A MODERN INTRODUCTION TO PARTICLE PHYSICS
Second Edition

FAYYAZUDDIN & RIAZUDDIN

World Scientific Publishing

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World Scientific
Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 912805

USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Fayyazuddin, 1930–

A modern introduction to particle physics / Fayyazuddin, Riazuddin -- 2nd ed.

p. cm.

Includes bibliographical references and index.

ISBN 9810238762 (alk. paper) ISBN 9810238770 (pbk)

1. Particles (Nuclear physics) I. Riazuddin. II. Title.

QC793.2 .F39 2000

539.7'2--dc21 00-035183

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

Printed in Singapore.

Thou seest not in the creation of All-
merciful any imperfection.
Return thy gaze; seest thou any fissure?
Then return thy gaze again, and again and thy
gaze comes back to thee dazzled, aweary
Koran, The Kingdom LXVII

To Masooda and Mumtaz

Preface to the first edition

Particle physics has been one of the frontiers of science since J. J. Thompson's discovery of the electron about one hundred years ago. Since then physicists have been concerned with (i) attempts to discover the ultimate constituents of matter, (ii) the fundamental forces through which the fundamental constituents interact, and (iii) seeking a unification of the fundamental forces.

At the present level of experimental resolution, the smallest units of matter appear to be leptons and quarks, which are spin 1/2 fermions. Hadrons (particles which feel the strong force) are composed of quarks. The evidence for this comes from the observed spectrum and static properties of hadrons and from high energy lepton-hadron scattering experiments involving large momentum transfers, which "prove" the actual existence of quarks within hadrons. As originally formulated, the quark model needed three flavors of quarks, up (u), down (d) and strangeness (s) not just u and d. The discoveries of the tau leptons and more flavors [charm (c) and bottom (b)] were to some extent welcomed and to some extent appeared to be there for no apparent reason since elementary building blocks of an atom are just u and d quarks and electrons. A charm quark was predicted to exist to remove all phenomenological obstacles to a proper and an elegant gauge theory of weak interaction. Without it, nonexistence of strangeness-changing neutral current posed a puzzle. This also restored the quark-lepton symmetry: for each pair of leptons of charges 0 and -1 there is a quark pair of charges 2/3 and $-1/3$. The existence of τ-leptons and discovery of the b quark (charge $-1/3$) demand the existence of another quark (charge $2/3$), called the top quark, to again restore the quark-lepton symmetry. Indeed, six quark flavors have been proposed to incorporate violation of CP invariance in weak interaction.

Quarks also have a hidden three valued degree of freedom known as color: each quark flavor comes in three colors. The antisymmetry of three-quark wave function of a baryon [e.g. proton] is attributed to color degree of freedom. The three number of colors also manifest themselves in π^0 decay and in the annihilation of lepton-antilepton into hadrons. We have encountered the following types of charges: gravitational, namely, mass, electric, flavor and color. The fundamental forces through which elementary fermions interact are then simply the forces of attraction or repulsion between these charges. The unification of forces is then sought by searching for a single entity of which the various charges are components in the sense that they can be transformed into one and another. In other words, they form generators of a gauge group G which is taken to be local so that a definite form of interaction between vector fields (which must exist and belong to the adjoint representation of G) and elementary fermions (which belong to the fundamental or trivial representation of G) is generated with a universal coupling constant. In this respect non-Abelian gauge field theories [Yang-Mills type] have played a major role. Here the field itself is a carrier of "charge" so that there are direct interactions between the field quanta.

Let us first discuss the strong quark interactions. The local gauge group is $SU_C(3)$ generated by three color charges, the field quanta are eight massless spin 1 color carrying gluons. The theory of quark interactions arising from the exchange of gluons is called quantum chromodynamics (QCD). The most striking physical properties of QCD are (i) the concept of a "running coupling constant $\alpha_s(q^2)$", depending on the amount of momentum transfer q^2. It goes to zero for high q^2 leading to asymptotic freedom and becomes large for low q^2, (ii) confinement of quarks and gluons in a hadron so that only color singlets can be produced and observed. Only the property (i) has a rigorous theoretical basis while the property (ii) finds support from hadron spectroscopy and lattice gauge simulations.

Weak and electromagnetic interactions result from a gauge group acting upon flavors. It is $SU_L(2) \times U(1)$ and is spontaneously broken rather than exact as was $SU_C(3)$.

The electroweak theory, together with the quark hypothesis and QCD, form the basis for the so called "Standard Model" of elementary particles. There have been many quantitative confirmations of the predictions of the standard model: existence of neutral weak current mediated by Z^0, discovery of weak vector bosons $W^{\pm}, Z^0$ at the predicated masses, precision determinations of electroweak parameters and coupling constants (e.g. $\sin^2\theta_W$ which comes out to be the same in all experiments) leading to one loop verification of the theory and providing constraints on the top quark and Higgs masses. Similarly there have been tests of QCD, verifying the running of the coupling constant $\alpha_s(q^2)$, q^2-dependence of structure functions in deep-inelastic lepton-nucleon scattering. Other evidences come from hadron spectroscopy and from high energy processes in which gluons play an essential role.

In spite of the above successes, many questions remain: replication of families and how many quarks and leptons are there? QCD does not throw any light on how many quark flavors there should be? Origin of fermion masses, which appear as free parameters since Higgs couplings with fermions contain as many arbitrary coupling constants as there are masses, is another unanswered question. Origin of CP violation at more fundamental level, rigorous basis of confinement and hadronization of quarks are other questions which await answers. Top quark and Higgs boson are still to be discovered.

Symmetry principles have played an important part in our understanding of particle physics. Thus Chapters 2-6 discuss global symmetries and flavor or classifications symmetries like SU(2) and SU(3) and quark model. Chapter 5 provides the necessary group theory and consequences of flavor SU(3). Chapters 2-6 together with Chapters 9, 10 and 11 on neutrino, weak interactions, properties of weak hadronic currents and chiral symmetry comprise mainly what is called old particle physics but include some new topics like neutrino oscillations and solar neutrino problem. These Chapters are included to provide necessary background to new particle physics, comprising mainly the standard model as defined above. The rest of the book is devoted to the standard model and the topics mentioned in paras 2-7 of the preface. Recently there has been an interface of particle physics with cosmology, providing not only an understanding of the history of very early universe but also shedding some light on questions such as dark matter and open or closed universe. Chapter 16 of the book is devoted to this interface.

Particle physics forms an essential part of physics curriculum. This book can be used as a text book, but it may also be useful for people working in the field. The book is so designed as to form one semester course for senior undergraduates (with suitable selection of the material) and one semester course for graduate students. Formal quantum field theory is not used; only a knowledge of non-relativistic quantum mechanics is required for some parts of the book. But for the remaining parts, the knowledge of relativistic quantum mechanics is essential. The familiarity with quantum field theory is an advantage and for this purpose two Appendicess which summarize the Feynman rules and renormalization group techniques, are added.

Initial incentive for this book came from the lectures which we have given at various places: Quaid-e-Azam University, Islamabad, Daresbury Nuclear Physics Laboratory (R), the University of Iowa (R), King Fahd University of Petroleum and Minerals, Dhahran (R) and King Abdulaziz University, Jeddah (F).

We have not prepared a bibliography of the original papers underlying the developments discussed in the book. Remedy for this can be found in the recent review articles and books listed at the end of each Chapter.

We wish to express our deep sense of appreciation to Dr. Ahmed Ali for critically reading the manuscript, for making many useful suggestions and for his help to update the data. We also wish to express our deep thanks to a colleague Mr. El hassan El aaud and a graduate student Mr. F. M. Al-Shamali [of one of us (R)], who drew diagrams and in general assisted in producing the final manuscript. In addition, the typing help provided by Mr. Mohammad Junaid at Research Institute of King Fahd University of Petroleum and Minerals was indispensable in getting the job done. Finally we wish to acknowledge the support of King Fahd University of Petroleum and Minerals for this project under Project No. PH/Particle/123.

We also take this opportunity to express our deep sense of gratitude to Prof. Abdus Salam, who first introduced us to this subject and for his encouragement throughout our work in this field.

Fayyazuddin
Riazuddin
March 4, 1992

Preface to the second edition

Our aim in producing this new edition is to bring the book up to date and as such many chapters have been throughly revised. In particular, the chapters on Neutrino Physics, Particle Mixing and CP–Violation and Weak Decays of Heavy Flavors have been mostly rewritten incorporating new material and new data. The heavy quark effective field theory has been included and a brief introductory section on supersymmetry and strings has been added. We wish to thank Ansar Fayyazuddin for writing this section.

A number of typographical errors have been corrected. Another change is that we have adopted a metric and notation for gamma matrices commonly used.*

Finally we wish to thank Mr. Amjad Hussain Gilani and Dr. Muhammad Nisar who did an excellent job in typing the manuscript; without their help it was difficult to put the manuscript in final shape.

Fayyazuddin
Riazuddin
Jan. 21, 2000

*see for example, J. D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics, McGraw–Hill Book Co., New York (1965).

Contents

Chapter 1
INTRODUCTION

1.1 Fundamental Force

Particle physics is concerned with the fundamental constituents of matter and the fundamental "forces" through which the fundamental constituents interact among themselves.

Until about 1932, only four particles, namely the proton (p), the neutron (n), the electron (e) and the neutrino (ν) were regarded as the ultimate constituents of matter. Of these four particles, two, the proton and the electron are electrically charged. The other two are electrically neutral. The neutron and proton form atomic nuclei, the electron and nucleus form atoms while the neutrino comes out in radioactivity, i.e. the neutron decays into a proton, an electron and a neutrino. Each of these particles, called a fermion, spins and exists in two spin (or polarization) states called left-handed (i.e. appears to be spinning clockwise as viewed by an observer that it is approaching) and right-handed (i.e. spinning anti-clockwise) spin states. One may add a fifth particle, the photon to this list. The photon is a quantum of electromagnetic field. It is a boson and carries spin 1, is electrically neutral and has zero mass, due to which it has only two spin directions or it has only transverse polarization. It is a mediator of electromagnetic force. A general feature of quantum field theory is that each particle has its own antiparticle with opposite charge and magnetic moment, but with same mass and spin. Accordingly we have four antiparticles viz., the antiproton ($\overline{p}$), the positron (e^+), the antineutron ($\overline{n}$)

and antineutrino ($\overline{\nu}$).

The four particles experience four types of forces:

i. The Gravitational Force

This is a force of attraction between two particles and is proportional to their gravitational charges, namely their masses. It is a long range force, controls the motion of planets and galaxies, governs the law of falling bodies, and determines the overall character of our Universe. The gravitational potential energy between two protons is given by the Newton's Law:

$$V = G_N \, \frac{m_p^2}{r}, \tag{1.1}$$

where G_N is the Newton's gravitational constant:

$$\begin{aligned} G_N &= 4.17 \times 10^{-5}\text{GeV-cm/gm}^2 \\ &= 0.67 \times 10^{-38}\text{GeV}^{-2}. \end{aligned} \tag{1.2}$$

For proton

$$\begin{aligned} m_p &\approx 1\text{GeV}, \; r = 10^{-13}\text{cm} \approx 5\text{GeV}^{-1}, \\ V &\approx 10^{-39}\text{GeV}. \end{aligned} \tag{1.3}$$

Thus we see that on microscopic scale, the gravitational potential energy is negligible. But we note that

$$\begin{aligned} \sqrt{G_N} &= \frac{1}{M_P} \approx 0.8 \times 10^{-19}\text{GeV}^{-1} \\ &= 5.3 \times 10^{-44}\text{s} \\ &= 1.6 \times 10^{-33}\text{cm}, \end{aligned} \tag{1.4a}$$

$$\frac{1}{\sqrt{G_N}} = M_P \approx 10^{19}\text{GeV}, \tag{1.4b}$$

where M_P is called the Planck mass. It is clear from Eq. (4b), that the gravitational interaction becomes significant at Planck mass or at a distance of order 10^{-33} cm. Assuming that this interaction is of the same order as the electromagnetic interaction (see below) ($\alpha = e_M^2/\hbar c = 1/137$, $e_M^2 = e^2/4\pi\varepsilon_0$) at Planck mass M_P, we conclude that the effective gravitational interaction at 1 GeV is given by

$$\alpha_{G_N} \approx \frac{(1\text{GeV})^2}{M_P^2}\alpha \approx 10^{-38}\alpha \approx 10^{-40}. \tag{1.5}$$

In particle physics, the gravitational interaction may be neglected at the present available energies.

ii. The Weak Nuclear Force

It is responsible for radioactivity, e.g.

$$n \rightarrow p + e^- + \overline{\nu}_e$$

and

$$O^{14} \rightarrow N^{14} + e^- + \overline{\nu}_e.$$

The latter process has half life of 71.4 sec. From the half life, we can determine its strength which is given by the Fermi constant [see Chap. 2]:

$$\begin{aligned} G_F &\approx 10^{-5}\text{GeV}^{-2} \\ \frac{1}{\sqrt{G_F}} &\approx 300 \text{ GeV}, \\ \sqrt{G_F} &\approx 0.7 \times 10^{-16}\text{cm}. \end{aligned} \tag{1.6}$$

This is the energy scale at which the weak interaction becomes significant i.e. of the same order as the electromagnetic interaction. At an energy scale of 1 GeV,

$$\alpha_W = \frac{(1 \text{ GeV})^2}{(300 \text{ GeV})^2}\alpha \approx 10^{-5}\alpha. \tag{1.7}$$

iii. The Electromagnetic Force

It acts between any two electrically charged particles, e.g. a negatively charged electron and a positively charged proton attract each other with a force which is proportional to their electric charges. It is responsible for the binding of atoms and mainly governs all known phenomena of life on earth. This force also manifests itself through the electromagnetic radiation in the form of light, radiowaves and X-rays. Photon is a quantum of electromagnetic force and it is the mediator of electromagnetic force, which is a long range force. The electromagnetic potential energy is given by

$$V = \frac{e_M^2}{r}. \tag{1.8}$$

For electron and proton bound in hydrogen atom, this force of attraction provides the binding energy of the electron in the hydrogen atom given by Bohr's formula

$$|E_1| = \frac{1}{2}\alpha^2\left(\mu c^2\right), \tag{1.9}$$

where μ is the reduced mass of the system. For hydrogen atom $\mu \approx m_e$ and $m_e c^2 \approx 0.5$MeV, giving the binding energy of the electron $|E_1| \approx 14$ eV. For a proton (p) - antiproton ($\bar{p}$) hypothetical atom ($\mu = m_p/2 \approx 1000\ m_e$) so that the binding energy provided by the electromagnetic potential is $\left|E_1^{p\bar{p}}\right| \approx 14$ keV. We see that the strength of electromagnetic interaction is determined by the dimensionless number $\alpha = e_M^2/\hbar c = 1/137$. In rationalized Gaussian units:

$$\alpha = \frac{e^2}{4\pi\hbar c} = \frac{e^2}{4\pi} = \frac{1}{137}.$$

iv. The Strong Nuclear Force

It is responsible for the binding of protons and neutrons in a nucleus. It is a strong force. We have seen that the electromagnetic binding energy for the $p\bar{p}$ atom is of the order of 14 keV, but the binding energy of deuteron (bound n p system) is about 2 MeV. Thus the strong nuclear force is about 100 times the electromagnetic force. It is a short range force effective over the nuclear dimension of the order of 10^{-13} cm.

Hence we conclude that the relative strengths of the four forces are in the order of

$$10^{-40} : 10^{-7} : 10^{-2} : 1 \tag{1.10}$$

The experimental results on the scattering of electron on nuclei can be explained by invoking electromagnetic interaction only. In fact the scattering of γ-rays on proton at low energy is given by the Thomson formula:

$$\sigma_\gamma = \frac{8\pi}{3}\alpha^2\left(\frac{1}{m_p}\right)^2 \simeq 4\pi\alpha^2\left(\frac{1}{m_p}\right)^2 \simeq 10^{-31}\text{cm}^2.$$

The neutrino participates in weak interactions only as reflected by the extreme smallness of the scattering cross-section of neutrino on proton, viz $\bar{\nu}_e p \to e^+ n$, which is given by $\sigma_W \simeq 10^{-43}\text{cm}^2$. Comparing the above cross-sections with the one for nucleon-nucleon scattering, which is of the order $\sigma_N \simeq 10^{-24}\text{cm}^2$, we see that the electron and neutrino do not experience strong interaction.

We now briefly and qualitatively discuss the ranges of the three basic forces. Due to quantum fluctuations, an electron can emit a photon and reabsorb it as depicted in Fig.1. Such a photon can exist only for a time

$$\Delta t \sim \frac{\hbar}{\Delta E} = \frac{1}{\omega} \tag{1.11a}$$

where $\Delta E =$ is the energy of the photon. Since the unobserved photon exists for a time $< \frac{1}{\omega}$, it can travel at most

$$R = \frac{c}{\omega}. \tag{1.11b}$$

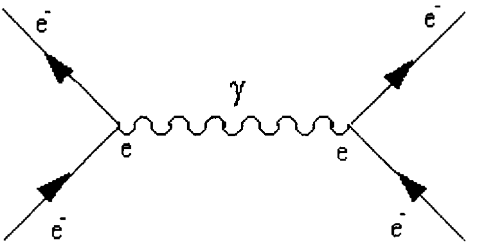


Figure 1 Electromagnetic force mediated by a photon.

Now ω can be arbitrarily small and therefore R can be arbitrarily large i.e. the distance over which a photon can transport electromagnetic force is arbitrarily large i.e. electromagnetic force has infinite range. This is expected from the Coulomb potential e_M^2/r.

If we assume that weak interaction is mediated by a vector boson W in analogy with electromagnetic interaction (Fig. 2), then since weak interaction is of short range, W must be massive. The

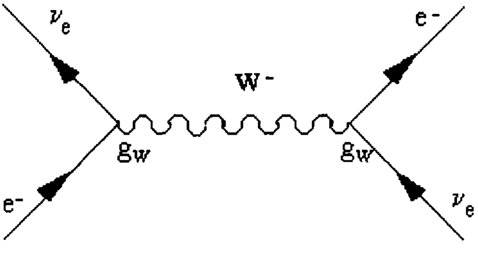


Figure 2 Weak interaction mediated by a vector boson W.

maximum distance to which the virtual $W-$boson is allowed to travel by the uncertainity relation is

$$R_W \approx \frac{\hbar c}{m_W\, c^2} \tag{1.12a}$$

The W−boson has been found experimentally in 1983 with a mass $m_W \approx 80$ GeV/c^2 as predicted by Salam and Weinberg when they unified weak and electromagnetic interactions (see below). Eq. (12a) then gives the range of the weak interaction as

$$R_W \approx \frac{197 \times 10^{-13}\ \text{MeV - cm}}{80 \times 10^3\ \text{MeV}} \approx 2 \times 10^{-16}\ \text{cm}. \qquad (1.12b)$$

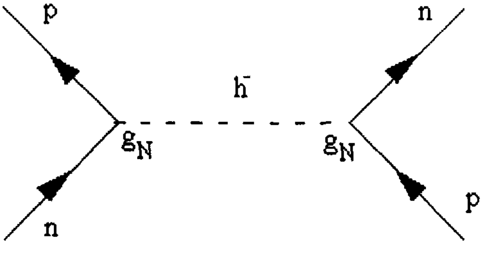


Figure 3 Strong nuclear force mediated by a particle of mass m_h.

If the strong nuclear force is mediated by a particle of mass m_h, as shown in Fig. 3, then its range is given by

$$R_h \approx \frac{\hbar c}{m_h\ c^2}. \qquad (1.13)$$

Since nuclear force has a range of 10^{-13} cm, $m_h \approx 100$ MeV/c^2. Yukawa in 1935, predicted the existence of pion by a similar argument. A particle of this mass was discovered in 1938, but it turned out that it was not the Yukawa particle, the pion; it did not interact strongly with matter and therefore is not responsible for strong nucleon force. It was actually the muon, while the pion was discovered in 1947 in the decay

$$\pi^- \longrightarrow \mu^- + \bar{\nu}_\mu,$$

where ν_μ is the neutrino corresponding to muon. The mass of m_π was found to be 140 MeV/c^2. Thus

$$R_h \approx 1.4 \times 10^{-13}\text{cm} \approx \sqrt{2}f, \tag{1.14}$$

where f is called the Fermi and is equal to 10^{-13} cm.

1.2 Classification of Matter: Leptons and Quarks

The electron and neutrino do not experience strong interactions. They are just two members of a family called leptons. The particles which also experience strong interactions are called hadrons. The proton and neutron are members of a much larger family of hadrons. There are six known leptons as given below:

Leptons	Mass[1]	Electric Charge	Life Time
ν_e, e^-	$m_{\nu_e} < 15$eV $m_e \approx 0.51$ MeV	$0, -1$	ν_e Stable $\tau_e > 4.3 \times 10^{23}$ yrs
ν_μ, μ^-	$m_{\nu_\mu} < 0.17$MeV $m_\mu \approx 105.6$ MeV	$0, -1$	ν_μ Stable $\tau_\mu = 2.197 \times 10^{-6}$ s
ν_τ, τ^-	$m_{\nu_\tau} < 18.2$MeV $m_\tau \approx 1777$ MeV	$0, -1$	ν_τ Stable $\tau_\tau = (290.0 \pm 1.2)$ $\times 10^{-15}$ s

Hadrons can be divided into two classes:

(a) **Baryons:** They are fermions with half integer spin i.e. $J = 1/2,\ 3/2$.

(b) **Mesons:** They are bosons with integral spin i.e. $J = 0, 1, 2$. Pions $\pi^\pm$, π^0 are the hadrons with lightest mass (140 MeV) with $J^P = 0^-$.

Hadrons found in nature are not fundamental constituents of matter. There are hundreds of them. The experiments, for example electron-proton scattering, clearly show that they have structure. All hadrons can be accounted for, if we assume that hadrons are made of some fundamental constituents called quarks. Quarks carry spin 1/2. One can explain the mass spectrum of hadrons, if one assumes that baryons are made of three quarks and mesons are made of quark and antiquark. Thus, baryon and mesons in the ground state are composite of $(qqq)_{L=0}$ and $(q\bar{q})_{L=0}$.

To summarize, according to present picture, the quarks and leptons, form the fundamental constituents of matter. Five quark flavors are needed to account for the known hadrons. The sixth flavor was expected on theoretical ground; it has been experimentally found. They are listed below:

Quark type (Flavor)	Electric charge	Mass [effective mass or constituent mass in a hadron]
(u, d)	(2/3, −1/3)	0.33 GeV
(c, s)	(2/3, −1/3)	(1.5 GeV, 0.5 GeV)
(t, b)	(2/3, −1/3)	(175 ± 5 GeV, 4.5 GeV)

It may be noted that matter has three layers or generations. The first generation (ν_e, e, u, d) is relevant for the matter in the universe: for example, the proton $p \sim uud$, the neutron $n \sim udd$ and the pion $\pi^+ \sim u\bar{d}$. The first generation of quarks u and d form an isodoublet i.e. they are assigned isospin $I = 1/2$ and $I_3 = \pm 1/2$, [that is why they are called up and down quarks]. The second and third generation of quarks are assigned new quantum numbers as follows: $s-$quark, strangeness $S = -1$, c-quark, charm, $C = +1$, b-quark, bottomness $B = -1$, τ-quark, topness $T = +1$. The second and third generation of quarks account for the strange hadrons, charmed hadrons and B-hadrons created in the laboratory in high energy collisions between hadrons of first generation. They are always created in pair, so that final state has $S = 0, C = 0, B = 0$ and $T = 0$ that is to say these quantum numbers are conserved in electromagnetic and strong interactions.

In high energy atomic collisions, we can split an atom into its constituents-atomic nucleus and electrons. In high energy nucleus

- nucleus collisions, we can split a nucleus into its constituents viz. neutrons and protons. But in high energy hadron - hadron collisions, a hadron is not split into its constituents viz., quarks. A hadron - hadron collision results not into free quarks but into hadrons. This leads us to the hypothesis of quark confinement i.e. quarks are always confined in a hadron. The quark - quark force, which keeps the quarks confined in a hadron, is a fundamental strong force on the same level as electromagnetic and weak nuclear forces which are the other two fundamental forces in nature. Its strength is characterized by a dimensionless coupling constant $\alpha_s = g_s^2/4\pi \approx 0.5$ at present energies. It is actually energy dependent. The strong nuclear force between protons and neutrons should then be a complicated interaction derivable from this basic quark-quark force. As for example, the fundamental force for an atomic system is electromagnetic force, the interatomic and intermolecular forces are derivable from the basic electromagnetic force.

1.3 Strong Color Charges

We have seen that the quarks form hadrons; the baryons and mesons in the ground state are composites of $(qqq)_{L=0}$ and $(q\bar{q})_{L=0}$. Quarks and (anti-quarks) are spin 1/2 fermions. Now q and $\bar{q}$ spins may be combined to form a total spin S, which is 0 or 1. Total spin for qqq system is 3/2 or 1/2. Further as q and $\bar{q}$ have opposite intrinsic parities, the parity of the $q\bar{q}$ system is $P = (-1)(-1)^L = -1$ for the ground state. Thus we have for the ground states [see Chap. 6]

Mesons	Baryons
$(q\bar{q})_{L=0}$	$(qqq)_{L=0}$
$S = 0, 1$	$S = 1/2, 3/2$
$J^P = 0^-, 1^-$	$J^P = 1/2^+, 3/2^+$

Examples:

Mesons	Baryons
$\pi,\ \rho$	$p,\ \Delta$
$\pi^+ = \left(u\bar{d}\right)\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$	$p\,(S = 1/2, S_Z = 1/2)$ $= (uud)\,(\uparrow\uparrow\downarrow)$
$\rho^+\,(S = 1, S_Z = 0)$: $\left(u\bar{d}\right)\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$ $\rho^+\,(S = 1, S_Z = 1)$: $\left(u\bar{d}\right)(\uparrow\uparrow)$	$\Delta^{++}\,(S = 3/2, S_Z = 3/2)$ $= (uuu)\,(\uparrow\uparrow\uparrow)$

There is a difficulty with the above picture; consider, for example the state, $|\Delta^{++}(S_Z = 3/2)\rangle \sim \left|u^\uparrow u^\uparrow u^\uparrow\right\rangle$. This state is symmetric in quark flavor and spin indices $(\uparrow)$. The space part of the wave function is also symmetric $(L = 0)$. Thus, the above state being totally symmetric violates the Pauli principle for fermions. Therefore, another degree of freedom (called color) must be introduced to distinguish the otherwise identical quarks: each quark flavor carries three different strong color charges, red(r), yellow(y) and blue(b) i.e.

$$q = q_a \qquad\qquad a = r, y, b$$

[Leptons do not carry color and that is why they do not take part in strong interactions]. Including the color, we write e.g.

$$\left|\Delta^{++}(S_Z = 3/2)\right\rangle = \frac{1}{\sqrt{6}}\sum \varepsilon_{abc}\left|u_a^\uparrow u_b^\uparrow u_c^\uparrow\right\rangle.$$

so that the wave function is now antisymmetric in color indices and satisfies the Pauli principle. Other examples, as far as quark content is concerned, are

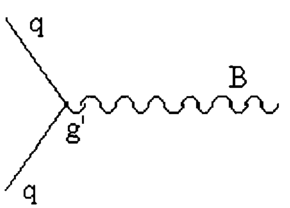
$$\begin{aligned} |p\rangle &= \frac{1}{\sqrt{6}}\sum \varepsilon_{abc}\,|u_a u_b d_c\rangle\,, \\ \left|\pi^+\right\rangle &= \frac{1}{\sqrt{3}}\sum_a \left|u_a \bar{d}_a\right\rangle, \end{aligned}$$

i.e. these states are color singlets. In fact, all known hadrons are color singlets. Thus, the color quantum number is hidden. This is the postulate of color confinement mentioned earlier and explains the non- existence of free quark (q) or such systems as (qq), ($q\bar{q}q$), and ($qqqq$). Actually nature has also assigned a more fundamental role to color charges as we briefly discuss below.

1.4 Fundamental Role of "Charges" and the Standard Model of Electroweak Unification and Strong Force

First thing to note is that the electromagnetic force and the strong nuclear force are each characterized by a dimensionless coupling constant and thus to achieve unification there has to be a "hidden" dimensionless coupling constant associated with the weak nuclear force which is related to the "observed" Fermi coupling constant by a mass scale. That this is so will be clear shortly. Secondly we know that the electromagnetic force is a gauge force describeable in terms of electric charge and the current associated with it. This force is mediated by electromagnetic radiation field whose quanta are spin 1 photons, the mediators of electromagnetic force. This is generalized: All fundamental forces are gauge forces describeable in terms of "charges" and their currents as summarized in Table 1.

Note that the coupling constants α, α_2, α_s in Table 1 are dimensionless but they are energy dependent due to quantum effects, a fact which is used in the unification of the forces. Note also that Q_3 is not identical with Q_{em}; thus the unification of electromagnetic and weak nuclear forces needs another charge, call it Q_B an associated mediator, call it B, which does not change flavor like photon:



$\alpha' = \frac{g'^2}{4\pi}$

Then the photon γ associated with the electric charge Q_{em} is a linear combination of the mediators B and W_3 bosons, associated with the charges Q_B and Q_3 respectively,

$$\gamma = \sin\theta_W W_3 + \cos\theta_W B, \tag{1.15a}$$

while the second orthogonal combination

$$Z = \cos\theta_W W_3 - \sin\theta_W B, \tag{1.15b}$$

is associated with a new charge Q_Z. Z is the mediator of a new interaction, called neutral weak interaction. The weak mixing angle θ_W is a fundamental parameter of the theory and in terms of it

$$Q_Z = Q_3 - \sin^2\theta_W Q_{em}, \tag{1.16}$$

The weak color charges $Q_W, Q_{\bar{W}}$ and Q_B generate the local group $SU_L(2) \times U(1)$where the subscript L on the weak isospin group SU(2) indicates that we deal with chiral fermions that is to say that the left handed fermions [i.e. those which appear to be spinning clockwise as viewed by an observer that they are approaching] are doublets under $\mathrm{SU}_L(2)$ [required by parity violation in weak interactions] while the right handed fermions (spinning anticlockwise) are singlets as indicated below:

	$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	e_R	u_R	d_R
I_{3L}	$\left(\frac{1}{2}, -\frac{1}{2}\right)$	$\left(\frac{1}{2}, -\frac{1}{2}\right)$			
Y_W	-1	$-1/3$	-2	$4/3$	$-2/3$

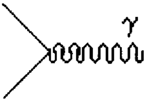
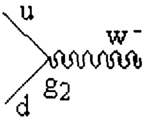
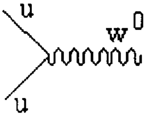
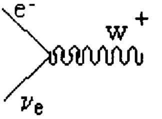
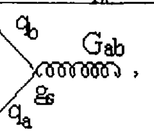
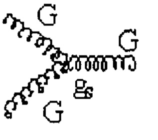
(1.17)

We have [I_{3L} is the same as Q_3 and $\frac{1}{2}Y_W$ is identical with Q_B]

$$Q_{em} = I_{3L} + \frac{1}{2}Y_W, \tag{1.18a}$$

giving

$$\frac{1}{e^2} = \frac{1}{g_2^2} + \frac{1}{g'^2}, \tag{1.18b}$$

Force	Charges	Mediators of force: Spin 1 gauge particles	Coupling between basic fermions and mediators
Electro-magnetic	Q_{em}	Photon (γ)	 $\alpha = \frac{e^2}{4\pi}$
Weak Nuclear	$Q_W, Q_{\bar{W}}$ $[Q_W, Q_{\bar{W}}]$ $= Q_3$ $\neq Q_{\text{em}}$	W^+, W^-, W^0 $W^\pm$change flavor as shown in the next column	   $\alpha_2 = \frac{g_2^2}{4\pi}$
Strong	3 color charges	8 color carrying gluons: G_{ab} a, b $= 1, 2, 3$	  $\alpha_s = \frac{g_s^2}{4\pi}$

so that we have the unification conditions

$$\sin^2\theta_W = \frac{e^2}{g_2^2}, \qquad \cos^2\theta_W = \frac{e^2}{g'^2}. \tag{1.18c}$$

Unlike photon, which is massless, the weak vector bosons W^+, W^- and Z^0 must be massive since we know that weak interactions are of short range. This is achieved by spontaneous breaking of gauge symmetry (SSB). For this purpose it is necessary to introduce a self-interacting complex scalar field

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix},$$

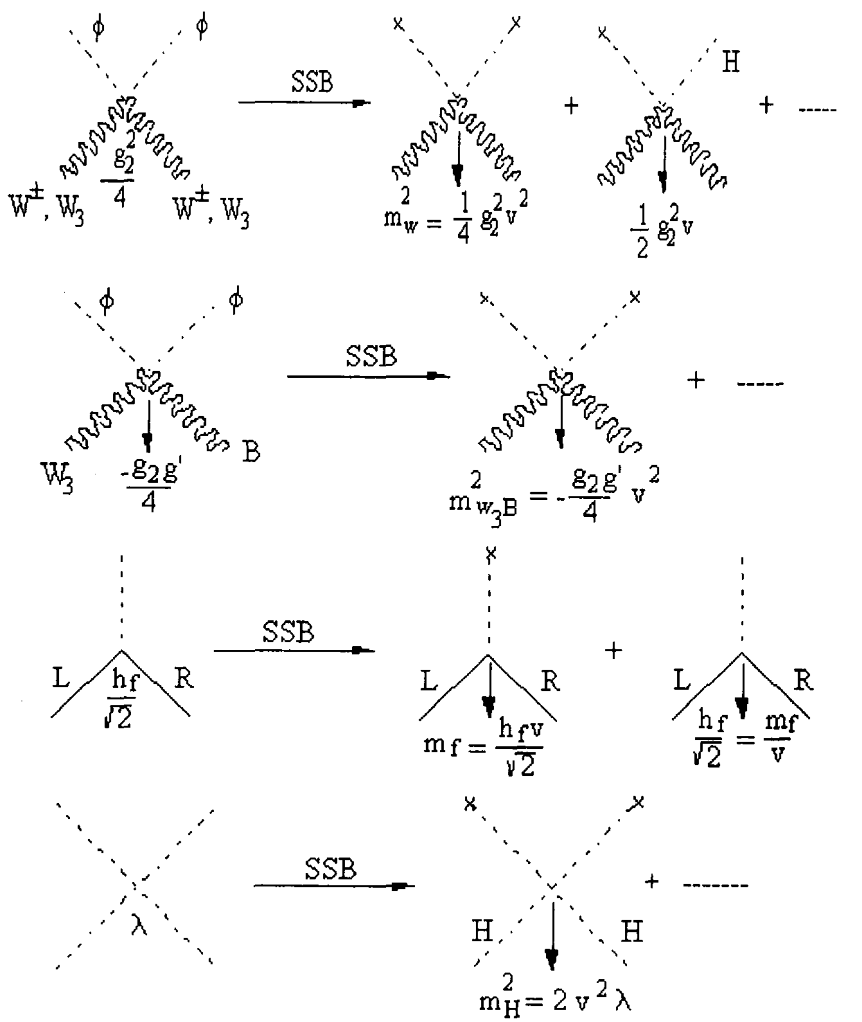
which is a doublet under $\mathrm{SU}_L(2)$ and has $Y_W = 1$. This so-called Higgs field also interacts with the chiral fermions introduced earlier as well as with gauge vector bosons, $W^\pm$, W_3 and B. The scalar field ϕ develops a non-zero vacuum expectation value:

$$\langle\phi\rangle = \langle 0\,|\phi|\,0\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix},$$

thereby breaking the gauge symmetry of the ground state $|0\rangle$. This amounts to rewriting

$$\phi = \begin{pmatrix} \phi^+ \\ \frac{\phi_1 + i\phi_2}{\sqrt{2}} + \frac{v}{\sqrt{2}} \end{pmatrix}$$

where ϕ^+ and hermitian fields ϕ_1 and ϕ_2 have zero vacuum expectation values. In contrast to the gauge invariant vertices shown in Table 1 [which are not affected by SSB], one starts with manifestly gauge invariant vertices involving ϕ and other fields and then translate them to physical amplitudes after SSB as pictorically shown on p. 16 [the dotted lines ending in X denotes $\langle\phi\rangle = v/\sqrt{2}$]:



Because of mixing between W_3 and B, these are not physical particles, the physical particles γ and Z are defined in Eq. (15). This requires diagonalization of the mass matrix for W_3-B sectors, which on diagonalization gives

$$m_A = 0, \qquad m_Z = \frac{m_W}{\cos\theta_W}, \tag{1.19a}$$

where from the above picture

$$m_W = \frac{1}{2} g_2 v. \tag{1.19b}$$

Further we note from the above picture that the mass of a fermion of flavor f and that of Higgs particle H are respectively given by

$$m_f = \frac{h_f v}{\sqrt{2}}, \qquad m_H = \sqrt{2v^2\lambda}. \tag{1.19c}$$

What has happened is that $\phi^{\pm}$ and ϕ_2 have provided the longitudinal degrees of freedom to $W^{\pm}$ and Z which have eaten them up while becoming massive. The remaining electrically neutral scalar field is called the Higgs field and its quantum is called the Higgs particle which we have denoted by H in the above picture. We note from Eq. (19a) that

$$\sin^2\theta_W = 1 - \frac{m_W^2}{m_Z^2}. \tag{1.20a}$$

The directly observed Fermi coupling constant in weak nuclear processes at low energies (i.e. $\ll m_W$) is given by [cf. Eq. (18c)]

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8m_W^2} = \frac{e^2}{8m_W^2 \sin^2\theta_W}, \tag{1.20b}$$

or

$$m_W = \left[\frac{\pi\alpha}{\sqrt{2}G_F \sin^2\theta_W}\right]^{1/2}, \tag{1.20c}$$

where $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$ is the fine structure constant and G_F is the Fermi constant ($\approx 10^{-5}$ GeV^{-2}).

The main predictions of the electroweak unification are

(i) existence of a new type of neutral weak interaction mediated by Z^0.

(ii) weak vector bosons W, Z whose masses are predicted by the relations (20a) and (20c), once $\sin^2\theta_W$ is determined, α and G_F being known.

(iii) existence of the Higgs particle with mass $m_H = \sqrt{2v^2\lambda}$, which is arbitrary since λ is not fixed.

The first prediction was verified more than 16 years ago and the phenomenology of neutral weak interaction gives

$$\sin^2\theta_W \approx 0.23. \tag{1.21}$$

One can now use this result to predict m_W and m_Z through the relations (20) to get

$$m_W \approx 80 \text{ GeV}, \quad m_Z \approx 92 \text{ GeV} \tag{1.22}$$

in agreement with their experimental values. The standard model is in very good shape experimentally. The third prediction is not yet tested and the present lower bound on m_H from Higgs searches at LEP is

$$m_H > 77.5 \text{ GeV}. \tag{1.23}$$

We also note that the electroweak unification energy scale is given by

$$\begin{aligned} \lambda_F \equiv v &= 2\frac{m_W}{g_2} \\ &= \left(\sqrt{2}G_F\right)^{-1/2} \\ &\approx 250 \text{ GeV}. \end{aligned} \tag{1.24}$$

We will briefly discuss the unification of the other two forces with the electroweak force after discussing the origin of the strong force between the two quarks below.

1.5 Strong Quark–Quark Force

We have already remarked:

(i) each quark flavor carries 3 colors.

(ii) only color singlets (colorless states) exist as free particles.

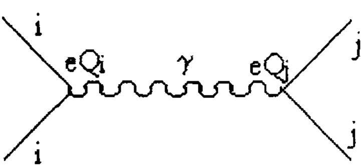
Strong color charges are the sources of the strong force between two quarks just as the electric charge is the source of electromagnetic interaction between two electrically charged particles. To carry the analogy further, we note the following:

Electromagnetic Force Between 2 Electrically Charged Particles

We deal with electrically neutral atoms.

Mediator of the electromagnetic force is electrically neutral massless spin 1 photon, the quantum of the electromagnetic field.

Exchange of photon gives the electric potential:



$V^e_{ij} = \frac{e^2}{4\pi}\frac{Q_iQ_j}{r}, r = |\mathbf{r}_i - \mathbf{r}_j|$

For an electron and proton

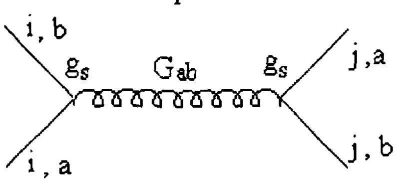
$V_{ij} = -\frac{\alpha}{r}, \alpha = \frac{e^2}{4\pi}$

Strong Color Force Between 2 Quarks

We deal with color singlet systems i.e. hadrons.

Mediators are eight massless spin 1 color carrying gauge vector bosons, called gluons.

Exchange of gluons gives the color electric potential:



$V^{q\bar{q}}_{ij} = -\frac{4}{3}\alpha_s\frac{1}{r}, \alpha_s = \frac{g_s^2}{4\pi}$,

for $\bar{q}q$ color singlet system (mesons) while for qqq color singlet system (baryons).

$V^{qq}_{ij} = -\frac{2}{3}\alpha_s\frac{1}{r}$

This attractive potential is responsible for the binding of atoms.	Note the very important fact that in both cases, we get an attractive potential. Without color, V_{ij}^{qq} would have been repulsive.
The theory here is called quantum electrodynamics (QED).	The theory here is called quantum chromodynamics (QCD).
Due to quantum (radiative) corrections, $\alpha\left(\sqrt{Q^2}\right)$ increases with increasing momentum transfer Q^2, for example $\alpha(m_e) \approx \frac{1}{137}$, $\alpha(m_W) \approx \frac{1}{128}$	Due to quantum (radiative) corrections, $\alpha\left(\sqrt{Q^2}\right)$decreases with increasing Q^2 [this is brought about by the self interaction of gluons (cf. Table 1)], for example $\alpha(m_\tau) \approx 0.35$, $\alpha_s(m_\Upsilon \simeq 10 \text{ GeV}) \approx 0.16$, $\alpha_s(m_Z) \approx 0.125$. That the effective coupling constant decreases at short distances is called the asymptotic freedom property of QCD.

The binding energy provided by one gluon exchange potential of the form mentioned above cannot be sufficient to confine the quarks in a hadron since as one can ionize an atom to knock out an electron, similarly a quark could be separated from a hadron if sufficient energy is supplied. Thus V^g, the one gluon exchange potential, can at best provide binding for quarks at short distances and cannot explain their confinement i.e. impossibility of separating a quark from a hadron. The hope here is that the self interaction of color carrying gluons may give rise to long distance behavior of the potential in QCD completely different from that in QED, where the electrically neutral photon has no self interaction. One hopes that the long range potential in QCD would increase with the distance so that the quarks would be confined in a hadron. Phenomenologically, a potential of form

$$V_{ij}(r) = V_{ij}^g(r) + V^c(r), \tag{1.25a}$$

where

$$V_{ij}^g(r) = -k_s \frac{\alpha_s}{r} + \cdots, \tag{1.25b}$$

(... denotes spin dependent terms, (see Chap. 7) and $k_s = 4/3(q\bar{q})$, $2/3(qqq)$) is the single gluon exchange potential while $V^c(r)$ is the confining potential (independent of the quark flavor), has been used in hadron spectroscopy with quite good success. Lattice gauge theories suggest

$$V^c(r) = kr, \tag{1.25c}$$

with $k \approx 0.25\ (\text{GeV})^2$, obtained from the quarkonium spectroscopy.

To sum up the most striking physical properties of QCD are asymptotic freedom and confinement of quarks and gluons. The quark hypothesis, the electroweak theory and QCD form the basis for the "Standard Model" of elementary particles to which most of the book is devoted while Chap. 18 is concerned with the interface of cosmology with particle physics.

We now briefly discuss the attempts to unify the other two forces with the electroweak force.

1.6 Grand Unification

The three strong color charges introduced earlier generate the gauge group $SU_C(3)$ while that of the electroweak interaction is $SU_L(2)\times U(1)$. Thus the standard model involves

$SU_C(3)$	$\times$	$SU_L(2)$	$\times$	$U(1)$
α_s		α_2		α'

where the associated coupling constants α_s, α_2 and α' are very different at the present energies. But these coupling constants are energy dependent due to quantum radiative corrections. Grand unification is an attempt to find a bigger group G:

$$G \supset SU_C(3) \times SU_L(2) \times U(1)$$

such that at some energy scale $q^2 = m_X^2$,

$$\begin{aligned} \alpha_s(m_X^2) &= \alpha_2(m_X^2) = \alpha'(m_X^2) \\ &= \alpha_G. \end{aligned} \tag{1.26}$$

This is possible because of the form of their energy dependence as calculated in quantum theory (renormalization group analysis), see Appendix B and the fact that two of the three coupling constants, namely, α' and α_2 are related at $\sqrt{Q^2} = m_W$ through the electroweak unification conditions given in Eq. (18c). As will be discussed in Chap. 17, the relation (26) holds at $\sqrt{Q^2} = m_X \approx 10^{15}$ GeV or less, which gives the grand unification (GUT) scale. The most dramatic consequence of popular GUT models is that the proton is not stable. How proton decay comes about can be seen as follows: in GUT, quarks and leptons share the same representation(s) and since gauge theories contain mediators linking all particles in a multiplet, there are additional mediators (apart from the ones mentioned in Table 1 called lepto-quarks X, Y (carrying charges $\pm$ 4/3 and $\pm$ 1/3)) which transform quarks having strong color charges to leptons as shown in Fig. 4. From dimensional analysis, life-time for proton decay is of the order [m_p is proton mass $\sim$ 1 GeV].

$$\frac{1}{\tau_p} \approx \alpha_G^2 \frac{m_p^5}{m_X^4}$$

which gives on using $\alpha_G^2 \approx 10^{-3}$ and $m_X \approx 10^{15}$ GeV (or less)

$$\tau_p \sim 10^{31} \text{ years}$$

(or less). This prediction is not yet borne out by experiment. In fact the world's largest (IBM) detector sensitive to the decay mode $p \to e^+\pi^0$ gives $\tau(p \to e^+\pi^0) > 5 \times 10^{32}$ years. In spite of this GUTS have some attractive features:

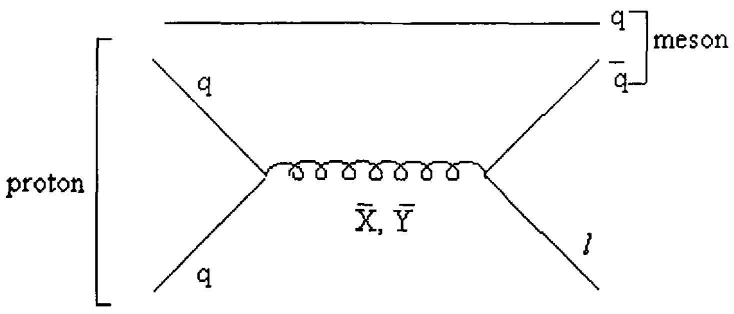


Figure 4 Decay of a proton via lepto-quarks.

(i) quark-lepton unification

(ii) relationships between quark and lepton masses

(iii) quantization of electric charge, for a simple group it is a consequence of the charge operator being a generator of the group and traceless. So for example, sum of charges in a multiplet containing quarks and leptons = 0, thus giving some relation between quark and lepton charges.

(iv) may in principle explain the baryon excess of the universe, $n_B/n_\gamma \approx 10^{-10}$ and that there is no evidence for existence of antibaryons (see Chap. 18).

But they still leave arbitrariness in Higgs sector needed to give masses to lepto-quarks and $W^\pm$, Z vector bosons, do not explain number of generations, do not explain fermions mass hierarchy typified by $m_t/m_u \approx 10^5$ and the gauge hierarchy problem $m_W/m_X \approx 10^{-12}$ in a natural way. These mass hierarchies are more naturally accommodated in supersymmetry (see Chap. 17).

1.7 Units and Notation

We shall use the natural units:

$$\hbar = c = 1.$$

We note that

$$\begin{aligned}
[\hbar] &= ML^2T = 6.582 \times 10^{-22}\text{MeV-s} \\
[c] &= LT^{-1} = 3 \times 10^{10}\text{cm/s} \\
[\hbar c] &= 197 \times 10^{-13}\text{MeV-cm.}
\end{aligned}$$

If $\hbar = c = 1$, then

$$v = \frac{c^2 p}{E} = \frac{p\,(\text{MeV/c})}{E\,(\text{MeV/c}^2)}\ (\text{in units of c})\,.$$

If we take $M = 1$ GeV,

$$\begin{aligned}
L &\sim \frac{1}{\text{GeV}} = \frac{\hbar c}{1000\ \text{MeV}} \approx 2 \times 10^{-14}\ \text{cm} \\
T &\sim \frac{1}{\text{GeV}} = \frac{\hbar}{1000\ \text{MeV}} \approx 6.58 \times 10^{-25}\ \text{s} \\
1\ \text{MeV} &= 1.6 \times 10^{-6}\text{erg} = 1.6 \times 10^{-13}J \\
1\ \text{gm} &= 5.61 \times 10^{23}\text{GeV} \\
1\ \text{GeV} &= 10^3\text{MeV,}
\end{aligned}$$

we will denote the position by a 4-vector $x\,(\mu = 0, 1, 2, 3)$:

$$\begin{aligned}
x^\mu &= (ct, \mathbf{x}) = (t, \mathbf{x}) \\
x_\mu &= (ct, -\mathbf{x}) = (t, -\mathbf{x}) = g_{\mu\nu}x^\nu \\
x^2 &= x_\mu x^\mu = t^2 - \mathbf{x}^2
\end{aligned}$$

with $g_{\mu\nu} = 0, \mu \neq \nu, g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$. On the light cone

$$x^2 = 0 \text{ i.e. } t^2 - \mathbf{x}^2 = 0.$$

The energy E and momentum $\mathbf{p}$ are represented by a 4-vector p:

$$\begin{aligned}
p^\mu &= (E/c, \mathbf{p}) = (E, \mathbf{p}) \\
\partial_\mu &= \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right) \\
\partial^\mu &= \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right) \\
\partial^\mu \partial_\mu &= \frac{\partial^2}{\partial t^2} - \nabla^2 = \Box^2 \\
p^2 &= p_\mu p^\mu = p_0^2 - \mathbf{p}^2 = E^2 - \mathbf{p}^2.
\end{aligned}$$

For a particle on the mass shell

$$E^2 = \mathbf{p}^2 + m^2$$

i.e.

$$p^2 = p_\mu p^\mu = m^2.$$

The scalar product

$$p.q = p^\mu q_\mu = E_p E_q - \mathbf{p} \cdot \mathbf{q}.$$

1.8 Bibliography

Bibliography for topics mentioned in Sections 1.5 - 1.6 which will either be not developed or briefly discussed in the subsequent chapters:

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Chapter 2
SCATTERING AND PARTICLE INTERACTION

Most of the information about the properties of particles and their interactions is extracted from the experiments involving scattering of particles. We, therefore, start this chapter by studying the kinematics of scattering processes.

2.1 Kinematics of a Scattering Process

Consider a typical 2-body scattering process

$$a + b \to c + d.$$

We denote the four momenta of particles a, b, c and d by p_a, p_b, p_c, p_d respectively. Energy momentum conservation gives:

$$p_a + p_b = p_c + p_d \tag{2.1a}$$

$$\mathbf{p}_a + \mathbf{p}_b = \mathbf{p}_c + \mathbf{p}_d \tag{2.1b}$$

$$E_a + E_b = E_c + E_d \tag{2.1c}$$

The reaction transition amplitude is a function of scalars (i.e. Lorentz invariants) formed out of the four vectors p_a, p_b, p_c and p_d. We assume Lorentz invariance in any process involving particles. The invariants are

$$s = (p_a + p_b)^2 = (p_c + p_d)^2 \tag{2.2a}$$
$$t = (p_a - p_c)^2 = (p_d - p_b)^2 \tag{2.2b}$$
$$u = (p_a - p_d)^2 = (p_c - p_b)^2. \tag{2.2c}$$

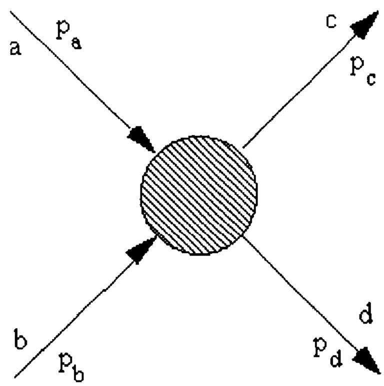

Figure 1 Two-body scattering: $a + b \to c + d$.

But only two of the three scalars are independent:

$$\begin{aligned} s + t + u &= 3p_a^2 + p_b^2 + p_c^2 + p_d^2 + 2p_a \cdot (p_b - p_c - p_d) \\ &= m_a^2 + m_b^2 + m_c^2 + m_d^2. \end{aligned} \tag{2.3}$$

In an actual scattering experiment, we have a projectile (let it be a) and a target (b), which is stationary in the laboratory frame. Thus

$$\begin{aligned} p_a &\equiv \left(E_a^L, \mathbf{p}_a^L\right) = (\nu_L, \mathbf{p}_L) \\ p_b &\equiv (m_b, 0) \\ p_c &\equiv \left(E_c^L, \mathbf{p}_c^L\right), \quad p_d \equiv \left(E_d^L, \mathbf{p}_d^L\right). \end{aligned} \tag{2.4}$$

Hence in the laboratory frame:

$$\begin{aligned} s &= (p_a + p_b)^2 \\ &= m_a^2 + m_b^2 + 2m_b\nu_L, \end{aligned} \tag{2.5a}$$

$$\begin{aligned} t &= (p_a - p_c)^2 \\ &= m_a^2 + m_c^2 - 2\nu_L E_c^L + 2\left|\mathbf{p}_L\right|\left|\mathbf{p}_c^L\right|\cos\theta_L. \end{aligned} \tag{2.5b}$$

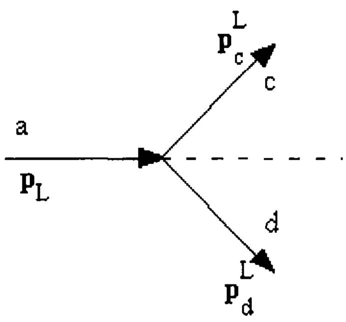

Figure 2 Two-body scattering in the laboratory frame.

or

$$\nu_L = \frac{s - m_a^2 - m_b^2}{2m_b} = \frac{p_a \cdot p_b}{m_b} \tag{2.6a}$$

$$\mathbf{p}_L^2 = -p_a^2 + \nu_L^2 = -m_a^2 + \nu_L^2 \tag{2.6b}$$

$$|\mathbf{p}_L| = \frac{\sqrt{\lambda\left(s,\ m_a^2,\ m_b^2\right)}}{2m_b}, \tag{2.6c}$$

where

$$\lambda\left(x, y, z\right) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{2.7}$$

Theoretically, it is convenient to consider a scattering process in the center of mass (c.m.) frame. In this frame:

$$\begin{aligned} p_a &\equiv \left(E_a, \mathbf{p}\right), \quad p_b \equiv \left(E_b, -\mathbf{p}\right) \\ p_c &\equiv \left(E_c, \mathbf{p}'\right), \quad p_d \equiv \left(E_d, -\mathbf{p}'\right). \end{aligned} \tag{2.8}$$

Thus we have

$$\begin{aligned} s &= \left(p_a + p_b\right)^2 = \left(p_c + p_d\right)^2 \\ &= \left(E_a + E_b\right)^2 = \left(E_c + E_d\right)^2 \equiv E_{cm}^2 \end{aligned} \tag{2.9a}$$

$$\begin{aligned} t &= m_a^2 + m_c^2 - 2E_aE_c + 2\left|\mathbf{p}\right|\left|\mathbf{p}'\right|\cos\theta \\ &= m_b^2 + m_d^2 - 2E_bE_d + 2\left|\mathbf{p}\right|\left|\mathbf{p}'\right|\cos\theta\cdot \end{aligned} \tag{2.9b}$$

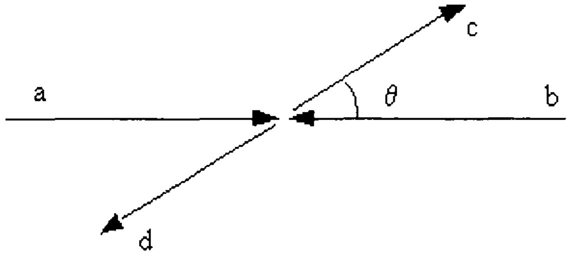

Figure 3 Two-body scattering in the centre of mass frame.

Now

$$\begin{aligned} s &= E_a^2 + E_b^2 + 2E_aE_b \qquad (2.10) \\ &= \left(\mathbf{p}^2 + m_a^2\right) + \left(\mathbf{p}^2 + m_b^2\right) + 2\sqrt{(\mathbf{p}^2 + \mathbf{m}_a^2)}\sqrt{(\mathbf{p}^2 + \mathbf{m}_b^2)}. \end{aligned}$$

From (10), we get

$$|\mathbf{p}| = \frac{\sqrt{\lambda\,(s,\ m_a^2,\ m_b^2)}}{2\sqrt{s}}. \qquad (2.11a)$$

Similarly by considering, $s = (E_c + E_d)^2$, we get

$$|\mathbf{p}'| = \frac{\sqrt{\lambda\,(s,\ m_c^2,\ m_d^2)}}{2\sqrt{s}}. \qquad (2.11b)$$

We also note that

$$\begin{aligned} E_a &= \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}, \quad E_b = \frac{s + m_b^2 - m_a^2}{2\sqrt{s}} \\ E_c &= \frac{s + m_c^2 - m_d^2}{2\sqrt{s}}, \quad E_d = \frac{s + m_d^2 - m_c^2}{2\sqrt{s}}. \qquad (2.12) \end{aligned}$$

For elastic scattering

$$c \equiv a, \quad d \equiv b$$

and

$$|\mathbf{p}| = |\mathbf{p}'|, \quad E_c = E_a, \quad E_d = E_b$$

$$t = -2\mathbf{p}^2(1-\cos\theta) = -4\mathbf{p}^2\sin^2\frac{\theta}{2}. \tag{2.13}$$

Thus we see that $-t$ is the square of momentum transfer.

Finally we derive a relation between the scattering angles θ and θ_L using Lorentz transformation. Let us take $\mathbf{p}_L$ and $\mathbf{p}$ along z-axis. The c.m. frame is moving relative to laboratory frame with a velocity:

$$\mathbf{v} = \frac{\mathbf{p}_L}{\nu_L + m_b}. \tag{2.14}$$

Lorentz transformation gives

$$\begin{aligned} p_c^L\cos\theta_L &= \gamma\left[p'\cos\theta + vE_c\right] \\ p_c^L\sin\theta_L &= p'\sin\theta \end{aligned} \tag{2.15}$$

$$E_c^L = \gamma\left[E_c + vp'\cos\theta\right].$$

Hence, we get

$$\tan\theta_L = \frac{p'\sin\theta}{\gamma\left[p'\cos\theta + vE_c\right]}, \tag{2.16a}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{\nu_L + m_b}{E_{cm}}. \tag{2.16b}$$

Equation (16b) follows from the relations:

$$p_L = \gamma\left[p + vE_a\right], \quad \nu_L = \gamma\left[E_a + vp\right], \quad m_b = \gamma\left[E_b - vp\right]. \tag{2.17}$$

2.2 Interaction Picture

In quantum mechanics, the transition rate from initial state $|i\rangle$ to final state $|f\rangle$ is given by

$$W = 2\pi \left| \langle f| \; V \; |i\rangle \right|^2 \rho_f(E_f), \tag{2.18}$$

where V is the interaction Hamiltonian viz.

$$H = H_0 + V. \tag{2.19}$$

The above formula is obtained when V is treated as small in first order perturbation theory and $|i\rangle$ and $|f\rangle$ are eigenstates of H_0 . $\rho_f(E_f)$ is the density of final states i.e. $\rho_f(E_f)dE_f$ = number of final states with energies between E_f and $E_f + dE_f$.

In order to define the transition rate in general, it is convenient to go to interaction picture, which we define below: The Schrödinger equation is given by

$$i\frac{d}{dt}\left|\Psi\left(t\right)\right\rangle_S = H\left|\Psi\left(t\right)\right\rangle_S . \tag{2.20}$$

We now go over to interaction picture by a unitary transformation

$$\left|\Psi\left(t\right)\right\rangle_I = e^{iH_0t}\left|\Psi\left(t\right)\right\rangle_S . \tag{2.21}$$

Then, using Eq. (20), we have

$$i\frac{d}{dt}\left|\Psi\left(t\right)\right\rangle_I = -H_0\left|\Psi\left(t\right)\right\rangle_I + e^{iH_0t}He^{-iH_0t}\left|\Psi\left(t\right)\right\rangle_I . \tag{2.22}$$

Now define

$$\begin{aligned} H_0^I\left(t\right) &= e^{iH_0t}H_0e^{-iH_0t} = H_0 && \text{(2.23a)}\\ H^I\left(t\right) &= e^{iH_0t}He^{-iH_0t} = H_0 + V_I(t), && \text{(2.23b)} \end{aligned}$$

where

$$V_I(t) = e^{iH_0t}Ve^{-iH_0t}. \tag{2.23c}$$

Hence we have from Eq. (22)

$$i\frac{d}{dt}\left|\Psi\left(t\right)\right\rangle_I = V_I(t)\left|\Psi\left(t\right)\right\rangle_I . \tag{2.24}$$

An operator $\widehat{A}$ in Schrödinger picture is related to operator $\widehat{A}_I\left(t\right)$ in interaction picture by a unitary transformation

$$\begin{aligned} \widehat{A}_I\left(t\right) &= e^{iH_0t}\ \widehat{A}\ e^{-iH_0t} && \text{(2.25a)}\\ i\frac{d\ \widehat{A}_I\left(t\right)}{dt} &= \left[\widehat{A}_I\left(t\right), H_0\right] . && \text{(2.25b)} \end{aligned}$$

2.3 Scattering Matrix (S–Matrix)

From the general principles of quantum mechanics, the probability of finding the system in state $|b\rangle$, when the system is in state $|\Psi(t)\rangle_I$, is given by $|C_b(t)|^2$ where

$$C_b(t) = \langle b|\Psi(t)\rangle_I . \tag{2.26}$$

Assume that $|\Psi(t)\rangle_I$ is generated from $|\Psi(t_0)\rangle_I$ by a linear operator $U(t, t_0)$:

$$\begin{aligned} |\Psi(t)\rangle_I &= U(t,\ t_o)\,|\Psi(t_0)\rangle_I && (2.27a) \\ U(t_0,\ t_o) &= 1. && (2.27b) \end{aligned}$$

Substituting Eq. (27a) in Eq. (24), we get

$$i\frac{\partial U(t,\ t_o)}{dt}\,|\Psi(t_0)\rangle_I = V_I(t)U(t,\ t_o)\,|\Psi(t_0)\rangle_I \tag{2.28a}$$

so that we obtain

$$i\frac{\partial U(t,\ t_o)}{dt} = V_I(t)U(t,\ t_o). \tag{2.28b}$$

We note that $U(t, t_0)$ depends only on the structure of physical system and not on the particular choice of the initial state $|\Psi(t_0)\rangle_I$. Thus

$$\begin{aligned} |\Psi(t)\rangle_I &= U(t,\ t_o)\,|\Psi(t_0)\rangle_I \\ &= U(t,\ t')\,|\Psi(t')\rangle_I \\ &= U(t,\ t')\ U(t',\ t_o)\,|\Psi(t_0)\rangle_I . \end{aligned} \tag{2.29}$$

Therefore,

$$U(t,\ t')\ U(t',\ t_o) = U(t,\ t_o) \tag{2.30a}$$

$$I = U(t_o, t_o) = U(t_o,\ t)U(t,\ t_o) \tag{2.30b}$$

$$U(t_o, t) = U^{-1}(t,\ t_o). \tag{2.30c}$$

Thus, the operator U satisfies the group properties.

The formal solution of differential equation (28b) is given by

$$U(t,t_o) = 1 - i\int_0^t V_I(t')U(t',\ t_o)dt'\cdot \tag{2.31}$$

This integral equation can be solved by iteration. Thus

$$\begin{aligned} U(t,t_o) &= 1 - i\int_0^t dt' V_I(t')\left[1 - i\int_0^t V_I(t'')U(t'',\ t_o)dt''\right] \\ &= 1 - i\int_0^t V_I(t')dt' + (-i)^2\int_0^t V_I(t')dt'\int_0^t V_I(t'')dt'' \\ &\quad + \cdots . \end{aligned} \tag{2.32}$$

Equation (32) is the basis of perturbation theory.

Now at $t = t_o \to -\infty$, the system is known to be in an eigenstate $|a\rangle$ of H_0. Hence the probability amplitude for transition to an eigenstate $|b\rangle$ of H_0 is given by

$$\begin{aligned} C_b(t) &= \langle b\,|\Psi(t)\rangle_I \\ &= \lim_{t_0\to-\infty} \langle b|\,U(t,\ t_o)\,|\Psi(t_0)\rangle_I \\ &= \lim_{t_0\to-\infty} \langle b|\,U(t,\ t_o)e^{iH_0t_0}\,|\Psi(t_0)\rangle_S\,. \end{aligned} \tag{2.33}$$

Now for $t_o \to -\infty$,

$$|\Psi(t_0)\rangle_S = |a, t_0\rangle = |a\rangle\, e^{-iE_at_0}. \tag{2.34}$$

Hence from Eq. (33), we get

$$C_b(t) = \langle b|\,U(t,\ -\infty)\,|a\rangle\,. \tag{2.35}$$

Our purpose is to calculate $C_b(t)$ for large t (since for $t \to \infty$, the system is an eigenstate of H_0) i.e.

$$\lim_{t\to\infty} C_b(t) = \lim_{t\to\infty} \langle b|\,U(t,\ -\infty)\,|a\rangle = \langle b|\,U(\infty,\ -\infty)\,|a\rangle\cdot \tag{2.36}$$

The operator

$$S = U(\infty,\ -\infty) \tag{2.37}$$

with matrix elements

$$S_{ba} = \langle b| U(\infty,\ -\infty) |a\rangle = \langle b| S |a\rangle \tag{2.38}$$

is called the S-matrix.

An important property of S-matrix is that it is a unitary operator. This follows from the conservation of probability. Now

$$\sum_b |C_b(\infty)|^2 = 1 \tag{2.39}$$

or

$$\sum_b \langle b| S |a\rangle\ \ \langle b| S |a\rangle^* = 1$$

or

$$\sum_b \langle a| S^\dagger |b\rangle\ \ \langle b| S |a\rangle = 1. \tag{2.40}$$

Hence

$$\langle a| S^\dagger S |a\rangle = 1$$

i.e.

$$S^\dagger S = \hat{1}. \tag{2.41}$$

Therefore, S is a unitary operator.

We can express the operator $U(t, t_0)$ explicitly in terms of the Hamiltonian. A formal solution of the Schrödinger equation (2.20) can be written as

$$|\Psi(t)\rangle_S = e^{-iH(t-t_0)} |\Psi(t_0)\rangle_S . \tag{2.42}$$

Then using Eqs. (21), (27) and (42), we have

$$e^{-iH_0t} U(t,\ t_o) e^{iH_0t} = e^{-iH(t-t_0)} \tag{2.43a}$$

or

$$U(t,\ t_o) = e^{iH_0t} e^{-iH(t-t_0)} e^{-iH_0t}. \tag{2.43b}$$

Therefore,

$$U(t,\ -\infty) = \lim_{t_0 \to -\infty} e^{iH_0t} e^{-iH(t-t_0)} e^{-iH_0t_0} \tag{2.44a}$$

and this limit is taken with the following prescription:

$$U(t,\ -\infty) = \lim_{\varepsilon\to 0} \varepsilon \int_{-\infty}^{0} e^{\varepsilon t'} e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'} dt'. \tag{2.44b}$$

Similarly,

$$U(\infty,\ t) = \lim_{\varepsilon\to 0} \varepsilon \int_{0}^{\infty} e^{-\varepsilon t'} e^{iH_0 t} e^{-iH(t'-t)} e^{-iH_0 t} dt'. \tag{2.44c}$$

Hence we have

$$\begin{aligned} U(0,\ -\infty)\,|a\rangle &= \left[\lim_{\varepsilon\to 0} \varepsilon \int_{-\infty}^{0} e^{\varepsilon t'} e^{iHt'} e^{-iE_a t'} dt'\right] |a\rangle\,, \\ &= \lim_{\varepsilon\to 0} \frac{i\varepsilon}{E_a - H + i\varepsilon}\,|a\rangle\,. \end{aligned} \tag{2.45}$$

It is clear from Eq. (45), that in the limit $\varepsilon \to 0$, the state

$$\left|a^+\right\rangle = U(0,\ -\infty)\,|a\rangle = |a\rangle + \frac{1}{E_a - H + i\varepsilon} V\,|a\rangle \tag{2.46}$$

is an eigenstate of H with eigenvalue E_a. The state $|a^+\rangle$ is called the incoming state or simply "in" state. The notation emphasises that the state $|a^+\rangle$ goes over to the unperturbed state $|a\rangle$ as $t \to -\infty$. Similarly, we define an "out" state

$$\begin{aligned} \left|a^-\right\rangle &= U(0,\ \infty)\,|a\rangle \\ &= \lim_{\varepsilon\to 0} \frac{-i\varepsilon}{E_a - H - i\varepsilon}\,|a\rangle \\ &= |a\rangle + \frac{1}{E_a - H - i\varepsilon} V\,|a\rangle\,. \end{aligned} \tag{2.47}$$

From Eq. (38), we get

$$\begin{aligned} S_{ba} &= \langle b|\,U(\infty,\ 0)U(0,\ -\infty)\,|a\rangle \\ &= \left\langle b^- \middle| a^+ \right\rangle \\ &= \left\langle b \middle| a^+ \right\rangle + \left\langle b \middle| V \frac{1}{E_b - H + i\varepsilon} \middle| a^+ \right\rangle \\ &= \left\langle b \middle| a^+ \right\rangle + \frac{1}{E_b - E_a + i\varepsilon} \left\langle b \middle| V \middle| a^+ \right\rangle. \end{aligned} \tag{2.48}$$

Now

$$\begin{aligned}\left\langle b\middle|a^{+}\right\rangle &= \left\langle b\middle|\frac{E_a - H_0 + i\varepsilon}{E_a - E_b + i\varepsilon}\middle|a^{+}\right\rangle \\ &= \left\langle b\middle|\frac{E_a - H + i\varepsilon}{E_a - E_b + i\varepsilon} + \frac{V}{E_a - E_b + i\varepsilon}\middle|a^{+}\right\rangle \\ &= \left\langle b\,|a\right\rangle + \frac{1}{E_a - E_b + i\varepsilon}\left\langle b\,|V|\,a^{+}\right\rangle,\end{aligned} \tag{2.49}$$

where we have used Eq. (46). Hence we have from Eqs. (48) and (49)

$$S_{ba} = \delta_{ba} - 2\pi i\delta\left(E_a - E_b\right)\left\langle b\,|V|\,a^{+}\right\rangle, \tag{2.50}$$

where we have used the fact

$$\pi\delta\left(E_a - E_b\right) = \lim_{\varepsilon\to 0}\frac{\varepsilon}{\left(E_b - E_a\right)^2 + \varepsilon^2}. \tag{2.51}$$

We define an operator T, called the T–matrix (transition matrix) with the matrix elements.

$$T_{ba} = \langle b|\,T\,|a\rangle = -\left\langle b\,|V|\,a^{+}\right\rangle, \tag{2.52}$$

so that

$$S_{ba} = \delta_{ba} + 2\pi i\delta\left(E_b - E_a\right)T_{ba}. \tag{2.53}$$

We note that

$$\begin{aligned}\langle b|\,T\,|a\rangle &= -\left\langle b\,|V|\,a^{+}\right\rangle, \\ &= -\langle b|\,V\,|a\rangle - \langle b|\,V\frac{1}{E_a - H + i\varepsilon}V\,|a\rangle\end{aligned} \tag{2.54}$$

or

$$T = -V - V\frac{1}{E_a - H + i\varepsilon}V. \tag{2.55}$$

In relativistic theory, where we treat energy and momentum on equal footing, we write for Eq. (53):

$$S_{fi} = \delta_{fi} + (2\pi)^4\, i\delta^4\left(p_f - p_i\right)T_{fi}, \tag{2.56}$$

where we have put $|a\rangle = |i\rangle$, $|b\rangle = |f\rangle$ to signify initial and final states and

$$\delta^4 (p_f - p_i) = \delta^3 (\mathbf{p}_f - \mathbf{p}_i) \, \delta (E_f - E_i) \,. \tag{2.57}$$

The δ-function ensures the energy momentum conservation in the transition. Then, using Eqs. (36) and (38), the transition probability for large t from a state $|i\rangle$ to state $|f\rangle$ for $i \neq f$ is given by

$$P = \lim_{t\to\infty} |C_f(t)|^2 = |\langle f|\, S\, |i\rangle|^2 = \sum (2\pi)^8 \, \delta^4 (p_f - p_i) \, \delta^4 (0) \, |T_{fi}|^2 \,. \tag{2.58}$$

Now

$$\begin{aligned} \delta^4 (p) &= \frac{1}{(2\pi)^4} \int e^{-ip^\mu x_\mu} d^4 x \\ \delta^4 (0) &= \frac{1}{(2\pi)^4} (\text{Volume}) \, t = \frac{Vt}{(2\pi)^4} \cdot \end{aligned} \tag{2.59}$$

Therefore, the transition rate per unit macroscopic volume is given by

$$W_{fi} = \frac{P}{Vt} = (2\pi)^4 \sum \delta^4 (p_f - p_i) \, |T_{fi}|^2 \cdot \tag{2.60}$$

To carry out sum over final states, we need to know the density of final states $\rho_f \; (E_f)$.

2.4 Phase Space

Consider first a single particle in one dimension confined in the region $0 \leq x \leq L$. The normalized eigenstate of momentum operator $\widehat{p}$ is given by

$$u_p (x) = \frac{1}{L^{1/2}} e^{ipx} . \tag{2.61}$$

The boundary condition that $u_p (x)$ is periodic in the range L gives

$$p = \left(\frac{2\pi}{L}\right) n \cdot \tag{2.62}$$

Thus

$$\frac{dn}{dE} = \left(\frac{L}{2\pi}\right)\frac{dp}{dE} = \rho(E) \tag{2.63}$$

i.e. the number of states within the interval E and $E+dE$ is given by $dn = \rho(E)dE$. In three dimensions, we have

$$\rho(E) = \frac{dn}{dE} = \left(\frac{L}{2\pi}\right)^3 \frac{d}{dE}\int d^3p = \left(\frac{L}{2\pi}\right)^3 p^2\frac{dp}{dE}\int d\Omega. \tag{2.64}$$

We now generalize to n particles in final state:

$$n = \left[\left(\frac{L}{2\pi}\right)^3\right]^{n-1}\int d^3p_1'\, d^3p_2' \cdots d^3p_{n-1}', \tag{2.65}$$

since

$$\mathbf{p}_i = \mathbf{p}_f = \mathbf{p}_1' + \mathbf{p}_2' + \cdots + \mathbf{p}_n' \tag{2.66}$$

and only $(n-1)$ momenta are independent. With the normalization $L = 2\pi$, we can write from Eq. (65)

$$n = \int \delta^3\left[\mathbf{p}_i - (\mathbf{p}_1' + \mathbf{p}_2' + \cdots + \mathbf{p}_n')\right] d^3p_1'\, d^3p_2' \cdots d^3p_n'. \tag{2.67}$$

Thus we can write

$$\begin{aligned}\rho_f(E) &= \int \delta\left[E - (E_1' + E_2' + \cdots + E_n')\right] \\ &\quad \times\delta^3\left[\mathbf{p}_i - (\mathbf{p}_1' + \mathbf{p}_2' + \cdots + \mathbf{p}_n')\right] \\ &\quad \times d^3p_1'\, d^3p_2' \cdots d^3p_n'.\end{aligned} \tag{2.68}$$

Hence the transition rate [cf. Eq. (60)]

$$\begin{aligned}W_f &= (2\pi)^4 \int d^3p_1'\, d^3p_2' \cdots d^3p_n' \\ &\quad \times \overline{\sum_{\text{final spins}}} |T_{fi}|^2\, \delta^4(p_1 + p_2 + \cdots + p_n' - p_i),\end{aligned} \tag{2.69}$$

where – denotes the average over initial spins if initial particles are unpolarized, otherwise we have to use the density matrix if initial particles are polarized.

Remarks:

(1) In the first order perturbation theory

$$T_{fi} = -\langle f|\, V\, |i\rangle \,. \tag{2.70}$$

(2) Our normalization of states is

$$\begin{aligned} \langle \mathbf{p}'\,|\mathbf{p}\rangle &= \int d^3x\ \langle \mathbf{p}'\,|\mathbf{x}\rangle\ \langle \mathbf{x}\,|\mathbf{p}\rangle \\ &= \frac{1}{(2\pi)^3}\int e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}d^3x = \delta\left(\mathbf{p}'-\mathbf{p}\right). \end{aligned} \tag{2.71}$$

The phase space $\int d^3p$ is not Lorentz invariant. Thus we consider the Lorentz invariant phase space

$$\begin{aligned} &\int d^4p'\ \delta\left(p'^2 - m^2\right)\theta\left(p_0'\right) \\ =& \int d^3p' \int dp_0'\ \frac{1}{2p_0'}\left[\delta\left(p_0' - E'\right) + \delta\left(p_0' + E'\right)\right]\ \theta\left(p_0'\right) \\ =& \int \frac{d^3p'}{2E'}. \end{aligned} \tag{2.72}$$

Now we write

$$\begin{aligned} |\mathbf{p}\rangle &= \int d^3p'\ \ |\mathbf{p}'\rangle\ \langle \mathbf{p}'\,|\mathbf{p}\rangle \\ &= \int \frac{d^3p'}{2E'}\ |\mathbf{p}'\rangle\ \left[2E'\ \langle \mathbf{p}'\,|\mathbf{p}\rangle\right] \\ &= \int \frac{d^3p'}{2E'}\sqrt{4\ p_0\ p_0'}\ \langle \mathbf{p}'\,|\mathbf{p}\rangle\ \ |\mathbf{p}'\rangle\,. \end{aligned} \tag{2.73}$$

It is clear from Eq. (73) , that $\langle p'|\,T\,|p\rangle$ is not Lorentz invariant , but $\sqrt{p_0\ p_0'}\ \langle p'|\,T\,|p\rangle$ is. Thus in general we write

$$T_{fi} = N'F_{fi}, \tag{2.74}$$

where N' is a multiple of factors like $1/E_r$. In fact it is convenient to take

$$N' = \left(\prod_r \frac{m_r}{(2\pi)^3\, E_r}\ \prod_s \frac{1}{(2\pi)^3\ 2E_s}\right)^{1/2}, \tag{2.75}$$

if there are r fermions and s bosons such that

$$r + s = n + m, \ N' = \left[\frac{1}{(2\pi)^{3/2}}\right]^{n+m} N, \tag{2.76}$$

m and n being the number of initial and final particles respectively. Hence finally the transition rate is

$$\begin{aligned} W_f &= \frac{(2\pi)^4}{(2\pi)^{3m}} \int \frac{d^3p'_1}{(2\pi)^3} \frac{d^3p'_2}{(2\pi)^3} \cdots \frac{d^3p'_n}{(2\pi)^3} \\ &\quad \times N^2 \delta^4 (p'_1 + p'_2 + \cdots + p'_n - p_i) \sum_{final\ spins} |F_{fi}|^2 \cdot \end{aligned} \tag{2.77}$$

In the first order perturbation theory

$$\left[\frac{1}{(2\pi)^{3/2}}\right]^{n+m} \prod_r \left(\frac{m_r}{E_r}\right)^{1/2} \prod_s \left(\frac{1}{2E_s}\right)^{1/2} F_{fi} = -\langle f | V | i \rangle \cdot \tag{2.78}$$

For example for $s = 0, r = 4$

$$\frac{1}{(2\pi)^6} \left(\frac{m_a \, m_b \, m_c \, m_d}{E_a \, E_b \, E_c \, E_d}\right)^{1/2} F_{fi} = -\langle f | V | i \rangle \cdot \tag{2.79}$$

Here m_a, m_b, m_c and m_d are the masses of four particles a, b, c and d involved in a scattering or a decay process.

2.5 Examples

2.5.1 Two-body scattering

Consider the scattering process

$$a + b \to c + d,$$

where a and c are bosons e.g. pions and b and d are fermions e.g. nucleons. The scattering cross section is given by

$$d\sigma = \frac{dW}{(\text{Flux})_{in}}, \tag{2.80}$$

where $(\text{Flux})_{in}$ is the incident flux defined as

$$(\text{Flux})_{in} = \rho_1\rho_2 v_{in} = \frac{v_{in}}{(2\pi)^6}. \tag{2.81}$$

$$v_{in} = \left|\frac{\mathbf{p}_a}{E_a} - \frac{\mathbf{p}_b}{E_b}\right|. \tag{2.82}$$

We calculate the scattering cross section in the c. m. frame. In this frame:

$$\mathbf{p}_a = -\mathbf{p}_b = \mathbf{p}_{ab} = \mathbf{p}, \quad \mathbf{p}_c = -\mathbf{p}_d = \mathbf{p}_{cd} = \mathbf{p}' \tag{2.83a}$$

$$E_{cm} = E_a + E_b = E_c + E_d. \tag{2.83b}$$

$$v_{in} = |\mathbf{p}| \frac{E_{cm}}{E_a E_b}. \tag{2.84}$$

Now from Eq. (77)

$$dW = \frac{(2\pi)^4}{(2\pi)^6} \int \frac{d^3p_c}{(2\pi)^3} \frac{d^3p_d}{(2\pi)^3} \left(\frac{m_b m_d}{4E_b E_d E_c E_a}\right) \times \delta^4 (p_c + p_d - p_a - p_b) \overline{\sum_{\text{spin}}} |F_{fi}|^2. \tag{2.85}$$

We can write

$$\delta^4 (p_c + p_d - p_a - p_b) = \delta^3 (\mathbf{p}_c + \mathbf{p}_d - \mathbf{p}_a - \mathbf{p}_b) \delta (E_c + E_d - E_a - E_b). \tag{2.86}$$

The integration over d^3p_d in Eq. (85) can be removed by the three-dimensional δ-function. Writing

$$d^3p_c = |\mathbf{p}'|^2 d|\mathbf{p}'| d\Omega', \tag{2.87}$$

we have from Eq. (85)

$$dW = \frac{m_b m_d}{4E_b E_a} \frac{1}{(2\pi)^8} \int |\mathbf{p}'|^2 d|\mathbf{p}'| d\Omega' \, \delta\left(E_{cm} - \sqrt{\mathbf{p}'^2 + m_c^2} - \sqrt{\mathbf{p}'^2 + m_d^2}\right) \frac{1}{\sqrt{\mathbf{p}'^2 + m_c^2}} \frac{1}{\sqrt{\mathbf{p}'^2 + m_d^2}} \overline{\sum_{\text{spins}}} |F_{fi}|^2. \tag{2.88}$$

Now using the formula

$$\int dx\, \delta\left[E - Y(x)\right]\, F(x) = \left[F(x)\, \frac{1}{Y'(x)}\right]_{E=Y(x)}, \tag{2.89}$$

we have from Eqs. (84) and (88)

$$\begin{aligned} d\sigma &= \frac{dW\ (2\pi)^6}{v_{in}} \\ &= \frac{m_b m_d}{4\,(2\pi)^2} \frac{|\mathbf{p}'|}{|\mathbf{p}|} \frac{1}{E_{cm}^2} |M|^2\, d\Omega', \end{aligned} \tag{2.90}$$

where we have put

$$|M|^2 = \overline{\sum_{spins}} |F_{fi}|^2 . \tag{2.91}$$

Hence we have

$$\frac{d\sigma}{d\Omega'} = \frac{m_b m_d}{E_{cm}^2} \frac{|\mathbf{p}'|}{|\mathbf{p}|} \frac{|M|^2}{16\pi^2}. \tag{2.92}$$

If a and c are also fermions, then

$$\frac{d\sigma}{d\Omega'} = \frac{m_a m_b m_c m_d}{E_{cm}^2} \frac{|\mathbf{p}'|}{|\mathbf{p}|} \frac{|M|^2}{4\pi^2}. \tag{2.93}$$

2.5.2 Three-body decay

Three-body phase space.

Consider a three-body decay

$$\begin{aligned} m &\rightarrow m_1 + m_2 + m_3 \\ K &= p_1 + p_2 + p_3. \end{aligned}$$

The decay rate [cf. Eq. (77)] is given by $[\rho_{in} = \frac{1}{(2\pi)^3}]$

$$\begin{aligned} d\Gamma &= \frac{dW}{\rho_{in}} \\ &= (2\pi)^4 \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} \int \frac{d^3p_3}{(2\pi)^3} \left(\frac{m m_1 m_2 m_3}{E\ E_1 E_2 E_3}\right) \\ &\quad \delta^3\left(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{K}\right) \delta\left(E_1 + E_2 + E_3 - E\right)\ |M|^2 , \end{aligned} \tag{2.94}$$

where for definiteness, we have taken all the particles to be fermions.

We evaluate Eq. (94) in the rest frame of particle m. In this frame $\mathbf{K} = 0$ and $E = m$. Hence we have

$$\begin{aligned} \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 &= 0 \\ E_1 + E_2 + E_3 &= m. \end{aligned} \tag{2.95}$$

From Eq. (94), removing the integration over d^3p_3 due to three-dimensional δ-function, we get

$$\begin{aligned} d\Gamma &= \frac{4\pi}{(2\pi)^5}(m_1 m_2 m_3)\int p_1^2 dp_1\ p_2^2 dp_2\ d\Omega_{12}\frac{1}{E_1 E_2 E_3} \\ &\quad \times\delta\left[E_1 + E_2 + \sqrt{(\mathbf{p}_1 + \mathbf{p}_2)^2 + m_3^2} - m\right]\ |M|^2 . \end{aligned} \tag{2.96}$$

After performing the angular integration over Ω_{12} , we obtain

$$\begin{aligned} d\Gamma &= \frac{2(2\pi)^2}{(2\pi)^5} m_1 m_2 m_3 \int \frac{|\mathbf{p}_1|\,|\mathbf{p}_2|\, E_1 E_2\ dE_1\ dE_2}{E_1 E_2}\frac{E_3}{|\mathbf{p}_1|\,|\mathbf{p}_2|}\left|\overline{M}\right|^2 \\ &= \frac{2 m_1 m_2 m_3}{(2\pi)^3}\int dE_1\ dE_2 \left|\overline{M}\right|^2, \end{aligned} \tag{2.97}$$

where $\left|\overline{M}\right|^2$ is the value $|M|^2$ after the angular integration has been performed. In order to evaluate the integral in Eq. (97), it is convenient to define the invariants:

$$\begin{aligned} s_{12} &= (K - p_3)^2 = (p_1 + p_2)^2 \\ s_{13} &= (K - p_2)^2 = (p_1 + p_3)^2 \\ s_{23} &= (K - p_1)^2 = (p_2 + p_3)^2 . \end{aligned} \tag{2.98}$$

In the rest frame of particle m, we have

$$\begin{aligned} s_{12} &= m^2 + m_3^2 - 2mE_3 \\ s_{13} &= m^2 + m_2^2 - 2mE_2 \\ s_{23} &= m^2 + m_1^2 - 2mE_1 \end{aligned} \tag{2.99}$$

$$s_{12} + s_{13} + s_{23} = m^2 + m_1^2 + m_2^2 + m_3^2. \tag{2.100}$$

On the other hand, in the center of mass frame of particles 1 and 2, we put

$$\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p} \quad \text{and} \quad \mathbf{p}_3 = \mathbf{q}. \tag{2.101}$$

In this frame, we denote the energies of particles 1, 2 and 3, by ω_1, ω_2, ω_3 respectively. Thus in this frame

$$\begin{aligned} s_{13} &= (\omega_1 + \omega_3)^2 - (\mathbf{p} + \mathbf{q})^2 = m_1^2 + m_3^2 - 2\mathbf{p}.\mathbf{q} + 2\omega_1\omega_3 \\ s_{23} &= (\omega_2 + \omega_3)^2 - (\mathbf{p} - \mathbf{q})^2 = m_2^2 + m_3^2 + 2\mathbf{p}.\mathbf{q} + 2\omega_2\omega_3 \\ s_{12} &= (\omega_1 + \omega_2)^2. \end{aligned} \tag{2.102}$$

For fixed s_{12}, the range of s_{23} is determined by letting $\mathbf{q}$ to be parallel or antiparallel to $\mathbf{p}$. Thus

$$(s_{23})_{\text{min}}^{\text{max}} = (\omega_2 + \omega_3)^2 - \left[\sqrt{\omega_3^2 - m_3^2} \mp \sqrt{\omega_2^2 - m_2^2}\right]^2. \tag{2.103}$$

We also note that we can express ω_1, ω_2 and ω_3 in terms of s_{12}.

$$\begin{aligned} \omega_1 &= \frac{s_{12} + m_1^2 - m_2^2}{2\sqrt{s_{12}}} \\ \omega_2 &= \frac{s_{12} - m_1^2 + m_2^2}{2\sqrt{s_{12}}} \\ \omega_3 &= \frac{m^2 - m_3^2 - s_{12}}{2\sqrt{s_{12}}}. \end{aligned} \tag{2.104}$$

In terms of the invariants s_{13} and s_{23}, Eq. (97) can be written

$$d\Gamma = \frac{2m_1m_2m_3}{(2\pi)^3\,(4m^2)} \int ds_{23}\, ds_{12} \left|\overline{M}\right|^2. \tag{2.105}$$

The scatter plot in s_{23} and s_{12} is called a Dalitz plot (Fig. 4). Phase space density is uniform across the plot i.e. if $\left|\bar{M}\right|^2$ is a constant, we have uniform distribution of events. Non uniform distribution of events over Dalitz plot will indicate a structure in $\left|\bar{M}\right|^2$ and would provide an important information about the dynamics underlying the process concerned.

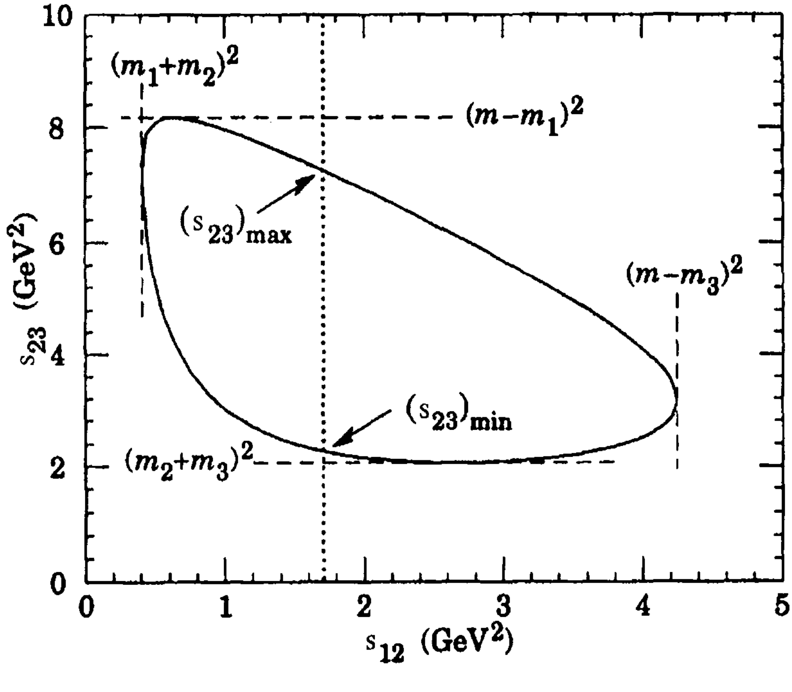

Figure 4 Dalitz plot for a three-body final state [ref. 5].

β-decay.

$$A \to B + e^- + \overline{\nu}_e, \qquad p = p_B + p_e + p_\nu$$

e.g.

$$O^{14} \to N^{14} + e^- + \overline{\nu}_e.$$

We obtain from Eq. (94)

$$\begin{aligned} d\Gamma &= \frac{4\pi}{(2\pi)^5} \int p_e^2 dp_e \; p_\nu^2 dp_\nu \; d\Omega_{e\nu} \; \delta\left(E_B + E_e + E_\nu - m_A\right) \\ &\times \frac{m_A \; m_B \; m_e \; m_\nu}{m_A \; E_B \; E_e \; E_\nu} \; |M|^2 . \end{aligned} \tag{2.106}$$

Now

$$p_\nu \; dp_\nu = E_\nu \; dE_\nu . \tag{2.107}$$

It is a very good approximation to neglect the recoil of the particle B, so that $\mathbf{p}_B \approx 0$ and $E_B = m_B$. For this case, the δ-function removes the integration over dE_ν and we get

$$\begin{aligned} d\Gamma &= \frac{4\pi}{(2\pi)^5} \; p_e^2 \; dp_e \; d\Omega_{e\nu} \; (m_A - m_B - E_e) \\ &\times \left((m_A - m_B - E_e)^2 - m_\nu^2\right)^{1/2} \left(\frac{m_e \; m_\nu}{E_e \; E_\nu} \; |M|^2\right) . \end{aligned} \tag{2.108}$$

Let us write

$$E_{\max} = E_e + E_\nu \approx m_A - m_B . \tag{2.109}$$

Then we get

$$\begin{aligned} d\Gamma &= \frac{4\pi}{(2\pi)^5} \; p_e^2 \; dp_e \; (E_{\max} - E_e) \left((E_{\max} - E_e)^2 - m_\nu^2\right)^{1/2} \\ &\times \left(\frac{m_e \; m_\nu}{E_e \; E_\nu} \; |M|^2\right) d\Omega_e \; d\Omega_\nu . \end{aligned} \tag{2.110}$$

In the first order perturbation theory,

$$|M|^2 = (2\pi)^{12} \overline{\sum_{\text{spin}} |\langle f| H_W |i\rangle|^2} \frac{E_e\, E_\nu}{m_e\, m_\nu}. \tag{2.111}$$

If the expression $\overline{\sum_{\text{spin}} |\langle f| H_W |i\rangle|^2}$ is averaged over angles between electron and neutrino, $d\Gamma$ can be integrated over $d\Omega_{e\nu}$ and we obtain

$$\begin{aligned} d\Gamma &= \frac{4\pi}{(2\pi)^5}\, p_e^2\, dp_e\, \left(E_{\max} - E_e\right) \\ &\quad \times \left((E_{\max} - E_e)^2 - m_\nu^2\right)^{1/2} \overline{(2\pi)^{12} \sum_{\text{spin}} |\langle f| H_W |i\rangle|} \cdot \end{aligned} \tag{2.112}$$

We make the simplest assumption that the averaged expression is independent of electron energy E_e . In this case

$$\frac{1}{p_e^2}\left(\frac{d\Gamma}{dp_e}\right) \propto (E_{\max} - E_e)\, \left[(E_{\max} - E_e)^2 - m_\nu^2\right]^{1/2}. \tag{2.113}$$

If we neglect the mass of the neutrino, then

$$K \equiv \left(\frac{d\Gamma}{p_e^2\, dp_e}\right)^{1/2} \propto (E_{\max} - E_e)\,. \tag{2.114}$$

From Eq. (114), we see that plot of $(d\Gamma/p_e^2\, dp_e)^{1/2}$versus E_e should be a straight line. This is called Fermi or Kurie plot. Figure 5 shows that it is indeed a straight line. Therefore, our assumption that the matrix elements $\langle f| H_W |i\rangle$ are independent of energy is correct. From Eq. (114), we get

$$\begin{aligned} \Gamma_\beta &= \frac{1}{(2\pi)^3} \left[\overline{(2\pi)^{12}\ |\langle f| H_W |i\rangle|^2}\right] \int_0^{p_e^{\max}} (E_{\max} - E_e)^2\, p_e^2\, dp_e \\ &= \frac{1}{(2\pi)^3}\, m_e^5 \left[\overline{(2\pi)^{12}\ |\langle f| H_W |i\rangle|^2}\right] f(\rho_0)\,, \end{aligned} \tag{2.115}$$

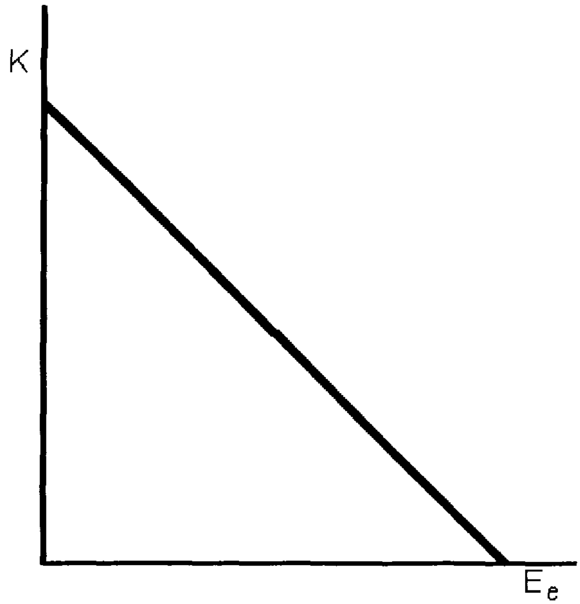

Figure 5 Fermi or Kurie plot.

where

$$f(\rho_0) = \int_0^{\rho_0} \rho^2 \left(\sqrt{\rho_0^2+1} - \sqrt{\rho^2+1}\right)^2 d\rho \qquad (2.116)$$

$$\rho = \frac{p_e}{m_e}, \qquad \rho_0 = \frac{p_e^{\max}}{m_e}.$$

This does not take into account Coulomb corrections due to Coulomb force which the electron experiences with the nucleus of charge Ze once it has left the nucleus. This can be taken into account in the integral $f(\rho_0)$ and the formula (115) remains valid. The life time for β-decay $\tau_\beta = 1/\Gamma_\beta$, but it is the half life $t_{1/2} = \tau_\beta(\ln 2)$ which is experimentally measured, while f is computed. $ft_{1/2}$ is called the ft value. It is assumed that H_W is universal i.e. the same for all decays (this assumption is supported

Table 2.1 Some characteristic ft values.

Decay		$t_{1/2}$	$T_e^{max} = E_{max} - m_e$ (MeV)	$ft_{1/2}$ (Sec)
$n \to p$	$1/2 \to 1/2$	10.6 min.	0.782	1100
$He^6 \to Li^6$	$0 \to 1$	0.813 sec.	3.50	810
$O^{14} \to N^{14}$	$0 \to 0$	71.4 sec.	1.812	3100
$H^3 \to He^3$	$1/2 \to 1/2$	12.33 yr.	18.6×10^{-3}	

by the experiments). ft values vary from about 10^3 to 10^{23} seconds. This variation is due to the phase space available in the final state characterized by E_{max} and hence by $f(\rho_0)$. Other cause of variation is due to the nuclear wave functions that enter into the calculation of matrix elements $\langle f| H_W |i\rangle$. Without the universality of H_W, an understanding of weak interaction would be hopeless. Some characteristic ft values are shown in Table 1.

We now consider the transition $O^{14} \to N^{14}$ $(O \to O)$ so that we do not have complications due to spin. Nuclei may be described by highly localized wave functions described by $\frac{1}{(2\pi)^{3/2}}U_i(r)$ and $\frac{1}{(2\pi)^{3/2}}U_f(r)$ which vanish for $r > 10^{-13}$ cm. Electron and neutrino can be described by plane waves as they carry large momenta. We take that H_W responsible for β-transitions is characterized by a parameter G_F which determines its strength. Thus

$$|\langle f| H_W |i\rangle|^2 = G_F^2 \left| \frac{1}{(2\pi)^6} \int U_f^*(r)\, U_i(r)\, e^{i\mathbf{p}_e \cdot \mathbf{r}}\, e^{i\mathbf{p}_\nu \cdot \mathbf{r}} d^3r \right|^2 . \tag{2.117}$$

Since $p_e/h \sim 10^{11}$ cm^{-1}, $r \approx 10^{-13}$ cm, it is a good approximation to replace the exponential in the integration by 1. This is called the allowed approximation. Thus we get from Eq. (117)

$$(2\pi)^{12}\ \overline{|\langle f| H_W |i\rangle|^2} = G_F^2. \tag{2.118}$$

Hence we obtain

$$\Gamma_\beta = \frac{G_F^2\, m_e^5}{2\pi^3}\ f(\rho_0) \tag{2.119}$$

$$G_F^2 = \frac{(2\pi^3\ \ln 2)}{ft}\ \frac{1}{m_e^5} \tag{2.120}$$

or

$$\left(G_F\ m_N^2\right)^2 = \frac{(2\pi^3\ \ln 2)}{ft}\left(\frac{m_N}{m_e}\right)^5 \frac{1}{m_N}. \tag{2.121}$$

Using $\frac{1}{m_N} \approx (0.7)\ 10^{-24}$ sec. and $ft = 3100$ sec. we get

$$G_F\ m_N^2 \approx 1.5 \times 10^{-5}. \tag{2.122}$$

More careful calculation gives

$$(2\pi)^{12}\ \left|\langle f|\ H_W\ |i\rangle\right|^2 = 2G_F^2, \tag{2.123}$$

so that

$$G_F\ m_N^2 \approx 10^{-5}. \tag{2.124}$$

Finally we note from Eq. (113) that a non-vanishing neutrino mass reveals itself as a downward deviation from a straight Kurie plot as the energy approaches its nominal ($m_\nu = 0$) kinematically allowed maximum T_e^{max}. We can write Eq. (113):

$$\left(\frac{d\Gamma}{dT_e}\right) \propto T_e^{3/2}\ \frac{(T_e + 2m_e)^{3/2}}{T_e + 2m_e}\ (T_e^{max} - T_e)\ \left[(T_e^{max} - T_e)^2 - m_\nu^2\right]^{1/2}, \tag{2.125}$$

where

$$T_e = E_e\ - m_e = \sqrt{\mathbf{p}_e^2 + m_e^2} - m_e. \tag{2.126}$$

We note that the effect of m_ν is near $T_e = T_e^{max}$, otherwise $(T_e^{max} - T_e)^2 \gg m_\nu^2$. If we put

$$\frac{T_e}{T_e^{max}} = x, \quad x_e = \frac{m_e}{T_e^{max}}, \tag{2.127}$$

then

$$\Gamma_{\nu=0} \propto (T_e^{max})^5\ \int_0^1 x^{3/2}\ (1-x)^2\ \frac{(x+2x_e)^{3/2}}{(x+x_e)} dx. \tag{2.128}$$

Hence it follows from Eqs. (125) and (128), that if $m_\nu \neq 0$, then the fraction of events $G(m_\nu)$ which will be absent at the end point is given by

$$G(m_\nu) = \frac{1}{\Gamma}\int_{T_e^{max}-m_\nu}^{T_e^{max}}\left(\frac{d\Gamma}{dT_e}\right)_{m_\nu=0} dT_e$$
$$\propto \frac{\left(T_e^{max}\right)^2 m_\nu^3}{\left(T_e^{max}\right)^5} = g\left(\frac{m_\nu}{T_e^{max}}\right)^3, \qquad (2.129)$$

where g is some constant. Hence it follows from Eq. (129), that in order to have $G(m_\nu)$ as large as possible, T_e^{max} be as small as possible. Thus we see from Table 1, that tritium (H^3) is most suitable to determine the mass m_ν of neutrino experimentally, since electrons from this decay have very low end-point energy (18.6 keV).

The distortion at the extreme end of the Kurie plot due to $m_\nu \neq 0$ is shown in Fig. 6. Thus in order to determine m_ν one has to look for such a distortion, but note that the deviation is in fact quite small. Moreover, the fraction of the events in the energy range of 18.5 keV $\leq Ee \leq$ 18.6 keV is only 3×10^{-7}. The experiment is hence quite difficult and even then it would be extremely difficult to determine m_ν better than 10 eV by this method. We shall come back to this point in Chap. 9.

2.6 Electromagnetic Interaction

A mono-chromatic electromagnetic wave is composed of N monoenergetic photons, each having energy and momentum, $E = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$. The electromagnetic field is described by a vector potential A with polarization vector ε. Electromagnetic waves are transverse waves so that $\mathbf{k} \cdot \boldsymbol{\varepsilon} = 0$ and these waves have two independent states of polarization. We can conveniently describe it as left-circularly or right-circularly polarized photon or we can say that a photon has two helicity states ± 1. Such a photon can be

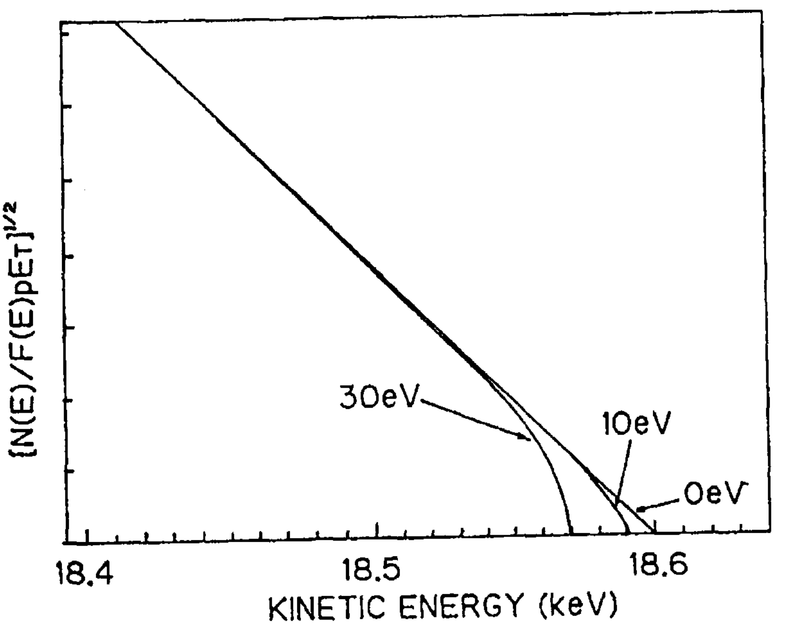

Figure 6 A schematic drawing of the Kurie plot with neutrino mass of 0, 10 eV and 30 eV.

described by a polarization vector

$$\boldsymbol{\varepsilon}_\pm = \frac{1}{\sqrt{2}}\left(\mp 1,\ -i,\ 0\right), \quad \varepsilon^0 = 0, \tag{2.130}$$

where we have taken the propagation vector $\mathbf{k}$ along z-axis.

The spin 1 matrices $\mathbf{S}$ are given by

$$\left(S_i\right)_{jk} = i\varepsilon_{ijk}\cdot \tag{2.131}$$

Writing them explicitly, we have

$$\begin{aligned} S_1 &= S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\ S_2 &= S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \\ S_3 &= S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.132}$$

If we write $\boldsymbol{\varepsilon}^+$ and $\boldsymbol{\varepsilon}^-$ as column matrices

$$\boldsymbol{\varepsilon}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}, \quad \boldsymbol{\varepsilon}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \tag{2.133}$$

it is easy to see that they are eigenstates of S_z with eigenvalues ± 1 respectively. We also note that

$$\begin{aligned} \boldsymbol{\varepsilon}_+^* \cdot \boldsymbol{\varepsilon}_+ &= 1 = \boldsymbol{\varepsilon}_-^* \cdot \boldsymbol{\varepsilon}_- \\ \boldsymbol{\varepsilon}_+^* \cdot \boldsymbol{\varepsilon}_- &= 0 = \boldsymbol{\varepsilon}_-^* \cdot \boldsymbol{\varepsilon}_+ \qquad (2.134) \\ \boldsymbol{\varepsilon}_\lambda^* \cdot \boldsymbol{\varepsilon}_\lambda &= \delta_{\lambda\lambda'}\ , \lambda, \lambda' = \pm 1\cdot \qquad (2.135) \end{aligned}$$

For a real photon, if we sum over polarizations (spin), we have

$$\sum_{\lambda=\pm 1} \varepsilon_{i\lambda}^* \varepsilon_{j\lambda} = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\cdot \tag{2.136}$$

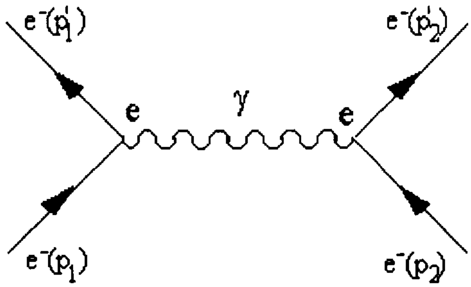

Figure 7 Electron-Electron scattering through exchange of a photon.

In quantum field theory, electromagnetic force between two electrons (or any charged particles) is assumed to be mediated by photons, the quanta of electromagnetic field. The simplest case is the exchange of a single photon as shown in Fig. 7.

The Coulomb potential between two charged particles is $e^2/4\pi r$ in rationalized Gaussian units. This is the Fourier transform of an amplitude $M(\mathbf{q})$ corresponding to the diagram shown in Fig. 7. Thus we write

$$\frac{e^2}{4\pi r} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\mathbf{q}.\mathbf{r}} M(\mathbf{q})\, d^3q. \tag{2.137}$$

In order to find $M(\mathbf{q})$, we note that

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{i\mathbf{q}.\mathbf{r}}}{\mathbf{q}^2}\,d^3q &= 2\pi\int_0^{\infty}\int_0^{\pi} e^{i|\mathbf{q}|r\cos\theta}\,\frac{1}{\mathbf{q}^2}\,|\mathbf{q}|^2\,d|\mathbf{q}|\sin\theta\,d\theta \\ &= \frac{4\pi}{r}\int_0^{\infty}\frac{\sin\,|\mathbf{q}|\,r}{|\mathbf{q}|}\,d|\mathbf{q}| \\ &= \frac{4\pi}{r}\int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{4\pi}{r}\frac{\pi}{2} = \frac{2\pi^2}{r}.\end{aligned} \tag{2.138}$$

Hence we have

$$M(\mathbf{q}) = \frac{e^2}{\mathbf{q}^2}. \tag{2.139}$$

This gives the matrix elements of the above diagram (Fig. 7) in momentum space in non-relativistic limit. A relativistic generalization of this is

$$\langle T\rangle = -\langle M\rangle = \frac{e^2}{(2\pi)^6}\sqrt{\frac{m_1\ m_2\ m_1'\ m_2'}{E_1\ E_2\ E_1'\ E_2'}}\ \frac{\langle J^\mu\rangle_1\ \langle J_\mu\rangle_2}{q^2} \tag{2.140}$$

i.e.

$$F = \frac{g_{\mu\nu}\ \langle J^\mu\rangle_1\ \langle J^\nu\rangle_2}{q^2}, \tag{2.141}$$

where $\langle J^\mu\rangle$ is the expectation value of the electromagnetic current. $g_{\mu\nu}/q^2$ is called the Feynman propagator of the photon. J_μ is given by

$$J^\mu = e\ \overline{\Psi}\ \gamma^\mu\ \Psi,$$

so that in free particle approximation

$$\langle J^\mu\rangle_i = e\overline{u}\left(\mathbf{p}_i'\right)\gamma^\mu u\left(\mathbf{p}_i\right), \quad i = 1, 2. \tag{2.142}$$

Thus

$$\begin{aligned} T &= \frac{1}{q^2}\ \overline{u}\left(\mathbf{p}_2'\right)e\gamma^\mu u\left(\mathbf{p}_2\right)\overline{u}\left(\mathbf{p}_1'\right)e\gamma_\mu u\left(\mathbf{p}_1\right) \\ &\quad\times\frac{1}{(2\pi)^6}\ \frac{m^2}{\sqrt{E_1\ E_2\ E_1'\ E_2'}}. \end{aligned} \tag{2.143}$$

In the non-relativistic limit $\frac{\mathbf{p}^2}{m} \approx 0$, $\frac{\mathbf{p}'^2}{m} \approx 0$, $E_1 = E_2 = E_1' = E_2' \approx m$,

$$\overline{u}\left(\mathbf{p}\right)\gamma^0 u\left(\mathbf{p}\right) \approx 1, \quad \overline{u}\left(\mathbf{p}\right)\boldsymbol{\gamma} u\left(\mathbf{p}\right) \approx 0,$$

$$q^2 = \left(p_1' - p_1\right)^2 = \left(E_1' - E_1\right)^2 - 4\mathbf{p}^2 \approx -4\mathbf{p}^2,$$

and $q^2 \to -\mathbf{q}^2$ so that we have from Eq. (140)

$$-T = M\left(\mathbf{q}\right) = \frac{e^2}{\mathbf{q}^2}. \tag{2.144}$$

2.7 Weak Interaction

If weak nuclear force is mediated by exchange of some particle, then this particle must have a finite mass, since weak nuclear force is a short range force. We assume that mediator of this force is a vector particle of finite mass. It, therefore, has three directions of polarization or it is a spin 1 particle with $M_z = \pm 1, 0$. These spin states can be expressed as

$$\begin{aligned} \boldsymbol{\varepsilon}_{\pm} &= \frac{1}{\sqrt{2}}(\mp 1,\ -i,\ 0),\ \varepsilon_{\pm}^{0} = 0 \\ \boldsymbol{\varepsilon}_{0} &= \left(0,\ 0,\ \frac{q_0}{m_W}\right),\ \boldsymbol{\varepsilon}_0^0 = \frac{i\,|\mathbf{q}|}{m_W}. \end{aligned} \tag{2.145}$$

In this representation

$$q = (q_0, 0,\ 0,\ |\mathbf{q}|),\ q^2 = m_W^2 \tag{2.146}$$

so that

$$q \cdot \varepsilon = 0. \tag{2.147}$$

It is easy to see that $\boldsymbol{\varepsilon}_{\pm}$, $\boldsymbol{\varepsilon}_0$ are eigenstates of S_z with eigenvalues ± 1, 0 respectively. For a spin 1 particle on the mass-shell

$$\sum_{\lambda=\pm 1,0} \varepsilon_\lambda^\mu \varepsilon_\lambda^\sigma = \varepsilon_+^{\mu^*} \varepsilon_+^\sigma + \varepsilon_-^\mu \varepsilon_-^{\sigma^*} + \varepsilon_0^\mu \varepsilon_0^{\sigma^*} = -g^{\mu\sigma} + \frac{q^\mu\, q^\sigma}{m_W^2}. \tag{2.148}$$

In order to estimate the strength of weak interaction, we evaluate the matrix elements of the scattering process

$$\nu_e + e^- \to \nu_e + e^-$$

as given by the diagram in Fig. 8. In analogy with Eq. (141), the scattering amplitude F is given by [the propagator $1/(q^2)$ is replaced by $1/(q^2 - m_W^2)$ as W-boson is massive]

$$F = \frac{\left\langle J^{W\mu}\right\rangle_1 \left\langle J_\mu^W\right\rangle_2}{(q^2 - m_W^2)} \tag{2.149a}$$

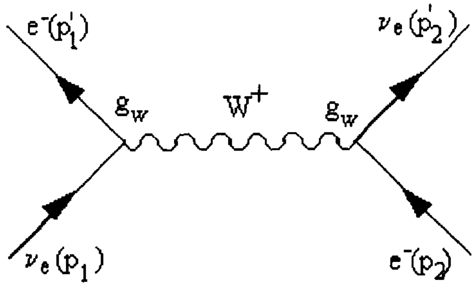

Figure 8 Neutrino-electron scattering through exchange of vector boson.

Now in contrast to electron, neutrino is a two-component object and its wave function is $(1-\gamma_5)u(p)$. Thus in analogy with Eq. (142)

$$\left\langle J^{W\mu}\right\rangle_i = g_W\ \overline{u}(\mathbf{p}_i')\ \gamma^\mu\ (1-\gamma_5)\ u(\mathbf{p}_i), \quad i=1,\ 2, \tag{2.149b}$$

where g_W is the strength of weak interaction just as e is the strength of the electromagnetic interaction. Thus for $q^2 << m_W^2$

$$F = -\frac{g_W^2}{m_W^2}\left[\overline{u}(\mathbf{p}_2')\ \gamma^\mu\ (1-\gamma_5)\ u(\mathbf{p}_2)\right]\ \left[\overline{u}(\mathbf{p}_1')\ \gamma_\mu\ (1-\gamma_5)\ u(\mathbf{p}_1)\right]. \tag{2.149c}$$

Using Eq. (A.48), we get

$$\begin{aligned} |F|^2 &= \frac{g_W^4}{m_W^4} A^\mu B_\mu A^{*\nu} B_\nu^* \qquad (2.150) \\ &= \frac{g_W^4}{m_W^4}\overline{\sum_{\text{spin}}}\sum_{\text{spin}}\left|\overline{u}(\mathbf{p}_2')\ \gamma^\mu\ (1-\gamma_5)\ u(\mathbf{p}_2)\right|^2 \\ &\quad \times\left|\overline{u}(\mathbf{p}_1')\ \gamma_\mu\ (1-\gamma_5)\ u(\mathbf{p}_1)\right|^2 \\ &= \frac{g_W^4}{m_W^4}\frac{1}{m_e m_\nu}\left[p_2'^\mu\ p_2^\nu + p_2'^\nu\ p_2^\mu - p_2\cdot p_2' g^{\mu\nu} + i\varepsilon^{\mu\nu\rho\sigma}\ p_{2\rho}\ p_{2\sigma}'\right] \end{aligned}$$

$$\times \frac{2}{m_e m_\nu} \left[p'_{1\mu} \, p_{1\nu} + p'_{1\nu} \, p_{1\mu} - p_1 \cdot p'_1 g_{\mu\nu} + i\varepsilon_{\mu\nu\alpha\beta} \, p_1^\alpha \, p_1'^\beta \right]$$

$$= \frac{g_W^4}{m_W^4} \frac{2}{m_e^2 m_\nu^2} \left(s - m_e^2 - m_\nu^2 \right)^2 , \tag{2.151}$$

where

$$s = (p_1 + p_2)^2 = (p'_1 + p'_2)^2 = E_{cm}^2 . \tag{2.152}$$

From Eqs. (93) and (150), we get

$$\sigma = \frac{2g_W^4}{\pi m_W^4} \frac{1}{s} \left(s - m_e^2 - m_\nu^2 \right)^2 . \tag{2.153}$$

If we neglect the lepton masses (viz. for $s >> m_e^2$), then we have

$$\sigma = \left(\frac{g_W^2}{4\pi} \right)^2 \left(\frac{8}{m_W^4} \right) 4\pi s. \tag{2.154}$$

Now $G_F/\sqrt{2} = g_W^2/m_W^2$ so that we have

$$\sigma = G_F^2 \, \frac{s}{\pi}. \tag{2.155}$$

Taking $\sigma \approx 10^{-38}$ cm^2 at $s = (1 \text{ GeV})^2$, we get

$$\left(\frac{10^{-38}}{4 \times 10^{-28}} \right) \pi \text{ GeV}^{-4} = G_F^2$$

$$G_F \approx 10^{-5} \text{ GeV}^{-2} \tag{2.156}$$

to be compared with Eq. (124). This shows the universality of the weak interaction since G_F is the same as obtained from the β-decay or from the scattering of neutrinos on leptons.

In unified electroweak theory [see Chap. 14]

$$g_W \sin\theta_W = \frac{e}{2\sqrt{2}}, \tag{2.157}$$

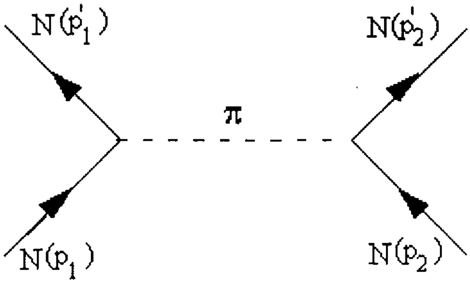

Figure 9 Nucleon-Nucleon scattering through pion exchange.

where $\sin\theta_W$ is a parameter of the theory. Experimentally, $\sin^2\theta_W \approx 1/4$. Thus we get

$$\frac{G_F}{\sqrt{2}} = \frac{e^2}{8m_W^2 \sin^2\theta_W} = 4\pi\frac{\alpha}{8m_W^2 \sin^2\theta_W}, \tag{2.158}$$

or

$$m_W = \left[\frac{\pi\alpha}{\sqrt{2}}\left(\sin^2\theta_W\, G_F\right)^{-1}\right]^{1/2}. \tag{2.159}$$

Using $\sin^2\theta_W \approx 1/4$, and Eq. (155), we get

$$m_W \approx 80 \text{ GeV}. \tag{2.160}$$

2.8 Hadronic Cross-section

Consider the $N - N$ scattering through the pion exchange. In particular consider the diagram (Fig. 9). Neglecting the spin of the nucleon

$$F \approx g_s^2 \frac{1}{q^2 - m_\pi^2}, \qquad q^2 = (p_1' - p_1)^2. \tag{2.161}$$

From Eqs. (93) and (160), we get

$$\frac{d\sigma}{d\Omega} = g_s^4 \frac{1}{(q^2 - m_\pi^2)^2} m_N^4 \frac{1}{4\pi^2}\frac{|\mathbf{p}'|}{|\mathbf{p}|}\frac{1}{E_{\text{cm}}^2}. \tag{2.162}$$

For elastic scattering $|\mathbf{p}'| = |\mathbf{p}|$, so that

$$\begin{aligned}\sigma &= \frac{g_s^4}{4\pi^2}\frac{m_N^4}{m_\pi^4}\frac{1}{s}\int_0^\pi \frac{2\pi\sin\theta d\theta}{\left[1+\frac{2}{m_\pi^4}\frac{|\mathbf{p}|^2}{}(1-\cos\theta)\right]^2} \\ &= \left(\frac{g_s^2}{4\pi}\right)^2\left(\frac{m_N}{m_\pi}\right)^2\left(\frac{4\pi}{m_\pi^2}\right)\left(\frac{4m_N^2}{s}\right)\frac{1}{1+\frac{s-4m_N^2}{m_\pi^2}}. \quad (2.163)\end{aligned}$$

Now

$$s = 4m_N^2\left(1+\frac{E_L^K}{2m_N}\right), \quad (2.164)$$

where E_L^K is the incident kinetic energy of the nucleon. Now $\frac{1}{m_\pi^2} \approx 2\times\left(10^{-13}\right)^2$ cm^2, $\left(\frac{m_N}{m_\pi}\right)^2 \approx 50$, thus

$$\sigma = \left(\frac{g_s^2}{4\pi}\right)^2\left(1.3\times 10^{-23}\text{cm}^2\right)\frac{1}{\left[1+14\frac{E_L^K}{m_N}\right]}. \quad (2.165)$$

For $E_L^K \ll \frac{m_\pi}{14} \approx 10$ MeV,

$$\sigma = \left(\frac{g_s^2}{4\pi}\right)^2\left(1.3\times 10^{-23}\text{cm}^2\right). \quad (2.166)$$

Experimentally $\sigma \approx 5\times 10^{-23}\text{cm}^2$, therefore,

$$\frac{g_s^2}{4\pi} \approx 1-2. \quad (2.167)$$

2.9 Problems

1. Show that for the scattering

$$e^-\ e^+ \to \gamma \to \pi^+(k_1)\ \pi^-(k_2)$$

the differential and total cross sections are given by ($s \gg m_e^2$):

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{s}|F(s)|^2\frac{\left(\frac{s}{4}-m_\pi^2\right)^{3/2}}{s^{3/2}}\left(1-\cos^2\theta\right) \\ \sigma &= \frac{8\pi}{3s}\alpha^2|F(s)|^2\frac{\left(\frac{s}{4}-m_\pi^2\right)^{3/2}}{s^{3/2}}\end{aligned}$$

where $s = q^2 = (k_1 + k_2)^2$ and $F(s)$ is the electromagnetic form factor of the pion, defined by $\left\langle 0|J_\mu^{em}|\pi^+(k_1)\pi^-(k_2)\right\rangle = F(s)(k_1 + k_2)_\mu$

Hint: See Appendix A.

2. Consider the decay

$$\omega \longrightarrow \pi^+\pi^-\pi^0.$$

Discuss the Dalitz plot for this decay.

Hint: From Lorentz invariance, the decay amplitude

$$F_\lambda \sim \varepsilon_{\lambda\mu\nu\rho}\, p_1^\mu \;\; p_2^\nu \;\; p_3^\rho,$$

where p_1, p_2 and p_3 are four momenta of pions.

2.10 Bibliography

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Chapter 3
SPACE–TIME SYMMETRIES

3.1 Invariance Principle

If the result of any experiment on some system is unchanged by a physical transformation of the apparatus, then the Hamiltonian or S-matrix describing that system is said to be invariant with respect to that transformation.

In quantum mechanics, such a transformation is described by a unitary transformation. Consider a particular experiment, for example, a transition from an initial state $|i\rangle$ to a final state $|f\rangle$. Such a transition is described by the matrix elements $\langle f| S |i\rangle$. If we change (transform) apparatus (e.g. displace or rotate), then invariance means

$$\langle f | S | i \rangle = \langle f^u | S | i^u \rangle = \langle f | U^\dagger S U | i \rangle \tag{3.1}$$

or

$$S = U^\dagger S U \tag{3.2a}$$

or

$$[S,\ U] = 0. \tag{3.2b}$$

Here

$$\begin{aligned} | i^u \rangle &= U | i \rangle \\ | f^u \rangle &= U | f \rangle \end{aligned} \tag{3.3}$$

are the transformed states. We see that the invariance under unitary transformation means that S-matrix commutes with it. Since

S-matrix is related to the Hamiltonian of the system, it follows that

$$[H,\ U] = 0 \tag{3.4}$$

if the system is invariant under U.

We consider two cases:

(a) U continuous:

U can be built out of infinitesimal transformations. Thus we need to consider an infinitesimal transformation:

$$U = 1 - i\,\varepsilon\,\widehat{F}, \tag{3.5}$$

where $\widehat{F}$ is a hermitian operator. $\widehat{F}$ can often be identified with an observable of the system, for example, the energy-momentum P_μ or the angular momentum J. $\widehat{F}$ is called the generator of the transformation represented by U. From Eq. (2b), we get

$$\left[S,\ \widehat{F}\right] = 0. \tag{3.6}$$

This means that $\widehat{F}$ is conserved. To see this, let $|i\rangle$ and $|f\rangle$ be eigenstates of $\widehat{F}$:

$$\begin{aligned} \widehat{F}\,|i\rangle &= F_i\,|i\rangle \\ \widehat{F}\,|f\rangle &= F_f\,|f\rangle\,. \end{aligned} \tag{3.7}$$

From Eq. (6), we have

$$\langle f|\left[S,\ \widehat{F}\right]|i\rangle = 0 \tag{3.8a}$$

or

$$(F_i - F_f)\,\langle f|\,S\,|i\rangle = 0. \tag{3.8b}$$

Hence, we get

$$F_i = F_f \quad \text{if} \quad \langle f|\,S\,|i\rangle \neq 0, \tag{3.9}$$

i.e. $\widehat{F}$ is conserved (eigenvalue of $\widehat{F}$ is conserved) in the transition $|i\rangle$ to $|f\rangle$. $\widehat{F}$ is then said to be a constant of motion. In Table 1, we give a list of some of the common transformations and their generators. Invariance under these transformations means that the corresponding generators are conserved.

Table 3.1

Transformation	Generator of the transformation	Unitary operator	Conservation Law
Displacement in [3] $\mathbf{x} \to \mathbf{x} + \mathbf{a}$	Momentum operator $\mathbf{p}$	$e^{i\widehat{\mathbf{p}}.\mathbf{a}}$	Momentum is conserved
Displacement in [4] $x_\mu \to x_\mu + a_\mu$	Energy-Momentum operator P_μ	$e^{-iP^\mu a_\mu}$	Energy-Momentum is conserved
Rotation in [3] $x_i \to x_i' = x_i + \varepsilon_{ij} x_j$ $\omega_i = \frac{1}{2}\varepsilon_{ijk}\ \varepsilon_{jk}$	Angular Momentum $\mathbf{J}$	$e^{-i\omega.\mathbf{J}}$	Angular Momentum is conserved

(b) U is discrete (e.g. space reflection)

$$U^2 = 1. \tag{3.10a}$$

Eigenvalues of U are

$$U' = \pm 1. \tag{3.10b}$$

Thus U is both unitary and hermitian. U can be regarded as an observable.

3.2 Parity

Consider a transformation corresponding to space reflection:

$$\mathbf{x} \to \mathbf{x}' = -\mathbf{x}. \tag{3.11}$$

The corresponding unitary operator is denoted by $\widehat{P}$, which acting on a wave function gives

$$\widehat{P}\ \Psi(\mathbf{x},\ t) = \Psi(-\mathbf{x},\ t)\,. \tag{3.12}$$

Now

$$\hat{P}^2 = 1, \tag{3.13}$$

so that $\hat{P}$ has two eigenvalues ± 1. If

$$\left[S,\ \hat{P}\right] = 0 \quad \text{or} \quad \left[H,\ \hat{P}\right] = 0 \tag{3.14}$$

then we say that parity is conserved. $\hat{P}$ does not commute with all types of H. In particular, the weak interaction Hamiltonian H_W does not commute with $\hat{P}$:

$$\left[H_W,\ \hat{P}\right] \neq 0 \tag{3.15}$$

i.e. parity is not conserved in weak processes.

Under parity operator $\hat{P}$

$$\mathbf{x} \to -\mathbf{x}, \quad \mathbf{p} \to -\mathbf{p} \tag{3.16}$$

but the orbital angular momentum

$$\mathbf{L} = \mathbf{x} \times \mathbf{p} \to \mathbf{L}, \tag{3.17a}$$

so that

$$\mathbf{J} \to \mathbf{J}, \quad \boldsymbol{\sigma} \to \boldsymbol{\sigma}. \tag{3.17b}$$

Such vectors are called axial vectors. Also under parity, the scalars:

$$\mathbf{x} \cdot \mathbf{p} \to \mathbf{x} \cdot \mathbf{p} \tag{3.18a}$$

$$(\mathbf{p}_1 \times \mathbf{p}_2) \cdot \mathbf{p}_3 \to -(\mathbf{p}_1 \times \mathbf{p}_2) \cdot \mathbf{p}_3 \tag{3.18b}$$

$$\mathbf{J} \cdot \mathbf{p} \to -\mathbf{J} \cdot \mathbf{p}. \tag{3.18c}$$

The scalars which change sign under parity are called pseudoscalars. All the three quantities are rotational invariant, but the last two have different behavior under $\hat{P}$.

A particle when it is in an orbital angular momentum state l has an orbital parity associated with it. In polar co-ordinates $\mathbf{x} \equiv (r, \theta, \phi)$, so that $\mathbf{x} \to -\mathbf{x}$ implies

$$r \to r, \quad \theta \to \pi - \theta, \quad \phi \to \pi + \phi. \tag{3.19}$$

Now we can write the wave function of a particle as

$$\Psi(\mathbf{x}) = R(r)Y_{lm}(\theta,\phi) \tag{3.20a}$$

$$Y_{lm}(\theta,\phi) = (-1)^m\left[\frac{(2l+1)\,(l-m)!}{4\pi\,(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\; e^{im\phi}. \tag{3.20b}$$

Under space inversion

$$P_l^m(\cos\theta) \to P_l^m(-\cos\theta) = (-1)^{l+m}P_l^m(\cos\theta) \tag{3.21a}$$

$$e^{im\phi} \to e^{im(\phi+\pi)} = (-1)^m e^{im\phi}, \tag{3.21b}$$

so that

$$Y_{lm}(\theta,\phi) \to (-1)^l\; Y_{lm}(\theta,\phi). \tag{3.21c}$$

We see that the orbital parity of a particle in an angular momentum state l is $(-1)^l$.

3.3 Intrinsic Parity

As far as orbital parity is concerned, it is independent of the species of particles and depends only on orbital angular momentum state of system of particles. When creation or annihilation of particles takes place, we have to assign an intrinsic parity to each particle. Consider, for example, a photon, the quantum of electromagnetic field represented by a vector potential.

$$\mathbf{A}\,(\mathbf{x}) = \boldsymbol{\varepsilon}\; f\,(x)\,, \tag{3.22}$$

where $\boldsymbol{\varepsilon}$ is the polarization vector and $f(x)$ is a scalar function. Now the interaction of a charged particle with electromagnetic field is introduced by the gauge invariant substitution:

$$\mathbf{p} \to \mathbf{p} - e\;\mathbf{A}\,(\mathbf{x})\,. \tag{3.23}$$

Since $\mathbf{x}$ and $\mathbf{p}$ change sign under $\widehat{P}$, it follows that

$$\mathbf{A}\,(\mathbf{x}) \to -\mathbf{A}\,(-\mathbf{x}) \tag{3.24a}$$

i.e.

$$\hat{P}\ \mathbf{A}(\mathbf{x})\ \hat{P}^{-1} = -\mathbf{A}(-\mathbf{x}). \tag{3.24b}$$

This means that under parity

$$\boldsymbol{\varepsilon} \longrightarrow -\boldsymbol{\varepsilon}. \tag{3.25}$$

The behavior of the polarization vector $\boldsymbol{\varepsilon}$ characterizes what we call the intrinsic parity of a photon. Thus we say that intrinsic parity of a photon is odd. Similarly for any particle a represented by a state vector $|a, \mathbf{p}\rangle$,

$$\hat{P}\,|a, \mathbf{p}\rangle = \eta_a^P\ |a, -\mathbf{p}\rangle, \tag{3.26}$$

where η_a^P is called the intrinsic parity of particle a. Note that $\eta_a^P = \pm 1$. We now show that the conservation of parity leads to multiplicative conservation law. Consider a reaction

$$a + b \longrightarrow c + d. \tag{3.27}$$

We can write the initial state

$$|\ i\ \rangle = |\ a\ \rangle\ |\ b\ \rangle\ |\ \text{relative motion}\ \rangle. \tag{3.28}$$

Here $|\ a\ \rangle$ and $|\ b\ \rangle$ describe the internal states of a and b, while the third factor describes their relative motion. This state can be described by a wave function $R(r)Y_{lm}\ (\theta,\ \phi)$. Since, we assume that parity is conserved in the reaction (27), it follows that the states $|\ i\ \rangle$ and $|\ f\ \rangle$ are eigenstates of $\hat{P}$, with eigenvalues η_i^P and η_f^P respectively. Now

$$\eta_i^P = \eta_a^P\,\eta_b^P\,(-1)^l \tag{3.29a}$$

$$\eta_f^P = \eta_c^P\,\eta_d^P\,(-1)^{l'}, \tag{3.29b}$$

where η_a^P, η_b^P, η_c^P, and η_d^P are intrinsic parities of a, b, c and d respectively and $(-1)^l$ and $(-1)^{l'}$ are their orbital parities in the initial and final states.

Parity conservation for the reaction (27) gives

$$\eta_i^P = \eta_f^P \tag{3.30a}$$

or

$$\eta_a^P\ \eta_b^P\ (-1)^l = \eta_c^P\ \eta_d^P\ (-1)^{l'} \tag{3.30b}$$

i.e. parity is conserved as a multiplicative quantum number.

However, the law of parity conservation is not universal, in particular it does not hold for weak interactions. Then it follows from Eq. (15) that it is not possible to find simultaneous eigenstates of H_W and $\hat{P}$. Thus if parity is not conserved, the energy eigenstates $|\ \Psi\ \rangle$ are not expected to be eigenstates of parity. In this case, we can write

$$|\Psi\rangle = |\Psi_{\text{regular}}\rangle + y\,|\Psi_{\text{irregular}}\rangle\,, \tag{3.31}$$

where $|\Psi_{\text{regular}}\rangle$ and $|\Psi_{\text{irregular}}\rangle$ have opposite parities. y is called the parity mixing amplitude and is a measure of the degree of parity non-conservation. Parity violation is maximum if $|y|^2 = 1$. Several experiments involving hadrons show that in hadronic interactions

$$|\ y\ |^2 < 10^{-13}.$$

Experiments involving atomic transitions show that parity is conserved to a high degree in electromagnetic interaction and that $|\ y\ |^2 < 10^{-14}$. For weak interactions, the parity violation is maximum viz. $|y|^2 = 1$. It follows that in order to determine the intrinsic parity of a particle, one cannot use weak interactions. Only by considering reactions involving hadronic or electromagnetic interactions, one can determine the intrinsic parity of a particle. Even then the intrinsic parity cannot be fixed uniquely and we have to use a convention viz. the intrinsic parity of a proton is +1 i.e.

$$\eta\,(\text{proton}) = +1. \tag{3.32}$$

Since proton and neutron form an isospin doublet, we also take

$$\eta\,(\text{neutron}) = +1. \tag{3.33}$$

Intrinsic Parity of Pion

We shall assume that the spin of pion is zero (we shall show later, how it comes out to be zero). Consider first the decay $\pi^0 \to 2\gamma$. Here we have two polarization vectors $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ corresponding to two γ - rays, whose momenta we take as $\mathbf{k}_1$ and $\mathbf{k}_2$, such that (gauge invariance) $\mathbf{k}_1.\boldsymbol{\varepsilon}_1 = 0$, $\mathbf{k}_2.\boldsymbol{\varepsilon}_2 = 0$. We also note that $\boldsymbol{\varepsilon}_1.\boldsymbol{\varepsilon}_2 = 0$. Now only the momentum $\mathbf{k} = \mathbf{k}_1 - \mathbf{k}_2$ is independent as $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 = 0$ in the rest frame of π^0. It is clear that the only invariant which we can form is $\mathbf{k} \cdot (\boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2)$, which is a pseudoscalar, showing that intrinsic parity of π^0 is -1.

Consider the capture of π^- at rest by deuteron. The dominant processes are

$$\begin{aligned} \pi^- + d &\to n + n \\ &\to n + n + \gamma. \end{aligned} \tag{3.34}$$

Parity conservation for the first reaction gives

$$\eta_\pi \, \eta_d \, (-1)^l = \eta_n \, \eta_n \, (-1)^{l'} = (-1)^{l'}, \tag{3.35}$$

where l is the relative orbital angular momentum of $\pi^- d$ and l' is that of two neutrons. There is evidence that π^- is captured in $l = 0$ orbital state. Thus from Eq. (35), we get

$$\eta_\pi \, \eta_d = (-1)^{l'}. \tag{3.36}$$

The deuteron is a bound state of a proton and neutron and has spin 1. The relative angular momentum of the two nucleons in deuteron is predominantly zero. Thus deuteron is a predominantly $^3S_1$ state i.e. for a deuteron $J^P = 1^+$. It follows that the total angular momentum of the initial state is $J = 1$. Conservation of angular momentum gives $J_{\text{final}} = 1$. The spin S of the two neutron system is either 0 or 1. Thus for $J = 1$, we have two possibilities: Triplet spin state ($S = 1$): $l' = 2, 1, 0$ i.e. the final state is $^3D_1$ or $^3P_1$ or $^3S_1$. For the singlet spin state ($S = 0$): $l' = 1$ and the final state is ${}^1P_1$. Now the Pauli exclusion principle requires that the final state must be antisymmetric. Since the triplet spin state is symmetric, the orbital state must be antisymmetric i.e. $l' = 1$ and allowed final state is ${}^3P_1$. For the spin singlet state, since it is antisymmetric, l' should be even. Thus ${}^1P_1$ state is not allowed by the Pauli exclusion principle. Hence we have the result that the final state must be ${}^3P_1$ so that from Eq. (36), we get

$$\eta_{\pi^-} = (-1)^{l'} = -1 \tag{3.37}$$

since $\eta_d = +1$. Thus for a pion $J^P = 0^-$ and it is called a pseudoscalar particle.

3.4 Parity Constraints on S-Matrix for Hadronic Reactions

3.4.1 Scattering of spin 0 particles on spin $\frac{1}{2}$ particles

Consider two-body elastic scattering of a spin 0 particle on a spin $\frac{1}{2}$ particle

$$\begin{array}{ccccccc} a & + & b & \longrightarrow & c & + & d \\ \mathbf{p}_1 & & (\mathbf{p}_2, \boldsymbol{\sigma}) & & \mathbf{p}_1' & & (\mathbf{p}_2', \boldsymbol{\sigma}) \end{array}$$

In the center of mass frame

$$\begin{aligned} \mathbf{p}_1 &= -\mathbf{p}_2 = \mathbf{p}_i \\ \mathbf{p}_1' &= -\mathbf{p}_2' = \mathbf{p}_f. \end{aligned} \tag{3.38}$$

For the elastic scattering $|\mathbf{p}_i| = |\mathbf{p}_f| = |\mathbf{p}| = p$. The initial and final states can be labelled as $|\, i\,\rangle = |\mathbf{p}_i,\ \boldsymbol{\sigma}\,\rangle, |\, f\,\rangle = |\mathbf{p}_f,\ \boldsymbol{\sigma}\,\rangle$. Under parity

$$\hat{P}\,|i\rangle = \eta_i^p\,|-\mathbf{p}_i, \boldsymbol{\sigma}\rangle\,, \quad \hat{P}\,|f\rangle = \eta_f^p\,|-\mathbf{p}_f, \boldsymbol{\sigma}\rangle\,. \tag{3.39}$$

The transition matrix elements

$$\begin{aligned} & \langle \mathbf{p}_f, \boldsymbol{\sigma}|\, T\, |\mathbf{p}_i, \boldsymbol{\sigma}\rangle \\ =\ & \langle \mathbf{p}_f, \boldsymbol{\sigma}|\, \hat{P}^\dagger \hat{P}\ T\ \hat{P}^\dagger \hat{P}\ \ |\mathbf{p}_i, \boldsymbol{\sigma}\rangle \\ =\ & \eta_f^p \eta_i^p\ \langle -\mathbf{p}_f, \boldsymbol{\sigma}|\, \hat{P}\ T\ \hat{P}^\dagger\ \ |-\mathbf{p}_i, \boldsymbol{\sigma}\rangle\,. \end{aligned} \tag{3.40}$$

Now invariance under P implies

$$\hat{P}\, T\, \hat{P}^{\dagger} = T. \tag{3.41}$$

Because of elastic scattering

$$\eta_i^p = \eta_f^p. \tag{3.42}$$

Therefore, we have from Eqs. (40)–(42)

$$\langle -\mathbf{p}_f, \boldsymbol{\sigma} |\, T\, | -\mathbf{p}_i, \boldsymbol{\sigma} \rangle = \langle \mathbf{p}_f, \boldsymbol{\sigma} |\, T\, | \mathbf{p}_i, \boldsymbol{\sigma} \rangle . \tag{3.43}$$

If we assume rotational invariance, then $\langle\, T\, \rangle$ can depend only on the rotational invariant quantities p, $\mathbf{p}_f \cdot \mathbf{p}_i$, $\boldsymbol{\sigma} \cdot \mathbf{p}_i$, $\boldsymbol{\sigma} \cdot \mathbf{p}_f$, $\boldsymbol{\sigma} \cdot (\mathbf{p}_i \times \mathbf{p}_f)$. We need not consider $\boldsymbol{\sigma}^2$ or higher powers of it, because $\boldsymbol{\sigma}^2 = 3$ and $(\boldsymbol{\sigma}\,.\,\mathbf{a})(\boldsymbol{\sigma}\,.\,\mathbf{b}) = \mathbf{a}\,.\,\mathbf{b} + i\,\boldsymbol{\sigma}\,.\,(\mathbf{a} \times \mathbf{b})$. Thus these quantities can be reduced to either a constant or $\boldsymbol{\sigma}$. In other words, assuming rotational invariance only, we can write in spin space

$$\begin{aligned} & \langle \mathbf{p}_f, \boldsymbol{\sigma} |\, T\, | \mathbf{p}_i, \boldsymbol{\sigma} \rangle \\ = \; & [A\,(p,\theta) + A_1\,(p,\theta)\, \boldsymbol{\sigma}.\mathbf{p}_i + A_2\,(p,\theta)\, \boldsymbol{\sigma}.\mathbf{p}_f \\ & + B\,(p,\theta)\, \boldsymbol{\sigma}.\,(\mathbf{p}_i \times \mathbf{p}_f)] . \end{aligned} \tag{3.44}$$

This is a 2×2 matrix in spin space. It is understood that the above matrix elements are to be taken between spin wave functions $\chi_f^{\dagger}$ and χ_i for the final and initial states. Thus using rotational invariance alone, we have $2^2 = 4$ independent amplitudes. If in addition we assume invariance under parity, then Eqs. (43) and (44) imply $A_1 = 0 = A_2$. Therefore, invariance under rotation and space-inversion gives

$$\begin{aligned} & \langle \mathbf{p}_f, \boldsymbol{\sigma} |\, T\, | \mathbf{p}_i, \boldsymbol{\sigma} \rangle \\ = \; & \chi_f^{\dagger}\, [A\,(p,\theta) + B\,(p,\theta)\, \boldsymbol{\sigma}.\,(\mathbf{p}_i \times \mathbf{p}_f)]\, \chi_i . \end{aligned} \tag{3.45}$$

This is an example which shows how a symmetry principle restricts the form of a transition matrix.

3.4.2 Decay of a spin 0^+ particle into three spinless particles each having odd parity

Consider the decay

$$A \to P_1 + P_2 + P_3,$$

where all the particles have spin 0. Consider the decay in the rest frame of particle A. We have

$$\mathbf{0} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3, \tag{3.46}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ are momenta of particles P_1, P_2 and P_3 respectively. The transition matrix elements for the decay is given by

$$M(\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3) = \langle\ P_1(\mathbf{p}_1)\ \ P_2(\mathbf{p}_2)\ \ P_3(\mathbf{p}_3)|T\ \ |\ A(0)\ \rangle. \tag{3.47}$$

Under parity

$$\begin{aligned} \hat{P}|\ A(0)\ \rangle &= |\ A(0)\ \rangle \\ \hat{P}\,|P_i(\mathbf{p}_i)\ \rangle &= -\,|P_i(-\mathbf{p}_i)\ \rangle, \qquad i = 1,2,3. \end{aligned} \tag{3.48}$$

Now

$$\begin{aligned} & M(\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3) \\ = & \ \langle\ P_1(\mathbf{p}_1)\ \ P_2(\mathbf{p}_2)\ \ P_3(\mathbf{p}_3)|\,\hat{P}^\dagger\hat{P}\ T\ \hat{P}^\dagger\hat{P}\,|\ A(0)\ \rangle \\ = & \ (-1)^3\langle\ P_1(-\mathbf{p}_1)\ \ P_2(-\mathbf{p}_2)\ \ P_3(-\mathbf{p}_3)|\,\hat{P}\ T\ \hat{P}^\dagger\,|\ A(0)\ \rangle. \end{aligned} \tag{3.49}$$

If parity is conserved

$$\hat{P}\ T\ \hat{P}^\dagger = T \tag{3.50}$$

and we have from Eqs. (3.49) and (3.50)

$$M\ (\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3) = -M\ (-\mathbf{p}_1,\ -\mathbf{p}_2,\ -\mathbf{p}_3). \tag{3.51}$$

Because of the rotational invariance, M can be a function of rotational invariant quantities $\mathbf{p}_1\cdot\mathbf{p}_2$, $\mathbf{p}_2\cdot\mathbf{p}_3$, $\mathbf{p}_3\cdot\mathbf{p}_1$ and $\mathbf{p}_1\cdot\ (\mathbf{p}_2 \times \mathbf{p}_3)$.

But the last invariant is zero, since $\mathbf{p}_3 = -(\mathbf{p}_1 + \mathbf{p}_2)$. Hence the rotational and space-inversion invariance implies

$$M\ (\mathbf{p}_1 \cdot \mathbf{p}_2,\ \mathbf{p}_2 \cdot \mathbf{p}_3,\ \mathbf{p}_3 \cdot \mathbf{p}_1) = -M\ (\mathbf{p}_1 \cdot \mathbf{p}_2,\ \mathbf{p}_2 \cdot \mathbf{p}_3,\ \mathbf{p}_3 \cdot \mathbf{p}_1)$$

or

$$M = 0.$$

Thus we have the result that the decay of a spinless particle with even parity to three pseudoscalar particles is forbidden if we assume invariance under space-inversion. On the other hand, decay of a spinless particle with odd parity to three pseudoscalar particles will be allowed under space-inversion invariance.

3.5 Time Reversal

Under time reversal

$$t \to -t, \quad \mathbf{x} \to \mathbf{x}. \tag{3.52a}$$

Therefore,

$$\mathbf{p} \to -\mathbf{p}, \quad \mathbf{L} \to -\mathbf{L}, \quad \boldsymbol{\sigma} \to -\boldsymbol{\sigma}. \tag{3.52b}$$

Let Π denote the operation which transforms quantum mechanical states and operators under the above transformation i.e. under $t \to -t$. First we show that Π cannot be a unitary operator. Under Π, the commutation relation

$$[\widehat{q}_i,\ \widehat{p}_j] = i\hbar\delta_{ij} \to -i\hbar\delta_{ij}, \tag{3.53}$$

is not invariant. Hence the transformation generated by Π cannot be unitary. But we want the above commutation relation to be invariant under Π. A way out of this difficulty is as follows: All $c-$numbers are simultaneously transformed into their complex conjugates. Such a transformation is called antiunitary. Then under Π,

$$\widehat{q}_i \to \Pi\ \widehat{q}_i\ \Pi^{-1} = \widehat{q}_i, \quad \widehat{p}_j \to \Pi\ \widehat{p}_j\ \Pi^{-1} = -\widehat{p}_j \tag{3.54}$$

$$i \to -i$$

and the commutation relation (53) remains invariant. Also, we note that

$$\Pi\ \mathbf{J}\ \Pi^{-1} = -\mathbf{J} \tag{3.55}$$

and the commutation relation

$$[J_i,\quad J_j] = i\varepsilon_{ijk}\ J_k \tag{3.56}$$

is preserved.

We now discuss the transformation of the transition matrix T under Π. If H_0 and V are invariant under time reversal, then

$$\begin{aligned} \Pi\ H_0\ \Pi^{-1} &= H_0 \\ \Pi\ V\ \Pi^{-1} &= V. \end{aligned} \tag{3.57}$$

Now [cf. Eq. (2.55)]

$$T = -V - V\frac{1}{E_a - H + i\varepsilon}V$$

and we have

$$\Pi\ T\ \Pi^{-1} = -V - V\frac{1}{E_a - H - i\mathbf{T}\cdot\boldsymbol{\varepsilon}}V \ = T^{\dagger}. \tag{3.58}$$

Invariance under time reversal implies

$$\begin{aligned} \langle f\ | T\ |\ i\rangle &= \langle f\ | \Pi^{-1}\Pi\ T\ \Pi^{-1}\Pi\ |\ i\rangle \\ &= \left\langle f^t\ \middle| T^{\dagger}\ \middle|\ i^t\right\rangle^* \\ &= \left\langle i^t\ \middle| T\ \middle|\ f^t\right\rangle. \end{aligned} \tag{3.59}$$

We now discuss the transformation of scattering states under time reversal. We specify a state $|\ a\ \rangle$ by its momentum, z-component of spin m_a, and α which denotes all other quantum numbers which may be necessary to specify the state. Thus we write

$$\begin{aligned} |\ a\ \rangle &= |\ \alpha,\ \mathbf{p}_\alpha,\ m_a\rangle \\ \left|\ a^{(+)}\ \right\rangle &= |\ \alpha,\ \mathbf{p}_\alpha,\ m_a\rangle_{\text{in}} \\ \left|\ a^{(-)}\ \right\rangle &= |\ \alpha,\ \mathbf{p}_\alpha,\ m_a\rangle_{\text{out}}\,. \end{aligned} \tag{3.60}$$

Under Π

$$\Pi \mid \alpha,\ \mathbf{p}_\alpha,\ m_a\rangle = \mid \alpha,\ -\mathbf{p}_\alpha,\ -m_a\rangle\,, \tag{3.61}$$

$$\begin{aligned}
&\Pi \mid \alpha,\ \mathbf{p}_\alpha,\ m_a\rangle_{\text{in}} \\
&= \mid \alpha,\ -\mathbf{p}_\alpha,\ -m_a\rangle + \frac{1}{E_a - H - i\varepsilon} V \mid \alpha,\ -\mathbf{p}_\alpha,\ -m_a\rangle \\
&= \mid \alpha,\ -\mathbf{p}_\alpha,\ -m_a\rangle_{\text{out}}\,,
\end{aligned} \tag{3.62}$$

where we have used Eqs. (2.46), (2.47) and (57) and the rule $i \to -i$.

Let us specify the initial and final states as

$$\begin{aligned}
\mid i\,\rangle &= \mid \alpha,\ \mathbf{p}_i,\ m_i\rangle \\
\mid f\,\rangle &= \mid \beta,\ \mathbf{p}_f,\ m_f\rangle\,.
\end{aligned} \tag{3.63}$$

Then

$$\begin{aligned}
\left| i^t \right\rangle &= \mid \alpha,\ -\mathbf{p}_i,\ -m_i\rangle \\
\left| f^t \right\rangle &= \mid \beta,\ -\mathbf{p}_f,\ -m_f\rangle\,.
\end{aligned} \tag{3.64}$$

Therefore, Eq. (59) gives

$$\langle \beta,\ \mathbf{p}_f,\ m_f | T \mid \alpha,\ \mathbf{p}_i,\ m_i\rangle = \langle\ \alpha,\ -\mathbf{p}_i,\ -m_i | T \mid \beta,\ -\mathbf{p}_f,\ -m_f\rangle\,. \tag{3.65}$$

This expresses the equality of two scattering processes obtained by reversing the momenta and spin-components and interchanging the initial and final states. This is known as reciprocity relation and is a consequence of invariance under time reversal. Since Π is not a unitary operator, therefore, it does not have observable eigenvalues. The states cannot be labelled by such eigenvalues. Therefore, invariance under Π cannot be tested by searching for time-parity forbidden decays. It can be tested by using the relation of the form given in Eq. (65). No violation of time reversal has been found in hadronic and electromagnetic interactions.

