b&. \end{aligned}$$

It follows that for all real numbers a and b,

$$e^a \cdot e^b = e^{a+b}.$$

(iii)
$$\frac{e^a}{e^b} = e^{a-b} \leftrightarrow$$

$$\ln\left(\frac{e^a}{e^b}\right) = \ln(e^{a-b}) \leftrightarrow$$

$$\ln(e^a) - \ln(e^b) = a - b \leftrightarrow$$

$$a - b = a - b.$$

It follows that for all real numbers a and b,

$$\frac{e^a}{e^b} = e^{a-b}.$$

(iv)
$$(e^a)^n = e^{na} \leftrightarrow$$

$$\ln((e^a)^n) = \ln(e^{na}) \leftrightarrow$$

$$n \ln(e^a) = na \leftrightarrow$$

$$na = na.$$

Therefore, for all real numbers a and natural numbers n, we have

$$(e^a)^n = e^{na}.$$

Definition 5.5.3 Suppose $b > 0$ and $b \neq 1$. Then we define the following:

(i) For each real number x, $b^x = e^{x \ln b}$.

(ii) $y = \log_b x = \dfrac{\ln x}{\ln b}$.

Theorem 5.5.3 (General Exponential Function) *Suppose $b > 0$ and $b \neq 1$. Then*

(i) $\ln(b^x) = x \ln b$, *for all real numbers x.*

(ii) $\dfrac{d}{dx}(b^x) = b^x \ln b$, *for all real numbers x.*

(iii) $b^{x_1} \cdot b^{x_2} = b^{x_1 + x_2}$, *for all real numbers x_1 and x_2.*

(iv) $\dfrac{b^{x_1}}{b^{x_2}} = b^{x_1 - x_2}$, *for all real numbers x_1 and x_2.*

(v) $(b^{x_1})^{x_2} = b^{x_1 x_2}$, *for all real numbers x_1 and x_2.*

(vi) $\displaystyle\int b^x \, dx = \frac{b^x}{\ln b} + c.$

Proof.

(i) $\ln(b^x) = \ln(e^{x \ln b}) = x \ln b$

(ii)
$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{x \ln b}) = e^{x \ln b} \cdot (\ln b) \quad \text{(by the chain rule)}$$
$$= b^x \ln b.$$

(iii)
$$b^{x_1} \cdot b^{x_2} = e^{x_1 \ln b} \cdot e^{x_2 \ln b}$$
$$= e^{(x_1 \ln b + x_2 \ln b)}$$
$$= e^{(x_1 + x_2) \ln b}$$
$$= b^{(x_1 + x_2)}$$

(iv)
$$\begin{aligned}\frac{b^{x_1}}{b^{x_2}} &= \frac{e^{x_1 \ln b}}{e^{x_2 \ln b}} \\ &= e^{x_1 \ln b - x_2 \ln b} \\ &= e^{(x_1 - x_2)\ln b} \\ &= b^{(x_1 - x_2)}.\end{aligned}$$

(v) By Definition 5.5.3 (i), we get

$$\begin{aligned}(b^{x_1})^{x_2} &= e^{x_2 \ln(b^{x_1})} \\ &= e^{x_2 \ln(e^{x_1 \ln b})} \\ &= e^{x_2 \cdot x_1 \ln b} \\ &= e^{(x_1 x_2)\ln b} \\ &= b^{x_1 x_2}.\end{aligned}$$

(vi) Since

$$\frac{d}{dx}(b^x) = b^x \ln b,$$

we get

$$\begin{aligned}\int b^x (\ln b)\ dx &= b^x + c, \\ \ln b \int b^x\ dx &= b^x + c, \\ \int e^x\ dx &= \frac{b^x}{\ln b} + D,\end{aligned}$$

where D is some constant. This completes the proof.

Theorem 5.5.4 *If $u(x) > 0$ for all x, and $u(x)$ and $v(x)$ are differentiable functions, then we define*

$$y = (u(x))^{v(x)} = e^{v(x)\ln(u(x))}.$$

Then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{d}{dx}\,(u(x))^{v(x)} = (u(x))^{v(x)}\left[v'(x)\ln(u(x)) + v(x)\,\frac{u'(x)}{u(x)}\right].$$

Proof. This theorem follows by the chain rule and the product rule as follows

$$\frac{d}{dx}\,[u^v] = \frac{d}{dx}\,[e^{v\ln u}] = e^{v\ln u}\left[v'\ln u + v\,\frac{u'}{u}\right] = u^v\left[v'\ln u + v\,\frac{u1}{u}\right].$$

Theorem 5.5.5 *The following differentiation formulas for the hyperbolic functions are valid.*

(i) $\frac{d}{dx}\,(\sinh x) = \cosh x$

(ii) $\frac{d}{dx}\,(\cosh x) = \sinh x$

(iii) $\frac{d}{dx}\,(\tanh x) = \operatorname{sech}^2 x$

(iv) $\frac{d}{dx}\,(\coth x) = -\operatorname{csch}^2 x$

(v) $\frac{d}{dx}\,(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

(vi) $\frac{d}{dx}\,(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Proof. We use the definitions and properties of hyperbolic functions given in Chapter 1 and the differentiation formulas of this chapter.

(i) $$\frac{d}{dx}\,(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x.$$

(ii) $$\frac{d}{dx}\,(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

(iii) $$\frac{d}{dx}\,(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{(\cosh x)(\cosh x) - \sinh(\sinh x)}{(\cosh x)^2}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{(\cosh x)^2} = \frac{1}{\cosh x)^2} = \operatorname{sech}^2 x$$

(iv) $$\frac{d}{dx}(\coth x) = \frac{d}{dx}(\tanh x)^{-1} = -1(\tanh x)^{-2}\cdot \operatorname{sech}^2 x$$
$$= -\frac{\cosh^2 x}{\sinh^2 x}\cdot\frac{1}{\cosh^2 x} = -\frac{1}{\sinh^2 x}$$
$$= -\operatorname{csch}^2 x.$$

(v) $$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}(\cosh x)^{-1} = -1(\cosh x)^{-2}\cdot\sinh x$$
$$= -\operatorname{sech} x\tanh x.$$

(vi) $$\frac{d}{dx}(\operatorname{csch} x) = \frac{d}{dx}(\sinh x)^{-1} = -1(\sinh x)^{-2}\cdot\cosh x$$
$$= -\coth x\ \operatorname{csch} x.$$

This completes the proof.

Theorem 5.5.6 *The following integration formulas are valid:*

(i) $\int \sinh x\ dx = \cosh x + c$ *(ii)* $\int \cosh x\ dx = \sinh x + c$

(iii) $\int \tanh x\ dx = \ln(\cosh x) + c$ *(iv)* $\int \coth x dx = \ln|\sinh x| + c$

(v) $\int \operatorname{sech} x\ dx = 2\arctan(e^x) + c$ *(vi)* $\int \operatorname{csch} x\ dx = \ln\left|\tanh\left(\frac{x}{2}\right)\right| + c$

Proof. Each formula can be easily verified by differentiating the right-hand side to get the integrands on the left-hand side. This proof is left as an exercise.

Theorem 5.5.7 *The following differentiation and integration formulas are valid:*

(i) $\dfrac{d}{dx}(\operatorname{arcsinh} x) = \dfrac{1}{\sqrt{1+x^2}}$ *(ii)* $\displaystyle\int \frac{dx}{\sqrt{1+x^2}} = \operatorname{arcsinh} x + c$

(iii) $\dfrac{d}{dx}(\operatorname{arccosh} x) = \dfrac{1}{\sqrt{x^2-1}}$ *(iv)* $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh} x + c$

(v) $\dfrac{d}{dx}(\operatorname{arctanh} x) = \dfrac{1}{1-x^2}, |x| < 1$ *(vi)* $\displaystyle\int \frac{1}{1-x^2}\, dx = \operatorname{arctanh} x + c$

Proof. This theorem follows directly from the following definitions:

(1) $\operatorname{arcsinh} x = \ln(x + \sqrt{1+x^2})$ (2) $\operatorname{arccosh} x = \ln(x + \sqrt{x^2-1})$

(3) $\operatorname{arctanh} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right), |x| < 1.$

The proof is left as an exercise.

Exercises 5.5

1. Prove Theorem 5.5.6.
2. Prove Theorem 5.5.7.
3. Show that $\sinh mx$ and $\cosh mx$ are linearly independent if $m \neq 0$. (Hint: Show that the Wronskian $W(\sinh mx, \cosh mx)$ is not zero if $m \neq 0$.)
4. Show that e^{mx} and e^{-mx} are linearly independent if $m \neq 0$.
5. Show that solution of the equation $y'' - m^2 y = 0$ can be expressed as $y = c_1 e^{mx} + c_2 e^{-mx}$.
6. Show that every solution of $y'' - m^2 y = 0$ can be written as $y = A \sinh mx + B \cosh mx$.
7. Determine the relation between c_1 and c_2 in problem 5 with A and B in problem 6.
8. Prove the basic identities for hyperbolic functions:

(i) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

(ii) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$.

(iii) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

(iv) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$.

(v) $\sinh 2x = 2 \sinh x \cosh x$.

(vi) $\cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \cosh 2x$.

(vii) $\cosh^2 x - \sinh^2 x = 1,\ 1 - \tanh^2 x = \text{ sech}^2 x,\ \coth^2 x - 1 = \text{ csch}^2 x$.

9. Eliminate the radical sign using the given substitution:

(i) $\sqrt{a^2 + x^2},\ x = a \sinh t$ (ii) $\sqrt{a^2 - x^2},\ x = \tanh t$

(iii) $\sqrt{x^2 - a^2},\ x = a \cosh t$.

10. Compute y' in each of the following:

(i) $y = 2\sinh(3x) + 4\cosh(2x)$ (ii) $y = 4\tanh(5x) - 6\coth(3x)$

(iii) $y = x \text{ sech }(2x) + x^2 \text{ csch }(5x)$ (iv) $y = 3\sinh^2(4x+1)$

(v) $y = 4\cosh^2(2x-1)$ (vi) $y = \sinh(2x)\cosh(3x)$

11. Compute y' in each of the following:

(i) $y = x^2 e^{-x^3}$ (ii) $y = 2^{x^2}$ (iii) $y = (x^2+1)^{\sin(2x)}$

(iv) $y = \log_{10}(x^2+1)$ (v) $y = \log_2(\sec x + \tan x)$ (vi) $y = 10^{(x^3+1)}$

12. Compute y' in each of the following:

(i) $y = x \ln x - x$ (ii) $y = \ln(x + \sqrt{x^2 - 4})$ (iii) $y = \ln(x + \sqrt{4 + x^2})$

(iv) $y = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$ (v) $y = \text{ arcsinh }(3x)$ (vi) $y = \text{ arccosh }(3x)$

13. Evaluate each of the following integrals:

(i) $\int \sinh(3x)\ dx$ (ii) $\int x^3 e^{x^2}\ dx$ (iii) $\int x^2 \ln(x+1)\ dx$

(iv) $\int x \sinh 2x\ dx$ (v) $\int x \cosh 3x\ dx$ (vi) $\int x 4^{x^2}\ dx$

14. Evaluate each of the following integrals:

(i) $\int \operatorname{arcsinh} x\ dx$ (ii) $\int \operatorname{arccosh} x\ dx$ (iii) $\int \operatorname{arctanh} x\ dx$

(iv) $\int \frac{dx}{\sqrt{4-x^2}}$ (v) $\int \frac{dx}{\sqrt{4+x^2}}$ (vi) $\int \frac{dx}{\sqrt{x^2-4}}$

15. Logarithmic Differentiation is a process of computing derivatives by first taking logarithms and then using implicit differentiation. Find y' in each of the following, using logarithmic differentiation.

(i) $y = \frac{(x^2+1)^3(x^2+4)^{10}}{(x^2+2)^5(x^2+3)^4}$ (ii) $y = (x^2+4)^{(x^3+1)}$

(iii) $y = (\sin x + 3)^{(4\cos x + 7)}$ (iv) $y = (3\sinh x + \cos x + 5)^{(x^3+1)}$

(v) $y = (e^{x^2}+1)^{(2x+1)}$ (vi) $y = x^2(x^2+1)^{(x^3+1)}$

In problems 16–30, compute $f'(x)$ each $f(x)$.

16. $f(x) = \int_1^x \sinh^3(t)dt$

17. $f(x) = \int_x^{x^2} \cosh^5(t)dt$

18. $f(x) = \int_{\sinh x}^{\cosh x} (1+t^2)^{3/2}dt$

19. $f(x) = \int_{\tanh x}^{\operatorname{sech} x} (1+t^3)^{1/2}dt$

20. $f(x) = \int_{\ln x}^{(\ln x)^2} (4+t^2)^{5/2}dt$

21. $f(x) = \int_{e^{x^2}}^{e^{x^2}} (1+4t^2)^{\pi}dt$

22. $f(x) = \int_{e^{\sin x}}^{e^{\cos x}} \frac{1}{(1+t^2)^{3/2}}\, dt$

23. $f(x) = \int_{2^x}^{3^x} \frac{1}{(4+t^2)^{5/2}}\, dt$

24. $f(x) = \int_{4^{2x}}^{5^{3x}} (1+2t^2)^{3/2} dt$

25. $f(x) = \int_{\log_2 x}^{\log_3 x} (1+5t^3)^{1/2} dt$

26. $f(x) = \int_{\operatorname{arcsinh} x}^{\operatorname{arccosh} x} \frac{1}{(1+t^2)^{3/2}}\, dt$

27. $f(x) = \int_{2^{x^2}}^{4^{x^3}} e^{t^2} dt$

28. $f(x) = \int_{4^{\sin x}}^{5^{\cos x}} e^{-t^2} dt$

29. $f(x) = \int_{\sinh(x^2)}^{\cosh(x^3)} e^{-t^3} dt$

30. $f(x) = \int_{\operatorname{arctanh} x}^{\operatorname{arccoth} x} \sin(t^2) dt$

In problems 31–40, evaluate the given integrals.

31. $\int \frac{e^{\arctan x}}{1+x^2}\, dx$

32. $\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\, dx$

33. $\int e^{\sin 2x} \cos 2x\, dx$

34. $\int x^2 e^{x^3} dx$

35. $\int \frac{e^{2x}}{1+e^{2x}}\, dx$

36. $\int e^x \cos(1+2e^x) dx$

37. $\int e^{3x} \sec^2(2+e^{3x}) dx$

38. $\int 10^{\cos x} \sin x\, dx$

39. $\int \frac{4^{\operatorname{arcsec} x}}{x\sqrt{x^2-1}}\, dx$

40. $\int x\ 10^{x^2+3}\, dx$

5.6 The Riemann Integral

In defining the definite integral, we restricted the definition to continuous functions. However, the definite integral as defined for continuous functions is a special case of the general Riemann Integral defined for bounded functions that are not necessarily continuous.

Definition 5.6.1 Let f be a function that is defined and bounded on a closed and bounded interval $[a, b]$. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of $[a, b]$. Let $C = \{c_i : x_{i-1} \leq c_i \leq x_i, i = 1, 2, \cdots, n\}$ be any arbitrary selection of points of $[a, b]$. Then the *Riemann Sum* that is associated with P and C is denoted $R(P)$ and is defined by

$$\begin{aligned} R(P) &= f(c_1)(x_1 - x_0) + f(c_2)(x_2 - x_1) + \cdots + f(c_n)(x_n + x_{n-1}) \\ &= \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}). \end{aligned}$$

Let $\Delta x_i = x_i - x_{i-1},\ i = 1, 2, \cdots, n$. Let $||\Delta|| = \max_{1 \leq i \leq n} \{\Delta x_i\}$. We write

$$R(P) = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

We say that

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = I$$

if and only if for each $\epsilon > 0$ there exists some $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} f(c_i)\Delta x_i - I \right| < \epsilon$$

whenever $||\Delta|| < \delta$ for all partitions P and all selections C that define the Riemann Sum.

If the limit I exists as a finite number, we say that f is (Riemann) integrable and write

$$I = \int_a^b f(x)\ dx.$$

Next we will show that if f is continuous, the Riemann integral of f is the definite integral defined by lower and upper sums and it exists. We first prove two results that are important.

Definition 5.6.2 A function f is said to be *uniformly continuous* on its domain D if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $|x_1 - x_2| < \delta$, for any x_1 and x_2 in D, then

$$|f(x_1) - f(x_2)| < \epsilon.$$

Definition 5.6.3 A collection $C = \{U_\alpha : U_\alpha \text{ is an open interval}\}$ is said to *cover* a set D if each element of D belongs to some element of C.

Theorem 5.6.1 *If $C = \{U_\alpha : U_\alpha$ is an open interval$\}$ covers a closed and bounded interval $[a, b]$, then there exists a finite subcollection $B = \{U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_n}\}$ of C that covers $[a, b]$.*

Proof. We define a set A as follows:

$$A = \{x : x \in [a, b] \text{ and } [a, x] \text{ can be covered by a finite subcollection of } C\}.$$

Since $a \in A$, A is not empty. A is bounded from above by b. Then A has a least upper bound, say $\text{lub}(A) = p$. Clearly, $p \leq b$. If $p < b$, then some U_α in C contains p. If $U_\alpha = (a_\alpha, b_\alpha)$, then $a_\alpha < p < b_\alpha$. Since $p = \ell\text{ub}(A)$, there exists some point a^* of A between a_α and p. There exists a subcollection $B = \{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ that covers $[a, a^*]$. Then the collection $B_1 = \{U_{\alpha_1}, \cdots, U_{\alpha_n}, U_\alpha\}$ covers $[a, b_\alpha)$. By the definition of A, A must contain all points of $[a, b]$ between p and b_α. This contradicts the assumption that $p = \ell\text{ub}(A)$. So, $p = b$ and $b \in A$. It follows that some finite subcollection of C covers $[a, b]$ as required.

Theorem 5.6.2 *If f is continuous on a closed and bounded interval $[a, b]$, then f is uniformly continuous on $[a, b]$.*

Proof. Let $\epsilon > 0$ be given. If $p \in [a, b]$, then there exists $\delta_p > 0$ such that $|f(x) - f(p)| < \epsilon/3$, whenever $p - \delta_p < x < p + \delta_p$. Let $U_p = \left(p - \frac{1}{3}\,\delta_p,\ p + \frac{1}{3}\,\delta_p\right)$. Then $C = \{U_p : p \in [a, b]\}$ covers $[a, b]$. By Theorem 5.6.1, some finite subcollection $B = \{U_{p_1}, U_{p_2}, \ldots, U_{p_n}\}$ of C covers $[a, b]$. Let $\delta = \frac{1}{3} \min\{\delta_{p_i} : i = 1, 2, \cdots, n\}$. Suppose that $|x_1 - x_2| < \delta$ for any two points x_1 and x_2 of $[a, b]$. Then $x_1 \in U_{p_i}$ and $x_2 \in U_{p_j}$ for some p_i and p_j. We note that

$$\begin{aligned}
|p_i - p_j| &= |(p_i - x_1) + (x_1 - x_2) + (x_2 - p_j)| \\
&\leq |p_i - x_i| + |x_1 - x_2| + |x_2 - p_j| \\
&< \frac{1}{3}\,\delta_{p_i} + \delta + \frac{1}{3}\,\delta_{p_j} \\
&\leq \max\{\delta_{p_i}, \delta_{p_j}\}.
\end{aligned}$$

It follows that both p_i and p_j are either in U_{p_i} or U_{p_j}. Suppose that p_i and p_j are both in U_{p_i}. Then

$$\begin{aligned}|x_2 - p_i| &= |(x_2 - x_1)| + (x_1 - p_i)| \\ &\le |x_2 - x_1| + |x_1 - p_i| \\ &< \delta + \frac{1}{3}\,\delta_{p_i} \\ &< \delta_{p_i}.\end{aligned}$$

So, x_1, x_2, p_i and p_j are all in U_{p_i}. Then

$$\begin{aligned}|f(x_1) - f(x_2)| &= |(f(x_1) - f(p_i)) + (f(p_i) - f(x_2))| \\ &\le |f(x_1) - f(p_i)| + |f(p_i) - f(x_2)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &< \epsilon.\end{aligned}$$

By Definition 5.6.2, f is uniformly continuous on $[a, b]$.

Theorem 5.6.3 *If f is continuous on $[a, b]$, then f is (Riemann) integrable and the definite integral and the Riemann integral have the same value.*

Proof. Let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ be a partition of $[a, b]$ and $C = \{c_i : x_{i-1} \le c_i \le x_i, i = 1, 2, \ldots, n\}$ be an arbitrary selection. For each $i = 1, 2, \ldots, n$ let

m_i = absolute minimum of f on $[x_{i-1}, x_i]$ obtained at c_i^*, $f(c_i^*) = m_i$;
M_i = absolute maximum of f on $[x_{i-1}, x_i]$ obtained at $c_i^{**}) = M_i$;
m = absolute minimum of f on $[a, b]$;
M = absolute maximum of f on $[a, b]$;
$R(P) = \displaystyle\sum_{i=1}^{n} f(c_i)\Delta x_i$,

Then for each $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}m(b-a) &\le \sum_{i=1}^{n} f(c_i^*)(x_i - x_{i-1}) \le \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \\ &\le \sum_{i=1}^{n} f(c_i^{**})(x_i - x_{i-1}) \le M(b-a).\end{aligned}$$

We recall that

$$L(P) = \sum_{i=1}^{n} f(c_i^*)\Delta x_i, \; R(P) = \sum_{i=1}^{n} f(c_i)\Delta x_i, U(P) = \sum_{i=1}^{n} f(c_i^{**})\Delta x_i.$$

We note that $L(P)$ and $U(P)$ are also Riemann sums and for every partition P, we have

$$L(P) \leq R(P) \leq U(P).$$

To prove the theorem, it is sufficient to show that

$$\text{lub}\{L(P)\} = \text{glb}\{U(P)\}.$$

Since f is uniformly continuous, by Theorem 5.6.2, for each $\epsilon > 0$ there is some $\delta > 0$ such that $|f(x)-f(y)| < \dfrac{\epsilon}{b-a}$ whenever $|x-y| < \delta$ for x and y in $[a, b]$. Consider all partitions P, selections $C = \{c_i\}, C^* = \{c_i^*\}, C^{**} = \{c_i^{**}\}$ such that

$$||\Delta|| = \max_{1\leq i\leq n}(x_i - x_{i-1}) < \frac{\delta}{3}.$$

Then, for each $i = 1, 2, \ldots, n$

$$\begin{aligned}
|f(c_i^{**}) - f(c_i^*)| &< \frac{\epsilon}{b-a} \\
|f(c_i^*) - f(c_i)| &< \frac{\epsilon}{b-a} \\
|f(c_i^{**}) - f(c_i)| &< \frac{\epsilon}{b-a} \\
|U(P) - L(P)| &= \left|\sum_{i=1}^{n}(f(c_i^{**}) - f(c_i^*))\Delta x_i\right| \\
&\leq \sum_{i=1}^{m} |f(c_i^{**}) - f(c_i^*)|\Delta x_i \\
&< \frac{\epsilon}{b-a} \; \sum_{i=1}^{n} \Delta x_i \\
&= \epsilon.
\end{aligned}$$

It follows that

$$\text{lub}\{L(P)\} = \lim_{||\Delta||\to 0} R(P)p = \text{glb}\{U(P)\} = I.$$

By definition of the definite integral, I equals the definite integral of $f(x)$ from $x = a$ to $x = b$, which is also the Riemann integral of f on $[a, b]$. We write

$$I = \int_a^b f(x)\ dx.$$

This proves Theorem 5.6.2 as well as Theorem 5.2.1.

Exercises 5.6

1. Prove Theorem 5.2.3. (Hint: For each partition $P = \{a = x_0 < x_1 < \ldots < x_n = b]$ of $[a, b]$,

$$\begin{aligned} g(b) - g(a) &= [g(x_n) - g(x_{n-1})] + [g(x_{n-1}) - g(x_{n-2})] + \ldots + [g(x_1) - g(x_0)] \\ &= \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})] \\ &= \sum_{i=1}^{n} g'(c_i)(x_i - x_{i-1}) \qquad \text{(by Mean Value Theorem)} \\ &= \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \\ &= R(P) \end{aligned}$$

for some selection $C = \{c_i : x_{i-1} < c_i < x_i, i = 1, 2, \cdots, n\}$.)

2. Prove Theorem 5.2.3 on the linearity property of the definite integral. (Hint:

$$\begin{aligned} \int_a^b [Af(x) + bg(x)]\ dx &= \lim_{||\Delta|| \to 0} \left\{ \sum_{i=1}^{n} [Af(c_i) + Bg(c_i)] \cdot [x_i - x_{i-1}) \right\} \\ &= \lim_{||\Delta|| \to 0} \left(A \sum_{i=1}^{n} f(c_i)\Delta x_i + B \sum_{i=1}^{n} g(c_i)\Delta x_i \right) \\ &= A \left(\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \right) + B \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} g(c_i)\Delta x_i \\ &= A \int_a^b f(x)dx + B \int_a^b g(x)dx.) \end{aligned}$$

3. Prove Theorem 5.2.4.

   (Hint: $[a, b] = [a, c] \cup [c, b]$. If $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ is a partition of $[a, b]$, then for some i, $P_1 = \{a = x_0 < \ldots < x_{i-1} < c < x_i < \ldots < x_n = b\}$ yields a partition of $[a, b]$; $\{a < x_0 < \cdots < x_{i-1} < c\}$ is a partition of $[a, c]$ and $\{c < x_i < \cdots < x_n = b\}$ is a partition of $[c, b]$. The addition of c to the partition does not increase $||\Delta||$.)

4. Prove Theorem 5.2.5.

   (Hint: For each partition P and selection C we have

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} g(c_i)(x_i - x_{i-1}).)$$

5. Prove that if f is continuous on $[a, b]$ and $f(x) > 0$ for each $x \in [a, b]$, then

$$\int_a^b f(x)\ dx > 0.$$

   (Hint: There is some c in $[a, b]$ such that $f(c)$ is the absolute minimum of f on $[a, b]$ and $f(c) > 0$. Then argue that

$$0 < f(c)(b - a) \leq L(P) \leq U(P)$$

   for each partition P.)

6. Prove that if f and g are continuous on $[a, b]$, $f(x) > g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)\ dx > \int_a^b g(x)\ dx.$$

   (Hint: By problem 5,

$$\int_a^b (f(x) - g(x))\ dx > 0.$$

   Use the linearity property to prove the statement.)

7. Prove that if f is continuous on $[a, b]$, then

$$\left| \int_a^b f(x)\ dx \right| \leq \int_a^b |f(x)| dx.$$

(Hint: Recall that $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a,b]$. Use problem 5 to conclude the result.)

8. Prove the Mean Value Theorem, Theorem 5.2.6.

   (Hint: Let

   $m =$ absolute minimum of f on $[a,b]$;

   $M =$ absolute minimum of f on $[a,b]$;

$$f_{av}[a,b] = \frac{1}{b-a}\int_a^b f(x)\ dx;$$
$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

   Then $m \leq f_{av}[a,b] \leq M$. By the intermediate value theorem for continuous functions, there exists some c on $[a,b]$ such that $f(c) = f_{av}[a,b]$.)

