Infinite Series

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1 Background

Although the rigor of this talk will not be stringent enough for mathematicians, the material is quite interesting and the techniques quite useful. I do not know infinite series well enough to say that I truly understand the topic, but I enjoy investigating it in the dozens of books on infinite series in my library and revelling in the amazing results. This ambivalent feeling about infinite series probably stems from the fact that I did not understand *anything* about infinite series in my calculus class 37 years ago and the other fact that I must now teach infinite series every year to my students. One of the highlights of the year (at least for me) in my class, after the Advanced Placement exam in May, is to see how the 28 year old Euler in 1735 solved a problem known as the Basel Problem. This problem had been proposed by Jakob Bernoulli in 1689 when he collected the all of the work on infinite series of the 17^{th} century in a volume entitled *Tractatus De Seriebus Infinitis.* Bernoulli's comment in this volume was that the evaluation "is more difficult than one would expect". Little did he know how difficult it really was and that it would take almost 50 years for the problem to be solved. In my class, we then see how the problem can be solved to the satisfaction of mathematicians today, using only knowledge gained in first year calculus. This problem is highlighted in the first volume of Polya's *Mathematics and Plausible Reasoning, Volume 1* [**6**] in his discussion on the use of analogy leading to discovery. It is this method of 'solving' the problem that this talk will address. The real proofs using elementary calculus are many, but not appropriate for this circle. If you have studied calculus then you will find proofs and references to proofs in [**2**], [**9**], and an extensive bibliography is given by E. L. Stark in *Mathematics Magazine*, 47 (1974) pp 197-202.

A word of caution is in order about entering fields without a proper foundation. It is easy to do things by example and never attain an understanding. This is the issue of *How* versus *Why* that Paul Zeitz gave a talk on last year. It is interesting to see the remarks of G. Chrystal in *Textbook of Algebra* [**3**] written in 1889. "A practice has sprung up of late (encouraged by demands for premature knowledge in certain examinations) of hurrying young students into the manipulation of the machinery of the Differential and Integral Calculus before they have grasped the preliminary notions of a *Limit* and of an *Infinite Series*, on which all the meaning and all the uses of the Infinitesimal Calculus are based. Besides being a sham, this is a sin against the spirit of mathematical progress." It is remarkable to see over one hundred years later that we are still fighting this battle. On the other hand, it is never too soon to begin thinking about big ideas and seeing how the great minds approached them. In this we will follow the great mathematician Pierre-Simon de Laplace:

"Read Euler, read Euler. He is the master of us all."

2 Geometric Series

The first thing that must be discussed when working with infinite series is the meaning of convergence of an infinite sequence of real numbers. A sequence is a function whose domain is the positive integers (sometimes the nonnegative integers). A sequence is denoted by $\{a_n\}$ where the a_n refers to the nth term of the sequence. For example, if $\{a_n\} = \{1/n\}$, then the 100^{th} term of the sequence is $1/100$. A sequence **converges** to a real number L if and only if only a finite number of terms of the sequence lie outside of any interval that has L as its midpoint. Put another way, given any interval with L as the midpoint, every term of the sequence after some term will lie in the interval. Every set of real numbers that is bounded has a least upper bound and a greatest lower bound. (This is known as the **completeness** property. The set of rational numbers does not possess this property.) A consequence of this property is that every bounded sequence that is nondecreasing (monotone increasing) or nonincreasing (monotone decreasing) converges. Which of the following sequences converge? $\{(.5)^n\}$, $\{n/(n+1)\}$, $\{1/n^2\}$, $\{(-1)^n\}$, $\{(-1)^n/n\}$.

An infinite series is formed by adding, successively, the terms of a sequence. If the sequence is $\{a_n\} = \{n\}$ then the series is $1 + 2 + 3 + \cdots$ which will be represented in sigma notation by $\sum_{k=1}^{n} k$. It is seen that this infinite series can exceed any given real number. On the other hand, if the sequence consists of terms that continue to get smaller and smaller, then it is not clear whether the sum will grow without bound.This is where the need for sequences comes in. For a given series, $\sum_{k=1}^{n} a_k$, form the sequence $\{S_n\}$, called the **sequence of partial sums**, where $S_1 = a_1, S_2 = a_1 + a_2, S_3 = a_1 + a_2 + a_3$, and $S_n = a_1 + a_2 + a_3 + \cdots + a_n$. In words, S_n is the sum of the first n terms of the sequence a_n. After all of this build up we can now say what it means for a series to converge. A **series converges** if and only if the sequence of partial sums converges. The limit of the sequence of partial sums is said to be the **sum of the series.** Practice using this definition with some geometric series. Recall that a sequence is *geometric* if the ratio of any term and the preceding term is a constant. For example, $\{(-1)^n\}$, $\{(1)^n\}$,$\{(1/2)^n\}$, $\{(-1/2)^n\}$, and $\{(3/2)^n\}$ and $\{2002(1/2)^n\}$ are all geometric sequences. Which of the sequences converge? To decide which of the of the corresponding infinite series converge, we need to find the sequence of partial sums. Problem 1 will get you started.

1. Find $\displaystyle\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \cdots + ar^{n-1}$.

2. Find $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$. For what values of r does the series have a sum.?

3. What geometric series has a sum of $\frac{1}{1-x}$?

4. What geometric series has a sum of $\frac{4}{3+2x}$?

5. Find $\displaystyle\sum_{k=1}^{2002} \frac{1}{2^k}$.

6. Find $\displaystyle\sum_{n=1}^{\infty} \frac{2001^n}{2002^n}$.

7. Find $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$.

8. (**Mandelbrot Competition March 2002)**
Find $\sum_{n=1}^{\infty} \frac{F_n}{3^n}$, where F_n is the nth Fibonacci number.