3.6 Applications

3.6.1 Detailed balance principle

Determination of spin of the pion

If we assume invariance under time reversal, we get Eq. (65). In addition, if we assume parity conservation, we have from Eq. (65).

$$\begin{aligned} & \langle \beta,\ \mathbf{p}_f,\ m_f | T \mid \alpha,\ \mathbf{p}_i,\ m_i \rangle \\ = & \langle\ \alpha,\ -\mathbf{p}_i,\ -m_i | \hat{P}^\dagger T\ \hat{P} \mid \beta,\ -\mathbf{p}_f,\ -m_f \rangle \\ = & \langle\ \alpha,\ \mathbf{p}_i,\ -m_i | T \mid \beta,\ \mathbf{p}_f,\ -m_f \rangle\ . \end{aligned} \tag{3.66}$$

If the spins are summed, then we can write

$$\begin{aligned} & \sum_{\text{spin}} |\langle \beta,\ \mathbf{p}_f,\ m_f | T \mid \alpha,\ \mathbf{p}_i,\ m_i \rangle|^2 \\ = & \sum_{\text{spin}} |\langle\ \alpha,\ \mathbf{p}_i,\ m_i | T \mid \beta,\ \mathbf{p}_f,\ m_f \rangle|^2\ . \end{aligned} \tag{3.67}$$

This is called the "semi detailed balance principle". We now apply the above result to two-body scattering

$$a + b \to c + d, \quad e.g. \quad p + p \to \pi^+ + d.$$

Then we get [cf. Eq. (2.92)]

$$\begin{aligned} & \frac{d\ \sigma}{d\Omega}(a + b \to c + d) \\ = & \frac{1}{16\pi^2} \frac{m_N^2}{E_{cm}^2} \frac{p_{cd}}{p_{ab}} \frac{1}{(2s_a + 1)(2s_b + 1)} \sum_{\text{spin}} |F_{ab\to cd}|^2 \end{aligned} \tag{3.68}$$

and

$$\begin{aligned} & \frac{d\ \sigma}{d\Omega}(c + d \to a + b) \\ = & \frac{1}{16\pi^2} \frac{m_N^2}{E_{cm}^2} \frac{p_{ab}}{p_{cd}} \frac{1}{(2s_c + 1)(2s_d + 1)} \sum_{\text{spin}} |F_{cd\to ab}|^2\ . \end{aligned} \tag{3.69}$$

But Eq. (67) gives

$$\sum_{\text{spin}} |F_{ab\to cd}|^2 = \sum_{\text{spin}} |F_{cd\to ab}|^2 . \tag{3.70}$$

Hence we have

$$\begin{aligned} &\frac{d\,\sigma}{d\Omega}\,(a+b\to c+d) \\ &= \frac{(2s_c+1)\,(2s_d+1)}{(2s_a+1)\,(2s_b+1)}\frac{p_{cd}^2}{p_{ab}^2}\frac{d\sigma}{d\Omega}\,(c+d\to a+b)\,. \end{aligned} \tag{3.71}$$

This is known as the principle of detailed balance. We now apply the above result to the reaction

$$p+p\to\pi^+ + d.$$

Then from Eq. (71), we get

$$\frac{d\,\sigma}{d\Omega}\left(p+p\to\pi^+ + d\right) = \frac{3\,(2s_\pi+1)}{4}\frac{p_\pi^2}{p_p^2}\frac{d\sigma}{d\Omega}\left(\pi^+ + d\to p+p\right), \tag{3.72}$$

where we have used the result that the proton spin $s_p = \frac{1}{2}$ and that the deuteron spin $s_d = 1$. For the total cross sections, we get

$$\sigma\ \left(p+p\to\pi^+ + d\right) = \frac{3}{4}\,(2s_\pi+1)\,\frac{p_\pi^2}{p_p^2}\ \sigma\ \left(\pi^+ + d\to p+p\right). \tag{3.73}$$

From the experimentally measured cross sections, we find $s_\pi = 0$ i.e. the spin of the pion is zero.

3.7 Unitarity Constraints

So far, assuming rotational invariance, we have discussed the constraints on the T-matrix imposed by space reflection and time reversal invariance. In this section, we discuss the constraints on the T-matrix due to the unitarity of the S-matrix.

Unitarity of the S-matrix gives

$$S\,S^{\dagger}=1 \tag{3.74a}$$

or

$$\langle j\mid S\,S^{\dagger}\mid i\rangle=\langle j\mid i\rangle=\delta_{ji}, \tag{3.74b}$$

where $\mid i\rangle$ and $\mid j\rangle$ are initial and final states. Introduce a complete set of states $\mid k\rangle$,

$$\sum_k \langle j\mid S\mid k\rangle\ \langle k\mid S^{\dagger}\mid i\rangle=\delta_{ji} \tag{3.75a}$$

or

$$\begin{aligned}&\sum_k \langle j\mid\left[1+i\ (2\pi)^4\,\delta^4\ (P_j-P_k)\,T\right]\mid k\rangle\\&\times\langle k\mid\left[1-i\ (2\pi)^4\,\delta^4\ (P_k-P_i)\ T^{\dagger}\right]\mid i\rangle\\=\ &\delta_{ji},\end{aligned} \tag{3.75b}$$

which gives

$$\begin{aligned}&-i\ (2\pi)^4\sum_k\left[\delta_{ki}\,\delta^4\ (P_j-P_k)\,T_{jk}-\delta_{jk}\,\delta^4\ (P_k-P_i)\left(T^{\dagger}_{ki}\right)\right]\\=\ &(2\pi)^8\sum_k\delta^4\ (P_j-P_i)\ \langle j\mid T\mid k\rangle\ \langle k\mid T^{\dagger}\mid i\rangle\ \delta^4\ (P_i-P_k)\end{aligned} \tag{3.76a}$$

or

$$-i\left[T_{ji}-T^{*}_{ij}\right]=(2\pi)^4\sum_k\ \langle j\mid T\mid k\rangle\ \langle k\mid T^{\dagger}\mid i\rangle\ \delta^4\ (P_i-P_k)\,. \tag{3.76b}$$

In Eq. (76b), $\sum_k$ means integration over momenta and sum over other quantum numbers. Only those states contribute which are allowed by energy-momentum conservation implied by the δ-function in Eq. (76b). For forward elastic scattering and no spin flip, $i=j$ and we get

$$2\,\mathrm{Im}\,T_{ii}=(2\pi)^4\sum_k\ \langle i\mid T\mid k\rangle\ \langle k\mid T^{\dagger}\mid i\rangle\ \delta^4\ (P_k-P_i)\,. \tag{3.77}$$

For two-body scattering viz.

$$a + b \rightarrow 1 + 2 + \cdots$$

the right-hand side of Eq. (77) is the transition rate W_i [cf. Eq. (2.69)], where $W_i(i = a + b)$ is given by

$$W_i = \sigma_i \ (\text{Flux})_{\text{in}} = \sigma_i \frac{1}{(2\pi)^6} \frac{|\mathbf{p}| \, E_{cm}}{E_a \ E_b}. \tag{3.78}$$

Expressing the T-matrix, in terms of the amplitude F, we have

$$T_{ii} = \left[\frac{1}{(2\pi)^{3/2}} \right]^4 N \ F_{ii}, \tag{3.79a}$$

where

$$N = \begin{bmatrix} \frac{m_a \ m_b}{E_a E_b}, & a \text{ and } b \text{ both fermions} \\ \frac{m_b}{2E_a E_b}, & a \text{ boson, } b \text{ fermion} \\ \frac{1}{4E_a E_b}, & a \text{ and } b \text{ both bosons} \end{bmatrix}. \tag{3.79b}$$

Hence, we have from Eq. (77):

$$2n \, \text{Im} \, F_{ii} = E_{cm} \ |\mathbf{p}| \ \sigma_{ab} = \frac{1}{2} \sqrt{\lambda \ (s, \ m_a^2, \ m_b^2)} \ \sigma_{ab}, \tag{3.80}$$

where F_{ii} is the forward elastic scattering amplitude, $\sigma_i = \sigma_{ab}$ is the total cross section for the reaction $a + b \rightarrow 1 + 2 + \cdots$ and

$$n = \begin{bmatrix} m_a \ m_b \\ \frac{m_b}{2} \\ \frac{1}{4} \end{bmatrix} \tag{3.81}$$

depending upon the nature of particles a and b. Equation (80) is known as the optical theorem. As a simple example, consider a and b to be spinless particles. Then we can express

$$\begin{aligned} F_{ij}(s, \theta) &= 8\pi \ s^{1/2} \sum_{L=0}^{\infty} (2L + 1) \, F_{ij,L}(s) \ P_L(\cos\theta) \\ &\equiv 8\pi \ s^{1/2} \ f_{ij}(s, \theta). \end{aligned} \tag{3.82}$$

If we put $f_{ii}(s,0) = f(0)$ we have from Eqs. (80)–(82)

$$\text{Im } f(0) = \frac{|\mathbf{p}|}{4\pi} \sigma_{ab} \tag{3.83}$$

the usual form of optical theorem in potential scattering.

Two-particle partial wave unitarity

Assume that for each channel k, three or more particles states can be neglected. We work in the center of mass frame, with initial state $i = a + b$, so that $\mathbf{p}_a = \mathbf{p}_b = \mathbf{p}$. We take $\mathbf{p}$ along z-axis. Two-body Lorentz invariant phase space is given by

$$n_k \, (2\pi)^4 \int \frac{d^3 \, p_{1k}}{(4\pi)^3 \, E_{1k}} \frac{d^3 \, p_{2k}}{(2\pi)^3 \, E_{2k}} \delta^4 \, (\, p_{1k} + p_{2k} - P_i) \, . \tag{3.84}$$

In the center of mass frame, $\mathbf{p}_{1k} = -\mathbf{p}_{2k} = \mathbf{p}_k$, where

$$p_k = |\mathbf{p}_k| = \frac{\sqrt{\lambda \, (s, \, m_{1k}^2, \, m_{2k}^2)}}{s^{1/2}} \tag{3.85}$$

$$N = \begin{bmatrix} \frac{1}{4}, & \text{both bosons} \\ \frac{m_{2k}}{2}, & \text{1st particle boson, 2nd one fermion} \\ m_{1k} \, m_{2k}, & \text{both fermions} \end{bmatrix} . \tag{3.86}$$

Then working out the integral (84), we get

$$n_k \, \frac{1}{4\pi^2} \frac{p_k}{s^{1/2}} \, d\Omega', \tag{3.87}$$

where $\Omega' \equiv (\theta', \phi')$ is the solid angle between $\mathbf{p}$ and $\mathbf{p}_k$. $\Omega \equiv (\theta, \phi)$ is the solid angle between $\mathbf{p}$ and $\mathbf{p}_{1j}$ where $\mathbf{p}_{1j}$ is the momentum of first particle in the state j. $\Omega'' \equiv (\theta'', \phi'')$ is the solid angle between $\mathbf{p}_k$ and $\mathbf{p}_{1j}$. For the two-particle states in channel k, the unitarity relation (76b) becomes, on using Eq. (87)

$$-i \left[F_{ji}(\Omega) - F_{ij}^*(-\Omega) \right] = \sum_k \frac{n_k \, p_k}{4\pi^2 \, s^{1/2}} \int F_{jk}(\Omega'') \, F_{ik}^*(-\Omega') \, d\Omega'. \tag{3.88}$$

We use the general relation (88) for two-particle unitarity for three important cases:

Case (i): Collision between spinless particles. In this case i, j, and k are simply channel indices. For this case, we can expand $F_{ij}\ (\theta)$ in terms of the Legendre polynomials of $\cos\theta$ [This is a consequence of rotational invariance; there can be no dependence on the magnetic quantum number m and hence no dependence on ϕ]. This expansion is given in Eq. (82). Similarly $F_{ik}(\Omega')$ is independent of ϕ' for spinless particles and can be expanded in terms of $P_L(\cos\theta')$. Likewise $F_{jk}\ (\Omega'')$ can be expanded in terms of $P_L(\cos\theta'')$. Hence, we have from Eq. (88)

$$\begin{aligned}
&-i8\pi\ s^{1/2}\ \sum_L \left(F_{ji,\ L}(s) - F^*_{ij,\ L}(s)\right)(2L+1)\ P_L(\cos\theta) \\
&= \frac{1}{4}\sum_k \frac{p_k}{4\pi^2\ s^{1/2}}\ 64\pi^2\ s \\
&\quad \times \sum_{L''}\sum_{L'}(2L''+1)\ (2L'+1)\ F_{jk,\ L''}(s)\ F^*_{ik,\ L'}(s) \\
&\quad \times \int P_{L''}(\cos\theta'')\ P_{L'}(\cos\theta')\ \sin\theta'\ d\theta'\ d\phi'. \qquad (3.89)
\end{aligned}$$

In order to evaluate the integral on the right-hand side of Eq. (89), we use the following formulae:

$$P_L(\cos\theta'') = \frac{4\pi}{(2L+1)}\sum_M Y^*_{LM}(\theta,0)\ Y_{LM}(\theta',\phi') \qquad (3.90a)$$

$$\begin{aligned}
\int_0^{2\pi} P_L(\cos\theta'')\ d\phi' &= \frac{4\pi}{(2L+1)}\sum_M \int_0^{2\pi} Y^*_{LM}(\theta,0)\ Y_{LM}(\theta',\phi')\ d\phi' \\
&= \frac{8\pi^2}{4\pi} P_L(\cos\theta)\, P_L(\cos\theta') \qquad (3.90b)
\end{aligned}$$

$$\int_{-1}^{1} d\ (\cos\theta')\ P_L(\cos\theta')\ P_{L'}(\cos\theta') = \frac{2}{2L+1}\delta_{L'L}. \qquad (3.90c)$$

We get from Eq. (89), using Eqs. (90)

$$\frac{1}{2i}\sum_L\ (2L+1)\left(F_{ji,\ L}(s) - F^*_{ij,\ L}(s)\right)P_L(\cos\theta)$$

$$= \sum_k p_k \sum_{L'} (2L'+1) \quad F_{jk,\, L'}(s) \quad F^*_{ik,\, L'}(s) \;\; P_{L'}(\cos\theta) \; . \tag{3.91}$$

Since the Legendre polynomials are linearly independent, we get the desired 2-body partial-wave unitarity relation

$$\frac{1}{2i}\left(F_{ji,\, L}(s) \; - \; F^*_{ij,\, L}(s)\right) = \sum_k p_k \; F_{jk,\, L}(s) \quad F^*_{ik,\, L}(s) \, . \tag{3.92}$$

If we are interested only in elastic scattering, we may drop indices i and j and we obtain

$$\mathrm{Im}\, F_L(s) = \sum_k p_k \;\; |F_{k,\, L}|^2 \; . \tag{3.93}$$

Occasionally all channels except the elastic one are closed at low energies. Then $p_k = p$ and we have

$$\mathrm{Im}\, F_L \; = p \;\; |F_L|^2 \; , \tag{3.94}$$

so that we can put

$$F_L = \frac{1}{p} \; e^{i\delta_L} \sin\delta_L, \tag{3.95}$$

where δ_L is a real function of s. We can also express

$$F_L = \frac{1}{2ip}\left(e^{2i\delta_L} - 1\right) = p^{-1}\left(\cot\delta_L - i\right)^{-1} . \tag{3.96}$$

The differential cross section is given by

$$\begin{aligned} \frac{d\;\sigma_{ij}}{d\Omega} &= \frac{1}{4s}\frac{p'}{p}\frac{1}{16\pi^2} \;\; |F_{ij}(s,\theta)|^2 \\ &= \frac{p'}{p}\left|\sum_L \;\; (2L+1)\, F_{ij,L}(s)\, P_L(\cos\theta)\right|^2 , \end{aligned} \tag{3.97}$$

where we have used Eq. (82). Using the orthogonality of Legendre polynomials, we get

$$\sigma_{ij} = 4\pi\frac{p'}{p}\sum_L \;\; (2L+1) \;\; |F_{ji,L}(s)|^2 \equiv \sum_L \;\; \sigma_{ji,L}, \tag{3.98a}$$

where

$$\sigma_{ji,L} = 4\pi\frac{p'}{p}\ (2L+1)\ |F_{ji,L}|^2 . \tag{3.98b}$$

For "purely elastic" region, where Eq. (95) applies, we have

$$\sigma_L = \frac{4\pi}{p^2}\ (2L+1)\sin^2\delta_L. \tag{3.99}$$

Case (ii): Particles a and b carry spin. Here it is convenient to introduce helicity. Let λ_1 and λ_2 be helicities of particle a and b respectively and let $\lambda = \lambda_1 - \lambda_2$. In the center of mass frame $\mathbf{p}_a = -\mathbf{p}_b = \mathbf{p}$. Let us take the vector $\mathbf{p} = (p, \theta, \phi)$. In the center of mass frame we represent the two particles state as $[\Omega = (\theta, \phi)]$

$$|p, \lambda_1, \lambda_2, \Omega\rangle = |\mathbf{p}, \lambda_1\rangle\ |-\mathbf{p}, \lambda_2\rangle\ (-1)^{s_2-\lambda_2} \tag{3.100}$$

The last factor in Eq. (100) is due to phase convention. Noting that $(\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2)$, $\mathbf{J}\cdot\mathbf{p} = \mathbf{J}_1\cdot\mathbf{p} - \mathbf{J}_2\cdot(-\mathbf{p})$, we have

$$\frac{\mathbf{J}\cdot\mathbf{p}}{|\mathbf{p}|}\,|p, \lambda_1, \lambda_2, \Omega\rangle\ = \lambda\ |p, \lambda_1, \lambda_2, \Omega\rangle\, . \tag{3.101}$$

Now

$$R\ |0,0\rangle = |\theta, \phi\rangle \tag{3.102}$$

where R is the rotation operator $e^{-i\theta\mathbf{n}\cdot\mathbf{J}/\hbar}\,[\mathbf{n} = (-\sin\phi, \cos\phi, 0)]$ and

$$\begin{aligned} R\ |J\ M\rangle\ &=\ \sum_{M'}\ |J\ M'\rangle\ \langle J\ M'\,|\,R\ |J\ M\rangle \\ &=\ \sum_{M'}\ |J\ M'\rangle\ d^J_{M'M}\ (\Omega) \end{aligned} \tag{3.103}$$

where $d^J_{M'M}\ (\Omega)$ are rotation matrices. Thus

$$\begin{aligned} \langle J\ M\ ,\lambda|\,\theta\phi\rangle\ &=\ \langle J\ M, \lambda\,|\,R\ |0\ 0\rangle \\ &=\ \sum_{M'}\langle J\ M, \lambda\,|\,R\ |J\ M', \lambda\rangle\,\langle J\ M', \lambda\,|\,0\ 0\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{M'} d^J_{MM'}(\Omega)\ \langle J\ M'\ ,\lambda|\,0\ 0\rangle \\
&= \sum_{M'} d^J_{MM'}(\Omega)\ \sqrt{\frac{2J+1}{4\pi}}\delta_{M'\lambda} \\
&= \sqrt{\frac{2J+1}{4\pi}}d^J_{M\lambda}(\Omega)\,. \qquad (3.104)
\end{aligned}
$$

Hence

$$
\begin{aligned}
|\theta,\phi,\lambda\rangle &= \sum_{JM} |J\ M,\lambda\rangle\ \langle J\ M\ ,\lambda|\,\theta\phi\rangle \\
&= \sum_{JM} \sqrt{\frac{2J+1}{4\pi}}\,|J\ M,\ \lambda\rangle\ d^J_{M\lambda}(\Omega)\,. \qquad (3.105)
\end{aligned}
$$

Thus we can write

$$
|\theta,\phi,\lambda_1,\lambda_2\rangle = \sum_{JM} |J\ M\ \lambda_1\lambda_2\rangle\sqrt{\frac{2J+1}{4\pi}}d^J_{M\lambda}(\Omega)\,. \qquad (3.106)
$$

We now consider the scattering process $a+b=c+d$. Let λ_1 and λ_2 be initial helicities and λ'_1 and λ'_2 be final helicities. $\lambda=\lambda_1-\lambda_2$ and $\lambda'=\lambda'_1-\lambda'_2$. We take initial momentum $\mathbf{p}=(p,0,0)$ and final momentum $\mathbf{p}'=(p',\theta,\phi)$. We can write the scattering amplitude on using Eq. (105)

$$
\begin{aligned}
&F_{ji}\ (\lambda'_1\lambda'_2,\lambda_1\lambda_2,\Omega) \\
&= \langle\theta\ \phi\ \lambda'_1\lambda'_2, j|\,F\ |0\ 0\ \lambda_1\lambda_2,\ i\rangle \\
&= \sum_{J'M'}\sum_{JM} d^{*J'}_{M'\lambda'}(\Omega)\sqrt{\frac{2J'+1}{4\pi}}d^J_{M\lambda}(0)\sqrt{\frac{2J+1}{4\pi}} \\
&\quad\times\langle J'M'\ \lambda'_1\lambda'_2,\ j|\,F\ |JM\ \lambda_1\lambda_2,i\rangle \\
&= \sum_{J'M'}\sum_{JM} d^{*J'}_{M'\lambda'}(\Omega)\sqrt{\frac{(2J'+1)(2J+1)}{(4\pi)^2}}\delta_{M\lambda}\ n\ \delta_{JJ'}\ \delta_{MM'}\ \delta_{M\lambda} \\
&\quad\times F^J_{ji}\ (\lambda'_1\lambda'_2,\lambda_1\lambda_2,s) \\
&= n\sum_J \frac{(2J+1)}{4\pi}F^J_{ji}\ (\lambda'_1\lambda'_2,\lambda_1\lambda_2,s)\,d^{J*}_{\lambda\lambda'}(\theta,\phi)\,. \qquad (3.107)
\end{aligned}
$$

Note that $n = 8\pi^2 \frac{s^{1/2}}{\sqrt{m_a\ m_b\ m_c\ m_d}}$ when all the particles are fermions. For spinless particles $n = 32\pi^2\ s^{1/2}$, $\lambda = \lambda' = 0$, $J = L$ and $d_{00}^{J*}(\theta,\phi) = \sqrt{\frac{4\pi}{2J+1}} Y_{L0}^{*}\ (\theta,\phi) = P_L(\cos\theta)$, we get back Eq. (82).

The differential scattering cross section for the process $a + b \to c + d$ is given by

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{|\mathbf{p}'|}{|\mathbf{p}|} \frac{1}{(2s_1+1)(2s_2+1)} \\ &\times \sum_{\lambda_1'\lambda_2'\lambda_1\lambda_2} \left| \sum_J (2J+1) F^J\ (\lambda_1'\lambda_2', \lambda_1\lambda_2, s)\, d_{\lambda\lambda'}^J(\theta) \right|^2 \end{aligned} \tag{3.108}$$

where we have used

$$d_{\lambda\lambda'}^J(\theta,\phi) = e^{i(\lambda-\lambda')\phi}\ d_{\lambda\lambda'}^J(\theta). \tag{3.109}$$

To proceed further we note the following properties of rotation matrices

$$d_{\lambda'\lambda}^J(-\Omega) = \left(d^{J^{-1}}(\Omega)\right)_{\lambda'\lambda} = \left(d^{J^\dagger}(\Omega)\right)_{\lambda'\lambda} = d_{\lambda\lambda'}^{J*}(\Omega) \tag{3.110}$$

$$\int d_{\lambda\lambda'}^{J*}(\Omega)\ d_{M\lambda'}^{J'}(\Omega)\ d\Omega = \frac{4\pi}{(2J+1)}\delta_{JJ'}\ \delta_{\lambda M} \tag{3.111}$$

$$d^J(\Omega'') = d^J(-\Omega')\ d^J(\Omega) = d^{J^\dagger}(\Omega')\ d^J(\Omega)$$

or

$$\begin{aligned} d_{\lambda\lambda'}^J(\Omega'') &= \sum_M \left(d^{J^\dagger}(\Omega')\right)_{\lambda M} \cdot \left(d^J(\Omega)\right)_{M\lambda'} \\ &= \sum_M d_{M\lambda}^{J*}(\Omega')\ d_{M\lambda'}^J(\Omega). \end{aligned} \tag{3.112}$$

Note that in Eq. (112) $d^J(\Omega'')$ has been expressed as product of two rotation matrices corresponding to $-\Omega'$,Ω. Then using Eqs. (107), (110)–(112), we get from Eq. (92), for the two-particle partial wave unitarity relation.

$$\begin{aligned}
&\frac{1}{2i}\left[F^J_{ji}\ (\lambda'_1\lambda'_2;\lambda_1\lambda_2;\ s) - F^{J^*}_{ij}\ (\lambda_1\lambda_2;\lambda'_1\lambda'_2;\ s)\right] \\
&= \sum_k p_k \sum_{\lambda_{k_1}}\sum_{\lambda_{k_2}} F^J_{jk}\ (\lambda'_1\lambda'_2;\lambda_{k_1}\lambda_{k_2};\ s)\, F^{J^*}_{ik}\ (\lambda_1\lambda_2;\lambda_{k_1}\lambda_{k_2};\ s)\,.
\end{aligned} \tag{3.113}$$

The two-particle elastic unitarity gives

$$\begin{aligned}
&\frac{1}{2i}\left[F^J\ (\lambda'_1\lambda'_2;\lambda_1\lambda_2;\ s) - F^{J^*}\ (\lambda_1\lambda_2;\lambda'_1\lambda'_2;\ s)\right] \\
&= p \sum_{\lambda''_1\lambda''_2} F^J\ (\lambda'_1\lambda'_2;\lambda''_1\lambda''_2;\ s)\, F^{J^*}\ (\lambda_1\lambda_2;\lambda''_1\lambda''_2;\ s)\,.
\end{aligned} \tag{3.114}$$

Assuming parity conservation, we get

$$\begin{aligned}
&F^J\ (-\lambda'_1,-\lambda'_2;-\lambda_1,-\lambda_2;\ s) \\
&= \eta\ (-1)^{s'_1+s'_2-s_1-s_2}\, F^J\ (\lambda'_1\lambda'_2;\lambda_1\lambda_2;\ s)\,,
\end{aligned} \tag{3.115}$$

where s_1, s_2, s'_1 and s'_2 are the spins of particle a and b in the initial and final states and η is the product of their intrinsic parities. Equation (115) shows that not all the amplitudes are independent. Time reversal invariance puts additional restrictions on the amplitudes F^J's namely

$$F^J_{ji}\ (\lambda'_1\lambda'_2;\lambda_1\lambda_2;\ s) = F^J_{ij}\ (\lambda_1\lambda_2;\lambda'_1\lambda'_2;\ s)\,. \tag{3.116}$$

For the elastic scattering:

$$F^J\ (\lambda'_1\lambda'_2;\lambda_1\lambda_2;\ s) = F^J\ (\lambda_1\lambda_2;\lambda'_1\lambda'_2;\ s)\,. \tag{3.117}$$

Finally using the orthogonality of d-matrices, we get integrated cross section for elastic scattering from Eq. (108)

$$\sigma = \sum_J \sigma_J\,,$$

where

$$\sigma_J = 4\pi \frac{(2J+1)}{(2s_1+1)(2s_2+1)} \sum_{\lambda_1'\lambda_2'\lambda_1\lambda_2} \left|F^J\ (\lambda_1'\lambda_2';\lambda_1\lambda_2;\ s)\right|^2. \tag{3.118}$$

In particular, when all the particles have spin 1/2, the S-wave unitarity gives ($s = 4p^2$)

$$\sigma_S = \frac{4\pi}{4}\frac{\sin^2\delta_0}{p^2} < \frac{\pi}{p^2}. \tag{3.119}$$

For special case for the elastic scattering of $a+b \to a+b$, where a carries spin s and b is spinless, we have $\lambda = \lambda_1 - \lambda_2 = \lambda_1$ and $\lambda' = \lambda_1' - \lambda_2' = \lambda_1'$. For this case we have from Eqs. (107), (108), (117), (113) and (114) (for a to be fermion)

$$F_{\lambda'\lambda}\ (\Omega) = \frac{4\pi\sqrt{s}}{m_a} \sum_J (2J+1)\ F^J_{\lambda'\lambda}\ (s)\ d^{J*}_{\lambda\lambda'}\ (\theta,\phi) \tag{3.120}$$

$$\frac{d\ \sigma}{d\Omega} = \frac{1}{(2s+1)} \sum_{\lambda'\lambda} \left|\sum_J (2J+1)\ F^J_{\lambda'\lambda}\ (s)\ d^{J*}_{\lambda\lambda'}\ (\theta)\right|^2 \tag{3.121}$$

$$\sigma_J = 4\pi\left(\frac{2J+1}{2s+1}\right) \sum_{\lambda'\lambda} \left|F^J_{\lambda'\lambda}\ (s)\right|^2 \tag{3.122}$$

$$\frac{1}{2i}\left[F^J_{\lambda'\lambda}\ (s) - F^{J*}_{\lambda\lambda'}\ (s)\right] = p\sum_{\lambda''} F^J_{\lambda'\lambda''}\ (s)\ F^{J*}_{\lambda'\lambda''}\ (s) \tag{3.123}$$

$$F^J_{\lambda'\lambda} = F^J_{-\lambda'-\lambda}. \tag{3.124}$$

We end this chapter with the following remarks. We have shown how the symmetry principles put restrictions on the S-matrix. In this way, we get the minimum set of observables to describe the experimental data. This approach is especially rewarding, when the underlying dynamics is not known, which is the case for the hadronic interactions.

3.8 Problems

Problem 1: Nucleon-Nucleon Scattering Amplitudes

$$N_a + N_b \to N_a + N_b.$$

In the center of mass frame

$$\begin{aligned} \mathbf{p}_a &= -\mathbf{p}_b = \mathbf{p} \\ \mathbf{p}'_a &= -\mathbf{p}'_b = \mathbf{p}'. \end{aligned}$$

Introduce three orthogonal unit vectors $\mathbf{l}, \mathbf{m}, \mathbf{n}$

$$\mathbf{l} = \frac{\mathbf{p} \times \mathbf{p}'}{|\mathbf{p} \times \mathbf{p}'|}, \quad \mathbf{m} = \frac{\mathbf{p}' - \mathbf{p}}{|\mathbf{p}' - \mathbf{p}|} \quad \mathbf{n} = \frac{\mathbf{p}' + \mathbf{p}}{|\mathbf{p}' + \mathbf{p}|}$$

$$l_i l_j + m_i m_j + n_i n_j = \delta_{ij} \quad i = 1, 2, 3.$$

Then T-matrix can be written as

$$\begin{aligned} \langle T \rangle &= \langle \alpha,\ \mathbf{p}',\ \sigma_1,\ \sigma_2 | T\ | \alpha,\ \mathbf{p},\ \sigma_1,\ \sigma_2 \rangle \\ &= [A_1 + B_1\ \boldsymbol{\sigma}_1 \cdot \mathbf{l} + C_1\ \boldsymbol{\sigma}_1 \cdot \mathbf{m} + D_1\ \boldsymbol{\sigma}_1 \cdot \mathbf{n}] \\ &\quad \times [A_2 + B_2\ \boldsymbol{\sigma}_2 \cdot \mathbf{l} + C_2\ \boldsymbol{\sigma}_2 \cdot \mathbf{m} + D_2\ \boldsymbol{\sigma}_2 \cdot \mathbf{n}]. \end{aligned}$$

It is understood that matrix elements are to be taken between the spin states $\chi^\dagger_{af}\ \chi^\dagger_{bf}$ and $\chi_{ai}\ \chi_{bi}$. Then using parity conservation and time reversal invariance, show that T can be written as

$$\begin{aligned} \langle T \rangle \ = \ & \left(\frac{m}{E}\right) [H_1 + H_2\ \boldsymbol{\sigma}_1 \cdot \mathbf{l}\ \boldsymbol{\sigma}_2 \cdot \mathbf{l} + iH_3\ (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{l} \\ & + iH'_3\ (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \cdot \mathbf{l} + H_4\ \boldsymbol{\sigma}_1 \cdot \mathbf{m}\ \boldsymbol{\sigma}_2 \cdot \mathbf{m} + H_5\ \boldsymbol{\sigma}_1 \cdot \mathbf{n}\ \boldsymbol{\sigma}_2 \cdot \mathbf{n}]. \end{aligned}$$

For identical nucleons like $p - p$ and $n - n$ scattering, $\langle T \rangle$ has to be symmetric under $1 \leftrightarrow 2$; hence $H'_3 = 0$. Show that

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1 \cdot \mathbf{l}\ \boldsymbol{\sigma}_2 \cdot \mathbf{l} + \boldsymbol{\sigma}_1 \cdot \mathbf{m}\ \boldsymbol{\sigma}_2 \cdot \mathbf{m} + \boldsymbol{\sigma}_1 \cdot \mathbf{n}\ \boldsymbol{\sigma}_2 \cdot \mathbf{n}.$$

By eliminating $\boldsymbol{\sigma}_1\cdot\mathbf{n}\ \boldsymbol{\sigma}_2\cdot\mathbf{n}$, using the above relation, express $\langle T\rangle$ as

$$\begin{aligned}\langle T\rangle &= \left(\frac{m}{E}\right)[G_1 + G_2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + G_3\, \boldsymbol{\sigma}_1\cdot\mathbf{m}\ \boldsymbol{\sigma}_2\cdot\mathbf{m} \\ &\quad + G_4\, i\,(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{l} + G_5\,\frac{1}{2}\,(\boldsymbol{\sigma}_1\cdot\mathbf{l}\ \boldsymbol{\sigma}_2\cdot\mathbf{l} + \boldsymbol{\sigma}_2\cdot\mathbf{l}\ \boldsymbol{\sigma}_1\cdot\mathbf{l}) \\ &\quad + G_6\, i\,(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{l}]\,.\end{aligned}$$

Using

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\,\mathbf{q}\cdot\mathbf{r}}\, G\left(\mathbf{q}^2\right) d^3q = V(r)\,,$$

show the most general form of 2-nucleon potential can be written in the form

$$V_{12} = V_c + V_s\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + V_T\, S_{12} + V_{LS}\, \boldsymbol{\sigma}\cdot\mathbf{L} + V_5\, Q_{12} + V_6\, (\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{L}$$

where

$$\begin{aligned}S_{12} &= 3\, \boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\ \boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}} - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \\ Q_{12} &= \frac{1}{2}\,[\boldsymbol{\sigma}_1\cdot\mathbf{L}\ \boldsymbol{\sigma}_2\cdot\mathbf{L} + \boldsymbol{\sigma}_2\cdot\mathbf{L}\ \boldsymbol{\sigma}_1\cdot\mathbf{L}] \\ \boldsymbol{\sigma} &= \boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2 = 2\mathbf{S} \\ \mathbf{L} &= (\mathbf{r}\times\mathbf{p})\,.\end{aligned}$$

Problem 2: Consider the elastic scattering $a + b \to a + b$, where a is spin half particle and b is spinless. For this case $J = L \pm 1/2$. Expressing the two independent amplitudes $F^J_{1/2\ 1/2}$ and $F^J_{1/2,\ -1/2}$ as

$$F^J_{1/2\ \pm 1/2} = \frac{1}{2}\left(f_{L_+} \pm f_{(L+1)_-}\right),$$

where $L_\pm$ correspond to $J = L \pm 1/2$, and then using Eq. (123), show that

$$\mathrm{Im}\, f_{L\pm} = p\,\left|f_{L\pm}\right|^2.$$

Hence one can write

$$f_{L+} = \frac{1}{p}\, e^{i\delta_{L+}}\sin\delta_{L+}$$

$$f_{L_-} = \frac{1}{p}\, e^{i\delta_{L-}} \sin \delta_{L_-} .$$

The scattering matrix [cf. Eq. (45)] can be written as

$$F_{M'M}\,(\theta,\phi) = \frac{4\pi s^{1/2}}{m_a}\;\chi^{\dagger}_{M'}\;\left[f + ig\;\boldsymbol{\sigma}\cdot\mathbf{n}\right]\chi_{\mu}$$

where $\mathbf{n} = \frac{\mathbf{p}\,\times\,\mathbf{p}'}{|\mathbf{p}\,\times\,\mathbf{p}'|}$. If $\mathbf{p}$ is along z-axis, then

$$\chi_{M'} = e^{-i\theta\;\sigma\cdot\mathbf{n}}\;\chi_{M}, \qquad \mathbf{n} = (-\sin\phi, \cos\phi, 0)\,.$$

Using the relations (where the prime denotes differentiation with respect to $\cos\theta$),

$$\begin{aligned} d^{J}_{1/2\;1/2}\;(\theta,\phi) &= d^{J}_{1/2\;1/2} = \frac{1}{J+1/2}\cos\frac{\theta}{2}\left(P'_{J+1/2} - P'_{J-1/2}\right) \\ &= d^{J}_{-1/2\;-1/2}\;(\theta) \end{aligned}$$

$$\begin{aligned} d^{J}_{1/2\;-1/2}\;(\theta,\phi) &= e^{-i\phi}\;d^{J}_{1/2\;-1/2}\;(\theta)] \\ &= \frac{e^{-i\phi}}{J+1/2}\sin\frac{\theta}{2}\left(P'_{J+1/2} + P'_{J-1/2}\right) \\ d^{J}_{-1/2\;1/2}\;(\theta,\phi) &= e^{-i\phi}\;d^{J}_{1/2\;-1/2}\;(\theta) \end{aligned}$$

show that

$$\begin{aligned} f\,(\theta) &= \frac{m_a}{4\pi s^{1/2}}\left[F_{1/2\;1/2}\;\cos\frac{\theta}{2} + e^{i\phi}\;F_{1/2\;-1/2}\;\sin\frac{\theta}{2}\right] \\ g\,(\theta) &= \frac{m_a}{4\pi s^{1/2}}\left[e^{i\phi}\;F_{1/2\;-1/2}\;\cos\frac{\theta}{2} - F_{1/2\;1/2}\;\sin\frac{\theta}{2}\right]. \end{aligned}$$

Now, using Eqs. (120) and (121), show that

$$f\,(\theta) = \sum_{L=0}^{\infty}\left[(L+1)\;f_{L_+} + Lf_{L_-}\right]P_L\,(\cos\theta)$$

$$
\begin{aligned}
g(\theta) &= \sum_{L=0}^{\infty} \left[\left(f_{L_+} - L f_{L_-} \right) \sin\theta \; P_L'(\cos\theta) \right] \\
\frac{d\sigma}{d\Omega} &= \frac{m_{\mathrm{a}}^2}{16\pi^2 s} \frac{1}{2} \sum_{MM'} |F_{MM'}|^2 = |f|^2 + |g|^2 .
\end{aligned}
$$

3.9 Bibliography

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Chapter 4
INTERNAL SYMMETRIES

Hadrons found in nature are not fundamental constituents of matter. There are hundreds of them. They can be divided into two classes: (a) baryons: they are fermions with half integer spin i.e. $J = 3/2, 1/2$; (b) mesons: they are bosons with integral spin i.e. $J = 0, 1, 2$. Some of the low lying mesons with $J^P = 0^-$and $J^P = 1^-$ are shown in Figs. 1a, 1b. Low lying baryons with $J^P = 1/2^+, 3/2^+$are shown in Figs. 2a, 2b. Hadrons with the same J^P are distinguished from each other by some internal quantum numbers. The assignment of these quantum numbers is meaningful, since these quantum numbers are additively conserved in hadronic interactions.

4.1 Selection Rules and Globally Conserved Quantum Numbers

A particle would decay into two or more lighter ones if the decay is allowed by energy-momentum conservation. The reason is that the entropy $S = k_B \ln$ (phase space). Since phase space for the lightest particles is largest and the entropy S tends to increase, the system tends to decay into the lightest particles, unless there is some selection rule to forbid that decay. But we know that certain decays, although allowed by energy-momentum and angular momentum conservation, do not take place. Thus there must be selection rules or conservation laws which forbid these decays.

We now list these "global" conservation laws:

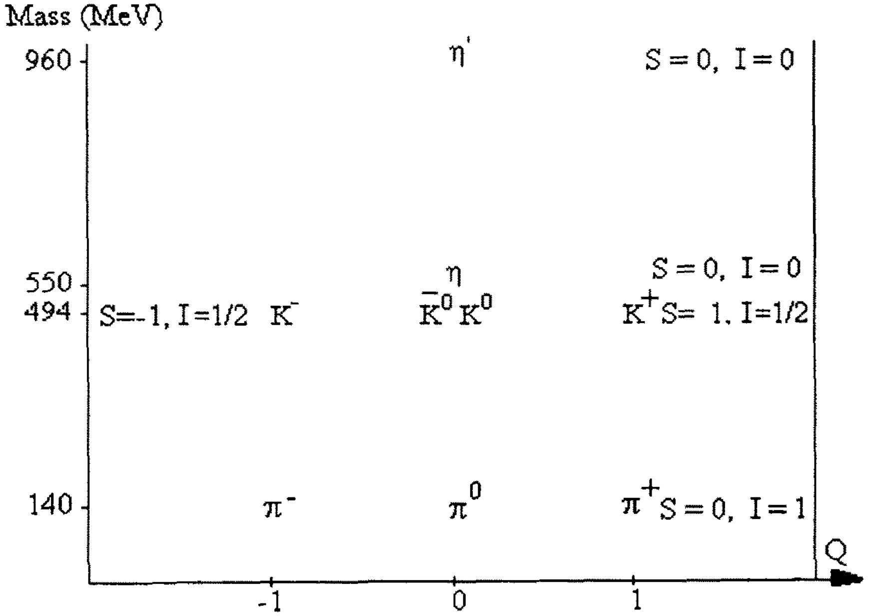

Figure 1 Lowest lying pseudoscalar mesons ($J^P = 0^-$).

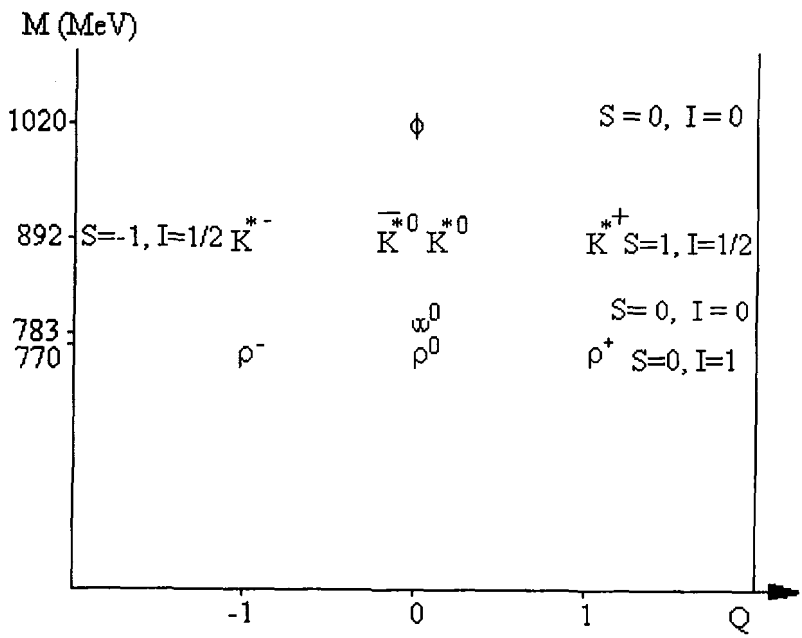

Figure 2 Lowest lying vector mesons ($J^P = 1^-$).

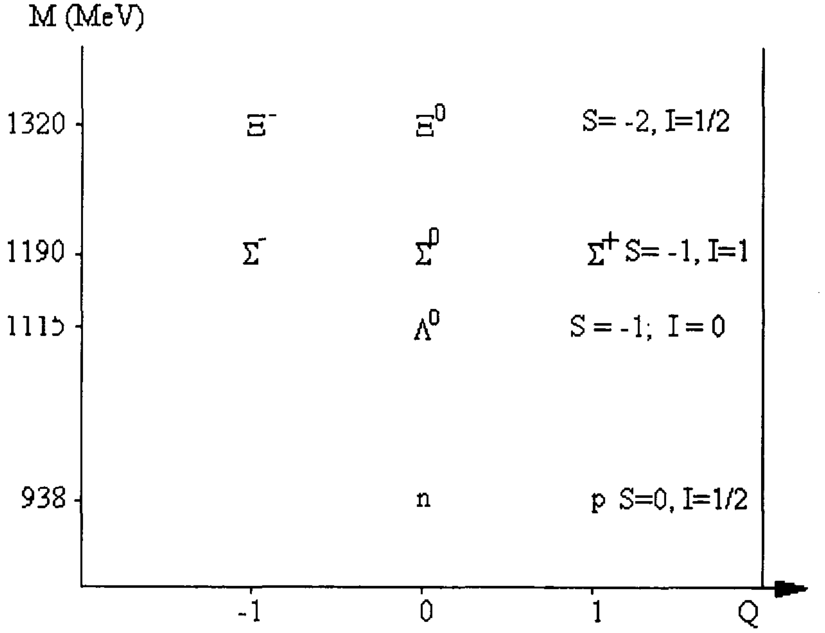

Figure 3 Lowest lying $1/2^+$ baryons.

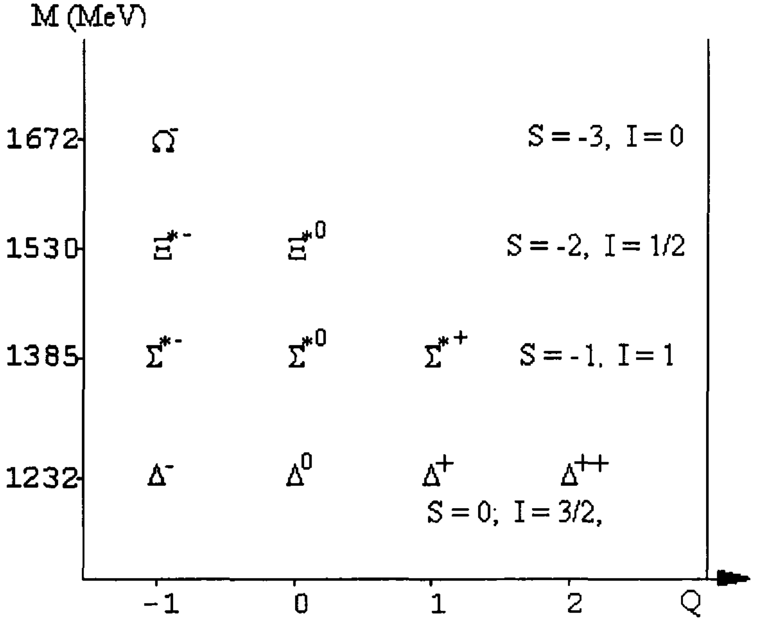

Figure 4 Lowest lying $3/2^+$ baryons.

(i) Electric charge conservation: The decay $e^- \to \nu + \gamma$ is not seen ($\tau_e > 4.3 \times 10^{23}$ years). This is a consequence of electric charge conservation: "Electric charge is additively conserved in any process". This in turn is a consequence of the invariance of Hamiltonian under the global gauge transformation: $U_Q(1)$:

$$|\Psi\rangle \to e^{i\hat{Q}\Lambda} \ |\Psi\rangle \tag{4.1a}$$

so that

$$\left[\hat{Q},\ H\right] = 0. \tag{4.1b}$$

The electric charge $\hat{Q}$ is a generator of $U_Q(1)$ global gauge group. If Λ is a function of space-time viz. $\Lambda = \Lambda(\mathbf{r}, t)$, then the gauge transformation is called local. Actually, electric charge has a dual rule; it is also a generator of the local gauge group, $U = e^{i\hat{Q}\Lambda(\mathbf{r},t)}$. It is a feature of local gauge group that corresponding to this transformation, there is a vector field A_μ coupled to the matter field Ψ, with a universal coupling whose strength is just the electric charge of the particle represented by the field Ψ. None of the other quantum numbers has this feature.

A closely related concept is the quantization of the electric charge, which at particle level is expressed as

$$q = N_q \ e \tag{4.2}$$

i.e. the electric charge q of any hadron or lepton is an integral multiple of elementary charge e. In particular $N_n = 0$ [$q_n = (-0.4 \pm 1.1) \times 10^{-21}\ e$] and $N_e + N_p = 0$ [$|(q_e + q_p)| < 1.0 \times 10^{-21}\ e$].

(ii) Baryon charge conservation:

The following decays

$$\begin{aligned} p &\to e^+ + \gamma \\ p &\to e^+ + \pi^0 \end{aligned}$$

although allowed by electric charge conservation are not seen experimentally ($\tau_p > 10^{32}$ years). This can be understood, if we assign a baryon charge B as follows:

$$B = \begin{cases} +1 \text{ for baryons} \\ -1 \text{ for antibaryons} \\ 0 \text{ for leptons and mesons} \end{cases} \tag{4.3}$$

and demand that B be additively conserved in any reaction

$$\Delta B = B_f - B_i = 0. \tag{4.4}$$

The corresponding global gauge transformation under which the Hamiltonian is invariant is given by

$$|\Psi\rangle \to e^{i\hat{B}\Lambda} \; |\Psi\rangle \,. \tag{4.5}$$

(iii) Lepton charge conservation:

Some decay modes of leptons are not seen. The absence of these decay modes is a consequence of non-conservation of lepton charge which is assigned as follows:

$$L = \begin{cases} +1 \text{ for leptons} \\ -1 \text{ for antileptons} \\ 0 \text{ for all other particles.} \end{cases} \tag{4.6}$$

Any reaction in which L is additively conserved ($\Delta L = 0$) is allowed; oherwise it is forbidden. Some examples are given below:

	n	$\to$	p	$+$	e^-	$+$	$\bar{\nu}_e$		Allowed
L	0		0		1		-1		$\Delta L = L_f - L_i = 0$
	$\bar{\nu}_e$	$+$	(Z, A)	$\to$	$(Z+1, A)$	$+$	e^-		not Allowed
L	-1		0		0		1		$\Delta L = 2$
	$\bar{\nu}_e$	$+$	(Z, A)	$\to$	$(Z-1, A)$	$+$	e^+		Allowed
L	-1		0		0		-1		$\Delta L = 0$

Further, the reaction [antineutrinos obtained from the decay of pile neutrons in a fission reactor $(n \to p + e^- + \bar{\nu}_e)$]

$$\bar{\nu}_e +^{37} Cl \to e^- +^{37} Ar$$

for which $\Delta L = 2$ is not seen, but the reaction using solar neutrinos,

$$\nu_e +^{37} Cl \to e^- +^{37} Ar$$

has been seen and is allowed by lepton charge conservation. Also the allowed reaction

$$\bar{\nu}_e + p \to e^+ + n$$

has been observed with expected cross section. The global gauge transformation, under which the Hamiltonian is invariant is given by

$$|\Psi\rangle \to e^{i\hat{L}\Lambda} \ |\Psi\rangle \,. \tag{4.7}$$

It was later discovered that the neutrino produced in the decay $\pi^+ \to \mu^+ \ \nu$ was not the same as ν_e since if it were so, a reaction of the type

$$\nu + (Z, A) \to (Z+1, \ A) + e^-$$

would have been observed. Instead what was observed was μ^- replacing e^-. This clearly shows that the neutrino accompanying μ^+ in π^+ decay is different from ν_e and is denoted by ν_μ. The muon number defined as

$$L_\mu = \begin{cases} +1 \text{ for } \mu^-, \ \nu_\mu \\ -1 \text{ for } \mu^+, \ \bar{\nu}_\mu \\ 0 \text{ for all other particles.} \end{cases}$$

is conserved in processes involving $\mu^\pm, \nu_\mu, \ \bar{\nu}_\mu$.

(iv) Strangeness and Hypercharge:

It is clear from Figs. 1–4, that hadrons with the same spin and parity occur in nature as multiplets. Consider, for example, $J^P = 0^-$ mesons. We distinguish the triplet of pions $(\pi^\pm, \pi^0)$, the doublets (K^+, K^0) and $\left(\bar{K}^0, K^-\right)$ by assigning a new quantum number, called strangeness: $S(\pi) = 0$, $S(K) = +1$ and $S\left(\bar{K}\right) = -1$. The singlets η and η' have strangeness $S = 0$. Similarly the baryons with $J^P = 1/2^+$ are assigned the strangeness quantum number as follows: For the doublet (p, n), $S = 0$, for the triplet $(\Sigma^{\pm}, \Sigma^0)$, $S = -1$, for the singlet (Λ^0), $S = -1$, and for the doublet (Ξ^0, Ξ^-), $S = -2$. Sometimes, it is convenient to write $Y = B + S$, where Y is called the hypercharge.

The quantum number S is additively conserved in hadronic interactions. In any process, involving hadronic interactions, ΔS must be zero. This immediately leads to the result that in hadronic collisions, the strange particles are produced in pairs:

$\pi^- + p$	$\rightarrow K^0 + \Lambda^0$	$\Delta S = 0$	(4.8)
	$\not\rightarrow K^- + \Sigma^+$	$\Delta S = -2$	
	$\not\rightarrow K^- + p$	$\Delta S = -1$	
	$\rightarrow n + K^+ + K^-$	$\Delta S = 0.$	

Experimentally, only the first and the last reactions are seen and the cross section for these reactions is typical of strong interactions. On the other hand, strange particles decay into ordinary particles by weak interactions:

$$\begin{aligned} \Lambda &\rightarrow p\,\pi^- \\ K^0 &\rightarrow \pi^+\pi^- . \end{aligned}$$

These decays have lifetimes of the order 10^{-10} seconds, characteristics of weak interactions. Thus strangeness is not conserved in weak interactions.

In strong interactions, since both quantum numbers B and S are conserved, it is clear that hypercharge is also conserved. The gauge transformation under which the Hamiltonian is invariant is given by

$$|\Psi\rangle \rightarrow e^{i\hat{Y}\Lambda}\ |\Psi\rangle\,. \tag{4.9}$$

It is interesting to note that the hypercharge of a multiplet is just equal to twice the average charge of that multiplet i.e.

$$Y = 2\ \langle Q\rangle = 2\ \langle q/e\rangle\,. \tag{4.10}$$

For example for the triplet of pions $(\pi^{\pm}, \pi^0)$, $\langle Q \rangle = 0$ and $Y = 0$, for the doublet (p, n), $\langle Q \rangle = 1/2$ and $Y = 1$, whereas for the doublet $(\bar{K}^0, K^-)$ or (Ξ^0, Ξ^-), $\langle Q \rangle = 1/2$ and $Y = -1$.

It is tempting to assign another quantum number, called isospin to each multiplet. For example we can assign $I = 1, I_3 = +1, 0, -1$ to the triplet of pions (π^+, π^0, π^-) and $I = 1/2$, $I_3 = 1/2$ and $-1/2$ to the doublet (p, n). We will discuss isospin in the next section. Here we summarize the conservation laws for internal quantum numbers Q, B, S and I.

Interactions

Quantum Number	Hadronic	Electromagnetic	Weak
Q	Yes	Yes	Yes
B	Yes	Yes	Yes
S or Y	Yes	Yes	No
Isospin	Yes	No	No

4.2 Isospin

We now introduce isospin. From Figs. 1 and 2, it is apparent that particles occur in nature as multiplets. In analogy with ordinary spin, we can regard proton and neutron as an isospin doublet (nucleon) $N = \begin{pmatrix} p \\ n \end{pmatrix}$, with $I = 1/2$ and $I_3 = \pm 1/2$.

The concept of isospin is meaningful only if in hadronic interactions isospin is conserved. This is indeed the case. Experiments on nucleon-nucleon scattering show that after subtracting the effect of Coulomb force in pp scattering, pp, np and nn hadronic forces are equal in strength and have the same range. That is nuclear forces do not depend on the charge of the particle and are thus charge independent. It is now known that all hadronic forces, not just the one between nucleons are charge independent.

The two states of nucleon N viz. p and n will have similar properties as far as hadronic forces are concerned. Without electromagnetic interaction, proton and neutron will have the same mass, but its presence makes their masses slightly different. This is supported by the fact that $(m_p - m_n) = -1.2$ MeV only i.e. about 0.1% of m_p.

Like ordinary angular momentum, we introduce a quantity isospin $\mathbf{I} \equiv (I_1, I_2, I_3)$ in isospin space. The operator $\hat{\mathbf{I}}$ satisfies the commutation relations of angular momentum J viz.

$$\left[\hat{I}_i,\ \hat{I}_j\right] = i\varepsilon_{ijk}\ \hat{I}_j, i = 1, 2, 3. \tag{4.11}$$

As a consequence of these commutation relations, it is possible to find a complete set of simultaneous eigenstates $|I\ I_3\rangle$ of $\hat{\mathbf{I}}^2$, and $\hat{I}_3$ with eigenvalues $I(I+1)$ and I_3:

$$\hat{\mathbf{I}}^2\ |I\ I_3\rangle = I(I+1)\ |I\ I_3\rangle \tag{4.12a}$$
$$\hat{I}_3\ |I\ I_3\rangle = I_3\ |I\ I_3\rangle\,. \tag{4.12b}$$

$\hat{I}_3$ has $(2I+1)$ eigenvalues

$$-I, \cdots, +I. \tag{4.13a}$$

The possible eigenvalues of I are

$$I = 0, 1/2, 1, 3/2, 2, \cdots. \tag{4.13b}$$

Thus, all the multiplets in Figs. 1 and 2 belong to an irreducible representation of the isospin group i.e. they have any of the possible eigenvalues of I given in Eq. (13b). For example, the proton and neutron states can be written as far as the isospin is concerned as

$$\begin{aligned} |p\rangle &= |1/2\ 1/2\rangle \\ |n\rangle &= |1/2\ -1/2\rangle\,, \end{aligned} \tag{4.14}$$

and the pions can be represented as

$$\begin{aligned} \left|\pi^+\right\rangle &= |1\ 1\rangle \\ \left|\pi^0\right\rangle &= |1\ 0\rangle \\ \left|\pi^-\right\rangle &= |1\ -1\rangle\,. \end{aligned} \tag{4.15}$$

The charge of a state is given by the relation

$$Q = \left(\frac{q}{e}\right) = I_3 + \langle Q \rangle = I_3 + \frac{1}{2}Y. \tag{4.16}$$

This is called the Gell-Mann-Nishijima relation.

Charge independence of hadronic force implies that this force does not distinguish any direction in isospin space that is to say that hadronic interactions are invariant under a rotation in isospin space in complete analogy with ordinary angular momentum. This means that the S-matrix or the hadronic part of the Hamiltonian H_h commutes with the rotation operator

$$U_I = e^{-i\boldsymbol{\alpha}.\hat{\mathbf{I}}}, \qquad \boldsymbol{\alpha} = \omega\ \mathbf{n} \tag{4.17}$$

in isospin space i.e.

$$[S,\ U_I] = 0, \text{ or } [H_h,\ U_I] \quad = 0. \tag{4.18}$$

$\hat{\mathbf{I}}$ is the generator of a rotation group in the isospin space. For an infinitesimal rotation

$$U_I = 1 - i\boldsymbol{\alpha}.\ \hat{\mathbf{I}}. \tag{4.19}$$

Hence, we have

$$\left[S,\ \hat{\mathbf{I}}\right] = 0, \text{ or } \left[H_h,\ \hat{\mathbf{I}}\right] = 0, \tag{4.20}$$

i.e. isospin is conserved in any process involving hadronic interactions. Thus we have the selection rules

$$\Delta\ |I|^2 = 0, \qquad \Delta\ I_3 = 0. \tag{4.21}$$

Since in the absence of electromagnetic interaction, the mass Hamiltonian H_M commutes with $\hat{\mathbf{I}}$, the eigenstates of H_M with the same I, i.e. $(2I+1)$ states with different values of I_3 , are degenerate in mass.

As an illustration of isospin conservation, we consider the $\pi - N$ scattering.

$$\begin{aligned} \pi^+ p &\rightarrow \pi^+ p \\ \pi^- p &\rightarrow \pi^- p \\ &\rightarrow \pi^0 n. \end{aligned}$$

We can write

$$\begin{aligned} \left|\pi^+ \, p\right\rangle &= |1\ 1\rangle\ |1/2\ 1/2\rangle = |1\ 1/2\ 1\ 1/2\rangle \\ \left|\pi^- \, p\right\rangle &= |1\ 1/2\ -1\ 1/2\rangle \\ \left|\pi^0 \, n\right\rangle &= |1\ 1/2\ 0\ -1/2\rangle\,. \end{aligned} \tag{4.22}$$

Now the scattering amplitude F is given by

$$\begin{aligned} &\left\langle \pi^- \, p \,|F|\, \pi^- \, p\right\rangle \\ = &\sum_{II_3}\sum_{I'I'_3} \left\langle \pi^- \, p\right| I'\ I'_3\ 1\ 1/2\rangle \\ &\times \langle I'\ I'_3\ 1\ 1/2\,|F|\, I\ I_3\ 1\ 1/2\rangle\ \langle I\ I_3\ 1\ 1/2 \left|\pi^- \, p\right\rangle \\ = &\sum_{II_3}\sum_{I'I'_3} \left\langle \pi^- \, p\right| I'\ I'_3\ 1\ 1/2\rangle\ F_I\ \delta_{II'}\ \delta_{I_3 I'_3}\ \langle I\ I_3\ 1\ 1/2 \left|\pi^- \, p\right\rangle \\ = &\sum_{I} \left\langle \pi^- \, p\right| I\ -1/2\ 1\ 1/2\rangle\ F_I\ \langle I\ -1/2\ 1\ 1/2 \left|\pi^- \, p\right\rangle. \end{aligned} \tag{4.23}$$

Using the Clebsch-Gordon coefficients, we have

$$\left\langle \pi^- \, p \,|F|\, \pi^- \, p\right\rangle = \frac{1}{3}\ F_{\frac{3}{2}} + \frac{2}{3}\ F_{\frac{1}{2}}. \tag{4.24a}$$

Similarly, we get.

$$\left\langle \pi^0 \, n \,|F|\, \pi^- \, p\right\rangle = \frac{\sqrt{2}}{3}\ F_{\frac{3}{2}} - \frac{\sqrt{2}}{3}\ F_{\frac{1}{2}} \tag{4.24b}$$

$$\left\langle \pi^+ \, p \,|F|\, \pi^+ \, p\right\rangle = F_{\frac{3}{2}}. \tag{4.24c}$$

Without using isospin invariance we have three independent amplitudes. With its use we have only two independent amplitudes. Thus

$$\begin{aligned}
\sigma_{\pi^+} &= \rho\left|F_{\frac{3}{2}}\right|^2 = \sigma^{(+)} &\quad (4.25a)\\
\sigma_{\pi^-} &\equiv \left[\sigma\left(\pi^- \, p \to \pi^- \, p\right) + \sigma\,\left(\pi^- \, p \to \pi^0 \, n\right)\right] \equiv \sigma^{(-)} + \sigma^{(0)} \\
&= \rho\left[\frac{1}{3}\left|F_{\frac{3}{2}}\right|^2 + \frac{2}{3}\left|F_{\frac{1}{2}}\right|^2\right]. &\quad (4.25b)
\end{aligned}$$

Here ρ is the kinematical factor. If $F_{3/2} \gg F_{1/2}$, then from Eq. (24)

$$\sigma^{(+)} : \sigma^{(-)} : \sigma^{(0)} = 9 : 1 : 2.$$

Experimentally, the cross-sections are in the ratio $(122\pm8) : (12.8\pm 1.10) : (25.6 \pm 1.3)$ for the kinetic energy of the pion from 120 MeV to 300 MeV. Thus it is clear that the scattering takes place predominantly in the $I = 3/2$ state for the above energy range.

Finally, we note that since the electric charge is always conserved, the conservation of I_3 implies Y-conservation and vice versa. To summarize, for hadronic interactions

$$\begin{aligned}
\Delta|\mathbf{I}|^2 &= 0 \\
\Delta(Q,\ B,\ Y) &= 0. &\quad (4.26)
\end{aligned}$$

4.2.1 Electromagnetic interaction and isospin

Because of Eq. (16), electromagnetic interaction breaks the rotational symmetry in the isospin space:

$$\left[H_{em},\ \hat{I}\right] \neq 0 \qquad (4.27a)$$

but

$$\left[H_{em},\ \hat{I}_3\right] = 0. \qquad (4.27b)$$

Hence H_{em} is invariant under an isospin rotation about the 3rd axis i.e. I_3 is still conserved by the electromagnetic interaction.

We can say that the isospin symmetry is broken by the electromagnetic interaction and a small mass difference between the members of an isospin multiplet may arise due to the electromagnetic interaction. Since

$$\left[H_{em},\ \hat{Q}\right] = 0, \tag{4.28}$$

therefore, it follows from Eq. (27b) that

$$\left[H_{em},\ \hat{Y}\right] = 0. \tag{4.29}$$

Hence for electromagnetic interaction, we have the selection rules:

$$\Delta I_3 = 0, \qquad \Delta Y = 0, \qquad \Delta B = 0 \tag{4.30a}$$

but

$$\Delta|\mathbf{I}|^2 \neq 0. \tag{4.30b}$$

4.2.2 Weak interaction and isospin

Consider the weak processes

$$\begin{aligned} \Lambda &\to p + \pi^- \\ n &\to p + e^- + \bar{\nu}_e. \end{aligned}$$

Clearly I_3 is not conserved in weak interactions and hence $\mathbf{I}^2$ is also not conserved. It follows that Y is also not conserved, since Q is conserved. Thus for weak interactions, we have the selection rules:

$$\Delta|\mathbf{I}|^2 \neq 0, \qquad \Delta Y \neq 0, \qquad \Delta B = 0. \tag{4.31}$$

4.3 Resonance Production

We now consider the reaction shown in Fig. 5. We have three particles in the final state, produced incoherently. Let us consider the pair of particles $(n\pi^+)$, $(n\pi^-)$ and $(\pi^+\ \pi^-)$. We define the invariant mass of each system designated by (12), (13) and (23):

$$\begin{aligned} s_{12} &= (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 && (4.32a) \\ s_{13} &= (E_1 + E_3)^2 - (\mathbf{p}_1 + \mathbf{p}_3)^2 && (4.32b) \\ s_{23} &= (E_2 + E_3)^2 - (\mathbf{p}_2 + \mathbf{p}_3)^2 && (4.32c) \end{aligned}$$

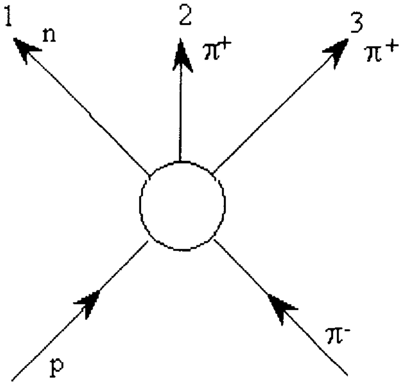

Figure 5 The reaction $\pi^- p \to n\pi^+\pi^-$, $\pi^- p \to p\pi^0\pi^-$.

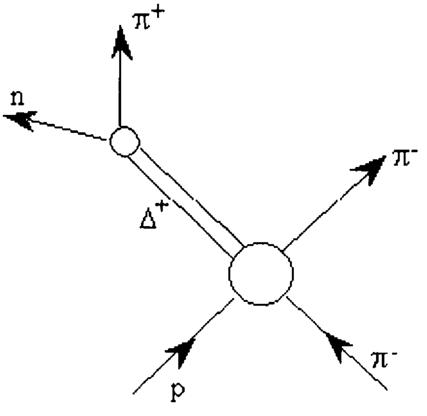

Figure 6 The pion production through resonance $\pi^- p \to \Delta^+\pi^- \to n\pi^+\pi^-$.

If the reaction proceeds as in Fig. 5, n, π^+ and π^- will have energy and momentum statistically distributed. The number of $(n\ \pi^+)$ pairs with an invariant mass $\sqrt{s_{12}}$, $N(s_{12})$ can also be calculated. $N(s_{12})$ can be plotted as a function of $\sqrt{s_{12}}$ and the result is called a phase space spectrum as shown in Fig. 7. If the reaction takes place as shown in Fig. 6, i.e. with π^+'s strongly correlated with the n's, then energy-momentum conservation demands

$$\begin{aligned} E_\Delta &= E_1 + E_2 \\ \mathbf{p}_\Delta &= \mathbf{p}_1 + \mathbf{p}_2 \\ m_\Delta &= \left[E_\Delta^2 - p_\Delta^2\right]^{1/2} = \sqrt{s_{12}}. \end{aligned} \tag{4.33}$$

In this case the final $n\ \pi^+$ results from the decay of a quasi-stable particle Δ^+, called a resonance. In this situation, $N(s_{12})$ shows a strong peak at $\sqrt{s_{12}} = m_\Delta$ (Fig. 7). The finite width of the peak shows that the particle is very short lived, the life time $\tau = \frac{1}{\Gamma}$, Γ being the width of the resonance. Actually a broad peak is seen experimentally at $\sqrt{s_{12}} = m_\Delta = 1238$ MeV with the full width at half maximum $\Gamma_\Delta \approx 120$ MeV.

Similarly, if we consider the pair $n\ \pi^-$, one finds a peak due to Δ^-. The $(\pi^+\pi^-)$ invariant mass distribution, $N(s_{23})$, also shows a broad peak at about $\sqrt{s_{23}} = 750$ MeV, due to the ρ^0 resonance.

Δ-resonance

We now discuss the quantum numbers of the Δ-resonance. We first determine its isospin. The resonance Δ is seen both in $\pi^- p$ and $\pi^+ p$ scattering. Since for $\pi^+ p$, $I = 3/2$ is the only possibility, it follows that its isospin must be 3/2. This is confirmed in the $\pi^+ p$ and $\pi^- p$ scattering experiments at energies at which multiple mesons production is insignificant viz. the processes:

$$\begin{aligned} \pi^+ p &\to \pi^+ p \\ \pi^- p &\to \pi^- p \\ &\to \pi^0 n. \end{aligned}$$

If the $I = 3/2$ channel dominates in the above processes, we then

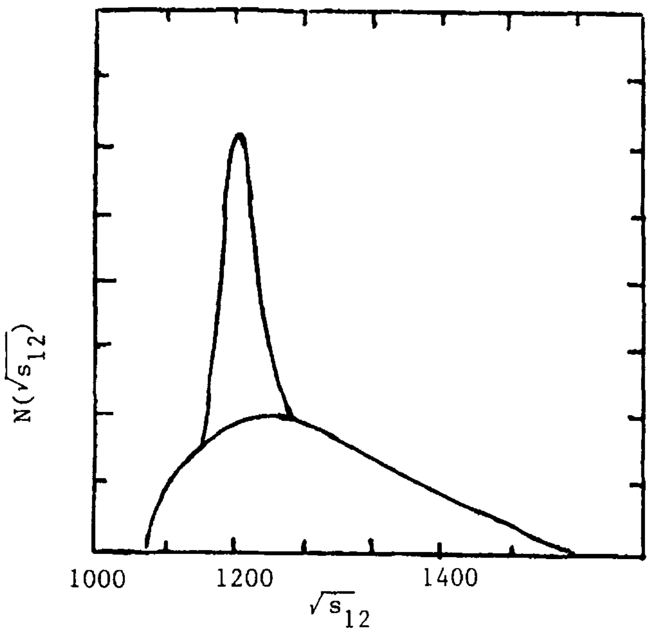

Figure 7 Phase space plot for $(n\pi^+)$ pairs.

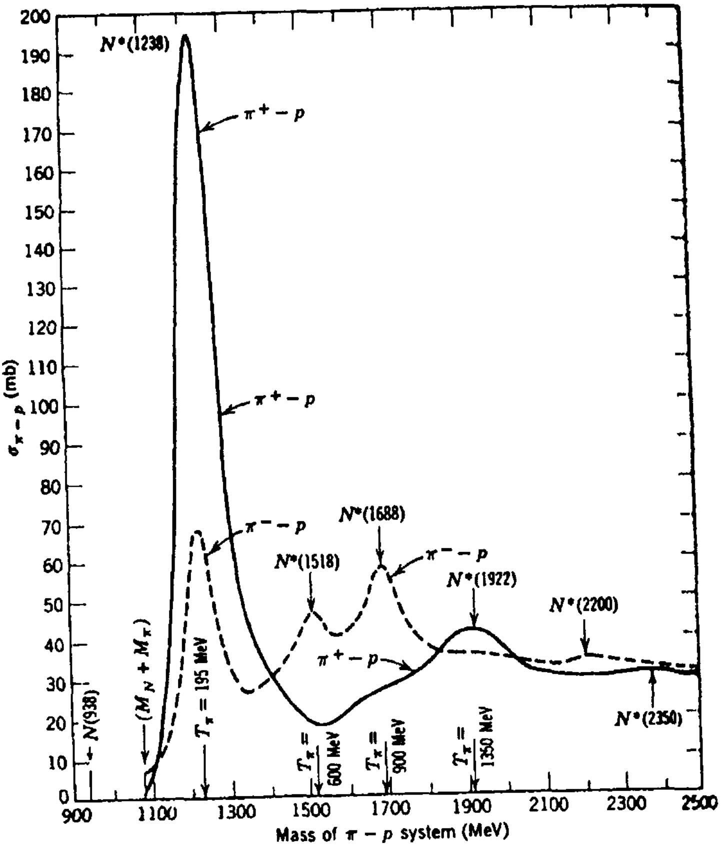

Figure 8 The resonance scattering for π^+p and π^-p channels.

have from Eq. (25) $\sigma_{\pi^+}/\sigma_{\pi^-} = 3$, at the resonance energy. This is what is borne out experimentally, showing unambiguously that the resonance channel is $I = 3/2$ (see Fig. 8).

Spin of Δ

We first consider two-body scattering

$$a + b \to R \to a' + b' \tag{4.34}$$

through a resonance R. Suppose the spin of R is J. Consider the decay

$$R \to a + b.$$

Let $\mathbf{p}$ be the momentum in the center of mass frame of particles a and b. Let λ_1 and λ_2 be their helicities. Now $|\mathbf{p}| = p$ and its direction is given by $\omega \equiv (\theta, \phi)$. We can write the helicity state [cf. Eq. (3.105)]

$$|\lambda_1\ \lambda_2\ \omega\rangle = \sum_{J'M'} \sqrt{\frac{2\ J' + 1}{4\pi}} d^{J'}_{M'\lambda}\ (\theta, \phi)\, |J'M', \lambda\rangle\,, \tag{4.35}$$

where $\lambda = \lambda_1 - \lambda_2$. Therefore, the decay amplitude is given by

$$F_\lambda(\omega) = \sum_{J'M'} \sqrt{\frac{2\ J' + 1}{4\pi}} d^{J^{*\prime}}_{M\lambda}\ \langle J'M', \lambda\, |F|\, JM\rangle\,. \tag{4.36}$$

We now take R and a to be fermions and b a boson. Now

$$\langle J'M', \lambda\, |F|\, JM\rangle = \delta_{JJ'}\ \delta_{MM'}\ F^J_\lambda\ (s)\sqrt{4\pi}. \tag{4.37}$$

Therefore,

$$\begin{aligned} F_\lambda\,(\omega) &= \sqrt{2J+1} d^{J^*}_{M\lambda}\ (\theta, \phi)\ F^J_\lambda\ (s) \\ &= \sqrt{2J+1} e^{i(\lambda - M)\phi}\ F^J_\lambda\ (s)\ d^{J^*}_{M\lambda}(\theta). \end{aligned} \tag{4.38}$$

Now

$$d\Gamma = (2\pi)^4 \int \frac{d^3 p_a}{(2\pi)^3} \frac{d^3 p_b}{(2\pi)^3} \frac{m_R\ m_a}{2E_R\ E_a\ E_b} \overline{\sum_{\text{spin}}} \left|F_\lambda\,(\omega)\right|^2, \tag{4.39a}$$

or

$$\frac{d\Gamma}{d(\cos\theta)} = \frac{4\ m_a}{16\pi\ s^{1/2}}\ |\mathbf{p}| \overline{\sum_{\text{spin}}}\ (2J+1)\ \left|F_\lambda^J\,(s)\right|^2 \left|d_{M\lambda}^J(\theta)\right|^2. \quad (4.39b)$$

Therefore

$$\Gamma = \frac{m_a}{2\pi\ s^{1/2}}\ |\mathbf{p}| \overline{\sum_{\text{spin}}}\ \left|F_\lambda^J\,(s)\right|^2, \quad (4.40)$$

where we have used the orthogonality of d-functions. When R, a and b all are bosons, we get

$$\Gamma = \frac{1}{8\pi\ s}\ |\mathbf{p}| \overline{\sum_{\text{spin}}}\ \left|F_\lambda^J\,(s)\right|^2. \quad (4.41)$$

For a resonance scattering as in Eq. (34), the invariant scattering amplitude is given by

$$F(ab \to R \to a'b') = \sum_M F(ab \to R)\ F(R \to a'b')\ \phi_R(s), \quad (4.42)$$

where $\phi_R(s)$ is the resonance factor. Now using Eq. (38), we have

$$\begin{aligned} F(ab \quad &\to \quad R \to a'b') = \sum_M\ d_{M\lambda}^J(\omega')\ d_{M\lambda'}^{J*}(\omega'')\ (2J+1) \\ \times F_\lambda^J\ (ab \quad &\to \quad R)\ F_{\lambda'}^J\ (R \to a'b')\ \phi_R(s), \end{aligned} \quad (4.43)$$

where $\omega' \equiv (\theta', \phi')$ and $\omega'' \equiv (\theta'', \phi'')$ are the polar and azimuthal angles of particles a and a' with respect to some fixed direction. Using the group property of d-functions

$$\sum_M\ d_{M\lambda}^J(\omega')\ d_{M\lambda'}^{J*}(\omega'') = d_{\lambda\lambda'}^{J*}(\theta, \phi), \quad (4.44)$$

where θ and ϕ are the polar and azimuthal angles of the particle a' relative to a. Hence we have

$$\begin{aligned} F(ab \quad &\to \quad R \to a'b') = (2J+1)\ d_{\lambda\lambda'}^{J*}(\theta, \phi) \\ \times F_\lambda^J\ (ab \quad &\to \quad R)\ F_{\lambda'}^J\ (R \to a'b')\ \phi_R(s). \end{aligned} \quad (4.45)$$

Now comparing it with [cf. Eq. (3.120) for the $J^{\underline{th}}$ partial wave]

$$F_{\lambda'\lambda}(\omega) = \frac{4\pi\sqrt{s}}{\sqrt{m_a\, m_a'}}\ (2J+1)\ F^J_{\lambda'\lambda}(s)\ d^{J*}_{\lambda\lambda'}(\theta,\phi), \tag{4.46}$$

we have

$$F^J_{\lambda'\lambda}(s) = \frac{\sqrt{m_a\, m_a'}}{4\pi\sqrt{s}} F^J_\lambda\ (ab \to R)\ F^J_{\lambda'}\ (R \to a'b')\ \phi_R(s). \tag{4.47}$$

Now the partial wave cross section in the angular momentum state J is given by

$$\sigma_J = 4\pi \frac{2J+1}{(2S_a+1)\ (2S_b+1)} \frac{|\mathbf{p}'|}{|\mathbf{p}|} \sum_{\lambda\lambda'} \left|F^J_{\lambda'\lambda}(s)\right|^2. \tag{4.48}$$

Using

$$\left|F^J_\lambda\ (ab \to R)\right|^2 = \left|F^J_\lambda\ (R \to ab)\right|^2 \tag{4.49}$$

and Eqs. (40), (47) and (48), we get

$$\sigma_J = \frac{4\pi}{|\mathbf{p}|^2} \frac{2J+1}{(2S_a+1)\ (2S_b+1)} \left[\frac{\Gamma\ (R \to ab)\ \Gamma\ (R \to a'b')}{4}\ |\phi_R(s)|^2\right]. \tag{4.50}$$

The resonance factor is given in the Breit-Wigner form:

$$|\phi_R(s)|^2 = \left[\frac{1}{(\sqrt{s}-m_R)^2 + \frac{\Gamma^2}{4}}\right]. \tag{4.51}$$

Hence we have

$$\sigma_J = \frac{\pi}{|\mathbf{p}|^2} \frac{2J+1}{(2S_a+1)\ (2S_b+1)} \left[\frac{\Gamma\ (R \to ab)\ \Gamma\ (R \to a'b')}{(\sqrt{s}-m_R)^2 + \frac{\Gamma^2}{4}}\right]. \tag{4.52}$$

Consider now the process

$$\pi^+ p \to \Delta^{++} \to \pi^+ p. \tag{4.53}$$

From Eq. (52), we get

$$\sigma_J(m_\Delta) = \frac{2\pi}{|\mathbf{p}|^2}\ (2J+1)\,.$$

Experimentally, near the resonance

$$\sigma_J \approx \frac{8\pi}{|\mathbf{p}|^2}, \tag{4.54}$$

giving $J = 3/2$.

It is also possible to determine the spin of a resonance by angular distribution of its decay products. This we illustrate by considering the Δ-resonance viz. $\Delta^{++} \to \pi^+ p$. Take the z-axis along the direction of the nucleon (or pion) in their center of mass frame, so that $l_z^i = 0$ (i-refers to $\pi^+ p$ in the initial state and l refers to orbital angular momentum). Since pion is spinless, $M^i = \pm 1/2$. If J is the spin of Δ-resonance, then $M = \pm 1/2$, by angular momentum conservation. Now from Eq. (39b), the angular distribution of $p\pi^+$ in the final state is given by

$$I(\theta) \propto \sum_{M,\lambda} \left|F_\lambda^J(s)\right|^2 \left|d_{M\lambda}^J(\theta)\right|^2 . \tag{4.55}$$

Thus for $J = 1/2,\ M = +1/2, -1/2,$

$$I(\theta) \propto \left(\left|F_{1/2}^{1/2}(s)\right|^2 + \left|F_{-1/2}^{1/2}(s)\right|^2\right)\left(\left|d_{1/2\ 1/2}^{1/2}(\theta)\right|^2 + \left|d_{1/2\ -1/2}^{1/2}(\theta)\right|^2\right). \tag{4.56}$$

Using [Problem 3.2], we have

$$I(\theta) \propto \left[\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right]. \tag{4.57}$$

Thus the angular distribution is isotropic.

For $J = 3/2$ and $M = \pm 1/2$, we have

$$I(\theta) \propto \left(\left|F_{1/2}^{3/2}(s)\right|^2 + \left|F_{-1/2}^{3/2}(s)\right|^2\right)\left(\left|d_{1/2\ 1/2}^{3/2}(\theta)\right|^2 + \left|d_{1/2\ -1/2}^{3/2}(\theta)\right|^2\right). \tag{4.58}$$

Again using [Problem 3.2], we have

$$I(\theta) \propto \left(1 + 3\cos^2\theta\right). \tag{4.59}$$

We note that $I(-\theta) = I(\theta)$. The observed angular distribution of the protons or the pions at the resonance agrees with the prediction of Eq. (59), showing that $J = 3/2$ for the Δ. The above derivation clearly shows that the angular distribution depends only on the value of J, and not on the parity i.e. orbital angular momentum which never enters in the helicity representation used above.

4.4 Charge Conjugation

It is a general feature of relativistic quantum mechanics that corresponding to a particle, there is an antiparticle which has the same mass and spin as its particle. We treat particle and antiparticle on equal footing. We, therefore, postulate an operator U_c, which changes a particle into its antiparticle. The operator U_c is a unitary operator. Thus, for example

$$U_c \left|\pi^+\right\rangle = \left|\pi^-\right\rangle \tag{4.60a}$$

$$U_c \left|p\right\rangle = \left|\bar{p}\right\rangle. \tag{4.60b}$$

In general, for a charged particle

$$U_c \left|Q,\ \mathbf{p},\ \mathbf{s}\right\rangle = \left|-Q,\ \mathbf{p},\ \mathbf{s}\right\rangle, \tag{4.61}$$

where $|Q,\ \mathbf{p},\ \mathbf{s}\rangle$ represents a single particle state with charge Q, momentum $\mathbf{p}$ and spin $\mathbf{s}$. Now

$$\hat{Q}\ |Q,\ \mathbf{p},\ \mathbf{s}\rangle = Q\ |Q,\ \mathbf{p},\ \mathbf{s}\rangle \tag{4.62a}$$

$$U_c\,\hat{Q}\ |Q,\ \mathbf{p},\ \mathbf{s}\rangle = Q\ |-Q,\ \mathbf{p},\ \mathbf{s}\rangle \tag{4.62b}$$

$$\begin{aligned}\hat{Q}\,U_c\ |Q,\ \mathbf{p},\ \mathbf{s}\rangle &= \hat{Q}\ |-Q,\ \mathbf{p},\ \mathbf{s}\rangle \\ &= -Q\ |-Q,\ \mathbf{p},\ \mathbf{s}\rangle. \end{aligned} \tag{4.62c}$$

Therefore, we have

$$U_c\,\hat{Q} + \hat{Q}\,U_c = 0, \tag{4.63a}$$

$$\left[U_c\,\hat{Q}\right]_+ = 0, \tag{4.63b}$$

i.e. U_c and Q do not commute. Hence it is not possible to find simultaneous eigenstates of U_c and $\hat{Q}$. In general, for any additive internal quantum number, such as Q, I_3, B, Y and L,

$$U_c \; |Q,\ I_3,\ B,\ Y,\ L\rangle = |-Q,\ -I_3,\ -B,\ -Y,\ -L\rangle \tag{4.64}$$

and consequently,

$$[U_c,\ Q_i] \neq 0, \tag{4.65}$$

where

$$Q_i = \hat{I}_3,\ \hat{B},\ \hat{Y},\ \text{or}\ \hat{L}.$$

Now

$$\begin{aligned} U_c \; |B\rangle &= |-B\rangle \\ U_c^2 \; |B\rangle &= U_c \; |-B\rangle = |B\rangle\,. \end{aligned} \tag{4.66}$$

Therefore,

$$U_c^2 = 1 \tag{4.67}$$

and eigenvalues of U_c are ± 1, i.e. U_c is a discrete transformation.

It follows from Eq. (64) that states with $Q \neq 0,\ B \neq 0,\ Y \neq 0$, etc. cannot be eigenstates of U_c. Only states with $Q = 0,\ B = 0,\ Y = 0,\ I_3 = 0$ can be eigenstates of U_c. For them it is possible to define the charge conjugation parity η_c:

$$U_c \; |B = 0\rangle = \eta_c \; |B = 0\rangle\,, \tag{4.68}$$

where

$$\eta_c^2 = 1 \quad \text{or} \quad \eta_c = \pm 1 \tag{4.69}$$

η_c is a multiplicatively conserved quantum number in any process which conserves C-parity. The C-parity is either $+1$ or -1.

Charge conjugation is an internal symmetry. If

$$[U_c,\ H] = 0, \quad \text{or} \quad [U_c,\ S] = 0, \tag{4.70}$$

we say that the corresponding interaction is invariant under charge conjugation U_c. While strong and electromagnetic interactions are invariant under U_c, weak interactions are not

$$[U_c,\ H_{\text{weak}}] \neq 0\,. \tag{4.71}$$

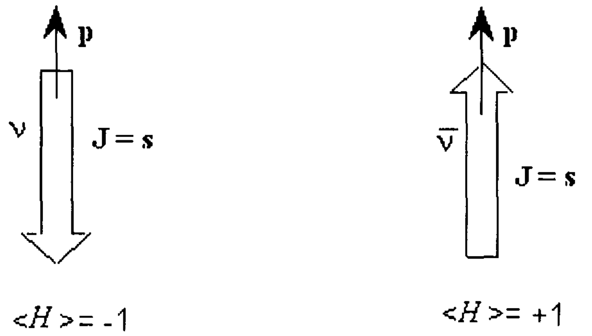

Figure 9 The neutrino with helicity -1 and antineutrino with helicity $+1$.

This is clear from the fact that neutrinos and antineutrinos which come out in β-decay of nuclei have opposite polarizations or helicities $[H = 2\mathbf{s} \cdot \mathbf{p}/ |\mathbf{p}|]$. If charge conjugation were conserved in weak interactions, neutrino and antineutrino would have the same helicity.

How to test charge conjugation in hadronic interactions? Consider for example, the reactions

$$\begin{aligned} \bar{p} + p &\rightarrow \pi^+ + h \\ &\rightarrow \pi^- + \bar{h}, \end{aligned}$$

where h $\left(\bar{h}\right)$ denote all other hadrons with $B = 0$ and with positive (negative) electric charge. Now

$$\begin{aligned} \left\langle \bar{p}\, p \left| S \right| \pi^+\, h \right\rangle &= \left\langle \bar{p}\, p \left| U_c^{-1}\, U_c\, S\, U_c^{-1}\, U_c \right| \pi^+\, h \right\rangle \\ &= \left\langle p\, \bar{p} \left| S \right| \pi^-\, \bar{h} \right\rangle, \end{aligned} \tag{4.72}$$

where we have assumed that S is invariant under U_c:

$$U_c\, S\, U_c^{-1} = S. \tag{4.73}$$

Thus C-invariance requires that positive and negative pions have the same energy spectrum. Comparison of π^+ and π^- distributions show no difference, the result is stated as

$$\left|\frac{C - \text{nonconserving amplitude}}{C - \text{conserving amplitude}}\right| \leq 0.01.$$

As we have discussed, γ, π^0 and η^0 can be eigenstates of U_c. We now determine the C-parity of these states. Now under U_c, the electromagnetic current j_μ^{em}:

$$j_\mu^{em} \xrightarrow{U_c} - j_\mu^{em}. \tag{4.74}$$

But the electromagnetic field A_μ satisfies the equation

$$\Box^2 A_\mu = j_\mu^{em}. \tag{4.75}$$

Thus from Eq. (75), it follows that

$$A_\mu \xrightarrow{U_c} - A_\mu. \tag{4.76}$$

Since a photon is a quantum of electromagnetic field, it follows that the C-parity of photon is -1 viz.

$$\eta_c\ (\gamma) = -1. \tag{4.77}$$

The decays $\pi^0 \rightarrow 2\gamma$ and $\eta^0 \rightarrow 2\gamma$ have been observed. Hence if these reactions proceed via electromagnetic interaction, it then follows from C-conservation that

$$\begin{aligned} \eta_c\left(\pi^0\right) &= +1 \\ \eta_c\left(\eta^0\right) &= +1. \end{aligned} \tag{4.78}$$

Since $\pi^0 \rightarrow 3\gamma$ and $\eta^0 \rightarrow 3\gamma$ can proceed via electromagnetic interaction, but have never been seen, these decays are strictly forbidden due to C-conservation in electromagnetic interaction. We conclude that the electromagnetic interaction is invariant under U_c.

$$[U_c,\ H_{\text{em}}] = 0. \tag{4.79}$$

Consider now the positronium, the bound states of e^- and e^+. Let us consider $e^- - e^+$ in definite $(l,\ s)$ state. Now e^- and e^+ are identical fermions which differ only in their electric charges. We can use a generalized Pauli principle for the positronium viz. "under total exchange of particles (which consists of changing simultaneously Q, $\mathbf{r}$ and s labels), the state should change sign or be antisymmetric". Under exchange of space co-ordinates, we get a factor $(-1)^l$, under spin co-ordinate exchange, we get a factor $(-1)^{s+1}$ ($s = 0$ for spin singlet state and $s = 1$, for spin triplet state), exchange of electric charge gives a factor η_c. We require the state to be antisymmetric, i.e.

$$(-1)^l\,(-1)^{s+1}\,\eta_c = -1 \tag{4.80}$$

or

$$\eta_c = (-1)^{l+s} \tag{4.81}$$

which gives the charge conjugation parity of the positronium in (l, s) state.

The positronium $(e^- - e^+)$ can decay into $n\ \gamma$ by electromagnetic interaction. C-parity conservation gives

$$(-1)^{l+s} = (-1)^n. \tag{4.82}$$

From Eq. (82), we get the following selection rules:

$l = 0 = s$	$^1S_0 \to 2\gamma$	Allowed
	$^1S_0 \to 3\gamma$	strictly forbidden
$l = 0$	$^3S_1 \to 2\gamma$	strictly forbidden
$s = 1$	$^3S_1 \to 3\gamma$	Allowed

Similarly for $(p - \bar{p})$ and quark-antiquark systems: $\eta_c = (-1)^{l+s}$.

Now for $(\pi^+ - \pi^-)$ system for which $B = 0,\ Y = 0,\ Q = 0$, generalized Pauli principle requires that the state should be symmetric (even) under total exchange of pions that is

$$(-1)^l\ \eta_c = 1 \tag{4.83}$$

or

$$\eta_c = (-1)^l. \tag{4.84}$$

Similarly for $\pi^0 - \pi^0$ system we get $\eta_c = (-1)^l$. For this case, since two π^0's are identical particle, ordinary Pauli principle requires that $(-1)^l =$ even i.e. they must be in an orbital state with l even. Thus η_c must be $+1$ for $\pi^0 - \pi^0$ system, whereas η_c depends upon l value for $\pi^+\pi^-$ system.

4.5 G-Parity

For strong interactions, both isospin and C-parity are conserved. For hadrons, it is convenient to define a new operator $\hat{G} =$ charge conjugation $+180^o$ rotation around 2nd axis in isospin space. It follows that strong interactions are invariant under G, but

$$[\hat{G},\ H_{\text{em}}] \neq 0 \tag{4.85a}$$

$$[\hat{G},\ H_{\text{weak}}] \neq 0 \tag{4.85b}$$

i.e. electromagnetic and weak interactions are not invariant under G.

Under 180^o rotation around the 2nd axis in isospin space, we have

$$\begin{aligned} |\pi_1\rangle &\rightarrow -|\pi_1\rangle \\ |\pi_2\rangle &\rightarrow |\pi_2\rangle \\ |\pi_3\rangle &\rightarrow -|\pi_3\rangle\,. \end{aligned} \tag{4.86}$$

Therefore, we get

$$\left|\pi^{\pm,0}\right\rangle \rightarrow -\left|\pi^{\mp,0}\right\rangle. \tag{4.87}$$

Under charge conjugation

$$\left|\pi^{\pm}\right\rangle \xrightarrow{U_{\mathcal{C}}} \left|\pi^{\mp}\right\rangle \tag{4.88a}$$

$$\left|\pi^{0}\right\rangle \xrightarrow{U_{\mathcal{C}}} \left|\pi^{0}\right\rangle . \tag{4.88b}$$

Thus we have

$$\left|\pi^{\pm}\right\rangle \xrightarrow{\hat{G}} -\left|\pi^{\pm}\right\rangle \tag{4.89a}$$

and

$$\left|\pi^{0}\right\rangle \xrightarrow{\hat{G}} -\left|\pi^{0}\right\rangle . \tag{4.89b}$$

Thus the G-parity of pions is $G(\boldsymbol{\pi}) = -1$. The nucleon state $|N\rangle$, under 180^{o} rotation about 2nd axis in isospin space transforms as

$$\begin{aligned} |N_R\rangle &= e^{i\,\tau_2\,\pi/2}\;|N\rangle \\ &= \left(\cos\frac{\pi}{2} + i\;\tau_2 \sin\frac{\pi}{2}\right)\;|N\rangle \\ &= i\;\tau_2\;|N\rangle \end{aligned} \tag{4.90}$$

i.e.

$$\begin{aligned} |p\rangle &\xrightarrow{I_{2(\pi)}} |n\rangle \\ |n\rangle &\xrightarrow{I_{2(\pi)}} -|p\rangle . \end{aligned} \tag{4.91}$$

But

$$\begin{aligned} |p\rangle &\xrightarrow{U_{\mathcal{C}}} |\bar{p}\rangle \\ |n\rangle &\xrightarrow{U_{\mathcal{C}}} |\bar{n}\rangle . \end{aligned} \tag{4.92}$$

Therefore,

$$|p\rangle \xrightarrow{G} |\bar{n}\rangle \tag{4.93a}$$

$$|n\rangle \xrightarrow{G} -|\bar{p}\rangle . \tag{4.93b}$$

Only states with $B = 0$ and $Y = 0$ for which isospin I is integer can be eigenstates of $\hat{G}$. Only for such states we can define G-parity G. In general G-parity of a state with isospin I is given by

$$G = \eta_c\,(-1)^{I} . \tag{4.94}$$

Thus for fermion-antifermion system, the G-parity is given by

$$G = (-1)^{l+s+I} = \eta_c\,(-1)^I. \tag{4.95}$$

For $(\pi^+\pi^-)$ system

$$G = (-1)^l\,(-1)^I = (-1)^{l+I} = 1. \tag{4.96}$$

4.6 Problems

1 Consider pion nucleon scattering

$$\pi^i + N \to \pi^j + N$$

where i and j are isospin indices of the incoming and outgoing pions respectively. Using isospin invariance, show that in isospin space, the scattering amplitude A can be written

$$A_{ji} = \left\{A^{(+)}\,\delta_{ji} + A^{(-)}\frac{1}{2}\,[\tau_j,\tau_i]\right\}.$$

Show that isospin $\frac{3}{2}$ and $\frac{1}{2}$ projection operators are given by

$$P_{3/2} = \frac{2+\mathbf{t}\cdot\boldsymbol{\tau}}{3}, \qquad P_{1/2} = \frac{1-\mathbf{t}\cdot\boldsymbol{\tau}}{3}$$

where $\boldsymbol{\tau}$ are Pauli matrices and $\mathbf{t}$ are isospin matrices for $I = 1$. Further show that

$$\left(P_{3/2}\right)_{ji} = \frac{1}{3}\left\{2\,\delta_{ji} - \frac{1}{2}\,[\tau_j,\tau_i]\right\}$$
$$\left(P_{1/2}\right)_{ji} = \frac{1}{3}\left\{\delta_{ji} + \frac{1}{2}\,[\tau_j,\tau_i]\right\}.$$

Hint: $(t_k)_{ji} = i\varepsilon_{jki}.$

2 Show that for the decay

$$i \to f + \gamma,$$

either $\Delta I = 0$ or $|\Delta I| = 1$.

Hence show that for the decay $\eta \to \pi^+\pi^-\gamma$, pions are in $I = 1$ state and l is odd, but for the decay $\omega \to \pi^+\pi^-\gamma$, pions are in $I = 0$ or 2 state and l is even.

3 Show that the decay $\omega \to \pi^+\pi^-$ is forbidden in strong interaction, but is allowed by electromagnetic interaction. What are the values of isospin I and orbital angular momentum for the pions ?

4 Show that $\eta \to \pi^+\pi^-\pi^0$ is forbidden in strong interaction but is allowed by electromagnetic interaction. Determine the possible values of isospin for the final pions.

5 Derive Eq. (94).

4.7 Bibliography

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Chapter 5
UNITARY GROUPS AND SU(3)

5.1 Unitary Groups and SU(3)

Consider a vector ϕ_i, $i = 1, 2, \cdots, N$ in an N-dimensional vector space. An arbitrary transformation in this space is

$$\phi'_i = a_i^j \, \phi_j \qquad i, j = 1, 2, \cdots, N. \tag{5.1}$$

For a unitary group $U(N)$ in N dimensions,

$$a_i^{*k} \, a_j^k = \left(a^\dagger\right)_k^i \, a_j^k = \delta_j^i. \tag{5.2}$$

For the group SU(N), we also have

$$\det \, a = 1. \tag{5.3}$$

The basic assumption of all the group theoretical approaches to classification of hadrons is that particles belong to an irreducible representation of some group (in our case $SU(N)$) and form a multiplet and thus have the same space-time properties, especially the mass, spin and parity. The basic mathematical problem is the investigation of the representations of a group. There are two approaches to this investigation (i) global way, (ii) infinitesimal way. For continuous groups it is convenient to restrict to (ii). For the general infinitesimal transformation:

$$\phi'_i = \left[\delta_i^j + \varepsilon_i^j\right] \phi_j. \tag{5.4}$$

Then conditions (2) and (3) give

$$\begin{aligned} \varepsilon_i^{*j} &= -\varepsilon_j^i \\ \varepsilon_i^i &= 0. \end{aligned} \tag{5.5}$$

The unitary transformation corresponding to (4) may be written as

$$U(a) = 1 - \varepsilon_i^j \; A_j^i + O\left(\varepsilon^2\right). \tag{5.6}$$

A_j^i are called the generators of the group $U(N)$ and characterize the group completely. The $N \times N$ unitary complex matrices $U(a)$ form the representation of $U(N)$. Hence there are N^2 arbitrary real parameters and thus there are N^2 generators of the group $U(N)$. For $SU(N)$ we have N^2-1 generators because of the unimodularity condition.

The matrices $U(a)$ have the group property:

$$\begin{aligned} U(b)\; U(a) &= U(c) \\ U(a) &= U(a)\; U(1),\; U(1) = 1 \\ U^{-1}(b)\; U(a)\; U(b) &= U\left(b^{-1}ab\right) \\ U^{\dagger}(a)\; U(a) &= 1. \end{aligned} \tag{5.7}$$

It is easy to see that Eqs. (5) and (6) give

$$\left(A_j^i\right)^{\dagger} = A_i^j. \tag{5.8}$$

By taking a to be infinitesimal transformation, it is easy to derive (see problem 5) the commutation relations

$$\left[A_i^j,\; A_k^l\right] = \delta_k^j\; A_i^l - \delta_i^l\; A_k^j. \tag{5.9}$$

For the transformation (4), we have

$$\phi_k' \equiv U^{-1}(a)\;\; \phi_k\; U(a) = \phi_k + \varepsilon_i^j\left[A_j^i, \phi_k\right]. \tag{5.10}$$

But we can write

$$\begin{aligned} \phi'_k &= U_k^l \, \phi_l. \\ \phi'_k &= \left(\delta_k^l + \varepsilon_i^j \left(M_j^i\right)_k^l\right) \phi_l. \end{aligned} \tag{5.11}$$

Comparing Eqs. (4) and (11), we get

$$\left(M_j^i\right)_k^l = \delta_k^i \delta_j^l. \tag{5.12}$$

Hence from Eq. (10) we have

$$\begin{aligned} \left[A_j^i, \phi_k\right] &= \left(M_j^i\right)_k^l \phi_l \\ &= \delta_k^i \delta_j^l \, \phi_l = \delta_k^i \, \phi_j. \end{aligned} \tag{5.13}$$

The matrices M_j^i $\left[\left(M_j^i\right)^\dagger = M_i^j\right]$ give the representation of the group $U(N)$ for the fundamental representation ϕ_i. We define a vector

$$\phi^i = \phi_i^*.$$

It belongs to the representation $\bar{N}$ of $U(N)$, whereas vector ϕ_i belongs to the representation N of $U(N)$. Thus ϕ_i transforms as

$$\begin{aligned} \phi^i &\rightarrow \phi'^i \equiv \phi_i'^* = \left(\delta_i^j - \varepsilon_i^{*j}\right) \phi_j^* \\ &= \left(\delta_j^i - \varepsilon_j^i\right) \phi^j. \end{aligned} \tag{5.14}$$

Hence it follows that

$$\left[A_j^i, \phi^k\right] = -\delta_j^k \, \phi^i. \tag{5.15}$$

Now if we consider a tensor T_l^k, it transforms as $\phi^k \phi_l$, so that we get

$$\left[A_j^i, T_l^k\right] = \delta_l^i \, T_j^k - \delta_j^k \, T_l^i. \tag{5.16}$$

Thus the tensor T_j^i transforms in the same way as the generator A_j^i.

Let us now restrict ourselves to $SU(N)$. The generators of $SU(N)$ must be traceless. Hence we can write its generators as

$$F_j^i = A_j^i - \frac{1}{N}\delta_j^i \, A_k^k, \tag{5.17}$$

so that

$$\begin{aligned} U(a) &= 1 - \varepsilon_j^i \, F_j^i \\ \left(F_j^i\right)^\dagger &= F_i^j \\ F_i^i &= 0. \end{aligned} \tag{5.18}$$

Since A_k^k is a $U(N)$ invariant, the commutation relation for F_j^i remains the same as in Eq. (9) viz.

$$\left[F_j^i, \, F_l^k\right] = \delta_l^i \, F_j^k - \delta_j^k \, F_l^i. \tag{5.19}$$

The matrices M_j^i must now be traceless, hence we have

$$\left(M_j^i\right)_l^k = \delta_l^i \, \delta_j^k - \frac{1}{N}\delta_j^i \, \delta_l^k. \tag{5.20}$$

Thus we have instead of Eqs. (13) and (15),

$$\begin{aligned} \left[F_j^i, \, \phi_k\right] &= \delta_k^i \, \phi_j - \frac{1}{N}\delta_j^i \, \phi_k \\ \left[F_j^i, \, \phi^k\right] &= -\delta_j^k \, \phi^i + \frac{1}{N}\delta_j^i \, \phi^k. \end{aligned} \tag{5.21}$$

We now confine to $SU(3)$. It is convenient to express eight generators F_j^i, $(i, j = 1, 2, 3)$ in terms of hermitian operators F_A, $(A = 1, \cdots, 8)$ introduced by Gell-Mann. The relationship between F_A and F_j^i is as follows:

$$\begin{aligned} F_2^1 &= F_1 - i\,F_2, \quad F_1^2 = F_1 + i\,F_2, \quad \frac{1}{2}\left(F_1^1 - F_2^2\right) = F_3, \\ F_3^1 &= F_4 - i\,F_5, \quad F_1^3 = F_4 + i\,F_5, \quad F_3^2 = F_6 - i\,F_7, \\ F_2^3 &= F_6 - i\,F_7, \quad F_3^3 = -\frac{2}{\sqrt{3}}F_8, \end{aligned}$$

From the commutation relation (19) for F^i_j, one can show that F_A's satisfy the standard commutation relation of a Lie group:

$$\begin{aligned}[F_A,\ F_B] &= C^D_{AB} F_D \\ &= i f_{ABC}\ F_C,\end{aligned} \tag{5.22}$$

where the structure constants f_{ABC} are real and antisymmetric. F_A's also satisfy the Jacobi identity

$$[F_A,\ [F_B,\ F_C]] + [F_B,\ [F_C,\ F_A]] + [F_C,\ [F_A,\ F_B]] = 0. \tag{5.23}$$

Infinitesimal unitary transformation generated by F_A is

$$U = 1 - i\ \varepsilon_A\ F_A,$$

ε_A being infinitesimal real parameters. For an inifinitesinal transformation, the vectors ϕ_i and ϕ^i transform as

$$\begin{aligned}\phi'_i &= U^j_i\ \phi_j \\ &= \left[\delta^j_i + \frac{i}{2}\varepsilon_A\,(\lambda_A)^j_i\right]\phi_j,\end{aligned} \tag{5.24}$$

$$\begin{aligned}\phi'^{\,i} &= U^{*\,j}_i \phi^j = \left[\delta^j_i - \frac{i}{2}\varepsilon_A\,(\lambda^*_A)^j_i\right]\phi^j \\ &= \left[\delta^j_i - \frac{i}{2}\varepsilon_A\,(\lambda_A)^i_j\right]\phi^j,\end{aligned} \tag{5.25}$$

since the matrices λ_A are hermitian. The matrices λ_A are related to M^i_j in the same way as F_A are related to F^i_j . Thus

$$M^1_2 = \frac{1}{2}\left(\lambda_1 - i\lambda_2\right),\ M^2_1 = \frac{1}{2}\left(\lambda_1 + i\lambda_2\right), \cdots, M^3_3 = -\frac{1}{\sqrt{3}}\lambda_8. \tag{5.26}$$

Now for $SU(3)$, the matrix elements of matrices M^i_j are given by

$$\left(M^i_j\right)^k_l = \delta^i_l\ \delta^k_j - \frac{1}{3}\delta^i_j\ \delta^k_l. \tag{5.27}$$

Using Eqs. (16) and (17), we can explicitly write 3×3 matrices λ_A. They are

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}.
\end{aligned} \tag{5.28}$$

Obviously the matrices $\frac{\lambda_A}{2}$ satisfy the same commutation relations as the generators F_A, so that

$$[\lambda_A,\ \lambda_B] = 2i\ f_{ABC}\ \lambda_C. \tag{5.29a}$$

They are traceless and have the following properties:

$$Tr(\lambda_A\ \lambda_B) = 2\delta_{AB} \tag{5.29b}$$

$$[\lambda_A,\ \lambda_B]_+ = 2\ d_{ABC}\ \lambda_C + \frac{4}{3}\delta_{AB}, \tag{5.30}$$

where d_{ABC} are real and are totally symmetric. Defining $\lambda_0 = \sqrt{\frac{2}{3}}I$, the commutation and anticommutation relations can be written as

$$\begin{aligned}
[\lambda_A,\ \lambda_B] &= 2i\ f_{ABC}\ \lambda_C \\
[\lambda_A,\ \lambda_B]_+ &= 2\ d_{ABC}\ \lambda_C \\
Tr(\lambda_A\ \lambda_B) &= 2\delta_{AB} \\
d_{0BC} &= \sqrt{\frac{2}{3}}\ \delta_{AB}, \qquad f_{0BC} = 0,
\end{aligned} \tag{5.31}$$

where $A, B, C = 0, 1, \cdots, 8$. Thus λ_A are closed both under commutation and anticommutation. We also note that λ_2, λ_5, λ_7 are antisymmetric while the rest of them are symmetric. We express this fact by writing

$$\lambda_A^T = \eta_A \, \lambda_A \qquad \text{(not summed)}\,, \tag{5.32}$$

where $\eta_A = -1$, for $A = 2, 5, 7$ and $+1$ otherwise. The following identities follow from Eqs. (31) and (32):

$$\begin{aligned} \eta_A \, \eta_B \, \eta_C \, f_{ABC} &= -f_{ABC} \\ \eta_A \, \eta_B \, \eta_C \, d_{ABC} &= d_{ABC} \text{ (repeated indices not summed)} \end{aligned} \tag{5.33}$$

i.e. f_{ABC} (d_{ABC}) is zero if even (odd) number of indices take the value 2, 5 or 7. The values of f_{ABC} and d_{ABC} have been tabulated by Gell-Mann and are reproduced in Table 1. The role of F_A is the same in $SU(3)$ as that of isospin **I** in $SU(2)$ and for this reason F_A's are sometimes called component of F-spin.

5.2 Particle Representations in Flavor SU(3)

Out of the eight tensor generators F_j^i of $SU(3)$, the set F_1^1, F_2^1, F_1^2 and F_2^2 form the generators of the subgroup $SU(2)\times\, U(1)$. We have $SU(3) \supset SU(2) \times U(1) \supset SU(2)$. It is convenient to classify states in an $SU(3)$ representation by making use of this fact. The generators of the $SU(2) \times U(1)$ subgroup which are conveniently taken to correspond to isospin and hypercharge are

$$\begin{aligned} I_+ &= F_1^2,\ I_- = F_2^1,\ I_3 = \frac{1}{2}\left(F_1^1 - F_2^2\right) \\ Y &= F_1^1 + F_2^2 = -F_3^3, \end{aligned} \tag{5.34}$$

in the case of $SU(3)$ group. There are thus two diagonal operators in $SU(3)$, namely I_3 and $Y \cdot SU(3)$ is, therefore, a group of rank 2. Further if we define the electric charge as

$$Q = F_1^1 \text{ in } SU(3), \tag{5.35}$$

Table 5.1 Values of f_{ABC} and d_{ABC}

ABC	f_{ABC}	ABC	d_{ABC}
123	1	118	$1/\sqrt{3}$
147	1/2	146	1/2
156	−1/2	157	1/2
246	1/2	228	$1/\sqrt{3}$
257	1/2	247	−1/2
345	1/2	256	1/2
367	−1/2	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	1/2
678	$\sqrt{3}/2$	355	1/2
		366	−1/2
		377	−1/2
		448	$-1/\left(2\sqrt{3}\right)$
		558	$-1/\left(2\sqrt{3}\right)$
		668	$-1/\left(2\sqrt{3}\right)$
		778	$-1/\left(2\sqrt{3}\right)$
		888	$-1/\sqrt{3}$
		d_{0AB}	$\sqrt{2/3}\delta_{AB}$

Eq. (24) give the Gell-Mann-Nishijima relation

$$Q = I_3 + \frac{Y}{2}. \tag{5.36}$$

The fundamental representation is a vector which we write as q_i. Let us take

$$q_i \equiv \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{bmatrix}, \tag{5.37}$$

as the field operator which creates a u-quark, or a d-quark or an s-quark viz.

$$\bar{u}\,|0\rangle = |u\rangle\,,\ \bar{d}\,|0\rangle = |d\rangle\,,\ \bar{s}\,|0\rangle = |s\rangle\,. \tag{5.38}$$

The field operators q_i belong to the representation **3** of $SU(3)$, whereas the field operators

$$q^i = q_i^* = \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} \equiv \begin{bmatrix} u \\ d \\ s \end{bmatrix} \tag{5.39}$$

belong to the representation $\bar{\mathbf{3}}$ of $SU(3)$. q^i create antiquarks or annihilate quarks. From Eq. (21), we have

$$\left[F^i_j,\ q_k\right] = \delta^i_k\ q_j - \frac{1}{3}\delta^i_j\ q_k. \tag{5.40}$$

In the matrix notation, we can write the field operators q_i and q^i as row and column matrix respectively viz.

$$\begin{aligned} \bar{q} &= \left(\bar{u}\ \bar{d}\ \bar{s}\right) \\ q &= \begin{pmatrix} u \\ d \\ s \end{pmatrix}. \end{aligned} \tag{5.41}$$

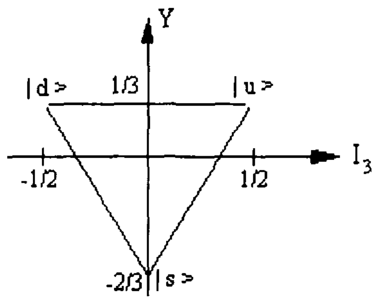


Figure 1 Weight diagram for **3**.

Then it follows from Eqs. (24) and (25):

$$\begin{aligned}[F_A,\, q] &= -\frac{\lambda_A}{2} q \\ [F_A,\, \bar{q}] &= \bar{q}\frac{\lambda_A}{2}.\end{aligned} \tag{5.42}$$

Hence we see from Eqs. (34), (36), (40) or (42), that the quark states or simply quarks belong to the triplet representation of $SU(3)$ and have the following quantum numbers:

		I_3	Y	Q
$\|u\rangle$		$1/2$	$1/3$	$2/3$
	$I = 1/2$			
$\|d\rangle$		$-1/2$	$1/3$	$-1/3$
$\|s\rangle$	$I = 0$	0	$-2/3$	$-1/3$

It is convenient to plot each state of the triplet representation on an $I - Y$ plot as shown in Fig. 1. Such a diagram is called the weight diagram.

The $\bar{\mathbf{3}}$ representation of $SU(3)$ is not equivalent to **3**; it transforms as $q^i = q_i^*$. It is the hypercharge which distinguishes **3** and $\bar{\mathbf{3}}$. Antiquarks belong to the $\bar{\mathbf{3}}$ representation of $SU(3)$ and the weight diagram is shown in Fig. 2.

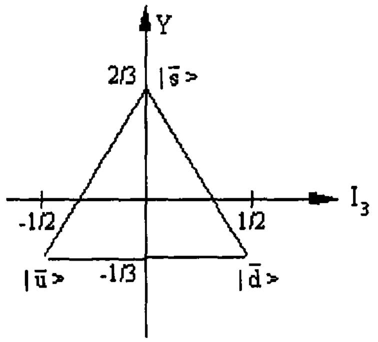


Figure 2 Weight diagram for $\bar{3}$.

Mesons:

Quarks are taken to be spin 1/2 particles. To build observed particles from quarks, it is convenient to assign a baryon number 1/3 to quarks. Thus

$$\begin{aligned} q_i \left|0\right\rangle &: \; B = 1/3 \\ q^i \left|0\right\rangle &: \; B = -1/3. \end{aligned}$$

Consider

$$\begin{aligned} q^i q_j &= \left(q^i q_j - \frac{1}{3}\delta^i_j q^k q_k\right) + \frac{1}{3}\delta^i_j q^k q_k \\ &= P^i_j \text{ (octet)} \qquad \text{Singlet} \\ \mathbf{\bar{3}} \otimes \mathbf{3} &= \qquad \mathbf{8} \qquad \oplus \qquad \mathbf{1} \end{aligned}$$

P^i_j can be regarded as a field operator for pseudoscalar mesons. Thus

$$P^i_j \left|0\right\rangle \equiv \left|P^i_j\right\rangle = \left(q^i q_j - \frac{1}{3}\delta^i_j q^k q_k\right)\left|0\right\rangle \tag{5.43}$$

has baryon number zero and is an octet. It may be taken to represent octet of pseudoscalar mesons π, K and η. We write (in our

Table 5.2 Pseudoscalar Mesons $J^P = 0^-$ [cf. Eqs. (43) and (44)]

State and its quark content	$\vert D, Y, I, I_3\rangle$
$\vert P_1^2\rangle = \vert\pi^+\rangle = \vert u\ \bar{d}\rangle$	$\left\vert\frac{\pi_1+i\pi_2}{\sqrt{2}}\right\rangle : -\vert 8,0,1,1\rangle$
$\left\vert\frac{P_1^1-P_2^2}{\sqrt{2}}\right\rangle = \vert\pi^0\rangle = \left\vert\frac{u\ \bar{u}-d\ \bar{d}}{\sqrt{2}}\right\rangle$	$\vert\pi_3\rangle : \vert 8,0,1,0\rangle$
$\vert P_2^1\rangle = \vert\pi^-\rangle = \vert d\ \bar{u}\rangle$	$\left\vert\frac{\pi_1-i\pi_2}{\sqrt{2}}\right\rangle : \vert 8,0,1,-1\rangle$
$\vert P_1^3\rangle = \vert K^+\rangle = \vert u\ \bar{s}\rangle$	$\left\vert\frac{\pi_4+i\pi_5}{\sqrt{2}}\right\rangle : \vert 8,1,1/2,1/2\rangle$
$\vert P_2^3\rangle = \vert K^0\rangle = \vert d\ \bar{s}\rangle$	$\left\vert\frac{\pi_6+i\pi_7}{\sqrt{2}}\right\rangle : \vert 8,1,1/2,-1/2\rangle$
$\vert P_3^2\rangle = \left\vert\bar{K}^0\right\rangle = \left\vert s\ \bar{d}\right\rangle$	$\left\vert\frac{\pi_6-i\pi_7}{\sqrt{2}}\right\rangle : \vert 8,-1,1/2,1/2\rangle$
$\vert P_3^1\rangle = \vert K^-\rangle = \vert s\ \bar{u}\rangle$	$\left\vert\frac{\pi_4-i\pi_5}{\sqrt{2}}\right\rangle : \vert 8,-1,1/2,-1/2\rangle$
$\left\vert-\frac{3}{\sqrt{6}}P_3^3\right\rangle = \vert\eta_8\rangle = \left\vert\frac{u\ \bar{u}+d\ \bar{d}-2s\bar{s}}{\sqrt{6}}\right\rangle$	$\vert\pi_8\rangle : \vert 8,0,0,0\rangle$

D in the last column denotes the dimension of the $SU(3)$ representation, in this case 8. The negative signs in front of certain states appear because of our phase convention.

notation upper index is row index and lower index is column index)

$$P_j^i = \frac{1}{\sqrt{2}}\,(\lambda_A)_j^i\,\pi_A \tag{5.44}$$

where identification is shown in Table 2. Hence in a matrix notation, the pseudoscalar mesons $J^P = 0^-$ can be represented by a matrix:

$$P = \begin{pmatrix} \frac{1}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{2}}\pi^0 & \pi^+ & K^+ \\ \pi^- & \frac{1}{\sqrt{6}}\eta_8 - \frac{1}{\sqrt{2}}\pi^0 & K^0 \\ K^- & \overline{K^0} & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}. \tag{5.45}$$

The singlet pseudoscalar meson η_1 is given by

$$|\eta_1\rangle = \left|\frac{u\bar{u}\ +\ d\bar{d} + s\bar{s}}{\sqrt{3}}\right\rangle = \frac{u\bar{u}\ +\ d\bar{d} + s\bar{s}}{\sqrt{3}}\,|0\rangle : |1,0,0,0\rangle\,.$$

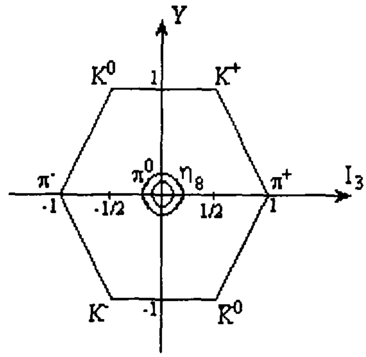


Figure 3 Weight diagram for pseudoscalar meson octet.

Another possible set of candidates for the octet of bosons is vector mesons $J^P = 1^-$:

$$\begin{aligned} \rho^+\rho^0\rho^- \quad I &= 1,\ Y=0 \\ K^{*+}K^{*0} \quad I &= 1/2,\ Y=1 \\ \bar{K}^{*0}K^{*-} \quad I &= 1/2,\ Y=-1 \\ \omega_8 \quad I &= 0,\ Y=0 \end{aligned}$$

A singlet vector boson is denoted as ω_1. In broken $SU(3)$, a singlet meson can mix with the eighth component of an octet. For example, ω_8 and ω_1 can mix and physical particles are mixtures of them and are denoted by ω and ϕ. The weight diagram for mesons is given in Fig. 3.

Baryons:

We now consider baryons. Baryons have $B = 1$ and they must be constructed out of 3 quarks. For this purpose we proceed as follows. We write

$$q_j\, q_k = \frac{1}{2}\,(q_j\, q_k + q_k\, q_j) + \frac{1}{2}\,(q_j\, q_k - q_k\, q_j)$$

$$= \frac{1}{\sqrt{2}}S_{jk} + \frac{1}{\sqrt{2}}A_{jk}, \tag{5.46}$$

where the symmetric tensor

$$S_{jk} = \frac{1}{\sqrt{2}}\left(q_j\ q_k + q_k\ q_j\right) \tag{5.47}$$

has six independent components. The antisymmetric tensor

$$A_{jk} = \frac{1}{\sqrt{2}}\left(q_j\ q_k - q_k\ q_j\right) \tag{5.48}$$

has three independent components. Now a vector T^i belonging to the representation $\bar{\mathbf{3}}$ can be written in terms of A_{lm} as

$$T^i = \varepsilon^{ilm}\ A_{lm}$$

or

$$A_{jk} = \frac{1}{2}\varepsilon_{ijk}\ T^i. \tag{5.49}$$

Hence we have the result

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$$

and

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\mathbf{6} \otimes \mathbf{3}) + (\bar{\mathbf{3}} \otimes \mathbf{3})\,.$$

We first consider $\bar{\mathbf{3}} \otimes \mathbf{3}$:

$$T^i\ q_j = \left(T^i\ q_j - \frac{1}{3}\ \delta^i_j\ T^k\ q_k\right) + \frac{1}{3}\ \delta^i_j\ T^k\ q_k \tag{5.50}$$

viz.

$$\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1}.$$

We write the octet operator for baryons as

$$\bar{B}^i_j = \left(T^i\ q_j - \frac{1}{3}\ \delta^i_j\ T^k\ q_k\right), \tag{5.51a}$$

where

$$T^i = \varepsilon^{ilm}\; A_{lm} = \frac{1}{2\sqrt{2}}\varepsilon^{ilm}\,(\; q_l\; q_m - \; q_m\; q_l)\,. \qquad (5.51\text{b})$$

For the singlet representation, we have

$$\begin{aligned}\frac{1}{2}\frac{1}{\sqrt{3}}T^k\; q_k &= \frac{1}{2\sqrt{3}}\varepsilon^{klm}\; A_{lm}\; q_k \\ &= \frac{1}{2\sqrt{6}}\varepsilon^{klm}\,(\; q_l\; q_m - \; q_m\; q_l)\, q_k. \qquad (5.52)\end{aligned}$$

We now consider $\mathbf{6}\otimes\mathbf{3}$: It is given by

$$\begin{aligned}S_{ij}\; q_k &= S_{ij}\; q_k + S_{jk}\; q_i + S_{ki}\; q_j - S_{jk}\; q_i - S_{ki}\; q_j \\ &= \tilde{T}_{\{ijk\}} - S_{jk}\; q_i - S_{ki}\; q_j, \qquad (5.53\text{a})\end{aligned}$$

where

$$\tilde{T}_{\{ijk\}} = S_{ij}\; q_k + S_{jk}\; q_i + S_{ki}\; q_j \qquad (5.53\text{b})$$

is completely symmetric tensor and has 10 independent components. Now we show that

$$-\,(S_{jk}\; q_i + S_{ki}\; q_j) + 2S_{ij}\; q_k = \varepsilon_{kjl}\;\varepsilon^{lmn}\; S_{in}\; q_m + \varepsilon_{kil}\;\varepsilon^{lmn}\; S_{jn}\; q_m. \qquad (5.54)$$

Proof:

$$\begin{aligned}R.H.S &= \left(\delta_k^m\;\delta_j^n - \delta_k^n\;\delta_j^m\right) S_{in}\; q_m + (\delta_k^m\;\delta_i^n - \delta_k^n\;\delta_i^m)\, S_{jn}\; q_m \\ &= S_{ij}\; q_k - S_{ik}\; q_j + S_{ji}\; q_k - S_{jk}\; q_i \\ &= -\,(S_{jk}\; q_i + S_{ki}\; q_j) + 2S_{ij}\; q_k = L.H.S.\end{aligned}$$

Hence from Eqs. (53) and (54), we get

$$\begin{aligned}S_{ij}\; q_k &= \frac{1}{3}\tilde{T}_{\{ijk\}} + \frac{1}{3}\left[\varepsilon_{kjl}\;\varepsilon^{lmn}\; S_{in}\; q_m + \varepsilon_{kil}\;\varepsilon^{lmn}\; S_{jn}\; q_m\right] \\ &= \frac{1}{3}\tilde{T}_{\{ijk\}} + \frac{1}{3}\left[\varepsilon_{kjl}\;\delta_i^r + \varepsilon_{kil}\;\delta_j^r\right]\varepsilon^{lmn}\; S_{rn}\; q_m. \qquad (5.55)\end{aligned}$$

Hence we have

$$\mathbf{6} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8}.$$

We write the decuplet representation:

$$\begin{aligned} \bar{T}_{\{ijk\}} &= \frac{1}{\sqrt{3}} \tilde{T}_{\{ijk\}} \\ &= \frac{1}{\sqrt{3}} \left[S_{ij}\ q_k + S_{jk}\ q_i + S_{ki}\ q_j \right] \end{aligned} \tag{5.56}$$

and the octet $(8')$ representation:

$$\begin{aligned} \bar{B}'^l_r &= \frac{1}{\sqrt{3}} \varepsilon^{lmn}\ S_{rn}\ q_m \\ \bar{B}'^l_l &= 0. \end{aligned} \tag{5.57}$$

Hence finally we have the result

$$\begin{aligned} \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} &= (\mathbf{6} \oplus \bar{\mathbf{3}}) \otimes \mathbf{3} \\ &= (\mathbf{6} \otimes \bar{\mathbf{3}}) \oplus (\bar{\mathbf{3}} \otimes \mathbf{3}) \\ &= \mathbf{10} \oplus \mathbf{8}' \oplus \mathbf{8} \oplus \mathbf{1}. \end{aligned}$$

Baryon States:

(i) **Octet Representation 8:**

From Eqs. (51), we have

$$\begin{aligned} \bar{B}^i_j\ |0\rangle &= \left| B^i_j \right\rangle \\ &= \frac{1}{2\sqrt{2}} \Big[\varepsilon^{ilm}\ (q_l\ q_m - q_m\ q_l) \\ &\quad - \frac{1}{3}\ \delta^i_j\ \varepsilon^{klm} (q_l\ q_m - q_m\ q_l)\, q_k \Big]\ |0\rangle \end{aligned} \tag{5.58}$$

and for representation $8'$ [cf. Eq.(57)]:

$$\bar{B}'^i_j\ |0\rangle = \left| B'^i_j \right\rangle = \frac{1}{\sqrt{3}} \varepsilon^{ikl}\ S_{jl}\ q_k\ |0\rangle . \tag{5.59}$$

Table 5.3 Baryons $J^P = \frac{1}{2}^+$

State: **8**	Quark Content	Q	I	I_3	Y
$\lvert p\rangle = \bar{B}_1^3 \lvert 0\rangle$	$\frac{1}{\sqrt{2}} \lvert [u,d]\,u\rangle$	1	$\frac{1}{2}$	$\frac{1}{2}$	1
$\lvert n\rangle = \bar{B}_2^3 \lvert 0\rangle$	$\frac{1}{\sqrt{2}} \lvert [u,d]\,d\rangle$	0	$\frac{1}{2}$	$\frac{-1}{2}$	1
$\lvert \Sigma^+\rangle = \bar{B}_1^2 \lvert 0\rangle$	$\frac{1}{\sqrt{2}} \lvert [u,s]\,u\rangle$	1	1	1	0
$\lvert \Sigma^0\rangle = \frac{1}{\sqrt{2}}\left(\bar{B}_1^1 - \bar{B}_2^2\right) \lvert 0\rangle$	$\frac{1}{2} \lvert [d,s]\,u + [u,s]\,d\rangle$	0	1	0	0
$\lvert \Sigma^-\rangle = \bar{B}_2^1 \lvert 0\rangle$	$\frac{1}{\sqrt{2}} \lvert [d,s]\,d\rangle$	-1	1	-1	0
$\lvert \Lambda^0\rangle = -\frac{3}{\sqrt{6}} \bar{B}_3^3 \lvert 0\rangle$	$\frac{1}{\sqrt{12}} \lvert 2\,[u,d]\,s - [d,s]\,u - [s,u]\,d\rangle$	0	0	0	0
$\lvert \Xi^-\rangle = \bar{B}_3^1 \lvert 0\rangle$	$\frac{1}{\sqrt{2}} \lvert [d,s]\,s\rangle$	-1	$1/2$	$-1/2$	-1
$\lvert \Xi^0\rangle = \bar{B}_3^2 \lvert 0\rangle$	$\frac{1}{\sqrt{2}} \lvert [s,u]\,s\rangle$	0	$1/2$	$1/2$	-1

State : **8′**	Quark content
$\bar{B}_1^{\prime 3} \lvert 0\rangle$	$\frac{1}{\sqrt{6}} \left\lvert \left([u,d]_+\, u - 2uud\right)\right\rangle$
$\bar{B}_2^{\prime 3} \lvert 0\rangle$	$-\frac{1}{\sqrt{6}} \left\lvert \left([u,d]_+\, d - 2ddu\right)\right\rangle$
$\bar{B}_1^{\prime 2} \lvert 0\rangle$	$\frac{1}{\sqrt{6}} \left\lvert \left([u,s]_+\, u - 2uus\right)\right\rangle$
$\frac{1}{\sqrt{2}}\left(\bar{B}_1^{\prime 1} - \bar{B}_2^{\prime 2}\right) \lvert 0\rangle$	$\frac{1}{\sqrt{12}} \left\{\left\lvert (-2\,[u,d]_+\, s + [u,s]_+\, d + [d,s]_+\, u)\right\rangle\right\}$
$\bar{B}_2^{\prime 1} \lvert 0\rangle$	$\frac{1}{\sqrt{6}} \left\lvert \left([d,s]_+\, d - 2dds\right)\right\rangle$
$-\frac{3}{\sqrt{6}} \bar{B}_3^{\prime 3} \lvert 0\rangle$	$-\frac{1}{2} \left\lvert \left([s,d]_+\, u - [s,u]_+\, d\right)\right\rangle$
$\bar{B}_3^{\prime 1} \lvert 0\rangle$	$\frac{1}{\sqrt{6}} \left\lvert \left(2ssd - [d,s]_+\, s\right)\right\rangle$
$\bar{B}_3^{\prime 2} \lvert 0\rangle$	$\frac{1}{\sqrt{6}} \left\lvert \left([s,u]_+\, s - 2ssu\right)\right\rangle$

The octet of baryons are then identified as given in Table 3. Hence from Eq. (58) and Table 3, we see that known eight $J^P = 1/2^+$ baryons can be represented as 3×3 matrices

$$B^i_j = \begin{pmatrix} \frac{1}{\sqrt{6}}\Lambda^0 + \frac{1}{\sqrt{2}}\Sigma^0 & \Sigma^+ & p \\ \Sigma^- & \frac{1}{\sqrt{6}}\Lambda^0 - \frac{1}{\sqrt{2}}\Sigma^0 & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda^0 \end{pmatrix} \tag{5.60a}$$

$$\bar{B}^i_j = \begin{pmatrix} \frac{1}{\sqrt{6}}\bar{\Lambda}^0 + \frac{1}{\sqrt{2}}\bar{\Sigma}^0 & \bar{\Sigma}^- & \bar{\Xi}^- \\ \bar{\Sigma}^+ & \frac{1}{\sqrt{6}}\bar{\Lambda}^0 - \frac{1}{\sqrt{2}}\bar{\Sigma}^0 & \bar{\Xi}^0 \\ \bar{p} & \bar{n} & -\frac{2}{\sqrt{6}}\bar{\Lambda}^0 \end{pmatrix}. \tag{5.60b}$$

Note that

$$\bar{B}^i_j = B^{*j}_i \, \gamma^0, \tag{5.61}$$

where the symbol $*$ denotes complex conjugation with respect to $SU(3)$ but hermitian conjugation for the field operators. The weight diagram for the octet representation is shown in Fig. 4.

(i) **Singlet representation 1:**

From Eq. (52), we have

$$\begin{aligned} \Lambda^0_1 &= \frac{1}{2}\frac{1}{\sqrt{6}}\,\varepsilon^{klm}\,(q_l\,q_m - q_m\,q_l)\,q_k\,\;|0\rangle \\ &= \frac{1}{2}\frac{2}{\sqrt{6}}\,\left\{\left[\bar{d},\ \bar{s}\right]\,\bar{u} + [\bar{s},\ \bar{u}]\,\bar{d} + \left[\bar{u},\ \bar{d}\right]\,\bar{s}\right\}\,|0\rangle\,. \\ &= \frac{1}{\sqrt{6}}\,|[d,\ s]\ u + [s,\ u]\ d + [u,\ d]\ s\rangle\,. \end{aligned} \tag{5.62}$$

(ii) **Decuplet representation 10:**

From Eq. (56), we have

$$|T_{ijk}\rangle = \frac{1}{\sqrt{3}}\,\{S_{ij}\;q_k + S_{jk}\;q_i + S_{ki}\;q_j\}\;|0\rangle\,. \tag{5.63}$$

The detailed identification of the states of decuplet representation are given in Table 4. The weight diagram for the decuplet of baryons is shown in Fig. 5.

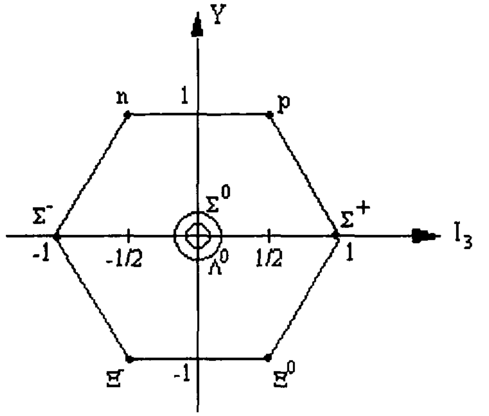


Figure 4 Weight diagram for $\frac{1}{2}^+$ baryon octet.

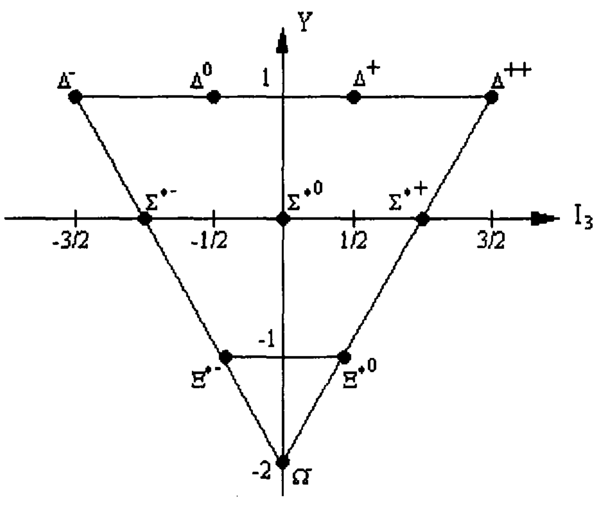


Figure 5 Weight diagram for $\frac{3}{2}^+$ baryon decuplet.

Table 5.4 Decuplet $J^P = 3/2^+$ [cf. Eq. (63)]

State	Quark Content	Q	I	I_3	Y
$\|\Delta^{++}\rangle \equiv \frac{1}{\sqrt{6}}\|T_{111}\rangle$	$\|uuu\rangle$	2	$\frac{3}{2}$	$\frac{3}{2}$	1
$\|\Delta^{+}\rangle \equiv \frac{1}{\sqrt{2}}\|T_{112}\rangle$	$\frac{1}{\sqrt{3}}\|udu + duu + uud\rangle$	1	$\frac{3}{2}$	$\frac{1}{2}$	1
$\|\Delta^{0}\rangle \equiv \frac{1}{\sqrt{2}}\|T_{122}\rangle$	$\frac{1}{\sqrt{3}}\|udd + ddu + dud\rangle$	0	$\frac{3}{2}$	1	1
$\|\Delta^{-}\rangle \equiv \frac{1}{\sqrt{2}}\|T_{222}\rangle$	$\|ddd\rangle$	-1	$\frac{3}{2}$	$\frac{-3}{2}$	1
$\left\|\Sigma^{*^+}\right\rangle \equiv \frac{1}{\sqrt{2}}\|T_{113}\rangle$	$\frac{1}{\sqrt{3}}\|uus + usu + suu\rangle$	1	1	1	0
$\left\|\Sigma^{*^0}\right\rangle \equiv \frac{1}{\sqrt{2}}\|T_{123}\rangle$	$\frac{1}{\sqrt{6}}\left\| \begin{matrix} uds + dus + dsu \\ +sdu + sud + usd \end{matrix} \right\rangle$	0	1	0	0
$\left\|\Sigma^{*^-}\right\rangle \equiv \frac{1}{\sqrt{2}}\|T_{322}\rangle$	$\frac{1}{\sqrt{3}}\|sdd + dds + dsd\rangle$	-1	1	-1	0
$\left\|\Xi^{*^0}\right\rangle \equiv \frac{1}{\sqrt{2}}\|T_{133}\rangle$	$\frac{1}{\sqrt{3}}\|uss + ssu + sus\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$	-1
$\left\|\Xi^{*^-}\right\rangle \equiv \frac{1}{\sqrt{2}}\|T_{233}\rangle$	$\frac{1}{\sqrt{3}}\|dss + ssd + sds\rangle$	-1	$\frac{1}{2}$	$\frac{-1}{2}$	-1
$\|\Omega^{-}\rangle \equiv \frac{1}{\sqrt{6}}\|T_{333}\rangle$	$\|sss\rangle$	-1	0	0	-2

5.3 U-Spin

We have labelled the states within an irreducible representation of $SU(3)$ uniquely by the eigenvalues of $\mathbf{I}^2$, I_3 and Y. The reason is that $SU(3)$ contains the direct product of $SU(2)_I \times U(1)_Y$ i.e.

$$SU(3) \supset SU(2)_I \times U(1)_Y.$$

The generators of $SU(2)_I$ and $U(1)_Y$ are identified with the generators of $SU(3)$ as given in Eq. (34).

However we can take a different decomposition. For example the generators

$$U_+ = F_2^3,\ U_- = F_3^2,\ U_3 = \frac{1}{2}\left(F_2^2 - F_3^3\right) \tag{5.64}$$

are the generators of group $SU(2)_U$. These generators commute with the generator

$$Q = F_1^1. \tag{5.65}$$

Thus we can decompose $SU(3)$ as follows

$$SU(3) \supset SU(2)_U \times U(1)_Q.$$

Therefore, it is possible to label the states within an irreducible representation $SU(3)$ by the eigenvalues of U^2, U_3 and Q. The generators of $SU(2)_U$ commutes with the generator $Q = F_1^1$, thus U-spin is very useful when dealing with electromagnetic interactions. Just as each isospin multiplet is associated with a definite hypercharge, each U-spin multiplet has a definite charge.

5.4 Irreducible Representations of SU(3)

We have already encountered two irreducible representations:

$$\begin{aligned} \text{triplet} &= q_i \\ \text{octet} &= P_j^i = q^i q_j - \frac{1}{3}\delta_j^i\, q^k q_k. \end{aligned}$$

The octet representation is a regular or an adjoint representation of $SU(3)$ because P^i_j transforms in the same way as the generators F^i_j.

Now we look at more general representations of $SU(3)$. The general prescription for finding the basic tensors $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ for an irreducible representation of $SU(3)$ is:

1. Construct tensors $T^{j_1\cdots j_q}_{i_1\cdots i_p}$.
2. Symmetrize among $i_1 \cdots i_p$ and $j_1 \cdots j_q$ indices.
3. Substract traces so that all contractions give zero, e.g.

$$T^{i\ j_2\cdots j_q}_{i\ i_2\cdots i_p} = 0, \ \text{etc.}$$

The linearly independent components of tensor T then supply an irreducible representation of $SU(3)$ which is designated as (p,q). The dimensionality of such a representation can be easily computed. First let us calculate the number of independent components for a symmetric tensor with p lower (or q upper) indices. We note that each index can take only the value 1, 2 or 3. Thus the number of independent components are the same as the number of ways of separating p identical objects with two identical partitions:

$$\frac{(p+2)!}{p!\,2!} = \frac{(p+2)\,(p+1)}{2}.$$

Thus a tensor which is symmetric in p lower indices and q upper indices has

$$B(p,q) = \frac{(p+2)\,(p+1)\,(q+2)\,(q+1)}{4}$$

independent components. But the trace condition shows that a symmetric object with $p-1$ lower indices and $q-1$ upper indices vanishes. This gives $B(p-1,\ q-1)$ conditions. Hence a symmetric traceless tensor has dimensions

$$\begin{aligned} D(p,q) &= B(p,q) - B(p-1,\ q-1) \\ &= (p+1)\,(q+1)\left(\frac{p+q}{2}+1\right). \end{aligned} \qquad (5.66)$$

Thus we have for example:

Representation	Dimensionality
(p,q)	$D(p,q)$
$(0,0)$	1 : Singlet
$(1,0)$	3 : Triplet
$(0,1)$	3 : Triplet
$(1,1)$	8 : Octet
$(3,0)$	10 : Decuplet
$(2,2)$	27 : 27 plet

Young's Tableaux

By taking the direct product of basic representation **3** with itself, we can generate the representations of higher dimensions. These representations are however reducible. We now discuss a general method to decompose these reducible representations into irreducible representations. We have already discussed some simple examples.

We represent the fundamental representation **3** by a box i.e. associate index i with a box.

$$\begin{array}{|c|}\hline \\ \hline \end{array} : \mathbf{3} : q_i. \tag{5.67}$$

We note that the representation $\bar{\mathbf{3}}$ is antisymmetric combination of two **3**'s viz.

$$\begin{aligned} T^i &= \varepsilon^{ijk}(q_j\, q_k - q_k\, q_j)\frac{1}{\sqrt{2}} \\ &= \varepsilon^{ijk}\, A_{jk}, \\ A_{ij} &= \frac{1}{2}\varepsilon_{ijk}\, T^k. \end{aligned} \tag{5.68}$$

This can be represented by a column of two boxes

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} : \bar{\mathbf{3}} : T^i. \tag{5.69}$$

Since a tensor index takes on only three values $i = 1, 2, 3$, a column in Young's tableau can have at most three boxes

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline \end{array} : \varepsilon_{ijk}. \tag{5.70}$$

It is completely antisymmetric and it corresponds to the trivial singlet representation. We note that

$$1 = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} = \cdots. \tag{5.71}$$

Consider the (p, q) representation. It is a tensor whose components are

$$T^{j_1 \cdots j_q}_{i_1 \cdots i_p} \tag{5.72}$$

symmetric in lower and upper indices and traceless. We can lower the upper indices with ε tensors, obtaining an object with $p + 2q$ lower indices:

$$\begin{aligned} & t_{i_1 \ldots i_p \; k_1 l_1 \ldots k_q l_q} \\ = \; & \varepsilon_{j_1 k_1 l_1} \ldots \varepsilon_{j_q k_q l_q} \; T^{j_1 \cdots j_q}_{i_1 \cdots i_p}. \end{aligned} \tag{5.73}$$

It is clear that t is antisymmetric in each pair $k_x \longleftrightarrow l_x, \; x = 1 \cdots q$. Since there are $p + 2q$ indices, we arrange $p + 2q$ boxes as follows:

$$\begin{array}{|c|c|c|c|c|c|}\hline k_1 & \cdots & k_q & i_1 & \cdots & i_p \\ \hline l_1 & \cdots & l_q \\ \cline{1-3} \end{array} \tag{5.74}$$

This is a Young Tableau. The most general Young Tableau has only two rows. An irreducible representation is completely specified by two integers (p, q). Note that $(\overline{p, q}) = (q, p)$. It is clear from Eq. (71) that tableaux of the form

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline & & & k_1 & \cdots & k_q & i_1 & \cdots & i_p \\ \hline & & & l_1 & \cdots & l_q \\ \cline{1-6} & & \\ \cline{1-3} \end{array} \tag{5.75}$$

still correspond to the representation (p, q). Comparison with the Young Tableau gives the following rule for preparing a tensor with the right symmetry properties to give a state in (p, q): First, symmetrize indices in each row of the tableau. Then antisymmetrize the indices in each column. If we have more general tableau with columns of more than two boxes, the rules for forming a tensor are the same as before. Assign an index to each box. Then symmetrize the indices in each row and finally antisymmetrize the indices in each column.

Some of the common irreducible representations of $SU(3)$ are shown in the Table 5.

Decomposition of Product Representations

We now consider the decomposition of the direct product of irreducible representations (p, q) and (r, s) corresponding to tableaux A and B.

(5.76)

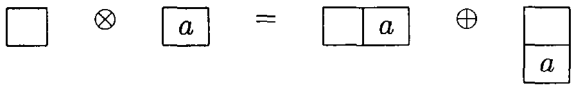
We now give a recipe for the decomposition of the direct product of (p, q) and (r, s) with the aid of Young Tableaux. Put a's in the top row of B and b's in the second row. Take boxes with a from B and add them to A, each in a different column, to form new tableaux. Then, take the boxes with b and add them to form tableaux, again each box in a different column, with one additional restriction given below. On reading the added symbols from right to left and from top to bottom, the number of a's must be greater than or equal to that of b's i.e. forget all tableaux which concave upwards or towards the lower left. This avoids double counting of tensors. The tableaux formed in this way correspond to irreducible representations in $(p, q) \otimes (r, s)$. We now give several examples to illustrate how this recipe works.

Table 5.5 Irreducible representations of $SU(3)$.

(p,q)	$D(p,q)$	Tabular	Tensor
	1	$\begin{array}{\|c\|}\hline i \\ \hline j \\ \hline k \\ \hline\end{array}$	$\mathbf{1}$:ε_{ijk}
$(1,0)$	3	$\begin{array}{\|c\|}\hline i \\ \hline\end{array}$	$\mathbf{3}$:q_i
$(0,1)$	3	$\begin{array}{\|c\|}\hline j \\ \hline k \\ \hline\end{array}$	$\overline{\mathbf{3}} : T^i = \varepsilon^{ijk}\ t_{jk}$
$(2,0)$	6	$\begin{array}{\|c\|c\|}\hline i & j \\ \hline\end{array}$	$\mathbf{6}$:T_{ij}
$(0,2)$	6	$\begin{array}{\|c\|c\|}\hline k_1 & k_2 \\ \hline l_1 & l_2 \\ \hline\end{array}$	$\overline{\mathbf{6}} : T^{ij} = \varepsilon^{ik_1l_1}\varepsilon^{jk_2l_2}\ t_{k_1l_1k_2l_2}$
$(1,1)$	8	$\begin{array}{\|c\|c\|}\hline k & j \\ \hline l \\ \cline{1-1}\end{array}$	$\mathbf{8}$:$T^i_j = e^{ikl}\ t_{klj}$
$(3,0)$	10	$\begin{array}{\|c\|c\|c\|}\hline i & j & k \\ \hline\end{array}$	$\mathbf{10}$:T_{ijk}
$(0,3)$	10	$\begin{array}{\|c\|c\|c\|}\hline l_1 & l_2 & l_3 \\ \hline m_1 & m_2 & m_3 \\ \hline\end{array}$	$\overline{\mathbf{10}} :\ T^{ij} = \varepsilon^{il_1m_1}\varepsilon^{jl_2m_2}\varepsilon^{kl_3m_3} \times t_{l_1m_1l_2m_2l_3m_3}$
$(2,1)$	15	$\begin{array}{\|c\|c\|c\|}\hline l & k & j \\ \hline m \\ \cline{1-1}\end{array}$	$\mathbf{15}$:$T^i_{kj} = e^{ilm}\ t_{lmkj}$
$(2,2)$	27	$\begin{array}{\|c\|c\|c\|c\|}\hline m_1 & m_2 & l & k \\ \hline n_1 & n_2 \\ \cline{1-2}\end{array}$	$\mathbf{27} :\ T^{ij}_{lk} = \varepsilon^{il_1m_1}\varepsilon^{jl_2m_2} \times t_{m_1n_1m_2n_3lk}$

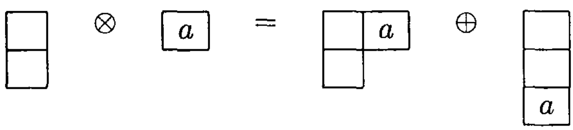
Examples

(i)



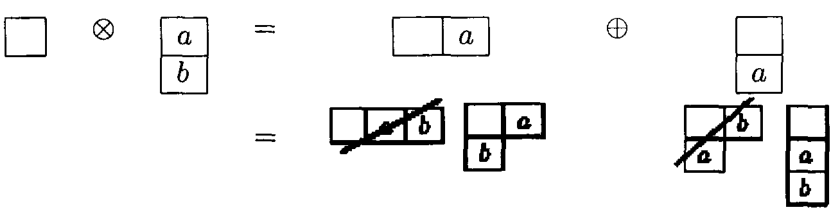
$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}} \tag{5.77}$$

(ii)



$$\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1} \tag{5.78}$$

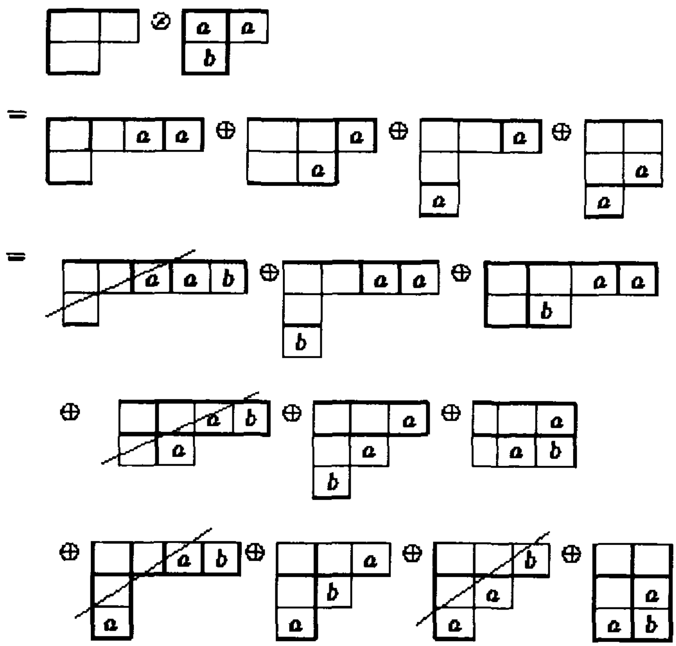
(iii)



$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1} \tag{5.79}$$

We discard the first and the third tableaux in Eq. (112) as they do not satisfy the constraint that number of a's greater than or equal to the number of b's as we go from right to left or top to bottom.

(iv)



(5.80)

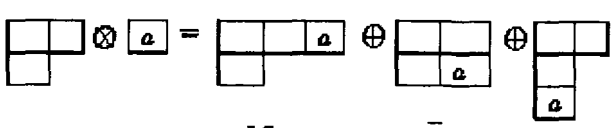
$$\mathbf{8} \otimes \mathbf{8} = \mathbf{10} \oplus \mathbf{27} \oplus \mathbf{8} \oplus \overline{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{1}$$

or

$$(1,1) \otimes (1,1) = (3,0) \oplus (2,2) \oplus (1,1) \oplus (0,3) \oplus (1,1) \oplus (0,0).$$

The slashed tableaux are discarded because they do not satisfy the constraint a's $\geq b$'s.

(v)



$\mathbf{15}$ $\bar{\mathbf{6}}$ $\mathbf{3}$

$$\mathbf{8} \otimes \mathbf{3} = \mathbf{15} \oplus \bar{\mathbf{6}} \oplus \mathbf{3} \tag{5.81}$$

or

$$(1,1)\otimes(1,0)=(2,1)\oplus(0,2)\oplus(1,0). \tag{5.82}$$

To summarize, an arbitrary irreducible representation of $SU(3)$ is denoted by two integers, each positive or zero: (p,q). The corresponding irreducible tensor is denoted by $T^{j_1\cdots j_q}_{i_1\cdots i_p}$. It transforms as $\phi^*_{j_1}\cdots\phi^*_{j_q}\cdot\phi_{i_1}\cdots\phi_{i_p}$. Each component of the tensor is an eigenstate of I_3 and Y and possibly of I^2. If it is not an eigenstate of I^2, such a state can be formed by a linear combination of states with components having the same I_3 and Y. The basic states occurring in (p,q) can be completely labelled by three quantities I, I_3, and Y, which form a complete commuting set within an irreducible representation. The values of I and Y that appear in (p,q) are given in Table 6. We note that highest state i.e. the one with I_{max} has

$$\begin{aligned} I_3 &= \frac{1}{2}(p+q) \\ Y &= \frac{1}{3}(p-q). \end{aligned} \tag{5.83}$$

5.5 SU(N)

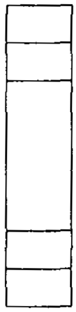
We now discuss Young's tableaux for $SU(N)$. Again we assign an index to a box. Thus fundamental representation $N(\phi_i,\ i=1\cdots N)$ is represented by a box:

$$\square : N \tag{5.84}$$

Table 5.6 Isospin I and hypercharge Y for the states in representation (p, q).

(p,q)	Y	I	Numberof states	I_3 and Y for the highest state
$(1,0)$	$\frac{1}{3}$ $\frac{-2}{3}$	$\frac{1}{2}$ 0	2 1	$\frac{1}{2}$, $\frac{1}{3}$
$(0,1)$	$\frac{2}{3}$ $\frac{-1}{3}$	0 $\frac{1}{2}$	1 2	$\frac{1}{2}$, $\frac{-1}{3}$
$(1,1)$	1 0 -1	$\frac{1}{2}$ $1, 0$ $\frac{1}{2}$	2 $3+1=4$ 2	1, 0
$(3,0)$	1 0 -1 -2	$\frac{3}{2}$ 1 $\frac{1}{2}$ 0	4 3 2 1	$\frac{3}{2}$, 1
$(2,1)$	$\frac{4}{3}$ $\frac{1}{3}$ $\frac{-2}{3}$ $\frac{-5}{3}$	1 $\frac{3}{2}, \frac{1}{2}$ $1, 0$ $\frac{1}{2}$	3 $4+2=6$ $3+1=4$ 2	$\frac{3}{2}, \frac{1}{3}$
$(2,2)$	2 1 0 -1 -2	1 $\frac{3}{2}, \frac{1}{2}$ $2, 1, 0$ $\frac{3}{2}, \frac{1}{2}$ 1	3 $4+2=6$ $5+3+1=9$ $4+2=6$ 3	2, 0

The tensor $\varepsilon_{i_1 i_2 \cdots i_N}$ is represented by a column of N boxes

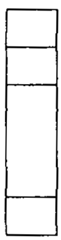


$: \mathbf{1}. \qquad (5.85)$

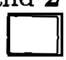
It describes the singlet representation **1** of $SU(N)$. Now $\varepsilon_{i_1 i_2 \cdots i_N}$ is a completely antisymmetric tensor:

$$\varepsilon_{i_1 i_2 \cdots i_N} = \left\{ \begin{array}{c} 0, \text{ if any of the two indices are equal} \\ \pm 1, \text{ if } i_1 \cdots i_N \text{ is an even (odd)} \\ \text{permutation of } 1, 2, \cdots N. \end{array} \right\} \qquad (5.86)$$

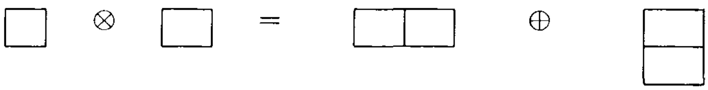
The N dimensional representation $\bar{N}$ is described by a column of $(N-1)$ boxes.