9. Prove the Fundamental Theorem of Calculus, First Form, Theorem 5.2.6.

   (Hint:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x} f(t)\ dx + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_x^{x+h} f(t)dt\right] \\ &= \lim_{h\to 0} f(c), (\text{for some } c,\ x \leq c \leq x+h;) \\ &= f(x) \end{aligned}$$

   where $x \leq c \leq x+h$, by Theorem 5.2.6.)

10. Prove the Leibniz Rule, Theorem 5.2.8.

    (Hint:

$$\int_{\alpha(x)}^{\beta(x)} f(t)dt = \int_a^{\beta(x)} f(t)dt - \int_a^{\alpha(x)} f(t)dt$$

    for some a. Now use the chain rule of differentiation.)

11. Prove that if f and g are continuous on $[a, b]$ and g is nonnegative, then there is a number c in (a, b) for which

$$\int_a^b f(x)g(x)\ dx = f(c)\int_a^b g(x)\ dx.$$

(Hint: If m and M are the absolute minimum and absolute maximum of f on $[a, b]$, then $mg(x) \leq f(x)g(x) \leq Mg(x)$. By the Order Property,

$$m\int_a^b g(x)\ dx \leq \int_a^b f(x)g(x) \leq M\int_a^b g(x)\ dx$$

$$m \leq \frac{\int_a^b f(x)g(x)\ dx}{\int_a^b g(x)\ dx} \leq M \quad \left(\text{if } \int_0^b g(x)\ dx \neq 0\right).$$

By the Intermediate Value Theorem, there is some c such that

$$f(x) = \frac{\int_a^b f(x)g(x)\ dx}{\int_a^b g(x)\ dx} \text{ or}$$

$$\int_a^b f(x)g(x)\ dx = f(c)\int_a^b g(x)\ dx.$$

If $\displaystyle\int_a^b g(x)\ dx = 0$, then $g(x) \not\equiv 0$ on $[a, b]$ and all integrals are zero.)

Remark 20 The number $f(c)$ is called the weighted average of f on $[a, b]$ with respect to the weight function g.

5.7 Volumes of Revolution

One simple application of the Riemann integral is to define the volume of a solid.

Theorem 5.7.1 *Suppose that a solid is bounded by the planes with equations $x = a$ and $x = b$. Let the cross-sectional area perpendicular to the x-axis at x be given by a continuous function $A(x)$. Then the volume V of the solid is given by*

$$V = \int_a^b A(x)\ dx.$$

Proof. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of $[a, b]$. For each $i = 1, 2, 3, \cdots, n$, let

V_i = volume of the solid between the planes with equations $x = x_{i-1}$ and $x = x_i$,

m_i = absolute minimum of $A(x)$ on $[x_{i-1}, x_i]$,

M_i = absolute maximum of $A(x)$ on $[x_{i-1}, x_i]$,

$\Delta x_i = x_i - x_{i-1}$.

Then

$$m_i \Delta x_i \leq V_i \leq M_i \Delta x_i, m_i \leq \frac{V_i}{\Delta x_i} \leq M_i.$$

Since $A(x)$ is continuous, there exists some c_i such that $x_{i-1} \leq c_i \leq M_i$ and

$$m_i \leq A(c_i) = \frac{V_i}{\Delta x_i} \leq M_i$$

$$V_i = A(c_i)\Delta x_i$$

$$V = \sum_{i=1}^{n} A(c_i)\Delta x_i.$$

It follows that for each partition P of $[a, b]$ there exists a Riemann sum that equals the volume. Hence, by definition,

$$V = \int_a^b A(x)\, dx.$$

Theorem 5.7.2 *Let f be a function that is continuous on $[a, b]$. Let R denote the region bounded by the curves $x = a,\ x = b,\ y = 0$ and $y = f(x)$. Then the volume V obtained by rotating R about the x-axis is given by*

$$V = \int_a^b \pi (f(x))^2 dx.$$

Proof. Clearly, the volume of the rotated solid is between the planes with equations $x = a$ and $x = b$. The cross-sectional area at x is the circle generated by the line segment joining $(x, 0)$ and $(x, f(x))$ and has area $A(x) = \pi(f(x))^2$. Since f is continuous, $A(x)$ is a continuous function of x. Then by Theorem 5.7.1, the volume V is given by

$$V = \int_a^b \pi (f(x))^2\, dx.$$

Theorem 5.7.3 *Let f and R be defined as in Theorem 5.7.2. Assume that $f(x) > 0$ for all $x \in [a, b]$, either $a \geq 0$ or $b \leq 0$, so that $[a, b]$ does not contain 0. Then the volume V generated by rotating the region R about the y-axis is given by*

$$V = \int_a^b (2\pi x f(x))\ dx.$$

Proof. The line segment joining $(x, 0)$ and $(x, f(x))$ generates a cylinder whose area is $A(x) = 2\pi x f(x)$. We can see this if we cut the cylinder vertically at $(-x, 0)$ and flattening it out. By Theorem 5.7.1, we get

$$V = \int_a^b 2\pi x f(x)\ dx.$$

Theorem 5.7.4 *Let f and g be continuous on $[a, b]$ and suppose that $f(x) > g(x) > 0$ for all x on $[a, b]$. Let R be the region bounded by the curves $x = a,\ x = b,\ y = f(x)$ and $y = g(x)$.*

(i) The volume generated by rotating R about the x-axis is given by

$$\int_a^b \pi[(f(x))^2 - (g(x))^2]\ dx.$$

(ii) If we assume R does not cross the y-axis, then the volume generated by rotating R about the y-axis is given by

$$V = \int_a^b 2\pi x[f(x) - g(x)]dx.$$

(iii) If, in part (ii), R does not cross the line $x = c$, then the volume generated by rotating R about the line $x = c$ is given by

$$V = \int_a^b 2\pi|c - x|[f(x) - g(x)]dx.$$

Proof. We leave the proof as an exercise.

Remark 21 There are other various horizontal or vertical axes of rotation that can be considered. The basic principles given in these theorems can be used. Rotations about oblique lines will be considered later.

Example 5.7.1 Suppose that a pyramid is 16 units tall and has a square base with edge length of 5 units. Find the volume of V of the pyramid.

graph

We let the y-axis go through the center of the pyramid and perpendicular to the base. At height y, let the cross-sectional area perpendicular to the y-axis be $A(y)$. If $s(y)$ is the side of the square $A(y)$, then using similar triangles, we get

$$\frac{s(y)}{5} = \frac{16-y}{16}, s(y) = \frac{5}{16}(16-y)$$
$$A(y) = \frac{25}{256}(16-y)^2.$$

Then the volume of the pyramid is given by

$$\begin{aligned}
\int_0^{16} A(y)dy &= \int_0^{16} \frac{25}{256}(16-y)^2 dy \\
&= \frac{25}{256}\left[\frac{(16-y)^3}{-3}\right]_0^{16} \\
&= \frac{25}{256}\left[\frac{(16)^3}{3}\right] = \frac{(25)(16)}{3} \\
&= \frac{400}{3} \text{ cubic units.} \\
\mathit{Check}: \ V &= \frac{1}{3}(\text{base side})^2 \cdot \ \text{height} \\
&= \frac{1}{3}(25)\cdot 16 \\
&= \frac{400}{3}.
\end{aligned}$$

Example 5.7.2 Consider the region R bounded by $y = \sin x,\ y = 0,\ x = 0$ and $x = \pi$. Find the volume generated when R rotated about

(i) x-axis (ii) y-axis (iii) $y = -2$ (iv) $y = 1$
(v) $x = \pi$ (vi) $x = 2\pi$.

(i) By Theorem 5.7.2, the volume V is given by

$$\begin{aligned} V &= \int_0^\pi \pi \sin^2 x\ dx \\ &= \pi \cdot \left[\frac{1}{2}\ (x - \sin x \cos x) \right]_0^\pi \\ &= \frac{\pi^2}{2}. \end{aligned}$$

graph

(ii) By Theorem 5.7.3, the volume V is given by (integrating by parts)

$$\begin{aligned} V &= \int_0^\pi 2\pi x \sin x\ dx \quad ; \quad (u = x,\ dv = \sin x\ dx) \\ &= 2\pi[-x \cos x + \sin x]_0^\pi \\ &= 2\pi[\pi] \\ &= 2\pi^2. \end{aligned}$$

graph

(iii) In this case, the volume V is given by

$$\begin{aligned} V &= \int_0^\pi \pi(\sin x + 2)^2 \ dx \\ &= \int_0^\pi \pi[\sin^2 x + 4\sin x + 4] \ dx \\ &= \pi \left[\frac{1}{2} \ (x - \sin x \cos x) - 4\cos x + 4x \right]_0^\pi \\ &= \pi \left[\frac{1}{2} \ \pi + 8 + 4\pi \right] \\ &= \frac{9}{2}\pi^2 + 8\pi. \end{aligned}$$

graph

(iv) In this case,

$$V = \int_0^\pi \pi[1^2 - (1 - \sin x)^2] \ dx.$$

graph

$$\begin{aligned}
V &= \int_0^{\pi} \pi[1 - 1 + 2\sin x - sin^2 x]dx \\
&= \pi\left[-2\cos x - \frac{1}{2}\ (x - \sin x \cos x)\right]_0^{\pi} \\
&= \pi\left[4 - \frac{1}{2}\ (\pi)\right] \\
&= \frac{\pi(8-\pi)}{2}.
\end{aligned}$$

(v)

$$\begin{aligned}
V &= \int_0^{\pi} (2\pi(\pi - x)\sin x]\ dx \\
&= 2\pi \int_0^{\pi} [\pi \sin x - x\sin x]\ dx \\
&= 2\pi[-\pi\cos x + x\cos x - \sin x]_0^{\pi} \\
&= 2\pi[2\pi - \pi] \\
&= 2\pi^2.
\end{aligned}$$

graph

(vi)

$$\begin{aligned}
V &= \int_0^{\pi} 2\pi(2\pi - x)\sin x\ dx \\
&= 2\pi[-2\pi\cos x + x\cos x - \sin x]_0^{\pi} \\
&= 2\pi[4\pi - \pi] \\
&= 6\pi^2.
\end{aligned}$$

graph

Example 5.7.3 Consider the region R bounded by the circle $(x-4)^2+y^2=4$. Compute the volume V generated when R is rotated around

(i) $y=0$ (ii) $x=0$ (iii) $x=2$

graph

(i) Since the area crosses the x-axis, it is sufficient to rotate the top half to get the required solid.

$$V = \int_2^6 \pi y^2\ dx = \pi \int_2^6 [4-(x-4)^2]\ dx$$
$$= \pi \left[4x - \frac{1}{3}\ (x-4)^3\right]_2^6 = \pi \left[16 - \frac{8}{3} - \frac{8}{3}\right] = \frac{32}{3}\pi.$$

This is the volume of a sphere of radius 2.

(ii) In this case,

$$V = \int_2^6 2\pi x(2y)\ dx = 4\pi \int_2^6 x[\sqrt{4-(x-4)^2}]dx\ ; x-4 = 2\sin t$$
$$dx = 2\cos t dt$$
$$= 4\pi \int_{-\pi/2}^{\pi/2} (4+2\sin t)(2\cos t)(2\cos t)dt$$
$$= 4\pi \int_{-\pi/2}^{\pi/2} (16\cos^2 t + 8\cos^2 t \sin t)\ dx$$
$$= 4\pi \left[16 \cdot \frac{1}{2}\ (t + \sin t \cos t) - \frac{8}{3}\ \cos^3 t\right]_{-\pi/2}^{\pi/2}$$
$$= 4\pi[8(\pi)]$$
$$= 32\pi^2$$

(iii) In this case,

$$
\begin{aligned}
V &= \int_2^6 2\pi(x-2)2y\ dx \\
&= 4\pi \int_2^6 (x-2)\sqrt{4-(x-4)^2}\ dx\ ; x-4 = 2\sin t \\
&\qquad\qquad\qquad\qquad\qquad\qquad dx = 2\cos t dt \\
&= 4\pi \int_{-\pi/2}^{\pi/2} (2+2\sin t)(2\cos t)(2\cos t)dt \\
&= 4\pi \int_{-\pi/2}^{\pi/2} (8\cos^2 t + 8\cos^2 t\sin t)dt \\
&= 4\pi \left[4(t+\sin t\cos t) - \frac{8}{3}\ \cos^3 t\right]_{-\pi/2}^{\pi/2} \\
&= 4\pi[4\pi] \\
&= 16\pi^2
\end{aligned}
$$

Exercises 5.7

1. Consider the region R bounded by $y = x$ and $y = x^2$. Find the volume generated when R is rotated around the line with equation

   (i) $x = 0$ (ii) $y = 0$ (iii) $y = 1$ (iv) $x = 1$
   (v) $x = 4$ (vi) $x = -1$ (vii) $y = -1$ (viii) $y = 2$

2. Consider the region R bounded by $y = \sin x,\ y = \cos x,\ x = 0,\ x = \frac{\pi}{2}$. Find the volume generated when R is rotated about the line with equation

   (i) $x = 0$ (ii) $y = 0$ (iii) $y = 1$ (iv) $x = \frac{\pi}{2}$

3. Consider the region R bounded by $y = e^x,\ x = 0,\ x = \ln 2,\ y = 0$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $x = \ln 2$ (iv) $y = -2$
   (v) $y = 2$ (iv) $x = 2$

4. Consider the region R bounded by $y = \ln x,\ y = 0,\ x = 1,\ x = e$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $x = 1$ (v) $x = e$

   (v) $y = 1$ (vi) $y = -1$

5. Consider the region R bounded by $y = \cosh x,\ y = 0,\ x = -1,\ x = 1$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 2$ (iii) $x = 1$ (iv) $y = -1$

   (v) $y = 6$ (vi) $x = 0$

6. Consider the region R bounded by $y = x,\ y = x^3$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $x = -1$ (iv) $x = 1$

   (v) $y = 1$ (vi) $y = -1$

7. Consider the region R bounded by $y = x^2,\ y = 8 - x^2$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $y = -4$ (iv) $y = 8$

   (v) $x = -2$ (vi) $x = 2$

8. Consider the region R bounded by $y = \sinh x,\ y = 0,\ x = 0,\ x = 2$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $x = 2$ (iv) $x = -2$

   (v) $y = -1$ (vi) $y = 10$

9. Consider the region R bounded by $y = \sqrt{x},\ y = 4,\ x = 0$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $x = 16$ (iv) $y = 4$

10. Compute the volume of a cone with height h and radius r.

5.8 Arc Length and Surface Area

The Riemann integral is useful in computing the length of arcs. Let f and f' be continuous on $[a, b]$. Let C denote the arc

$$C = \{(x, f(x)) : a \le x \le b\}.$$

Let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ be a partition of $[a, b]$. For each $i = 1, 2, \ldots, n$, let

graph

$$\begin{aligned}
\Delta x_i &= x_i - x_{i-1} \\
\Delta y_i &= f(x_i) - f(x_{i-1}) \\
\Delta s_i &= \sqrt{(f(x_i) - f(x_{i-1}))^2 + (x_i - x_i)^2} \\
||\Delta|| &= \max_{1 \le i \le n} \{\Delta x_n\}.
\end{aligned}$$

Then Δs_i is the length of the line segment joining the two points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$. Let

$$A(P) = \sum_{i=1}^{n} \Delta s_i.$$

Then $A(P)$ is called the polygonal approximation of C with respect to the portion P.

Definition 5.8.1 Let $C = \{(x, f(x)) : x \in [a, b]\}$ where f and f' are continuous on $[a, b]$. Then the arc length L of the arc C is defined by

$$L = \lim_{||\Delta|| \to 0} Ap = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} \sqrt{(f(x_i) - f(x_{i-1}))^2 + (x_i - x_{i-1})^2}.$$

Theorem 5.8.1 *The arc length L defined in Definition 5.8.1 is given by*

$$L = \int_a^b \sqrt{(f'(x))^2 + 1}\ dx.$$

Proof. By the Mean Value Theorem, for each $i = 1, 2, \dots, n$,

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$$

for some c_i such that $x_{i-1} < c_i < x_i$. Therefore, each polynomial approximation Ap is a Riemann Sum of the continuous function

$$\sqrt{(f'(x))^2 + 1}$$

$$A(P) = \sum_{i=1}^{n} \sqrt{(f'(c_i))^2 + 1}\ \Delta x_i,$$

for some c_i such that $x_{i-1} < c_i < x_i$.

By the definition of the Riemann integral, we get

$$L = \int_a^b \sqrt{(f'(x))^2 + 1}\ dx.$$

Example 5.8.1 Let $C = \{(x, \cosh x) : 0 \le x \le 2\}$. Then the arc length L of C is given by

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \sinh^2 x}\ dx \\ &= \int_0^2 \cosh x\ dx \\ &= [\sinh x]_0^2 \\ &= \sinh 2. \end{aligned}$$

Example 5.8.2 Let $C = \left\{\left(x, \frac{2}{3}\ x^{3/2}\right) : 0 \le x \le 4\right\}$. Then the arc length L of the curve C is given by

$$\begin{aligned} L &= \int_0^4 \sqrt{1 + \left(\frac{2}{3} \cdot \frac{3}{2}\ x^{1/2}\right)^2}\ dx \\ &= \int_0^4 (1 + x)^{1/2}\ dx \\ &= \left[\frac{2}{3}\ (1 + x)^{3/2}\right]_0^4 \\ &= \frac{2}{3}\ [5\sqrt{5} - 1]. \end{aligned}$$

Definition 5.8.2 Let C be defined as in Definition 5.8.1.

(i) The surface area S_x generated by rotating C about the x-axis is given by

$$S_x = \int_a^b 2\pi |f(x)| \sqrt{(f'(x))^2 + 1}\ dx.$$

(ii) The surface area S_y generated by rotating C about the y-axis

$$S_y = \int_a^b 2\pi |x| \sqrt{(f'(x))^2 + 1}\ dx.$$

Example 5.8.3 Let $C = \{(x, \cosh x) : 0 \leq x \leq 4\}$.

(i) Then the surface area S_x generated by rotating C around the x-axis is given by

$$\begin{aligned} S_x &= \int_0^4 2\pi \cosh x \sqrt{1 + \sinh^2 x}\ dx \\ &= 2\pi \int_0^4 \cosh^2 x\ dx \\ &= 2\pi \left[\frac{1}{2}\ (x + \sinh x \cosh x) \right]_0^4 \\ &= \pi[4 + \sinh 4 \cosh 4]. \end{aligned}$$

(ii) The surface area S_y generated by rotating the curve C about the y-axis is given by

$$\begin{aligned} S_y &= \int_0^4 2\pi x \sqrt{1 + \sinh^2 x}\ dx \\ &= 2\pi \int_0^4 x \cosh x\ dx\ ;\ (u = x,\ dv = \cosh x\ dx) \\ &= 2\pi[x \sinh x - \cosh x]_0^4 \\ &= 2\pi[4 \sinh 4 - \cosh 4 + 1] \end{aligned}$$

Theorem 5.8.2 *Let* $C = \{(x(t), y(t)) : a \le t \le b\}$. *Suppose that* $x'(t)$ *and* $y'(t)$ *are continuous on* $[a, b]$.

(i) The arc length L *of* C *is given by*

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\; dt.$$

(ii) The surface area S_x *generated by rotating* C *about the* x*-axis is given by*

$$S_x = \int_a^b 2\pi |y(t)| \sqrt{(x'(t))^2 + (y'(t))^2}\; dt.$$

(iii) The surface area S_y *generated by rotating* C *about the* y*-axis is given by*

$$S_y = \int_a^b 2\pi |x(t)| \sqrt{(x'(t))^2 + (y'(t))^2}\; dt.$$

Proof. The proof of this theorem is left as an exercise.

Example 5.8.4 Let $C = \left\{(e^t \sin t, e^t \cos t) : 0 \le t \le \frac{\pi}{2}\right\}$. Then

$$\begin{aligned} ds &= \sqrt{(x'(t))^2 + (y'(t))^2}\; dt \\ &= \sqrt{(e^t(\sin t + \cos t))^2 + (e^t(\cos t - \sin t))^2}\; dt \\ &= \{e^{2t}(\sin^2 t + \cos^2 t + 2 \sin t \cos t + \cos^2 t + \sin^2 t - 2 \cos t \sin t)\}^{1/2}\; dt \\ &= e^t \sqrt{2}\; dt. \end{aligned}$$

(i) The arc length L of C is given by

$$\begin{aligned} L &= \int_0^{\pi/2} \sqrt{2} e^t dt \\ &= \sqrt{2}\left[e^t\right]_0^{\pi/2} \\ &= \sqrt{2}\left(e^{\pi/2} - 1\right). \end{aligned}$$

(ii) The surface area S_x obtained by rotating C about the x-axis is given by

$$\begin{aligned} S_x &= \int_0^{\pi/2} 2\pi(e^t \cos t)(\sqrt{2}e^t dt) \\ &= 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t} \cos t dt \\ &= 2\sqrt{2}\pi \left[\frac{e^{2t}}{5}\ (2\cos t + \sin t)\right]_0^{\pi/2} \\ &= 2\sqrt{2}\pi \left[\frac{e^{\pi}}{5}\ (1) - \frac{2}{5}\right] \\ &= \frac{2\sqrt{2}\pi}{5}\ (e^{\pi} - 2). \end{aligned}$$

(iii) The surface area S_y obtained by rotating C about the y-axis is given by

$$\begin{aligned} S_y &= \int_0^{\pi/2} 2\pi(e^t \sin t)(\sqrt{2}e^t dt) \\ &= 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t} \sin t dt \\ &= 2\sqrt{2}\pi \left[\frac{e^{2t}}{5}\ [2\sin t - \cos t]\right]_0^{\pi/2} \\ &= 2\sqrt{2}\pi \left[\frac{2e^{\pi}}{5} + \frac{1}{5}\right] = \frac{2\sqrt{2}\pi}{5}\ (2e^{\pi} + 1). \end{aligned}$$

Exercises 5.8 Find the arc lengths of the following curves:

1. $y = x^{3/2},\ 0 \le x \le 4$
2. $y = \dfrac{1}{3}\ (x^2 + 2)^{3/2},\ 0 \le x \le 1$
3. $C = \left\{(4(\cos t + t\sin t),\ 4(\sin t - t\cos t)) : 0 \le t \le \dfrac{\pi}{2}\right\}$
4. $x(t) = a(\cos t + t\sin t),\ y(t) = a(\sin t - t\cos t),\ 0 \le t \le \dfrac{\pi}{2}$
5. $x(t) = \cos^3 t, y(t) = \sin^3 t,\ 0 \le t \le \pi/2$

6. $y = \dfrac{1}{2}\, x^2,\ 0 \le t \le 1$

7. $x(t) = t^3,\ y(t) = t^2,\ 0 \le t \le 1$

8. $x(t) = 1 - \cos t,\ y(t) = t - \sin t,\ 0 \le t \le 2\pi$

9. In each of the curves in exercises 1-8, set up the integral that represents the surface area generated when the given curve is rotated about

   (a) the x-axis

   (b) the y-axis

10. Let $C = \{(x, \cosh x) : -1 \le x \le 1\}$

    (a) Find the length of C.

    (b) Find the surface area when C is rotated around the x-axis.

    (c) Find the surface area when C is rotated around the y-axis.

In exercises 11–20, consider the given curve C and the numbers a and b. Determine the integral that represents:

(a) Arc length of C

(b) Surface area when C is rotated around the x-axis.

(c) Surface area when C is rotated around the y-axis.

(d) Surface area when C is rotated around the line $x = a$.

(e) Surface area when C is rotated around the line $y = b$.

11. $C = \{(x, \sin x) : 0 \le x \le \pi\};\ a = \pi, b = 1$

12. $C = \left\{(x, \cos x) : 0 \le x \le \dfrac{\pi}{3}\right\};\ a = \pi,\ b = 2$

13. $C = \{(t, \ln t) : 1 \le t \le e\};\ a = 4,\ b = 3$

14. $C = \{(2 + \cos t, \sin t) : 0 \le t \le \pi\};\ a = 4,\ b = -2$

15. $C = \left\{(t, \ln \sec t) : 0 \le t \le \dfrac{\pi}{3}\right\};\ a = \pi,\ b = -3$

16. $C = \{(2x, \cosh 2x) : 0 \le x \le 1\};\ a = -2,\ b = \dfrac{1}{2}$

17. $C = \left\{ (\cos t, 3 + \sin t) : -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}$; $a = 2,\ b = 5$

18. $C = \left\{ (e^t \sin 2t, e^t \cos 2t) : 0 \leq t \leq \frac{\pi}{4} \right\}$; $a = -1,\ b = 3$

19. $C = \{(e^{-t}, e^t) : 0 \leq t \leq \ln 2\}$; $a = -1,\ b = -4$

20. $C = \{(4^{-t}, 4^t) : 0 \leq t \leq 1\}$; $a = -2,\ b = -3$

Chapter 6

Techniques of Integration

6.1 Integration by formulae

There exist many books that contain extensive lists of integration, differentiation and other mathematical formulae. For our purpose we will use the list given below.