9. Find $\sum_{n=1}^{\infty} \frac{5n+1}{3^n}$.

10. Find $\sum_{n=1}^{\infty} \frac{n^2}{4^k}$.

11. Find $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$. [answer 33/8]

12. Find $\sum_{n=1}^{\infty} \frac{n^6}{2^n}$. [answer 9366]

3 Telescoping Series

The topic of telescoping series came up last week in Andrew Dudzik's talk on Generating Functions. The method of changing series whose terms are rational functions into telescoping series is that of transforming the rational functions by the method of partial fractions. For example, let $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$. Solve for A and B to find $\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1}$. Use this idea to solve the finite series in part a. Then find the limit of the sequence of partial sums in order to find the sum of the infinite telescoping series in part b.

1. (a) Find $\sum_{k=1}^{2002} \frac{1}{k(k+1)}$.

   (b) Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

2. (a) Find $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}$.

   (b) Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.

3. (a) Find $\sum_{k=1}^{n} \frac{3}{k(k+3)}$.

   (b) Find $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.

4 Maclaurin Series

The subject of Maclaurin Series is one that will be carefully justified in Calculus. However, the idea is so wonderful that it is difficult to keep it a secret. All of the functions that one develops in advanced algebra and precalculus can be approximated to any degree of accuracy by polynomial functions and thereby evaluated by just adding, subtracting and multiplying. It is this very technique that allows your calculators to evaluate those functions without having any of the values stored away wasting memory. The Maclaurin Series will converge rapidly for numbers near zero. The series can be shifted to any other number if the required input is not near zero. In this case, the series are known as Taylor Series. Some examples of familiar functions are the following.

I. $e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad x \in \Re$

II. $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad x \in \Re$

III. $\cos x = \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad x \in \Re$

IV. $\ln|x+1| = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x \le 1$

V. $\arctan x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad -1 \le x \le 1$

VI. $(x+1)^k = x^k + kx^{k-1} + \frac{k(k-1)}{1 \cdot 2}x^{k-2} + \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3}x^{k-3} + \cdots \qquad k \in \Re$

To get some feel for these series make the following substitutions for x.

1. In [I], let $x = 1$ to find the sum of $\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$
2. In [IV], let $x = 1$ to find the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.
3. In [V], let $x = 1$ to find the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
4. In [I] let $x = i\theta$ where $i^2 = -1$. Separate the real and imaginary parts and use [II] and [III] to show that $e^{\pi i} = -1$ or $e^{\pi i} + 1 = 0$. This is considered by many to be one of the most beautiful of all mathematical formulae with its collecting the five most important constants in mathematics.
5. (a) Show $\arctan x - \arctan y = \arctan \frac{x-y}{1+xy}$.
   (b) Show $\arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4}$.
   (c) Use the fact that $\arctan x$ is odd and replace y with $-x$ in part (a). Use this formula twice to show that $4 \arctan \frac{1}{5} = \arctan \frac{120}{119}$, so that $4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$.
   (d) Use the first seven terms of $\arctan \frac{1}{5}$ and the first three terms of $\arctan \frac{1}{239}$ to compute π to 10 decimal places by adding only 10 terms.
6. **(BAMM 2001)** Find $\sqrt[3]{1729}$ to 4 decimal places in 40 seconds. (Hint: recall that 1729 is the famous taxicab number that Ramanujan stated was the smallest integer that can be written as a sum of two cubes in two ways; $9^3 + 10^3$ and $1^3 + 12^3$.) Now use [VI] with $x = 12^3$ and $k = 1/3$.

5 Harmonic Series

The harmonic series is the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$. The terms get smaller and smaller, and if you add 250,000,000 terms the sum is still less than 20. Therefore, you might be surprised to find out that the sequence of partial sums is unbounded and the series diverges. It was proved to diverge around 1350 by Oresme, forgotten and rediscovered again in the late 17^{th} century. An interesting question is how much of the sequence must be removed before the series converges?

1. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ can be made larger than any real number by choosing an appropriate n.

2. Show that when all the terms with denominators that are not prime are removed, the series still diverges. In other words the sum of the reciprocals of the prime numbers diverges. For proofs see [**5**] [**6**] or [**14**].

3. Show that when all the terms that contain the digit nine are deleted, the series converges. In fact, the sum is less than 90. See *An Intriguing Series* in [**7**].

4. $2002 = 2 \cdot 7 \cdot 11 \cdot 13$. Find the sum of all the unit fractions that have denominators with only factors from the set $\{2, 7, 11, 13\}$. That is, find the following sum:
   $\frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{8} + \frac{1}{11} + \frac{1}{13} + \ frac114 + \frac{1}{16} + \frac{1}{22} + \frac{1}{26} + \frac{1}{28} + \cdots$

6 p-Series and the Exact Value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$

The term p-series refers to series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$. It can be easily shown in calculus, using the integral test, that the p-series diverges for $p \leq 1$ and converges for $p > 1$. We have already seen that $p = 1$ leads to a divergent series. Now we will finally investigate the Basel Problem, the p-series for $p = 2$.

1. Since $\frac{1}{n^2} < \frac{2}{n(n+1)}$, show by using a telescoping sum how Jakob Bernoulli was able to prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$

2. Euler showed that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

   (a) Find the exact sum of $\frac{1}{15} + \frac{1}{63} + \frac{1}{80} + \frac{1}{255} + \frac{1}{624} + \cdots$. (The reciprocals of the numbers that are one less than perfect squares which simultaneously are other powers, i.e. $16 = 4^2 = 2^4$ so $16 - 1 = 15$ is a denominator in the series.)

   (b) Find the ratio of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ to $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$, where $p > 1$.

   (c) Find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

   (d) Find $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

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