$: \bar{N}. \qquad (5.87)$

Hence we see that for $SU(2)$, **2** and $\bar{\mathbf{2}}$ are equivalent representation and both will be represented by . Only for $N \geq 3$, N and $\bar{N}$ are distinct representations.

We now discuss the decomposition of the product of representation N by itself into irreducible representations of $SU(N)$.



$$N \qquad N \quad = \quad \tfrac{1}{2}N\,(N+1) \qquad \tfrac{1}{2}N(N-1). \tag{5.88}$$

Thus N^2 components decompose into two irreducible representations of dimensions $\frac{N\,(N+1)}{2}$ and $\frac{N\,(N-1)}{2}$ viz.

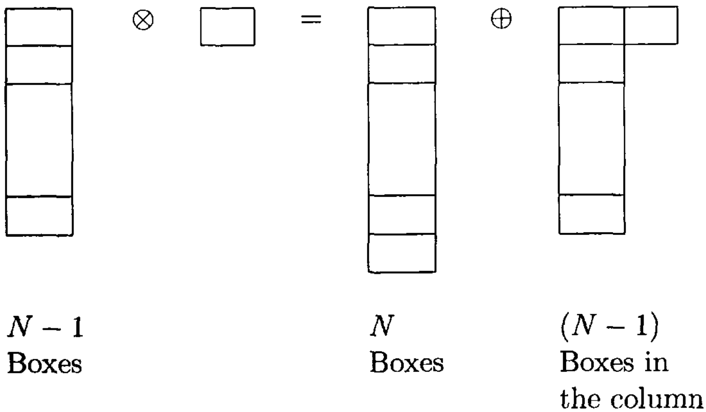
$$\phi_i\,\phi_j = \frac{1}{\sqrt{2}}\left(S_{ij} + A_{ij}\right), \tag{5.89}$$

where

$$S_{ij} = \frac{1}{\sqrt{2}}\left(\phi_i\,\phi_j + \phi_j\,\phi i\right). \tag{5.90a}$$

$$A_{ij} = \frac{1}{\sqrt{2}}\left(\phi_i\,\phi_j - \phi_j\,\phi i\right). \tag{5.90b}$$

We can regard S_{ij} as an $N \times N$ matrix, but since it is a symmetric matrix, it has only $\frac{N^2-N}{2} + N = \frac{1}{2}N\,(N+1)$ independent elements and this gives the dimension of symmetric representation S_{ij}. Again if we regard A_{ij} as $N \times N$ matrix, we can easily see that it has $\frac{N^2-N}{2} = \frac{1}{2}N\,(N-1)$ independent elements and this gives the dimension of antisymmetric representation A_{ij}.



$$\bar{N} \otimes N = 1 \oplus \text{Adjoint representation of dimension } N^2 - 1. \tag{5.91}$$

Thus

$$\phi^i\ \phi_j = T^i_j + \frac{1}{N}\ \delta^i_j\ \phi^k\ \phi_k, \tag{5.92a}$$

where

$$T^i_j = \phi^i\ \phi_j - \frac{1}{N}\ \delta^i_j\ \phi^k\ \phi_k. \tag{5.92b}$$

The adjoint representation has the same dimension as the number of generators of $SU(N)$. For example for $SU(6)$: $\bar{\mathbf{6}} \otimes\ \mathbf{6} =\ \mathbf{1} \oplus\ \mathbf{35}$.

We now give a general recipe to calculate the dimension of irreducible representations in the decomposition of the product of representation N by itself. To calculate the dimension of an array of boxes there is a recipe which involves calculation of $\frac{\text{Numerator}}{\text{Denominator}}$.

Numerator: Insert N in each of the diagonal boxes starting from the top left hand corner of the tableaux.

N	$N+1$	$N+2$
$N-1$	N	$N+1$
$N-2$	$N-1$	N

(5.93)

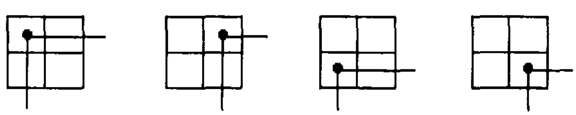
Along the diagonals immediately above and below insert $N+1$ and $N-1$ respectively. In the next diagonals insert $N+2$ and so on. The numerator is equal to the product of all these numbers. For example for the tableaux

N	$N+1$
$N-1$	N

(5.94)

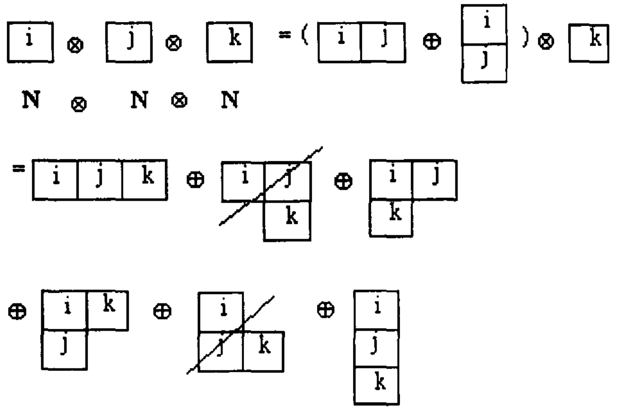
the numerator $= N^2\ (N+1)\ (N-1) = N^2\ (N^2-1)$.

Denominator: The denominator is given by the "product of hooks". We associate each box with a value of the hook. To find it, draw a line entering the row in which the box lies from the right. On entering the box, this line turns downwards through an angle of 90^o and then proceeds along the column until it leaves the diagram. The value of hook associated with that box is then the total number of boxes that the line has passed through, including the box in question. The product of hooks is the denominator. We illustrate this by the following example. Consider the tableau (94). The hooks associated with each box are shown in Eq. (95).



3 2 2 1 (5.95)

We see that the denominator $= 3 \times 2 \times 2 \times 1 = 12$. Hence the tableau (94) corresponds to an irreducible representation of dimension $N^2\left(N^2-1\right)/12$. Let us now consider some more examples:



(5.96)

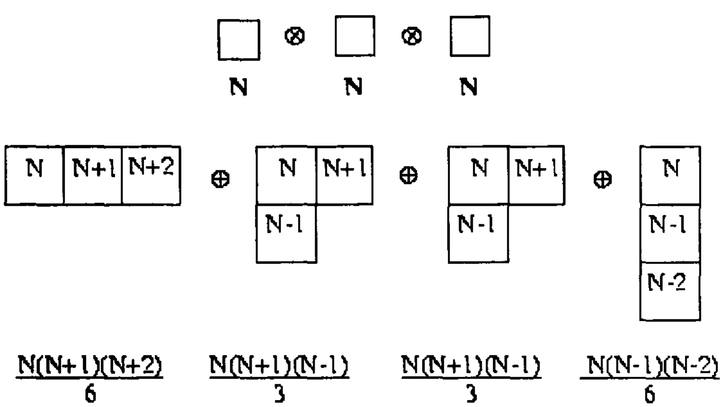
To avoid double counting, we discard the slashed tableaux. Thus we can write

$$\phi_i\ \phi_j\ \phi_k \sim T_{\{ijk\}} + T_{[ij]k} + T_{[ik]j} + T_{[ijk]}. \tag{5.97}$$

Note further that

$$T_{[ij]k} + T_{[ki]j} + T_{[jk]i} = 0. \tag{5.98}$$

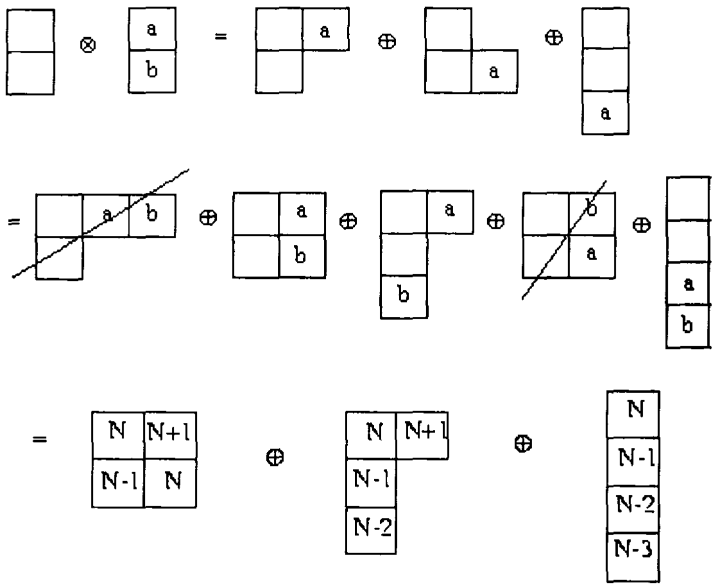
In order to find the dimension of these representations, we note that



(5.99)

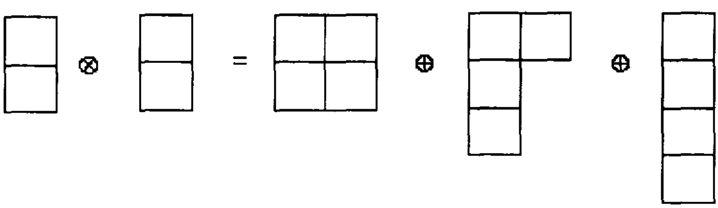
For example for SU(6) : $\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56} \oplus \mathbf{70} \oplus \mathbf{70} \oplus \mathbf{20}$.

(2)



$$\frac{N^2(N^2-1)}{12} \qquad \frac{N(N+1)(N-1)(N-2)}{4\text{x}2\text{x}1} \qquad \frac{N(N-1)(N-2)(N-3)}{4\text{x}3\text{x}2\text{x}1} \tag{5.100}$$

Hence we have



$$\frac{N(N-1)}{2} \quad \frac{N(N-1)}{2} \quad \frac{N^2(N^2-1)}{12} \quad \frac{N(N^2-1)(N-2)}{8} \quad \frac{N(N-1)(N-2)(N-3)}{24} \tag{5.101}$$

For example for SU(3) : $\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} = \bar{\mathbf{6}} \oplus \mathbf{3}$ and for SU(5) : $\mathbf{10} \otimes \mathbf{10} = \mathbf{50} \oplus \overline{\mathbf{45}} \oplus \bar{\mathbf{5}}$.

5.6 Applications of Flavor SU(3)

1. SU(3) Invariant BBP Couplings

If O_A is an octet operator, the matrix elements of this operator between the states $|8,B\rangle$ and $|8,C\rangle$ can be written as

$$\langle 8,C\,|O_A|\,8,B\rangle = i\; f_{ABC}\; F + d_{ABC}\; D. \tag{5.102}$$

That there are two independent couplings follow from the fact that as noted previously 8⊗8 contains 8 twice. In particular if O_A is pseudoscalar meson octet operator P_A, and $|8,B\rangle$, $|8,C\rangle$ are octet of baryon states, the BBP couplings can be written as

$$g_{ABC} = 2g\;\left[i\; f_{ABC}\; f + d_{ABC}\; d\right]. \tag{5.103}$$

For example

$$\begin{aligned} g_{\pi^0 pp} &= 2g\;\left[i\; f_3 \frac{4+i5}{\sqrt{2}}\frac{4-i5}{\sqrt{2}}\; f + d_3 \frac{4+i5}{\sqrt{2}}\frac{4-i5}{\sqrt{2}} d\right] \\ &= g\,(f+d) = -g_{\pi^0 nn} = \frac{g_{\pi^- pn}}{\sqrt{2}}. \end{aligned} \tag{5.104}$$

We normalize $g_{\pi^0 pp} = g$, so that $f+d=1$. Then

$$g_{Kp\Lambda} = g\left(-\sqrt{3}f - \frac{1}{\sqrt{3}}d\right) = -\frac{g}{\sqrt{3}}\,(3f+d) = -\frac{g}{\sqrt{3}}\,(1+2f)\,. \tag{5.105}$$

In this way we can calculate all the relevant couplings:

$$\begin{aligned} g_{\pi\Lambda\Sigma} &= \frac{2}{\sqrt{3}}(1-f)\;g,\;\; g_{\pi\Sigma\Sigma} = 2\;f\;g,\;\; g_{\pi\Xi\Xi} = -(1-2f)\;g \\ g_{KN\Lambda} &= -\frac{1}{\sqrt{3}}(1+2f)\;g,\;\; g_{KN\Sigma} = -(1-2f)\;g \\ g_{K\Lambda\Xi} &= -\frac{1}{\sqrt{3}}(1-4f)\;g,\;\; g_{K\Sigma\Xi} = -\;g \\ g_{\eta_8 NN} &= -\frac{1}{\sqrt{3}}(1-4f)\;g,\;\; g_{\eta_8\Lambda\Lambda} = -\frac{2}{\sqrt{3}}(1-f)\;g \\ g_{\eta_8\Sigma\Sigma} &= \frac{2}{\sqrt{3}}(1-f)\;g,\;\; g_{\eta_8\Xi\Xi} = -\frac{1}{\sqrt{3}}(1+2f)\;g. \end{aligned} \tag{5.106}$$

Experimentally

$$\begin{aligned} \frac{g_{\pi NN}^2}{4\pi} &\equiv \frac{g^2}{4\pi} \approx 14 \\ f &\geq 0.35, \qquad r \equiv \frac{d}{f} = \frac{1-f}{f} \leq 1.85. \end{aligned} \qquad (5.107)$$

2. VPP Coupling

Here we take $O_A = V_A$, the vector meson octet and $|8, B\rangle$ and $|8, C\rangle$ are octet of pseudoscalar meson states. Now under charge conjugation C:

$$\begin{aligned} V_A &\rightarrow -\eta_A V_A, \text{ (no summation over } A) \\ |8, B\rangle &\rightarrow \eta_B \,|8, B\rangle\,, \end{aligned} \qquad (5.108)$$

where

$$\eta_B = \begin{pmatrix} +1, & B = 1, 3, 4, 6, 8 \\ -1, & B = 2, 5, 7 \end{pmatrix}. \qquad (5.109)$$

Hence the invariance under charge conjugation gives

$$\begin{aligned} [\gamma_F \; i \; f_{ABC} + \gamma_D \; d_{ABC}] &\equiv \langle 8, C\,|V_A|\,8, B\rangle \\ &\rightarrow -\eta_A \; \eta_B \; \eta_C \; \langle 8, C\,|V_A|\,8, B\rangle \\ &= -\eta_A \; \eta_B \; \eta_C \; [\gamma_F \; i \; f_{ABC} + \gamma_D \; d_{ABC}]. \end{aligned} \qquad (5.110)$$

But [cf. Eq. (30)]

$$\begin{aligned} \eta_A \; \eta_B \; \eta_C \; f_{ABC} &= -f_{ABC} \\ \eta_A \; \eta_B \; \eta_C \; d_{ABC} &= d_{ABC}. \end{aligned} \qquad (5.111)$$

Therefore, we have

$$\gamma_D = -\gamma_D \quad \text{or} \quad \gamma_D = 0.$$

Hence VPP has only F-type coupling. Thus

$$\langle 8, C\,|V_A|\,8, B\rangle = i \; f_{ABC} \; 2\gamma, \qquad (5.112)$$

where we have put $\gamma_F = 2\gamma$. For example $V_3 = \rho^0$:

$$\left\langle 8, \frac{1-i\,2}{\sqrt{2}} \left| V_3 \right| 8, \frac{1+i\,2}{\sqrt{2}} \right\rangle = i\ f_3\ \frac{1+i\,2}{\sqrt{2}} \frac{1-i\,2}{\sqrt{2}}\ 2\gamma = 2\gamma. \quad (5.113)$$

Thus $\gamma_{\rho\pi\pi} = 2\gamma$. It is straightforward to calculate the VPP coupling for other members of the octet, which are given below:

$$\gamma_{\rho\pi\pi} = 2\gamma,\ \gamma_{K^{*+}\pi^0 K^+} = \gamma = \frac{1}{\sqrt{2}} \gamma_{K^{*+}\pi^+ K^0},\ \gamma_{\omega_8 K \bar{K}} = \sqrt{3}\gamma. \quad (5.114)$$

The decay width of decay $V \to PP$ is given by

$$\Gamma(V \to PP) = \frac{\gamma_{VPP}^2}{4\pi} \frac{2}{3} \left(\frac{p_{cm}^3}{m_V^2} \right). \quad (5.115)$$

Hence we have

$$\Gamma(\rho \to \pi\pi) = \frac{\gamma^2}{4\pi} \frac{8}{3} \left(\frac{p_{\pi\pi}^3}{m_V^2} \right) = 149.1 \pm 2.9 \text{ MeV}. \quad (5.116)$$

This gives $\frac{\gamma^2}{4\pi} \approx 0.74$. Now

$$\begin{aligned} \Gamma_{\text{tot}}\left(K^{*+} \to K\pi\right) &= \Gamma\left(K^{*+} \to \pi^0 K^+\right) + \Gamma\left(K^{*+} \to \pi^+ K^0\right) \\ &= \frac{\gamma^2}{4\pi}(1+2)\frac{2}{3}\frac{p_{K\pi}^3}{m_{K^*}^2}, \end{aligned}$$

and

$$\frac{\Gamma_{\text{tot}}\left(K^{*+} \to K\pi\right)}{\Gamma(\rho \to \pi\pi)} = \frac{3}{4} \frac{p_{K\pi}^3 / m_{K^*}^2}{p_{\pi\pi}^3 / m_\rho^2} \approx 0.29. \quad (5.117)$$

This gives $\Gamma_{\text{tot}}\left(K^{*+} \to K\pi\right) \approx 44.5$ MeV to be compared with the experimental value 49.8 ± 0.8 MeV.

In broken $SU(3)$, ω_8 can mix with the singlet ω_1, so that the physical particles ω and ϕ are linear combinations of ω_8 and ω_1:

$$\begin{aligned} \phi &= \omega_8 \cos\theta - \omega_1 \sin\theta \\ \omega &= \omega_8 \sin\theta + \omega_1 \cos\theta \\ \begin{pmatrix} \phi \\ \omega \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \omega_8 \\ \omega_1 \end{pmatrix}. \end{aligned} \quad (5.118)$$

We now show that $\omega_1 \to PP$ is forbidden by charge conjugation invariance. The invariant coupling in this case is

$$\omega_{1\mu}\, P^i_j\, \partial_\mu\, P^j_i$$

which changes sign under charge conjugation. Hence

$$\Gamma\left(\phi \to K^+K^-\right) = \cos^2\theta\, \Gamma\left(\omega_8 \to K\bar{K}\right).$$

Therefore,

$$\Gamma\left(\phi \to K^+K^-\right) = \cos^2\theta\, \frac{\gamma^2}{4\pi}\left(3 \times \frac{2}{3}\right)\left(\frac{p^3_{KK}}{m^2_\phi}\right), \tag{5.119}$$

and

$$\frac{\Gamma\left(\phi \to K^+K^-\right)}{\Gamma\left(\rho \to \pi\pi\right)} = \frac{3}{4}\cos^2\theta\,\left(\frac{p^3_{KK}/m^2_\phi}{p^3_{\pi\pi}/m^2_\rho}\right) = 0.013, \tag{5.120}$$

where we have used $\cos^2\theta = 2/3$ [cf. Eq. (153)]. This gives $\Gamma\left(\phi \to K^+K^-\right) = 1.95$ MeV to be compared with the experimental value 2.1 MeV.

5.7 Mass Splitting in Flavor SU(3)

In exact $SU(3)$, the particles belonging to an irreducible representation of $SU(3)$ must have the same mass. But we note that all members of a supermultiplet do not have the same mass. This means that $SU(3)$ is not an exact symmetry of strong interactions, but is a broken symmetry of these interactions. This means that the interaction Hamiltonian consists of two parts viz.

$$H = H_0 + H_1, \tag{5.121a}$$

where

$$\left[F^i_j,\ H_0\right] = 0 \tag{5.121b}$$

$$\left[F^i_j,\ H_1\right] \neq 0 \tag{5.121c}$$

i.e. H_0 is $SU(3)$ invariant, but H_1 breaks the $SU(3)$ symmetry. If we take H_1 such that

$$[\mathbf{I},\ H_1] = 0, \qquad [Y,\ H_1] = 0, \tag{5.122}$$

H_1 still preserves the isospin symmetry and hypercharge is conserved in its presence. The first of Eqs. (122) holds only in the absence of electromagnetic interaction. In order that $SU(3)$ to be meaningful, H_1 must be at least an order of magnitude weaker than H_0.

The simplest general form of H_1 in $SU(3)$ which satisfies Eqs. (121c) and (122) is

$$H_1 \sim T_3^3 \text{ or } \lambda_8. \tag{5.123}$$

To get H_1 from the quark model, the mass Hamiltonian for quarks is given by

$$H_q = m_u\ \bar{u}\ u + m_d\ \bar{d}\ d + m_s\ \bar{s}\ s,$$

where m_u, m_d and m_s are masses of u-quark, d-quark and s-quark respectively. In the exact $SU(3)$ limit, $m_u = m_d = m_s$. If $SU(3)$ is broken but isospin symmetry $SU(2)$ is still exact, then $m_u = m_d \neq m_s$. Now we can write

$$\begin{aligned} H_q &= \bar{m}\ \left(\bar{u}\ u + \bar{d}\ d\right) + m_s\ \bar{s}\ s + (m_u\ - m_d)\,\frac{\bar{u}\ u - \bar{d}\ d}{2} \\ &= \frac{2\bar{m}\ + m_s}{3}\left(\bar{u}\ u + \bar{d}\ d + \bar{s}\ s\right) \\ &\quad + \frac{\bar{m}\ - m_s}{\sqrt{3}}\frac{1}{\sqrt{3}}\left(\bar{u}\ u + \bar{d}\ d - 2s\bar{s}\ \right) \\ &\quad + (m_u\ - m_d)\,\frac{1}{2}\left(\bar{u}\ u - \bar{d}\ d\right), \end{aligned} \tag{5.124}$$

where

$$\begin{aligned} \bar{m} &= \frac{m_u\ + m_d}{2},\ \frac{2\bar{m}\ + m_s}{3} = \frac{m_u\ + m_d + m_s}{3} \equiv m_q \\ \frac{(\bar{m}\ - m_s)}{\sqrt{3}} &= \frac{m_u\ + m_d - 2m_s}{2\sqrt{3}} \equiv \frac{\lambda}{2}. \end{aligned} \tag{5.125}$$

Hence we can write

$$H_q = m_q\ \bar{q}\ q + \lambda\ \bar{q}\ \frac{\lambda_8}{2} q + (m_u\ - m_d)\ \bar{q}\ \frac{\lambda_3}{2} q. \tag{5.126}$$

This also shows that $SU(3)$ symmetry breaking term transforms as λ_8 under $SU(3)$.

It was shown by Okubo that for any irreducible representation (p, q) of $SU(3)$, the matrix elements of tensor T_3^3 are given by

$$\left\langle (p,q)I,\ Y \left| T_3^3 \right| (p,q)I,\ Y \right\rangle = a + bY + c\left[\frac{Y^2}{4} - I\,(I+1)\right], \tag{5.127}$$

where a, b, c are independent of quantum numbers I and Y but in general depend on (p, q). Thus we can write the mass formula for particles in a multiplet of $SU(3)$ as

$$m = m_0 + \Delta m = a + bY + c\left[\frac{Y^2}{4} - I\,(I+1)\right]. \tag{5.128}$$

Let us apply this formula to baryon octet. Then from Eq. (128), we get

$$\frac{m_N + m_\Xi}{2} = \frac{3m_\Lambda + m_\Sigma}{4} \tag{5.129}$$

whereas for pseudoscalar meson octet, we get

$$m_K^2 = \frac{3\ m_{\eta_8}^2 + m_\pi^2}{4}. \tag{5.130}$$

In Eq. (130), we have used squared masses, as in the Lagrangian for bosons, the square of boson masses appear. Equations (129) and (130) are well known Gell-Mann-Okubo mass formulae.

For the decuplet

$$I = \frac{Y}{2} + 1, \tag{5.131}$$

and Eq. (128) reduces to

$$m = a' + b'Y \tag{5.132}$$

and we obtain the equal spacing rule for the decuplet:

$$m_\Omega - m_{\Xi^*} = m_{\Xi^*} - m_{\Sigma^*} = m_{\Sigma^*} - m_\Delta. \tag{5.133}$$

The mass relations (129) and (133) are well satisfied experimentally and are regarded as a great success of $SU(3)$. Similarly for vector bosons we get

$$m_{K^*}^2 = \frac{3\, m_{\omega_8}^2 + m_\rho^2}{4}. \tag{5.134}$$

Since due to mixing between ω_8 and the singlet ω_1, the physical particles are ϕ and ω, the formula (134) is not directly applicable. We will come to this formula later. Similar remarks are applicable to the mass formula (130).

For octet and decuplet representations of $SU(3)$, one can easily derive the mass formula as follows. We note from Eq. (126) that

$$\begin{aligned} H_1 &\sim \bar{q}\, \frac{\lambda_8}{2}\, q = q_i \left(\frac{\lambda_8}{2}\right)^i_j q^j \\ &= \left(\frac{\lambda_8}{2}\right)^i_j O^j_i = (T_8)^i_j\ O^j_i \end{aligned} \tag{5.135}$$

where O^i_j is an octet operator viz.

$$O^i_j = q^i\, q_j - \frac{1}{3}\delta^i_j\, q^k\, q_k \tag{5.136}$$

and

$$T_8 = \frac{\lambda_8}{2}. \tag{5.137}$$

Hence we see that H_1 transforms under $SU(3)$ as

$$\begin{aligned} H_1 &\sim (T_8)^i_j\ O^j_i = \left(M^3_3\right)^i_j\ O^j_i = \left(\delta^3_j\, \delta^i_3 - \frac{1}{3}\delta^3_3\, \delta^i_j\right) O^j_i \\ &= O^3_3. \end{aligned} \tag{5.138}$$

Thus to first order in λ , the mass splitting for the state $|A\rangle$ of an $SU(3)$ multiplet is given by

$$\begin{aligned} \Delta m &= \lambda \ \langle A|H_I|\,A\rangle \\ &= \lambda \ \left\langle A\middle|O_3^3\middle|\,A\right\rangle . \end{aligned} \tag{5.139}$$

Let us apply it to baryon octet:

$$\begin{aligned} O_3^3 &= O_F\left(\bar{B}_3^i \ B_i^3 - \bar{B}_j^3 \ B_3^j\right) + O_D\left(\bar{B}_3^i \ B_i^3 + \bar{B}_j^3 \ B_3^j\right) \\ &= O_F\left(\bar{\Xi}^-\Xi^- + \bar{\Xi}^0\Xi^0 - \bar{p}p - \bar{n}n\right) \\ &\quad + O_D\left[\bar{\Xi}^-\Xi^- + \bar{\Xi}^0\Xi^0 + \bar{p}p + \bar{n}n + 2\left(-\frac{2}{\sqrt{6}}\bar{\Lambda}\right)\left(-\frac{2}{\sqrt{6}}\Lambda\right)\right]. \end{aligned} \tag{5.140}$$

Hence we have

$$\begin{aligned} m_p &= m_0 + \lambda\left(-O_F + O_D\right) = m_n \\ m_\Sigma &= m_0 \\ m_\Lambda &= m_0 + \frac{4}{3}\lambda \ O_D \\ m_\Xi &= m_0 + \lambda\left(O_F + O_D\right). \end{aligned} \tag{5.141}$$

This gives the Gell-Mann-Okubo mass formula (129) for baryons.

For the decouplet, we have

$$\lambda \ O_3^3 = \lambda \ \bar{T}^{ij3} T_{ij3}. \tag{5.142}$$

This is the only possibility as T_{ijk} is a completely symmetric tensor.

$$\begin{aligned} \lambda \ O_3^3 &= \left[\bar{T}^{113}T_{113} + 2\bar{T}^{123}T_{123} + 2\bar{T}^{133}T_{133} \right. \\ &\quad \left. + 2\bar{T}^{233}T_{233} + \bar{T}^{223}T_{223} + \bar{T}^{333}T_{333}\right] \\ &= \lambda\left[2 \ \bar{\Sigma}^{*+} \ \Sigma^{*+} + 2 \ \bar{\Sigma}^{*0} \ \Sigma^{*0} + 2 \ \bar{\Sigma}^{*-} \ \Sigma^{*-} \right. \\ &\quad \left. + 4\bar{\Xi}^{*-}\Xi^{*-} + 4\bar{\Xi}^{*0}\Xi^{*0} + 6\bar{\Omega}^-\Omega^-\right]. \end{aligned} \tag{5.143}$$

This gives

$$\begin{aligned} m_{\Delta} &= m_0 \\ m_{\Sigma^*} &= m_0 + 2\lambda \\ m_{\Xi^*} &= m_0 + 4\lambda \\ m_{\Omega^-} &= m_0 + 6\lambda, \end{aligned} \tag{5.144}$$

and hence we have the mass relation (133) for the decuplet.

For octet of vector mesons

$$\begin{aligned} O_3^3 &= O_D\left[V_3^i\, V_i^3 + V_j^3\, V_3^j\right] \\ &= O_D\left[\bar{K}^{*^+}K^{*^+} + \bar{K}^{*^-}K^{*^-} + \bar{K}^{*^0}K^{*^0} + 2\left(-\frac{2}{\sqrt{6}}\,\omega_8\right)\right. \\ &\qquad \left. \times\left(-\frac{2}{\sqrt{6}}\,\omega_8\right)\right]. \end{aligned} \tag{5.145}$$

Hence we have

$$\begin{aligned} m_\rho^2 &= m_0^2 \\ m_{K^*}^2 &= m_0^2 + \lambda\, O_D \\ m_{\omega_8}^2 &= m_0^2 + \frac{4}{3}\lambda\, O_D. \end{aligned} \tag{5.146}$$

This gives the octet formula (134) for vector mesons. Now ω_8 and ω_1 mix, when $SU(3)$ is broken, the mass matrix in ω_8 and ω_1 basis can be written

$$M^2 = \begin{pmatrix} m_8^2 & m_{18}^2 \\ m_{18}^2 & m_1^2 \end{pmatrix}. \tag{5.147}$$

Using Eq. (118), we can diagonalize it:

$$U^T M^2 U = \begin{pmatrix} m_\phi^2 & 0 \\ 0 & m_\omega^2 \end{pmatrix}, \tag{5.148}$$

where

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{5.149}$$

This gives

$$\begin{aligned} m_\phi^2 + m_\omega^2 &= m_8^2 + m_1^2 & (5.150a) \\ m_\phi^2 - m_\omega^2 &= \left(\cos^2\theta - \sin^2\theta\right)\left(m_8^2 - m_1^2\right) \\ &\quad +4\sin\theta\cos\theta\ m_{18}^2, & (5.150b) \end{aligned}$$

$$\begin{aligned} \tan 2\theta &= \frac{2\ m_{18}^2}{m_8^2 - m_1^2} & (5.151a) \\ \tan^2\theta &= \frac{m_\phi^2 - m_8^2}{m_8^2 - m_\omega^2} = \frac{3\ m_\phi^2 - 4\ m_{K^*}^2 + m_\rho^2}{4\ m_{K^*}^2 - m_\rho^2 - 3\ m_\omega^2}. & (5.151b) \end{aligned}$$

Now using $m_{K^*} = 892$ MeV, $m_\rho = 770$ MeV, $m_\omega = 783$ MeV and $m_\phi = 1020$ MeV, we have $m_8 \equiv m_{\omega 8} = 930$ MeV, $m_1 \equiv m_{\omega 1} = 880$ MeV and

$$\tan\theta \approx 0.84, \qquad \theta \approx 40^0. \tag{5.152}$$

It is tempting to take

$$\tan\theta \approx \frac{1}{\sqrt{2}} \approx 0.71, \qquad \theta \approx 35.3^0. \tag{5.153}$$

For this case $\sin\theta \approx \frac{1}{\sqrt{3}}$, $\cos\theta \approx \frac{\sqrt{2}}{\sqrt{3}}$ and

$$\begin{aligned} |\phi\rangle &= \frac{\sqrt{2}}{\sqrt{3}}|\omega_8\rangle - \frac{1}{\sqrt{3}}|\omega_1\rangle = -\,|s\ \bar{s}\rangle & (5.154a) \\ |\omega\rangle &= \frac{1}{\sqrt{3}}|\omega_8\rangle + \frac{\sqrt{2}}{\sqrt{3}}|\omega_1\rangle = \left|\frac{u\ \bar{u} + d\ \bar{d}}{\sqrt{2}}\right\rangle. & (5.154b) \end{aligned}$$

Hence we have

$$m_\omega^2 = m_\rho^2 \tag{5.155a}$$

and from Eqs. (151b) and (153)

$$m_\phi^2 - m_\omega^2 = 2\left(m_{K^*}^2 - m_\rho^2\right). \tag{5.155b}$$

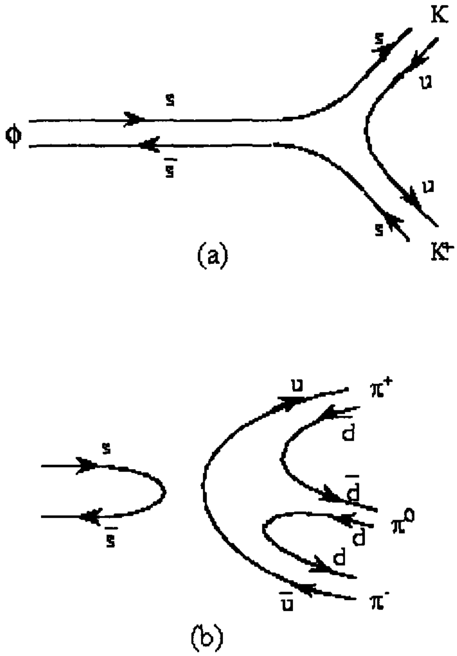


Figure 6 a): $\phi \to K\bar{K}$ decay allowed by OZI rule, b): ϕ decay suppressed by OZI rule

Equation (153) gives the "ideal mixing". With this mixing ϕ is made up of $s\ \bar{s}$ i.e. of strange quarks only. Experimentally it is observed that $\phi \to \rho\pi$ or 3π is very much suppressed as compared with $\phi \to K\bar{K}$. Note that $\rho\pi$ or 3π do not contain any strange quark. The suppression of ϕ decay into non-strange particles is explained by the so-called Okubo-Zweig-Iizuka rule (OZI rule): "**The decays which correspond to disconnected quark diagrams are forbidden**". Thus the decay in Fig. 6a is allowed but the decay in Fig. 6b is forbidden. There is no theoretical basis for the OZI rule. No strong interaction selection rule forbids the decay of $\phi \to \rho\pi$ or 3π. But experimentally this rule seems to be well satisfied. The small decay width for $\phi \to \rho\pi$ (3π) can be explained by some deviation from the "ideal mixing" which allows small admixture of non-strange quarks in ϕ.

5.8 Problems

1. Show that for a vector operator $O_i (i = 1, 2)$ under $SU(2)$

$$[I_A,\ O_i] = O_j \left(\frac{\tau_A}{2}\right)_i^j, \qquad A = 1, 2, 3.$$

Given

$$\langle \alpha, 3/2, -1/2 \,| O_1 |\, \beta, 1, -1 \rangle = F,$$

find

$$\langle \alpha, 3/2, -3/2 \,| O_2 |\, \beta, 1, -1 \rangle .$$

The states are labelled as $|\alpha, I, I_3\rangle$.

2. Suppose that $(\tau_A)^i_j \equiv (\tau_A)_{ij}$ and $(\sigma_A)^\alpha_\beta \equiv (\sigma_A)_{\alpha\beta}$ are Pauli matrices in two different two dimensional spaces. In the four dimensional product space, define the basis vectors

$$\begin{aligned} |\mu = 1\rangle &= |i = 1\rangle\ |\alpha = 1\rangle\,,\ |\mu = 2\rangle = |i = 1\rangle\ |\alpha = 2\rangle \\ |\mu = 3\rangle &= |i = 2\rangle\ |\alpha = 1\rangle\,,\ |\mu = 4\rangle = |i = 2\rangle\ |\alpha = 2\rangle\,. \end{aligned}$$

Define
$T_{AB} = \tau_A \otimes \sigma_B;\ (T_{AB})^\mu_\nu = (\tau_A)^i_j\ (\sigma_B)^\alpha_\beta\,,$
$\mu, \nu = 1, \cdots, 4, \qquad A, B = 1, 2, 3.$
Evaluate
$T_{21} = (\tau_2 \otimes \sigma_1)\,,$ as a 4×4 matrix.

3. A second ranked mixed tensor T^i_j transforms as $\phi^*_i\ \phi_j$, under the unitary transformation

$$\phi'_i = a^j_i\ \phi_j,$$

show that

$$\left[F^i_j,\ T^k_l\right] = \delta^i_l\ T^k_j - \delta^k_j\ T^i_l.$$

4. (a). Using the following relation for $SU(3)$:

$$\left[F^i_j,\ q_k\right] = \delta^i_k\ q_j - \frac{1}{3}\delta^i_j\ q_k,$$

show that

$$F^2_1\ |d\rangle = |u\rangle\,, \qquad F^1_3\ |u\rangle = |s\rangle\,.$$

(b). Using the relation

$$\left[F^i_j,\ \bar{B}^k_l\right] = \delta^j_l\ \bar{B}^k_j - \delta^k_j\ \bar{B}^i_l$$

and Eq. (60b) of the text, show that

$$F^1_2\ \left|\Sigma^+\right\rangle = -\sqrt{2}\left|\Sigma^0\right\rangle, \qquad F^2_3\ |p\rangle = -\left|\Sigma^+\right\rangle.$$

5. From the group property

$$U^{-1}(b)\ U(a)\ U(b) = U(b^{-1}ab),$$

derive the commutation relation for the generators of the unitary group $U(N)$:

$$\left[A^j_i,\ A^l_k\right] = \delta^j_k\ A^l_i - \delta^l_i\ A^j_k.$$

6. Show that λ_1, λ_2 and λ_3 generate an $SU(2)$ subalgebra of $SU(3)$. Show that the representations generated by the remaining λ's or their linear combinations transform as doublets and singlet representations of $SU(2)$.

7. Show that λ_2, λ_5 and λ_7 generate an $SU(2)$ subalgebra of $SU(3)$. Show that the representation generated by the linear combinations of remaining λ's transform as 5-dimensional representation of $SU(2)$.
Hint: $\lambda_5 \mp i\lambda_2$ act as raising and lowering operators.

8. Find the matrix generators λ_A $(A = 1 \cdots 15)$ for the group $SU(4)$.

9. The following assignments for 3 quarks are given instead of usual ones:

	B	S	I	I_3
u'	1	-2	1/2	1/2
d'	1	-2	1/2	$-1/2$
s'	1	-3	0	0

Find the charge Q and hypercharge Y for each quark in this case. Mesons can be constructed as $\bar{q}'q$ as before. If baryons are constructed as $\bar{q}'q'q'$, can the above assignment of quarks work? If not, discuss the difficulties encountered.

10. Find the U-spin eigenstates for the baryon octet and decuplet. Plot them on Q versus U_3 plot.

11. As far as $SU(3)$ is concerned, magnetic moment operator transforms as T_1^1 which is singlet under U-spin. Using this fact and the U-spin multiplets found above, show that the baryon octet magnetic moments are related as follows:

$$\begin{aligned}
\mu_{\Sigma^+} &= \mu_p, \quad \mu_{\Sigma^-} = \mu_{\Xi^-}, \quad \mu_{\Xi^0} = \mu_n = \frac{1}{2}\left(3\mu_\Lambda - \mu_{\Sigma^0}\right), \\
\mu_{\Sigma^0-\Lambda^0} &= -\frac{\sqrt{3}}{2}\left(\mu_\Lambda - \mu_\Sigma\right).
\end{aligned}$$

12. In $SU(3)$, find

$$\mathbf{10} \otimes \mathbf{8}, \qquad \overline{\mathbf{10}} \otimes \mathbf{10}, \qquad \mathbf{8} \otimes \bar{\mathbf{3}}.$$

13. In $SU(5)$, show that

$$\begin{aligned}
\bar{\mathbf{5}} \otimes \mathbf{5} &= \mathbf{24} \oplus \mathbf{1}, & \mathbf{10} \otimes \mathbf{10} &= \bar{\mathbf{5}} \oplus \mathbf{50} \oplus \overline{\mathbf{45}} \\
\bar{\mathbf{5}} \otimes \mathbf{10} &= \mathbf{5} \oplus \overline{\mathbf{45}}, & \overline{\mathbf{10}} \otimes \mathbf{10} &= \mathbf{1} \oplus \mathbf{24} \oplus \mathbf{75}.
\end{aligned}$$

14. Consider the representation 6 of $SU(3)$. Write down the particle content of this representation in terms of quarks. If $SU(3)$ breaking Hamiltonian H_I transforms as O_3 or T_8, write down the mass formula for these particles.

15. Draw the weight diagrams for the 15 plet and 27 plet representations of $SU(3)$

16. Consider the O^- nonet. Experimental masses are

$$m_\pi = 137 \text{ MeV}, \quad m_K = 496 \text{ MeV},$$
$$m_\eta = 549 \text{ MeV}, \quad m_{\eta'} = 958 \text{ MeV}.$$

From the octet mass formula, find m_{η_8}. Compare it with m_η. Assuming that discrepancy between the two values is entirely due to $\eta_1 - \eta_8$ mixing in broken $SU(3)$, so that

$$|\eta'\rangle = \cos\theta \; |\eta_1\rangle + \sin\theta \; |\eta_8\rangle \,, \quad |\eta\rangle = -\sin\theta \; |\eta_1\rangle + \cos\theta \; |\eta_8\rangle \,,$$

find from the experimental masses and m_{η_8} , the values of m_{η_1} and the mixing angle θ. If we write

$$|\eta\rangle = \cos\phi \; |\eta_{ns}\rangle - \sin\phi \; |\eta_s\rangle \,, \quad |\eta'\rangle = \sin\phi \; |\eta_{ns}\rangle + \cos\phi \; |\eta_s\rangle \,,$$

where

$$|\eta_{ns}\rangle = \frac{1}{\sqrt{2}} \left|\bar{u}\, u + \bar{d}\, d\right\rangle, \quad |\eta_s\rangle = \frac{1}{\sqrt{2}} \left|\bar{s}\, s\right\rangle,$$

show that

$$\phi = \tan^{-1}\sqrt{2} + \theta.$$

17. You are given an octet operator

$$O = \cos\theta \; O_{1+i2} + \sin\theta \; O_{4+i5},$$

determine the $SU(3)$ matrix elements for the transitions:

$$\begin{array}{cccc} n \to p \;, & \Sigma^- \to n \;, & \Sigma \to \Lambda \;, & \Sigma^0 \to p \\ \Xi^- \to \Xi^0 \;, & \Xi^0 \to \Lambda \;, & \Xi^0 \to \Sigma \;, & \Xi^- \to \Sigma^+ \end{array}$$

in terms of F, D and θ.

18. Write down the $D\ B\ P$ couplings in the $SU(3)$ limit for the process

$$D \to B\ p$$

where

$$\begin{aligned} D &: \text{Bayron decuplet} \quad J^P = 3/2^+ \\ B &: \text{Baryon octet} \quad J^P = 1/2^+ \\ P &: \text{Meson octet} \quad J^P = O^-. \end{aligned}$$

Hence show that for the energetically allowed decays, they are in the following ratios:

$\Delta^{++} \to p\pi^+$	:	$\Sigma^{*+} \to \Lambda\pi$	:	$\Sigma^* \to \Sigma\pi$	:	$\Xi^{*0} \to \Xi^-\pi^+$	:	$\Xi^{*-} \to \Xi^{*0}\pi^0$
$-\sqrt{6}$	:	$\sqrt{3}$	:	1	:	$\sqrt{2}$	:	-1

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Chapter 6
SU(6) AND QUARK MODEL

6.1 SU(6)

Quarks have spin 1/2. The well known baryons with spin 1/2 and spin 3/2 are in the octet and decuplet representations of flavor SU(3). We note that within each representation the mass splitting between adjacent members is of the same order. For example

$$\begin{aligned} m_\Lambda - m_N &\approx 170\text{ MeV}, \quad m_\Xi - m_\Sigma = 125\text{ MeV}; \\ m_{\Sigma^*} - m_\Delta &\approx 153\text{ MeV}, \quad m_\Omega - m_{\Xi^*} = 142\text{ MeV}. \end{aligned}$$

It is tempting to put these two representations in an irreducible representation of the group higher than SU(3). But octet and decuplet representations have different spins. This means that the proposed group cannot commute with angular momentum (spin). The proposed group must contain SU(3) $\times$ SUσ(2) as its subgroup. This might cause some trouble, since we are combining an internal symmetry with a space-time symmetry. It does cause trouble but this does not show up until one tries to make the theory relativistic.

We note that spin 3/2 baryon decuplet has (10×4) states and spin 1/2 baryon octet has (8×2) states. Thus, we look for an irreducible representation with 56 dimensions. Such a representation occurs in the decomposition of the product of representation **6** of SU(6) by itself viz.

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56} \oplus \mathbf{70} \oplus \mathbf{70} \oplus \mathbf{20}. \tag{6.1}$$

The representation **56** is completely symmetric irreducible representation of SU(6). The six quark states (in this section we will not write $|\ \rangle$ explicitly) $(u\uparrow u\downarrow\ \ d\uparrow d\downarrow\ \ s\uparrow s\downarrow)$ can be put in the fundamental representation **6**. We denote such a state as $\Psi_{i\alpha}$: $\alpha = 1, 2$; $i = 1, 2, 3$. In matrix notation we write

$$\Psi \equiv \begin{pmatrix} u\uparrow & d\uparrow & s\uparrow \\ u\downarrow & d\downarrow & s\downarrow \end{pmatrix}. \tag{6.2}$$

Now SU(3), SU(2) and SU(3)×SU(2) are subgroups of SU(6). The representation **6** splits under these subgroups as shown in table below:

Subgroups of SU(6)	Quarks Representation	Generators
$SU(3)$	$\underset{(\mathbf{3},\ \mathbf{1})}{(u\uparrow d\uparrow s\uparrow)}, \underset{(\mathbf{3},\ \mathbf{1})}{(u\downarrow d\downarrow s\downarrow)}$	$\frac{1}{2}\lambda_A \otimes 1$ $A = 1 \cdots 8$
$SU(2)$	$\underset{(\mathbf{1},\ \mathbf{2})}{(u\uparrow u\downarrow)}, \underset{(\mathbf{1},\ \mathbf{2})}{(d\uparrow d\downarrow)}, \underset{(\mathbf{1},\ \mathbf{2})}{(s\uparrow s\downarrow)}$	$1 \otimes \frac{1}{2}\,\sigma_n$ $n = 1, 2, 3$
$SU(3) \times SU(2)$	$(\mathbf{3},\ \mathbf{2})$	$\left(\frac{\lambda_A}{2} \otimes \frac{\sigma_n}{2}\right)$

Thus, we see that SU(6) has 35 generators. Hence the adjoint representation of SU(6) has dimension 35 and is given in the following decomposition:

$$\mathbf{6} \otimes \bar{\mathbf{6}} = \mathbf{35} \oplus \mathbf{1}. \tag{6.3}$$

The representations **56** and **35** split under the subgroup SU(3) ×SU(2) as follows:

$$\begin{aligned} \mathbf{56} \quad : \quad & [(\mathbf{3},\ \mathbf{2}) \otimes (\mathbf{3},\ \mathbf{2}) \otimes (\mathbf{3},\ \mathbf{2})]_{\text{symmetric}} \\ = \quad & \underset{\substack{\text{baryon} \\ \text{decuplet}}}{(\mathbf{10},\ \mathbf{4})} \quad + \quad \underset{\substack{\text{baryon} \\ \text{octet}}}{(\mathbf{8},\ \mathbf{2})} \end{aligned} \tag{6.4}$$

$$\mathbf{35} \;:\; \left[(\mathbf{3},\ \mathbf{2}) \otimes \left(\overline{\mathbf{3}},\ \mathbf{2}\right)\right]$$

$$= \begin{array}{cccc} (\mathbf{8},\ \mathbf{3}) \oplus (\mathbf{1},\ \mathbf{3}) & \oplus\ (\mathbf{8},\ \mathbf{1}) & \oplus\ (\mathbf{1},\ \mathbf{1}) \\ \text{nonet} & \text{octet of} & \text{singlet} \\ \text{of vector} & \text{pseudoscalar} & \text{pseudoscalar} \\ \text{mesons} & \text{mesons} & \text{meson} \end{array} \tag{6.5}$$

SU(6) Wave Function for Mesons

The mesons are composite of $q\overline{q}$. The lowest lying mesons have

$$(q\overline{q})_{L=0} \qquad \text{and} \qquad P = (-1)\,(-1)^0 = -1.$$

The spin wave functions are given by:

$$\text{Spin singlet state :} \quad \chi_A = \frac{1}{\sqrt{2}}\,|\uparrow\downarrow - \downarrow\uparrow\rangle \tag{6.6a}$$

$$\text{Spin triplet states:} \quad \chi_S^{1,0,-1} = |\uparrow\uparrow\rangle\,,\, \frac{1}{\sqrt{2}}\,|\uparrow\downarrow + \downarrow\uparrow\rangle\,,|\downarrow\downarrow\rangle\,. \tag{6.6b}$$

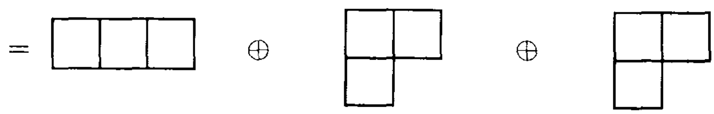
The spin singlet state is antisymmetric, it gives $J^P = 0^-$, whereas the spin triplet states are symmetric and gives $J^P = 1^-$. Thus we can write for 0^- and 1^- mesons the state functions as given in Tables 1 and 2 respectively.

Lowest lying baryons are made up of three quarks: $(qqq)_{L=0}$, $P = (-1)^0(1)^3 = 1$. Here we have to combine three spin $1/2$'s. In this case we have the following decomposition:

$$\underset{\mathbf{2}}{\square} \otimes \underset{\mathbf{2}}{\square} \otimes \underset{\mathbf{2}}{\square}$$

Table 6.1 Pseudoscalar meson states: $(q\bar{q})_{L=0}$, $J^P = 0^-$.

Particle	SU(6) State
π^+	$\frac{1}{\sqrt{2}}\left(u^\uparrow \bar{d}^\downarrow - u^\downarrow \bar{d}^\uparrow\right)$
π^0	$\frac{1}{2}\left(u^\uparrow \bar{u}^\downarrow - u^\downarrow \bar{u}^\uparrow - d^\uparrow \bar{d}^\downarrow + d^\downarrow \bar{d}^\uparrow\right)$
K^+	$\frac{1}{\sqrt{2}}\left(u^\uparrow \bar{s}^\downarrow - u^\downarrow \bar{s}^\uparrow\right)$
K^0	$\frac{1}{\sqrt{2}}\left(d^\uparrow \bar{s}^\downarrow - d^\downarrow \bar{s}^\uparrow\right)$
K^-	$\frac{1}{\sqrt{2}}\left(s^\uparrow \bar{u}^\downarrow - s^\downarrow \bar{u}^\uparrow\right)$
$\overline{K}^0$	$\frac{1}{\sqrt{2}}\left(s^\uparrow \bar{d}^\downarrow - s^\downarrow \bar{d}^\uparrow\right)$
η_8	$\frac{1}{\sqrt{12}}\left(u^\uparrow \bar{u}^\downarrow - u^\downarrow \bar{u}^\uparrow + d^\uparrow \bar{d}^\downarrow - d^\downarrow \bar{d}^\uparrow - 2s^\uparrow \bar{s}^\downarrow + 2s^\downarrow \bar{s}^\uparrow\right)$
η_1	$\frac{1}{\sqrt{6}}\left(u^\uparrow \bar{u}^\downarrow - u^\downarrow \bar{u}^\uparrow + d^\uparrow \bar{d}^\downarrow - d^\downarrow \bar{d}^\uparrow + s^\uparrow \bar{s}^\downarrow - s^\downarrow \bar{s}^\uparrow\right)$



4 Completely Symmetric Spin3/2 ⊕ **2** Mixed Symmetry Spin1/2 ⊕ **2** Mixed Symmetry Spin1/2 (6.7)

It is convenient to combine first two spin 1/2's. For this case we have $S = 0$ and spin wave function χ_A (Eq. 6a) and $S = 1$ and spin wave functions χ_S (Eq. 6b). We now combine spin 0 with spin 1/2 and we get the spin $S = 1/2$ and the following wave function χ_{MA}:

$$\chi_{MA}^{1/2} = \frac{1}{\sqrt{2}}\left|(\uparrow\downarrow - \downarrow\uparrow)\uparrow\right\rangle, \quad \chi_{MA}^{-1/2} = \frac{1}{\sqrt{2}}\left|(\uparrow\downarrow - \downarrow\uparrow)\downarrow\right\rangle. \quad (6.8)$$

We now combine spin 1 with the remaining spin 1/2. For this case we get $S = 3/2$ and $S = 1/2$. The spin wave functions for this case are given in Table 3. In this table, the numerical coefficients are Clebsch-Gordon Coefficients in combining spin 1 and spin 1/2.

The state function for the completely symmetric represen-

Table 6.2 Vector meson states: $(q\bar{q})_{L=0}$, $J^P = 1^-$.

Particle	SU(6) State
	$J_z = S_z = 0$
ρ^+	$\frac{1}{\sqrt{2}}\left(u^\uparrow \bar{d}^\downarrow + u^\downarrow \bar{d}^\uparrow\right)$
ρ^0	$\frac{1}{2}\left(u^\uparrow \bar{u}^\downarrow + u^\downarrow \bar{u}^\uparrow - d^\uparrow \bar{d}^\downarrow - d^\downarrow \bar{d}^\uparrow\right)$
ρ^-	$\frac{1}{\sqrt{2}}\left(d^\uparrow \bar{u}^\downarrow + d^\downarrow \bar{u}^\uparrow\right)$
K^{*+}	$\frac{1}{\sqrt{2}}\left(u^\uparrow \bar{s}^\downarrow + s^\downarrow \bar{u}^\uparrow\right)$
K^{*0}	$\frac{1}{\sqrt{2}}\left(d^\uparrow \bar{s}^\downarrow + d^\downarrow \bar{s}^\uparrow\right)$
K^{*-}	$\frac{1}{\sqrt{2}}\left(s^\uparrow \bar{u}^\downarrow + s^\downarrow \bar{u}^\uparrow\right)$
$\overline{K}^{*0}$	$\frac{1}{\sqrt{2}}\left(s^\uparrow \bar{d}^\downarrow + s^\downarrow \bar{d}^\uparrow\right)$
ω	$\frac{1}{2}\left(u^\uparrow \bar{u}^\downarrow + u^\downarrow \bar{u}^\uparrow + d^\uparrow \bar{d}^\downarrow + d^\downarrow \bar{d}^\uparrow\right)$
ϕ	$\frac{1}{\sqrt{2}}\left(s^\uparrow \bar{s}^\downarrow + s^\downarrow \bar{s}^\uparrow\right)$

	$J_z = 1$	$J_z = -1$
ρ^+	$u^\uparrow \bar{d}^\uparrow$	$u^\downarrow \bar{d}^\downarrow$
ρ^0	$\frac{1}{\sqrt{2}}\left(u^\uparrow \bar{u}^\uparrow - d^\uparrow \bar{d}^\uparrow\right)$	$\frac{1}{\sqrt{2}}\left(u^\downarrow \bar{u}^\downarrow - d^\downarrow \bar{d}^\downarrow\right)$
ρ^-	$d^\uparrow \bar{u}^\uparrow$	$d^\downarrow \bar{u}^\downarrow$
K^{*+}	$u^\uparrow \bar{s}^\uparrow$	$u^\downarrow \bar{s}^\downarrow$
K^{*0}	$d^\uparrow \bar{s}^\uparrow$	$d^\downarrow \bar{s}^\downarrow$
K^{*-}	$s^\uparrow \bar{u}^\uparrow$	$s^\downarrow \bar{u}^\downarrow$
$\overline{K}^{*0}$	$s^\uparrow \bar{d}^\uparrow$	$s^\downarrow \bar{d}^\downarrow$
ω	$\frac{1}{\sqrt{2}}\left(u^\uparrow \bar{u}^\uparrow + d^\uparrow \bar{d}^\uparrow\right)$	$\frac{1}{2}\left(u^\downarrow \bar{u}^\downarrow + d^\downarrow \bar{d}^\downarrow\right)$
ϕ	$s^\uparrow \bar{s}^\uparrow$	$s^\downarrow \bar{s}^\downarrow$

Table 6.3 Spin wave functions for $S = 3/2$ and $S = 1/2$ resulting in the combination of spin 1 and spin 1/2.

	$S_z = 3/2$	$S_z = 1/2$	$S_z = -1/2$	$S_z = -3/2$
$\chi_S^{3/2}$ Symmetric:	$\lvert\uparrow\uparrow\uparrow\rangle$,	$\frac{1}{\sqrt{3}}\lvert\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow\rangle$,	$\frac{1}{\sqrt{3}}\lvert\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow\rangle$,	$\lvert\downarrow\downarrow\downarrow\rangle$
$\chi_{MS}^{1/2}$(Mixed symmetry: symmetric in 1 and 2)		$\frac{1}{\sqrt{3}}\left\lvert -\frac{(\uparrow\downarrow+\downarrow\uparrow)\uparrow}{\sqrt{2}} + \sqrt{2}\uparrow\uparrow\downarrow\right\rangle$,	$\frac{1}{\sqrt{3}}\left\lvert \frac{(\uparrow\downarrow+\downarrow\uparrow)\downarrow}{\sqrt{2}} - \sqrt{2}\downarrow\downarrow\uparrow\right\rangle$,	

tation **56** of SU(6) can be written:

$$\Phi_S\,\chi_S + \frac{1}{\sqrt{2}}\left[\Phi_{MS}\,\chi_{MS} + \Phi_{MA}\,\chi_{MA}\right], \tag{6.9}$$

where [cf. Eqs. (5.96), (5.91) and (5.92)]

$$\Phi_S = |T_{ijk}\rangle \tag{6.10a}$$

$$\Phi_{MA} = \overline{B}^i_j\ |0\rangle\,, \quad \Phi_{MS} = -\ \overline{B'}^i_j\ |0\rangle\,. \tag{6.10b}$$

The spin state functions χ_S, χ_{MS} and χ_{MA} are given in Table 3 and Eq. (8). Using Tables 5.3 and 3, we can write the state function $\Phi_S\,\chi_S$ for the decuplet. For example,

$$\begin{aligned}
&\left|\Delta^+,\ S_z = 1/2\right\rangle \\
&= \frac{1}{\sqrt{3}}(u\ u\ d + u\ d\ u + d\ u\ u) \\
&\quad\times\frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \\
&= \frac{1}{3}\begin{bmatrix} u^\uparrow u^\uparrow d^\downarrow + u^\uparrow d^\uparrow u^\downarrow + d^\uparrow u^\uparrow u^\downarrow \\ +u^\uparrow u^\downarrow d^\uparrow + u^\uparrow d^\downarrow u^\uparrow + d^\uparrow u^\downarrow u^\uparrow \\ +u^\downarrow u^\uparrow d^\uparrow + u^\downarrow d^\uparrow u^\uparrow + d^\downarrow u^\uparrow u^\uparrow \end{bmatrix}.
\end{aligned} \tag{6.11}$$

Table 6.4 SU(6) states for the decuplet of baryons $J^P = 3/2^+$, $S_z = 3/2$.

Particle State	SU(6) State Functions
$\|\Delta^{++}, S_z = 3/2\rangle$	$u^\uparrow u^\uparrow u^\uparrow$
$\|\Delta^{+}, S_z = 3/2\rangle$	$\frac{1}{\sqrt{3}}\left(u^\uparrow u^\uparrow d^\uparrow + u^\uparrow d^\uparrow u^\uparrow + d^\uparrow u^\uparrow u^\uparrow\right)$
$\|\Delta^{0}, S_z = 3/2\rangle$	$\frac{1}{\sqrt{3}}\left(d^\uparrow d^\uparrow u^\uparrow + d^\uparrow u^\uparrow d^\uparrow + u^\uparrow d^\uparrow d^\uparrow\right)$
$\|\Delta^{-}, S_z = 3/2\rangle$	$d^\uparrow d^\uparrow d^\uparrow$
$\left\|\Sigma^{*+}, S_z = 3/2\right\rangle$	$\frac{1}{\sqrt{3}}\left(s^\uparrow u^\uparrow u^\uparrow + u^\uparrow s^\uparrow u^\uparrow + u^\uparrow u^\uparrow s^\uparrow\right)$
$\left\|\Sigma^{*0}, S_z = 3/2\right\rangle$	$\frac{1}{\sqrt{6}}\begin{pmatrix} s^\uparrow d^\uparrow u^\uparrow + s^\uparrow u^\uparrow d^\uparrow + d^\uparrow s^\uparrow u^\uparrow \\ +u^\uparrow s^\uparrow d^\uparrow + u^\uparrow d^\uparrow s^\uparrow + d^\uparrow u^\uparrow s^\uparrow \end{pmatrix}$
$\left\|\Sigma^{*-}, S_z = 3/2\right\rangle$	$\frac{1}{\sqrt{3}}\left(s^\uparrow d^\uparrow d^\uparrow + d^\uparrow s^\uparrow d^\uparrow + d^\uparrow s^\uparrow s^\uparrow\right)$
$\left\|\Xi^{*0}, S_z = 3/2\right\rangle$	$\frac{1}{\sqrt{3}}\left(s^\uparrow s^\uparrow u^\uparrow + s^\uparrow u^\uparrow s^\uparrow + u^\uparrow s^\uparrow s^\uparrow\right)$
$\left\|\Xi^{*-}, S_z = 3/2\right\rangle$	$\frac{1}{\sqrt{3}}\left(s^\uparrow s^\uparrow d^\uparrow + s^\uparrow d^\uparrow s^\uparrow + d^\uparrow s^\uparrow s^\uparrow\right)$
$\|\Omega^{-}, S_z = 3/2\rangle$	$s^\uparrow s^\uparrow s^\uparrow$

$$\left|\Omega^-,\ S_z = \frac{3}{2}\right\rangle = s\ s\ s\,|\uparrow\uparrow\uparrow\rangle = \left|s^\uparrow s^\uparrow s^\uparrow\right\rangle . \tag{6.12}$$

Similarly we can calculate all the other states. These states for $S_z = 3/2$ are given in Table 4.

For baryon octet $1/2^+$ states, we use Tables 5.2 and 3 and Eq. (8). We explicitly calculate the state $|p,\ S_z = 1/2\rangle$. It is given by

$$\begin{aligned} & |p,\ S_z = 1/2\rangle \\ = & \ \frac{1}{\sqrt{2}}\left\{\left(\frac{-1}{\sqrt{6}}\right)[(u\ d + d\ u)\,u - 2u\ u\ d] \right. \\ & \times\left(\frac{-1}{\sqrt{6}}\right)[(\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow] \\ & \left. + \left(\frac{1}{\sqrt{2}}\right)\left[(u\ d - d\ u)\,u\left(\frac{1}{\sqrt{2}}\right)[(\uparrow\downarrow - \downarrow\uparrow)\uparrow]\right]\right\} \end{aligned} \tag{6.13}$$

$$= \frac{1}{6\sqrt{2}}\left[\begin{array}{c} \left(u^\uparrow d^\downarrow + u^\downarrow d^\uparrow\right) u^\uparrow - 2u^\uparrow d^\uparrow u^\downarrow \\ + \left(d^\uparrow u^\downarrow + d^\downarrow u^\uparrow\right) u^\uparrow - 2d^\uparrow u^\uparrow u^\downarrow - 2u^\uparrow u^\downarrow d^\uparrow \\ -2u^\downarrow u^\uparrow d^\uparrow + 4u^\uparrow u^\uparrow d^\downarrow + 3u^\uparrow d^\downarrow u^\uparrow \\ -3u^\downarrow d^\uparrow u^\uparrow - 3d^\uparrow u^\downarrow u^\uparrow + 3d^\downarrow u^\uparrow u^\uparrow \end{array}\right]$$

$$= \frac{1}{\sqrt{18}}\left[\begin{array}{c} 2u^\uparrow d^\downarrow u^\uparrow + 2u^\uparrow u^\uparrow d^\downarrow + 2d^\downarrow u^\uparrow u^\uparrow \\ -u^\uparrow u^\downarrow d^\uparrow - u^\uparrow d^\uparrow u^\downarrow - u^\downarrow d^\uparrow u^\uparrow \\ -d^\uparrow u^\downarrow u^\uparrow - d^\uparrow u^\uparrow u^\downarrow - u^\downarrow u^\uparrow d^\uparrow \end{array}\right]. \tag{6.14}$$

In a similar manner we can calculate the rest of the states. They are given in Table 5.

Finally we give the state functions for the representations **70**, **70** and **20**. They are as follows:

Representation **70** : MS

$\Phi_S\ \chi_{MS}$: $(\mathbf{10},\ \mathbf{2})$: 20

$\Phi_{MS}\ \chi_S$: $(\mathbf{8},\ \mathbf{4})$: 32

$\frac{1}{\sqrt{2}}\left(-\Phi_{MS}\ \chi_{MS} + \Phi_{MA}\ \chi_{MA}\right)$: $(\mathbf{8},\ \mathbf{2})$: 16

$\Phi_A\ \chi_{MA}$: $(\mathbf{1},\ \mathbf{2})$: 2, $\Phi_A = |\Lambda_1^0\rangle$[cf. Eq. (5.95)].

Representation **70** : MA

$\Phi_S\ \chi_{MA}$: $(\mathbf{10},\ \mathbf{2})$: 20

$\Phi_{MA}\ \chi_S$: $(\mathbf{8},\ \mathbf{4})$: 32

$\frac{1}{\sqrt{2}}\left(\Phi_{MS}\ \chi_{MA} + \Phi_{MA}\ \chi_{MS}\right)$: $(\mathbf{8},\ \mathbf{2})$: 16

$\Phi_A\ \chi_{MA}$: $(\mathbf{1},\ \mathbf{2})$: 2

Representation **20**

$\Phi_A\ \chi_S$: $(\mathbf{1},\ \mathbf{4})$: 4

$\frac{1}{\sqrt{2}}\left(\Phi_{MS}\ \chi_{MA} - \Phi_{MA}\ \chi_{MS}\right)$: $(\mathbf{8},\ \mathbf{2})$: 16

We will not give the detailed identification for these states.

6.2 Magnetic Moments of Baryons

Magnetic moment operator is given by

$$\hat{\boldsymbol{\mu}} = g\mu_0\, \mathbf{J}\ /\ \hbar. \tag{6.15}$$

We define the magnetic moment μ of a particle of mass m:

$$\mu = g\mu_0 J, \tag{6.16}$$

Table 6.5 SU(6) states for the octet of bayrons $J^P = 1/2^+$.

Particle State	SU(6) State functions	Remarks
$\|p,\ S_z = 1/2\rangle$	$\frac{1}{\sqrt{18}}\begin{bmatrix} 2u^\uparrow d^\downarrow u^\uparrow + 2u^\uparrow u^\uparrow d^\downarrow \\ +2d^\downarrow u^\uparrow u^\uparrow - u^\uparrow u^\downarrow d^\uparrow \\ -u^\uparrow d^\uparrow u^\downarrow - u^\downarrow d^\uparrow u^\uparrow \\ -d^\uparrow u^\downarrow u^\uparrow - d^\uparrow u^\uparrow u^\downarrow \\ -u^\downarrow u^\uparrow d^\uparrow \end{bmatrix}$	
$\|n,\ S_z = 1/2\rangle$	$\frac{1}{\sqrt{18}}\begin{bmatrix} -2d^\uparrow u^\downarrow d^\uparrow - 2d^\uparrow d^\uparrow u^\downarrow \\ -2u^\downarrow d^\uparrow d^\uparrow + d^\uparrow d^\downarrow u^\uparrow \\ +d^\uparrow u^\uparrow d^\downarrow + d^\downarrow u^\uparrow d^\uparrow \\ +u^\uparrow d^\downarrow d^\uparrow + u^\uparrow d^\uparrow d^\downarrow \\ +d^\downarrow d^\uparrow u^\uparrow \end{bmatrix}$	Change $u \longleftrightarrow d$ and over all sign in p
-$\|\Sigma^+,\ S_z = 1/2\rangle$	$\frac{1}{\sqrt{18}}\begin{bmatrix} 2u^\uparrow s^\downarrow u^\uparrow + 2u^\uparrow u^\uparrow s^\downarrow \\ +2s^\downarrow u^\uparrow u^\uparrow - u^\uparrow u^\downarrow s^\uparrow \\ -u^\uparrow s^\uparrow u^\downarrow - u^\downarrow s^\uparrow u^\uparrow \\ -s^\uparrow u^\downarrow u^\uparrow - s^\uparrow u^\uparrow u^\downarrow \\ -u^\downarrow u^\uparrow s^\uparrow \end{bmatrix}$	Change $d \to s$ in p
$\|\Sigma^0,\ S_z = 1/2\rangle$	$\frac{1}{\sqrt{36}}\begin{bmatrix} 2d^\uparrow s^\downarrow u^\uparrow - d^\downarrow s^\uparrow u^\uparrow \\ -s^\uparrow d^\downarrow u^\uparrow + 2s^\downarrow d^\uparrow u^\uparrow \\ +2u^\uparrow d^\uparrow s^\downarrow - u^\downarrow d^\downarrow s^\uparrow \\ -u^\uparrow s^\uparrow d^\downarrow + 2u^\uparrow s^\downarrow d^\uparrow \\ +2s^\downarrow u^\uparrow d^\uparrow - s^\uparrow u^\uparrow d^\downarrow \\ -d^\downarrow u^\uparrow s^\uparrow + 2d^\uparrow u^\uparrow s^\downarrow \\ -s^\uparrow u^\downarrow d^\uparrow - u^\downarrow s^\uparrow d^\uparrow \\ -d^\uparrow s^\uparrow u^\downarrow - d^\uparrow u^\downarrow s^\uparrow \\ -u^\downarrow d^\uparrow s^\uparrow - s^\uparrow d^\uparrow u^\downarrow \end{bmatrix}$	

Particle State	SU(6) State functions	Remarks
$\|\Sigma^-,\ S_z = 1/2\rangle$	$\frac{1}{\sqrt{18}}\begin{bmatrix} 2d^\uparrow s^\downarrow d^\uparrow + 2d^\uparrow d^\uparrow s^\downarrow \\ +2s^\downarrow d^\uparrow d^\uparrow - d^\uparrow d^\downarrow s^\uparrow \\ -d^\uparrow s^\uparrow d^\downarrow - d^\downarrow s^\uparrow d^\uparrow \\ -s^\uparrow d^\downarrow d^\uparrow - s^\uparrow d^\uparrow d^\downarrow \\ -d^\downarrow d^\uparrow s^\uparrow \end{bmatrix}$	Change $u \to d$ and over all sign in $[-\Sigma^+]$
$\|\Lambda,\ S_z = 1/2\rangle$	$-\frac{1}{2\sqrt{12}}\begin{bmatrix} 2u^\uparrow d^\downarrow s^\uparrow - 2u^\downarrow d^\uparrow s^\uparrow \\ -2d^\uparrow u^\downarrow s^\uparrow + 2d^\downarrow u^\uparrow s^\uparrow \\ +2s^\uparrow u^\uparrow d^\downarrow - 2s^\downarrow u^\downarrow d^\uparrow \\ -2s^\uparrow d^\uparrow u^\downarrow + 2s^\uparrow d^\downarrow u^\uparrow \\ +2d^\downarrow s^\uparrow u^\uparrow - 2d^\uparrow s^\uparrow u^\downarrow \\ -2u^\downarrow s^\uparrow d^\uparrow + 2u^\uparrow s^\uparrow d^\downarrow \end{bmatrix}$	
$\|\Xi^0,\ S_z = 1/2\rangle$	$\frac{1}{\sqrt{18}}\begin{bmatrix} 2s^\uparrow u^\downarrow s^\uparrow + 2s^\uparrow s^\uparrow u^\downarrow \\ +2u^\downarrow s^\uparrow s^\uparrow - s^\uparrow s^\downarrow u^\uparrow \\ -s^\uparrow u^\uparrow s^\downarrow - s^\downarrow u^\uparrow s^\uparrow \\ -u^\uparrow s^\downarrow s^\uparrow - u^\uparrow s^\uparrow s^\downarrow \\ -s^\downarrow s^\uparrow u^\uparrow \end{bmatrix}$	Change $d \to s$ and over all sign in n
$\|\Xi^-,\ S_z = 1/2\rangle$	$\frac{1}{\sqrt{18}}\begin{bmatrix} -2s^\uparrow d^\downarrow s^\uparrow - 2s^\uparrow s^\uparrow d^\downarrow \\ -2d^\downarrow s^\uparrow s^\uparrow + s^\uparrow s^\downarrow d^\uparrow \\ +s^\uparrow d^\uparrow s^\downarrow + s^\downarrow d^\uparrow s^\uparrow \\ +d^\uparrow s^\downarrow s^\uparrow + d^\uparrow s^\uparrow s^\downarrow \\ +s^\downarrow s^\uparrow d^\uparrow \end{bmatrix}$	Change $u \to -d$ in Ξ^0

where

$$\mu_0 = e\hbar \ / \ 2mc \tag{6.17}$$

and J is the angular momentum viz. the eigenvalue of J^2 is $J(J+1)\hbar^2$. For electron, $J = 1/2$, $g = -2$, i.e.,

$$\mu_e = -e\hbar \ / \ 2m_e c. \tag{6.18}$$

For a spin 1/2 particle, $\mathbf{J} = 1/2\hbar\boldsymbol{\sigma}$. Thus for a quark, the magnetic moment operator is given by

$$\begin{aligned}\widehat{\mu}_q &= 2Q_q\left(\frac{e\hbar}{2m_q c}\right)\frac{1}{2}\sigma_q \\ &= \mu_q\sigma_q,\end{aligned} \tag{6.19}$$

where

$$\mu_q = Q_q\left(\frac{e\hbar}{2m_q c}\right) \tag{6.20}$$

is the magnetic moment of the quark.

The magnetic moment operator for a baryon of $J^P = 1/2^+$ in the quark model is given by

$$\widehat{\mu}_B = \sum_q \mu_q\sigma_q. \tag{6.21}$$

We need to calculate the expectation value of $\widehat{\mu}_{Bz}$ viz.

$$\mu_B = \langle\widehat{\mu}_{Bz}\rangle = \sum_q \mu_q \langle\sigma_{qz}\rangle\,. \tag{6.22}$$

We now explicitly calculate the magnetic moment of the proton. For the proton

$$\widehat{\mu}_{pz} = \mu_u\sigma_{uz} + \mu_d\sigma_{dz} + \mu_u\sigma_{uz}. \tag{6.23}$$

Using the proton state $|p, \sigma_z = 1\rangle$ as given in Eq. (13), we have (we will not write $\sigma_z = 1$ explicitly in the state)

$$\widehat{\mu}_{pz}\ |p\rangle = \frac{1}{\sqrt{18}} \begin{bmatrix} 2\left(\mu_u - \mu_d + \mu_u\right) u^{\uparrow} d^{\downarrow} u^{\uparrow} \\ +2\left(\mu_u + \mu_u - \mu_d\right) u^{\uparrow} u^{\uparrow} d^{\downarrow} \\ +2\left(-\mu_d + \mu_u + \mu_u\right) d^{\downarrow} u^{\uparrow} u^{\uparrow} \\ -\left(\mu_u - \mu_u + \mu_d\right) u^{\uparrow} u^{\downarrow} d^{\uparrow} \\ -\left(\mu_u + \mu_d - \mu_u\right) u^{\uparrow} d^{\uparrow} u^{\downarrow} \\ -\left(-\mu_u + \mu_d + \mu_u\right) u^{\downarrow} d^{\uparrow} u^{\uparrow} \\ -\left(\mu_d - \mu_u + \mu_u\right) d^{\uparrow} u^{\downarrow} u^{\uparrow} \\ -\left(\mu_d + \mu_u - \mu_u\right) d^{\uparrow} u^{\uparrow} u^{\downarrow} \\ -\left(-\mu_u + \mu_u + \mu_d\right) u^{\downarrow} u^{\uparrow} d^{\uparrow} \end{bmatrix} \tag{6.24}$$

Hence

$$\begin{aligned} \mu_p &= \langle p|\, \widehat{\mu}_{pz}\ |p\rangle \\ &= \frac{1}{18}\left[12\left(2\mu_u - \mu_d\right) + 6\mu_d\right] \\ &= \frac{4}{3}\mu_u - \frac{1}{3}\ \mu_d. \end{aligned} \tag{6.25}$$

Similarly using Table 5, we can calculate the magnetic moments for the rest of the baryons in the octet.

However, we can use simplified state functions to calculate the magnetic moments. In this calculation the order in which quarks appear is important. For the proton, we write the state function

$$|p\rangle = |u\ u\ d\rangle\ \chi_{MS}^{1/2} = |u\ u\ d\rangle \left(-\frac{1}{\sqrt{6}}\right) |[(\uparrow\downarrow + \downarrow\uparrow) \uparrow -2 \uparrow\uparrow\downarrow]\rangle \tag{6.26}$$

$$\sigma_z(1)\ |[(\uparrow\downarrow + \downarrow\uparrow) \uparrow -2 \uparrow\uparrow\downarrow]\rangle = |[(\uparrow\downarrow - \downarrow\uparrow) \uparrow -2 \uparrow\uparrow\downarrow]\rangle \tag{6.27a}$$

$$\sigma_z(2)\ |[(\uparrow\downarrow + \downarrow\uparrow) \uparrow -2 \uparrow\uparrow\downarrow]\rangle = |[(- \uparrow\downarrow + \downarrow\uparrow) \uparrow -2 \uparrow\uparrow\downarrow]\rangle \tag{6.27b}$$

$$\sigma_z(3)\ |[(\uparrow\downarrow + \downarrow\uparrow) \uparrow -2 \uparrow\uparrow\downarrow]\rangle = |[(\uparrow\downarrow + \downarrow\uparrow) \uparrow +2 \uparrow\uparrow\downarrow]\rangle\,. \tag{6.27c}$$

Hence

$$\begin{aligned}\widehat{\mu}_{pz}\ \chi_{MS}^{1/2} &= \left[\mu_u\sigma_z(1)+\mu_u\sigma_z(2)+\mu_d\sigma_z(3)\right]\chi_{MS}^{1/2} \\ &= \left(-\frac{1}{\sqrt{6}}\right)\begin{bmatrix} -4\mu_u\ \left|\uparrow\uparrow\downarrow\right\rangle \\ +\mu_d\ \left|\left[\left(\uparrow\downarrow+\downarrow\uparrow\right)\uparrow+2\uparrow\uparrow\downarrow\right]\right\rangle\end{bmatrix}.\end{aligned} \tag{6.28}$$

Therefore,

$$\begin{aligned}\mu_p &= \left\langle\widehat{\mu}_{pz}\right\rangle_p=\frac{1}{6}\left[8\mu_u+(1+1-4)\,\mu_d\right] \\ &= \frac{4}{3}\mu_u-\frac{1}{3}\ \mu_d.\end{aligned}$$

For $|\Lambda^0\rangle$, the simplified state function is given by

$$\left|\Lambda^0\right\rangle=-\left|u\ d\ s\right\rangle\ \chi_{MA}^{1/2}=-\left|u\ d\ s\right\rangle\ \frac{1}{\sqrt{2}}\left|\left(\uparrow\downarrow-\downarrow\uparrow\right)\uparrow\right\rangle \tag{6.29}$$

$$\widehat{\mu}_{\Lambda z}=\mu_u\sigma_z(1)+\mu_d\sigma_z(2)+\mu_s\sigma_z(3) \tag{6.30}$$

$$\widehat{\mu}_{\Lambda z}\chi_{MA}^{1/2}=\frac{1}{\sqrt{2}}\begin{bmatrix}\mu_u\ \left|\left(\uparrow\downarrow+\downarrow\uparrow\right)\uparrow\right\rangle+\mu_d\ \left|\left(-\uparrow\downarrow-\downarrow\uparrow\right)\uparrow\right\rangle \\ +\mu_s\ \left|\left(\uparrow\downarrow-\downarrow\uparrow\right)\uparrow\right\rangle\end{bmatrix}. \tag{6.31}$$

Therefore,

$$\mu_\Lambda=\left\langle\widehat{\mu}_{\Lambda z}\right\rangle_\Lambda=\frac{1}{2}\left[0+2\ \mu_s\right]=\mu_s. \tag{6.32}$$

For $|\Sigma^0\rangle$, the simplified state function is

$$\left|\Sigma^0\right\rangle=\left|u\ d\ s\right\rangle\ \chi_{MS}^{1/2}=\left|u\ d\ s\right\rangle\left(-\frac{1}{\sqrt{6}}\right)\left|\left[\left(\uparrow\downarrow+\downarrow\uparrow\right)\uparrow-2\uparrow\uparrow\downarrow\right]\right\rangle \tag{6.33}$$

$$\begin{aligned}\widehat{\mu}_{\Sigma^0 z}\chi_{MS}^{1/2} &= -\frac{1}{\sqrt{6}}\{\mu_u\ \left|\left[\left(\uparrow\downarrow-\downarrow\uparrow\right)\uparrow-2\uparrow\uparrow\downarrow\right]\right\rangle \\ &\quad+\mu_d\ \left|\left[\left(-\uparrow\downarrow+\downarrow\uparrow\right)\uparrow-2\uparrow\uparrow\downarrow\right]\right\rangle \\ &\quad+\mu_s\ \left|\left[\left(\uparrow\downarrow+\downarrow\uparrow\right)\uparrow+2\uparrow\uparrow\downarrow\right]\right\rangle\}.\end{aligned} \tag{6.34}$$

Therefore,

$$\begin{aligned}
\mu_{\Sigma^0} &= \langle \widehat{\mu}_{\Sigma^0 z} \rangle_{\Sigma^0} = \frac{1}{6}\left[\mu_u\,(4) + \mu_d\,(4) + \mu_s\,(1+1-4)\right] \\
&= \frac{2}{3}\mu_u + \frac{2}{3}\mu_d - \frac{1}{3}\mu_s. \qquad (6.35)
\end{aligned}$$

From Eqs. (31) and (33), we get

$$\begin{aligned}
\mu_{\Sigma^0-\Lambda^0} &= \left\langle \Sigma^0 \middle| \widehat{\mu}_{\Lambda z} \middle| \Lambda \right\rangle \\
&= \frac{1}{\sqrt{12}}\left[2\mu_u - 2\mu_d\right] \\
&= \frac{1}{\sqrt{3}}\left[\mu_u - \mu_d\right] = \mu_{\Lambda^0-\Sigma^0}. \qquad (6.36)
\end{aligned}$$

The magnetic moments for the rest of the baryons in the octet can be written from Eq. (24) as follows:

$$\mu_n \ : \quad (\mu_u \longleftrightarrow \mu_d) = \frac{4}{3}\,\mu_d - \frac{1}{3}\,\mu_u. \qquad (6.37)$$

$$\mu_{\Sigma^+} \ : \quad (\mu_d \longleftrightarrow \mu_s) = \frac{4}{3}\,\mu_u - \frac{1}{3}\,\mu_s. \qquad (6.38)$$

$$\mu_{\Sigma^-} \ : \quad (\mu_u \longleftrightarrow \mu_d \text{ in } \mu_{\Sigma^+}) = \frac{4}{3}\,\mu_d - \frac{1}{3}\,\mu_s. \qquad (6.39)$$

$$\mu_{\Xi^0} \ : \quad (\mu_d \longleftrightarrow \mu_s \text{ in } \mu_n) = \frac{4}{3}\,\mu_s - \frac{1}{3}\,\mu_u. \qquad (6.40)$$

$$\mu_{\Xi^-} \ : \quad (\mu_u \longleftrightarrow \mu_d \text{ in } \mu_{\Xi^0}) = \frac{4}{3}\,\mu_s - \frac{1}{3}\,\mu_d. \qquad (6.41)$$

In order to compare these magnetic moments with their experimental values, we introduce the following quantities:

$$\mu_0 = \frac{e\hbar}{2\overline{m}c}, \qquad \overline{m} = \frac{m_u + m_d}{2}. \qquad (6.42)$$

We can write

$$\mu_0 = \mu_N\left(\frac{m_p}{\overline{m}}\right), \qquad (6.43a)$$

where

$$\mu_N = \frac{e\hbar}{2m_pc}. \tag{6.43b}$$

Here μ_N is the nucleon magneton. Thus we can write the magnetic moments of u, d and s quarks in terms of μ_N:

$$\mu_u = \frac{2}{3}\left(\frac{m_p}{m_u}\right)\mu_N \tag{6.44a}$$

$$\mu_d = -\frac{1}{3}\left(\frac{m_p}{m_d}\right)\mu_N \tag{6.44b}$$

$$\mu_s = -\frac{1}{3}\left(\frac{m_p}{m_s}\right)\mu_N. \tag{6.44c}$$

We will now assume isospin symmetry i.e. will take $m_u = m_d = \overline{m}$. We see that we have two unknown numbers $\overline{m}$ and m_s. These numbers we fix from the experimental values of μ_p and μ_Λ. From Eqs. (24), (31) and (43), we obtain

$$\mu_p = \frac{m_p}{\overline{m}}\,\mu_N = 2.793\,\mu_N. \tag{6.45}$$

$$\mu_\Lambda = -\frac{1}{3}\frac{m_p}{m_s}\,\mu_N = -0.613\,\mu_N. \tag{6.46}$$

On the right hand side of Eqs. (44) and (45), we have put their experimental values. From Eqs. (44) and (45), we get

$$\overline{m} = m_u = m_d \approx 336 \text{ MeV} \tag{6.47a}$$

$$m_s \approx 510 \text{ MeV}. \tag{6.47b}$$

It is interesting to compare these values with those obtained from the naive quark model. Now proton is made up of uud quarks and Λ is made up of uds quarks:

$$3\,\overline{m} = m_p, \qquad \overline{m} \approx 313 \text{ MeV} \tag{6.48a}$$

$$2\,\overline{m} + m_s = m_\Lambda, \qquad m_s = \frac{3m_\Lambda - 2m_p}{3} = 490 \text{ MeV}. \tag{6.48b}$$

The masses of u, d and s quarks given in Eq. (46) or (47) are called the constituent quark masses. These are effective masses of the quarks confined in a hadron. The constituent quark masses are quite different from those appearing in the Hamiltonian or the Lagrangian. These masses are called current quark masses.

Using Eq. (46) and (43), we get

$$\mu_u \approx 1.862\ \mu_N \tag{6.49a}$$
$$\mu_d \approx -0.991\ \mu_N \tag{6.49b}$$
$$\mu_s \approx -0.613\ \mu_N. \tag{6.49c}$$

Using Eqs. (48) the predictions of quark model for the baryon magnetic moments as given in Eqs. (24), (31), (34), (35) and (36)-(40) are tabulated in Table 6 along with their experimental values. If we put $m_u = m_d = m_s$, in Eqs. (24), (31), (34), (35) and (36)-(40), we get the SU(6) predictions

$$\begin{aligned} \mu_p &= \mu_{\Sigma^+} = -\frac{3}{2}\ \mu_n = -3\ \mu_\Lambda = -3\ \mu_{\Sigma^-} = 3\ \mu_{\Sigma^0} \\ &= -\frac{3}{2}\ \mu_{\Xi^0} = -3\ \mu_{\Xi^-} = \sqrt{3}\ \mu_{\Sigma^0-\Lambda^0}. \end{aligned} \tag{6.50}$$

We conclude this section by the following observations:

1) The quark model is simpler than SU(6).

2) It is more predictive than SU(6). It gives information about the scale of magnetic moments.

3) It gives good account of some corrections to SU(6) relations.

From Table 6, we see the agreement between quark model values of baryon magnetic moments and their experimental values is not bad.

6.3 Radiative Decays of Vector Mesons

For a quark and antiquark system, the Hamiltonian is given by

$$H = \frac{\hat{\mathbf{p}}_1^2}{2m_1} + \frac{\hat{\mathbf{p}}_2^2}{2m_2} + V(\mathbf{r}_1, \mathbf{r}_2)$$

Table 6.6 Magnetic moments of baryons: Quark model predictions and comparison with their experimental values.

Magnetic moment	Quark model values (in μ_N)	Experimental values (in μ_N)
μ_p	input	2.793
μ_n	$-1.862 : \frac{4}{3}\mu_d - \frac{1}{3}\mu_u$	-1.913
μ_Λ	input	-0.613 ± 0.004
μ_{Σ^+}	$2.687 : \frac{4}{3}\mu_u - \frac{1}{3}\mu_s$	2.458 ± 0.010
μ_{Σ^-}	$-1.037 : \frac{4}{3}\mu_d - \frac{1}{3}\mu_s$	1.160 ± 0.025
μ_{Σ^0}	$0.785 : \frac{2}{3}\mu_u + \frac{2}{3}\mu_d - \frac{1}{3}\mu_s$	–
μ_{Ξ^0}	$-1.438 : \frac{4}{3}\mu_s - \frac{1}{3}\mu_u$	-1.250 ± 0.014
μ_{Ξ^-}	$-0.507 : \frac{4}{3}\mu_s - \frac{1}{3}\mu_d$	-0.6507 ± 0.0025
$\mu_{\Sigma^0-\Lambda^0}$	$1.647 : \frac{1}{\sqrt{3}}(\mu_u - \mu_d)$	1.61 ± 0.08

$$= \frac{(\boldsymbol{\sigma}_1.\widehat{\mathbf{p}}_1)(\boldsymbol{\sigma}_1.\widehat{\mathbf{p}}_1)}{2m_1} + \frac{(\boldsymbol{\sigma}_2.\widehat{\mathbf{p}}_2)(\boldsymbol{\sigma}_2.\widehat{\mathbf{p}}_2)}{2m_2} + V(\mathbf{r}_1, \mathbf{r}_2)\,. \quad (6.51)$$

To introduce electromagnetic interaction, we make the gauge invariant replacement

$$\widehat{\mathbf{p}} \rightarrow \widehat{\mathbf{p}} - e\,Q\,\mathbf{A}(\mathbf{r}, t)\,, \quad (6.52)$$

where $\mathbf{A}(\mathbf{r}, t)$ is the electromagnetic field, eQ is the electric charge of the quark and $\widehat{\mathbf{p}} = -i\nabla$. From Eqs. (50) and (51), we get

$$H = \sum_{i=1}^{2} \left[\begin{array}{c} \frac{1}{2m_i}\widehat{\mathbf{p}}_i^2 - \frac{e\,Q_i}{2m_i}(\boldsymbol{\sigma}_i \cdot \widehat{\mathbf{p}}_i)(\boldsymbol{\sigma}_i \cdot \mathbf{A}(\mathbf{r}_i, t)) \\ -\frac{e\,Q_i}{2m_i}(\boldsymbol{\sigma}_i \cdot \mathbf{A}(\mathbf{r}_i, t))\,\boldsymbol{\sigma}_i \cdot \widehat{\mathbf{p}}_i + \frac{e^2\,Q_i^2}{2m_i}\,\mathbf{A}^2(\mathbf{r}_i, t) \end{array} \right] + V(\mathbf{r}_1, \mathbf{r}_2)\,. \quad (6.53)$$

Using the identities

$$(\boldsymbol{\sigma} \cdot \widehat{\mathbf{p}})(\boldsymbol{\sigma} \cdot \mathbf{A}) + (\boldsymbol{\sigma} \cdot \mathbf{A})(\boldsymbol{\sigma} \cdot \widehat{\mathbf{p}}) = \widehat{\mathbf{p}} \cdot \mathbf{A} + \mathbf{A}.\widehat{\mathbf{p}} + i\boldsymbol{\sigma} \cdot (-i\nabla \times \mathbf{A}) \quad (6.54a)$$

$$\widehat{\mathbf{p}} \cdot \mathbf{A} = \mathbf{A}.\widehat{\mathbf{p}} - \nabla \cdot \mathbf{A} \quad (6.54b)$$

and the gauge condition

$$\nabla \cdot \mathbf{A} = 0, \quad (6.54c)$$

we write Eq. (52):

$$H = H_0 + H_{\text{int}}, \tag{6.55}$$

where

$$H_0 = \sum_{i=1}^{2} \frac{1}{2m_i} \mathbf{p}_i^2 + V(\mathbf{r}_1, \mathbf{r}_2) \tag{6.56}$$

$$H_{\text{int}} = -e \sum_i \frac{Q_i}{2m_i} \left[2\mathbf{A}(\mathbf{r}_i, t) \cdot \widehat{\mathbf{p}}_i + i\boldsymbol{\sigma}_i \cdot \left(-i\nabla_i \times \mathbf{A}(\mathbf{r}_i, t) \right) \right]. \tag{6.57}$$

In Eq. (56), we have neglected the second order term e^2. Now

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{\sqrt{2V}} \sum_{\mathbf{k}'} \sum_{\lambda'} \frac{1}{\sqrt{\omega'}} \begin{bmatrix} \varepsilon^{\lambda'} a_{\lambda'}(\mathbf{k}') \; e^{i\mathbf{k}'\cdot\mathbf{r}} e^{-i\omega' t} \\ +\varepsilon^{*\lambda'} a^{\dagger}_{\lambda'}(\mathbf{k}') \; e^{-i\mathbf{k}'\cdot\mathbf{r}} e^{i\omega' t} \end{bmatrix} \tag{6.58}$$

where $\varepsilon^{\lambda'}$ is the polarization vector, $a_{\lambda'}(\mathbf{k}')$ and $a^{\dagger}_{\lambda'}(\mathbf{k}')$ are the annihilation and creation operators for the photon respectively. They satisfy the commutation relation

$$\left[a_\lambda(\mathbf{k}) \; a^{\dagger}_{\lambda'}(\mathbf{k}') \right] = \delta_{\lambda\lambda'} \; \delta(\mathbf{k} - \mathbf{k}'). \tag{6.59}$$

We now consider the emission of a photon viz. the process

$$a \to b + \gamma. \tag{6.60}$$

We note that

$$a_\lambda(\mathbf{k}) \left| a \right\rangle = 0 \tag{6.61a}$$

$$\left\langle b \, \gamma \right| = \left\langle b \right| a_\lambda(\mathbf{k}) \tag{6.61b}$$

$$\begin{aligned} \left\langle b \, \gamma \right| a^{\dagger}_{\lambda'}(\mathbf{k}') &= \left\langle b \,\right| a_\lambda(\mathbf{k}) \; a^{\dagger}_{\lambda'}(\mathbf{k}') \\ &= \left\langle b \,\right| \left[\delta_{\lambda\lambda'} \; \delta(\mathbf{k} - \mathbf{k}') - a^{\dagger}_{\lambda'}(\mathbf{k}') \; a_\lambda(\mathbf{k}) \right]. \end{aligned} \tag{6.61c}$$

It is clear from Eq. (60) that only second half of Eq. (57) contributes and the matrix elements for the process (59) are given by

$$H_{ba} = -e\sum_i \langle b \mid \frac{Q_i}{2m_i}\frac{1}{\sqrt{2V\omega}}e^{-i\mathbf{k}\cdot\mathbf{r}_i} \times \left[2\boldsymbol{\varepsilon}^{*\lambda}\cdot\hat{\mathbf{p}}_i - i\boldsymbol{\sigma}_i\cdot\left(\mathbf{k}\times\boldsymbol{\varepsilon}^{*\lambda}\right)\right]|a\rangle\ e^{i\omega t}, \tag{6.62}$$

where we have used

$$-\nabla\times\mathbf{A}\propto(-i)^2\ \mathbf{k}\times\boldsymbol{\varepsilon}^{\lambda^*}. \tag{6.63}$$

In Eq. (61), the term with 2 $\boldsymbol{\varepsilon}^{\lambda^*}\cdot\hat{\mathbf{p}}_i$ gives the electric transition and the term $\boldsymbol{\sigma}_i\left(\mathbf{k}\times\boldsymbol{\varepsilon}^{*\lambda'}\right)$ gives the magnetic transition.

We now make the dipole approximation so that in the expansion

$$e^{-i\mathbf{k}\cdot\mathbf{r}_i} = 1 - i\ \mathbf{k}\cdot\mathbf{r}_i + \cdots, \tag{6.64}$$

we retain only the first term. Then

$$H_{ba}^{E_1} = -e\ \sum_i \frac{1}{\sqrt{2V\omega}}\langle b|\, Q_i\frac{\hat{\mathbf{p}}_i}{m_i}\,|a\rangle\cdot\boldsymbol{\varepsilon}^{\lambda^*}\ e^{i\omega t}. \tag{6.65}$$

Now

$$i\frac{\hat{\mathbf{p}}_i}{m_i} = [\mathbf{r}_i,\ H_0] + 0(e). \tag{6.66}$$

We go to the center of mass (c.m.) frame and introduce

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{6.67a}$$

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2} \tag{6.67b}$$

$$\frac{1}{\mu} = \frac{1}{m_1}+\frac{1}{m_2}. \tag{6.67c}$$

In the c.m. frame $\mathbf{R}=0$, so that

$$[\mathbf{R},\ H] = 0. \tag{6.68}$$

Therefore, we have from Eqs. (64)-(66):

$$H_{ba}^{E_1} = \frac{ie\mu\omega}{\sqrt{2V\omega}}\left[\langle b|\left(\frac{Q_1}{m_1}-\frac{Q_2}{m_2}\right)\mathbf{r}\,|a\rangle\cdot\boldsymbol{\varepsilon}^{\lambda^*}\right]\ e^{i\omega t}, \tag{6.69}$$

where we have used the fact that $|a\rangle$ and $|b\rangle$ are eigenstates of H_0 with eigenvalues E_a and E_b:

$$H_0 |a\rangle = E_a |a\rangle , \tag{6.70a}$$

$$H_0 |b\rangle = E_b |a\rangle , \tag{6.70b}$$

$$E_a - E_b = \omega. \tag{6.70c}$$

We shall make use of Eq. (68) later. Here we consider the magnetic transition in dipole approximation i.e. allowed M1 transition. For M1 transition we get from Eq. (61)

$$H_{ba}^{M_1} = \frac{i\, e}{\sqrt{2V\omega}} \langle b| \sum_i \frac{Q_i}{2m_i}\sigma_i \cdot \left(\mathbf{k} \times \boldsymbol{\varepsilon}^{\lambda^*}\right) |a\rangle \; e^{i\omega t}. \tag{6.71}$$

We consider the decays of the form

$$V \to P + \gamma$$

$$^3S_1 \to {}^1S_0 + \gamma. \tag{6.72}$$

For the transition $^3S_1 \to ^1 S_0$, $\Delta L = 0$ and there is no change in parity. Therefore, it is $M1$ transition and the Hamiltonian given in Eq. (70) is relevant for the decay (71). Now we can write

$$\begin{aligned}\boldsymbol{\sigma} \cdot \left(\mathbf{k} \times \boldsymbol{\varepsilon}^{\lambda^*}\right) &= \sigma_z \left(\mathbf{k} \times \boldsymbol{\varepsilon}^{\lambda^*}\right)_z + \sqrt{2}\, s_+ \left(\mathbf{k} \times \boldsymbol{\varepsilon}^{\lambda^*}\right)_- \\ &\quad + \sqrt{2}\, s_- \left(\mathbf{k} \times \boldsymbol{\varepsilon}^{\lambda^*}\right)_+ ,\end{aligned} \tag{6.73a}$$

where

$$s_+ = \frac{1}{2}\left(\sigma_x + i\sigma_y\right), \quad s_- = \frac{1}{2}\left(\sigma_x - i\sigma_y\right), \tag{6.73b}$$

$$\left(\mathbf{k} \times \boldsymbol{\varepsilon}^{\lambda^*}\right)_\pm = \frac{1}{\sqrt{2}}\left[\left(\mathbf{k} \times \boldsymbol{\varepsilon}^{\lambda^*}\right)_x \pm i \left(\mathbf{k} \times \boldsymbol{\varepsilon}^{\lambda^*}\right)_y\right]. \tag{6.73c}$$

If we take the matrix elements between $V(S_z = 0)$ and $P(S_z = 0)$, we need to consider $\sigma_{iz}\left(\mathbf{k} \times \boldsymbol{\varepsilon}^{\lambda^*}\right)_z$ i.e. we have to calculate the matrix elements of the operator

$$\hat{\mu}_z = \sum_i \left(Q_i/2m_i\right) \sigma_{iz} \tag{6.74a}$$

between the states

$$|V,\ S_z = 0\rangle \quad \text{and} \quad |P\rangle . \tag{6.74b}$$

Using the state functions given in Tables 1 and 2, we can easily calculate the matrix elements $\langle \widehat{\mu}_z \rangle$. We explicitly calculate $\langle \widehat{\mu}_z \rangle$ for the transition $\omega^0 \to \pi^0$. It is convenient to write

$$\left|\omega^0,\ S_z = 0\right\rangle = \frac{1}{\sqrt{2}}\left|\ u\overline{u} +\ d\overline{d}\right\rangle\ \chi_S^0. \tag{6.75}$$

Then

$$\begin{aligned} &\widehat{\mu}_z \left|\omega^0,\ S_z = 0\right\rangle \\ = &\ \frac{1}{2}\left(-\frac{2}{3m_u}\sigma_{\overline{u}z} + \frac{2}{3m_u}\sigma_{uz} + \frac{1}{3m_d}\sigma_{\overline{d}z} - \frac{1}{3m_d}\sigma_{dz}\right)\left|\omega^0,\ S_z = 0\right\rangle \\ = &\ -\frac{2}{3m_u}\frac{1}{\sqrt{2}}\left|\ u\overline{u}\right\rangle\ \chi_A^0 + \frac{1}{3m_d}\frac{1}{\sqrt{2}}\left|\ d\overline{d}\right\rangle\ \chi_A^0. \end{aligned} \tag{6.76}$$

Now

$$\left|\pi^0\right\rangle = \frac{1}{\sqrt{2}}\left|\ u\overline{u} - d\overline{d}\ \right\rangle\ \chi_A^0. \tag{6.77}$$

Hence we get

$$\left\langle \pi^0\right| \widehat{\mu}_z \left|\omega^0,\ S_z = 0\right\rangle = -\frac{1}{6}\left(\frac{2}{m_u} + \frac{1}{m_d}\right). \tag{6.78}$$

Similarly we can calculate $\langle \widehat{\mu}_z \rangle$ for other members of the octet. They are given in Table 7.

We now calculate the decay rate for $V \to P\gamma$. According to Fermi Golden Rule, the decay rate is given by

$$\Gamma = 2\pi \left|\langle P|\, H_{\text{int}}^{M1}\, |V\rangle\right|^2 \rho(E). \tag{6.79}$$

If we consider the decay of the vector meson at rest, then

$$E_V = m_V\ ; \qquad 0 = \mathbf{k} + \mathbf{k}_P, \quad |\mathbf{k}| = \omega$$

Table 6.7 The matrix elements $\langle P|\,\hat{\mu}_z\,|V,\ S_z=0\rangle$ for M1 transition for the decay $V \to P+\gamma$.

Transition	Matrix elements $\langle P\|\,\hat{\mu}_z\,\|V,\ S_z=0\rangle$
$\omega^0 \to \pi^0$	$-\frac{1}{6}\left(\frac{2}{m_u}+\frac{1}{m_d}\right)$
$\rho^0 \to \pi^0$	$-\frac{1}{6}\left(\frac{2}{m_u}-\frac{1}{m_d}\right)$
$\rho^\pm \to \pi^\pm$	$-\frac{1}{6}\left(\frac{2}{m_u}-\frac{1}{m_d}\right)$
$\omega^0 \to \eta_{ns}$	$\frac{1}{4}\left(-\frac{4}{3m_u}+\frac{2}{3m_d}\right)$
$\rho^0 \to \eta_{ns}$	$\frac{1}{4}\left(-\frac{4}{3m_u}-\frac{2}{3m_d}\right)$
$\phi \to \eta_s$	$\frac{1}{3m_s}$
$K^{*+} \to K^+$	$\frac{1}{6}\left(\frac{1}{m_s}-\frac{2}{m_u}\right)$
$K^{*0} \to K^0$	$\frac{1}{6}\left(\frac{1}{m_s}+\frac{1}{m_d}\right)$

$$E_P = \sqrt{\omega^2 + m_P^2}$$

and

$$\begin{aligned}\rho(E) &= \int \delta\left(m_V - E_P - \omega\right)\frac{V}{(2\pi)^3}\omega^2 d\omega d\Omega \\ &= \frac{V\omega^2}{(2\pi)^3}\frac{E_P}{m_V}d\Omega, \quad [m_V = E_P + \omega]\,. \end{aligned} \tag{6.80}$$

Now $\langle P|\,H_{\text{int}}^{M1}\,|V\rangle$ is given in Eq. (70) with $a=V$ and $b=P$. In order to calculate Γ, we have to average over the initial spins of vector meson V and sum over the final spins of the photon. The vector meson has three spin orientations $S_z=+1,0,-1$. Instead of calculating $\left\langle H_{\text{int}}^{M1}\right\rangle$ for $S_z=\pm1,0$ and then taking the average, it is more convenient to calculate $\left\langle H_{\text{int}}^{M1}\right\rangle$ for $S_z=0$ and forget about the spin average. Thus from Eqs. (70) and (73), we get

$$\left|\langle P|\,H_{\text{int}}^{M1}\,|V\rangle\right|^2 = \sum_{\lambda=1,\ 2}\frac{e^2}{2V\omega}\left|\langle P|\,\hat{\mu}_z\,|V,\ S_z=0\rangle\right|^2\left|\left(\mathbf{k}\times\boldsymbol{\varepsilon}_z^{\lambda*}\right)\right|^2. \tag{6.81}$$

From now on we will not write $S_z = 0$ explicitly in $|V\rangle$. We note the following properties of the polarization vector ε^λ :

$$\boldsymbol{\varepsilon}^\lambda \cdot \boldsymbol{\varepsilon}^{\lambda'} = \delta_{\lambda\lambda'}$$

$$\mathbf{k} \cdot \boldsymbol{\varepsilon}^\lambda = 0, \qquad \lambda = 1, 2$$

$$\sum_\lambda \varepsilon_n^{*\lambda} \varepsilon_{n'}^\lambda = \left(\delta_{nn'} - \frac{k_n\, k_{n'}}{k^2} \right), \quad n, n' = 1, 2, 3. \tag{6.82}$$

Using Eq. (81), we have

$$\sum_{\lambda=1,\ 2} \left| \left(\mathbf{k} \times \boldsymbol{\varepsilon}^{*\lambda} \right)_z \right|^2 = k^2 \left(1 - \cos^2\theta \right) \tag{6.83}$$

and

$$\int d\Omega\ k^2 \left(1 - \cos^2\theta \right) = \frac{8\pi}{3} k^2. \tag{6.84}$$

Hence from Eqs. (78)-(80) and (83), we get

$$\Gamma = \frac{4\alpha}{3} \left| \langle P | \,\widehat{\mu}_z\, | V \rangle \right|^2 k^3 \frac{E_P}{m_V}. \tag{6.85}$$

For the decay

$$P \to V + \gamma, \tag{6.86}$$

we only sum over the spin of vector meson and do not take the average. Hence for this decay, we have

$$\Gamma \left(P \to V + \gamma \right) = 4\alpha\ \left| \langle V | \,\widehat{\mu}_z\, | P \rangle \right|^2 k^3 \frac{E_V}{m_P}. \tag{6.87}$$

We note that a relativistic treatment of the phase space gives the expressions (84) and (86) without the factor E_P/m_V and E_V/m_P respectively. Thus we can write Eq. (84):

$$\Gamma = \frac{4\alpha}{3} \left| \langle P | \,\widehat{\mu}_z\, | V \rangle \right|^2 k^3 \Omega^2, \tag{6.88}$$

where Ω is the overlap integral. It is of order 1, but it may differ from 1, if we take into account the distortion of wave function due

Table 6.8 Quark model prediction for V→ P + γ with $\Omega = 0.735$.

Decay	k (in MeV)	Γ (in keV)	Γ Experimental (in keV)
$\omega^0 \to \pi^0 + \gamma$	380	(9.6) (123) Ω^2 = (640)	$716.6 \begin{matrix} +50.7 \\ -49.8 \end{matrix}$
$K^{*\pm} \to K^\pm + \gamma$	307	124 Ω^2 = (67)	50.3 ± 5.3
$K^{*0} \to K^0 + \gamma$	309	190 Ω^2 = (103)	114.5 ± 11.8

to symmetry breaking introduced by the quark mass differences. Ω may vary from process to process. We assume that this variation is not large. Then we can fix Ω by using one decay, which we take $\rho^\pm \to \pi^\pm + \gamma$. Using Eq. (87), Table 7, $m_u = m_d = 336$ MeV and $k = 372$ MeV, we get

$$\Gamma\left(\rho^\pm \to \pi^\pm + \gamma\right) = (123 \text{ keV})\ \Omega^2 \tag{6.89}$$

But

$$\Gamma_{\text{exp}}\left(\rho^\pm \to \pi^\pm + \gamma\right) = (67.1 \pm 8.8)\ \text{keV}. \tag{6.90}$$

From Eqs. (88) and (89), we get

$$\Omega \approx 0.735. \tag{6.91}$$

Using this value of Ω and $m_u/m_s = 0.66$, we can compare the predictions of quark model using Table 7 and Eq. (87) with their experimental values. This is given in Table 8.

We *notice from Table 8 that agreement between the predictions and experiment is only fair.* This is understandable since the relativistic corrections become important for hadrons involving light quarks [see, for instance Ref. 6 in the bibliography].

6.4 Problems

1. In quark model, using SU(6) wave functions, show that the Fermi matrix element for n $\rightarrow$ p transition:

$$\left\langle p,\ S_z = \frac{1}{2}\right| \sum_q \tau_q^+ \left| n,\ S_z = \frac{1}{2}\right\rangle = 1.$$

Find the Gamow - Teller matrix element

$$\left\langle p,\ S_z = \frac{1}{2}\right| \sum_q \tau_q^+ \sigma_{qz} \left| n,\ S_z = \frac{1}{2}\right\rangle.$$

2. Show that the transition moment between Δ^+ and p is given by

$$\left\langle p,\ S_z = \frac{1}{2}\right| \widehat{\mu}_z \left| \Delta^+,\ S_z = \frac{1}{2}\right\rangle = \frac{2\sqrt{2}}{3}\mu_p.$$

3. Calculate the decay rates for the following decays in quark model:

$$\phi \rightarrow \eta + \gamma$$

$$\begin{aligned} \eta' &\rightarrow \rho^0 + \gamma \\ &\rightarrow \omega^0 + \gamma \end{aligned}$$

and compare them with their experimental values (54.9 $\pm$6.5) keV, (72 $\pm$13) keV, (72 $\pm$13) keV and (6.5 $\pm$1.0) keV respectively. [You may take $\eta_8 - \eta_1$ mixing angle as $\theta = -11^0$.]

4. Consider $M1$ transition decay

$$\Sigma^0 \rightarrow \Lambda^0 + \gamma.$$

Calculate its decay rate in the non-relativistic quark model and compare it with its experimental value

$$\tau = (5.8 \pm 1.3) \times 10^{-20} \text{ sec.}$$

Hint: M1 transition operator is

$$\sum_q \frac{Q_q}{2m_q}\sigma_q \cdot \left(\mathbf{k} \times \boldsymbol{\varepsilon}^{*\lambda}\right) = \hat{\mu}_z \left(\mathbf{k} \times \boldsymbol{\varepsilon}^{*\lambda}\right)_z + \sqrt{2}\ \hat{\mu}_+ \left(\mathbf{k} \times \boldsymbol{\varepsilon}^{*\lambda}\right)_- + \sqrt{2}\ \hat{\mu}_- \left(\mathbf{k} \times \boldsymbol{\varepsilon}^{*\lambda}\right)_+ ,$$

where

$$\hat{\mu}_z = \sum_q \frac{Q_q}{2m_q}\sigma_{qz}, \quad \hat{\mu}_\pm = \sum_q \frac{Q_q}{2m_q}\sigma_{q\pm}$$

$$\left|\Lambda,\ S_z = \pm\frac{1}{2}\right\rangle = -\left|u\ d\ s\right\rangle\ \chi_{MA}^{\pm 1/2}$$

$$\left|\Sigma^0,\ S_z = \pm\frac{1}{2}\right\rangle = \left|u\ d\ s\right\rangle\ \chi_{MS}^{\pm 1/2}.$$

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Chapter 7

COLOR, GAUGE PRINCIPLE AND QUANTUM CHROMODYNAMICS

7.1 Evidence for Color

As we have discussed in the introduction in order that 3 quark wave function of lowest lying baryons satisfy the Pauli principle, each quark flavor carries three color charges, red (r), yellow (y) and blue (b) i.e.

$$q_a \qquad\qquad a = r, y, b.$$

Leptons do not carry color and that is the reason why they do not experience strong interactions. Thus each quark belongs to a triplet representation of color SU(3), which we write as $SU_C(3)$. Now SU(3) has the remarkable property that $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{10}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}$ and $\bar{\mathbf{3}}\otimes\mathbf{3} = \mathbf{8}\oplus\mathbf{1}$, so that baryons which are bound states of 3 quarks belong to the singlet representation, which is totally antisymmetric as required by the Pauli principle and mesons which are bound states of $q\bar{q}$ belong to the singlet representation which is totally symmetric. This assignment takes into account the fact that all known hadrons are color singlets. Thus the color is hidden. This is the postulate of color confinement and explains the non-existence of free quarks.

Evidence for color also comes from $\pi^0 \to 2\gamma$ decay. Since π^0 is bound state of $q\bar{q}$ i.e. $|\pi^0\rangle = \frac{1}{\sqrt{2}}\left|u\bar{u} - d\bar{d}\right\rangle$, one can imagine that the decay takes place as shown in Fig. 1. The matrix elements M for the π^0 - decay, without and with color [where we have to sum over the 3 colors for the quarks in the above diagrams] are respectively proportional to

$$
\begin{aligned}
M &\propto \frac{1}{\sqrt{2}}\left[\left(\frac{2}{3}\right)^2-\left(-\frac{1}{3}\right)^2\right]e^2=\frac{1}{\sqrt{2}\,3}e^2 \\
M &\propto \frac{1}{\sqrt{2}}\left[\left(\frac{2}{3}\right)^2-\left(-\frac{1}{3}\right)^2\right]3e^2=\frac{1}{\sqrt{2}}e^2.
\end{aligned}
$$

In fact the above quark triangle diagrams predict

$$
M=e^2F=\frac{e^2}{2\pi^2}\frac{S_\pi}{f_\pi} \tag{7.1a}
$$

where

$$
S_\pi=\begin{cases}\frac{1}{3\sqrt{2}} & \text{without color}\\ \frac{1}{\sqrt{2}} & \text{with color}\end{cases}, \tag{7.1b}
$$

and f_π is the pion decay constant and is determined from the decay $\pi^+ \to \mu^+ + \nu_e$ [see Chapter 11]; its value is 132 MeV. Hence the decay rate is given by

$$
\begin{aligned}
\Gamma(\pi^0\to 2\gamma) &= 4\pi\alpha^2|F|^2\frac{m_{\pi^0}^3}{16} \\
&= \frac{\alpha^2}{16\pi^3 f_\pi^2}S_\pi^2 m_{\pi^0}^3.
\end{aligned} \tag{7.2}
$$

With $S_\pi = \frac{1}{\sqrt{2}}$, this gives $\Gamma(\pi^0 \to 2\gamma) = 7.58$ eV in very good agreement with the experimental value $\Gamma_{exp} = 7.74\pm0.50$. Without color Γ_{th} will be a factor of 9 less in complete disagreement with the experimental value.

Another evidence for color comes from measuring the ratio of e^-e^+ annihilation processes

$$
R=\frac{\sigma(e^-e^+\to \text{ hadrons})}{\sigma(e^-e^+\to\mu^-\mu^+)} \tag{7.3}
$$

in the large center of mass energy $\sqrt{s}=\sqrt{(p_1+p_2)^2}$ limit, where p_1 and p_2 are the momenta of e^- and e^+ respectively. To the

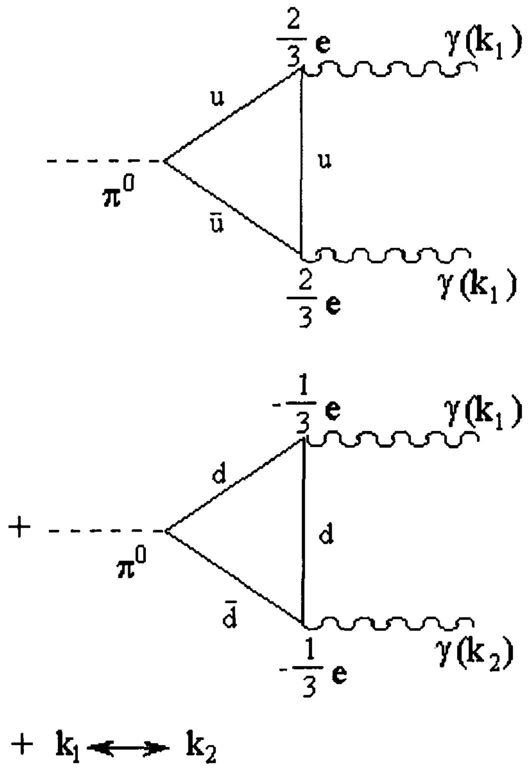


Figure 1 Triangle diagrams for $\pi^0 \to 2\gamma$ through its constituents.

lowest order in electromagnetic interaction, Eq. (A.77b) gives in the asymptotic region ($s \gg m_e^2,\, m_\mu^2$)

$$\sigma(e^- e^+ \to \mu^- \mu^+) = \frac{4\pi}{3}\alpha^2 \frac{1}{s}. \tag{7.4}$$

Now for the inclusive process $e^- e^+ \to$ hadrons, we expect this to take place via $e^- e^+ \to q\bar{q}$ and quarks (antiquarks) fragment into hadrons [see Fig. 2], so that

$$(e^- e^+ \to \text{ hadrons}) = \sum_q \sigma(e^- e^+ \to q\bar{q})$$

where the analogue of Eq. (4) gives in the asymptotic region [$s \gg m_e^2,\, m_q^2$]

$$\sigma(e^- e^+ \to q\bar{q}) = \frac{4\pi}{3}\alpha[3e_q^2]\frac{1}{s}, \tag{7.5}$$

where e_q (in units of e) are the electric charges of the quarks which enter the photon-$q\bar{q}$ vertex [see Fig. 2] and the factor 3 arises because we have to sum over 3 colors for each quark flavor q. This gives in the asymptotic region

$$R = 3\sum_q e_q^2. \tag{7.6}$$

For example, above the bottom quark threshold (see Chapter 8) i.e. for $\sqrt{s}$ in the range $2m_b < \sqrt{s} \ll m_Z$ [so that weak interaction effects can be neglected],

$$3\sum_q e_q^2 = 3\left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9}\right) = \frac{11}{3},$$

which is confirmed by experimental measurement of R above $\sqrt{s} > 2m_b$ [see Fig. 3].

Actually nature has also assigned a more fundamental role to color charges. We know that electromagnetic force is a gauge force; here we postulate that strong force is also a gauge force. In order to discuss the gauge force, we first state the gauge principle.

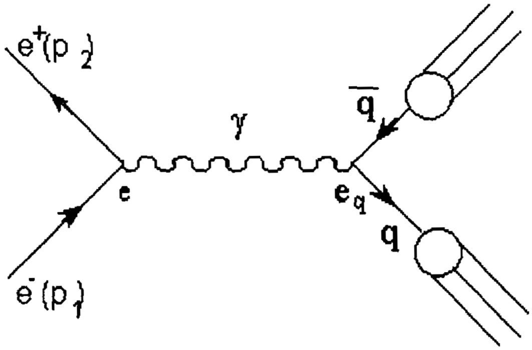


Figure 2 One photon exchange diagram for hadron production in e^-e^+ annihilation.

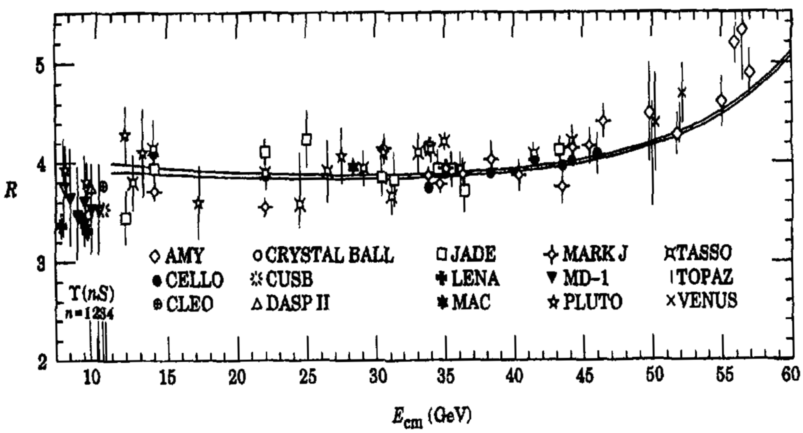


Figure 3 Compilation of R-values from different e^-e^+ experiments

7.2 Gauge Principle

Suppose a physical system described by a wave function $\Psi(x)$, $x \equiv (t, \mathbf{r})$ has the property that under a phase transformation

$$\Psi(x) \rightarrow \Psi'(x) = e^{ie\Lambda}\Psi(x) \tag{7.7}$$

(with Λ constant), the wave equation satisfied by Ψ or the corresponding Lagrangian is invariant. Now if we demand that it remains invariant when Λ is a function of space-time, then we shall show that it is necessary to introduce a vector boson which is coupled to a vector current with universal coupling e. We call such a phase transformation, local gauge transformation and the vector boson associated with it is a mediator of force whose strength is determined by the charge e.

This is best illustrated by considering a non-relativistic particle of charge e and mass m described by a complex wave function $\Psi(x)$. Consider a space-time dependent phase transformation given in Eq. (7), with Λ as a function of x and e the electric charge. For this case the physical law is given by the Schrödinger equation

$$-\frac{1}{2m}\nabla^2\Psi = i\frac{\partial\Psi}{\partial t}. \tag{7.8}$$

This is not invariant under the local gauge transformation (7). In order to restore gauge invariance, it is necessary to postulate a vector field $A_\mu \equiv (\phi, \mathbf{A})$ and make the substitutions

$$\begin{aligned} \nabla &\rightarrow \nabla - ie\mathbf{A} \\ \frac{\partial}{\partial t} &\rightarrow \frac{\partial}{\partial t} + ie\phi \end{aligned}$$

or

$$\partial_\mu \rightarrow \partial_\mu + ieA_\mu. \tag{7.9}$$

Equation (8) now becomes

$$-\frac{1}{2m}(\nabla - ie\mathbf{A})^2\Psi = i\left(\frac{\partial}{\partial t} + ie\phi\right)\Psi. \tag{7.10}$$

This equation is invariant under the transformation (7), provided that $\mathbf{A}$ and ϕ simultaneously undergo the transformations:

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A} + \nabla\Lambda \\ \phi &\rightarrow \phi - \frac{\partial}{\partial t}\Lambda \end{aligned}$$

or

$$A_\mu \rightarrow A_\mu - \partial_\mu \Lambda. \tag{7.11}$$

$A_\mu \equiv (\phi, -\mathbf{A})$ are the electromagnetic potentials. From the present point of view, the necessity for the existence of the electromagnetic potential $A_\mu(x)$ is a consequence of assuming invariance under the local gauge transformation. The electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are related to the vector potential A_μ as follows:

$$\begin{aligned} \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \\ \mathbf{B} &= \nabla \times \mathbf{A}. \end{aligned} \tag{7.12}$$

They are clearly invariant under the gauge transformations (11).

The Lagrangian density which gives Eq. (10) is given by

$$\begin{aligned} L &= -\frac{1}{2m}\nabla\Psi^* \cdot \nabla\Psi + \frac{1}{2i}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right) \\ &\quad -e(\rho\phi - \mathbf{j}\cdot\mathbf{A}) + \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \end{aligned} \tag{7.13}$$

where

$$\begin{aligned} \rho &= \Psi^*\Psi, \\ \mathbf{j} &= \frac{1}{2im}\left(\Psi^*\nabla\Psi - (\nabla\Psi^*)\Psi\right) - \frac{e}{2m}\mathbf{A}\Psi^*\Psi. \end{aligned} \tag{7.14a}$$

L is clearly invariant under the gauge transformations (7) and (11). ρ and $\mathbf{j}$ satisfy the equation of continuity

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0. \tag{7.14b}$$

This implies that the charge

$$Q = \int \rho(x) d^3x \tag{7.14c}$$

is conserved. Note also that the last term in Eq. (13) can be written in manifestly covariant form $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor. The term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Lagrangian density for pure electromagnetic field.

7.2.1 Aharanov and Bohm experiment

We now discuss the question of testing the applicability of the gauge principle in electromagnetism. Taking the vector potential $\mathbf{A}$ to be independent of time and putting $V = e\phi$, we try solution of Eq. (10) in the following form

$$\Psi(\mathbf{r}, t) = \Psi^0(\mathbf{r}, t)e^{i\gamma\mathbf{r}} \tag{7.15a}$$

where

$$\gamma(\mathbf{r}) = e\int^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\ell'. \tag{7.15b}$$

Here Ψ can be regarded as a wave function of a particle that goes from one place to another along a certain route where a field $\mathbf{A}$ is present while Ψ^0 is the wave function for the same particle along the same route but with $\mathbf{A} = 0$. It is easy to see that

$$\begin{aligned} \mathbf{D}\Psi &\equiv (\nabla - ie\mathbf{A})\Psi \\ &= e^{i\gamma(\mathbf{r})}\nabla\Psi^0 \\ \mathbf{D}^2\Psi &= e^{i\gamma(\mathbf{r})}\nabla^2\Psi^0. \end{aligned}$$

Thus (15) is a solution of Eq. (10) when $\mathbf{A}(\mathbf{r}) \neq 0$ if $\Psi^0(\mathbf{r}, t)$ satisfies

$$-\frac{1}{2m}\nabla^2\Psi^0 + V\Psi^0 = i\frac{\partial\Psi^0}{\partial t}. \tag{7.16}$$

The solution (15) has some striking physical consequences as shown in the two slit electron interferometer experiment proposed by Aharanov and Bohm [Fig. 4].

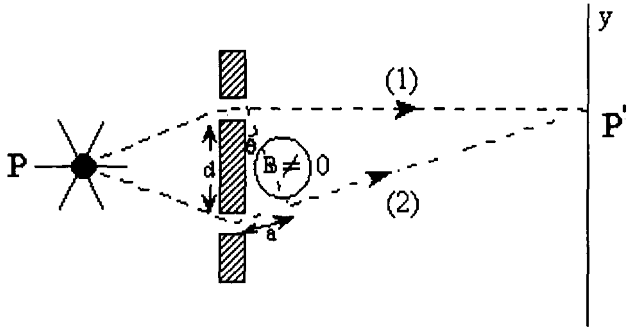


Figure 4 Double slit electron interferometer to test Aharanov-Bohm effect.

In this experiment the magnetic field $\mathbf{B}$ (pointing in a horizontal direction out of the paper) is produced by a long solenoid of small cross-section and is confined to the interior of the solenoid so that the two electron beams (1) and (2) can go above and below the $\mathbf{B} \neq 0$ region but stay within the $B = 0$ region and finally meet in the interference region P'. In the interference region, the wave function for the electron is

$$\Psi = \Psi_1 + \Psi_2$$

so that

$$|\Psi|^2 = |\Psi_1^0|^2 + |\Psi_2^0|^2 + 2|\Psi_1^0|\,|\Psi_2^0| \cos[\gamma_1(\mathbf{r}) - \gamma_2(\mathbf{r})] \tag{7.17a}$$

where

$$\gamma_1 = \gamma_1^0 + e \int_{(1)P}^{P'} \mathbf{A}(\mathbf{r}') \cdot d\ell'. \tag{7.17b}$$

$$\gamma_2 = \gamma_2^0 + e \int_{(2)P}^{P'} \mathbf{A}(\mathbf{r}') \cdot d\ell'. \tag{7.17c}$$

Here γ_1^0 and γ_2^0 are the phases of the wave functions Ψ_1^0 and Ψ_2^0 in the absence of **A**. The interference pattern is determined by the phase difference

$$\begin{aligned} \delta(B \neq 0) &= \gamma_1 - \gamma_2 \\ &= \gamma_1^0 - \gamma_2^0 + e \oint_C \mathbf{A}(\mathbf{r}') \cdot d\mathbf{l}\,. \\ &= \delta(B = 0) + \Delta, \end{aligned} \tag{7.18a}$$

where C is the closed path PP′P and

$$\Delta = e \oint_C \mathbf{A}(\mathbf{r}') \cdot d\mathbf{l}' = e \int_S \mathbf{B} \cdot d\boldsymbol{\sigma} = e\Phi. \tag{7.18b}$$

In Eq. (18) we have used Stokes theorem and put $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and Φ is the magnetic flux through the surface S bounded by the closed path C. Note the important fact that the phase difference Δ is gauge invariant while the individual phases γ_1 and γ_2 are not. Note also the remarkable fact that the amount of interference can be controlled by varying magnetic flux even though in the idealized experimental arrangement, electrons never enter the region $B \neq 0$.

Now referring to Fig. 4

$$\frac{\text{Phase difference}}{2\pi} = \frac{\text{Path difference}}{\lambda}$$

$$\frac{\delta}{2\pi} = \frac{a}{\lambda} = \frac{1}{\lambda} d \sin\theta \approx \frac{d}{\lambda}\frac{y}{L}$$

where L is the distance of the screen from the slits. Thus from Eq. (18) we see that the diffraction maximum of the interference pattern for $B \neq 0$ is shifted from that for $B = 0$ by the amount Δy given by

$$\Delta y = e\Phi \left(\frac{L}{d} \frac{\lambda}{2\pi} \right). \tag{7.19}$$

This shift in the diffraction maximum, being gauge invariant, should be measurable. In fact the existence and magnitude of Aharanov-Bohm effect has been confirmed to within 5% of the theoretical prediction (19) by two qualitatively different experimental arrangements - one involving an electron biprism interferometer while the second used a Josephson-junction interferometer.

The following comments are in order.

(i) Measurement of Aharanov-Bohm effect not only varifies the gauge principle in electromagnetism but also quantum mechanics itself since classically the dynamical behavior of electrons is controlled by Lorentz force which is zero when the electrons go through magnetic field free region; yet in quantum mechanics observable effects are seen and depend on the magnetic field in a region inaccessible to the electrons.

(ii) The vector potential **A** rather than the fields plays a crucial role as the basic dynamical variable in quantum mechanics.

(iii) When $\Delta = 2n\pi$ or $\Phi = n\phi_0\,[\phi_0 = 2\pi/e = 4.135 \times 10^{-7}$ gauss $\text{cm}^2]$, the shift vanishes.

7.2.2 Gauge principle for relativistic quantum mechanics

We now discuss the gauge principle for relativistic quantum mechanics. The spin 1/2 particle is described by Dirac equation with the Lagrangian density:

$$L = \bar{\Psi}(x)i\gamma^\mu\partial_\mu\Psi(x) - m\bar{\Psi}(x)\Psi(x). \tag{7.20}$$

In order that the Lagrangian density L be invariant under the gauge transformation (7), we must introduce a vector field $A_\mu(x)$ satisfying Eq. (11) and replace in Eq. (20) $\partial_\mu\Psi$ by

$$\partial_\mu\Psi(x) \to (\partial_\mu + ieA_\mu)\Psi \equiv D_\mu\Psi. \tag{7.21}$$

D_μ is called the co-variant derivative. The gauge invariant Lagrangian density is given by

$$L = \bar{\Psi}(x)i\gamma^\mu(\partial_\mu + ieA_\mu)\Psi - m\bar{\Psi}(x)\Psi(x) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{7.22}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.23}$$

It is easy to see that under the transformation (11), $F_{\mu\nu}$ is invariant. Under the transformations (7) and (11),

$$D_\mu\Psi \to e^{ie\Lambda(x)}D_\mu\Psi, \tag{7.24}$$

so that $\bar{\Psi}D_\mu\Psi$ is gauge invariant, and so is $m\bar{\Psi}\Psi$. From Eq. (22), we see that the interaction of matter field Ψ with the electromagnetic field A_μ is given by

$$L_{\text{int}} = -e\bar{\Psi}\gamma^\mu\Psi A_\mu = -J^\mu_{em}A_\mu, \tag{7.25a}$$

where

$$J^\mu_{em} = e\bar{\Psi}\gamma^\mu\Psi,\ \partial_\mu J^\mu_{em} = 0 \tag{7.25b}$$

is the electromagnetic current. We conclude that the gauge principle viz. the invariance of fundamental physical law under the gauge transformation gives correctly the form of interaction of a charged particle with electromagnetic field. To sum up the consequences of the electromagnetic force as a gauge force are as follows:

(i) It is universal viz. any charged particle is coupled with the electromagnetic field A with a universal coupling strength given by e, the electric charge of the particle.

(ii) J^μ_{em} is conserved.

(iii) The electromagnetic field is a vector and hence the associated quantum, the photon, has spin 1.

(iv) The photon must be massless, since the mass term $\mu^2 A^\mu A_\mu$ is not invariant under the gauge transformation. Thus unbroken gauge symmetry gives rise to long range force mediated by a massless gauge boson i.e. photon.

(v) The covariant derivative D_μ is an operator whose commutator is

$$[D_\mu, D_\nu] = ieF_{\mu\nu}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.26}$$

7.3 Quantum Chromodynamics (QCD)

We now generalize the ideas of Sec. 2 to the case where there is more than one type of states, e.g. q_a $(a = 1, 2, 3)$ and where there exist transformations $[SU_C(3)]$ between the different states

$$q_a \to q'_a = U^b_a q_b, \tag{7.27a}$$

with

$$\begin{aligned} U(x) &= \exp\left[\frac{i}{2}\lambda_A \Lambda_A(x)\right], \\ UU^\dagger &= 1, \qquad det\, U = 1 \end{aligned} \tag{7.27b}$$

and repeated indices imply summation. Here q_a $(a = 1, 2, 3)$ for a particular quark flavor q form the fundamental representation of the color SU(3) group and λ_A, $A = 1 \cdots 8$, are the eight matrix generators of the group $SU_C(3)$ [see Chap. 5 for the form of these matrices. Although in Chap. 5 we discussed flavor SU(3) but the mathematics is the same].

Quarks are spin 1/2 particles. The Lagrangian density for free quarks

$$L = \bar{q}^a i\gamma^\mu \partial_\mu q_a - \bar{q}^a m q_a, \tag{7.28a}$$

where

$$q_a = \begin{pmatrix} u_a \\ d_a \\ s_a \end{pmatrix} \text{ and } m = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} \tag{7.28b}$$

is clearly invariant under the SU(3) transformation (27) with Λ constant. If we now require that the Lagrangian density (28) be invariant under the gauge transformation (27), with $\Lambda(x)$ as function of space-time, then as we have seen in Sec. 2, we must replace ∂_μ by its co-variant derivative which in the present case takes the form

$$D_\mu = \left(\partial_\mu - \frac{i}{2} g_s \boldsymbol{\lambda} \cdot \mathbf{G}_\mu\right) = \left(\partial_\mu - \frac{i}{2} g_s \lambda_A G_{A\mu}\right) \tag{7.29}$$

where g_s is a scale parameter, the coupling constant and $\mathbf{G}_{A\mu}$ are vector gauge fields, their number being equal to the generators of $SU_C(3)$ group, namely 8. Then we note the important fact that the covariant derivatives satisfy the commutation relation

$$\begin{aligned}
[D_\mu, D_\nu] &= -ig_s\frac{\lambda_B}{2}[\partial_\mu, G_{B\nu}] - ig_s\frac{\lambda_A}{2}[G_{A\mu}, \partial_\nu] \\
&\quad +(-ig_s)^2\left[\frac{\lambda_A}{2}, \frac{\lambda_B}{2}\right]G_{A\mu}G_{B\nu} \\
&= -ig_s\frac{\lambda_C}{2}\{\partial_\mu G_{C\nu} - \partial_\nu G_{C\mu} + g_s f_{ABC}G_{A\mu}G_{B\nu}\} \\
&= -ig_s\{\partial_\mu G_\nu - \partial_\nu G_\mu - ig_s[G_\mu, G_\nu]\} \\
&= -ig_s G_{\mu\nu}, \qquad (7.30)
\end{aligned}$$

where in the matrix notation

$$\begin{aligned}
\frac{1}{2}\boldsymbol{\lambda}\cdot\mathbf{G}_\mu &= \frac{1}{2}\lambda_A G_{A\mu} \equiv G_\mu, \qquad (7.31a) \\
\frac{1}{2}\boldsymbol{\lambda}\cdot\boldsymbol{\Lambda}(x) &= \frac{1}{2}\lambda_A\Lambda_A \equiv \Lambda, \qquad (7.31b) \\
G_{\mu\nu} \equiv \frac{1}{2}\boldsymbol{\lambda}\cdot\mathbf{G}_{\mu\nu} &= \partial_\mu G_\nu - \partial_\nu G_\mu - ig_s[G_\mu, G_\nu] \\
&= D_\mu G_\nu - D_\nu G_\mu \qquad (7.31c) \\
G_{A\mu\nu} &= G_{A\nu} - \partial_\nu G_{A\mu} + g_s f_{ABC}G_{B\mu}G_{C\nu} \qquad (7.31d) \\
Tr(G_{\mu\nu}G_{\mu\nu}) &= \mathrm{Tr}\left(\frac{1}{2}\lambda_A G_{A\mu\nu}\frac{1}{2}\lambda_B G_{B\mu\nu}\right) \\
&= \frac{1}{4}\mathrm{Tr}(\lambda_A\lambda_B)G_{A\mu\nu}G_{B\mu\nu} \\
&= \frac{1}{2}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}_{\mu\nu}. \qquad (7.31e)
\end{aligned}$$

Note the important fact that $G_{\mu\nu}$ in Eq. (30) provides the generalization of $F_{\mu\nu}$ [cf. Eq. (26) in Abelian case] for the present non-Abelian case. The two differ in the appearance of the last term in Eq. (31c) or (31d). This is because the gauge fields themselves carry color charges in contrast to photons which are electrically neutral in the electromagnetic case. Now if we replace the Lagrangian density (28a) by

$$L = \bar{q}^a i\gamma^\mu \left(\partial_\mu - \frac{i}{2} g_s \lambda_A G_{A\mu}\right)_a^b q_b - \bar{q}^a m q_a - \frac{1}{4} G_A^{\mu\nu} G_{A\mu\nu} \tag{7.32a}$$

or in the matrix notation by

$$L = \bar{q}\, i\gamma^\mu \left(\partial_\mu - i g_s G_\mu\right) q - \bar{q} m q - \frac{1}{2} \mathrm{Tr}(G^{\mu\nu} G_{\mu\nu}), \tag{7.32b}$$

then the Lagrangian density (32) is invariant under the infinitesimal gauge transformation [cf. Eq. (27b)]

$$q \to \left(1 + \frac{i}{2} \lambda \cdot \Lambda(x)\right) q, \tag{7.33}$$

provided that the vector fields $G_{A\mu}$ undergo the simultaneous transformation

$$G_\mu \to G_\mu + i\left[\Lambda,\, G_\mu\right] + \frac{1}{g_s} \partial_\mu \Lambda \tag{7.34a}$$

or

$$G_{A\mu} \to G_{A\mu} - f_{ABC} \Lambda_B G_{C\mu} + \frac{1}{g_s} \partial_\mu \Lambda_A. \tag{7.34b}$$

To see this, we note that under these transformations

$$D_\mu q \to \left(1 + \frac{i}{2} \lambda \cdot \Lambda(x)\right] D_\mu q \tag{7.35a}$$

$$G_{A\mu\nu} \to G_{A\mu\nu} - f_{ABC} \Lambda_B G_{C\mu\nu}. \tag{7.35b}$$

It is then trivial to show that the Lagrangian (32) is gauge invariant.

For the finite gauge transformation (27), we have the gauge invariance provided that the gauge fields G_μ simultaneously undergo the transformation

$$G_\mu \to U G_\mu U^\dagger + \frac{i}{g_s} U \partial_\mu U^\dagger . \tag{7.36}$$

Under these transformations:

$$D_\mu q \;\to\; U(D_\mu q) \tag{7.37a}$$
$$G_{\mu\nu} \;\to\; U G_{\mu\nu} U^\dagger \tag{7.37b}$$

and hence the Lagrangian (32) is gauge invariant.

The eight gauge vector bosons $G_{A\mu}$ are called gluons. They are mediators of strong interaction between quarks just as photons are mediators of electromagnetic force between electrically charged particles. The gauge transformation given in Eq. (27) is called the non-Abelian gauge transformation, whereas the gauge transformation (7) is called the Abelian gauge transformation. The non-Abelian gauge transformation was first considered by Yang and Mills and gauge bosons are sometimes called Yang-Mills fields.

7.3.1 Conserved current

In order to discuss the conserved current associated with gauge fields, we discuss a general method. Suppose we have a set of fields which we denote by $\phi_a(x)$. The Lagrangian is a function of these fields ϕ_a and $\partial_\mu \phi_a$:

$$L = L(\phi_a, \partial_\mu \phi_a). \tag{7.38}$$

Consider an infinitesimal gauge transformation

$$\phi_a(x) \to \phi_a(x) + i\Lambda_A(x)(T_A)_a^b \phi_b . \tag{7.39}$$

T_A are matrices corresponding to the non-Abelian gauge group and the representation to which the fields $\phi_a(x)$ belong. From Eq. (38),

$$\delta L = \sum_\phi \frac{\partial L}{\partial \phi_a} \delta \phi_a + \sum_\phi \frac{\partial L}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a). \tag{7.40}$$

Using the Euler-Lagrange equations

$$\frac{\partial L}{\partial \phi_a} - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi_a)} \right) = 0 \tag{7.41}$$

and the fact that $\delta(\partial_\mu \phi_a) = \partial_\mu \delta(\phi_a)$, we have

$$\begin{aligned} \delta L &= \sum_\phi \left[\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi_a)} \right) \delta\phi_a + \frac{\partial L}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right] \\ &= \sum_\phi \partial_\mu \left[\frac{\partial L}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right] . \end{aligned} \tag{7.42}$$

On using Eq. (39) so that $\delta\phi_a = i\Lambda_A (T_A)^b_a \phi_b$, we get

$$\delta L = \sum_\phi \partial_\mu \left[\frac{\partial L}{\partial(\partial_\mu \phi_a)} i\Lambda_A (T_A)^b_a \phi_b \right] . \tag{7.43}$$

If we take Λ_A as constant i.e. independent of x, then we can rewrite Eq. (43) as

$$\delta L = \partial_\mu \sum_\phi i \left[\frac{\partial L}{\partial(\partial_\mu \phi_a)} (T_A)^b_a \phi_b \right] \Lambda_A \equiv -\partial_\mu F^\mu_A \Lambda_A, \tag{7.44}$$

where

$$F^\mu_A = -\sum_\phi i \left(\frac{\partial L}{\partial(\partial_\mu \phi_a)} \right) (T_A)^b_a \phi_b. \tag{7.45}$$

Hence we have the Nöether's theorem. If the Lagrangian is invariant under the gauge transformation (39) with constant Λ_A, i.e. $\delta L = 0$, then the current given in Eq. (45) is conserved.

Let us apply this to the QCD Lagrangian (32). Here ϕ_a correspond to $G_{B\mu}$ and q_a. Now, for the gauge vector bosons which belong to the adjoint representation of $\mathrm{SU}_C(3)$, we have $i(T_A)^C_B = -f_{BAC}$ and for the quarks which belong to the triplet representation of $SU_C(3)$, $T_A = \frac{1}{2}\lambda_A$. Then using the expression (31d) for $G_{A\mu\nu}$ in the Lagrangian (32), Eq. (45) gives

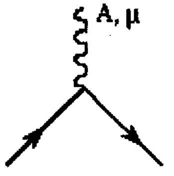
$$\begin{aligned} F_A^\mu &= \frac{1}{2}\bar{q}\gamma^\mu\lambda_A q + f_{ABC}G_B^{\mu\nu}G_{C\nu} \\ &= \frac{1}{2}\bar{q}^a\gamma^\mu\left(\lambda_A\right)_a^b q_b - f_{ABC}G_{B\nu}G_C^{\mu\nu}. \end{aligned} \tag{7.46}$$

The current F_A^μ is universally coupled to the gauge fields $G_{A\mu}$ with universal coupling g_s. Now the interaction part of the Lagrangian (32) is given by

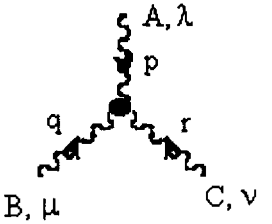
$$\begin{aligned} L_{int} &= g_s G_{A\mu}\bar{q}^a\gamma^\mu\left(\frac{\lambda_A}{2}\right)_a^b q_b \\ &\quad -g_s G_A^\mu f_{ABC}G_B^\nu\left[\frac{1}{2}\left(\partial_\mu G_{C\nu} - \partial_\nu G_{C\mu}\right) + g_s f_{CDE}G_{D\mu}G_{E\nu}\right]. \end{aligned} \tag{7.47}$$

The last term of Eq. (47) represents the self interaction of gauge bosons among themselves as they carry the color charges. This term is very important in QCD and is responsible for the asymptotic freedom of QCD.

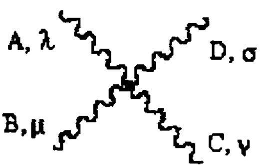
From Eq. (47), the q q G, G G G and G G G G vertices in the momentum space can be represented graphically as shown below



$$ig_s\gamma^\mu\left(\frac{\lambda_A}{2}\right)$$



$$g_s f_{ABC}\left[(p-q)^\nu g^{\lambda\mu} + (q-r)^\lambda g^{\mu\nu} + (r-p)^\mu g^{\nu\lambda}\right]$$



$$\begin{aligned}&-ig_s^2 f_{ABE}f_{CDE}\left(g^{\lambda\nu}g^{\mu\sigma}-g^{\lambda\sigma}g^{\mu\nu}\right)\\&-ig_s^2 f_{ACE}f_{BDE}\left(g^{\lambda\mu}g^{\nu\sigma}-g^{\lambda\sigma}g^{\mu\nu}\right)\\&-ig_s^2 f_{ADE}f_{CBE}\left(g^{\lambda\nu}g^{\mu\sigma}-g^{\lambda\mu}g^{\sigma\nu}\right)\end{aligned}$$

The Feynman rules for the QCD Lagrangian are discussed in Appendix B.

7.3.2 Experimental determinations of $\alpha_s(q^2)$ and asymptotic freedom of QCD

The important physical properties of QCD are

(i) the gluons, being mediators of strong interaction between quarks, are vector particles and carry color; both of these properties are supported by hadron spectroscopy discussed in the next section,

(ii) asymptotic freedom which implies that the effective coupling constant $\alpha_s = g_s^2/4\pi$ decreases logarithmically at short distances or high momentum transfers, a property which has a rigorous theoretical basis. This is the basis for perturbative QCD which is relevant for processes involving large momentum transfers,

(iii) confinement which implies that potential energy between color charges increases linearly at large distances so that only color singlet states exist, a property not yet established but find support from lattice simulations and qualitative pictures (see next section) and from quarkonium spectroscopy to be discussed in Chap. 8.

In this section, we discuss the present evidence for QCD being asymptotic free. First we note that due to quantum radiative corrections, α_s evolves with the characteristic energy of the process in which it appears. Actually these corrections give

$$g_s(Q^2) = g_{s0}\left[1 + g_{s0}^2 b_0 \ln\frac{\lambda^2}{Q^2} + \cdots\right], \qquad (7.48a)$$

where $\lambda^2 \gg Q^2$ and must be introduced so that the integrals involved in these corrections are convergent. Here $\cdots$ denotes higher order corrections and $\sqrt{Q^2}$ is the momentum carried by a gluon at quark-quark-gluon vertex which defines $g_s(Q^2)$. It is convenient to rewrite Eq. (48a) as

$$\frac{1}{g_s^2(Q^2)} = \frac{1}{g_{s0}^2}\left[1 - 2g_{s0}^2 b_0 \ln\frac{\lambda^2}{Q^2} + O(g_s^4)\right]. \qquad (7.48b)$$

This gives

$$\alpha_s^{-1}(Q^2) - \alpha_{s0}^{-1} = -8\pi b_0 \ln\frac{\lambda^2}{Q^2}. \qquad (7.48c)$$

We now eliminate the unobserved "bare" coupling constant α_{s0} and the cut-off λ^2 by making a subtraction at $Q^2 = \mu^2$. Thus we obtain

$$\alpha_s^{-1}(Q^2) - \alpha_s^{-1}(\mu^2) = b \ln\frac{Q^2}{\mu^2} \qquad (7.48d)$$

with $b = 8\pi b_0$. Or

$$\alpha_s(Q^2) = \frac{1}{\alpha_s^{-1}(\mu^2) + b \ln Q^2/\mu^2}. \tag{7.48e}$$

The constant b is evaluated in Appendix B and is given by

$$b = \frac{1}{4\pi}\left(11 - \frac{2}{3}n_f\right) \tag{7.48f}$$

where n_f is the number of effective quark flavors. Another way of writing Eq. (48e) is

$$\alpha_s^{-1}(Q^2) = b \ln \frac{Q^2}{\Lambda_{QCD}^2} \tag{7.48g}$$

where

$$\alpha_s^{-1}(\mu) - b \ln \mu^2 = -b \ln \Lambda_{QCD}^2.$$

Thus finally we have

$$\alpha_s(Q^2) = \frac{4\pi}{\left(11 - \frac{2}{3}n_f\right) \ln \frac{Q^2}{\Lambda_{QCD}^2}} \tag{7.48h}$$

and we see the running of $\alpha_s(Q^2)$ with Q^2. Λ_{QCD} is the QCD scale factor which effectively defines the energy scale at which the running coupling constant attains its maximum value. Λ_{QCD} can be determined from experiment. For $\frac{2}{3}n_f < 11$, it is clear from Eq. (48b) or (48h) that $\alpha_s(Q^2)$ decreases as Q^2 increases and approaches zero as $Q^2 \to \infty$ or $r \to 0$. This is known as the asympotic freedom property of QCD. This is due to the factor 11 in Eq. (48f) or (48h) and arises due to the self-interaction of gluons (see Appendix B).

We now discuss the experimental determination of the coupling constant $\alpha_s(Q^2)$ at various values of Q^2 from different reactions, starting from the lowest value of $\sqrt{Q^2}$.

The rates of quarkonium decay, in particular the ratio of rates $\Gamma_{\mu\mu}/\Gamma_{ggg}$ [see, in particular Eq. (8.45)] provides a determination of $\alpha_s\left(\sqrt{Q^2}\right)$ at $m_{J/\psi}$ and m_Υ for the charmonium and bottomonium states and give

$$\alpha_s(m_{J/\psi}) = 0.216 \pm 0.024,\ \alpha_s(m_\Upsilon) = 0.178 \pm 0.005 \tag{7.49a}$$

where we have used

$$R_\mu\begin{pmatrix} J/\psi \\ \Upsilon \end{pmatrix} = \frac{\Gamma\left(\begin{matrix} J/\psi \\ \Upsilon \end{matrix} \to \text{hadrons}\right)}{\Gamma\left(\begin{matrix} J/\psi \\ \Upsilon \end{matrix} \to \mu^+\mu^-\right)} = \begin{pmatrix} 14.5 \pm 1.6 \\ 32.5 \pm 0.9 \end{pmatrix}.$$

The value of α_s obtained from the scaling violations in deep inelastic lepton-nucleon scattering [see Chap. 14] gives

$$\alpha_s\left(\sqrt{Q^2} = 2.6 \text{ GeV}\right) = 0.264 \bullet 0.101. \tag{7.49b}$$

The order α_s corrections to the total hadronic cross-section in e^-e^+ annihilation in the ratio (3) modify it from Eq. (6) to

$$R = 3\sum_q e_q^2\left[1 + \frac{\alpha_s(\sqrt{s})}{\pi} + 1.411\left(\frac{\alpha_s(\sqrt{s})}{\pi}\right)^2 + \cdots\right].$$

By fitting the value of R at $\sqrt{s} = 34$ GeV shown in Fig. 3 one obtains

$$\alpha_s(34 \text{ GeV}) = 0.142 \pm 0.03. \tag{7.49c}$$

Finally from the semi-leptonic branching ratio R_τ for the inclusive decay $\tau \to \nu_\tau +$ hadrons, one obtains

$$\alpha_s(m_\tau) = 0.35 \pm 0.03.$$

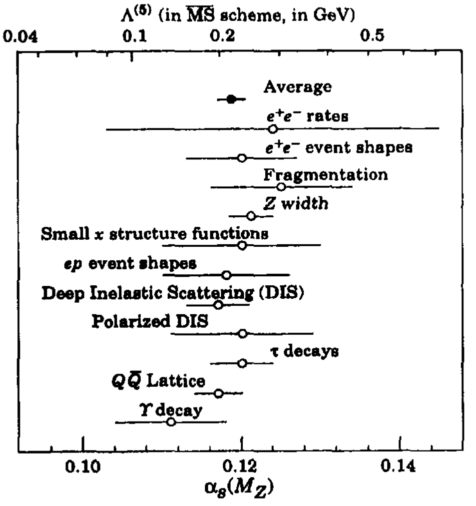


Figure 5 Summary of the values of $\alpha_s(m_Z)$ and $\Lambda^{(5)}$ from various processes. The values shown indicate the process and the measured value of α_s extrapolated upto $\mu = m_Z$. The error shown is the total error including theoretical uncertainties.