1. $\int af(u)du = a\int f(u)du$

2. $\int \left(\sum_{i=1}^{n} a_i f_i(u)\right) du = \sum_{i=1}^{n}\left(\int a_i f_i(u)du\right)$

3. $\int u^n du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$

4. $\int u^{-1} du = \ln|u| + C$

5. $\int e^{au} du = \dfrac{e6au}{a} + C$

6. $\int a^{bu} du = \dfrac{a^{bu}}{b\ln a} + C, \ a > 0, \ a \neq 1$

7. $\int \ln|u| du = u\ln|u| - u + C$

8. $\displaystyle\int \sin(au)du = \frac{-\cos(au)}{a} + C$

9. $\displaystyle\int \cos(au)du = \frac{\sin(au)}{a} + C$

10. $\displaystyle\int \tan(au)du = \frac{\ln|\sec(au)|}{a} + C$

11. $\displaystyle\int \cot(au)du = \frac{\ln|\sin(au)|}{a} + C$

12. $\displaystyle\int \sec(au)du = \frac{\ln|\sec(au)+\tan(au)|}{a} + C$

13. $\displaystyle\int \csc(au)du = \frac{\ln|\csc(au)-\cot(au)|}{a} + C$

14. $\displaystyle\int \sinh(au)du = \frac{\cosh(au)}{a} + C$

15. $\displaystyle\int \cosh(au)du = \frac{\sinh(au)}{a} + C$

16. $\displaystyle\int \tanh(au)du = \frac{\ln|\cosh(au)|}{a} + C$

17. $\displaystyle\int \coth(au)du = \frac{\ln|\sinh(au)|}{a} + C$

18. $\displaystyle\int \operatorname{sech}(au)du = \frac{2}{a}\ \arctan(e^{au}) + C$

19. $\displaystyle\int \operatorname{csch}(au)\ du = \frac{2}{a}\ \operatorname{arctanh}(e^{au}) + C$

20. $\displaystyle\int \sin^2(au)du = \frac{u}{2} - \frac{\sin(au)\cos(au)}{2a} + C$

21. $\displaystyle\int \cos^2(au)du = \frac{u}{2} + \frac{\sin(au)\cos(au)}{2a} + C$

22. $\displaystyle\int \tan^2(au)du = \frac{\tan(au)}{a} - u + C$

23. $\displaystyle\int \cot^2(au)du = -\frac{\cot(au)}{a} - u + C$

24. $\displaystyle\int \sec^2(au)du = \frac{\tan(au)}{a} + C$

25. $\displaystyle\int \csc^2(au)du = -\frac{\cot(au)}{a} + C$

26. $\displaystyle\int \sinh^2(au)du = -\frac{u}{2} + \frac{\sinh(2au)}{4a} + C$

27. $\displaystyle\int \cosh^2(au)du = \frac{u}{2} + \frac{\sinh(2au)}{4a} + C$

28. $\displaystyle\int \tanh^2(au)du = u - \frac{\tanh(au)}{a} + C$

29. $\displaystyle\int \coth^2(au)du = u - \frac{\coth(au)}{a} + C$

30. $\displaystyle\int \operatorname{sech}^2(au)du = \frac{\tanh(au)}{a} + C$

31. $\displaystyle\int \operatorname{csch}^2(au)du = \frac{-\coth(au)}{a} + C$

32. $\displaystyle\int \sec(au)\tan(au)du = \frac{\sec(au)}{a} + C$

33. $\displaystyle\int \csc(au)\cot(au)du = -\frac{\csc(au)}{a} + C$

34. $\displaystyle\int \operatorname{sech}(au)\tanh(au)du = -\frac{\operatorname{sech}(au)}{a} + C$

35. $\displaystyle\int \operatorname{csch}(au)\coth(au)du = -\frac{\operatorname{csch}(au)}{a} + C$

36. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\ \arctan\left(\frac{u}{a}\right) + C$

37. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{a}\ \operatorname{arctanh}\left(\frac{u}{a}\right) + C = \frac{1}{2a}\ \ln\left|\frac{a+u}{a-u}\right| + C$

38. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \text{arcsinh}\left(\frac{u}{a}\right) + C$

39. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C,\ |a| > |u|$

40. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \text{arccosh}\left(\frac{u}{a}\right) + C,\ |u| > |a|$

41. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\ \text{arcsec}\left(\frac{u}{a}\right) + C,\ |u| > |a|$

42. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ \text{arcsech}\left(\frac{u}{a}\right) + C,\ |a| > |u|$

43. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ \text{arccsch}\left(\frac{u}{a}\right) + C$

44. $\displaystyle\int \frac{u\ du}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2} + C$

45. $\displaystyle\int \frac{u\ du}{a^2-u^2} = -\ln\sqrt{a^2-u^2} + C,\ |a| > |u|$

46. $\displaystyle\int \frac{u\ du}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2} + C$

47. $\displaystyle\int \frac{u\ du}{\sqrt{a^2-u^2}} = -\sqrt{a^2-u^2} + C,\ |a| > |u|$

48. $\displaystyle\int \frac{u\ du}{\sqrt{u^2-a^2}} = \sqrt{u^2-a^2} + C,\ |u| > |a|$

49. $\displaystyle\int \arcsin(au)du = u\arcsin(au) + \frac{1}{a}\ \sqrt{1-a^2u^2} + C,\ |a||u| < 1$

50. $\displaystyle\int \arccos(au)du = u\arccos(au) - \frac{1}{a}\ \sqrt{1-a^2u^2} + C,\ |a||u| < 1$

51. $\displaystyle\int \arctan(au)du = u\arctan(au) - \frac{1}{2a}\ \ln(1+a^2u^2) + C$

52. $\displaystyle\int \text{arccot}\,(au)du = u\text{arccot}\,(au) + \frac{1}{2a}\ \ln(1+a^2u^2) + C$

53. $\displaystyle\int \operatorname{arcsec}(au)du = u\operatorname{arcsec}(au) - \frac{1}{a}\ \ln\left|au + \sqrt{a^2u^2-1}\right| + C,\ au > 1$

54. $\displaystyle\int \operatorname{arccsc}(au)du = u\operatorname{arccsc}(au) + \frac{1}{a}\ \ln\left|au + \sqrt{a^2u^2-1}\right| + C,\ au > 1$

55. $\displaystyle\int \operatorname{arcsinh}(au)du = u\operatorname{arcsinh}(au) - \frac{1}{a}\ \sqrt{1+a^2u^2} + C$

56. $\displaystyle\int \operatorname{arccosh}(au)du = u\operatorname{arccosh}(au) - \frac{1}{a}\ \sqrt{-1+a^2u^2} + C,\ |a||u| > 1$

57. $\displaystyle\int \operatorname{arctanh}(au)du = u\operatorname{arctanh}(au) + \frac{1}{2a}\ \ln(-1+a^2u^2) + C,\ |a||u| \neq 1$

58. $\displaystyle\int \operatorname{arccoth}(au)du = u\operatorname{arccoth}(au) + \frac{1}{2a}\ \ln(-1+a^2u^2) + C,\ |a||u| \neq 1$

59. $\displaystyle\int \operatorname{arcsech}(au)du = u\operatorname{arcsech}(au) + \frac{1}{a}\ \arcsin(au) + C,\ |a||u| < 1$

60. $\displaystyle\int \operatorname{arccsch}(au)du = u\operatorname{arccsch}(au) + \frac{1}{a}\ \ln\left|au + \sqrt{a^2u^2+1}\right| + C$

61. $\displaystyle\int e^{au}\sin(bu)du = \frac{e^{au}[a\sin(bu) - b\cos(bu)]}{a^2+b^2} + C$

62. $\displaystyle\int e^{au}\cos(bu)du = \frac{e^{au}[a\cos(bu) + b\sin(bu)]}{a^2+b^2} + C$

63. $\displaystyle\int \sin^n(u)du = \frac{-1}{n}\ \left[\sin^{n-1}(u)\cos(u)\right] + \frac{n-1}{n}\ \int \sin^{n-2}(u)du$

64. $\displaystyle\int \cos^n(u)du = \frac{1}{n}\ \left[\cos^{n-1}(u)\sin(u)\right] + \frac{n-1}{n}\int \cos^{n-2}(u)du$

65. $\displaystyle\int \tan^n(u)du = \frac{\tan^{n-1}(u)}{n-1} - \int \tan^{n-2}(u)du$

66. $\displaystyle\int \cot^n(u)du = -\frac{\cot^{n-1}(u)}{n-1} - \int \cot^{n-2}(u)du$

67. $\displaystyle\int \sec^n(u)du = \frac{1}{n-1}\ \left[\sec^{n-2}(u)\tan(u)\right] + \frac{n-2}{n-1}\int \sec^{n-2}(u)du$

68. $\displaystyle\int \csc^n(u)du = \frac{-1}{n-1}\left[\csc^{n-2}(u)\cot(u)\right] + \frac{n-2}{n-1}\int \csc^{n-2}(u)du$

69. $\displaystyle\int \sin(mu)sin(nu)du = \frac{\sin[(m-n)u]}{2(m-n)} - \frac{\sin[(m+n)u]}{2(m+n)} + C,\ m^2 \neq n^2$

70. $\displaystyle\int \cos(mu)\cos(nu)du = \frac{\sin[(m-n)u]}{2(m-n)} + \frac{\sin[(m+n)u]}{2(m+n)} + C,\ m^2 \neq n^2$

71. $\displaystyle\int \sin(mu)\cos(nu)du = \frac{\cos[(m-n)u]}{2(m-n)} - \frac{\cos[(m+n)u]}{2(m+n)} + C,\ m^2 \neq n^2$

Exercises 6.1

1. Define the statement that $g(x)$ is an antiderivative of $f(x)$ on the closed interval $[a, b]$

2. Prove that if $g(x)$ and $h(x)$ are any two antiderivatives of $f(x)$ on $[a, b]$, then there exists some constant C such that $g(x) = \ln(x) + C$ for all x on $[a, b]$.

In problems 3–30, evaluate each of the indefinite integrals.

3. $\displaystyle\int x^5 dx$

4. $\displaystyle\int \frac{4}{x^3}\,dx$

5. $\displaystyle\int x^{-3/5}dx$

6. $\displaystyle\int 3x^{2/3}dx$

7. $\displaystyle\int \frac{2}{\sqrt{x}}\,dx$

8. $\displaystyle\int t^2\sqrt{t}\,dt$

9. $\displaystyle\int \left(t^{-1/2} + t^{3/2}\right)dt$

10. $\displaystyle\int (1+x^2)^2 dx$

11. $\displaystyle\int t^2(1+t)^2 dt$

12. $\displaystyle\int (1+t^2)(1-t^2)dt$

13. $\displaystyle\int \left(\frac{1}{t^{1/2}} + \sin t\right)dt$

14. $\displaystyle\int (2\sin t + 3\cos t)dt$

15. $\displaystyle\int 3\sec^2 t\ dt$

16. $\displaystyle\int 2\csc^2 x\ dx$

17. $\displaystyle\int 4\sec t\tan t\ dt$

18. $\displaystyle\int 2\csc t\cot t\ dt$

19. $\displaystyle\int \sec t(\sec t + \tan t)dt$

20. $\int \csc t(\csc t - \cot t)dt$

21. $\int \frac{\sin x}{\cos^2 x}\,dx$

22. $\int \frac{\cos x}{\sin^2 x}\,dx$

23. $\int \frac{\sin^3 t - 3}{\sin^2 t}\,dt$

24. $\int \frac{\cos^3 t + 2}{\cos^2 t}\,dt$

25. $\int \tan^2 t\;dt$

26. $\int \cot^2 t\;dt$

27. $\int (2\sec^2 t + 1)dt$

28. $\int \frac{2}{t}\,dt$

29. $\int \sinh t\;dt$

30. $\int \cosh t\;dt$

31. Determine $f(x)$ if $f'(x) = \cos x$ and $f(0) = 2$.

32. Determine $f(x)$ if $f''(x) = \sin x$ and $f(0) = 1,\; f'(0) = 2$.

33. Determine $f(x)$ if $f''(x) = \sinh x$ and $f(0) = 2,\; f'(0) = -3$.

34. Prove each of the integration formulas 1–77.

6.2 Integration by Substitution

Theorem 6.2.1 *Let $f(x), g(x), f(g(x))$ and $g'(x)$ be continuous on an interval $[a, b]$. Suppose that $F'(u) = f(u)$ where $u = g(x)$. Then*

(i) $\int f(g(x))g'(x)dx = \int f(u)du = F(g(x)) + C$

(ii) $\int_a^b f(g(x))g'(x)dx = \int_{u=g(u)}^{u=g(b)} f(u)du = F(g(b)) - F(g(a))$.

Proof. See the proof of Theorem 5.3.1.

Exercises 6.2 In problems 1–39, evaluate the integral by making the given substitution.

1. $\displaystyle\int 3x(x^2+1)^{10}dx,\ u=x^2+1$

2. $\displaystyle\int x\sin(1+x^2)dx,\ u=1+x^2$

3. $\displaystyle\int \frac{\cos(\sqrt{t})}{\sqrt{t}}\,dt,\ x=\sqrt{t}$

4. $\displaystyle\int \frac{3x^2}{(1+x^3)^{3/2}}\,dx,\ u=1+x^3$

5. $\displaystyle\int \frac{2e^{\arcsin x}}{\sqrt{1-x^2}}\,dx,\ u=\arcsin x$

6. $\displaystyle\int \frac{3e^{\arccos x}}{\sqrt{1-x^2}}\,dx$

7. $\displaystyle\int x\ 4^{x^2}dx,\ u=4^{x^2}$

8. $\displaystyle\int 10^{\sin x}\cos x\ dx,\ u=\sin x$

9. $\displaystyle\int \frac{4^{\arctan x}}{1+x^2}\,dx,\ u=4^{\arctan x}$

10. $\displaystyle\int \frac{(1+\ln x)^{10}}{x}\,dx,\ u=1+\ln x$

11. $\displaystyle\int \frac{5^{\operatorname{arcsec} x}}{x\sqrt{x^2-1}}\,dx,\ u=\operatorname{arcsec} x$

12. $\displaystyle\int (\tan 2x)^3\sec^2 2x\ dx,\ u=\tan 2x$

13. $\displaystyle\int (\cot 3x)^5\csc^2 3x\ dx,\ u\ =\cot 3x$

14. $\displaystyle\int \sin^{21} x\cos x\ dx,\ u=\sin x$

15. $\displaystyle\int \cos^5 x\sin x\ dx,\ u=\cos x$

16. $\displaystyle\int (1+\sin x)^{10}\cos x\ dx,\ u=1+\sin x$

17. $\displaystyle\int \sin^3 x\ dx,\ u=\cos x$

18. $\displaystyle\int \cos^3 x\ dx,\ u=\sin x$

19. $\displaystyle\int \tan^3 x\ dx,\ u=\tan x$

20. $\displaystyle\int \cot^3 x\ dx,\ u=\cot x$

21. $\displaystyle\int \sec^4 x\ dx,\ u=\tan x$

22. $\displaystyle\int \csc^4 x\ dx,\ u=\cot x$

23. $\displaystyle\int \sin^3 x\cos^3 x\ dx,\ u=\sin x$

24. $\displaystyle\int \sin^3 x\cos^3 x\ dx,\ u=\cos x$

25. $\displaystyle\int \tan^4 x\ dx,\ u=\tan x$

26. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx,\ u=\ln x$

27. $\displaystyle\int \frac{x\cos(\ln(1+x^2))}{1+x^2}\,dx,\ u=\ln(1+x)^2$

28. $\displaystyle\int \tan^3 x\sec^4 x\ dx,\ u=\sec x$

29. $\displaystyle\int \cot^3 x\csc^4 x\ dx,\ u=\csc x$

30. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}, x=2\sin t$

31. $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}, x=3\cos t$

32. $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}, x=2\sinh t$

33. $\displaystyle\int \frac{dx}{\sqrt{x^2-9}}, x=3\cosh t$

34. $\displaystyle\int \frac{dx}{4+x^2}, x=2\tan t$

35. $\displaystyle\int \frac{dx}{4-x^2}, x=2\tanh t$

36. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}, x=2\sec t$

37. $\displaystyle\int 4e^{\sin(3x)}\cos(3x)dx,\ u=\sin 3x$

38. $\displaystyle\int x\ 3^{(x^2+4)}dx,\ u=3^{x^2+4}$

39. $\displaystyle\int 3\ e^{\tan 2x}\sec^2 x\ dx,\ u=\tan 2x$

40. $\displaystyle\int x\sqrt{x+2}\ dx,\ u=x+2$

Evaluate the following definite integrals.

41. $\displaystyle\int_0^1 (x+1)^{30}dx$

42. $\displaystyle\int_1^2 x(4-x^2)^{1/2}dx$

43. $\displaystyle\int_0^{\pi/4} \tan^3 x\sec^2 x\ dx$

44. $\displaystyle\int_0^1 x^3(x^2+1)^3dx$

45. $\displaystyle\int_0^2 (x+1)(x-2)^{10}dx$

46. $\displaystyle\int_0^8 x^2(1+x)^{1/2}dx$

47. $\displaystyle\int_0^{\pi/6} \sin(3x)dx$

48. $\displaystyle\int_0^{\pi/4} \cos(2x)dx$

49. $\displaystyle\int_0^{\pi/4} \sin^3 2x\cos 2x\ dx$

50. $\displaystyle\int_0^{\pi/6} \cos^4 3x\sin 3x\ dx$

51. $\displaystyle\int_0^1 \frac{e^{\arctan x}}{1+x^2}\ dx$

52. $\displaystyle\int_0^{1/2} \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\ dx$

53. $\displaystyle\int_2^3 \frac{e^{\operatorname{arcsec} x}}{x\sqrt{x^2-1}}\, dx$

54. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^2}}$

6.3 Integration by Parts

Theorem 6.3.1 *Let $f(x), g(x), f'(x)$ and $g'(x)$ be continuous on an interval $[a, b]$. Then*

(i) $\displaystyle\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$

(ii) $\displaystyle\int_a^b f(x)g'(x)dx = (f(b)g(b) - f(a)g(a)) - \int_a^b g(x)f'(x)dx$

(iii) $\displaystyle\int udv = uv - \int vdu$

where $u = f(x)$ and $dv = g'(x)dx$ are the parts of the integrand.

Proof. See the proof of Theorem 5.4.1.

Exercises 6.3 Evaluate each of the following integrals.

1. $\displaystyle\int x \sin x\ dx$

2. $\displaystyle\int x \cos x\ dx$

3. $\displaystyle\int x \ln x\ dx$

4. $\displaystyle\int x\ e^x\ dx$

5. $\displaystyle\int x\ 4^x\ dx$

6. $\displaystyle\int x^2 \ln x\ dx$

7. $\displaystyle\int x^2 \sin x\ dx$

8. $\displaystyle\int x^2 \cos x\ dx$

9. $\displaystyle\int x^2 e^x\ dx$

10. $\displaystyle\int x^2\ 10^x\ dx$

11. $\int e^x \sin x \, dx$ (Let $u = e^x$ twice and solve.)

12. $\int e^x \cos x \, dx$ (Let $u = e^x$ twice and solve.)

13. $\int e^{2x} \sin 3x \, dx$ (Let $u = e^{2x}$ twice and solve.)

14. $\int x \sin(3x) dx$

15. $\int x^2 \cos(2x) dx$

16. $\int x^2 e^{4x} dx$

17. $\int x^3 \ln(2x) dx$

18. $\int x \sec^2 x \, dx$

19. $\int x \csc^2 x \, dx$

20. $\int x \sinh(4x) dx$

21. $\int x^2 \cosh x \, dx$

22. $\int x \cos(5x) dx$

23. $\int \sin(\ln x) dx$

24. $\int \cos(\ln x) dx$

25. $\int x \arcsin x \, dx$

26. $\int x \arccos x \, dx$

27. $\int x \arctan x \, dx$

28. $\int x \ \mathrm{arcsec}\, x \, dx$

29. $\int \arcsin x \, dx$

30. $\int \arccos x \, dx$

31. $\int \arctan x \, dx$

32. $\int \mathrm{arcsec}\, x \, dx$

Verify the following integration formulas:

33. $\displaystyle\int \sin^n(ax)dx = -\frac{\sin^{n-1}(ax)\cos(ax)}{na} + \frac{n-1}{n}\int(\sin^{n-2}ax)dx$

34. $\displaystyle\int \cos^n(ax)dx = \frac{1}{na}\ \cos^{n-1}(ax)\sin(ax) + \frac{n-1}{n}\int(\cos^{n-2}ax)dx$

35. $\displaystyle\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$

36. $\displaystyle\int x^n \sin x\ dx = -x^n\cos x + n\int x^{n-1}\cos x\ dx$

37. $\displaystyle\int x^n \cos x\ dx = x^n\sin x - n\int x^{n-1}\sin x\ dx$

38. $\displaystyle\int e^{ax}\sin(bx)dx = \frac{1}{a^2+b^2}\ e^{ax}[a\sin(bx) - b\cos(bx)] + C$

39. $\displaystyle\int e^{ax}\cos(bx)\ dx = \frac{1}{a^2+b^2}\ e^{ax}[a\cos(bx) + b\sin(bx)] + C$

40. $\displaystyle\int x^n \ln x\ dx = \frac{1}{n+1}\ x^{n+1}\ln x - \frac{1}{(n+1)^2}\ x^{n+1} + C,\ n \neq -1,\ x > 0$

41. $\displaystyle\int \sec^n x\ dx = \frac{1}{n-1}\ \sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\ dx,\ n \neq 1,\ n > 0$

42. $\displaystyle\int \csc^n x\ dx = \frac{-1}{n-1}\ \csc^{n-2}x\cot x + \frac{n-2}{n-1}\int \csc^{n-2}x\ dx,\ n \neq 1, n > 0$

Use the formulas 33–42 to evaluate the following integrals:

43. $\displaystyle\int \sin^4 x\ dx$

44. $\displaystyle\int \cos^5 x\ dx$

45. $\displaystyle\int x^3 e^x dx$

46. $\displaystyle\int x^4 \sin x\ dx$

47. $\displaystyle\int x^3 \cos x\ dx$

48. $\displaystyle\int e^{2x}\sin 3x\ dx$

49. $\displaystyle\int e^{3x}\cos 2x\ dx$

50. $\displaystyle\int x^5 \ln x\ dx$

51. $\displaystyle\int \sec^3 x\ dx$

52. $\displaystyle\int \csc^3 x\ dx$

Prove each of the following formulas:

53. $\displaystyle\int \tan^n x\ dx = \frac{1}{n-1}\ \tan^{n-1} x - \int \tan^{n-2} x\ dx,\ n \neq 1$

54. $\displaystyle\int \cot^n x\ dx = \frac{1}{n-1}\ \cot^{n-1} x - \int \cot^{n-2} x\ dx,\ n \neq 1$

55. $\displaystyle\int \sin^{2n+1} x\ dx = -\int (1-u^2)^n du,\ u = \cos x$

56. $\displaystyle\int \cos^{2n+1} x\ dx = -\int (1-u^2)^n du,\ u = \sin x$

57. $\displaystyle\int \sin^{2n+1} x\ \cos^m x\ dx = -\int (1-u^2)^n u^m du,\ u = \cos x$

58. $\displaystyle\int \cos^{2n+1} x\ \sin^m x\ dx = \int (1-u^2)^n u^m du,\ u = \sin x$

59. $\displaystyle\int \sin^{2n} x\ \cos^{2m} x\ dx = \int (\sin x)^{2n}(1-\sin^2 x)^m dx$

60. $\displaystyle\int \tan^n x\ \sec^{2m} x\ dx = \int u^n(1+u^2)^{m-1} du,\ u = \tan x$

61. $\displaystyle\int \cot^n x\ \csc^{2m} x\ dx = -\int u^n(1+u^2)^{m-1} du,\ u = \cot x$

62. $\displaystyle\int \tan^{2n+1} x\ \sec^m x\ dx = \int (u^2-1)^n u^{m-1} du,\ u = \sec x$

63. $\displaystyle\int \cot^{2n+1} x\ \csc^m x\ dx = -\int (u^2-1)^n u^{m-1} du,\ u = \csc x$

64. $\displaystyle\int \sin mx \cos nx\ dx = -\frac{1}{2}\ \left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right] + C;\ m^2 \neq n^2$

65. $\displaystyle\int \sin mx \sin nx\ dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right] + C;\ m^2 \neq n^2$

66. $\displaystyle\int \cos mx \cos nx\ dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right] + C;\ m^2 \neq n^2$

6.4 Trigonometric Integrals

The trigonometric integrals are of two types. The integrand of the first type consists of a product of powers of trigonometric functions of x. The integrand of the second type consists of $\sin(nx)\cos(mx), \sin(nx)\sin(mx)$ or $\cos(nx)\cos(mx)$. By expressing all trigonometric functions in terms of sine and cosine, many trigonometric integrals can be computed by using the following theorem.

Theorem 6.4.1 *Suppose that m and n are integers, positive, negative, or zero. Then the following reduction formulas are valid:*

1. $\displaystyle\int \sin^n x\ dx = \frac{-1}{n}\ \sin^{n-1} x \cos x + \frac{(n-1)}{n} \int \sin^{n-2} x\ dx,\ n > 0$

2. $\displaystyle\int \sin^{n-2} x\ dx = \frac{1}{n-1}\ \sin^{n-1} x \cos x + \frac{n}{n-1} \int \sin^n x\ dx,\ n \leq 0$

3. $\displaystyle\int (\sin x)^{-1}\ dx = \int \csc x\ dx = \ln|\csc x - \cot x| + c \text{ or } -\ln|\csc x + \cot x| + c$

4. $\displaystyle\int \cos^n x\ dx = \frac{1}{n}\ \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x\ dx,\ n > 0$

5. $\displaystyle\int \cos^{n-2} x\ dx = \frac{-1}{n-1}\ \cos^{n-1} x \sin x + \frac{n}{n-1} \int \cos^n x\ dx,\ n \leq 0$

6. $\displaystyle\int (\cos x)^{-1}\ dx = \int \sec x\ dx = \ln|\sec x + \tan x| + c$

7. $$\begin{aligned}\int sin^n x \cos^{2m+1} x\ dx &= \textstyle\int \sin^n x(1-\sin^2 x)^m \cos x\ dx \\ &= \textstyle\int u^n(1-u^2)^m du,\ u = \sin x,\ du = \cos x\ dx\end{aligned}$$

8. $\displaystyle\int sin^{2n+1} x \cos^m x\ dx = \int \cos^m x(1-\cos^2 x)^n \sin x\ dx$
$\displaystyle = -\int u^m (1-u^2)^n du,\ u = \cos x,\ du = -\sin x\ dx$

9. $\displaystyle\int \sin^{2n} x \cos^{2m} x\ dx = \int (1-\cos^2 x)^n \cos^{2m} x\ dx$
$\displaystyle = \int (1-\sin^2 x)^m \sin^{2n} x\ dx$

10. $\displaystyle\int \sin(nx)\cos(mx)dx = \frac{-1}{2}\left[\frac{\cos(m+n)x}{m-n} + \frac{\cos(m-n)x}{m-n}\right] + c,\ m^2 \neq n^2$

11. $\displaystyle\int \sin(mx)\sin(mx)\ dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right] + c,\ m^2 \neq n^2$

12. $\displaystyle\int \cos(mx)\cos(mx)\ dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right] + c,\ m^2 \neq n^2$

Corollary. The following integration formulas are valid:

13. $\displaystyle\int \tan^n u\ du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\ d$

14. $\displaystyle\int \sec^n u\ du = \frac{1}{n-1}\ \sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$

15. $\displaystyle\int \csc^n u\ du = \frac{-1}{n-1}\ \csc^{n-2} x \cot x + \frac{n-2}{n-1}\ \int \csc^{n-2} x\ dx$

Exercises 6.4 Evaluate each of the following integrals.

1. $\displaystyle\int \sin^5 x\ dx$

2. $\displaystyle\int \cos^4 x\ dx$

3. $\displaystyle\int \tan^5 x\ dx$

4. $\displaystyle\int \cot^4 x\ dx$

5. $\displaystyle\int \sec^5 x\ dx$

6. $\displaystyle\int \csc^4 x\ dx$

7. $\int \sin^5 x \cos^4 x \, dx$

8. $\int \sin^3 x \cos^5 x \, dx$

9. $\int \sin^4 x \cos^3 x \, dx$

10. $\int \sin^2 x \cos^4 x \, dx$

11. $\int \tan^5 x \sec^4 x \, dx$

12. $\int \cot^5 x \csc^4 x \, dx$

13. $\int \tan^4 x \sec^5 x \, dx$

14. $\int \cot^4 x \csc^5 x \, dx$

15. $\int \tan^4 x \sec^4 x \, dx$

16. $\int \cot^4 x \csc^4 x \, dx$

17. $\int \tan^3 x \sec^3 x \, dx$

18. $\int \cot^3 x \csc^3 x \, dx$

19. $\int \sin 2x \cos 3x \, dx$

20. $\int \sin 4x \cos 4x \, dx$

21. $\int \sin 3x \cos 3x \, dx$

22. $\int \sin 2x \sin 3x \, dx$

23. $\int \sin 4x \sin 6x \, dx$

24. $\int \sin 3x \sin 5x \, dx$

25. $\int \cos 3x \cos 5x \, dx$

26. $\int \cos 2x \cos 4x \, dx$

27. $\int \cos 3x \cos 4x \, dx$

28. $\int \sin 4x \cos 4x \, dx$

6.5 Trigonometric Substitutions

Theorem 6.5.1 ($a^2 - u^2$ Forms). *Suppose that* $u = a \sin t,\ a > 0$. *Then*

$$du = a\cos t dt,\ a^2 - u^2 = a^2\cos^2 t,\ \sqrt{a^2-u^2} = a\cos t,\ t = \arcsin(u/a),$$
$$\sin t = \frac{u}{a},\ \cos t = \frac{\sqrt{a^2-u^2}}{a},\ \tan t = \frac{u}{\sqrt{a^2-u^2}},$$
$$\cot t = \frac{\sqrt{a^2-u^2}}{u},\ \sec t = \frac{a}{\sqrt{a^2-u^2}},\ \csc t\ \frac{a}{u}.$$

graph

The following integration formulas are valid:

1. $\displaystyle\int \frac{udu}{a^2-u^2} = -\frac{1}{2}\ \ln|a^2-u^2| + c$

2. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ \ln\left|\frac{a-u}{a+u}\right| + c = \frac{1}{a}\ \text{arctanh}\left(\frac{u}{a}\right) + c$

3. $\displaystyle\int \frac{udu}{\sqrt{a^2-u^2}} = -\sqrt{a^2-u^2} + c$

4. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + c$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{1}{a}\ \ln\left|\frac{a}{u} - \frac{\sqrt{a^2-u^2}}{u}\right| + c$

6. $\displaystyle\int \sqrt{a^2-u^2}\ du = \frac{a^2}{2}\ \arcsin\left(\frac{u}{a}\right) + \frac{1}{2}\ u\sqrt{a^2-u^2} + c$

Proof. The proof of this theorem is left as an exercise.

Theorem 6.5.2 ($a^2 + u^2$ Forms). *Suppose that* $u = a\tan t,\ a > 0$. *Then*

$$du = a\sec^2 t dt, a^2 + u^2 = a^2\sec^2 t, \sqrt{a^2+u^2} = a\sec t, t = \arctan\left(\frac{u}{a}\right),$$

$$\sin t = \frac{u}{\sqrt{a^2+u^2}},\ \cos t = \frac{a}{\sqrt{a^2+u^2}},\ \tan t = \frac{u}{a}$$

$$\csc t = \frac{\sqrt{a^2+u^2}}{u},\ \sec t = \frac{\sqrt{a^2+u^2}}{a},\ \cot t = \frac{a}{u}.$$

graph

Proof. The proof of this theorem is left as an exercise.

The following integration formulas are valid:

1. $\displaystyle\int \frac{udu}{a^2+u^2} = \frac{1}{2}\ \ln\left|a^2+u^2\right| + c$
2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\ \arctan\left(\frac{u}{a}\right) + c$
3. $\displaystyle\int \frac{udu}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2} + c$
4. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left|u + \sqrt{a^2+u^2}\right| + c$
5. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{1}{a}\ \ln\left|\frac{\sqrt{a^2+u^2}}{u} - \frac{a}{u}\right| + c$
6. $\displaystyle\int \sqrt{a^2+u^2}\ du = \frac{1}{2}\ u\sqrt{a^2+u^2} + \frac{a^2}{2}\ \ln\left|u + \sqrt{a^2+u^2}\right| + c$

Theorem 6.5.3 *($u^2 - a^2$ Forms) Suppose that* $u = a\sec t,\ a > 0$. *Then*

$$du = a\sec t\tan t\ dt,\ u^2 - a^2 = a^2\tan^2 t,\ \sqrt{u^2-a^2} = a\tan t, t = \mathit{arcsec}\left(\frac{u}{a}\right),$$

$$\sin t = \frac{\sqrt{u^2-a^2}}{u},\ \cos t = \frac{a}{u},\ \tan t = \frac{\sqrt{u^2-a^2}}{a},$$

$$\csc t = \frac{u}{\sqrt{u^2-a^2}},\ \sec t = \frac{u}{a},\ \cot t = \frac{a}{\sqrt{u^2-a^2}}.$$

graph

Proof. The proof of this theorem is left as an exercise.

The following integration formulas are valid:

1. $\displaystyle\int \frac{udu}{u^2-a^2} = \frac{1}{2}\ \ln\left|u^2-a^2\right| + c$

2. $\displaystyle\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ \ln\left|\frac{u-a}{u+a}\right| + c$

3. $\displaystyle\int \frac{udu}{\sqrt{u^2-a^2}} = \sqrt{u^2-a^2} + c$

4. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + c$

5. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\ \mathrm{arcsec}\left(\frac{u}{a}\right) + c$

6. $\displaystyle\int \sqrt{u^2-a^2}\ du = \frac{1}{2}\ u\sqrt{u^2-a^2} - \frac{a^2}{2}\ \ln\left|u+\sqrt{u^2-a^2}\right| + c$

Exercises 6.5 Prove each of the following formulas:

1. $\displaystyle\int \frac{u\ du}{a^2-u^2} = -\frac{1}{2}\ \ln|a^2-u^2| + C$

2. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ \ln\left|\frac{a-u}{a+u}\right| + C$

3. $\displaystyle\int \frac{u\ du}{\sqrt{a^2-u^2}} = -\sqrt{a^2-u^2} + C$

4. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C,\ a > 0$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{1}{a}\ \ln\left|\frac{a}{u} - \frac{\sqrt{a^2-u^2}}{u}\right| + C$

6. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{a^2}{2}\,\arcsin\left(\frac{u}{a}\right) + \frac{1}{2}\,u\sqrt{a^2-u^2} + C,\ a>0$

7. $\displaystyle\int \frac{u\,du}{a^2+u^2} = \frac{1}{2}\,\ln\left|a^2+u^2\right| + C$

8. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\,\arctan\left(\frac{u}{a}\right) + C$

9. $\displaystyle\int \frac{u\,du}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2} + C$

10. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left|u+\sqrt{a^2+u^2}\right| + C$

11. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{1}{a}\,\ln\left|\frac{\sqrt{a^2+u^2}}{u} - \frac{a}{u}\right| + C$

12. $\displaystyle\int \sqrt{a^2+u^2}\,du = \frac{1}{2}\,u\sqrt{a^2+u^2} + \frac{a^2}{2}\,\ln\left|u+\sqrt{a^2+u^2}\right| + C$

13. $\displaystyle\int \frac{u\,du}{u^2-a^2} = \frac{1}{2}\,\ln\left|u^2-a^2\right| + C$

14. $\displaystyle\int \frac{du}{u^2-a^2} = \frac{1}{2a}\,\ln\left|\frac{u-a}{u+a}\right| + C$

15. $\displaystyle\int \frac{u\,du}{\sqrt{u^2-a^2}} = \sqrt{u^2-a^2} + C$

16. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

17. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\,\text{arcsec}\left(\frac{u}{a}\right) + C$

18. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{1}{2}\,u\sqrt{u^2-a^2} - \frac{a^2}{2}\,\ln\left|u+\sqrt{u^2-a^2}\right| + C$

Evaluate each of the following integrals:

19. $\displaystyle\int \frac{x\ dx}{\sqrt{4-x^2}}$ 20. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$ 21. $\displaystyle\int \frac{x\ dx}{4-x^2}$

22. $\displaystyle\int \frac{dx}{4-x^2}$ 23. $\displaystyle\int \frac{x\ dx}{9+x^2}$ 24. $\displaystyle\int \frac{dx}{9+x^2}$

25. $\displaystyle\int \frac{x\ dx}{\sqrt{9+x^2}}$ 26. $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$ 27. $\displaystyle\int \frac{x\ dx}{x^2-16}$

28. $\displaystyle\int \frac{dx}{x^2-16}$ 29. $\displaystyle\int \frac{x\ dx}{\sqrt{x^2-16}}$ 30. $\displaystyle\int \frac{dx}{\sqrt{x^2-16}}$

31. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}$ 32. $\displaystyle\int \frac{dx}{x\sqrt{9-x^2}}$ 33. $\displaystyle\int \frac{dx}{x\sqrt{x^2+16}}$

34. $\displaystyle\int \sqrt{9-x^2}\ dx$ 35. $\displaystyle\int \sqrt{4-9x^2}$ 36. $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\ dx$

37. $\displaystyle\int \frac{x^2}{\sqrt{4+x^2}}\ dx$ 38. $\displaystyle\int \frac{x^2}{\sqrt{x^2-16}}\ dx$ 39. $\displaystyle\int \frac{dx}{(9+x^2)^2}$

40. $\displaystyle\int \frac{dx}{(9-x^2)^2}$ 41. $\displaystyle\int \frac{dx}{(x^2-16)^2}$ 42. $\displaystyle\int \frac{dx}{(4+x^2)^{3/2}}$

43. $\displaystyle\int \frac{\sqrt{4+x^2}}{x}$ 44. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\ dx$ 45. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$

46. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$ 47. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}}$ 48. $\displaystyle\int \frac{dx}{x^2-2x+5}$

49. $\displaystyle\int \frac{dx}{x^2-4x+12}$ 50. $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}}$ 51. $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+12}}$

52. $\displaystyle\int \frac{dx}{4x-x^2}$ 53. $\displaystyle\int \frac{dx}{\sqrt{x^2-2x+5}}$ 54. $\displaystyle\int \frac{x\ dx}{x^2-4x-12}$

55. $\displaystyle\int \frac{x\ dx}{\sqrt{x^2-2x+5}}$ 56. $\displaystyle\int \frac{x}{x^2+4x+13}\ dx$ 57. $\displaystyle\int (5-4x-x^2)^{1/2}dx$

58. $\displaystyle\int \frac{2x+7}{x^2+4+13}\,dx$ 59. $\displaystyle\int \frac{x+3}{\sqrt{x^2+2x+5}}\,dx$ 60. $\displaystyle\int \frac{dx}{\sqrt{4x^2-1}}$

61. $\displaystyle\int \frac{x+4}{\sqrt{9x^2+16}}\,dx$ 62. $\displaystyle\int \frac{x+2}{\sqrt{16-9x^2}}\,dx$ 63. $\displaystyle\int \frac{e^{2x}dx}{(5-e^{2x}+e^{4x})^{1/2}}$

64. $\displaystyle\int \frac{e^{3x}dx}{(e^{6x}+4e^{3x}+3)^{1/2}}$

6.6 Integration by Partial Fractions

A polynomial with real coefficients can be factored into a product of powers of linear and quadratic factors. This fact can be used to integrate rational functions of the form $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials that have no factors in common. If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then by long division we can express the rational function by

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

where $q(x)$ is the quotient and $r(x)$ is the remainder whose degree is less than the degree of $Q(x)$. Then $Q(x)$ is factored as a product of powers of linear and quadratic factors. Finally $r(x)/Q(x)$ is split into a sum of fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

and

$$\frac{B_1x+c_1}{ax^2+bx+c} + \frac{B_2x+c_2}{(ax^2+bxc)^2} + \cdots + \frac{B_mx+c_m}{(ax^2+bx+c)^m}.$$

Many calculators and computer algebra systems, such as Maple or Mathematica, are able to factor polynomials and split rational functions into partial fractions. Once the partial fraction split up is made, the problem of integrating a rational function is reduced to integration by substitution using linear or trigonometric substitutions. It is best to study some examples and do some simple problems by hand.

Exercises 6.6 Evaluate each of the following integrals:

1. $\displaystyle\int \frac{dx}{(x-1)(x-2)(x+4)}$

2. $\displaystyle\int \frac{dx}{(x-4)(10+x)}$

3. $\displaystyle\int \frac{dx}{(x-a)(x-b)}$

4. $\displaystyle\int \frac{dx}{(x-a)(b-x)}$

5. $\displaystyle\int \frac{dx}{(x^2+1)(x^2+4)}$

6. $\displaystyle\int \frac{dx}{(x-1)(x^2+1)}$

7. $\displaystyle\int \frac{2x\ dx}{x^2-5x+6}$

8. $\displaystyle\int \frac{x\ dx}{(x+3)(x+4)}$

9. $\displaystyle\int \frac{x+1}{(x+2)(x^2+4)}\ dx$

10. $\displaystyle\int \frac{(x+2)dx}{(x+3)(x^2+1)}$

11. $\displaystyle\int \frac{2\ dx}{(x^2+4)(x^2+9)}$

12. $\displaystyle\int \frac{dx}{(x^2-4)(x^2-9)}$

13. $\displaystyle\int \frac{x^2\ dx}{(x^2+4)(x^2+9)}$

14. $\displaystyle\int \frac{x\ dx}{(x^2-4)(x^2-9)}$

15. $\displaystyle\int \frac{dx}{x^4-16}$

16. $\displaystyle\int \frac{x\ dx}{x^4-81}$

6.7 Fractional Power Substitutions

If the integrand contains one or more fractional powers of the form $x^{s/r}$, then the substitution, $x = u^n$, where n is the least common multiple of the denominators of the fractional exponents, may be helpful in computing the integral. It is best to look at some examples and work some problems by hand.

Exercises 6.7 Evaluate each of the following integrals using the given substitution.

1. $\displaystyle\int \frac{4x^{3/2}}{1+x^{1/3}}\ dx;\ x = u^6$

2. $\displaystyle\int \frac{dx}{1+x^{1/3}};\ x = u^3$

3. $\displaystyle\int \frac{dx}{\sqrt{1+e^{2x}}}$; $u^2 = 1 + e^{2x}$

4. $\displaystyle\int \frac{dx}{x\sqrt{x^3-8}}$; $u^2 = x^3 - 8$

Evaluate each of the following by using an appropriate substitution:

5. $\displaystyle\int \frac{x\ dx}{\sqrt{x+2}}$

6. $\displaystyle\int \frac{x^2 dx}{\sqrt{x+4}}$

7. $\displaystyle\int \frac{1}{4+\sqrt{x}}\ dx$

8. $\displaystyle\int \frac{x\ dx}{1+\sqrt{x}}$

9. $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}$

10. $\displaystyle\int \frac{x^{2/3}}{8+x^{1/2}}$

11. $\displaystyle\int \frac{1}{x^{2/3}+1}\ dx$

12. $\displaystyle\int \frac{dx}{1+\sqrt{x}}$

13. $\displaystyle\int \frac{x\ dx}{1+x^{2/3}}$

14. $\displaystyle\int \frac{1+\sqrt{x}}{2+\sqrt{x}}\ dx$

15. $\displaystyle\int \frac{1-\sqrt{x}}{1+x^{3/2}}\ dx$

16. $\displaystyle\int \frac{1+\sqrt{x}}{1-x^{3/2}}\ dx$

6.8 Tangent $x/2$ Substitution

If the integrand contains an expression of the form $(a+b\sin x)$ or $(a+b\cos x)$, then the following theorem may be helpful in evaluating the integral.

Theorem 6.8.1 *Suppose that* $u = \tan(x/2)$. *Then*

$$\sin x = \frac{2u}{1+u^2},\ \cos x = \frac{1-u^2}{1+u^2}\ \ \textit{and}\ \ dx = \frac{2}{1+u^2}\ du.$$

Furthermore,

$$\int \frac{dx}{a+b\sin x} = \int \frac{(2/(1+u^2))du}{a+b\left(\frac{2u}{1+u^2}\right)} = \int \frac{2du}{a(1+u^2)+2bu}$$
$$\int \frac{dx}{a+b\cos x} = \int \frac{(2/(1+u^2))du}{a+b\left(\frac{1-u^2}{1+u^2}\right)} = \int \frac{2du}{a(1+u^2)+b(1-u^2)}.$$

Proof. The proof of this theorem is left as an exercise.

Exercises 6.8

1. Prove Theorem 6.8.1

Evaluate the following integrals:

2. $\displaystyle\int \frac{dx}{2+\sin x}$

3. $\displaystyle\int \frac{dx}{\sin x+\cos x}$

4. $\displaystyle\int \frac{dx}{\sin x-\cos x}$

5. $\displaystyle\int \frac{dx}{2\sin x+3\cos x}$

6. $\displaystyle\int \frac{dx}{2-\sin x}$

7. $\displaystyle\int \frac{dx}{3+\cos x}$

8. $\displaystyle\int \frac{dx}{3-\cos x}$

9. $\displaystyle\int \frac{\sin x\ dx}{\sin x+\cos x}$

10. $\displaystyle\int \frac{\cos x\ dx}{\sin x-\cos x}$

11. $\displaystyle\int \frac{(1+\sin x)dx}{(1-\sin x)}$

12. $\displaystyle\int \frac{1-\cos x}{1+\cos x}\ dx$

13. $\displaystyle\int \frac{2-\cos x}{2+\cos x}\ dx$

14. $\displaystyle\int \frac{2+\cos x}{2-\sin x}\ dx$

15. $\displaystyle\int \frac{2-\sin x}{3+\cos x}\ dx$

16. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

6.9 Numerical Integration

Not all integrals can be computed in the closed form in terms of the elementary functions. It becomes necessary to use approximation methods. Some of the simplest numerical methods of integration are stated in the next few theorems.

Theorem 6.9.1 (Midpoint Rule) *If f, f' and f'' are continuous on $[a,b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{f''(c)}{24}\,(b-a)^3.$$

Proof. The proof of this theorem is omitted.

Theorem 6.9.2 (Trapezoidal Rule) *If f, f' and f'' are continuous on $[a,b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = (b-a)\left[\frac{1}{2}\,(f(a)+f(b))\right] - \frac{f''(c)}{12}\,(b-a)^3.$$

Proof. The proof of this theorem is omitted.

Theorem 6.9.3 (Simpson's Rule) *If $f, f', f'', f^{(3)}$ and $f^{(4)}$ are continuous on $[a,b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = \frac{b-a}{6}\,\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{f^{(4)}(c)}{2880}\,(b-a)^5.$$

These basic numerical formulas can be applied on each subinterval $[x_i, x_{i+1}]$ of a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of the interval $[a,b]$ to get composite numerical methods. We assume that $h = (b-a)/n,\ x_i = a + ih,\ i = 0, 1, 2, \cdots, n$.

Proof. The proof of this theorem is omitted.

Theorem 6.9.4 (Composite Trapezoidal Rule) *If f, f' and f'' are continuous on $[a,b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right] - \frac{b-a}{12}\,h^2 f''(c).$$

Proof. The proof of this theorem is omitted.

Theorem 6.9.5 (Composite Simpson's Rule) *If $f, f', f'', f^(3)$ and $f^{(4)}$ are continuous on $[a,b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + f(b)\right] - \frac{b-a}{180}\,h^4 f^{(4)}(c).$$

where n is an even natural number.

Proof. The proof of this theorem is omitted.

Remark 22 In practice, the composite Trapezoidal and Simpson's rules can be applied when the value of the function is known at the subdivision points $x_i, i = 0, 1, 2, \cdots, n$.

Exercises 6.9 Approximate the value of each of the following integrals for a given value of n and using

(a) Left-hand end point approximation: $\sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$

(b) Right-hand end point approximation: $\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$

(c) Mid point approximation: $\sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)(x_i - x_{i-1})$

(d) Composite Trapezoidal Rule

(e) Composite Simpson's Rule

1. $\int_1^3 \frac{1}{x}\, dx,\ n = 10$
2. $\int_2^4 \frac{1}{\sqrt{x}}\, dx,\ n = 10$
3. $\int_0^1 \frac{1}{1+\sqrt{x}}\, dx,\ n = 10$
4. $\int_1^2 \frac{1}{1+x^2}\, dx,\ n = 10$
5. $\int_0^1 \frac{1+\sqrt{x}}{1+x}\, dx,\ n = 10$
6. $\int_0^2 x^3\, dx,\ n = 10$
7. $\int_0^2 (x^2 - 2x)\, dx,\ n = 10$
8. $\int_0^1 (1+x^2)^{1/2} dx,\ n = 10$
9. $\int_0^1 (1+x^3)^{1/2} dx,\ n = 10$
10. $\int_0^1 (1+x^4)^{1/2} dx,\ n = 10$

Chapter 7

Improper Integrals and Indeterminate Forms

7.1 Integrals over Unbounded Intervals

Definition 7.1.1 Suppose that a function f is continuous on $(-\infty, \infty)$. Then we define the following improper integrals when the limits exist

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx \quad (1)$$

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx \quad (2)$$

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx \quad (3)$$

provided the integrals on the right hand side exist for some c. If these improper integrals exist, we say that they are convergent; otherwise they are said to be divergent.

Definition 7.1.2 Suppose that a function f is continuous on $[0, \infty)$. Then the Laplace transform of f, written $\mathcal{L}(f)$ or $F(s)$, is defined by

$$\mathcal{L}(f) = F(s) = \int_0^\infty e^{-st} f(t)^{dt}.$$

Theorem 7.1.1 *The Laplace transform has the following properties:*

$$\mathcal{L}(c) = \frac{c}{s} \tag{4}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a} \tag{5}$$

$$\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2} \tag{6}$$

$$\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2} \tag{7}$$

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2} \tag{8}$$

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2} \tag{9}$$

$$\mathcal{L}(t) = \frac{1}{s^2} \tag{10}$$

Proof.

(i) $\mathcal{L}(c) = \int_0^\infty ce^{-st}dt$

$$= \left.\frac{ce^{-st}}{-s}\right|_0^\infty$$

$$= \frac{c}{s}.$$

(ii) $\mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-st}dt$

$$= \int_0^\infty e^{-(s-a)t}dt$$

$$= \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty$$

$$= \frac{1}{s-a}$$

provided $s > a$.

(iii) $$\mathcal{L}(\cosh at) = \int_0^\infty \left(\frac{e^{at}+e^{-at}}{2}\right) e^{-st}dt$$

$$= \frac{1}{2}[\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{s}{s^2-a^2}, s > |a|.$$

(iv) $$\mathcal{L}(\sinh at) = \int_0^\infty \frac{1}{2}\,(e^{at} - e^{-at})e^{-st}dt$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right], s > |a|$$

$$= \frac{a}{s^2-a^2}, s > |a|.$$

(v) $$\mathcal{L}(\cos\omega t) = \int_0^\infty \cos\omega t e^{-st}dt$$

$$= \frac{1}{\omega^2+s^2}\left[e^{-st}(-s\cos\omega t + \omega\sin\omega t)\right]_0^\infty$$

$$= \frac{s}{\omega^2+s^2}.$$

(vi) $$\mathcal{L}(\sin\omega t) = \int_0^\infty \sin\omega t e^{-st}dt$$

$$= \frac{1}{\omega^2+s^2}\left[e^{-st}(-s\sin\omega t - \omega\cos\omega t)\right]_0^\infty$$

$$= \frac{\omega}{\omega^2+s^2}.$$

(vii) $\mathcal{L}(t) = \int_0^\infty te^{-st}dt; \quad (u = t, dv = e^{-st}dt)$

$$= \left.\frac{te^{-st}}{-s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\,dt$$

$$= \left.\frac{e^{-st}}{-s^2}\right|_0^\infty$$

$$= \frac{1}{s^2}.$$

This completes the proof of Theorem 7.1.1.

Theorem 7.1.2 *Suppose that f and g are continuous on $[a, \infty)$ and $0 \leq f(x) \leq g(x)$ on $[a, \infty)$.*

(i) If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.

(ii) If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ diverges.

Proof. The proof of this follows from the order properties of the integral and is omitted.

Definition 7.1.3 For each $x > 0$, the Gamma function, denoted $\Gamma(x)$, is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

Theorem 7.1.3 *The Gamma function has the following properties:*

$$\Gamma(1) = 1 \qquad (11)$$

$$\Gamma(x+1) = x\Gamma(x) \qquad (12)$$

$$\Gamma(n+1) = n!, \ n = \text{ natural number} \qquad (13)$$

Proof.

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty e^{-t}dt \\
&= -e^{-t}\big|_0^\infty \\
&= 1 \\
\Gamma(x+1) &= \int_0^\infty t^x e^{-t}dt;\ (u = t^x, dv = e^{-t}dt) \\
&= -t^x e^{-t}\big|_0^\infty + x\int_0^\infty t^{x-1}e^{-t}dt \\
&= x\Gamma(x),\ x > 0 \\
\Gamma(2) &= 1\Gamma(1) = 1 \\
\Gamma(3) &= 2\Gamma(2) = 1 \cdot 2 = 2! \\
\text{If } \Gamma(k) &= (k-1)!,\ \text{then} \\
\Gamma(k+1) &= k\Gamma(k) \\
&= k((k-1)!) \\
&= k!.
\end{aligned}$$

By the principle of mathematical induction,

$$\Gamma(n+1) = n!$$

for all natural numbers n. This completes the proof of this theorem.

Theorem 7.1.4 *Let f be the normal probability distribution function defined by*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}$$

where μ is the constant mean of the distribution and σ is the constant standard deviation of the distribution. Then the improper integral

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

Let F be the normal distribution function defined by

$$F(x) = \int_{-\infty}^{x} f(x)dx.$$

Then $F(b) - F(a)$ represents the percentage of normally distributed data that lies between a and b. This percentage is given by

$$\int_a^b f(x)dx.$$

Furthermore,

$$\int_{\mu+a\sigma}^{\mu+b\sigma} f(x)dx = \int_a^b \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}dx.$$

Proof. The proof of this theorem is omitted.

Exercises 7.1 None available.

7.2 Discontinuities at End Points

Definition 7.2.1 (i) Suppose that f is continuous on $[a, b)$ and

$$\lim_{x\to b^-} f(x) = +\infty \quad \text{or} \quad -\infty.$$

Then, we define

$$\int_a^b f(x)dx = \lim_{x\to b^-} \int_a^x f(x)dx.$$

If the limit exists, we say that the improper integral converges; otherwise we say that it diverges.

(ii) Suppose that f is continuous on $(a, b]$ and

$$\lim_{x\to a^+} f(x) = +\infty \quad \text{or} \quad -\infty.$$

Then we define,

$$\int_a^b f(x)dx = \lim_{x\to a^+} \int_x^b f(x)dx.$$

If the limit exists, we say that the improper integral converges; otherwise we say that it diverges.