Figure 5 shows the values of $\alpha_s(m_z)$ deduced from the various experiments. Figure 6 clearly shows the experimental evidance for the running of $\alpha_s(\mu)$ i.e. decrease of the coupling constant as $\mu = \sqrt{Q^2}$ increases as indicated by Eq. (48). An average of the values in Fig. 5 gives

$$\alpha_s(m_Z) = 0.119 \pm 0.002$$

which corresponds to

$$\Lambda_{QCD} = 219^{+25}_{-23}\,\text{MeV}. \tag{7.50}$$

The LEP / SLC value for $\alpha_s(m_Z)$ is 0.124 ± 0.004.

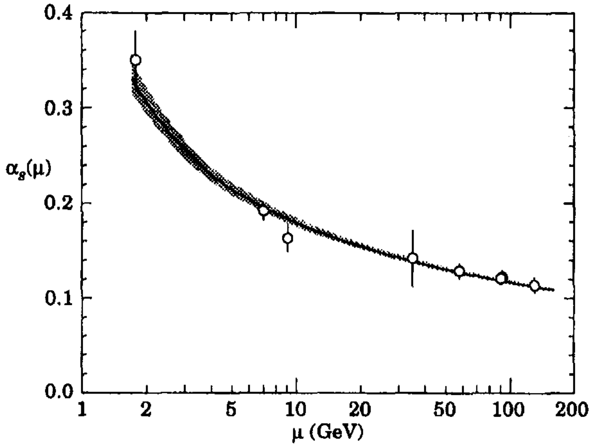


Figure 6 Summary of the values of $\alpha_s(\mu)$ at the values of μ where they are measured. The figure shows clearly the decrease in $\alpha_s(\mu)$ with increasing μ.

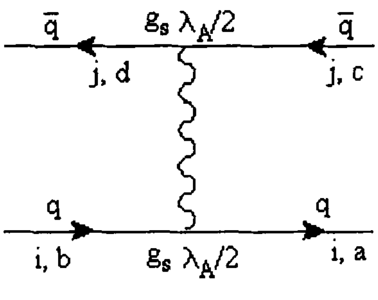


Figure 7 Diagram generating one–gluon exchange potential for $q\bar{q}$ system.

7.4 Hadron Spectroscopy

7.4.1 One gluon exchange potential

All known hadrons are color singlets. Just as an exchange of photon gives force of repulsion between like charges and force of attraction between unlike charges, the exchange of gluon gives force of attraction between color singlet states. The exchange of gluons can provide binding between quarks in a hadron.

For $q\bar{q}$ system (meson), the color electric potential due to one gluon exchange diagram [see Fig. 7] is given by:

$$V_{ij} = -g_s^2 \frac{1}{4\pi r} \sum_{A=1}^{8} \left(\frac{\lambda_A}{2}\right)_b^a \left(\frac{\lambda_A}{2}\right)_c^d \frac{1}{\sqrt{3}}\delta_a^c \frac{1}{\sqrt{3}}\delta_d^b. \tag{7.51}$$

The factors $\frac{1}{\sqrt{3}}\delta_a^c$ and $\frac{1}{\sqrt{3}}\delta_d^b$ in the initial and final states arise due to normalized color singlet totally symmetric wave function for the $q\bar{q}$ system. The minus sign arises due to the coupling of a vector particle to the antiquark. Here i, j are flavor indices and a, b, c, d are color indices. Since $Tr(\lambda_A\lambda_B) = 2\delta_{AB}$, $Tr(\lambda_A\lambda_A) = 16$, we get

$$V_{ij} = -\frac{4}{3}\frac{\alpha_s}{r}, \alpha_s = \frac{g_s^2}{4\pi}. \tag{7.52}$$

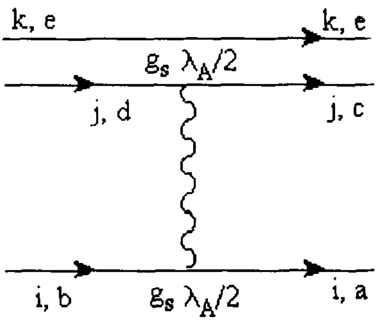


Figure 8 Diagram generating one-gluon exchange two-body potential for three quarks (baryon) system.

For three quarks system (baryon), one gluon exchange diagram (Fig. 8) gives the following two-body potential

$$V_{ij} = g_s^2 \frac{1}{4\pi r} \frac{\varepsilon_{eac}}{\sqrt{6}} \frac{\varepsilon^{ebd}}{\sqrt{6}} \left(\frac{\lambda_A}{2}\right)^c_d \left(\frac{\lambda_A}{2}\right)^a_b . \tag{7.53a}$$

The factors $\frac{\varepsilon_{eac}}{\sqrt{6}}$ and $\frac{\varepsilon_{ebd}}{\sqrt{6}}$ arise due to the fact that three-quark color wave function is totally antisymmetric in color indices. Using $\varepsilon_{eac}\varepsilon^{ebd} = \delta^b_a\delta^d_c - \delta^d_a\delta^b_c$, and $Tr\lambda_A = 0$, we get

$$V_{ij} = -\frac{2}{3}\frac{\alpha_s}{r}. \tag{7.53b}$$

Note the important fact that in both cases, we get an attractive potential. We also note that $V_{ij}^{q\bar{q}} = 2V_{ij}^{qq}$ for color singlet states. Thus we can write the two-body one-gluon exchange potential as

$$V_{ij} = k_s \frac{\alpha_s}{r}, k_s = \left\{ \begin{array}{cc} -\frac{4}{3} & q\bar{q} \\ -\frac{2}{3} & qq \end{array} \right\} \tag{7.54}$$

Since the running coupling constant α_s becomes smaller as we decrease the distance, the effective potential V_{ij} approaches the lowest order one-gluon exchange potential given in Eq. (53) as $r \to 0$.

Now in momentum space, we can write the potential in QCD perturbation theory for small distances ($r < 0.1$ fm) as

$$V(\mathbf{q}^2) = k_s 4\pi\alpha_s(\mathbf{q}^2)/\mathbf{q}^2, \qquad (7.55)$$

where V(r) is the Fourier transform of V($\mathbf{q}^2$) and $\mathbf{q}^2$ is the momentum conjugate to r. The running coupling $\alpha_s(\mathbf{q}^2)$ in QCD is given by Eq. (48h).

We conclude that for short distances, one can use the one gluon exchange potential, taking into account the running coupling constant $\alpha_s(\mathbf{q}^2)$.

7.4.2 Long range QCD motivated potential

The second regime, i.e. for large r, QCD perturbation theory breaks down and we have the confinement of the quarks. Thus unlike the short range part of the potential, the long range part cannot be calculated on perturbative QCD as the QCD constants become large in this region. Perturbative QCD gives no hint of intrinsically nonperturbative phenomena such as color confinement. One may look for the origin of this yet unsatisfactorily explained phenomena. There are many pictures which support the existence of a linear confining term. One of these is discussed below:

The string picture of hadrons:

This picture is depicted in Figs. 9 and 10. A string carries color indices at its ends. Gauge invariance implies that each site must be a color-singlet. Thus, an allowed configuration of a quark and an antiquark on adjacent sites is the one in which the quark and antiquark are linked by a string so that the color index of quark (antiquark) and the color index of the string at that end are contracted to form a color singlet. When a quark and an antiquark are far apart, many strings have to be excited to connect the two sites [see Fig. 10]. When there is enough energy available to create a new $q\bar{q}$ pair, the system breaks up permitting the formation of two color singlets. Calculation based on this theory shows that the energy stored in this configuration is:

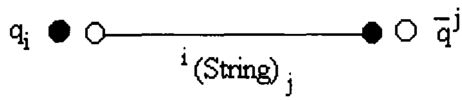


Figure 9 String picture of $q\bar{q}$.

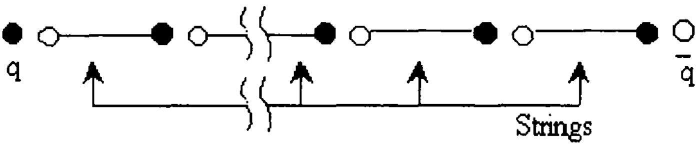


Figure 10 String separation of a quark-antiquark pair.

$$E = T_0 \frac{L}{a} \qquad \text{for } L \gg a,$$

where L is the quark-antiquark separation and T_0 is the string tension. To isolate a quark for example, the antiquark in the above illustration has to be removed to infinity; it clearly takes an infinite amount of energy to do this. This is the basis of color confinement. The confining potential is of the form:

$$V(r) \sim \text{ constant } \times r,$$

for $r > 1/M$, where M is a typical hadronic mass scale. Thus $\frac{1}{M}$ is of order of the hadron size of 1 fm = 5 GeV^{-1} so that $M \approx$ 200 MeV. The confining potential is spin and flavor independent. This picture is supported by the observation that hadrons of a given internal symmetry quantum number but different spins obey a simple spin (J) - mass (M) straight line relation i.e. we say that they lie on linear Regge trajectories, an example of which is displayed in Fig. 11.

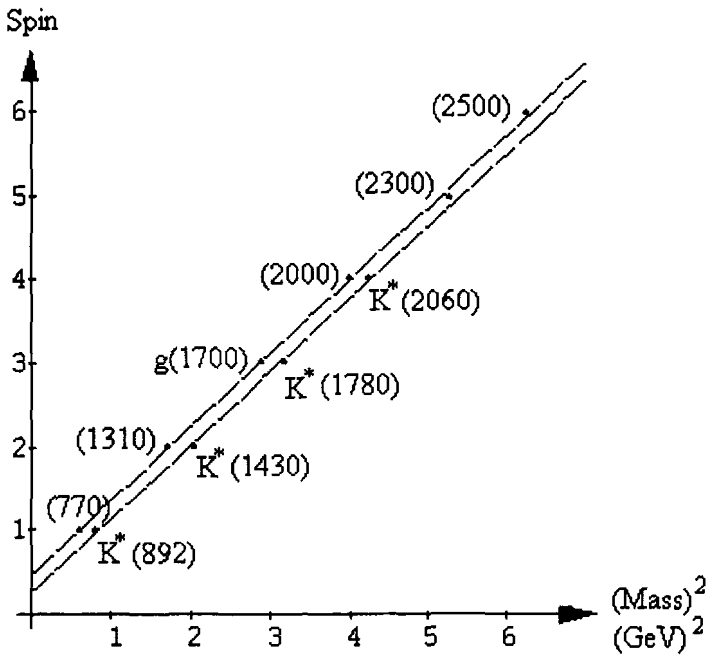


Figure 11 Regge trajectories for non-strange ($I = 1$) and strange ($I = 1/2$) bosons.

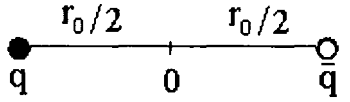
For the families of hadrons composed entirely of light quarks, the above mentioned relation between J and M^2 for Regge trajectories is given by:

$$J(M^2) = \alpha_0 + \alpha' M^2, \tag{7.56a}$$

with

$$\alpha' \approx 0.8 - 0.9(\mathrm{GeV/c^2})^{-2}. \tag{7.56b}$$

The connection between linear energy density and the linear Regge trajectory is provided by the string model formulated by Nambu. We consider a massless (and for simplicity spinless) quark and antiquark connected by a string of length r_0, which is characterized by an energy per unit length σ. The situation is sketched below:



For a given value of length r_0, the largest achievable angular momentum J occurs when the ends of the string move with the velocity of light. In these circumstances, the speed at any point along the string at a distance r from the center will be: $(\beta = v/c)$

$$\beta(r) = 2r/r_0.$$

The total mass of the system is then:

$$M = 2\int_0^{r_0/2} \frac{dr\,\sigma}{\sqrt{1-\beta(r)^2}} = \sigma r_0 \frac{\pi}{2}, \tag{7.57a}$$

while the orbital angular momentum of the string is:

$$J = 2\int_0^{r_0/2} \frac{dr\,\sigma\, r\beta(r)}{\sqrt{1-\beta(r)^2}} = \sigma r_0^2 \frac{\pi}{8}. \tag{7.57b}$$

Using the relation (57a), one finds that:

$$J = \frac{M^2}{2\pi\sigma}, \tag{7.58a}$$

which corresponds to a linear Regge trajectory with

$$\alpha' = \frac{1}{2\pi\sigma}. \tag{7.58b}$$

This connection yields:

$$\sigma = \begin{cases} 0.18 \text{ GeV}^2 \\ 0.20 \text{ GeV}^2 \end{cases} \text{for } \alpha = \begin{cases} 0.9 \text{ GeV}^{-2} \\ 0.8 \text{ GeV}^{-2} \end{cases}. \tag{7.59}$$

This heuristic estimate of the energy density suggests that at a separation of the order of 1 fm, we may characterize the interquark interaction by the linear potential

$$V(r) = \sigma r. \tag{7.60}$$

The lattice gauge theory calculations also support the linear form for the long range part of the QCD potential.

Thus phenomenological potential of the form

$$V_{ij}(r) = V_{ij}^{G}(r) + V_{ij}^{C}(r) \tag{7.61}$$

can be used for heavy quarks. The Cornell potential

$$V(r) = -\frac{K}{r} + \frac{r}{a^2} + C, \tag{7.62a}$$

where

$$K = 0.48,\ a = 2.34(\text{GeV})^{-1} \quad \text{and} \quad C = -0.25 \tag{7.62b}$$

has been used successfully to describe mass spectrum of charmonium and bottomonium systems [see Chap. 8]. Note that value of $a\left(\equiv \frac{1}{\sqrt{\sigma}}\right)$ in Eq. (62b) is consistent with the value of σ stated above [cf. Eq. (59)]. The purely phenomenological potentials of the form and

$$V(r) = a + br^{0.1} \tag{7.63a}$$

and

$$V(r) = C \ln r \tag{7.63b}$$

have also been used successfully for $c\bar{c}$ and $b\bar{b}$ systems.

7.4.3 Spin-spin interaction

Finally, we note that a spin 1/2 charged particle of charge eQ_i has a magnetic momentum $e\mu_i = \frac{eQ_i}{2m_i}\sigma_i$. In quantum mechanics, the energy splitting between $S-$states (zero orbital angular momentum) is given by two-particle operator (Fermi contact term)

$$H_{ij}^{M} = -\frac{8\pi}{3}\alpha \boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j \delta^3(\mathbf{r}_i - \mathbf{r}_j). \tag{7.64}$$

Similarly in QCD, we have eight color-magnetic moments

$$g_s \boldsymbol{\mu}_A^i = \frac{g_s}{2m_i}\left(\frac{\lambda_A}{2}\right)\boldsymbol{\sigma}, \qquad A = 1, \cdots, 8. \tag{7.65}$$

The analogous two-particle interaction for QCD is then given by

$$H_{ij} = -\frac{8\pi}{3}\alpha_s \boldsymbol{\mu}_A^{(i)} \cdot \boldsymbol{\mu}_A^{(j)} \delta^3(\mathbf{r}_i - \mathbf{r}_j). \tag{7.66}$$

Again for a color singlet system

$$H_{ij} = -\frac{8\pi}{3}\alpha_s k_s \frac{\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j}{4m_i m_j}\delta^3(\mathbf{r}), \tag{7.67}$$

Eq. (67) would immediately give $m(^3S_1) > m(^1S_0)$ [for example $m_\rho > m_\pi$] in agreement with the experimental result. This supports the fact that gluons are spin 1 particles.

7.5 The Mass Spectrum

The one gluon exchange potential is obtained by summing over all possible quark indices in V_{ij}^G in a multiquark system like $q\bar{q}$ and qqq. Thus

$$\begin{aligned} V^G &= \frac{1}{2}\sum_{i\neq j} V_{ij}^G \\ &= \frac{1}{2}\left[\sum_{i>j} V_{ij}^G + \sum_{i<j} V_{ij}^G\right] \\ &= \frac{1}{2}\left[\sum_{i>j}\left(V_{ij}^G + V_{ji}^G\right)\right] \\ &= \sum_{i>j} V_{ij}^G. \end{aligned} \tag{7.68}$$

The potential V_G for $S-$states is found to be [in non-relativistic limit keeping terms up to (p^2/m^2)]

$$V_G = k_s\alpha_s \sum_{i>j}\left[\frac{1}{r} - \frac{1}{2m_i m_j}\left(\frac{\mathbf{p}_i \cdot \mathbf{p}_j}{r} + \frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{p}_i)\cdot\mathbf{p}_j}{r^3}\right)\right.$$

$$-\frac{\pi}{2}\delta^3(\mathbf{r})\left(\frac{1}{m_i^2}+\frac{1}{m_j^2}+\frac{16\mathbf{s}_i\cdot\mathbf{s}_j}{3m_im_j}\right)\Bigg]. \tag{7.69}$$

The first term on the right hand side is the potential in the extreme non relativistic limit $\left(\frac{v}{c}\approx 0\right)$; spin dependent term is due to the color magnetic moments interaction as mentioned previously.

For $S-$states,

$$\begin{aligned}\left\langle\Psi_s\left|\delta^3(\mathbf{r})\right|\Psi_s\right\rangle &= \int\Psi_s^*(\mathbf{r})\delta^3(\mathbf{r})\Psi_s(\mathbf{r})d^3r \\ &= |\Psi_s(0)|^2. \end{aligned} \tag{7.70}$$

Now our Hamiltonian, including the rest masses of the quarks can be written as

$$H(\mathbf{r})=\sum_i m_i+\sum_i\frac{\hat{\mathbf{p}}_i^2}{2m_i}+V_C(r)+V_G(r), \tag{7.71}$$

where

$$\hat{\mathbf{p}}_i^2=-\hbar^2\nabla_i^2. \tag{7.72}$$

Here $V_C(r)$ is the confining potential, $V_G(r)$ is the one gluon exchange potential given in Eq. (69), i is the quark flavor index, i.e. $i=u,d,s$ for ordinary hadrons. We will take $m_u=m_d$. In order to discuss the mass spectrum of hadrons, we have to take the expectation value of the Hamiltonian $H(\mathbf{r})$ with respect to the relevant wave functions of the hadrons. The wave function is the product of three parts viz. unitary spin, spin and space parts. For $s-$wave, we write the space function as $\Psi_s(\mathbf{r})$. Let us first take the expectation value of $H(\mathbf{r})$ with respect to $\Psi_s(\mathbf{r})$, we have

$$\begin{aligned} M &\equiv \langle\Psi_s|H|\Psi_s\rangle \\ &= \sum_i m_i+\sum_i\frac{a}{m_i}+A_0 \\ &\quad +k_s\alpha_s\left[b-\frac{c}{m_im_j}-d\left(\frac{1}{m_i^2}+\frac{1}{m_j^2}+\frac{16\mathbf{s}_i\cdot\mathbf{s}_j}{3m_im_j}\right)\right], \end{aligned} \tag{7.73}$$

where

$$\begin{aligned} a &= \left\langle \Psi_s \left| \hat{\mathbf{p}}_i^2 \right| \Psi_s \right\rangle && (7.74a)\\ A_0 &= \langle \Psi_s | V_C(r) | \Psi_s \rangle && (7.74b)\\ d &= \frac{\pi}{2} \left\langle \Psi_s \left| \delta^3(\mathbf{r}) \right| \Psi_s \right\rangle && (7.74c)\\ b &= \left\langle \Psi_s \left| \frac{1}{r} \right| \Psi_s \right\rangle && (7.74d)\\ c &= \frac{1}{2} \left\langle \Psi_s \left| \frac{1}{r^3} \left[r^2 \mathbf{p}_i \cdot \mathbf{p}_j \mathbf{r} + \mathbf{r} \cdot (\mathbf{r} \cdot \mathbf{p}_i) \mathbf{p}_j \right] \right| \Psi_s \right\rangle . && (7.74e) \end{aligned}$$

Note that the mass operator M is still an operator in unitary spin and spin space. The parameters a, A_0, d, b and c may be different for $L = 0(q\bar{q})$ meson and $L = 0(qqq)$ baryon systems. We first apply the mass formula (73) to pseudoscalar meson system.

7.5.1 Meson mass spectrum

From Eq. (73), the mass operator for $S-$wave mesons can be written as

$$\begin{aligned} M &= M_0 + m_1 + m_2 + a \left[\frac{1}{m_1} + \frac{1}{m_2} \right] + \frac{\bar{c}}{m_1 m_2} \\ &\quad + \bar{d} \left[\frac{1}{m_1^2} + \frac{1}{m_2^2} + \frac{16 \mathbf{s}_1 \cdot \mathbf{s}_2}{3 m_1 m_2} \right], \qquad (7.75) \end{aligned}$$

where

$$\begin{aligned} M_0 &= A_0 + k_s \alpha_s b \\ \bar{c} &= -k_s \alpha_s c \qquad (7.76) \\ \bar{d} &= -k_s \alpha_s d. \end{aligned}$$

Indices 1 and 2 refer to the constituent antiquark and quark respectively. For vector gluon $k_s = -\frac{4}{3}$. Now

$$\mathbf{s}_1 \cdot \mathbf{s}_2 = \begin{cases} \frac{1}{4} & \text{spin triplet state } S = 1\text{: vector meson} \\ -\frac{3}{4} & \text{spin singlet state } S = 0\text{: pseudoscalar meson.} \end{cases}$$

Thus if $k_s = -\frac{4}{3}$, as for vector gluons, it is clear from Eqs. (75) and (76) that

$$m(^3S_1) > m(^1S_0) \tag{7.77}$$

i.e. vector meson mass is greater than the corresponding pseudoscalar mass in agreement with experimental observation. If gluons were scalar particles, then $\mathbf{s}_1 \cdot \mathbf{s}_2$ term would be absent so that $m(^3S_1) = m(^1S_0)$ in disagreement with the experimental observation. For pseudoscalar gluons, $k_s = \frac{4}{3}$, since pseudoscalar coupling is the same for antiquarks. In this case we would have $m(^3S_1) < m(^1S_0)$, again in disagreement with the experimental result. We conclude that the experimental results about meson spectrum support the fact that gluons are vector particles and are thus quanta of QCD.

From Eq. (75), we can write down the masses of vector and pseudoscalar mesons. For example (with $m_u = m_d$):

$$m_\rho = M_0 + 2m_u + \frac{2a}{m_u} + \frac{\bar{c}}{m_u^2} + \bar{d}\left[\frac{2}{m_u^2} + \frac{4}{3m_u^2}\right] \tag{7.78a}$$

$$m_\pi = M_0 + 2m_u + \frac{2a}{m_u} + \frac{\bar{c}}{m_u^2} + \bar{d}\left[\frac{2}{m_u^2} - \frac{4}{m_u^2}\right]. \tag{7.78b}$$

For K^* and K, replace $2m_u$ by $m_u + m_s$, $2/m_u$ by $\left(\frac{1}{m_u} + \frac{1}{m_s}\right)$, $\frac{1}{m_u^2}$ by $\frac{1}{m_u m_s}$ and $2/m_u^2$ by $\left(\frac{1}{m_u^2} + \frac{1}{m_s^2}\right)$ in m_ρ and m_π respectively. $m_\rho = m_\omega$ and for m_ϕ replace m_u by m_s in the expression for m_ρ. From Eq. (78), we have the following results

$$m_\omega = m_\rho \tag{7.79}$$

$$m_\rho - m_\pi = \frac{16}{3m_u^2}\bar{d} = \frac{64}{9}\alpha_s d\frac{1}{m_u^2} \tag{7.80}$$

$$m_K^* - m_K = \frac{16}{3m_u m_s}\bar{d} = \frac{64}{9}\alpha_s d\frac{1}{m_u m_s} \tag{7.81}$$

$$\frac{m_K^* - m_K}{m_\rho - m_\pi} = \frac{m_u}{m_s} \approx 0.66 \text{ (Expt } 0.64), \tag{7.82}$$

where we have used for m_u and m_s, the values of constituent quark masses [$m_u = 336$ MeV, $m_s = 510$ MeV] obtained for the magnetic moments of baryons (see Chap. 6). We also obtain

$$\begin{aligned} 2m_K^* - \frac{m_\rho + m_\omega}{2} &= m_\phi - \left(\bar{c} + \frac{4}{3}\bar{d}\right)\left(\frac{m_s - m_u}{m_u m_s}\right)^2 \\ &= m_\phi - \frac{4}{3}\alpha_s\left(c + \frac{4}{3}d\right)\left(\frac{m_s - m_u}{m_u m_s}\right)^2 \\ &= m_\phi + O(\lambda^2), \end{aligned} \tag{7.83}$$

where $\lambda = \frac{m_s - m_u}{m_s + m_u}$ is the SU(3) symmetry breaking parameter. Hence to order λ, we recover the Gell-Mann–Okubo mass formula with ideal mixing angle between ω_8 and ω_1.

For pseudoscalar mesons η_{ns} and η_s, we get

$$m_{\eta_{ns}} = m_\pi \tag{7.84a}$$

$$m_{\eta_s} = (2m_K - m_\pi) + O(\lambda^2). \tag{7.84b}$$

These formulae are badly broken. Thus the above analysis breaks down for $J = 0$ mesons, η and η'. The reason for this is that our Hamiltonian does not take into account quark-antiquark annihilation into gluons. The lowest order annihilation diagram is shown in Fig. 12. This diagram contributes only to $^1S_0$ state, because of charge conjugation conservation. Since gluons do not carry any flavor, therefore it contributes to $I = Y = 0$, $^1S_0$ states only. This diagram is relevant only for η and η' mesons, and is of order $O(\alpha_s^2)$. For $I = Y = 0$ vector bosons, the diagram with three-gluon exchange contributes, which is of order $O(\alpha_s^3)$ and hence can be neglected.

We now take into account the diagram of Fig. 12 for pseudoscalar mesons. If $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$ can annihilate with an amplitude A, which we assume to be SU(3) invariant, then there will be an additional contribution to the mass matrix, which in the $u\bar{u}$, $d\bar{d}$

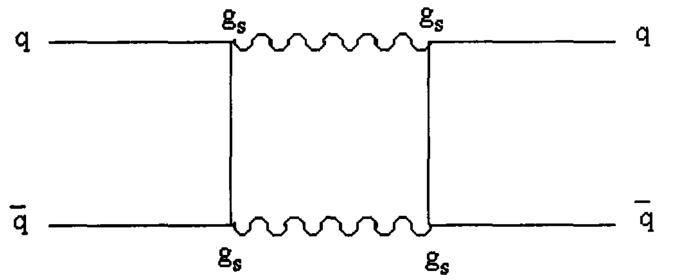


Figure 12 The $q\bar{q}$ annihilation diagram for $^1S_0$ state through two gluons.

and $s\bar{s}$ basis is given by

$$M_{ann} = \begin{pmatrix} A & A & A \\ A & A & A \\ A & A & A \end{pmatrix}. \tag{7.85}$$

Taking into account Eq. (84) and the fact that $|\eta_s\rangle = |s\bar{s}\rangle$, $|\eta_{ns}\rangle = \frac{1}{\sqrt{2}}\left|(u\bar{u} + d\bar{d})\right\rangle$, $|\pi^0\rangle = \frac{1}{\sqrt{2}}\left|(u\bar{u} - d\bar{d})\right\rangle$, we get in π^0, η_{ns} and η_s basis, the mass matrix

$$\begin{pmatrix} m_\pi & 0 & 0 \\ 0 & m_\pi + 2A & \sqrt{2}A \\ 0 & \sqrt{2}A & 2m_K - m_\pi + A \end{pmatrix}. \tag{7.86}$$

From Eq. (86), we note that we have to diagonalize the mass matrix

$$M \to M + M_{ann} = \begin{pmatrix} m_\pi + 2A & \sqrt{2}A \\ \sqrt{2}A & 2m_K - m_\pi + A \end{pmatrix}. \tag{7.87}$$

For this purpose, we define the physical states as (see Problem 5.15)

$$\begin{aligned} |\eta\rangle &= \cos\phi\,|\eta_{ns}\rangle - \sin\phi\,|\eta_s\rangle \\ |\eta'\rangle &= \sin\phi\,|\eta_{ns}\rangle + \cos\phi\,|\eta_s\rangle\,. \end{aligned} \tag{7.88}$$

Then the mass eigenvalues are given by

$$\begin{aligned} m_\eta m_{\eta'} &= m_\pi(2m_K - m_\pi) + A(4m_K - m_\pi) \\ m_\eta + m_{\eta'} &= m_{\eta_{ns}} + m_{\eta_s} = 2m_K + 3A. \end{aligned} \tag{7.89}$$

Using the experimental values for η and η', we can determine A. The mass scale A comes out to be ≈ 172 MeV, a rather low value compared to m_η and $m_{\eta'}$ which is both interesting and reasonable.

To conclude, we have shown that mass spectrum of vector mesons can be explained successfully. With the addition of annihilation diagram, the pseudoscalar meson mass spectrum can also be understood.

7.5.2 Baryon mass spectrum

In order to discuss the mass spectrum of the baryons, it is convenient to first calculate the matrix elements of the spin operator

$$\Omega_{ss} = \sum_{i>j} \frac{1}{m_i m_j} \mathbf{s}_i \cdot \mathbf{s}_j \tag{7.90}$$

between spin states. The eigenvalues of $\mathbf{s}_i \cdot \mathbf{s}_j$ are $1/4$ and $-3/4$ for spin triplet and singlet states respectively. Therefore,

$$\begin{aligned} \mathbf{s}_i \cdot \mathbf{s}_j \left|\uparrow\uparrow\right\rangle &= \frac{1}{4}\left|\uparrow\uparrow\right\rangle \\ \mathbf{s}_i \cdot \mathbf{s}_j \left[\frac{1}{\sqrt{2}}\left|(\uparrow\downarrow + \downarrow\uparrow)\right\rangle\right] &= \frac{1}{4\sqrt{2}}\left|(\uparrow\downarrow + \downarrow\uparrow)\right\rangle \\ \mathbf{s}_i \cdot \mathbf{s}_j \left|\downarrow\downarrow\right\rangle &= \frac{1}{4}\left|\downarrow\downarrow\right\rangle \end{aligned} \tag{7.91a}$$

$$\mathbf{s}_i \cdot \mathbf{s}_j \left[\frac{1}{\sqrt{2}}\left|(\uparrow\downarrow - \downarrow\uparrow)\right\rangle\right] = -\frac{3}{4\sqrt{2}}\left|(\uparrow\downarrow - \downarrow\uparrow)\right\rangle. \tag{7.91b}$$

From Eq. (91), we get

$$\begin{aligned} \mathbf{s}_i \cdot \mathbf{s}_j \left|i^\uparrow j^\downarrow\right\rangle &= -\frac{1}{4}\left|i^\uparrow j^\downarrow\right\rangle + \frac{1}{2}\left|i^\downarrow j^\uparrow\right\rangle \\ \mathbf{s}_i \cdot \mathbf{s}_j \left|i^\downarrow j^\uparrow\right\rangle &= -\frac{1}{4}\left|i^\downarrow j^\uparrow\right\rangle + \frac{1}{2}\left|i^\uparrow j^\downarrow\right\rangle. \end{aligned} \tag{7.92}$$

The spin wave functions for baryons are given in Table 6.3 and Eq. (6.8). Using these wave functions, we get with the help of Eqs. (91) and (92) for $\frac{1}{2}^+$ baryons with $s_z = \frac{1}{2}$:

$$
\begin{aligned}
&\Omega_{ss}\,|p\rangle \\
=\;& \sum_{i>j} \frac{1}{m_i m_j} \mathbf{s}_i \cdot \mathbf{s}_j \,|uud\rangle \left(-\frac{1}{\sqrt{6}}\right) |(\uparrow\downarrow + \downarrow\uparrow)\uparrow -2\,|\uparrow\uparrow\downarrow\rangle \\
=\;& \frac{1}{m_u^2}\,|uud\rangle \left(-\frac{1}{\sqrt{6}}\right) \left\{\frac{1}{4}\,|(\uparrow\downarrow + \downarrow\uparrow)\uparrow\rangle - 2\,|\uparrow\uparrow\downarrow\rangle\right. \\
&+ \left[\frac{1}{4}\,|\uparrow\downarrow\uparrow\rangle - \frac{1}{4}\,|\downarrow\uparrow\uparrow\rangle + \frac{1}{2}\,|\uparrow\uparrow\downarrow\rangle - 2\left(-\frac{1}{4}\,|\uparrow\uparrow\downarrow\rangle + \frac{1}{2}\,|\downarrow\uparrow\uparrow\rangle\right)\right] \\
&+ \left.\left[-\frac{1}{4}\,|\uparrow\downarrow\uparrow\rangle + \frac{1}{2}\,|\uparrow\uparrow\downarrow\rangle + \frac{1}{4}\,|\downarrow\uparrow\uparrow\rangle - 2\left(-\frac{1}{4}\,|\uparrow\uparrow\downarrow\rangle + \frac{1}{2}\,|\uparrow\downarrow\uparrow\rangle\right)\right]\right\} \\
=\;& |uud\rangle\,\frac{1}{m_u^2}\left(-\frac{3}{4}\right)\frac{1}{\sqrt{6}}\,|(\uparrow\downarrow + \downarrow\uparrow)\uparrow -2\,\uparrow\uparrow\downarrow\rangle \\
=\;& -\frac{3}{4m_u^2}\,|p\rangle\,.
\end{aligned}
\tag{7.93a}
$$

Similarly we get

$$\Omega_{ss}\,|\Lambda\rangle = -\frac{3}{4m_u^2}\,|\Lambda\rangle \tag{7.93b}$$

$$\Omega_{ss}\left|\Sigma^0\right\rangle = \frac{1}{4}\left(\frac{1}{m_u^2} - \frac{4}{m_u m_s}\right)\left|\Sigma^0\right\rangle \tag{7.93c}$$

$$\Omega_{ss}\left|\Xi^0\right\rangle = \frac{1}{4}\left(\frac{1}{m_u^2} - \frac{4}{m_u m_s}\right)\left|\Xi^0\right\rangle, \tag{7.93d}$$

where we have used

$$
\begin{aligned}
|\Lambda\rangle &= -\,|uds\rangle\,\chi_{MS}^{1/2} \\
\left|\Sigma^0\right\rangle &= |uds\rangle\,\chi_{MS}^{1/2} \\
\left|\Xi^0\right\rangle &= |ssu\rangle\,\chi_{MS}^{1/2}.
\end{aligned}
\tag{7.94}
$$

For $\frac{3}{2}^+$ baryons, we take $s_z = 3/2$ and calculate the matrix elements of Ω_{ss}. Now

$$\begin{aligned}\Omega_{ss}\left|\Delta^{++}\right\rangle &= \sum_{i>j}\frac{1}{m_i m_j}\mathbf{s}_i\cdot\mathbf{s}_j\left|uuu\right\rangle\left|\uparrow\uparrow\uparrow\right\rangle \\ &= \frac{3}{4}\frac{1}{m_u^2}\left|\Delta^{++}\right\rangle . \end{aligned}\tag{7.95a}$$

Similarly we get

$$\Omega_{ss}\left|\Sigma^{*+}\right\rangle = \frac{1}{4}\left(\frac{2}{m_u m_s}+\frac{1}{m_u^2}\right)\left|\Sigma^{*+}\right\rangle \tag{7.95b}$$

$$\Omega_{ss}\left|\Xi^{*0}\right\rangle = \frac{1}{4}\left(\frac{2}{m_u m_s}+\frac{1}{m_s^2}\right)\left|\Xi^{*0}\right\rangle \tag{7.95c}$$

$$\Omega_{ss}\left|\Omega^{-}\right\rangle = \frac{3}{4}\frac{1}{m_s^2}\left|\Omega^{-}\right\rangle , \tag{7.95d}$$

where we have used

$$\left|\Sigma^{*+}\right\rangle = \left|uus\right\rangle\left|\uparrow\uparrow\uparrow\right\rangle \tag{7.96a}$$

$$\left|\Xi^{*0}\right\rangle = \left|ssu\right\rangle\left|\uparrow\uparrow\uparrow\right\rangle \tag{7.96b}$$

$$\left|\Omega^{-}\right\rangle = \left|sss\right\rangle\left|\uparrow\uparrow\uparrow\right\rangle . \tag{7.96c}$$

Since the spin–spin interaction term from Eqs. (73) and (90) is,

$$\frac{16}{3}\left(-\frac{2}{3}\alpha_s\right)(-d)\Omega_{ss} , \tag{7.97}$$

we have from Eqs. (93) and (95):

$$m\left(J=\frac{3}{2}\right) > m\left(J=\frac{1}{2}\right)$$

in agreement with experimental observations. For gluons with color, $k_s = -2/3$; if gluons do not carry color, then $k_s = 1$ instead of $-2/3$ and we would get results in contradiction with experimental values. This supports that the vector gluons carry color.

The spin dependent term Ω_{ss} splits the masses of baryons with the same quark content, but with different spin. Thus, we get from Eqs. (93), (95) and (97):

$$\begin{aligned} m_\Delta - m_p &= 8\frac{\bar{d}}{m_u^2} \\ m_\Sigma - m_\Lambda &= \frac{16}{3}\frac{\bar{d}}{m_u^2}\left(1 - \frac{m_u}{m_s}\right) \\ m_{\Sigma^*} - m_\Sigma &= 8\frac{\bar{d}}{m_u m_s} \\ m_{\Xi^*} - m_\Xi &= 8\frac{\bar{d}}{m_u m_s} \end{aligned} \tag{7.98}$$

where $\bar{d} = \frac{2\alpha_s d}{3}$. From Eqs. (98), we get

$$\frac{m_{\Xi^*} - m_\Xi}{m_{\Sigma^*} - m_\Sigma} = 1 \qquad \text{(expt 1.12)} \tag{7.99a}$$

$$\frac{2m_{\Sigma^*} + m_\Sigma - 3m_\Lambda}{2(m_\Delta - m_p)} = 1 \qquad \text{(expt 1.04)} \tag{7.99b}$$

$$\frac{m_\Sigma - m_\Lambda}{m_\Delta - m_p} = \frac{2}{3}\left(1 - \frac{m_u}{m_s}\right) = 0.23 \qquad \text{(expt 0.26)}. \tag{7.99c}$$

In the above derivation, the effects of wave function distortion due to symmetry breaking by quark effective masses have been neglected. These effects will give slight deviations from unity in the relations (99a, b).

We now discuss the baryon masses of same spin, using Eqs. (73), (93) and (98). We can write the baryon mass formula:

$$\begin{aligned} m = {} & (m_1 + m_2 + m_3) + a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right) + A_0 \\ & + 3\bar{b} + \bar{c}\left(\frac{1}{m_1 m_2} + \frac{1}{m_2 m_3} + \frac{1}{m_3 m_1}\right) \\ & + \bar{d}\sum_{i>j}\left(\frac{1}{m_i^2} + \frac{1}{m_j^2}\right) + \frac{16}{3}\bar{d}\left\langle B\left|\Omega_{ss}\right|B\right\rangle , \end{aligned} \tag{7.100}$$

where $\bar{c} = \frac{2}{3}\alpha_s c$, $\bar{b} = -\frac{2}{3}\alpha_s b$. Introducing the SU(3) breaking parameter $\lambda = (m_s - m_u)/(m_s + m_u)$ and writing $m_0 = \frac{1}{2}(m_s + m_u)$ and retaining only the first order term in λ, we get

$$\begin{aligned} m_p &= A + \lambda\left[-3B - 8\frac{\bar{d}}{m_0^2}\right] \\ m_\Lambda &= A + \lambda\left[-B - 8\frac{\bar{d}}{m_0^2}\right] \\ m_{\Sigma}' &= A + \lambda\left[-B + \frac{8}{3}\frac{\bar{d}}{m_0^2}\right] \\ m_\Xi &= A + \lambda\left[B - \frac{8}{3}\frac{\bar{d}}{m_0^2}\right], \end{aligned} \tag{7.101}$$

where

$$\begin{aligned} A &= 3\left(m_0 + \bar{b} + \frac{a}{m_0} + \frac{\bar{c}}{m_0^2} + \frac{2}{3}\frac{\bar{d}}{m_0^2}\right) + A_0 \\ B &= m_0 - \frac{a}{m_0} - \frac{2\bar{c}}{m_0^2} - \frac{4\bar{d}}{m_0^2}. \end{aligned} \tag{7.102}$$

From Eq. (101), we get the Gell-Mann-Okubo mass formula

$$\frac{m_p + m_\Xi}{2} = \frac{m_\Sigma + 3m_\Lambda}{2}. \tag{7.103}$$

We conclude that both the meson and baryon mass spectra can be explained quite well in QCD. In this simple picture, we have used non-relativistic quantum mechanics for u, d and s quarks. Although this approximation is not so good for these quarks (as their masses are less than 1/2 GeV) and at this energy scale QCD perturbation theory may not be a good approximation, even then the results are good.

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E. For Figs. 3 and 5 and 6

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Chapter 8
HEAVY FLAVORS

8.1 Discovery of Charm

The J/Ψ was discovered in 1974 in the reaction

$$p + Be \longrightarrow e^+e^- + X$$

at $\sqrt{s} = 7.6$ GeV. A narrow peak at $m(e^+e^-) = 3.1$ GeV was found. It was also seen in e^+e^- collision at $\sqrt{s} = 3.105$ GeV in the following reactions

$$\begin{aligned} e^-e^+ &\longrightarrow e^-e^+ \\ e^-e^+ &\longrightarrow \mu^-\mu^+ \\ e^-e^+ &\longrightarrow \text{hadrons.} \end{aligned}$$

The width of the resonance was very narrow. It was less than the energy spread of the beam, $\Gamma \leq 3$ MeV. For this reason, the width cannot be read off directly from resonance curve. The resonant cross section for any final state f :

$$e^-e^+ \longrightarrow J/\Psi \longrightarrow f$$

is given by the Breit-Wigner formula [cf. Eq.(4.52)]:

$$\sigma_{ef} = \frac{\pi}{k^2}\frac{2J+1}{(2s_1+1)(2s_2+1)}\frac{\Gamma_e\Gamma_f}{(\sqrt{s}-m)^2+\frac{\Gamma^2}{4}} \tag{8.1}$$

where J is the spin of the resonance, m is its mass, $s_1 = s_2 = 1/2$ is the spin of electron or positron and

$$s = E_{cm}^2 = 4(k^2 + m_e^2) \approx 4k^2. \tag{8.2}$$

Here $k = |\mathbf{k}|$ is the center of mass momentum. Γ is the total width, Γ_e and Γ_f are the partial widths into e^-e^+ and f respectively. We can write Eq.(1) as

$$\sigma_{ef} = \frac{\pi}{s}(2J+1)\frac{\Gamma_e \Gamma_f}{(\sqrt{s}-m)^2 + \frac{\Gamma^2}{4}}. \tag{8.3}$$

Since the resonance is very narrow, Γ is very small and it is a good approximation to replace the denominator in Eq.(3) by the δ-function $\frac{2\pi}{\Gamma}\delta(\sqrt{s}-m)$ and then integration of Eq. (3) gives

$$\int \sigma_{ef} d\sqrt{s} = \pi(2J+1)\frac{2\pi\Gamma_e\Gamma_f}{m^2\Gamma}. \tag{8.4}$$

Now $\sum_f \sigma_{ef} = \sigma_{tot}$, $\sum_f \Gamma_f = \Gamma$ and assuming $\Gamma_e = \Gamma_\mu$, we have for the process

$$\begin{aligned} e^-e^+ &\longrightarrow \Psi \longrightarrow \mu^-\mu^+ \\ \int \sigma_{ef} d\sqrt{s} &= 2\pi^2(2J+1)\frac{\Gamma_e^2}{m^2\Gamma}. \end{aligned} \tag{8.5}$$

We also have for the total cross section

$$\int (\Sigma\sigma_{ef}) d\sqrt{s} = 2\pi^2(2J+1)\frac{\Gamma_e}{m^2}. \tag{8.6}$$

Assuming the spin of the resonance J/Ψ, $J = 1$, we determine the widths $\Gamma_e = \Gamma_\mu$ and the total decay with Γ. Since $\Gamma = \Gamma_e + \Gamma_\mu + \Gamma_h$, we can also determine the hadronic decay with Γ_h. The experimental values for these decay widths are given below:

$$\begin{aligned} m(J/\Psi) &= 3096.88 \pm 0.04 \text{ MeV}, \\ \Gamma_e &= \Gamma_\mu = 5.26 \pm 0.37 \text{ keV}, \\ \Gamma &= 87 \pm 5 \text{ keV}. \end{aligned}$$

The J/Ψ spin-parity can be determined from a study of the interference between $e^-e^+ \longrightarrow \gamma \longrightarrow \mu^-\mu^+$ and $e^-e^+ \longrightarrow \Psi \longrightarrow \mu^-\mu^+$. The cross section for the QED process $e^-e^+ \longrightarrow \gamma \longrightarrow \mu^-\mu^+$ is well known [Eq. (78) of Appendix A] and is given by $(s \gg m_e^2, m_\mu^2)$

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{4s}(1+\cos^2\theta) \\ \sigma &= \frac{4\pi\alpha^2}{s}. \end{aligned} \tag{8.7}$$

If the spin-parity of J/Ψ is that of photon viz. 1^-, then the angular distribution would not change by the interference between QED amplitude and the resonant amplitude. In fact, experimentally, it was found to be $(1+\cos^2\theta)$ near the resonance, clearly establishing the spin-parity of J/Ψ to be 1^-.

8.1.1 Isospin

Experimentally the decay $J/\Psi \to p\bar{p}$ occurs with a branching ratio $\Gamma_{p\bar{p}}/\Gamma = (0.214 \pm 0.010)\%$, which is too large to be explained by the electromagnetic effects. Now $p\bar{p}$ can have only $I = 0$ or $I = 1$. Thus the isospin of J/Ψ is either 0 or 1. If J/Ψ has $I = 1$, then the decay $J/\Psi \to \rho^0\pi^0$ is forbidden while for $I = 0$ (see problem 1), we have

$$\frac{\Gamma(J/\Psi \to \rho^0\pi^0)}{\Gamma(J/\Psi \to \rho^-\pi^+) + \Gamma(J/\Psi \to \rho^+\pi^-)} = \frac{1}{2} \tag{8.8}$$

to be compared with the experimental value of 0.494 ± 0.068. Thus the isospin of J/Ψ is 0. Now G-parity is given by $G = (-1)^I C$, where C is the charge conjugation parity of J/Ψ. Since $I = 0$, therefore, $G = C$. The allowed decay $J/\Psi \to \rho^0\pi^0$ fixes its C–parity to be $C = (-1)(+1) = -1$. Hence $G = -1$ for J/Ψ.

8.1.2 SU(3) classification

Due to C-invariance, the VPP coupling is F-type (see Sec. 5.8.2) which is not possible if V is an SU(3) singlet. Thus an SU(3) singlet vector meson cannot decay into two pseudoscalar mesons belonging to the same SU(3) multiplet. In particular if J/Ψ is an SU(3) singlet, then $J/\Psi \to K\bar{K}$ is forbidden while $J/\Psi \to K^*\bar{K}$ or $J/\Psi \to \bar{K}^*K$ is allowed by C-invariance. Experimentally one finds

$$\frac{\Gamma\left(J/\Psi \to K^+K^-\right)}{\Gamma\left(J/\Psi \to K\bar{K}^*\right)} \approx 2.6 \times 10^{-2} \tag{8.9}$$

which shows that J/Ψ is an SU(3) singlet. If J/Ψ is an SU(3) singlet, then its invariant coupling with PV is given by

$$\begin{aligned} \Psi Tr(PV) &= \Psi[\rho^0\pi^0 + \rho^-\pi^+ + \rho^+\pi^- + K^{*-}K^+ + K^{*+}K^- \\ &\quad + \bar{K}^{*0}K^0 + K^{*0}\bar{K}^0 + \frac{2}{6}\omega_8\eta_8 + \frac{4}{6}\,\omega_8\eta_8]\,. \end{aligned} \tag{8.10}$$

Hence we have

$$\begin{aligned} \frac{\Gamma\left(J/\Psi \to \rho\pi\right)}{\Gamma\left(J/\Psi \to K\bar{K}^*\right)} &= 1 \text{ (phase space correction)} \\ &= 1.2 \end{aligned} \tag{8.11}$$

to be compared with the experimental value 1.39 ± 0.12. To summarize, the J/Ψ resonance is an SU(3) singlet with $J^{PC} = 1^{--}$, $G = -1$ and $I = 0$.

8.2 Charm

Although J/Ψ itself does not carry any new quantum number, its unusually narrow width in spite of large available phase space suggests that it is a bound state of $c\bar{c}$, where c is a quark with a flavor which is outside the three flavors u, d and s of SU(3). This new flavor is called charm. The quark c is assigned a new quantum number $C = 1$ and $C = 0$ for u, d and s quarks. Thus to take this quantum number into account, the Gell-Mann-Nishijima relation would be modified to

$$Q = I_3 + \frac{1}{2}(Y + C). \tag{8.12}$$

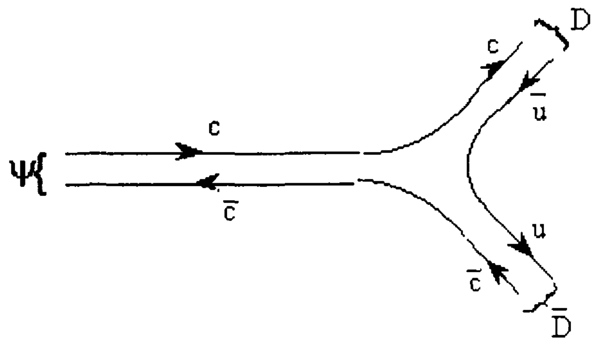

Figure 1 allowed by OZI

For the charmed quark c, $C = 1$, $I_3 = 0$, $Y = B = 1/3$. Thus the charge of charmed quark is 2/3 and its mass $m_c \approx \frac{1}{2}m_{J/\Psi} = 1.55$ GeV.

The narrow width of J/Ψ (87 keV compared to 100 MeV for ρ) can be qualitatively understood by the OZI rule, just as the suppression of $\phi \to 3\pi$ compared to $\phi \to K\bar{K}$ is explained (see Sec. 5.5.9) by this rule. Thus the decay depicted in Fig. 1 is allowed but that shown in Fig. 2 is suppressed by OZI rule. But the decay $J/\psi \to D\bar{D}$ shown in Fig. 1 is not allowed energetically since $m_{J/\Psi} < 2m_D$.

8.2.1 Charmed mesons

The charmed quark c can form bound states with $\bar{q}$, where $q = u, d, s$. The low lying bound states such as $c\bar{q}$ have been found experimentally and are listed in Table 1.

The $C = 1$ states $(D^+, D^0)\, D_s^+$ form an SU(3) triplet $(\bar{\mathbf{3}})$; (D^+, D^0) form an isospin doublet. Similarly $C = -1$ states $q\bar{c}$: $(\bar{D}^0, D^-)D_s^-$ form an SU(3) triplet $(\mathbf{3})$.

The states D^* and D_J^* are unstable and decay strongly and

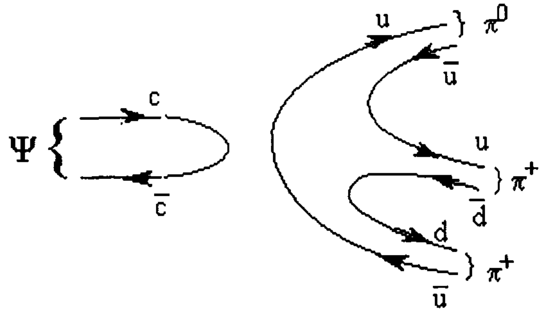

Figure 2 J/ψ suppressed by OZI rule.

Table 8.1

Charmed meson	Quark content	Mass (MeV)	Life Time ($10^{-12}s$)/ width	J^P
D^0	$c\bar{u}$	1864.6 ± 0.5	0.415 ± 0.004	O^-
D^+	$c\bar{d}$	1869.3 ± 0.5	1.057 ± 0.015	O^-
D_s^+	$c\bar{s}$	1968.5 ± 0.6	0.467 ± 0.017	O^-
D^{*0}	$c\bar{u}$	2006.7 ± 0.5	$\Gamma < 2.1$ MeV	1^-
D^{*+}	$c\bar{d}$	2010.0 ± 0.5	$\Gamma \leq 0.131$ MeV	1^-
D_s^{*+}	$c\bar{s}$	2112.4 ± 0.7	$\Gamma \leq 1.9$ MeV	1^-
D_1^0	$c\bar{u}$	2422.2 ± 1.8	$\Gamma = 18.9^{+4.6}_{-3.5}$ MeV	1^+
D_1^+	$c\bar{d}$	–	–	1^+
D_{s1}^+	$c\bar{s}$	2535.35 $\pm 0.34 \pm 0.5$	$\Gamma < 2.3$ MeV	1^+
D_2^{*0}	$c\bar{u}$	2458.9 ± 2.0	$\Gamma = 23 \pm 5$ MeV	2^+
D_2^{*+}	$c\bar{d}$	2459 ± 4	$\Gamma = 25^{+8}_{-7}$ MeV	2^+
D_{s2}^{*+}	$c\bar{s}$	2573.5 ± 1.7	$\Gamma = 15^{+5}_{-4}$ MeV	2^+

Table 8.2

Bottom meson	Quark content	Mass (MeV)	Life-time ($10^{-12}\,s$)	J^P
B^-	$b\bar{u}$	5278.9 ± 1.8	1.65 ± 0.04	O^-
$\bar{B}^0$	$b\bar{d}$	5279.2 ± 1.8	1.56 ± 0.04	O^-
$\bar{B}^0_s$	$b\bar{s}$	5369.3 ± 2.0	1.54 ± 0.07	O^-
B^{*-}	$b\bar{u}$	5324.8 ± 1.8	–	1^-
$\bar{B}^{*0}$	b$\bar{d}$	5324.8 ± 1.8	–	1^-
$\bar{B}^{*0}_s$	$b\bar{s}$	5416.3 ± 3.3	–	1^-
$\bar{B}^*_J$	$b\bar{u}$, $b\bar{d}$	5698 ± 12	$\Gamma =$ 128 ± 18 MeV	2^+ or 1^+
$\bar{B}^*_{sJ}$	$b\bar{s}$	5853 ± 15	$\Gamma = 47 \pm 22$ MeV	2^+ or 1^+

radiatively. For example

$$\begin{aligned} D^* &\to D\pi \\ &\to D\gamma \\ D^*_2 &\to D\pi,\ D^*\pi \\ &\to D\gamma \end{aligned}$$

8.2.2 The fifth quark flavor: Bottom mesons

Fifth quark was discovered, when in 1977 the upsilon meson $\Upsilon(J^{PC} = 1^{--})$ was found experimentally as a narrow resonance at Fermi Lab. with mass $\sim$ 9.5 GeV. This was later confirmed in e^+e^- experiments at DESY and CESR which determined its mass to be 9460 ± 10 MeV and also its width. The updated parameters of this resonance [from the Particle Data Group Tables] are mass 9460.37 ± 0.21 MeV and width 52.5 ± 1.8 keV. Again the narrow width in spite of large phase space available suggests the existence of a fifth quark flavor called beauty, with a new quantum number

$B = -1$ for the bottom (b) quark. With this assignment the formula $Q = I_3 + 1/2(Y + B + C)$ would give the charge of b quark the value $-1/3(I_3 = 0)$. The mass of b quark is expected to be around 4.9 GeV as suggested by the Υ mass which is regarded as a ${}^3S_1$ bound state of $b\bar{b}$.

Thus one would expect particles with $B = \pm 1$, such as $b\bar{q}$ or $q\bar{b}$. The lowest lying bound states $b\bar{q}$ and $q\bar{b}$ have been found experimentally, they are given in Table 2. The $B = -1$ states $(\bar{B}^0,\ B^-)\bar{B}_s^0$ form an SU(3) triplet ($\bar{\mathbf{3}}$) and $B = +1$ states $(B^+,\ B^0)B_s^0$ form another triplet ($\mathbf{3}$).

8.2.3 The sixth quark flavor: The top

The top quark t with $Q = 2/3$ and new flavor $T = 1$ was expected on theoretical grounds. It was first found experimentally in 1996; its mass is $m_t = 175 \pm 6$ GeV. Since $(t,\ b)$ form a weak doublet, it decays weakly to $W^+ + b$, i.e.

$$t \rightarrow W^+ + b.$$

The predicted decay (see Eq. (15.27)) rate is

$$\Gamma(t \rightarrow W^+ + b) = \frac{G_F}{8\pi\sqrt{2}} m_t^3 \left(1 - \frac{m_W^2}{m_t^2}\right)^2 \left(1 + 2\frac{m_W^2}{m_t^2}\right) \quad (8.13)$$

where we have neglected the b quark mass compared to m_W and m_t. Taking $m_t = 175$ GeV, $m_W = 80$ GeV, $G_F = 1.166 \times 10^{-5}$ GeV^{-2}, we get

$$\Gamma \approx 1.56 \ \text{GeV}. \quad (8.14)$$

If QCD correction is taken into account, then

$$\Gamma \approx 1.43\, GeV \quad (8.15)$$

which gives the life time of τ to be

$$\tau = 4.60 \times 10^{-25}\,\text{s}. \quad (8.16)$$

Thus t quark decays before it can form bound states such as $t\bar{t}$ and $t\bar{q}$.

8.3 Heavy Baryons

Since u, d, s belong to the triplet representation of SU(3), the charmed and bottom baryons with spin parity $\frac{1}{2}^+$ belong to either triplet representation $\bar{3}$ or sextet representation 6 of SU(3). Using the Pauli principle, the unitary spin and spin wave functions of spin $\frac{1}{2}^+$ baryons can be written as

$$A_{ij} = \frac{1}{\sqrt{2}}(q_i q_j - q_j q_i)\, Q\, \chi_{MA} \tag{8.17}$$

$$S_{ij} = \frac{1}{\sqrt{2}}(q_i q_j + q_j q_i)\, Q\, \chi_{MS}, \tag{8.18}$$

where $i,\, j = 1,\, 2,\, 3\, (q_1 = u,\, q_2 = d,\, q_3 = s,\, Q = c$ or $b)$ and the spin wave functions χ_{MA} and χ_{MS} are given in Eq.(6.8) and Table (6.3) respectively. Note that A_{ij} belongs to triplet representation $\bar{3}$ of SU(3). In particular we have an isospin singlet and isospin doublet:

$$A_{12} = \Lambda_c^0(\Lambda_b^0),\; A_{13} = -\Xi_c^+(\Xi_b^0),\; A_{23} = \Xi_c^0(\Xi_b^-) \tag{8.19}$$

S_{ij} belongs to the sextet representation of SU(3). In particular, we have an isospin triplet, an isospin doublet and an isospin singlet:

$$\begin{aligned} S_{11} &= \sqrt{2}\Sigma_c^{++}(\Sigma_b^+), \\ S_{12} &= \Sigma_c^+(\Sigma_b^0), \\ S_{22} &= \sqrt{2}\Sigma_c^0(\Sigma_b^-), \\ S_{13} &= \Xi_c^{\prime +}(\Xi_b^{\prime 0}), \\ S_{23} &= \Xi_c^{\prime 0}(\Xi_b^{\prime -}), \\ S_{33} &= \sqrt{2}\Omega_c^0(\Omega_b^-). \end{aligned} \tag{8.20}$$

The spin $\frac{3}{2}^+$ baryons also belong to the sextet representation of $SU(3)$. They are given by Eq.(18), with χ_{MS} replaced by χ_s where the spin wave functions χ_s are given in Table (6.3). The six spin $\frac{3}{2}^+$ baryons are labelled as

$$\Sigma_c^{*++}(\Sigma_b^{*+}),\; \Sigma_c^{*+}(\Sigma_b^{*0}),\; \Sigma_c^{*0}(\Sigma_b^{*-}),$$

$$\Xi_c^{*+}(\Xi_b^{*0}),\ \Xi_c^{*0}(\Xi_b^{*-}),\ \Omega_c^{*+}(\Omega_b^{*0}).$$

In addition to $C = +1$ and $B = -1$ baryons considered above, we also have the following baryons with $C = 2$ and $B = -2$ belonging to the triplet representation of $SU(3)$ with spin parity $(3/2)^+$:

$$\begin{aligned} &\Xi_{cc}^{*++} = ccu\chi_s,\ \Xi_{cc}^{*+} = ccd\chi_s,\ \Omega_{cc}^{*+} = ccs\chi_s \\ &\Xi_{bb}^{*0} = bbu\chi_s,\ \Xi_{bb}^{*-} = bbu\chi_s,\ \Omega_{bb}^{*-} = bbs\chi_s. \end{aligned} \tag{8.21}$$

Finally we have singlets with $C = 3$ and $B = -3$, namely

$$\Omega_{ccc}^{*++} = ccc\chi_s,\ \Omega_{bbb}^{*-} = bbb\chi_s. \tag{8.22}$$

Experimentally only one bottom baryon has been detected so far. Its mass and life time are $m_{\Lambda_b} = 5641 \pm 50$ MeV and $\tau = (1.14 \pm 0.08) \times 10^{-12}\,s$. Some of the charmed baryons have been discovered experimentally, they are given in Table 3.

8.4 Quarkonium

The bound system of heavy quarks $Q\bar{Q}$, $Q = c, b$, is called quarkonium e.g. charmonium $c\bar{c}$, bottomonium $b\bar{b}$,

Since quarks are fermions with spin 1/2, their bound system can be written as $(Q\bar{Q})_{L,S}$. Now S can have two values 0 and 1 with spin wave function antisymmetric and symmetric respectively. If we regard Q and $\bar{Q}$ as identical fermions which differ only in their charges, then we can state generalized Pauli principle: The wave function is antisymmetric with the exchange of particles Q and $\bar{Q}$. Under particle exchange, we get with space coordinates exchange, a factor $(-1)^L$, with spin coordinates exchange, a factor $(-1)^{S+1}$ and with charge exchange, a factor C (C is called C-parity). Hence Pauli principle gives

$$(-1)^{L+S+1}C = -1. \tag{8.23}$$

Table 8.3

Baryons	Quark content	Mass (MeV)	Life-time $\tau(10^{-12}\,s)$	Dominant decays
Λ_c^+	udc	2284.9 ± 0.6	0.206 ± 0.012	weak
Ξ_c^+	usc	2465.6 ± 1.4	$0.35^{+0.07}_{-0.04}$	weak
Ξ_c^0	dsc	2470.3 ± 1.8	$0.098^{+0.023}_{-0.015}$	weak
Σ_c^{++}	uuc	2452.8 ± 0.6	–	$\Lambda_c^+\pi^+$
Σ_c^+	udc	2453.6 ± 0.9	–	$\Lambda_c^+\pi^0$
Σ_c^0	ddc	2452.1 ± 0.6	–	$\Lambda_c^+\pi^-$
$\Xi_c'^0$	dsc	$\sim 2580(?)$	–	$\Xi_c^0\gamma$
$\Xi_c'^+$	usc	$\sim 2580(?)$	–	$\Xi_c^+\gamma$
Ω_c^0	ssc	2704 ± 4	0.064 ± 0.020	weak
Σ_c^{*++}	uuc	2519.4 ± 1.5	–	$\Lambda_c^+\pi^+$
Σ_c^{*+}	udc	?	–	$\Lambda_c^+\pi^0$
Σ_c^{*0}	ddc	2517.5 ± 1.4	–	$\Lambda_c^+\pi^-$
Ξ_c^{*+}	usc	2644.6 ± 2.1	–	$\Xi_c^+\pi^0$, $\Xi_c^0\pi^+$
Ξ_c^{*0}	dsc	2643.8 ± 1.8	–	$\Xi_c^0\pi^0$, $\Xi_c^+\pi^-$
Ω_c^{*0}	ssc	?	–	$\Omega_c^0\gamma$

Therefore,

$$C = (-1)^{L+S}. \tag{8.24}$$

Hence we have the result

$$C = \begin{cases} -1 & L+S \quad \text{odd} \\ +1 & L+S \quad \text{even.} \end{cases} \tag{8.25}$$

Also for $(Q\bar{Q})$ system, the parity

$$P = (-1)(-1)^L = (-1)^{L+1}. \tag{8.26}$$

Let us now use the spectroscopic notation,

$$\mathrm{L} \;=\; \begin{array}{ccccc} 0, & 1, & 2, & 3, & \cdots \\ \mathrm{S}, & \mathrm{P}, & \mathrm{D}, & \mathrm{E}, & \cdots . \end{array}$$

A state is completely specified as

$$n\ {}^{2S+1}L_J,$$

where n is the principal quantum number and J is the total angular momentum. Thus for $L = 0$, we have the following states

$$\begin{array}{llll} n & {}^1S_0 & C = +1, & n = 1,\, 2,\, \cdots \\ n & {}^1S_0 & C = -1, & n = 1,\, 2,\, \cdots . \end{array}$$

The ground state is therefore a hyperfine doublet $1\,{}^1S_0(0^{-+})$ and $1\,{}^3S_1(1^{--})$. For $L = 1$, we have the following states

$$\begin{array}{lllll} n & {}^1P_J & J = +1, & C = -1, & 1^{+-} \\ n & {}^3P_J & J = 0,\, 1,\, 2 & C = 1, & 0^{++},\, 1^{++},\, 2^{++}. \end{array}$$

Finally, we note that for $L = 2$, we have the following states

$$\begin{array}{lllll} n & {}^1D_J & J = 2, & C = +1, & 2^{-+} \\ n & {}^3D_J & J = 1,\, 2,\, 3 & C = -1, & 1^{--},\, 2^{--},\, 3^{--}. \end{array}$$

It is interesting to see that the state $^3D_1$ has the same quantum number as $^3S_1$. They can therefore mix, but the mixing is expected to be small.

The states ^3P_J and $^1P_1$ is a hyperfine quartet (degenerate), but this degeneracy is removed due to hyperfine splitting. The low lying states listed above are shown in Figs. 3 and 4. Most of these states have been discovered experimentally and they are listed in Tables 4 and 5. The transitions and decays of charmonium states are shown in Fig. 5. Similar transitions and decays occur for bottomonium bound states.

From Fig. 5, we note that both $M1$ and $E1$ radiative transitions are possible:

$$\begin{aligned} J/\Psi &\rightarrow \eta_c + \gamma \\ \Psi' &\rightarrow \eta_c + \gamma \text{ M1 transitions} \\ \Psi' &\rightarrow \eta_c' + \gamma \text{ (no parity change)} \\ &\rightarrow \Psi + \gamma \\ \Psi' &\rightarrow \chi + \gamma \text{ E1 transitions} \\ \chi &\rightarrow \eta_c + \gamma \text{ (parity changes).} \end{aligned}$$

From Eq.(6.87) and Table (6.7), we get (for example)

$$\begin{aligned} \Gamma(\Psi' \rightarrow \eta_c\gamma) &= \frac{4\alpha}{3}\left[\frac{2}{3}\frac{1}{m_c}\right]^2 k^3\,\Omega \\ &= 2.7 \text{ keV}\,\Omega \end{aligned} \quad (8.27)$$

where Ω is the overlap integral defined as

$$\Omega_{n'n} = \int_0^\infty e^{i\mathbf{q}\cdot\mathbf{r}}\phi_{n'00}(\mathbf{r})\phi_{n00}(\mathbf{r})\,d^3\mathbf{r} \quad (8.28)$$

and $\mathbf{q} = (m_{sp}/m)\mathbf{k}$, $\mathbf{k}$ is the momentum carried by photon, m_{sp} is the mass of the spectator quark and m is the mass of the bound state. For $\Omega = 1$, Γ is about a factor of three larger than the experimental value.

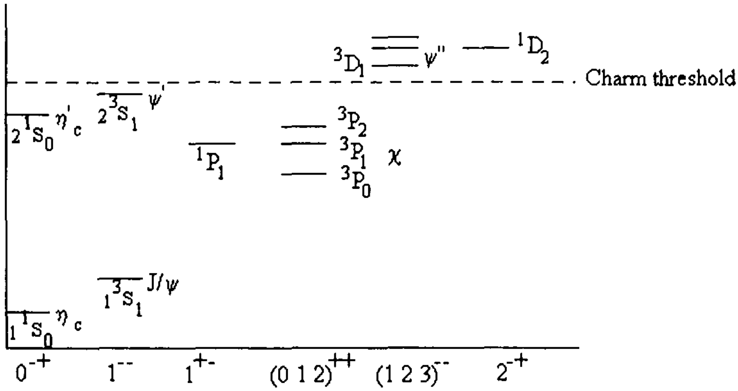

Figure 3 The charmonium spectrum ($c\bar{c}$ bound state).

Table 8.4 The ψ family $(c\bar{c})$ bound states.

	State	Mass (MeV)	Γ_{tot} (MeV)	Γ_{ee} (keV)
$1\,^1S_0$	η_c	2979.8 ± 2.1	$13.2^{+3.8}_{-3.2}$	–
$1\,^3S_0$	J/Ψ	3096.88 ± 0.04	(87 ± 5) $\times 10^{-3}$	5.26 ± 0.37
$^3P_0$	χ_{co}	3417.3 ± 2.8	14 ± 5	–
$^3P_1$	χ_{c1}	3510.53 ± 0.12	0.88 ± 0.14	–
$^3P_2$	χ_{c2}	3556.17 ± 0.13	2.0 ± 0.18	–
$2\,^3S_1$	Ψ'	3686 ± 0.09	(277 ± 31) $\times 10^{-3}$	2.14 ± 0.21
$^3D_1$	$\Psi(3770)$	3769.9 ± 2.5	23.6 ± 2.7	0.26 ± 0.04
$3\,^3S_1$	Ψ''	4040.0 ± 10.0	52 ± 10	0.75 ± 0.15
$?\,^3S_1$	Ψ'''	4159 ± 20	78.0 ± 20.0	0.77 ± 0.23
$?\,^3S_1$	Ψ''''	4415 ± 6	43 ± 15	0.47 ± 0.10

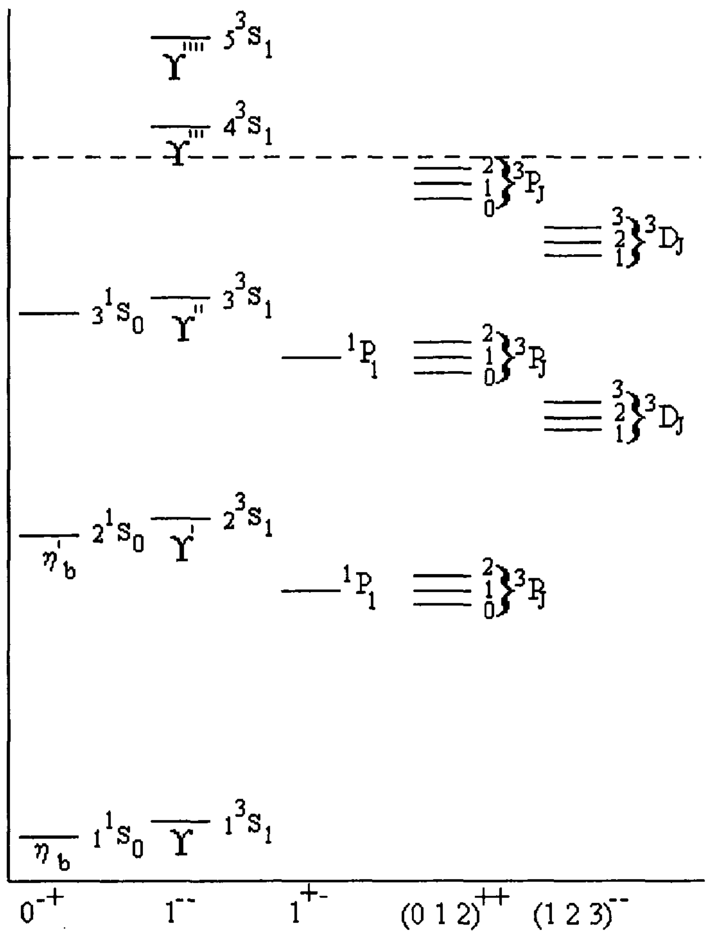
Υ'''' 5^3S_1
Υ''' 4^3S_1
^3P_J
^3D_J
3^1S_0
Υ'' 3^3S_1
$^1P_1$
^3P_J
^3D_J
η'_b 2^1S_0
Υ' 2^3S_1
$^1P_1$
^3P_J
η_b 1^1S_0
Υ 1^3S_1
0^{-+}
1^{--}
1^{+-}
$(0\ 1\ 2)^{++}$
$(1\ 2\ 3)^{--}$

Figure 4

Table 8.5 The Υ family ($b\bar{b}$) bound states.

	State	Mass (MeV)	Γ_{tot} (keV)	Γ_{ee} (keV)
$1\,^1S_0$	η_b	-	-	-
$1\,^1S_0$	Υ	9460.37 ± 0.21	52.5 ± 1.8	1.32 ± 0.05
$^3P_0$	χ_{b0}	9859.8 ± 1.3	–	–
$^3P_1$	χ_{b1}	9891.9 ± 0.7	–	–
$^3P_2$	χ_{b2}	9913.2 ± 0.6	–	–
$2\,^3S_1$	Υ'	10023.30 ± 0.31	44 ± 7	0.520 ± 0.032
$^3P_0$	χ'_{b0}	10232.1 ± 0.6	–	–
$^3P_1$	χ'_{b1}	10255.2 ± 0.5	–	–
$^3P_2$	χ'_{b2}	10268.5 ± 0.74	–	–
$3\,^3S_1$	Υ''	10355.3 ± 0.5	26.3 ± 3.5	0.48 ± 0.08
$4\,^3S_1$	Υ'''	10580.0 ± 3.5	10 ± 4 MeV	0.248 ± 0.031
$5\,^3S_1$	Υ''''	10865 ± 8	110 ± 13 MeV	0.31 ± 0.07
$6\,^3S_1$	Υ'''''	11019 ± 8	79 ± 16 MeV	0.130 ± 0.030

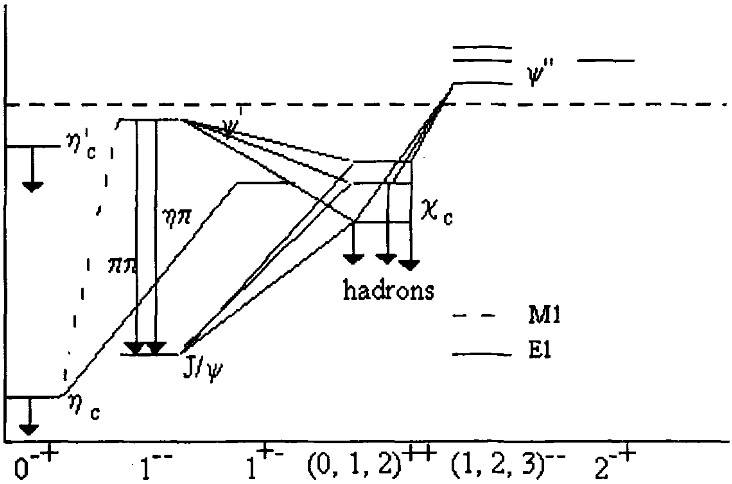

Figure 5 The transitions and decays of charmonium states.

For $E1$ transitions $nS_1 \to n'P_J$ and $nP_J \to n'S_1$ ($J = 0, 1, 2$) the decay widths can be written (cf. Eq.(6.68))

$$\Gamma_{nS_1 \to n'P_J} = \frac{4\alpha}{3}\left[\frac{2J+1}{3}\right]\left|M_{n'n}\right|^2 k^3 \quad (8.29)$$

$$\Gamma_{nP_J \to n'S_1} = \frac{4\alpha}{3}\left|M_{n'n}\right|^2 k^3 \quad (8.30)$$

where

$$M_{n'n} = \langle Q \rangle \Omega_{n'n}, \; \Omega_{n'n} = (1/\sqrt{3}) \times \int_0^\infty [j_0(qr) - 2j_2(qr)] R_{n'0}(r) R_{n1}(r) r^3 dr. \quad (8.31)$$

Note that j_0 and j_2 are spherical Bessel functions and R_{nl} are radial wave functions. In order to predict these decay widths one needs to know the radial wave functions, i.e. some potential model is needed.

Finally, we note that there are 22 states below B threshold as compared with eight states below charm threshold. This is a

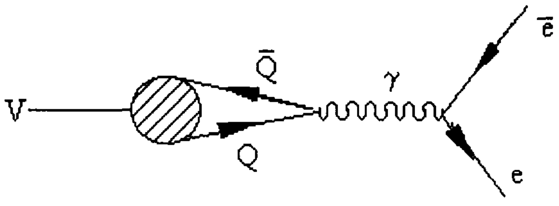

Figure 6

consequence of the fact that interquark potential is flavor independent (as expected in QCD) so that $E_{n2} - E_{n1}$ is the same for $c\bar{c}$ and $b\bar{b}$. (Note that charm threshold is at about 3.74 GeV whereas B threshold is at about 10.55 GeV.)

8.5 Leptonic Decay Width of Quarkonium

The decays of ${}^3S_1(Q\bar{Q})$ state (V) into charged leptons proceeds through the virtual photon as shown in Fig. 6.

The scattering cross section for the $Q\bar{Q} \to l\bar{l}$ is given by Eq.(A.78)

$$\begin{aligned}\sigma &= \frac{4\pi\alpha^2}{3}\langle Q\rangle^2 \frac{1}{s}\frac{\beta_l}{\beta_Q} \\ &\times \left[1 + \frac{2-\beta_Q^2-\beta_l^2}{2} + \frac{\left(1-\beta_Q^2\right)\left(1-\beta_l^2\right)}{4}\right] \qquad (8.32)\end{aligned}$$

where

$$\beta_l = \frac{\sqrt{s-4m_l^2}}{\sqrt{s}}, \qquad \beta_Q = \frac{\sqrt{s-4m_Q^2}}{\sqrt{s}}$$
$$s = E_{cm}^2 \qquad (8.33)$$

and Q is the charge of the quark Q. Now the cross section σ can be written as

$$\sigma = \frac{3}{4}\sigma_t + \frac{1}{4}\sigma_s, \tag{8.34}$$

where σ_t is the cross section for ${}^3S_1$ state and σ_s is the cross-section for ${}^1S_0$ state. Since the photon is coupled to a conserved vector current, therefore it contributes only to spin triplet state. Thus $\sigma_s = 0$. Hence the decay rate in the limit $\beta_l \to 1$ $(s = 4m_Q^2 \gg 4m_l^2)$ is given by

$$\begin{aligned}\Gamma &= (\text{incident flux})\,\sigma_t \\ &= 2\beta_Q|\Psi_s(0)|^2\frac{4}{3}\sigma,\end{aligned} \tag{8.35}$$

where the incident flux $= \rho_{in}(2\beta_Q) = 2|\Psi_s(0)|^2\beta_Q$. Hence from Eqs. (32) and (34), we get

$$\Gamma\left[{}^3S_1(V) \to l^+l^-\right] = \frac{16\pi\alpha^2}{3}\langle Q\rangle^2\frac{|\Psi_s(0)|^2}{m_V^2}, \tag{8.36}$$

where we have put $s = 4m_Q^2 \approx m_V^2$ and $\beta_Q \approx 0$ (in the non-relativistic limit).

Taking into account the color $|V> = \frac{1}{\sqrt{3}}\sum_a|\bar{Q}_aQ_a>$, we multiply Eq.(35) by a factor of three. Hence we have

$$\Gamma^0\left[V \to l^+l^-\right] = 16\pi\alpha^2\langle Q\rangle^2\frac{|\Psi_s(0)|^2}{m_V^2}. \tag{8.37}$$

It may be pointed out that before comparing experimental leptonic widths with their theoretical predictions, the vacuum polarization contributions to the leptonic decay width have to be removed so that

$$\Gamma^0 = \Gamma^{exp}(1-\Pi)^2$$

where $(1-\Pi)^2 = 0.958, 0.932$ for charmonium and bottomonium respectively and then it is Γ^0 which is to be compared with the theoretical predictions.

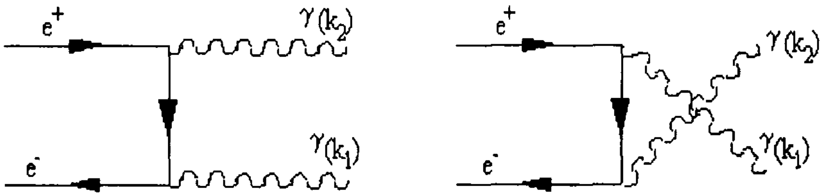

Figure 7 Positronium ($^1S_0$ state) decay into two photons.

8.6 Hadronic Decay Width

The decays of quarkonium states $^3S_1$ and $^1S_0$ to ordinary hadrons are suppressed by the OZI rule. The narrowness of their decay widths can be explained as follows. By C-conservation $^3S_1$ state can decay in the lowest order to three gluons and thus its hadronic decay width is proportional to $\alpha_s^3 \times$ (probability of conversion of gluons into hadrons). Since color is confined so this probability is unity. Similarly the decay of $^1S_0$ into hadrons is proportional to α_s^2, since by C-conservation it can decay into two gluons. Here analogy with positronium is in order. Positronium in $^1S_0$ state (para positronium) decay into two photons via the diagram (Fig. 7).

In the low energy limit the cross section for the above process is given by

$$\sigma = \frac{\pi}{\beta}\left(\frac{\alpha}{m_e}\right)^2 . \tag{8.38}$$

Since $\sigma_t = 0$, we get using Eq.(34)

$$\sigma_s = 4\sigma = \frac{4\pi}{\beta}\left(\frac{\alpha}{m_e}\right)^2 . \tag{8.39}$$

Hence the decay rate

$$\Gamma\left[{}^1S_0(e^-e^+) \to 2\gamma\right] = |\beta\Psi_s(0)|^2\, 4\,\sigma = 16\pi\frac{\alpha^2}{4m_e^2}|\Psi_s(0)|^2 . \tag{8.40}$$

For $(Q\bar{Q})\,{}^1S_0$ state decaying into 2γ, we replace

$$e^4 \to \left[\sqrt{3}Q^2e^2\right]^2 = 3Q^4e^4$$

and $4m_e^2 \to 4m_Q^2 \approx m_P^2$. Hence we get

$$\Gamma\left[{}^1S_0(m_P) \to 2\gamma\right] = \frac{16\pi\alpha^2}{m_P^2}3Q^4|\Psi_s(0)|^2. \tag{8.41}$$

For $\eta_c \to 2$ gluons, we replace α^2 by $\frac{2}{3}\alpha_s^2$ in Eq. (40) [see problem 2], so that we get the hadronic decay rate

$$\Gamma\left[\eta_c \to \text{ hadrons}\right] = \frac{32\pi}{3}\frac{\alpha_s^2(m_{\eta_c})}{m_{\eta_c}^2}|\Psi_s(0)|^2. \tag{8.42}$$

The decay rate for ${}^3S_1(e^-e^+)$ system going to 3γ is given by

$$\Gamma\left[{}^3S(e^-e^+) \to 3\gamma\right] = \frac{64\pi}{9\pi}(\pi^2-9)\frac{\alpha^3}{4m_e^2}|\Psi_s(0)|^2. \tag{8.43}$$

For the decay of ${}^3S_1(\bar{Q}Q) \to 3g$, we replace α^3 by $5\alpha_s^3/18$ [see problem 2] and $(2m_e)^2 = (2m_Q)^2 \approx m_V^2$ in Eq.(43). Hence we get

$$\begin{aligned}\Gamma\left[{}^3S_1(V) \to \text{ hadrons}\right] &= \Gamma\left[{}^3S_1 \to 3g\right] \\ &= \frac{160\pi(\pi^2-9)}{81\pi}\frac{\alpha_s^3}{m_V^2}|\Psi_s(0)|^2.\end{aligned} \tag{8.44}$$

We now apply the above results to ϕ, J/Ψ and Υ decays. From Eqs.(44) and (37), we get

$$\alpha_s^3(m_V) = \frac{81\pi\alpha^2 < Q >^2}{10(\pi^2-9)}\frac{\Gamma\left(V \to \text{ hadrons}\right)}{\Gamma\left(V \to e^-e^+\right)}. \tag{8.45}$$

From Eq.(45), we get

$$\alpha_s(m_\phi) \approx 0.44, \quad \alpha_s(m_\Psi) \approx 0.22, \quad \alpha_s(m_\Upsilon) \approx 0.18,$$

where we have used $\Gamma(\phi \to \text{ non-strange mesons}) \approx 653$ keV, $\Gamma(J/\Psi \to \text{ hadrons}) \approx 76.5$ keV, $\Gamma(\Upsilon \to \text{ hadrons}) \approx 50$ keV, $\Gamma(\phi \to e^+e^-) \approx 1.37$ keV, $\Gamma(J/\Psi \to e^+e^-) \approx 5.26$ keV, $\Gamma(\Upsilon \to e^+e^-) \approx 1.32$ keV. From this we see a realization of the asymptotic freedom of QCD, the coupling $\alpha_s(q^2)$ falls with the increase of q^2.

Finally from Eqs.(42), (44) and (37), we have [with $\alpha_s(m_{\eta_c}) = \alpha_s(m_\Psi)$]

$$\Gamma(\eta_c \to \text{ hadrons}) = \frac{27\pi}{5(\pi^2-9)} \frac{m_\Psi^2}{m_{\eta_c}^2} \frac{1}{\alpha_s(m_\Psi)} \Gamma(J/\Psi \to \text{ hadrons}) \tag{8.46}$$

$$\begin{aligned} \Gamma(\eta_c \to \text{ hadrons}) &= \frac{3}{2}\left[\frac{\alpha_s(m_\Psi)}{\alpha}\right]^2 \frac{m_\Psi^2}{m_{\eta_c}^2} \Gamma(J/\Psi \to e^+e^-) \\ &\approx 7.6\, MeV \end{aligned} \tag{8.47}$$

where we have used $\alpha_s(m_\Psi) \approx 0.22$. This value is lower than the experimental value $\Gamma_{tot} \approx 13.2^{+3.8}_{-3.2}$ MeV for η_c.

8.7 Non-Relativistic Treatment of Quarkonium

From a theoretical point of view, heavy quark system (quarkonium) is interesting because this is a relatively simple system. To a good approximation, the quark motion in this bound state should be non-relativistic. Thus we can use the Schrödinger equation for $Q\bar{Q}$ system:

$$-\frac{\hbar^2}{2\mu}\nabla^2\Psi(\mathbf{r}) + [V(r) - E]\Psi(\mathbf{r}) = 0 \tag{8.48}$$

μ is the reduced mass of $Q\bar{Q}$ system i.e. $\mu = \frac{1}{2}m_Q$. For central potential, we can use the wave function:

$$\Psi(\mathbf{r}) = R(r)\, Y_{lm}(\theta, \phi). \tag{8.49}$$

The radial wave function $R(r)$ satisfies the equation

$$-\frac{\hbar^2}{2\mu}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right] R(r) - \left[E - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2}\right] R(r) = 0. \tag{8.50}$$

If we define a radial function

$$\chi(r) = r\,R(r), \tag{8.51}$$

then $\chi(r)$ satisfies the equation

$$\frac{d^2\chi}{dr^2} + \left[\frac{2\mu}{\hbar^2}(E - V(r)) - \frac{l(l+1)}{r^2}\right]\chi = 0. \tag{8.52}$$

The wave function $\chi(r)$ is normalized as

$$\begin{aligned}\int_0^\infty [\chi(r)]^2 dr &= \int_0^\infty [R(r)]^2 r^2 dr \\ &= 1\end{aligned} \tag{8.53}$$

with the boundary conditions

$$\chi(0) = 0. \tag{8.54}$$

For $S-$waves:

$$\left.\begin{aligned}\chi(r) &\to 0 \\ R(r) &\to 0.\end{aligned}\right\} \quad r \to \infty \tag{8.55}$$

For $S-$waves:

$$\chi'(0) = R(0) = \sqrt{4\pi}\Psi_s(0). \tag{8.56}$$

We now prove two important results:
1.

$$|\Psi_s(0)|^2 = \frac{\mu}{2\pi\hbar^2}\left\langle\frac{dV}{dr}\right\rangle. \tag{8.57}$$

Proof.
From Eq. (52) for $l = 0$

$$\frac{\chi''}{\chi} = -\frac{\mu}{2\pi\hbar^2}(E - V). \tag{8.58}$$

Therefore,

$$\frac{d}{dr}\left[\frac{\chi''}{\chi}\right]=\frac{2\mu}{\hbar^2}\frac{dV}{dr}\,. \tag{8.59}$$

Taking the expectation value [note $\chi(r)$ is real], we get

$$\int_0^\infty \chi\frac{d}{dr}\left[\frac{\chi''}{\chi}\right]\chi dr=\frac{2\mu}{\hbar^2}\int_0^\infty \chi\,\frac{dV}{dr}\,\chi dr. \tag{8.60}$$

Integrating left hand side by parts, we get

$$\begin{aligned} l.h.s. &= \left[\frac{\chi''}{\chi}\chi^2\right]_0^\infty-\int_0^\infty \frac{\chi''}{\chi}2\chi\chi' dr \\ &= \left[\frac{\chi''}{\chi}\chi^2-\chi'^2\right]_0^\infty \\ &= [\chi'(0)]^2=[R(0)]^2, \end{aligned} \tag{8.61}$$

where we have used the boundary conditions (54) - (56). Hence Eq. (60) gives

$$|\Psi_s(0)|^2=\frac{\mu}{2\pi\hbar^2}\left\langle\frac{dV}{dr}\right\rangle.$$

2. Virial Theorem

$$\langle T\rangle=\frac{1}{2}\left\langle r\frac{dV}{dr}\right\rangle. \tag{8.62}$$

Proof.

From Eq. (59), we have

$$\int_0^\infty \chi\, r\,\frac{d}{dr}\left(\frac{\chi''}{\chi}\right)\chi dr=\frac{2\mu}{\hbar^2}\left\langle r\frac{dV}{dr}\right\rangle. \tag{8.63}$$

Integrating left hand side by parts and using Eqs. (54) and (55), we get

$$\text{l.h.s.}=2\int_0^\infty \chi'^2 dr. \tag{8.64}$$

Therefore,

$$2\int_0^\infty \chi'^2 dr = \frac{2\mu}{\hbar^2}\left\langle r\frac{dV}{dr}\right\rangle . \tag{8.65}$$

Now from Eq. (58),

$$\left\langle \frac{\chi''}{\chi}\right\rangle = -\frac{2\mu}{\hbar^2}\left[E- < V >\right]. \tag{8.66}$$

But

$$\begin{aligned}\left\langle \frac{\chi''}{\chi}\right\rangle &= \int_0^\infty \chi\chi'' dr \\ &= |-\chi\chi'|_0^\infty - \int_0^\infty \chi'^2 dr \\ &= -\int_0^\infty \chi'^2 dr. \end{aligned} \tag{8.67}$$

Hence from Eqs. (65) – (67), we get

$$E - \langle V\rangle = \frac{1}{2}\left\langle r\frac{dV}{dr}\right\rangle \tag{8.68}$$

or

$$\langle T\rangle = \frac{1}{2}\left\langle r\frac{dV}{dr}\right\rangle .$$

Let us apply Eq. (62) to one gluon exchange potential $V(r) = -\frac{4}{3}\alpha_s\frac{1}{r}$. For this case

$$\left\langle \frac{p^2}{2\mu}\right\rangle = \frac{2}{3}\alpha_s\left\langle \frac{1}{r}\right\rangle = \frac{2}{3}\alpha_s\frac{1}{a}, \tag{8.69}$$

where $a = 3/4\mu\alpha_s$ is the Bohr radius. Thus, we get

$$\frac{v}{c} = \frac{2}{3}\alpha_s . \tag{8.70}$$

As α_s decreases with mass, for sufficiently high mass $v/c \ll 1$ and one can treat dynamics non-relativistically.

For the special case of power law potential

$$V(r) = A + \lambda r^{\nu}, \tag{8.71}$$

one can obtain interesting results by studying the scaling of Schrödinger equation (52). Put $\rho = \beta r$, where β is some parameter such that it makes ρ dimensionless. Let us put $\chi(r) = u(\rho)$ and $\bar{E} = E - A$. Then we get from Eq.(52)

$$-\frac{d^2}{d\rho^2}u(\rho) = \left[\frac{2\mu}{\hbar^2}\frac{1}{\beta^2}\bar{E} - \frac{2\mu}{\hbar^2}\lambda\frac{1}{\beta^{2+\nu}}\rho^{\nu} - \frac{l(l+1)}{\rho^2}\right]u(\rho). \tag{8.72}$$

Put $|\lambda| = \frac{2\mu}{\hbar^2}\beta^{2+\nu}$, this gives

$$\beta = \left[\frac{2\mu|\lambda|}{\hbar^2}\right]^{1/2+\nu}. \tag{8.73}$$

Then we put

$$\varepsilon = \frac{2\mu}{\hbar^2}\left[\frac{2\mu|\lambda|}{\hbar^2}\right]^{1/2+\nu}\bar{E}, \tag{8.74}$$

where ε is dimensionless. If we write sgn$(\lambda) = \lambda/|\lambda|$, we obtain Eq.(71) as

$$\frac{d^2}{d\rho^2}u(\rho) + \left[\varepsilon - sgn(\lambda)\rho^{\nu} - \frac{l(l+1)}{\rho^2}\right]u(\rho) = 0 \tag{8.75}$$

which depends only upon pure numbers. We now study the consequences of Eq. (75).

(i) Lenghts and quantities with the dimensions of length depend upon constituent quark mass $m = 2\mu$ and coupling strength $|\lambda|$ as

$$L \sim \frac{1}{\beta} \sim (\mu|\lambda|)^{-1/2+\nu}. \tag{8.76}$$

Table 8.6

	Coulomb like $\nu = -1$	Simple harmonic oscillator $\nu = 2$	linear $\nu = 1$	log $\nu = 0$	Power law $\nu = 0.1$
$\|\Psi_s(0)\|^2$	μ^3	$\mu^{3/4}$	μ	$\mu^{3/2}$	$\mu^{1.43}$
ΔE	μ	$\mu^{-1/2}$	$\mu^{-1/3}$	constant	$\mu^{-0.048}$

Particle density at the origin of coordinates

$$|\Psi_s(0)|^2 \sim L^{-3} \propto (\mu|\lambda|)^{-1/2+\nu}. \tag{8.77}$$

(ii) Level spacing between energy levels depends on μ and $|\lambda|$ as

$$\Delta E \sim \frac{1}{\mu}(\mu|\lambda|)^{2/2+\nu} \propto \mu^{-\nu/2+\nu}|\lambda|^{2/2+\nu}. \tag{8.78}$$

The "power law" potential corresponding to the limiting value $\nu \to 0$ is simply the logarithmic potential.

$$V(r) = C \ln \frac{r}{r_0}. \tag{8.79}$$

We summarize these results for the power law potentials in Table 6.

Observations

$$m_{\Upsilon'} - m_{\Upsilon} = m_{\Psi'} - m_{\Psi} \tag{8.80}$$

implies either $\nu = 0$ or ν is very small. In fact Martin has shown that the potential

$$V(r) = -8.04\,\text{GeV} \; + 6.870(r/1\,\text{GeV}^{-1})^{0.1} \tag{8.81}$$

gives a good fit to quarkonium mass spectrum.

The logarithmic potential

$$V(r) = (0.71\,\mathrm{GeV})\ln\left(\frac{r}{r_0}\right) \tag{8.82}$$

also gives good fit to the data. The two forms are numerically indistinguishable for $0.1\,fm \leq r \leq 1\,fm$.

If we plot $|\Psi_s(0)|^2$ for the vector bosons ρ, ω, ϕ, Ψ, Υ versus μ in a log – log plot, a straight line fit is possible i.e.

$$|\Psi_s(0)|^2 \sim \mu^p \tag{8.83}$$

with $p \sim 1.6$. Again this supports the power law potential with ν very small, i.e., $\nu \sim 0.1$.

It is clear from Tables 4 and 5 that both for charmonium and bottomonium, the low lying bound state energy spectrum satisfies the rule

$$E_{1s} < E_{1p} < E_{2s}. \tag{8.84}$$

In particular we find for $c\bar{c}$

$$\begin{aligned} \bar{m}(1\,^3P) - \bar{m}(1\,S) &= 457\,\mathrm{MeV} \\ m(2\,^3S_1) - m(1\,^3S_1) &= \Psi' - J/\Psi = 589\,\mathrm{MeV} \\ m(1\,^3S_1) - m(1\,^1S_0) &= J/\Psi - \eta_c = 117\,\mathrm{MeV} \\ m(2\,^3S_1) - m(2\,^1S_0) &= \Psi' - \eta_c' = 82\,\mathrm{MeV}, \end{aligned} \tag{8.85}$$

where

$$\bar{m}(S) = \frac{3m(^3S_1) + m(^1S_0)}{4} \tag{8.86}$$

$$\bar{m}(^3P) = \frac{5m(^3P_2) + 3m(^3P_0) + m(^3P_0)}{9}. \tag{8.87}$$

For Coulomb potential, the energy spectrum satisfies the rule

$$E_{1s} < E_{2s} = E_{2p} < E_{3s} = E_{3p} = E_{3d} \tag{8.88}$$

and for the harmonic oscillator potential

$$E_{1s} < E_{1p} < E_{2s} = E_{1d} < E_{2p}. \tag{8.89}$$

Further, the harmonic oscillator potential gives the level spacing as follows:

$$E_{1p} - E_{1s} = E_{2s} - E_{1p} = \frac{1}{2}(E_{2s} - E_{1s}). \tag{8.90}$$

Thus although oscillator potential is a confining potential, the level spacing is not in agreement with the experimental results.

The QCD inspired Cornell potential [cf. Eq. (7.62a)]

$$V(r) = C - \frac{K}{r} + \frac{r}{a^2}$$

reproduces the mass spectrum for $c\bar{c}$ and $b\bar{b}$ bound states quite well (see problem 3).

Thus we see that the quarkonium spectroscopy is consistent with a potential that increases linearly at large distances, thereby supporting the color confinement. We also saw in this chapter (as well as in Chap. 7) a realization of other striking property of QCD, namely the running of the QCD coupling constant $\alpha_s(q^2)$ with q^2.

8.8 Problems

1. Show that if J/Ψ has isospin $I = 0$,

$$\frac{\Gamma\left(\Psi \to \rho^0\pi^0\right)}{\Gamma\left(\Psi \to \rho^-\pi^+\right) + \Gamma\left(\Psi \to \rho^+\pi^-\right)} = \frac{1}{2}.$$

Hint: The $\rho\pi$ final state has $I = 0$, 1 or 2. But we are interested in $I = 0$. Using C. G. coefficients

$$\begin{aligned} \left|\rho^-\pi^+\right\rangle &= \frac{1}{\sqrt{3}}\left|0,0\right\rangle + \cdots \\ \left|\rho^0\pi^0\right\rangle &= -\frac{1}{\sqrt{3}}\left|0,0\right\rangle + \cdots \end{aligned}$$

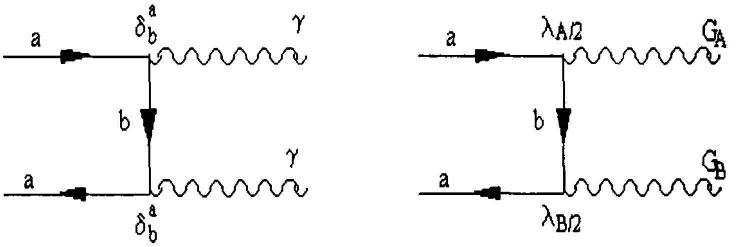

Figure 8

2. For $(q\bar{q})_{\text{color singlet}} \to 2\gamma$ or 2 gluons, as shown in Fig. 8, where $a, b = 1, 2, 3$ are 3 colors of quarks, $A, B = 1, \cdots, 8$ are eight colors of gluons, show that

$$\frac{M(2g)}{M(2\gamma)} = \frac{\alpha_s}{\alpha Q_q^2} \frac{\frac{1}{2}\delta_{AB}}{3}$$

and hence show that

$$\frac{\Gamma(2g)}{\Gamma(2\gamma)} = \frac{2}{9} \frac{\alpha_s^2}{\alpha^2 Q_q^4}.$$

For $(q\bar{q})_{\text{color singlet}} \to 3\gamma$ or 3 gluons coupled symmetrically in gluon color, show that

$$\frac{\Gamma(3g)}{\Gamma(3\gamma)} = \frac{5}{54} \frac{\alpha_s^3}{\alpha^3 Q_q^6}.$$

Hint: Use $\frac{1}{16} d_{ABC} d_{ABC} = \frac{5}{6}$.

3. By writing

$$E = < H > = < \Psi|H|\Psi > = 2m + \frac{1}{2\mu} < \mathbf{p}^2 > + < V(r) >$$

where $\mu = m/2$ and $V(r) = C - \frac{K}{r} + \frac{1}{a^2} r$, evaluate the energy eigenvalues E_{1s}, E_{1p} and E_{2s} by variational principle.

Hint: Write $\bar{\varepsilon} = (2\mu b^4)^{1/3}\bar{E} = (2\mu b^4)^{1/3}[< H > -2m - C]$, $(E = \bar{E} + C)$ and $\Psi = \frac{u(r)}{r}Y_{lm}(\theta, \phi)$ and express

$$\bar{\varepsilon} = \frac{\int \left[-u\frac{d^2u}{dy^2} + \left(y - \frac{\eta}{y} + \frac{l(l+1)}{y^2}\right)u^2\right]dy}{\int u^2 dy}$$

where $y = (2\mu/b^2)^{1/3}r$ and $\eta = (4\mu^2 b^2)^{1/3}K$. [All these quantities are dimensionless and are therefore also suitable for numerical solution on a computer.]

Using the trial wave functions

$$\begin{aligned} 1S &: \; u = Nye^{-1/2\beta^2 y^2}, \\ 2S &: \; Ny\left(1 - \frac{2}{3}\beta^2 y^2\right)e^{-1/2\beta^2 y^2} \\ 1P &: \; u = Ny^2 e^{-1/2\beta^2 y^2}, \end{aligned}$$

minimize $\bar{\varepsilon}$ in order to determine the parameter β for each wave function. Then find $\bar{\varepsilon}$. For numerical purpose, use $m_c = 1.52$ GeV, $K = 0.48$, $a = 2.34$ GeV^{-1}. Compare your results with the experimental values.

Using the equation

$$|\Psi_{ns}(0)|^2 = \frac{\mu}{2\pi}\left\langle\frac{dV}{dr}\right\rangle_{ns} = \frac{\mu}{2\pi}\left[\frac{1}{a^2} + K\left\langle\frac{1}{r^2}\right\rangle_{ns}\right],$$

evaluate $|\Psi_{ns}(0)|^2$ by using the above wave functions for $1S$ or $2S$ states in order to determine $< 1/r^2 >_{ns}$ with the parameters β's determined in the first part of the problem. Hence evaluate the mass difference $\Psi - \eta_c$ and $\Psi' - \eta_c'$ and compare your results with the experimental values.

4. Using the formula (36) in the text and the experimental table

Vector Meson	m_V (MeV)	Γ (keV)
ρ	770	6.77 ± 0.32
ω	783	0.60 ± 0.02
ϕ	1020	1.37 ± 0.05
Ψ	3097	4.72 ± 0.35

find $|\Psi_{ns}(0)|^2_{\rho,\omega,\phi,\Psi}$ and using the values thus found and the dependence of $|\Psi_{ns}(0)|^2$ on $\mu = m_q/2$ discussed in Table 6 for various potentials, find which potential(s) are favored.

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Chapter 9
HEAVY QUARK EFFECTIVE THEORY (HQET)

9.1 Effective Lagrangian

The QCD Lagrangian (7.32) for light quarks is chiral invariant in the limit $m_{u,d,s} \to 0$. For a heavy quark c, b, or t, the chiral symmetry does not hold. However, QCD has asymptotic freedom which *implies that the effective coupling constant* α_s *decreases* logarithmically at short distances or high momentum transfers. This is the basis for perturbative QCD i.e. above a certain mass scale μ, the perturbative QCD is applicable. The size of a hadron is of the order of Λ_{QCD}, where $\Lambda_{\text{QCD}} \sim 0.2$ GeV (see Chap. 7). Thus for a bound state of quarks or (quark–antiquark), we are in the nonperturbative regime i.e. in the confinement region.

Consider for example a bound state of light-heavy quark-antiquark, viz. $q\bar{Q}$ or $Q\bar{q}$. In the limit $m_Q \to \infty$, heavy quark (anti quark) can be taken as a static source of field in which light antiquark (quark) moves. The situation is like hydrogen atom. In the limit $m_Q \to \infty$, the Hamiltonian for the light degree of freedom in analogy with H atom can be written to order v^2/c^2

$$H = \frac{\hat{p}^2}{2m_q} + V_c(r) - \frac{\hat{p}^4}{8m_q^2} - \frac{1}{4m_q^2}\sigma_q.(\mathbf{E}^c \times \hat{\mathbf{p}}) - \frac{1}{8m_q^2}\nabla \cdot \mathbf{E}^c, \quad (9.1)$$

where $\mathbf{E}^c$ is the color electric field, $V_c(r)$ is related to $\mathbf{E}^c$ by $\mathbf{E}^c = -\frac{dV_c}{dr}\frac{\mathbf{r}}{r}$, and $\hat{\mathbf{p}} = -i\nabla$. Although λ_{QCD}/m_Q is small, but it is still finite. The effective heavy quark theory provides a framework to take into account $1/m_Q$ corrections.

The starting point is to define a four velocity

$$v_\mu = \frac{dx_\mu}{d\tau}, \qquad \mathbf{u} = \frac{d\mathbf{x}}{d\tau}\frac{d\tau}{dt} = \frac{\boldsymbol{v}}{\gamma} \tag{9.2}$$

$$v_0 = \gamma = v^0$$

so that

$$v^2 = v^\mu v_\mu = \gamma^2 - \gamma^2 \mathbf{u}^2 = \gamma^2(1-\mathbf{u}^2) = 1. \tag{9.3}$$

It is convenient to define

$$\not{v} = \gamma^\mu v_\mu, \qquad \not{v}^2 = v^2 = 1. \tag{9.4}$$

The Dirac equation for a heavy quark is given by

$$(i\gamma^\mu D_\mu - m)\Psi = 0 \tag{9.5}$$

where D_μ is the covariant derivative

$$D_\mu = \frac{\partial}{\partial x_\mu} - ig_s G_\mu \tag{9.6}$$

$$G_\mu = \lambda_A/2G_\mu^A.$$

We define the projection operators

$$P_\pm = \frac{1}{2}\left(1 \pm \not{v}\right). \tag{9.7}$$

Note that in the rest frame $\boldsymbol{v} = 0$

$$P_\pm = \frac{1}{2}(1 \pm \gamma^0)$$

i.e. it projects out upper and lower components of Ψ. Write

$$\begin{aligned} \Psi &= P_+\Psi + P_-\Psi \\ &= e^{-imv.x}[h_{+v} + h_{-v}] \\ &= e^{-im\not{v}v.x}h_{+v} + e^{im\not{v}v.x}h_{-v}. \end{aligned} \tag{9.8}$$

We note that

$$h_{\pm v} = e^{imv.x} P_{\pm} \Psi$$

$$\not{v} h_{+v} = h_{+v}, \qquad \not{v} h_{-v} = -h_{-v} \tag{9.9}$$

$$P_{+}\gamma^{\mu} = \gamma^{\mu} P_{-} + v^{\mu}$$

$$P_{-}\gamma^{\mu} = \gamma^{\mu} P_{+} - v^{\mu}. \tag{9.10}$$

Using Eqs. (8)-(10), we obtain from Eq. (5)

$$(i\gamma \cdot D + iv \cdot D)h_{-v} + iv \cdot Dh_{+v} = 0 \tag{9.11}$$

$$(i\gamma \cdot D - iv \cdot D)h_{+v} - (2m + iv \cdot D)h_{-v} = 0. \tag{9.12}$$

Note that h_{+v} and h_{-v} are not decoupled. We show that to order $1/m^2$, the equations for h_{+v} and h_{-v} are decoupled. From Eq. (12), we obtain

$$\begin{aligned} h_{-v} &= \frac{i\gamma \cdot D - iv \cdot D}{2m + iv \cdot D} h_{+v} \\ &= \frac{1}{2m}[1 - \frac{iv \cdot D}{2m} + \cdots][i\gamma \cdot D - iv \cdot D]h_{+v}. \end{aligned} \tag{9.13}$$

Thus from Eqs. (11) and (13) to order $1/m$, we get

$$[i\gamma \cdot D + iv \cdot D]\frac{i\gamma \cdot D - iv \cdot D}{2m} h_{+v} + iv \cdot Dh_{+v} = 0. \tag{9.14}$$

Now

$$\begin{aligned} \gamma \cdot D\, \gamma \cdot D &= \gamma^{\mu}\gamma^{\nu} D_{\mu} D_{v} \qquad (9.15) \\ &= D^2 - \frac{i}{2}\sigma^{\mu v}[D_{\mu}, D_{v}] \\ &= D^2 - \frac{g_s}{2}\sigma^{\mu v} G_{\mu v}. \end{aligned}$$

Hence from Eq. (14), we obtain

$$[iv \cdot D - \frac{D^2}{2m} + \frac{g_s}{4m}\sigma^{\mu v} G_{\mu v} - \frac{(iv \cdot D)^2}{2m}]h_{+v} = 0. \tag{9.16}$$

This is the Pauli form of Dirac equation to order $1/m$. The corresponding Lagrangian for the field h_{+v} is given by

$$\mathcal{L}_{eff} = \bar{h}_{+v}[iv \cdot D - \frac{D^2 + (iv \cdot D)^2}{2m} + \frac{g_s}{4m}\sigma^{\mu v}G_{\mu v}]h_{+v}. \quad (9.17a)$$

Note that h_{+v} annihilates a heavy quark. In the limit $m \to \infty$

$$\mathcal{L}_{eff} = \bar{h}_{+v} iv \cdot D h_{+v}. \quad (9.17b)$$

Now from the relation

$$-i\partial_\mu \Psi(x) = [\hat{P}_\mu, \Psi(x)] \quad (9.18)$$

it follows through the transformation (8) that

$$-i\partial_\mu h_{+v} = mv_\mu h_{+v} + [\hat{P}_\mu, h_{+v}]. \quad (9.19)$$

This shows that a derivative acting on h_{+v} corresponds to a factor of the residual momentum k_μ carried by the heavy quark

$$-k_\mu = mv_\mu - p_\mu \quad (9.20)$$

so that k_μ indicates how much heavy quark is offmass shell. In the limit $m \to \infty$ (no recoil limit) with v_μ and k_μ fixed

$$v_\mu^{quark} \equiv \frac{p_\mu}{m} = v_\mu + \frac{k_\mu}{m} \to v_\mu. \quad (9.21)$$

One would expect the heavy quark to carry most of the momentum of the $\bar{q}Q$ bound state, but not all:

$$p_\mu^B = p_\mu + l_\mu = mv_\mu^{quark} + l_\mu \quad (9.22)$$

where $p_\mu^B = m_B v_\mu$ is the momentum of the bound system and l_μ is that which is carried by the light degree of freedom. Now $m_B = m + m_q - B$ where B is the binding energy supplied by the interaction through gluon. Thus from Eq. (22)

$$v_\mu^{quark} = \frac{m_B}{m}v_\mu - \frac{l_\mu}{m} \quad (9.23)$$

so that again $v_\mu^{quark} \to v_\mu$ as $m \to \infty$ and a comparison of Eq.(23) with Eq.(21) shows that in the limit $m \to \infty$, the interaction with gluons can change k_μ but not $v_\mu^{quark} = v_\mu$; the velocity of the heavy quark can be altered only by an external current which absorbs "infinite momentum". Thus in a hadron, the light degrees of freedom are independent of the heavy quark mass i.e. residual motion of the heavy quark in a hadron can be taken into account by adding the effective Hamiltonian for heavy quark Q from the $\mathcal{L}_{eff}$ given in Eq. (17) to the Hamiltonian for the light quark given in Eq. (1). We will come to this point later, when we discuss the masses of heavy hadrons.

The third term in the Lagrangian (17a) undergoes short-distance QCD corrections and as such is multiplied by the renormalization factor

$$Z_Q(\mu) = \left[\frac{\alpha_s(\mu)}{\alpha_s(m)}\right]^{-9/25} \tag{9.24}$$

with $\quad Z_Q(\mu = m) = 1.$

The heavy quark propagator in QCD can be written in HQET using Eq. (20):

$$i\delta_b^a \frac{m + \gamma \cdot p}{p^2 - m^2 + i\varepsilon} = i\delta_b^a \frac{1 + \not{v}}{2(v \cdot k + i\varepsilon)} \tag{9.25}$$

where we have neglected the term $k^2/m \to 0$. The gluon heavy quark vertex can be written from the Lagrangian (17) and is given by

$$ig_s v^\mu (T_s)_b^a . \tag{9.26}$$

The following relations are useful

$$\begin{aligned} \not{v}\gamma^\mu + \gamma^\mu \not{v} &= 2v^\mu \\ \not{v}\gamma_\mu \not{v} &= 2v_\mu \not{v} - \gamma_\mu . \end{aligned} \tag{9.27}$$

From Eq. (27), one can write, on using Eq. (9),

$$\bar{h}_{+v}\gamma^\mu h_{+v} = \bar{h}_{+v} v^\mu h_{+v} . \tag{9.28}$$

9.2 Spin Symmetry of Heavy Quark

In the limit $m \to \infty$, the Lagrangian $\mathcal{L}_{eff}$ given in Eq.(17) has additional symmetries not present in the full QCD Lagrangian. One such symmetry namely the spin symmetry of heavy quark is reflected in the fact that the first term in the Lagrangian (17) viz. Eq.(17b) makes no reference to the Dirac structure at all which can couple to the spin degrees of h_{+v}.

More explicitly define the spin:

$$s^i = -s_i = -\gamma_5 \not{v} \gamma \cdot e_i \tag{9.29}$$

where

$$\begin{aligned} e_i \cdot v &= v_\mu e_i^\mu = 0, \\ e_{j\mu} e_k^\mu &= -\delta_{jk}. \end{aligned} \tag{9.30}$$

In the rest frame of h_v

$$\begin{pmatrix} \boldsymbol{v} = 0 \\ v_0 = 1 \end{pmatrix}, e_i^\mu = \delta_i^\mu.$$

Thus in the rest frame

$$\begin{aligned} s_i &= \gamma_5 \left(\gamma^0 v_0\right) \gamma_\mu \delta_i^\mu \\ &= v_0 \gamma^0 \gamma_5 \gamma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = -s^i \end{aligned} \tag{9.31}$$

i.e. we get the usual definition of the spin. We note that the Lagrangian $\mathcal{L}_{eff}$ given in Eq.(17a) is invariant under the infinitesimal transformation

$$\begin{aligned} \delta h_{+v} &= i\boldsymbol{\theta} \cdot \mathbf{s} h_{+v} \\ \delta \bar{h}_{+v} &= -i\boldsymbol{\theta} \cdot \mathbf{s} \bar{h}_{+v}. \end{aligned} \tag{9.32}$$

Now the Noether current is given by

$$\begin{aligned} \mathbf{J}^\mu &= -i \frac{\partial \mathcal{L}}{\partial \left(D_\mu h_{+v}\right)} \mathbf{s} h_{+v} \\ &= \bar{h}_{+v} v^\mu \mathbf{s} h_{+v}. \end{aligned} \tag{9.33}$$

Hence the spin operator is given by

$$\begin{aligned}\mathbf{S} &= \int \mathbf{J}^0(\mathbf{x},t)\, d^3x \\ &= v_0 \int \bar{h}_{+v} \mathbf{s} h_{+v} d^3x. \end{aligned} \tag{9.34}$$

We note that

$$[S_i, h_{+v}] = -s_i h_{+v}. \tag{9.35}$$

We conclude that the Lagrangian $\mathcal{L}_{eff}$ in Eq.(17a) is invariant under SU(2) of heavy quark spin symmetry. It means that pseudoscalar and vector meson states $|P(v)\rangle$ and $|V(v,\varepsilon)\rangle$, containing same heavy quark, with momentum $p_\mu^B = m_B v_\mu$ can be related to each other:

$$S_3(v)|P(v)\rangle = -i|V(v,\varepsilon)\rangle, \; S^3(v)|P(v)\rangle = i|V(v,\varepsilon)\rangle. \tag{9.36}$$

Thus their masses are degenerate in this limit. This degeneracy is lifted by the third term in the Lagrangian (17) giving e.g. for $B^*(1^-)$ and $B(0^-)$ mesons, the mass difference $(m_B^* - m_B)$ which scales like $1/m_b$.

The second symmetry of the Lagrangian (17) in the limit $m \to \infty$ arises when we introduce two distinct flavors h_1 $(e.g. b)$and h_2 $(e.g. c)$. Since the first term in Eq.(17) makes no reference to masses m_i $(i = 1, 2)$, and since mass is the only property which can distinguish between quarks of different flavors in QCD, the effective theory has a symmetry under which $h_1(v) \leftrightarrow h_2(v)$. It may be emphasized that this symmetry does not in any way depend on $m_1 = m_2$ but only on $m_i \gg \Lambda$, where Λ is a scale parameter such that $1/\Lambda$ determines the size of light degrees of freedom in the bound state and is a few hundred MeV; it may vary from process to process. Note also that the flavor symmetry holds between heavy quark fields of same velocity and not with the same momentum. This flavor symmetry together with the spin symmetry mentioned above gives rise to SU(4) symmetry for the system $\begin{pmatrix} h_1(v) \\ h_2(v) \end{pmatrix}$ and has been used to relate the matrix elements of flavor changing effective currents which mediate weak decays of mesons containing heavy quarks. Since we will be dealing with h_{+v} only, we will drop the subscript $+$ in what follows. We first note that

$$\begin{aligned} \left[S_i^v, \bar{h}_{v'}\Gamma h_v\right] &= -\bar{h}_{v'}\Gamma s_i h_v \\ \left[S_i^{v'}, \bar{h}_{v'}\Gamma h_v\right] &= \bar{h}_{v'} s_i \Gamma h_v. \end{aligned} \tag{9.37}$$

These equations follow from Eq.(35). Their use is as follows:

Consider the transition $B^-(v) \to D\left(v'\right)$. Then from Eq.(37), we obtain

$$\left\langle D^0\left(v'\right)\right| \left[S_i^{v'}, \bar{c}_{v'}\Gamma b_v\right] \left|B^-(v)\right\rangle = \left\langle D^0\left(v'\right)\right| \left[\bar{c}_{v'} s_i \Gamma b_v\right] \left|B^-(v)\right\rangle . \tag{9.38}$$

Now using Eq. (36), we get

$$i\left\langle D^{*0}\left(v'\right)\right| \bar{c}_{v'}\Gamma b_v \left|B^-(v)\right\rangle = \left\langle D^0\left(v'\right)\right| \left[\bar{c}_{v'} s_3 \Gamma b_v\right] \left|B^-(v)\right\rangle \tag{9.39}$$

where $s_3 = \gamma_5 \not{v}' \gamma^\rho \varepsilon_\rho^*$, ε_ρ is polarization vector for D^{*0}. For $\Gamma = \gamma^\mu(1-\gamma_5)$, we have

$$s_3\gamma^\mu(1-\gamma_5) = \gamma_5 \not{v}' \gamma^\rho \varepsilon_\rho^* \, \gamma^\mu (1-\gamma_5). \tag{9.40}$$

Now using

$$\gamma^\lambda\gamma^\rho\gamma^\mu = g^{\lambda\rho}\gamma^\mu - g^{\lambda\mu}\gamma^\rho + g^{\rho\mu}\gamma^\lambda + i\varepsilon^{\lambda\rho\mu\sigma}\gamma_5\gamma_\sigma \tag{9.41}$$

we get

$$\begin{aligned} & i\left\langle D^{*0}\left(v'\right)\right| \left[\bar{c}_{v'}\gamma^\mu(1-\gamma_5) b_v\right] \left|B^-(v)\right\rangle \\ = & -\left(v'^\mu\varepsilon^{*\sigma} - v'^\sigma\varepsilon^{*\mu} - i\varepsilon^{\mu\lambda\rho\sigma} v'_\lambda \varepsilon_\rho^*\right) \\ & \times \left\langle D^0\left(v'\right)\right| \left[\bar{c}_{v'}\gamma_\sigma(1-\gamma_5) b_v\right] \left|B^-(v)\right\rangle \\ = & -\left(v'^\mu\varepsilon^{*\sigma} - v'^\sigma\varepsilon^{*\mu} - i\varepsilon^{\mu\lambda\rho\sigma} v'_\lambda \varepsilon_\rho^*\right) \\ & \times \frac{1}{\sqrt{4v_0v_0'}}\left[\xi_0\left(v\cdot v'\right)\left(v+v'\right)_\sigma\right] \end{aligned} \tag{9.42}$$

where we have used the fact that since $\bar{c}\gamma^{\mu}b$ is a symmetry current, so that

$$\left\langle D^0\left(v'\right)\right|\left[\bar{c}_{v'}\gamma_{\mu}b_v\right]\left|B^-(v)\right\rangle = \frac{1}{\sqrt{4v_0v_0'}}\left[\xi_0\left(v\cdot v'\right)\left(v+v'\right)_{\mu}\right] \tag{9.43}$$

with

$$\xi_0\left(v^2\right) = \xi_0(1) = 1. \tag{9.44}$$

Another application of Eq. (37) is for the matrix elements of the current $\bar{q}\gamma^{\mu}(1-\gamma_5)b$ $(q = u, d, s)$ between the vacuum and B meson state viz.

$$\langle 0|\left[S_i^v, \bar{q}\Gamma b\right]|B_q\rangle = -\langle 0|\bar{q}\Gamma s_i b|B_q\rangle. \tag{9.45}$$

Hence we get

$$i\langle 0|\bar{q}\Gamma b\left|B_q^*\right\rangle = -\langle 0|\bar{q}\Gamma\left(\gamma_5\not{v}\gamma\cdot\varepsilon\right)b|B_q\rangle. \tag{9.46}$$

Thus for $\Gamma = \gamma^{\mu}(1-\gamma_5)$ on using Eqs. (40) and (41), we obtain

$$\begin{aligned} i\sqrt{\frac{1}{m_{B_q^*}}}\left(f_{B_q^*}\varepsilon_{\mu}\right) &= if_{B_q}m_{B_q}v^2\varepsilon_{\mu} = if_{B_q}\sqrt{m_{B_q}}\varepsilon_{\mu} \\ f_{B_q^*} &= \sqrt{m_{B_q B_q^*}}f_{B_q} = m_{B_q}f_{B_q} \end{aligned} \tag{9.47}$$

where we have used

$$\langle 0|\bar{q}\gamma^{\mu}\gamma_5 b|B_q(p)\rangle = \sqrt{\frac{1}{2p_0}}ip^{\mu}f_{B_q} = \sqrt{\frac{m_B}{2v_0}}if_{B_q}v^{\mu} \tag{9.48}$$

and

$$\langle 0|\bar{q}\gamma^{\mu}b|B_q(p)\rangle = \sqrt{\frac{1}{2p_0}}\varepsilon^{\mu}f_{B_q^*} = \sqrt{\frac{1}{2m_{B^*}v_0}}\varepsilon^{\mu}f_{B_q^*}. \tag{9.49}$$

The results obtained in Eqs. (42) and (47) will be used in Chap. 16, where we will discuss the semileptonic decays of B mesons involving the vector and axial form factors in the transitions $B\to D, D^*$.

Similar results can also be derived by the following procedure (called trace technique). For vector and pseudoscalar meson fields, we can write ($P = B$ or D)

$$H_a = \frac{1+\not{v}}{2}\left[i\gamma^\mu P^*_{a\mu} - \gamma_5 P_a\right], \tag{9.50}$$

where $a = 1, 2, 3$ for u, d and s quarks and $P^*_{a\mu}$ and P_a are annihilation operators normalized as

$$\left\langle 0\left|P^*_{a\mu}\right| Q\,\bar{q}_a\left(1^-\right)\right\rangle = \varepsilon_\mu \tag{9.51}$$

$$\left\langle 0\left|P_a\right| Q\,\bar{q}_a\left(0^-\right)\right\rangle = 1. \tag{9.52}$$

We define the adjoint field

$$\bar{H}_a = \gamma^0 H_a^\dagger \gamma^0 = \left[-\bar{P}^*_{a\mu} i\gamma^\mu + \bar{P}_a\gamma_5\right]\frac{1+\not{v}}{2}. \tag{9.53}$$

We note that

$$\begin{aligned} \not{v} H_a &= H_a \\ H_a \not{v} &= -H_a. \end{aligned} \tag{9.54}$$

The spin symmetry which relates P_a and $\mathrm{P}^*_{a\mu}$ is automatically incorporated in Eq. (50).

In case of spinor field $\overline{\Psi}(x)$, the wave function can be written

$$\left\langle 0\left|\overline{\Psi}(x)\right|p\right\rangle \sim e^{-ipx}u(p). \tag{9.55}$$

Then in view of Eqs. (50)-(52), we can write for mesons containing a heavy quark

$$\left\langle 0\left|H_a\right| P_a\left(v\right)\right\rangle = -\frac{1+\not{v}}{2}\gamma_5 \tag{9.56}$$

$$\left\langle 0\left|H_a\right| P^*_a\left(v,\varepsilon\right)\right\rangle = \frac{1+\not{v}}{2} i\gamma.\varepsilon. \tag{9.57}$$

We now apply the above considerations for the matrix elements $\langle 0^-(v')|J_\lambda|0^-(v)\rangle$ and $\langle 1^-(v',\varepsilon^*)|J_\lambda|0^-(v)\rangle$ where $J_\lambda = V_\lambda - A_\lambda$. Using Eqs. (50)-(53), (56) and (57), we get

$$\begin{aligned} &\left\langle 0^-(v')\left|J^\lambda\right|0^-(v)\right\rangle \\ &= \frac{1}{\sqrt{4v_o v'_o}}\left[-\xi(v\cdot v')\right] Tr\left[-\gamma^5\frac{1+\not{v}'}{2}\gamma^\lambda\frac{1+\not{v}}{2}\gamma^5\right] \\ &= \frac{1}{\sqrt{4v_o v'_o}}\xi(v\cdot v')(v+v')^\lambda \end{aligned} \tag{9.58}$$

$$\begin{aligned} &\left\langle 1^-\left(v',\varepsilon_\mu^*\right)\left|J^\lambda\right|0^-(v)\right\rangle \\ &= \frac{1}{\sqrt{4v_o v'_o}}\left[-\xi(v\cdot v')\right]\varepsilon_\mu^* \times \\ &\quad Tr\left[-i\gamma^\mu\frac{1+\not{v}'}{2}\gamma^\lambda(1-\gamma_5)\times\frac{1+\not{v}}{2}\gamma_5\right] \\ &= \frac{-i}{\sqrt{4v_o v'_o}}\xi(v\cdot v')\times\left[i\varepsilon^{\lambda\rho\mu\sigma}v'_\rho\varepsilon_\mu^* v_\sigma + (1+v'\cdot v)\varepsilon^{*\lambda} - v\cdot\varepsilon^* v'_\lambda\right]. \end{aligned} \tag{9.59}$$

Note that since only vector current contributes in Eq.(58), the form factor $\xi(-v\cdot v')$ is normalized as $\xi(1)=1$. Comparing Eqs. (58) and (59) with Eqs. (40) and (46), we see that trace technique gives exactly the same results for $B\to D(D^*)$ transitions as previously obtained.

9.3 Mass Spectroscopy for Hadrons with One Heavy Quark

We now discuss the effective Lagrangian (17a) in relation to hadronic masses containing one heavy quark. We introduce the following notation: Write a pseudoscalar (vector) heavy meson as $P_q(P_q^*)$, $P = B$ or D, $q = u$, d or s and we take $m_u = m_d$. The heavy baryon is written as B_Q, $Q = b$ or c, B=$\Lambda, \Xi, \Sigma, \Xi'$ or Ω ; the light quark content and spin configurations are contained in these symbols.

Define

$$\overline{a}\,(P) = \langle P|\, \overline{h}_v D^2 h_v\, |P\rangle \tag{9.60a}$$

$$\overline{d}\,(P) = Z_Q\,(\mu)\, \langle P|\, g_s \overline{h}_v \sigma^{\mu\nu} G_{\mu\nu} h_v\, |P\rangle\,. \tag{9.60b}$$

We take $\overline{a}(P)$ and $\overline{d}(P)$ independent of the light quark flavor q. We also assume

$$\overline{a}\,(B) = \overline{a}\,(D) = \overline{a} \tag{9.61a}$$

$$\frac{\overline{m}_B}{m_b} Z_b^{-1}\,(\mu)\, \overline{d}\,(B) = \frac{\overline{m}_D}{m_c} Z_c^{-1}\,(\mu)\, \overline{d}\,(D) = \overline{d}. \tag{9.61b}$$

These assumptions imply that interquark interactions are flavor independent. To understand the physical meaning of these terms we go to the rest frame of Q,$\boldsymbol{v} = 0$. In this frame

$$iv \cdot D = g_s G_{\circ} \tag{9.62}$$

$$-D^2 = (\boldsymbol{\nabla} - ig_s \mathbf{G}) \cdot (\boldsymbol{\nabla} - ig_s \mathbf{G}) = \mathbf{D}^2 \tag{9.63}$$

$$\sigma_{\mu\nu} G_{\mu\nu} = 2 \begin{pmatrix} 0 & -i\boldsymbol{\sigma} \cdot \mathbf{E}^c \\ i\boldsymbol{\sigma} \cdot \mathbf{E}^c & 0 \end{pmatrix} + 2 \begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{B}^c & 0 \\ 0 & \boldsymbol{\sigma} \cdot \mathbf{B}^c \end{pmatrix} \tag{9.64a}$$

where

$$G_{4j} = -E_j^c\ ,\ B_i^c = \frac{1}{2} \varepsilon_{ijk} G_{jk} \tag{9.64b}$$

are the colour electric and magnetic fields respectively. Thus $\overline{h}_v \mathbf{D}^2 h_v$ is gauge invariant extension of kinetic energy term representing the residual motion of heavy quark in a hadron and term $\overline{h}_v \boldsymbol{\sigma}_Q \cdot \mathbf{B}^c h_v$ describes the color magnetic coupling of the heavy-quark spin to the gluon field ($\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$)[we have exhibited the subscript Q with $\boldsymbol{\sigma}$].

Matching Eq.(62) with V_c in Eq.(1), we can write the effective Hamiltonian for a bound hadron containing one heavy quark Q as

$$H = H_q + H_Q \tag{9.65}$$

where H_Q takes care of the residual motion of the heavy quark and in view of Eqs.(63) and (64), it is obtained from $\mathcal{L}_{eff}$ in Eq. $(17a)$ to order $1/m_Q$ as follows:

$$H_Q = -\frac{\mathbf{D}^2}{2m_Q} - \frac{\boldsymbol{\sigma}_Q \cdot \mathbf{B}^c}{2m_Q} \left(Z_Q \left(\mu \right) g_s \right). \tag{9.66}$$

Note that the second term on the right hand side of Eq.(66) represents the interaction of color magnetic field $\mathbf{B}^c$ with color magnetic moment of the heavy quark $\boldsymbol{\mu}_Q = \frac{\sigma_Q}{2m_Q} \left(Z_Q \left(\mu \right) g_s \right)$. The Hamiltonian (65) gives the mass of the heavy meson P_q as

$$m_{P_q} = m_Q + \bar{\Lambda}_q + \frac{\bar{a}}{2m_Q} - \frac{\bar{d}\,(P)}{2m_Q} \tag{9.67}$$

where we have put

$$\bar{\Lambda}_q = \langle H_q \rangle + m_q. \tag{9.68}$$

To proceed further, we note that the term $\boldsymbol{\sigma}_Q \cdot \mathbf{B}^c$ gives rise to color magnetic moment interaction of the type $\boldsymbol{\mu}_q \cdot \boldsymbol{\mu}_Q$ which is proportional to $\boldsymbol{\sigma}_q \cdot \boldsymbol{\sigma}_Q$ and as is well known $\langle \boldsymbol{\sigma}_q \cdot \boldsymbol{\sigma}_Q \rangle$ is -3 or 1 for spin singlet $({}^1S_0)$ or spin triplet $({}^3S_1)$ states respectively. Hence relative to Eq.(67) for the heavy meson P_q^*, we obtain

$$m_{P_q^*} = m_Q + \overline{\Lambda}_q + \frac{\bar{a}}{2m_Q} + \frac{\bar{d}\,(P)}{6m_Q}. \tag{9.69}$$

We obtain from Eqs. (67) and (69) the mass relations

$$m_{B_d^*} - m_{B_d} = m_{B_s^*} - m_{B_s} = \frac{2}{3}\frac{\bar{d}\,(B)}{m_b} \tag{9.70a}$$

$$m_{D_d^*} - m_{D_d} = m_{D_s^*} - m_{D_s} = \frac{2}{3}\frac{\bar{d}\,(D)}{m_c} \tag{9.70b}$$

$$m_{B_s} - m_{B_d} = \left(\overline{\Lambda}_s - \overline{\Lambda}_d \right) + O\left(\frac{1}{m_b} \right) \tag{9.71a}$$

$$m_{D_s} - m_{D_d} = \left(\overline{\Lambda}_s - \overline{\Lambda}_d \right) + O\left(\frac{1}{m_c} \right) \tag{9.71b}$$

$$\overline{m}_B - \overline{m}_D = \left(m_b - m_c \right) \left[1 - \frac{\bar{a}}{2m_b m_c} \right] \tag{9.71c}$$

where

$$\overline{m}_P = \frac{3m_{P^*} + m_P}{4}. \tag{9.72}$$

Experimentally (in MeV)

$$\begin{aligned} m_{D_d^*} - m_{D_d} &= 142.12 \pm 0.07,\ m_{D_s^*} - m_{D_s} = 143.8 \pm 0.4 \\ m_{B_d^*} - m_{B_d} &= 45.7 \pm 0.4,\ m_{B_s^*} - m_{B_s} = 47.0 \pm 2.6 \\ m_{B_s} - m_{B_d} &= 90.2 \pm 2.2,\ m_{D_s} - m_{D_d} = 99.2 \pm 0.5 \\ \overline{m}_B &= 5.313 GeV,\ \overline{m}_D = 1.971 \text{ GeV}. \end{aligned}$$

From Eqs.(70) and (61b), we also obtain

$$\frac{\overline{m}_B\left(m_{B^*} - m_B\right)}{\overline{m}_D\left(m_{D^*} - m_D\right)} = \frac{\frac{\overline{m}_B \bar{d}(B)}{m_b}}{\frac{\overline{m}_D \bar{d}(D)}{m_c}} = \frac{Z_c(\mu)}{Z_b(\mu)} = \left[\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right]^{9/25}. \tag{9.73a}$$

Using the experimental values for the masses, we obtain

$$\frac{\alpha_s(m_b)}{\alpha_s(m_c)} \simeq 0.69 \tag{9.73b}$$

Eq. (74b) is compatible with $\alpha_s(m_b) \simeq 0.22$, $\alpha_s(m_c) \simeq 0.32$ [see Chap. 16] used in discussing the decays of D and B mesons. If we use $m_b = 4.9$, $m_c = 1.5$ (in GeV), then from Eq. (70), we get

$$\frac{\bar{d}(B)}{m_b} \simeq 0.07 \text{ GeV},\ \bar{d}(B) \simeq 0.34 \text{ GeV}^2 \tag{9.74a}$$

$$\frac{\bar{d}(D)}{m_c} \simeq 0.213 \text{ GeV},\ \bar{d}(D) \simeq 0.32 \text{ GeV}^2. \tag{9.74b}$$

It may be noted that we cannot use Eq.(72) to determine $\bar{a}$, in a meaningful way since it is very sensitive to quark masses m_b, m_c which are not well known. Finally we note from Eqs.(67), (69) and (73) that

$$\bar{m}_B - m_b = \bar{\Lambda}_d + O\left(\frac{1}{m_b}\right) \tag{9.75a}$$

which gives for $m_b \simeq 4.9$ GeV,

$$\bar{\Lambda}_d \simeq 0.41 \text{ GeV}. \tag{9.75b}$$

With the help of Eqs. (7.91) and (7.92), we can derive the mass formulae for heavy baryons using Eqs (65) and (66). We obtain for the baryons Λ_Q , Σ_Q and Σ_Q^*, the masses

$$m_{\Lambda_Q} = m_Q + \widetilde{\Lambda}_d - \frac{3}{4m_u m_d}\widetilde{\lambda} + \frac{\widetilde{a}}{2m_Q} \tag{9.76}$$

$$m_{\Sigma_Q} = m_Q + \widetilde{\Lambda}_d - \frac{3}{4m_u m_d}\widetilde{\lambda} + \frac{\widetilde{a}}{2m_Q} - \frac{\widetilde{d}(\Sigma_Q)}{m_Q} \tag{9.77a}$$

$$m_{\Sigma_Q^*} = m_Q + \widetilde{\Lambda}_d + \frac{3}{4m_u m_d}\widetilde{\lambda} + \frac{\widetilde{a}}{2m_Q} + \frac{\widetilde{d}(\Sigma_Q)}{2m_Q} \tag{9.77b}$$

where $\widetilde{a}$ and $\widetilde{d}$ are given in Eqs. (60) with P replaced by B_Q and

$$\widetilde{\Lambda}_d = \langle H_q \rangle + m_d + m_u \tag{9.78}$$

and parameter $\widetilde{\lambda}$ arises from the color magnetic moment interaction of light quarks in a heavy baryon[cf. Eq. (7.66)]:

$$H_{q_1 q_2} = -\frac{8\pi}{3}\left(-\frac{2}{3}\alpha_s\right)\frac{\mathbf{S}_{q_1}\cdot\mathbf{S}_{q_2}}{m_{q_1} m_{q_2}}\delta^3(\mathbf{r}). \tag{9.79}$$

From Eq.(80), one can write

$$\langle H_{q_1 q_2}\rangle = \tilde{\lambda}\left\langle \frac{\mathbf{S}_{q_1}\cdot\mathbf{S}_{q_2}}{m_{q_1} m_{q_2}}\right\rangle \tag{9.80a}$$

where

$$\tilde{\lambda} = \frac{16}{3}\left(\frac{2}{3}\alpha_s\right)\frac{\pi}{2}\left\langle \overline{\Psi}\left|\delta^3(\mathbf{r})\right|\overline{\Psi}\right\rangle. \tag{9.80b}$$

The masses for other baryons can be obtained from m_{Λ_Q} , m_{Σ_Q} and $m_{\Sigma_Q^*}$ by appropriate replacement in the light flavor index. From Eqs. (77) and (78), we get

$$m_{\Sigma_Q^*} - m_{\Sigma_Q} = m_{\Xi_Q^*} - m'_{\Xi_Q} = m_{\Omega_Q^*} - m_{\Omega_Q} = \frac{3\widetilde{d}(\Sigma_Q)}{2m_Q} \tag{9.81}$$

$$\frac{1}{2}\left(m_{\Sigma_Q}+m_{\Omega_Q}\right)-m'_{\Xi_Q}=\frac{1}{2}\left(m_{\Sigma^*_Q}+m_{\Omega^*_Q}\right)-m_{\Xi^*_Q}=0 \quad (9.82)$$

$$\left(m_{\Lambda_b}-m_{\Lambda_c}\right)=(m_b-m_c)\left[1-\frac{\tilde{a}}{2m_bm_c}\right]. \quad (9.83)$$

The present experimental value for the charmed baryons are given in Table (8.3) . From this table we find

$$m_{\Sigma^*_c}-m_{\Sigma_c}=m_{\Xi^*_c}-m'_{\Xi_c}\simeq 65 \text{ MeV}. \quad (9.84)$$

Thus the mass relation (82) is well satisfied. From Eq. (82) then we obtain

$$m_{\Omega^*_c}=m_{\Omega_c}+65 \text{ MeV}\simeq 2765 \text{ MeV}. \quad (9.85)$$

We also note from Eqs. (75) , (82) and (85) that

$$\frac{\tilde{d}(\Sigma_c)}{\overline{d}(D)}\simeq 0.2. \quad (9.86)$$

Further from Eqs.(72) and (84), we get

$$\frac{m_b-m_c}{2m_bm_c}\left[\overline{a}-\tilde{a}\right]=\left[\left(m_{\Lambda_b}-m_{\Lambda_c}\right)-\left(\overline{m}_B-\overline{m}_D\right)\right]. \quad (9.87)$$

Now $m_{\Lambda_b}-m_{\Lambda_c}\simeq 3.34$ GeV$\simeq \overline{m}_B-\overline{m}_D$, therefore $\tilde{a}=\overline{a}$. Using $\tilde{a}=\overline{a}$, we get from Eqs. (71), (72), (77), and (78):

$$\overline{m}_{\Lambda_b}-\overline{m}_B=\overline{m}_{\Lambda_c}-\overline{m}_D=\tilde{\Lambda}_d-\overline{\Lambda}_d \quad (9.88)$$

where

$$\overline{m}_{\Lambda_Q}=\frac{1}{4}\left[m_{\Lambda_Q}+m_{\Sigma_Q}+2m_{\Sigma^*_Q}\right]. \quad (9.89)$$

Thus using $\overline{m}_{\Lambda_c}=2.443$ GeV,we get

$$\tilde{\Lambda}_d-\overline{\Lambda}_d=0.47 \text{ GeV}. \quad (9.90)$$

Also from Eqs. (77) and (78)

$$m_{\Sigma_Q}-m_{\Lambda_Q}=\frac{\tilde{\lambda}}{m_um_d}-\frac{\overline{d}(\Sigma_Q)}{m_Q}. \quad (9.91)$$

Thus in particular for charmed baryons, we have

$$\begin{aligned} \frac{\tilde{\lambda}}{m_u m_d} &= \left(m_{\Sigma c} - m_{\Lambda_c}\right) + \frac{2}{3}\left(m^{\star}_{\Sigma c} - m_{\Sigma_c}\right) \\ &\simeq 168 + 43 \simeq 211 \text{ MeV} \end{aligned} \tag{9.92}$$

which is not very much different from $\frac{\bar{d}(D)}{m_c} \simeq 213$ MeV (see Eq. 75).

The success of the mass formulae Eqs. (70) and (82) cannot be taken as verification of HQET, since similar formulae also hold for light hadrons (see Eqs. 7.80, 7.81, 7.82, 7.98, 7.99). If we follow the approach of Chap. 7 for baryons and put

$$\tilde{d} = \frac{\tilde{\lambda}}{m_q} \tag{9.93}$$

then Eq. (83) remains unchanged. Using $m_{\Sigma c} - m_{\Lambda_c} = 168$ MeV, we obtain

$$\begin{aligned} m_{\Sigma^* c} - m_{\Sigma_c} &\simeq 67 \text{ MeV} \\ m_{\Sigma_b^*} - m_{\Sigma_b} &\simeq 20 \text{ MeV} \qquad (9.94) \\ m_{\Sigma_b} - m_{\Lambda_b} &\simeq 199\text{MeV} \qquad (9.95) \end{aligned}$$

i.e. the results similar to the ones obtained in HQET.

9.4 The P-wave Heavy Mesons: Mass Spectroscopy

So far we have discussed only $S-$wave heavy mesons. We now discuss $P-$wave mesons for which experimental evidence is available. Since the spin of heavy quark is decoupled, it is natural to couple orbital angular momentum $\mathbf{L}$ with $\mathbf{S}_q$ in the heavy quark limit. Thus we define

$$\mathbf{j} = \mathbf{L} + \mathbf{S}_q. \tag{9.96}$$

Now the total angular momentum $\mathbf{J}$ of the bound $q\bar{Q}$ system is given by

$$\mathbf{J} = \mathbf{j} + \mathbf{S}_Q \tag{9.97}$$

we note that

$$2\langle \mathbf{L}\cdot\mathbf{S}_q\rangle = \left[j(j+1)-l(l+1)-\frac{3}{4}\right] \tag{9.98}$$

$$2\left\langle \mathbf{j}\cdot\mathbf{S}_Q\right\rangle = \left[J(J+1)-j(j+1)-\frac{3}{4}\right]. \tag{9.99}$$

Hence for $l=0$ states, $\mathbf{J}=\mathbf{S}_q+\mathbf{S}_Q$, and we have $J=0$ or 1. The corresponding ${}^1S_0$ and ${}^3S_1$ states (D, D^*) and (B, B^*) have already been considered. We will suppress the subscript q [i.e. light flavor index] and concentrate on D_J mesons (for B_J, replace D by B) for which $l=1, j=3/2$ or $1/2$. Thus

$$J = j+1/2,\ j-1/2$$

i.e. we have the states

$$\begin{array}{lll} j = 3/2 & J=2,1 & D_2^*,\ D_1 \\ j = 1/2 & J=1,0 & D_1^*,\ D_0 \end{array}$$

It is useful to write down the angular momentum part of the wave functions for the four P–states. According to the angular momentum scheme outlined above, P state can be labeled as $|JMjs_Q\rangle$. We can write these states for D_J mesons:

$$\begin{aligned} |D_2^*, M=\pm 1\rangle &= \frac{1}{\sqrt{2}}\left[Y_{10}\chi_+^{+1}+Y_{11}\chi_+^{0}\right] \\ &\quad \frac{1}{\sqrt{2}}\left[Y_{10}\chi_+^{-1}+Y_{1-1}\chi_+^{0}\right] \\ |D_2^*, M=0\rangle &= \frac{1}{\sqrt{6}}\left[Y_{1-1}\chi_+^{+1}+2Y_{10}\chi_+^{0}+Y_{11}\chi_+^{-1}\right] \end{aligned} \tag{9.100a}$$

$$\begin{aligned} |D_1, M=\pm 1\rangle &= \frac{1}{\sqrt{6}}\left[-Y_{10}\chi_+^{+1}+Y_{11}\left(\chi_+^{0}+2\chi_-^{0}\right)\right] \\ &\quad \frac{1}{\sqrt{6}}\left[Y_{10}\chi_+^{-1}+Y_{1-1}\left(-\chi_+^{0}+2\chi_-^{0}\right)\right] \end{aligned}$$

$$|D_1, M=0\rangle = \frac{1}{\sqrt{6}}\left[-Y_{1-1}\chi_+^{+1} + 2Y_{10}\chi_-^0 + Y_{11}\chi_+^{-1}\right] \tag{9.100b}$$

$$\begin{aligned}|D_1^*, M=\pm 1\rangle &= \frac{1}{\sqrt{3}}\left[-Y_{10}\chi_+^{+1} + Y_{11}\left(\chi_+^0 - \chi_-^0\right)\right] \\ & \quad \frac{1}{\sqrt{3}}\left[Y_{10}\chi_+^{-1} + Y_{1-1}\left(-\chi_+^0 - \chi_-^0\right)\right] \\ |D_1^*, M=0\rangle &= \frac{1}{\sqrt{3}}\left[-Y_{1-1}\chi_+^{+1} - Y_{10}\chi_-^0 + Y_{11}\chi_+^{-1}\right]\end{aligned} \tag{9.101a}$$

$$|D_0, M=0\rangle = \frac{1}{\sqrt{3}}\left[Y_{1-1}\chi_+^{+1} - Y_{10}\chi_+^0 + Y_{11}\chi_+^{-1}\right]. \tag{9.101b}$$

We first discuss the masses of P–wave mesons. The four P-states are degenerate. The degeneracy between $j = 3/2$ and $j = 1/2$ states is removed by the spin-orbit coupling term $\mathbf{S}_q \cdot \mathbf{L}$ in the Hamiltonian H_q given in Eq.(1). Thus we have using Eq. (99):

$$m_{j=3/2} - m_{j=1/2} = \left(\frac{1}{2} + 1\right)\bar{\lambda}_1 = \frac{3}{2}\bar{\lambda}_1 \tag{9.102a}$$

where

$$m_{j=3/2} = \frac{5m_{D_2^*} + 3m_{D_1}}{8} \tag{9.102b}$$

$$m_{j=1/2} = \frac{3m_{D_1^*} + m_{D_0}}{4} \tag{9.102c}$$

$$\bar{\lambda}_1 = \frac{1}{4m_q^2}\left\langle \frac{1}{r}\frac{dV_c}{dr}\right\rangle \tag{9.102d}$$

(subscript 1 on $\bar{\lambda}$ refers to $l = 1$ state). The degeneracy between the doublet D_2^* and D_1 and the doublet D_1^* and D_0 is removed by the term $\boldsymbol{\sigma}_Q \cdot \mathbf{B}^c$ in the Hamiltonian (65). For P–wave this term induces the color magnetic moment interaction of the type

$$S_{12} = \left[12\left(\mathbf{S}_q \cdot \mathbf{n}\right)\left(\mathbf{S}_Q \cdot \mathbf{n}\right) - 4\mathbf{S}_q \cdot \mathbf{S}_Q\right] \tag{9.103}$$

where $\mathbf{n}$ is a unit vector $\frac{\mathbf{r}}{r}$. Then using the angular wave functions for the states D_2^*, D_1, D_1^*, and D_0 given in Eqs. (101) and (102), we have

$$\langle S_{12}\rangle_{D_2^*} = -2/5, \qquad \langle S_{12}\rangle_{D_1} = \frac{2}{3} \tag{9.104a}$$

$$\langle S_{12}\rangle_{D_1^*} = \frac{4}{3}, \qquad \langle S_{12}\rangle_{D_0} = -4. \tag{9.104b}$$

Hence we can write the masses for these states. We write explicitly the mass formulae for D_2^* and D_1 mesons.

$$m_{D_{2q}^*} = m_c + \bar{\Lambda}_{1q} + \frac{1}{2}\bar{\lambda}_{1q} + \frac{\bar{a}_1}{2m_c} - \frac{3}{5}\frac{\bar{d}_1(D)}{2m_c} \tag{9.105}$$

$$m_{D_{1q}} = m_c + \bar{\Lambda}_{1q} + \frac{1}{2}\bar{\lambda}_{1q} + \frac{\bar{a}_1}{2m_c} + \frac{\bar{d}_1(D)}{2m_c} \tag{9.106}$$

where the parameters $\bar{a}_1$ and $\bar{d}_1$ refer to P-state similar to $\bar{a}$ and $\bar{d}$ for S-state. For $m_{D_{1q}^*}$ and $m_{D_{0q}}$, replace $\frac{1}{2}\bar{\lambda}_{1q}$ by $-\bar{\lambda}_{1q}$, $\bar{d}_1(D)$ by $2\bar{d}_1(D)$ and $-6\bar{d}_1(D)$ respectively in Eq. (107). From Eqs. (106) and (107), we obtain

$$m_{D_2^*} - m_{D_1} = m_{D_{s2}^*} - m_{D_{s1}} = -\frac{8}{5}\frac{\bar{d}_1(D)}{2m_c} \tag{9.107}$$

$$\begin{aligned} m_{D_1^*} - m_{D_0} &= m_{D_{1s}^*} - m_{D_{0s}} = 8\frac{\bar{d}_1(D)}{2m_c} \\ &= -5\left(m_{D_2^*} - m_{D_1}\right). \end{aligned} \tag{9.108}$$

Needless to say that for b-flavor $P-$states, replace D by B and m_c by m_b. Using the experimental values for the masses, we find

$m_{D_2^*} - m_{D_1} = 40$ MeV $= m_{D_{2s}^*} - m_{D_{s1}}$. Thus relation (108) is well satisfied. From Eqs. (108),(70), and (109) we obtain

$$\begin{aligned}
\frac{\bar{d}_1(D)}{2m_c} &= -25 \text{ MeV} & (9.109)\\
\frac{\bar{d}_1(D)}{\bar{d}(D)} &= -\frac{5}{12}\frac{m_{D_2^*} - m_{D_1}}{m_{D^*} - m_D} \simeq -0.12 & (9.110)\\
m_{D_1^*} - m_{D_0} &= -200 \text{ MeV}. & (9.111)
\end{aligned}$$

Also from Eq. (106), we get

$$\left(\bar{\Lambda}_{1s} - \bar{\Lambda}_{1d}\right) + \frac{1}{2}\left(\bar{\lambda}_{1s} - \bar{\lambda}_{1d}\right) = m_{D_{2s}^*} - m_{D_{2d}^*} = 113 \text{ MeV}. \quad (9.112)$$

On the other hand, for the $S-$states $m_{Ds} - m_{D_d} = (99.2 \pm 0.50$ MeV) implying that $\bar{\lambda}_{1s} = \bar{\lambda}_{1d}$ i.e. independent of light flavor.

From Eq. (112), we conclude that if we use the same $\bar{d}_1$ for $j = 3/2$ and $j = 1/2$ states then $m_{D_1^*} < m_{D_0}$. If as expected $m_{D_1^*} > m_{D_0}$, then $\bar{d}_1\,(j = 1/2)$ must have opposite sign to that of $\bar{d}_1\,(j = 3/2)$; in that case there is no reason to believe that they have the same magnitude also. It is hard to imagine that the interaction which removes the degeneracy between $j = 3/2$ multiplet and $j = 1/2$ multiplets is strongly dependent on their j values. However, when we take into account the relative motion of heavy quark it is not reasonable to neglect the spin orbit coupling $\frac{\mathbf{S}\cdot\mathbf{L}}{2m_q m_Q}$. We now take this term into account. Then from the wave functions given in Eqs. (101) and (102) we find

$$\begin{aligned}
\langle \mathbf{S}\cdot\mathbf{L}\rangle_{D_2^*} &= 1, & \langle \mathbf{S}\cdot\mathbf{L}\rangle_{D_1} &= -\frac{1}{3} & (9.113)\\
\langle \mathbf{S}\cdot\mathbf{L}\rangle_{D_1^*} &= -\frac{2}{3}, & \langle \mathbf{S}\cdot\mathbf{L}\rangle_{D_0} &= -2. & (9.114)
\end{aligned}$$

Thus the contribution of this term can be written respectively for D_2^*, D_1, D_1^*, and D_0 as:

$$\frac{3\bar{c}_1}{4m_c}, \frac{-\bar{c}_1}{4m_c}, -\frac{\bar{c}_1}{2m_c}, \frac{-3\bar{c}_1}{2m_c}. \quad (9.115)$$

Hence we get

$$m_{D_2^*} - m_{D_1} = \frac{8}{5}\frac{\bar{d}_1}{2m_c}\left[-1 + \frac{5}{4}\frac{\bar{c}_1}{\bar{d}_1}\right] \tag{9.116}$$

$$m_{D_1^*} - m_{D_0} = 8\frac{\bar{d}_1}{2m_c}\left[1 + \frac{1}{4}\frac{\bar{c}_1}{\bar{d}_1}\right]. \tag{9.117}$$

If we put $\bar{c}_1 = 4\bar{d}_1$, i.e. the same strength for the tensor interaction and the spin orbit interaction, we obtain

$$\begin{aligned} m_{D_2^*} - m_{D_1} &= \frac{32}{5}\frac{\bar{d}_1(D)}{2m_c} \\ m_{D_1^*} - m_{D_0} &= 16\frac{\bar{d}_1(D)}{2m_c}. \end{aligned} \tag{9.118}$$

Hence

$$\frac{m_{D_1^*} - m_{D_0}}{m_{D_2^*} - m_{D_1}} = \frac{5}{2}. \tag{9.119}$$

Therefore

$$m_{D_1^*} - m_{D_0} = 100 \text{ MeV} \tag{9.120}$$

and

$$\frac{\bar{d}_1(D)}{\bar{d}(D)} = \frac{2 \times 5}{3 \times 32}\left(\frac{m_{D_2^*} - m_{D_1}}{m_{D^*} - m_D}\right) \simeq 0.03. \tag{9.121}$$

There is no experimental evidence for D_1^*, and D_0 . Since they are broad resonances, it is not easy to test Eq. (120) or (121). However if the spin orbit interaction for the relative motion is not taken into account and tensor interaction is not strongly dependent on the j -values, then as we have seen $m_{D_0} - m_{D_1^*} = 200$ MeV; hence D_0 can decay into D_1^* by emission of the pion and this decay is a P-wave decay and can be distinguished from the S−wave decay $D_0 \to D\pi$. This is an interesting possibility which can be tested experimentally.