Exercises 7.2

1. Suppose that f is continuous on $(-\infty, \infty)$ and $g'(x) = f(x)$. Then define each of the following improper integrals:

(a) $\displaystyle\int_a^{+\infty} f(x)dx$

(b) $\displaystyle\int_{-\infty}^{b} f(x)dx$

(c) $\displaystyle\int_{-\infty}^{+\infty} f(x)dx$

2. Suppose that f is continuous on the open interval (a, b) and $g'(x) = f(x)$ on (a, b). Define each of the following improper integrals if f is not continuous at a or b:

   (a) $\displaystyle\int_a^{x} f(x)dx,\ a \leq x < b$

   (b) $\displaystyle\int_x^{b} f(x)dx,\ a < x \leq b$

   (c) $\displaystyle\int_a^{b} f(x)dx$

3. Prove that $\displaystyle\int_0^{+\infty} e^{-x}dx = 1$

4. Prove that $\displaystyle\int_0^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}$

5. Prove that $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \pi$

6. Prove that $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1}$, if and only if $p > 1$.

7. Show that $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}dx = 2\int_0^{\infty} e^{-x^2}dx$. Use the comparison between e^{-x} and e^{-x^2}. Show that $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}dx$ exists.

8. Prove that $\displaystyle\int_0^{1} \frac{dx}{x^p}$ converges if and only if $p < 1$.

9. Evaluate $\int_0^{+\infty} e^{-x}\sin(2x)dx$.

10. Evaluate $\int_0^{+\infty} e^{-4x}\cos(3x)dx$.

11. Evaluate $\int_0^{+\infty} x^2e^{-x}dx$.

12. Evaluate $\int_0^{+\infty} xe^{-x}dx$.

13. Prove that $\int_0^{\infty} \sin(2x)dx$ diverges.

14. Prove that $\int_0^{\infty} \cos(3x)dx$ diverges.

15. Compute the volume of the solid generated when the area between the graph of $y = e^{-x^2}$ and the x-axis is rotated about the y-axis.

16. Compute the volume of the solid generated when the area between the graph of $y = e^{-x}$, $0 \le x < \infty$ and the x-axis is rotated
    (a) about the x-axis
    (b) about the y-axis.

17. Let A represent the area bounded by the graph $y = \frac{1}{x}$, $1 \le x < \infty$ and the x-axis. Let V denote the volume generated when the area A is rotated about the x-axis.
    (a) show that A is $+\infty$
    (b) show that $V = \pi$
    (c) show that the surface area of V is $+\infty$.
    (d) Is it possible to fill the volume V with paint and not be able to paint its surface? Explain.

18. Let A represent the area bounded by the graph of $y = e^{-2x}, 0 \le x < \infty$, and $y = 0$.

(a) Compute the area of A.

(b) Compute the volume generated when A is rotated about the x-axis.

(c) Compute the volume generated when A is rotated about the y-axis.

19. Assume that $\int_0^{+\infty} \sin(x^2)dx = \sqrt{(\pi/8)}$. Compute $\int_0^{+\infty} \frac{\sin x}{\sqrt{x}}\, dx$.

20. It is known that $\int_{-\infty}^{+\infty} e^{-x^2} = \sqrt{\pi}$.

(a) Compute $\int_0^{+\infty} e^{-x^2} dx$.

(b) Compute $\int_0^{+\infty} \frac{e^{-x}}{\sqrt{x}}\, dx$.

(c) Compute $\int_0^{+\infty} e^{-4x^2} dx$.

Definition 7.2.2 Suppose that $f(t)$ is continuous on $[0,\infty)$ and there exist some constants $a > 0$, $M > 0$ and $T > 0$ such that $|f(t)| < Me^{at}$ for all $t \geq T$. Then we define the Laplace transform of $f(t)$, denoted $\mathcal{L}\{f(t)\}$, by

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$$

for all $s \geq s_0$. In problems 21–34, compute $\mathcal{L}\{f(t)\}$ for the given $f(t)$.

21. $f(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$

22. $f(t) = t$

23. $f(t) = t^2$

24. $f(t) = t^3$

25. $f(t) = t^n, n = 1, 2, 3, \cdots$

26. $f(t) = e^{bt}$

27. $f(t) = te^{bt}$

28. $f(t) = t^n e^{bt}, n = 1, 2, 3, \cdots$

29. $f(t) = \dfrac{e^{at} - e^{bt}}{a - b}$

30. $f(t) = \dfrac{ae^{at} - be^{bt}}{a - b}$

31. $f(t) = \dfrac{1}{b}\ \sin(bt)$

32. $f(t) = \cos(bt)$

33. $f(t) = \dfrac{1}{b}\ \sinh(bt)$

34. $f(t) = \cosh(bt)$

Definition 7.2.3 For $x > 0$, we define the Gamma function $\Gamma(x)$ by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt.$$

In problems 35–40 assume that $\Gamma(x)$ exists for $x > 0$ and $\displaystyle\int_0^{+\infty} e^{-x^2} = \frac{1}{2}\sqrt{\pi}$.

35. Show that $\Gamma(1/2) = \sqrt{\pi}$

36. Show that $\Gamma(1) = 1$

37. Prove that $\Gamma(x+1) = x\Gamma(x)$

38. Show that $\Gamma\left(\dfrac{3}{2}\right) = \dfrac{\sqrt{\pi}}{2}$

39. Show that $\Gamma\left(\dfrac{5}{2}\right) = \dfrac{3}{4}\sqrt{\pi}$

40. Show that $\Gamma(n+1) = n!$

In problems 41–60, evaluate the given improper integrals.

41. $\displaystyle\int_0^{+\infty} 2xe^{-x^2}dx$

42. $\displaystyle\int_1^{+\infty} \frac{dx}{x^{3/2}}$

43. $\displaystyle\int_4^{+\infty} \frac{dx}{x^{5/2}}$

44. $\displaystyle\int_1^{+\infty} \frac{4x}{1+x^2}\ dx$

45. $\displaystyle\int_1^{+\infty} \frac{x}{(1+x^2)^{3/2}}\ dx$

46. $\displaystyle\int_{16}^{+\infty} \frac{4}{x^2-4}\ dx$

47. $\displaystyle\int_2^{+\infty} \frac{1}{x(\ln x)^2}\ dx$

48. $\displaystyle\int_2^{+\infty} \frac{1}{x(\ln x)^p}\ dx, p > 1$

49. $\displaystyle\int_{-\infty}^{1} 3xe^{-x^2}dx$

50. $\displaystyle\int_{-\infty}^{2} e^{x}\ dx$

51. $\displaystyle\int_{0}^{\infty} \frac{2}{e^x + e^{-x}}\ dx$

52. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+9}$

53. $\displaystyle\int_{0}^{2} \frac{dx}{\sqrt{4-x^2}}$

54. $\displaystyle\int_{0}^{4} \frac{x}{\sqrt{16-x^2}}\ dx$

55. $\displaystyle\int_{0}^{5} \frac{x}{(25-x^2)^{2/3}}\ dx$

56. $\displaystyle\int_{2}^{+\infty} \frac{dx}{x\sqrt{x^2-4}}$

57. $\displaystyle\int_{0}^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$

58. $\displaystyle\int_{0}^{\infty} \frac{dx}{\sqrt{x}(x+25)}$

59. $\displaystyle\int_{0}^{\infty} \frac{e^{-x}}{\sqrt{1-(e^{-x})^2}}\ dx$

60. $\displaystyle\int_{0}^{+\infty} x^2e^{-x^3}dx$

7.3

Theorem 7.3.1 (Cauchy Mean Value Theorem) *Suppose that two functions f and g are continuous on the closed interval $[a,b]$, differentiable on the open interval (a,b) and $g'(x) \neq 0$ on (a,b). Then there exists at least one number c such that $a < c < b$ and*

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. See the proof of Theorem 4.1.6.

Theorem 7.3.2 *Suppose that f and g are continuous and differentiable on an open interval (a,b) and $a < c < b$. If $f(c) = g(c) = 0,\ g'(x) \neq 0$ on (a,b) and*

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = L.$$

Proof. See the proof of Theorem 4.1.7.

Theorem 7.3.3 (L'Hôpital's Rule) *Let* lim *represent one of the limits*

$$\lim_{x\to c}, \ \lim_{x\to c^+}, \ \lim_{x\to c^-}, \ \lim_{x\to +\infty}, \quad or \quad \lim_{x\to -\infty}.$$

Suppose that f and g are continuous and differentiable on an open interval (a,b) except at an interior point c, $a<c<b$. Suppose further that $g'(x)\neq 0$ on (a,b), $\lim f(x)=\lim g(x)=0$ or $\lim f(x)=\lim g(x)=+\infty$ or $-\infty$. If

$$\lim \frac{f'(x)}{g'(x)} = L, +\infty \quad or \quad -\infty$$

then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$

Proof. The proof of this theorem is omitted.

Definition 7.3.1 (Extended Arithmetic) For the sake of convenience in dealing with indeterminate forms, we define the following arithmetic operations with real numbers, $+\infty$ and $-\infty$. Let c be a real number and $c>0$. Then we define

$$+\infty+\infty=+\infty,\ -\infty-\infty=-\infty,\ c(+\infty)=+\infty,\ c(-\infty)=-\infty$$
$$(-c)(+\infty)=-\infty,\ (-c)(-\infty)=+\infty,\ \frac{c}{+\infty}=0,\ \frac{-c}{+\infty}=0,\ \frac{c}{-\infty}=0,$$
$$\frac{-c}{-\infty}=0, (+\infty)^c=+\infty,\ (+\infty)^{-c}=0,\ (+\infty)(+\infty)=+\infty,\ (+\infty)(-\infty)=-\infty,$$
$$(-\infty)(-\infty)=+\infty.$$

Definition 7.3.2 The following operations are indeterminate:

$$\frac{0}{0},\ \frac{+\infty}{+\infty},\ \frac{+\infty}{-\infty}\ \frac{-\infty}{-\infty},\ \frac{-\infty}{+\infty},\ \infty-\infty,\ 0\cdot\infty, 0^0,\ 1^\infty,\ \infty^0.$$

Remark 23 The L'Hôpital's Rule can be applied directly to the $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$ forms. The forms $\infty-\infty$ and $0\cdot\infty$ can be changed to the $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ by using arithmetic operations. For the 0^0 and 1^∞ forms we use the following procedure:

$$\lim (f(x))^{g(x)} = \lim e^{g(x)\ln(f(x))} = e^{\lim\ \frac{\ln(f(x))}{(1/g(x))}}.$$

It is best to study a lot of examples and work problems.

Exercises 7.3

1. Prove the Theorem of the Mean: Suppose that a function f is continuous on a closed and bounded interval $[a, b]$ and f' exists on the open interval (a, b). Then there exists at least one number c such that $a < c < b$ and

(1) $\dfrac{f(b) - f(a)}{b - a} = f'(c)$ (2) $f(b) = f(a) + f'(c)(b - a)$.

2. Prove the Generalized Theorem of the Mean: Suppose that f and g are continuous on a closed and bounded interval $[a, b]$ and f' and g' exist on the open interval (a, b) and $g'(x) \neq 0$ for any x in (a, b). Then there exists some c such that $a < c < b$ and

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

3. Prove the following theorem known as l'Hôpital's Rule: Suppose that f and g are differentiable functions, except possibly at a, such that

$$\lim_{x\to a} f(x) = 0, \quad \lim_{x\to a} g(x) = 0, \quad \text{and} \quad \lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

4. Prove the following theorem known as an alternate form of l'Hôpital's Rule: Suppose that f and g are differentiable functions, except possibly at a, such that

$$\lim_{x\to a} f(x) = \infty, \quad \lim_{x\to a} g(x) = \infty, \quad \text{and} \quad \lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

5. Prove that if f' and g' exist and

$$\lim_{x\to+\infty} f(x) = 0, \quad \lim_{x\to+\infty} g(x) = 0, \quad \text{and} \quad \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L.$$

6. Prove that if f' and g' exist and

$$\lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to+\infty} g(0) = 0, \quad \text{and} \quad \lim_{x\to-\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = L.$$

7. Prove that if f' and g' exist and

$$\lim_{x\to+\infty} f(x) = \infty, \quad \lim_{x\to+\infty} g(x) = \infty, \quad \text{and} \quad \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L.$$

8. Prove that if f' and g' exist and

$$\lim_{x\to-\infty} f(x) = \infty, \quad \lim_{x\to-\infty} g(x) = \infty, \quad \text{and} \quad \lim_{x\to-\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L.$$

9. Suppose that f' and f'' exist in an open interval (a, b) containing c. Then prove that

$$\lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

10. Suppose that f' is continuous in an open interval (a,b) containing c. Then prove that
$$\lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h}=f'(c).$$

11. Suppose that $f(x)$ and $g(x)$ are two polynomials such that
$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n,\ a_0\neq 0,$$
$$g(x)=b_0x^m+b_1x^{m-1}+\cdots+b_{m-1}x+b_m,\ b_0\neq 0.$$
Then prove that
$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\begin{cases}0 & \text{if } m>n\\ +\infty \text{ or } -\infty & \text{if } m<n\\ a_0/b_0 & \text{if } m=n\end{cases}$$

12. Suppose that f and g are differentiable functions, except possibly at c, and
$$\lim_{x\to c}f(x)=0,\quad \lim_{x\to c}g(x)=0 \quad\text{and}\quad \lim_{x\to c}g(x)\ln(f(x))=L.$$
Then prove that
$$\lim_{x\to c}(f(x))^{g(x)}=e^L.$$

13. Suppose that f and g are differentiable functions, except possibly at c, and
$$\lim_{x\to c}f(x)=+\infty,\quad \lim_{x\to c}g(x)=0 \quad\text{and}\quad \lim_{x\to c}g(x)\ln(f(x))=L.$$
Then prove that
$$\lim_{x\to c}(f(x))^{g(x)}=e^L.$$

14. Suppose that f and g are differentiable functions, except possibly at c, and
$$\lim_{x\to c}f(x)=1,\quad \lim_{x\to c}g(x)=+\infty \quad\text{and}\quad \lim_{x\to c}g(x)\ln(f(x))=L.$$
Then prove that
$$\lim_{x\to c}(f(x))^{g(x)}=e^L.$$

15. Suppose that f and g are differentiable functions, except possibly at c, and

$$\lim_{x\to c} f(x) = 0, \quad \lim_{x\to c} g(x) = +\infty \quad \text{and} \quad \lim_{x\to c} \frac{f(x)}{(1/g(x))} = L.$$

Then prove that

$$\lim_{x\to c} f(x)g(x) = L.$$

16. Prove that $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$.

17. Prove that $\lim_{x\to 0} (1-x)^{\frac{1}{x}} = \frac{1}{e}$.

18. Prove that $\lim_{x\to +\infty} \frac{x^n}{e^x} = 0$ for each natural number n.

19. Prove that $\lim_{x\to 0^+} \frac{\sin x - x}{x \sin x} = 0$.

20. Prove that $\lim_{x\to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x = 1$.

In problems 21–50 evaluate each of the limits.

21. $\lim_{x\to 0} \frac{\sin(x^2)}{x^2}$

22. $\lim_{x\to 0} \frac{1-\cos x^2}{x^2}$

23. $\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$

24. $\lim_{x\to 0} \frac{\tan(mx)}{\tan(nx)}$

25. $\lim_{x\to 0} \frac{e^{3x}-1}{x}$

26. $\lim_{x\to 0} (1+2x)^{3/x}$

27. $\lim_{h\to 0} \frac{\ln(x+h)-\ln(x)}{h}$

28. $\lim_{h\to 0} \frac{e^{x+h}-e^x}{h}$

29. $\lim_{x\to 0} (1+mx)^{n/x}$

30. $\lim_{x\to \infty} \frac{\ln(100+x)}{x}$

31. $\lim\limits_{x\to 0} (1+\sin mx)^{n/x}$

32. $\lim\limits_{x\to 0^+} (\sin x)^x$

33. $\lim\limits_{x\to 0^+} (x)^{\sin x}$

34. $\lim\limits_{x\to\infty} \dfrac{x^4-2x^3+10}{3x^4+2x^3-7x+1}$

35. $\lim\limits_{x\to 0^+} \tan(2x)\ln(x)$

36. $\lim\limits_{x\to +\infty} x\sin\left(\dfrac{2\pi}{x}\right)$

37. $\lim\limits_{x\to 0} (x+e^x)^{2/x}$

38. $\lim\limits_{x\to\infty} \left(\dfrac{3+2x}{4+2x}\right)^x$

39. $\lim\limits_{x\to 0} (1+\sin mx)^{n/x}$

40. $\lim\limits_{x\to 0^+} (x)^{\sin(3x)}$

41. $\lim\limits_{x\to 0^+} (e^{3x}-1)^{2/\ln x}$

42. $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2}-\dfrac{\cos 4x}{x^2}\right)$

43. $\lim\limits_{x\to 0^+} \dfrac{\cot(ax)}{\cot(bx)}$

44. $\lim\limits_{x\to +\infty} \dfrac{\ln x}{x}$

45. $\lim\limits_{x\to 0^+} \dfrac{x}{\ln x}$

46. $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x}-\dfrac{2}{\ln x}\right)$

47. $\lim\limits_{x\to +\infty} \dfrac{2x+3\sin x}{4x+2\sin x}$

48. $\lim\limits_{x\to +\infty} x(b^{1/x}-1), b>0,\ b\neq 1$

49. $\lim\limits_{h\to 0} \left(\dfrac{b^{x+h}-b^x}{h}\right), b>0, b\neq 1$

50. $\lim\limits_{h\to 0} \dfrac{\log_b(x+h)-\log_b x}{h}, b>0, b\neq 1$

51. $\lim\limits_{x\to 0} \dfrac{(e^x-1)\sin x}{\cos x-\cos^2 x}$

52. $\lim\limits_{x\to +\infty} x\ln\left(\dfrac{x+1}{x-1}\right)$

53. $\lim\limits_{x\to 0^+} \dfrac{\sin 5x}{1-\cos 4x}$

54. $\lim\limits_{x\to 1} \dfrac{2x-3x^6+x^7}{(1-x)^3}$

55. $\lim\limits_{x\to +\infty} e^x\ln\left(\dfrac{e^x+1}{e^x}\right)$

56. $\lim\limits_{x\to 0} \dfrac{\tan x-\sin x}{x^3}$

57. $\lim\limits_{x\to 0} \dfrac{x^3 \sin 2x}{(1-\cos x)^2}$

58. $\lim\limits_{x\to 0} \dfrac{5^x - 3^x}{x^2}$

59. $\lim\limits_{x\to 0} \dfrac{1}{x} \ln\left(\dfrac{1+x}{1-x}\right)$

60. $\lim\limits_{x\to 0} \dfrac{\arctan x - x}{x^3}$

61. $\lim\limits_{x\to 0} \dfrac{\sin(\pi \cos x)}{x \sin x}$

62. $\lim\limits_{x\to +\infty} \dfrac{\ln(1 + xe^{2x})}{x^2}$

63. $\lim\limits_{x\to +\infty} \dfrac{(\ln x)^n}{x}, n = 1, 2, \cdots$

64. $\lim\limits_{x\to +\infty} \dfrac{1}{\sqrt{x}} \ln\left(\dfrac{x + e^{2x}}{x}\right)$

65. $\lim\limits_{x\to +\infty} \dfrac{\ln x}{(1 + x^3)^{1/2}}$

66. $\lim\limits_{x\to 0^+} \dfrac{\ln(\tan 3x)}{\ln(\tan 4x)}$

67. $\lim\limits_{x\to 0^+} (1 - 3^{-x})^{-2x}$

68. $\lim\limits_{x\to 0} \left(\dfrac{\sin x}{x}\right)^{1/x^2}$

69. $\lim\limits_{x\to +\infty} (e^{-x} + e^{-2x})^{1/x}$

70. $\lim\limits_{x\to +\infty} \left(\cos\left(\dfrac{3}{x}\right)\right)^{x^2}$

71. $\lim\limits_{x\to 0^+} \left(\ln\left(\dfrac{1}{x}\right)\right)^{x}$

72. $\lim\limits_{x\to +\infty} \left(1 + \dfrac{1}{2x}\right)^{x^2}$

73. $\lim\limits_{x\to +\infty} \left(1 + \dfrac{1}{2x}\right)^{3x+\ln x}$

74. $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin 2x}\right)$

75. $\lim\limits_{x\to +\infty} x\left(\sqrt{x^2 + b^2} - x\right)$

76. $\lim\limits_{x\to 0} \left(\dfrac{1}{x \sin x} - \dfrac{1}{x^2}\right)$

77. $\lim\limits_{x\to 2} \left(\dfrac{1}{x-2} - \dfrac{5}{x^2 + x - 6}\right)$

78. $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \ln\left(\dfrac{1}{x}\right)\right)$

79. $\lim\limits_{x\to 0} \left(\cot x - \dfrac{1}{x}\right)$

80. $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\tan^2 x}\right)$

81. $\lim_{x\to 0}\left(\frac{e^{-x}}{x}-\frac{1}{e^x-1}\right)$

82. $\lim_{x\to\infty}\frac{x-\sin x}{x}$

83. $\lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{\sin x}$

84. $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

85. $\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x}$

86. $\lim_{x\to+\infty}\frac{\ln(\ln x)}{\ln(x-\ln x)}$

87. $\lim_{x\to 0^+}\left(\frac{1}{x^2}-\frac{1}{x\ln x}\right)$

88. $\lim_{x\to+\infty}\frac{1}{x}\int_1^x\frac{\ln t}{1+t}\,dt$

89. $\lim_{x\to+\infty}(\ln(1+e^x)-x)$

90. $\lim_{x\to+\infty}\frac{1}{x^2}\left(\int_0^x\sin^2 x\,dx\right)$

91. Suppose that f is defined and differentiable in an open interval (a,b). Suppose that $a<c<b$ and $f''(c)$ exists. Prove that

$$f''(c)=\lim_{x\to c}\frac{f(x)-f(c)-(x-c)f'(c)}{((x-c)^2/2!)}.$$

92. Suppose that f is defined and $f', f'', \cdots, f^{(n-1)}$ exist in an open interval (a,b). Also, suppose that $a<c<b$ and $f^{(n)}(c)$ exists

(a) Prove that

$$f^{(n)}(c)=\lim_{x\to c}\frac{f(x)-f(c)-(x-c)f'(c)-\cdots-\frac{(x-c)^{n-1}}{(n-1)!}f^{n-1}(c)}{\frac{(x-c)^n}{n!}}.$$

(b) Show that there is a function $E_n(x)$ defined on (a,b), except possibly at c, such that

$$f(x)=f(c)+(x-c)f'(c)+\cdots+\frac{(x-c)^{n-1}}{(n-1)!}f^{(n-1)}(x)$$
$$+\frac{(x-c)^n}{n!}f^{(n)}(c)+E_n(x)\frac{(x-c)^nE_n(x)}{n!}$$

and $\lim_{n\to c} E_n(x) = 0$. Find $E_2(x)$ if $c = 0$ and

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$$

(c) If $f'(c) = \cdots = f^{(n-1)}(c) = 0$, n is even, and f has a relative minimum at $x = c$, then show that $f^{(n)}(c) \geq 0$. What can be said if f has a relative maximum at c? What are the sufficient conditions for a relative maximum or minimum at c when $f'(c) = \cdots = f^{(n-1)}(c) = 0$? What can be said if n is odd and $f'(c) = \cdots = f^{(n-1)}(c) = 0$ but $f^{(n)}(c) \neq 0$.

93. Suppose that f and g are defined, have derivatives of order $1, 2, \cdots, n-1$ in an open interval $(a, b), a < c < b, f^{(n)}(c)$ and $g^{(n)}(c)$ exist and $g^{(n)}(c) \neq 0$. Prove that if f and g, as well as their first $n - 1$ derivatives are 0, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

Evaluate the following limits:

94. $\lim_{x\to 0} \left(\frac{x^2 \sin \frac{1}{x}}{x} \right)$

95. $\lim_{x\to 0} \frac{\cos\left(\frac{\pi}{2} \cos x\right)}{\sin^2 x}$

96. $\lim_{x\to 1} x^{\left(\frac{1}{1-x}\right)}$

97. $\lim_{x\to 0^+} x(\ln(x))^n, n = 1, 2, 3, \cdots$

98. $\lim_{x\to 1^+} \frac{x^x - x}{1 - x + \ln x}$

99. $\lim_{x\to +\infty} \frac{x^{3/2} \ln x}{(1 + x^4)^{1/2}}$

100. $\lim_{x\to +\infty} x^n \ln\left(\frac{1 + e^x}{e^x}\right), \ n = 1, 2, \cdots$

101. $\lim_{x\to 0} \frac{x \int_0^x e^{-t^2} dx}{1 - e^{-x^2}}$

7.4 Improper Integrals

1. Suppose that f is continuous on $(-\infty, \infty)$ and $g'(x) = f(x)$. Then define each of the following improper integrals:

Chapter 8

Infinite Series

8.1 Sequences

Definition 8.1.1 An infinite sequence (or sequence) is a function, say f, whose domain is the set of all integers greater than or equal to some integer m. If n is an integer greater than or equal to m and $f(n) = a_n$, then we express the sequence by writing its range in any of the following ways:

1. $f(m), f(m+1), f(m+2), \ldots$
2. $a_m, a_{m+1}, a_{m+2}, \ldots$
3. $\{f(n) : n \geq m\}$
4. $\{f(n)\}_{n=m}^{\infty}$
5. $\{a_n\}_{n=m}^{\infty}$

Definition 8.1.2 A sequence $\{a_n\}_{n=m}^{\infty}$ is said to converge to a real number L (or has limit L) if for each $\epsilon > 0$ there exists some positive integer M such that $|a_n - L| < \epsilon$ whenever $n \geq M$. We write,

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty.$$

If the sequence does not converge to a finite number L, we say that it diverges.

Theorem 8.1.1 *Suppose that c is a positive real number,* $\{a_n\}_{n=m}^{\infty}$ *and* $\{b_n\}_{n=m}^{\infty}$ *are convergent sequences. Then*

(i) $\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty} a_n$

(ii) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

(iii) $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

(iv) $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$

(v) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, *if* $\lim_{n\to\infty} b_n \neq 0$.

(vi) $\lim_{n\to\infty}(a_n)^c = \left(\lim_{n\to\infty} a_n\right)^c$

(vii) $\lim_{n\to\infty}(e^{a_n}) = e^{\lim_{n\to\infty} a_n}$

(viii) Suppose that $a_n \leq b_n \leq c_n$ *for all* $n \geq m$ *and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then

$$\lim_{n\to\infty} b_n = L.$$

Proof. Suppose that $\{a_n\}_{n=m}^{\infty}$ converges to a and $\{b_n\}_{n=m}^{\infty}$ converges to b. Let $\epsilon_1 > 0$ be given. Then there exist natural numbers N and M such that

$$|a_n - a| < \epsilon_1 \text{ if } n \geq N, \tag{1}$$

$$|b_n - b| < \epsilon_1 \text{ if } n \geq M. \tag{2}$$

Part (i) Let $\epsilon > 0$ be given and $c \neq 0$. Let $\epsilon_1 = \frac{\epsilon}{2|c|}$ and $n \geq N + M$. Then by the inequalities (1) and (2), we get

$$\begin{aligned} |ca_n - ca| &= |c|\ |a_n - a| \\ &< |c|\ \epsilon_1 \\ &< \epsilon. \end{aligned}$$

This completes the proof of Part (i).