9.5 Decays of P-wave Mesons

We now discuss the strong decays of P-wave mesons. Parity and angular momentum conservation restricts these decays to the following modes: $D_2^* \to (D\pi)_{l=2}$, $D_2^* \to (D^*\pi)_{l=2}$, D_1, $D_1^* \to (D^*\pi)_{l=0,2}$, and $D_0 \to (D\pi)_{l=0}$. Note that D_1, $D_1^* \to D\pi$ is forbidden due to parity conservation.

It is convenient to express the decay width in terms of the helicity amplitudes (see Eq. 4.41)

$$\Gamma^J = \frac{|\mathbf{p}_\pi|}{8\pi s} \overline{\sum_\lambda} \left|F_\lambda^J(s)\right|^2. \tag{9.122}$$

In the rest frame of the decaying particle the helicity amplitudes which contribute are $F_0^2, F_\pm^2, F_0^1, F_{\pm 1}^1$ and F_0^0. In the heavy quark limit the helicity amplitudes are related as follows:

$j = 3/2$ multiplet:

$$F_0^1 = -2F_\pm^1 = F_0^2 = \frac{2}{\sqrt{3}} F_{\pm 1}^2 \tag{9.123}$$

$j = 1/2$ multiplet:

$$\acute{F}_0^1 = -\acute{F}_{\pm 1}^1 = \acute{F}_0^0. \tag{9.124}$$

The simplest way to see this is as follows. The emission of pion by D_J would not affect the velocity of heavy quark. Thus it is the operator $\mathbf{S}_q \cdot \mathbf{n}$ which is relevant for these decays. If we select the direction of quantization along z-axis, (i.e. L_z is taken along z-axis) then for the helicity amplitudes, the operator $S_{3q}\sqrt{\frac{4\pi}{3}}\, Y_{10}$ contributes. Then using the wave functions in Eqs. (101) and (102), it is easy to derive Eqs. (124) and (125) by considering the matrix elements of the type

$$F_\lambda^J = f \left\langle D^*(D), \lambda \left| S_{3q} Y_{10} \right| D_J, \lambda \right\rangle \tag{9.125}$$

where f is the reduced amplitude. Since hadronic decays of D_2^* are pure $D-$wave, it follows that $D_1 \to D^*\pi$ is also $D-$wave. For these decays, therefore $F_{\lambda}^{J}(s) \sim |\mathbf{p}_\pi|^2$. Similarly since the decay of D_0 is pure S-wave, it follows that the decay of D_1^* is also pure S-wave. The above restrictions are consequence of relations (124) and (125) which hold in the heavy quark spin symmetry limit. Hence for the decays $D_2^* \to D\pi$, $D_2^* \to D^*\pi$, and $D_1 \to D^*\pi$, we get

$$\frac{\Gamma\left(D_2^* \to D\pi\right)}{\Gamma\left(D_2^* \to D^*\pi\right)} = \frac{2}{3}\frac{|\mathbf{p}_\pi|_{D\pi}^5}{|\mathbf{p}_\pi|_{D^*\pi}^5} \approx 2.5\,(2.4 \pm 0.7) \tag{9.126}$$

$$\begin{aligned} \frac{\Gamma\left(D_1 \to D^*\pi\right)}{\Gamma\left(D_2^* \to D^*\pi\right)} &= \frac{5}{3}\frac{|\mathbf{p}_\pi|_{D_1 D^*\pi}^5}{|\mathbf{p}_\pi|_{D^*\pi}^5} \approx 1.1 &(9.127\text{a})\\ \frac{\Gamma\left(D_1 \to D^*\pi\right)}{\Gamma\left(D_2^* \to D\pi\right)} &= \frac{5}{3}\text{ (phase space)} \approx 0.44 &(9.127\text{b}) \end{aligned}$$

where we have used from the experimental data, $|\mathbf{p}_\pi|_{D\pi} = 503$ MeV, $|\mathbf{p}_\pi|_{D^*\pi} = 387$ MeV, $|\mathbf{p}_\pi|_{D_1 D^*\pi} = 355$ MeV. In Eq. (127), the number in parenthesis is experimental value. Thus we see that this prediction of heavy quark spin symmetry is well satisfied. Experimnetally

$$\begin{aligned} \Gamma_{D_2^*} &\equiv \Gamma\left(D_2^0 \to D^*\pi^- + D^*\pi^0 + D\pi^- + D^0\pi^0\right) &(9.128)\\ &= 23 \pm 5 \text{ MeV}. \end{aligned}$$

From Eq. (128) we get

$$\frac{\Gamma_{D_1}}{\Gamma_{D_2^*}} \simeq 0.31 \text{ MeV} \tag{9.129}$$

which gives

$$\Gamma_{D_1} \simeq 7.1 \pm 1.5 \text{ MeV } \left(18.9^{+4.6}_{-3.5} \text{ MeV}\right). \tag{9.130}$$

This is in complete disagreement with the experimental value. This shows that the decay $D_1 \to D^*\pi$ is not pure $D-$wave; there may be a component of $S-$wave. The $S-$wave widths are usually large, a small component of $S-$wave may be possible due to symmetry breaking, since heavy quark spin symmetry is not exact. This may be tested for B_2^* and B_1 decays where the symmetry breaking effects are expected to be small.

The decays $D_1^* \to D^*\pi$ and $D_0 \to D\pi$ are $S-$wave decays; thus the decay widths are expected to be large i.e. in the range of few hundreds of MeV. No experimental data are available even on the masses of D_1^* and D_0.

To sum up: from the analysis of mass spectrum one cannot conclude that heavy quark spin symmetry is well established; additional experimental data on the masses of heavy hadrons are needed. Except for one prediction on the decay widths viz.

$$\frac{\Gamma\left(D_2^* \to D\pi\right)}{\Gamma\left(D_2^* \to D^*\pi\right)} \approx 2.5$$

which agrees with the experimental values, the other predictions on the decay widths of heavy hadrons have to wait fot their verification till the experimental data are available.

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The references to the original literature can be found in the above reviews.

Chapter 10
NEUTRINO

10.1 Introduction

Experimental puzzles in the past have led to some important discoveries in Physics. Neutrino, which has spin 1/2, was invented in 1930 by Pauli as the explanation of such a puzzle, namely the conservation of angular momentum and energy in β-decay

$$n \to p + e^-,$$

require such a particle, so that

$$n \to p + e^- + \bar{\nu}_e. \tag{10.1}$$

Its direct observation was made much later. The electron type anti-neutrinos are thus produced by the decay of pile neutrons in a fission reactor. These can be captured in hydrogen giving the reaction:

$$\bar{\nu}_e + p \to e^+ + n, \tag{10.2}$$

whose cross-section was measured by Reines and Cowan

$$\sigma_{exp} = (11 \pm 2.5) \times 10^{-44}\text{cm}^2 \tag{10.3}$$

to be compared with the theoretical value

$$\sigma_{th} = (11 \pm 1.6) \times 10^{-44}\text{cm}^2. \tag{10.4}$$

Note the extreme smallness of the cross-section. It is a reflection of the fact that neutrino has only weak interaction.

10.2 Mass

The question of neutrino mass is one of long standing. In the context of the standard model of unified electro-weak interactions (Chap. 13), there is no understanding of the origin of masses of elementary fermions. In this category the question of neutrino mass also arises. It has an added importance for the following reasons:

(a) Among the elementary fermions, only the neutrinos occur asymmetrically in one (LH, left handed) helicity state (see Sec. 11.1.1) i.e. appear to be spinning clockwise as viewed by an observer. This is still an unsolved puzzle. This fact together with lepton number conservation imply that $m_\nu = 0$ (see below). However, there is no local gauge symmetry to guarantee the masslessness of neutrino and lepton number conservation in contrast to the photon where both the masslessness of photon and charge conservation are consequences of local gauge invariance of Maxwell's equations. One may thus expect a finite mass for neutrino. But the intriguing question is why $m(\nu_e) \ll m(e)$.

(b) The interesting phenomena of neutrino oscillations is possible if one or more of neutrinos have non-vanishing mass.

(c) Non-vanishing neutrino mass has important implications in Astrophysics. It is a candidate for hot dark matter. It affects history, structure and fate of the universe as we shall see in Chap. 18.

Experimentally the question of neutrino mass is still open. This is because

(i) m_ν is small and a smaller quantity is more difficult to measure with high precision than a bigger quantity.

(ii) Neutrino has only weak interaction with matter which implies in practice that no direct measurement of m_ν is possible.

10.2.1 Constraints on neutrino mass

a) Direct Limits

We first confine to $\bar{\nu}_e \cdot \bar{\nu}_e$ comes out in $\beta-$decay of Tritium

$$3_H \to 3_{He} + e^- + \bar{\nu}_e. \tag{10.5}$$

Electrons from this decay has a very low end-point energy (18.6 keV). As such this process is ideal to look for a possible finite mass of neutrino. If $m_\nu = 0$,

$$\left[\frac{d\Gamma}{p_e^2 dp_e}\right]^{1/2} \propto (E_{max} - E_e). \tag{10.6}$$

Kurie plot is thus a straight line. If $m_\nu \neq 0$,

$$\left(\frac{d\Gamma}{p_e^2 dp_e}\right)^{1/2} \propto (E_{max} - E_e)^{1/2}\left[(E_{max} - E_e)^2 - m_\nu^2\right]^{1/4}. \tag{10.7}$$

This equation also illustrates why the end point energy range is important for determining $m_\nu = 0$. This gives a distortion at the extreme end of the Kurie plot (see Fig. 2.5). Thus one has to look for such a distortion, but note that the deviation is in fact quite small and the experiment is thus quite difficult. An added complication is the presence of final state ionic and/or molecular effects that are not well understood. Anyway, the present limit on ν_e mass is

$$m_{\nu_e} < 5.1 \text{ eV} : m_{\nu_e} \ll m_e.$$

Direct limits on the other two types of neutrinos are

$$\pi \to \mu\nu_\mu : m_{\nu_\mu} < 170 \text{ keV} : m_{\nu_\mu} \ll m_\mu$$
$$\tau \to 5\pi\nu_\tau : m_{\nu_\tau} < 18.2 \text{ MeV} : m_{\nu_\tau} \ll m_\tau$$

Thus there is no definitive evidence that ν's have mass, but still the question of neutrino mass is interesting in particle and astrophysical theories as remarked earlier.

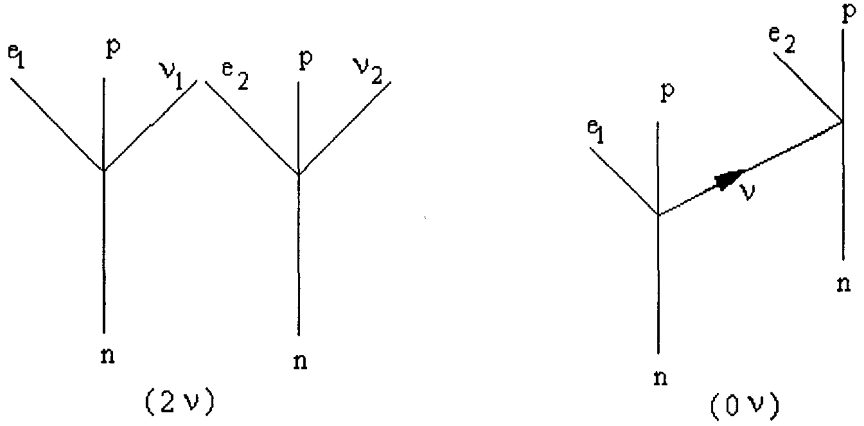

Figure 1 Basic reactions in double β-decay.

b) Double β-Decay

The double β-decay is another way to look for a finite mass of neutrino. Two kinds of double β-decay can be considered:

$$\begin{array}{lll} (2\nu) & (A,\, Z) \rightarrow (A,\, Z+2) + 2e^- + 2\bar{\nu}_e & \\ (0\nu) & \qquad\quad\; \rightarrow (A,\, Z+2) + 2e^- . & \end{array} \tag{10.8}$$

Usually the neutrinos are assumed to be Dirac particles, that is, neutrino ν and its anti-neutrino $\bar{\nu}$ are distinct. There is another picture of neutrinos, called Majorana in which ν and $\bar{\nu}$ are identical. This implies

$$\begin{aligned} n &\rightarrow p + e^- + \bar{\nu}_L \equiv p + e^- + \nu_L \\ \nu_L + n &\rightarrow p + e^- , \end{aligned} \tag{10.9}$$

so that $(2n) \rightarrow (2p) + 2e^-$ as shown in Fig. 1. The important physics issues in (0ν) double β-decay are:

(i) Lepton number must not be conserved, which is possible if neutrinos are Majorana particles: $\nu \equiv \bar{\nu}$

(ii) Helicity of the neutrino cannot be exactly -1, this can be satisfied if $m_\nu \neq 0$.

Thus $(0\nu)\beta\beta$–decay is especially interesting in determining m_ν as half life

$$T_{1/2} \propto Q^{-5} < m_\nu >^{-2}, \tag{10.10}$$

where Q is the Q-value of the reaction involved.

There is now distinct evidence of $(2\nu)\,\beta\beta$–decay:

$$\begin{aligned} {}^{82}Se &\rightarrow {}^{82}Kr \\ T_{1/2} &= \left(1.1^{+0.8}_{-0.3}\right) \times 10^{20}\,\text{Yr}. \end{aligned} \tag{10.11}$$

Incidently this is the rarest natural decay process ever observed directly in a laboratory. This would help to provide a standard by which to test the double β-decay matrix elements of nuclear theory. From the limit on half-life on $(0\nu)\beta\,\beta$ decay process ${}^{76}Ge \rightarrow {}^{76}Se + 2e^-$,

$$T_{1/2} > 1.1 \times 10^{25}\,\text{Yr}. \tag{10.12}$$

This implies

$$\langle m_{\nu_e} \rangle \leq 0.68 \ \text{eV}, \tag{10.13}$$

which depends on the calculated nuclear matrix elements. Actually, if there is a mixing among neutrinos (see Sec. 4 below), then $\langle m_\nu \rangle = \sum_i \lambda_i |U_{ei}|^2 m_{\nu_i}$, where λ_i is a possible sign since Majorana neutrinos are CP eigenstates and U_{ei} arises due to two vertices.

c) Astrophysical Constraints

As will be shown in Chap. 18, the mass density of all fairly light ($m_\nu < 1$ MeV) stable neutrinos is

$$\begin{aligned} \rho_\nu^0 &= \sum_i \left(\frac{n_\nu^0}{n_\gamma^0}\right) n_\gamma^0 m_{\nu_i} \\ &= \frac{3}{11} n_\gamma^0 \sum_i m_{\nu_i} \\ &= \sum_i 2 m_{\nu_i}(eV) \times 10^{-31}\,\text{gm/cm}^3, \end{aligned} \tag{10.14}$$

where $n_\gamma^0 = 400$ cm^{-3} is the present photon number density. Now the average mass density of the universe is

$$\rho_0 = \Omega_0 \rho_{c0}, \tag{10.15}$$

where $\Omega_0 \leq 1$ and ρ_{c0} is the critical density

$$\rho_{c0} = \frac{3H_0^2}{8\pi G_N}. \tag{10.16}$$

Here H_0 is the Hubble parameter, $H_0 = 3 \times 10^{-18} h_0$ sec^{-1}, with $h_o \simeq 0.5 - 0.8$ and G_N is Newton's gravitational constant. Thus

$$\rho_{c0} = 2 \times 10^{-29} h_0^2 \,\text{gm/cm}^3.$$

The sum of the masses of all stable meutrinos is thus constrained by

$$\rho_\nu^0 \leq \rho_0$$

i.e.

$$\sum_i 2m_{\nu_i}(eV) \times 10^{-31}\,\text{gm/cm}^3 \;\;\leq\;\; 2 \times 10^{-29}\Omega_0 h_0^2\ \text{gm/cm}^3 \tag{10.17}$$

i.e.

$$\sum_i m_{\nu_i} \leq 100\Omega_0 h_0^2 \ \text{eV}. \tag{10.18}$$

The observed age of the universe yields $\Omega_0 h_0^2 \leq 0.4$ so that

$$\sum_i m_{\nu_i} \leq 40 \ \text{eV}. \tag{10.19}$$

We may also mention here that the big-bang nucleosynthesis puts constraints on the mass of any meta stable (m.s) neutrinos which are

$$m_{\nu_{m.s}}\ (\text{Dirac}) \ > 32 \ \ \text{MeV or} \ < 0.95\ \text{MeV}$$
$$m_{\nu_{m.s}}\ (\text{Majorana}) \ > 25 \ \ \text{MeV or} \ < 0.37\ \text{MeV}$$

Both the lower limits are in conflict with $m_{\nu_\tau} < 18.2$ MeV, mentioned earlier, implying that m_{ν_τ} must actually be below 1 MeV.

10.2.2 *Dirac and Majorana masses*

It is a general feature of weak interactions that only left handed neutrino ν_L takes part in it (see Chap. 11). Let us write a Dirac spinor ψ as

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{10.20}$$

In a representation in which γ_5 is diagonal,

$$\psi_L = \frac{1-\gamma_5}{2}\psi = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \psi_R = \frac{1+\gamma_5}{2}\psi = \begin{pmatrix} 0 \\ \eta \end{pmatrix}. \tag{10.21}$$

The Dirac equation for the two component spinors ξ and η can be written as

$$\begin{aligned} \left(i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} - i\frac{\partial}{\partial t}\right)\xi &= -m_D\eta \\ \left(-i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} - i\frac{\partial}{\partial t}\right)\eta &= -m_D\xi. \end{aligned} \tag{10.22a}$$

These equations can also be written in the form

$$\begin{aligned} i\bar{\sigma}^\mu\partial_\mu\xi - m_D\eta &= 0 \\ i\sigma^\mu\partial_\mu\eta - m_D\xi &= 0 \end{aligned} \tag{10.22b}$$

where

$$\sigma^\mu = (1, \boldsymbol{\sigma}), \ \bar{\sigma}^\mu = (1, -\boldsymbol{\sigma}). \tag{10.22c}$$

Under charge conjugation C (particle $\to$ antiparticle) $\psi \to \psi^c = -i\gamma^2\psi^*$ [see Appendix A], so that

$$\begin{aligned} \xi \to \xi^c &= -i\sigma^2\eta^* \\ \eta \to \eta^c &= i\sigma^2\xi^*. \end{aligned} \tag{10.23}$$

For massless neutrino, ξ and η decouple and we have from Eq. (22)

$$\left(i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} - i\frac{\partial}{\partial t}\right)\xi = 0 \tag{10.24a}$$

$$\left(-i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}-i\frac{\partial}{\partial t}\right)\eta \;=\; 0. \tag{10.24b}$$

The plane wave solution of Eq. (24a) is given by

$$\xi(x)=w(\mathbf{p})e^{-ip\cdot x}=w(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}. \tag{10.25}$$

Then from Eq. (24a), we get

$$[\boldsymbol{\sigma}\cdot\mathbf{p}+E]\,w(\mathbf{p})=0 \tag{10.26}$$

with $E^2=\mathbf{p}^2$. Let us denote the positive energy spinor by $u(\mathbf{p})$ and negative energy ($E=-|\mathbf{p}|$) spinor by $v(\mathbf{p})$. Thus we get

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}u(\mathbf{p}) \;=\; -u(\mathbf{p}) \tag{10.27}$$

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}v(\mathbf{p}) \;=\; v(\mathbf{p}), \tag{10.28}$$

where $v(\mathbf{p})=i\sigma^2u^*(\mathbf{p})$. Hence we get the important result: if neutrino is massless, we have a left-handed (helicity negative) neutrino and a right-handed antineutrino. This is what is realized in nature. If we start with η-field, then we have opposite case: a right-handed neutrino and left-handed antineutrino. This case is not realized in nature.

Let us write $\xi=\nu_L$ and $\eta=\nu_R$, then as is clear from Eq. (22) it is the mass which links ν_L to ν_R while Eq.(23) can be written as

$$\nu_L^c \;=\; -i\sigma^2\nu_R^* \tag{10.29}$$

$$\nu_R^c \;=\; i\sigma^2\nu_L^*. \tag{10.30}$$

Hence for a massless neutrino, we will have ν_L and $\nu_R^c=i\sigma^2\nu_L^*$ i.e. a left-handed neutrino and a right-handed antineutrino.

If we allow both a finite mass and lepton number non-conservation, then for an electrically neutral lepton, the Lagrangian is

$$L=\bar{\Psi}\left(i\gamma^\mu\partial_\mu-m_D\right)\Psi+\frac{m_M}{2}\left(\Psi^TC^{-1}\Psi-\bar{\Psi}C\bar{\Psi}^T\right). \tag{10.31}$$

The second term in Eq. (31) is the Majorana mass term and violates lepton number conservation: $\Delta L = 2$. Let us define the new fields ϕ_1 and ϕ_2:

$$\begin{aligned}\phi_1 &= \frac{1}{\sqrt{2}}\left(\xi - i\sigma^2\eta^*\right) \\ \phi_2 &= -\frac{i}{\sqrt{2}}\left(\xi + i\sigma^2\eta^*\right).\end{aligned} \tag{10.32}$$

It then follows from Eq. (23) that under charge conjugation

$$\phi_{1,2} \xrightarrow{C} \pm\phi_{1,2} \tag{10.33}$$

i.e. $\phi_{1,2}$ are eigenstates of C with eigenvalues $+1$ and -1 respectively. In terms of ϕ_1 and ϕ_2, Eq. (31) becomes [see Eq. (A.107)]

$$\begin{aligned}L &= \Big[i\phi_1^\dagger\bar{\sigma}^\mu\partial_\mu\phi_1 - \left(\frac{m_D + m_M}{2}\phi_1^T(-i\sigma^2)\phi_1 + \text{h.c.}\right) \\ &\quad + i\phi_2^\dagger\bar{\sigma}^\mu\partial_\mu\phi_2 - \left(\frac{m_D - m_M}{2}\phi_2^T(-i\sigma^2)\phi_2 + \text{h.c.}\right)\Big].\end{aligned} \tag{10.34}$$

If we start with ξ and η or equivalently ν_L and ν_R, then we can have two Majorana particles of masses $(m_D \pm m_M)/2$. If we start with ν_L only, $m_D = 0$, we have a Majorana neutrino of mass m_M. In this case Eq. (34) reduces to

$$L = i\nu_L^\dagger\bar{\sigma}^\mu\partial_\mu\nu_L - \frac{m_M}{2}\left(\nu_L^T(-i\sigma^2)\nu_L + \text{h.c.}\right). \tag{10.35}$$

We get an important result: a two-component neutrino (ν_L) cannot have a Dirac mass; it can have only Majorana mass, which violates lepton number conservation. Thus one helicity state (-1 for neutrino) together with lepton number conservation implies that $m_\nu = 0$. It may be mentioned that if neutrino is massless, there is no distinction between Majorana and Dirac (Weyl) neutrino.

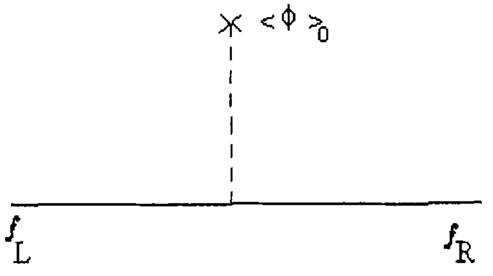

Figure 2 Fermion mass generation

10.2.3 Fermion masses in the standard model (SM) and see-saw mechanism

The fermion masses in the standard model are generated through a Yukawa coupling of fermions with a Higgs scalar (see Chap.13):

$$\mathcal{L} = -g_f \bar{f}_R \phi f_L + \text{h.c.} \tag{10.36}$$

where ϕ develops a vacuum expectation value as shown in Fig.2. Here f_L and ϕ are doublets while f_R is a singlet under the standard model group $SU(2) \otimes U(1)$, e.g.

$$f_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \quad f_R = e_R, \; N_R.$$

The above mechanism gives

$$\mathcal{L}^{\text{Dirac}}_{\text{mass}} = -g_f < \phi_0 > \bar{f}_R f_L + \text{h.c.}, \tag{10.37}$$

leading to the Dirac mass

$$m_f = g_f < \phi >_0 . \tag{10.38}$$

Thus $m_D(\nu_\ell) = g_\ell < \phi >_0$ and one thus expects $m_D(\nu_\ell) \sim m_D(\ell)$, say within a factor of 10 or so. Also for the neutrino it is convenient to write the mass Lagrangian in the two-component basis:

$$\begin{aligned}\mathcal{L}^D_{\text{mass}} &= -m_D\left[\bar{\nu}_L N_R + \bar{N}_R\nu_L\right] \\ &= -\frac{1}{2}m_D\left[\bar{\nu}_L N_R + \bar{\nu}^c_L N^c_R + h.c.\right]. \qquad (10.39)\end{aligned}$$

The Majorana mass term can be generated through an effective Lagrangian:

$$\mathcal{L}_{\text{eff}} = -\frac{G}{M}\bar{\nu}^c_L \nu_L \phi\phi + h.c. \qquad (10.40)$$

giving a Majorana mass

$$m_\nu = \frac{G}{M}\left\langle \phi^2 \right\rangle_0 \qquad (10.41)$$

and

$$\mathcal{L}^{\text{Majorana}}_{\text{mass}}(\nu) = -\left(m\bar{\nu}^c_L\nu_L + h.c.\right). \qquad (10.42)$$

The above mass generation can be pictured as a two-step process shown in Fig.3. This process also gives a Majorana mass to N_R

$$\mathcal{L}^{\text{Majorana}}_{\text{mass}}(N) = -\left(M\bar{N}^c_R N_R + h.c.\right). \qquad (10.43)$$

We may remark here that with $G \approx 1$, $\sqrt{2} < \phi >_o \simeq 246\,\text{GeV}$ as in the stanadard model, Eq.(41) gives $m_\nu \approx 10^{-6}\,\text{eV}$ for $M \approx 10^{19}\,\text{GeV}$ (Planck mass scale).

Referring to Eqs. (39), (42) and (43), the mass matrix in 2 - component basis needs diagonalization. Denoting by prime fields before diagonalization, we have in 2 - component basis

$$\mathcal{L}_M = -\left(\bar{\nu}'^c_L \bar{N}'^c_R\right)\begin{pmatrix} m & m_{D/2} \\ m_{D/2} & M \end{pmatrix}\begin{pmatrix} \nu'_L \\ N'_R \end{pmatrix}. \qquad (10.44)$$

It is useful to consider various limits:

$$\begin{aligned}\text{Majorana} &: \ m_D \to 0 \\ \text{Dirac} &: \ m,\, M \to 0 \\ \text{Seesaw} &: \ m \to 0,\ m_D \ll M\end{aligned}$$

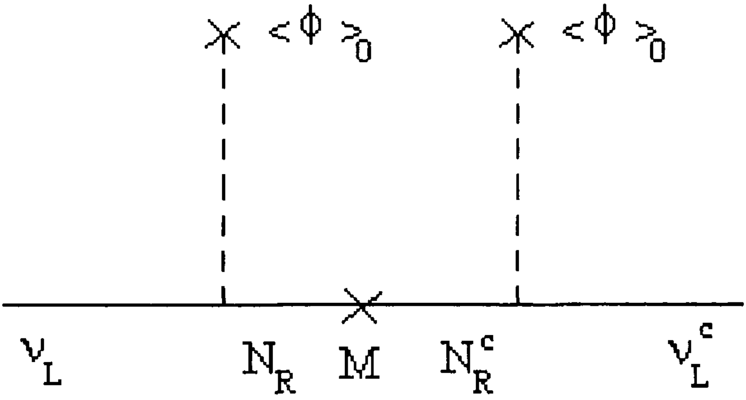

Figure 3 Majorana mass generation

The diagonalization of the mass matrix (44) in the seesaw limit gives

$$\begin{aligned} \nu_L &= \nu'_L - \frac{m_D}{2M} N'^c_R \\ N_R &= \frac{m_D}{2M}\nu'^c_L + N'_R. \end{aligned} \tag{10.45}$$

Hence we have two Majorana neutrinos ν_L and N_R with masses

$$\begin{aligned} m_\nu &\simeq m_D^2/4M \ll m_D \\ m_N &\simeq M. \end{aligned} \tag{10.46}$$

Depending upon M, ν_L could be extremely light and N_R correspondingly heavy. To summarize, in the Dirac case, one must answer the question why

$$(m_{\nu_\ell})_{Dirac} \ll m_\ell \tag{10.47}$$

while in the Majorana case the seesaw mechanism sidesteps this question; here one has

$$(m_{\nu_\ell})_{Majorana} \simeq \frac{m_\ell^2}{4M} \ll m_\ell \tag{10.48}$$

by requiring the existence of a large scale M, associated with some new physics. Below we give some typical scales indicative of new physics and the corresponding neutrino masses, which may be relevant for neutrino oscillations (to be discussed below) and dark matter:

M (GeV)	m_{ν_e} (eV)	m_{ν_μ} (eV)	m_{ν_τ} (eV)
$M_{\text{Planck}}(10^{19})$	10^{-14}	4×10^{-10}	10^{-7}
$M_{GUT}(10^{16})$	10^{-11}	4×10^{-7}	10^{-4}
$M_R(10^{12})$	10^{-7}	4×10^{-3}	1
10^6	10^{-1}	4×10^3	10^6

10.3 Neutrino Oscillations

If neutrinos are massless, then the neutrinos ν_e, ν_μ, ν_τ, which enter the weak interaction Lagrangian are also the mass eigenstates. If anyone of them have a mass, then it may be that the mass eigenstates which we denote by $\nu_i (i = 1, 2, 3)$ are different from flavor eigenstates ν_w, $(w = e,\ \mu,\ \tau)$. In this case, we can get neutrino oscillations. The phenomenon of neutrino oscillations can provide a mechanism to measure extremely small neutrino masses. We note that two sets of states $|\nu_w >$ and $|\nu_i >$ are connected with each other by a unitary transformation:

$$|\nu_w\rangle = \sum_i U_{wi} |\nu_i\rangle \tag{10.49}$$

$$\sum_w U_{iw} U^*_{jw} = \delta_{ij}. \tag{10.50}$$

Now

$$H(k) |\nu_i\rangle = E_i |\nu_i\rangle \tag{10.51}$$

$$E_i = (k^2 + m_i^2)^{1/2} \approx k + \frac{m_i^2}{2k}, \tag{10.52}$$

since $k \gg m_i$ and we take the extreme relativistic limit. Now at time t, $|\nu(t) >$ satisfies the Schrödinger equation:

$$i \frac{d}{dt} |\nu(t) >= H |\nu(t) > . \tag{10.53}$$

In ν_i basis, H is a diagonal matrix with eigenvalues E_1, E_2 and E_3. Thus

$$|\nu_i(t)>= e^{-iE_it}|\nu_i(0)>\equiv e^{-iE_it}|\nu_i> . \tag{10.54}$$

Hence from Eq. (49), we can write

$$|\nu_w(t)>=\sum_i U_{wi}e^{-iE_it}|\nu_i> \tag{10.55}$$

and

$$\begin{aligned} <\nu_{w'}|\nu_w>_t &= \sum_i U_{wi}e^{-iE_it}<\nu_{w'}|\nu_i> \\ &= \sum_i U_{wi}e^{-iE_it}U^*_{w'i}. \end{aligned} \tag{10.56}$$

Thus the probability that at time t, the neutrino of type w is converted to the neutrino of type w' is given by

$$\begin{aligned} P_{w'w} &= |\langle\nu_{w'}|\nu_w\rangle_t|^2 \\ &= \sum_i\sum_j (U_{wi}U^*_{w'i})\left(U_{wj}U^*_{w'j}\right)\cos(E_i-E_j)t. \end{aligned} \tag{10.57}$$

Neglecting CP-violating phases so that U is real, it is convenient to rewrite it as

$$P_{w'w}=\delta_{w'w}-4\sum_{j>i}U_{wi}U_{w'j}U_{wj}U_{w'j}\sin^2(\pi L/\lambda_{ij}) \tag{10.58}$$

where L is the distance travelled after which ν_w is converted to $\nu_{w'}$ and

$$\lambda_{ij}=\frac{4\pi E_\nu}{\Delta_{ij}}=2.47\,m\left(\frac{E_\nu}{MeV}\right)\frac{eV^2}{\Delta_{ij}} \tag{10.59}$$

where we have used the relation $L = ct$,

$$\begin{aligned} \lambda_{ij} &= \frac{2\pi c}{E_i - E_j}, \\ (E_i - E_j)t &= \frac{L(m_i^2 - m_j^2)}{2E_\nu} \\ &= \frac{L\Delta_{ij}}{2E_\nu}. \end{aligned} \tag{10.60}$$

As a consequence of CPT and CP invariance

$$P_{\nu_{w'}\nu_w} = P_{\bar{\nu}_{w'}\bar{\nu}_w} = P_{\nu_w\nu_{w'}} = P_{\bar{\nu}_w\bar{\nu}_{w'}}. \tag{10.61}$$

The form of transition probability (58) depends on the spectrum of Δm^2 or Δ_{ij} chosen and the explicit form of U. If Δm^2 is chosen such that $\lambda >> L$, then the oscillator term $\sin^2 \frac{\pi L}{\lambda} \to 0$. On the other hand, if $\lambda \ll L$, one has a large number of oscillations and $\sin^2 \frac{\pi L}{\lambda}$ averages out to $\frac{1}{2}$.

For the conversion of ν_e to ν_x ($x = \mu$ or τ),

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{10.62}$$

and

$$P_{\nu_e \to \nu_x} = \sin^2 2\theta \sin^2 \left[1.27 \frac{\Delta m^2}{E_\nu} L\right] \tag{10.63}$$

while the survival probability is $P_{\nu_e \to \nu_e} = 1 - P_{\nu_e \to \nu_x}$. Here θ is the vacuum mixing angle. $P_{\nu_e \to \nu_e}$ and $P_{\nu_e \to \nu_x}$ oscillate with L as shown in Fig. 4. The amplitude of the oscillations is determined by the mixing angle; the wavelength of the oscillations is λ.

To look for oscillations, one needs

- Low energy neutrinos
- Long path length
- Large flux

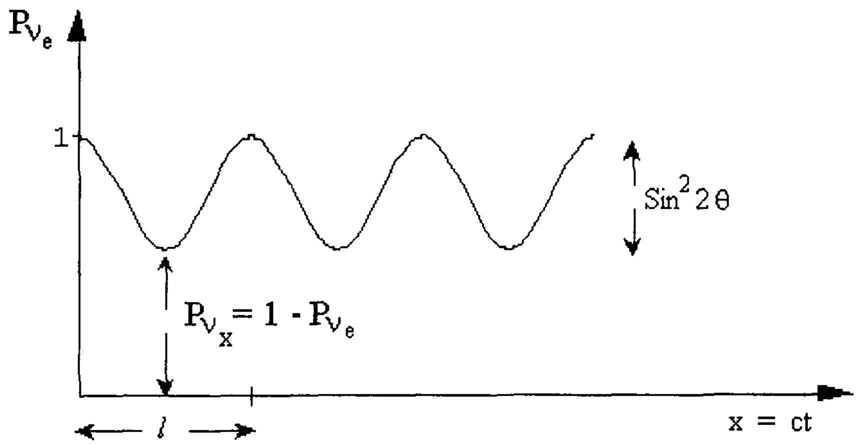

Figure 4 The neutrino oscillations

10.3.1 Evidence for neutrino oscillations

One looks for neutrino oscillations in two types of experiments:

(i) Disappearance experiments

Reactors are source of $\bar{\nu}_e$ through neutron β decay $n \rightarrow p + e^- + \bar{\nu}_e$ and experiment looks for a possible decrease in the $\bar{\nu}_e$ flux as a function of distance from the reactor, $\bar{\nu}_e \rightarrow X$ [if converted to $\bar{\nu}_\mu$, say, one would see nothing, $\bar{\nu}_\mu$ could have produced μ^+ but does not have sufficient energy to do so].

(ii) Appearance experiments

Here one searches for a new neutrino flavor, absent in the initial beam, which can arise from oscillations. All terestrial experiments (except one, see below) are consistent with no neutrino oscillations and provide exclusion regions in the $\Delta m^2 - \sin^2 2\theta$ plane (see Fig. 5).

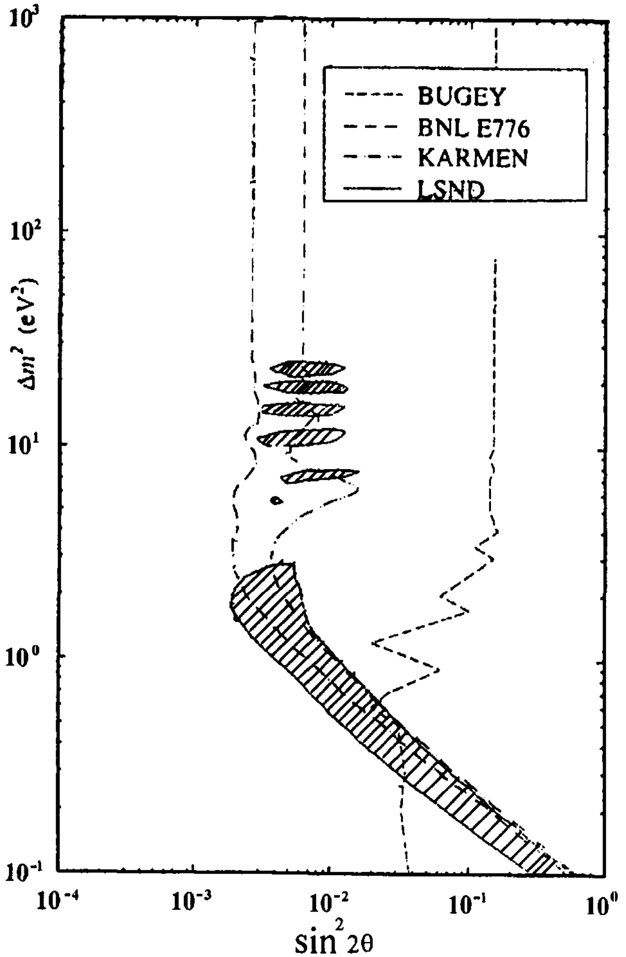

Figure 5 Plot of LSND Δm^2 vs $\sin^2 2\theta$ forward region (shaded) limited by solid curves. The region excluded by the BNL E776 ($E \sim 1 - 10$ GeV, $L \sim 1$ km) and KARMEN $\nu_\mu \to \nu_e$ appearance experiments are bounded by the dashed-dotted and dash-dot-dotted curves respectively. The dashed lines represents the results of Bugey ($E \sim 5$ MeV, $L \sim 40$ m) experiments.

There are now three claims of "evidence" for oscillations and hence indirectly for non-vanishing neutrino masses.

a) Los Alamos liquid scintillation dectector (LSND) $\bar{\nu}_\mu \to \bar{\nu}_e$ oscillations

In this experiment neutrinos are produced in decays at rest of π^+ and μ^+ :

$$\pi^+ \to \mu^+\nu_\mu, \mu^+ \to e^+\nu_e\bar{\nu}_\mu, \; \bar{\nu}_\mu \to \bar{\nu}_e.$$

In this experiment $E \approx 30 - 60$ MeV, and $L \simeq 30$ m. The LSND detector searched for $\bar{\nu}_e$ by observing e^+ as signal in the process $\bar{\nu}_e p \to e^+ n$. 22 such events were found against the expected background of 4.6 ± 0.6 events. If negative results of other expriments are also taken into account, then from Fig. 5, one obtains the following allowed values of oscillation parameters: π

$$\begin{aligned} 0.25 \text{ eV}^2 &< \Delta m^2 < 2.3 \text{ eV}^2 \\ 0.002 &< \sin^2 2\theta < 0.04. \end{aligned} \tag{10.64}$$

These data, which indicate rather large values of Δm^2 in $\nu_\mu \to \nu_e$ channel, need confirmation from other experiments, e.g., KARMEN which would reach sensitivity of LSND experiment in about 2 years.

b) Atmospheric neutrino anomaly

Atmospheric neutrinos are produced in decays of pions (kaon's) that are produced in the interaction of cosmic rays with the atmosphere:

$$\begin{aligned} p + A &\to \pi^\pm + A', \\ \pi^\pm &\to \mu^\pm \nu_\mu \, (\bar{\nu}_\mu) \\ &\to e^\pm \nu_e \, (\bar{\nu}_e) \, \bar{\nu}_\mu \, (\nu_\mu) \end{aligned}$$

These neutrinos are detected through the reactions $\nu_\mu + n \to \mu^- + p$, $\bar{\nu}_\mu + p \to \mu^+ + n$ and $\nu_e + n \to e^- + p$, $\bar{\nu}_e + p \to e^+ + n$ and are respectively called μ–like and e–like events. One would expect the ratio

$$\frac{N(\nu_\mu)}{N(\nu_e)} \equiv \frac{N(\nu_\mu + \bar{\nu}_\mu)}{N(\nu_e + \bar{\nu}_e)} \simeq 2.$$

However this ratio has been measured in several detectors and it is found that [MC denotes the Monte Carlo Simulated ratio]

$$R \equiv \frac{(N(\nu_\mu)/N(\nu_e))\, obs}{(N(\nu_\mu)/N(\nu_e))_{MC}} < 1$$

and that it depends on the zenith angle as well, implying neutrino oscillations. The latest atmospheric ν data is summarized below:

$$\begin{aligned} R &= 0.63 \pm 0.03 \pm 0.05 \text{ (Super-Kamiokande sub-GeV)} \\ R &= 0.65 \pm 0.05 \pm 0.08 \text{ (Super-Kamiokande multi-GeV)} \end{aligned}$$

The zenith angle distribution of R is shown in Fig. 6. One would not expect up/down asymmetry i.e. between the number of events arising from the neutrinos coming from below the earth and going upward through its center to the detector and those arising from neutrinos coming from up, since we are in a "spherical shell of ν's". However, for multi GeV one finds for this asymmetry:

$$\begin{aligned} \text{up/down } (e - \text{like}) &= 0.93 \pm 0.1 \pm 0.02 \\ \text{up/down } (\mu - \text{like}) &= 0.54^{+0.06}_{-0.05}. \end{aligned} \qquad (10.65)$$

The former is consistent with no oscillations. The latter is a 6σ discrepancy. The result (65) is consistent with $\nu_\mu \leftrightarrow \nu_\tau$ oscillations which would imply that the former ratio to be unity. The conversion probability $P_{\nu_\mu - \nu_\tau}$ as given in Eq. (63) fits the data quite well for

$$\Delta m^2 = 2.5 \times 10^{-3}\,\text{eV}^2,\ \sin^2 2\theta = 1.0. \qquad (10.66)$$

Several long - base line neutrino oscillation experiments that will allow an investigation of the atmospheric neutrino range of Δm^2 to other channels are at present under preparation.

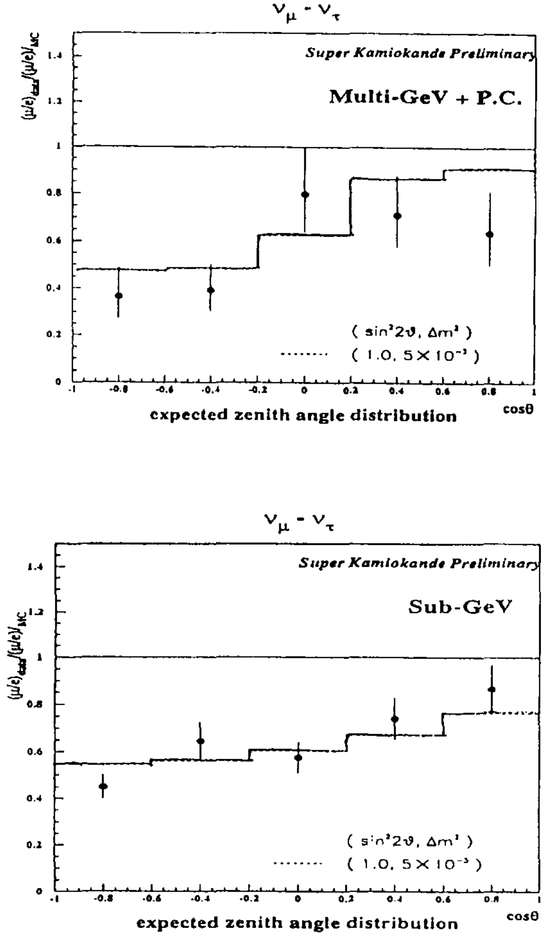

Figure 6 Zenith-angle distribution of R with neutrino oscillations parameters corresponding to the best fit values to the Super-Kamiokande data.

(c) Solar neutrinos

Electron type antineutrinos are produced by the decay of pile neutrons in a fission reactor: $n \to p + e^- + \bar{\nu}_e$. Electron type neutrinos are, on the other hand, produced from reactions in the sun, called solar neutrinos. The energy of the sun is generated in the reactions of pp and CNO cycles. Energy is generated through nuclear burning involving the transitions of four protons into ^4He :

$$4p \to \, ^4He + 2e^+ + 2\nu_e + Q \tag{10.67}$$

where $Q = 26.7$ MeV is the energy release in the above transition. Thus the generation of the energy of the sun is accompanied by the emission of ν_e's. The total flux of the neutrinos is connected to the luminosity of the sun $L_\odot$ by the relation:

$$Q \sum_i \left(l - 2\frac{\bar{E}_i}{Q} \right) \Phi_i = \frac{L_\odot}{2\pi R^2} \tag{10.68}$$

where R is the sun-earth distance, Φ_i is the total flux of neutrinos from the source i, and $\bar{E}_i$ is the average energy.

The most important sources of solar neutrinos in the *pp* cycle, which dominates cooler stars, particularly the sun, are the following reactions:

$$\begin{aligned}
pp \to \, ^2He^+\nu_e \quad &: \quad E_\nu < 0.42 \;\; \text{MeV} \\
ppe^- \to \, ^2H\nu_e \quad &: \quad E_\nu = 1.442 \;\; \text{MeV} \\
^7B_e\, e^- \to \, ^7Li\, \nu_e \quad &: \quad E_\nu \sim 0.86 \;\; \text{MeV} \\
^8B \to \, ^8B_e^* e^+\nu_e \quad &: \quad E_\nu < 15 \;\; \text{MeV}
\end{aligned}$$

On the other hand, the CNO cycle dominates hot stars and following reactions are sources of ν_e's:

$$\begin{aligned}
^{13}N \;\; &\to \;\; ^{13}Ce^+\nu_e \\
^{15}O \;\; &\to \;\; ^{15}Ne^+\nu_e \, .
\end{aligned}$$

The first reaction in the *pp* cycle is the main source of solar neutrinos. The third reaction is a source of monochromatic neutrinos. This reaction contributes about 10% to the total flux of solar neutrinos. The fourth reaction contributes only about 10^{-4} to the

Table 10.1 The standard solar model predictions of neutrino fluxes and observed rates.

	Homestake [SNU]	Kamiokande $[10^6 cm^{-1} s^{-}]$	SAGE and GALLEX [SNU
$E_{th}(MeV)$	0.814	9.3, 7.5 and 7.0	0.232
Mode	$\nu_e +^{37} Cl \rightarrow e^- +^{37} Ar$	$\nu + e^- \rightarrow \nu + e^-$	$\nu_e +^{71} Ga \rightarrow e^- +^{71} Ge$
Sensitive to	$^8B\nu$'s($\sim 90\%$) but also to $^7B_e\,\nu$'s	$^8B\,\nu$'s	all 3 sources
Observed rate	2.54 ± 0.20	2.89 ± 0.42 $2.45 \pm 0.06^{+0.25}_{-0.09}$ Super-Kamiokande	70.3 ± 7.0
BPSSM (Expected)	$9.3^{+1.2}_{-1.4}$	6.62 ± 1.06	137^{+8}_{-7}
Ratio	0.273 ± 0.03	0.42 ± 0.07 $0.368 \pm 0.01^{+0.037}_{-0.013}$ Super-Kamiokande	0.51 ± 0.06

$1\,SNU = 10^{-36}$ interactions per target atom per sec
BPSSM: Bahcall-Pinsonneault, Phys. Rev. Lett. **78** (1997) 67.

total flux but it is the main source of high energy solar neutrinos (up to 15 MeV).

Due to different detection thresholds, solar neutrinos from different sources can be detected in different reactions. Thus the solar neutrinos with energy > 0.814 MeV can be detected in ^{37}Cl and those > 0.233 MeV in ^{71}Ga. A discrepancy exists between the standard solar model (SSM) predictions of neutrino fluxes and rates observed in terrestial experiments as shown Table 1. We see from this table that in all experiments the observed event rate is significantly smaller than the rate predicted by the standard solar model. We may thus conclude that solar neutrinos are detected, thereby establishing the solar fusion. That the observed event rate for solar neutrinos production is smaller than the predicted rate provides a cricumstantial evidence for new physics as will be discussed in the next section.

10.4 Possible Particle Physics Solutions of Solar Neutrino Problem

If the experiments are correct, it is very unlikely that non-standard solar models can fit the solar neutrino data. However, there are possible particle-physics solutions, some of which are listed below:

(i) Vacuum oscillations (involving 2 or 3 ν's)

(ii) Matter induced oscillations (involving 2 or 3 ν's)

(iii) Sterile neutrino

(iv) Magnetic moment transitions

Magnetic moment transitions need large neutrino magnetic moment which surpass upper limits on them from astrophysics [see Sec. 5]. The possibilities (i) to (iii) involve new physics (non-standard neutrino properties) in terms of modest extension of the standard electroweak theory in which neutrinos have small masses and lepton flavor is not conserved leading to neutrino oscillations.

10.4.1 Vacuum oscillations

Vacuum oscillations of ν_e to ν_x give the survival probability :

$$P_{vac}(\nu_e \to \nu_e) = 1 - \sin^2 2\theta \sin^2 \left[\frac{\pi R}{\lambda}(1 - \frac{r}{R})\right] \tag{10.69}$$

where R is the earth–sun distance ($\simeq 1.5 \times 10^{13}$cm) while r gives the production location. The results of a fit including all the latest data is shown in Fig. 7. The best fit gives the oscillation parameters

$$\Delta m^2 = 6.0 \times 10^{-11} eV^2, \ \sin^2 2\theta = 0.96\,. \tag{10.70}$$

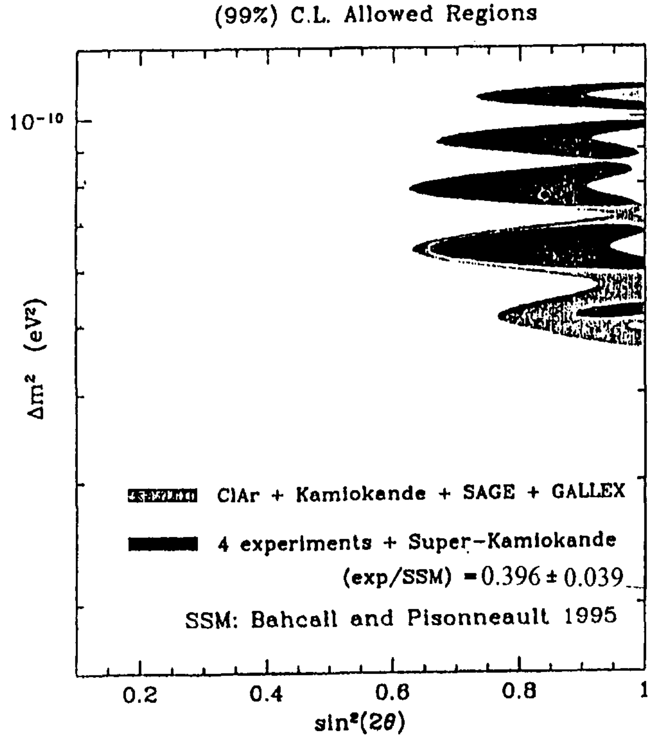

Figure 7 Region in the Δm^2 vs $\sin^2 2\theta$ plane for the vacuum solution in the solar neutrino problem

10.4.2 Possible explanation in terms of resonant matter oscillations: Mikheyev-Smirnov-Wolfenstein [MSW] effect

First we write the Hamiltonian in ν_e, ν_x basis [$x = \mu$ or τ or s(sterile ν)] :

$$H_\nu(k) = UHU^{-1} \tag{10.71}$$

where H is diagonal in $\nu_1 - \nu_2$ basis (cf. Sec. 3):

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4k} \begin{pmatrix} -\Delta m^2 & 0 \\ 0 & \Delta m^2 \end{pmatrix} \tag{10.72}$$

and

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{10.73}$$

while $E_i \approx k + \frac{m_i^2}{2k}$ and $E_2 - E_1 = \frac{m_2^2 - m_1^2}{2k} = \frac{\Delta m^2}{2k}$. Then

$$H_\nu(k) = \text{const.} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{-\Delta m^2 \cos 2\theta}{2k} & \frac{\Delta m^2 \sin 2\theta}{4k} \\ \frac{\Delta m^2 \sin 2\theta}{4k} & 0 \end{pmatrix} \tag{10.74}$$

where the first part of Eq. (74) is irrelevant for oscillations. Now in traversing matter, neutrinos interact with electrons and nucleons of intervening material and their forward coherent scattering induces an effective potential energy. Such contributions of weak interaction in matter to H_ν arise due to Feynman diagrams shown in Fig. 8. The first diagram contributes equally to ν_e, ν_μ and ν_τ and as such is not relevant $\nu_\mu \leftrightarrow \nu_\mu$ or ν_τ oscillations. This gives the effective Hamiltonian[see Chap.13]:

$$\frac{2G_F}{\sqrt{2}} \left[\bar{f}_L \gamma^\mu \left(I_{3L} - Q\sin^2\theta_W\right) f_L\right] \left[\bar{\nu}\gamma_\mu(1-\gamma_5)\nu\right] \tag{10.75}$$

where $f = e^-$, p or n for which respectively $I_{3L} = -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$ and $Q = -1, 1, 0$. The second diagram after Fierz rearrangement gives the effective Hamiltonian:

$$\frac{G_F}{\sqrt{2}} \left\langle \psi_e \middle| e^- \gamma^\mu (1-\gamma_5) e \middle| \psi_e \right\rangle \bar{\nu}_e \gamma_\mu (1-\gamma_5) \nu_e \tag{10.76}$$

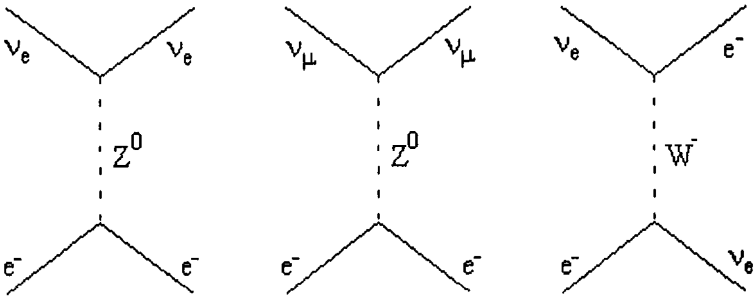

Figure 8 Feynman diagrams for neutral current (n.c.) and charged current (c.c) weak interactions which contribute to H_ν for oscillations in matter

where ψ_e denotes the state of the medium. These diagrams give the potential energy

$$V_{\nu_e} = \sqrt{2}G_F(n_e^{c.c.} - \frac{1}{2}n_n^{n.c.})$$

$$V_{\nu_{\mu,\tau}} = -\sqrt{2}G_F\frac{1}{2}(n_n^{c.c.}) \tag{10.77}$$

$$V_{\nu_s} = 0$$

where n_e denotes the number of electrons per unit volume and n_n that of neutrons. Then the Hamiltonian in the matter is $[k \simeq E]$

$$\begin{aligned} H_M(k) &= H_\nu(k) + H_W \\ &= \begin{pmatrix} \frac{-\Delta m^2 \cos 2\theta}{2E} + \sqrt{2}G_F n & \frac{\Delta m^2 \sin 2\theta}{4E} \\ \frac{\Delta m^2 \sin 2\theta}{4E} & 0 \end{pmatrix} \end{aligned} \tag{10.78}$$

where

$$\begin{aligned} n &= n_e \text{ for } \nu_e \leftrightarrow \nu_\mu \text{ or } \nu_\tau \\ &= n_e - \frac{1}{2}n_n \text{ for } \nu_e \leftrightarrow \nu_s \end{aligned}$$

The diagonalization to ν_1, ν_2 basis gives:

$$\begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix} = \begin{pmatrix} \cos\theta_M & \sin\theta_M \\ -\sin\theta_M & \cos\theta_M \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{10.79}$$

with

$$\begin{aligned} \sin 2\theta_M &= \sin 2\theta \frac{l_M}{l_V}, \\ \cos 2\theta_M &= (\cos 2\theta - A)\frac{l_M}{l_V} && (10.80) \\ \Delta E &= E_2 - E_1 = \frac{1}{2l_M} && (10.81) \end{aligned}$$

where

$$\begin{aligned} A &= 2\sqrt{2} G_F n \frac{E}{\Delta m^2} && (10.82) \\ l_M &= \frac{E}{\Delta m^2}\left[(A - \cos 2\theta)^2 + \sin^2 2\theta\right]^{-1/2}, && (10.83) \\ l_V &= \frac{E}{\Delta m^2}. && (10.84) \end{aligned}$$

For constant density n, the considerations of Sec. 4 give the conversion probability

$$P(\nu_e \to \nu_x) = \sin^2 2\theta_M \sin^2\left[1.27\frac{L}{l_M}\right]. \tag{10.85}$$

The following are useful limits:

$$\begin{aligned} &(i) \quad n \to 0, \quad l_M \to l_V, \quad \theta_M = \theta, \ \Delta E = \frac{\Delta m^2}{2E} \\ &(ii) \quad n \to \infty, \ H_M \to \begin{pmatrix} \sqrt{2} G_F n & \to 0 \\ \to 0 & 0 \end{pmatrix} && (10.86) \\ &\qquad \Delta E = 2\sqrt{2} G_F n \\ &\qquad \theta_M = \frac{\pi}{2}, \ \nu_e = \nu_2, \ \nu_x = \nu_1 \end{aligned}$$

$$
\begin{aligned}
(iii) \qquad & n = n_{\rm res} \text{ defined by } \sqrt{2}G_F(n)_{\rm res} = \frac{\Delta m^2 \cos 2\theta}{2E}, \\
& l_M \to \frac{E}{\Delta m^2 \sin 2\theta} \\
& H_M \to \begin{pmatrix} 0 & \frac{\Delta m^2 \sin 2\theta}{4E} \\ \frac{\Delta m^2 \sin 2\theta}{4E} & 0 \end{pmatrix} \\
& \Delta E_{\rm res} = \frac{\Delta m^2 \sin 2\theta}{2E} \\
& \theta_M = \frac{\pi}{4}, \, \nu_2 = \frac{\nu_e + \nu_x}{\sqrt{2}}, \, \nu_1 = \frac{\nu_e - \nu_x}{\sqrt{2}}.
\end{aligned}
\tag{10.87}
$$

Using the above limits, the plot of E versus n is shown in Fig. 9.

Suppose ν_e is created at $n_0 > n_{res}$ say at the center of the sun, and then it propagates out. If there is no level crossing (shown by dotted lines in Fig.9, then $\nu(n=0) \simeq \nu_x$ and undetectable. This conversion of ν_e into ν_x is the cause of the depletion of observable neutrinos. Now neutrinos of any energy will not go through the resonance. The resonance condition for any given neutrino energy E is:

$$n_{\rm res} E = \cos 2\theta \frac{\Delta m^2}{2\sqrt{2}G_F}. \tag{10.88}$$

We may remark here that for $\nu_e \to \nu_\mu$ or ν_τ conversion,

$$n = n_e = \left(\frac{\rho}{m_N}\right) Y \tag{10.89}$$

where Y denotes the number of electrons per nucleon and is $1/2$ for ordinary matter. Then, the resonance condition (88) can be written as

$$
\begin{aligned}
\rho_{\rm res} &= \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2}G_F} \frac{m_N}{Y} \frac{1}{E} \\
&= 1.3 \times 10^7 \, g/cc \frac{1}{2Y} \cos 2\theta \left[\frac{\Delta m^2}{(\text{eV})^2}\right] \left(\frac{\text{MeV}}{E}\right).
\end{aligned}
\tag{10.90}
$$

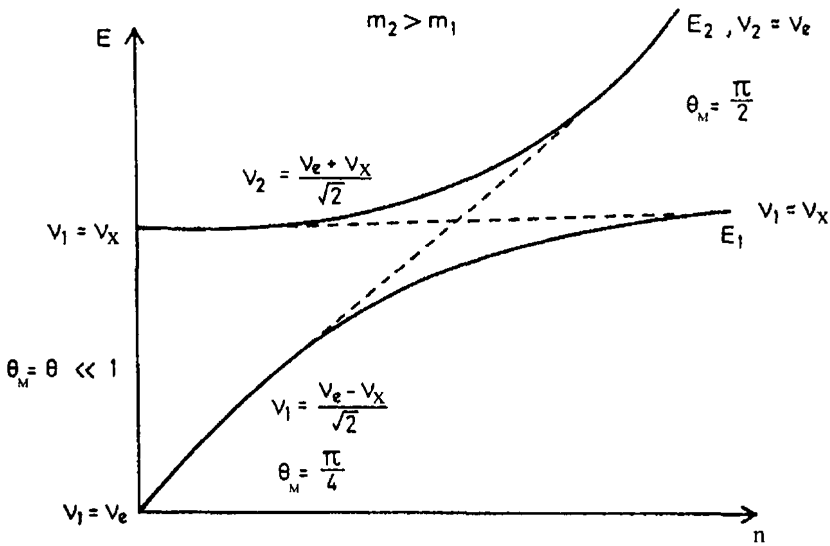

Figure 9 Plot of neutrino energy E versus density n , showing conversion of ν_e to ν_x in matter

For $\rho_{res} \geq \rho$ (center of sun) $= 100\, g/cc$, we have

$$\left(\frac{E}{\text{MeV}}\right) \leq 1.3 \times 10^5 \frac{\Delta m^2}{(\text{eV})^2}. \tag{10.91}$$

Thus, for example, for $\Delta m^2 \geq 6 \times 10^{-6}\,\text{eV}^2$, we will not have resonance for $E \leq 0.4\,\text{MeV}$ and the resonance will be at least at $E = 0.8$ MeV. In this case the resonance will not affect pp neutrino for which $E_{max} = 0.44$ MeV but can eliminate ^7B neutrinos.

10.5 Evolution of Flavor Eigenstates in Matter

The evolution of flavor eigenstates in matter is governed by the equation:

$$i\frac{\partial}{\partial x}\begin{pmatrix} \nu_e(x) \\ \nu_x(x) \end{pmatrix} = H(x)\begin{pmatrix} \nu_e(x) \\ \nu_x(x) \end{pmatrix} \tag{10.92}$$

where $H(x)$ is given in Eq.(78). Note that the x dependence arises due to the x dependence of the density n for varying density case. Using

$$\begin{pmatrix} \nu_e(x) \\ \nu_x(x) \end{pmatrix} = U(x) \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix} \tag{10.93}$$

with

$$U(x) = \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{pmatrix}, \tag{10.94}$$

we have

$$i\frac{\partial}{\partial x}\begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix} = U^{-1}HU \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix} - iU^{-1}\frac{\partial U}{\partial x}\begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix} \tag{10.95}$$

where

$$\begin{aligned} U^{-1}HU &= \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \\ &= \frac{E_1+E_2}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{\Delta E}{2} & 0 \\ 0 & \frac{\Delta E}{2} \end{pmatrix} \end{aligned} \tag{10.96}$$

and

$$U^{-1}\frac{\partial U}{\partial x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\theta'_M(x). \tag{10.97}$$

Noting that the first part of Eq. (96) is irrelevant for oscillations and using Eq. (81) we have

$$i\frac{\partial}{\partial x}\begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix} = \begin{pmatrix} -\frac{1}{4l_M(x)} & -i\theta'_M(x) \\ i\theta'_M(x) & \frac{1}{4l_M(x)} \end{pmatrix}\begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix}. \tag{10.98}$$

For the constant density case, $\theta'_M(x) = 0$ and l_M is independent of x, so that Eq. (98) has simple solutions

$$\begin{aligned} \nu_1(x) &= \nu_1(0)\exp\left(i\frac{x}{4l_M}\right) \\ \nu_2(x) &= \nu_2(0)\exp\left(-i\frac{x}{4l_M}\right) \end{aligned} \tag{10.99}$$

where we have taken $x = 0$ as the initial point. Then Eq.(93) gives

$$\begin{aligned} \nu_e(x) &= \cos\theta(x)\nu_1(0)\exp\left(i\frac{x}{4l_M}\right) \\ &\quad + \sin\theta(x)\nu_2(0)\exp\left(-i\frac{x}{4l_M}\right) \\ &= \cos\theta(x)\cos\theta_M^0\nu_e(0)\exp\left(i\frac{x}{4l_M}\right) \\ &\quad + \sin\theta(x)\sin\theta_M^0\nu_e(0)\exp\left(-i\frac{x}{4l_M}\right) \end{aligned} \tag{10.100}$$

where we have used the boundary condition $\nu_x(0) = 0$ [cf. Eq. (79)]. Then the electron neutrino survival probability averaged over the detector position L (from the solar surface) is given by

$$\begin{aligned} P(\nu_e \longrightarrow \nu_e) &= \cos^2\theta_V\cos^2\theta_M^0 + \sin^2\theta_V\sin^2\theta_M^0 \\ &= \frac{1}{2} + \frac{1}{2}\cos 2\theta_V\cos 2\theta_M^0 \end{aligned} \tag{10.101}$$

where $\theta_V \equiv \theta$ is the vacuum mixing angle. In general when the density n is a function of x one has to solve Eq. (98) and as a result $P(\nu_e \rightarrow \nu_e)$ is given by the Parke formula:

$$P(\nu_e \rightarrow \nu_e) = \frac{1}{2} + \left(\frac{1}{2} - P_j\right)\cos 2\theta\cos 2\theta_M^\circ \tag{10.102}$$

where θ_M^0 is the initial mixing angle and $P_j \equiv \exp\left(-\frac{\pi}{2}\gamma\right)$ is the Landau-Zener formula. Here

$$\gamma = \frac{\Delta m^2}{E}\frac{\sin^2 2\theta}{\cos 2\theta}\left(\frac{1}{n}\frac{dn}{dx}\right)^{-1}_{res}$$

and is called the adiabaticity. In the adiabataic limit $\gamma \gg 1$ and $P_j \rightarrow 0$ and we recover the relation (101). The survival probability $P(\nu_e \rightarrow \nu_e)$ as a function of E_ν is displayed for large and small mixing solutions in Fig.10.

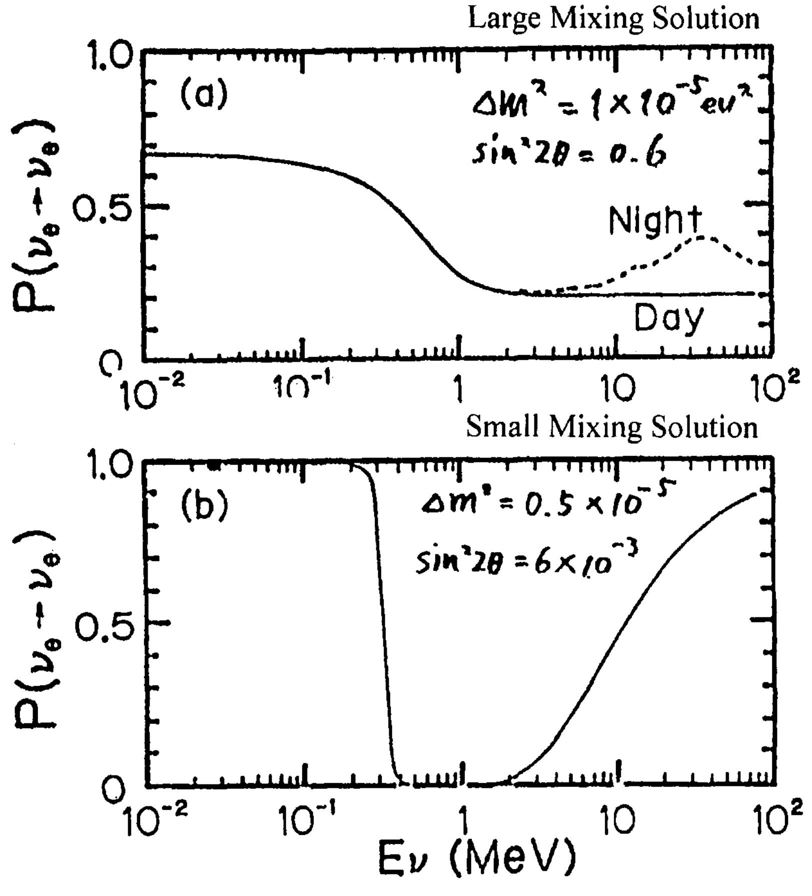

Figure 10 Survival probability as a function of E_ν for large angle and small-angle solution

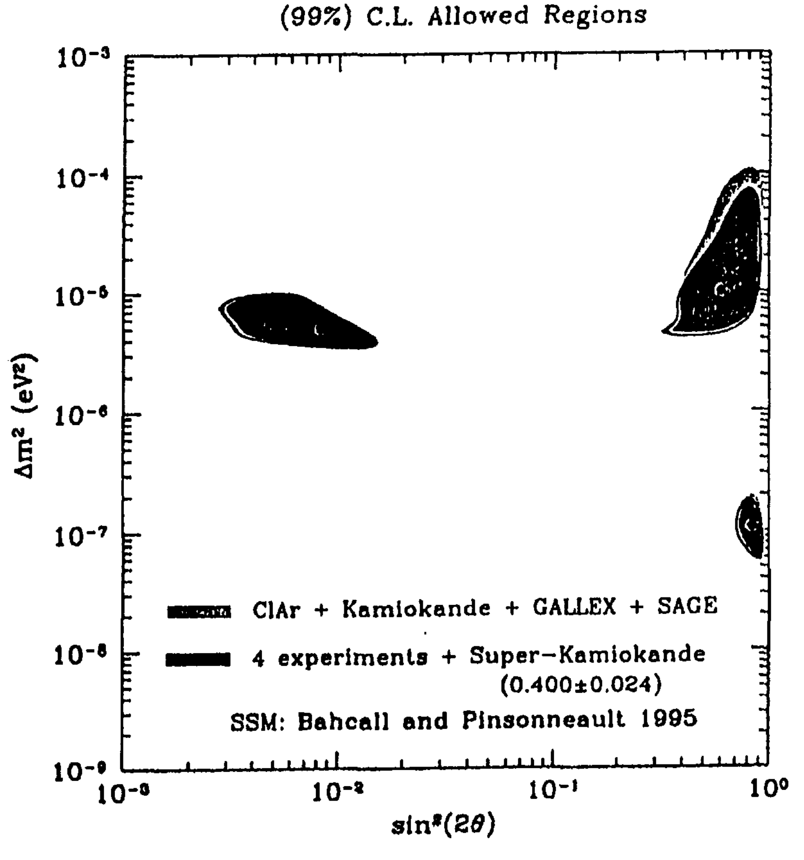

Figure 11 99% C.L. allowed regions in the $\Delta m^2 - \sin^2 2\theta$ plane for the MSW solution to the solar neutrino problem

We now outline the standard analysis. First determine from Kamiokande the flux of ν_e from ^8B that reaches the earth. Then one can understand Cl and Ga results with

$$\begin{aligned} 8_B &\simeq 0.40\, SSM \\ 7_B &\simeq 0.00\, SSM \\ pp &\simeq SSM \end{aligned}$$

for the small mixing solution (see Fig.10).

One can thus conclude that there exist neutrino oscillations that almost totally convert $7_{Be}\, \nu_e$'s into ν_x that have little effect on $pp\, \nu_e$'s. This leads to the solution shown in Figs.11 and 12. The best-fit solutions are

1) small-angle MSW:-

$$\begin{array}{lll} & Active & Sterile \\ \Delta m^2(eV^2) & = 5.4 \times 10^{-6} & 3.5 \times 10^{-6} \\ \sin^2 2\theta & = 7.9 \times 10^{-3} & 10^{-2} \end{array}$$

2) large-angle MSW:

$$\begin{aligned} \Delta m^2(eV^2) &= 1.7 \times 10^{-5} \\ \sin^2 2\theta &= 0.69 \end{aligned}$$

3) Vacuum oscillations [cf. Fig. 7]:

$$\begin{aligned} \Delta m^2(eV^2) &= 6.0 \times 10^{-11} \\ \sin^2 2\theta &= 0.96 \end{aligned}$$

The above different interpretations may be distinguished by new experiments: Super–Kamiokande, SNO, GNO and Borexino. Solar model independent tests of the oscillations may then be feasible. We may mention here that no evidence for "Day-Night" effect

$$\frac{D-N}{D-N} = -0.023 \pm 0.020 \pm 0.014$$

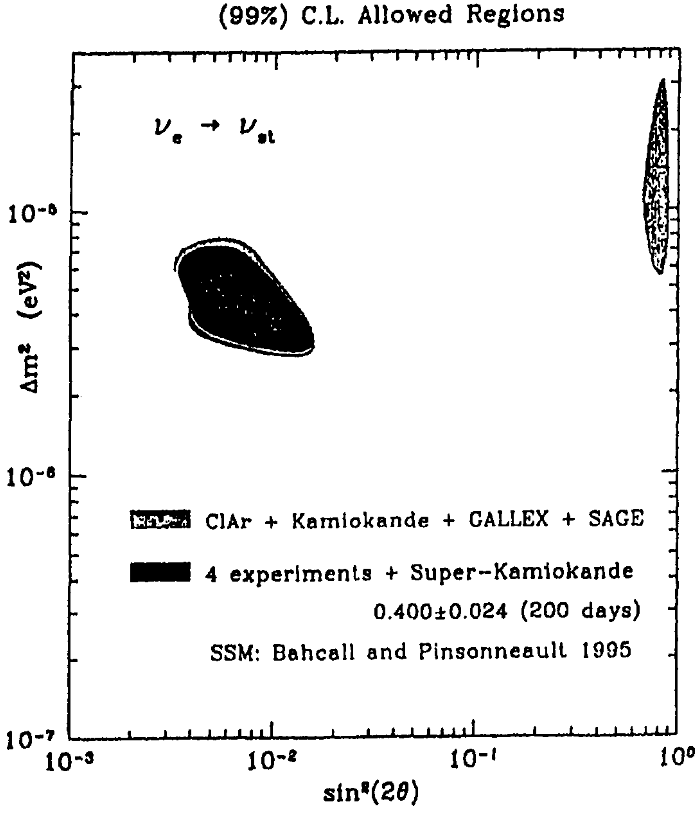

Figure 12 99% C.L. allowed regions in the $\Delta m^2 - \sin^2 2\theta$ MSW solution with $\nu_e - \nu_s$ conversion for the solar neutrino problem

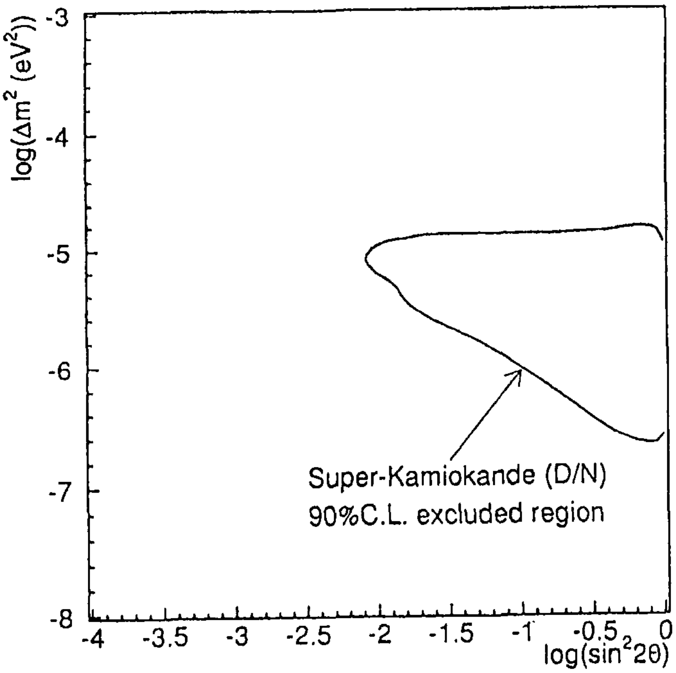

Figure 13 99% C.L. excluded regions found from Day–Night effect for the MSW solution found in Super-Kamiokande has already excluded the heart of the large-angle mixing solution (see Fig.13)

In summary it appears that

$$\Delta m^2_{LSND} \gg \Delta m^2_{Atm} \gg \Delta m^2_{\odot}. \tag{10.103}$$

Phenomenological analyses of neutrino oscillations find it difficult to accommodate the above heirarchy of mass ranges in a three-generation picture unless one of the experiments is sacrificed. For example, if we ignore LSND experiment, a possible solution is

$$\begin{aligned} \sin^2 2\theta_{\mu\tau} &\simeq 1,\ \Delta m^2_{\mu\tau} \simeq 5\times 10^{-3} \\ \sin^2 2\theta_{\mu e} &\simeq 10^{-2},\ \Delta m^2_{\mu e} \simeq 5\times 10^{-6} \end{aligned} \tag{10.104}$$

consistent with the mass pattern $m(\nu_\tau) \gg m(\nu_\mu) \geq m(\nu_e)$. Some suggest the remedy by introducing a fourth sterile neutrino, which may however, be disfavored by big bang nucleosynthesis (see Chap. 18).

We may conclude that neutrinos have masses, neutrinos mix and oscillate, mixing angles are small, solar neutrinos are detected, and the solar fusion is established. None of the above has been convincingly proven. Nevertheless we can say that the neutrino physics provides a circumstantial evidence for physics beyond the standard model. New experiments will test new physics and establish new mass scale(s) indicative of it.

10.6 Neutrino Magnetic Moment

With the definition

$$\mu_\nu = \kappa \frac{e\hbar}{2m_e} = \kappa \mu_B, \tag{10.105}$$

where μ_B is Bohr Magneton, magnetic moment interaction is

$$H_{mag} = \mu_\nu \boldsymbol{\sigma} \cdot \mathbf{B}. \tag{10.106}$$

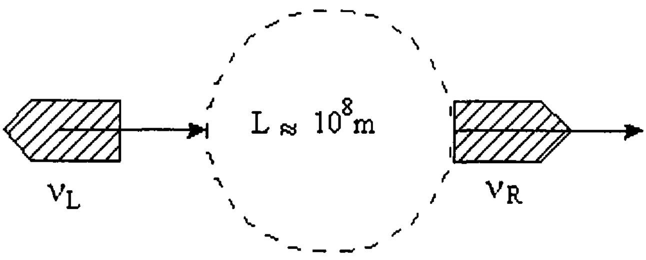

Figure 14 The conversion of ν_L into ν_R in the solar magnetic field.

Here **B** is the solar magnetic field. The neutrino spin would then process in the magnetic field, some left handed (LH) neutrinos would become RH and sterile to the detector as shown in Fig. 14. The conversion probability is determined by

$$\kappa \mu_B B \left(\frac{L}{\hbar c}\right). \tag{10.107}$$

Now the solar magnetic field in the convective zone of thickness $L \approx 2 \times 10^8 m$ is $B = (1-5) \times 10^3$ gauss, so that the conversion probability is

$$\kappa(5.79 \times 10^{-9} eV/G)(1-5) \times 10^3 G \frac{2 \times 10^8 m}{[3 \times 10^8 m/s][6.6 \times 10^{-16} eV.s]}$$

$$\approx \kappa(0.6-3)10^{10}. \tag{10.108}$$

This is $O(1)$ if $\kappa = (0.3-1) \times 10^{-10}$ giving $\mu_\nu \approx (0.3-1) \times 10^{-10} \mu_B$.

In the standard model,

$$(\mu_\nu)_{SM} = 3eG_F \frac{m_\nu}{\sqrt{2} 8\pi^2} \tag{10.109}$$

i.e.

$$(\mu_\nu)_{SM} \sim 3 \times 10^{-19} \mu_B (m_\nu/eV). \tag{10.110}$$

So if $\mu_\nu \approx 10^{-10}\mu_B$, this would definitely indicate physics beyond the standard model. Thus the question of dipole moment of neutrino is very important. What are the other limits on it? The best laboratory limit on m_ν comes from reactor experiments. In addition to the usual electroweak scattering via $W^\pm$ and Z^0 bosons exchange, the process

$$\bar{\nu}_e + e \to \bar{\nu}_e + e$$

could proceed via magnetic scattering which is large in the forward direction and for small E_ν. Consistency with measured cross-section requires

$$\mu_{\nu_e} < 10^{-10}\mu_B. \tag{10.111}$$

More stringent limits have, however, been quoted from astrophysics:

(1) Nucleosynthesis in the Early Universe

Presence of μ_ν mediates $\nu_L e^- \to \nu_R e^-$ scattering. If this occurs frequently in the era before the decoupling of the neutrinos, it doubles the neutrino species and increases the expansion rate of the universe, causing overabundance of helium. To avoid this,

$$\mu_\nu < 8.5 \times 10^{-11}\mu_B. \tag{10.112}$$

(2) Stellar Cooling

Magnetic scattering of neutrinos produced in thermonuclear reactions may occur, flipping the helicity $[\nu_L \to \nu_R]$ so that the outer regions of the star will no longer be opaque to neutrinos and cooling will proceed much faster. Applied to helium burning star in order that

$$\varepsilon_{\text{exotic}} < \varepsilon_{H_e}$$

where $\varepsilon_{\text{exotic}}$ denotes energy loss due to process of the above types while ε_{H_e} denotes energy generation rate. This gives

$$\mu < 10^{-11}\mu_B. \tag{10.113}$$

(3) Limit on μ_ν from Supernova 1987A

Neutrinos produced in the initial collapse state have high energies $\sim$ 100 MeV. These high energy neutrinos could escape following spin-flip magnetic scattering $[\nu_L \rightarrow \nu_R]$. Furthermore, a proportion can process back $\nu_R \rightarrow \nu_L$ in the galactic magnetic fields and the result on earth could be a signal of high energy ($\sim$ 100 MeV) neutrino interactions in the underground detector with a high rate [note that $\sigma \sim E^2$ in $\bar{\nu}_e + p \rightarrow e^+ + n$]. The observance of no signal implies

$$\mu_\nu \leq 10^{-12}\mu_B. \tag{10.114}$$

In view of the above upper limits on μ_ν, the neutrino spin precession mechanism does not appear to be a viable solution to the solar neutrino problem.

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Chapter 11
WEAK INTERACTIONS

11.1 $V - A$ Interaction

In analogy with electromagnetic interaction $J_\mu A^\mu$, Fermi proposed for β-decay the interaction $J^\mu J_\mu$, viz.

$$H_{int} = G\left[\bar{\Psi}_1(x)\gamma_\mu\Psi_2(x)\right]\left[\bar{\Psi}_3(x)\gamma^\mu\Psi_4(x)\right] + h.c. \tag{11.1}$$

The above interaction is for the process

$$2 \to 1 + 3 + \bar{4}\,(e.g.\, n \to p + e^- + \bar{\nu}_e).$$

The interaction (1) can be generalized using five Dirac bilinear co-variants. Thus the most general non-derivative four-fermion interaction can be written as

$$\begin{aligned} H_{int} &= \sum_i \left[\bar{\Psi}_1(x)\Gamma_i\Psi_2(x)\right]\left[\bar{\Psi}_3(x)\Gamma^i(C_i - C_i'\gamma_5)\Psi_4(x)\right] \\ &\quad + h.c. \end{aligned} \tag{11.2}$$

where Γ_i ($i = S, V, T, A, P$) are the five Dirac independent matrices: 1, γ_μ, $\sigma_{\mu\nu}, \gamma_\mu\gamma_5$, γ_5. In writing Eq. (2), we have taken into account the parity violation in β-decay.

For a massless Dirac particle, if Ψ is a solution of Dirac equation, then $\pm\gamma_5\Psi$ is also its solution. Without loss of generality, we take only negative sign. Suppose particle 4 is massless, then the bilinear

$$\bar{\Psi}_3(x)\Gamma^i\Psi_4(x) \to -\bar{\Psi}_3(x)\Gamma^i\gamma_5\Psi_4(x).$$

Hence for this case $C_i = C_i'$. Thus we can write Eq. (2) as

$$H_{int} = \sum_i \left[\bar{\Psi}_1 \Gamma_i \Psi_2\right] \left[\bar{\Psi}_3 C_i \Gamma^i (1-\gamma_5)\Psi_4\right] + h.c. \tag{11.3}$$

If we identify particle 4 with the neutrino, we have the result that only left handed neutrino takes part in weak processes. This is what is observed experimentally (see below). Thus irrespective of the fact whether neutrino is massless or not, Eq. (3) will hold if we take into account the fact that only left handed neutrinos take part in weak processes. Suppose we impose the chiral transformation for the field Ψ_3 viz. $\Psi_3 \to -\gamma_5 \Psi_3$, then if H_{int} is to be invariant under such a transformation, we have

$$C_S = C_P = C_T = 0.$$

Hence Eq. (3) becomes

$$\begin{aligned} H_{int} &= \left[C_V \bar{\Psi}_1 \gamma_\mu \Psi_2 - C_A \bar{\Psi}_1 \gamma_\mu \gamma_5 \Psi_2\right] \left[\bar{\Psi}_3 \gamma^\mu (1-\gamma_5)\Psi_4\right] \\ &= \frac{G_F}{\sqrt{2}} \left[\bar{\Psi}_1 \gamma_\mu (1-\varepsilon\gamma_5)\Psi_2\right] \left[\bar{\Psi}_3 \gamma^\mu (1-\gamma_5)\Psi_4\right], \end{aligned} \tag{11.4}$$

where we put

$$C_V = \frac{G_F}{\sqrt{2}}, \ \frac{C_A}{C_V} = \varepsilon. \tag{11.5}$$

Further we note that if we impose the chiral transformation on fields Ψ_1 or Ψ_2, we have

$$C_V = C_A \tag{11.6}$$

i.e. $\varepsilon = 1$ or $V - A$ theory. We conclude that if one requires invariance of the four-fermion interaction under the chirality transformation of each field separately, we have the $V - A$ theory.

We have written Eq. (2) in the order 1 2 3 4. We can go to the order 3 2 1 4 by Fierz reordering theorem:

$$K_i(3214) = \sum_{j=1}^{5} \lambda_{ij} K_j(1234). \tag{11.7}$$

The coefficients λ_{ij} are given by the matrix

$$\lambda_{ij} = -\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}, \tag{11.8}$$

where

$$K_i(1234) = \left[\bar{\Psi}_1\Gamma_i\Psi_2\right]\left[\bar{\Psi}_3\Gamma^i\Psi_4\right]. \tag{11.9}$$

It is obvious that

$$K_i(3214) = K_i(1432).$$

If we denote by S, V, T, A, P the five quadrilinears appearing in the order (1 2 3 4) and S', V', T', A', P' when they appear in the order (3 2 1 4), then from Eqs. (7) and (8) we get

$$\begin{aligned} V' - A' &= V - A \\ S' - T' + P' &= S - T + P \end{aligned} \tag{11.10}$$

i.e. these combinations are invariant under Fierz rearrangement.

11.1.1 *Helicity of the neutrino*

To obtain a direct measurement of neutrino helicity, the following reaction was studied

$$\begin{aligned} {}^{152}Eu_{(J^P=0^-)} \quad \longrightarrow \quad & {}^{152}Sm^*_{(J^P=1^-)} + \nu_e \\ & \qquad \downarrow \\ & \qquad \longrightarrow \left({}^{152}Sm_{(J^P=0^+)} + \gamma\right). \end{aligned}$$

The main point of this experiment is that we can select those γ rays from the decay of the excited state which go opposite to the ν_e direction (i.e., in the direction of the recoil nucleus) by having them resonance-scatter from a target of ${}^{152}Sm$. Balancing the spin along the upward z direction (ν_e is assumed to be emitted along this direction), one finds that the helicity of the downward γ-ray will be the same as that for the upward ν_e. By measuring the circular polarization of γ-ray, the experiment fixed the helicity of the γ-ray as negative, indicating a left-handed ν_e. Thus it is established that only left-handed neutrinos take part in weak processes.

11.2 Classification of Weak Processes

(i) Purely leptonic processes

The well known example is μ-meson decay

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu.$$

In this process four well known particles μ^-, e^-, ν_e, ν_μ, called leptons, take part. The decay process is described by $V-A$ interaction [cf. Eqs. (4) and (6)].

$$\begin{aligned} -H_{int} = L_W^\ell &= -\frac{G_F}{\sqrt{2}} \{\bar{\nu}_\mu \gamma^\mu (1-\gamma_5)\mu\} \{\bar{e}\gamma_\mu (1-\gamma_5)\nu_e\} \\ &= -\frac{G_F}{\sqrt{2}} L^\mu_{(\mu)} L^\dagger_{(e)\mu}. \end{aligned} \tag{11.11a}$$

$L^\mu_{(\mu)}$ and $L_{(e)\mu}$ are lepton currents associated respectively with μ meson and its associated neutrino ν_μ and e^- and ν_e

$$L^\mu_{(\mu)} = \bar{\nu}_\mu \gamma^\mu (1-\gamma_5)\mu \tag{11.11b}$$

$$L_{(e)\mu} = \bar{\nu}_e \gamma_\mu (1-\gamma_5) e. \tag{11.11c}$$

The γ_μ and γ_5 ($\equiv i\gamma^0\gamma^1\gamma^2\gamma^3$) appearing above are the usual Dirac matrices. We write the lepton current as

$$L^\mu = L^\mu_{(\mu)} + L^\mu_{(e)}. \tag{11.12}$$

Here $L^\dagger_\mu$ denotes the hermitian conjugate of L_μ. One can also picture the process (1) as being mediated by a vector boson W_μ, the so-called weak vector boson. This is shown in Fig. 1 below:

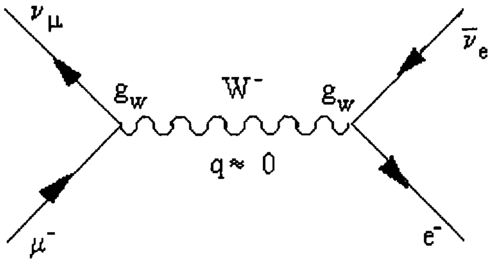

Figure 1 The muon decay mediated by a W−boson.

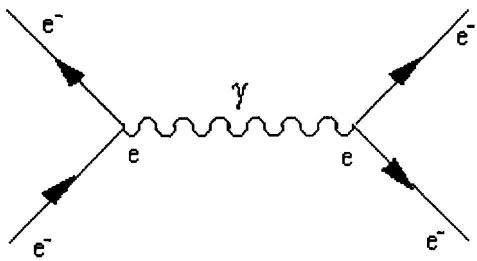

Figure 2 Electromagnetic interaction mediated by a photon.

Thus all leptonic weak processes can be described by interaction of the form

$$L_W = -g_W L_\mu W^{-\mu} + h.c. \tag{11.13}$$

where h.c. denotes the hermitian conjugate. Note that Eq. (13) is analogous to electromagnetic interaction of say electron which is mediated by photon and is shown in Fig. 2.

The interaction responsible for the process shown in Fig. 2 is the usual electromagnetic interaction

$$L^{e.m.} = -e j_\mu^{e.m.} a^\mu, \tag{11.14}$$

where a^μ is the photon field and $j_\mu^{e.m.}$ is the electromagnetic current:

$$j_\mu^{e.m.} = \bar{e}\gamma_\mu e. \tag{11.15}$$

Note the similarity between Eqs. (13), (14), (11b,c) and (15) respectively. Both the electromagnetic and weak currents are vector in character, the appearance of γ_5 in weak current is due to the fact that parity is not conserved in weak interaction, in fact it is violated maximally. The coupling of electromagnetic current with the photon is characterized by electric charge (related to the fine structure constant α by $\frac{e^2}{4\pi} = \alpha = 1/137$) while that of weak current with the weak vector boson field W_μ is characterized by g_W (related to the Fermi coupling constant G_F by $\frac{g_W^2}{4\pi} = \frac{G_F}{4\pi\sqrt{2}m_W^2}$).

(ii) Semileptonic Processes

Some examples of these processes are given below

$$\begin{aligned}
n &\to p + e^- + \bar{\nu}_e \\
\pi^+ &\to e^+ + \nu_e,\ \mu^+ + \nu_\mu \\
\pi^- &\to e^- + \bar{\nu}_e,\ \mu^- + \bar{\nu}_\mu \\
\Sigma^- &\to \Lambda^0 + e^- + \bar{\nu}_e \\
\Sigma^- &\to n + e^- + \bar{\nu}_e \\
\Sigma^0 &\to p + e^- + \bar{\nu}_e \\
K^+ &\to \pi^0 + e^+ + \nu_e \\
K^- &\to \pi^0 + e^- + \bar{\nu}_e.
\end{aligned} \tag{11.16}$$

From these processes, one notes the following rules:

1. The hadronic charge changes by one unit i.e. $\Delta Q = \pm 1$.

2. In the first four processes, strangeness does not change, in the last four processes it changes by one unit.

For hadrons, Gell-Mann-Nishijima relation

$$Q = I_3 + \frac{Y}{2}$$

implies that for $\Delta Q = \pm 1$, either $\Delta I_3 = \pm 1$, $\Delta Y = 0$ or $\Delta I_3 = \pm 1/2$, $\Delta Y = \pm 1$, if we assume that $\Delta Y = 2$ processes are suppressed. The processes of first kind are called hypercharge conserving processes and those of second kind are called hypercharge changing processes. In all the processes listed above, we see that either $\Delta Y = 0$ or $\Delta Y = \pm 1$; no weak process with $|\Delta Y| > 1$ is seen with the same strength as $|\Delta Y| \geq 1$ transitions. Thus we have the selection rule $\Delta Y = 0, \pm 1$, $\Delta Q = \Delta Y$.

Since there are so many hadrons in nature, therefore to deal with semi-leptonic decays of each of them would be very tedious. Thus we use the simple picture of hadrons made up of quarks. The main thing about the quarks is that they are regarded as truly elementary similar to leptons. Their weak and electromagnetic interactions would then be like those of leptons. Thus in analogy with Eqs. (15) and (11), their electromagnetic and weak currents are respectively

$$j_\mu^{e.m.} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s \tag{11.17}$$

while

$$J_\mu^h = \bar{u}\gamma_\mu(1-\gamma_5)d', \tag{11.18}$$

where

$$d' = \cos\theta_c d + \sin\theta_c s, s' = -\sin\theta_c d + \cos\theta_c s. \tag{11.19}$$

Here θ_c is the Cabibbo angle; its value is $\theta_c = 13^o$ or $\sin\theta_c = 0.22$. This is introduced since it is seen experimentally that decay rates for $|\Delta Y| = 1$ semi-leptonic decays are suppressed by a factor of about $1/16$ compared to those for $\Delta Y = 0$ processes. We shall

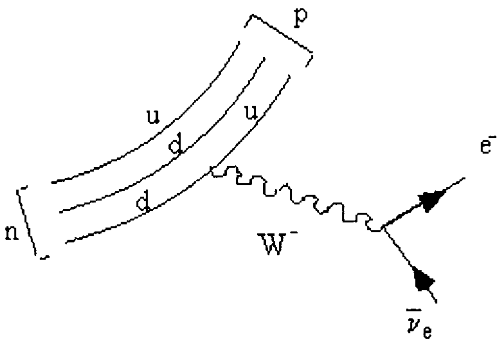

Figure 3 Quark level process for neutron β-decay.

deal with s' in Chap. 13. Then in analogy with Eq. (11) or (13) the interaction responsible for fundamental processes like

$$\begin{aligned} d &\to u + e^- + \bar{\nu}_e \\ s &\to u + e^- + \bar{\nu}_e \end{aligned} \tag{11.20}$$

would be

$$L_W^{h.l.} = -\frac{G_F}{\sqrt{2}} J_\mu^h L_{(e)}^{\mu\dagger}$$

or

$$L_W = -g_W J_\mu^h W^{-\mu} + h.c. \tag{11.21}$$

In this picture neutron β-decay, $\Lambda - \beta$-decay and $\bar{K}^0 \to \pi^+ + e^- + \bar{\nu}_e$, for example, would be pictured as shown in Figs. 3-5.

Note the very important fact that both the leptonic and hadronic weak currents in (11b, c) and (18) are charged i.e. they carry one unit of charge and the hadronic weak currents (18) satisfy the selection rules $|\Delta Y| \leq 1$ and $\Delta Q = \Delta Y$. We also note that in terms of flavor SU(3) notation we can write

$$J_\mu^h = \cos\theta_c J_\mu^+ + \sin\theta_c J_\mu^1 \tag{11.22}$$

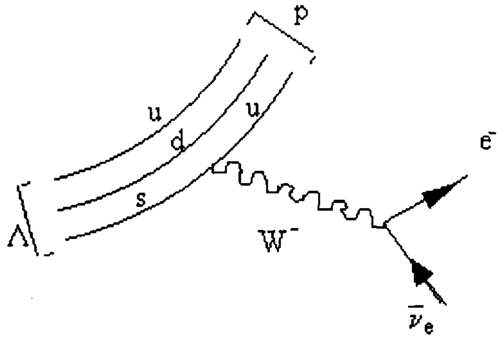

Figure 4 Quark level process for neutron $\Lambda - \beta$-decay.

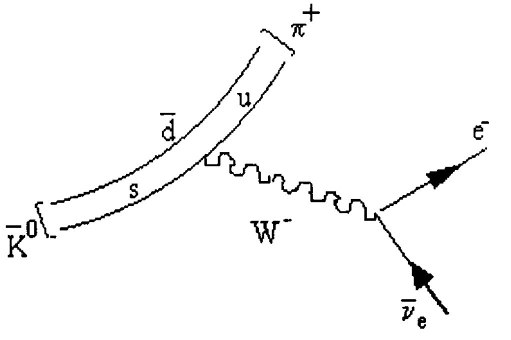

Figure 5 Quark level process for $\bar{K}^0 \to \pi^+ e^- \bar{\nu}_e$.

where weak hadronic current is a linear combination of vector and axial vector currents involving respectively γ_μ and $\gamma_\mu\gamma_5$ and are given by

$$\begin{aligned} J_\mu^\pm &= V_\mu^\pm - A_\mu^\pm \\ &= \bar{q}\frac{1}{2}(\lambda_1 \pm i\lambda_2)\gamma_\mu(1-\gamma_5)q \qquad (11.23a) \\ J_\mu^1 &= \bar{q}\frac{1}{2}(\lambda_4 + i\lambda_5)\gamma_\mu(1-\gamma_5)q. \qquad (11.23b) \end{aligned}$$

Note also that

$$\begin{aligned} J_{3\mu} &= \bar{q}\frac{1}{2}\lambda_3\gamma_\mu(1-\gamma_5)q \qquad (11.24a) \\ J_{8\mu} &= \bar{q}\frac{1}{2}\lambda_8\gamma_\mu(1-\gamma_5)q \qquad (11.24b) \\ J_\mu^{em} &= V_{3\mu} + \frac{1}{\sqrt{3}}V_{8\mu}. \qquad (11.24c) \end{aligned}$$

Here $q = \begin{pmatrix} u \\ d \\ c \end{pmatrix}$. The heavy quarks and s' will be considered in Chap. 13.

(iii) Non-Leptonic Processes

Here no leptons are involved. The well known non-leptonic processes are:

$$\begin{aligned} \Lambda &\to p\pi^-(\Lambda_-^0) \\ \Lambda &\to p\pi^0(\Lambda_0^0) \qquad (11.25a) \end{aligned}$$

$$\begin{aligned} \Sigma^- &\to n\pi^-(\Sigma_-^-) \\ \Sigma^+ &\to p\pi^0(\Sigma_0^+) \qquad (11.25b) \\ \Sigma^+ &\to n\pi^+(\Sigma_+^+) \end{aligned}$$

$$\begin{aligned}\Xi^- &\rightarrow \Lambda\pi^-(\Xi^-_-) \\ \Xi^0 &\rightarrow \Lambda\pi^0(\Xi^0_0) \end{aligned} \qquad (11.25c)$$

or

$$\begin{aligned} K^0, \bar{K}^0 &\rightarrow \pi^+\pi^-, \pi^0\pi^0 \\ K^\pm &\rightarrow \pi^\pm, \pi^0, \text{ etc.} \end{aligned} \qquad (11.26)$$

Note that all these decays are strangeness changing ($|\Delta S| = 1$). Let us concentrate on the decays (25), the so-called non-leptonic decays of hyperons. If we consider the decaying particle in its rest frame, the conservation of angular momentum J gives

$$J_{in} \equiv \frac{1}{2} = J_{\text{final}} \equiv \ell + S,$$

where ℓ is the relative orbital angular momentum of the pion and the baryon in final state. Since spin $s = 1/2$, ℓ can be 0 or 1. The pion being pseudoscalar (having odd intrinsic parity), the relative parity of final state with respect to the initial state is

$$\begin{aligned} P_f &= (-1)^0(-1) = -1 \text{ odd for } \ell = 0 \\ &= (-1)^1(-1) = +1 \text{ even for } \ell = 1. \end{aligned}$$

The $s-$wave ($\ell = 0$) decays are parity violating while p-wave ($\ell = 1$) decays are parity conserving. Accordingly decays (25) are governed by two amplitudes, parity violating ($s-$wave) and parity conserving ($p-$wave). We can write the Lagrangian responsible for non-leptonic decays as

$$\begin{aligned} L_W^{(h)} &= L_W^{h(p.v)} + L_W^{h(p.c)} \\ &= -\frac{G_F}{\sqrt{2}} J_\mu^h J^{h\mu\dagger} + h.c., \end{aligned} \qquad (11.27)$$

where J_μ^h is given in (18). The $|\Delta S| = 1$ component of (27) behaves as

$$\frac{G_F}{\sqrt{2}} \sin\theta_c \cos\theta_c \{\bar{s}\gamma_\mu(1-\gamma_5)u\} \cdot \{\bar{u}\gamma^\mu(1-\gamma_5)d\}. \qquad (11.28)$$

Now u and d belong to isospin doublet $I = 1/2$ while s is isospin singlet $I = 0$. Thus from the combination of angular momentum rules (isospin behaves like angular momentum) first term in curly brackets in (28) has $I = 1/2$ while the second term in curly brackets has $I = 0,\ 1$. Thus the interaction contains both $\Delta I = 1/2$ and $3/2$ parts. Experimentally $\Delta I = 1/2$ part predominates over $\Delta I = 3/2$ and then (25a), (25b) and (25c) respectively get related among themselves. We shall come to these relations later.

11.2.1 μ-decay

Consider the μ-decay

$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e.$$

From Eq. (4), we can write the interaction as

$$H_{int} = \frac{G_F}{\sqrt{2}} \left[\bar{e}\gamma_\mu(1 - \varepsilon\gamma_5)\mu\right] \left[\bar{\nu}_\mu \gamma^\mu (1 - \gamma_5)\nu_e\right]. \tag{11.29}$$

The interaction written in this order is called the charge retention order. It is easier to deal with this order in calculations. Here we have assumed 2-component neutrinos (left- handed ν_μ and right-handed $\bar{\nu}_e$) but have allowed $V - \varepsilon A$ interaction, where for $V - A$, $\varepsilon = 1$ and in that case by Fierz rearrangement we get Eq. (11a).

From Eq. (29), we can write the T-matrix for μ^- decay:

$$\begin{aligned} T \;=\; & \frac{-1}{(2\pi)^6}\sqrt{\frac{m_\mu m_e m_{\nu_\mu} m_{\nu_e}}{p_{10}p_{20}k_{10}k_{20}}}\frac{G_F}{\sqrt{2}} \\ & \times \left[\bar{u}(p_2)\gamma_\lambda(1 - \varepsilon\gamma_5)u(p_1)\right]\left[\bar{u}(k_2)\gamma^\lambda(1 - \gamma_5)v(k_1)\right] \end{aligned} \tag{11.30}$$

where p_1, p_2, k_1 and k_2 are the four momenta of μ^-, e^-, ν_μ and $\bar{\nu}_e$ respectively and $u(p_1), u(p_2), u(k_1)$ and $v(k_2)$ are Dirac spinors. From Eq. (30), we get

$$\begin{aligned} d\Gamma \;=\; & \frac{1}{(2\pi)^5}\delta^4\left(p_1 - p_2 - k_1 - k_2\right)\frac{m_\mu m_e m_{\nu_\mu} m_{\nu_e}}{p_{10}p_{20}k_{10}k_{20}} \\ & \times \left|M\right|^2 d^3p_2 d^3k_1 d^3k_2 \end{aligned} \tag{11.31}$$

where

$$|M|^2 = \overline{\sum_{spin}} |F|^2 = \left(\frac{G_F^2}{2}\right) \overline{\sum_{spin}} |\bar{u}(p_2)\gamma_\lambda(1-\varepsilon\gamma_5)u(p_1)|^2 \times \left|\bar{u}(k_2)\gamma^\lambda(1-\gamma_5)v(k_1)\right|^2 . \tag{11.32}$$

We can easily calculate $|M|^2$ using the standard trace techniques. Neglecting the neutrino masses, we get

$$\begin{aligned} |M|^2 = & \frac{G_F^2}{m_\mu m_e m_{\nu_\mu} m_{\nu_e}} \\ & \times \left[\left(1+\varepsilon^2\right)(p_2 \cdot k_2\ p_1 \cdot k_1 + p_2 \cdot k_1\ p_1 \cdot k_2)\right. \\ & - \left(1-\varepsilon^2\right) m_\mu m_e k_1 \cdot k_2 \\ & +2\varepsilon\,(p_1 \cdot k_2\ p_2 \cdot k_1 - p_1 \cdot k_1\ p_2 \cdot k_2)\,. \end{aligned} \tag{11.33}$$

Since neutrinos are not observed, we integrate over $d^3k_1 d^3k_2$. Performing these integrations, and writing $d^3p_2 = 4\pi p_e E_e dE_e$, we get

$$d\Gamma = m_\mu \frac{p_e E_e dE_e}{24\pi^3} G_F^2 (1+|\varepsilon|^2) \times \left[3W - 2E_e - \frac{m_e^2}{E_e} + 6\eta \frac{m_e}{E_e}(W - E_e)\right], \tag{11.34}$$

where

$$\eta = \frac{1}{2}\frac{|\varepsilon|^2 - 1}{|\varepsilon|^2 + 1} \tag{11.35}$$

$$W = \frac{m_\mu^2 + m_e^2}{2m_\mu}. \tag{11.36}$$

In evaluating the final result (34), we have gone to the rest frame of the muon:

$$\begin{aligned} E_\mu &= m_\mu \\ m_\mu &= E_e + E_1 + E_2 \\ O &= \mathbf{p}_e + \mathbf{k}_1 + \mathbf{k}_2. \end{aligned} \tag{11.37}$$

11.2.2 Remarks

(1) It is always possible to take C_V as real and take $C_A = \varepsilon C_V$ (ε complex).

(2) The electron spectrum does not distinguish between $\varepsilon = +1\,(V-A)$ or $\varepsilon = -1\,(V+A)$ interaction.

(3) Any deviation from $\varepsilon = \pm 1$ can be determined by measuring η in the electron spectrum. Since η is the coefficient of $\left(\frac{m_e(W-E_e)}{E_e}\right)$, it plays a minor role except at low electron energies, where measurements are difficult. The best experimental value of η is

$$\eta = -0.007 \pm 0.013 \tag{11.38}$$

which is consistent with zero.

(4) It is instructive to write the electron energy spectrum (34) as

$$d\Gamma = \frac{m_\mu p_e E_e dE_e}{12\pi^3} G_F^2 \left(1 + |\varepsilon|^2\right)$$
$$\times \left[3(W - E_e) + 2\rho\left(\frac{4}{3}E_e - W - \frac{1}{3}\frac{m_e^2}{E_e}\right) + 3\eta\frac{m_e}{E_e}(W - E_e)\right], \tag{11.39}$$

where $\rho = 3/4$; ρ is called the Michel parameter. In fact the most general interaction without assuming two-component neutrinos gives the electron spectrum of the form within the square brackets. The experimental value of $\rho = 0.7518 \pm 0.0026$ is in excellent agreement with $\rho = 3/4$ as given by $V - \varepsilon A$ theory. We conclude that the two-component neutrino hypothesis is in an excellent agreement with the experimental results. Finally integrating Eq. (34), we obtain

$$\Gamma = \tau_\mu^{-1} = G_F^2 P, \tag{11.40a}$$

where

$$P = \left[1 - \frac{8m_e^2}{m_\mu^2}\right]\frac{m_\mu^5}{192\pi^3}. \tag{11.40b}$$

If we include $O(\alpha)$ radiative corrections

$$\tau_\mu^{-1} = G_F^2 P \left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right) + \left(1 + \frac{2\alpha}{3\pi}\ln\frac{m_\mu}{m_e}\right)\right], \qquad (11.40c)$$

where the fine structure constant $\alpha = \frac{1}{137.036}$. The Fermi constant G_F determined from (40c), using the experimental value for $\tau_\mu = 2.19703 \times 10^{-6}$ sec, is

$$G_F = 1.16637 \times 10^{-5}\ \text{GeV}^{-2}. \qquad (11.41)$$

Decay of polarized muon

We have seen that the electron spectrum cannot determine the sign of ε. In order to determine ε, we consider the decay of polarized muon. Let n_μ be the polarization vector of muon. We note that

$$\begin{aligned} n^2 &= n_\mu n^\mu = -1, \\ n \cdot p_1 &= 0. \end{aligned} \qquad (11.42)$$

In the rest frame of the muon $m_\mu n_0 = 0$; thus $n_0 = 0$ and $n \equiv (0,\ \mathbf{n})$. For this case in taking the trace, we put

$$u(p_1)\,\bar{u}\ (p_1) = \left[\left(\frac{\not p_1 + m_\mu}{2m_\mu}\right)\left(\frac{\gamma_5\ \gamma \cdot n + 1}{2}\right)\right]. \qquad (11.43)$$

Using the standard trace techniques, and performing the integrations over $d^3k_1\, d^3k_2$, the differential spectrum in the asymmetry angle for μ^- decay is

$$\begin{aligned} d\Gamma &= \left(\frac{d\Omega_\gamma}{4\pi}\right)\frac{G_F^2(1+|\varepsilon|^2)}{12\pi^3}\, m_\mu\, p_e^2\, dE_e\, \xi \cos\gamma \\ &\quad \times \left[W - 2E_e + \frac{m_e^2}{m_\mu^2}\right], \end{aligned} \qquad (11.44)$$

where γ is the angle between the electron momentum and the μ−spin direction and

$$\xi = \frac{2\ R_e\ \varepsilon}{1 + |\varepsilon|^2}. \qquad (11.45)$$

It is instructive to write Eq. (44) in the form

$$d\Gamma = -\left(\frac{d\Omega_\gamma}{4\pi}\right)\frac{G_F^2}{6\pi^3}\left(1+|\varepsilon|^2\right) m_\mu p_e^2 dE_e \xi\cos\gamma \times\left[(W-E_e)+2\delta\left(\frac{4}{3}E_e - W - \frac{1}{3}\frac{m_e^2}{m_\mu}\right)\right], \quad (11.46)$$

where $\delta = 3/4$ for two-component neutrinos viz. for $V - \varepsilon A$ theory. For a general interaction without assuming two-component neutrino, the asymmetry distribution in angle γ is of the form given within the square brackets. The experimental value of δ is 0.749 ± 0.004 in excellent agreement with two-component neutrino hypothesis.

The experimental value of ξ is given by

$$\xi P_\mu = 1.003 \pm 0.008. \quad (11.47)$$

11.2.3 Semi-leptonic processes

For a semi-leptonic weak process we can write the interaction Hamiltonian as [cf. Eq. (21)].

$$-L_W^{h.l.} = H_{\text{int}} = \frac{G_F}{\sqrt{2}} J_\lambda(x)\left[\bar{e}(x)\,\gamma^\lambda(1-\gamma_5)\,\nu_e(x)\right] + h.c. \quad (11.48)$$

To first order in weak interaction, the T-matrix for a semi-leptonic process of the type

$$A \to B + e^- + \bar{\nu}_e$$

is given by

$$T = -\frac{G_F}{\sqrt{2}}\langle B|J_\lambda|A\rangle \frac{1}{(2\pi)^3}\sqrt{\frac{m_\nu m_e}{k_0 k_0'}}\left[\bar{u}(k')\,\gamma^\lambda(1-\gamma_5)\,v(k)\right], \quad (11.49)$$

where k' and k are four momenta of electron and antineutrino. We denote four momenta of A and B by p and p'.

(a) Baryon Decays

We consider the case when A and B are spin 1/2 baryons and $\langle B|J_\lambda|A\rangle = \langle B|V_\lambda - A_\lambda|A\rangle$. From Lorentz invariance, the most general structure of these matrix elements is given by $[q = p' - p]$

$$\langle B(p')|V_\lambda|A(p)\rangle = \frac{1}{(2\pi)^3}\sqrt{\frac{m_A m_B}{p_0 p_0'}}\bar{u}_B(p')$$

$$\times\left[g_V\left(q^2\right)\gamma_\lambda + i f_V\left(q^2\right)\sigma_{\lambda\nu}q^\nu + h_V\left(q^2\right)q_\lambda\right]u_A(p) \qquad (11.50a)$$

$$\langle B(p')|A_\lambda|A(p)\rangle = \frac{1}{(2\pi)^3}\sqrt{\frac{m_A m_B}{p_0 p_0'}}\bar{u}_B(p')$$

$$\times\left[g_A\left(q^2\right)\gamma_\lambda\gamma_5 + f_A\left(q^2\right)\gamma_5 q_\lambda - i\sigma_{\lambda\nu}\, q^\nu\gamma_5 h_A\left(q^2\right)\right]u_A(p). \qquad (11.50b)$$

Since the momentum transfer $q = p' - p$ is very small compared to the mass of A or B for the processes we are considering, we can write

$$\langle B(p')|J_\lambda|A(p)\rangle = \frac{1}{(2\pi)^3}\sqrt{\frac{m_A m_B}{p_0 p_0'}}\bar{u}_B(p') \left[g_V\,\gamma_\mu - g_A\,\gamma_\mu\gamma_5\right]u_A(p). \qquad (11.51)$$

Now we shall take A and B as members of the spin 1/2 baryon octet and then

$$J_\lambda = \cos\theta_c\, J_\lambda^+ + \sin\theta_c\, J_\lambda^1, \qquad J_\lambda^\dagger = \cos\theta_c\, J_\lambda^- + \sin\theta_c\, J_\lambda^{1\dagger}, \qquad (11.52)$$

where $J_\lambda^\pm$ and J_λ^1, $J_\lambda^{1\dagger}$ are $1 \pm i2$ and $4 \pm i5$ components of octet of currents $J_{i\lambda}$ $(i = 1, \cdots, 8)$. As shown in Chap. 5

$$\langle B_k|A_{i\lambda}|B_j\rangle \equiv \frac{1}{(2\pi)^3}\sqrt{\frac{m_B m_B}{p_0 p_0'}}\bar{u}(p')\gamma_\lambda\gamma_5 u(p) g_A^{ijk}, \qquad (11.53a)$$

where

$$g_A^{ijk} = if_{ijk}F + d_{ijk}D. \tag{11.53b}$$

Since $F_i = \int V_{i0}(0, \mathbf{x})\, d^3x$ is a generator of SU(3), it follows that

$$\langle B_k |V_{i\lambda}| B_j\rangle \equiv \frac{1}{(2\pi)^3}\sqrt{\frac{m_B^2}{p_0 p_0'}}\bar{u}(p')\gamma_\lambda u(p) g_V^{ijk}, \tag{11.54a}$$

where

$$g_V^{ijk} = if_{ijk}. \tag{11.54b}$$

Thus if we neglect the momentum transfer q^2, ($q^2 \approx 0$), the matrix elements $\langle B_k |J_{i\lambda}| B_j\rangle$ are essentially determined in terms of Cabibbo angle θ_c and the two reduced matrix elements F and D. Using Eqs. (53) and (54), the matrix elements of these decays are given below:

Decay $B \to B' l \nu_e$	Vector current g_V	Axial vector current g_A	Ratio g_A/g_V	Expt. value of g_A/g_V
$n \to p$	$\cos\theta_c$	$\cos\theta_c\,(F+D)$	$F+D$	1.2670 ± 0.0035
$\Lambda \to p$	$-\sqrt{\frac{3}{2}}\sin\theta_c$	$-\sqrt{3/2}\sin\theta_c \times \left(F+\frac{1}{3}D\right)$	$F+\frac{1}{3}D$	0.718 ± 0.015
$\Sigma^- \to n$	$-\sin\theta_c$	$\sin\theta_c\,(F-D)$	$F-D$	-0.340 ± 0.017
$\Xi^- \to \Lambda$	$\sqrt{\frac{3}{2}}\sin\theta_c$	$\sqrt{3/2}\sin\theta_c \times \left(F-\frac{1}{3}D\right)$	$F-\frac{1}{3}D$	0.25 ± 0.05

In order to test the octet hypothesis, we note that if we determine F and D from the first two decays, we find $F - D$ and $F - \frac{1}{3}D$ for the third and fourth decays in agreement with their experimental values. The parametrization given in Eqs. (53b) and (54) is in excellent agreement with experiment. Using the first two entries of the above table, we find $F = 0.444 \pm 0.015$, $D = 0.823 \pm 0.015$.

As an example to show how g_A/g_V is determined we consider the case of neutron $\beta-$ decay $n \to p + e^- + \bar{\nu}_e$ in detail, where from Eqs. (51) and (52) we have

$$\begin{aligned}\left\langle p(p') \left| J_\mu^{(+)} \right| n(p) \right\rangle &= \frac{1}{(2\pi)^3}\sqrt{\frac{m_p m_n}{p_0 p_0'}}\bar{u}(p') \\ &\quad \times \left[g_V \gamma_\mu - g_A \gamma_\mu \gamma_5\right] u(p) \qquad (11.55)\end{aligned}$$

with $g_V = \cos\theta_c$, $g_A = \cos\theta_c\,(F + D)$. In the rest frame of neutron, we write $k' \equiv (E_e,\ \mathbf{p}_e)$, $k \equiv (E_\nu,\ \mathbf{p}_\nu)$, $\mathbf{p} = 0$, $\mathbf{p}' + \mathbf{p}_e + \mathbf{p}_\nu = 0$. Since q is very small as compared to neutron and proton masses, we can treat them non-relativistically. Then

$$\begin{aligned}\left\langle p(p') \left| J_0^{(+)} \right| n(p) \right\rangle &= \frac{1}{(2\pi)^3}\ \chi_p^+\ g_V\ \chi_n \\ \left\langle p(p') \left| J_i^{(+)} \right| n(p) \right\rangle &= -\frac{1}{(2\pi)^3}\ \chi_p^+\ g_A\ \sigma_i\ \chi_n. \qquad (11.56)\end{aligned}$$

Let us write the leptonic part as

$$L^\mu = \bar{u}(k')\ \gamma^\mu\,(1 - \gamma_5)\, v\,(k)\,. \qquad (11.57)$$

The amplitude F [cf. Eq. (49) and Eq. (2.75)] is given by

$$F = -\frac{G_F}{\sqrt{2}}\ \chi_p^+ \left[g_V L^0 + g_A\ \boldsymbol{\sigma}\cdot\ \mathbf{L}\right]\chi_n. \qquad (11.58)$$

We now sum over proton spin and lepton spin and define the neutron spin $\mathbf{S}_n$ as

$$\mathbf{S}_n = \frac{1}{2}\chi_n^+\ \sigma\ \chi_n. \qquad (11.59)$$

Using Eq. (2.110), we get for the probability distribution

$$\begin{aligned}d\Gamma &= \frac{G_F^2}{(2\pi)^5}\ p_e^2(E_{\max} - E_e)\ A\left[1 + \lambda\frac{\mathbf{p}_e\cdot\mathbf{p}_\nu}{E_e\ E_\nu}\right. \\ &\quad \left. + \hat{\mathbf{S}}_n\cdot\left(A'\frac{\mathbf{p}_e}{E_e} + B\frac{\mathbf{p}_\nu}{E_\nu} + D\frac{\mathbf{p}_e\times\mathbf{p}_\nu}{E_e\ E_\nu}\right)\right] dp_e\ d\Omega_e\ d\Omega_\nu \\ &\qquad (11.60)\end{aligned}$$

where $\hat{\mathbf{S}}_n$ is the direction of the neutron spin and

$$\begin{aligned} A &= \left[|g_V|^2 + 3|g_A|^2\right] \\ \lambda &= \frac{1}{A}\left[|g_V|^2 - |g_A|^2\right] \\ A' &= -\frac{2}{A}\left[|g_A|^2 - Re\ g_V\ g_A^*\right] \\ B &= \frac{2}{A}\left[|g_A|^2 + Re\ g_V\ g_A^*\right] \\ D &= \frac{2}{A} Im\ g_V\ g_A^*. \end{aligned} \tag{11.61}$$

The experimental data give the following values of these correlation functions,

$$\begin{aligned} \lambda &= -0.102 \pm 0.005 \\ A' &= -0.1162 \pm 0.0012 \\ B &= 0.990 \pm 0.008 \\ D &= (-0.5 \pm 1.4) \times 10^{-3}. \end{aligned} \tag{11.62}$$

If we write $x = |g_A/g_V|$, then the value of λ gives

$$x = |g_A/g_V| = 1.261 \pm 0.004. \tag{11.63}$$

The very fact that B is nearly 1 implies the maximum parity violation in β–decay. The value of A' [assuming g_V and g_A are relatively real, see below] gives $|g_A/g_V| = 1.267 \pm 0.014$, consistent with Eq. (63). A non-zero value of D would imply time reversal violation in β-decay. The experimental value of D is nearly zero and show that time reversal invariance holds. If we write $g_A/g_V = -xe^{i\phi}$, where for $\phi = 0$ or π, T invariance holds, we obtain

$$\phi = (180.07 \pm 0.18)^0. \tag{11.64}$$

Finally, from Eq. (60), we obtain for the decay width Γ:

$$\Gamma = \frac{AG_F^2}{2\pi^3} m_e^5 f(\rho_0), \tag{11.65}$$

where

$$A = |g_V|^2 + 3|g_A|^2$$

and

$$f(\rho_0) = \int_0^{\rho_0} d\rho\, \rho^2 \left(\sqrt{\rho_0^2 + 1} - \sqrt{\rho^2 + 1}\right), \quad \rho_0 = \frac{p_e^{max}}{m_e}. \quad (11.66)$$

Since charged particles are involved, this expression of $f\tau = f\Gamma^{-1}$ is subject to radiative corrections, which are normally incorporated into the factor f along with the first order Coulomb corrections. These corrections change f by about 5%. The average value from direct neutron life-time measurements is

$$\tau = 888.6 \pm 3.5 \text{ sec.} \quad (11.67)$$

Knowing $|g_A/g_V|$, one can determine $G_V = G_F g_V$ from Eq. (65). G_V can also be determined from the superallowed $O^+ \to O^+$ pure Fermi decays for which $F\tau$ value is

$$F\tau = \frac{2\pi^3}{m_e^5} \frac{1}{G_V^2 |M_F|^2}, \quad (11.68a)$$

where

$$M_F = \left\langle \begin{array}{c} \Psi_f \\ 0^+, I = 1 \end{array} \middle| I_\pm \middle| \begin{array}{c} \Psi_i \\ 0^+, I = 1 \end{array} \right\rangle. \quad (11.68b)$$

F here is different from f for the neutron $\beta-$ decay and it must account for the stronger Coulomb effect and for the much more subtle radiative effects associated with the higher electric charge. The quantity $(F\tau)_{AV} = 3070.6 \pm 1.6$ sec from the $O^+ \to O^+$decays together with the phase space factor F from Wilkinson and the value of $|g_A/g_V|$ given in Eq. (63) gives $\tau = 894 \pm 37$ sec, to be compared with the direct neutron life-time measurement given in Eq. (67).

Finally the Cabibbo angle

$$\cos\theta_c = \frac{G_V^\beta}{G_V^\mu}\left(1 + \Delta\beta - \Delta\mu\right)^{-1/2} \tag{11.69}$$

where $G_V^\mu = 1.16637\ (13) \times 10^{-5}\ GeV^{-2}$ while $\Delta\beta$ and $\Delta\mu$ are the "inner" radiative corrections to both nucleon and muon $\Delta\beta$-decay with $\Delta\beta - \Delta\mu = 0.023\ (2)$. This gives

$$|V_{ud}| = \cos\theta_c = 0.9744 \pm 0.0010. \tag{11.70}$$

$\sin\theta_c$ is determined from hyperon β-decays and is given by

$$|V_{us}| = \sin\theta_c = 0.2176 \pm 0.0026 \tag{11.71}$$

whereas from K_{e3} decay its value is 0.2196 ± 0.0023

(b) Pseudoscalar Meson Decays

(i) Pion Decay

$$\pi^- \longrightarrow \ell^- + \bar{\nu}_e, \qquad \ell = e, \mu.$$

For this decay, the T–matrix is given by

$$\begin{aligned} T &= -\frac{G_F}{\sqrt{2}}\cos\theta_c \left\langle 0 \left| J_\lambda^+ \right| \pi^- \right\rangle \\ &\quad \times \frac{1}{(2\pi)^3}\sqrt{\frac{m_\ell\, m_\nu}{k_0' k_0}}\, u(k')\ \gamma^\lambda \left(1 - \gamma_5\right) v\left(k\right). \end{aligned} \tag{11.72}$$

Here, we have $p = k' + k$. Now from Lorentz invariance

$$\left\langle 0 \left| J_\lambda^+ \right| \pi^- \right\rangle = -\left\langle 0 \left| A_\lambda^+ \right| \pi^- \right\rangle = -\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{1}{p_0}}\, i f_\pi\ p_\lambda. \tag{11.73}$$

Using the standard techniques of Chap. 2, the decay rate Γ can be easily calculated. We obtain

$$\Gamma\left(\pi^- \rightarrow \ell^- + \nu_e\right) = \frac{G_F^2 \cos^2\theta_c}{8\pi}\ f_\pi^2\ m_\ell^2\ m_\pi \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2. \tag{11.74}$$

It thus follows that pion decays mainly to muon, its decay to electron is suppressed by a factor m_e^2/m_μ^2 (phase space). In the same way, we can write down the decay rate of $K^- \to \ell^- + \bar{\nu}_\ell$; it is given by

$$\Gamma\left(K^- \to \ell^- + \bar{\nu}_\ell\right) = \frac{G_F^2 \sin^2\theta_c}{8\pi}\, f_K^2\, m_\ell^2\, m_K \left(1 - \frac{m_\ell^2}{m_K^2}\right)^2 . \quad (11.75)$$

From the experimental values of the decay rates for pion and kaon we can determine f_π and f_K. We get $f_\pi \approx 131\ MeV$ and $f_K/f_\pi \approx 1.22$. From the particle data group:
$f_\pi \approx (130.7 \pm 0.1 \pm 0.36)$ MeV, $f_K \approx (159.8 \pm 1.4 \pm 0.44)$ MeV.

Remarks

Suppose pion decay occurs through a vector boson W. Then we can write the decay amplitude F:

$$F = -g_W\ i\ f_\pi\ p^\mu\ \frac{-g_{\mu\lambda} + \frac{p_\mu p_\lambda}{m_W^2}}{p^2 - m_W^2} \bar{u}(k')\ \gamma^\lambda (1 - \gamma_5)\, v\,(k)\,. \quad (11.76)$$

We write the W–propagator in the following form

$$\begin{aligned} & \frac{1}{p^2 - m_W^2}\left[\left(-g_{\mu\lambda} + \frac{p_\mu p_\lambda}{p^2}\right) + p_\mu p_\lambda \frac{p^2 - m_W^2}{p^2\ m_W^2}\right] \\ = & \left(-g_{\mu\lambda} + \frac{p_\mu p_\lambda}{p^2}\right)\frac{1}{p^2 - m_W^2} + \frac{p_\mu p_\lambda}{p^2 m_W^2}. \end{aligned} \quad (11.77)$$

The first part of Eq. (77) gives the transverse part of the propagator and second part gives its longitudinal part. If we substitute Eq. (77) into Eq. (76), we find that the first part of Eq. (77) gives zero and the entire contribution comes from the second part. We get

$$\begin{aligned} F &= -\frac{g_W^2}{m_W^2} i\ f_\pi\ p_\lambda\ \bar{u}(k')\ \gamma^\lambda (1 - \gamma_5)\, v\,(k) \\ &= -\frac{g_W^2}{m_W^2}\ i f_\pi\ m_\ell\ \bar{u}(k')\ (1 - \gamma_5)\, v\,(k)\,. \end{aligned} \quad (11.78)$$

Here we have used the Dirac equation $\bar{u}(k')\ (\gamma \cdot k' - m_\ell) = 0$ and $p = k' + k$. Thus we note that the longitudinal part behaves as if the decay has taken place through a scalar particle of zero mass with effective coupling g_W^2/m_W^2. We also note that it gives a contribution proportional to the lepton mass which is reflected in the formula (74). This is called helicity suppression.

(ii) Strangeness Changing Semi-Leptonic Decays

As an example of these decays we consider the decay.

$$K^- \to \pi^0 + \ell^- + \bar{\nu}_\ell, \qquad \ell = e, \mu.$$

We first note the rule: $\Delta Q = \Delta S = 1$. The T-matrix is given by

$$\begin{aligned} T &= -\frac{G_F}{\sqrt{2}} \sin\theta_c \left\langle \pi^0 \left| J_\lambda^1 \right| K^- \right\rangle \\ &\quad \times \frac{1}{(2\pi)^3} \sqrt{\frac{m_\ell\, m_\nu}{k_0'\, k_0}} \left[\bar{u}(k')\, \gamma^\lambda (1 - \gamma_5)\, v(k) \right]. \end{aligned} \quad (11.79)$$

The Lorentz structure of the hadronic matrix elements is given by

$$\begin{aligned} \left\langle \pi^0 \left| J_\lambda^1 \right| K^- \right\rangle &= \left\langle \pi^0 \left| V_\lambda^1 \right| K^- \right\rangle \\ &= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2p_0\, 2p_0'}} \\ &\quad \times \left[f_+\left(q^2\right)(p + p')_\lambda + f_-\left(q^2\right)(p - p')_\lambda \right], \end{aligned} \quad (11.80)$$

where p and p' are four-momenta of K^- and π^0, $q = (p' - p)$ and k' and k are four momenta of ℓ^- and ν_ℓ respectively. In the rest frame of K^-, we have $m_K = \omega + E_\ell + E_\nu$, $\mathbf{p}_\pi + \mathbf{p}_\ell + \mathbf{p}_\nu = 0$. Using the standard techniques of Chap. 2, we get

$$\frac{d\Gamma}{dE_\ell\, d\omega} = \frac{1}{4\pi^3} G_F^2 \sin^2\theta_c \left| f_+\left(q^2\right) \right|^2 \left[A + B\, \mathrm{Re}\,\xi + C \left|\xi\right|^2 \right], \quad (11.81)$$

where

$$\begin{aligned}
A &= \left\{m_K \ [2E_\ell \ E_\nu - m_K \ (W-\omega)] + \frac{m_\ell^2}{4}(W-\omega) - m_\ell^2 \ E_\nu\right\} \\
B &= \left[E_\nu - \frac{1}{2}(W-\omega)\right] \ m_\ell^2 \\
C &= \frac{1}{4}[W-\omega] \ m_\ell^2 \\
W &= \frac{m_K^2 + m_\pi^2 - m_\ell^2}{2 \ m_K} \\
\xi &= f_-\left(q^2\right) / \ f_+\left(q^2\right). \qquad (11.82)
\end{aligned}$$

For electron, we can neglect its mass i.e. we put $m_e^2 \approx 0$. Then Eq. (81) is much simplified. In this case, we get for the electron spectrum

$$\frac{d\Gamma}{dE_e} = \frac{1}{4\pi^3} G_F^2 \sin\theta_c \ |f_+|^2 \ m_K \ E_e^2 \frac{(W-E_e)^2}{(m_K - 2E_e)}. \qquad (11.83)$$

Here we have put $f_+(q^2) \approx f_+(0) = f_+$. For this case we obtain

$$\Gamma\left(K_{e3}^{\pm}\right) = \frac{G_F^2 \sin^2\theta_c}{768\pi^3} |f_+|^2 \ m_K^2 \ (0.573). \qquad (11.84)$$

In the SU(3) limit $\langle \pi^0 | V_\lambda^1 | K^- \rangle \propto i f_{4+i5 \frac{6-i7}{12} 3}$ so that $f_+(0) = \frac{1}{\sqrt{2}}$.

Consider the neutral Kaon decays:

$$\begin{aligned}
K^0 &\rightarrow \pi^- + \ell^+ + \nu_\ell, \qquad \Delta S = \Delta Q \\
K^0 &\rightarrow \pi^+ + \ell^- + \nu_\ell, \qquad \Delta S = -\Delta Q.
\end{aligned}$$

For the first case the hadronic matrix elements are given by

$$\left\langle \pi^- \left| J_\lambda^{1\dagger} \right| K^0 \right\rangle$$

where

$$J_\lambda^1 = J_{4+i5\lambda}, \qquad J_\lambda^{1\dagger} = J_{4-i5\lambda}.$$

$J_\lambda^{1\dagger}$ creates negative charge and $S=-1$. For $\Delta S = -\Delta Q$, no such current can be written down in this conventional theory. For more details for semi-leptonic $K-$decays see Ref. 2.

11.2.4 Hadronic weak decays

(a) Non-Leptonic Decays of Hyperons

Consider the decay

$$B\,(p) \to B'\,(p') + \pi\,(k)\,.$$

The Lorentz structure of the T–matrix for this process is given by

$$T = \frac{1}{(2\pi)^{9/2}}\sqrt{\frac{m\;m'}{2k_0\;p_0\;p_0'}}\;\bar{u}(p')\,[\;A - B\gamma_5]\,u\,(p)\,. \tag{11.85}$$

The amplitudes A and B are functions of scalars: $s = (p'+k)^2$, $t = (p-p')^2$. A is called the parity violating $(p.v)$ [or s–wave] amplitude and B is called the parity conserving $(p.c)$ [or p–wave] amplitude. In the rest frame of baryon B

$$\mathbf{p}' = -\mathbf{k}, \quad |\mathbf{p}'| = |\mathbf{k}| = k, \quad \mathbf{p}' = k\mathbf{n},$$

$$m = p_0' + k_0, \quad p_0' = \sqrt{k^2 + m'^2}, \quad k_0 = \sqrt{k^2 + m_\pi^2}$$

$$s = m^2, \quad t = m^2 + m'^2 - 2mp_0'$$

$$k = \frac{1}{2m}\sqrt{\left[m^2 - (m' - m_\pi)^2\right]\,\left[m^2 - (m' + m_\pi)^2\right]}$$

$$p_0' = \frac{m^2 + m'^2 - m_\pi^2}{2m}. \tag{11.86}$$

In this frame, the amplitudes A and B are constants. In the rest frame of B

$$u\,(p) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,\chi, \quad u\,(p') = \frac{1}{\sqrt{2m'\,(p_0' + m')}}\begin{pmatrix} m' + p_0' \\ \boldsymbol{\sigma}\cdot\mathbf{p}' \end{pmatrix}\chi, \tag{11.87}$$

where χ is a constant 2-component spinor. Using Eq. (87), we may write the T-matrix

$$T = \chi^+\;M\;\chi, \tag{11.88a}$$

where

$$M = \frac{1}{(2\pi)^{9/2}} \frac{1}{\sqrt{2\,k_0}} \left[a_s + a_p\, \boldsymbol{\sigma} \cdot \mathbf{n}\right], \qquad (11.88b)$$

$$a_s = \sqrt{\frac{(p_0' + m')}{2\,p_0'}} A, \quad a_p = \frac{k}{\sqrt{2\,p_0'\,(p_0' + m')}} B. \quad (11.88c)$$

We note that the *p.v.* amplitude A is essentially the $s-$wave amplitude and the *p.c.* amplitude B accounts for the $p-$wave amplitude.

The decay width is given by

$$d\Gamma = (2\pi)^7\, \delta^4 (p - p' - k) \left[\frac{1}{2} Tr\, \left(M M^\dagger\right)\right] d^3p'\, d^3k. \qquad (11.89)$$

Performing the integration, we get the decay width

$$\Gamma = \frac{k\, p_0'}{2\pi m} \left[|a_s|^2 + |a_p|^2\right]. \qquad (11.90)$$

We now consider the decay of polarized baryon B. Let $\mathbf{S}$ be the polarization (spin) of B. Let $\mathbf{s}$ be the polarization of decayed baryon B'. In the rest frame of B', s gives the spin of B'. The decay probability in this case is given by

$$\begin{aligned} dW &= (2\pi)^7\, \delta^4 (p - p' - k) \\ &\quad \times \frac{1}{2} \left\{Tr\left[(1 + \boldsymbol{\sigma} \cdot \mathrm{s})\, M\, (1 + \boldsymbol{\sigma} \cdot \mathrm{S})\right] M^\dagger\right\} d^3p' d^3k. \end{aligned} \qquad (11.91)$$

The trace can be easily evaluated and the transition rate is proportional to

$$R = 1 + \alpha \mathbf{S} \cdot \mathbf{n} + \mathbf{s} \cdot \left[(\alpha + \mathbf{S} \cdot \mathbf{n})\, \mathbf{n} + \beta\, (\mathbf{S} \times \mathbf{n}) + \gamma\, \mathbf{n} \times (\mathbf{S} \times \mathbf{n})\right], \qquad (11.92)$$

where

$$\alpha = \frac{2 R_e\; a_s^*\; a_p}{|a_s|^2 + |a_p|^2}, \qquad \beta = \frac{2\, \mathrm{Im}\;\, a_s^*\; a_p}{|a_s|^2 + |a_p|^2}$$

$$\gamma = \frac{|a_s|^2 - |a_p|^2}{|a_s|^2 + |a_p|^2}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \tag{11.93}$$

Because of the last constraint, we can write

$$\begin{aligned} \beta &= \left(1-\alpha^2\right)^{1/2} \sin\phi \\ \gamma &= \left(1-\alpha^2\right)^{1/2} \cos\phi \\ \phi &= \tan^{-1}(\beta/\gamma) . \end{aligned} \tag{11.94}$$

One also defines

$$\Delta = -\tan^{-1}(\beta/\alpha) .$$

If we do not observe the polarization of B', we put $\mathbf{s} = 0$ and we get

$$dW/\Gamma = \frac{d\Omega_S}{4\pi}\left[1 + \alpha\ \mathbf{S}\cdot\mathbf{n}\right]. \tag{11.95}$$

Hence we can write the angular distribution

$$I_B(\theta) = \text{Const}\ \left[1 + \alpha\ S\cos\theta\right], \tag{11.96}$$

where θ is the angle between the hyperon spin S and the decayed baryon momentum direction n. If $\alpha = 0$, the angular distribution is isotropic. $\alpha\ = 0$ implies either $a_s = 0$ or $a_p = 0$. For this case parity is conserved. The anisotropy in angular distribution implies nonconservation of parity. From the angular distribution we can determine the product αS. Since the polarization S of baryon is not generally known, it is difficult to measure α by this method. Further information about α can be obtained from the polarization of decayed baryon B'. From Eq. (92), we obtain the polarization of decayed bayron B'.

$$\langle \mathrm{s} \rangle = \frac{1}{1+\alpha\ \mathbf{S}\cdot\mathbf{n}}\left\{(\alpha + \mathbf{S}\cdot\mathbf{n})\,\mathbf{n} + \beta\,(\mathbf{S}\times\mathbf{n}) + \gamma\ \mathbf{n}\times(\mathbf{S}\times\mathbf{n})\right\}. \tag{11.97}$$

In particular if the original baryon B is unpolarized viz. $\mathbf{S} = 0$, we get

$$\langle \mathbf{s} \rangle = \alpha \, \mathbf{n}. \tag{11.98}$$

This equation implies that the baryon B' obtained from the decay of unpolarized baryon B is longitudinally polarized. Thus a measurement of this polarization allowed a direct determination of α. The experimental values for α, β and γ are given in the Table 1.

Now a non-zero value for β implies the violation of time reversal invariance in these decays. From Table 1, it is clear that $\beta = (1 - \alpha^2)^{1/2} \sin\phi$ is consistent with zero. Thus the time reversal invariance holds in these decays. P invariance implies either $a_s = 0$ or $a_p = 0$, so that $\alpha = 0$, $\beta = 0$. But Table 1 shows that α is non-zero. C invariance implies $\alpha = 0$, $\beta \neq 0$; hence from Table 1, it follows that C invariance is also violated. The consequences of T and C invariance quoted above hold if we neglect the final state interactions.

(b) $\Delta I = 1/2$ Rule for Hyperon Decays

The effective weak Hamiltonian responsible for $|\Delta S| = 1$ non-leptonic decays in the conventional theory is given in Eq. (28), namely

$$\begin{aligned} H_W^{\text{eff}} &= \frac{G_F}{\sqrt{2}} \sin\theta_c \cos\theta_c \left[J_\lambda^+ \left(J^{1\lambda}\right)^\dagger + \text{h.c.} \right] \\ &\equiv \frac{G_F}{\sqrt{2}} \sin\theta_c \cos\theta_c \, H_W, \end{aligned} \tag{11.99}$$

where

$$H_W = \left[J_\lambda^+ \left(J^{1\lambda}\right)^\dagger + \text{h.c.} \right]. \tag{11.100}$$

Now $J_\lambda^+ \sim \bar{u}\,\gamma_\lambda\,(1+\gamma_5)\,d$ has $I = 1$, $I_3 = +1$, $J_\lambda^1 \sim \bar{s}\,\gamma_\lambda\,(1+\gamma_5)\,u$ has $I = \frac{1}{2}$, $I_3 = +\frac{1}{2}$. Thus in general H_W has a mixture of $\Delta I = 1/2$ and $\Delta I = 3/2$. However, the most striking effect of these decays is the approximate validity of $\Delta I = 1/2$ rule. The decays

Table 11.1

Decay	α	ϕ	γ (derived)	Δ (derived)
$\Lambda^0_- : \Lambda \to p\pi^-$	0.642 ± 0.013	$(-6.5 \pm 3.5)^0$	0.76	$(8 \pm 4)^0$
$\Lambda^0_0 : \Lambda \to n\pi^0$	0.65 ± 0.05	–	–	-
$\Sigma^+_0 : \Sigma^+ \to p\pi^0$	$-0.980^{+0.017}_{-0.015}$	$(36 \pm 34)^0$	0.16	$(187 \pm 6)^0$
$\Sigma^+_+ : \Sigma^+ \to n\pi^+$	0.068 ± 0.013	$(167 \pm 20)^0$	-0.97	$\left(-73^{+133}_{-10}\right)^0$
$\Sigma^-_- : \Sigma^- \to n\pi^-$	-0.068 ± 0.008	$(10 \pm 15)^0$	0.98	$\left(249^{+12}_{-120}\right)^0$
$\Xi^0_0 : \Xi^0 \to \Lambda + \pi^0$	-0.411 ± 0.022	$(21 \pm 12)^0$	0.85	$\left(218^{+12}_{-19}\right)^0$
$\Xi^-_- : \Xi^- \to \Lambda + \pi^-$	-0.456 ± 0.014	$(4 \pm 4)^0$	0.89	$(188 \pm 8)^0$

with $\Delta I = 3/2$ are suppressed. A satisfactory understanding of this rule is still lacking.

We now examine the consequences of $\Delta I = 1/2$ rule in non-leptonic hyperon decays and its approximate experimental validity. Consider first the decays

$$\begin{array}{lll} \Lambda_-^0 & : \Lambda \to p + \pi^- & \Delta I_3 = 1/2 \\ \Lambda_0^0 & : \Lambda \to n + \pi^0 & \Delta I_3 = -1/2 \end{array}$$

$$\Delta I = 1/2,\ 3/2, \cdots$$

The simplest possibility is $\Delta I = 1/2$. Assuming this to be the case, the only possible isospinor which one can form is

$$\bar{N}\, \boldsymbol{\tau} \cdot \boldsymbol{\pi}\, \Lambda = \left(\bar{p}\, \pi^0 + \sqrt{2}\, \bar{n}\, \pi^-,\ \sqrt{2}\, \bar{p}\, \pi^+ - \bar{n}\, \pi^0 \right) \Lambda. \quad (11.101)$$

Then for $\Delta Q = 0$, we have

$$\Lambda_-^0 = -\sqrt{2}\Lambda_0^0. \quad (11.102)$$

Hence we get

$$\Gamma\left(\Lambda_-^0\right) = 2\Gamma\left(\Lambda_0^0\right), \quad (11.103a)$$

$$\alpha_{\Lambda_-^0} = \alpha_{\Lambda_0^0}. \quad (11.103b)$$

It is clear from Table 1, $\alpha_{\Lambda_+^0} \approx \alpha_{\Lambda_0^0}$; experimentally

$$\frac{\Gamma\left(\Lambda_+^0\right)}{\Gamma\left(\Lambda_0^0\right)} \approx \frac{63.9}{35.8} = 1.78. \quad (11.103c)$$

Thus $\Delta I = 1/2$, rule is a good approximation, $\Delta I = 3/2$ amplitude is very much suppressed for Λ-decays. An exactly similar argument gives

$$\Xi_-^- = -\sqrt{2}\, \Xi_0^0, \quad (11.104a)$$

which implies

$$\Gamma\left(\Xi_-^-\right)/\Gamma\left(\Xi_0^0\right) = 2\left(Expt: \frac{2.90}{1.639} \approx 1.77\right) \quad (11.104b)$$

$$\alpha_{\Xi_-^-}/\alpha_{\Xi_0^0} = 1\left(Expt: \frac{0.456}{0.411} \approx 1.11\right). \quad (11.104c)$$

For $\sum$-decays, assuming $\Delta I = 1/2$, the only isospinors which we can form are

$$a\,\bar{N}\;(\mathbf{\Sigma}\cdot\boldsymbol{\pi}) + i\,b\,\bar{N}\;(\mathbf{\Sigma}\times\boldsymbol{\pi})\cdot\boldsymbol{\tau}. \tag{11.105a}$$

Writing only the part for which total charge is zero, we have

$$a\,\bar{n}\,\left(\Sigma^-\pi^+ + \Sigma^+\pi^- + \Sigma^0\pi^0\right) + b\,\left(\sqrt{2}\,\bar{p}\,\Sigma^0\pi^+ - \sqrt{2}\,\bar{p}\,\Sigma^+\pi^0\right.$$
$$\left. - \bar{n}\,\Sigma^+\pi^- + \bar{n}\,\Sigma^+\pi^-\right). \tag{11.105b}$$

Thus we get

$$\begin{aligned} \Sigma^-_- &= a+b \\ \Sigma^+_+ &= a-b \\ \Sigma^0_- &= \sqrt{2}\,b \\ \Sigma^+_0 &= -\sqrt{2}\,b. \end{aligned} \tag{11.106}$$

From Eq. (106), we get

$$\Sigma^+_+ - \Sigma^-_- = \sqrt{2}\Sigma^+_0. \tag{11.107}$$

The prediction can be tested as follows: In the (a_s, a_p) plane if we regard Σ^+_+, Σ^-_- and $\sqrt{2}\Sigma^+_0$ as vectors, then they should form a closed triangle.

To sum up, in case the $\Delta I = 1/2$ rule holds, out of 7 decays listed in Eq. (25) only four are independent. In the language of flavor SU(3) [cf. Chap. 5], the dominance of $\Delta I = 1/2$ rule is generalized to octet dominance. This can be seen as follows:

u, d, s, belong to 3 representation of SU(3).

$\bar{u}$, $\bar{d}$, $\bar{s}$, belonging to $\bar{3}$ representation of SU(3).

Now

$$3 \otimes \bar{3} = 8 \oplus 1.$$

Thus J^h_μ in Eq. (27) belongs to an octet representation of SU(3). Hence H^h_{int} in Eq. (27) or (28) contains

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus +\overline{10} \oplus 27.$$

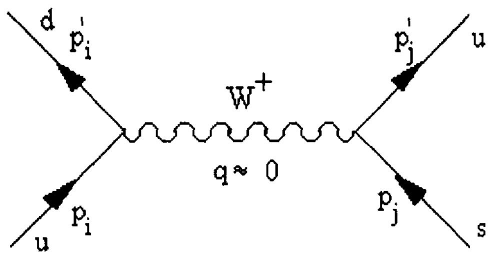

Figure 6 $W-$boson exchange graph for the reaction $u + s \to d + u$.

It can be seen that only 8 and 27 are relevant for the decays (25). Thus H^h_{int} contains both 8 and 27 where 8 corresponds to $\Delta I = 1/2$ only while 27 contains $\Delta I = 3/2$ as well. Thus in the language of SU(3), generalization of $\Delta I = 1/2$ rule is the octet dominance. The octet dominance for the current-current interaction implies an additional relation (called Lee-Sugawara relation) between $s-$wave decay amplitudes of (25)

$$2A\left(\Xi^-_-\right) + A\left(\Lambda^0_-\right) = +\sqrt{3}\, A\, \left(\Sigma^+_0\right). \tag{11.108}$$

(c) Non–leptonic Hyperon Decays in Non–Relativistic Quark Model

One can recover not only the $\Delta I = 1/2$ rule but also the right order of magnitude of the scale required to reproduce the $s-$ and $p-$wave fits of non-leptonic hyperon decays. Consider the weak vector boson exchange graph of Fig. 6 as the analogue of the gluon exchange quark-quark scattering graph considered in Chap. 7 which quite successfully described the quark spectroscopy.

The matrix elements for the process shown in Fig. 6 are of the form

$$M \xrightarrow[m_W^2 \gg q^2]{} \frac{1}{2}\frac{g_W^2}{m_W^2} \sin\theta_c \cos\theta_c \left\{ \bar{u}(p_i')\gamma^\mu \, (1-\gamma_5)\, \alpha_i^- \, u(p_i) \right.$$

$$\times \ \bar{u}(p_j')\gamma_\mu \left(1-\gamma_5\right) \beta_i^+ u(p_j) + i \longleftrightarrow j \Big\} , \tag{11.109}$$

where $q = p_i - p_i' = p_j' - p_j$. u's are Dirac spinors in Dirac space but are column vectors involving u, d, s quarks in ordinary flavor SU(3) space. α_i^- and β_j^+ are operators which transform a $u-$like state into a $d-$like state and a $s-$like state into a $u-$like state respectively. We take the leading non-relativistic limit of the above matrix elements. In the leading non-relativistic approximation, only γ^0 and $\gamma^i \ \gamma_5$ have nonzero limits. Thus only parity conserving ($p.c$) part of M survives in the leading non-relativistic approximation and we have in this limit

$$M^{p\cdot c} \sim \frac{1}{\sqrt{2}} G_F \sin\theta_c \cos\theta_c \sum_{i\rangle j} \left(\alpha_i^- \beta_j^+ + \beta_i^+ \alpha_j^-\right) \cdot \left(1 - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right)$$

$$M^{p\cdot v} = 0. \tag{11.110}$$

The latter corresponds to a general result that $\langle B' \left| (JJ)^{p\cdot v} \right| B\rangle = 0$ as a consequence of CP and SU(3) invariance. The Fourier transform of Eq. (110) gives the effective H_W as

$$\begin{aligned} H_W^{p\cdot c} &= \frac{1}{\sqrt{2}} G_F \sin\theta_c \cos\theta_c \sum_{i>j} \left(\alpha_i^- \beta_j^+ + \beta_i^+ \alpha_j^-\right) \\ &\quad \times \left(1 - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right) \delta^3 \left(\mathbf{r}\right). \end{aligned} \tag{11.111}$$

Now it has been shown [see Sec. 12.4.2] that in the current-algebra approach the question of $\Delta I = 1/2$ rule or octet dominance for non-leptonic decays of baryons hinges on the matrix elements

$$\langle B_s \left| H_W^{p\cdot c} \right| B_r\rangle \sim a_{rs} \bar{u} u, \tag{11.112}$$

which essentially determine both $s-$ and $p-$wave amplitudes. Here u is a Dirac spinor for B_r or B_s which denotes a baryon like Λ, Σ, Ξ, n, or p. Therefore, we have to take the matrix elements of Eq. (111) between the baryon states B_r and B_s. We regard the baryon state B_r or B_s as made up of three quarks. We take the spatial wave function for such states to be the same for the octet of baryons p, n, Λ, $\Sigma^{\pm}$, Σ^0, Ξ^0, Ξ^- and denote it by Ψ_0. Thus writing

$$d' = \left\langle \Psi_0 \left| \delta^3 (\mathbf{r}) \right| \Psi_0 \right\rangle = \left| \Psi_0 (0) \right|^2 , \qquad (11.113)$$

where $\mathbf{r} = \mathbf{r}_i - \mathbf{r}_j$ $(i \neq j)$, we have to calculate the matrix elements of the operator

$$\sum_{i>j} \left(\alpha_i^- \beta_j^+ + \beta_i^+ \alpha_j^- \right) (1 - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)$$

between the spin-unitary spin wave functions of the states p, n, Σ^+, Σ^0, Λ, Ξ^0, given in Chap. 6. We obtain

$$\begin{aligned} a_{\Lambda n} &= \frac{G_F}{\sqrt{2}} \sin\theta_c \cos\theta_c d' \left(+\sqrt{6} \right) & (11.114\text{a}) \\ a_{\Sigma^+ p} &= \frac{G_F}{\sqrt{2}} \sin\theta_c \cos\theta_c d' \left(-6 \right) & (11.114\text{b}) \\ &= -\sqrt{2} a_{\Sigma^0 n} & (11.114\text{c}) \\ a_{\Xi^0 \Lambda^0} &= \frac{G_F}{\sqrt{2}} \sin\theta_c \cos\theta_c d' \left(-2\sqrt{6} \right) & (11.114\text{d}) \\ a_{\Xi^- \Sigma^-} &= 0. & (11.114\text{e}) \end{aligned}$$

The relation $a_{\Sigma^+ p} = -\sqrt{2} a_{\Sigma^0 n}$ expressed in Eq. (114c) ensures the $\Delta I = 1/2$ rule (or octet dominance) and hence $A\left(\Sigma_+^+\right) = 0$ (which is good experimentally) in current algebra approach [see Eq. (117) below].

Once the octet dominance for a_{rs} is established we can parametrize a_{rs} in the SU(3) limit as

$$a_{rs} = \sqrt{2} \ \left(2F' \ i \ f_{6rs} + 2D' \ d_{6rs} \right). \qquad (11.115)$$

Then the relations (114) immediately give

$$\frac{D'}{F'} = -1. \qquad (11.116)$$

Now using the current algebra relations [see Sec. 12.4.2] for the $s-$wave amplitudes one has

$$\begin{aligned}
A\left(\Lambda^0_-\right) &= -\frac{1}{f_\pi} a_{\Lambda n} = -\sqrt{2}\, A\ \left(\Lambda^0_0\right) \\
A\left(\Xi^-_-\right) &= -\frac{1}{f_\pi}\ a_{\Xi^0\Lambda} = -\sqrt{2} A\left(\Xi^0_0\right) \\
A\left(\Sigma^+_0\right) &= \frac{1}{\sqrt{2}\ f_\pi}\ a_{\Sigma^+ p} \\
A\left(\Sigma^+_+\right) &= -\frac{1}{f_\pi}\left(a_{\Sigma^+ p} + \sqrt{2}\ a_{\Sigma^0 n}\right) \\
A\left(\Sigma^-_-\right) &= \left(\frac{\sqrt{2}}{f_\pi}\right) a_{\Sigma^0 n}.
\end{aligned} \tag{11.117}$$

Here f_π is the constant which enters in $\pi^- \to \mu^- + \bar{\nu}_\mu$ decay. Then using Eqs. (114) and (117), we have the relations (107) and (108). Using the value of d' as determined by the constituent quark spectroscopy [cf. Chap. 7] ,

$$\begin{aligned}
a_{\Sigma^+ p} &= \frac{-27\ G_F\ \sin\theta_c \cos\theta_c}{8\sqrt{2}\pi\alpha_s} \left(m_\Sigma - m_\Lambda\right) \left(\frac{\hat{m}^2}{1 - \hat{m}/m_s}\right)_{\text{constituent}} \\
&\approx -105\ eV
\end{aligned} \tag{11.118}$$

for the accepted value of α_s $(q^2 \approx 1GeV) \approx 0.5$. This is almost the phenomenological octet dominance scale, which together with $D'/F' \approx -0.86$ [not very far from the prediction (116)], are required to fit the $s-$ and $p-$wave amplitudes of hyperon decays.

11.3 Problems

(1) Show that the electron spectrum in the decay of b-quark

$$b \to c + e^- + \bar{\nu}_e,$$

using $V - A$ theory is given by (neglecting the electron mass)

$$\frac{d\Gamma}{dx} = \frac{G_F^2}{96\pi^3}\, m_b^5 \frac{\left(y_m - y\right)^2\, y^2}{\left(1 - y\right)^2} \left[(3 - 2y) + \frac{\left(1 - y_m\right)\left(3 - y\right)}{\left(1 - y\right)}\right],$$

where

$$y = \frac{2E_e}{m_b}, \qquad y_m = 1 - \frac{m_c^2}{m_b^2}.$$

Similarly, show that for c-quark decay

$$c \to s + e^+ + \nu_e$$

the electron spectrum is given by

$$\begin{aligned} \frac{d\Gamma}{dx} &= \frac{G_F^2}{16\pi^3}\, m_c^5\, y^2 \frac{(y_m - y)^2}{(1-y)} \\ y &= \frac{2E_e}{m_c}, \qquad y_m = 1 - \frac{m_s^2}{m_c^2}. \end{aligned}$$

Hint: For $b \to c + e^- + \bar{\nu}_e$, the matrix elements are

$$T = -\frac{G_F}{\sqrt{2}} \left[\bar{u}(p_2)\, \gamma_\lambda\, (1-\gamma_5)\, u(p_1)\right] \left[\bar{u}(p_1)\, \gamma^\lambda\, (1-\gamma_5)\, v(k_2)\right].$$

Use Eqs. (31) and (32) with the replacements $(m_\mu,\, m_e,\, m_{\nu m},\, m_{\nu e}) \to (m_e,\, m_c,\, m_e,\, m_{\nu e})$, $\varepsilon = 1$ so that

$$\begin{aligned} |M|^2 &= \frac{G_F^2}{m_b\, m_c\, m_e\, m_\nu}\, 4\, p_1 \cdot k_2\, p_2 \cdot k_1 \\ &= \frac{G_F^2}{m_b\, m_c\, m_e\, m_\nu}\, 2\, m_b\, E_\nu \left[m_b^2 - m_c^2 - 2\, m_b\, E_\nu\right], \end{aligned}$$

in the rest frame of b. Performing d^3p_2 integration, write $d^3k_1 d^3k_2 = k_1^2\, dk_1\, dk_2\, d\Omega$ and use

$$\delta\left(m_b - E_e - E_\nu - \sqrt{k_1^2 + k_2^2 + 2k_1k_2\cos\theta + m_c^2}\right)$$

to perform the angular integration to obtain

$$\frac{d\Gamma}{dE_e\, dE_\nu} = \frac{4G_F^2}{(2\pi)^3} E_\nu \left[m_b^2 - m_c^2 - 2\, m_b\, E_\nu\right]$$

where from

$$E_\nu = \frac{m_b^2 - m_c^2 - 2\, m_b\, E_e}{2\,(m_b - m_e + E_e \cos\theta)}$$

one has

$$\begin{aligned}
(E_\nu)_{\min} &= \frac{m_b^2 - m_c^2 - 2\, m_b\, E_e}{2m_b} = \frac{m_b}{2}\,(y_b - y) \\
(E_\nu)_{\max} &= \frac{(m_b^2 - m_c^2) - 2\, m_b\, E_e}{2m_b - 2E_e} = \frac{m_b\,(y_b - y)}{(1-y)}.
\end{aligned}$$

The integration of E_ν gives the result $\frac{d\Gamma}{dy}$.

For the second problem, the matrix elements are

$$T = -\frac{G_F}{\sqrt{2}}\,[\bar{u}(p_2)\;\gamma_\lambda\,(1-\gamma_5)\,u(p_1)]\left[\bar{u}(k_2)\;\gamma^\lambda\,(1-\gamma_5)\,v(k_1)\right].$$

Results from the first can be obtained by changing $k_2 \longleftrightarrow k_1,\ m_b \to m_c,\ m_c \to m_s$

$$\begin{aligned}
|M|^2 &= \frac{G_F^2}{m_c\, m_s\, m_e\, m_\nu}\,[\,4\, p_1\cdot k_1\; p_2\cdot k_2] \\
&= \frac{G_F^2}{m_c\, m_s\, m_e\, m_\nu}\;(2\, m_c\, E_e)\left[m_c^2 - m_s^2 - 2\, m_c\, E_e\right]
\end{aligned}$$

and then follow the same steps as in the first part.

(2) Consider the decay

$$K \to 3\pi.$$

Show that decay rate can be expressed as

$$\Gamma = \frac{1}{2^6\pi^3 m_K}\frac{Q^2}{6\sqrt{3}}\int dx\; dy\;\, |A|^2$$

$$0 \le x^2 + y^2 \le 1,$$

where A is the decay amplitude,

$$x = \sqrt{3}\frac{T_2 - T_1}{Q},\quad y = \frac{3T_3 - Q}{Q}$$

T_1, T_2 and T_3 are kinetic energies of pions. Then the energies $\omega_1, \omega_2, \omega_3$ of pions are given by $\omega_i = T_i + m_\pi$ and $Q = T_1 + T_2 + T_3 = \omega_1 + \omega_2 + \omega_3 - 3m_\pi = m_K - 3m_\pi$.

The events in Dalitz plot can be expressed by taking

$$A_j = A_j(0)\left[1 + \frac{2\sigma_j}{3}\frac{m_K}{m_\pi^2}(2\omega_3 - \omega_1 - \omega_2)\right]$$

where j stands for any decay channel of K.

(3) Show that if the three pions in the decay of $K \to 3\pi$ are in $I = 1$ states, then

$$\Gamma\left(K_2^0 \to \pi^+\pi^-\pi^0\right) = 2\Gamma\left(K^+ \to \pi^+\pi^0\pi^0\right) \qquad (1)$$

$$\begin{aligned} & \Gamma\left(K^+ \to \pi^+\pi^+\pi^-\right) - \Gamma\left(K^+ \to \pi^+\pi^0\pi^0\right) \\ = \; & \Gamma\left(K_2^0 \to \pi^0\pi^0\pi^0\right). \end{aligned} \qquad (2)$$

Equations (1) and (2) are the necessary conditions for $\Delta I = 1/2$ rule to hold. But they are not sufficient since $I = 1$ state can be reached also by $\Delta I = 3/2$.

Show that for totally symmetric $I = 1$ states

$$\begin{aligned} \Gamma\left(K^+ \to \pi^+\pi^+\pi^-\right) &= 4\Gamma\left(K^+ \to \pi^+\pi^0\pi^0\right), \\ \Gamma\left(K_2^0 \to \pi^0\pi^0\pi^0\right) &= \frac{3}{2}\Gamma\left(K_2^0 \to \pi^+\pi^-\pi^0\right). \end{aligned}$$

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Chapter 12
PROPERTIES OF WEAK HADRONIC CURRENTS AND CHIRAL SYMMETRY

12.1 Introduction

In Chap. 11, we have introduced an octet of vector and axial vector currents

$$V_{i\lambda} = \bar{q}\frac{\lambda_i}{2}\gamma_\lambda q \tag{12.1}$$

$$A_{i\lambda} = \bar{q}\frac{\lambda_i}{2}\gamma_\lambda\gamma_5 q, \tag{12.2}$$

where

$$J^{\pm}_\lambda = V_{1\pm i2\lambda} + A_{1\pm i2\lambda} \tag{12.3}$$

$$J^1_\lambda,\ J^{1\dagger}_\lambda = V_{4\pm i5\lambda} + A_{4\pm i5\lambda} \tag{12.4}$$

take part in $|\Delta Y| = 0$ and $|\Delta Y| = 1$ semi-leptonic processes respectively. The electromagnetic current is given by

$$V^{em}_\lambda = V_{3\lambda} + \frac{1}{\sqrt{3}}V_{8\lambda} \tag{12.5}$$

where the first part is the third component of an isovector while the second part is an isoscalar. Now $H^{em}_{int} \sim V^{em}_\lambda a^\lambda$. Since photon field a^λ has C-parity -1 and the intrinsic parity of the photon is -1, we see that CP of V^{em}_λ is $+1$. From this we can generalize that CP of vector current V_λ is $+1$. The parity of axial-vector current A_λ is $+1$ and since the weak Hamiltonian is CP invariant, the C-parity of A_λ must be $+1$.

12.2 Conserved Vector Current Hypothesis (CVC)

The hypothesis of conserved vector current (CVC) states that $V_\lambda^\pm$ and $V_{3\lambda}(= J_\lambda^{em},\ \Delta I = 1)$ are respectively $1+i2$, $1-i2$ and 3 members of an isospin current, which is conserved by strong interaction. The generators of the isospin group $SU_I(2)$ are then given by

$$I_i = \int V_{i0}(\mathbf{x},\, t)\, d^3x, \ \ i = 1,\, 2,\, 3. \tag{12.6}$$

The first consequence of CVC ($\partial^\lambda V_\lambda^\pm = 0$) is that the form factor $h_V(q^2) = 0$ in Eq. (11.50a) where A and B are respectively taken as neutron and proton. [Note: When invariance under $SU(2)$ is assumed, $m_p = m_n = m_N$.]

In order to discuss the other consequences of CVC, we note from Eqs. (6) and (11.50a) that

$$\begin{aligned}&\langle p(p')\,|I_+|\,n(p)\rangle \\ &= \frac{1}{(2\pi)^3}\sqrt{\frac{m_N^2}{p_0p_0'}}\bar{u}(p')\left[g_V(q^2)\gamma_0 + i f_V(q^2)\sigma_{0\nu}q^\nu\right]u(p)\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}.\end{aligned} \tag{12.7}$$

Since I_+ is conserved in the absence of electromagnetism, $I_+(t)$ is a constant of motion i.e. $I_+(t) = I_+(0) = I_+$, we can take $t = 0$ and

$$\frac{1}{(2\pi)^3}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}} = \delta^3(\mathbf{q}). \tag{12.8}$$

Now

$$\begin{aligned}\bar{u}(p)\gamma_0 u(p) &= \frac{p_0}{m_N} \\ I_+\,|n(p)\rangle &= |p(p)\rangle\end{aligned} \tag{12.9}$$

and thus

$$\langle p(p')\,|I_+|\,n(p)\rangle = \delta^3(\mathbf{p}' - \mathbf{p}) = \delta^3(\mathbf{q})\,. \tag{12.10}$$

Hence it follows from Eq. (7) that

$$g_V(0) = 1. \tag{12.11}$$

Thus in the absence of electromagnetism, the vector coupling constant in nuclear β-decay is not renormalized and is equal to its "bare" value. Noting that [$J_\lambda^Y = \frac{1}{\sqrt{3}} V_{8\lambda}$ in SU(3)]

$$\begin{aligned} \left[J_\lambda^Y(x), I_+\right] &= 0, \\ [V_{3\lambda}, I_+] &= V_{1+i2\lambda}(x), \end{aligned} \tag{12.12}$$

and

$$I_+ |n\rangle = |p\rangle\,, \ \langle p| I_+ = \langle n| \tag{12.13}$$

it follows that

$$\begin{aligned} \left\langle p \left| V_\lambda^+ \right| n \right\rangle &= \langle p \left| [V_{3\lambda}, I_+] \right| n\rangle \\ &= \langle p \left| [V_\lambda^{em}, I_+] \right| n\rangle \\ &= \langle p \left| V_\lambda^{em} \right| p\rangle - \langle n \left| V_\lambda^{em} \right| n\rangle\,. \end{aligned} \tag{12.14}$$

Now Lorentz invariance gives the electromagnetic form factors of proton and neutron as

$$\begin{aligned} \langle p(p') \left| V_\lambda^{em} \right| p(p)\rangle &= \frac{1}{(2\pi)^3}\sqrt{\frac{m_N^2}{p_0 p_0'}} \\ &\quad \times \bar{u}(p') \left[F_1^p(q^2)\gamma_\lambda + i\frac{F_2^p(q^2)}{2m_N}\sigma_{\lambda\nu}q^\nu \right] u(p) \end{aligned} \tag{12.15}$$

$$\begin{aligned} \langle n(p') \left| V_\lambda^{em} \right| n(p)\rangle &= \frac{1}{(2\pi)^3}\sqrt{\frac{m_N^2}{p_0 p_0'}} \\ &\quad \times \bar{u}(p') \left[F_1^n(q^2)\gamma_\lambda + i\frac{F_2^n(q^2)}{2m_N}\sigma_{\lambda\nu}q^\nu \right] u(p) \end{aligned} \tag{12.16}$$

where [since $\int d^3x\, V_0^{em}(\mathbf{x}, 0)$ is the electric charge in unit of e] it follows, on using Eqs. (8)–(10) that

$$F_1^p(0) = 1,\ F_1^n(0) = 0. \tag{12.17}$$

Since $\sigma_{\lambda\nu}q^\nu$ gives Pauli type interaction, it also follows that

$$F_2^p(0) = \kappa_p,\ \ F_2^n(0) = \kappa_n \tag{12.18}$$

where κ_p and κ_n are the anomalous magnetic moments of proton and neutron respectively. $\kappa_p = 1.792$ and $\kappa_n = -1.913$ in units of nuclear magneton. Hence we get from Eq. (11.50a) and Eqs. (14)-(16) that

$$\begin{aligned} g_V(q^2) &= F_1^p(q^2) - F_1^n(q^2) = F_1^V(q^2) \\ f_V(q^2) &= \frac{1}{2m_N}\left[F_2^p(q^2) - F_2^n(q^2)\right] = \frac{F_2^V(q^2)}{2m_N}, \end{aligned} \tag{12.19}$$

where F_1^V and F_2^V are the isovector electromagnetic nucleon form factors. Their normalization follows from Eqs. (17) and (18).

$$F_1^V(0) = 1,\ \ F_2^V(0) = (\kappa_p - \kappa_n). \tag{12.20}$$

Thus in particular

$$\begin{aligned} g_V(0) &= 1, \\ f_V(0) &= \frac{\kappa_p - \kappa_n}{2m_N}. \end{aligned} \tag{12.21}$$

Using SU(3), we can write the matrix elements of vector current $V_{i\lambda}$, $i = 1, \cdots, 8$ for an octet of baryons (assuming $q^2 \approx 0$):

$$\langle B_k(p')\,|V_{i\lambda}|\,B_j(p)\rangle = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m_B^2}{p_0p_0'}}\,\bar{u}(p')[if_{ijk}\gamma_\lambda]u(p) \tag{12.22}$$

namely the relation (11.54).

12.3 Partially Conserved Axial Vector Current Hypothesis (PCAC)

From Eq. (11.73), we have

$$\begin{aligned}\left\langle 0\left|\partial^\lambda A_\lambda^+(x)\right|\pi^-\right\rangle &= -ip^\lambda\left\langle 0\left|A_\lambda^+\right|\pi^-\right\rangle e^{-ip\cdot x}\\ &= \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2p_0}}f_\pi m_\pi^2 e^{-ip\cdot x}. \qquad (12.23)\end{aligned}$$

If the axial vector current A_λ^+ is conserved, then either $f_\pi = 0$ or $m_\pi^2 = 0$. Since for a physical pion $m_\pi^2 \neq 0$, f_π must be zero and pion decay is forbidden. Thus A_λ^+ is not conserved. Now $\partial^\lambda A_\lambda^+$ has the same quantum numbers as those for a pion. If we now put

$$\partial^\lambda A_\lambda^+ = f_\pi m_\pi^2\, \pi^- \qquad (12.24)$$

then

$$\left\langle 0\left|\pi^-(x)\right|\pi^-\right\rangle = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2p_0}}e^{-ip\cdot x}. \qquad (12.25)$$

Here $\pi^-(x)$ is the pion field operator which creates π^+ or destroys π^-. Equation (24) is called the PCAC hypothesis. We note from Eq. (23), that in the limit $m_\pi^2 \to 0$, the axial vector current is conserved. This implies that strong interactions have an approximate symmetry which is exact in the limit of zero pion mass. Such a symmetry is called chiral symmetry. Chiral symmetry manifests itself in the existence of massless pseudoscalar mesons called Nambu-Goldstone bosons.

We shall come to this point again later. Here we discuss one of the important consequences of PCAC. We apply PCAC to neutron β-decay. From Eq. (11.50b), we have

$$\begin{aligned}&\left\langle p(p')\left|A_\lambda^+\right|n(p)\right\rangle\\ &= \frac{1}{(2\pi)^3}\sqrt{\frac{m_p m_n}{p_0 p_0'}}\\ &\quad\times \bar{u}(p')\left[g_A(q^2)\gamma_\lambda\gamma_5 + f_A(q^2)\gamma_5 q_\lambda - ih_A(q^2)\gamma_5\sigma_{\lambda\nu}q^\nu\right]u(p).\end{aligned} \qquad (12.26)$$

We note that pion pole contributes to the form factor $f_A(q^2)$ only. It does not contribute to $g_A(q^2)$ nor $h_A(q^2)$. Separating out the pion pole contribution, we write

$$f_A(q^2) = -\frac{\sqrt{2}g_{\pi NN}f_\pi}{q^2 - m_\pi^2} + \bar{f}_A(q^2) \qquad (12.27)$$

where $\bar{f}_A(q^2)$ is the remaining part of $f_A(q^2)$. From Eqs. (26) and (27), we get

$$\begin{aligned} &\left\langle p(p') \left| \partial^\lambda A_\lambda^+ \right| n(p) \right\rangle \\ = \; & \frac{1}{(2\pi)^3}\sqrt{\frac{m_p m_n}{p_0 p_0'}}\,\bar{u}(p')i\gamma_5 u(p) \\ & \times \left[2m_N g_A(q^2) - q^2\,\frac{\sqrt{2}g_{\pi NN}f_\pi}{q^2 - m_\pi^2} + q^2\,\bar{f}_A(q^2) \right]. \qquad (12.28) \end{aligned}$$

Now if we assume that in the limit $m_\pi^2 \to 0$, the axial vector current is conserved, we get,

$$2m_N g_A\left(q^2\right) - \sqrt{2}g_{\pi NN}f_\pi + q^2\,\bar{f}_A(q^2) = 0. \qquad (12.29)$$

At $q^2 = 0$, this gives

$$g_A = \frac{g_{\pi NN}f_\pi}{\sqrt{2}m_N}. \qquad (12.30)$$

This is called the Goldberger-Treiman (G–T) relation. Thus G–T relation is exact in the chiral symmetry limit when pion mass is zero and the axial vector current is conserved. This relation can be easily tested as all the quantities in Eq. (30) are experimentally known. This relation is valid within 6% agreement with experiment. On the other hand, we note that

$$\begin{aligned} \left\langle p(p') \left| \partial^\lambda A_\lambda^+ \right| n(p) \right\rangle \; = \; & \frac{1}{(2\pi)^3}\sqrt{\frac{m_p m_n}{p_0 p_0'}}\,\bar{u}(p')i\gamma_5 u(p) \\ & \times \left[2m_N g_A(q^2) + q^2\,f_A(q^2) \right] . \qquad (12.31) \end{aligned}$$

Using PCAC, viz. Eq. (24), we get

$$-\frac{\sqrt{2}g_{\pi NN}f_\pi m_\pi^2}{q^2-m_\pi^2}=2m_N g_A(q^2)+q^2 f_A(q^2). \tag{12.32}$$

Evaluating it at $q^2=0$, $m_\pi^2\neq 0$, we again get the G–T relation. We conclude that the success of the G-T relation implies that deviations from chiral symmetry or equivalently from PCAC are indeed small.

Finally, using SU(3) we can write for $q^2\approx 0$ for an octet of baryons [cf. Eq. (11.53)].

$$\begin{aligned}&\langle B_k(p')\,|A_{i\lambda}|\,B_j(p)\rangle\\ =\;&\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m_B^2}{p_0p_0'}}\bar{u}(p')\left[\gamma_\lambda\gamma_5\left(if_{ijk}F+d_{ijk}D\right)\right]u(p).\end{aligned} \tag{12.33}$$

In particular for neutron β-decay, we get

$$g_A=F+D, \tag{12.34}$$

where

$$(2\pi)^3\frac{p_0}{m_N}\langle p\,|A_{3\lambda}|\,p\rangle=\frac{1}{2}g_A\bar{u}(p)\gamma_\lambda\gamma_5u(p). \tag{12.35}$$

We define a four-vector

$$s^\lambda=\bar{u}(p)\gamma^\lambda\gamma^5u(p). \tag{12.36}$$

We note that

$$p\cdot s=0,\ \ s^2=-1. \tag{12.37}$$

The vector s^λ thus gives the spin of the proton. To see it explicitly we go to the rest frame of the proton. In this frame, we get from Eq. (37), $s_0=0$, $\mathbf{s}^2=1$. From Eq. (36), we get

$$\mathbf{s}=\chi^+\sigma\chi. \tag{12.38}$$

In quark model, we can write the axial-vector current $A_{i\mu} = \bar{q}\gamma_\mu\gamma_5\frac{\lambda_i}{2}q$
We define the quantity Δq as

$$(2\pi)^3\frac{p_0}{m}\langle p\,|\bar{q}\gamma_\lambda\gamma_5 q|\,p\rangle = \Delta q s_\lambda. \tag{12.39}$$

In particular for $A_{3\lambda} = \frac{1}{2}(\bar{u}\gamma_\lambda\gamma_5 u - \bar{d}\gamma_\lambda\gamma_5 d)$, we have

$$(2\pi)^3\frac{p_0}{m}\langle p\,|A_{3\lambda}|\,p\rangle = \frac{1}{2}(\Delta u - \Delta d)s_\lambda \tag{12.40}$$

so that

$$\Delta u - \Delta d = g_A = F + D. \tag{12.41}$$

12.4 Current Algebra and Chiral Symmetry

Isospin conservation implies that strong interactions are invariant under SU(2) group generated by the charges:

$$I_i(t) = \int V_{i0}(\mathbf{x}, t)d^3x, \; i = 1, 2, 3. \tag{12.42}$$

In the same way we can define the axial charges

$$I_i^5(t) = \int A_{i0}(\mathbf{x}, t)d^3x, \; i = 1, 2, 3. \tag{12.43}$$

The generators of the isospin group SU(2) satisfy the commutation relations

$$[I_i(t), I_j(t)] = i\varepsilon_{ijk}I_k(t). \tag{12.44}$$

Since $I_i^5(t)$'s belong to the adjoint representation of SU(2) group, we have

$$\left[I_i(t), I_j^5(t)\right] = i\varepsilon_{ijk}I_k^5(t). \tag{12.45}$$

We obtain a closed algebraic system by requiring that

$$\left[I_i^5(t), I_j^5(t)\right] = i\varepsilon_{ijk}I_k(t). \tag{12.46}$$

The last relation constitutes a major theoretical assumption. The commutation relations (44)-(46) represent the algebra of the group

SU(2)×SU(2) generated by the vector and axial vector charges. This group is called the chiral SU(2) group.

Let us now write the part of the QCD Lagrangian [cf. Eq. (7.32)] which involves u and d quarks:

$$L_{u,d} = i\bar{q}\gamma^\mu D_\mu q - \frac{m_u + m_d}{2}\bar{q}q - \frac{m_u - m_d}{2}\left(\bar{u}u - \bar{d}d\right), \quad (12.47)$$

where $q = \begin{pmatrix} u \\ d \end{pmatrix}$ is an isodoublet field and we have suppressed color indices. For $m_u = m_d$ this Lagrangian is invariant under the isospin transformation

$$q \to Uq, \quad (12.48)$$

where U is a special unitary matrix, $\exp\left[i\frac{\tau_i}{2}\Lambda_i\right]$, Λ_i being constant. The associated vector current $V_{i\mu} = \bar{q}\frac{\tau_i}{2}\gamma_\mu q$ is conserved. The existence of nearly degenerate isospin multiplets of hadrons shows clearly that $|m_u - m_d|$ is small compared to hadron mass scale (~1 GeV). Setting $m_u = m_d = m$, we can write

$$L_{u,d} = i\bar{q}_L\gamma^\mu D_\mu q_L + i\bar{q}_R\gamma^\mu D_\mu q_R - m(\bar{q}_L q_R + \bar{q}_R q_L), \quad (12.49)$$

where we have split q into "left-handed" and "right-handed" components

$$q_{L,R} = \frac{1 \mp \gamma_5}{2}q.$$

It is clear that in the limit $m = 0$, the Lagrangian (49) would be invariant under independent 'chiral' isospin transformations on q_L and q_R:

$$q_L = U_L q_L, \ q_R \to U_R q_R$$

and not only $V_{i\mu}$ but also the axial vector current $\bar{q}\gamma_\mu\gamma_5\frac{\tau_i}{2}q$ would be conserved. We note that the mass term $m\left(\bar{q}_L q_R + \bar{q}_R q_L\right)$ or in general the coupling to scalar and pseudoscalar fields

$$\left[\phi\bar{q}\begin{pmatrix} 1 \\ \gamma_5 \end{pmatrix} q = \phi\left(\bar{q}_L q_R \pm \bar{q}_R q_L\right)\right]$$

would break chiral symmetry. This also demonstrates that the forces between the quarks have to be vector in nature [mediated by spin 1 gluons, cf. the term $\bar{q}\gamma_\mu \boldsymbol{\lambda} \cdot \mathbf{G}^\mu q$ in Eq. (47) or Eq. (49)]. As we shall see later $m_u \sim 5$ MeV, $m_d \sim 10$ MeV (these are called current quark masses, not to be confused with constituent quark masses of order 300 MeV [cf. Chap. 6]) are small compared to the hadron scale of $O(1$ GeV) so that chiral symmetry is nearly exact.

Now if $A_{i\lambda}$ were conserved, the axial charge I_i^5 would commute with the Hamiltonian:

$$[I_i^5, H] = 0. \tag{12.50}$$

Hence if we define

$$I_i^5 \left|X_j\right\rangle = i\varepsilon_{ijk} \left|Y_k\right\rangle , \tag{12.51}$$

use of Eq. (50) would imply that the states $|Y_k\rangle$ are degenerate in mass with $|X_j\rangle$ even though they have opposite parity. This is because I_i^5 has negative parity. This condition can be realized in either of the two ways:

1. The Wigner-Weyl realization of SU(2) symmetry, in which case $|Y_k\rangle$ would consist of "parity doublets" of $|X_j\rangle$ e.g. if $|X_j\rangle$ were pseudoscalar mesons, $|Y_k\rangle$ would be scalar mesons degenerate in mass with the pseudoscalar mesons. This is not what occurs in nature and therefore chiral symmetry is not realized in nature in this way in contrast to the ordinary isospin symmetry which is realized in this way.

2. Spontaneously broken symmetry realization of SU(2), in which case $|Y_k\rangle$ would consist of $|X_j\rangle$ plus an odd number of pions with vanishing four-momentum (called soft pions), the pion being a massless "Nambu-Goldstone" boson. In particular

$$I_i^5 \left|0\right\rangle = -\frac{i}{2}\frac{f_\pi}{\sqrt{2}} \left|\pi_i(0)\right\rangle \neq 0, \tag{12.52}$$

the first part being valid only for single-pion transitions, while

$$I_i \left|0\right\rangle = 0. \tag{12.53}$$

As we shall see m_π^2 would involve $(m_u + m_d)/2$ as a factor and so a measure of explicit chiral symmetry breaking is provided by $m_\pi^2/m_\rho^2 \approx 0.03$, ρ being the non-strange (non Nambu-Goldstone) boson next to pion. The notion of (approximate) spontaneously broken chiral symmetry has been found useful in hadron physics and has given rise to many predictions involving soft pions which are in good agreement with the data [see bibliography]. One such prediction is the Goldberger-Treiman relation (30):

$$\left[\frac{m_A g_A \sqrt{2}}{f_\pi g_{\pi NN}}\right] - 1 = 0 \tag{12.54}$$

to be compared with the experimental value 0.06 ± 0.01 of the left-hand side.

The above considerations can be easily generalized to SU(3). Thus the QCD Lagrangian (7.32) shows an approximate global symmetry in the limit $m_q \to 0$, this Lagrangian is invariant under the group SU(3)×SU(3) generated by the charges associated with the weak currents $J_{i\mu}$. Thus the generators of the group are ($i = 1, \cdots, 8$).

$$\begin{aligned} F_i &= \int V_{i0}(\mathbf{x}, t) d^3x \\ F_i^5 &= \int A_{i0}(\mathbf{x}, t) d^3x. \end{aligned}$$

They satisfy the commutation relations

$$[F_i, F_j] = i f_{ijk} F_k \tag{12.55}$$

$$\left[F_i, F_j^5\right] = i f_{ijk} F_k^5 \tag{12.56}$$

$$\left[F_i^5, F_j^5\right] = i f_{ijk} F_k. \tag{12.57}$$

The commutation relations (55) and (56) follow from flavor SU(3), the commutation relation (57) is a new assumption. Equivalently if we define

$$F_i^L = \frac{1}{2}\left(F_i - F_i^5\right), \; F_i^R = \frac{1}{2}\left(F_i + F_i^5\right) \tag{12.58}$$

we get

$$\begin{aligned}\left[F_i^L, F_j^L\right] &= i f_{ijk} F_k^L \\ \left[F_i^R, F_j^R\right] &= i f_{ijk} F_k^R \\ \left[F_i^L, F_j^R\right] &= 0.\end{aligned} \tag{12.59}$$

Symmetry generated by the above group is called the chiral symmetry. If (R_1, R_2) is a multiplet of group SU(3)×SU(3), then under parity

$$(R_1, R_2) \to (R_2, R_1). \tag{12.60}$$

For example $(8,1) \to (1,8), (3,3^*) \to (3^*,3)$. This means that if this symmetry is realized as a classification symmetry, we must have parity doublets. This is not the case in nature. No parity doublets are found. This implies that the chiral symmetry is realized in the Nambu-Goldstone mode that is to say, there are eight bosons which in the chiral limit have zero mass. As we have already seen, pions are the Nambu-Goldstone bosons which in the chiral SU(2)×SU(2) limit are massless. The eight pseudoscalar mesons are identified with Nambu-Goldstone bosons of chiral group.

The algebra generated by F_i and F_i^5 is called the chiral algebra. This algebra has rather rich physical content because generators of the symmetry group can be identified with observables. The matrix elements can be measured in electroweak interactions. This in fact provides evidence for chiral symmetry [see bibliography].

12.4.1 Explicit breaking of chiral symmetry

As already seen the chiral symmetry is spontaneously broken [cf. Eq. (52)]. Another way of expressing it is that

$$\langle 0 |\bar{q}q| 0\rangle \neq 0 \Rightarrow F_i^5 |0\rangle \neq 0. \tag{12.61}$$

To see this, we note that in the quark model, we have the following commutation relations:

$$\left[F_i^5, S_j\right] = i d_{ijk} P_k \qquad i = 0, 1, ..., 8$$

$$\begin{aligned} \left[F_i^5, P_j\right] &= -id_{ijk}S_k \\ [F_i, S_j] &= if_{ijk}S_k \\ [F_i, P_j] &= if_{ijk}P_k \end{aligned} \tag{12.62}$$

where

$$S_i = \bar{q}\frac{\lambda_i}{2}q, \;\; P_i = \bar{q}\frac{\lambda_i}{2}\gamma_5 q \tag{12.63}$$

are respectively the scalar and pseudoscalar densities. We note from Eqs. (62) that

$$\left\langle 0 \left| \left[P_1 + iP_2, F_{1-i2}^5\right] \right| 0 \right\rangle = i2\sqrt{\frac{2}{3}} \left\langle 0 \left| S_0 \right| 0 \right\rangle + i\frac{2}{\sqrt{3}} \left\langle 0 \left| S_8 \right| 0 \right\rangle . \tag{12.64}$$

Now we expect that flavor SU(3) is realized in the usual way and is not spontaneously broken [cf. Eq. (53)]. This implies that

$$\left\langle 0 \left| S_8 \right| 0 \right\rangle = \frac{1}{\sqrt{3}} \left\langle 0 \left| \left[F_{4+i5}, S_{4-i5}\right] \right| 0 \right\rangle = 0 \tag{12.65a}$$

as

$$F_{4\pm i5} \left|0\right\rangle = 0. \tag{12.65b}$$

Thus, if

$$\left\langle S_0 \right\rangle_0 = \left\langle \bar{u}u + \bar{d}d + \bar{s}s \right\rangle_0 \neq 0 \tag{12.66}$$

then we have from Eq. (64):

$$F_{1-i2}^5 \left|0\right\rangle \neq 0, \tag{12.67}$$

the condition for spontaneously broken symmetry [cf. Eq. (52)]. Let us write

$$\left\langle \bar{u}u \right\rangle_0 = \left\langle \bar{d}d \right\rangle_0 = \left\langle \bar{s}s \right\rangle_0 = -v(\text{say}). \tag{12.68}$$

Hence we have the result that $\left\langle S_0 \right\rangle_0 \neq 0$ which implies that chiral symmetry is spontaneously broken and $\left\langle S_8 \right\rangle_0 = 0$ implying that flavor SU(3) is not spontaneously broken.

We can write the QCD Hamiltonian density [cf. Eq. (7.32)] as

$$
\begin{aligned}
\mathcal{H} &= \mathcal{H}_0 + \left(m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s\right) \\
&= \mathcal{H}_0 + \sqrt{\frac{2}{3}}\left(2\bar{m} + m_s\right)S_0 + \frac{2}{\sqrt{3}}\left(\bar{m} - m_s\right)S_8 + \left(m_u - m_d\right)S_3 \\
&\equiv \mathcal{H}_0 + \mathcal{H}'. \qquad (12.69)
\end{aligned}
$$

The Hamiltonian density $\mathcal{H}_0$ is chiral invariant. Here $\bar{m} = (1/2)(m_u + m_d)$. Now

$$\frac{dF_i^5}{dt} = -i\left[F_i^5, H\right] = -i\left[F_i^5, H'\right], \qquad (12.70)$$

where

$$H(t) = \int d^3x \mathcal{H}(t, \mathbf{x}).$$

The (charge) continuity equation

$$
\begin{aligned}
\frac{dF_i^5}{dt} &= \int d^3x \left(\frac{\partial A_{i0}(t,\mathbf{x})}{\partial t} + \nabla \cdot \mathbf{A}_i(t,\mathbf{x})\right) \\
&= \int d^3x \partial^\mu A_{i\mu} \qquad (12.71)
\end{aligned}
$$

then converts Eq. (71) into

$$\partial^\lambda A_{i\lambda} = -i\left[F_i^5, \mathcal{H}'\right]. \qquad (12.72)$$

From Eq. (72), we have

$$\left\langle 0\left|\left[F_j^5, \partial^\lambda A_{i\lambda}\right]\right|0\right\rangle = -i\left\langle 0\left|\left[F_j^5, \left[F_i^5, \mathcal{H}'\right]\right]\right|0\right\rangle. \qquad (12.73)$$

Using Eq. (52), namely

$$
\begin{aligned}
F_j^5 |0\rangle &= -i\frac{f_\pi}{2\sqrt{2}}|\pi_j\rangle \\
\langle 0| F_j^5 &= \frac{i}{2\sqrt{2}}\langle \pi_j| f_\pi, \qquad (12.74)
\end{aligned}
$$

we obtain

$$\frac{i}{2}\frac{f_\pi}{\sqrt{2}}\left[\left\langle \pi_j\left|\partial^\lambda A_{i\lambda}\right|0\right\rangle + \left\langle 0\left|\partial^\lambda A_{i\lambda}\right|\pi_j\right\rangle\right] = -i\left\langle 0\left|\left[F_j^5,\left[F_i^5,\mathcal{H}'\right]\right]\right|0\right\rangle . \tag{12.75}$$

The use of PCAC relation $\partial^\lambda A_{i\lambda} = \left(f_\pi/\sqrt{2}\right) m_i^2 \pi_i$, then gives [$m_{ij}^2$ is symmetric in i and j].

$$m_{ij}^2 = -\frac{2}{f_\pi^2}\left\langle 0\left|\left[F_j^5,\left[F_i^5,\mathcal{H}'\right]\right]\right|0\right\rangle \tag{12.76}$$

where $\mathcal{H}'$ [cf. Eq. (69)] is

$$\mathcal{H}' = \sqrt{\frac{2}{3}}\left(2\bar{m}+m_s\right)S_0 + \frac{2}{\sqrt{3}}\left(\bar{m}-m_s\right)S_8 + \left(m_u-m_d\right)S_3 . \tag{12.77}$$

Substituting Eq. (77) into Eq. (76) and using Eqs. (68) and (62), one obtains

$$\begin{aligned}
m_{\pi^0}^2 &= m_{\pi^+}^2 = \frac{2}{f_\pi^2}\left(m_u+m_d\right)v \\
m_{K^+}^2 &= \frac{2}{f_\pi^2}\left(m_u+m_s\right)v,\ m_{K^0}^2 = \frac{2}{f_\pi^2}\left(m_d+m_s\right)v \\
m_{\pi^0\eta}^2 &= m_{\eta\pi^0}^2 = \frac{2}{\sqrt{3}f_\pi^2}\left(m_u-m_d\right)v \\
m_\eta^2 &= \frac{2}{3f_\pi^2}\left(m_u+m_d+4m_s\right)v.
\end{aligned} \tag{12.78}$$

Let Δ be the electromagnetic contribution due to photon exchange to $m_{\pi^\pm}^2$. Since π^+, K^+ form a U-spin multiplet the electromagnetic contribution to $m_{K^\pm}^2$ is also Δ while it is zero for $m_{\pi^0}^2$, $m_{K^0}^2$, m_η^2, so that adding Δ in Eq. (78) for π^+, K^+, we get

$$\begin{aligned}
\frac{m_d}{m_u} &= \frac{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2}{2m_{\pi^0}^2 - m_{\pi^+}^2 + m_{K^+}^2 - m_{K^0}^2} \approx 1.8 \\
\frac{m_s}{m_d} &= \frac{m_{K^0}^2 + m_{K^+}^2 - m_{\pi^+}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} \approx 20.1.
\end{aligned} \tag{12.79}$$

Here we have used the explicit breaking of chiral symmetry in calculating the current quark mass ratios in terms of masses of pseudoscalar mesons. When quark masses go to zero pseudoscalar mesons become zero mass Nambu-Goldstone bosons required by spontaneously broken chiral symmetry.

12.4.2 An application of chiral symmetry to non-leptonic decays of hyperons

Consider the matrix elements [where B_r and B_s are members of the same baryon octet]:

$$\left\langle B_s(p')\left|\left[F_i^5, H_W\right]\right| B_r(p)\right\rangle = \left\langle B_s(p')\left|F_i^5 H_W - H_W F_i^5\right| B_r(p)\right\rangle \tag{12.80}$$

where $i = 1, 2, 3$. Using Eq. (74) and its hermitian conjugate, we can write it as

$$\begin{aligned}
&\left\langle B_s(p')\left|\left[F_i^5, H_W\right]\right| B_r(p)\right\rangle \\
&= \frac{i}{2}\frac{f_\pi}{\sqrt{2}}\left[\langle B_s(p')\,\pi_i(0)\,|H_W|\,B_r(p)\rangle + \langle B_s(p')\,|H_W|\,\pi_i(0)B_r(p)\rangle\right] \\
&= i\frac{f_\pi}{\sqrt{2}}\langle B_s(p')\,\pi_i(0)\,|H_W|\,B_r(p)\rangle\,.
\end{aligned} \tag{12.81}$$

In other words in the limit $q_\mu = (p - p')_\mu \to 0$ [called the soft pion limit], if the matrix elements $\langle B_s(p')\,\pi_i(q)\,|H_W|\,B_r(p)\rangle$ are non singular, then Eq. (81) gives

$$\lim_{q\to 0}\langle B_s(p')\,\pi_i(q)\,|H_W|\,B_r(p)\rangle = -i\frac{\sqrt{2}}{f_\pi}\left\langle B_s(p')\left|\left[F_i^5, H_W\right]\right| B_r(p)\right\rangle. \tag{12.82}$$

Now $H_W = H_W^{p.c} + H_W^{p.v}$ [cf. Chap. 11] and it can be shown that for $s-$waves $[H_W^{p.v}]$, the amplitude on the left hand side of Eq. (82) is non-singular [see below] and we have

$$\lim_{q\to 0}\langle B_s(p')\,\pi_i(q)\,|H_W^{p.v}|\,B_r(p)\rangle = -i\frac{\sqrt{2}}{f_\pi}\left\langle B_s(p')\left|\left[F_i^5, H_W^{p.v}\right]\right| B_r(p)\right\rangle. \tag{12.83}$$

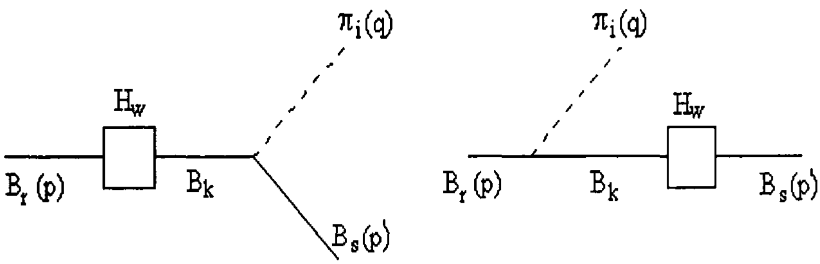

Figure 1 Pole diagram in hyperon decay.

For p-waves $[H_W^{p.c}]$, one can apply the result (82) to

$$
\begin{aligned}
& \lim_{q\to 0}\left[\langle B_s(p')\,\pi_i(q)\,|H_W^{p.c}|\,B_r(p)\rangle\right.\\
& \left.-\ \langle B_s(p')\,\pi_i(q)\,|H_W^{p.c}|\,B_r(p)\rangle_{\rm Born}\right]\\
=\ & -i\frac{\sqrt{2}}{f_\pi}\left\langle B_s(p')\left|\left[F_i^5, H_W^{p.c}\right]\right|B_r(p)\right\rangle
\end{aligned}
\tag{12.84}
$$

where the Born terms are shown in Fig. 1. These are singular for $H_W^{p.c}$ in the limit $q_\mu \to 0$ where $m_B = m'_B$ as they behave like $1/|m_B - m'_B|$ but for $H_W^{p.v}$ they behave like $1/|m_B + m'_B|$ and are non-singular. Now as we have seen in Chap. 11 [cf. Eq. (11.28)], the $|\Delta S| = 1$ non-leptonic Hamiltonian is

$$
H_W = \frac{G_F}{\sqrt{2}}\sin\theta_c\cos\theta_c\,[\bar{s}\gamma^\mu(1+\gamma_5)u]\,[\bar{u}\gamma_\mu(1+\gamma_5)d]\,. \tag{12.85}
$$

This being the product of two left handed currents $[F_R = F_i + F_i^5]$ satisfy

$$[F_i^R, H_W] = 0$$

or

$$[F_i^5, H_W] = -[F_i, H_W]$$

i.e.

$$
\left[F_i^5, H_W^{p.v,p.c}\right] = -\left[F_i, H_W^{p.c,p.v}\right] \tag{12.86}
$$

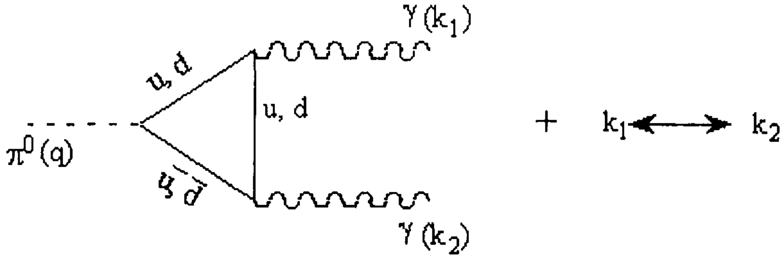

Figure 2 Triangle diagram for $\pi^0 \to 2\gamma$ decay.

Furthermore F_i (being the generator of SU(3) flavor group) acting on$|B_r>$ or $|B_s>$ produces a member of the same octet. To illustrate this point, consider for example, $|B_r>=|\Lambda>$ and $<B_s| =<p|$, $i = \frac{1+i2}{\sqrt{2}}$. Then

$$F_{1+i2}\,|\Lambda\rangle = 0 \text{ and } \langle n| = \langle p|\,F_{1+i2}.$$

Thus for $s-$wave from Eqs. (83) and (86)

$$\left\langle p\,(p')\,\pi^-(q)\,|H_w^{p.v}|\,\Lambda^0\,(p)\right\rangle = \frac{i}{f_\pi}\,\langle n\,|H_w^{p.c}|\,\Lambda\rangle\,. \tag{12.87}$$

Also as shown in Chap. 11, in the exact SU(3) limit $\langle B_s\,|H_w^{p.v}|\,B_r\rangle = 0$. Thus the p-wave non-leptonic decays are given by the Born terms which are also determined by $\langle B_s\,|H_w^{p.c}|\,B_r\rangle$ as far as weak vertices are concerned. These were the results which we employed in Sec. 3.3c of Chap. 11.

12.5 Axial Anomaly

As seen in Chap. 7, $\pi^0 \to 2\gamma$ is given by the triangle graph of Fig. 2. In the chiral limit ($m_u = m_d = 0$), this triangle graph gives a finite value for the $\pi^0 \to 2\gamma$ amplitude:

$$M(\pi^0 \to 2\gamma) = \varepsilon^{\mu *}(k_1)\varepsilon^{\nu *}(k_2)\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta F_{\pi^0\gamma\gamma}(q^2), \tag{12.88}$$

with

$$F_{\pi^0\gamma\gamma}(0) = N_c[e_u^2 - e_d^2]\left(\frac{-\alpha}{\pi}\right)\frac{g_{\pi qq}}{m_q}, \tag{12.89}$$

where N_c is the number of colors, e. g. 3, $e_u = 2/3$, $e_d = -1/3$ while the Goldberger-Trieman relation for $\langle q|A_{3\mu}|q\rangle$ with $A_{3\mu} = \frac{1}{2}\left(\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu d\right)$ gives $(f_\pi/\sqrt{2})g_{\pi qq} = m_q$ so that Eq. (88) gives

$$F_{\pi\gamma\gamma}(0) = \frac{-\sqrt{2}\alpha}{\pi f_\pi}. \tag{12.90}$$

It is important to remark that the result (90) is unaltered by radiative corrections to the quark triangle and Eq. (90) is independent of the masses of fermions in the loop. Equation (90) gives

$$\Gamma_{\pi\gamma\gamma} \equiv F_{\pi\gamma\gamma}^2\frac{m_{\pi^0}^3}{64\pi} = 7.58 \text{ eV} \tag{12.91}$$

which is remarkably close to experiment with only 2% PCAC correction to the amplitude.

The above result is often stated in terms of contribution to the amplitude due to an axial-vector "anomalous" divergence:

$$\partial^\lambda A_{3\lambda} = \frac{\alpha}{4\pi}F_{\mu\nu}\widetilde{F}^{\mu\nu}, \tag{12.92}$$

where $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ [a_μ being electromagnetic potential] and $\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. Note that Eq. (92) does not arise from equations of motion (72). That is why it is called "anomalous" divergence. Combining Eqs. (72) and (92), we have

$$\partial^\lambda A_{i\lambda} = -\left[F_i^5, \mathcal{H}'\right] + \delta_{i3}\frac{\alpha}{4\pi}F_{\mu\nu}\widetilde{F}^{\mu\nu}. \tag{12.93}$$

The first term on the right-hand side of Eq. (93) vanishes in the chiral limit but it is not so for the second term. The PCAC relation for $A_{3\lambda}$ thus becomes

$$\partial^\lambda A_{3\lambda}(x) = \frac{f_\pi}{\sqrt{2}}m_\pi^2\pi^0(x) + \frac{\alpha}{4\pi}F_{\mu\nu}\widetilde{F}^{\mu\nu}. \tag{12.94}$$

The "anomalous" divergence equations for η_8 and η_0 are

$$\partial^\lambda A_{k\lambda} = \frac{\alpha}{4\pi} S_P^k F_{\mu\nu} \tilde{F}^{\mu\nu}, \tag{12.95}$$

where $k = 8$ or 0 and

$$\begin{aligned} S_{\eta_8} &= N_c \left(\frac{1}{\sqrt{3}}\right)\left[e_u^2 + e_d^2 - 2e_s^2\right] = \frac{1}{\sqrt{3}} \\ S_{\eta_0} &= N_c \left(\sqrt{\frac{2}{3}}\right)\left[e_u^2 + e_d^2 + e_s^2\right] = 2\frac{2}{\sqrt{3}}. \end{aligned} \tag{12.96}$$

Similar considerations show that in QCD, the flavor SU(3) singlet current

$$A_{0\mu} = \bar{q}\frac{\lambda_0}{2}\gamma_\mu\gamma_5 q = \frac{1}{2}\sqrt{\frac{2}{3}}\left[\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s\right]$$

has "anomalous" divergence

$$\partial^\lambda A_{0\lambda} = \sqrt{\frac{2}{3}}\frac{3\alpha_s}{4\pi}\mathbf{G}_{\mu\nu} \cdot \widetilde{\mathbf{G}}^{\mu\nu} \tag{12.97}$$

where

$$\widetilde{G}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} G_{\alpha\beta} \tag{12.98}$$

and $\mathbf{G}_{\mu\nu}$ involving gluon field has been defined in Chap. 7 [cf. Eq. (7.31c)]. Thus

$$\begin{aligned} \partial^\lambda A_{0\lambda} &= \sqrt{\frac{2}{3}}\left[m_u \bar{u} i\gamma_5 \bar{u} + m_d \bar{d} i\gamma_5 d + m_s \bar{s} i\gamma_5 s\right] \\ &\quad + \sqrt{\frac{2}{3}}\frac{3\alpha_s}{4\pi}\mathbf{G}_{\mu\nu} \cdot \widetilde{\mathbf{G}}^{\mu\nu}. \end{aligned} \tag{12.99}$$

It is clear from Eq. (99) that the SU(3) singlet current is not conserved in chiral SU(3)⊗SU(3) limit. An application of this will be considered in Chap. 14.

12.6 QCD Sum Rules

We have seen in Chap. 7 that the asymptotic freedom property of QCD makes it possible to calculate processes at short distances or for large q^2, q^2 being the square of the momentum transfer. On the other hand, bound states of quarks and gluons (hadrons or hadron resonances) arise because of large distance confinement effects, i.e. strong coupling effects, which cannot be treated in perturbation theory. The idea of QCD sum rules is to calculate resonance parameters (masses, width) in terms of QCD parameters (α_s, quarks masses and number of other matrix elements which are introduced to parametrize the non-perturbative effects). We have also seen previously that in the absence of quark masses, the QCD Lagrangian shows a global chiral symmetry i.e. it is invariant under a global $SU_L(3) \times SU_R(3)$ group. But this chiral symmetry is spontaneously broken i.e. the ground state is not invariant under this symmetry. This gives rise to [$q = u$, d, s] [cf. Eq. (61)]

$$\langle 0 \,|\bar{q}q|\, 0\rangle \neq 0$$

leading to an octet of zero mass pseudoscalar mesons (so-called Nambu-Goldstone bosons; such bosons acquire masses when QCD Lagrangian is explicitly broken by the quark mass terms). The non-vanishing of the above quark condensate is a non-perturbative effect and gives rise to power corrections to asymptotic freedom effect, which is logarithmic. The essential point of the QCD sum rules i.e. to relate QCD and non-perturbative parameters of the above type with resonance parameters, is illustrated by the simplest of sum rules i.e. for a two-point function:

$$\Delta\left(q^2\right) = \frac{1}{\pi}\int \frac{Im\Delta\left(q^2\right)}{s-q^2} = \sum_i C_i\left(q^2\right)\langle 0\,|O_i|\,0\rangle\,. \qquad (12.100)$$

The left-hand side is saturated with resonance so that

$$\text{l.h.s.} = \sum_i \frac{g_i^2}{m_i^2 - q^2} \qquad (12.101)$$

where (g_i, m_i) are resonance parameters. The right-hand side is useful only for large q^2 in which limit the perturbative QCD allows us to calculate the coefficients $C_i(q^2)$ in the operator product expansion. In practice we want to saturate l.h.s. by a few low lying resonances. Thus we should use some weighting factor to suppress large s contributions on l.h.s. This is done by using Borel transform of the sum rule, which introduces a weighting factor involving a mass parameter M^2, which should be sufficiently large to suppress non leading terms on r.h.s. of Eq. (101) but not too large in order to suppress contribution from higher hadron states on l.h.s. Thus the problem in practice reduces to finding a region of stability point for M^2 so that a small variation in M^2 will not affect the physical parameters. In this way from QCD sum rules for two-point and three-point functions, a large number of constraints on hadron spectrum have been obtained providing not only a consistency check but also a useful phenomenological information on resonance as well as QCD parameters and on $\langle 0|\bar{q}q|0\rangle$. For details see the bibliography.

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Chapter 13
ELECTROWEAK UNIFICATION

13.1 Introduction

The Fermi theory of β-decay cannot be the fundamental theory of weak interactions. It leads to many difficulties; it is non renormalizable theory. In this theory the scattering cross section for the process $\nu_\mu + e^- \to \nu_e + \mu^-$ is given by Eq. (2.155):

$$\sigma_s = \frac{G_F^2}{\pi} s. \tag{13.1}$$

The above scattering is purely $S-$wave. Now Eq. (3.118) $[\lambda_1 = \lambda_2 = \pm 1/2]$ gives $\sigma_s = \frac{4\pi}{2} \left|2F^0\right|^2$ [the factor 2 in the denominator is average over initial electron spin], where $F^0 = \frac{\eta_0 \ e^{2i\delta_0} - 1}{2ip}$. Now the maximum absorption occurs when $\eta_0 = 0$, so that $\left|F^0\right|^2 \leq \frac{1}{4p^2} = \frac{1}{s}$. Thus the partial wave unitarity gives

$$\sigma_s = 8\pi \left|F^0\right|^2 \leq \frac{8\pi}{s} \tag{13.2}$$

so that from Eq. (1)

$$\frac{G_F^2}{\pi} s \leq \frac{8\pi}{s}$$

or

$$\frac{G_F \ s}{2\sqrt{2} \ \pi} \leq 1. \tag{13.3}$$

Hence Fermi theory breaks down for $s > (2\sqrt{2}/G_F) = (0.9 \text{ TeV})^2$. Therefore, we need a cut-off Λ_F signifying new physics beyond Λ_F

where from Eq. (3)

$$\Lambda_F^{PWU} \leq 0.9 \text{ TeV}. \tag{13.4}$$

Here PWU signifies that this has been obtained from partial wave unitarity. On the other hand if weak interactions are mediated through vector boson W, then instead of Eq. (1), we get

$$\begin{aligned} \sigma_s &= \left(\frac{g_W^2}{4\pi}\right)^2 \frac{32\ \pi\ s}{(s+m_W^2)\ m_W^2} \\ &= \frac{G_F^2}{\pi} \frac{s}{\left(1+\frac{s}{m_W^2}\right)} \end{aligned} \tag{13.5}$$

which is finite for all energies, approaching the limiting value

$$\sigma \longrightarrow \left(\frac{G_F^2}{\pi}\right)\ m_W^2.$$

Thus we see from Eq. (1) that the W–boson mass m_W provides the cut-off Λ_F. As we shall see $m_W \approx 80$ GeV, so that $m_W \ll \Lambda_F = 0.9$ TeV.

The charged weak interactions like electromagnetic interaction are vector in character $(V - A)$ and if the mediators of these interactions are vector bosons, then the universality of weak interactions suggests that the underlying theory of these interactions is a gauge theory. Since weak interactions have short range, the vector bosons associated with them must be massive. But the mass term is not gauge invariant. However, if the gauge symmetry is spontaneously broken, then the gauge vector bosons acquire mass. In this way all the desirable features of a gauge theory like universality and renormalizability are preserved.

13.2 Spontaneous Gauge Symmetry Breaking

Before discussing the gauge theory of weak interactions, we consider a simple model to illustrate the idea of spontaneous symmetry breaking. Consider a simple Lagrangian

$$\begin{aligned} L &= \bar{\Psi}\, i\gamma^{\mu}\, \partial_{\mu}\, \Psi + \partial^{\mu}\, \bar{\phi}\, \partial_{\mu}\, \phi - h\bar{\Psi}_L\, \Psi_R\, \phi \\ &\quad -h\bar{\Psi}_R\, \Psi_L\, \bar{\phi} - \mu^2\, \bar{\phi}\, \phi - \lambda\, \left(\bar{\phi}\, \phi\right)^2 \\ &= \bar{\Psi}_L\, i\gamma^{\mu}\, \partial_{\mu}\, \Psi_L + \bar{\Psi}_R\, i\gamma^{\mu}\, \partial_{\mu}\, \Psi_R + \partial^{\mu}\, \bar{\phi}\, \partial_{\mu}\, \phi \\ &\quad -h\bar{\Psi}_L\, \phi\, \Psi_R - h\bar{\Psi}_R\, \bar{\phi}\, \Psi_L - V(\phi) \end{aligned} \tag{13.6}$$

where

$$\Psi_L = \frac{1}{2}(1-\gamma_5)\,\Psi \tag{13.7a}$$

$$\Psi_R = \frac{1}{2}(1+\gamma_5)\,\Psi \tag{13.7b}$$

are left handed and right handed fermion fields respectively. ϕ is a complex scalar field interacting with fermion having a coupling strength h. $V(\phi)$ is given by

$$V(\phi) = \mu^2\, \bar{\phi}\, \phi + \lambda\, \left(\bar{\phi}\, \phi\right)^2. \tag{13.8}$$

Consider the gauge transformations

$$\begin{aligned} \Psi_L &\longrightarrow e^{i\Lambda_1(x)}\, \Psi_L, \\ \Psi_R &\longrightarrow e^{i\Lambda_1(x)}\, \Psi_R, \\ \phi &\longrightarrow \phi \end{aligned} \tag{13.9a}$$

$$\begin{aligned} \Psi_L &\longrightarrow e^{i\Lambda_2(x)}\, \Psi_L, \\ \Psi_R &\longrightarrow e^{-i\Lambda_2(x)}\, \Psi_R, \\ \phi &\longrightarrow e^{2i\Lambda_2(x)}\, \phi. \end{aligned} \tag{13.9b}$$

Obviously the Lagrangian (6) is invariant under the gauge transformations (9) if Λ_1 and Λ_2 are constants. The gauge group corresponding to gauge transformations (9) is U(1)⊗U(1). If we require the Lagrangian (6) to be local gauge invariant, then we must introduce two massless gauge fields A_μ and B_μ, which transform as

$$\begin{aligned} A_\mu &\rightarrow A_\mu - \frac{1}{e}\partial_\mu \Lambda_1 && (13.10a) \\ B_\mu &\rightarrow B_\mu + \frac{1}{g}\partial_\mu \Lambda_2. && (13.10b) \end{aligned}$$

Then we can write the gauge invariant Lagrangian, by replacing ∂_μ by the covariant derivatives:

$$\begin{aligned} \partial_\mu \Psi_L &\rightarrow (\partial_\mu + ieA_\mu - igB_\mu)\Psi_L, \\ \partial_\mu \Psi_R &\rightarrow (\partial_\mu + ieA_\mu + igB_\mu)\Psi_R, \\ \partial_\mu \phi &\rightarrow (\partial_\mu - 2igB_\mu)\phi \end{aligned}$$

in Eq. (6). Hence the gauge invariant Lagrangian is given by

$$\begin{aligned} L = & -\frac{1}{4}A^{\mu\nu} A_{\mu\nu} - \frac{1}{4}B^{\mu\nu} B_{\mu\nu} + \bar{\Psi}_L \, i\gamma^\mu \, (\partial_\mu + ieA_\mu - igB_\mu)\Psi_L \\ & -h\left(\bar{\Psi}_L\phi\Psi_R + \bar{\Psi}_R\bar{\phi}\Psi_L\right) + \bar{\Psi}_R \, i\gamma^\mu (\partial_\mu + ieA_\mu + igB_\mu)\Psi_R \\ & + (\partial^\mu + 2igB^\mu)\bar{\phi}(\partial_\mu - 2igB_\mu)\phi - V(\phi). \qquad (13.11) \end{aligned}$$

In Eq. (8), it is usual to choose $\lambda > 0$, since $V(\phi)$ would have no minimum if $\lambda < 0$. If in $V(\phi)$, $\mu^2 > 0$, then we have the ordinary scalar particles of mass μ and $V(\phi)$ has a local minimum at $\phi = 0$. Then the model is not interesting. However, if $\mu^2 < 0$, then $V(\phi)$ has a local minimum at $|\phi|^2 = -\frac{\mu^2}{2\lambda}$ or $\phi = \pm\sqrt{-\frac{\mu^2}{2\lambda}}$. This is shown in Fig. 1. By convention, we select the positive sign. This is a classical approximation to the vacuum expectation value of ϕ

$$\langle 0 |\phi| 0\rangle = \sqrt{-\frac{\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}}. \qquad (13.12)$$

Although the Lagrangian (11) is invariant under the local gauge transformations (9) and (10), the non-vanishing expectation

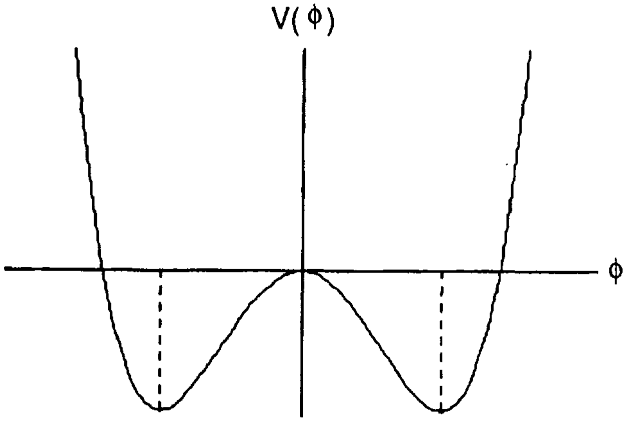

Figure 1 Effective potential $V(\phi)$ for $\mu^2 < 0$, showing local minima.

value of ϕ means that the gauge symmetry is broken i.e. the vacuum or the ground state is not invariant under the gauge transformation (9b). This can be seen as follows. From Eq. (9b)

$$U^{-1}(\Lambda_2)\,\phi U(\Lambda_2) = e^{2i\Lambda_2}\phi. \tag{13.13}$$

Therefore,

$$\left\langle 0 \left| U^{-1}(\Lambda_2)\,\phi U(\Lambda_2) \right| 0 \right\rangle = e^{2i\Lambda_2}\langle 0|\phi|0\rangle\,. \tag{13.14}$$

If the vacuum is gauge invariant, then

$$U(\Lambda_2)\,|0\rangle = |0\rangle \tag{13.15}$$

and Eq. (14) gives

$$\langle 0|\phi|0\rangle = e^{2i\Lambda_2}\langle 0|\phi|0\rangle\,. \tag{13.16}$$

Thus if $\langle 0|\phi|0\rangle \neq 0$, then $e^{2i\Lambda_2} = 1$ for any Λ_2, a contradiction. Hence $U(\Lambda_2)\,|0\rangle \neq |0\rangle$, and the gauge symmetry is spontaneously broken i.e. $U_1 \times U_2 \to U_1$, U_1 is unbroken.

We now show, how spontaneous symmetry breaking leads to the massive vector boson $B\mu$. It is convenient to define a new field ϕ':

$$\begin{aligned} \phi &= \phi' + \frac{v}{\sqrt{2}} = \frac{\phi_1 + i\phi_2}{\sqrt{2}} + \frac{v}{\sqrt{2}} \\ \langle \phi' \rangle &= 0, \end{aligned} \tag{13.17}$$

where ϕ_1 and ϕ_2 are hermitian fields with zero expectation values. The Lagrangian (11), in terms of the fields ϕ_1 and ϕ_2 has the form

$$\begin{aligned} L \;=\; & -\frac{1}{4}A^{\mu\nu}\, A_{\mu\nu} - \frac{1}{4}B^{\mu\nu}\, B_{\mu\nu} + \frac{1}{2}\left(4g^2v^2\right) B^\mu B_\mu - 2gvB^\mu\partial_\mu\phi_2 \\ & +\bar{\Psi}\left(i\gamma^\mu\partial_\mu - \frac{hv}{\sqrt{2}}\right)\Psi - e\bar{\Psi}\gamma^\mu\Psi A_\mu - g\bar{\Psi}\gamma^\mu\gamma_5\Psi B_\mu \\ & -\frac{h}{\sqrt{2}}\bar{\Psi}\left(\phi_1 + i\gamma_5\phi_2\right)\Psi + \frac{1}{2}\left(\partial_\mu\phi_1\right)^2 - \frac{1}{2}\left(2\lambda v^2\right)\phi_1^2 \\ & +\frac{1}{2}\left(\partial_\mu\phi_2\right)^2 + 2g^2B^\mu B_\mu\left(\phi_1^2 + \phi_2^2\right) \\ & +2g^2B^\mu\left(\phi_2\partial_\mu\phi_1 - \phi_1\partial_\mu\phi_2\right) + 4g^2vB^\mu B_\mu\phi_1 \\ & -v\lambda\phi_1\left(\phi_1^2 + \phi_2^2\right) - \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2\right)^2. \end{aligned} \tag{13.18}$$

From the Lagrangian (18), we derive some interesting results. If the gauge group U_2 is a global gauge group, then we do not require the vector boson B_μ and from the Lagrangian (18), we see that the scalar field ϕ_1 has acquired a mass $\sqrt{2\lambda v^2}$, the scalar (pseudo) field ϕ_2 is massless, the fermion field Ψ has also acquired a mass $\frac{hv}{\sqrt{2}}$ and the vector field A_μ corresponding to unbroken gauge symmetry U_1 is massless. Hence we have the Goldstone-Nambu theorem. A spontaneous breakdown of global symmetry leads to a massless scalar particle. But when U_2 is a local gauge symmetry, then due to the presence of the term $2\ g\ v\ B^\mu\partial_\mu\phi_2$ a straightforward interpretation of (18) is not possible. But we can eliminate this term by a field dependent gauge transformation. Actually what happens is that $\partial_\mu\ \phi_2$ combines with B_μ (which has only transverse components) to form a single massive spin 1 field, $\partial_\mu\ \phi_2$ now becomes longitudinal mode of spin 1 field. This can explicitly be seen as follows: Choose the gauge function $\Lambda_2(x)$ to be $\frac{\eta(x)}{2v}$. Then under the gauge transformations

$$\begin{aligned}
\Psi_L &= e^{\frac{i\eta(x)}{2v}}\hat{\Psi}_L \quad , \quad \Psi_R = e^{\frac{-i\eta(x)}{2v}}\hat{\Psi}_R \\
A_\mu &= \hat{A}_\mu \quad , \quad B_\mu = \hat{B}_\mu + \frac{1}{2vg}\partial_\mu \eta(x) \\
\phi(x) &= \frac{1}{\sqrt{2}}\left[v+\rho(x)\right] e^{\frac{i\eta(x)}{V}},
\end{aligned} \tag{13.19}$$

the Lagrangian (18) becomes (removing ˆ):

$$\begin{aligned}
L &= -\frac{1}{4}A^{\mu\nu}\,A_{\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} + \frac{1}{2}\left(4g^2v^2\right)B^\mu B_\mu \\
&\quad +\bar{\Psi}\left(i\gamma^\mu\partial_\mu - \frac{hv}{\sqrt{2}}\right)\Psi - e\bar{\Psi}\gamma^\mu\Psi A_\mu - g\bar{\Psi}\gamma^\mu\gamma_5\Psi B_\mu \\
&\quad -\frac{h}{\sqrt{2}}\bar{\Psi}\Psi\rho + \frac{1}{2}\left(\partial_\mu\rho\right)^2 - \frac{1}{2}\left(2v^2\lambda\right)\rho^2 \\
&\quad +2g^2B^\mu B_\mu\left(\rho^2+2v\rho\right) - v\lambda\rho^3 - \frac{1}{4}\lambda\rho^4.
\end{aligned} \tag{13.20}$$

It is clear from Eq. (20), that the would be Goldstone boson field $\eta(x)$ has been transformed away; it has been eaten away by the field B_μ to give a longitudinal component. This mechanism is called the Higgs-Kibble mechanism. The massive scalar particle ρ is called the Higgs particle. To summarize: (1) No massless scalar boson appears. (2) A_μ which is associated with unbroken gauge symmetry (electric charge conservation) has zero mass. (3) The vector boson B_μ has acquired a mass $m_B = 2gv$. (4) The fermion field has acquired a mass $m_f = \frac{hv}{\sqrt{2}}$. (5) Both the masses of B_μ and Ψ arise due to the same symmetry breaking mechanism. (6) A massive scalar particle with mass$\sqrt{2\lambda v^2}$ appears. This particle is called Higgs particle. Presence of Higgs scalar is an essential feature of spontaneously broken gauge symmetry.

13.3 Renormalizability

We give here few remarks about the renormalizability of a gauge theory. Now the fields A_μ and B_μ cannot be determined uniquely by field equations. In order to quantize these fields, one has to fix a gauge that is to say break gauge invariance. For the photon field A_μ, a term added to the Lagrangian for this purpose is $-\frac{1}{2}\xi^{-1}\left(\partial^\mu A_\mu\right)^2$. Photon propagator is then given by

$$i\left[-g_{\mu\nu}+(1-\xi)\frac{k_\mu k_\nu}{k^2}\right]\frac{1}{k^2}.$$

For the field B_μ, the gauge fixing term is

$$-\frac{1}{2}\xi^{-1}\left(\partial^\mu B_\mu-\xi m_B\phi_2\right)^2.$$

It is so chosen that it cancels awkward looking mixing term $B^\mu\partial_\mu\phi_2$ in the Lagrangian (18). ξ is a parameter which determines the gauge. The propagator for the vector boson B_μ is given by

$$i\left(-g_{\mu\nu}+(1-\xi)\frac{k_\mu k_\nu}{k^2-\xi m_B^2}\right)\frac{1}{k^2-m_B^2}.$$

The field ϕ_2 has its propagator

$$\frac{i}{k^2-\xi^2\, m_B^2}.$$

These form of propagators are expected to give a renormalizable theory for any finite value of ξ since they have good high k^2 behavior, falling like $\frac{1}{k^2}$. This is called R-gauge. The fields B_μ and ϕ_2 separately have no physical significance. In particular the poles at $k^2=\xi^2\, m_B^2$ are unphysical and are canceled out in any S-matrix element, which is also independent of ξ. In the limit $\xi\to\infty$, the B-meson propagator becomes

$$i\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m_B^2}\right)\frac{1}{k^2-m_B^2}$$

and ϕ_2 propagator vanishes. This is called unitary (U) gauge. The renormalizability is not obvious in this gauge.

13.4 Electroweak Unification

As we have discussed in Chap. 11, the leptonic charged current of weak interactions has the form $\bar{\nu}_e\gamma_\mu(1-\gamma_5)\,e = 2\bar{\nu}_{eL}\gamma^\mu e_L$. The corresponding hadronic charged weak current can be written as $\bar{u}\gamma_\mu(1-\gamma_5)\,d' = 2\bar{u}_L\gamma_\mu d'_L$. Here d' means that it is not mass eigenstate. This suggests that we consider

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} u \\ d' \end{pmatrix}_L$$

as left handed doublets in a weak isospin space. The weak currents are then associated with weak isospin raising and lowering operators

$$J_\mu^{(+)} = \bar{\Psi}_L\frac{\tau_+}{2}\gamma_\mu\Psi_L, \qquad J_\mu^{(-)} = \bar{\Psi}_L\frac{\tau_-}{2}\gamma_\mu\Psi_L, \tag{13.21}$$

where Ψ_L is any of the above doublets, $\tau_+ = (\tau_1 + i\tau_2)$ and $\tau_- = (\tau_1 - i\tau_2)$. Let the charges associated with these currents be Q_+ and Q_- . These charges generate an $SU_L(2)$ algebra

$$[Q_+, Q_-] = 2Q_3. \tag{13.22}$$

The current associated with the charge Q_3 is given by

$$J_\mu^3 = \bar{\Psi}_L\frac{1}{2}\tau_3\gamma_\mu\Psi_L. \tag{13.23}$$

The gauge transformation corresponding to the group $\mathrm{SU}_L(2)$ is

$$\Psi_L(x) \to \Psi_L(x) = \exp\left(i\frac{\boldsymbol{\tau}}{2}\cdot\boldsymbol{\Lambda}(x)\right)\Psi_L(x). \tag{13.24}$$

Then the Lagrangian

$$L = \bar{\Psi}_L i\gamma^\mu D\mu\Psi_L - \frac{1}{4}\mathbf{W}^{\mu\nu}\cdot\mathbf{W}_{\mu\nu}, \tag{13.25}$$

where

$$D_\mu = \partial_\mu + ig\frac{1}{2}\boldsymbol{\tau}\cdot\mathbf{W}_\mu = \partial_\mu + igW_\mu, \qquad (13.26)$$
$$\left(W_\mu = \frac{1}{2}\boldsymbol{\tau}.\mathbf{W}_\mu\right)$$

$$\mathbf{W}_{\mu\nu} = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu - g\mathbf{W}_\mu\times\mathbf{W}_\nu \qquad (13.27a)$$
$$W_{\mu\nu} \equiv \frac{1}{2}\boldsymbol{\tau}\cdot\mathbf{W}_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$$
$$= \partial_\mu W_\nu - \partial_\nu W\mu + ig\left[W_\mu, W_\nu\right], \qquad (13.27b)$$

is invariant under the gauge transformations:

$$\Psi_L(x) \to U\,\Psi_L(x)$$
$$W_\mu \to UW_\mu U^\dagger - \frac{i}{g}U\partial_\mu U^\dagger \qquad (13.28a)$$

where U is given in Eq.(24). For Λ infinitesimal, we get

$$\Psi_L(x) \to \left(1 + i\frac{\boldsymbol{\tau}}{2}\cdot\boldsymbol{\Lambda}(x)\right)\Psi_L(x)$$
$$\mathbf{W}_\mu \to \mathbf{W}_\mu - \boldsymbol{\Lambda}\times\mathbf{W}_\mu - \frac{1}{g}\partial_\mu\boldsymbol{\Lambda}. \qquad (13.28b)$$

The gauge group $\mathrm{SU}_L(2)$ leads to a neutral current J_μ^3 which is neither observed experimentally nor is identical with the electromagnetic current. It is possible to unify weak and electromagnetic forces into a single gauge force, if we extend the gauge group to $\mathrm{SU}_L(2)\times\mathrm{U}_Y(1)$. For this group we have two gauge couplings g and g' associated with $\mathrm{SU}_L(2)$ and $\mathrm{U}_Y(1)$ respectively. The weak hypercharge Y is defined by the relation $Q = t_3 + \frac{1}{2}Y = \frac{1}{2}\tau_3 + \frac{1}{2}Y$. The gauge vector bosons $W^\pm$, W^0 belong to the adjoint representation of $\mathrm{SU}_L(2)$ and vector boson B_μ is associated with $\mathrm{U}_Y(1)$.

Fermions belong to either fundamental representation [doublet] or trivial representation [singlet]. The structure of charged

weak currents suggest the following assignments:
Ist generation

	$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$,	e^-_R	$\begin{pmatrix} u \\ d' \end{pmatrix}_L$,	u_R,	d_R
Y	-1	-2	$1/3$	$4/3$	$-2/3$

2nd generation

$$\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \quad \mu^-_R \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L, \quad c_R, \quad s_R$$

3rd generation

$$\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \quad \tau^-_R \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L, \quad t_R, \quad b_R$$

In order to break the gauge symmetry spontaneously so that weak vector bosons acquire their mass, we need a Higgs doublet ϕ:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \qquad Y = 1. \tag{13.29}$$

The Lagrangian invariant under the local gauge transformations

$$\begin{aligned} \Psi_L &\rightarrow \exp\left(\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\Lambda} + \frac{i}{2}Y_L\Lambda_0\right)\Psi_L \\ \Psi_R &\rightarrow \exp\left(\frac{i}{2}Y_R\Lambda_0\right)\Psi_R \end{aligned} \tag{13.30}$$

is given by

$$\begin{aligned} L &= \bar{\Psi}_L i\gamma^\mu\left(\partial_\mu + \frac{i}{2}g\boldsymbol{\tau}\cdot\mathbf{W}_\mu + \frac{i}{2}g'Y_L B_\mu\right)\Psi_L \\ &\quad + \bar{\Psi}_{Ra} i\gamma^\mu\left(\partial\mu + \frac{i}{2}g'Y_{Ra}B_\mu\right)\Psi_{Ra} \\ &\quad + \left(\partial^\mu\bar{\phi} - \frac{i}{2}g\bar{\phi}\boldsymbol{\tau}\cdot\mathbf{W}^\mu - \frac{i}{2}g'\bar{\phi}B^\mu\right) \end{aligned}$$

$$\times\left(\partial_\mu\phi+\frac{i}{2}g\boldsymbol{\tau}\cdot\mathbf{W}_\mu\phi+\frac{i}{2}g'B_\mu\phi\right)$$
$$-h_1\left[\bar{\Psi}_L\phi\Psi_{R1}+\bar{\Psi}_{R1}\bar{\phi}\Psi_L\right]-h_2\left[\bar{\Psi}_L\phi\Psi_{R2}+\bar{\Psi}_{R2}\tilde{\phi}\Psi_L\right]$$
$$-\frac{1}{4}\mathbf{W}^{\mu\nu}\cdot\mathbf{W}_{\mu\nu}-\frac{1}{4}B^{\mu\nu}\cdot B_{\mu\nu}-V(\phi) \qquad (13.31)$$

where $W_{\mu\nu}$ is given in Eq. (27a) and

$$B_{\mu\nu} = \partial_\mu B_\nu-\partial_\nu B_\mu \qquad (13.32)$$
$$V(\phi) = \mu^2\bar{\phi}\phi+\lambda\left(\bar{\phi}\phi\right)^2 \qquad (13.33)$$
$$\tilde{\phi} = i\tau_2\bar{\phi}=\begin{pmatrix}\bar{\phi}_0^*\\ -\phi^-\end{pmatrix} \qquad (13.34)$$

$a=1,2$ with $\Psi_{R1}=e_R$ or $d_R, \Psi_{R2}=u_R$. Under the infinitesimal gauge transformations (30), vector fields W_μ transform as given in Eq. (28), but B_μ transforms as

$$B_\mu\to B_\mu-\frac{1}{g'}\partial_\mu\Lambda_0. \qquad (13.35)$$

In order to break the gauge symmetry spontaneously, assume that

$$\langle\phi\rangle_0=\begin{pmatrix}0\\ \frac{v}{\sqrt{2}}\end{pmatrix}, \qquad (13.36)$$

where $v=\sqrt{-\mu^2/\lambda}$, $\langle\phi\rangle_0=\langle 0\left|\phi\right|0\rangle$. In this way, not only $SU_L(2)$ is broken but $U_Y(1)$ is also broken, but it leaves the group $U(1)$ corresponding to electric charge unbroken viz. $SU_L(2)\times U_Y(1)$ is broken to $U_Q(1)$. We can now write Eq. (29) as

$$\phi=\begin{pmatrix}\phi^+\\ \frac{(\phi_1+i\phi_2)}{\sqrt{2}}+\frac{v}{\sqrt{2}}\end{pmatrix}, \qquad (13.37)$$

where ϕ^+ and hermitian fields ϕ_1 and ϕ_2 have zero vacuum expectation values. We can select a gauge such that ϕ^+ and ϕ_2 disappear from the theory. Instead $\partial_\mu \phi^\pm$ and $\partial_\mu \phi_2$ provide longitudinal components to $W^\pm$ and one of neutral vector bosons respectively. Thus out of the four gauge vector bosons, three become massive and the remaining one remains massless. This massless vector boson is the photon corresponding to unbroken $U_Q(1)$ symmetry. All this amounts to replacing ϕ given in Eq. (37) by ($\phi_1 = H$)

$$\phi = \begin{pmatrix} 0 \\ \frac{H+v}{\sqrt{2}} \end{pmatrix}. \tag{13.38}$$

With Eq. (38), the following term of the Lagrangian (31)

$$\left[\partial^\mu \bar{\phi} - \frac{i}{2} g \bar{\phi} \boldsymbol{\tau} \cdot \mathbf{W}^\mu - \frac{i}{2} g' \bar{\phi} B^\mu\right] \left[\partial_\mu \phi + \frac{i}{2} g \tau \cdot W_\mu + \frac{i}{2} g' B_\mu \phi\right]$$

gives

$$\begin{aligned} & L^{W-H} \\ = \; & \frac{1}{2}\partial^\mu H \partial_\mu H + \frac{g^2}{8}\left(H^2 + 2vH + v^2\right)\left(2W_\mu^+ W^{\mu -} + W^{3\mu} W_{3\mu}\right) \\ & + \frac{g'^2}{8}\left(H^2 + 2vH + v^2\right) B^\mu B_\mu \\ & - \frac{gg'}{4}\left(H^2 + 2vH + v^2\right) W^{3\mu} B_\mu, \end{aligned} \tag{13.39}$$

where $W_\mu^\pm = (W_{1\mu} \mp i W_{2\mu})/\sqrt{2}$. From this equation, it is clear that vector bosons $W_\mu^\pm$ have acquired a mass:

$$m_W^2 = \frac{1}{4} g^2 v^2. \tag{13.40a}$$

For the neutral vector bosons, the mass terms in Eq. (39) give the matrix

$$M^2 = \frac{1}{4}\begin{pmatrix} g^2 v^2 & -gg' v^2 \\ -gg' v^2 & g'^2 v^2 \end{pmatrix}. \tag{13.40b}$$

Since det $(M^2) = 0$, therefore one of the eigenvalues of M^2 is zero. The mass matrix (40b) can be diagonalized by defining the physical fields A_μ, Z_μ:

$$\begin{aligned} A_\mu &= \cos\theta_W B_\mu + \sin\theta_W W_{3\mu}, \\ Z_\mu &= -\sin\theta_W B_\mu + \cos\theta_W W_{3\mu}. \end{aligned} \tag{13.41}$$

Then we get

$$m_A^2 = 0, \qquad A_\mu : \text{ photon} \tag{13.42}$$

$$\begin{aligned} m_Z^2 &= \frac{1}{4}\left(g^2 + g'^2\right) v^2 \\ &= \frac{1}{4} g^2 v^2 \left(\frac{1}{\cos^2\theta_W}\right) \end{aligned} \tag{13.43}$$

where

$$\tan\theta_W = \frac{g'}{g} \tag{13.44}$$

and the parameter

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = 1. \tag{13.45}$$

The fermion masses are given by

$$m_i = h_i \frac{v}{\sqrt{2}} \tag{13.46}$$

and Higgs boson mass is given by

$$m_H^2 = 2\lambda v^2 = -2\mu^2. \tag{13.47}$$

From Eq. (31), using Eqs. (41), (44), (40a) and (46), the Lagrangian for the fermions can be written as:

$$\begin{aligned} L_F &= \bar{\Psi}_i \left(i\gamma^\mu\, \partial_\mu - m_i - \frac{g m_i}{2 m_W} H\right) \Psi_i \\ &\quad - \frac{g}{2\sqrt{2}} \bar{\Psi}_i\, \gamma^\mu \left(1 - \gamma_5\right) \left(T^+ W_\mu^+ + T^- W_\mu^-\right) \Psi_i \\ &\quad - e\, \bar{\Psi}_i\, \gamma^\mu\, Q_i\, \Psi_i\, A_\mu + \frac{g}{2\cos\theta_W} \bar{\Psi}_i\, \gamma^\mu \left(g_{Vi} - \gamma^5 g_{Ai}\right) \Psi_i\, Z_\mu \end{aligned} \tag{13.48}$$

where $\Psi_i = \begin{pmatrix} \nu_i \\ l_i \end{pmatrix}$ and $\begin{pmatrix} u_i \\ d_i' \end{pmatrix}$, $e = g' \cos\theta_W = g \sin\theta_W = \frac{gg'}{\sqrt{g^2+g'^2}}$, $d_i' = V_{ij}\ d_j\,(V : \text{CKM matrix})$, m_i is the mass of ith fermion and Q_i is its charge. g_{Vi} and g_{Ai} are given by

$$g_{Vi} = \left(T_i^3 - 2Q_i \sin\theta_W\right), g_{Ai} = T_i^3 \quad (13.49\text{a})$$

$$T^{\pm} = \frac{1}{2}\tau^{\pm}, \qquad T^3 = \frac{1}{2}\tau^3 \quad (13.49\text{b})$$

We note that the interaction part of the Lagrangian can be written as

$$L_{int} = -g\sin\theta_W\ J^{\mu}_{em}\ A_\mu - \frac{g}{2\sqrt{2}}\left(J^{+\mu}\ W^{+}_{\mu} + \text{h.c.}\right) - \frac{g}{\cos\theta_W} J^{Z\mu}\ Z_\mu \quad (13.50\text{a})$$

where

$$\begin{aligned}
J^{\mu}_{em} &= \bar{\Psi}_i\ \gamma^\mu\ Q_i\ \Psi_i \\
&= \left(-\bar{e}\ \gamma^\mu\ e + \frac{2}{3}\bar{u}\ \gamma^\mu\ u - \frac{1}{3}\bar{d}\ \gamma^\mu\ d\right) + \cdots \quad (13.50\text{b}) \\
J^{+\mu} &= \frac{1}{2}\ \bar{\Psi}_i\ \gamma^\mu\left(1-\gamma^5\right)\tau^+\Psi_i \\
&= \left(\bar{\nu}_e\ \gamma^\mu\left(1-\gamma^5\right)e + \bar{u}\ \gamma^\mu\left(1-\gamma^5\right)d'\right) + \cdots \quad (13.50\text{c}) \\
J^{Z\mu} &= \frac{1}{2}\ \bar{\Psi}_i\left(g_{Vi}\ \gamma^\mu - g_A\ \gamma^\mu\ \gamma^5\right)\Psi_i \\
&= \frac{1}{2}J^{3\mu} - \sin^2\theta_W\ J^{\mu}_{em} \\
&= \frac{1}{2}\left[\bar{\Psi}_i\left(\gamma^\mu\left(1-\gamma^5\right)\frac{\tau_3}{2} - 2\sin^2\theta_W\ Q_i\ \gamma^\mu\right)\Psi_i\right] \\
&= \frac{1}{4}\Big[\bar{\nu}_e\gamma^\mu\left(1-\gamma^5\right)\nu_e - \bar{e}\ \gamma^\mu\left(1-\gamma^5\right)e \\
&\quad + \bar{u}\ \gamma^\mu\left(1-\gamma^5\right)u - \bar{d}\ \gamma^\mu\left(1-\gamma^5\right)d \\
&\quad -4\sin^2\theta_W\left(-\bar{e}\ \gamma^\mu\ e + \frac{2}{3}\bar{u}\ \gamma^\mu\ u - \frac{1}{3}\bar{d}\ \gamma^\mu\ d\right)\Big] \\
&\quad + \cdots \quad (13.50\text{d})
\end{aligned}$$

where ellipses in Eqs. (50) indicate repitition for the second and third generations.

For low momentum transfer phenomena, $q^2 \ll m_W^2,\ m_Z^2$, we can write

$$\frac{g^2}{8m_W^2} = \frac{G_F}{\sqrt{2}} \tag{13.51}$$

$$m_W^2 = \frac{\sqrt{2}e^2}{8G_F \sin^2\theta_W} = \frac{\pi\alpha}{\sqrt{2}G_F \sin^2\theta_W} \tag{13.52}$$

$$m_W = \frac{37.3 \text{ GeV}}{\sin^2\theta_W} \geq 37.3 \text{ GeV} \tag{13.53}$$

$$\rho = 1; \qquad m_Z = \frac{m_W}{\cos\theta_W} = \frac{74.6 \text{ GeV}}{\sin^2\theta_W} \geq 74.6 \text{ GeV}. \tag{13.54}$$

Note that $\rho = 1$ is a consequence of the fact that Higgs scalar ϕ is an $SU_L(2)$ doublet. The effective neutral current coupling [see Eq. (49)] is

$$\frac{g^2}{m_Z^2\ \cos^2\theta_W} = \rho\frac{g^2}{m_W^2} = 8\rho\frac{G_F}{\sqrt{2}}. \tag{13.55}$$

Finally, we note that for the Higgs vacuum expectation value v , using Eqs. (39) and (51), we get

$$v^2 = \frac{1}{\sqrt{2}G_F} \approx (246 \text{ GeV})^2. \tag{13.56}$$

This gives the weak interaction scale i.e. the energy scale after which the weak interactions become as strong as electromagnetic interaction.

The fermion masses are given by

$$m_i^2 = h_i^2\frac{v^2}{2} = h_i^2\frac{1}{2\sqrt{2}G_F} \tag{13.57a}$$

$$h_i^2 = 2\sqrt{2}\,G_F\, m_i^2 \tag{13.57b}$$

i.e. the Yukawa couplings are very weak.

We conclude this section with the following remarks:

1. A definite prediction of electroweak unification is the existence of weak neutral current J_μ^Z with the same effective coupling as charged currents $J_\mu^\pm$. This current has been found experimentally.

2. The existence of vector bosons $W^\pm$, Z, with definite masses given in Eqs. (53) and (54).

3. The theory has one free parameter $\sin^2\theta_W$.

At low energies $q^2 \ll m_W^2$, one test of the model is to determine $\sin^2\theta_W$ from different classes of experiments. If $\sin^2\theta_W$ comes out to be the same in all these experiments, it will support the model. The true test of the model is the existence of vector bosons. This requires much higher energies. We first discuss low energy consequences of the electroweak unification. The vector bosons $W^\pm$ and Z have been found experimentally with masses predicted by the model.

13.4.1 Experimental consequences of the electroweak unification

Low energy phenomena $q^2 \ll m_W^2$: From the Lagrangian (50), for low momentum transfer phenomena $[q^2 \ll m_W^2,\ m_Z^2]$ we can write the effective Lagrangians for charged and neutral currents:

$$L_{eff}^{CC} = \frac{G_F}{\sqrt{2}} J^{+\mu}\, J_\mu^- \tag{13.58}$$

$$L_{eff}^{NC} = \rho \frac{G_F}{\sqrt{2}} 8 J^{Z\mu} J_\mu^{Z\dagger}. \tag{13.59}$$

It is convenient to write J_μ^Z:

$$J_\mu^Z = J_\mu^Z(\nu) + J_\mu^Z(e) + J_\mu^Z(h)\,, \tag{13.60}$$

where

$$J_\mu^Z(\nu) = \frac{1}{4}\left[\bar{\nu}\gamma_\mu(1-\gamma_5)\,\nu\right] \tag{13.61}$$

Table 13.1

	e	u	d
ε_L	$-\frac{1}{2}+\sin^2\theta_W$	$\frac{1}{2}-\frac{2}{3}\sin^2\theta_W$	$-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W$
ε_R	$\sin^2\theta_W$	$-\frac{2}{3}\sin^2\theta_W$	$\frac{1}{3}\sin^2\theta_W$
g_A	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
g_V	$-\frac{1}{2}+2\sin^2\theta_W$	$\frac{1}{2}-\frac{4}{3}\sin^2\theta_W$ $C_{1u}=2g_A^e\ g_V^u$ $C_{2u}=2g_V^e\ g_A^u$	$-\frac{1}{2}+\frac{2}{3}\sin^2\theta_W$ $C_{1d}=2g_A^e\ g_V^d$ $C_{2d}=2g_V^e\ g_A^d$

$$2J_\mu^Z(e) = \left[\varepsilon_L(e)\,\bar{e}\gamma_\mu(1-\gamma_5)\,e + \varepsilon_R(e)\,\bar{e}\gamma_\mu(1+\gamma_5)\,e\right] \qquad (13.62)$$

$$\begin{aligned} 2J_\mu^Z(h) &= \sum\left[\varepsilon_L(i)\,\bar{q}_i\,\gamma_\mu(1-\gamma_5)\,q_i + \varepsilon_R(i)\,\bar{q}_i\,\gamma_\mu(1+\gamma_5)\,q_i\right], \\ & \quad i = u, d, s, \ldots. \end{aligned} \qquad (13.63)$$

Since the net strangeness of the proton is zero, we will assume that strange quark s and heavy flavor quarks c, b etc. make negligible contribution to $J_\mu^Z(h)$ for proton and neutron targets. Then we can write the effective Lagrangians for various neutral current processes as follows:

$$L^{\nu e} = \rho\frac{G_F}{\sqrt{2}}\bar{\nu}\,\gamma^\mu(1-\gamma_5)\,\nu\left(2J_\mu^Z(e)\right) \qquad (13.64)$$

$$L^{\nu h} = \rho\frac{G_F}{\sqrt{2}}\bar{\nu}\,\gamma^\mu(1-\gamma_5)\,\nu\left(2J_\mu^Z(h)\right) \qquad (13.65)$$

$$\begin{aligned} L^{eh} &= -\rho\frac{G_F}{\sqrt{2}}\sum_i - \Big[C_{1i}\ \bar{e}\ \gamma^\mu\ \gamma^5\ e\ \bar{q}_i\ \gamma_\mu\ q_i \\ &\qquad + C_{2i}\ \bar{e}\ \gamma^\mu\ e\ \bar{q}_i\ \gamma_\mu\ \gamma_5\ q_i\Big]. \end{aligned} \qquad (13.66)$$

From Eqs. (50), we can determine the parameters $\varepsilon_L(e)$, $\varepsilon_R(e)$, $\varepsilon_L(i)$, $\varepsilon_R(i)$, C_{1i}, and $C_{2i}(e)$, $(i = u, d)$. They are given in Table 1.

13.4.2 Need for radiative corrections

Before we discuss the experiments in support of the standard model, let us summarize here the three parameters (not counting Higgs meson mass m_H and the fermion masses) which the minimal model (with $\rho = 1$) has: (a) fine structure constant $\alpha = 1/137.0359895(61)$ determined from the Josephson effect (b) the Fermi coupling constant $G_F = 1.166389(22) \times 10^{-5}$ GeV^{-2} determined from the muon life-time {including lepton mass and $O(\alpha)$ radiative corrections [cf. Eq. (11.40c)]}. (c) $\sin^2\theta_W$, determined from neutral current processes or the W and Z masses. Now a best fit to the neutral current neutrino reactions data gives

$$\sin^2\theta_W = 0.2255 \pm 0.0021. \tag{13.67}$$

This implies that the theory without radiative corrections gives through the relations [cf. Eqs. (52) and (54)]

$$m_W^2 = \left(\frac{\pi\alpha}{\sqrt{2}G_F}\right)/\sin^2\theta_W = \frac{A_0^2}{\sin^2\theta_W} \tag{13.68a}$$

$$m_Z^2 = \frac{m_W^2}{\cos^2\theta_W}, \tag{13.68b}$$

where

$$A_0 = \left(\frac{\pi\alpha}{\sqrt{2}G_F}\right)^{1/2} = (37.2802 \text{ GeV}), \tag{13.69}$$

$$m_W = 78.42 \text{ GeV}, \qquad m_Z = 89.14 \text{ GeV}. \tag{13.70}$$

These values are to be compared with the experimental ones $m_W = 80.39 \pm 0.06$ GeV and $m_Z = 91.1867 \pm 0.002$ GeV. This shows a need for radiative corrections. First we note that the two coupling contants g and g' which determine the strength of weak interactions are related to e through $e = g\, g'/(g^2 + g'^2)$. Since most measurements are made at Z peak, therefore most convenient mass scale for these couplings is at m_Z. Thus one should take into consideration the running of QED coupling countant [see Appendix B] which gives the value of α at m_Z :

$$\begin{aligned}
\alpha\left(m_Z\right) &= \frac{\alpha}{1-\Delta\alpha}=\frac{\alpha}{1-\Pi_{\gamma\gamma}\left(z\right)} \\
&= \frac{\alpha}{1-\frac{\alpha}{3\pi}\sum_f Q_f^2\, N_{cf}\left(-\frac{5}{3}+\ln\frac{m_Z^2}{m_f^2}\right)}, \qquad (13.71)
\end{aligned}$$

where $f=e,\mu,\tau,\ u,d,s,c$ and b and $N_{cf}=3$ for quarks and 1 for leptons. Equation (71) can be directly evaluated for leptons, since their masses are well known. For the light hadronic part, quark masses are not available as reasonable input parameters. The 5-flavor contribution to $\Pi_{\gamma\gamma}$ is extracted from the experimental data on $e^+e^-\rightarrow$ hadrons. The best estimate leads to

$$\begin{aligned}
\alpha\left(m_Z\right) &= \frac{1}{128.88\pm 0.09} \\
\Delta\alpha &= 1-\frac{\alpha}{\alpha\left(m_Z\right)}=0.0595\pm 0.0007. \qquad (13.72)
\end{aligned}$$

Thus knowing G_F, $\alpha\left(m_Z\right)$ and m_Z, one should be in a position to predict all electroweak observables, including the mixing angle $\sin^2\theta_W=\frac{e^2}{g^2}$. However, the lowest order relation $m_W\,/\,m_Z=\cos\theta_W$ and some other lowest order relations are affected by the fermion loops in gauge bosons propagaters. Thus the relation (68b) is modified to

$$\frac{m_W^2}{m_Z^2\cos^2\theta_W}=\rho=\left(1+\Delta\rho\right). \qquad (13.73)$$

The leading contribution to $\Delta\rho$ comes from the top quark loop ($m_b\ll m_t$) to the self energies of W and Z bosons (see Fig. 2):

$$\Delta\rho=\frac{3G_F}{8\pi^2\sqrt{2}}m_t^2=0.0096\pm 0.006 \qquad (13.74)$$

for $m_t=175\pm 5$ GeV. By contrast, the Higgs boson contribution to $\Delta\rho$ at the one loop level is logarithmic:

$$\left(\Delta\rho\right)_{\text{Higgs}}=-\frac{G_F\,m_W^2}{\pi^2\sqrt{2}}\tan^2\theta_W\left[\frac{11}{3}\ln\frac{m_H^2}{m_W^2}-\frac{5}{6}\right]. \qquad (13.75)$$

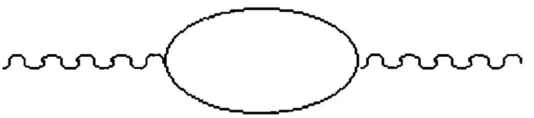

Figure 2 Top and bottom quarks loop contribution to W and Z boson self energies.

Radiative corrections are scheme dependent, leading to $\sin^2\theta_W$ values which differ by small factors which depend on m_t and m_H. A useful scheme [called the on-shell scheme] is to take tree level formula $\sin^2\theta_W = 1 - m_W^2 \,/\, m_Z^2$ as the definition of renormalized $\sin^2\theta_W$ to all orders in perturbation theory i.e. $\sin^2\theta_W \to s_W^2 = 1 - m_W^2 \,/\, m_Z^2$.

Now the tree level expression (69) is modified to

$$A_0^2 \to A_0^2 \frac{\alpha(m_Z)}{\alpha} = A_0^2 \frac{1}{1-\Delta\alpha}$$

while using Eq. (73),

$$\begin{aligned}
\sin^2\theta_W \quad &\to \quad s_W^2 = 1 - \frac{m_W^2}{m_Z^2} = 1 - \frac{m_W^2}{m_Z^2\cos^2\theta_W}\cos^2\theta_W \\
&= \left(1-\cos^2\theta_W\right) + \cos^2\theta_W\left(1 - \frac{m_W^2}{m_Z^2\cos^2\theta_W}\right) \\
&= \sin^2\theta_W - \Delta\rho\cos^2\theta_W. \qquad (13.76)
\end{aligned}$$

Thus the tree level expressions (68) are modified to

$$\left(1 - \frac{m_W^2}{m_Z^2}\right) m_W^2 \equiv m_W^2\ s_W^2 = m_Z^2\ s_W^2\ c_W^2 = \frac{A_0^2}{1-\Delta r} \qquad (13.77)$$

where

$$\begin{aligned}\Delta r &= \Delta\alpha - \Delta\rho\cot^2\theta_W + (\Delta r)_{\text{remainder}} \\ &= \Delta\alpha - \frac{c_W^2}{s_W^2}\Delta\rho + (\Delta r)_{\text{remainder}} \qquad (13.78)\end{aligned}$$

where $(\Delta r)_{\text{remainder}}$ contains all possible contributions not included in the fermionic contributions $\Delta\alpha$ and $\Delta\rho$.

Likewise, taking into account radiative corrections, the relationship between the $W^{\pm}$-boson mass, s_W^2 and G_F^2 in the standard model gets modified

$$m_W^2 = \frac{A^2}{s_W^2\,(1-\Delta r_W)} \qquad (13.79)$$

where $A^2 = A_0^2\frac{\alpha(Z)}{\alpha} = A_0^2\frac{1}{1-\Delta\alpha}$, so that from Eqs. (77)-(79)

$$\begin{aligned}(1-\Delta r) &= (1-\Delta\alpha)\,(1-\Delta r_W)\,, \qquad (13.80)\\ \left(1-\frac{m_W^2}{m_Z^2}\right)\frac{m_W^2}{m_Z^2} &= s_W^2\;c_W^2 = A_0^2\frac{\alpha\,(m_Z)}{\alpha\;m_Z^2}\frac{1}{1-\Delta r_W} \\ &= \frac{\pi\alpha\,(m_Z)}{\sqrt{2}G_F\;m_Z^2\,(1-\Delta r_W)} \\ &= \frac{s_0^2\;c_0^2}{1-\Delta r_W} \qquad (13.81)\end{aligned}$$

where we have used Eq. (69) and have defined

$$s_0^2\;c_0^2 = \frac{\pi\alpha\,(m_Z)}{\sqrt{2}G_F\;m_Z^2}. \qquad (13.82)$$

The relation (79) defines Δr_W, which is completely determined by purely weak correction, once $\alpha\,(m_Z)$ is specified. For LEP physics, $\sin^2\theta_W$ is usually defined from $Z\to\mu^+\;\mu^-$ effective vertex. At the tree level we have [cf. Table 1]

$$\begin{aligned}Z &\to f\;\bar{f} : \frac{g}{2\cos\theta_W}\bar{f}\;\gamma_\mu\left(g_V^f - g_A^f\;\gamma_5\right)f \\ &= \left(\sqrt{2}G_F\;m_Z^2\right)^{1/2}\bar{f}\left[\left(I_3^f - 2Q_f\;\sin^2\theta_W\right)\gamma_\mu - I_3^f\;\gamma_\mu\;\gamma_5\right]f.\end{aligned} \qquad (13.83)$$

The weak (non QED) connections to the $Z \to f\,\bar{f}$ effective vertex can be conveniently written as

$$\left(\sqrt{2}G_F\, m_Z^2\, \rho_f\right)^{1/2} \bar{f}\left[\left(I_3^f - 2Q_f\, s_f^2\right)\gamma_\mu - I_3^f\gamma_\mu\,\gamma_5\right] f \qquad (13.84)$$

where $\rho_f = \frac{1}{1-\Delta\rho}$, $\Delta\rho$ being given in Eq. (74) and [cf. Eq. (76)]

$$\begin{aligned} s_W^2 &= s_f^2 - c_f^2 \frac{\Delta\rho}{1-\Delta\rho} \\ s_f^2 &= s_W^2 + c_W^2\, \Delta\rho. \end{aligned} \qquad (13.85)$$

Thus $\left[I_3^\mu = \frac{-1}{2},\ Q_\mu = -1\right]$

$$g_V^\mu / g_A^\mu = 1 - 4s_f^2 \qquad (13.86)$$

$$g_A^\mu = -\frac{1}{2}\rho_f^{1/2} = -\frac{1}{2}\left(1 + \frac{\Delta\rho}{2}\right). \qquad (13.87)$$

The radiative correction functions Δr_W, $(\rho_f - 1)$ become the basis on which the corrected standard model and experiments are confronted. These functions involve, among other parameters, the top quark mass m_t whose value due to quadratic dependence of some of these functions on m_t is numerically important and the Higgs boson mass which enters only through logarithm and hence is not effectively bounded.

Now if we use the LEP value $s_f^2 = 0.23189 \pm 0.00024$ for leptons, we can obtain $s_W^2 = 0.2245 \pm 0.0010$ from Eq. (85), with $\Delta\rho$ given in Eq. (74) and hence m_W through the relation ($m_Z = 91.187$) $s_W^2 = 1 - m_W^2/m_Z^2$: $m_W = 80.30 \pm 0.05$ GeV which is consistent with the direct vector boson mass measurements: $m_W = 80.39 \pm 0.06$ GeV and the standard model best fit value: $m_W = 80.372$ GeV obtained from Eq. (77) (including higher order terms) from m_Z, G_F, α and m_t, m_H.

Finally including m_t, m_W from the direct experimental measurements, together with s_W^2 from neutrino scattering, global fits of

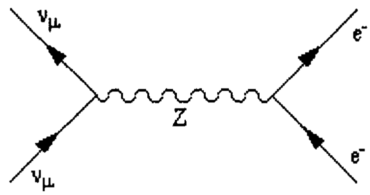

Figure 3 ν_μ–electron scattering through Z–boson exchange.

the standard model parameters to electroweak precision data give

$$\begin{aligned} m_t &= 171.1 \pm 4.9 \text{ GeV} \\ m_H &= 76 + \begin{matrix} +85 \\ -47 \end{matrix} \text{ GeV} \\ \alpha_s(m_Z) &= 0.119 \pm 0.003 \text{ GeV}. \end{aligned} \tag{13.88}$$

The upper limit on m_H at the 95 % CL is $m_H < 262$ GeV, where the theoretical uncertainty is included.

13.4.3 Experiments which determine $sin^2\theta_W$

We now discuss three sets of experiments to determine $\sin^2\theta_W$.
1. Consider the process

$$\nu_\mu + e^- \to \nu_\mu + e^- .$$

This process can occur only through Z exchange (Fig. 3). The effective Lagrangian for this process is given by Eq. (64). For this case the laboratory cross-section for $E_\nu \gg m_e$ give

$$\sigma^{\nu_\mu,\bar{\nu}_\mu} = \frac{G_F^2\, m_e E_\nu}{2\pi}\left[\left(g_V^e \pm g_A^e\right)^2 + \frac{1}{3}\left(g_V^e \mp g_A^e\right)^2\right], \tag{13.89}$$

where the upper (lower) sign refers to $\nu_\mu\,(\bar{\nu}_\mu)$, E_ν is the incident energy and $G_F^2\ m_e/2\pi = 4.31 \times 10^{-42} cm^2/GeV$. The expressions for g_V^e and g_A^e in terms of $\sin^2\theta_W$ are given in Table 1. The most accurate leptonic measurements of $\sin^2\theta_W$ are from the ratio

$$R = \frac{\sigma_{\nu_\mu e}}{\sigma_{\bar{\nu}_\mu e}} = \frac{3 - 12\sin^2\theta_W + 16\sin^4\theta_W}{1 - 4\sin^2\theta_W + 16\sin^4\theta_W}. \tag{13.90}$$

The most precise experiment (Charm II) determined not only $\sin^2\theta_W$ but $g^e_{V,A}$ as well. The experimental results are

$$\begin{aligned} g^e_V &= -0.035 \pm 0.017 \\ g^e_A &= -0.503 \pm 0.017 \\ \sin^2\theta_W &= 0.2326 \pm 0.0084. \end{aligned} \tag{13.91}$$

2. The deep inelastic neutrino scattering $\nu_\mu + N \to \nu_\mu + X$ (isoscalar target), gives a precise determination of $\sin^2\theta_W$ on the mass shell i.e. s^2_W. The relevant Lagrangian is given in Eq. (65). The ratio $R_\nu \equiv \sigma^{NC}_{\nu N}/\sigma^{C}_{\nu N}$ of neutral to charged cross-section has been measured to 1% accuracy. A simple zeroth order approximation gives

$$R_\nu = g_L^2 + g_R^2\, r, \qquad R_{\bar{\nu}} = g_L^2 + g_R^2 \,/\, r, \tag{13.92a}$$

where

$$\begin{aligned} g_L^2 &= \varepsilon_L(u)^2 + \varepsilon_L(d)^2 \approx \frac{1}{2} - \sin^2\theta_W + \frac{5}{9}\sin^4\theta_W \\ g_R^2 &= \varepsilon_R(u)^2 + \varepsilon_R(d)^2 \approx \frac{5}{9}\sin^4\theta_W \end{aligned} \tag{13.92b}$$

and $r \equiv \sigma^C_{\bar{\nu}N} \,/\, \sigma^C_{\nu N}$ is the ratio of $\bar{\nu}$ and ν charged current cross-sections, which can be measured directly. In parton model $r \approx \left(\frac{1}{3} + \varepsilon\right) / \left(1 + \frac{1}{3}\varepsilon\right)$, where $\varepsilon \approx 0.125$ is the ratio of the fraction of the nucleon's momentum carried by antiquarks to that carried by quarks. Now from Eq. (92) on using Eq. (73) and (76) we can write

$$R_\nu = \frac{1}{2} - s_W^2 + (1+r)\frac{5}{9}s_W^4 + s_W^2\left(-1 + \frac{10}{9}(1+r)\right)\Delta\rho \tag{13.93a}$$

$$R_{\bar{\nu}} = \frac{1}{2} - s_W^2 + \left(1 + \frac{1}{r}\right)\frac{5}{9}s_W^4 + s_W^2\left(-1 + \frac{10}{9}\left(1 + \frac{1}{r}\right)\right)\Delta\rho \tag{13.93b}$$

$$R_\nu - rR_{\bar{\nu}} = (1-r)\left[\frac{1}{2} - s_W^2\,(1+\Delta\rho)\right]. \qquad (13.94)$$

It is clear from Eq. (93a), that dependence of R_ν on $\Delta\rho$ is weak. Hence this equation is useful to determine s_W^2. Using the experimental values $R_\nu = 0.317 \pm 0.003$, $r = 0.440$, and $\Delta\rho = 0.0096$ we obtain from Eq. (93a) $s_W^2 = 0.2242 \pm 0.0022$, and (with $m_Z = 91.187$) $m_W = 80.32 \pm 0.11$ GeV. The recent value quoted for s_W^2 from νN scattering is 0.2255 ± 0.9021, which gives $m_W = 80.25 \pm 0.11$ GeV fully consistent with the directly measured value for $m_W = 80.39 \pm 0.06$ GeV. We note from Eq. (94) that if we plot R_ν versus $R_{\bar{\nu}}$ it gives a straight line with a slope determined by r. This provides an accurate method to determine $\rho\ s_W^2$ from the experimental data.

3. Parity violating deep inelastic eD scattering: The relevant Lagrangian for this process through Z exchange, which is parity violating, is given in Eq. (66). There is an interference with the parity conserving process through photon exchange. This gives us the parity-violating asymmetry

$$A = \frac{\sigma_R - \sigma_L}{\sigma_R + \sigma_L}, \qquad (13.95)$$

where $\sigma_{R,L}$ is the cross-section for the deep inelastic scattering of a right (left)-handed electron $e_{R,L}N \to eX$. In the quark parton model (see Chap.14 for this model)

$$\frac{A}{q^2} = a_1 + a_2\frac{1-(1-y)^2}{1+(1-y)^2}, \qquad (13.96a)$$

where $q^2 < 0$ is the momentum transfer and this essentially comes through the photon propagator which appears in the photon exchange process. Here y is the fractional energy transfer from the electron to hadrons. For the deuteron or other isoscalar target neglecting the s quark and antiquarks,

$$a_1 = \frac{3G_F}{5\sqrt{2}\pi\alpha}\left(C_{1u} - \frac{1}{2}C_{1d}\right) \approx \frac{3G_F}{5\sqrt{2}\pi\alpha}\left(-\frac{3}{4} + \frac{5}{3}\sin^2\theta_W\right)$$

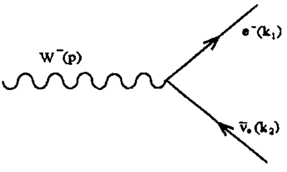

Figure 4 W–boson decay.

$$a_2 = \frac{3G_F}{5\sqrt{2}\pi\alpha}\left(C_{2u} - \frac{1}{2}C_{2d}\right) \approx \frac{9G_F}{5\sqrt{2}\pi\alpha}\left(\sin^2\theta_W - \frac{1}{4}\right) \tag{13.96b}$$

where we have used Table 1 in the second step of these formulae. The experimental values for a_1 and a_2 can be used to determine the mixing angle $\sin^2\theta_W$.

13.5 Decay Widths of W and Z Bosons

Consider the decay $W^- \to e^- + \bar{\nu}_e$ shown in Fig. 4: From Eqs. (49) and (50b), the decay amplitude F is given by

$$F = \frac{-g}{2\sqrt{2}}\bar{u}(k_1)\gamma^\lambda\left(1-\gamma^5\right)v(k_2)\cdot\varepsilon_\lambda, \tag{13.97a}$$

where ε_λ is the polarization of W-boson. From Eq. (97), the decay width can be easily calculated and is given by [in the limit when we neglect the lepton masses as compared with m_W] :

$$\begin{aligned}\Gamma\left(W^- \to e^- + \bar{\nu}_e\right) &= \left(\frac{g^2}{8}\right)\frac{m_W}{3(2\pi)^2}\cdot 2\pi \\ &= \frac{G_F\, m_W^3}{6\sqrt{2}\pi}.\end{aligned} \tag{13.97b}$$

We can also calculate the hadronic decays of W^- from the basic processes like $W^- \to \bar{u}\, d,\ \bar{c}\, s$. Again we get an expression like

(97b), except that we multiply it by a factor $N_c = 3\left(1 + \frac{\alpha_s(m_W)}{\pi}\right) \approx 3.12$, where the factor 3 is due to color and the factor in the parentheses is a QCD correction. In this case we also neglect quark masses as compared with W-mass. This is a good approximation with the exception of τ-quark which channel is not open as $m_t = 175 \pm 5$ GeV. Since in weak interactions a linear combination of mass eigenstates d, s and b enters, therefore, we have to multiply the decay rates by square of such factors as $|V_{ud}|^2$, $|V_{cs}|^2$, $|V_{us}|^2$, etc. [see Sec.13.10]. The link of one generation to succeeding generations is very weak, therefore, we will put $|V_{ud}|^2 \approx \cos^2\theta_c \approx 1$, $|V_{cs}|^2 \approx \cos^2\theta_c \approx 1$, $|V_{us}|^2 \approx \sin^2\theta_c \approx 0$, $|V_{cb}|^2 \approx \sin^4\theta_c \approx 0$. Hence the relative decay widths for three generations are given by:

$e^-\bar{\nu}_e$	$\mu^-\bar{\nu}_\mu$	$\tau^-\bar{\nu}_\tau$	$\bar{u}\ d$	$\bar{c}\ s$
1	1	1	$3\left[1 + \frac{\alpha_s}{\pi}\right]$	$3\left[1 + \frac{\alpha_s}{\pi}\right]$

Thus we get

$$\begin{aligned} \Gamma\left(W^+ \to l^+\nu_l\right) &\approx 227.5 \pm 0.3 \text{ MeV}, \\ \Gamma\left(W^+ \to u_i d_i\right) &\approx (708 \pm 1) \text{ MeV} \\ \Gamma_W^{\text{tot}} &\approx 2.098 \text{ GeV}, \end{aligned} \tag{13.98}$$

to be compared with the experimental value 2.097 ± 0.003 GeV for Γ_W^{tot}.

For the decay $Z \to f + \bar{f}$, the decay amplitude F is given by [from Eqs. (49) and (50c)]

$$F = \frac{-g}{\cos\theta_W}\bar{u}(k_1)\left[g_{Vf}\ \gamma^\mu - g_{Af}\ \gamma_\mu\ \gamma^5\right]v(k_2)\,\varepsilon_\mu \tag{13.99}$$

and the decay width is given by

$$\begin{aligned} \Gamma\left(Z \to f\bar{f}\right) &= \frac{g^2}{8\cos^2\theta_W m_Z^2}\frac{m_Z^3}{6\pi}\left[g_{Vf}^2 + g_{Af}^2\right]N_c^f \\ &= \frac{G_F}{\sqrt{2}}\frac{m_Z^3}{6\pi}\left[g_{Vf}^2 + g_{Af}^2\right]N_c^f \end{aligned} \tag{13.100}$$

where $N_c^f = 1$ or $3\,(1+\alpha_s/\pi)$ for $f =$ lepton (l) or quark q. First we note that $g_{Af} = -\frac{1}{2}$, $g_{Vf} = \left(-1/2 + |Q_f|\,2\sin^2\theta_W\right)$. In the presence of radiative corrections, we have [cf. Eqs. (84), (86) and (87)]

$$g_{Af} = -\frac{1}{2}\sqrt{\rho_f} \approx -\frac{1}{2}\left(1+\frac{1}{2}\Delta\rho\right), \quad x = \frac{g_{Vf}}{g_{Af}} = 1 - 4\,|Q_f|\,s_f^2. \tag{13.101}$$

Thus we get

$$\Gamma\left(Z \to l\bar{l}\right) = \frac{G_F}{\sqrt{2}}\frac{m_Z^3}{24\pi}\left[1+\Delta\rho\right]\left[1+\left(1-4s_f^2\right)^2\right] \tag{13.102a}$$

$$\Gamma\left(Z \to q\bar{q}\right) = \frac{G_F}{\sqrt{2}}\frac{m_Z^3}{24\pi}N_c\left[1+\Delta\rho\right]\left[1+\left(1-4\,|Q_f|\,s_f^2\right)^2\right]. \tag{13.102b}$$

It is convenient to write

$$s_f^2 = (1+\Delta k)\,s_o^2 \tag{13.103}$$

where Δk signifies non QED corrections and s_0^2 is given in Eq. (82) and has the value 0.2311 for $m_Z = 91.187$ GeV. Then

$$\Gamma\left(Z \to f\bar{f}\right) = \frac{G_F}{\sqrt{2}}\frac{m_Z^3}{24\pi}\left[1+\Delta\rho\right]N_{cf} \times\left[1+\left(1-4\,|Q_f|\,(1+\Delta k)\,s_0^2\right)^2\right]. \tag{13.104}$$

Let us write

$$\Gamma_0\left(Z \to l\bar{l}\right) = \frac{G_F}{\sqrt{2}}\frac{m_Z^3}{24\pi}\left[1+\left(1-4s_0^2\right)^2\right] \tag{13.105a}$$

$$\Gamma_0\left(Z \to q\bar{q}\right) = \frac{G_F}{\sqrt{2}}\frac{m_Z^3}{24\pi}\left[1+\left(1-4\,|Q_q|\,s_0^2\right)^2\right]3\left(1+\frac{\alpha_s}{\pi}\right). \tag{13.105b}$$

From Eqs. (105), we get on using $s_0^2 = 0.2311$,

$$\begin{aligned}
\Gamma_0(Z \to \nu\bar{\nu}) &= 165.9 \text{ MeV} && (13.106a)\\
\Gamma_0\left(Z \to e^- e^+\right) &= \Gamma_0\left(Z \to \bar{\mu}\mu^+\right) = \Gamma_0\left(Z \to \tau^-\tau^+\right) \\
&= 83.4\,(83.91 \pm 0.10) \text{ MeV} && (13.106b)\\
\Gamma_0(Z \to u\bar{u}) &= \Gamma_0(Z \to c\bar{c}) = 296.9 \text{ MeV} && (13.106c)\\
\Gamma_0\left(Z \to d\bar{d}\right) &= \Gamma_0(Z \to s\bar{s}) = \Gamma_0\left(Z \to b\bar{b}\right) = 382.6 \text{ MeV}. && (13.106d)
\end{aligned}$$

Thus

$$\begin{aligned}
\Gamma_0(Z \to \text{hadrons}) &= 1742 \text{ MeV} = 1.742\,(1.7432 \pm 0.0023) \text{ GeV} && (13.107a)\\
\Gamma_0(Z \to \text{invisible}) &= \Gamma_Z - (\Gamma_{had} + 3\Gamma_{e^+e^-}) \\
&= (2.494) - (1.742 + 0.250) \\
&= 0.502 \text{ GeV} = 502 \text{ MeV} && (13.107b)
\end{aligned}$$

where experimentally $\Gamma_Z = 2.4939 \pm 0.0024$ GeV. The values given in parenthesis are experimental values. Now $3\Gamma_0(Z \to \nu\bar{\nu}) = 498$ MeV; hence one concludes from Eq. (107b) that $N_\nu = 3$ i.e. there are three generations of neutrinos or three generations of fermions. It may be noted that in calculating the decay widths we have put fermion mass as zero. This is a very good approximation; the decay width for $Z \to b\bar{b}$ may need some improvement if m_b is not neglected

Even the theoretical values as given by the Born approximation Γ_0 are not bad, the small discrepancy can be explained by taking into account the radiative corrections given in Eq. (106). We can also use Eq. (104) to constrain Δk and $\Delta\rho$ by using the experimental value for $\Gamma(Z \to e^+ e^-)$.

Another important observable is forward-backward asymmetry measured at LEP. Consider the process $e^-e^+ \to f\bar{f}$ as depicted in Fig. 5. The cross-section for $e^-e^+ \to f\bar{f}$ can be

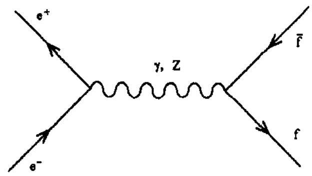

Figure 5 Production of Z in $e^- e^+$ collision and its decay into $f\bar{f}$ pair.

easily calculated by using Eq. (99). It is given by (in the limit $s \gg 4m_e^2,\ 4m_f^2,\ \beta_e \approx 1,\ \beta_f \approx 1$)

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\alpha^2 N_c^f}{4s}\left\{\left(1+\cos^2\theta\right)\left[Q_f^2 - 2\frac{Q_f\ v_e v_f}{16\sin^2\theta_W\cos^2\theta_W}\operatorname{Re}\chi(s)\right.\right. \\ &\quad \left. + \frac{(v_e^2+a_e^2)(v_f^2+a_f^2)}{(16\sin^2\theta_W\cos^2\theta_W)^2}\mid\chi(s)\mid^2\right] \\ &\quad + \cos\theta\left[-\frac{4Q_f\ a_e a_f}{16\sin^2\theta_W\cos^2\theta_W}\operatorname{Re}\chi(s)\right. \\ &\quad \left.\left. + \frac{8v_e v_f a_e a_f}{(16\sin^2\theta_W\cos^2\theta_W)^2}\mid\chi(s)\mid^2\right]\right\}, \end{aligned} \tag{13.108a}$$

where

$$\chi(s) = \frac{s}{s - m_Z^2 + i m_Z\,\Gamma_Z}, \tag{13.108b}$$

$$\begin{aligned} v_e &= 2g_{Ve} = -1 + 4\sin^2\theta_W, \qquad a_e = 2g_{Ae} = -1 \\ v_f &= 2g_{Vf} = 2T_{3L}^f - 4Q_f\sin^2\theta_W, \qquad a_f = 2g_{Af} = 2T_{3L}^f. \end{aligned} \tag{13.108c}$$

Near and on the peak, integrated cross section is dominated by Z-exchange and we get from Eq. (100):

$$\sigma_f = \sigma^0_{\text{peak}}\left|\frac{s\Gamma_Z\ /\ m_Z}{s - m_Z^2 + i m_Z\,\Gamma_Z}\right|^2 (1+\delta(s)) \tag{13.109a}$$

where

$$\sigma^0_{\text{peak}} = \frac{12\pi}{m_Z^2}\frac{\Gamma_e\Gamma_f}{\Gamma_Z^2} \tag{13.109b}$$

and the effect of radiative corrections are contained in $\delta(s)$, the large effects due to initial $e^{\pm}$ bremsstrahlung are represented in δ. The other radiative corrections which lead to improved Born approximation have already been discussed in Secs. 3.2 and 4.

The LEP data is fitted with an additional modification i.e. by replacing $s - m_Z^2 + im_Z\Gamma_Z$ by $s - m_Z^2 + i\frac{s}{m_Z}\Gamma_Z$. Note that the expression for Γ_f is given in Eq. (100). Thus by measuring σ^0_{peak} for a particular final state e.g. e^+e^- itself, one can directly obtain Γ_{ee}/Γ_Z or $\frac{\Gamma_{ee}\ /\ \Gamma_{f\bar{f}}}{\Gamma_Z}$ and therefore $\Gamma_{f\bar{f}}$. These widths have already been discussed in the beginning of this section.

The forward-backward asymmetry is defined as:

$$A_{FB} = \frac{\int_0^{\pi/2}\left(\frac{d\sigma}{d\Omega}\right)d\Omega - \int_{\pi/2}^{\pi}\left(\frac{d\sigma}{d\Omega}\right)d\Omega}{\int_0^{\pi}\left(\frac{d\sigma}{d\Omega}\right)d\Omega}. \tag{13.110}$$

It is clear that this asymmetry is given by $\cos\theta$ term in Eq. (108a). Near and on the Z-peak, we get from Eqs. (110) and (108c)

$$A_{FB} = 3\frac{\left(g_{V_e}g_{A_e}\right)\left(g_{V_f}g_{A_f}\right)}{\left(g_{V_e}^2 + g_{A_e}^2\right)\left(g_{V_f}^2 + g_{A_f}^2\right)}. \tag{13.111}$$

For leptons

$$A_{FB}^l = \frac{3g_{V_l}^2\ /\ g_{A_l}^2}{(1 + g_{Vl}^2\ /\ g_{Al}^2)^2}. \tag{13.112}$$

Taking into account the radiative corrections [cf. Eqs. (101) and (103)], we get from Eq. (112)

$$\begin{aligned} A_{FB}^l &= \frac{3\left(1 - 4s_f^2\right)^2}{\left[1 + \left(1 - 4s_f^2\right)\right]^2} \\ &= \frac{3\left(1 - 4\left(1 + \Delta k\right)s_o^2\right)^2}{\left[1 + 4\left(1 + \Delta k\right)s_o^2\right]^2}, \end{aligned} \tag{13.113}$$

where $s_o^2 = 0.23116$ and

$$A^l_{FB_O} = \frac{3\left(1-4s_o^2\right)^2}{\left[1+\left(1-4s_o^2\right)^2\right]^2} = 0.01685\,(0.01683 \pm 0.00096)\,. \tag{13.114}$$

The value in parentheses is the experimental value. The agreement is quite good.

Finally we discuss the longitudinal polarization of a fermion in the process $e^-e^+ \to f\bar{f}$. Near and on Z-peak, the cross sections for positive and negative helicities are given by

$$\frac{d\sigma^{(+)}}{d\Omega} - \frac{d\sigma^{(-)}}{d\Omega} = \frac{2\alpha^2}{4s}\frac{1}{\left(16\sin^2\theta_W\cos^2\theta_W\right)^2}\left|\frac{s}{s-m_Z^2+i\frac{s}{m_Z}\Gamma_Z}\right|^2$$
$$\times\left[-2\left(v_e^2+a_e^2\right)a_f\,v_f\left(1+\cos^2\theta\right)-4\left(v_f^2+a_f^2\right)a_e v_e\cos\theta\right] \tag{13.115a}$$

$$\frac{d\sigma^{(+)}}{d\Omega} + \frac{d\sigma^{(-)}}{d\Omega} = 2\left(\frac{d\sigma}{d\Omega}\right). \tag{13.115b}$$

Hence the polarization at $s = m_Z^2$ is given by

$$-A_f = \frac{\int_o^\pi\left(\frac{d\sigma^{(+)}}{d\Omega}-\frac{d\sigma^{(-)}}{d\Omega}\right)d\Omega}{2\int_o^\pi\left(\frac{d\sigma}{d\Omega}\right)d\Omega} = -\frac{2v_f\,a_f}{v_f^2+a_f^2}, = -2\frac{g_{Vf}/g_{Af}}{1+g_{Vf}^2/g_{Af}^2}. \tag{13.116}$$

We can also write Eq. (116) in terms of effective mixing angle s_f:

$$A_l = 2\frac{1-4s_f^2}{1+\left(1-4s_f^2\right)^2}. \tag{13.117}$$

Using the value $A_\tau = 0.1431 \pm 0.0046$ we obtain $s_f^2 = 0.23201 \pm 0.00057$ to be compared with $s_0^2 = 0.23116 \pm 0.00022$.

13.6 Tests of Yang-Mills Character of Gauge Bosons

The vector bosons self-couplings are given by the Lagrangian (31)

$$L_W = -\frac{1}{4}\left[\partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu - g\left(\mathbf{W}_\mu \times \mathbf{W}_\nu\right)\right]^2. \qquad (13.118a)$$

This gives the trilinear $W^+W^-W_3$ coupling as

$$\begin{aligned} L_W &= i\frac{g}{2}\left[\left(\partial_\mu W_{3\nu} - \partial_\nu W_{3\mu}\right)\left(W^{-\mu}W^{+\nu} - W^{+\mu}W^{-\nu}\right)\right. \\ &\quad + \left(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+\right)\left(W^{3\mu}W^{-\nu} - W_\nu^3 W^{-\mu}\right) \\ &\quad \left. - \left(\partial_\mu W_\nu^- - \partial_\nu W_\mu^-\right)\left(W^{3\mu}W^{+\nu} - W_\nu^3 W^{+\mu}\right)\right]. \end{aligned} \qquad (13.118b)$$

Using $W_\nu^3 = \sin\theta_W A_\nu + \cos\theta_W Z_\nu$, the above equation gives for the $W^+W^-\gamma$ and W^+W^-Z vertices

$$\begin{aligned} L &= \begin{pmatrix} -g\sin\theta_W \\ -g\cos\theta_W \end{pmatrix}\left[g_{\alpha\beta}\left(k_1 + k_2\right)_\gamma - k_{2\alpha}\, g_{\gamma\beta} - k_{1\beta}\, g_{\gamma\alpha}\right. \\ &\quad \left. + \left(k_1 - k_2\right)_\alpha\, g_{\beta\gamma} - \left(k_1 - k_2\right)_\beta g_{\alpha\gamma}\right], \end{aligned} \qquad (13.118c)$$

where α, β and γ are the indices of polarization vectors of W^-, W^+, W^3 respectively. On the other hand from Eq. (39), the Higgs coupling to gauge bosons is given by

$$L_{W-H} = \frac{g^2}{8}\left(H + v\right)^2\left[2W_\mu^+ W^{-\mu} + \frac{1}{\cos^2\theta_W} Z_\mu Z^\mu\right] \qquad (13.119a)$$

and the Yukawa coupling of Higgs to leptons is given by

$$L_{llH} = \frac{1}{\sqrt{2}} h_l \bar{l}\, l H, \qquad (13.119b)$$

where

$$h_l = \frac{\sqrt{2}}{v} m_l = \left(2\sqrt{2}G_F\right)^{1/2} m_l. \qquad (13.120)$$

One process in which the trilinear couplings can be tested directly is

$$e^+ + e^- \to W^+ + W^- .$$

In the lowest order of g, the diagrams shown in Fig. 6 contribute to this process.

We are interested in the high energy behavior of the amplitude M. The bad behavior comes from the longitudinal polarization of W's. For this case $\mu = 0$. The longitudinal polarization vector ε_μ^L for a W-boson of four-momentum k_μ is given by

$$\begin{aligned} \varepsilon_\mu^L &= \frac{1}{m_W}\left(|\mathbf{k}| \ , \ k_o\hat{\mathbf{k}}\right) \\ &= \frac{k_\mu}{m_W} + \frac{m_W}{k_o + |\mathbf{k}|}\left(-1, -\hat{\mathbf{k}}\right). \end{aligned} \tag{13.121}$$

It is the first term in Eq. (121) viz. $\frac{k_\mu}{m_W}$ which gives the worst high energy behavior. The amplitude may grow with high energy due to this term, if it is not compensated. In fact as $E \to \infty$ the diagrams of Fig. 6 give

$$M_{LL}(a) = -\frac{g^2}{4m_W^2}\bar{v}(p)\,\gamma\cdot k\,(1-\gamma_5)\,u(p') - \frac{g^2 m_e}{4m_W^2}\bar{v}(p)\ u(p') \tag{13.122}$$

$$M_{LL}(b) = \frac{e^2}{m_W^2}\bar{v}(p)\,\gamma\cdot k\ u(p') \tag{13.123}$$

$$\begin{aligned} M_{LL}(c) &= -\frac{g^2}{4m_W^2}\bar{v}(p)\,\gamma\cdot k \\ &\quad \left(\left(-1 + 2\sin^2\theta_W\right)(1-\gamma_5) + 2\sin^2\theta_W\,(1+\gamma_5)\right)u(p'). \end{aligned} \tag{13.124}$$

It is clear from Eqs. (122) and (123) that there is no possibility of cancellation between $M_{LL}(a)$ and $M_{LL}(b)$ even if $e = g\sin\theta_W$. The third diagram, arises due to trilinear couplings - a feature of gauge theory. All the three diagrams cancel the bad high energy

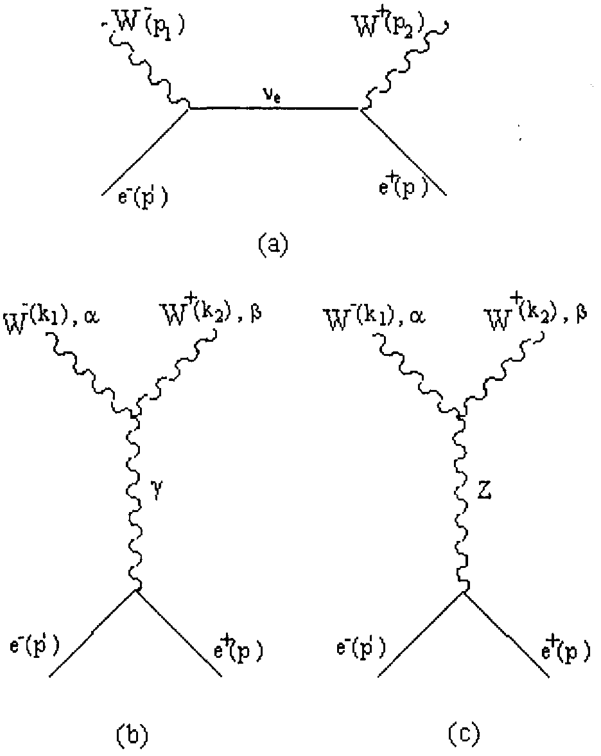

Figure 6 Production of W^-W^+ pair in e^-e^+ collision through ν_e, γ and Z boson exchange.

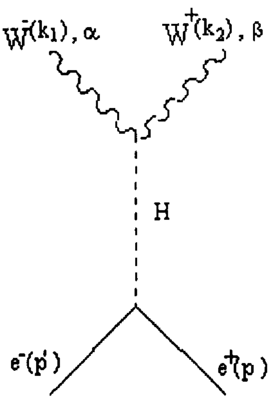

Figure 7 Production of W^-W^+ pair in e^-e^+ collision through Higgs boson exchange.

behavior except for the last term in Eq. (122), which gives S-wave cross-section for $s \gg m_W^2, \sigma_s = \frac{G_F^2}{4\pi}s$. This is in conflict with the unitarity constraint [Eq. (2)] $\sigma_s \le \frac{8\pi}{s}$. This conflict thus starts at

$$s = \frac{4\sqrt{2}}{G_F}\pi = (1.2 \text{ TeV})^2 . \tag{13.125}$$

However, even this term is canceled by the diagram (Fig. 7) due to Higgs exchange. This is because this diagram gives for $s \gg m_H^2$:

$$M_{LL}(d) = \frac{g^2 m_e}{4m_W^2}\bar{v}(p)\, u(p') . \tag{13.126}$$

Thus there is no trouble with the high energy behavior in the standard model if $m_H^2 < (1.2 \text{ TeV})^2$. There is similar cancellation for the amplitude M_{LT}, which for each individual diagram goes as constant when $s \to \infty$. $\sigma(e^+e^- \to W^- W^+)$ depends crucially on gauge cancellation discussed above. For example, $\sigma(\nu$ - exchange) for $s \gg m_W^2 \approx \frac{\pi\alpha^2 s}{96\sin^4\theta_W m_W^2}$, this would be the only contribution

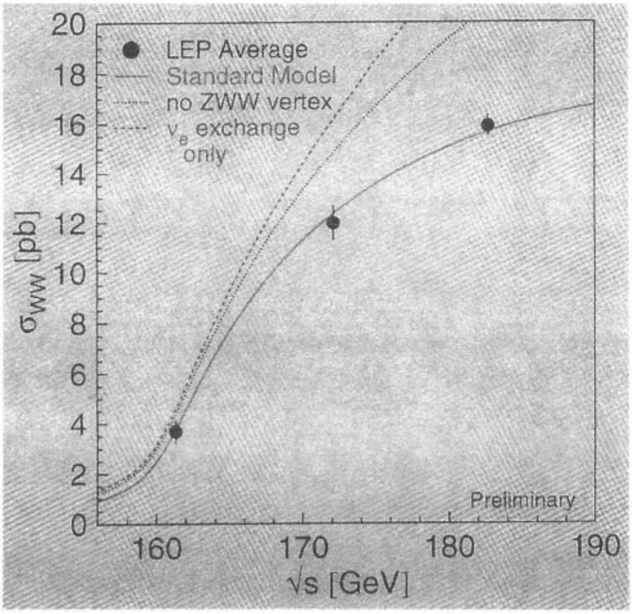

Figure 8 Behavior of σ_{SM} with energy E.

without $W^- W^+\gamma$ and $W^- W^+Z$ vertices. On the other hand, with the above cancellation

$$\sigma_{s\to\infty}(SM) = \frac{\pi\alpha^2}{2\sin^4\theta_W}\frac{1}{s}\ln\frac{s}{m_W^2}. \tag{13.127}$$

The cross section also contains the threshold factor $\sqrt{1-\frac{4m_W^2}{s}}$ which tends to 1 as $s \to \infty$. Thus the cross section grows near the threshold and then falls like $\frac{1}{s}$ at large values of $\sqrt{s} \gg m_W$. The cross section is $\sim 10^{-35}\text{cm}^2$ at its maximum which occurs at about 40 GeV above W^+W^- threshold. The situation is shown in Fig. 8.

13.7 Higgs Boson Mass

The Higgs potential

$$V(\phi) = \mu^2\phi^2 + \lambda\phi^4, \qquad \phi^2 = \bar{\phi}\phi \tag{13.128}$$

goes over to

$$V(H) = \frac{1}{2}\left(2\lambda v^2\right) H^2 + \lambda v H^4 - \frac{1}{4}\lambda H^4 - \frac{1}{4}\lambda v^4 \tag{13.129a}$$

when the symmetry is spontaneously broken; $\mu^2 = -\lambda v^2\,(\lambda > 0)$. Thus we see that the Higgs boson mass

$$m_H^2 = 2\lambda v^2 \equiv \frac{1}{2}\left|\frac{\partial^2 V}{\partial\phi^2}\right|_{\phi=\frac{v}{\sqrt{2}}} \tag{13.129b}$$

is arbitrary. We now discuss theoretical bounds on the Higgs boson mass.

13.8 Upper Bound

(a) Unitarity:

We have seen in Sec. 13.6 that the Higgs boson contribution to the cross section for the process

$$e^- + e^+ \to W_L^+ + W_L^-$$

is given by

$$\sigma_S = \frac{1}{4\pi}\left(\frac{G_F}{\sqrt{2}}\right)^2 s. \tag{13.130}$$

Then comparing it with Eq. (2), we get

$$\left(\frac{1}{4\pi}\right)\left(\frac{G_F}{\sqrt{2}}\right)^2 s \le \frac{8\pi}{s}. \tag{13.131}$$

This requires a "cut-off" (signaling new physics beyond Λ_{SB}):

$$\Lambda_{SB}^{PWU} \le \left(4\sqrt{2}\pi/G_F\right)^{1/2} = (1.2\ \text{TeV}). \tag{13.132a}$$

To avoid this conflict, the Higgs mass m_H should be such that

$$m_H < \Lambda_{SB}^{PWU} = (1.2 \text{ TeV}) . \tag{13.132b}$$

We saw at the beginning of this chapter that $m_W \ll \Lambda_{SB}^{PWU}$. Whether similar thing happens or not for m_H only experiments will tell.

(b) Finiteness of couplings:

The Higgs-self coupling λ is not asymptotically free. In $\lambda\, \phi^4$ theories, the renormalized group equation gives (see appendix)

$$\frac{d}{d \ln q^2} \lambda \left(q^2\right) = \frac{3}{4\pi^2} \lambda \left(q^2\right) . \tag{13.133}$$

This gives

$$\lambda \left(q^2\right) = \frac{\lambda \left(v^2\right)}{1 - \frac{3}{4\pi^2} \lambda \left(v^2\right) \ln \frac{q^2}{v^2}} . \tag{13.134}$$

The minus sign in this equation indicates that the Higgs coupling is not asymptotically free. In fact it implies that regardless of how small $\lambda\left(v^2\right)$ is, $\lambda\left(q^2\right)$ will eventually blow up at some large energy scale $q = \Lambda$. In order to avoid this and to guarantee positivity of $\lambda\left(\Lambda\right) : \lambda\left(\Lambda\right) < \infty$, $\lambda\left(v^2\right) < \frac{4\pi^2}{3} \frac{1}{\ln \frac{\Lambda^2}{v^2}}$ giving m_H^2 : $\left[v^2 = \frac{1}{\sqrt{2} G_F}\right]$

$$m_H^2 \equiv 2\lambda(v^2) v^2 = \sqrt{2} \frac{\lambda \left(v^2\right)}{G_F} < \frac{4\sqrt{2}\pi^2}{3G_F} \frac{1}{\ln \frac{\Lambda^2}{v^2}} . \tag{13.135}$$

The upper bound on m_H is related logarithmically to the scale Λ up to which the standard model is assumed to be valid. For some values of Λ, the upper bound on m_H is given below

Λ	m_H
1 TeV	753 GeV
10^{16} GeV	159 GeV
10^{19} GeV	144 GeV

Thus we see that if we assume the standard model to be valid up to Planck scale , then $m_H \leq 144$ GeV.

We note that the top quark Yukawa coupling $h_t = \sqrt{2}m_t/v$, for $m_t = 175$ GeV, can be of order 1. Top quark coupling modifies the renormalization group equation for the Higgs boson coupling λ. Top loop corrections reduce λ for increasing top-Yukawa coupling. Such an effect is shown in Fig. 9. We also note that non-perturbative effects as the quartic Higgs coupling becomes large have also been estimated, mostly in the context of the lattice-Higgs model. Again a cut off on the parameter $\lambda(v)$ provides an upper bound on m_H : $m_H < 700$ GeV. Finally the precision electroweak data give as previously noted

$$m_H = 76 \begin{array}{c} +85 \\ -47 \end{array} \text{GeV}. \tag{13.136}$$

13.9 Higgs Boson Searches

The search for Higgs is one of the objectives of new accelerators. The dominant production mechanism in $LEP2$, is $e^-e^+ \to Z \to ZH$. The tree level cross section is given by

$$\sigma\left(e^-e^+ \to ZH\right) = \frac{G_F^2\, m_Z^4}{96\pi s}\left[v_e^2 + a_e^2\right]\lambda^{1/2}\frac{\lambda + 12m_Z^2\,/\,s}{\left(1 - m_Z^2\,/\,s\right)^2}. \tag{13.137}$$

Here the standard model couplings [cf. Eqs.(113) and (126)]

$$\begin{aligned} g_{Ze^-e^+} &= \left(v_e^2 + a_e^2\right) = \left[\left(1 - 4\sin^2\theta_W\right)^2 + 1\right] \\ g_{ZZH} &= \frac{g m_Z}{\cos\theta_W} \end{aligned} \tag{13.138}$$

have been used in deriving Eq. (137). λ is the phase space factor

$$\lambda \equiv \left[1 - \frac{\left(m_H + m_Z\right)^2}{s}\right]\left[1 - \frac{\left(m_H - m_Z\right)^2}{s}\right]. \tag{13.139}$$

With the present LEP energies a lower limit has been established on Higgs mass

$$m_H \geq 98 \text{ GeV}. \tag{13.140}$$

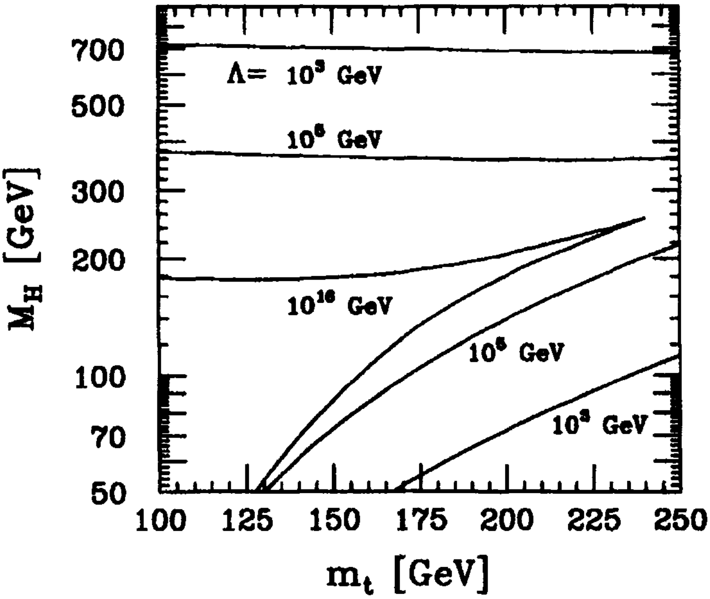

Figure 9 Bounds on the mass of the Higgs boson in the SM. Here Λ denotes the energy scale at which the Higgs boson system of the SM would become strongly interacting (upper bound); The lower bound follows from the requirement of vacuum stability. [9]

$LEP2$ running at energies $\sim$ 200 GeV should enable the Higgs to be discovered if $m_H \leq 110$ GeV.

We now discuss the decays of Higgs, since Higgs searches involve its decay widths. For $H \to f\bar{f}$, the coupling involved is

$$h_f^2 = \frac{2m_f^2}{v^2} = 2\sqrt{2}m_f^2 G_F, \tag{13.141}$$

giving

$$\Gamma\left(H \to f\bar{f}\right) = N_c^f \frac{G_F\, m_f^2}{4\sqrt{2}\pi} m_H\, \beta_f^3 \tag{13.142a}$$

where

$$\begin{aligned} N_c^f &= 1 \qquad \text{for} \qquad f = l^\pm \\ &= 3 \qquad \text{for} \qquad f = q \end{aligned}$$

$$\beta_f = \left(1 - 4m_f^2 \,/\, m_H^2\right)^{1/2}. \tag{13.142b}$$

It may be noted that there are important QCD corrections for $H \to q\bar{q}$. The bulk of QCD radiative corrections can be mapped into the scale dependence of the quark mass, evaluated at the Higgs mass i.e. use m_f at m_H i.e. $m_f(m_H)$ in Eq. (142) [see Eq. (B.47)].

For $H \to W^+W^-$ and $Z\bar{Z}$, the couplings are given by [cf. Eq.(119)]

$$g_{WWH} = g\, m_W, \qquad g_{ZZH} = \frac{g\, m_Z}{\cos\theta_W}, \tag{13.143a}$$

where $\frac{g^2}{8m_W^2} = \frac{G_F}{\sqrt{2}}$. These give the widths

$$\Gamma\left(H \to W^+W^-\right) = \frac{G_F}{8\sqrt{2}\pi} m_H^2 \left(1 - x_W\right)^{1/2} r_W \tag{13.143b}$$

$$\Gamma\left(H \to ZZ\right) = \frac{G_F}{16\sqrt{2}\pi} m_H^3 \left(1 - x_Z\right)^{1/2} r_Z, \tag{13.143c}$$

where

$$\begin{aligned} r_{W,Z} &= 1 - x_{W,Z} + \frac{3}{4}x_{W,Z}^2 \\ x_{W,Z} &= \frac{4m_{W,Z}^2}{m_H^2}. \end{aligned} \tag{13.144}$$

It is useful to remember that for $m_H \approx 1.4$ TeV,

$$\Gamma(H \to VV) \approx \frac{1}{2}m_H^3 G_F \sim m_H. \tag{13.145}$$

To sum up the standard model is in very good shape, but Higgs boson H is still a missing link.

13.10 GIM Mechanism

Since in weak interactions the flavor quantum numbers are not conserved, weak interaction eigenstates of different generations, d', s' and b' are not identical with mass eigenstates d, s and b. These states are linear combinations of d, s and b. Thus we can write

$$\begin{aligned} d' &= V_{ud}\, d + V_{us}\, s + V_{ub}\, b \\ s' &= V_{cd}\, d + V_{cs}\, s + V_{cb}\, b \\ b' &= V_{td}\, d + V_{ts}\, s + V_{tb}\, b. \end{aligned} \tag{13.146}$$

The quarks of one generation are linked to those of the succeeding generations with decreasing strength. Thus for example $V_{ub} \ll V_{us} < V_{ud}$. This is illustrated by the following diagram [Fig. 10]:

If we confine ourselves to ordinary and strange hadrons, then we can safely put $V_{ub} = 0$, but we cannot ignore V_{cs}, since charmed quark is linked to strange quark with maximum strength.