Part (ii) Let $\epsilon > 0$ be given and $\epsilon_1 = \dfrac{\epsilon}{2}$. Let $m \geq N + M$. Then by the inequalities (1) and (2), we get

$$\begin{aligned}|(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &< \epsilon_1 + \epsilon_1 \\ &= \epsilon.\end{aligned}$$

This completes the proof of Part (ii).

Part (iii)

$$\begin{aligned}\lim_{n\to\infty}(a_n - b_n) &= \lim_{n\to\infty}\ (a_n + (-1)b_n) \\ &= \lim_{n\to\infty}\ a_n + \lim_{n\to\infty}\ [(-1)b_n] \quad \text{(by Part (ii))} \\ &= \lim_{n\to\infty}\ a_n + (-1)\lim_{n\to\infty}\ b_n \quad \text{(by Part (i))} \\ &= a + (-1)b \\ &= a - b.\end{aligned}$$

Part (iv) Let $\epsilon > 0$ be given and $\epsilon_1 = \min\left(1, \dfrac{\epsilon}{1 + |a| + |b|}\right)$. If $n \geq N + M$, then by the inequalities (1) and (2) we have

$$\begin{aligned}|a_n b_n - ab| &= |[(a_n - a) + a][(b_n - b) + b] - ab| \\ &= |(a_n - a)(b_n - b) + (a_n - a)b + a(b_n - b| \\ &\leq |a_n - a|\ |b_n - b| + |b|\ |a_n - a| + |a|\ |b_n - b| \\ &< \epsilon_1^2 + |b|\epsilon_1 + |a|\epsilon_1 \\ &= \epsilon_1(\epsilon_1 + |b| + |a|) \\ &\leq \epsilon_1(1 + |b| + |a|) \\ &\leq \epsilon.\end{aligned}$$

Part (v) First we assume that $b > 0$ and prove that

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}.$$

By taking $\epsilon_1 = \dfrac{1}{2}\,b$ and using inequality (2) for $n \geq M$, we get

$$|b_n - b| < \frac{1}{2}\,b,\ -\frac{1}{2}\,b < b_n - b < \frac{1}{2}\,b,$$
$$\frac{1}{2}\,b < b_n < \frac{3}{2}\,b,\ 0 < \frac{2}{3b} < \frac{1}{b_n} < \frac{2}{b}.$$

Then, for $n \geq M$, we get

$$\begin{aligned}\left|\frac{1}{b_n} - \frac{1}{b}\right| &= \left|\frac{b - b_n}{b - nb}\right| \\ &= |b_n - b| \cdot \frac{1}{b} \cdot \frac{1}{b_n} \\ &< |b_n - b| \cdot \frac{2}{b^2}. \end{aligned} \tag{3}$$

Let $\epsilon > 0$ be given. Choose $\epsilon_2 = \min\left(\dfrac{b}{2}, \dfrac{\epsilon b^2}{2}\right)$. There exists some natural number N such that if $n \geq N$, then

$$|b_n - b| < \epsilon_2. \tag{4}$$

If $n \geq N + M$, then the inequalities (3) and (4) imply that

$$\begin{aligned}\left|\frac{1}{b_n} - \frac{1}{b}\right| &< |b_n - b|\ \frac{2}{b^2} \\ &< \epsilon_2 \frac{2}{b^2} \\ &\leq \epsilon. \end{aligned}$$

It follows that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{b_n} &= \frac{1}{b} \\ \lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) &= \lim_{n\to\infty} (a_n) \cdot \lim_{n\to\infty} \left(\frac{1}{b_n}\right) \\ &= a \cdot \left(\frac{1}{b}\right) \\ &= \frac{a}{b}. \end{aligned}$$

If $b < 0$, then

$$\begin{aligned}\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) &= \lim_{n\to\infty} (-a_n) \cdot \lim_{n\to\infty} \left(\frac{1}{-b_n}\right) \\ &= (-a)\left(\frac{1}{-b}\right) \\ &= \frac{a}{b}.\end{aligned}$$

This completes the proof of Part (v).

Part (vi) Since $f(x) = x^c$ is a continuous function,

$$\lim_{n\to\infty} (a_n)^c = \left(\lim_{n\to\infty} a_n\right)^c = a^c.$$

Part (vii) Since $f(x) = e^x$ is a continuous function,

$$\lim_{n\to\infty} e^{a_n} = e^{\lim_{n\to\infty} a_n} = e^a.$$

Part (viii) Suppose that $a_n \leq b_n \leq c_n$ for all $n \geq m$ and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n = L.$$

Let $\epsilon > 0$ be given. Then there exists natural numbers N and M such that

$$\begin{aligned}|a_n - L| &< \frac{\epsilon}{2}\,, \quad \frac{-\epsilon}{2} < a_n - L < \frac{\epsilon}{2} \quad \text{for } n \geq N, \\ |c_n - L| &< \frac{\epsilon}{2}\,, \quad \frac{-\epsilon}{2} < c_n - L < \frac{\epsilon}{2} \quad \text{for } n \geq M.\end{aligned}$$

If $n \geq N + M$, then $n > N$ and $n > M$ and, hence,

$$-\frac{\epsilon}{2} < a_n - L \leq b_n - L \leq c_n - L < \frac{\epsilon}{2}.$$

It follows that

$$\lim_{n\to\infty} b_n = L.$$

This completes the proof of this theorem.

8.2 Monotone Sequences

Definition 8.2.1 Let $\{t_n\}_{n=m}^{\infty}$ be a given sequence. Then $\{t_n\}_{n=m}^{\infty}$ is said to be

(a) *increasing* if $t_n < t_{n+1}$ for all $n \geq m$;

(b) *decreasing* if $t_{n+1} < t_n$ for all $n \geq m$;

(c) *nondecreasing* if $t_n \leq t_{n+1}$ for all $n \geq m$;

(d) *nonincreasing* if $t_{n+1} \leq t_n$ for all $n \geq m$;

(e) *bounded* if $a \leq t_n \leq b$ for some constants a and b and all $n \geq m$;

(f) *monotone* if $\{t_n\}_{n=m}^{\infty}$ is increasing, decreasing, nondecreasing or nonincreasing.

(g) a *Cauchy sequence* if for each $\epsilon > 0$ there exists some M such that $|a_{n_1} - a_{n_2}| < \epsilon$ whenever $n_1 \geq M$ and $n_2 \geq M$.

Theorem 8.2.1 *(a) A monotone sequence converges to some real number if and only if it is a bounded sequence.*

(b) A sequence is convergent if and only if it is a Cauchy sequence.

Proof.
Part (a) Suppose that $a_n \leq a_{n+1} \leq B$ for all $n \geq M$ and some B. Let L be the least upper bound of the sequence $\{a_n\}_{n=m}^{\infty}$. Let $\epsilon > 0$ be given. Then there exists some natural number N such that

$$L - \epsilon < a_N \leq L.$$

Then for each $n \geq N$, we have

$$L - \epsilon < a_N \leq a_n \leq L.$$

By definition $\{a_n\}_{n=m}^{\infty}$ converges to L.

Similarly, suppose that $B \leq a_{n+1} \leq a_n$ for all $n \geq M$. Let L be the greatest lower bound of $\{a_n\}_{n=m}^{\infty}$. Then $\{a_n\}_{n=m}^{\infty}$ converges to L. It follows that a bounded monotone sequence converges. Conversely, suppose that a monotone sequence $\{a_n\}_{n=m}^{\infty}$ converges to L. Let $\epsilon = 1$. Then there exists some natural number N such that if $n \geq N$, then

$$\begin{aligned} |a_n - L| &< \epsilon \\ -\epsilon < a_n - L &< \epsilon \\ L - \epsilon < a_n &< L + \epsilon. \end{aligned}$$

The set $\{a_n : m \leq n \leq N\}$ is bounded and the set $\{a_n : n \geq N\}$ is bounded. It follows that $\{a_n\}_{n=m}^{\infty}$ is bounded. This completes the proof of Part (a) of the theorem.

Part (b) First, let us suppose that $\{a_n\}_{n=m}^{\infty}$ converges to L. Let $\epsilon > 0$ be given. Then $\frac{\epsilon}{2} > 0$ and hence there exists some natural number N such that for all natural numbers $p \geq N$ and $q \geq N$, we have

$$\begin{aligned} |a_p - L| < \frac{\epsilon}{2} \quad &\text{and} \quad |a_q - L| < \frac{\epsilon}{2} \\ |a_p - a_q| &= |(a_p - L) + (L + a_q)| \\ &\leq |a_p - L| + |a_1 - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

It follows that $\{a_n\}_{n=m}^{\infty}$ is a Cauchy sequence.

Next, we suppose that $\{a_n\}_{n=m}^{\infty}$ is a Cauchy sequence. Let $S = \{a_n : m \leq n < \infty\}$. Suppose $\epsilon > 0$. Then there exists some natural number N such that for all $p \geq 1$

$$|a_{N+p} - a_N| < \frac{\epsilon}{2}, \; a_N - \frac{\epsilon}{2} < a_{N+p} < a_N + \frac{\epsilon}{2} \tag{1}$$

It follows that S is a bounded set. If S is an infinite set, then S has some limit point q and some subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=m}^{\infty}$ that converges to q. Since $\epsilon > 0$, there exists some natural number M such that for all $k \geq M$, we have

$$|a_{n_k} - q| < \frac{\epsilon}{2} \tag{2}$$

Also, for all $k \geq N + M$, we get $n_k \geq k \geq N + M$ and

$$\begin{aligned}|a_k - q| &= |a_k - a_{n_k} + a_{n_k} - q|\\ &\leq |a_{n_k} - a_k| + |a_{n_k} - q|\\ &< \frac{\epsilon}{2} + \epsilon 2 \qquad \text{(by (1) and (2))}\\ &= \epsilon.\end{aligned}$$

It follows that the sequence $\{a_n\}_{n=m}^{\infty}$ converges to q. If S is a finite set, then some a_k is repeated infinite number of times and hence some subsequences of $\{a_n\}_{n=m}^{\infty}$ converges to a_k. By the preceding argument $\{a_n\}_{n=m}^{\infty}$ also converges to a_k. This completes the proof of this theorem.

Theorem 8.2.2 *Let $\{f(n)\}_{n=m}^{\infty}$ be a sequence where f is a differentiable function defined for all real numbers $x \geq m$. Then the sequence $\{f(n)\}_{n=m}^{\infty}$ is*

(a) increasing if $f'(x) > 0$ for all $x > m$;

(b) decreasing if $f'(x) < 0$ for all $x > m$;

(c) nondecreasing if $f'(x) \geq 0$ for all $x > m$;

(d) nonincreasing if $f'(x) \leq 0$ for all $x > m$.

Proof. Suppose that $m \leq a < b$. Then by the Mean Value Theorem for derivatives, there exists some c such that $a < c < b$ and

$$\begin{aligned}\frac{f(b) - f(a)}{b - a} &= f'(c),\\ f(b) &= f(a) + f'(c)(b - a).\end{aligned}$$

The theorem follows from the above equation by considering the value of $f'(c)$. In particular, for all natural numbers $n \geq m$,

$$f(n + 1) = f(n) + f'(c),$$

for some c such that $n < c < n + 1$.

Part (a). If $f'(c) > 0$, then $f(n + 1) > f(n)$ for all $n \geq m$.
Part (b). If $f'(c) < 0$, then $f(n + 1) < f(n)$ for all $n \geq m$.
Part (c). If $f'(c) \geq 0$, then $f(n + 1) \geq f(n)$ for all $n \geq m$.
Part (d). If $f'(c) \leq 0$, then $f(n + 1) \leq f(n)$ for all $n \leq m$.

This completes the proof of this theorem.

8.3 Infinite Series

Definition 8.3.1 Let $\{t_n\}_{n=1}^{\infty}$ be a given sequence. Let

$$s_1 = t_1,\ s_2 = t_1 + t_2,\ s_3 = t_1 + t_2 + t_3, \cdots,\ s_n = \sum_{k=1}^{n} t_k,$$

for all natural number n. If the sequence $\{s_n\}_{n=1}^{\infty}$ converges to a finite number L, then we write

$$L = t_1 + t_2 + t_3 + \cdots = \sum_{k=1}^{\infty} t_k.$$

We call $\sum_{k=1}^{n} t_k$ an infinite series and write

$$\sum_{k=1}^{\infty} t_k = \lim_{n\to\infty} \sum_{k=1}^{n} t_k = L.$$

We say that L is the sum of the series and the series *converges* to L. If a series does not converge to a finite number, we say that it *diverges*. The sequence $\{s_n\}_{n=1}^{\infty}$ is called the sequence of the nth partial sums of the series.

Theorem 8.3.1 *Suppose that a and r are real numbers and $a \neq 0$. Then the geometric series*

$$a + ar + ar^2 + \cdots = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r},$$

if $|r| < 1$. The geometric series diverges if $|r| \geq 1$.

Proof. For each natural number n, let

$$s_n = a + ar + \cdots + ar^{n-1}.$$

On multiplying both sides by r, we get

$$\begin{aligned} rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ s_n - rs_n &= a - ar^n \\ (1-r)s_n &= a(1-r^n) \\ s_n &= \frac{a}{1-r} - \left(\frac{a}{1-r}\right) r^n. \end{aligned}$$

If $|r| < 1$, then

$$\lim_{n\to\infty} s_n = \frac{a}{1-r} - \frac{a}{1-r} \lim_{n\to\infty} r^n = \frac{a}{1-r}.$$

If $|r| > 1$, then $\lim\limits_{n\to\infty} r^n$ is not finite and so the sequence $\{s_n\}_{n=1}^{\infty}$ of nth partial sums diverges.

If $r = 1$, then $s_n = na$ and $\lim\limits_{n\to\infty} na$ is not a finite number.

This completes the proof of the theorem.

Theorem 8.3.2 (Divergence Test) *If the series* $\sum\limits_{k=1}^{\infty} t_k$ *converges, then* $\lim\limits_{n\to\infty} t_n = 0$. *If* $\lim\limits_{n\to\infty} t_n \neq 0$*, then the series diverges.*

Proof. Suppose that the series converges to L. Then

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k \right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} a_k - \lim_{n\to\infty} \sum_{k=1}^{n-1} a_k \\
&= L - L \\
&= 0.
\end{aligned}$$

The rest of the theorem follows from the preceding argument. This completes the proof of this theorem.

Theorem 8.3.3 (The Integral Test) *Let f be a function that is defined, continuous and decreasing on* $[1, \infty)$ *such that* $f(x) > 0$ *for all* $x \geq 1$. *Then*

$$\sum_{n=1}^{\infty} f(n) \quad \text{and} \quad \int_1^{\infty} f(x)dx$$

either both converge or both diverge.

Proof. Suppose that f is decreasing and continuous on $[1, \infty)$, and $f(x) > 0$ for all $x \geq 1$. Then for all natural numbers n, we get,

$$\sum_{k=2}^{n+1} f(k) \leq \int_1^{n+1} f(x)dx \leq \sum_{k=1}^{n} f(k)$$

graph

It follows that,

$$\sum_{k=2}^{\infty} f(k) \leq \int_1^{\infty} f(x)dx \leq \sum_{k=1}^{\infty} f(k).$$

Since $f(1)$ is a finite number, it follows that

$$\sum_{k=1}^{\infty} f(k) \quad \text{and} \quad \int_1^{\infty} f(x)dx$$

either both converge or both diverge. This completes the proof of the theorem.

Theorem 8.3.4 *Suppose that $p > 0$. Then the p-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $0 < p \leq 1$. In particular, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof. Suppose that $p > 0$. Then

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p}\, dx &= \int_1^{\infty} x^{-p} dx \\
&= \left.\frac{x^{1-p}}{1-p}\right|_1^{\infty} \\
&= \frac{1}{1-p} \left(\lim_{x \to \infty} x^{1-p} - 1 \right).
\end{aligned}$$

It follows that the integral converges if $p > 1$ and diverges if $p < 1$. If $p = 1$, then

$$\int_1^{\infty} \frac{1}{x}\, dx = \ln x \Big|_1^{\infty} = \infty.$$

Hence, the p-series converges if $p > 1$ and diverges if $0 < p \leq 1$. This completes the proof of this theorem.

Exercises 8.1

1. Define the statement that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L.

2. Suppose the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L and the sequences $\{b_n\}_{n=1}^{\infty}$ converges to M. Then prove that

   (a) $\{ca_n\}_{n=1}^{\infty}$ converges to cL, where c is constant.

   (b) $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$.

   (c) $\{a_n - b_n\}_{n=1}^{\infty}$ converges to $L - M$.

   (d) $\{a_n b_n\}_{n=1}^{\infty}$ converges to LM.

   (e) $\left\{\dfrac{a_n}{b_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{L}{M}$, if $M \neq 0$.

3. Suppose that $0 < a_n \leq a_{n+1} < M$ for each natural number n. Then prove that

   (a) $\{a_n\}_{n=1}^{\infty}$ converges.

   (b) $\{-a_n\}_{n=1}^{\infty}$ converges.

   (c) $\left\{a_n^k\right\}_{n=1}^{\infty}$ converges for each natural number k.

4. Prove that $\left\{\dfrac{x^n}{n!}\right\}_{n=1}^{\infty}$ converges to 0 for every real number x.

5. Prove that $\left\{\dfrac{n!}{n^n}\right\}_{n=1}^{\infty}$ converges to 0.

6. Prove that for each natural number $n \geq 2$,

   (a) $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \ln(n) < 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1}$.

   (b) $\dfrac{1}{2^p} + \dfrac{1}{3^p} + \cdots + \dfrac{1}{n^p} < \int_1^n \dfrac{1}{t^p}\,dt < 1 + \dfrac{1}{2^p} + \cdots + \dfrac{1}{(n-1)^p}$ for each $p > 0$.

   (c) $\left\{\displaystyle\sum_{k=1}^{n} \frac{1}{k^p}\right\}_{n=1}^{\infty}$ converges if and only if $\left\{\displaystyle\int_1^{\infty} \frac{1}{t^p}\,dt\right\}$ converges. Determine the numbers p for which $\left\{\displaystyle\sum_{n=1}^{n} \frac{1}{k^p}\right\}_{n=1}^{\infty}$ converges.

7. Prove that $\left\{\sum_{k=0}^{n} r^k\right\}_{n=1}^{\infty}$ converges if and only if $|r| < 1$.

8. Prove that $\left\{\sum_{k=1}^{n} \frac{1}{k}\right\}_{n=1}^{\infty}$ diverges.

9. Prove that $\left\{\sum_{k=2}^{n} \frac{1}{k \ln k}\right\}_{n=2}^{\infty}$ diverges.

10. Prove that for each natural number $m \geq 2$,

    (a) $\int_1^m (\ln t)dt < \ln(m!) < \int_1^{m+1} (\ln t)dt$

    (b) $m(\ln(m) - 1) < \ln(m!) < (m+1)(\ln(m+1) - 1)$.

    (c) $\frac{m^m}{e^{m-1}} < m! < \frac{(m+1)^{m+1}}{e^m}$.

    (d) $\lim_{m \to +\infty} (m!)^{1/m} = +\infty$.

    (e) $\lim_{m \to +\infty} \frac{(m!)^{1/m}}{m} = \frac{1}{e}$

11. Prove that $\{(-1)^n\}_{n=1}^{\infty}$ does not converge.

12. Prove that $\left\{\frac{\sin(1/n)}{(1/n)}\right\}_{n=1}^{\infty}$ converges to 1.

13. Prove that $\left\{\frac{\sin n}{n}\right\}_{n=1}^{\infty}$ converges to zero.

8.4 Series with Positive Terms

Theorem 8.4.1 (Algebraic Properties) *Suppose that* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *are convergent series and* $c > 0$*. Then*

(i) $\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$

(ii) $\sum_{k=1}^{\infty}(a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$

(iii) $\sum_{k=1}^{\infty} c\ a_k = c\sum_{k=1}^{\infty} a_k$

(iv) If m is any natural number, then the series

$$\sum_{k=1}^{\infty} c_k \quad \textit{and} \quad \sum_{k=m}^{\infty} c_k$$

either both converge or both diverge.

Proof.
Part (i)

$$\begin{aligned}\sum_{k=1}^{\infty}(a_k \pm b_k) &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}(a_k \pm b_k)\right)\\ &= \left(\lim_{n\to\infty}\sum_{k=1}^{n} a_k\right) \pm \left(\lim_{n\to\infty}\sum_{k=1}^{n} b_k\right)\\ &= \sum_{k=1}^{\infty} a_k \pm \sum_{k=1}^{\infty} b_k.\end{aligned}$$

Part (ii) This part also follows from the preceding argument.

Part(iii) We see that

$$\begin{aligned}\sum_{k=1}^{\infty} c\ a_k &= \lim_{n\to\infty}\left(\sum_{k=1}^{n} c\ a_k\right)\\ &= c\left(\lim_{n\to\infty}\sum_{k=1}^{n} a_k\right)\\ &= c\sum_{k=1}^{\infty} a_k.\end{aligned}$$

Part (iv) We observe that

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{m-1} a_k + \sum_{k=1}^{\infty} a_k.$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} a_k &= \lim_{n\to\infty} \sum_{k=1}^{n} a_k \\ &= \sum_{k=1}^{m-1} a_k + \lim_{n\to\infty} \sum_{k=m}^{n} a_k.\end{aligned}$$

It follows that the series

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \sum_{k=m}^{\infty} a_k$$

either both converge or both diverge. This completes the proof of this theorem.

Theorem 8.4.2 (Comparison Test) *Suppose that* $0 < a_n \leq b_n$ *for all natural numbers* $n \geq 1$.

(a) If there exists some M *such that* $\sum_{k=1}^{n} a_k \leq M$, *for all natural numbers* n, *then* $\sum_{k=1}^{\infty} a_k$ *converges. If there exists no such* M, *then the series diverges.*

(b) If $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

(c) If $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

(d) If $c_n > 0$ *for all natural numbers* n, *and*

$$\lim_{n\to\infty} \frac{c_n}{a_n} = L, \ 0 < L < \infty,$$

then the series $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} c_k$ *either both converge or both diverge.*

Proof. Let $A_n = \sum_{k=1}^{n} a_k,\ B_n = \sum_{k=1}^{n} b_k,\ 0 < a_n \le b_n$ for all natural numbers n. The sequences $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ are strictly increasing sequence. Let A represent the least upper bound of $\{A_n\}_{n=1}^{\infty}$ and let B represent the least upper bound of $\{B_n\}_{n=1}^{\infty}$

Part (a) If $A_n \le M$ for all natural numbers, then $\{A_n\}_{n=1}^{\infty}$ is a bounded and strictly increasing sequence. Then A is a finite number and $\{A_n\}_{n=1}^{\infty}$ converges to A and

$$A = \sum_{k=1}^{\infty} a_k.$$

Part (b) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} b_k = B$ and $A_n \le B_n \le B$ for all natural numbers n. By Part (a), $\sum_{k=1}^{\infty} a_k$ converges to A.

Part (c) If $\sum_{k=1}^{\infty} a_k$ diverges, then the sequence $\{A_n\}_{n=1}^{\infty}$ diverges. Since $\{A_n\}_{n=1}^{\infty}$ is strictly increasing and divergent, for every M there exists some m such that

$$M < A_n \le B_n$$

for all natural numbers $n \ge m$. It follows that $\{B_n\}_{n=1}^{\infty}$ diverges.

Part (d) Suppose that $0 < a_n$ and $0 < c_n,\ 0 < L < \infty,\ \epsilon = \dfrac{L}{2}$ and

$$\lim_{n\to\infty} \frac{c_n}{a_n} = L.$$

Then there exists some natural number m such that

$$\left| \frac{c_n}{a_n} - L \right| < \frac{1}{2}$$

for all natural numbers $n \geq m$. Hence, for all $n \geq m$, we have

$$-\frac{L}{2} < \frac{c_n}{a_n} - L < \frac{L}{2}, \quad \frac{L}{2} < \frac{c_n}{a_n} < \frac{3}{2}\, L$$
$$\left(\frac{L}{2}\right) a_n \leq c_n \leq \left(\frac{3}{2}\, L\right) a_n.$$
$$\left(\frac{L}{2}\right) \sum_{k=m}^{n} a_k \leq \sum_{k=m} mc_k \leq \left(\frac{3}{2}\, L\right) \sum_{k=m}^{n} a_k$$

If $\left\{\sum_{k=1}^{n} a_k\right\}_{n=1}^{\infty}$ diverges, then $\left\{\frac{L}{2} \sum_{k=m}^{m} a_k\right\}$ diverges and, hence $\left\{\sum_{k=m}^{n} c_k\right\}_{n=m}^{\infty}$ and $\left\{\sum_{k=1}^{n} c_k\right\}_{k=1}^{\infty}$ both diverge.

If $\left\{\sum_{k=1}^{n} a_k\right\}_{k=1}^{\infty}$ converges, then $\left\{\left(\frac{3}{2}\, L\right) \sum_{k=m}^{n} a_k\right\}_{n=m}^{\infty}$ converges and, hence, $\left\{\sum_{k=m} c_k\right\}_{n=m}^{\infty}$ and $\left\{\sum_{k=1}^{n} c_k\right\}_{n=1}^{\infty}$ both converge.

This completes the Proof of Theorem 8.4.2.

Theorem 8.4.3 (Ratio Test) *Suppose that* $0 < a_n$ *for every natural number* n *and*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

Then the series $\sum_{k=1}^{\infty} a_k$

(a) converges if $r < 1$*;*

(b) diverges if $r > 1$*;*

(c) may converge or diverge if $r = 1$*; the test fails.*

Proof. Suppose that $0 < a_n$ for every natural number n and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

Let $\epsilon > 0$ be given. Then there exists some natural number M such that

$$\left|\frac{a_{n+1}}{a_n} - r\right| < \epsilon, \quad -\epsilon + r < \frac{a_{n+1}}{a_n} < r + \epsilon$$
$$(r - \epsilon)a_n < a_{n+1} < (r + \epsilon)a_n \tag{1}$$

for all natural numbers $n \geq M$.

Part (a) Suppose that $0 \leq r < 1$ and $\epsilon = (1 - r)/2$. Then for each natural number k, we have

$$a_{m+k} < (r + \epsilon)^k a_m = \left(\frac{1 + r}{2}\right)^k a_m \ldots \tag{2}$$

Hence, by (2), we get

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{m-1} a_n + \sum_{k=0}^{\infty} a_{m+k} \\
&< \sum_{n=1}^{m-1} a_n + \sum_{k=0}^{\infty} \left(\frac{1 + r}{2}\right)^k a_m \\
&= \sum_{n=1}^{m-1} a_n + \frac{a_m}{1 - \left(\frac{1+r}{2}\right)} \\
&= \sum_{n=1}^{m-1} a_n + \frac{2a_m}{1 - r} \\
&< \infty.
\end{aligned}$$

It follows that the series $\sum_{n=1}^{\infty} a_n$ converges.