As a first approximation, we can ignore the third generation completely and can put $V_{ud} = \cos\theta_c$, $V_{us} = \sin\theta_c$, as given by Cabibbo theory. Thus we can write $d' = d\cos\theta_c + s\sin\theta_c$. In the weak neutral current, we have a term of the form

$$\bar{d}'\,\gamma_\mu\,(1-\gamma_5)\,d'$$

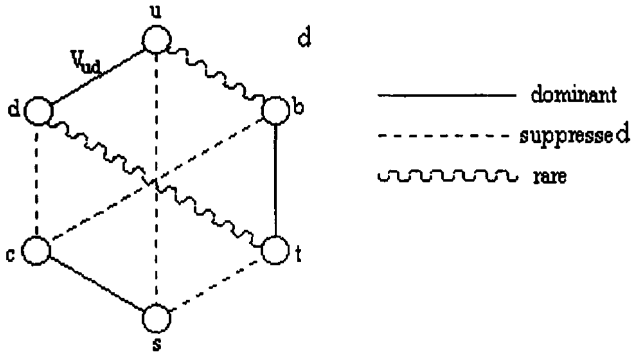

Figure 10 Relative strengths of flavor changing transitions.

$$
\begin{aligned}
= \quad & \cos^2\theta_c\, \bar{d}\,\gamma_\mu\,(1-\gamma_5)\,d + \sin^2\theta_c\,\bar{s}\,\gamma_\mu\,(1-\gamma_5)\,s \\
& + \sin\theta_c\cos\theta_c\left[\bar{d}\,\gamma_\mu\,(1-\gamma_5)\,s + \bar{s}\,\gamma_\mu\,(1-\gamma_5)\,d\right]
\end{aligned}
\tag{13.147}
$$

which arises from the doublet $\begin{pmatrix} u \\ d' \end{pmatrix}$. The above term can give rise to the following processes (Figs. 11a, 11b). It is clear from Figs. 11 that both the processes

$$
\begin{aligned}
K^+ &\rightarrow \pi^+ + \nu + \bar{\nu} \\
K^+ &\rightarrow \pi^0 + e^+ + \nu_e
\end{aligned}
$$

occur with equal strength. But experimentally

$$
\frac{\Gamma\,(K^+ \rightarrow \pi^+\nu\bar{\nu})}{\Gamma\,(K^+ \rightarrow \pi^0 e^+\nu_e)} < 1.2 \times 10^{-5}
$$

i.e. the strangeness changing neutral current is very much suppressed compared with the strangeness changing charged current. Here the charmed quark c comes to the rescue. If we put $V_{cd} = -\sin\theta_c$ and $V_{cs} = \cos\theta_c$, then $s' = -d\sin\theta_c + s\cos\theta_c$ and we get a

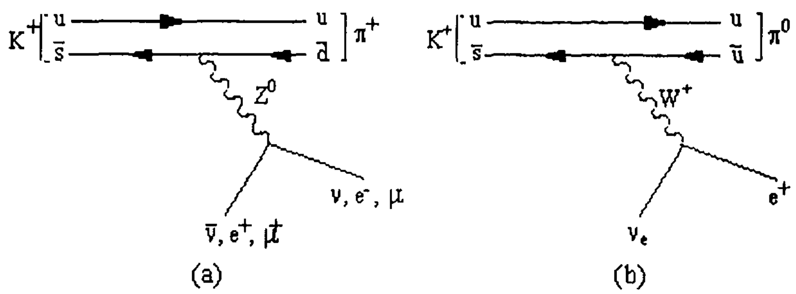

Figure 11 Decay $K^+ \to \pi^+ + \nu + \bar{\nu}$ through neutral current and $K^+ \to \pi^0 + e^+ + \nu_e$ through charged current.

term

$$\begin{aligned} & \bar{s}' \, \gamma_\mu \, (1 - \gamma_5) \, s' \\ = \quad & \sin^2\theta_c \, \bar{d} \, \gamma_\mu \, (1 - \gamma_5) \, d + \cos^2\theta_c \, \bar{s} \, \gamma_\mu \, (1 - \gamma_5) \, s \\ & - \sin\theta_c \cos\theta_c \left[\bar{d} \, \gamma_\mu \, (1 - \gamma_5) \, s + \ \bar{s} \, \gamma_\mu \, (1 - \gamma_5) \, d \right] \end{aligned} \tag{13.148}$$

from the doublet $\begin{pmatrix} c \\ s' \end{pmatrix}$. From Eqs. (147) and (148), it is clear that strangeness changing terms are canceled and J_μ^Z does not contain any strangeness changing term. This mechanism to eliminate the strangeness changing neutral current in tree approximation was suggested by Glashow, Iliapoulas and Maiani (GIM) before the experimental discovery of charm.

The $\Delta S = 2$, $K^0 \to \bar{K}^0$ transition shown in Fig. 12 is second order in G_F. With GIM mechanism, a complete cancellation between u and c couplings occur if $m_c = m_u$. With the known experimental value for this transition, a limit on the mass of m_c can be put and it was predicted that m_c must be less than a few GeV and this is what was found later experimentally.

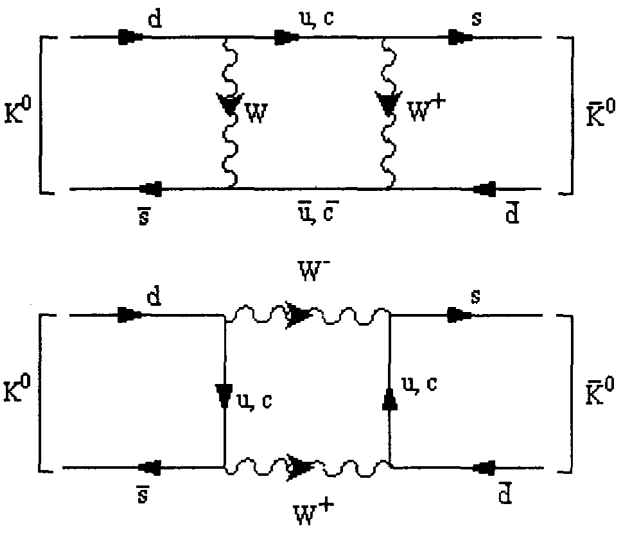

Figure 12 Box diagrams for $\Delta S = 2$, $K^0 - \bar{K}^0$ transitions.

13.11 Cabibbo–Kobayashi–Maskawa Matrix

Three generations of fermions are linked with each other by weak interactions. The states d', s', and b' are not mass eigenstates. They are related to mass eigenstates d, s and b as follows:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = V \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{13.149}$$

where V is a 3×3 matrix:

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{13.150}$$

called 'Cabibbo–Kobayashi–Maskawa' (CMK) matrix. The hadronic charged weak current can be written as

$$J_\mu^W(h) = (\bar{u}, \bar{c}, \bar{t})\, \gamma_\mu \,(1 - \gamma_5)\, V \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{13.151}$$

and $J_\mu^3\,(h)$ which is a part of the neutral current:

$$\begin{aligned} J_\mu^3(h) &= (\bar{u}, \bar{c}, \bar{t})\, \gamma_\mu \,(1 - \gamma_5) \begin{pmatrix} u \\ c \\ t \end{pmatrix} \\ &\quad + \left(\bar{d}, \bar{s}, \bar{b}\right) \gamma_\mu \,(1 - \gamma_5)\, V^\dagger V \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \end{aligned} \tag{13.152}$$

We want weak neutral currents to be flavor diagonal as flavor changing neutral currents are very much suppressed. Hence we must have

$$V^\dagger V = V V^\dagger = 1, \tag{13.153}$$

i.e. V must be a unitary matrix. Thus this matrix has nine real parameters. These parameters are the same in number as unitary group U_3. Now U_3 has three diagonal matrices, so that we can write

$$V = e^{i\theta\lambda_0}\, e^{i\alpha\lambda_3}\, e^{i\beta\lambda_8}\, C\, e^{i\alpha'\lambda_3} e^{i\beta'\lambda_8}, \tag{13.154}$$

where C is a 3×3 unitary matrix with 4 real parameters. The five parameters θ, α, β, α' and β' can be absorbed into redefinitions of phases of u, c, t and d, s, b quarks. Thus we can write

$$V = R_2 R_1 \tilde{C} R_3 \tag{13.155}$$

where R_1, R_2 and R_3 are 3×3 rotation matrices:

$$\begin{aligned} R_1 &= \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix}, \\ R_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix}, \end{aligned} \tag{13.156}$$

and $\tilde{C}$ is a unitary matrix which can be written as

$$\tilde{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix}. \tag{13.157}$$

Hence we have

$$V = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3\, e^{i\delta} & c_1 c_2 s_3 + s_2 c_3\, e^{i\delta} \\ s_1 s_2 & -c_1 s_2 c_3 - c_2 s_3\, e^{i\delta} & -c_1 s_2 s_3 + c_2 c_3\, e^{i\delta} \end{pmatrix}, \tag{13.158}$$

where

$$c_i = \cos\theta_i, \qquad s_i = \sin\theta_i. \tag{13.159}$$

There is an arbitrary phase δ, which makes the Lagrangian density non-real. Thus the Lagrangian density violates time-reversal invariance. By CPT theorem, it violates CP invariance. Thus there is an attractive possibility of accommodating CP violation in 3 generation model; this cannot be done in two generation model [see chapter 15].

If we consider the three generations, the fermion mass matrix for u, c, t and d, s, b quarks can be written as

$$\begin{aligned} L^q_{\text{mass}} &= \frac{v}{\sqrt{2}}\left[\bar{\Psi}^u_{L_i}\, h_{ij}\, q^u_{R_j} + \bar{\Psi}'^d_{L_i}\, \tilde{h}_{ij}\, q^d_{R_j}\right] + h.c. \\ &= \left[(\bar{u}_L, \bar{c}_L, \bar{t}_L)\, M_u \begin{pmatrix} u_R \\ c_R \\ u_R \end{pmatrix}\right] \\ &\quad + \left[\left(\bar{d}'_L, \bar{s}'_L, \bar{b}'_L\right) \tilde{M} \begin{pmatrix} d'_R \\ s'_R \\ b'_R \end{pmatrix}\right] + h.c. \end{aligned} \tag{13.160}$$

Without any loss of generality, we can take M_u to be diagonal matrix viz.

$$M_u = \begin{pmatrix} m_u & & \\ & m_c & \\ & & m_t \end{pmatrix}. \tag{13.161}$$

It is clear from Eq. (160) that

$$V^\dagger \tilde{M} V = M_d, \tag{13.162}$$

where M_d is now diagonal matrix.

Below we give the experimental values of CKM matrix elements:

Matrix element	Experimental value
$\lvert V_{ud}\rvert$	0.9745 to 0.976
$\lvert V_{us}\rvert$	0.217 to 0.224
$\lvert V_{ub}\rvert$	0.0018 to 0.0045
$\lvert V_{cd}\rvert$	0.217 to 0.224
$\lvert V_{cs}\rvert$	0.9737 to 0.9753
$\lvert V_{cb}\rvert$	0.036 to 0.042
$\lvert V_{td}\rvert$	0.004 to 0.013
$\lvert V_{ts}\rvert$	0.035 to 0.042
$\lvert V_{tb}\rvert$	0.9991 to 0.9994

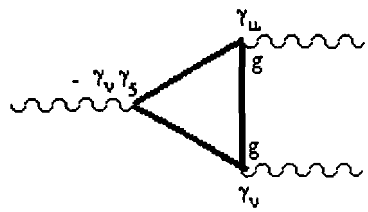

Figure 13 Axial vector current anomaly from the triangle graph.

Note that $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2$ is consistent with 1 as required by unitarity.

13.12 Axial Anomaly

For a theory to be renormalizable, it is essential that vector and axial vector currents are conserved. In electroweak gauge theories, before spontaneous symmetry breaking, fermions are massless and it is, therefore, expected that axial vector current is also conserved. But this is not so, in fact as seen in Chap.12, axial vector current receives anomalous contribution from the triangle graph: a closed fermion loop with one axial-vector vertex and two vector vertices as shown in Fig. 13. This anomalous contribution is equivalent to the statement that in the zero fermion mass limit, the divergence of axial vector current is given by

$$\partial^\mu A_\mu = \frac{g^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{13.163}$$

where $F_{\mu\nu}$ is the field tensor of the vector field and g is the coupling constant as shown in Fig. 13.

The contribution from Δ graph arises only if Δ graph is odd in axial couplings. This contribution is independent of fermion masses and is unaltered by radiative corrections. In QED, such graphs do not cause any trouble as photon is not coupled to axial

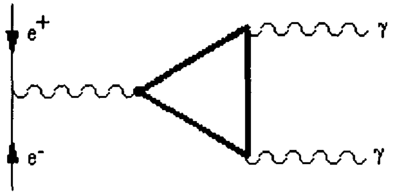

Figure 14 Process $e^-e^+ \to 2\gamma$ through Δ graph.

current. Nor does it cause any problem if one or more of the currents is associated with a global symmetry of the theory. In such a case, it can even be useful as for example the case for $\pi^0 \to 2\gamma$, which arises due to the anomaly as discussed in Chap. 12.

In electroweak theory, such graphs are not absent. For example, in the process $e^-e^+ \to \gamma\gamma$ shown in Fig. 14, the Δ graph can cause trouble as it would give bad high energy behavior. To ensure the renormalizability of electroweak theory, it is, therefore, essential to ensure the cancellation of Δ anomalies.

Consider a gauge group G, where the coupling of the fermions to gauge bosons is given by

$$L_{\text{int}} = \bar{\Psi}_L i\gamma^\mu \left(\partial_\mu + ig\Lambda_a^L W_{a\mu}\right) \Psi_L + \bar{\Psi}_R i\gamma^\mu \left(\partial_\mu + ig\Lambda_a^R W_{a\mu}\right) \Psi_R. \tag{13.164}$$

The current coupled to gauge bosons is given by

$$J_\mu^a = \frac{1}{2}\bar{\Psi}\gamma_\mu (1-\gamma_5)\Lambda_a^L \Psi + \frac{1}{2}\bar{\Psi}\gamma_\mu (1+\gamma_5)\Lambda_a^R \Psi. \tag{13.165}$$

Here Λ_a^L and Λ_a^R are hermitian matrices; they satisfy the following commutation relations

$$\begin{aligned} \left[\Lambda_a^L, \Lambda_b^L\right] &= i f_{abc}\, \Lambda_c^L \\ \left[\Lambda_a^R, \Lambda_b^R\right] &= i f_{abc}\, \Lambda_c^R. \end{aligned} \tag{13.166}$$

Ψ_L and Ψ_R need not transform in the same way under G as is the case in electroweak group. $\Lambda_a^L \neq \Lambda_a^R$ in general. The Δ-anomaly (being independent of fermion masses) is proportional to

$$A^L_{abc} - A^R_{abc}$$

where - sign arises since it is odd axial vector vertices which give anomaly and it has to be symmetric in two indices say a and b . Thus [{ } denotes anticommutator]

$$A^L_{abc} = Tr\left(\left\{\Lambda^L_a,\ \Lambda^L_b\right\}\Lambda^L_c\right) \tag{13.167a}$$

$$A^R_{abc} = Tr\left(\left\{\Lambda^R_a,\ \Lambda^R_b\right\}\Lambda^R_c\right). \tag{13.167b}$$

Theory is thus anomaly free when

$$Tr\left(\left\{\Lambda^L_a,\ \Lambda^L_b\right\}\Lambda^L_c\right) - Tr\left(\left\{\Lambda^R_a,\ \Lambda^R_b\right\}\Lambda^R_c\right) = 0 \tag{13.168}$$

for all values of a, b, c.

Examples

(i) Vector or vector like gauge theory:

$$A^L = A^R \neq 0. \tag{13.169}$$

For such theories either

$$\Lambda^L_a = \Lambda^R_a \tag{13.170}$$

or

$$\Lambda^L_a = U^{-1}\Lambda^R_a U, \tag{13.171}$$

where U is a fixed unitary matrix. The gauge current is given by

$$\begin{aligned} J^a_\mu &= \bar{\Psi}_L\gamma_\mu\Lambda^L_a\Psi_L + \bar{\Psi}_R\gamma_\mu\Lambda^R_a\Psi_R \\ &= \bar{\Psi}_L\gamma_\mu\Lambda^L_a\Psi_L + \bar{\Psi}_R\gamma_\mu U\Lambda^L_a U^{-1}\Psi_R \\ &= \bar{\Psi}\gamma_\mu\Lambda_a\Psi, \end{aligned} \tag{13.172}$$

where

$$\Psi = \Psi_L + U^{-1}\Psi_R, \qquad \Lambda_a = \Lambda^L_a, \tag{13.173}$$

is a pure vector. Note that in general the redefinition of Ψ generates γ_5 terms in the fermion mass matrix. Such a theory is called vector-like.

In QCD, the left handed and right handed quarks belong to the fundamental representation 3 of $SU_c(3)$. Thus it is a vector theory and is anomaly free.

(ii) $A^L = A^R = 0$.

In this case, fermion representation is such that anomalies cancel separately for left handed and right handed fermions. This is the case for example for $SU(2)$. For the fundamental representation 2 of $SU(2)$, $\Lambda_a = \frac{1}{2}\tau_a$ and since $\tau_a\tau_b + \tau_b\tau_a = 2\delta_{ab}$,

$$A_{abc} = Tr\left[\{\tau_a, \tau_b\}\,\tau_c\right] = 0. \tag{13.174}$$

The representation 2 is a real representation in $SU(2)$. But this is not the case for $SU(n)$, $n > 2$, e.g. representation 3 of $SU(3)$ is not equivalent to 3^*. Thus $SU(n)$, $n > 2$ is not safe in general. However, fermions belonging to an octet representation of $SU(3)$ are anomaly free since octet representation is real. This can be seen as follows:

If Λ_a form a representation, $-\Lambda_a^*$ also form a representation. The negative sign arises, since matrices Λ_a^* satisfy the commutation relation

$$[\Lambda_a^*,\ \Lambda_b^*] = -i\ f_{abc}\Lambda_c^*. \tag{13.175}$$

Hence $-\Lambda_a^*$ form a representation conjugate to Λ_a. If (as in the case for real representation),

$$\Lambda_a = -U^{-1}\Lambda_a^* U, \tag{13.176}$$

where U is a unitary matrix, then

$$\begin{aligned} A_{abc} &= Tr\left[\{\Lambda_a^*,\ \Lambda_b^*\}\,\Lambda_c^*\right]. \\ &= -A_{abc}. \end{aligned} \tag{13.177}$$

Thus in general real representations are safe. They do not produce axial anomaly. However, a safe representation need not be real.

(iii) The standard model $SUc(3) \times SU(2) \times U(1)$.

We need to consider $SU(2)\times U(1)$ only as $SUc(3)$ is anomaly free. The matrices $\Lambda_a^L$and Λ_a^R are given by

$$\begin{aligned}\Lambda_a^L &: \frac{1}{2}\tau_a^L, \qquad \frac{1}{2}Y_L, \qquad \Lambda_a^R : \frac{1}{2}Y_R \\ Q &= \frac{1}{2}\tau_3 + \frac{1}{2}Y. \end{aligned} \tag{13.178}$$

Now

$$T_R\left(\left\{\tau_a^L,\ \tau_b^L\right\}\tau_c^L\right) = 0 \tag{13.179}$$

so that from Eq. (168), we have to show that

$$\begin{aligned} & Tr\left(\left\{\tau_a^L, \tau_b^L\right\}Y_L\right) \\ = \ & 2\delta_{ab}\ Tr\ Y_L \\ = \ & 2\delta_{ab}\ Tr\left[2Q - \tau_3\right] = 4\delta_{ab}\ Tr\ Q = 0 \end{aligned} \tag{13.180}$$

and

$$Tr\left[Y_L^3\right] - Tr\left[Y_R^3\right] = 0 \tag{13.181}$$

for the cancellation of anomalies. Now

$$Tr\left[Y_R^3\right] = 8Tr\left[Q^3\right] \tag{13.182}$$

$$\begin{aligned} Tr\left[Y_L^3\right] &= Tr\left[8Q^3 + 6Q\ \tau_3^2 - 6Q^2\ \tau_3 - \tau_3^3\right] \\ &= 8TrQ^3 + 6TrQ - 6Tr\left(Q^2\ \tau_3\right). \end{aligned} \tag{13.183}$$

But

$$\begin{aligned} Tr\left[Q^3\right] &\propto TrQ \\ Tr\left[Q^2\tau_3\right] &\propto Tr\tau_3 = 0. \end{aligned} \tag{13.184}$$

Hence for the cancellation of anomaly, we must have

$$Tr\ Q = 0. \tag{13.185}$$

Now

$$Tr\ Q = [0 - 1 + 3(\frac{2}{3} - \frac{1}{3})] = 0. \tag{13.186}$$

Hence in the standard model, lepton anomalies cancel quark anomalies. Note that in the cancellation of anomalies, color plays a crucial role. Left-handed fermions anomalies cancel among themselves and so do the right-handed fermions anomalies.

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Chapter 14
DEEP INELASTIC SCATTERING

14.1 Introduction

Lepton-nucleon scattering is an excellent tool to study the structure of nucleon. Electron (muon) scattering clearly shows that nucleon has a structure. Consider for example the scattering

$$e + p \to e' + X.$$

Let E be the energy of the incident electron e and E' be the energy of the scattered electron. Let $q = k - k'$ be the momentum transfer. Then in the lab. frame, the four momenta P, k and k' of the target (proton), initial electron and the scattered electron are given by

$$\begin{aligned} P &\equiv (M, O), \quad k = (E, \mathbf{k}) \\ k' &\equiv (E', \mathbf{k}'). \end{aligned}$$

Neglecting the mass of the lepton, we have

$$q^2 = (k - k')^2 = -2EE'(1 - \cos\theta) = -4EE' \sin^2\frac{\theta}{2}$$

$$\mathbf{k} \cdot \mathbf{k}' = EE' \cos\theta. \tag{14.1a}$$

We define another invariant ν:

$$M\nu = P \cdot q. \tag{14.1b}$$

In the lab. frame

$$\nu = q_0 = (E - E'). \tag{14.1c}$$

We also define the invariant mass:

$$s = P_X^2 = (q + P)^2 = q^2 + M^2 + 2M\nu. \tag{14.1d}$$

Note that $2M\nu + q^2 \geq 0$; for elastic scattering $2M\nu = -q^2$.

The elastic scattering of electrons on spinless proton can be written in terms of Mott cross section:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left|F(q^2)\right|^2 \tag{14.2a}$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2 \cos^2\frac{\theta}{2}}{4E^2 \sin^4\frac{\theta}{2}}. \tag{14.2b}$$

The structure of the proton manifests itself in term of the form factor $F(q^2)$. In elastic scattering proton recoils as a whole and the scattering is coherent. The form factor $F(q^2)$ measures the charge distribution of the proton, viz.

$$\begin{aligned} F(q^2) &= \int e^{-i\mathbf{q}\cdot\mathbf{r}} \rho(r) d^3r \\ &= \int e^{-i\mathbf{q}\cdot\mathbf{r}} \rho(r) r^2 dr\, d\Omega. \end{aligned} \tag{14.3a}$$

If we expand $F(q^2)$ in powers of q^2, we get

$$\begin{aligned} F(0) &= \int \rho(r) d^3r = 1 \\ \left.\frac{\partial F(q^2)}{\partial q^2}\right|_{q^2=0} &= -2\pi \int r^4 \rho(r) dr \left\langle \cos^2\theta \right\rangle = -\frac{1}{6}\left\langle r^2 \right\rangle. \end{aligned} \tag{14.3b}$$

$\langle r^2 \rangle$ is called the mean square charge radius.

It is convenient to write the Mott cross section in the form

$$\begin{aligned}\left(\frac{d\sigma}{dq^2}\right)_{\text{Mott}} &= -\left(\frac{4\pi\alpha^2}{q^4}\right)\frac{E'}{E}\cos^2\frac{\theta}{2} \\ &= -\frac{4\pi\alpha^2}{q^4}\left[1+\frac{q^2}{2mE}+\frac{q^2}{4E^2}\right]. \qquad (14.4)\end{aligned}$$

This is the scattering cross section for the scattering of electrons on spinless (structureless) particles of mass m. The scattering cross section for the scattering of electrons on structureless spin 1/2 particles can be calculated using the standard trace techniques and is given by $[Q^2 = -q^2]$

$$\begin{aligned}\frac{d\sigma}{dQ^2} &= \frac{4\pi\alpha^2}{Q^4}\frac{E'}{E}\cos^2\frac{\theta}{2}\left[1+\frac{Q^2}{2m^2}\tan^2\frac{\theta}{2}\right] \\ &= \frac{4\pi\alpha^2}{Q^4}\left[1-\frac{Q^2}{2mE}-\frac{Q^2}{4E^2}+\frac{Q^4}{8m^2E^2}\right]. \qquad (14.5)\end{aligned}$$

14.2 Deep-Inelastic Lepton-Nucleon Scattering

We now consider the inelastic scattering of electrons on nucleons (see Fig. 1). For this case the matrix elements are

$$T = \frac{e^2}{q^2}\frac{1}{(2\pi^3)}\frac{m_e}{\sqrt{EE'}}\bar{u}_e(k')\gamma_\mu u_e(k)\left\langle X\left|j^\mu_{e.m}\right|P\right\rangle. \qquad (14.6a)$$

The cross-section is given by [cf. Chap. 2]

$$\begin{aligned}d\sigma &= \frac{1}{v_{in}}\frac{d^3k}{(2\pi)^3}\frac{d^3P_X}{(2\pi)^3}\frac{e^4}{q^4}(2\pi)^4 \\ &\times\ \delta(P_X+k'-k-P)\frac{m_e^2}{EE'}L_{\mu\nu}W^{\mu\nu}, \qquad (14.6b)\end{aligned}$$

where [see the Appendix A]

$$v_{in} = \frac{|\mathbf{k}|}{E}$$

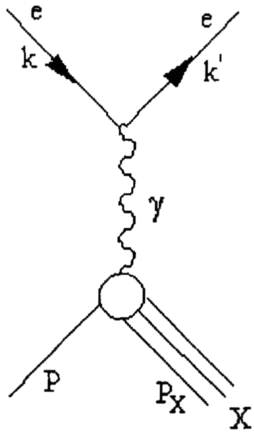

Figure 1 Inelastic charged lepton-proton scattering.

$$L_{\mu\nu} = \frac{1}{2m_e^2}\left[k_\mu k'_\nu - g_{\mu\nu} k \cdot k' + k_\nu k'_\mu\right] \tag{14.6c}$$

and

$$\begin{aligned} W^{\mu\nu}(q, p, S) &= (2\pi)^6 \frac{P_0}{M} \sum_n (2\pi)^4 \delta^4(P + q - P_X) \\ & \quad \times \langle PS | j^\mu_{em} | X \rangle \langle X | j^\nu_{em} | PS \rangle . \end{aligned} \tag{14.6d}$$

Here S denotes the spin of the target and $\sum_n$ denotes the sum over all the quantum numbers of state X and integration over $d^3 P_X$. Then the differential cross-section is given by

$$\frac{d^2\sigma}{d\Omega dE'} = E'(E'^2 - m_e^2)^{1/2} \frac{1}{(2\pi)^3} \frac{e^4}{q^4} \frac{m_e^2}{EE'} L_{\mu\nu} W^{\mu\nu}. \tag{14.6e}$$

Assuming invariance under C, P and T and conservation of the electromagnetic current $\partial^\mu j^{em}_\mu = 0$, the Lorentz structure of

$W^{\mu\nu}$ is

$$\begin{aligned}\frac{MW^{\mu\nu}}{2\pi} &= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(q^2, \nu) \\ &+ \frac{1}{M\nu}\left(P^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right) F_2(q^2, \nu) \\ &+ \frac{i}{\nu}\varepsilon^{\mu\nu\alpha\beta} q_\alpha S_\beta g_1(\nu, q^2) \\ &+ \frac{i}{\nu}\varepsilon^{\mu\nu\alpha\beta} q_\alpha \left(S_\beta - \frac{q\cdot S}{P\cdot q} P_\beta\right) g_2(\nu, q^2). \qquad (14.7)\end{aligned}$$

Here $S^2 = S^\mu S_\mu = -1$, $S\cdot P = 0$ and F_1 and F_2 are spin averaged structure functions: $MW_1 \equiv F_1$ and $\nu W_2 \equiv F_2$ while the remaining two are spin dependent structure functions. In Fig. 2, we show the plot of $Q^2(= -q^2)$ versus $2M\nu$ where we have defined the variables:

$$x = \frac{Q^2}{2M\nu},\ y = \frac{\nu}{E} = \frac{E - E'}{E}$$

$$0 \leq y \leq 1. \qquad (14.8a)$$

Now

$$(P+q)^2 \geq M^2,$$

so that

$$2P\cdot q - Q^2 \geq 0 \text{ or } 0 \leq x \leq 1. \qquad (14.8b)$$

If hadron masses are not important, F's could not depend on Q^2 and one might expect that scale invariance holds in the asymptotic (Bjorken) limit $Q^2, \nu \to \infty$ with x fixed. In the "naive" quark model (where the virtual photon interacts with point like constituents), in the limit of quark masses $\to 0$, there are no dimensions and this suggests that in the asymptotic limit the structure

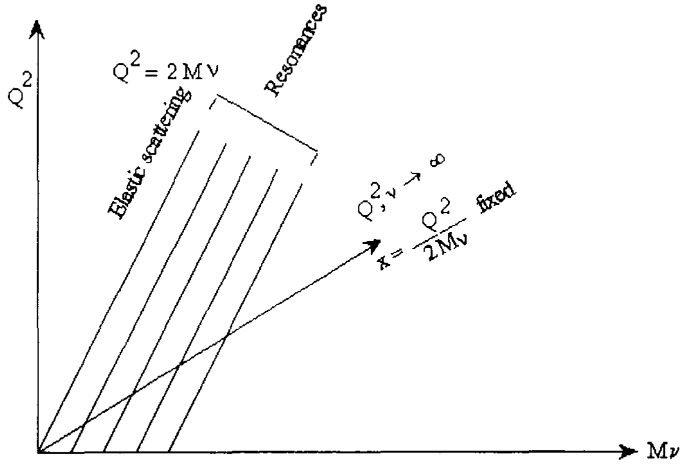

Figure 2 Plot of momentum transfer Q^2 versus energy transfer $\nu = E - E'$ in charged lepton-proton scattering, showing various kinematic regions.

functions scale:

$$\begin{aligned} MW_1(\nu, Q^2) &\equiv F_1(\nu, Q^2) \to F_1(x) \\ \nu W_2(\nu, Q^2) &\equiv F_2(\nu, Q^2) \to F_2(x) \\ g_{1,2}(\nu, Q^2) &\to g_{1,2}(x). \end{aligned} \tag{14.9}$$

In QCD, however, this scaling is broken but only by logarithms of $Q^2/\Lambda^2_{\rm QCD}$.

From Eqs. (6) and (7), the spin averaged cross-section is given by

$$\frac{d^2\sigma}{d\Omega\, dE'} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm Mott} \left[W_2(\nu, Q^2) + 2\tan^2\frac{\theta}{2}W_1(\nu, Q^2)\right]. \tag{14.10a}$$

It is instructive to write this cross-section in the form

$$\frac{d^2\sigma}{dQ^2\, d\nu} = \left(\frac{d\sigma}{dQ^2}\right)_{\rm Mott} \left[W_2(\nu, Q^2) + 2\tan^2\frac{\theta}{2}W_1(\nu, Q^2)\right]. \tag{14.10b}$$

We now define right and left polarized cross-sections as

$$\sigma_{R,L} = \sigma \pm \Delta\sigma, \tag{14.11}$$

where $d^2\sigma/dQ^2\, d\nu$ is given in Eq. (10). In terms of the variables x, y and $K = (1 - \frac{Q^2}{\nu^2})[= 1 - \frac{2M^2x^2}{Q^2} \to 1$ in the scaling limit and is a measure of how close one is to the limit $Q^2 \to \infty]$, we have

$$\frac{d^2\sigma}{dx\, dy} = \frac{4\pi\alpha^2}{Q^4}ME\left[2xy^2F_1 + \left(2(1-y) + \frac{1}{2}y^2(K-1)\right)F_2\right] \tag{14.12}$$

and polarized asymmetry $\Delta\sigma = \sigma_R - \sigma_L/2$ is given by

$$\begin{aligned} \frac{d\Delta\sigma}{dx\, dy} &= \frac{4\pi\alpha^2}{Q^4}ME \\ &\times \left[\cos\beta\left\{2\left(1 - \frac{y}{2} + \frac{y^2}{4}(K-1)\right)g_1 - y(K-1)g_2\right\}\right]. \end{aligned} \tag{14.13}$$

At high energies $y \to 0$ and F_2 and g_1 dominate. It may be noted that g_2 has never been measured. In Eq. (13) β is the angle between $\mathbf{k}$ and spin quantization direction $\mathbf{S}$. If the target is longitudinally polarized $\beta = 0$.

The presence of the structure functions in Eq. (10) indicates that proton is not a point particle. The structure of the proton can be probed in two ways - one by elastic lepton-nucleon scattering and second by deep inelastic lepton-nucleon scattering. First we discuss the elastic scattering for which $\nu = Q^2/2M$. For this case the structure functions are given by

$$\begin{aligned} W_2 &= \left[F_1^2(Q^2) + \tau F_2^2(Q^2)\right] \delta\left(-\nu + \frac{Q^2}{2M}\right) \\ W_1 &= \tau \left[F_1^2(Q^2) + F_2^2(Q^2)\right] \delta\left(-\nu + \frac{Q^2}{2M}\right) \end{aligned} \tag{14.14}$$

where $\tau = Q^2/2M$. Thus from Eq. (10), we have

$$\begin{aligned} \frac{d\sigma}{dQ^2} &= \left(\frac{d\sigma}{dQ^2}\right)_{\text{Mott}} \left\{ \left[F_1^2(Q^2) + \tau F_2^2(Q^2)\right] \right. \\ &\quad \left. + 2\tau \tan^2 \frac{\theta}{2} \left[F_1(Q^2) + F_2(Q^2)\right]^2 \right\}. \end{aligned} \tag{14.15}$$

The form factors for the proton are normalized to $F_1^p(0) = 1$, $F_2^p(0) = \kappa_p$ and for the neutron $F_1^n(0) = 0$, $F_2^n(0) = \kappa_n$ where $\kappa_p = 1.792$ and $\kappa_n = -1.913$ are anomalous magnetic moments of the proton and the neutron respectively. Experimental data is analyzed in terms of Sachs form factors

$$\begin{aligned} G_E(Q^2) &= F_1(Q^2) - \tau F_2(Q^2) \\ G_M(Q^2) &= F_1(Q^2) + F_2(Q^2). \end{aligned} \tag{14.16}$$

These form factors are normalized as follows: $G_E^p(0) = 1$, $G_M^p(0) = \mu_p = 2.792$, $G_E^n(0) = 0$ and $G_M^n(0) = \mu_n$. In terms of G_E and G_M, the elastic scattering cross-section is given by

$$\frac{d\sigma}{dQ^2} = \left(\frac{d\sigma}{dQ^2}\right)_{\text{Mott}} \left\{ \frac{[G_E^2(Q^2) + \tau G_M^2(Q^2)]}{1+\tau} + 2\tau \tan^2 \frac{\theta}{2} G_M^2(Q^2) \right\}. \tag{14.17}$$

The experimental data is fitted remarkably well by a single form factor

$$\begin{aligned} G_E^p(Q^2) &= \frac{G_M^p(Q^2)}{\mu_p} = \frac{G_M^n(Q^2)}{\mu_n} = \frac{1}{[1+Q^2/m_V^2]^2} \\ G_E^n(q^2) &= 0, \end{aligned} \tag{14.18}$$

where $m_V^2 = 0.71\ \text{GeV}^2$. From Eq. (15), we get [cf. Eq. (3b)]

$$\left\langle r_E^2 \right\rangle_p = \frac{12}{m_V^2} = 0.66\,\text{fm}^2,\ \left\langle r_E^2 \right\rangle_n = 0. \tag{14.19}$$

Now Eqs. (14) and (15) clearly show that $\frac{d\sigma}{dQ^2} \to \left(\frac{d\sigma}{dQ^2}\right)_{\text{Mott}}$ as $Q^2 \to \infty$ i.e. cross section rapidly falls as Q^2 become large, clearly showing that the nucleon has a "diffused" structure in the elastic region.

But the behavior of the structure functions W_2 and W_1 is quite different in the deep inelastic region. The experimental data in this region indicate that the cross section stays large and is of the order of $\left(\frac{d\sigma}{dQ^2}\right)_{\text{Mott}}$, characteristics of a point particle. This clearly indicates that in this region the scattering is incoherent and is what one would expect if a nucleon consists of non-interacting or weakly interacting point like constituents called partons (quarks). This scattering region thus gives us information about the elementary constituents of nucleon, i.e. about their charges, spin and flavor. Moreover, the structure functions νW_2 and MW_1 show Bjorken scaling i.e. νW_2 and $MW_1 \to F_2(x)$ and $F_1(x)$ as Q^2, $\nu \to \infty$ where $x = \frac{2M\nu}{Q^2}$ is fixed. This is clearly indicated in Fig. 3 where $F_2(x)$ is plotted against Q^2 for various values of x.

The above characteristics lead to parton model of deep inelastic scattering which we now discuss.

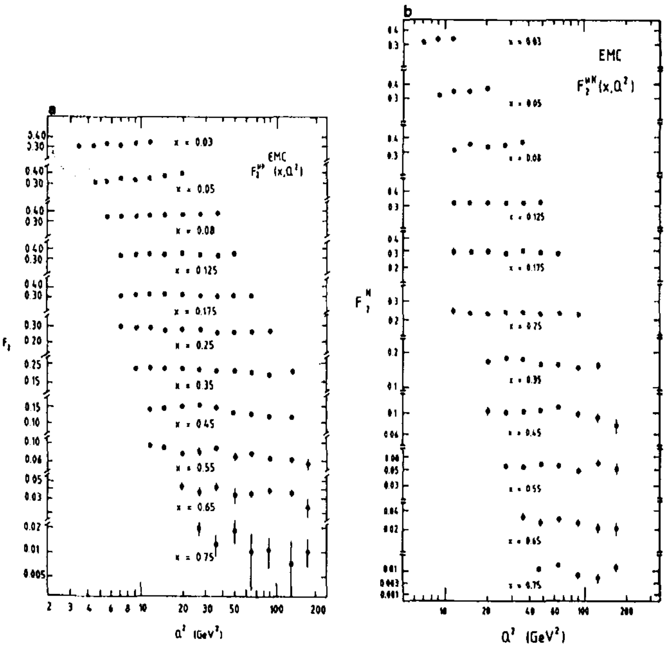

Figure 3 The structure function F_2 measured by the CERN muon experiments, (a) proton (b) nucleon in deuterium.

14.3 Parton Model

Partons are quarks (spin 1/2), antiquarks (spin 1/2) and gluons (spin 1). Gluons do not contribute here since they carry no electric charge. Thus we shall deal with spin 1/2 partons. If the target is a free quark of flavor i, of mass m and charge e_i, we have from Eq. (6d)

$$\begin{aligned} \frac{mW_i^{\mu\nu}}{2\pi} &= \frac{m}{2\pi}(2\pi)^6\frac{p_0}{m}\int(2\pi)^4\frac{d^3p_n}{(2\pi)^3} \\ & \sum \langle ps\,|j^\mu_{em}|\,p_n\rangle\,\langle p_n\,|j^\nu_{em}|\,ps\rangle\,\delta^4(p+q-p_n) \\ &= me_i^2\int m\frac{d^3p_n}{p_{n_0}}\bar{u}(ps)\gamma^\mu\frac{m+\not{p}_n}{2m}\gamma^\nu u(ps) \\ &\times\ \delta^4(p+q-p_n). \end{aligned} \tag{14.20a}$$

Now $\frac{d^3p_n}{p_{n_0}} = d^4p_n\delta[p_n^2-m^2]$ and we obtain from Eq. (20a)

$$\begin{aligned} \frac{m}{2\pi}W_i^{\mu\nu} &= me_i^2\bar{u}(ps)\gamma^\mu\left[\not{p}+\not{q}+m\right]\gamma^\nu u(ps) \\ &\times\ \delta(2p\cdot q-Q^2). \end{aligned} \tag{14.20b}$$

To proceed further, we make use of the following identities of Dirac matrices algebra [see Appendix A],

$$\begin{aligned} \gamma^\mu\gamma^\nu &= g^{\mu\nu}-i\sigma^{\mu\nu} \\ \gamma^\mu\gamma^\rho\gamma^\nu &= g^{\mu\rho}\gamma^\nu-g^{\mu\nu}\gamma^\rho+g^{\nu\rho}\gamma^\mu+i\varepsilon^{\mu\rho\nu\sigma}\gamma_5\gamma_\sigma \\ \bar{u}(ps)\gamma_5\gamma^\mu u(ps) &= -s^\mu \\ \bar{u}(ps)\gamma^\mu u(ps) &= \frac{p^\mu}{m} \\ \bar{u}(ps)i\sigma^{\mu\nu}u(ps) &= -\frac{i}{m}\varepsilon^{\mu\nu\alpha\beta}\gamma_5\gamma_\beta p_\alpha. \end{aligned} \tag{14.21}$$

Then Eq. (20b) becomes

$$\begin{aligned} &\frac{m}{2\pi}W_i^{\mu\nu} = e_i^2\delta(2p\cdot q-Q^2) \\ &\times\ \left[-g^{\mu\nu}p\cdot q+2p^\mu p^\nu+q^\mu p^\nu+p^\mu q^\nu+im\varepsilon^{\mu\nu\alpha\beta}q_\alpha s_\beta\right]. \end{aligned} \tag{14.22a}$$

Thus the comparison with Eq. (7) gives

$$\begin{aligned} F_{1i} &= \frac{1}{2}\delta(x-1)e_i^2, \quad F_{2i} = \delta(x-1)e_i^2 \\ g_{1i} &= \frac{1}{2}\delta(x-1)e_i^2, \quad g_{2i} = 0. \end{aligned} \tag{14.22b}$$

Hence from Eqs. (12) and (13) for a spin 1/2 parton i,

$$\frac{d\sigma_i}{dx\,dy} = \frac{4\pi\alpha^2}{Q^4}e_i^2 mE\left[1+(1-y)^2+\frac{1}{2}y^2(K-1)\right]\delta(x-1) \tag{14.23a}$$

$$\frac{d\Delta\sigma_i}{dx\,dy} = \frac{4\pi\alpha^2}{Q^4}e_i^2 mE\cos\beta\left[1-\frac{y}{2}+\frac{y^2}{4}(K-1)\right]\delta(x-1). \tag{14.23b}$$

The comparison of Eq. (23) with Eqs. (12) and (13) clearly shows that if we replace $\delta(1-x)$ in Eq. (23) by some distribution functions $F(x)$ and $g(x)$ we get Eqs. (12) and (13). Hence it follows that in the scaling region, the nucleon is behaving as if it consists of point-like constituents and the structure function $F_{2i}(x)$ or $F_{1i}(x)$ or $g_{1i}(x)$ gives us the x-distribution of point-like constituents inside the nucleon. The point-like constituents have been assumed to be free i.e. interaction between them can be neglected in the scaling region. This is compatible with QCD, as QCD is asymptotically free. More accurately one can write F_2 and F_1 as $F_2(x,\,Q^2)$, $F_1(x,\,Q^2)$; but the dependence on Q^2 is very weak (logarithmic). The following physical picture emerges. In the deep inelastic region, the virtual photon interacts in an incoherent manner and probes roughly the instantaneous construction of proton. In the center of mass frame of electron and proton, we can write (neglecting lepton mass):

$$k \equiv (P,\,0,\,0,P), \;\; P \equiv \left[(M^2+P^2)^{1/2} \simeq P\left(1+\frac{M^2}{2P}\right),\,0,\,0,-P\right]$$

$$q_0 = \frac{2M\nu - Q^2}{4P}.$$

Let us assume that the target (proton) has point-like constituents called partons of flavor, i. Neglecting any parton momentum transverse to the target, let us assume that the longitudinal momentum of a parton is given by $p = xP$. The time of interaction of photon is given by

$$\tau = \frac{1}{q_0} = \frac{4P}{2M\nu - Q^2} = \frac{2P}{M\nu(1-x)}.$$

The energy of a parton $= \sqrt{p_z^2 + p_\perp^2 + m^2} \approx xP\left(1 + \frac{p_\perp^2 + m^2}{2x^2P^2}\right)$, so that the lifetime of virtual parton states is

$$\begin{aligned} T &= \frac{1}{\sum xP\left(1 + \frac{p_\perp^2 + m^2}{2x^2P^2}\right) - P\left(1 + \frac{M^2}{2P^2}\right)} \\ &= \frac{2P}{\sum \frac{p_\perp^2 + m^2}{x} - M^2}. \end{aligned}$$

For x not going to 0 or 1, $\tau \ll T$ in the deep inelastic region so that one can consider the partons contained in the proton as free during the interaction. Hence in the deep inelastic region the photon interacts with the constituents of proton as depicted in Fig. 4.

If the target is built from partons of type i and the probability for a parton i to have momentum fraction x' to $x' + dx'$ is $f_i(x')$, then $p \cdot q = x'P \cdot q = M\nu x'$,

$$\begin{aligned} \nu_{\text{parton}} &= \frac{p \cdot q}{m} = \frac{M}{m}\nu x', \\ \delta(Q^2 - 2p \cdot q) &= \delta(Q^2 - 2M\nu x') = \frac{1}{2M\nu}\delta(x - x') \end{aligned}$$

and Eq. (22a) becomes

$$\begin{aligned} \frac{m}{2\pi} W_i^{\mu\nu} &= e_i^2 \left[-g^{\mu\nu}\frac{x'}{2} + \frac{1}{M\nu}x'^2 P^\mu P^\nu + \cdots + i\varepsilon^{\mu\nu\alpha\beta} q_\alpha s_\beta \frac{x'}{2\nu}\right] \\ &\times \delta(x - x'). \end{aligned} \tag{14.24}$$

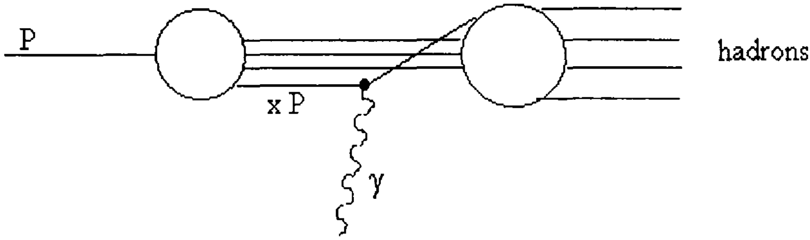

Figure 4 The parton model.

Since $\frac{d^2\sigma}{dQ^2\,d\nu} \propto W^{\mu\nu}$, we should write

$$\frac{dW^{\mu\nu}}{d\nu} = \frac{1}{d\nu_{\text{parton}}} \sum_i \int W_i^{\mu\nu} f_i(x')\,dx',$$

or, on using Eq. (24),

$$\begin{aligned} \frac{MW^{\mu\nu}}{2\pi} &= M \int \frac{d\nu}{\frac{M}{m} x'\,d\nu} \sum_i \frac{W_i^{\mu\nu}}{2\pi} f_i(x')\,dx' \\ &= \int \sum_i e_i^2 \left[-g^{\mu\nu}\frac{1}{2} + \frac{1}{M\nu} x' P^\mu P^\nu + \cdots \right] \\ &\times \ \delta(x - x') f_i(x')\,dx'. \end{aligned} \tag{14.25}$$

Using then the expression (7) for $MW^{\mu\nu}/2\pi$, it follows that in the parton model:

$$\begin{aligned} F_2(x) &= \sum_{i,\text{spin}} e_i^2 x\, f_i(x) \\ &= \sum_i e_i^2 x\ (f_{i\uparrow}(x) + f_{i\downarrow}(x)) \\ &= 2xF_1(x), \end{aligned} \tag{14.26}$$

while [cf. Eqs. (24) and (7)]

$$g_1(x) = \sum_i e_i^2 \,(f_{i\uparrow}(x) - f_{i\downarrow}(x)) \,. \tag{14.27}$$

$$g_2(x) = 0,$$

where ↑ and ↓ denote respectively parton spin parallel and antiparallel to proton spin S.

The relation $F_2(x) = 2xF_1(x)$, which is a consequence of parton having spin $\frac{1}{2}$, is well satisfied experimentally. For the proton target [denoting $f_i(x)$ conveniently by $q(x)+\bar{q}(x)$, $e_i \to e_q$ and with spin sum understood], we have from Eq. (26)

$$F_2^{ep} = \sum_{q=u,\,d,\,\cdots} xe_q^2\,[q(x)+\bar{q}(x)]\,. \tag{14.28a}$$

In other words

$$F_2^{ep} = x\left[\frac{4}{9}\left(u(x)+\bar{u}(x)\right)+\frac{1}{9}\left(d(x)+\bar{d}(x)\right)+\cdots\right]. \tag{14.28b}$$

Applying isospin conservation so that the $u(d)$ flavored parton distribution in the proton is the same as $d(u)$ flavored parton distribution in the neutron, whilst the s and $c\,\cdots$, distributions remain unchanged being isoscalar, the neutron structure function becomes:

$$F_2^{en} = x\left[\frac{4}{9}\left(d(x)+\bar{d}(x)\right)+\frac{1}{9}\left(u(x)+\bar{u}(x)\right)+\cdots\right]. \tag{14.28c}$$

In the above equations $u(x)$, $d(x)$, $\cdots$, are the probabilities that parton (antiparton) of flavor u, d, $\cdots$, carries a fraction x of the momentum of the proton or the neutron.

For an isosinglet target N, we get

$$\begin{aligned} F_2^{eN}(x) &\equiv \frac{1}{2}\left(F_2^{ep}(x)+F_2^{en}(x)\right) \\ &= x\left\{\frac{5}{18}\left[u(x)+\bar{u}(x)+d(x)+\bar{d}(x)\right]\right. \\ &\qquad \left.+\frac{1}{9}\left[s(x)+\bar{s}(x)\right]+\cdots\right\}. \end{aligned} \tag{14.28d}$$

Here $\cdots$ means the contributions of other quarks like c, b, t. Note that $\frac{5}{18}$ is just the average squared charge of the u, d quarks.

We have thus seen that the parton model leads to the Bjorken scaling of the structure functions in the deep inelastic scattering.

14.4 Deep Inelastic Neutrino-Nucleon Scattering

Let us consider the processes

$$\begin{aligned} \bar{\nu}_\ell + N &\rightarrow \ell^+ + X \\ \nu_\ell + N &\rightarrow \ell^- + X \end{aligned}$$

The matrix elements are given by

$$\begin{aligned} T_{\bar{\nu}} &= -\frac{G_F}{\sqrt{2}}\frac{1}{(2\pi)^3}\sqrt{\frac{m_\ell m_\nu}{EE'}} \\ &\quad \times \bar{v}(k)\gamma_\mu(1-\gamma_5)v(k')\left\langle X\left|J^\mu\right|P\right\rangle, \qquad (14.29a) \\ T_\nu &= -\frac{G_F}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m_\ell m_\nu}{EE'}} \\ &\quad \times \bar{u}(k')\gamma_\mu(1-\gamma_5)u(k)\left\langle X\left|J^{\mu\dagger}\right|P\right\rangle, \qquad (14.29b) \end{aligned}$$

where $J^\mu = V^\mu - A^\mu$. Then we have to replace in Eq. (6b) e^4/q^4 by $G_F^2/2$ and $L_{\mu\nu}$ by

$$L_{\mu\nu}^{\nu,\bar{\nu}} = \frac{2}{m_e m_\nu}\left[k_\mu k'_\nu - g_{\mu\nu} k\cdot k' k_\nu k'_\mu \pm i\varepsilon_{\mu\nu\alpha\beta}k^\alpha k'^\beta\right], \qquad (14.29c)$$

while $W_{\mu\nu}$ now contains for the spin averaged case three structure functions W_1, W_2 and W_3. The third function W_3 arises due to $V-A$ interference term and appears in Eq. (7) as $\frac{1}{2M\nu}\varepsilon^{\mu\nu\alpha\beta}P_\alpha q_\beta F_3$, with $\nu W_3 = F_3$. The cross-section is given by

$$\begin{aligned} \frac{d^2\sigma^{\bar{\nu},\nu}}{dQ^2\,d\nu} &= \frac{G_F^2}{2}\frac{1}{\pi}\frac{E'}{E}\left[W_2^{\bar{\nu},\nu}(\nu,\,Q^2)\cos^2\frac{\theta}{2} + 2W_1^{\bar{\nu},\nu}(\nu,\,Q^2)\sin^2\frac{\theta}{2}\right. \\ &\quad \left.\mp\frac{E+E'}{M}W_3^{\bar{\nu},\nu}(\nu,\,Q^2)\sin^2\frac{\theta}{2}\right]. \qquad (14.30) \end{aligned}$$

In order to discuss the scaling, we again express the cross-sections in terms of the variables x and y. The structure functions show the following scaling behavior in the deep inelastic region:

$$\begin{aligned} \nu W_2^{\bar{\nu},\nu}(\nu,\,Q^2) &\rightarrow F_2^{\bar{\nu},\nu}(x) \\ MW_1^{\bar{\nu},\nu}(\nu,\,Q^2) &\rightarrow F_1^{\bar{\nu},\nu}(x) \\ \nu W_3^{\bar{\nu},\nu}(\nu,\,Q^2) &\rightarrow F_3^{\bar{\nu},\nu}(x). \qquad (14.31) \end{aligned}$$

The cross section can be written

$$\frac{d^2\sigma^{\bar{\nu},\nu}}{dx\,dy} = \frac{G_F^2 ME}{\pi}\left[\left(1-y-\frac{M}{2E}xy\right)F_2^{\bar{\nu},\nu}(x) + \frac{y^2}{2}2xF_1^{\bar{\nu},\nu}(x) \mp \left(y-\frac{y^2}{2}\right)xF_3^{\bar{\nu},\nu}(x)\right]. \quad (14.32)$$

For the basic processes

$$\bar{\nu}_\ell + u \to \ell^+ + d, \qquad \bar{\nu}_\ell + \bar{d} \to \ell^+ + \bar{u}$$
$$\nu_\ell + d \to \ell^- + u, \qquad \nu_\ell + \bar{u} \to \ell^- + \bar{d}, \qquad (14.33a)$$

we get for quarks by substituting Eqs. (22b) without e_i^2 and $F_3 = \delta(x-1)$ in Eq. (32),

$$\frac{d^2\sigma^{\bar{\nu},\nu}}{dx\,dy} = \frac{G_F^2 mE}{\pi}\left[\left(1-y-\frac{m}{2E}xy\right)+\frac{y^2}{2}x \mp \left(y-\frac{y^2}{2}\right)x\right]\delta(1-x). \quad (14.33b)$$

For antiquarks, the signs of last term are interchanged. Thus for instance, for the processes

$$\nu_\mu \begin{pmatrix} d \\ \bar{u} \end{pmatrix} \to \mu^- \begin{pmatrix} u \\ \bar{d} \end{pmatrix}$$

we have for large E

$$\frac{d^2\sigma^\nu}{dx\,dy} = \frac{G_F^2 mE}{\pi}\left[\frac{1+(1-y)^2}{2} \pm \frac{1-(1-y)^2}{2}x\right]\delta(1-x)$$

and the relation for F_3 corresponding to the relation (28a) is

$$F_3(x) = 2\sum_q \left[q(x) - \bar{q}(x)\right]. \quad (14.34)$$

In view of Eq. (33a) [note that the role of e_q^2 in Eq. (28a) is taken over by the isospin raising and lowering operators, namely $I^{\pm}$], we get

$$F_2(x) = 2xF_1(x)$$

$$\begin{aligned} F_2^{\bar{\nu}p} &= 2x\left[u(x) + \bar{d}(x) + c(x) + \bar{s}(x) + t(x) + \bar{b}(x)\right] \\ F_3^{\bar{\nu}p} &= 2\left[u(x) - \bar{d}(x) + c(x) - \bar{s}(x) + t(x) - \bar{b}(x)\right], \end{aligned} \quad (14.35a)$$

$$\begin{aligned} F_2^{\nu p} &= 2x\left[d(x) + \bar{u}(x) + s(x) + \bar{c}(x) + b(x) + \bar{t}(x)\right] \\ F_3^{\nu p} &= 2\left[d(x) - \bar{u}(x) + s(x) - \bar{c}(x) + b(x) - \bar{t}(x)\right]. \end{aligned} \quad (14.35b)$$

The factor 2 is due to the fact that for weak decays we have both vector and axial vector currents. The corresponding values for neutron are obtained by replacing $u \leftrightarrow d$, $\bar{u} \leftrightarrow \bar{d}$ on the ground of isospin invariance. Hence for an isosinglet target N, we get (suppressing x)

$$\begin{aligned} F_2^{\nu N} &= 2x\left[\frac{1}{2}(u + \bar{u}) + \frac{1}{2}(d + \bar{d}) + s + b + \bar{c} + \bar{t}\right] \\ F_3^{\nu N} &= 2\left[\frac{1}{2}(u - \bar{u}) + \frac{1}{2}(d - \bar{d}) + s + b - \bar{c} - \bar{t}\right] \\ F_2^{\bar{\nu} N} &= 2x\left[\frac{1}{2}(u + \bar{u}) + \frac{1}{2}(d + \bar{d}) + c + t + \bar{s} + \bar{b}\right] \\ F_3^{\bar{\nu} N} &= 2\left[\frac{1}{2}(u - \bar{u}) + \frac{1}{2}(d - \bar{d}) + c + t - \bar{s} - \bar{b}\right]. \end{aligned} \quad (14.36)$$

If we assume that in a nucleon, the probability of having q and $\bar{q}$ ($q = s,\ c,\ b,\ t$) is the same or we neglect $s, \bar{s}, \cdots$, then we can write

$$\begin{aligned} F_2^{\nu N} &= \sum_q x\left[q + \bar{q}\right] = F_2^{\bar{\nu} N} \\ xF_3^{\nu N} &= \sum_q x\left[q - \bar{q}\right] = xF_3^{\bar{\nu} N}. \end{aligned} \quad (14.37)$$

We observe from Eq. (28d), neglecting the sea quark contribution of heavy quarks, and Eq. (37) that $[\ell = e \text{ or } \mu]$

$$\frac{F_2^{lN}}{\frac{5}{18}F_2^{\nu N}} = \left[1 - \frac{3}{5}\frac{s+\bar{s}}{\sum_q (q+\bar{q})}\right]. \tag{14.38}$$

This ratio has been experimentally tested as the left-hand side is 1.007 ± 0.063. This also shows that the strange quark sea contribution is very small. It verifies the charges of u and d valence quarks as their mean square is $\frac{5}{18}$.

14.5 Sum Rules

One can write a number of sum rules. First the momentum conservation gives

$$\int_0^1 dx \left[\sum_q x(q+\bar{q})\right] = 1 - \varepsilon, \tag{14.39}$$

where ε is the fraction of the momentum carried by the gluon constituents. Hence we get the sum rule

$$\int_0^1 F_2^{\nu N} dx = 1 - \varepsilon. \tag{14.40}$$

Experimentally, the left-hand side is 0.52 ± 0.03 giving the momentum fraction carried by the quarks. Thus the remaining momentum fraction, which is about 50%, is attributed to the gluon constituents.

Since the nucleon has quantum numbers S (strangeness) $= 0$, C (charm) $= 0$, B (bottom) $= 0$ and T (top) $= 0$, we have

$$0 = \int_0^1 dx\,[q(x) - \bar{q}(x)], \tag{14.41}$$

for $q = s, c, b$ and τ. On the other hand, the charges of proton and neutron give

$$\begin{aligned} 1 &= \int_0^1 dx \left[\frac{2}{3}(u-\bar{u}) - \frac{1}{3}(d-\bar{d})\right], \\ 0 &= \int_0^1 dx \left[\frac{2}{3}(d-\bar{d}) - \frac{1}{3}(u-\bar{u})\right]. \end{aligned} \tag{14.42}$$

We can combine them, so that we get

$$\begin{aligned} 1 &= \int_0^1 dx \left[(u - \bar{u}) - (d - \bar{d})\right], \\ 1 &= \frac{1}{3}\int_0^1 dx \left[(d - \bar{d}) + (u - \bar{u})\right], \end{aligned} \tag{14.43}$$

thus from Eqs. (35), (41) and (43), we have

$$\int_0^1 \left[F_2^{\bar{\nu}p} - F_2^{\nu p}\right]\frac{dx}{x} = 2, \tag{14.44}$$

and

$$\int_0^1 F_3^{\nu N}(x)\, dx = \int_0^1 dx \left[(u - \bar{u}) + (d - \bar{d})\right] = 3. \tag{14.45}$$

If we use Eq. (35b) and the corresponding equation for the neutron, we get the sum rule (44) in the form

$$\int_0^1 [F_2^{\nu n} - F_2^{\nu p}]\frac{dx}{x} = 2. \tag{14.46}$$

This is known as the Adler sum rule. It is an exact sum rule obtained from quark structure of electromagnetic and weak hadronic currents and is protected by conservation laws implied by Eqs. (41) and (42). It is difficult at present to verify it experimentally with good precision as it requires good low x data. On the other hand, the sum rule (45), known as the Gross-Llewellyn Smith sum rule, is modified by QCD corrections in the leading order to,

$$\int_0^1 F_3^{\nu N}(x)dx = 3\left(1 - \frac{\alpha_s(Q^2)}{\pi}\right). \tag{14.47}$$

The right-hand side of (47) for $\alpha_s(Q^2 \approx 3\ \text{GeV}^2) = 0.35 \pm 0.05$ is 2.66 ± 0.05 while experimentally the left-hand side is $2.50 \pm 0.018 \pm 0.078$, verifying the sum rule.

Another sum rule which follows from Eqs. (28b, c) is

$$\begin{aligned}
&\int_0^1 [F_2^{ep} - F_2^{en}] \frac{dx}{x} \\
&= \frac{1}{3}\int_0^1 dx \left[u(x) + \bar{u}(x) - d(x) - \bar{d}(x)\right] \\
&= \frac{1}{3}\int_0^1 dx \left[u(x) - \bar{u}(x) - d(x) + \bar{d}(x)\right] \\
&\quad + \frac{2}{3}\int_0^1 dx \left[\bar{u}(x) - \bar{d}(x)\right] \\
&= \frac{1}{3} + \frac{2}{3}\int_0^1 dx \left[\bar{u}(x) - \bar{d}(x)\right], \qquad (14.48)
\end{aligned}$$

on using Eq. (43). This is known as the Gottfried sum rule. Experimentally the left-hand side is 0.258 ± 0.017 implying that the second term on the right-hand side is not zero. Its non vanishing does not contradict any known principle.

There are two sum rules which involve the spin-dependent structure function $g_1(x)$. We note from Eq. (27) that

$$\int_0^1 g_1(x)dx = \frac{1}{2}\sum_q e_q^2 \Delta q, \qquad (14.49a)$$

where we have defined (for a nucleon target)

$$\Delta q = \int_0^1 \left\{[q_\uparrow(x) + \bar{q}_\uparrow(x)] - [q_\downarrow(x) + \bar{q}_\downarrow(x)]\right\} dx. \qquad (14.49b)$$

Here Δq is the quark contribution to the first moment of the structure function $g_1(x)$. There is also gluon contribution to it, this is due to the short-range interaction of photons with polarized gluons via the quark box diagram, shown in Fig. 5. To include this we replace Δq by

$$\Delta\tilde{q} = \Delta q - \frac{\alpha_s}{2\pi}\Delta G_q. \qquad (14.50)$$

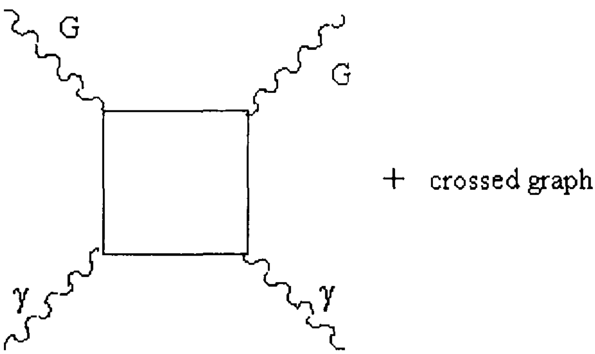

Figure 5 The photon-gluon scattering graph.

This separation is not unambiguous but has been found useful. For the proton target $\Delta\tilde{q}$ has been shown to be related to the matrix elements of the axial vector current $\bar{q}\gamma_\mu\gamma_5 q$

$$\langle p\,|\bar{q}\gamma_\mu\gamma_5 q|\,p\rangle = \Delta\tilde{q}(S_\mu), \;\; q = u,\, d,\, s \tag{14.51}$$

where $S^\mu = \bar{\Psi}\gamma^\mu\gamma_5\Psi$ is the spin of the proton, Ψ being the proton spinor. For the first moment of $g_1^p(x)$, the gluon contribution in relation (50) is related to the triangle axial anomaly [cf. Eq. (11.79)] in the divergence of the singlet current $\partial^\mu A_{0\mu}$

$$\begin{aligned} \langle p\,|\partial^\mu A_{0\mu}|\,p\rangle &= \sqrt{\frac{2}{3}}\frac{3\alpha_s}{4\pi}\left\langle p\left|Tr\left(G^{\mu\nu}\tilde{G}_{\mu\nu}\right)\right|p\right\rangle \\ &\quad +\sqrt{\frac{2}{3}}\left\langle p\left|\left[m_u\bar{u}i\gamma_5 u + m_d\bar{d}i\gamma_5 d + m_s\bar{s}i\gamma_5 s\right]\right|p\right\rangle \\ &= \sqrt{\frac{2}{3}}\frac{3\alpha_s}{4\pi}\left(-\Delta G\right)2m_p\bar{\Psi}i\gamma_5\Psi \\ &\quad +\sqrt{\frac{2}{3}}\left\langle p\left|\left[m_u\bar{u}i\gamma_5 u + m_d\bar{d}i\gamma_5 d + m_s\bar{s}i\gamma_5 s\right]\right|p\right\rangle . \end{aligned} \tag{14.52}$$

Note that the second term on the right-hand side is not an SU(3) singlet. The first term on right-hand side also contains a non-singlet part (that is why we have put a subscript q on ΔG in Eq. (50)) . For the proton target, Eq. (49) gives the sum rule

$$\begin{aligned}\int_0^1 g_1^p(x)\,dx &= \frac{1}{2}\left[\frac{4}{9}\Delta\tilde{u}+\frac{1}{9}\Delta\tilde{d}+\frac{1}{9}\Delta\tilde{s}+\cdots\right]\\ &= \frac{1}{12}\left\{\left(\Delta\tilde{u}-\Delta\tilde{d}\right)+\frac{1}{3}\left(\Delta\tilde{u}+\Delta\tilde{d}-2\Delta\tilde{s}\right)\right.\\ &\quad\left.+\frac{4}{3}\left(\Delta\tilde{u}+\Delta\tilde{d}+\Delta\tilde{s}\right)+\cdots\right\} \qquad (14.53)\end{aligned}$$

where $\cdots$ denotes isospin singlet sea contribution of heavy quarks and second and third terms are isospin singlets. Therefore, for the neutron target, only $\left(\Delta\tilde{u}-\Delta\tilde{d}\right)$ changes sign and we get in the isospin conservation limit

$$\int_0^1\left[g_1^p(x)-g_1^n(x)\right]\,dx=\frac{1}{6}\left(\Delta\tilde{u}-\Delta\tilde{d}\right)=\frac{1}{6}g_A \qquad (14.54)$$

since from Eq. (51), it is clear that

$$\begin{aligned}\frac{\Delta\tilde{u}-\Delta\tilde{d}}{2}(S_\mu) &= \langle p\,|A_{3\mu}|\,p\rangle_{Q^2=0}\\ &= \frac{1}{2}g_A(S_\mu). \qquad (14.55)\end{aligned}$$

Here g_A is the axial vector coupling constant determined from β-decay of the neutron. The sum rule (54) is known as the Bjorken sum rule. If the leading order QCD corrections are included, it then becomes

$$\begin{aligned}\Gamma_1^p-\Gamma_1^n &= \int_0^1\left[g_1^p(x)-g_1^n(x)\right]\,dx\\ &= \frac{1}{6}g_A\left(1-\frac{\alpha_s}{\pi}\right). \qquad (14.56)\end{aligned}$$

This sum rule obtained from quark structure of electromagnetic and weak hadronic currents, is regarded as a fundamental prediction

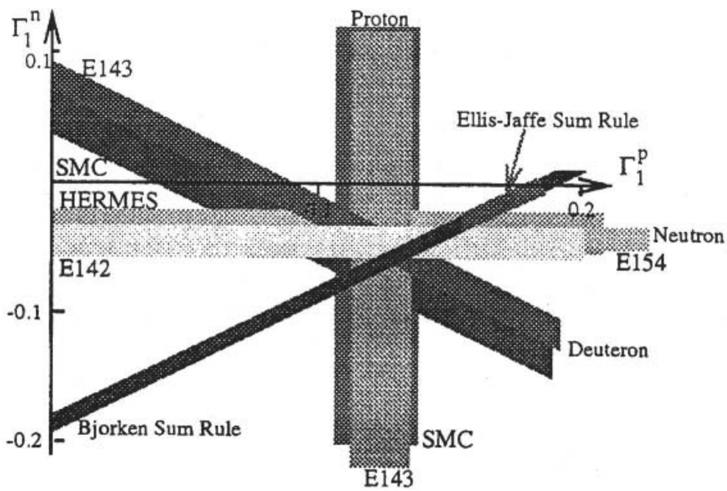

Figure 6 Plot of Γ_1^n versus Γ_1^p. The predictions of the Bjorken and Ellis-Jaffe sum rules are shown on the diagonal band from the lower left to the upper right of the figure. While the data and the Bjorken sum rule overlap within one sigma, the Ellis-Jaffe prediction is roughly two sigma away from the overlap region in the data. [16]

of QCD. For $g_A = 1.2670 \pm 0.0035$, $\alpha_s = 0.35 \pm 0.05$ one finds for the right hand side of Eq. (56), the value 0.187 ± 0.01. The experimental situation is best summarized in the Γ_1^n, Γ_1^p plane, Fig. (6) which illustrates that Ellis–Jaffe sum rule [see below] is violated by the experimental data where as Bjorken sum rule is compatible with the data.

One can obtain another sum rule involving only g_1^p if one assumes exact SU(3) flavor symmetry for semi-leptonic decays of baryon octet, so that

$$\begin{aligned} & \left(\Delta\tilde{u} + \Delta\tilde{d} - 2\Delta\tilde{s}\right)(S_\mu) \\ = \; & 2\sqrt{3}\,\langle p\,|A_{8\mu}|\,p\rangle_{Q^2=0} \\ = \; & g_A^8(S_\mu) = (3F - D)(S_\mu) \end{aligned} \tag{14.57}$$

where g_A , F and D have been defined in Chap. 11, namely

$$\begin{aligned} \Delta\tilde{u} - \Delta\tilde{d} = g_A &= 1.2670 \pm 0.0035, \\ F &= 0.463 \pm 0.023, \\ D &= 0.803 \pm 0.040. \end{aligned} \tag{14.58}$$

Thus neglecting the sea contribution of heavy quarks, one obtains from Eq. (53)

$$\Gamma_1^p = \int_0^1 g_1^p(x)dx = \frac{1}{12}\left[g_A + \frac{1}{3}\left(\frac{3F}{D} - 1\right) + \frac{4}{3}g_A^0\right] \tag{14.59}$$

where

$$g_A^0 = \Delta\tilde{u} + \Delta\tilde{d} + \Delta\tilde{s} = \Delta\tilde{\Sigma} \tag{14.60}$$

is unknown. Furthermore, if one assumes as is done in naive quark model

$$\langle p\,|\bar{s}\gamma_\mu\gamma_5 s|\,p\rangle = 0, \tag{14.61}$$

one has

$$g_A^0 \approx g_A^8 = (3F - D) \tag{14.62}$$

so that the sum rule (59) becomes

$$\Gamma_1^p = \int_0^1 g_1^p(x) = \frac{1}{12}\left[1 + \frac{5}{3}\frac{3F/D - 1}{F/D + 1}\right]. \tag{14.63}$$

This is known as the Ellis-Jaffe sum rule. With g_A and F/D given in Eq. (58), the right-hand side of Eq. (63) is 0.187 ± 0.003 in disagreement with the SMC ($Q^2 = 10$ GeV^2) data which gives

$$\Gamma_1^p = 0.139 \pm 0.01. \tag{14.64}$$

In view of the above disagreement the assumption (61) has been questioned. If one relaxes it, one does not have any prediction. However, one can use the sum rule (53), [neglecting $\cdots$] together with (64) to determine

$$\frac{1}{2}\left[\frac{4}{9}\Delta\tilde{u} + \frac{1}{9}\Delta\tilde{d} + \frac{1}{9}\Delta\tilde{s}\right] = 0.139 \pm 0.01. \tag{14.65}$$

This together with the values given in Eqs. (57) and (58) give

$$\begin{aligned} \Delta\tilde{u} &= 0.78 \pm 0.07 \\ \Delta\tilde{d} &= -0.48 \pm 0.08 \\ \Delta\tilde{s} &= -0.14 \pm 0.07, \end{aligned} \tag{14.66}$$

so that

$$\left(\Delta\tilde{u} + \Delta\tilde{d} + \Delta\tilde{s}\right) = 0.16 \pm 0.22 \tag{14.67}$$

which is consistent with zero. In other words [cf. Eq. (50)]

$$g_A^0 \equiv \Delta\tilde{\Sigma} = \Delta\Sigma - \frac{3\alpha_s}{2\pi}\Delta\tilde{G} = 0.16 \pm 0.22, \tag{14.68}$$

where $\Delta\Sigma = \Delta u + \Delta d + \Delta s$ is the quark contribution to the spin of the proton and $\Delta\tilde{G}$ is the singlet part of ΔG. Various estimates of $\Delta\Sigma$ indicate that $\Delta\Sigma \approx 0$, which implies that $\frac{\alpha_s}{2\pi}(-\Delta G) = 0.05 \pm 0.07$. Thus one can say that the quarks do not contribute to the spin of the proton (this is known as spin crisis for the proton) implying in view of the angular momentum sum rule $\frac{1}{2} = \Delta\Sigma + \Delta\tilde{G} + L_z$, that its spin is carried by gluons and/or orbital angular momentum of its constituents. $\Delta\Sigma \approx 0$ is in complete disagreement with NQM result which predicts $\Delta\Sigma = 1$.

It is important to measure both g_A^0 and $F_2^0(0)$ [the SU(3) singlet anomalous magnetic moment of the proton] experimentally in order to determine the flavor and spin content of the proton.

14.6 Deep-Inelastic Scattering Involving Neutral Weak Currents

For neutral weak currents mediated by $Z-$boson (see Chap. 13), the relevant Lagrangian for the processes

$$(\bar{\nu})\nu + N \to (\bar{\nu})\nu + X$$

is given in Eq. (13.64) [see also Table 13.1]. From Eq. (13.64) and Table 13.1, we get for the proton

$$\begin{aligned}
F_2^{\nu NC} &= 2\rho^2 x \left\{ \left[(\varepsilon_L(u))^2 + (\varepsilon_R(u))^2 \right] [u(x) + \bar{u}(x)] \right. \\
&\quad \left. + \left[(\varepsilon_L(d))^2 + (\varepsilon_R(d))^2 \right] \left[d(x) + \bar{d}(x) \right] \right\} . \\
F_3^{\nu NC} &= 2\rho^2 x \left\{ \left[(\varepsilon_L(u))^2 - (\varepsilon_R(u))^2 \right] [u(x) - \bar{u}(x)] \right. \\
&\quad \left. + \left[(\varepsilon_L(d))^2 - (\varepsilon_R(d))^2 \right] \left[d(x) - \bar{d}(x) \right] \right\} . \qquad (14.69) \\
F_{2,3}^{\bar{\nu} NC} &= F_{2,3}^{\nu NC} .
\end{aligned}$$

Thus from the experimental data on deep inelastic scattering, we can determine $\varepsilon_L(u)$, $\varepsilon_L(d)$, $\varepsilon_R(u)$ and $\varepsilon_R(d)$. This information have been used in Chap. 13. In writing Eqs. (69), we have neglected the contribution of strange and heavy quarks. For neutron, we can obtain the structure function by replacing $u(x) \leftrightarrow d(x)$ and $\bar{u}(x) \leftrightarrow \bar{d}(x)$.

We end this chapter by the remarks that the quark-parton model is simple and quite successful. A closer examination of Fig.

3 reveals a systematic deviation from exact Bjorken scaling, the structure function increases with increasing Q^2 at small x whereas it has opposite behavior for large x. The attempts to understand such deviations from the quark-parton model in terms of QCD are beyond the scope of this book.

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Chapter 15
PARTICLE MIXING AND CP–VIOLATION

15.1 Introduction

We have seen that neither parity P nor charge conjugation C is conserved in weak interaction. Let us consider the decay $\pi^+ \to \mu^+\nu$ in the rest frame of π where experimentally μ^+ is found to be polarized with helicity $\mathcal{H} = \mathbf{s}\cdot\mathbf{p}/|\mathbf{p}|$ to be negative. The application of charge conjugation operation C changes π^+ to π^-, $\mu^+ \to \mu^-$ and $\nu \to \bar{\nu}$ but does not change the helicity. Thus if weak interaction were invariant under C, one would find $\Gamma_{\pi^+\to\mu^+(-)\nu} = \Gamma_{\pi^-\to\mu^-(-)\bar{\nu}}$, where $(-)$ denotes negative helicity. Experimentally $\Gamma_{\pi^+\to\mu^+(-)\nu} \gg \Gamma_{\pi^-\to\mu^-(-)\bar{\nu}}$, showing that C is violated in weak interaction. If however, we now apply CP, then since helicity also changes sign we have $\Gamma_{\pi^-\to\mu^-(+)\bar{\nu}} = \Gamma_{\pi^+\to\mu^+(-)\nu}$ as seen experimentally. Thus CP is conserved here.

Let us now consider the $K^0 \to \bar{K}^0$ system. In hadronic and electromagnetic interactions, the hyperchange Y is conserved so that $K^0(Y=1) \longleftrightarrow \bar{K}^0(Y=-1)$ transitions are not possible. In a production process involving hadronic (or electromagnetic) interaction, K^0 and $\bar{K}^0$ appear as two distinctly different particles. In the presence of weak interaction, Y is no longer conserved and transitions between K^0 and $\bar{K}^0$ can occur, for example.

$$K^0 \underset{\text{weak}}{\to} \pi^+\pi^- \underset{\text{weak}}{\to} \bar{K}^0, |\Delta Y| = 2$$

Thus if we write $H = H_0 + H_W$, where $H_0 = H_{had} + H_{e.m}$, K^0 and $\bar{K}^0$, which are eigenstates of H_0, are no longer eigenstates of H.

A linear combinates of K^0 and $\bar{K}^0$ will be eigenstates of H. Such states cannot be eigenstates of C or P since neither is conserved in weak interaction; CP is a better choice. Choosing the CP phase

$$CP\left|K^0\right\rangle = -\left|\bar{K}^0\right\rangle$$

so that

$$CP\left|\bar{K}^0\right\rangle = -\left|K^0\right\rangle, \tag{15.1}$$

it is easy to see that

$$\left|K^0_{1,2}\right\rangle = \frac{1}{\sqrt{2}}\left[\left|K^0\right\rangle \mp \left|\bar{K}^0\right\rangle\right] \tag{15.2}$$

are eigenstates of CP with eigenvalues ± 1. Further if CP is conserved so that $[H, CP] = 0$, then

$$\begin{aligned} \left\langle K^0_2 \left|H\right| K^0_1\right\rangle &= \left\langle K^0_2 \left|(CP)^{-1} H\, CP\right| K^0_1\right\rangle \\ &= -\left\langle K^0_2 \left|H\right| K^0_1\right\rangle \end{aligned} \tag{15.3}$$

so that $\langle K^0_2 \left|H\right| K^0_1\rangle = 0 = \langle K^0_1 \left|H\right| K^0_2\rangle$, showing that H is diagonal in the basis provided by $|K^0_1\rangle$ and $|K^0_2\rangle$. Thus eigenstates of H can be chosen to be eigenstates of CP.

Now $\bar{K}^0$ is the antiparticle of K^0 ; they should have the same mass. But K^0_1 is not the antiparticle of K^0_2 and so they can have different properties. In fact due to weak interaction, K^0_1 and K^0_2 should have slightly different rest energies; experimentally $\left(m_{K_2} - m_{K_1}\right)/m_K \sim 10^{-14}$ and it is remarkable that such a small quantity is measured. What about their life times. Energetically kaons can decay into two or three pions. Consider 2π final state. As seen in section 4.6, C parity of 2π state is $(-1)^\ell$ where ℓ is the relative orbital angular of 2π system. Thus

$$\begin{aligned} CP\left|\pi^+\pi^-\right\rangle &= (-1)^\ell(-1)^2(-1)^\ell\left|\pi^+\pi^-\right\rangle \\ &= (-1)^{2\ell}\left|\pi^+\pi^-\right\rangle = \left|\pi^+\pi^-\right\rangle. \end{aligned}$$

Similarly

$$CP\left|\pi^0\pi^0\right\rangle = \left|\pi^0\pi^0\right\rangle. \tag{15.4}$$

Thus only K_1^0 can decay into 2π if CP is conserved in weak interaction and $K_2 \to 2\pi$ is forbidden. K_2^0 will have other modes, e.g. three pionic which can have $CP = -1$. Now decay energy available for 2π mode is about 220 MeV and for 3 pionic model it is about 90 MeV. Thus the phase space available for decay into three pions is considerably smaller than that for two pions, implying

$$\tau_1 \equiv \tau(K_1^0) \ll \tau(K_2^0) \equiv \tau_2.$$

Experimentally $\tau(K_1^0) = 0.893 \times 10^{-10}$ sec. and $\tau(K_2^0) = 0.517 \times 10^{-7}$ sec so that $\tau_1/\tau_2 = 1/580$.

As seen above, if CP is conserved, $K_2^0 \to \pi^+\pi^-$ is forbidden. But $K_2 \to \pi^+\pi^-$ occurs, showing that CP is not conserved. Numerically it is not a big effect.

$$\frac{A(K_2^0 \to \pi^+\pi^-)}{A(K_1^0 \to \pi^+\pi^-)} = 2.269 \times 10^{-3}. \tag{15.5}$$

This chapter is devoted to CP violation and particle mixing.

15.2 General Formalism

As seen above K^0 and $\bar{K}^0$ can mix. We now develop a general formalism for particle mixing. Let X^0 and $\bar{X}^0$ be two pseudoscalar particles ($X = K, B$ or $D; \bar{X}$ being the antiparticle of X). Let $|\Psi(t)\rangle$ be a state at time t. It is a coherent mixture of $|X^0\rangle$ and $\left|\bar{X}^0\right\rangle$

$$|\Psi(t)\rangle = a(t)\left|X^0\right\rangle + \bar{a}(t)\left|\bar{X}^0\right\rangle \tag{15.6}$$

where t is measured in the rest system of the particle X^0. Then the time evolution of the state

$$\Psi(t) = \begin{pmatrix} a(t) \\ \bar{a}(t) \end{pmatrix}$$

is given by

$$i\frac{d\Psi}{dt} = \mathfrak{m}\Psi \tag{15.7}$$

where $\mathfrak{m}$ is a 2 x 2 matrix in the space spanned by X^0 and $\bar{X}^0$ states and since the particles X^0 and $\bar{X}^0$ decay, $\mathfrak{m}$ is not hermitian and has the form

$$\mathfrak{m}_{\alpha'\alpha} = M_{\alpha'\alpha} - \frac{i}{2}\,\Gamma_{\alpha'\alpha} \tag{15.8}$$

with α, $\alpha' = X^0, \bar{X}^0$ (1,2). Note that Γ and M are hermitian

$$\begin{aligned} \Gamma^\dagger &= \Gamma,\ M^\dagger = M \\ \Gamma^*_{\alpha\alpha'} &= \Gamma_{\alpha'\alpha},\ M^*_{\alpha\alpha'} = M_{\alpha'\alpha}. \end{aligned} \tag{15.9}$$

If one now assumes CPT invariance, then

$$\left\langle X^0 \left|\mathfrak{m}\right| X^0 \right\rangle = \left\langle \bar{X}^0 \left|\mathfrak{m}\right| \bar{X}^0 \right\rangle$$

or

$$\mathfrak{m}_{11} = \mathfrak{m}_{22}. \tag{15.10}$$

It is worth proving this result; CPT invariance implies (see Section 3.5).

$$\begin{aligned} &\left\langle f_{out}\left|\mathfrak{m}\right|X^0\right\rangle \\ =\ &\left\langle f_{out}\left|T^{-1}P^{-1}C^{-1}\ \mathfrak{m}\left(CPT\right)\right| X^0\right\rangle \\ =\ &\eta_T^{X*}\eta_T^f\left\langle \tilde{f}_{in}\right|(CP)^\dagger\ \mathfrak{m}^\dagger(CP)^{-1\dagger}\left|\ \widetilde{\bar{X}}^0\right\rangle^* \\ =\ &\eta_T^{X*}\eta_T^f\left\langle \widetilde{\bar{X}}^0\left|(CP)^{-1}\ \mathfrak{m}(CP)\right|\ \tilde{f}_{in}\right\rangle \end{aligned} \tag{15.11}$$

where

$$T\left|X^0\right\rangle = \eta_T^X\left|\widetilde{X}^0\right\rangle, \left\langle f_{out}\right|T^{-1} = \left\langle \tilde{f}_{in}\right|\eta_T^{f*} \tag{15.12}$$

and $\sim$ means momenta and spins of the corresponding states are reversed. Since we are in the rest frame of X^0, T will reverse only the magnetic quantum number and so we can drop $\sim$. We may choose CP phase such that

$$\begin{aligned} CP\left|X^0\right\rangle &= -\left|\bar{X}^0\right\rangle \\ CP\left|f\right\rangle &= \eta^f_{CP}\left|\bar{f}\right\rangle . \end{aligned} \tag{15.13}$$

Then

$$\begin{aligned} \left\langle f_{out}\right| \mathfrak{m} \left|X^0\right\rangle &= -\,\eta^f_{CP}\,\eta^f_T\,\eta^{X*}_T\,\left\langle \bar{X}^0|\mathfrak{m}|\,\bar{f}_{in}\right\rangle \\ &= \eta_f\left\langle \bar{X}^0|\mathfrak{m}|\,\bar{f}_{in}\right\rangle \end{aligned} \tag{15.14}$$

where

$$\eta_f = -\eta^f_{CP}\,\eta^f_T\,\eta^{X*}_T .$$

Now

$$\begin{aligned} \left|f_{in}\right\rangle &= S_f\left|f_{out}\right\rangle \\ &= e^{2i\delta_s}\left|f_{out}\right\rangle \end{aligned} \tag{15.15}$$

where δ_s is the strong interaction phase for the state $|f\rangle$. Then Eq. (14) gives finally

$$\left\langle f_{out}|\mathfrak{m}|X^0\right\rangle = \eta_f\,e^{2i\delta_s}\,\left\langle \bar{X}^0|\mathfrak{m}|\bar{f}_{out}\right\rangle . \tag{15.16}$$

In particular for single particle states $|f_{in}\rangle = |f_{out}\rangle = |X^0\rangle$ [$\delta_s = 0$, $\eta^X_{CP} = -1$, according to our choice of the phase in Eq. (13) so that $\eta_f = 1$], Eq. (14) gives

$$\left\langle X^0|\mathfrak{m}|X^0\right\rangle = \left\langle \bar{X}^0|\mathfrak{m}|\bar{X}^0\right\rangle$$

proving (10) and giving

$$M_{11} = M_{22},\ \Gamma_{11} = \Gamma_{22} \tag{15.17}$$

that is particle - antiparticle have identical mass and same total width. Note that if we take $f = \bar{X}^0$ in Eq.(14), we get an identity so that with CPT invariance alone $\mathfrak{m}_{12}$ and $\mathfrak{m}_{21}$ are not related. However, if we assume CP invariance, then by an argument similar to the above, the relation (16) is replaced by

$$\left\langle f \ |\mathfrak{m}|X^0\right\rangle = -\eta^f_{CP}\left\langle \bar{f} \ |\mathfrak{m}|\bar{X}^0\right\rangle \tag{15.18}$$

so that for $f = \bar{X}^0[\eta^X_{CP} = -1]$

$$\left\langle \bar{X}^0|\mathfrak{m}|X^0\right\rangle = \left\langle X^0|\mathfrak{m}|\bar{X}^0\right\rangle,$$

and thus CP invariance implies

$$\mathfrak{m}_{21} = \mathfrak{m}_{12}. \tag{15.19}$$

We have the result that in the $X^0 - \bar{X}^0$ space $\mathfrak{m}$ is a 2×2 matrix of the form

$$\mathfrak{m} = \begin{pmatrix} A & B \\ C & A' \end{pmatrix} = \begin{pmatrix} M_{11} - i\frac{\Gamma_{11}}{2} & M_{12} - i\frac{\Gamma_{12}}{2} \\ M_{21} - i\frac{\Gamma_{21}}{2} & M_{22} - i\frac{\Gamma_{22}}{2} \end{pmatrix} \tag{15.20}$$

where CPT invariance alone (which we now assume) requires

$$A = A' \tag{15.21}$$

or

$$M_{11} = M_{22} \qquad\qquad \Gamma_{11} = \Gamma_{22}.$$

But hermiticity of the matrices M and Γ [see Eqs. (9)] gives

$$\begin{aligned} M_{12} &= M^*_{21}, \Gamma_{12} = \Gamma^*_{21} \\ M_{11} &= M^*_{11}, \Gamma_{11} = \Gamma^*_{11} \\ M_{22} &= M^*_{22}, \Gamma_{22} = \Gamma^*_{22}. \end{aligned} \tag{15.22}$$

Then the diagonalization of the matrix (20) gives the eigenvalues

$$\gamma_{1,2} = A \mp \sqrt{BC} = A \mp pq \tag{15.23}$$

where

$$\begin{aligned} p^2 &= B = M_{12} - \frac{i}{2}\Gamma_{12} \\ q^2 &= C = M_{21} - \frac{i}{2}\Gamma_{21} = M_{12}^* - \frac{i}{2}\Gamma_{12}^*. \end{aligned} \quad (15.24)$$

Then the corresponding eigenstates are

$$|X_{1,2}\rangle = \frac{1}{\sqrt{|p|^2 + |q|^2}} \left[p \left| X^0 \right\rangle \mp q \left| \bar{X}^0 \right\rangle \right]. \quad (15.25)$$

Hence we have the result

$$\begin{aligned} M_{11} - \frac{i}{2}\Gamma_{11} - pq &= \gamma_1 = m_1 - \frac{i}{2}\Gamma_1 \\ M_{11} - \frac{i}{2}\Gamma_{11} + pq &= \gamma_2 = m_2 - \frac{i}{2}\Gamma_2 \end{aligned} \quad (15.26)$$

so that

$$\begin{aligned} m_1 &= M_{11} - \operatorname{Re} pq \\ m_2 &= M_{11} + \operatorname{Re} pq \\ \Gamma_1 &= \Gamma_{11} + 2 \operatorname{Im} pq \\ \Gamma_2 &= \Gamma_{11} - 2 \operatorname{Im} pq. \end{aligned} \quad (15.27)$$

Thus finally we have

$$\begin{aligned} \Delta m &= m_2 - m_1 = 2 \operatorname{Re} pq \\ m &= \frac{m_1 + m_2}{2} = M_{11} \\ \Delta\Gamma &= \Gamma_2 - \Gamma_1 = -4 \operatorname{Im} pq \\ \Gamma &= \frac{1}{2}(\Gamma_1 + \Gamma_2) = \Gamma_{11}. \end{aligned} \quad (15.28)$$

Let us define

$$\frac{1-\varepsilon}{1+\varepsilon} = \frac{q}{p} = \sqrt{\frac{C}{B}} = \sqrt{\frac{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}{M_{12} - \frac{i}{2}\Gamma_{12}}}. \quad (15.29)$$

If CP is conserved, then $B = C$, $q = p(\varepsilon = 0)$ so that the mass eigenstates given in Eq. (25) become

$$|X_{1,2}\rangle = \frac{1}{\sqrt{2}}\left[\left|X^0\right\rangle \mp \left|\bar{X}^0\right\rangle\right] \tag{15.30}$$

which are now also the eigenstates of CP:

$$\begin{aligned} CP\,|X_1\rangle &= |X_1\rangle \\ CP\,|X_2\rangle &= -\,|X_2\rangle\,. \end{aligned} \tag{15.31}$$

It follows that CP–violation is determined by the parameter

$$\varepsilon = \frac{p-q}{p+q}. \tag{15.32}$$

Since the particles X^0 and $\bar{X}^0$ are unstable, it is the particles X_1 and X_2 defined in Eq. (25) which have definite masses m_1 and m_2 and decay widths Γ_1 and Γ_2 respectively. Let $|\Psi(t) >$ be a state at time t. In the X_1 and X_2 basis, we can write

$$|\Psi(t)\rangle = a(t)\,|X_1\rangle + b(t)\,|X_2\rangle \tag{15.33a}$$

$$i\frac{d}{dt}\,|\Psi(t)\rangle = \begin{pmatrix} m_1 - \frac{i}{2}\Gamma_1 & 0 \\ 0 & m_2 - \frac{i}{2}\Gamma_2 \end{pmatrix} |\Psi(t)\rangle\,. \tag{15.33b}$$

The solution is

$$\begin{aligned} a(t) &= a(0)\exp\left[-i\left((m_1 - \frac{i}{2}\Gamma_1)t\right)\right] \\ b(t) &= b(0)\exp\left[-i\left(m_2 - \frac{i}{2}\Gamma_2\right)t\right]. \end{aligned} \tag{15.34}$$

Suppose we start with X^0, viz. $|\Psi(0)\rangle = |X^0\rangle$, then from Eq. (25), we get

$$a(0) = b(0) = \frac{\sqrt{|p|^2 + |q|^2}}{2p}. \tag{15.35}$$

Hence from Eqs. (33a), (34) and (35), we get

$$\begin{aligned}
&|\Psi(t)\rangle \\
=\ & \frac{\sqrt{|p|^2+|q|^2}}{2p}\left\{\exp\left[(-im_1-\frac{1}{2}\Gamma_1)t\right]|X_1\rangle\right. \\
&\left.+\exp\left[\left(-im_2-\frac{1}{2}\Gamma_2\right)t\right]|X_2\rangle\right\}, \qquad (15.36a)
\end{aligned}$$

$$\begin{aligned}
&|\Psi(t)\rangle \\
=\ & \frac{1}{2}\left\{\exp(-im_1-\frac{1}{2}\Gamma_1)t+\exp\left(-im_2-\frac{1}{2}\Gamma_2\right)t\right\}\left|X^0\right\rangle \\
&-\frac{q}{p}\left[\exp\left(-im_1-\frac{1}{2}\Gamma_1\right)t-\exp\left(-im_2-\frac{1}{2}\Gamma_2\right)t\right]\left|\bar{X}^0\right\rangle. \qquad (15.36b)
\end{aligned}$$

Equation (36b) clearly shows the particle mixing. Similarly if we start with $\bar{X}^0$, we get at time t:

$$\begin{aligned}
&\left|\overline{\Psi}(t)\right\rangle \\
=\ & \frac{\sqrt{|p|^2+|q|^2}}{2q}\left(\exp\left[\left(-im_1-\frac{1}{2}\Gamma_1\right)t\right]|X_1\rangle\right. \\
&\left.-\exp\left[\left(-im_2-\frac{1}{2}\Gamma_2\right)t\right]|X_2\rangle\right), \qquad (15.37a)
\end{aligned}$$

$$\begin{aligned}
&\left|\overline{\Psi}(t)\right\rangle \\
=\ & \frac{1}{2}\left\{\frac{p}{q}\left[\exp\left(-im_1-\frac{1}{2}\Gamma_1\right)t-\exp\left(-im_2-\frac{1}{2}\Gamma_2\right)t\right]\left|X^0\right\rangle\right. \\
&\left.-\left[\exp\left(-im_1-\frac{1}{2}\Gamma_1\right)t+\exp\left(-im_2-\frac{1}{2}\Gamma_2\right)t\right]\left|\bar{X}^0\right\rangle\right\}. \qquad (15.37b)
\end{aligned}$$

From Eqs. (36) and (37) we can determine X^0 and $\bar{X}^0$ mixing. It is clear that if we start with X^0, then at time t, the probability of finding the particles X^0or $\bar{X}^0$ is given by [using Eq. (36b)].

$$\left|\left\langle X^0|\Psi(t)\right\rangle\right|^2 = \frac{1}{4}\left[e^{-\Gamma_1 t}+e^{-\Gamma_2 t}+2e^{-\Gamma t}\cos\Delta mt\right]$$

$$\left|\left\langle \bar{X}^0|\Psi(t)\right\rangle\right|^2 = \frac{1}{4}\left|\frac{1-\varepsilon}{1+\varepsilon}\right|^2\left[e^{-\Gamma_1 t}+e^{-\Gamma_2 t}-2e^{-\Gamma t}\cos\Delta m t\right]. \tag{15.38}$$

We define the mixing parameter r as

$$r = \frac{\int_0^T\left|\left\langle \bar{X}^0|\Psi(t)\right\rangle\right|^2 dt}{\int_0^T\left|\langle X^0|\Psi(t)\rangle\right|^2 dt}, \tag{15.39}$$

where T is a sufficiently long time. In the limit $T\to\infty$,using Eq. (38), we can easily determine:

$$r = \left|\frac{1-\varepsilon}{1+\varepsilon}\right|^2 \frac{x^2+y^2}{2+x^2-y^2}, \tag{15.40}$$

where $x = \Delta m/\Gamma$ and $y = \Delta\Gamma/2\Gamma$. If we start with $\bar{X}^0$, we can use Eq. (37b). Then we find

$$\overline{r} = \frac{\int_0^T\left|\left\langle X^0|\overline{\Psi}(t)\right\rangle\right|^2 dt}{\int_0^T\left|\left\langle \bar{X}^0|\overline{\Psi}(t)\right\rangle\right|^2 dt} \underset{T\to\infty}{\longrightarrow} \left|\frac{1+\varepsilon}{1-\varepsilon}\right|^2 \frac{x^2+y^2}{2+x^2-y^2}. \tag{15.41}$$

When $CP-$violation effects are neglected, then

$$r = \overline{r} = \frac{x^2+y^2}{2+x^2-y^2}. \tag{15.42}$$

The asymmetry parameter a

$$a = \frac{\overline{r}-r}{\overline{r}+r} = \frac{4\,\mathrm{Re}\,\varepsilon}{1+|\varepsilon|^2} \tag{15.43}$$

is a measure of $CP-$violation.

We define another parameter χ which is also a measure of particle mixing. Let χ be the probability of $X^0\to\bar{X}^0$, then

$$\begin{aligned}\chi &= \int_0^T\left|\left\langle \bar{X}^0\right|\Psi(t)\right\rangle\right|^2 dt\\ 1-\chi &= \int_0^T\left|\left\langle X^0\right|\Psi(t)\right\rangle\right|^2 dt.\end{aligned} \tag{15.44}$$

Thus

$$r = \frac{\chi}{1-\chi}; \chi = \frac{r}{1+r}. \tag{15.45}$$

Similarly, we get

$$\overline{r} = \frac{\overline{\chi}}{1-\overline{\chi}}; \overline{\chi} = \frac{\overline{r}}{1+\overline{r}}.$$

We note from the definitions $x = \frac{\Delta m}{\Gamma}, y = \frac{\Delta\Gamma}{2\Gamma}$

$$\begin{aligned} 0 &\leq x^2 \leq \infty \\ 0 &\leq y^2 \leq 1. \end{aligned} \tag{15.46}$$

Obviously

$$0 \leq r \leq 1. \tag{15.47}$$

We now discuss, how the mixing parameter r can be measured experimentally. Suppose that X^0 and $\bar{X}^0$ are produced in the reaction

$$e^- e^+ \rightarrow X^0 \bar{X}^0.$$

Taking into account the particle mixing, we have four possible final states $X^0 \bar{X}^0$, $\bar{X}^0 X^0$, $X^0 X^0$, $\bar{X}^0 \bar{X}^0$. Experimentally $X^0 \bar{X}^0$ and $\bar{X}^0 X^0$ are indistinguishable. We can define a parameter

$$R = \frac{N(X^0 X^0) + N(\bar{X}^0 \bar{X}^0)}{N(X^0 \bar{X}^0) + N(\bar{X}^0 X^0)} \tag{15.48}$$

which can be measured experimentally. $N(X^0 X^0)$ can be identified by some convenient final states (e.g. two charged leptons $l^- l^-$). If $X\bar{X}^0$ pair is produced incoherently (for example not through a resonance of definite spin and parity and $C-$parity), then

$$R = \frac{\overline{\chi}(1-\chi) + \chi(1-\overline{\chi})}{(1-\bar{\chi})(1-\chi) + \chi\overline{\chi}}. \tag{15.49}$$

Neglecting $CP-$violation effects, i.e. using $\overline{\chi} = \chi$, we get

$$\begin{aligned} R &= \frac{2\chi(1-\chi)}{(1-\chi)^2 + \chi^2} \\ &= \frac{2r}{1+r^2}. \end{aligned} \tag{15.50}$$

Now suppose that $X^0\bar{X}^0$ are produced through a resonance with $J^{PC} = 1^{--}$, for example

$$e^-e^+ \to \Upsilon \to B^0\bar{B}^0.$$

For this case we have to consider a state with $C = -1$ viz.

$$\left[|\Psi(t)\rangle \left|\bar{\Psi}(t)\right\rangle - \left|\bar{\Psi}(t)\right\rangle |\Psi(t)\rangle\right].$$

If the two decays take place at t_1 and t_2, then neglecting $CP-$ violation, we have from Eqs. (36) and (37):

$$\begin{aligned}
& |\Psi(t_1)\rangle \left|\overline{\Psi}(t_2)\right\rangle - \left|\overline{\Psi}(t_1)\right\rangle |\Psi(t_2)\rangle \\
= \; & (g_+(t_1)g_-(t_2) - g_-(t_1)g_+(t_2)) \left|X^0X^0\right\rangle \\
& + (g_-(t_1)g_-(t_2) - g_+(t_1)g_+(t_2)) \left|X^0\bar{X}^0\right\rangle \\
& + (g_+(t_1)g_+(t_2) - g_-(t_1)g_-(t_2)) \left|\bar{X}^0X^0\right\rangle \\
& + (g_-(t_1)g_+(t_2) - g_+(t_1)g_-(t_2)) \left|\bar{X}^0\bar{X}^0\right\rangle \qquad (15.51)
\end{aligned}$$

where

$$\begin{aligned}
g_\pm(t) &= \left[e^{-im_1t}\exp\left(-\frac{1}{2}\Gamma_1 t\right) \pm e^{-im_2t}\exp\left(-\frac{1}{2}\Gamma_2 t\right)\right] \\
&= e^{-im_1t}\exp\left(\frac{-1}{2}\Gamma t\right)\left[\exp(\frac{i}{2}\Delta mt)\exp(\frac{1}{4}\Delta\Gamma t)\right. \\
&\quad \left.\pm\exp(\frac{-i}{2}\Delta mt)\exp\left(-\frac{1}{4}\Delta\Gamma t\right)\right]. \qquad (15.52)
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\left.\begin{matrix} N\left(X^0X^0\right) \\ N\left(X^0\bar{X}^0\right)\end{matrix}\right\} \\
&= \int_0^\infty dt_1 \int_0^\infty dt_2 \left|g_\pm(t_1)g_-(t_2) - g_\mp(t_1)g_+(t_2)\right|^2 \\
&= 4\int_0^\infty dt_1 \int_0^\infty dt_2\; e^{-\Gamma(t_1+t_2)}\left[\exp\left(\frac{1}{2}\Delta\Gamma(t_2 - t_1)\right)\right.
\end{aligned}$$

$$+\exp\left(-\frac{1}{2}\Delta\Gamma(t_2-t_1)\right)\pm 2\,\mathrm{Re}\exp\left(-i\Delta m(t_2-t_1)\right)\Big]$$
$$= 4\left[\frac{2}{\Gamma^2-\frac{1}{4}(\Delta\Gamma)^2}\pm\frac{2}{\Gamma^2+(\Delta m)^2}\right]. \tag{15.53}$$

Noting from Eq. (51) that $N(X^0X^0) = N\left(\bar{X}^0\bar{X}^0\right)$ and $N(X^0\bar{X}^0) = N(\bar{X}^0X^0)$, we get [cf. Eq. (40) with $\varepsilon = 0$]

$$R = \frac{N(\bar{X}^0X^0)}{N(X^0X^0)} = \frac{(\Delta m)^2+\frac{1}{4}(\Delta\Gamma)^2}{2\Gamma^2+(\Delta m)^2-\frac{1}{4}(\Delta\Gamma)^2} = r. \tag{15.54}$$

15.3 CP–Violation in the Standard Model

Here we discuss how CP–violation can arise in the standard model of electroweak [SMEW] interaction. Three generations are known to exist:

$$\begin{pmatrix} u \\ d \end{pmatrix},\begin{pmatrix} c \\ s \end{pmatrix},\begin{pmatrix} t \\ b \end{pmatrix}\ \begin{matrix} i \\ q \end{matrix}$$

Since the mass eigenstates are not identical with weak eigenstates, the hadronic charged current can be written as [see Sec. 13.10].

$$J_\mu^W(H) = \sum_{\substack{i=u,c,t\\ q=d,s,b}} V_{iq}\bar{i}_L\,\gamma_\mu\,q_L \tag{15.55}$$

where V is the CKM matrix. This gives the charged current interaction Lagrangian

$$\begin{aligned} L_{cc} &= -\frac{g}{\sqrt{2}}\{W^\mu\sum_{iq} V_{iq}\bar{i}_L\gamma_\mu q_L \\ &\quad +W^{\mu\dagger}\sum_{iq} V_{iq}^*\bar{q}_L\gamma_\mu i_L\}. \end{aligned} \tag{15.56}$$

Now [see Sec. A.8]

$$\begin{aligned} &(CP)[W^\mu\,V_{iq}\,\bar{i}_L\,\gamma_\mu\,q_L](CP)^{-1} \\ =\ &W^{\mu\dagger}\,V_{iq}\,\bar{q}_L\gamma_\mu i_L\eta(W)\,\eta(q)\,\eta^*(i). \end{aligned} \tag{15.57}$$

Noting that η the phase $\eta(W)\ \eta(q)\ \eta^*(i)$ can be chosen to be $+1$, Eq. (56) gives

$$\begin{aligned} & CPL_{cc}(CP)^{-1} \\ = & -\frac{g}{\sqrt{2}}\left\{W^{\mu\dagger}\sum_{iq}V_{iq}\bar{q}_L\gamma_\mu i_L + W^\mu\sum_{\substack{i\\q}}V^*_{iq}\bar{i}_L\gamma_\mu q_L\right\}. \end{aligned} \tag{15.58}$$

This is identical with (56) except that

$$V_{iq} \rightarrow V^*_{iq}. \tag{15.59}$$

On the other hand

$$(CP)L_{nc}(CP)^{-1} = L_{nc} \tag{15.60}$$

where L_{nc} is the neutral current interaction Lagrangian, involving only diagonal couplings,

$$\begin{aligned} L_{nc} = & -\frac{g}{\cos\theta_W}Z^\mu\sum_{\substack{q=u,c,t\\d,s,b}}\left\{\left[I_3(q) - Q(q)\sin^2\theta_W\right]\bar{q}_L\gamma_\mu q_L\right. \\ & \left. -Q(q)\sin^2\theta_W\ \bar{q}_R\gamma_\mu q_R\right\}. \end{aligned} \tag{15.61}$$

Thus the neutral current in the interaction Lagrangian is necessarily CP invariant. On the other hand from Eqs. (56) and (57), it is clear that

$$(CP)L_{cc}(CP)^{-1} = L_{cc} \tag{15.62}$$

if and only if V is real $[V_{iq} = V^*_{iq}]$ or can be made real. Thus SMEW is capable of CP–violation.

Suppose we have N generations so that V is an $N\times N$ matrix and as such has N^2complex elements or $2N^2$ real parameters. But since V has to be unitary, it has N^2 real parameters. Then there is freedom to define any quark field by a phase, for example, $d \rightarrow e^{i\theta}d$, $W^\mu V_{id}\bar{i}_L\gamma_\mu d_L \rightarrow W^\mu V_{id}\bar{i}_L\gamma_\mu(e^{i\theta}d_L) = W^\mu(V_{id}e^{i\theta})(\bar{i}_L\gamma_\mu d_L)$ and $e^{i\theta}$ can be absorbed in the redefinition of V_{id} without changing the physics. Thus phase of any individual CKM matrix has no physical meaning [what counts is the relative phase]. Hence the number of phases which have no physical meaning [remember there are $2N$ fields] are $(2N-1)$. Therefore, number of independent parameters in $V_{N\times N}$ are

$$N^2-(2N-1)=(N-1)^2$$

$$\begin{array}{ll} 0 & N=1 \\ 1 & N=2 \\ 4 & N=3 \end{array} \tag{15.63}$$

One way of choosing the parameters is mixing angles and complex phases. Now if $V_{N\times N}$ were orthogonal matrix, then the number of independent parameters would be $N^2-N-\frac{N(N-1)}{2}=\frac{N(N-1)}{2}$, which give the number of mixing angles. Then the number of phases are

$$(N-1)^2-\frac{N(N-1)}{2}=\frac{(N-1)(N-2)}{2}$$

$$= \begin{array}{ll} 0 & N=1 \\ 0 & N=2 \\ 1 & N=3 \end{array}$$

Thus the SMEW interaction is capable of CP violation provided that V is at least 3×3 i.e. three mixing angles and one phase. In other words CP violation can be accommodated if number of generations is at least three. This observation was made before the third generation was discovered.

15.4 $B^0\bar{B}^0$ Mixing and CP–Violation

We now apply the general formalism to $B^0\bar{B}^0$ system. We can write

$$M_{12}-\frac{i}{2}\Gamma_{12}=\left\langle \bar{B}^0|H_{\text{eff}}^{\Delta B=2}|B^0\right\rangle . \tag{15.64}$$

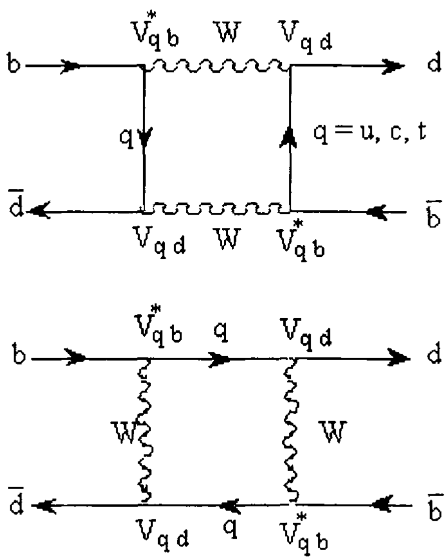

Figure 1 Box diagrams for $|\Delta B| = 2$ transition.

The $H_{\text{eff}}^{\Delta B=2}$ induces particle - antiparticle mixing involving the neutral B mesons, $\bar{B}_d^0 \to B_d^0$ and $\bar{B}_s^0 \to \bar{B}_s^0$. We shall deal with both of these transitions. The $H_{\text{eff}}^{\Delta B=2}$ for $\bar{B}_d^0 \to B_d^0$ transition can be extracted from the box diagrams of Fig.1 [for $\bar{B}_s^0 \to B_s^0$, change $d \to s$] as in the standard model $\Delta B = 2$ transitions arise from these diagrams. We note from

$$\Gamma_{12} = 2\pi \sum_f \left\langle B^0|H_W|f\right\rangle \left\langle f|H_W|\bar{B}^0\right\rangle \delta(E_f - m_B) \qquad (15.65)$$

that only B^0 and $\bar{B}^0$ decays contribute to Γ_{12}. The common final states for the B_d^0 and $\bar{B}_d^0$ decays are shown in Fig.2 while those for B_s^0 and $\bar{B}_s^0$ can be obtained by changing d to s.

We will not write $H_{\text{eff}}^{\Delta B=2}$ explicitly. Some of the main conclusions can be deduced from the general features of the diagrams:

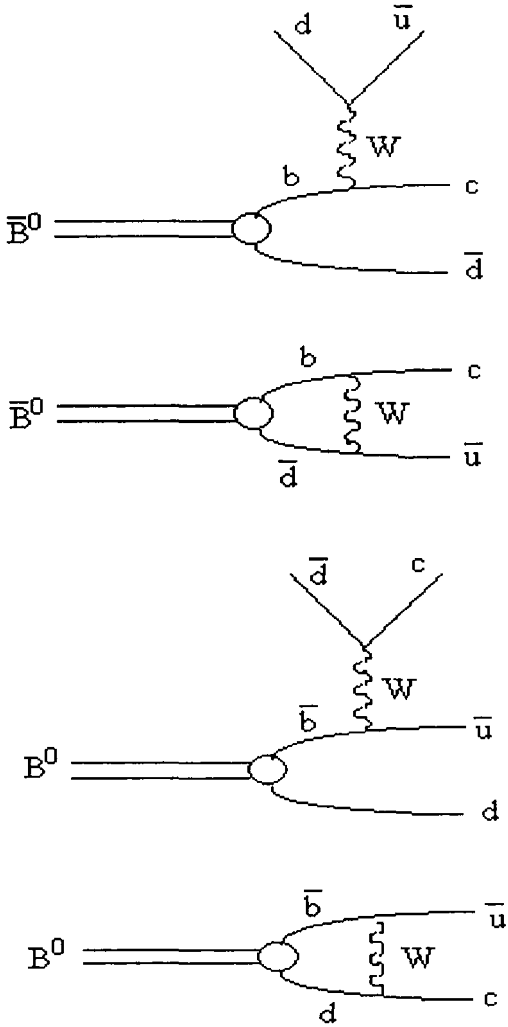

Figure 2 Diagrams showing the common final states for the B_d^0 and B_d^0 decays.

i) From Figs. 1 and 2, we note that M_{12} and Γ_{12} depend on $V_{qb}\, V^*_{qd}$ or $V_{qb}\, V^*_{qs}$ $(q = t, c, u)$. These parameters are given by the matrix elements of Cabibbo-Kobayashi-Maskawa (CKM) matrix parameterized in the Maiani-Wolfenstein way:

$$
\begin{aligned}
V &= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\
&\approx \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}
\end{aligned}
\tag{15.66}
$$

where $\lambda \approx \sin\theta_c \approx 0.22$ and $|A| = 0.90 \pm 0.18$ is determined from semileptonic B–decays; $\eta \neq 0$ if CP is not conserved. The unitarity of V gives

$$V^*_{ud}\, V_{ub} + V^*_{cb}\, V_{cd} + V^*_{td}\, V_{tb} = 0. \tag{15.67a}$$

To leading order in λ, this relation can be written, using Eq. (66), as

$$V^*_{ub} + V_{td} - \lambda\, V_{cb} = 0. \tag{15.67b}$$

The relation (67) can be represented by a triangle in the complex plane (Fig.3). In particular we note that

$$\sin\, 2\beta = \frac{2\eta\,(1-\rho)}{\eta^2 + (1-\rho)^2}\,. \tag{15.68}$$

ii) If the quarks, t, c, u have nearly the same mass, sum of their contributions would involve $(\sum_{q=u,c,t} V_{qd}\, V^*_{qb})^2$ which vanishes due to the unitarity condition (67). Since $m_t >> m_c$, it is clear that dominant contribution comes from exchanged t–quark.

iii) In the standard model all the complex phases enter through CKM matrix (see Eq. (66)). In particular

$$
\begin{aligned}
\arg\left[A(B_d \to \bar{B}_d)\right] &= \arg\left[(V_{td}\, V^*_{tb})^2\right] \\
&= \arg\left[\frac{V_{td}}{V^*_{td}}\frac{V^*_{tb}}{V_{tb}}\right]
\end{aligned}
$$

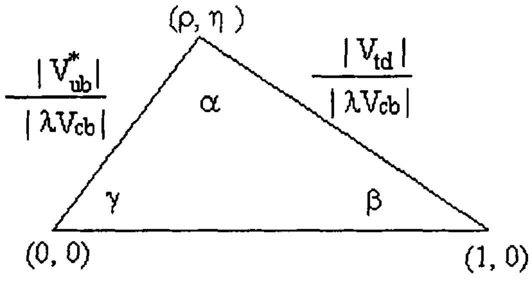

Figure 3 The CKM-unitarity triangle in the Wolfenstein parameterization.

which implies

$$\frac{V_{td}}{V_{td}^*}\frac{V_{tb}^*}{V_{tb}} = e^{-2i\delta_{CKM}^m} \tag{15.69a}$$

where

$$\delta_{\mathrm{CKM}}^m = arg(V_{td}^* V_{tb}) = \beta. \tag{15.69b}$$

From the above considerations, using Eqs. (66) and (67), one finds the main results from the box diagrams:

$B_d^0\bar{B}_d^0$ System:

$$M_{12} \propto (V_{tb}V_{td}^*)^2\, m_t^2 = [A\lambda^3\, (1-\rho\, + i\eta)]^2\, m_t^2 \tag{15.70a}$$

while [cf. Eqs. (65) and (67a)]

$$\begin{aligned}\Gamma_{12} &\propto (V_{cb}\, V_{cd}^* + V_{ub}\, V_{ud}^*)^2\, m_b^2 \\ &= (V_{tb}\, V_{td}^*)^2\, m_b^2 = [A\lambda^3\, (1-\rho\, + i\eta)]^2\, m_b^2. \end{aligned} \tag{15.70b}$$

Hence $|M_{12}| >> |\Gamma_{12}|$ and both M_{12} and Γ_{12} have the same phase in the leading order. Thus it follows from Eq. (69):

$$M_{12} = |M_{12}|e^{2i\beta},\ \Gamma_{12} = |\Gamma_{12}|e^{2i\beta}, \tag{15.71a}$$

and

$$\frac{|\Gamma_{12}|}{|M_{12}|} \propto \frac{m_b^2}{m_t^2}. \tag{15.71b}$$

$B_s^0 \bar{B}_s^0$ **Systems:**

$$\begin{aligned} M_{12} &\propto (V_{tb}\, V_{ts}^*)^2\, m_t^2 = A^2 \lambda^4 m_t^2, \\ \Gamma_{12} &\propto (V_{cb}\, V_{cs}^* + V_{ub}\, V_{us}^*)^2\, m_b^2 \approx A^2 \lambda^4 m_b^2. \end{aligned} \tag{15.72}$$

Hence we have

$$\begin{aligned} |M_{12}| &\gg |\Gamma_{12}| \\ \text{Im}\ M_{12} &= 0 \\ \text{Im}\ \Gamma_{12} &= 0 \end{aligned} \tag{15.73}$$

We also note that

$$\frac{|M_{12}|_{B_s}}{|M_{12}|_{B_d}} \approx \frac{1}{\lambda^2[(1-\rho)^2 + \eta^2]}. \tag{15.74}$$

Using Eqs. (24) and (71), we get for the $B_d^0 - \bar{B}_d^0$ system (noting that M_{12} and Γ_{12} have the same phase).

$$\begin{aligned} pq &= [(M_{12} - \frac{i}{2}\Gamma_{12})\ (M_{12}^* - \frac{i}{2}\Gamma_{12}^*)]^{1/2} \\ &= |M_{12}| - \frac{i}{2}|\Gamma_{12}|. \end{aligned} \tag{15.75}$$

From Eqs. (28) and (75), we then obtain

$$\begin{aligned} \Delta m_B &= 2|M_{12}| \\ \Delta\Gamma &= 2|\Gamma_{12}|. \end{aligned} \tag{15.76}$$

Hence [cf. Eq. (73)]

$$\frac{\Delta\Gamma}{\Delta m_B} \ll 1. \tag{15.77}$$

Now $|\Delta m_B| \sim 3.7 \times 10^{-10}$ MeV, and $\Gamma \sim 5.9 \times 10^{-10}$ MeV, hence we conclude that $\Delta m_B \sim \Gamma$ and

$$\frac{\Delta\Gamma}{\Gamma} \ll 1. \tag{15.78}$$

For the $B_s^0 - \bar{B}_s^0$ system we see from Eqs. (24), (28), (72) and (73) that Eq. (77) also holds for this system. There is general consensus that the same is also true for the inequality (78). So the mixing effects in these cases can be interpreted in terms of $\Delta m_B/\Gamma$.

From Eqs. (29) and (71), we have for the $B_d^0 - \bar{B}_d^0$ system

$$\frac{q}{p} = e^{-2i\beta}, \tag{15.79}$$

so that we get

$$\left|\frac{q}{p}\right| = 1,\ \text{Im}\left(\frac{q}{p}\right) = -\sin 2\beta \tag{15.80a}$$

and hence

$$\text{Re}\ \varepsilon\ = 0,\ \text{Im}\ \varepsilon\ =\ \tan\beta. \tag{15.80b}$$

Since sin β is expected to be small, we have

$$\text{Re}\,\varepsilon\ = 0,\ \text{Im}\ \varepsilon\ \approx\ \sin\ \beta. \tag{15.80c}$$

Similarly for the $B_s^0 - \bar{B}_s^0$ system $q/p \approx 1$ and $\varepsilon \approx 0$ [cf. Eqs. (29) and (73)]. Thus ε is expected to be small in the B system. In fact the theoretical estimates give

$$\begin{aligned} \varepsilon\ &=\ O(10^{-3}) \text{ for } B_d \\ &=\ O(10^{-4}) \text{ for } B_s. \end{aligned} \tag{15.81}$$

Thus it seems that the prospects for observation of $\Delta B = 2$, CP violating phenomena are essentially hopeless. However, in the B–system, CP–violation in B decays ($\Delta B = 1$) can be large, unlike in the kaon system [see Sec. 5].

We now discuss the CP–violation and the mixing in B^0 decay. First we consider the case in which the final states f and

$\bar{f}$ are not eigenstates of CP, $f \neq \bar{f}$. Let us define the decay amplitudes

$$\begin{aligned} A(f) &= \left\langle f|H_W \,|B^0\right\rangle \\ \overline{A}(f) &= \left\langle f|H_W \,|\bar{B}^0\right\rangle \end{aligned} \tag{15.82}$$

and the ratios

$$\begin{aligned} \frac{\left\langle f|H_W \,|\bar{B}^0\right\rangle}{\langle f|H_W \,|B^0\rangle} &= \overline{x}_f \\ \frac{\left\langle \bar{f} \,|H_W \,|B^0\right\rangle}{\left\langle \bar{f} \,|H_W \,|\bar{B}^0\right\rangle} &= x_{\bar{f}}. \end{aligned} \tag{15.83}$$

Then from the decay $B^0(t) \to f$ and its CP mirror $\bar{B}^0\ (t) \to \bar{f}$,we have from Eqs. (36b) and (37b), using Eqs. (28) and (78)

$$\begin{aligned} \Gamma_f(t) &= \Gamma\left(B^0(t) \ \to \ f\right) \ \propto |\langle f|H_w|\Psi(t)\rangle|^2 \\ &= e^{-\Gamma t} \left| \ \cos \ \frac{\Delta m}{2} tA(f) - i \ \frac{q}{p} \ \sin \ \frac{\Delta m}{2} t\bar{A}(f)\right|^2 \end{aligned} \tag{15.84a}$$

$$\overline{\Gamma}_{\bar{f}}(t) = e^{-\Gamma t} \left|\cos \ \frac{\Delta m}{2} t \ \bar{A}(\bar{f}) - i\frac{p}{q} \sin \frac{\Delta m}{2} tA(\bar{f})\right|^2 . \tag{15.84b}$$

On using Eqs. (82) and (83), we obtain

$$\begin{aligned} \Gamma_f(t)| \ &\propto \ |A(f)|^2 \frac{1}{2} \ e^{-\Gamma t} \left[(1 + |\bar{x}_f|^2) + (1 \ - |\bar{x}_f|^2) \cos \ \Delta mt \right. \\ &\left. \quad +2 \operatorname{Im}\left(q\bar{x}_{f/}p\right) \sin \Delta mt\right] \end{aligned} \tag{15.85a}$$

$$\begin{aligned} \overline{\Gamma}_{\bar{f}}(t)| \ &\propto \ |\bar{A}(\bar{f})|^2 \frac{1}{2} e^{-\Gamma t} \left\{(1 + |x_{\bar{f}}|^2) + (1 - |x_{\bar{f}}|^2) \cos \ \Delta mt \right. \\ &\left. \quad +2 \operatorname{Im}\left(p \ x_{\bar{f}}/q\right) \sin \ \Delta mt\right\}. \end{aligned} \tag{15.85b}$$

Now from Eq. (16) with $X = B$, $\mathfrak{m} = H_W$, CPT invariance gives

$$\begin{aligned} A(f) &\equiv \left\langle f|H_W|B^0\right\rangle = \eta_f\, e^{2i\delta_s} \left\langle \bar{B}^0\, |H_W\, |\bar{f}\right\rangle \\ &= \eta_f\, e^{2i\delta_s} \left\langle \bar{f}\, |H_W\, |\bar{B}^0\right\rangle^* \\ &= \eta_f\, e^{2i\delta_s}\, \bar{A}^*(\bar{f}). \end{aligned} \tag{15.86a}$$

Similarly

$$A(\bar{f}) = \eta_f\, e^{2i\bar{\delta}_s}\, \bar{A}^*(f). \tag{15.86b}$$

Thus from Eq. (83)

$$\begin{aligned} \overline{x}_f &= \frac{\bar{A}(f)}{A(f)} = \frac{e^{2i\bar{\delta}_s}\, A^*(\bar{f})}{e^{2i\delta_s}\, \bar{A}^*(\bar{f})} \\ &= e^{-2i(\delta_s - \bar{\delta}_s)}\, x^*_{\bar{f}}. \end{aligned} \tag{15.87}$$

Hence we can write

$$\begin{aligned} \overline{x}_f &= |\overline{x}_f| e^{-i(\delta_s - \bar{\delta}_s)}\, e^{-2i\phi_f} \\ x_{\bar{f}} &= |\overline{x}_f| e^{-i(\delta_s - \bar{\delta}_s)} e^{2i\phi_f} \end{aligned} \tag{15.88}$$

where ϕ_f is the weak phase of the decay transition. Then Eqs. (85), on using Eqs. (79), (86) and (88), give

$$\begin{aligned} \Gamma_f(t) &\propto |A(f)|^2\, \frac{1}{2}\, e^{-\Gamma t}\, \{(1 + |\overline{x}_f|^2) + (1 - |\overline{x}_f|^2) \cos\, \Delta m t \\ &\quad -2|\overline{x}_f| \sin\, \Delta m t \sin(2\phi + \phi_s)\} \\ \overline{\Gamma}_{\bar{f}}(t) &\propto |A(f)|^2\, \frac{1}{2}\, e^{-\Gamma t}\, \{(1 + |\overline{x}_f|^2) + (1 - |\overline{x}_f|^2) \cos\, \Delta m t \\ &\quad -2|\overline{x}_f| \sin\, \Delta m t \sin(-2\phi\, +\, \phi_s)\} \end{aligned} \tag{15.89}$$

where

$$2\phi = 2\beta + 2\phi_f,\ \phi_s = \delta_s - \overline{\delta}_s. \tag{15.90}$$

We have introduced four parameters $|A(f)|, |\overline{x}_f|$, ϕ and ϕ_s. Can these parameters be determined from the decay rates (89). The

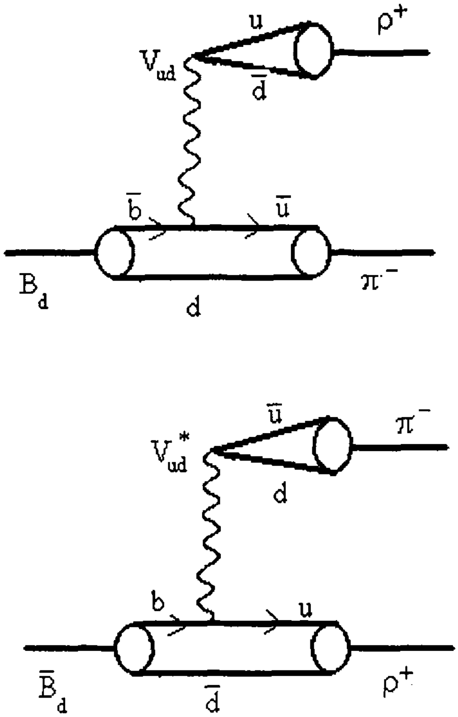

Figure 4 Quark level diagrams for $B \to \rho\pi$ decays.

answer is positive since the decay rates are not numbers but are functions of time.

As an illustration of the above considerations, consider the decays $B_d(t) \to \rho^+\pi^-(f)$ and $\bar{B}_d(t) \to \rho^-\pi^+$ $(\bar{f})$. The quark level diagrams for these decays are shown in Fig. 4. Thus

$$e^{i\delta^f_{CKM}} = \frac{V_{ud}}{|V_{ud}|}\frac{V^*_{ub}}{|V_{ub}|}, \; e^{i\bar{\delta}^f_{CKM}} = \frac{V^*_{ud}}{|V_{ud}|}\frac{V_{ub}}{|V_{ub}|} \tag{15.91}$$

so that $2\phi_f = \delta^f_{\text{CKM}} + \bar{\delta}^f_{\text{CKM}}$ and using Eq. (69b), we obtain

$$e^{2i\phi} = \left(\frac{V_{ud}}{|V_{ud}|}\frac{V_{ub}^*}{|V_{ub}|}\right)^2 \frac{V_{td}^*}{V_{td}}\frac{V_{tb}}{V_{tb}^*} = \frac{V_{ud}V_{ub}^*V_{tb}V_{td}^*}{V_{ud}^*V_{ub}V_{tb}^*V_{td}} \tag{15.92}$$

so that

$$\begin{aligned} \phi &= \arg[V_{ud}V_{ub}^*V_{tb}V_{td}^*] \\ &= \arg\left[\frac{V_{ud}}{V_{td}}\frac{V_{ub}^*}{V_{tb}^*}\right] \end{aligned} \tag{15.93}$$

or using Eq. (66) and Fig. 3

$$\phi \simeq \arg\left(\frac{V_{ub}^*}{V_{td}}\right) = \alpha. \tag{15.94}$$

We now apply the above formalisms when the final state $|f\rangle$ in B^0 decay is an eigenstate of CP

$$\left|\bar{f}\right\rangle = CP\,|f\rangle = \eta_{CP}^f\,|f\rangle\,,\ \eta_{CP}^f = \pm 1. \tag{15.95}$$

Then it follows from Eq. (83) that $x_{\bar{f}} = 1/\bar{x}_f$, so that Eq. (88) gives

$$|\bar{x}_f| = 1,\ \phi_s \equiv \delta_s - \bar{\delta}_s = 0, \text{ or } \pi \tag{15.96}$$

accordingly as $\eta_{CP}^f = \pm 1$ [cf. Eqs. (86)]. Then it follows from Eqs. (89)

$$\begin{aligned} \Gamma_f(t) &\propto |A(f)|^2\, e^{-\Gamma t}\,\{1 - \eta_{CP}^f\,\sin 2\phi\,\sin\,\Delta m t\} \\ \overline{\Gamma}_f(t) &\propto |A(f)|^2\, e^{-\Gamma t}\,\{1 + \eta_{CP}^f\,\sin 2\phi\,\sin\,\Delta m t\}. \end{aligned} \tag{15.97}$$

Then the asymmetry is given by

$$A_f(t) = \frac{\Gamma_f(t) - \overline{\Gamma}_f(t)}{\Gamma_f(t) + \overline{\Gamma}_f(t)} = -\eta_{CP}^f\,\sin\,(\Delta m t)\sin 2\phi. \tag{15.98}$$

As η_{CP}^f $(\pm)$ is known and Δm is already measured, this asymmetry measures ϕ, independent of strong phases. The time integrated CP asymmetry is

$$A_f = \frac{\int_0^\infty \left[\Gamma_f(t) - \overline{\Gamma}_f(t)\right] dt}{\int_0^\infty \left[\Gamma_f(t) + \overline{\Gamma}_f(t)\right] dt} \tag{15.99a}$$

which on using Eqs. (97) gives

$$\begin{aligned} A_f &= -\eta_{CP}^f \, \Gamma \sin 2\phi \, \frac{\Delta m}{\Gamma^2 + \Delta m^2} \\ &= -\eta_{CP}^f \, \sin 2\phi \, \frac{x}{1+x^2}. \end{aligned} \tag{15.99b}$$

For B^0 $(\bar{B}^0) \rightarrow \psi \, K_s$, $\eta_{CP}^f = \eta_{CP}\,(\psi)\, \eta_{CP}\,(K_s) = (+1)(+1) = 1$ and in the standard model from the transition $b \rightarrow c\bar{c}s$, it is easy to see that $\phi_f = 0$ since the CKM elements involved are $V_{cb}^* \, V_{cs}$ which according to Eq. (66) do not involve any CKM phase. Thus according to Eq. (90)

$$\phi = \beta. \tag{15.100}$$

Thus from Eq. (98) the asymmetry is given directly in terms of CKM phase β (independent of strong phase).

$$A_{\psi Ks}\,(t) = -\sin 2\beta \, \sin \, \Delta m t \tag{15.101}$$

and the time integrated asymmetry is

$$A_{\psi Ks} = -\sin 2\beta \, \frac{x}{1+x^2}. \tag{15.102}$$

If $\beta \simeq 10^0$ and $x_d \simeq 0.7$ (see below), then the asymmetry $|A_{\psi Ks}| \simeq 0.16$, which is much larger than $\varepsilon \simeq 10^{-3}$ in kaon decays (see Sec. 5).

Finally we consider the decays [for example $X^- = D^-, X^+ = D^+$]

$$\begin{aligned} B^0 &\rightarrow \ell^+\nu\, X^- : f \\ \bar{B}^0 &\rightarrow \ell^-\bar{\nu}\, X^+ \; : \bar{f}. \end{aligned}$$

In the standard model $\bar{B}^0$ decay into $\ell^+\nu\ X^-$ and B^0 decay into $\ell^-\bar{\nu}\ X^+$ are forbidden. Then $\bar{A}(f) = 0 = A(\bar{f})$ i.e. $\bar{x}_f = 0 = x_{\bar{f}}$ so that from Eq. (85).

$$\begin{aligned} \Gamma\left(B^0(t) \rightarrow f\right) &\propto \frac{1}{2}\,|A(f)|^2 e^{-\Gamma t}\,[1 + \cos\ \Delta m t] \\ &= \Gamma\left(\bar{B}^0(t) \rightarrow \bar{f}\right). \end{aligned} \tag{15.103}$$

On the other hand from Eq. (84a) [replacing f by $\bar{f}$]

$$\begin{aligned} \Gamma\left(B^0(t) \rightarrow \bar{f}\right) &\propto |A(f)|^2 e^{-\Gamma t}\,|-i\frac{q}{p}\ \sin\ \frac{\Delta m}{2}\ t|^2 \\ &= \frac{1}{2}|A(f)|^2\ e^{-\Gamma t}\ [1 - \cos\ \Delta m t] \end{aligned} \tag{15.104}$$

where we have used $|\bar{A}(\bar{f})| = |A(f)|$. Integrating Eqs. (103) and (104)

$$\begin{aligned} \delta &\equiv \frac{\int_0^\infty \Gamma(B^0(t) \rightarrow \bar{f}\ )\ dt}{\int_0^\infty \Gamma(B^0(t) \rightarrow f) dt} \\ &= \frac{\Delta m_B^2}{2\Gamma^2 + \Delta m_B^2} \\ &= \frac{x^2}{2 + x^2} = r \quad \text{[cf. Eq. (40)]}. \end{aligned} \tag{15.105}$$

Hence a nonzero value of δ would indicate B^0 and $\bar{B}^0$ mixing as in the standard model $x_{\bar{f}} = 0 = \bar{x}_f = 0$. But if $x_{\bar{f}}$ and $\bar{x}_f$ are not zero due to some exotic mechanism, then $\delta \neq 0$, even if there is no mixing.

The particle data group give for the $B_d^0 - \bar{B}_d^0$ system the value

$$\begin{aligned} \chi_d &= \frac{r_d}{1+r_d} \\ &= \frac{\Gamma(\mu^+ X^-)}{\Gamma(\mu^+ X^-) + \Gamma(\mu^- X^+)} \\ &= 0.172 \pm 0.010 \end{aligned} \tag{15.106a}$$

and

$$x_d = 0.723 \pm 0.032. \tag{15.106b}$$

This gives a clear proof of the $B_d^0 - \bar{B}_d^0$ mixing in the standard model. However, in contrast to $K^0 - \bar{K}^0$case (see the next section) δ does not give any information about CP–violation.

The value x_d provides a determination of V_{td} in terms of m_t and $f_B\sqrt{B_B}$ ($\equiv \xi_B$) which parameterizes the hadronic matrix elements of the four quark operator [cf. Fig. 1] ($\bar{d}_{\rm La}\ \gamma^\mu\ b_{\rm La}$) ($\bar{d}_{\rm Lc}$ $\gamma_\mu\ b_{\rm Lc}$) between $\bar{B}^0$ and B^0. This can be seen as follows: First we note that the dominant contribution to M_{12} comes from the top quark in Fig.1 and it has been shown that [cf. Eqs. (76), (77) and (105)]

$$\begin{aligned} x_d &= \frac{\Delta m_{B_d}}{\Gamma_{B_d}} = 2|M_{12}|_{B_d}\ \tau_{B_d} \\ &= \frac{G_F^2 m_{B_d}}{6\pi^2} m_W^2 (\xi_{B_d}^2)\eta_B \tau_{B_d} F\left(\frac{m_t^2}{m_W^2}\right) |V_{td}^* V_{tb}|^2, \end{aligned} \tag{15.107a}$$

where

$$F(x) = \frac{1}{4} + \frac{9}{4(1-x)} - \frac{3}{2}\frac{1}{(1-x)^2} - \frac{3}{2}\frac{x^2 \ln x}{(1-x)} \tag{15.107b}$$

where η_B is a QCD correction factor. The constant ξ_{B_d} is model dependent and is estimated to be in the range (0.20 ± 0.04) GeV.

The measured value of x_d thus provides a determination of V_{td} in terms of m_t and ξ_{B_d} , yielding

$$|V_{td}| = (9.0 \pm 2.6) \times 10^{-3} \tag{15.108}$$

which lies within the standard model unitarity constraint $V_{td} < 0.013$ [cf. Particle Data Group].

For $B_s^0 - \bar{B}_s^0$ mixing parameter

$$x_s = \frac{\Delta m_{B_s}}{\Gamma_{B_s}} = 2|M_{12}|\ \tau_{B_s}, \tag{15.109a}$$

we have [using the analog of Eq. (107a) for x_s]

$$\begin{aligned} \frac{x_s}{x_d} &= \frac{\eta_{B_s}}{\eta_{B_d}} \left(\frac{\xi_{B_s}}{\xi_{B_d}}\right)^2 \frac{m_{B_s}}{m_{B_d}} \frac{\tau_{B_s}}{\tau_{B_d}} \left|\frac{V_{ts}}{V_{td}}\right|^2 \\ &= \frac{\eta_{B_s}}{\eta_{B_s}} \frac{m_{B_s}}{m_{B_d}} \left(\frac{\xi_{B_s}}{\xi_{B_d}}\right)^2 \frac{\tau_{B_s}}{\tau_{B_d}} \frac{1}{\lambda^2[(1-\rho)^2 + \ \eta^2]}. \end{aligned} \tag{15.109b}$$

First we note that since $x_s/x_d \sim 1/\lambda^2 = 1/\sin^2\theta_c \approx 21$, x_s is expected to be large. The ratio (109b) is a useful quantity since it leaves the square of the ratio of CKM matrix elements, multiplied by a factor which reflects flavor SU(3) breaking effects which we lump into a parameter ξ_s^2. The particle data group gives the lower limit for x_s, $x_s > 14$, which together with x_d given in Eq. (106b) yields $x_s/x_d > 19.4$. This bound has been used to restrict the allowed $\rho - \eta$ region for some representative values of ξ_s^2. This results in the range

$$\begin{aligned} 0.2 &< \eta < 0.4 \\ 0 &< \rho < 0.4 \end{aligned} \tag{15.110}$$

with the best solution around $\rho = 0.11$, $\eta = 0.33$.

Coming back to the ratio (x_s/x_d) in Eq. (109b), we see that most of the models give $(\xi_{B_s}/\xi_{B_d})^2 = 1.2 - 2$. Thus taking $\tau_{B_s} \approx \tau_{B_d}$, $m_{B_s} \approx m_{B_d}$, $(\xi_{B_s}/\xi_{B_d}) \approx 1$ and constraint (110), we can safely conclude that $x_s >> 2$ so that Eq. (105) gives $r_s = 1$ and hence from Eq. (45), $\chi_s = 0.5$. Any marked deviation from $\chi_s = 0.5$ would indicate some new physics beyond the standard model. The particle data group gives $\chi_s > 0.4975$, consistent with the above standard model value.

We conclude this section with the remarks that there is clear experimental evidence for $B^0 - \bar{B}^0$ mixing. The experimental determination of asymmetry parameter $A_{\Psi_{K_s}}$[cf. Eq. (102)] can give us information about the angle β.

15.5 CP–Violation in $K^0\bar{K}^0$ System

We now apply the general formalism developed in Sec. 15.2 to the $K^0\bar{K}^0$ system. Here we denote K_1 and K_2 as K_S and K_L. First we discuss hypercharge oscillations. Suppose that at $t = 0, K^0$ $(Y = 1)$ is produced by the reaction $\pi^- p \to K^0\Lambda^0$. The initial state is then pure $Y = 1$. It is clear from Eq. (36b) [with $X = K$] that a kaon beam which has been produced in a pure $Y = 1$ state has changed into one containing both parts with $Y = 1$ and $Y = -1$. Experimentally $\bar{K}^0$ can be verified through the observation of hadronic signature such as $\bar{K}^0\, p \to \pi^+\, \Lambda^0$ since $\pi^+\, \Lambda^0$ can only be produced by $\bar{K}^0$ and not by K^0. The probability of finding $Y = -1$ component at time t in the kaon produced at $t = 0$ in a pure $Y = 1$ state is given by Eq. (38) $[|\varepsilon| << 1]$.

$$P(K^0 \to \bar{K}^0, t) \simeq \frac{1}{4}\left\{\exp\left(-\frac{t}{\tau_S}\right) + \exp\left(-\frac{t}{\tau_L}\right) - 2\exp\left[-\frac{1}{2}\left(\frac{t}{\tau_S} + \frac{t}{\tau_L}\right)\right]\cos\Delta m t\right\}. \quad (15.111)$$

If kaons were stable $(\tau_S, \tau_L \to \infty)$, then

$$P(K^0 \to \bar{K}^0, t) = \frac{1}{2}[1 - \cos\Delta m t]$$

which shows that a state produced as pure $Y = 1$ state at $t = 0$ continuously oscillates between $Y = 1$ and $Y = -1$ states with circular frequency $\omega = \Delta m/h$. Kaons, however, decay and oscillations are damped. By measuring the period of oscillation given by $2\pi/(\Delta m/h)$, one can determine Δm.

We now discuss CP–violation in K^0 and $\bar{K}^0$ system. From CKM matrix [cf. Eq. 66], we have

$$\begin{aligned} V_{ts}V_{td}^* &= -A^2\lambda^5(1-\rho+i\eta) \\ V_{cs}V_{cd}^* &= \left(1-\frac{1}{2}\lambda^2\right)\left[-\lambda+A^2\lambda^5\left(1-\rho+i\eta\right)\right] \\ V_{us}V_{ud}^* &= \lambda\left(1-\frac{1}{2}\lambda^2\right) \end{aligned} \quad (15.112)$$

where we have used $V_{cd} = -\lambda + A^2\lambda^5\ (1-\rho+i\eta)$ as given by the unitarity of CKM matrix. It is clear that top quark contribution to $\mathrm{Re}\,M_{12}$ as compared with that of the c quark is of the order of $\frac{\lambda^{10}m_t^2}{\lambda^2 m_c^2} \approx 5\times\ 10^{-6}\ \frac{m_t^2}{m_c^2}$ and hence is negligible for $m_t \approx 175\ GeV$. Thus since M_{12} and $\Gamma_{12} \propto (V_{cs}V_{cd}^*)^2$ we conclude from Eq. (112):

$$\begin{aligned} \mathrm{Im}\ \Gamma_{12} &\ll \mathrm{Re}\ \Gamma_{12} \\ \mathrm{Im}\,M_{12} &\ll \mathrm{Re}\,M_{12}. \end{aligned} \quad (15.113)$$

Now from Eq. (24), we have

$$\left.\begin{matrix} p \\ q \end{matrix}\right\} = (\mathrm{Re}\,M_{12} - \frac{i}{2}\ \mathrm{Re}\,\Gamma_{12})^{1/2}\left[1\ \pm\ \frac{i\,\mathrm{Im}\,M_{12} + \mathrm{Im}\ \Gamma_{12}}{\mathrm{Re}\,M_{12} - \frac{i}{2}\,\mathrm{Re}\,\Gamma_{12}}\right]^{1/2} \quad (15.114a)$$

so that

$$\frac{2\varepsilon}{1+\varepsilon^2} = \frac{p^2-q^2}{p^2+q^2} = \frac{i\,\mathrm{Im}\,M_{12} +\ \frac{1}{2}\,\mathrm{Im}\ \Gamma_{12}}{\mathrm{Re}\,M_{12} - \frac{i}{2}\,\mathrm{Re}\,\Gamma_{12}}. \quad (15.114b)$$

Thus we get from Eq. (28), using the approximations (113)

$$\begin{aligned} \Delta m &\equiv m_L - m_S = 2\,\mathrm{Re}\,pq = 2\,\mathrm{Re}\,M_{12} \\ \Delta\Gamma &= \Gamma_L - \Gamma_S\ = -4\,\mathrm{Im}\,pq\ = 2\,\mathrm{Re}\ \Gamma_{12}. \end{aligned} \quad (15.115)$$

Hence we get

$$\varepsilon \approx \frac{i\,\mathrm{Im}\,M_{12} + \frac{1}{2}\mathrm{Im}\ \Gamma_{12}}{\Delta m - \frac{i}{2}\,\Delta\Gamma}. \tag{15.116}$$

The parameter ε determines $CP-$violation in $K^0 - \bar{K}^0$ mixing. However, CP violation can also occur in the decay amplitudes $K^0 \to 2\pi$ and $\bar{K}^0 \to 2\pi$. Now two pions in the final state can either be in $I = 0$ or $I = 2$ state. The dominant decay amplitude is for $I = 0$ due to $\Delta I = \frac{1}{2}$ rule, $|A_2/A_0| \simeq \frac{1}{22}$. We define the decay amplitudes:

$$\begin{aligned} \langle 2\pi_{\text{out}}, I &= 0|H|K^0\rangle = A_0\, e^{i\delta_0} \\ \langle 2\pi_{\text{out}}, I &= 2|H|K^0\rangle = A_2\, e^{i\delta_2}. \end{aligned} \tag{15.117}$$

Here δ_0 and δ_2 are the phase shifts for $I = 0$ and $I = 2$ $\pi\pi$ scattering. We take the $S-$matrix for these states $S = e^{2i\delta_0}$ and $e^{2i\delta_2}$.

Now CPT invariance viz. Eq. (86a) [with B^0 replaced by K^0 and phase η_f chosen to be -1] gives

$$\begin{aligned} \langle 2\pi, I &= 0|H|\bar{K}^0\rangle = -A_0^*\, e^{i\delta_0} \\ \langle 2\pi, I &= 2|H|\bar{K}^0\rangle = -A_2^*\, e^{i\delta_2}. \end{aligned} \tag{15.118}$$

Using the Clebsch-Gordon (CG) coefficients, we have

$$\begin{aligned} A(K^0 &\to \pi^+\pi^-) = \frac{1}{\sqrt{3}} e^{i\delta_0}\left[\sqrt{2}\,A_0 + FA_2\right] \\ A(\bar{K}^0 &\to \pi^+\pi^-) = -\frac{1}{\sqrt{3}}\, e^{i\delta_0}\left[\sqrt{2}A_0^* + FA_2^*\right] \\ A(K^0 &\to \pi^0\pi^0) = \frac{1}{\sqrt{3}}\, e^{i\delta_0}\left[A_0 - \sqrt{2}\,FA_2\right] \\ A(\bar{K}^0 &\to \pi^0\pi^0) = -\frac{1}{\sqrt{3}}\, e^{i\delta_0}\left[A_0^* - \sqrt{2}FA_2^*\right], \end{aligned} \tag{15.119}$$

where $F = e^{i(\delta_2-\delta_0)}$. Ignoring the irrelevant overall phase factor $e^{i\delta_0}$, we have from Eqs.(25), (29) and (119)

$$\begin{aligned}\eta_{+-} &\equiv \frac{A(K_L \to \pi^+\pi^-)}{A(K_S \to \pi^+\pi^-)} \\ &= \frac{\varepsilon + i\,\frac{\operatorname{Im} A_0}{\operatorname{Re} A_0} + i\,\frac{F}{\sqrt{2}}\,\frac{\operatorname{Im} A_2}{\operatorname{Re} A_0} + \varepsilon\,\frac{F}{\sqrt{2}}\,\frac{\operatorname{Re} A_2}{\operatorname{Re} A_0}}{1 + \frac{F}{\sqrt{2}}\,\frac{\operatorname{Re} A_2}{\operatorname{Re} A_0} + i\varepsilon\left(\frac{\operatorname{Im} A_0}{\operatorname{Re} A_0} + \frac{F}{\sqrt{2}}\,\frac{\operatorname{Im} A_2}{\operatorname{Re} A_0}\right)} \\ &\simeq \tilde{\varepsilon} + \varepsilon' \end{aligned} \tag{15.120}$$

$$\begin{aligned}\eta_{00} &\equiv \frac{A(K_L \to \pi^0\pi^0)}{A(K_S \to \pi^0\pi^0)} \\ &\simeq \tilde{\varepsilon} - 2\varepsilon' \end{aligned} \tag{15.121}$$

where we have neglected corrections of order $\varepsilon\,\frac{\operatorname{Re} A_2}{\operatorname{Re} A_0}$, $\varepsilon\,\frac{\operatorname{Im} A_0}{\operatorname{Re} A_0}$ and $\varepsilon\,\frac{\operatorname{Im} A_2}{\operatorname{Re} A_0}$ (in fact the last two are completely negligible, much smaller than the first one) and we have defined

$$\tilde{\varepsilon} = \varepsilon + i\,\frac{\operatorname{Im} A_0}{\operatorname{Re} A_0} \tag{15.122a}$$

$$\varepsilon' = \frac{i}{\sqrt{2}}\, e^{i(\delta_2-\delta_0)} \left(\frac{\operatorname{Re} A_2}{\operatorname{Re} A_0}\right) \left[\frac{\operatorname{Im} A_2}{\operatorname{Re} A_2} - \frac{\operatorname{Im} A_0}{\operatorname{Re} A_0}\right]. \tag{15.122b}$$

The quantities $\operatorname{Im} A_0$, $\operatorname{Im} A_2$, and $\operatorname{Im}\,\tilde{\varepsilon}$ depend on the choice of phase convention. It is possible by a choice of phase convention to set $\operatorname{Im} A_0 = 0$, known as Wu and Yang phase convention, in which case $\tilde{\varepsilon} = \varepsilon$. Note, however, the value of ε' is independent of phase convention. Its nonzero value would demonstrate direct CP–violation. We now adopt Wu–Yang convention so that

$$\tilde{\varepsilon} = \varepsilon \tag{15.123a}$$

and

$$\varepsilon' = \frac{i}{\sqrt{2}} e^{i(\delta_2-\delta_0)} \frac{\operatorname{Im} A_2}{\operatorname{Re} A_0}. \tag{15.123b}$$

If we retain only the dominant two-pion contribution to the unitarity relation (65) for Γ_{12}, we get on using Eq. (119)

$$\begin{aligned}\frac{\Gamma_{12}}{\Gamma_{11}} &\simeq -\frac{1}{3}\Big[(\sqrt{2}\,A_0 + FA_2)(\sqrt{2}A_0 + F^* A_2) \\ &\quad + (A_0 - \sqrt{2}\,FA_2)(A_0 - \sqrt{2}\,F^* A_2)\Big] / \\ &\quad \frac{1}{3}(2|A_0|^2 + |A_2|^2).\end{aligned} \tag{15.124}$$

This gives

$$\mathrm{Im}\,\frac{\Gamma_{12}}{\Gamma_{11}} = -3\left(\mathrm{Re}\,\frac{A_2}{A_0}\right)\frac{\mathrm{Im}\left(\frac{A_2}{A_0}\right)}{1+\frac{1}{2}\frac{|A_2|^2}{A_0^2}} \tag{15.125}$$

where $\Gamma_{11} = \frac{\Gamma_L + \Gamma_S}{2} \simeq \Gamma_S/2 \simeq -\frac{\Delta\Gamma}{2}$, since $\Gamma_S >> \Gamma_L$ ($\Gamma_S/\Gamma_L \simeq 580$). Hence

$$\left|\mathrm{Im}\,\frac{\Gamma_{12}}{\Delta\Gamma}\right| \approx \frac{3}{\sqrt{2}}|\varepsilon'|\left|\mathrm{Re}\,\frac{A_2}{A_0}\right|$$

which is completely negligible. Hence we get from Eq. (116)

$$\tan\,\theta_\epsilon \approx -\frac{2\Delta m}{\Delta\Gamma} \tag{15.126}$$

which is clear from its derivation is a consequence of CPT invariance and the unitarity relation (65) approximated by dominant two-pion contribution. Putting the experimental values

$$\begin{aligned}\Delta m &= (m_L - m_S) \\ &= 0.474\,\Gamma_S \\ \Delta\Gamma &= \Gamma_L - \Gamma_S = -0.998\,\Gamma_S\end{aligned} \tag{15.127}$$

we obtain the phase of ε

$$\phi_\varepsilon = 43.49 \pm 0.08^0 \tag{15.128}$$

while Eq. (123b) gives

$$\phi_{\varepsilon'} = \delta_2 - \delta_0 + \frac{\pi}{2} \simeq 48 \pm 4^0 \tag{15.129}$$

where the numerical value is based on an analysis of $\pi\ \pi$ scattering. Finally writing Eqs. (120) and (121) as

$$\begin{aligned} \eta_{+-} &\equiv |\eta_{+-}|\, e^{i\phi_{+-}} \approx (\varepsilon + \varepsilon') \\ \eta_{00} &\equiv |\eta_{00}| e^{i\phi_{00}} \approx (\varepsilon - 2\varepsilon') \end{aligned} \tag{15.130}$$

the experimental measurements give:

$$\begin{aligned} |\eta_{+-}| &= (2.285 \pm 0.019) \times 10^{-3} \\ \phi_{+-} &= (43.5 \pm 0.6)^0 \end{aligned} \tag{15.131}$$

$$\begin{aligned} |\eta_{00}| &= (2.275 \pm 0.019) \times 10^{-3} \\ \phi_{00} &= (43.5 \pm 1.0)^0 \end{aligned} \tag{15.132}$$

$$\Delta\phi = \phi_{00} - \phi_{+-} = (-0.1 \pm 0.8)^0. \tag{15.133}$$

Since ε' involves the $\Delta I = \frac{1}{2}$ rule suppression factor $|A_2/A_0| \approx \frac{1}{22}$. [cf. Eq. (123b)] one has $|\varepsilon'| << |\varepsilon|$. Then from Eq. (130)

$$\begin{aligned} R &= |\eta_{00}/\eta_{+-}|^2 = |\frac{\varepsilon - 2\varepsilon'}{\varepsilon + \varepsilon'}|^2 \\ &\approx 1 - 6\,\mathrm{Re}\left(\frac{\varepsilon'}{\varepsilon}\right). \end{aligned} \tag{15.134}$$

The measurement of R provides a test for $|\Delta S| = 1$, CP violation. Recall here that $q/p = (1-\varepsilon)\,/\,(1+\ \varepsilon)$, $K_L \sim K_2 + \varepsilon\ K_1$ [cf. Eqs. (2) and (25)]. Thus if $\varepsilon' = 0$, $CP-$violation arises entirely through the mass matrix i.e. through $|\Delta S| = 2$ transitions $K^0 \longleftrightarrow \bar{K}^0$ allowed by second order weak process. This is accommodated in superweak theory.