Part (b) Suppose that $1 < r$, $\epsilon = (r - 1)/2$. Then by (1) we get

$$a_n < \frac{3r - 1}{2} a_n < a_{n+1}$$

for all $n \geq m$. It follows that

$$0 < a_m \leq \lim_{k \to \infty} a_{m+k} = \lim_{n \to \infty} a_n.$$

By the Divergence test, the series $\sum_{n=1}^{\infty} a_n$ diverges.

Part (c) For both series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

But, by the p-series test, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Thus, the ratio test fails to test the convergence or divergence of these series when $r = 1$.

This completes the proof of Theorem 8.4.3.

Theorem 8.4.4 (Root Test) *Suppose that* $0 < a_n$ *for each natural number* n *and*

$$\lim_{n\to\infty} (a_n)^{1/n} = r.$$

Then the series $\sum_{k=1}^{\infty} a_k$

(a) converges if $r < 1$*;*

(b) diverges if $r > 1$*;*

(c) may converge or diverge if $r = 1$*; the test fails.*

Proof. Suppose that $0 < a_n$ for each natural number n and

$$\lim_{n\to\infty} (a_n)^{1/n} = r.$$

Let $\epsilon > 0$ be given. Then there exists some natural number m such that

$$\left|(a_n)^{1/n} - r\right| < \epsilon$$
$$r - \epsilon < (a_n)^{1/n} < r + \epsilon \ldots \qquad (3)$$

for all natural numbers $n \geq m$.

Part (a) Suppose $r < 1$ and $\epsilon = \frac{1+r}{2}$. Then, by (3), for each natural number $n \geq m$, we have

$$(a_n)^{1/n} < \frac{1+r}{2} \quad \text{and} \quad a_n < \left(\frac{1-r}{2}\right)^n.$$

it follows that

$$\begin{aligned}\sum_{n=1}^{\infty} a_k &= \sum_{n=1}^{m-1} a_n + \sum_{n=m}^{\infty} a_n \\ &< \sum_{n=1}^{m-1} a_n + \sum_{n=m}^{\infty} \left(\frac{1+r}{2}\right)^n \\ &= \sum_{n=1}^{m-1} a_n + \left(\frac{1+r}{2}\right)^m \left(\frac{1}{1-\left(\frac{1+r}{2}\right)}\right) \\ &= \sum_{n=1}^{m-1} a_n + \left(\frac{1+r}{2}\right)^m \left(\frac{2}{1-r}\right) \\ &< \infty.\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} a_k$ converges.

Part (b) Suppose $r > 1$ and $\epsilon = (r-1)/2$. Then, by (3), for each natural number $n \geq m$, we have

$$\begin{aligned}1 &< \frac{1+r}{2} = r + \epsilon < (a_n)^{1/n} \\ 1 &< \left(\frac{1+r}{2}\right)^n < a_n.\end{aligned}$$

It follows that $\lim_{n\to\infty} a_n \neq 0$ and, by the Divergence test, the series $\sum_{n=1}^{\infty} a_n$ diverges.

Part (c) For each of the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ we have $r = 1$, where

$$r = \lim_{n\to\infty} (a_n)^{1/n}.$$

But the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test. Therefore, the test fails to determine the convergence or divergence for these series when $r = 1$. This completes the proof of Theorem 8.4.4.

Exercises 8.2

1. Define what is meant by $\sum_{k=1}^{\infty} a_k$.

2. Define what is meant by the sequence of nth partial sums of the series $\sum_{k=1}^{\infty} a_k$.

3. Suppose that $a \neq 0$. Prove that $\sum_{k=0}^{\infty} ar^k$ converges to $\frac{a}{1-r}$ if $|r| < 1$.

4. Prove that the series $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$ converges to $\frac{3}{4}$.

5. Prove that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges to $\frac{1}{p-1}$ if $p > 1$ and diverges otherwise.

6. Prove that $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is an increasing sequence and the series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ diverges.

7. Prove that $\sum_{k=0}^{\infty} (-1)^k x^k$ converges to $\frac{1}{1+x}$ if $|x| < 1$.

8. Prove that $\sum_{k=0}^{\infty} x^{2k}$ converges to $\frac{1}{1-x^2}$ if $|x| < 1$.

9. Prove that $\sum_{k=0}^{\infty} (-1)^k x^{2k}$ converges to $\frac{1}{1+x^2}$ if $|x| < 1$.

10. Prove that if $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Is the converse true? Explain your answer.

11. Suppose that if $\sum_{k=0}^{\infty} a_k$ converges to L and $\sum_{k=0}^{\infty} b_k$ converges to M. Prove that

(a) $\sum_{k=0}^{\infty} (c\ a_k)$ converges to cL for each constant c.

(b) $\sum_{k=0}^{\infty} (a_k + b_k)$ converges to $L + M$.

(c) $\sum_{k=0}^{\infty} (a_k - b_k)$ converges to $L - M$.

(d) $\sum_{k=0}^{\infty} a_k b_k$ may or may not converge to LM.

12. Prove that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $\int_1^{\infty} \frac{1}{t^p}\,dt$ converges. Determine the values of p for which the series converges.

13. Suppose that $f(x)$ is continuous and decreasing on the interval $[a, +\infty)$. Let $a_k = f(k)$ for each natural number k. Then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_a^{\infty} f(x)dx$ converges.

14. Suppose that $0 \leq a_k \leq a_{k+1}$ for each natural number k, and $s_n = \sum_{k=1}^{n} a_k$. Prove that if $s_n \leq M$ for some M and all natural numbers n, then $\sum_{k=1}^{\infty} a_k$ converges.

15. Suppose that $0 \leq a_k \leq b_k$ for each natural number k. Prove that

(a) if $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(b) if $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

(c) if $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(d) if $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ may or may not converge.

16. Suppose that $0 < a_k$ for each natural number k. Prove that if $\lim_{k\to\infty} (a_{k+1}/a_k) < 1$, then $\sum_{k=1}^{\infty} a_k$ converges.

17. Suppose that $0 < a_k$ for each natural number k. Prove that if $\lim_{k\to\infty} (a_{k+1}/a_k) > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

18. Suppose that $0 < a_k$ for each natural number k. Prove that if $\lim_{k\to\infty} (a_{k+1}/a_k) = 1$, then $\sum_{k=1}^{\infty} a_k$ may or may not converge.

19. Suppose that $0 < a_k$ and $0 < b_k$ for each natural number k. Prove that if $0 < \lim_{k\to\infty} (a_k/b_k) < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.

20. Suppose that $0 < a_k$ for each natural number k. Prove that if $\lim_{k\to\infty} (a_k)^{1/k} < 1$, then $\sum_{k=1}^{\infty} a_k$ converges.

21. Suppose that $0 < a_k$ for each natural number k. Prove that if $\lim_{k\to\infty} (a_k)^{1/k} > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

22. Suppose that $0 < a_k$ for each natural number k. Prove that if $\lim_{k\to\infty} (a_k)^{1/k} = 1$, then $\sum_{k=1}^{\infty} a_k$ may or may not converge.

23. A series $\sum_{k=1}^{\infty} a_k$ is said to converge absolutely if $\sum_{k=1}^{\infty} |a_k|$ converges. Suppose that $\lim_{k\to\infty} |a_{k+1}/a_k| = p$. Prove that

(a) $\sum_{k=1}^{\infty} a_k$ converges absolutely if $p < 1$.

(b) $\sum_{k=1}^{\infty} a_k$ does not converge absolutely if $p > 1$.

(c) $\sum_{k=1}^{\infty} a_k$ may or may not converge absolutely if $p = 1$.

24. A series $\sum_{k=1}^{\infty} a_k$ is said to converge absolutely if $\sum_{k=1}^{\infty} |a_k|$ converges. Suppose that $\lim_{k\to\infty} (|a_k|)^{1/k} = p$. Prove that

(a) $\sum_{k=1}^{\infty} a_k$ converges absolutely if $p < 1$.

(b) $\sum_{k=1}^{\infty} a_k$ does not converge absolutely if $p > 1$.

(c) $\sum_{k=1}^{\infty} a_k$ may or may not converge absolutely if $p = 1$.

25. Prove that if $\sum_{k=1}^{\infty} a_k$ converges absolutely, then it converges. Is the converse true? Justify your answer.

26. Suppose that $a_k \neq 0, b_k \neq 0$ for any natural number k and $\lim_{k\to\infty} \left|\frac{a_k}{b_k}\right| = p$. Prove that if $0 < p < 1$, then the series $\sum_{k=1}^{\infty} a_k$ converges absolutely if and only if $\sum_{k=1}^{\infty} b_k$ converges absolutely.

27. A series $\sum_{k=1}^{\infty} a_k$ is said to converge conditionally if $\sum_{k=1}^{\infty} a_k$ converges but

$\sum_{k=1}^{\infty} |a_k|$ diverges. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally or absolutely.

28. Suppose that $0 < a_k$ and $|a_{k+1}| < |a_k|$ for every natural number k. Prove that if $\lim_{k \to +\infty} a_k = 0$, then the series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ and $\sum_{k=1}^{\infty} (-1)^k a_k$ are both convergent. Furthermore, show that if s denotes the sum of the series, then s is between the nth partial sum s_n and the $(n+1)$st partial sum s_{n+1} for each natural number n.

29. Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{3^n}$ converges absolutely or conditionally.

30. Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{n^{10}}$ converges absolutely or conditionally.

In problems 31–62, test the given series for convergence, conditional convergence or absolute convergence.

31. $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{5^n}$

32. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{5^n}{n!}$

33. $\sum_{n=1}^{\infty} (-1)^n n \left(\frac{4}{5}\right)^n$

34. $\sum_{n=1}^{\infty} (-1)^{n+1} n^2 \left(\frac{4}{5}\right)^n$

35. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$

36. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/2}}$

37. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}, 0 < p < 1$

38. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}, 1 < p$

39. $\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)}{n^2+2}$

40. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n+1)^2}{3^n}$

41. $\sum_{n=1}^{\infty}(-1)^{n+1}\ \dfrac{(n+2)^2}{(n+1)^3}$

42. $\sum_{n=1}^{\infty}(-1)^{n-1}\ \dfrac{\left(\frac{3}{2}\right)^n}{n^2}$

43. $\sum_{n=1}^{\infty}\dfrac{(-1)^n(4/3)^n}{n^4}$

44. $\sum_{n=1}^{\infty}\dfrac{(-4)^n}{(n!)n}$

45. $\sum_{n=1}^{\infty}(-3)^n\dfrac{n}{(2n)!}$

46. $\sum_{n=1}^{\infty}(-1)^n\dfrac{(n+1)!}{1\cdot 3\cdot 5\cdots(2n+1)}$

47. $\sum_{n=1}^{\infty}(-1)^n\dfrac{(n!)^2 2^n}{(2n)!}$

48. $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(n-1)}{n^{3/2}}$

49. $\sum_{n=1}^{\infty}(-1)^n(n!)^2\dfrac{4^n}{(2n)!}$

50. $\sum_{n=1}^{\infty}(-1)^n\dfrac{2\cdot 4\cdots(2n+2)}{1\cdot 4\cdot 7\cdots(3n+1)}$

51. $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{5^{n+1}}{2^{4n}}$

52. $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(n+1)}{(n+3)}$

53. $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(n+2)}{n^{5/4}}$

54. $\sum_{n=1}^{\infty}(-1)^n\dfrac{(n+2)}{n^{7/4}}$

55. $\sum_{n=1}^{\infty}(-1)^n\dfrac{(3n^2+2n-1)}{2n^3}$

56. $\sum_{n=2}^{\infty}\dfrac{(-1)^n}{n(\ln n)}$

57. $\sum_{n=2}^{\infty}(-1)^n\dfrac{(\ln n)}{n}$

58. $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(\ln n)}{n^2}$

59. $\sum_{n=1}^{\infty}(-1)^n\dfrac{n!}{n^p}, 0<p<1$

60. $\sum_{n=1}^{\infty}(-1)^n\dfrac{n^p}{n!}, 0<p<1$

61. $\sum_{n=1}^{\infty}(-1)^n\dfrac{n!}{n^p}, 1<p$

62. $\sum_{n=1}^{\infty}(-1)^n\dfrac{n^p}{n!}, 1<p$

63. Suppose that $0 < a_k$ for each natural number k and $\displaystyle\sum_{k=1}^{\infty} a_k$ converges. Prove that $\displaystyle\sum_{k=1}^{\infty} a_k^p$ converges for every $p > 1$.

64. Suppose that $0 < a_k$ for each natural number k and $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges. Prove that $\displaystyle\sum_{k=1}^{\infty} a_k^p$, for $0 < p < 1$.

65. Suppose that $0 < r < 1$ and $|a_{k+1}/a_k| < r$ for all $k \geq N$. Prove that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges absolutely.

66. Prove that $\displaystyle\sum_{k=1}^{\infty}(-1)^n \frac{a^n}{3+b^n}$ converges absolutely if $0 < a < b$.

8.5 Alternating Series

Definition 8.5.1 Suppose that for each natural number n, b_n is positive or negative. Then the series $\sum_{k=1}^{\infty} b_k$ is said to converge

(a) *absolutely* if the series $\sum_{k=1}^{\infty} |b_k|$ converges;

(b) *conditionally* if the series $\sum_{k=1}^{\infty} b_k$ converges but $\sum_{k=1}^{\infty} |b_k|$ diverges.

Theorem 8.5.1 *If a series converges absolutely, then it converges.*

Proof. Suppose that $\displaystyle\sum_{k=1}^{\infty} |b_k|$ converges. For each natural number k, let $a_k = b_k + |b_k|$ and $c_k = 2|b_k|$. Then $0 \leq a_k \leq c_k$ for each k. Since

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} 2|b_k| = 2\sum_{k=1}^{\infty} |b_k|,$$

the series $\sum_{k=1}^{\infty} c_k$ converges. by the comparison test $\sum_{k=0}^{\infty} a_k$ also converges. It follows that

$$\begin{aligned}\sum_{k=1}^{\infty} b_k &= \sum_{k=1}^{\infty}(a_k - |b_k|) \\ &= \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} |b_k|\end{aligned}$$

and the series $\sum_{k=1}^{\infty} b_k$ converges. This completes the proof of the theorem.

Definition 8.5.2 Suppose that for each natural number $n, a_n > 0$. Then an *alternating* series is a series that has one of the following two forms:

(a) $a_1 - a_2 + a_3 - \cdots + (-1)^{n+1}a_n + \cdots = \sum_{k=1}^{n}(-1)^{k+1}a_k$

(b) $-a_1 + a_2 - a_3 + \cdots + (-1)^n a_n + \cdots = \sum_{k=1}^{\infty}(-1)^k a_k.$

Theorem 8.5.2 *Suppose that* $0 < a_{n+1} < a_n$ *for all natural numbers* m*, and* $\lim_{n\to\infty} a_n = 0$*. Then*

(a) $\sum_{n=1}^{\infty}(-1)^n a_n$ *and* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *both converge.*

(b) $\left|\sum_{k=1}^{\infty}(-1)^{k+1}a_n - \sum_{k=1}^{n}(-1)^{k+1}a_n\right| < a_{n+1}$*, for all* n*;*

(c) $\left|\sum_{k=1}^{\infty}(-1)^k a_k - \sum_{k=1}^{\infty}(-1)^k a_k\right| < a_{n+1}$*, or all* n*.*

Proof.

Part (a) For each natural number n, let $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$. Then,

$$\begin{aligned} s_{2n+2} - s_{2n} &= \sum_{k=1}^{2n+2}(-1)^{k+1}a_k - \sum_{k=1}^{2n}(-1)^{k+1}a_k \\ &= (-1)^{2n+3}a_{2n+2} + (-1)^{2n+2}a_{2n+1} \\ &= a_{2n+1} - a_{2n+2} > 0. \end{aligned}$$

Therefore, $s_{2n+2} > s_{2n}$ and $\{s_{2n}\}_{n=1}^{\infty}$ is an increasing sequence. Similarly,

$$s_{2n+3} - s_{2n+1} = (-1)^{2n+4}a_{2n+3} - (-1)^{2n+2}a_{2n+1} = a_{2n+3} - a_{2n+1} < 0.$$

Therefore, $s_{2n+3} < s_{2n+1}$ and $\{s_{2n+1}\}_{n=0}^{\infty}$ is a decreasing sequence. Furthermore,

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{2n+1}a_{2n} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} < a_1. \end{aligned}$$

Thus, $\{s_{2n}\}_{n=1}^{\infty}$ is an increasing sequence which is bounded above by a_1. Therefore, $\{s_{2n}\}_{n=1}^{\infty}$ converges to some number $s \leq a_1$. Then

$$\begin{aligned} \lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{2n+1} \\ &= \lim_{n\to\infty} s_{2n} \\ &= s. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} s_n = s$$

and the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_k$ converges to s and the series $\sum_{n=1}^{\infty}(-1)^{n}a_k$ converges to $-s$.

Part (b) In the proof of Part (a) we showed that

$$s_{2n} < s < s_{2n+1} < s_{2n-1} \ldots \tag{1}$$

for each natural number n. It follows that

$$0 < s - s_{2n} < s_{2n+1} - s_{2n} = a_{2n+1}$$

and
$$\left|\sum_{k=1}^{\infty}(-1)^{k+1}a_k - \sum_{k=1}^{2n}(-1)^{k+1}a_k\right| < a_{2n+1}.$$
Similarly,
$$\begin{aligned} s_{2n} - s_{2n-1} &< s - s_{2n-1} \\ s_{2n-1} - s_{2n} &> s_{2n-1} - s \\ s - s_{2n-1} &< s_{2n-1} - s_{2n} = a_{2n} \\ \left|\sum_{k=1}^{\infty}(-1)^{k+1}a_k - \sum_{k=1}^{2n-1}(-1)^{k+1}a_k\right| &< a_{2n}. \end{aligned}$$
It follows that for all natural numbers n,
$$\left|\sum_{k=1}^{\infty}(-1)^{k+1}a_k - \sum_{k=1}^{n}(-1)^{k+1}a_k\right| < a_{n+1}.$$

Part (c) $\left|\sum_{k=1}^{\infty}(-1)^{k}a_k - \sum_{k=1}^{n}(-1)^{k}a_k\right|$
$$\begin{aligned} &= \left|(-1)\left\{\sum_{k=1}^{\infty}(-1)^{k+1}a_k - \sum_{k=1}^{n}(-1)^{k+1}a_k\right\}\right| \\ &= \left|\sum_{k=1}^{\infty}(-1)^{k+1}a_k - \sum_{k=1}^{n}(-1)^{k+1}a_k\right| < a_{2n+1}. \end{aligned}$$
This concludes the proof of this theorem.

Theorem 8.5.3 *Consider a series* $\sum_{k=1}^{\infty} a_k$. *Let*
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \ , \ \lim_{n\to\infty}|a_n|^{1/n} = M$$

(a) If $L < 1$*, then the series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*

(b) If $L > 1$*, then the series* $\sum_{k=1}^{\infty} a_k$ *does not converge absolutely.*

(c) If $M < 1$*, then the series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*

(d) If $M > 1$, then the series $\sum_{k=1}^{\infty} a_k$ does not converge absolutely.

(e) If $L = 1$ or $M = 1$, then the series $\sum_{k=1}^{\infty} a_k$ may or may not converge absolutely.

Proof. Suppose that for a series $\displaystyle\sum_{k=1}^{\infty} a_k$,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \quad \text{and} \quad \lim_{n\to\infty} |a_n|^{1/n} = M.$$

Part (a) If $L < 1$, then the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges to the ratio test, since

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1.$$

Hence, the series $\displaystyle\sum_{k=1}^{\infty} a_k$ converges absolutely.

Part (b) As in Part (a), the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ diverges by the ratio test if $L > 1$, since

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1.$$

Part (c) If $M < 1$, then the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges by the root test, since $\displaystyle\lim_{n\to\infty} |a_k|^{1/n} = M < 1$.

Part (d) If $M > 1$, then the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ diverges by the root test as in Part (c).

Part (e) For the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$, $L = M = 1$, but $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p-series test. Thus, $L = 1$ and $M = 1$ fail to determine convergence or divergence.

This completes the proof of Theorem 8.5.3.

Exercises 8.3 Determine the region of convergence of the following series.

71. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2n}$

72. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{3^n n^2}$

73. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n!}$

74. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!(x-1)^n}{5^n}$

75. $\displaystyle\sum_{n=0}^{\infty} (-2)^n x^n$

76. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{2^n n^2}$

77. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(x+1)^n}{3^n n^3}$

78. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n^{3/2}}$

79. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x)^n}{n!}$

80. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n)!}$

81. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)!(x-1)^n}{4^n}$

82. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (2n)! x^n}{n!}$

83. $\displaystyle\sum_{n=1}^{\infty} n^2 (x+1)^n$

84. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!(x-1)^n}{1 \cdot 3 \cdots 5 \cdots (2n+1)}$

85. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2 (x-1)^n}{3^n (2n)!}$

86. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n x^n}{2^{3n}}$

87. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{(n+1)\ln(n+1)}$

88. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+1) 2^n (x+1)^n}{n+2}$

89. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (\ln n) 3^n x^n}{4^n n^2}$

90. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)}\, x^n$

8.6 Power Series

Definition 8.6.1 If $a_0, a_1, a_2, \ldots$ is a sequence of real numbers, then the series $\sum_{k=1}^{\infty} a_k x^k$ is called a *power series* in x. A positive number r is called the *radius of convergence* and the interval $(-r, r)$ is called the *interval of convergence* of the power series if the power series converges absolutely for all x in $(-r, r)$ and diverges for all x such that $|x| > r$. The end point $x = r$ is included in the interval of convergence if $\sum_{k=1}^{\infty} a_k r^k$ converges. The end point $x = -r$ is included in the interval of convergence if the series $\sum_{k=1}^{\infty}(-1)^k a_k r^k$ converges. If the power series converges only for $x = 0$, then the radius of convergence is defined to be zero. If the power series converges absolutely for all real x, then the radius of convergence is defined to be ∞.

Theorem 8.6.1 *If the series* $\displaystyle\sum_{n=1}^{\infty} c_n x^n$ *converges for* $x = r \neq 0$*, then the series* $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ *converges absolutely for all numbers* x *such that* $|x| < |r|$.

Proof. Suppose that $\displaystyle\sum_{n=0}^{\infty} c_n r^n$ converges. Then, by the Divergence Test,

$$\lim_{n\to\infty} c_n r^n = 0.$$

For $\epsilon = 1$, there exists some natural number m such that for all $n \geq m$,

$$|c_n r^n| < \epsilon = 1.$$

Let

$$M = \max\{|c_n r^n| + 1 : 1 \leq n \leq m\}.$$

Then, for each x such that $|x| < |r|$, we get $|x/r| < 1$ and

$$\begin{aligned}
\sum_{n=0}^{\infty} |c_n x^n| &= \sum_{n=0}^{\infty} |c_n r^n| \cdot \left|\frac{x}{r}\right|^n \\
&\leq \sum_{n=0}^{\infty} M \left|\frac{x}{r}\right|^n \\
&= \frac{M}{1 - \left|\frac{x}{r}\right|} < \infty.
\end{aligned}$$

By the comparison test the series $\sum_{n=0}^{\infty} |c_n x^n|$ converges for x such that $|x| < |r|$. This completes the proof of Theorem 8.6.1.

Theorem 8.6.2 *If the series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges for some* $x - a = r \neq 0$, *then the series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges absolutely for all* x *such that* $|x-a| < |r|$.

Proof. Let $x - a = u$. Suppose that $\sum_{n=0}^{\infty} c_n u^n$ converges for some $u = r$. Then by Theorem 8.6.1, the series $\sum_{n=0}^{\infty} c_n u^n$ converges absolutely for all u such that $|u| < |r|$. It follows that the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely for all x such that $|x-a| < |r|$. This completes the proof of the theorem.

Theorem 8.6.3 *Let* $\sum_{n=0}^{\infty} c_n x^n$ *be any power series. Then exactly one of the following three cases is true.*

(i) The series converges only for $x = 0$.

(ii) The series converges for all x.

(iii) There exists a number R *such that the series converges for all* x *with* $|x| < R$ *and diverges for all* x *with* $|x| > R$.

Proof. Suppose that cases (i) and (ii) are false. Then there exist two nonzero numbers p and q such that $\sum_{n=0}^{\infty} c_n p^n$ converges and $\sum_{n=0}^{\infty} c_n q^n$ diverges. By Theorem 8.6.1, the series converges absolutely for all x such that $|x| < |p|$. Let

$$A = \{p : \sum_{n=0}^{\infty} c_n p^n \text{ converges}\}.$$

The set A is bounded from above by q. Hence A has a least upper bound, say R. Clearly $|p| \leq R < q$ and hence R is a positive real number. Furthermore, $\sum_{n=0}^{\infty} c_n x^n$ converges for all x such that $|x| < R$ and diverges for all x such that $|x| > R$. We define R to be 0 for case (i) and R to be ∞ for case (ii). This completes the proof of Theorem 8.6.3.

Theorem 8.6.4 *Let* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *be any power series. Then exactly one of the following three cases is true:*

(i) The series converges only for $x = a$ *and the radius of convergence is* 0.

(ii) The series converges for all x *and the radius of convergence is* ∞.

(iii) There exists a number R *such that the series converges for all* x *such that* $|x-a| < R$ *and diverges for all* x *such that* $|x-a| > R$.

Proof. Let $u = x - a$ and use Theorem 8.6.3 on the series $\sum_{n=0}^{\infty} c_n u^n$. The details of the proof are left as an exercise.

Theorem 8.6.5 *If* $R > 0$ *and the series* $\sum_{n=0}^{\infty} c_n r^n$ *converges for* $|x| < R$*, then the series* $\sum_{n=1}^{\infty} n c_n x^{n-1}$*, obtained by term-by-term differentiation of* $\sum_{n=0}^{\infty} c_n x^n$*, converges absolutely for* $|x| < R$.

Proof. For each x such that $|x| < R$, choose a number r such that $|x| < r < R$. Then $\sum_{n=0}^{\infty} c_n x^n$ converges, $\lim_{n\to\infty} c_n r^n = 0$ and hence $\{c_n r^n\}_{n=0}^{\infty}$ is bounded. There exists some M such that $|c_n r^n| \leq M$ for each natural number n. Then

$$\begin{aligned}\sum_{n=1}^{\infty} |n c_n x^{n-1}| &= \sum_{n=1}^{\infty} n|c_n r^n| \cdot \frac{1}{r} \cdot \left|\frac{x}{r}\right|^{n-1} \\ &\leq \frac{M}{r} \sum_{n=1}^{\infty} n\left|\frac{x}{r}\right|^{n-1}.\end{aligned}$$

The series $\sum_{n=1}^{\infty} n\left|\frac{x}{r}\right|^{n-1}$ converges by the ratio test, since $\left|\frac{x}{r}\right| < 1$. It follows that $\sum_{n=1}^{\infty} nc_n x^{n-1}$ converges absolutely for all x such that $|x| < R$. This completes the proof of this theorem.

Theorem 8.6.6 *If $R > 0$ and the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for all x such that $|x-a| < R$, then the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ may be differentiated with respect to x any number of times and each of the differential series converges for all x such that $|x-a| < R$.*

Proof. Let $u = x-a$. Then $\sum_{n=0}^{\infty} c_n u^n$ converges for all u such that $|u| < R$. By Theorem 8.6.5, the series $\sum_{n=1}^{\infty} nc_n u^{n-1}$ converges for all u such that $|u| < R$. This term-by-term differentiation process may be repeated any number of times without changing the radius of convergence. This completes the proof of this theorem.

Theorem 8.6.7 *Suppose that $R > 0$ and $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and R is radius of convergence of the series $\sum_{n=0}^{\infty} c_n x^n$. Then $f(x)$ is continuous for all x such that $|x| < R$.*

Proof. For each number c such that $-R < c < R$, we have

$$\begin{aligned}\left|\frac{f(x)-f(c)}{x-c}\right| &= \left|\sum_{n=0}^{\infty} c_n\left(\frac{x^n-c^n}{x-c}\right)\right| \\ &= \left|\sum_{n=1}^{\infty} c_n\ na_n^{n-1}\right| \\ &\le \sum_{n=1}^{\infty} n\left|c_n a_n^{n-1}\right|\end{aligned}$$

for some a_n between c and x, for each natural number n, by the Mean Value Theorem. By Theorem 8.6.6, the series $\sum_{n=1}^{\infty} n\,|c_n a_n|^{n-1}$ converges. Hence,

$$\begin{aligned}\lim_{x\to c} |f(x) - f(c)| &= \lim_{x\to c} |x - c| \left\{ |c_0 - c| + \sum_{n=1}^{\infty} n \left|c_n a_n^{n-1}\right| \right\} \\ &= 0 \cdot \left\{ |c_0 - c| + \sum_{n=1}^{\infty} n \left|c_n a_n^{n-1}\right| \right\} \\ &= 0.\end{aligned}$$

Hence, $f(x)$ is continuous at each number c such that $-R < c < R$. This completes the proof of this theorem.

Theorem 8.6.8 *Suppose that $R > 0$, $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and R is the radius of convergence of the series $\sum_{n=0}^{\infty} c_n x^n$. For each x such that $|x| < R$, we define*

$$F(x) = \int_0^x f(t)dt.$$

Then, for each x such that $|x| < R$, we get

$$F(x) = \sum_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}.$$

Proof. Suppose that $|x| < |r| < R$. Then

$$\begin{aligned}
\lim_{m\to\infty}\left|F(x) - \sum_{n=0}^{m} c_n \frac{x^{n+1}}{n+1}\right| &= \lim_{n\to\infty}\left|\int_0^x f(t)dt - \sum_{n=0}^{m} c_n \int_0^x t^n\right| \\
&= \lim_{m\to\infty}\left|\int_0^x \left\{f(t) - \sum_{n=0}^{m} c_n t^n\right\} dt\right| \\
&= \lim_{m\to\infty}\left|\int_0^x \left\{\sum_{n=m+1}^{\infty} c_n t^n\right\} dt\right| \\
&\le \lim_{m\to\infty}\int_0^x \left\{\sum_{n=m+1}^{\infty} |c_n t^n|\right\} dt \\
&\le \lim_{m\to\infty}\int_0^x \left\{\sum_{n=m+1}^{\infty} |c_n r^n|\right\} dt \\
&\le \lim_{m\to\infty}\left(\sum_{n=m+1}^{\infty} |c_n r^n|\right)\left|\int_0^x 1\ dt\right| \\
&= 0 \cdot |x| \\
&= 0,
\end{aligned}$$

since $\sum_{n=0}^{\infty} |c_n r^n|$ converges.

It follows that

$$\begin{aligned}
\int_0^x f(t)dt &= \int_0^x \left(\sum_{n=0}^{\infty} c_n t^n\right) dt \\
&= \sum_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}.
\end{aligned}$$

This completes the proof of the this theorem.

Theorem 8.6.9 *Suppose that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for all $|x| < R$, where $R > 0$ is the radius of convergence of the series $\sum_{n=0}^{\infty} c_n x^n$. Then $f(x)$ has continuous* *derivatives of all orders for $|x| < R$ that are obtained by successive term-by-term differentiations of* $\sum_{n=0}^{\infty} c_n x^n$.

Proof. For each $|x| < R$, we define

$$g(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}.$$

Then, by Theorem 8.6.5, R is the radius of convergence of the series $\sum_{n=1}^{\infty} nc_n x^{n-1}$. By Theorem 8.6.7, $g(x)$ is continuous. Hence,

$$c_0 + \int_0^x g(x)dx = c_0 + \sum_{n=1}^{\infty} c_n x^n = f(x).$$

By the fundamental theorem of calculus, $f'(x) = g(x)$. This completes the proof of this theorem.

Definition 8.6.2 The radius of convergence of the power series

$$\sum_{k=1}^{\infty} a_k (x-a)^k$$

is

(a) zero, if the series converges only for $x = a$;

(b) r, if the series converges absolutely for all x such that $|x - a| < r$ and diverges for all x such that $|x - a| > r$.

(c) ∞, if the series converges absolutely for all real number x.

If the radius of convergence of the power series in $(x - a)$ is r, $0 < r < \infty$, then the *interval of convergence* of the series is $(a - r, a + r)$. The end points $x = a + r$ or $x = a - r$ are included in the interval of convergence if the corresponding series $\sum_{k=1}^{\infty} a_k r^k$ or $\sum_{k=1}^{\infty} (-1)^k a_k r^k$ converges, respectively. If $r = \infty$, then the interval of convergence is $(-\infty, \infty)$.

Exercises 8.4 In problem 1–12, determine the Taylor series expansion for each function f about the given value of a.

1. $f(x) = e^{-2x}, \ a = 0$
2. $f(x) = \cos(3x), \ a = 0$
3. $f(x) = \ln(x), \ a = 1$
4. $f(x) = (1 + x)^{-2}, \ a = 0$
5. $f(x) = (1 + x)^{-3/2}, a = 0$
6. $f(x) = e^x, \ a = 2$
7. $f(x) = \sin x, \ a = \dfrac{\pi}{6}$
8. $f(x) = \cos x, a = \dfrac{\pi}{4}$
9. $f(x) = \sin x, \ a = \dfrac{\pi}{3}$
10. $f(x) = x^{1/3}, a = 8$
11. $f(x) = \sin\left(x - \dfrac{1}{2}\right), a = 0$
12. $f(x) = \cos\left(x - \dfrac{1}{2}\right), a = 0$

In problems 13-20, determine $\displaystyle\sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}$.

13. $f(x) = e^{x^2}, a = 0, n = 3$
14. $f(x) = x^2 e^{-x}, a = 0, n = 3$
15. $f(x) = \dfrac{1}{1 - x^2}, a = 0, n = 2$
16. $f(x) = \arctan x, a = 0, n = 3$
17. $f(x) = e^{2x}\cos 3x, a = 0, n = 4$
18. $f(x) = \arcsin x, a = 0, n = 3$
19. $f(x) = \tan x, a = 0, n = 3$
20. $f(x) = (1 + x)^{1/2}, a = 0, n = 5$

8.7 Taylor Polynomials and Series

Theorem 8.7.1 (Taylor's Theorem) *Suppose that $f, f', \cdots, f^{(n+1)}$ are all continuous for all x such that $|x - a| < R$. Then there exists some c between a and x such that*

$$f(x) = P_n(x) + R_n(x)$$

where

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}, \ R_n(x) = f^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!}.$$

The polynomial $P_n(x)$ is called the nth degree Taylor polynomial approximation of f. The term $R_n(x)$ is called the Lagrange form of the remainder.

Proof. We define a function g of a variable z such that

$$g(z) = [f(x) - f(z)] - \frac{f'(z)(x-z)}{1!} - \frac{f''(z)(x-z)^2}{2!} - \cdots - \frac{f^{(n)}(z)(x-z)^n}{n!} - R_n(x)\frac{(x-z)^{n+1}}{(x-a)^{n+1}}.$$

Then

$$g(a) = f(x) - \left\{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\ (x-a)^k + R_n(x)\right\} = 0,$$

and

$$g(x) = f(x) - f(x) = 0.$$

By the Mean Value Theorem for derivatives there exists some c between a and x such that $g'(c) = 0$. But

$$\begin{aligned} g'(z) &= -f'(z) - [-f'(z) + f''(z)(x-z)] - \left[-f''(z)(x-z) + \frac{f'''(z)(x-z)^2}{2!}\right] - \cdots \\ &\quad - \left[-\frac{f^n(z)(x-z)^{n-1}}{n!} + \frac{f^{(n+1)}(z)(x-z)^n}{n!}\right] + R_n(x)\frac{(n+1)(x-z)^n}{(x-a)^{n+1}} \\ &= -f^{(n+1)}(z)\frac{(x-z)^n}{n!} + R_n(x)\frac{(n+1)(x-z)^n}{(x-a)^{n+1}} \\ g'(c) &= 0 = -f^{(n+1)}(c)\frac{(x-c)^n}{n!} + R_n(x)\frac{(n+1)(x-c)^n}{(x-a)^{n+1}}. \end{aligned}$$

Therefore,

$$R_n(x) = \frac{(x-a)^{n+1}}{n+1} \cdot \frac{f^{(n+1)}(c)}{n!} = f^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!}$$

as required. This completes the proof of this theorem.

Theorem 8.7.2 (Binomial Series) *If m is a real number and $|x| < 1$, then*

$$\begin{aligned} (1+x)^m &= 1 + \sum_{k=1}^{\infty} \frac{m(m-1)\cdots(m-k+1)}{k!}\ x^k \\ &= 1 + mx + \frac{m(m-1)}{2!}\ x^2 + \frac{m(m-1)(m-2)}{3!}\ x^3 + \cdots . \end{aligned}$$

This series is called the binomial series. If we use the notation

$$\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$$

then $\binom{m}{k}$ *is called the binomial coefficient and*

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k.$$

If m *is a natural number, then we get the binomial expansion*

$$(1+x)^m = 1 + \sum_{k=1}^{m} \binom{m}{k} x^k.$$

Proof. Let $f(x) = (1+x)^m$. Then for all natural numbers n,

$$\begin{aligned} f'(x) &= m(1+x)^{m-1}, f''(x) = m(m-1)(1+x)^{m-2}, \cdots, \\ f^{(n)}(x) &= m(m-1)\cdots(m-n+1)(1+x)^{m-n}. \end{aligned}$$

Thus, $f^{(n)}(0) = m(m-1)\cdots(m-n+1)$, and

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \binom{m}{n} x^n \end{aligned}$$

where $\binom{m}{0} = 1$ and $\binom{m}{n} = m(m-1)\cdots(m-n+1)$ is called the nth binomial coefficient. By the ratio test we get

$$\begin{aligned} &\lim_{n\to\infty} \left| \frac{m(m-1)\cdots(m-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{m(m-1)\cdots(m-n+1)x^n} \right| \\ &= |x| \left| \lim_{n\to\infty} \left(\frac{m-n}{n+1} \right) \right| \\ &= |x| \left| \lim_{n\to\infty} \frac{\frac{m}{n} - 1}{1 + \frac{1}{n}} \right| \\ &= |x|, \end{aligned}$$

and, hence, the series converges for $|x| < 1$.

This completes the proof of the theorem.

Theorem 8.7.3 *The following power series expansions of functions are valid.*

1. $(1-x)^{-1} = 1 + \sum_{k=1}^{\infty} x^k$ *and* $(1+x)^{-1} = 1 + \sum_{k=1}^{\infty} (-1)^k x^k, \ |x| < 1.$

2. $e^x = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}, \ e^{-x} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!}, \ |x| < \infty.$

3. $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \ |x| < \infty.$

4. $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \ |x| < \infty.$

5. $\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k-1}}{(2k+1)!}, \ |x| < \infty.$

6. $\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \ |x| < \infty.$

7. $\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}, \ -1 < x \leq 1.$

8. $\frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}, \ -1 < x < 1.$

9. $\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}, \ -1 \leq x \leq 1.$

10. $\arcsin x = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k \frac{x^{2k+1}}{2k+1}, \ |x| \leq 1.$

Proof.
Part 1. By the geometric series expansion, for all $|x| < 1$, we have

$$\frac{1}{1-x} = 1 + \sum_{k=1}^{\infty} x^k \quad \text{and} \quad \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + \sum_{k=1}^{\infty} (-1)^k x^k.$$

Part 2. If $f(x) = e^x$, then $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for each $n = 0, 1, 2, \cdots$. Thus

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

By the ratio test the series converges for all x.

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Part 3. Let $f(x) = \sin x$. Then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$ and $f^{(4)}(x) = \sin x$. It follows that, for each $n = 0, 1, 2, 3, \cdots$, we have

$$f^{(4n)}(0) = 0,\ f^{(4n+1)}(0) = 1,\ f^{(4n+2)}(0) = 0 \quad \text{and} \quad f^{(4n+3)}(0) = -1.$$

Hence,

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

By the ratio test, the series converges for all $|x| < \infty$:

$$\begin{aligned} &\lim_{n\to\infty} \left| (-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= x^2 \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} \\ &= 0. \end{aligned}$$

Part 4. By term-by-term differentiation we get

$$\cos x = (\sin x)' = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, |x| < \infty.$$

Part 5. For all $|x| < \infty$, we get

$$\begin{aligned}\sinh x &= \frac{1}{2}\ (e^x - e^{-x}) \\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.\end{aligned}$$

Part 6. By differentiating term-by-term, we get

$$\cosh x = (\sinh x)' = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, l \quad |x| < \infty.$$

Part 7. For each $|x| < 1$, by performing term by integration, we get

$$\begin{aligned}\ln(1+x) &= \int_0^x \frac{1}{1+x}\ dx \\ &= \int_0^{\infty} \left(\sum_{n=0}^{\infty}(-1)^n x^n\right)\ dx \\ &= \sum_{n=0}^{\infty}(-1)^n\ \frac{x^{n+1}}{n+1}.\end{aligned}$$

Part 8. By Part 7, for all $|x| < 1$, we get

$$\begin{aligned}\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) &= \frac{1}{2}\ [\ln(1+x) - \ln(1-x)] \\ &= \frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n\ \frac{x^{n+1}}{n+1} - \sum_{n=0}^{\infty}(-1)^n\ \frac{(-x)^{n+1}}{n+1}\right] \\ &= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\ (1-(-1)^{n+1})x^{n+1}\right] \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}.\end{aligned}$$

Recall that $\operatorname{arctanh} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

Part 9. For each $|x| \leq 1$, we perform term-by-term integration to get

$$\begin{aligned}
\arctan x &= \int_0^x \frac{1}{1+x^2}\, dx \\
&= \int_0^x \left(\sum_{k=0}^{\infty}(-1)^k x^{2k}\right) dx \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)}.
\end{aligned}$$

Part 10. By performing term-by-term integration of the binomial series, we get

$$\begin{aligned}
\arcsin x &= \int_0^x \frac{1}{\sqrt{1-x^2}}\, dx \\
&= \int_0^x (1-x^2)^{-1/2} dx \\
&= \int_0^x \left(\sum_{k=0}^{\infty}\binom{-1/2}{k}(-x^2)^k\right) dx \\
&= \sum_{k=0}^{\infty}\binom{-1/2}{k}(-1)^k \frac{x^{2k+1}}{(2k+1)}.
\end{aligned}$$

This series converges for all $|x| \leq 1$.

This completes the proof of this theorem.

8.8 Applications

Chapter 9

Analytic Geometry and Polar Coordinates

A double right-circular cone is obtained by rotating a line about a fixed axis such that the line intersects the axis and makes the same angle with the axis. The intersection point of the line and the axis is called a vertex. A conic section is the intersection of a plane and the double cone. Some of the important conic sections are the following: parabola, circle, ellipse and a hyperbola.

9.1 Parabola

Definition 9.1.1 A *parabola* is the set of all points in the plane that are equidistant from a given point, called the *focus*, and a given line called the *directrix*. A line that passes through the focus and is perpendicular to the directrix is called the *axis* of the parabola. The intersection of the axis with the parabola is called the *vertex*.

Theorem 9.1.1 *Suppose that $v(h,k)$ is the vertex and the line $x = h - p$ is the directrix of a parabola. Then the focus is $F(h+p,k)$ and the axis is the horizontal line with equation $y = k$. The equation of the parabola is*

$$(y-k)^2 = 4p(x-h).$$

Theorem 9.1.2 *Suppose that $v(h,k)$ is the vertex and the line $y = k - p$ is the directrix of a parabola. Then the focus is $F(h, k + p)$ and the axis is the vertical line with equation $x = h$. The equation of the parabola is*

$$(x - h)^2 = 4p(y - k).$$

9.2 Ellipse

Definition 9.2.1 An *ellipse* is the locus of all points, the sum of whose distances from two fixed points, called foci, is a fixed positive constant that is greater than the distance between the foci. The midpoint of the line segment joining the two foci is called the center. The line segment through the foci and with end points on the ellipse is called the major axis. The line segment, through the center, that has end points on the ellipse and is perpendicular to the major axis is called the minor axis. The intersections of the major and minor axes with the ellipse are called the vertices.

Theorem 9.2.1 *Let an ellipse have center at (h,k), foci at $(h \pm c, k)$, ends of the major axis at $(h \pm a, k)$ and ends of the minor axis at $(h, k \pm b)$, where $a > 0,\ b > 0,\ c > 0$ and $a^2 = b^2 + c^2$. Then the equation of the ellipse is*

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

The length of the major axis is $2a$ and the length of the minor axis is $2b$.

Theorem 9.2.2 *Let an ellipse have center at (h,k), foci at $(h, k \pm c)$, ends of the major axis at $(h, k \pm a)$, and the ends of the minor axis at $(h \pm b, k)$, where $a > 0,\ b > 0,\ c > 0$ and $a^2 = b^2 + c^2$. Then the equation of the ellipse is*

$$\frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1.$$

The length of the major axis is $2a$ and the length of the minor axis is $2b$.

Remark 24 If $c = 0$, then $a = b$, foci coincide with the center and the ellipse reduces to a circle.

9.3 Hyperbola

Definition 9.3.1 A hyperbola is the locus of all points, the difference of whose distances from two fixed points, called foci, is a fixed positive constant that is less than the distance between the foci. The mid point of the line segment joining the two foci is called the center. The line segment, through the foci, and with end points on the hyperbola is called the major axis. The end points of the major axis are called the vertices.

Theorem 9.3.1 *Let a hyperbola have center at* (h, k)*, foci at* $(h \pm c, k)$*, vertices at* $(h \pm a, k)$*, where* $0 < a < c$*,* $b = \sqrt{c^2 - a^2}$*, then the equation of the hyperbola is*

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

Theorem 9.3.2 *Let a hyperbola have center at* (h, k)*, foci at* $(h, k \pm c)$*, vertices at* $(h, k \pm a)$*, where* $0 < a < c$*,* $b = \sqrt{c^2 - a^2}$*, then the equation of the hyperbola is*

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

9.4 Second-Degree Equations

Definition 9.4.1 The transformations

$$\begin{Bmatrix} x = x' \cos\theta - y' \sin\theta \\ y = x' \sin\theta + y' \cos\theta \end{Bmatrix}$$

and

$$\begin{Bmatrix} x' = x \cos\theta + y \sin\theta \\ y' = -x \sin\theta + y \cos\theta \end{Bmatrix}$$

are called rotations. The point $P(x, y)$ has coordinates (x', y') in an $x'y'$-coordinate system obtained by rotating the xy-coordinate system by an angle θ.

Theorem 9.4.1 *Consider the equation* $ax^2 + bxy + cy^2 + dx + ey + f = 0,\ b \neq 0$*. Let* $\cot 2\theta = (a - c)/b$ *and* $x'y'$*-coordinate system be obtained* *through rotating the xy-coordinate system through the angle θ. Then the given second degree equation*

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

becomes

$$a'x'^2 + c'y'^2 + d'x + e'y + f' = 0$$

where

$$\begin{aligned} a' &= a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta \\ c' &= a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta \\ d' &= d\cos\theta + e\sin\theta \\ e' &= -d\sin\theta + e\cos\theta \\ f' &= f \end{aligned}$$

Furthermore, the given second degree equation represents

(i) an ellipse, a circle, a point or no graph if $b^2 - 4ac < 0$;

(ii) a hyperbolic or a pair of intersecting lines if $b^2 - 4ac > 0$;

(iii) a parabola, a line, a pair of parallel lines, or else no graph if $b^2 - 4ac = 0$.

9.5 Polar Coordinates

Definition 9.5.1 Each point $P(x, y)$ in the xy-coordinate plane is assigned the polar coordinates (r, θ) that satisfy the following relations:

$$x^2 + y^2 = r^2, \; y = r\cos\theta, \; y = r\sin\theta.$$

The origin is called the *pole* and the positive x-axis is called the *polar axis*. The number r is called the *radial* coordinate and the angle θ is called the *angular* coordinates. The polar coordinates of a point are not unique as the rectangular coordinates are. In particular,

$$(r, \theta) \equiv (r, \theta + 2n\pi) \equiv (-r, \theta + (2m + 1)\pi)$$

where n and m are any integers. There does exist a unique polar representation (r, θ) if $r \geq 0$ and $0 \leq \theta < 2\pi$.

9.6 Graphs in Polar Coordinates

Theorem 9.6.1 *A curve in polar coordinates is symmetric about the*

(a) x-axis if (r, θ) *and* $(r, -\theta)$ *both lie on the curve;*

(b) y-axis if (r, θ) *and* $(r, \pi - \theta)$ *both lie on the curve;*

(c) origin if (r, θ), $(r, \theta + \pi)$ *and* $(-r, \theta)$ *all lie on the curve.*

Theorem 9.6.2 *Let* e *be a positive number. Let a fixed point* F *be called the focus and a fixed line, not passing through the focus, be called a directrix. If* P *is a point in the plane, let* PF *stand for the distance between* P *and the focus* F *and let* PD *stand for the distance between* P *and the directrix. Then the locus of all points* P *such that* $PF = ePD$ *is a conic section representing*

(a) an ellipse if $0 < e < 1$;

(b) a parabola if $e = 1$;

(c) a hyperbola if $e > 1$;

The number e *is called the eccentricity of the conic.*

In particular an equation of the form

$$r = \frac{ek}{1 \pm e \cos \theta}$$

represents a conic with eccentricity e*, a focus at the pole (origin), and a directrix perpendicular to the polar axis and* k *units to the right of the pole, in the case of* $+$ *sign, and* k *units to the left of the pole, in the case of* $-$ *sign.*

Also, an equation of the form

$$r = \frac{ek}{1 \pm e \sin \theta}$$

represents a conic with eccentricity e*, a focus at the pole, and a directrix parallel to the polar axis and* k *units above the pole, in the case of* $+$ *sign, and* k *units below the pole, in the case of* $-$ *sign.*

9.7 Areas in Polar Coordinates

Theorem 9.7.1 *Let $r = f(\theta)$ be a curve in polar coordinates such that f is continuous and nonnegative for all $\alpha \leq \theta \leq \beta$ where $\alpha \leq \beta \leq 2\pi + \alpha$. Then the area A bounded by the curves $r = f(\theta)$, $\theta = \alpha$ and $\theta = \beta$ is given by*

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

Theorem 9.7.2 *Let $r = f(\theta)$ be a curve in polar coordinates such that f and f' are continuous for $\alpha \leq \theta \leq \beta$, and there is no overlapping, the arc length L of the curve from $\theta = \alpha$ to $\theta = \beta$ is given by*

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\ d\theta \\ &= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta \end{aligned}$$

9.8 Parametric Equations

Definition 9.8.1 A parametrized curve C in the xy-plane has the form

$$C = \{(x, y) : x = f(t), y = g(t), t \in I\}$$

for some interval I, finite or infinite.

The functions f and g are called the coordinate functions and the variable t is called the parameter.

Theorem 9.8.1 *Suppose that $x = f(t), y = g(t)$ are the parametric equations of a curve C. If $f'(t)$ and $g'(t)$ both exist and $f'(t) \neq 0$, then*

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}.$$

Also, if $f''(t)$ and $g''(t)$ exist, then

$$\frac{d^2y}{dx^2} = \frac{f'(g)g''(t) - g'(t)f''(t)}{(f'(t))^2}.$$

At a point $P_0(f(t_0), g(t_0))$, the equation of *(a) the tangent line is*

$$y - g(t_0) = \frac{g'(t_0)}{f'(t_0)}\ (x - f(t_0))$$

(b) the normal line is

$$y - g(t_0) = -\frac{f'(t_0)}{g'(t_0)}\ (x - f(t_0))$$

provided $g'(t_0) \neq 0$ *and* $f'(t_0) \neq 0$.

Theorem 9.8.2 *Let* $C = \{(x, y) : x = f(t), y = g(t), a \leq t \leq b\}$ *where* $f'(t)$ *and* $g'(t)$ *are continuous on* $[a, b]$. *Then the arc length* L *of* C *is given by*

$$\begin{aligned} L &= \int_a^b [(f'(t))^2 + (g'(t))^2]^{1/2} dt \\ &= \int_a^b \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} dt. \end{aligned}$$

Theorem 9.8.3 *Let* $C = \{(x, y) : x = f(t), y = g(t), a \leq t \leq b\}$, *where* $f'(t)$ *and* $g'(t)$ *are continuous on* $[a, b]$.

(a) If C *lies in the upper half plane or the lower half plane and there is no overlapping, then the surface area generated by revolving* C *around the* x*-axis is given by*

$$\int_a^b 2\pi g(t) \sqrt{(f'(t))^2 + (g'(t))^2}\ dt.$$

(b) If $0 \leq f(t)$ *on* $[a, b]$, *(or* $f(t) \leq 0$ *on* $[a, b]$*) and there is no overlapping, then the surface area generated by revolving* C *around the* y*-axis is*

$$\int_a^b 2\pi f(t) \sqrt{(f'(t))^2 + (g'(t))^2}\ dt.$$

Definition 9.8.2 Let $C = \{(x(t), y(t)) : a \leq t \leq b\}$ for some interval I. Suppose that $x'(t), y'(t), x''(t)$ and $y''(t)$ are continuous on I.

(a) The arc length $s(t)$ is defined by

$$s(t) = \int_a^t [(x'(t))^2 + (y'(t))^2]^{1/2} dt.$$

(b) The angle of inclination, ϕ, of the tangent line to the curve C is defined by

$$\phi(t) = \arctan\left(\frac{y'(t)}{x'(t)}\right) = \arctan\left(\frac{dy}{dx}\right).$$

(c) The curvature $\kappa(t)$, read kappa of t, is defined by

$$\left|\frac{d\phi}{ds}\right| = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{[(x'(t))^2 + (y'(t))^2]^{3/2}}.$$

(d) The radius of curvature, R, is defined by

$$R(t) = \frac{1}{\kappa(t)}.$$