In case $\mathrm{Re}(\varepsilon'/\varepsilon)$ is not zero, CP violation must occur in a decay amplitude [specifically viz. Im $A_2 \neq 0$ cf. Eq. (123b)] as well as through the mass matrix. The present experimental value for ε'/ε is

$$\frac{\varepsilon'}{\varepsilon} = (1.5 \pm 0.8) \times 10^{-3}. \tag{15.135}$$

Finally the phases ϕ_{+-} and ϕ_{00} can be used to test CPT symmetry. From Eqs. (130), we find

$$\Delta\phi \simeq 3\,\mathrm{Re}\left(\frac{\varepsilon'}{\varepsilon}\right)\tan(\phi_\varepsilon - \phi_{\varepsilon'}). \tag{15.136}$$

Using Eqs. (128), (129) and (135), one can limit the magnitude of the right hand side (which has negative sign) to be under 0.06^0 showing the experimental value for the phase $\Delta\phi$ in Eq. (133) to be consistent with CPT, although further accuracy will be desired.

Finally what are theoretical expectations for the ratio ε'/ε. First we note that in the standard model, the tree level diagrams shown in Fig.5 involve CKM elements $\lambda_u = V_{ud}^* V_{us}$ and λ_u^* so that $A(K_L \to \pi^+\pi^-)$ involves $(\lambda_u - \lambda_u^*) = \mathrm{Im}\ \lambda_u = 0$ [cf. Eq. (66)]. Thus $(\varepsilon'/\varepsilon)$ arises from the ratio of the so-called "penguin" diagram shown in Fig. 6 to the box diagram shown in Fig.1 with b replaced by s. The $CP-$violation is determined by $\sum_i \mathrm{Im}\ \lambda_i = \mathrm{Im}\ \lambda_t$, [cf. Eq. (66)] where $\lambda_i = V_{id}\ V_{is}^*$, $i = u, c, t$. This involves t quark which belongs to the third generation and which has very small mixing with the first and the second generation. This also explains why $CP-$violation is so small in kaon decays. The theoretical prediction for Re $(\varepsilon'/\varepsilon)$ is not precise [for $m_t > m_W$] but most theoretical calculations give

$$\mathrm{Re}\left(\frac{\varepsilon'}{\varepsilon}\right) < 3\times\ 10^{-3} \tag{15.137}$$

since this ratio depends on various parameters which are not as yet well determined. This is consistent with its experimental value given in Eq. (135).

Finally we discuss the CP asymmetry in leptonic decays of kaon. Let us define the decay amplitudes: $(l = e, \mu)$

$$\begin{array}{ll} K^0 \to \pi^- + l^+ + \nu : f & \frac{\Delta S}{\Delta Q} = 1 \\ \bar{K}^0 \to \pi^- + l^+ + \nu : g & \frac{\Delta S}{\Delta Q} = -1. \end{array} \tag{15.138}$$

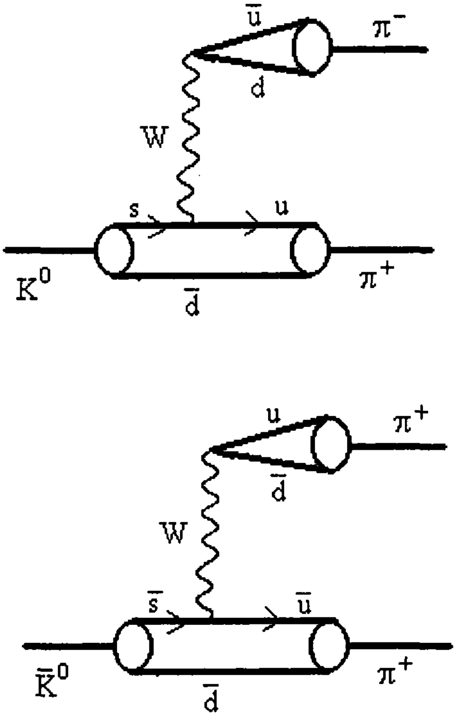

Figure 5 Tree level diagrams for $K \to 2\pi$ decays.

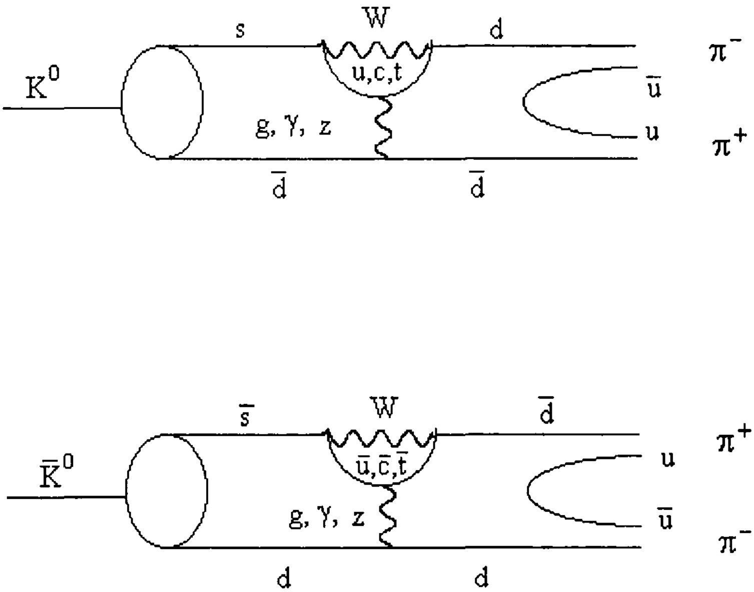

Figure 6 Penguin diagrams for $K \to 2\pi$ decays.

Then CPT invariance gives

$$\begin{aligned} \bar{K}^0 &\rightarrow \pi^+ + l^- + \bar{\nu} : f^* \\ K^0 &\rightarrow \pi^+ + l^- + \bar{\nu} : g^*. \end{aligned} \tag{15.139}$$

Hence from Eq. (25), we get

$$\begin{aligned} A\left(K_L^0 \rightarrow \pi^- + l^+ + \nu\right) &= \frac{pf + qg}{\sqrt{|p|^2 + |q|^2}} \\ A\left(K_L^0 \rightarrow \pi^+ + l^- + \bar{\nu}\right) &= \frac{pg^* + qf^*}{\sqrt{|p|^2 + |q|^2}}. \end{aligned} \tag{15.140}$$

Thus the $CP-$asymmetry parameter δ_l can be written as

$$\begin{aligned} \delta_\ell &\equiv \frac{\Gamma(K_L^0 \rightarrow \pi^- l^+ \nu) - \Gamma(K_L^0 \rightarrow \pi^+ l^- \bar{\nu})}{\Gamma(K_L^0 \rightarrow \pi^- l^+ \nu) + \Gamma(K_L^0 \rightarrow \pi^+ l^- \bar{\nu})} \\ &= \frac{[|p|^2 - |q|^2]\,[|f|^2 - |g|^2]}{[|p|^2 + |q|^2]\,[|f|^2 + |g|^2] + 2(pq^* f g^* + p^* q f^* g)}. \end{aligned} \tag{15.141}$$

Now using Eq. (29), we get

$$\delta_l \approx 2\,\mathrm{Re}\,\varepsilon \,\frac{1 - |x_l|^2}{1 + |x_l|^2 + 2\,\mathrm{Im}\,\varepsilon\,\mathrm{Im}\,x_l} \tag{15.142}$$

where we have put

$$x_l = \frac{g}{f}\,. \tag{15.143}$$

In the standard model, $x_l = 0$. Hence we get

$$\delta_l \approx 2\,\mathrm{Re}\ \varepsilon\ . \tag{15.144}$$

The experimental average for this quantity is

$$\delta_l = (3.27 \pm 0.12) \times 10^{-3}. \tag{15.145}$$

Using $|\varepsilon| \approx |\eta_{+-}|$ and the experimental values of η_{+-} amd ϕ_ε given respectively in Eqs. (131) and (128), we get $2\,\mathrm{Re}\,\varepsilon = (3.32 \pm 0.03) \times 10^{-3} = \delta_l$ in excellent agreement with Eq. (145).

We conclude this section by noting that from Eq. (40), the mixing parameter r for the kaon:

$$r_K = \frac{x_K^2 + y_K^2}{2 + x_K^2 - y_K^2} \approx 1 \qquad (15.146)$$

since for kaon

$$\begin{aligned} x_K &= \frac{\Delta m}{\Gamma} \approx \frac{2\Delta m}{\Gamma_s} = 1 \\ y_K &= \frac{\Delta\Gamma}{2\Gamma} \approx -\frac{\Gamma_S}{\Gamma_S} = -1. \end{aligned} \qquad (15.147)$$

Thus mixing parameter r_K has the maximum value viz. unity to be compared with $r_d = 0.20 \pm 0.01$ [cf. Eq. (106)].

15.6 CP–Violation in Hyperon Non–Leptonic Decays

Because of the uncertainities in ε', it does not as yet test direct CP–violation in the standard model. There is thus a need to study CP violation outside the kaon system, we have already studied CP violation in B decays. We now consider the same in hyperon decays: $\Lambda \to p\pi^-$ and $\Xi^- \to \Lambda\pi^-$. These are described by the amplitude [cf. Sec. 11.2.3] $a_s + a_p\, \boldsymbol{\sigma} \cdot \mathbf{n}, \mathbf{n} = \mathbf{p}'/|\mathbf{p}'|$, $\mathbf{p}'$ being the momentum of the final baryon. The observables are decay rate Γ, asymmetry parameter α, the tranverse polarization of final baryon β and the longitudinal polarization of final baryon γ defined in Sec. 11.2.3. Let us now construct CP–odd observables i.e. compare $B \to B'\pi^-$ with $\bar{B} \to \bar{B}'\pi^+$. CP symmetry predicts

$$\overline{\Gamma} = \Gamma, \overline{\alpha} = -\alpha, \bar{\beta} = -\beta. \qquad (15.148)$$

Thus to leading order, CP-odd observables are

$$\begin{aligned} \delta\Gamma &= \frac{\Gamma - \overline{\Gamma}}{\Gamma + \overline{\Gamma}} \\ \delta\alpha &= \frac{\alpha + \overline{\alpha}}{\alpha - \overline{\alpha}} \\ \delta\beta &= \frac{\beta + \overline{\beta}}{\beta - \overline{\beta}}. \end{aligned} \qquad (15.149)$$

Such asymmetries can be measured in the proposed Super Lear in

$$p\overline{p} \rightarrow \Lambda\overline{\Lambda} \rightarrow p_f\pi^- \, \overline{p}_f\pi^+ \tag{15.150}$$

where one studies the asymmetry

$$\bar{A} = \frac{N_p^+ - N_p^- + N_{\overline{p}}^+ - N_{\overline{p}}^-}{N_{\text{total}}} = \mathcal{P}_\Lambda \alpha_\Lambda \delta\alpha_\Lambda . \tag{15.151}$$

Here $N_p^\pm$ is the number of protons with $(\mathbf{p}_i \times \mathbf{p}_\Lambda) \cdot \mathbf{p}_f$ greater than or less than zero. $\mathcal{P}_\Lambda$ denotes the polarization of Λ. Similarly in the reaction

$$\begin{aligned} p\overline{p} &\rightarrow \Xi\bar{\Xi} \\ &\rightarrow \Lambda\pi^- \, \bar{\Lambda}\pi^+ \\ &\rightarrow p_f \, \pi^-\pi^- \, \overline{p}_f \, \pi^+\pi^+ \end{aligned} \tag{15.152}$$

the relevant asymmetry is

$$\begin{aligned} \bar{B} &= \frac{\bar{N}_p^+ - \bar{N}_p^- + \bar{N}_{\overline{p}}^+ - \bar{N}_{\overline{p}}^-}{N_{\text{total}}} \\ &= \frac{\pi}{8} \mathcal{P}_\Xi \alpha_\Lambda \beta_\Xi (\delta\alpha_\Lambda + \delta\beta_\Xi) \end{aligned} \tag{15.153}$$

where $\bar{N}_{p^\pm}$ denotes number of events with $\mathcal{P}_\Xi \cdot (\mathbf{p}_f \times \mathbf{p}_\Lambda)$ greater than or less than zero.

We now discuss the isospin analysis. First we note that assuming CPT only, Eq. (86) gives

$$\begin{aligned} a_\ell(I) &\equiv \left\langle f_{\ell J}^{out} | H_W | B \right\rangle = \eta_f \, e^{2i\delta_\ell(I)} \left\langle \bar{f}_{\ell I}^{out} \, | H_W | \bar{B} \right\rangle^* \\ &= \eta_f \, e^{2i\delta_\ell(I)} \overline{a}_\ell^* \, (I) \end{aligned} \tag{15.154}$$

where $\delta_\ell \, (I)$ are strong phases. Selecting the phase η_f as $(-1)^{\ell+1}$, Eq. (154) gives

$$\overline{a}_\ell(I) = (-1)^{\ell+1} e^{2i\delta_\ell(I)} \, \overline{a}_\ell^* \, (I). \tag{15.155}$$

Denoting by $\phi_\ell(I)$ CP odd phases, we define

$$\begin{aligned} a_s &= \sum_I e^{i(\delta_s^I+\phi_s^I)} S_I \\ a_p &= \sum_I e^{i(\delta_p^I+\phi_p^I)} P_I . \end{aligned} \tag{15.156}$$

Then from Eq. (155)

$$\begin{aligned} \bar{a}_s &= -\sum_I e^{i(\delta_s^I-\phi_s^I)} S_I \\ \bar{a}_p &= \sum_I e^{i(\delta_p^I-\phi_p^I)} P_I . \end{aligned} \tag{15.157}$$

To leading order, we obtain

$$\begin{aligned} \delta\Gamma &= \sqrt{2}\,\frac{S_{33}}{S_{11}} \sin\ (\delta_3^s - \delta_1^s)\ \sin\ (\phi_3^s - \phi_1^s) \\ &= 0, \text{ for } \Xi, \end{aligned}$$

since there is only one isospin final state in Ξ decay and neglecting $\frac{S_{33}}{S_{11}}$ and $\frac{P_{33}}{P_{11}}$, which are very small ($\sim \frac{1}{25}$) if $\Delta I = \frac{1}{2}$ rule dominates, we get

$$\begin{aligned} \delta\alpha &= -\tan\ (\delta_1^p - \delta_1^s)\ \sin(\phi_1^p - \phi_1^s) \\ \delta\beta &= \cot\ (\delta_1^p - \delta_1^s)\sin(\ \phi_1^p - \phi_1^s)\,. \end{aligned}$$

Thus in order to get non-vanishing $\delta\Gamma$, $\delta\alpha$ and $\delta\beta$, the following conditions must be satisfied: (i) the amplitudes must have CP violating phases (ii) there must be final state phases (iii) there must be two or more decay channels (iv) the CP phases and final state phases must be different in different channels. The expectations in the standard model are

	$\delta\Gamma$	$\delta\alpha$	$\delta\beta$
$\Lambda \to p\pi^-$	10^{-6}	10^{-4}	3×10^{-3}
$\Xi \to \Lambda\pi^-$	0	10^{-4}	10^{-3}

These estimates have considerable uncertainty and are model dependent. The above observables can also be used to study the effects of extensions of the standard model.

15.7 Problems

1) Show that if CP is conserved, then

$$K_2^0 \rightarrow \pi^+\pi^-\pi^0$$

is allowed for pions in $I = 1, 3$ states with $\ell = L =$ even and

$$K_1^0 \rightarrow \pi^+\pi^-\pi^0$$

is allowed for pions in $I = 0, 2$ states with $\ell = L =$ odd, where L is the relative orbital angular momentum of $\pi^+\pi^-$ system and ℓ is that of π^0 relative to the center of mass of π^+ and π^-.
Show that

$$K_2^0 \rightarrow \pi^0\pi^0\pi^0$$

is forbidden but

$$K_1^0 \rightarrow \pi^0\pi^0\pi^0$$

is allowed.
2) From the unitary triangle, show that

$$\begin{aligned}
\sin 2\alpha &= \frac{2\eta\left(\eta^2+\rho^2-\rho\right)}{\left(\eta^2+\rho^2\right)\left[(1-\rho)^2+\eta^2\right]} \\
\sin 2\beta &= \frac{2\eta\,(1-\rho)}{(1-\rho)^2+\eta^2} \\
\sin 2\gamma &= \frac{2\rho\eta}{\eta^2+\rho^2}.
\end{aligned}$$

3) Consider the decays $B^\pm \rightarrow \pi^0 K^\pm$. Draw the tree level and penguin diagrams for these decays. Denoting the contributions from these diagrams respectively by

$$A(f)e^{i\delta^f_{CKM}}\, e^{i\delta_s},\ A'(f)e^{i\delta'^f_{CKM}}\, e^{i\delta'_s}$$

where $f = \pi^0 K^+$ and $\delta_{\rm CKM}$ is the phase of some product of CKM elements and δ_s are strong phases. Using CPT invariance, write the corresponding contributions for $B^- \to \bar{f}$, $\bar{f} \equiv \pi^0 K^-$. Show that CP violating asymmetry,

$$\frac{\Gamma(B^+ \to f) - \Gamma(B^- \to \bar{f})}{\Gamma(B^+ \to f) + \Gamma(B^- \to \bar{f})} \propto \sin\phi \sin\phi_s$$

where $\phi = (\delta^f_{\rm CKM} - \delta'^f_{\rm CKM})$ and $\phi_s \equiv (\delta_s - \delta'_s)$. Note that if ϕ_s is absent, CP is not violated; in fact both CKM and strong phases are needed.

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Chapter 16
WEAK DECAYS OF HEAVY FLAVORS

In the standard model, three generations of matter replicate themselves with increasing mass scale. We have already discussed the first and second generation leptons (e, ν_e), (μ, ν_μ). In this chapter, we first discuss the weak decays of τ lepton (the third generation lepton). Later we study the heavy flavors viz. decays of D and B mesons.

The study of heavy flavors provides us an opportunity to discover any deviation from the standard model. However, we will find that the standard model works quite well for heavy flavors. We begin with τ-decays. The mass of τ lepton is $1777.00\ {}^{+0.30}_{-0.27}$ MeV and its mean life is $(291.0 \pm 1.5) \times 10^{-15}$ s. The upper limit on the mass of ν_τ is 24 MeV.

16.1 Leptonic Decays of τ Lepton

In the standard model, the third generation leptons (τ, ν_τ) behave exactly in the same manner as (μ, ν_μ). Because τ lepton mass is 1777 MeV, τ can decay into light mesons (π's and K's). As far as the decay $\tau^- \to \nu_\tau + e^- + \bar{\nu}_e$ is concerned, in the standard model it should have exactly the same structure as that for the decay $\mu^- \to \nu_\mu + e^- + \overline{\nu}_e$. Now e^- and ν_e are common in both these decays. The e^- and $\bar{\nu}_e$ enter in the effective Lagrangian for muon decay in the form $\overline{e}\gamma_\mu(1-\gamma_5)\nu_e$, it should occur in this form in the effective Lagrangian for τ-decay. Hence the most general form for the T-matrix is given by:

$$\begin{aligned} T &= \frac{-G_F}{\sqrt{2}} \frac{1}{(2\pi)^6} \left(\frac{m_\tau m_e m_{\nu\tau} m_{\nu e}}{p_{10} p_{20} k_{10} k_{20}} \right)^{1/2} \\ &\quad \times \left[\overline{u}(\mathbf{p}_2) \gamma^\lambda (1 - \varepsilon \gamma^5) u(\mathbf{p}_1) \right] \left[\overline{u}(\mathbf{k}_1) \gamma_\lambda (1 - \gamma_5) v(\mathbf{k}_2) \right] \end{aligned} \quad (16.1)$$

where p_1, p_2, k_1, and k_2, are four momenta of τ, ν_τ, e and $\overline{\nu}_e$ respectively. In the standard model, $\varepsilon = +1$. Thus any deviation from the standard model should manifests itself with a value of ε different from 1.

Using the standard techniques, we can easily calculate the electron energy spectrum

$$\frac{d\Gamma}{dx} \approx \frac{G_F^2 m_\tau^5}{384\pi^3} x^2 \left[(1+\varepsilon)^2 (3-2x) + 6(1-\varepsilon)^2 (1-x) \right]. \quad (16.2)$$

In deriving the above expression, we have taken neutrinos to be massless and have put $m_e/m_\tau \approx 0$ and $x = 2E_e/m_\tau$. It is convenient to put Eq. (2) in the form

$$\frac{1}{\Gamma} \frac{d\Gamma}{dx} \approx 4x^2 \left[3(1-x) + 2\rho(\frac{4}{3}x - 1) \right], \quad (16.3)$$

where

$$\Gamma \approx \frac{G_F^2 \, m_\tau^5}{192\pi^3} \frac{(1+\varepsilon^2)}{2} \quad (16.4)$$

and

$$\rho = \frac{3}{8} \frac{(1+\varepsilon)^2}{1+\varepsilon^2}. \quad (16.5)$$

Equation (4) gives the decay rate for the decay $\tau^- \to e^- + \nu_\tau + \overline{\nu}_e$. Equation (5) gives the Michel parameter ρ. In the standard model [V - A theory] $\varepsilon = +1, \rho = \frac{3}{4}$. The experimental value for ρ is 0.742 ± 0.027 in agreement with the theoretical value of 0.75. This reinforces our assumption that (τ, ν_τ) are sequential leptons.

Using $\varepsilon = 1$, we get from Eq. (4)

$$\frac{\Gamma(\tau \to \nu_\tau + e + \overline{\nu}_e)}{\Gamma(\mu \to \nu_\mu + e + \overline{\nu}_e)} = \frac{m_\tau^5}{m_\mu^5} \frac{(1 + \delta_{rad}^\tau)}{(1 + \delta_{rad}^\mu)}. \qquad (16.6)$$

Since $(1 + \delta_{rad}^\tau)/(1 + \delta_{rad}^\mu) \approx 1$, one can write for the branching ratio

$$BR(\tau \to \nu_\tau + e + \overline{\nu}_e) = \left(\frac{\tau_\tau}{\tau_\mu}\right)\left(\frac{m_\tau}{m_\mu}\right)^5 BR(\mu \to \nu_\mu + e + \overline{\nu}_e). \qquad (16.7a)$$

Using the experimental values, we get

$$BR(\tau \to \nu_\tau + e + \overline{\nu}_e) = (17.82 \pm 0.09)\% \qquad (16.7b)$$

to be compared with the experimental average $(17.83 \pm 0.08)\%$.

If we neglect m_μ/m_τ, we get for the decay $\tau \to \nu_\tau + e + \overline{\nu}_\mu$, the same expressions as in Eqs. (2), (3), (4) and (6). However, taking into account the finite value of m_μ/m_τ we get

$$\Gamma(\tau \to \nu_\tau + \mu + \overline{\nu}_\mu) = \frac{G_F^2\, m_\tau^5}{192\pi^3} K\left(\frac{m_\mu^2}{m_\tau^2}\right), \qquad (16.8)$$

where

$$K(y) = 1 - 8y + 8y^3 - y^4 - 12y^2 \ln y. \qquad (16.9)$$

Using the experimental values of m_μ and m_τ, we get from Eqs. (8) and (4)

$$\begin{aligned} BR(\tau \to \nu_\tau + \mu + \overline{\nu}_\mu) &\approx (0.9726 \pm 0.0001) BR(\tau \to \nu_\tau + e + \overline{\nu}_e) \\ &= (17.33 \pm 0.09)\% \qquad (16.10) \end{aligned}$$

to be compared with the experimental average $(17.35 \pm 0.1)\%$. Thus we see that $e - \mu - \tau$ universality is satisfied to an excellent degree of accuracy.

16.2 Semi-Hadronic Decays of τ Lepton

We consider a general decay

$$\tau(k) \to X(p_X) + \nu_\tau(k'),$$

where X is any number of hadrons allowed by energy conservation. The T-matrix is given by

$$T = -\frac{G'}{\sqrt{2}} \left\langle 0 \left| J_\mu^W \right| X \right\rangle \; \bar{u}(\mathbf{k}')\gamma^\mu(1-\gamma_5)\; u(\mathbf{k}) \frac{1}{(2\pi)^3}\sqrt{\frac{m_\tau m_{\nu_\tau}}{k_0 k_0'}}. \tag{16.11}$$

The decay rate is given by

$$\Gamma = \frac{(2\pi)^3}{(2\pi)^5}\frac{m_{\nu_\tau}}{2} \int d^3p_X \int \frac{d^3k'}{k_0'\, p_{X_0}} \left|F\right|^2 \delta(p_X - k - k') \tag{16.12}$$

where

$$\left|F\right|^2 = \frac{G'^2}{\sqrt{2}}(2\pi)^3(2p_{X_0}) \left\langle 0 \left| J_\mu^W \right| X \right\rangle \left\langle X \left| J_\lambda^{W\dagger} \right| 0 \right\rangle L^{\mu\lambda}. \tag{16.13}$$

Note that G' is the effective decay constant, $L_{\mu\lambda}$ is the leptonic part given by

$$L_{\mu\lambda} = \frac{2}{m_\tau m_{\nu_\tau}} \frac{1}{2}\left[k'_\mu k_\lambda + k'_\lambda k_\mu - g_{\mu\lambda}\, k'.k - i\varepsilon_{\mu\lambda\rho\sigma}\, k^\rho\, k'^\sigma\right]. \tag{16.14}$$

The weak current $J_\mu^W = V_\mu^W - A_\mu^W$. Since the interference term $V^\mu A_\mu$ does not contribute, we can separately consider the vector and axial vector parts. Using the Lorentz invariance and CVC (the spin over final hadrons is summed), we can write quite generally ($q = k - k'$):

$$\begin{aligned} &(2\pi)^3 \int \left\langle 0 \left| V_\mu^W \right| X \right\rangle \left\langle X \left| V_\lambda^{W\dagger} \right| 0 \right\rangle \, d^3p_X \delta\left(p_X - q\right) \\ = \;& \theta\left(q_0\right)\left(-q^2 g_{\mu\lambda} + q_\mu q_\lambda\right)\rho_V\left(q^2\right). \end{aligned} \tag{16.15}$$

Similarly we can write

$$(2\pi)^3 \int \left\langle 0 \left| A_\mu^W \right| X \right\rangle \left\langle X \left| A_\lambda^{W\dagger} \right| 0 \right\rangle d^3p_X \delta \left(p_X - q\right)$$
$$= \theta \left(q_0\right) \left[\left(-q^2 g_{\mu\lambda} + q_\mu q_\lambda\right) \rho_A \left(q^2\right) + q_\mu q_\lambda \sigma_A \left(q^2\right)\right]. \quad (16.16)$$

In writing Eq. (16), we have not used the conservation of axial vector current. The form factor $\sigma_A(q^2)$ arises due to non-conservation of A_μ. From Eqs. (12), (13), (15) and (16), we get

$$\Gamma_V = \frac{G'^2}{2} \frac{m_\tau^3}{8\pi} \int^{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \rho_V(s) ds, \quad (16.17)$$

$$\Gamma_A = \frac{G'^2}{2} \frac{m_\tau^3}{8\pi} \int^{m_\tau^2} \left[\left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \rho_A(s) + \left(1 - \frac{s}{m_\tau^2}\right)^2 \sigma_A(s)\right] ds \quad (16.18)$$

where

$$s = q^2 = (k + k')^2. \quad (16.19)$$

Special Cases:

1. $$\tau^- \to \pi^- + \nu_\tau$$

Here

$$\rho_A(s) = \rho_V(s) = 0, \qquad \sigma_A(s) = f_\pi^2 \, \delta(s - m_\pi^2), \quad (16.20)$$

$G'^2 = G_F^2 \cos^2\theta_c$. f_π is the pion decay constant and is defined by the matrix element

$$\left\langle 0 \left| A_\mu^W \right| \pi \right\rangle = i f_\pi \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_0}} \, q_\mu. \quad (16.21)$$

Hence from Eq. (18), we get

$$\Gamma_\pi = \frac{G_F^2}{16\pi} \cos^2\theta_c \, f_\pi^2 \, m_\tau^3 \left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2. \quad (16.22)$$

2. $$\tau^- \to \rho^- + \nu_\tau$$

Here

$$\rho_\nu(s) = f_\rho^2 \delta(s - m_\rho^2), \tag{16.23}$$

where f_ρ is defined by

$$\left\langle 0 \left| V_\mu^W \right| \rho^- \right\rangle = f_\rho \frac{m_\rho}{(2\pi)^{3/2}} \frac{\varepsilon_\mu}{\sqrt{2q_0}}. \tag{16.24}$$

Hence from Eq. (17), we get

$$\Gamma_\rho = \frac{G_F^2}{16\pi} \cos^2\theta_c \, f_\rho^2 \, m_\tau^3 \left(1 - \frac{m_\rho^2}{m_\tau^2}\right)^2 \left(1 + \frac{2m_\rho^2}{m_\tau^2}\right). \tag{16.25}$$

3. $$\tau^- \to a_1^- + \nu_\tau$$

Here

$$\rho_A(s) = f_{a_1}^2 \delta(s - m_{a_1}^2). \tag{16.26}$$

Hence from Eq. (18), we get

$$\Gamma_{a_1} = \frac{G_F^2 \cos^2\theta_c}{16\pi} f_{a_1}^2 m_\tau^3 \left(1 - \frac{m_{a_1}^2}{m_\tau^2}\right)^2 \left(1 + \frac{2m_{a_1}^2}{m_\tau^2}\right). \tag{16.27}$$

Let us compare these results with their experimental values. From Eqs. (22), (25) and (27), we have

$$\frac{\Gamma_\pi}{\Gamma_e} = (12\pi^2)\left(\frac{f_\pi^2}{m_\tau^2}\right)\cos^2\theta_c \left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2. \tag{16.28}$$

$$\frac{\Gamma_\rho}{\Gamma_e} = (12\pi^2)\left(\frac{f_\rho^2}{m_\tau^2}\right)\cos^2\theta_c \left(1 - \frac{m_\rho^2}{m_\tau^2}\right)^2 \left(1 + 2\frac{m_\rho^2}{m_\tau^2}\right). \tag{16.29}$$

$$\frac{\Gamma_{a_1}}{\Gamma_e} = (12\pi^2)\left(\frac{f_{a_1}^2}{m_\tau^2}\right)\cos^2\theta_c \left(1 - \frac{m_{a_1}^2}{m_\tau^2}\right)^2 \left(1 + 2\frac{m_{a_1}^2}{m_\tau^2}\right). \tag{16.30}$$

Using $f_\pi = 132$ MeV, $\cos\theta_c = 0.97$ and the experimental values for masses, we get

$$BR(\tau^- \to \nu_\tau \pi^-) \approx (0.605)\left(\frac{f_\pi}{132}\right)^2 B_e \qquad (11.31 \pm 0.15)\% \tag{16.31a}$$

$$BR(\tau^- \to \nu_\tau \rho^-) = (0.557)\left(\frac{f_\rho}{132}\right)^2 B_e \qquad (24.94 \pm 0.16)\% \tag{16.31b}$$

$$BR(\tau^- \to \nu_\tau + a_1^-) = (0.329)\left(\frac{f_{a_1}}{132}\right)^2 B_e \qquad (17.65 \pm 0.32)\%. \tag{16.31c}$$

Using $B_e = (17.83 \pm 0.08)\%$, we see that to get the experimental values given in the parentheses in Eqs. (31), we should use

$$f_\pi = 134 \text{ MeV}, \qquad f_\rho = 208 \text{ MeV}, \qquad f_{a_1} = 229 \text{ MeV}. \tag{16.32}$$

We now show that the above values of the decay constants f_π and f_ρ as extracted from τ decay are consistent with those determined from light flavors physics showing the inner consistency of the standard model. To see this we first note that the $KSRF$ relation and the Weinberg first sum rule give respectively

$$f_\rho = \sqrt{2} f_\pi, \qquad f_{a1} = \left(\frac{m_\rho}{m_{a1}}\right) f_\rho = 0.62 f_\rho. \tag{16.33}$$

On the other hand the decay width for $\rho^0 \to e^+e^-$ is given by

$$\Gamma\left(\rho \to e^+e^-\right) = \frac{4\pi\alpha^2}{3}\frac{f_\rho^2}{m_\rho}\left(\frac{1}{2}\right). \tag{16.34}$$

The comparison with its experimental value 6.77 keV gives $f_\rho \approx 216$ MeV in agreement with that in Eq. (32) and the latter value is also consistent with the $KSRF$ value $f_\rho \approx 190$ MeV.

On the other hand the Weinburg sum rule gives $f_{a_1} \approx 130$ MeV, incompatible with that in Eq. (32).

One can improve the theoretical predictions of $\Gamma(\tau^- \to \rho^-\nu_\tau \to \pi^-\pi^0\nu_\tau)$ and $\Gamma(\tau^- \to a_1^-\nu_\tau \to \pi^-\rho^0\nu_\tau)$ by taking into account finite decay widths of ρ and a_1 mesons (see problems 1 and 2). However, for hadronic decays $(\tau^- \to f^- + \nu_\tau)$ which proceed through the vector current only, one can use CVC to relate $\Gamma(\tau^- \to f^- + \nu_\tau)$ to the scattering cross section for the process:

$$e^-e^+ \to \gamma \to f^0.$$

The cross section of this process is given by Eq. (A.79)

$$\sigma_f(s) = \frac{16\pi^3\alpha^2}{s}\rho_\gamma(s). \tag{16.35}$$

Using CVC, we get

$$\rho(s) = 2\rho_\gamma(s) = \frac{2s\ \sigma_f(s)}{16\pi^3\alpha^2}. \tag{16.36}$$

Hence we have

$$\begin{aligned}\Gamma(\tau^- \quad \to \quad f^-\nu_\tau) &= \frac{G_F^2\cos^2\theta_c}{128\pi^4\alpha^2}m_\tau^3 \\ &\times \int^{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2\left(1 + \frac{2s}{m_\tau^2}\right) s\ \sigma_f(s)\ ds. \end{aligned}\tag{16.37}$$

Let us apply this to the decay

$$\begin{aligned}\tau^- \quad &\to \quad \pi^-\pi^0\pi^0\pi^0 + \nu_\tau \\ &\to \quad \pi^-\pi^-\pi^+\pi^0 + \nu_\tau. \end{aligned}\tag{16.38}$$

Then we get from Eq. (37)

$$\begin{aligned}&\Gamma\left(\tau^- \to \pi^-\pi^0\pi^0\pi^0\nu_\tau\right) + \Gamma\left(\tau^- \to \pi^-\pi^-\pi^+\pi^0\nu_\tau\right) \\ = \ &\frac{G_F^2\cos^2\theta_c}{128\pi^4\alpha^2}m_\tau^3\int^{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2\left(1 + \frac{2s}{m_\tau^2}\right) \\ &\times s\left[\sigma_{\pi^-\pi^-\pi^+\pi^+}(s) + \sigma_{\pi^-\pi^+\pi^0\pi^0}(s)\right] ds. \end{aligned}\tag{16.39}$$

Since the decay proceeds via $I = 1$ weak currents, one also obtains an additional relation

$$\Gamma\left(\tau^- \to \pi^- 3\pi^0 \nu_\tau\right) = \frac{G_F^2 \cos^2\theta_c}{128\pi^4\alpha^2} m_\tau^3 \int^{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \times s \left[\frac{1}{2}\sigma_{2\pi^- 2\pi^+}(s)\right] ds. \qquad (16.40)$$

We have discussed above the dominant decay modes of τ^- viz. $\tau^- \to \nu_\tau\, e^- \nu_e$, $\tau^- \to \nu_\tau \mu^- \nu_\mu$, $\tau^- \to \nu_\tau (2\pi)^-, \tau^- \to \nu_\tau (3\pi)^-$ and $\tau^- \to \nu_\tau (4\pi)^-$. The other small decay modes can also be estimated. All these decay rates occur at the expected rates. The agreement between the theoretical and experimental values is good for each exclusive decay mode.

Finally if we add the decay rates for all exclusive channels for 1 prong events [$\tau^- \to \nu_\tau$ (particle)$^-$ neutrals (≥ 0)], we get the value $(84.96 \pm 0.14)\%$ to be compared with direct inclusive one prong branching ratio $B_1 = (85.53 \pm 0.14)\%$. Thus there is no discrepancy between the two branching ratios. τ- decays are well understood in the standard model.

16.3 Weak Decays of Heavy Flavors

In the standard model, the hadronic charged weak current can be written as (see Chap. 13)

$$J_\mu^W = (\bar{u}\,\bar{c}\,\bar{t})\,\gamma_\mu\,(1 - \gamma_5)\,V \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \qquad (16.41)$$

where

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \qquad VV^\dagger = 1 \qquad (16.42)$$

is the Cabibbo-Kobayashi-Maskawa (CKM) matrix. As we have discussed in Chap.13, the matrix V has only four real parameters [see Eq. (13.156)].

Since $|V_{cd}| \ll |V_{cs}|$, for the Cabbibo favored decays of the charmed mesons ($c \to s + W^+$), we have the selection rule:

$$\Delta Q = \Delta C = \Delta S \tag{16.43a}$$

whereas the decays with

$$\Delta Q = \Delta C, \qquad \Delta S = 0 \tag{16.43b}$$

are suppressed. The decays for which

$$\Delta Q = \Delta C = -\Delta S \tag{16.43c}$$

are strictly forbidden in the lowest order. Thus we expect, D mesons ($c\bar{u}$, $c\bar{d}$) to decay predominantly into states with strangeness $S = -1$ (K^-+ anything) and D_s mesons ($c\bar{s}$) to decay predominantly into states with strangeness $S = 0$ $\left[D_s \to \phi\,(\pi\text{'s})^+,\ K^{*+}\bar{K},\ \bar{K}^{*0}K^+,\ (\pi\text{'s})^+\right]$.

Since $|V_{ub}| \ll |V_{cb}|$, for the Cabibbo favored decay of B mesons ($\bar{b} \to \bar{c} + W^+$), we have the selection rule

$$\Delta Q = \Delta B = \Delta C \tag{16.44a}$$

whereas the decays with

$$\Delta Q = \Delta B, \qquad \Delta C = 0 \tag{16.44b}$$

are suppressed. The decays for which

$$\Delta Q = \Delta B = -\Delta C \tag{16.44c}$$

are strictly forbidden in the lowest order. Thus we expect B ($u\bar{b}$, $d\bar{b}$) meson to decay predominantly into states with $C = -1$ and B_s ($s\bar{b}$) to decay predominantly into states with $C = -1$, $S = -1$.

16.3.1 Leptonic decays of D and B mesons

The decay constants f_D, f_{D_s}, f_B, f_{B_s} can in principle be determined from the leptonic decays of D and B mesons. Thus for example, using Eq. (41), we can write

$$\frac{\Gamma\left(D^+ \to \mu^+\nu_\mu\right)}{\Gamma\left(\pi^+ \to \mu^+\nu_\mu\right)} = \frac{f_D^2}{f_\pi^2}\frac{|V_{cd}|^2}{|V_{ud}|^2}\frac{m_\pi}{m_D}\frac{p_D^2}{p_\pi^2}, \tag{16.45a}$$

where

$$p_D = \frac{1}{2}\, m_D\left(1 - \frac{m_\mu^2}{m_D^2}\right), \qquad p_\pi = \frac{1}{2}\, m_\pi\left(1 - \frac{m_\mu^2}{m_\pi^2}\right). \tag{16.45b}$$

In order to determine f_D, we need $|V_{ud}|^2$ and $|V_{cd}|^2$. Now $|V_{ud}|^2$ can be determined with a great degree of accuracy from nuclear β-decay. Its value is given by [see Chap. 11].

$$|V_{ud}| = 0.9750 \pm 0.0007. \tag{16.46a}$$

$|V_{cd}|$ has been determined from ν and $\bar{\nu}$ production of charm in deep inelastic scattering. Its value is

$$|V_{cd}| = 0.221 \pm 0.003. \tag{16.46b}$$

Using $|V_{ud}| \approx 0.97$, $|V_{cd}| \approx 0.22$, and $f_\pi = 132$ MeV and the experimental values

$$\begin{aligned} \Gamma\left(\pi^+ \to \mu^+ + \nu_\mu\right) &= 2.53 \times 10^{-14} \text{ MeV}, \\ \Gamma\left(D^+ \to \mu^+ + \nu_\mu\right) &< 4.48 \times 10^{-13} \text{ MeV}, \end{aligned}$$

we get

$$f_D < 288 \text{ MeV}. \tag{16.47}$$

Needless to say we can write similar expressions for the leptonic decays of $D_s^\pm$, $B^\pm$ and $B_s^\pm$. For $D_s^\pm$ decay,

$$|V_{cs}| = 0.9743 \pm 0.0007 \tag{16.48a}$$

and

$$\Gamma\left(D_s^+ \to \mu^+ + \nu_\mu\right) = (13 \pm 6) \times 10^{-12}\ \text{MeV}, \tag{16.48b}$$

so that from Eq. (45), we get

$$247\ \text{MeV} < f_{D_s} < 406\ \text{MeV}. \tag{16.48c}$$

For $B^+ \to \mu^+ \nu_\mu$, we can express the branching ratio:

$$BR\left(B^+ \to \mu^+ \nu_\mu\right) = \left[1.27 \times 10^{-7}\right] \left(\frac{f_B}{f_\pi}\right) \left|\frac{V_{ub}}{0.003}\right|^2. \tag{16.49}$$

Thus one can directly determine $f_B\,|V_{ub}|$ from this branching ratio whenever the experimental data are available.

16.3.2 Semi-leptonic decays of D and B mesons

The prototype of these decays is K_{e3} decay ($K^- \to \pi^0 + e^- + \bar{\nu}_e$). In fact from this decay and hyperon decays, $|V_{us}|$ has been determined:

$$|V_{us}| = 0.2205 \pm 0.0018. \tag{16.50}$$

It is theoretically simplest to begin with semileptonic decays of heavy flavors. We start with the decay

$$D^- \to X^0 + e^- + \bar{\nu}_e : \quad p = p_X + k_1 + k_2$$

where X^0 is any number of hadrons consistent with energy conservation and allowed by the selection rules.

The T-matrix for this decay is given by

$$\begin{aligned} T &= -\frac{G_F\, V_{cs}}{\sqrt{2}} \left\langle X \left| J_\mu^W \right| D \right\rangle \frac{1}{(2\ \pi)^3} \sqrt{\frac{m_e\ m_\nu}{k_{10}\ k_{20}}} \\ &\quad \times \left[\bar{u}\left(\mathbf{k}_1\right)\ \gamma^\mu\ \left(1-\gamma^5\right)\ v\left(\mathbf{k}_2\right)\right]. \end{aligned} \tag{16.51}$$

Since experimentally, we observe only charged leptons, we sum over hadrons. Thus for the inclusive semileptonic decays, we get for the decay rate:

$$\Gamma = \frac{G_F^2\ \left|V_{cs}\right|^2}{2\,(2\pi)^6\ 2\ p_0} \int \frac{d^3\ k_1}{k_{10}} \frac{d^3\ k_2}{k_{20}} \left(m_e\ m_\nu\right) L^{\mu\lambda}\ A_{\mu\lambda} \tag{16.52a}$$

where

$$L_{\mu\lambda} = \frac{2}{m_e\, m_\nu}\left[k_{1\mu}\, k_{2\lambda} + k_{1\lambda}\, k_{2\mu} - g_{\mu\lambda}\, k_1 \cdot k_2 - i\varepsilon_{\mu\lambda\rho\sigma}\, k_1^\rho\, k_2^\sigma\right], \tag{16.52b}$$

$$\begin{aligned} A_{\mu\lambda} &= (2\pi)^4 \int d^3\, p_X\, \delta^4 (p - q - p_X) \\ &\quad \times \left\langle D \left| J_\lambda^{W\dagger} \right| p_X \right\rangle \left\langle p_X \left| J_\mu^W \right| D \right\rangle \qquad (16.52c) \\ &= \int d^4\, z\, e^{-iqz} \left\langle D \left| \left[J_\lambda^{W\dagger}(z), J_\mu^W(0) \right] \right| p \right\rangle \\ &= \frac{1}{(2\pi)^3\, 2\, p_0} \left[\frac{1}{m_D^2} p_\lambda\, p_\mu\, 2\pi\, f_2\left(\nu, Q^2\right) - g_{\lambda\mu}\, 2\pi\, f_1\left(\nu, Q^2\right) \right. \\ &\quad \left. - \frac{1}{2\, m_D^2}\, i\varepsilon_{\lambda\mu\alpha\beta}\, p^\alpha\, q^\beta\, 2\pi\, f_3\left(\nu, Q^2\right) + \cdots \right]. \qquad (16.52d) \end{aligned}$$

Here

$$q = (\, k_1 + k_2), \qquad q^2 = 2E_e E_\nu \left(1 - \cos\theta\right)$$

$$\nu = \frac{p \cdot q}{m_D} = E_e + E_\nu. \tag{16.53}$$

Note in writing Eq. (52), we have neglected those form factors which give contribution proportional to lepton mass (m_e). In Eq. (53), we have also put $m_e = m_\nu = 0$, and $k_{10} = E_e$ and $k_{20} = E_\nu$. Hence we get from Eqs. (52) and (53)

$$\begin{aligned} &\frac{d\Gamma}{dE_e\, d\nu\, d\Omega} \\ &= \frac{G_F^2\, |V_{cs}|^2}{(2\pi)^4} \frac{8}{2\, m_D} E_e^2 E_\nu^2 \\ &\quad \times \left[\frac{1}{4}(1 + \cos\theta)\, f_2\left(\nu, q^2\right) + \frac{1}{2}(1 - \cos\theta)\, f_1\left(\nu, q^2\right) \right. \\ &\quad \left. + \frac{1}{4\, m_D}(E_e - E_\nu)\frac{1}{2}(1 - \cos\theta)\, f_3\left(\nu, q^2\right) \right]. \qquad (16.54) \end{aligned}$$

This is a general expression for the semileptonic decay rate of D meson. But this expression is not useful since we do not know

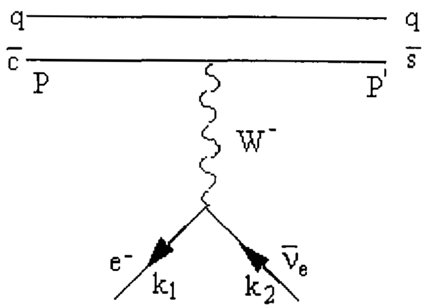

Figure 1 Dominant (spectator) diagram for Cabibbo favored semileptonic decays of D mesons.

the form factors f_1, f_2 and f_3. In order to determine these form factors one has to use some models. The simplest model is the spectator quark model. According to this model the decay proceeds as shown in Fig. 1. It is assumed in this model that the quarks in the final states fragment into hadrons with unit probability. In this model, one can easily calculate the tensor $A_{\mu\lambda}$ using the usual trace techniques. Noting that $P = xp$, $P' = P - q$, we get

$$\begin{aligned} A_{\mu\lambda} &= 2\frac{(2\pi)^4}{(2\pi)^6}\frac{1}{xp_0}\delta\left(2xp\cdot q - m_c^2 + m_s^2 - q^2\right) \\ &\quad \times \left[2x^2\, p_\mu\, p_\lambda + g_{\lambda\mu}\left(-m_c^2 + xp\cdot q\right)\right. \\ &\quad \left. - x\, i\varepsilon_{\mu\lambda\alpha\beta}\, p^\alpha\, q^\beta\right]. \end{aligned} \tag{16.55}$$

Thus comparing it with Eq. (52), we obtain

$$\begin{aligned} f_2\left(\nu, q^2\right) &= 8x\, m_D^2\, \delta\left(2xp\cdot q - m_c^2 + m_s^2 - q^2\right) \\ f_1\left(\nu, q^2\right) &= -\frac{4}{x}\left(-m_c^2 + xp\cdot q\right)\,\delta\left(2xp\cdot q - m_c^2 + m_s^2 - q^2\right) \\ f_3\left(\nu, q^2\right) &= -8\, m_D^2\, \delta\left(2xp\cdot q - m_c^2 + m_s^2 - q^2\right). \end{aligned} \tag{16.56}$$

Hence from Eq. (54) and noting here that $x = m_D/m_c \approx 1$, we get

$$\begin{aligned} \frac{d\Gamma}{dy} &= \frac{G_F^2\ |V_{cs}|^2}{16\pi^3} m_c^5\ y^2 \frac{(y_c - y)^2}{(1-y)} \\ \Gamma &= \frac{G_F^2\ m_c^5}{192\pi^3} |V_{cs}|^2\ F\left(\frac{m_s}{m_c}\right), \end{aligned} \tag{16.57}$$

where

$$y = \frac{2\ E_e}{m_c}, \qquad y_c = 1 - \frac{m_s^2}{m_c^2}$$

$$\begin{aligned} F\left(\frac{m_s}{m_c}\right) &= 1 - 8\left(\frac{m_s}{m_c}\right)^2 + 8\left(\frac{m_s}{m_c}\right)^6 - \left(\frac{m_s}{m_c}\right)^8 \\ &\quad -24\left(\frac{m_s}{m_c}\right)^4 \ln\left(\frac{m_s}{m_c}\right). \end{aligned} \tag{16.58}$$

Equations (57) are exactly the same as one gets (see problem 11.1 with G_F^2 replaced by $G_F^2\ |V_{cs}|^2$) for the decay

$$\bar{c} \to \bar{s} + e^- + \bar{\nu}_e\ \left(c \to s + e^+ + \nu_e\right).$$

Similarly for the (inclusive) semileptonic decay of B mesons viz.

$$B^+ \to X^0 + e^+ + \nu_e\ \left(B^- \to X^0 + e^- + \bar{\nu}_e\right),$$

the basic process is

$$\bar{b} \to \bar{c} + e^+ + \nu_e\ \left(b \to c + e^- + \bar{\nu}_e\right),$$

and we get (see problem 11.1 with G_F^2 replaced by $G_F^2\ |V_{bc}|^2$).

$$\begin{aligned} \frac{d\Gamma}{dy} &= \frac{G_F^2\ |V_{bc}|^2}{96\pi^3} m_b^5\ y^2 \frac{(y_b - y)^2}{(1-y)^2} \left[(3-2y) + \frac{(1-y_b)(3-y)}{(1-y)}\right] \\ \Gamma &= \frac{G_F^2 m_b^5\ |V_{cs}|^2}{192\pi^3}\ F\left(\frac{m_c}{m_b}\right), \end{aligned} \tag{16.59}$$

where

$$y = \frac{2\,E_e}{m_b}, \qquad y_b = 1 - \frac{m_c^2}{m_b^2}. \tag{16.60}$$

Note that the difference between Eqs. (57) and (59) is due to the fact that V - A interference term [the third term in Eq. (54)] has opposite sign in the two cases. We conclude this section with the remarks that according to this picture, we get

$$\begin{aligned} \Gamma\left(D^+ \to X^0 e^+ \nu_e\right) &= \Gamma\left(D^0 \to X^- e^+ \nu_e\right) \\ \Gamma\left(B^+ \to X^0 e^+ \nu_e\right) &= \Gamma\left(B^0 \to X^- e^+ \nu_e\right). \end{aligned} \tag{16.61}$$

For the decay

$$D_s^+ \to X^0 + e^+ + \nu_e$$

we get for $\frac{d\Gamma}{dx}$ and Γ exactly the same expressions as those given in Eq. (57). Similar remarks are applicable to the decay $B_s^0 \to X^- + e^+ + \nu_e$. Hence the quark model predicts

$$\Gamma\left(D^+ \to X^0 e^+ \nu_e\right) = \Gamma\left(D^0 \to X^- e^+ \nu_e\right) = \Gamma\left(D_s^+ \to X^0 e^+ \nu_e\right), \tag{16.62}$$

$$\Gamma\left(B^+ \to X^0 e^+ \nu_e\right) = \Gamma\left(B^0 \to X^- e^+ \nu_e\right) = \Gamma\left(B_s^0 \to X^- e^+ \nu_e\right). \tag{16.63}$$

The experimental branching ratios for these decays are:

$$\begin{aligned} \left(D^+\right)_{SL} &\equiv BR\left(D^+ \to X^0 e^+ \nu_e\right) = (17.2 \pm 1.9)\ \% \\ \left(D^0\right)_{SL} &\equiv BR\left(D^0 \to X^- e^+ \nu_e\right) = (7.7 \pm 1.2)\ \% \\ \left(D_s^+\right)_{SL} &\equiv BR\left(D_s^+ \to X^0 e^+ \nu_e\right) < 20\ \%. \end{aligned} \tag{16.64}$$

For B mesons the experimental values are

$$\begin{aligned} \left(B^+\right)_{SL} &\equiv BR\left(B^+ \to X^0 e^+ \nu_e\right) = (10.1 \pm 2.3)\ \% \\ \left(B^0\right)_{SL} &\equiv BR\left(B^0 \to X^- e^+ \nu_e\right) = (10.3 \pm 1)\ \%. \end{aligned} \tag{16.65}$$

Now the experimental values for τ_{D^+}, τ_{D^0}, $\tau_{D_s^+}$, τ_{B^+} and τ_{B^0} are respectively $(1.057 \pm 0.015) \times 10^{-12}$, $(0.415 \pm 0.004) \times 10^{-12}$, (0.467

$\pm 0.017) \times 10^{-12}$, $(1.62 \pm 0.06) \times 10^{-12}$, $(1.56 \pm 0.06) \times 10^{-12}$. Thus we see that $\tau_{D^0} \approx \tau_{D_s^+} \neq \tau_{D^+}$, $\tau_{D^+} \approx 2.5\ \tau_{D^0}$ i.e. isospin is badly broken for D^+ and D^0. Also we note that $\tau_{B^+} \approx \tau_{B^0}$. Thus one would expect using Eq. (62)

$$\frac{(D^+)_{SL}}{(D^0)_{SL}} = \frac{BR\,(D^+ \to X^0 e^+ \nu_e\,)}{BR\,(D^0 \to X^- e^+ \nu_e\,)} = \frac{\tau_{D^+}}{\tau_{D^0}} \approx 2.5. \qquad (16.66)$$

This is consistent with the experimental value given in Eq. (64). We can conclude that Eqs. (62) and (63) are well verified experimentally.

In the end, we note from Eq. (57) that we can get an estimate of $|V_{cs}|$, using the $BR(D \to Xe\nu_e)$ from the experiment if we know the quark masses m_s and m_c.

16.3.3 (Exclusive) semileptonic decays of D and B mesons

Our general formulation can be used to calculate the decay rate for semileptonic decays of the type

$$P\,(p) = M\,(p') + l + \nu$$

where $P = D$ or B. M is a pseudoscalar meson P' or a vector meson V . In order to discuss these decays, we first define the form factors

$$\begin{aligned}
&\langle P'\,(p')\,|\bar{q}\,\gamma_\mu\,Q|\,P\,(p)\rangle \\
=\;& \frac{1}{(2\pi)^3}\frac{1}{\sqrt{4\,p_0\,p_0'}}\left[F_+\,(t)\,(p+p')_\mu + F_-\,(t)\,(p-p')_\mu\right] \\
=\;& \frac{1}{(2\pi)^3}\frac{1}{\sqrt{4\,p_0\,p_0'}}\left[\left((p+p')_\mu + \frac{m_P^2 - m_{P'}^2}{t} q_\mu\right) F_1\,(t)\right. \\
&\left. + \left(\frac{m_P^2 - m_{P'}^2}{t} q_\mu\right) F_0\,(t)\right] \qquad (16.67a)
\end{aligned}$$

where

$$t \;=\; q^2 = (p-p')^2\,,$$

$$\begin{aligned} q_\mu &= (p-p')_\mu \\ F_1(t) &= F_+(t), \\ F_0(t) &= F_+(t) + \frac{t}{m_P^2 - m_{P'}^2} F_-(t). \end{aligned} \tag{16.67b}$$

Here m_P is the mass of P and $m_{P'}$ is the mass of P'. For the vector meson V, the form factors are defined:

$$\begin{aligned} &\langle V(p',\varepsilon)|\bar{q}\,\gamma_\mu\,\gamma_5\,Q|P(p)\rangle \\ &= \frac{1}{(2\pi)^3}\frac{-i}{\sqrt{4\,p_0\,p_0'}}\left\{\left[(m_P+m_V)\,\varepsilon_\mu^* - \frac{(m_P+m_V)}{t}q\cdot\varepsilon^* q_\mu\right]A_1(t)\right. \\ &- \left[\frac{1}{m_P+m_V}(p+p')_\mu - \frac{(m_P-m_V)}{t}\,q_\mu\right]q\cdot\varepsilon^* A_2(t) \\ &- \left.\left[2m_V\frac{q\cdot\varepsilon^*}{t}\,q_\mu\right]A_0(t)\right\} \end{aligned} \tag{16.68a}$$

$$\begin{aligned} &= \frac{1}{(2\pi)^3}\frac{-i}{\sqrt{4\,p_0\,p_0'}} \\ &\times\left[(m_P+m_V)\,\varepsilon_\mu^*\,A(t) - \frac{1}{m_P+m_V}\,q\cdot\varepsilon^*(p+p')_\mu\;A_+(t)\right. \\ &- \left.\frac{1}{m_P+m_V}q\cdot\varepsilon^* q_\mu\;A_-(t)\right] \end{aligned} \tag{16.68b}$$

$$\langle V(p',\varepsilon)|\bar{q}\,\gamma_\mu\,Q|P(p')\rangle = \frac{1}{(2\pi)^3}\frac{1}{\sqrt{4\,p_0\,p_0'}}\frac{2V(t)}{m_P+m_V}\,\varepsilon_{\mu\nu\lambda\sigma}\,\varepsilon^{\nu*}\,p'^{\lambda}p^{\sigma}. \tag{16.69}$$

However for the transition $B \to D$ or $B \to D^*$, the heavy quark spin symmetry gives the following relations among the form factors [cf. Eqs. (9.42), (9.58) and (9.59)]

$$\begin{aligned} F_1(t) &= F_+(t) = \frac{m_B+m_D}{2\sqrt{m_B\,m_D}}\,\xi(v\cdot v') \\ F_0(t) &= \frac{\sqrt{m_B\,m_D}}{(m_B+m_D)}\,[1+v\cdot v']\;\xi(v\cdot v') \end{aligned} \tag{16.70}$$

$$
\begin{aligned}
V(t) &= \frac{m_B + m_{D^*}}{2\sqrt{m_B\, m_{D^*}}}\, \xi(v\cdot v') \\
&= A_2(t) = A_+(t) = -A_-(t) = -A_0(t) \qquad (16.71a) \\
A_1(t) &= A(t) = \frac{\sqrt{m_B m_{D^*}}}{m_B + m_{D^*}}\left[1 + v\cdot v'\right]\, \xi(v\cdot v')\,. \qquad (16.71b)
\end{aligned}
$$

Note that

$$
\begin{aligned}
t &= m_B^2 + m_{D^*}^2 - 2m_B\, m_{D^*}\, v\cdot v' \\
&= m_B^2 + m_{D^*}^2 - 2m_B\, m_{D^*}\, w \qquad (16.72)
\end{aligned}
$$

where

$$w = v\cdot v'. \qquad (16.73)$$

At $w = 1$, $t = (m_B - m_{D^*})^2 = t_{\max}$, the form factor $\xi(w)$ is normalized as

$$\xi(1) = 1, \qquad \xi(t_{\max}) = 1. \qquad (16.74)$$

Having defined the matrix elements and form factors, we give in the following table the exclusive semileptonic decays, which we are considering:

Q	q	Current	$\lvert V_{qQ}\rvert$	Decay
c	s	$\bar{s}\,\gamma_\mu\,(1-\gamma_5)\,c$	$\lvert V_{cs}\rvert = 0.9743 \pm 0.0007$	$D \to K l\nu$ $\to K^* l\nu$
c	d	$\bar{d}\,\gamma_\mu\,(1-\gamma_5)\,c$	$\lvert V_{cd}\rvert = 0.221 \pm 0.003$	$D \to \pi l\nu$ $\to \rho l\nu$
c	s	$\bar{s}\,\gamma_\mu\,(1-\gamma_5)\,c$	$\lvert V_{cs}\rvert$	$D_s \to \eta l\nu$ $\to \phi l\nu$
b	c	$\bar{c}\,\gamma_\mu\,(1-\gamma_5)\,b$	$\lvert V_{cb}\rvert = 0.039 \pm 0.002$	$B \to D l\nu$ $\to D^* l\nu$
b	u	$\bar{u}\,\gamma_\mu\,(1-\gamma_5)\,b$	$\lvert V_{ub}\rvert = (0.08 \pm 0.002)\,\lvert V_{cb}\rvert$	$B \to \pi l\nu$ $\to \rho l\nu$

Now for the semileptonic decay $P \to M\ l\nu$, since the meson M is on the mass shell, we can integrate our master equation (54) over ν using the delta function $\delta\left(2m_p\nu - m_P^2 - t + m_M^2\right)$. We obtain

$$\begin{aligned}\frac{d\,\Gamma}{dt\;dE_\ell} &= \frac{G^2}{(2\pi)^3}\frac{1}{8m_P^2}\left\{\left[4\sqrt{K^2+t} - E_\ell - t\right] f_2(t) + 2t\ f_1(t)\right.\\ &\quad\left. + \frac{t}{m_P}\left[2E_\ell - \sqrt{K^2+t}\right] f_3(t)\right\},\end{aligned} \tag{16.75}$$

where K is the momentum of meson M in the rest frame of P, $t = q^2$ and $G = G_F\left|V_{qQ}\right|$. Integrating Eq. (75) over the lepton energy E_ℓ we get

$$\frac{d\,\Gamma}{dt} = \frac{G^2}{(2\pi)^3}\frac{1}{8m_P^2}\left[\frac{2}{3}K^3\ f_2(t) + 2tK\ f_1(t)\right]. \tag{16.76}$$

Note that in Eqs. (75) and (76), we have neglected those form factors which give contribution proportional to lepton mass m_l.

First we consider the case when M is a pseudoscalar meson i.e. $M = P'$. In this case we get from Eqs. (67) and (52c)

$$f_2(t) = 4\ m_P^2\left|F_+(t)\right|^2, \qquad f_1(t) = 0. \tag{16.77}$$

Hence

$$\frac{d\,\Gamma}{dt} = \frac{G^2}{24\pi^3}K^3\ \left|F_+(t)\right|^2. \tag{16.78}$$

For the vector case i.e. $M = V$, we get from Eqs. (68), (69) and (52c) after straightforward but some what lengthy calculation

$$\begin{aligned}\frac{d\,\Gamma}{dt} &= \frac{G^2}{96\pi^3}\,m_V\ \sqrt{w^2-1}\ \left|A_1(t)\right|^2\ \left(m_P+m_V\right)^2\\ &\quad\times\left\{\left(w^2-1\right)\left[1 + 4\ m_V\left(m_V - m_P w\right)\frac{r_2(t)}{\left(m_V+m_P\right)^2}\right.\right.\\ &\quad\left. + 4m_P^2\ m_V^2\ \left(w^2-1\right)\frac{r_2^2(t)}{\left(m_V+m_P\right)^4}\right]\\ &\quad\left. + 4\frac{t\ m_V^2}{\left(m_V+m_P\right)^4}\left(w+1\right)\ r_V^2(t)\left[\left(w+1\right)+4w\right]\right\}\end{aligned} \tag{16.79}$$

where we have used

$$\begin{aligned} K_V &= m_V\,\sqrt{w^2-1}, \qquad E_V = m_V\,w \\ t &= m_P^2 + m_V^2 - 2m_P m_V\,w \qquad (16.80) \\ r_2(t) &= \frac{A_2(t)}{A_1(t)}, \qquad r_V(t) = \frac{V(t)}{A_1(t)}. \qquad (16.81) \end{aligned}$$

For the transition $B \to D^*$, it is convenient to define

$$\begin{aligned} R_1(t) &= \left[1 - \frac{t}{(m_B + m_{D^*})^2}\right] \frac{V(t)}{A_1(t)} \\ R_2(t) &= \left[1 - \frac{t}{(m_B + m_{D^*}^2)^2}\right] \frac{A_2(t)}{A_1(t)}. \qquad (16.82) \end{aligned}$$

Note that in the heavy quark symmetry limit $R_1(t) = R_2(t) = 1$. Using Eqs. (81) and (82), we get from Eq. (79) with $m_P \to m_B$, $m_V \to m_{D^*}$,

$$\begin{aligned} \frac{d\,\Gamma}{d\,w} &= \frac{G_F^2\,|V_{cb}|^2}{48\pi^3}\,m_{D^*}^3\,\sqrt{w^2-1}\,(w+1)^2 \\ &\times \Big\{\Big[m_B^2\big(w^2-1\big) + 2m_B\,(m_{D^*} - m_B\,w)\,(w-1)\,R_2(t) \\ &+ m_B^2\,(w-1)^2\,R_2^2(t)\Big] \\ &+ \left[\left(m_B^2 + m_{D^*}^2 - 2m_B\,m_{D^*}\right)\left(1 + \frac{4w}{w+1}\right)\right] \\ &\times R_1^2(t)\Big\}\,\mathcal{F}^2(w). \qquad (16.83) \end{aligned}$$

In the symmetry limit, $R_1(t) = R_2(t) = 1$, and we obtain

$$\frac{d\,\Gamma}{d\,w} = \frac{G_F^2\,|V_{cb}|^2}{48\pi^3}\,m_{D^*}^3\,\sqrt{w^2-1}\,(w+1)^2$$

$$\times \left\{(m_B - m_{D^*})^2 + \frac{4w\,(m_B^2 + m_{D^*}^2 - 2m_B\,m_{D^*}\,w)}{w+1}\right\}\mathcal{F}^2(w)\,. \qquad (16.84)$$

For $B \to D$ transition, we obtain from Eqs. (78) and (70)

$$\frac{d\,\Gamma}{d\,w} = \frac{G_F^2\ |V_{cb}|^2}{48\pi^3}\left(m_B + m_D\right)^2\ m_D^3\ \left(w^2 - 1\right)^{3/2}\ \mathcal{G}^2\left(w\right). \quad (16.85)$$

In the heavy quark symmetry limit

$$\mathcal{G}\left(w\right) = \mathcal{F}\left(w\right) = \xi\left(w\right). \quad (16.86)$$

Let us first consider the semileptonic decays of D-mesons. The experimental results on the form factors are as follows:

$$D^+ \to \bar{K}^0\ell^+\nu_\ell : \quad F_+\ (0) = 0.74 \pm 0.03 \quad (16.87)$$

$$\begin{aligned} D^+ &\to \bar{K}^{*^0}\ell^+\nu_\ell : \quad V\ (0) = 1.0 \pm 0.3 \\ A_1\left(0\right) &= 0.55 \pm 0.03 \\ A_2\left(0\right) &= 0.40 \pm 0.08 \\ r_2\left(0\right) &= 0.73 \pm 0.15 \\ r_V\left(0\right) &= 1.90 \pm 0.25 \end{aligned} \quad (16.88)$$

$$\begin{aligned} D_s^+ &\to \phi\ell^+\nu_\ell : \quad V\ (0) = 0.9 \pm 0.3 \\ A_1\left(0\right) &= 0.62 \pm 0.06 \\ A_2\left(0\right) &= 1.0 \pm 0.3 \\ r_2\left(0\right) &= 1.6 \pm 0.4 \\ r_V\left(0\right) &= 1.5 \pm 0.5 \end{aligned} \quad (16.89)$$

These form factors are of importance for two-body nonleptonic decays of D-mesons in the factorization ansatz (see the next section).

There is no model independent way to determine these form factors. First we note that the form factors for various decays are related by SU(3) as follows

$$-\sqrt{2}F_+\ \left(D^+ \to \pi^0\right) = F_+\ \left(D^0 \to \pi^-\right)$$

$$\begin{aligned}
&= F_+ \left(D^+ \to \bar{K}^0\right) \\
&= F_+ \left(D^0 \to K^-\right) \\
&= -\sqrt{3/2}\, F_+ \left(D_s^+ \to \eta_8\right) \\
&= F_+ \left(D_s^+ \to K^0\right). \qquad (16.90)
\end{aligned}$$

For $D \to V$ transitions, we have similar relations with $\pi \to \rho$ and $K \to K^*$ and since the nonet symmetry holds for vector mesons, we have

$$\begin{aligned}
V \left(D^0 \to \rho^-\right) &= V \left(D^+ \to \bar{K}^{*^0}\right) \\
&= V \left(D^0 \to K^{*^-}\right) \\
&= V \left(D_s^+ \to K^{*^0}\right) \\
&= -V \left(D_s^* \to \phi\right)
\end{aligned}$$

and

$$V \left(D_s^+ \to \omega\right) = 0. \qquad (16.91)$$

Needless to say that similar relations hold for axial vector form factors A_1 and A_2.

Most of the models agree that the form factors F_+ and V are dominated by vector bosons D^* and D_s^*. Hence for the Cabbibo favored decays $D^0 \to K^- \ell^+ \nu$, $K^{*^-} \ell^+ \nu$ and $D_s^* \to \phi \ell^+ \nu$, the vector mesons dominance shown in Fig. 2 gives

$$\begin{aligned}
F_+^{D^0 \to K^-}(t) &= g_{D_s^*\, DK} \frac{f_{D_s^*}\, m_{D_s^*}}{m_{D_s^*}^2 - t} \\
&= \lambda_D \frac{f_{D_s^*}}{f_K} \frac{m_{D_s^*}^2}{m_{D_s^*}^2 - t} \qquad (16.92) \\
V^{D^0 \to K^{*^-}}(t) &= \left(\frac{m_D + m_{K^*}}{2}\right) \frac{g_{D_s^* DK^*}\, f_{D_s^*}\, m_{D_s^*}}{m_{D_s^*}^2 - t} \\
&= \frac{(m_D + m_{K^*})}{m_{D_s^*}} \lambda_D \frac{f_{D_s^*}}{f_K} \frac{m_{D_s^*}^2}{m_{D_s^*}^2 - t} \qquad (16.93)
\end{aligned}$$

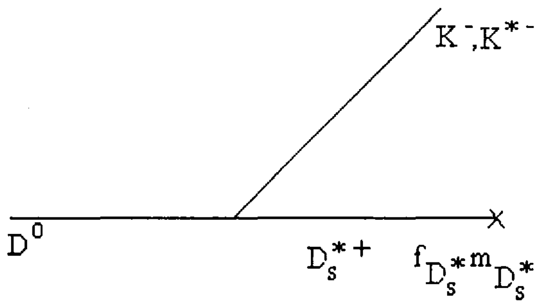

Figure 2 Vector meson dominance for two-body decays of D^0

where we have parametrized $g_{D_s^* DK}$and $g_{D_s^* DK^*}$ as follows

$$g_{D_s^* DK} = \lambda_D \frac{m_{D_s^*}}{f_K} \quad (16.94)$$

$$g_{D_s^* DK^*} = \lambda_D \frac{2}{f_K}. \quad (16.95)$$

For $\lambda_D = 1$, Eq. (94) follows from SU(4) (flavor + spin) current algebra and Eq. (95) with $\lambda_D = 1$ is also a consequence of SU(4) algebra, vector meson dominance and the $KSRF$ relation. Using SU(3), we can write

$$V^{D_s^* \to \phi}(t) = -\frac{m_{D_s} + m_\phi}{m_{D_s^*}} \lambda_D \frac{f_{D_s^*}}{f_s} \frac{m_{D_s^*}^2}{m_{D_s^*}^2 - t}. \quad (16.96)$$

Note that all the form factors for $D_s^* \to \phi$ have negative sign relative to $D^0 \to K^-$, so that an overall minus sign in front of Eq. (96) can be ignored. Comparing Eqs. (92), (93) and (96) with Eqs. (87), (88) and (89), we have

$$\lambda_D \frac{f_{D_s^*}}{f_K} = 0.74 \pm 0.03 \quad (16.97)$$

$$\lambda_D \frac{m_D + m_{K^*}}{m_{D_s^*}} \frac{f_{D_s^*}}{f_K} = 1.0 \pm 0.3 \tag{16.98}$$

$$\lambda_D \frac{m_{D_s} + m_\phi}{m_{D_s^*}} \frac{f_{D_s^*}}{f_s} = 0.9 \pm 0.3. \tag{16.99}$$

Now using $\frac{m_D+m_{K^*}}{m_{D_s^*}} = 1.30$, $\frac{m_{D_s}+m_\phi}{m_{D_s^*}} = 1.41$, $f_K = 1.25\ f_\pi$, $f_s = 1.2\ f_\pi$, we get from Eqs. (98) and (99) respectively

$$\lambda_D \frac{f_{D_s^*}}{f_K} = 0.8 \pm 0.2 \tag{16.100}$$

$$\lambda_D \frac{f_{D_s^*}}{f_K} = \frac{f_s}{f_K} (0.6 \pm 2) \approx (0.6 \pm 2)\,. \tag{16.101}$$

Thus within the experimental errors, Eqs. (97)-(99) are consistent with each other. On the other hand, if we use $f_{D_s^*} = 275$ MeV, $f_\pi = 132$ MeV, $f_K / f_\pi = 1.25$, we get from Eq. (97)

$$\lambda_D = 0.44.$$

However, since $f_{D_s^*}$ is also not well determined, it is safe to say that λ_D is of order 1/2. The value of λ_D can be determined from the experimental value of the decay width for the decay $D^{*+} \to D^0\pi^+$. The decay width is given by

$$\begin{aligned}
\Gamma\left(D^{*+} \to D^0\pi^+\right) &= \frac{g_{D^*D\pi}}{6\pi} \frac{p^3}{m_{D^*}^2} \\
&= \frac{\lambda_D^2}{6\pi} \frac{p^3}{f_\pi^2} \\
&= \lambda_D^2\ (181)\ \text{keV} \\
&= 45\ \text{keV},
\end{aligned} \tag{16.102}$$

for $\lambda_D = 1/2$. Experimental upper limit on Γ is $\Gamma\left(D^{*+} \to D^0\pi^+\right) <$ 89 MeV. With improved experimental numbers in Eqs. (87)-(89) and for Γ, better information on these form factors can be obtained.

For $B \to D$, D^* transitions, the heavy quark spin symmetry gives $R_1(t) = R_2(t) = 1$; but it does not give the form factors

$\mathcal{F}(w)$ and $\mathcal{G}(w)$ at any w except at $w = 1$. Taking into account symmetry breaking corrections to the heavy quark limit, it is found

$$\begin{aligned} \mathcal{F}(1) &= 0.924 \pm 0.27 \\ \mathcal{G}(1) &= 1.00 \pm 0.07. \end{aligned} \tag{16.103}$$

A more refined analysis of symmetry breaking gives

$$\begin{aligned} R_1 &\approx 1 + \frac{4\alpha_s(m_c)}{3\pi} + \frac{\bar{\Lambda}}{2m_c} \\ R_2 &\approx 1 - \frac{\bar{\Lambda}}{2m_c}. \end{aligned} \tag{16.104}$$

If we use, $\alpha_s(m_c) \approx 0.34$, $\bar{\Lambda} \approx 0.41$ GeV, [cf. Eq. (9.91)] $m_c \approx 1.5$ GeV we obtain $R_1 \approx 1.3$, $R_2 \approx 0.9$. The data can be fitted by assuming R_1 and R_2 as constants and by writing

$$\mathcal{F}(w) = \mathcal{F}(1)\left[1 - \rho_{A_1}^2 (w-1)\right]. \tag{16.105}$$

The fit to the data gives

$$\begin{aligned} R_1 &= 1.18 \pm 0.30 \pm 0.12 \\ R_2 &= 0.71 \pm 0.22 \pm 0.07 \\ \rho_{A_1}^2 &= 0.91 \pm 0.15 \pm 0.06. \end{aligned} \tag{16.106}$$

Thus we see that the values of R_1 and R_2 are in good agreement with the predictions of HQET given in Eq. (104). R_1 and R_2 are rather insensitive to the form assumed for $\mathcal{F}(w)$. However, the value of $\rho_{A_1}^2$ is sensitive to the form of $\mathcal{F}(w)$.

16.3.4 Non-leptonic decays of B and D mesons

At tree level, the $\Delta B = 1$ non-leptonic weak decays are described by a single W-exchange as shown in Fig. 3, which represents the decay $b \to u + q' + \bar{q}\,(q = u$ or c; $q' = d$ or $s)$. Here α and β are color indices. The effective Lagrangian for this decay is given by

$$\begin{aligned} L_{eff} &= \frac{G_F}{\sqrt{2}} \sum_{q=u,c} V_{cb}\, V_{qq'}^* \, \left[\bar{c}^{\alpha}\, \gamma^{\mu}\, (1-\gamma_5)\, b_{\alpha}\right] \\ &\quad \times \left[\bar{q}'^{\beta}\, \gamma_{\mu}\, (1-\gamma_5)\, q_{\beta}\right] + c \to u. \end{aligned} \tag{16.107}$$

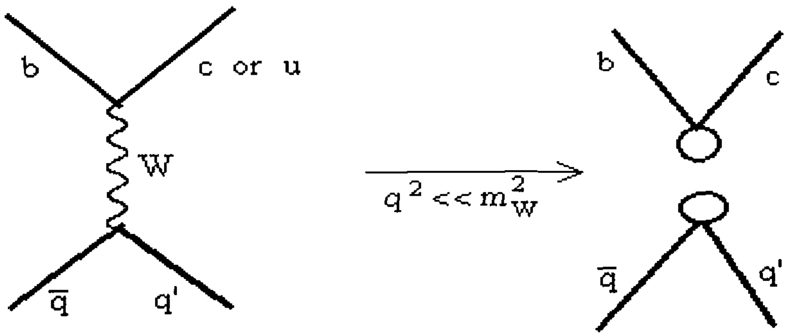

Figure 3 Decay of $b \to c(u) + q' + \bar{q}$ through W–exchange

Since quarks carry color, the QCD corrections must be taken into account. Under QCD renormalization, the Lagrangian (107) becomes [see Fig. 4]:

$$\begin{aligned} L_{\text{eff}} &= \frac{G_F}{\sqrt{2}} \left[\sum_{q=u,c} V_{cb}\, V^*_{qq'}\, (C_1 O_1^c + C_2 O_2^c) \right. \\ & \left. + \sum_{q=u,c} V_{ub}\, V^*_{qq'}\, (C_1 O_1^u + C_2 O_2^u) \right] \end{aligned} \tag{16.108}$$

where C_i are Wilson coefficients evaluated at the renormalization scale μ; the current-current operators $O_{1,2}$ are

$$\begin{aligned} O_1^c &= (\bar{c}^\alpha\, b_\alpha)_{V-A} \left(\bar{q}'^\beta\, q_\beta\right)_{V-A} \\ O_2^c &= (\bar{c}^\alpha\, b_\beta)_{V-A} \left(\bar{q}'^\beta\, q_\alpha\right)_{V-A} \end{aligned} \tag{16.109}$$

and O_i^u are obtained through replacing c by u. Here

$$(\bar{c}^\alpha\, b_\beta)_{V-A} = \bar{c}^\alpha\, \gamma_\mu\, (1 - \gamma_5)\, b_\beta \text{ etc.}$$

For $c \to s + u + \bar{q}$ $(q = d \text{ or } s)$, replace b by c, c by s and q' by u. Note that strong interaction due to hard gluon corrections has been

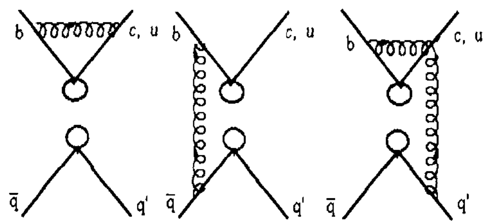

Figure 4 QCD corrections to the Fig. 3, ○'s denote standard operator insertions

taken into account in the Wilson coefficients $C_1(\mu)$ and $C_2(\mu)$; without these corrections $C_1 = 1$ and $C_2 = 0$. The long range QCD effects are taken into account by the matrix elements of operators $O_{1,2}$ between hadronic states. They manifest themselves in the form factors. To the leading logarithmic approximation (LLA), the Wilson coefficients $C_\pm = C_1 \pm C_2$ [note that QCD corrections mix the operators O_1 and O_2, requiring diagonalization which result in $O_\pm = \frac{O_1 \pm O_2}{2}$, $C_\pm = C_1 \pm C_2$] are given by [see Appendix B]

$$C_\pm(\mu) = \left[\frac{\alpha_s(m_W)}{\alpha_s(\mu)}\right]^{\gamma_\pm} C_\pm(m_W) \tag{16.110a}$$

where $C_\pm(m_W) = 1$ (in the approximation we are using) and $\gamma_\pm = \gamma_\pm^0/2\beta_0$, with $\gamma_\pm^0 = \pm 6\frac{N_c \mp 1}{N_c}$ so that for $N_c = 3$ [$\beta_0 = 4\pi b$]

$$\gamma_+ = \frac{2}{\beta_0} = \frac{2}{11 - \frac{2}{3}n_f}, \quad \gamma_- = -2\,\gamma_+, \quad C_+^2\, C_- = 1 \tag{16.110b}$$

and n_f is the number of active flavors (in the region between μ and m_W). At $\mu = m_b = 4.9$ GeV we get from Eqs. (110), taking $\alpha_s(m_W) \approx 0.12$, $n_f = 5$ and $\alpha_s(m_b) = 0.22$, $\alpha_s(m_c) = 0.34$,

$$C_1(m_b) \approx 1.11, \qquad C_2(m_b) \approx -0.26. \tag{16.111}$$

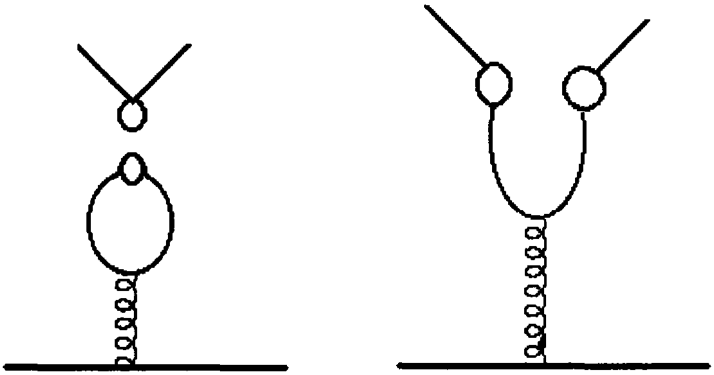

Figure 5 QCD penguin diagram

They are not very much different from the Wilson coefficients at $\mu = 2.5$ GeV in next-to-leading logarithmic (NLL) precision:

$$C_1 = 1.117, \qquad C_2 = -0.257. \tag{16.112}$$

At $\mu = m_c$, we get from Eqs. (110a)

$$C_1\,(m_c) \approx 1.24, \qquad C_2\,(m_c) \approx -0.48. \tag{16.113}$$

In addition to the current-current operators $O_i (i = 1, 2)$ in Eq. (108), originating in the usual W-exchange and subsequent QCD corrections, there are QCD penguin operators $O_i (i = 3 \cdots 6)$ originating in the QCD penguin diagrams shown in Fig. 5. These operators add to the effective Lagrangian (108) the following term

$$-V_{tb}\, V_{tq}^* \sum_{i=3}^{6} C_i\, O_i \tag{16.114}$$

where

$$O_{3,5} \;=\; (\bar{q}^{\alpha}\, b_{\alpha})_{V-A} \sum_{q'} \left(\bar{q}'^{\beta}\, q'_{\beta}\right)_{V\mp A}$$

$$O_{4,6} = \left(\bar{q}^{\beta}\, b_{\alpha}\right)_{V-A} \sum_{q'} \left(\bar{q}'^{\alpha}\, q'_{\beta}\right)_{V\mp A}. \tag{16.115}$$

However, the Wilson coefficients here are much smaller than C_1 and C_2 :

$$C_3 = 0.017, \qquad C_4 = -0.044, \qquad C_5 = 0.011, \qquad C_6 = -0.056. \tag{16.116}$$

Thus generally we shall not consider the penguin operators.

Now for the Lagrangian (108), or the corresponding one for $c \to s + u + \bar{q}$, it is simple to write the decay widths for $D, B \to$ hadrons in the spectator quark model for the Cabbibo favored decays:

$$\begin{aligned} & \Gamma\left[D\ (c\,\bar{q}) \to \bar{q}\ (c \to s + d + \bar{u})\right] \\ = \ & \Gamma\left(c \to s + d + \bar{u}\right) \\ = \ & \frac{G_F^2\, m_c^5}{192\pi^3}\left[|V_{cs}|^2\,|V_{ud}|^2\ F\left(\frac{m_s}{m_c}\right)\right]\left[3C_1^2 + 2C_1C_2 + 3C_2^2\right] \end{aligned} \tag{16.117}$$

$$\begin{aligned} & \Gamma\left[\bar{B}_d\ \left(b\bar{d}\right) \to \bar{d}\begin{pmatrix} (b \to c + d + \bar{u}) \\ \to c + s + \bar{c} \end{pmatrix}\right] \\ = \ & \Gamma\begin{pmatrix} b \to c + d + \bar{u} \\ \to c + s + \bar{c} \end{pmatrix} \\ = \ & \frac{G_F^2}{192\pi^3}\left[3C_1^2 + 2C_1\ C_2 + 3C_2^2\right]\left[|V_{cb}|^2\ |V_{ud}|^2\ F\left(m_c/m_b\right)\right] \\ & +0.12\ \Gamma\left[b \to c + d + \bar{u}\right] \end{aligned} \tag{16.118}$$

where we have used

$$\frac{B\left(\bar{B} \to X_{c\bar{c}s}\right)}{B\left(\bar{B} \to X_{c\bar{u}d}\right)} \approx 0.12. \tag{16.119}$$

We also take

$$\frac{B\left(b \to X\ \tau\ \bar{\nu}_{\tau}\right)}{B\left(b \to X\ e\ \bar{\nu}_e\right)} \approx 0.24. \tag{16.120}$$

Hence from Eqs. (62), (117) and (63), (118), (119) and (120) we get

$$\begin{aligned}\left(D^0\right)_{SL} &= \left(D^+\right)_{SL} = \frac{1}{2+\left(3C_1^2+2C_1\ C_2+3C_2^2\right)} = \left(D_s^+\right)_{SL} \\ &\approx 16\ \% \qquad (16.121a)\\ \frac{\Gamma_{D^0}}{\Gamma_{D^+}} &= 1 \qquad (16.121b)\end{aligned}$$

and

$$\begin{aligned}\left(\bar{B}^0\right)_{SL} &= \left(B^-\right)_{SL} = \frac{1}{2.24+1.2\left(3C_1^2+2C_1\ C_2+3C_2^2\right)} \\ &= \left(B_s\right)_{SL} \approx 15\ \% \qquad (16.122a)\\ \frac{\Gamma_{\bar{B}^0}}{\Gamma_{B^-}} &= 1 \qquad (16.122b)\end{aligned}$$

where in Eqs. (121) and (122), we have used the values of C_1 and C_2 given in Eqs. (113) and (112) respectively.

Let us first discuss $(D)_{SL}$, while Eq. (121) is consistent with the experimental value for $(D^+)_{SL}$ but it is about a factor of two greater than $(D^0)_{SL}$. Also we note that the experimental values, namely

$$\frac{(D^+)_{SL}}{(D^0)_{SL}} \approx 2.2 \pm 0.4 \qquad (16.123)$$

$$\frac{\Gamma_{D^0}}{\Gamma_{D^+}} = \frac{\tau_{D^+}}{\tau_{D^0}} \approx 2.55 \pm 0.04 \qquad (16.124)$$

are in disagreement with the predictions of spectator quark model given in Eqs. (121). A possible explanation is as follows: There are two spectator diagrams shown in Fig. 6, similar to Fig. 3 for B decays. Figures 6a and 6b correspond to the charged and neutral current operators in the Lagrangian (108) and are multiplied by the coefficients C_1 and C_2 respectively. For D^+, $\bar{q}=\bar{d}$, and in the two diagrams we have the same final states (for example $\bar{K}^0\pi^+$). The two diagrams destructively interfere, so that $\Gamma\left(D^+ \to \text{hadron}\right) \propto$

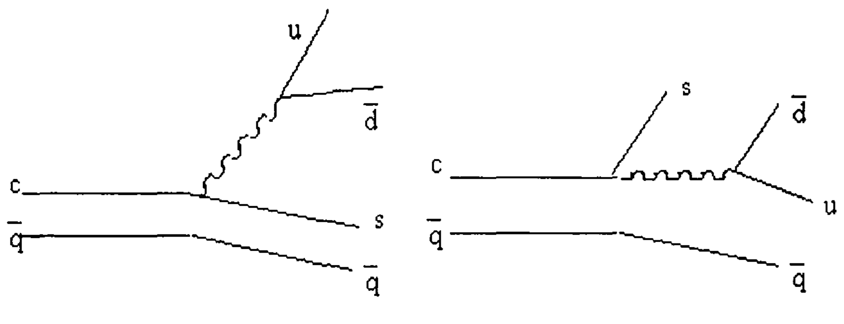

Figure 6 Spectator diagrams corresponding to the effective (a) charged (b) neutral current operators.

$3(C_1+C_2)^2$. On the other hand for D^0, $\bar{q}=\bar{u}$ and we have different final states for Figs. 6a and 6b (for example $K^-\pi^+$ for Fig. 6a and $\bar{K}^0\pi^0$ for Fig. 6b). Thus in this case $\Gamma(D^0 \to \text{hadrons}) \propto 3(C_1^2+C_2^2)$. Hence we have

$$\frac{\Gamma_{D^0}}{\Gamma_{D^+}} = \frac{2+3(C_1^2+C_2^2)}{2+3(C_1+C_2)^2} = \frac{7.3}{3.73} \approx 2 \qquad (16.125a)$$

$$\frac{(D^+)_{SL}}{(D^0)_{SL}} \approx 2. \qquad (16.125b)$$

Thus whereas Eq. (125b) is in agreement with Eq. (123), Eq. (125a) has still some discrepancy. Numerically

$$\left(D^+\right)_{SL} \approx 27\ \%, \qquad \left(D^0\right)_{SL} \approx 14\ \%. \qquad (16.126)$$

Both these values are not in agreement with their experimental values given in Eq. (64).

The spectator approach is expected to be a better approximation for B decay as b quark is heavy. We now discuss the comparison of the prediction of spectator model for the semi-leptonic branching ratio B_{SL} with its experimental value. First we note that the naive quark model gives $B_{SL} \approx 15\ \%$ [cf Eq. 122a]. Charm

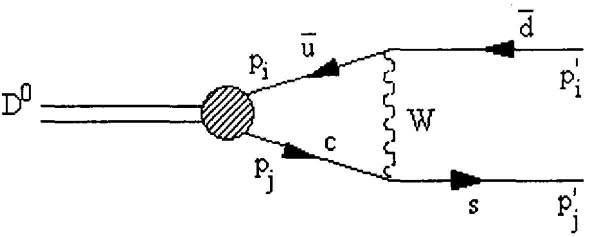

Figure 7 W-exchange quark level diagram for hadronic D^0 decay.

mass corrections to $\Gamma(b \to c\bar{c}s)$ have been found to be large and can reduce the theoretical prediction for B_{SL} :

$$B_{SL} = (11.7 \pm 1.4 \pm 1.0)\ \% \tag{16.127}$$

The *CLEO* and Argus collaboration have measured the branching ratio B_{SL} :

$$B_{SL} = (10.3 \pm 0.39)\ \%. \tag{16.128}$$

The corresponding *LEP* number measured from the $Z^0 \to b\bar{b}$ decay rate are consistent with (128). There does not appear to be any significant discrepancy between Eqs. (127) and (128).

16.3.5 Scattering and annihilation diagrams

There are two kinds of mechanism for the hadronic decays of heavy mesons which we have not considered. They are depicted for Cabibbo favored decays of D^0 and D_s mesons in Figs. 7 and 8 respectively.

The basic processes depicted in Figs. 7 and 8 are respectively $c+\bar{u} \to s+\bar{d}$ and $c+\bar{s} \to u+\bar{d}$ where for the second process, the analogue of Eq. (108) gives the effective Lagrangian

$$\begin{aligned} L_{eff} &= \frac{G_F}{\sqrt{2}}\xi \left[C_1 (\bar{s}^\alpha\, c_\alpha)_{V-A} \left(\bar{u}^\beta\, d_\beta\right)_{V-A} \right. \\ &\quad \left. + C_2 (\bar{s}^\alpha\, c_\beta)_{V-A} \left(\bar{u}^\beta\, d_\alpha\right)_{V-A}\right] \end{aligned} \tag{16.129}$$

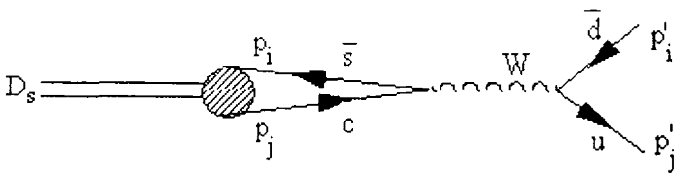

Figure 8 W-annihilation quark level diagram for hadronic D_s^0 decay.

where $\xi = |V_{cs}|\ |V_{ud}|$ while for the first process the effective Langrangian is obtained by Fierz rearrangement. Thus the T-matrix for the process $c + \bar{u} \to s + \bar{d}$ is given by

$$\begin{aligned}
&T_{\text{exch}} \\
\sim\ & \frac{G_F}{\sqrt{2}}\xi\frac{1}{3}\left(C_1\delta_{\alpha\beta}\ \delta_{\alpha\beta} + C_2\delta_{\alpha\alpha}\ \delta_{\beta\beta}\right)\left[\bar{u}\left(\mathbf{p}_j'\right)\ \gamma^{\mu}\left(1-\gamma^5\right)u\left(\mathbf{p}_j\right)\right] \\
& \times\left[-\bar{v}\left(\mathbf{p}_i\right)\ \gamma_{\mu}\ \left(1-\gamma_5\right)\ v\left(\mathbf{p}_i'\right)\right] \\
=\ & -\frac{G_F}{\sqrt{2}}\ \xi\ \left(C_1 + 3C_2\right)\ \left[\bar{u}\left(\mathbf{p}_j'\right)\ \gamma^{\mu}\ \left(1-\gamma^5\right)\ u\left(\mathbf{p}_j\right)\right] \\
& \times\left[\bar{u}\left(\mathbf{p}_i'\right)\ \gamma_{\mu}\ \left(1+\gamma_5\right)\ u\left(\mathbf{p}_i\right)\right], \qquad (16.130\text{a})
\end{aligned}$$

where we expressed the v-spinor in terms of u-spinor by the relation $v = C^{\dagger}\bar{u}^T$. Similarly for the annihilation diagram (Fig. 8), Eq. (129) gives the T-matrix:

$$\begin{aligned}
&T_{\text{ann}} \\
\sim\ & \frac{G_F}{\sqrt{2}}\xi\frac{1}{3}\left(C_1\delta_{\alpha\alpha}\ \delta_{\beta\beta} + C_2\delta_{\alpha\beta}\ \delta_{\alpha\beta}\right)\left[\bar{v}\left(\mathbf{p}_i\right)\ \gamma^{\mu}\left(1-\gamma^5\right)u\left(\mathbf{p}_j\right)\right] \\
& \times\left[\bar{u}\left(\mathbf{p}_j'\right)\ \gamma_{\mu}\ \left(1-\gamma_5\right)\ v\left(\mathbf{p}_i'\right)\right] \\
=\ & -\frac{G_F}{\sqrt{2}}\ \xi\ \left(3C_1 + C_2\right)\ \left[\bar{u}\left(\mathbf{p}_j'\right)\ \gamma^{\mu}\ \left(1-\gamma^5\right)\ u\left(\mathbf{p}_j\right)\right] \\
& \times\left[\bar{v}\left(\mathbf{p}_i\right)\ \gamma_{\mu}\ \left(1-\gamma_5\right)\ v\left(\mathbf{p}_i'\right)\right] \qquad (16.130\text{b})
\end{aligned}$$

where we have used the Fierz rearrangement in going from the first line to the second line. Thus we conclude that both diagrams give the same results apart from the color factors. In taking the nonrelativistic limit it is convenient to express v-spinor in terms of u-spinor and use the relation $\bar{v}\,\Gamma_i\,v = \varepsilon_i\,u\,\Gamma_i\,\bar{u}$, with $\varepsilon_i = \pm 1$, $\Gamma_i = \gamma_\lambda, \gamma_\lambda\gamma_5$. We note that m_i', $m_j' = m$ (the mass of u and d quark), $m_i = m_c \gg m$, $m_j = m_s$ or m_d; $E_i' = E_j' = E$, $m_c + E_j = 2E$, where in the nonrelativistic limit, we have put $E_i = m_c$ and we also put $m_s = m_d$. Using Pauli representation of Dirac matrices, it is a straightforward but long calculation to obtain the cross section σ for the scattering or annihilation processes shown in Figs. 7 and 8. Suppressing the color factor, we get for the singlet and triplet scattering cross sections respectively

$$\sigma_S = \frac{|\xi|^2}{8\pi} G_F^2 \left(8m^2\right) \frac{1}{v}, \qquad (16.131a)$$

$$\sigma_T = \frac{|\xi|^2}{8\pi} G_F^2 \left(\frac{8}{3} m_c^2\right) \frac{1}{v}, \qquad (16.131b)$$

where v is the incoming velocity in the initial state. Now defining the decay width as

$$\Gamma = v\ |\Psi_s(0)|^2\ \sigma, \qquad (16.132)$$

we get for the triplet state

$$\Gamma\left({}^3S_1 \to u\,\bar{d}\right) = \frac{1}{8\pi}\ G_F^2\ \xi^2\ \frac{8}{3}\ m_{D^*}^2\ |\Psi_s(0)|^2, \qquad (16.133)$$

where we have put $m_{D^*}^2 = (m_c + m_s)^2 \approx m_c^2$. For the singlet state, we get

$$\Gamma\left({}^1S_0 \to u\,\bar{d}\right) = \frac{1}{8\pi}\ G_F^2\ \xi^2\ 8\ m^2\ |\Psi_s(0)|^2. \qquad (16.134)$$

Note the important fact that the decay width for the singlet state (D) is proportional to the square of the light quark masses; in the spectator quark model it is proportional to m_c^2. This is called helicity suppression.

Inserting back the color factors we have finally

$$\Gamma^{D}_{exch} = \frac{G_F^2}{8\pi} |V_{cs}|^2 |V_{ud}|^2 \left(8\ m^2\right) |\Psi_s(0)|^2\ (C_1+3C_2)^2, \tag{16.135}$$

$$\Gamma^{D_s}_{ann} = \frac{G_F^2}{8\pi} |V_{cs}|^2 |V_{ud}|^2 \left(8\ m^2\right) |\Psi_s(0)|^2 (3C_1+C_2)^2. \tag{16.136}$$

It is clear from Eqs. (135) and (136), that both the exchange and annihilation diagrams are helicity suppressed, but Γ_{exch} is color suppressed and Γ_{ann} is color enhanced.

It is intresting to see that for the annihilation diagram, one can get the same result just by writing the T-matrix for the $D_s \to$ hadrons in the form

$$T = \frac{G_F}{\sqrt{2}} \xi \frac{\delta_{\alpha\alpha}}{\sqrt{3}} \left(C_1 + \frac{1}{3}C_2\right) \left\langle X \left| J^{W\mu} \right| 0 \right\rangle \left\langle 0 \left| J_\mu^{W\dagger} \right| D_s \right\rangle \tag{16.137}$$

where $J_\mu^{W\dagger}$ and $J^{W\mu}$ are color singlet currents with appropriate quantum numbers. Then

$$\Gamma^{D_s}_{ann} = \frac{G_F^2}{2} |\xi|^2 \frac{(3C_1+C_2)^2}{3} (2\pi)^7 \sum_{\text{spin}} \int d^3p_X\ \delta(p-p_X) \times \left|\left\langle 0 \left| J_\mu^{W\dagger} \right| D_s \right\rangle\right|^2 \left|\left\langle X \left| J^{W\mu} \right| 0 \right\rangle\right|^2. \tag{16.138}$$

Now from Lorentz invariance

$$\left\langle 0 \left| J_\mu^{W\dagger} \right| D_s \right\rangle = i\ \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2p_0}}\ f_{D_s}\, p_\mu, \tag{16.139}$$

while

$$\sum_{\text{spin}} \int d^3p_X\ \delta(p_X - p) \left\langle 0 \left| J_\mu^{W\dagger} \right| X \right\rangle \left\langle X \left| J^{W\mu} \right| 0 \right\rangle = \frac{1}{(2\pi)^3} \theta(p_0) \left[\left(-p^2\ g_{\mu\lambda} + p_\mu\, p_\lambda\right) \rho\left(p^2\right) + p_\mu\, p_\lambda\ \sigma\left(p^2\right)\right]. \tag{16.140}$$

Hence we get

$$\Gamma_{ann}^{D_s} = \frac{G_F^2}{2}|\xi|^2 \frac{(3C_1+C_2)^2}{3}\frac{2\pi}{2\, m_{D_s}}\, f_{D_s}^2\, m_{D_s}^4\, \sigma\left(m_D^2\right). \tag{16.141}$$

Now from dimensional consideration

$$\sigma\left(m_D^2\right) = \frac{2\, m^2}{4\pi^2\, m_{D_s}^2}. \tag{16.142}$$

Thus we obtain

$$\Gamma_{ann}^{D_s} = \frac{G_F^2}{4\pi}\, |V_{cs}|^2\, |V_{ud}|^2\, f_{D_s}^2\, m_{D_s}\, \left(m^2\right)\frac{(3C_1+C_2)^2}{3}. \tag{16.143}$$

One gets eactly the same results if in Eq. (137) one replaces $|X\rangle$ by $\left|u\bar{d}\right\rangle$ (see problem 3). Comparing Eq. (143) with Eq. (136) we get

$$f_{D_s}^2 = \frac{12\left|\Psi_s(0)\right|^2}{m_{D_s}}. \tag{16.144}$$

It is interesting to note that the vacuum saturation of the T-matrix for $D_s \to$ hadrons viz. $\left\langle X\left|J^{W\mu}\right|0\right\rangle\ \left\langle 0\left|J_\mu^{W\dagger}\right|D_s\right\rangle$ gives the same results as the annihilation diagram.

From Eqs. (117), (135) and (136) and (143), we get

$$\frac{\Gamma_{ann}^{D_s}}{\Gamma_{sp}^{D_s}} = 16\pi^2\frac{m_d^2\, f_{D_s}^2\, m_{D_s}}{m_c^5\, F(m_s/m_c)}\frac{(3C_1+C_2)^2}{(3C_1^2+2C_1\, C_2+3C_2^2)}, \tag{16.145}$$

$$\frac{\Gamma_{exch}^{D^0}}{\Gamma_{sp}^{D^0}} = 8\pi^2\frac{(m_d^2+m_s^2)\, f_D^2\, m_D}{m_c^5\, F(m_s/m_c)}\frac{(C_1+3C_2)^2}{3\,(C_1^2+C_2^2)}, \tag{16.146}$$

where in Eq. (146) the factor $3\,(C_1^2+C_2^2)$ appears for the reason discussed earlier in connection with Eq. (125). Using $C_1 =$

1.24, $C_2 = -0.48$, $m_c \approx 1.50$ GeV, $F(m_s/m_c) \approx 0.47$, $f_D = 200\ MeV$ and $f_{D_s} = 240$ MeV, we get from Eq. (144)

$$\frac{\Gamma_{ann}^{D_s}}{\Gamma_{sp}^{D_s}} \approx 0.63, \tag{16.147}$$

while $\Gamma_{exch}^{D^0}/\Gamma_{sp}^{D^0}$ is negligible. The annihilation diagram gives negligible contribution to D^+ decays. Taking into account Eq. (147)

$$\Gamma(D_s \to \text{hadrons}) = (1.6)\ \Gamma_{sp} \tag{16.148}$$

where Γ_{sp} is given in Eq. (117) and [cf. Eq. (121)]

$$\left(D_s^+\right)_{SL} = \frac{1}{2 + 1.6\left[3C_1^2 + 2C_1\ C_2 + 3C_2^2\right]} \approx 12\ \% \tag{16.149}$$

$$\frac{\tau_{D_s^+}}{\tau_{D^0}} \approx \frac{3\left(C_1^2 + C_2^2\right)}{1.6\left[3C_1^2 + 2C_1\ C_2 + 3C_2^2\right]} \approx 0.82. \tag{16.150}$$

While the prediction (149) is consistent with the experimental limit $(D_s^+)_{SL} < 20$ %, the prediction (150) is not consistent with its experimental value ≈ 1.12.

We conclude that the contribution of the annihilation diagram is helicity suppressed, but enhancement by a factor of $192\pi^2$ due to phase space and that due to color factor more than compensate the helicity suppression. However, there are still problems to explain the ratios $\tau_{D^+}/\tau_{D^0} \approx 2.5$ and $\tau_{D_s^+}/\tau_{D^0} \approx 1.12$ as was discussed in Sec. 3.4.

Finally the annihilation diagram for B decays are Cabibbo suppressed and they may be neglected.

16.4 Problems

1. Taking into account finite width for ρ meson and using Eq. (17), show that

$$\begin{aligned}&\Gamma\left(\tau^- \to \rho^- \nu_\tau \to \pi^- \pi^0 \nu_\tau\right)\\ = &\frac{G_F^2 \cos^2\theta_c}{384\,\pi^3} m_\tau^3 \int_{2m_\pi^2}^{m_\tau^2} |F_\pi(s)|^2 \\ &\times \left(1-\frac{s}{m_\tau^2}\right)^2 \left(1+\frac{2s}{m_\tau^2}\right)\left(1-\frac{4\,m_\pi^2}{s}\right)^{3/2} ds, \qquad \text{(A)}\end{aligned}$$

where

$$F_\pi(s) = \frac{f_{\rho\pi\pi}\, f_\rho}{\left[\left(s-m_\rho^2\right)+i\, m_\rho\, \Gamma\right]}.$$

Hint:

$$\left[2\pi\delta\left(s-m_\rho^2\right) \to \frac{2\, m_\rho\, \Gamma}{\left(s-m_\rho^2\right)+m_\rho^2\, \Gamma^2}\right].$$

Considering the process

$$e^-e^+ \to \gamma \to \pi^+\pi^-,$$

show that the cross section is given by ($s \gg 4m_e^2$)

$$\sigma_{\pi^+\pi^-}(s) = \frac{\pi}{3\,s}\,\alpha^2\, |F_\pi(s)|^2 \left(1-\frac{4\,m_\pi^2}{s}\right)^{3/2}$$

where $F_\pi\,(s)$ is the electromagnetic form factor of pion:

$$\begin{aligned}\left\langle\pi^+\pi^-\left|J_\lambda^{em}\right|0\right\rangle &\propto F_\pi(q^2)\,(p_1-p_2)_\lambda\\ s &= q^2=(p_1+p_2)^2.\end{aligned}$$

Using Eq. (37), show that we get back Eq. (A). From Eq. (A), find the decay rate for $\tau^- \to \pi^-\pi^0\,\nu_\tau$ through ρ-resonance.

2. Taking into account finite width of a_1 meson, and Eq. (18), show that (taking $m_\pi = 0$)

$$\begin{aligned}
&\Gamma\left(\tau^- \to a_1^-\, \nu_\tau \to \pi^-\, \rho^0\, \nu_\tau\right) \\
=\ &\Gamma\left(\tau^- \to \pi^0\, \rho^-\, \nu_\tau\right) \\
=\ &\frac{G_F^2\, \cos^2\theta_c}{96\,\pi^3}\, m_\tau^3 \int_{m_\rho^2}^{m_\tau^2} ds\, \left|F_{\rho\pi}(s)\right|^2 \\
&\times \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \left\{ 1 + \frac{s}{8\, m_\rho^2} \right. \\
&\left. \times \left[\left(1 + \frac{m_\rho^2}{s}\right)^2 + 2r\, \left(1 - \frac{m_\rho^2}{s}\right)^2 + r^2 \frac{\left(1 - \frac{m_\rho^2}{s}\right)^4}{\left(1 + \frac{m_\rho^2}{s}\right)^2} \right] \right\} \qquad \text{(B)}
\end{aligned}$$

where

$$F_{\rho\pi}(s) = \frac{f_{a_1}\, F_{a_{1\rho\pi}}}{\left[\left(s - m_{a_1}^2\right) + i\, m_{a_1}\, \Gamma\right]}.$$

The $a_{1\rho\pi}$ couplings are defined by the decay amplitude T:

$$T \propto 2\, m_{a_1} F_{a_{1\rho\pi}} \left[\eta \cdot \varepsilon + \frac{r}{k \cdot q} (\eta \cdot k)(\varepsilon \cdot q) \right]$$

where η_μ, ε_μ are polarization vectors of a_1 and ρ, q and k are their four momenta. In order to derive (B), first show that

$$\begin{aligned}
&\Gamma(a_1 \to \rho\, \pi) \\
=\ &\frac{\left(F_{a_{1\rho\pi}}\right)^2}{6\pi}\, m_{a_1} \left(1 - \frac{m_\rho^2}{m_{a_1}^2}\right) \left\{ 1 + \frac{m_{a_1}^2}{8\, m_\rho^2} \right. \\
&\left. \times \left[\left(1 + \frac{m_\rho^2}{m_{a_1}^2}\right)^2 + 2r\, \left(1 - \frac{m_\rho^2}{m_{a_1}^2}\right)^2 + r^2 \frac{\left(1 - \frac{m_\rho^2}{m_{a_1}^2}\right)^4}{\left(1 + \frac{m_\rho^2}{m_{a_1}^2}\right)^2} \right] \right\}.
\end{aligned}$$

Using the experimental numbers for $\Gamma(\tau^- \to \pi^- \rho^0 \nu_\tau)$ and $\Gamma(a_1 \to \rho\,\pi)$, determine $F_{a_1\rho\pi}$ and r, using $f_{a_1}^2 = f_\rho^2 = 2F_\pi^2\, m_\rho^2$.

3. Using Eqs. (137) and (138) and writing

$$\begin{aligned} &\left\langle X \left| J_\mu^W \right| 0 \right\rangle \\ = &\left\langle \mu \bar{d} \left| J_\mu^W \right| 0 \right\rangle \\ = &\frac{1}{(2\,\pi)^3} \sqrt{\frac{m^2}{p_{10}\,p_{20}}}\ \left[\bar{u}(p_1)\ \gamma_\mu\ (1-\gamma_5)\ v(p_2)\right] \end{aligned}$$

$$\int d^3\,p_X\ \delta(p - p_X) \to \int d^3\,p_1\ d^3\,p_2\ \delta(p - p_1 - p_2),$$

show that

$$\Gamma\left(D_s \to u\bar{d}\right) = \frac{(3C_1 + C_2)^2}{3} \frac{G_F^2}{4\,\pi}\ |V_{cs}|^2\ |V_{ud}|^2 f_{D_s}^2\ m_{D_s}\ \left(m^2\right).$$

4. Writing

$$\left\langle 0 \left| J_\mu^{W\dagger} \right| D_s^* \right\rangle \sim\ f_{D_s^*}\ \varepsilon_\mu$$

where ε_μ is the polarization of D_s^*, show that ,

$$\Gamma\left(D_s^* \to u\bar{d}\right) = \frac{(3C_1 + C_2)^2}{3} \frac{G_F^2}{4\,\pi}\ |V_{cs}|^2\ |V_{ud}|^2\, \frac{1}{3} f_{D_s^*}^2\ m_{D_s^*}.$$

Comparing it with Eq. (133) when multiplied by the color factor $(3C_1 + C_2)^2$, show that

$$f_{D_s^*}^2 = 12\ \left|\Psi_s(0)\right|^2 m_{D^*}.$$

Hence show that

$$f_{D_s^*} = \left(m_{D_s}\ m_{D_s^*}\right)^{1/2}\ f_{D_s}.$$

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Chapter 17

GRAND UNIFICATION, SUPERSYMMETRY AND STRINGS

17.1 Grand Unification

As we have seen all fundamental forces are of gauge nature. Thus they may be deduced from some generalized gauge principle. Ingradients of gauge models are

(i) Choice of gauge group

(ii) Choice of fundamental representations

(iii) If gauge symmetry is spontaneously broken, choice of Higgs sector which generate mass parameters.

Gauge principle restricts the form of interaction. Also gauge model may be renormalizable if its fermion content is such that the model is anomaly free. At low energies we have a spontaneously broken SU(2)×U(1) gauge group for electroweak forces and an exact $SU_C(3)$ gauge group for the strong quark–gluon forces. Thus the standard model involves

$$\begin{array}{ccccccc} G_1 & \equiv & SU(2) & \times & U(1) & \times & S_cU(3) \\ & & g_2 & & g' & & g_s > g_2 > g' \end{array}$$

The fermion content of G_1 for the first generation is

$$\begin{array}{llll} \begin{pmatrix} u \\ d \end{pmatrix}^a_L & u^a_R & d^a_R & a = r, y, b \\ (2,3) & (1,3) & (1,3) & \\ \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L & e_R & & \\ (2,1) & (1,1) & & \end{array}$$

Thus we have 15 two-component fermion states per generation. The electroweak part of G_1 is spontaneously broken

$$G_1 \equiv SU(2)_L \times U(1) \times SU_c(3) \to G_2 \equiv U_{em}(1) \times SU_c(3).$$

Also the experimental data show that

$$\rho \equiv \left(\frac{m_W}{m_z \cos\theta_W}\right)^2 \approx 1$$

which implies that $\mathrm{SU}_L(2)\times\mathrm{U}(1)$ breaking predominantly occurs only through an $\mathrm{SU}_L(2)$ Higgs doublet or doublets. Despite the fact that the above picture is capable of providing a current phenomenological description of all the observed "low energy physics", many questions given below remain:

(i) 3 independent coupling constants

(ii) no charge quantization because of $U(1)$ factor

(iii) no relation between lepton and quark masses

(iv) why are 3 generations identical in representation content but vastly different in mass ?

(v) why is the intergeneration mixing small ?

(vi) no principle limiting the number of $SU(2)$ generations – e, μ, τ, $\cdots$.

Could the situation be improved ? Grand unification of electroweak and strong quark–gluon forces answer some of these questions but say nothing about the generation problem. The basic hypothesis is that there exists a simple group G

$$G \supset G_1 \equiv SU_L(2) \times U(1) \times SU_c(3)$$

which is characterized by a single coupling constant and that all interactions are generated by G. Quarks and leptons are in general members of the same multiplets of the group G. Then at some energy scale, G suffers a breakdown to G_1:

$$\begin{array}{ccl} G & \rightarrow & G_1 \rightarrow G_2 \equiv U_{em}(1) \times SU_c(3) \\ M_X & \gg & m_W \qquad \approx 100 \text{ GeV} \end{array}$$

The rank of $G \geq 4$ since the rank of G_1 is 4 and some possibilities for G are

(i) $G \equiv SU(n) \supset SU(n-3) \times U(1) \times SU_c(3)$
e.g. $SU(5)$

(ii) $G \equiv SO(n) \supset SO(n-6) \times U(1) \times SU_c(3)$
e.g.

$$SO(10) \supset SO(4) \times U(1) \times SU_c(3)$$

or

$$SU_L(2) \times SU_R(2)$$

There may be intermediate steps before reaching the right-hand side. Another possibility is $SO(10) \supset SU(5) \times U(1)$.

(iii) Exceptional groups
$E_6 \supset SU(3) \times SU(3) \times SU_c(3)$
$E_7 \supset SU(6) \times SU_c(3)$

(iv) Any semi–simple group

$$G = G' \times G'' \times \cdots$$

with an additional reflection symmetry will also do.

17.1.1 Q^2 evolution of gauge coupling constants and the grand unification mass scale

At presently available energies g_s, g_2 and g' are very different. How then can we have G with a single coupling constant ? This is possible since due to quantum radiative corrections g's are Q^2 dependent. Thus if we have a grand unification theory (GUT), there must be a point Q^2 where g_s, g_2 and g' coincide. To see how this comes about, let us consider the Q^2 evolution equation for the effective coupling constant in a general gauge theory [see Appendix B for more details].

$$\frac{d\alpha_g^{-1}}{d \ln Q^2} = b + \ldots, \tag{17.1}$$

for $Q^2 >$ masses of fermions and gauge bosons but $Q^2 < M_X^2$ and

$$b = \frac{1}{4}\left(\frac{11}{3}C_2(G) - \frac{4}{3}T_f\right). \tag{17.2}$$

For $SU_c(3), C_2(G) = 3$,

$$\begin{aligned} T_f \delta_{AB} &= Tr\left(\frac{\lambda^A}{2}\frac{\lambda^B}{2}\right)[\text{number of } SU_c(3) \text{ triplets}] \\ &= \frac{1}{2}\delta_{AB} n_f, \end{aligned} \tag{17.3}$$

where n_f is the number of quark flavors, known to be six. For $SU_L(2)$ in the electroweak group,

$$\begin{aligned} C_2(G) &= 2,\ T_f \delta_{rs} = Tr\,(\tau_r/2\ \tau_s/2) \\ &\times \left[\frac{1}{2} \text{ number of left-handed doublets}\right], \end{aligned} \tag{17.4a}$$

where $\frac{1}{2}$ comes from the fact that we have only left-handed couplings. Thus

$$T_f \delta_{rs} = \frac{1}{2}\delta_{rs}\frac{1}{2}\left(2n_f\right). \tag{17.4b}$$

Here $2n_f = 12$ appears since each generation has one lepton doublet and 3 quark doublets (one for each color). For $U(1)$ group of electroweak

$$C_2 = 0,\ T_f = \frac{1}{2}\sum_f \left(\frac{1}{2}Y\right)^2, \tag{17.5a}$$

because each fermion has either left-handed or right-handed coupling. Thus

$$\begin{aligned} T_f &= \frac{1}{2}\left\{\frac{n_f}{2}\frac{1}{4}\left[3\cdot\frac{1}{9}+3\cdot\frac{1}{9}+1+1+3\cdot\frac{16}{9}+3\cdot\frac{4}{9}+4\right]\right\} \\ &= \frac{1}{2}\frac{5}{3}n_f. \end{aligned} \tag{17.5b}$$

It is convenient to introduce $g_1 = \sqrt{5/3}g'$. Thus for $Q^2 \gg m_W^2$, m_Z^2, m_t^2

$$\frac{d\alpha_s^{-1}}{d\ln Q^2} = b_s,\ b_s = \frac{1}{4\pi}\left(\frac{33}{3}-\frac{2}{3}n_f\right) > 0 \tag{17.6}$$

$$\frac{d\alpha_2^{-1}}{d\ln Q^2} = b_2,\ b_2 = \frac{1}{4\pi}\left(\frac{22}{3}-\frac{2}{3}n_f\right) > 0 \tag{17.7}$$

$$\frac{d\alpha_1^{-1}}{d\ln Q^2} = b_1,\ b_1 = \frac{1}{4\pi}\left(-\frac{2}{3}n_f\right) < 0. \tag{17.8}$$

These renormalization group (RG) equations have solution

$$\alpha_i^{-1}\left(Q^2\right) = \alpha_i^{-1}\left(m_Z^2\right) + b_i \ln\frac{Q^2}{m_Z^2}. \tag{17.9}$$

Hence as Q^2 increases

1. $\alpha_s\left(Q^2\right)$ decreases

2. $\alpha_2(Q^2)$ also decreases but less rapidly than $\alpha_s(Q^2)$

3. $\alpha_1(Q^2)$ increases.

Thus, since $\alpha_1 < \alpha_2 < \alpha_s$ at available energies, at some $Q^2 = m_X^2, \alpha_s, \alpha_2$ and α_1 should coincide

$$C_3^2\alpha_s\left(M_X^2\right) = C_2^2\alpha_2\left(M_X^2\right) = C_1^2\alpha_1\left(M_X^2\right) = \alpha_G, \qquad (17.10)$$

where C_3, C_2 and C_1 are group theory numbers (so that the generators of the group are properly normalized) and are of order 1. For example for SU(5), $C_3^2 = C_2^2 = C_1^2 = 1$. M_X is called the grand unification mass scale at which one has only one free coupling constant α_G. Since the gauge coupling constants are supposed to merge into one in GUT, the value of $\sin^2\theta_W$, which measures the relative strengths of α_1 and α_2 at $Q^2 = m_Z^2$, namely

$$\frac{\alpha_1(m_Z^2)}{\alpha_2(m_Z^2)} = \frac{5}{3}\tan^2\theta_W, \ \alpha_2\left(m_Z^2\right) = \frac{\alpha(m_Z^2)}{\sin^2\theta_W}, \qquad (17.11)$$

enters into the determination of α_G and M_X. Whether the three coupling constants meet at a single point $Q^2 = M_X^2$ depends on the gauge group G. It may be noted that to include the contribution of Higgs doublet one adds $-\frac{1}{6}n_H$ in the expression given in Eqs. (7) and (8), where n_H is the number of Higgs doublets.

17.1.2 General consequences of GUTs

The general consequences which one would expect from GUTs are

1. G being simple, the charge operator will be a generator of the group and traceless. So if it acts on any representation of G containing quarks and leptons, it would give some relation between quark and lepton charges (sum of charges in each multiplet = 0) i.e. we would have charge quantization.

2. The fact that quarks and leptons share the same representation(s) of G, there would be relationship between quark and lepton masses.

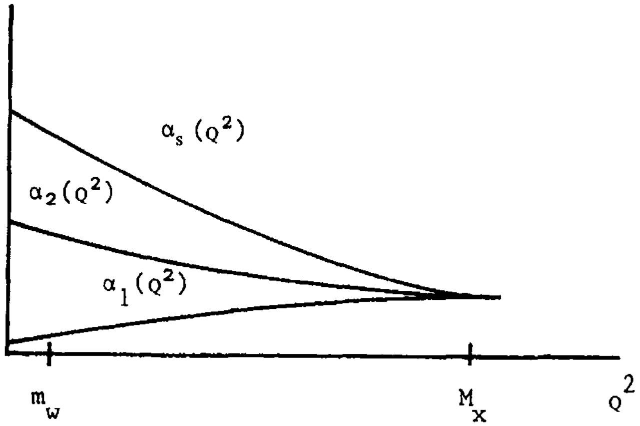

Figure 1 Behavior of $\alpha_s(Q^2)$, $\alpha_2(Q^2)$ and $\alpha_1(Q^2)$ versus Q^2.

3. Since quarks and leptons share the same representation(s) of G and since gauge theories contain vector bosons linking all particles in a multiplet, there would in general be some interation changing quarks into leptons, thereby violating baryon charge (B) and lepton charge (L) conservation. At present energy scale $E \ll E_{GUT} \approx M_X$, we have effective B and L conservation but this conservation cannot be exact.

In general B violating forces will make proton unstable and so one has to watch that protons do not decay too quickly, the present experimental limit on proton decay is

$$\begin{aligned}\tau_p &\geq 1.6 \times 10^{25} \text{ years (independent of modes)} \\ &> 10^{31} \text{ to } 5 \times 10^{32} \text{ years (mode dependent).} \qquad (17.12)\end{aligned}$$

Using $\alpha_3(m_Z)$ and $\alpha(m_Z)$ in $\tilde{M}_S$ renormalization scheme adopted for the definition of the coupling constants:

$$\begin{aligned}\alpha_3(m_Z) &= \alpha_s(m_Z) = 0.1214 \pm 0.0031 \\ \alpha^{-1}(m_Z) &= 127.88 \pm 0.09 \qquad (17.13)\end{aligned}$$

as inputs, one can predict both M_X and $\sin^2\theta_W\,(m_Z)$ [cf. Eqs.(10) and (11) and RG equations] in GUT models such as $SU(5)$ with no extra scales between the electroweak scale and the GUT scale M_X. Typical predictions are $\sin^2\theta_W = 0.215 \pm 0.003$ and $M_X \approx \left(2\ \begin{matrix}+2\\-1\end{matrix}\right) \times 10^{14}$ GeV. This value of M_X in turn gives $\tau(p \to e^+\pi^0) \approx 4 \times 10^{29\pm0.7\pm1.2}$ years which contradicts the experimental limit $\tau(p \to e^+\pi^+) \geq 5 \times 10^{32}$ years. Likewise the above predicted value of $\sin^2\theta_W\,(m_Z)$ differ from the presently determined value of $\sin^2\theta_W\,(m_Z)$.

$$\sin^2\theta_W\,(m_Z) = 0.23124 \pm 0.00017 \qquad (17.14)$$

by six standard derivations. The same mismatch between theory (single – breaking GUT models) and low energy measurements given in Eqs. (13) and (14) is observed if one uses the three effective coupling constants from their measured values to the GUT scale and above. This is shown in Fig. 2, which shows that the three couplings evolved to the GUT scale do not meet at a point. This observation and the others discussed in Sec. 1.2 perhaps point to the presence of new physics between the electroweak scale and the GUT scale. One such candidate is supersymmetry (SUSY), with a SUSY breaking scale somewhere between the electroweak scale and $O(1)$ TeV.

One consequence of supersymmetry (see next section) is that bosonic particles are naturally paired with fermionic ones. Each minimal pairing is called a supermultiplet. For example: a left-handed fermion, its right-handed antiparticle, a complex boson and its conjugate form a chiral supermultiplet. On the other hand a massless vector field and a left-handed fermion form a vector super-multiplet – two transversally polarized vector boson states, plus the left handed fermion and its antiparticle. Thus for $\mathcal{N} = 1$ supersymmetry one has the following helicity states

$$\text{chiral: } (1/2, 0)\,, \text{ gauge: } (1, 1/2)\,, \text{ graviton: } (2, 3/2)$$

Thus in the minimal supersymmetric extension of the standard model, we have the following particles

Particle	Spin	Spartner	Spin
quark: q	1/2	squark: $\tilde{q}$	0
lepton: l	1/2	slepton: $\tilde{l}$	0
photon: γ	1	photino: $\tilde{\gamma}$	1/2
weak vector boson: W	1	wino: $\tilde{W}$	1/2
weak vector boson: Z	1	zino: $\tilde{Z}$	1/2
Higgs: H	0	higgsino: $\tilde{H}$	1/2
gluon: G	1	gluino: $\tilde{G}$	1/2

Due to the presence of supersymmetric particles the RG coefficients b's given in Eqs. (6)-(8) are modified. This modification leads to a solution such that the couplings do meet at a point [see Fig. 3]. The unification scale in such extensions is higher than the value of M_X discussed above in the context of $SU(5)$ model. This would imply a longer time for the proton, evading the present experimental bound. Supersymmetry is needed from another point of view, which is discussed in the next section.

Before we end this section, we may mention that another popular GUT model, $SO(10)$ [rotation group in ten dimensions in internal space with spinor representations], when broken in a single descent to $SU_L(2) \times U(1) \times SU_C(3)$ is also in conflict with the limit (12). However, in contrast to $SU(5)$, $SO(10)$ admits various symmetry breaking patterns, some containing new intermediate mass scales. One such chain of symmetry breaking is

$$\begin{aligned} & SO(10) \underset{M_X}{\rightarrow} SU_L(2) \times SU_R(2) \times SU_C(4) \\ & \underset{m_R}{\rightarrow} SU_L(2) \times U(1) \times SU_C(3) \\ & \underset{m_L}{\rightarrow} U_{em}(1) \times SU_C(3) \end{aligned}$$

where $SU_C(4)$ is the Pati-Salam group. Here it is possible to avoid the conflict with the limit (12). However, there is no prediction for

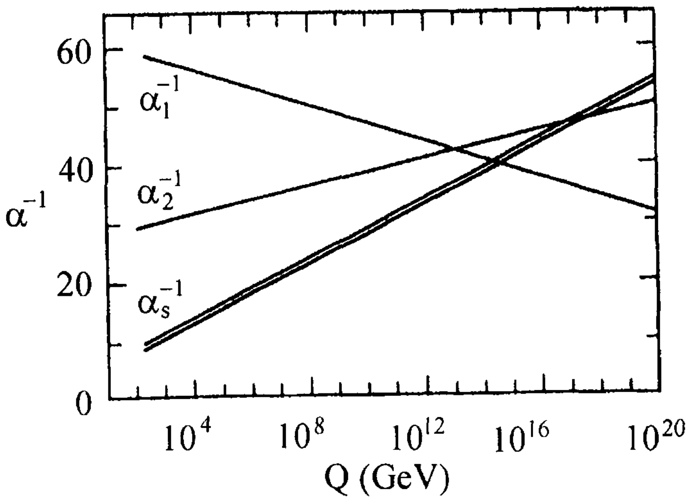

Figure 2 Running of the three gauge couplings in minimal $SU(5)$ GUT showing disagreement with a single unification point. [ref. 5]

$\sin^2\theta_W$; in fact its value is used to fix the intermediate mass scale m_R which is of the order of 10^{13} GeV and being so large has no observable consequences.

To conclude GUTs have several attractive features mentioned above, but their predictive power is limited. However, the idea that quarks and leptons can be treated on an equal footing, and that both lepton and baryon number violations are possible in such unified theories, is now an integral part of GUT models and their extension.

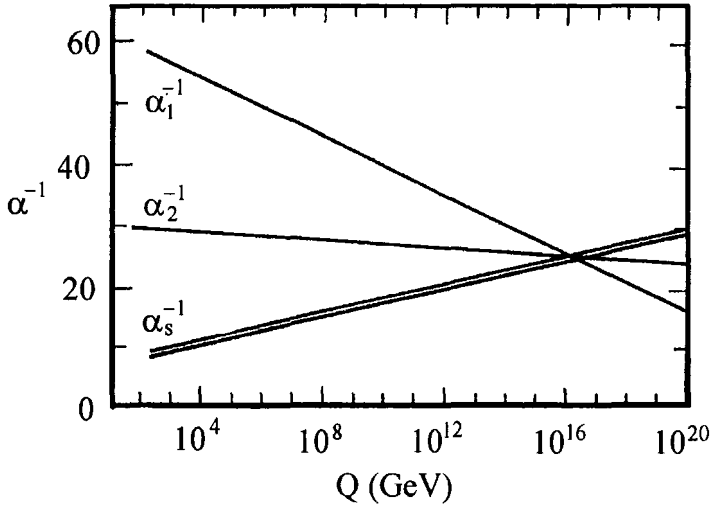

Figure 3 Running of the three gauge couplings in minimal supersymmetric extension of the standard model. [ref. 5]

17.2 Supersymmetry and Strings

17.2.1 Introduction

One of the main puzzles in quantum theory is how to reconcile General Relativity with quantum mechanics. The usual method of taking the classical Lagrangian and quantizing it fails because of insurmountable difficulties in making sense of the renormalization program, which has been so successful in other quantum field theories.

In most situations the domains in which quantum field theories are interesting and the domains in which General Relativity is relevant have no overlap. General Relativity is used when dealing with massive bodies of interest at large distance scales in astrophysics and cosmology and quantum mechanics is used at short distance scales. However, there are situations where both theories become relevant. For instance, close to a black hole quantum effects become relevant as evidenced by Hawking radiation. When one begins to probe distances of the order of the Planck scale one expects that quantum gravitational effects will become important.

The impasse in the field theoretic approach to gravity can be circumvented by using string theory. String theory is a novel program which replaces the plethora of particles that exist by a single string! In this approach the vibrational modes of the string correspond to different particles. Whereas in field theory it seems virtually impossible to include dynamical gravity, in string theory quite the opposite situation prevails: one cannot have string theory without gravity! This is because in the spectrum of string theory there is always a massless spin 2 field, which is naturally identified as the graviton.

Another feature of string theory is that it requires supersymmetry. Even though there is no conclusive evidence at the present time that supersymmetry is a symmetry of the world, supersymmetry is a favored way of resolving some problems in phenomenology beyond the Standard Model. Issues such as the fine-tuning problem due to a fundamental Higgs are naturally avoided in supersymmetric theories since the normally large radiative corrections due to a fundamental scalar Higgs are suppressed due to the presence of its fermionic partner, the Higgsino. Similarly the hierarchy problem can also be resolved in this framework. Supersymmetry is thus seen by many as a positive feature of string theory since the theory requires it and one doesn't have to introduce it by hand.

17.2.2 Supersymmetry

Space-time supersymmetry is a symmetry which generalizes ordinary Poincaré symmetry by augmenting the usual generators with fermionic generators. They satisfy certain commutation relations with the bosonic generators and anti-commutation relations with the remaining fermionic ones:

$$\begin{aligned} [Q_{\alpha i}, P_\mu] &= 0, \\ [Q_{\alpha i}, J_{\mu\nu}] &= \frac{1}{2}(\sigma_{\mu\nu})_\alpha^\beta Q_{\beta i}, \\ \{Q_{\alpha i}, Q_{\beta j}\} &= -\delta_{ij}(\gamma_\mu C)_{\alpha\beta} P_\mu + C_{\alpha\beta} Z_{ij} + (\gamma_5 C)_{\alpha\beta} Z'_{ij}. \end{aligned} \tag{17.15}$$

P_μ are generators of translations and $J_{\mu\nu}$ are Lorentz generators. Together they generate the Poincaré group. The fermionic generators Q are in the Majorana representation and C is the charge conjugation matrix so that:

$$Q_{\alpha i} = C_{\alpha\beta} \overline{Q}_i^\beta . \tag{17.16}$$

The index i runs over the number of supersymmetries $i = 1, ..., \mathcal{N}$. In the simplest case $\mathcal{N} = 1$, the other cases are known as extended supersymmetries. The Z and Z' are so-called central charges, they are anti-symmetric in the indices i, j and commute with everything. They only exist when one has extended supersymmetry.

One of the consequences of supersymmetry is that bosonic particles are naturally paired with fermionic ones so that the number of on-shell degrees of freedom of fermions and bosons are the same. Each minimal pairing consistent with a certain amount of supersymmetry is called a "multiplet". For instance, in four dimensions the smallest amount of supersymmetry has four real fermionic generators and is referred to as $\mathcal{N} = 1$ supersymmetry. In this case one can have an $\mathcal{N} = 1$ "vector multiplet" which consists of a spin 1 gauge boson along with its supersymmetric partner, a Majorana fermion. The fermions and bosons both have two on-shell degrees of freedom. In addition to the vector multiplet one can have a "chiral multiplet" consisting of a complex scalar and its partner, a Weyl fermion. Again the degrees of freedom are the same, i.e. two. One can have up to sixteen real supersymmetries (usually refered to as $\mathcal{N} = 4$ supersymmetry) without introducing anything above spin 1 in four dimensions. Beyond that one has to include higher spin degrees of freedom. Another useful limit to remember is that if one restricts the highest spin of the fields to 2, corresponding to the graviton, the maximum amount of supersymmetry is generated by 32 real fermionic generators (often referred to as $\mathcal{N} = 8$ supergravity). The highest space-time dimension in which a supersymmetric theory can be written down with fields with highest spin equal to 2, is 11 dimensions. This is why eleven dimensional supergravity plays a distinguished role in supersymmetric physics.

When supersymmetry is an exact symmetry, the bosonic and fermionic partners in a multiplet have the same mass. Clearly, this is not seen in nature. For instance, there is no experimentally observed scalar with the same mass as the electron which would qualify as the electron's supersymmetric partner. Phenomenological models then have to break supersymmetry. The mechanism of supersymmetry breaking is not well understood, however, once one assumes that superymmetry is broken at some high energy scale, one can incorporate in low energy models the breaking by simply introducing terms which break it. The number of such terms can be restricted to soft-breaking terms which are relevant in the infrared. These terms push the masses of the (as yet) unobserved supersymmetric partners of the known fields up, to account for their unobserved status while carefully avoiding contradictions with well measured data.

Supersymmetry is a vast area of research which deserves and has received book-length accounts* In the next subsection we will content ourselves with a simple example to illustrate the ideas touched on in our exposition.

Supersymmetric Yang-Mills: An Example

To illustrate the basic ideas of supersymmetry we analyze a toy model: $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. As mentioned earlier, in a minimally supersymmetric model containing a vector field we need to introduce fermions with as many on-shell degrees of freedom as the vector field. A vector meson in d dimensions has $d-2$ physical degrees of freedom, whereas a fermion field with n components has $n/2$ on-shell degrees of freedom. In four dimensions we need to find a fermion field with 2 on-shell degrees of freedom to match the vector field's physical polarizations. Both Weyl and Majorana fermion have 2 real on-shell degrees of freedom. Consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{i}{2}\overline{\psi}^a\gamma^\mu(D_\mu\psi)^a, \tag{17.17}$$

where a sum over repeated indices is implied. a is a group theory index and runs over the generators of the gauge group since all fields transform in the adjoint representation of the gauge group:

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\ (D_\mu\psi)^a &= \partial_\mu + g f^{abc} A^b_\mu \psi^c. \end{aligned} \tag{17.18}$$

The fermionic field ψ is taken to be a Majorana field:

$$\psi^a_\alpha = C_{\alpha\beta}\overline{\psi}^{\beta a}. \tag{17.19}$$

This Lagrangian is invariant under the Poincaré group and local gauge transformation, in addition it enjoys a fermionic sym-

*See for instance, J. Wess and J. Bagger, "Supersymmetry and Supergravity" Princeton University Press (1992).

metry:

$$\begin{aligned} \delta A_\mu^a &= \frac{i}{2}\bar{\varepsilon}\gamma_\mu\psi^a \\ \delta\psi^a &= -\frac{1}{4}F_{\mu\nu}^a\gamma^{\mu\nu}\varepsilon. \end{aligned} \tag{17.20}$$

ε is an "infinitesimal" spinor which anti-commutes with fermionic fields and commutes with bosonic fields. And $\gamma_{\mu\nu} = i\sigma_{\mu\nu}$ as defined in Appendix A. This fermionic symmetry combined with the Poincaré symmetry is known as $\mathcal{N} = 1$ supersymmetry.

We can derive equal-time (anti-)commutation relations for the fields ψ and A_μ. There is a subtlety which needs to be mentioned here. Since the field A_0 does not have a conjugate momentum one cannot quantize it in the usual way, more sophisticated methods are called for. In the following we pick the gauge $A_0 = 0$ and agree to impose the equation of motion of the A_0 field (Gauss' law) by hand on all physical states. In this gauge we can write down the following equal-time commutation relations:

$$\begin{aligned} \left\{\psi_\alpha^a(x), \psi^{*\beta b}(y)\right\} &= \delta^{(3)}(x-y)\delta_\alpha^\beta\delta^{ab} \\ \left[F_{0i}^a(x), A_j^b(y)\right] &= i\delta_{ij}\delta^{ab}\delta^{(3)}(x-y). \end{aligned} \tag{17.21}$$

Using these commutation relations and using the Majorana condition, we can write down the generators of supersymmetry in terms of the fields:

$$Q_\alpha = -\frac{1}{4}\int d^3x F_{\mu\nu}^a(\gamma^{\mu\nu}\gamma^0)_\alpha^\beta\psi_\beta^a. \tag{17.22}$$

One can easily verify that these generators generate the above supersymmetry transformations in the gauge $A_0 = 0$:

$$\begin{aligned} [\bar{\varepsilon}Q, \psi^a] &= \bar{\varepsilon}\{Q, \psi^a\} = -\frac{1}{4}F_{\mu\nu}^a\gamma^{\mu\nu}\varepsilon \\ [\bar{\varepsilon}Q, A_i^a] &= \bar{\varepsilon}[Q, A_i^a] = \frac{i}{2}\bar{\varepsilon}\gamma_i\psi^a \end{aligned} \tag{17.23}$$

$\mathcal{N} = 1$ super Yang-Mills (SYM) has some properties in common with ordinary QCD. For instance, the one-loop beta function of this theory is given by:

$$\Lambda\frac{dg}{d\Lambda} = -\frac{3}{16\pi^2}(f^{acd}f^{acd})g^3. \tag{17.24}$$

The beta function is negative implying that the theory is asymptotically free just like ordinary QCD. Also like QCD, it is believed that SYM is confining and develops a mass gap. In addition, SYM has instantons which contribute to correlation functions.

17.3 String Theory and Duality

There are five known string theories, which are called the Type I, Type IIA, Type IIB, Heterotic SO(32), and Heterotic $E_8 \times E_8$ string theories. They are at first sight very different. For instance, the Type I and the two Heterotic theories have half the supersymmetries of the Type II theories. Similarly, the Type I and Heterotic theories have non-abelian gauge groups while the others don't. One key feature that they do have in common is that they are all formulated in 10 dimensions.

In 1995, the groundbreaking work of Hull, Townsend and Witten unified these theories. They argued that, while naively the theories had distinct properties, in many cases they were non-perturbatively the same. Many of these properties can be understood by thinking of these theories as limits of a single theory: "M-theory".

The key concept unifying the string theories is called "duality". The basic idea is simple. Consider a physical system which has two distinct descriptions A and B, say. A is then said to be dual to B, and vice versa. If the two descriptions are different, as they must for duality to be non-trivial, there must be mechanisms by which their apparent disparity can be overcome. Also, their region of validity must be such that one doesn't find any obvious contradiction. There are many different dualities. We list a few to illustrate the concept.

Strong-weak coupling duality. This is a very powerful type of duality which relates a theory A, say, at strong coupling to another theory B at weak coupling. An example of this duality is provided by the Type I and SO(32) Heterotic theories in 10 dimensions. Their couplings are inversely related. Thus when one of them is strongly coupled the other is weakly coupled. Another example is that of the Type IIB theory which is self-dual under strong-weak duality. This means that the weakly coupled theory is the same as the strongly coupled theory with some fields interchanged.

T-duality. In its most general form T-duality relates string theories on different manifolds to each other. An example is of Type IIA on $R^9 \times S^1$ (where S^1 is a circle) which is dual to Type IIB on $R^9 \times S^1$. The radii of the two circles are related by $R_A = \alpha'/R_B$ (α' is the string tension which is the same as the 10 dimensional Planck length squared). Here we find that two distinct string theories on different manifolds (different because of their radii) are dual. Similarly, we have that Heterotic string theory on $R^6 \times T^4$ (T^4 is the four dimensional torus) is dual to Type IIA on $R^6 \times$K3 (K3 is a Ricci flat manifold of complex dimension 2).

Perhaps the most amazing dualities involve M-theory. Very little is known about M-theory and yet it is a powerful tool in string theory. The defining feature of M-theory is that at low energies it is accurately described by 11 dimensional supergravity. One duality states that M-theory on a circle of radius R is the same as type IIA string theory in 10 dimensions with coupling constant $g_s = (R/l_p)^{3/2}$ (where l_p is the 11 dimensional Planck length). A surprising consequence of this identification is that strongly coupled type IIA string theory develops a new dimension (since in that limit R becomes large)! Another, similar, duality states that M-theory on a line segment is equivalent to $E_8 \times E_8$ Heterotic string theory.

One of the appeals of duality is that it allows one to formulate the notion of non-perturbative string theory by changing the description. A key method used in establishing duality is to work with the various supergravities which capture the low-energy dynamics of string theories. The field content of supergravity consists of the massless modes of the string theory in question. For instance, the Type IIA supergravity describes the low-energy dynamics of Type IIA string theory. It has a number of massless fields of which the bosonic fields are as follows:

$$\begin{array}{ll} \phi & \text{scalar dilaton} \\ g_{\mu\nu} & \text{graviton} \\ B_{\mu\nu} & \text{anti-symmetric 2-tensor} \\ A_\mu & \text{abelian gauge field} \\ A_{\mu\nu\rho} & \text{anti-symmetric 3-tensor} \end{array} \tag{17.25}$$

(17.26)

The anti-symmetric fields all couple to extended objects known as p-branes. Just as a gauge field couples to a point particle, an antisymmetric (p+1)-tensor couples to a p-brane. An important example is $B_{\mu\nu}$ which couples to the Type IIA fundamental string.

We can compare the above field content to that of 11 dimensional supergravity. The massless bosonic content of 11 dimensional supergravity is:

$$\begin{array}{ll} G_{\mu\nu} & \text{graviton} \\ C_{\mu\nu\rho} & \text{anti-symmetric 3-tensor} \end{array} \tag{17.27}$$

At first sight it seems to bare little resemblance to the type IIA field content. Recall, however, that M-theory on $R^9 \times S^1$ is supposed to be equivalent to Type IIA string theory. When we compactify on S^1 and take the radius to be small we can ignore the dependence of the fields on the compact coordinate, as is usual when one performs dimensional reduction. From the ten dimensional point of view we can make the following identifications:

$$\begin{aligned} \exp 4\phi/3 &= G_{11,11} \\ A_\mu &= G_{11,\mu} \\ g_{\mu\nu} &= G_{\mu\nu} \qquad (17.28) \\ B_{\mu\nu} &= C_{\mu\nu,11} \\ A_{\mu\nu\rho} &= C_{\mu\nu\rho}. \qquad (17.29) \end{aligned}$$

Thus we see that all the fields are accounted for. The dilaton serves as a coupling constant in type IIA supergravity. The usual string-frame dilaton is related. We see immediately that when the dilaton is large the radius of the circle becomes large and type IIA supergravity becomes a poor approximation for 11 dimensional supergravity. We understand this to mean that Type IIA is a perturbative theory which is non-perturbatively equivalent to M-theory on S^1.

The spectrum of p-branes is different in the two theories, but they too are related as above. We illustrate this identification with a few examples. Type IIA string theory has 0-branes which couple to the gauge field A_μ, in M-theory they correspond to momentum modes along S^1. Since momentum is quantized in the S^1 direction in integer units of $2\pi/R$, where R is the radius of the compact direction, the number of units is naturally identified with the number of 0-branes. A striking difference is that M-theory contains no strings. It does, however, have a 2-brane (membrane) which when wrapped on the S^1 appears as a string in 10 dimensions as long as one is justified in ignoring scales smaller than the radius of the compact direction.

All string dualities have to satisfy consistency checks of the above kind. Fortunately there are many tests one can perform. Here the importance of a distinguished set of states known as BPS states are particularly useful. BPS states preserve some fraction of the total space-time supersymmetry, by virtue of which they are the lowest mass states in their class and are guaranteed to be stable. Many of their properties can be established exactly, even when the theory is strongly coupled.

17.4 Some Important Results

Many new insights have been gained using duality. Although these areas do not directly touch on finding phenomenologically viable models, some do demonstrate the ability to study phenomena which generically exist in realistic models. We briefly discuss some of these below.

In the last few years, using duality, considerable progress has been made in our understanding of gauge theories, particularly supersymmetric gauge theories. Significant results include the demonstration of confinement and chiral symmetry breaking in four dimensional gauge theories.

String theories have been used to study black holes. One of the most exciting new results concerns the problem of black hole entropy. The Beckenstein-Hawking entropy is a thermodynamic quantity which satisfies a generalized version of the second law of thermodynamics. It has recently been given a statistical mechanical basis by relating it to microscopic states of a black hole.

Recently, progress has been made in finding a connection between gravity and field theory. One manifestation of this has been a proposal that a quantum mechanics model known as Matrix theory captures the dynamics of M-theory. Many checks have been performed to test the ability of Matrix theory to reproduce supergravity calculations with success. Another approach known as the Maldacena conjecture has led to a radically new connection between conformal field theories and supergravity in AdS backgrounds.

17.5 Conclusions

We have given just a flavor of the vast and rapidly growing area of supersymmetry and string theory dualities. The interested reader should consult review articles and books for a thorough introduction to the subject. A good place to start is the recent book by Polchinski (J. Polchinski, "String Theory" Vols. 1 and 2, Cambridge University Press (1998)).

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Chapter 18
COSMOLOGY AND PARTICLE PHYSICS

18.1 Cosmological Principle and Expansion of the Universe

On a sufficiently large scale, universe is homogeneous and isotropic. This is called the cosmological principle. A coordinate system in which matter is at rest at any moment is called a co-moving coordinate system. An observer in this coordinate system is called a co-moving observer. Any co-moving observer will see around himself a uniform and isotropic universe. Cosmological principle implies the existence of a universal cosmic time, since all observers see the same sequence of events with which to synchronize their clocks. In particular they all start their clocks with big bang.

A homogeneous and isotropic universe is described by the Friedmann–Robertson–Walker (F–R–W) metric

$$d\,s^2 = c^2 dt^2 - R^2\,(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\; d\phi^2\right)\right]. \quad (18.1)$$

r, θ, ϕ are co-moving coordinates and the scale factor $R(t)$ is a scale factor for distances in co-moving coordinates and describes the expansion. k is related to the 3-space curvature. With suitable choice of units for r, k has the values $+1, 0$, or -1 corresponding to the closed, flat or open universe respectively. For $k = 1$, the spatial universe can be regarded as the surface of a sphere of radius $R(t)$ in four dimensional Euclidean space. This can be seen as follows.

Consider a sphere in four dimensional Euclidean space

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2. \tag{18.2a}$$

The line element is

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2. \tag{18.2b}$$

From Eq. (2a), we get

$$x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + x_4 dx_4 = 0. \tag{18.2c}$$

The fourth element dx_4^2 can be eliminated in Eq. (2b), using Eqs. (2a) and (2c), and we obtain

$$dl^2 = \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 \left(d\theta^2 + \sin^2\theta \; d\phi^2\right), \tag{18.3}$$

where we have used the spherical polar coordinates $x_1 = r\cos\phi\sin\theta$, $x_2 = r\sin\phi\sin\theta$, $x_3 = r\cos\theta$. Putting $r' = \frac{r}{R}$ and then removing the prime, we get

$$dl^2 = R^2(t)\left[\frac{dr^2}{1 - r^2} + r^2\left(d\theta^2 + \sin^2\theta \; d\phi^2\right)\right]. \tag{18.4}$$

It is instructive to use the spherical polar coordinates in four dimensional Euclidean space

$$\begin{aligned} x_1 &= R\sin\chi\sin\theta\cos\phi \\ x_2 &= R\sin\chi\sin\theta\sin\phi \\ x_3 &= R\sin\chi\cos\theta \\ x_4 &= R\cos\chi. \end{aligned} \tag{18.5}$$

Then we get

$$dl^2 = R^2\left[d\chi^2 + \sin^2\chi \; \left(d\theta^2 + \sin^2\theta \; d\phi^2\right)\right] \tag{18.6a}$$

$$dV = R^3 \sin^2\chi\sin\theta \; d\chi \; d\theta \; d\phi. \tag{18.6b}$$

Now using $r = R\sin\chi$, we get back Eq. (3). The radius of the sphere and its volume are given by

$$\int_0^r \frac{dr}{\sqrt{1-\frac{r^2}{R^2}}} = R\sin^{-1}\left(\frac{r}{R}\right). \tag{18.7}$$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} R^3\sin^2\chi\sin\theta\, d\chi\; d\theta\; d\phi = 2\pi^2 R^3.$$

The cosmological principle implies [cf. Eq. (1)]

$$\ell = R(t)\; r. \tag{18.8}$$

Thus the velocity of expansion is given by

$$\begin{aligned} v &= \frac{dl}{dt} = \dot{R}(t)\; r \\ &= \frac{\dot{R}(t)}{R(t)} R(t)\; r = Hl, \end{aligned} \tag{18.9}$$

where

$$H = \frac{\dot{R}(t)}{R(t)} \tag{18.10}$$

is called the Hubble parameter. Let us denote by t_0 the present time and t_e the time at which the light was emitted from a distant galaxy. Correspondingly we denote the detected wavelength by λ and emitted (laboratory) wavelength by λ_e of some electromagnetic spectral line. We define the redshift

$$z = \frac{\Delta\lambda}{\lambda_e} = \frac{\lambda_0 - \lambda_e}{\lambda_e}$$

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{R(t_0)}{R(t_e)}. \tag{18.11}$$

The redshift is experimentally observed and it clearly shows that the universe is expanding.

The highest redshift so far discovered $z = 4.89$ so that the Lyman–alpha line appears in the red part of the spectrum arround 7200 Å. This implies that $\frac{R(t_0)}{R(t_e)} = (1+z) = 5.89$. In the matter dominated universe $R \sim t^{2/3}$ (see below). This gives [with $t_0 \approx 1.5 \times 10^{10}$ yrs, the present age of the universe] $t_e \simeq \frac{1}{14}(1.5 \times 10^{10}\text{ yrs}) \simeq 10^9$ yrs. The existence of these high z–objects implies that by the time the universe was about 10^9 yrs old, some galaxies (or at least their inner region) had already been formed.

For small time intervals since emission compared to H_0^{-1}, Eq. (11) takes the form

$$z \simeq \Delta t \frac{\dot{R}_0}{R_0} \simeq \frac{l}{c}\frac{\dot{R}_0}{R_0} = \frac{l}{c}H_0, \tag{18.12}$$

where l is the distance to the source.

18.2 The Standard Model of Cosmology

The model is described by two differential equations

$$\dot{R}^2 + kc^2 - \frac{\Lambda\, c^2 R^2}{3} = \frac{8\pi G}{3}\rho\, R^2 \tag{18.13}$$

$$d\,\left(\rho R^3 c^2\right) + p d\left(R^3\right) = 0. \tag{18.14}$$

Here the second term in Eq. (13) is due to the curvature, the third term contains cosmological constant Λ. Cosmological constant Λ is very small ($|\Lambda| < 3\times 10^{-52}\, m^{-2}$) and this term is usually neglected except in the inflationary phase of expansion. G is the Newtonian gravitational constant. In the units $\hbar = c = 1$, $\frac{1}{\sqrt{G}} = M_P$ (the Planck mass) $\approx 1.2\times 10^{19}$ GeV. Equation (14) expresses the energy conservation. Here ρ is the density of the universe and p is the isotropic pressure. Note that Eq. (13) can also be put in the form

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\,\rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda\, c^2}{3}. \tag{18.15a}$$

Differentiating Eq. (13) and using Eq. (14), one gets

$$\frac{\ddot{R}}{R} = \frac{1}{3}\,\Lambda\, c^2 - \frac{4\pi G}{3\, c^2}\,\left(\rho\, c^2 + 3p\right). \tag{18.15b}$$

In addition we need the equation of state. We take this to be that for an ideal gas

$$p = nk_BT, \tag{18.16}$$

where n is the particle density and k_B is the Boltzmann constant. $k_B = 0.86 \times 10^{-10}$ MeV/K (K: Kelvin). If we take $k_B = 1$, then the temperature is measured in MeV. In particular 0.86 MeV= 10^{10}K.

From Eq. (13) ($\Lambda = 0$), we have

$$kc^2 = R^2\,(t)\,\frac{8\pi\, G}{3}\,\left(\rho\,(t) - \rho_c\,(t)\right), \tag{18.17}$$

where

$$\rho_c\,(t) = \frac{3}{8\pi\, G}\left[\frac{\dot{R}\,(t)}{R\,(t)}\right]^2 = \frac{3\, H^2\,(t)}{8\pi\, G} \tag{18.18}$$

is called the critical density. It is convenient to define the density parameter of the universe

$$\Omega = \frac{\rho}{\rho_c}. \tag{18.19}$$

Then from Eq. (17), we get

$$\begin{aligned} kc^2 &= R^2\,(t)\;\; H^2\,(t)\;\; (\Omega - 1) \\ &= R_0^2\,(t)\;\; H_0^2\,(t)\;\; (\Omega_0 - 1)\,. \end{aligned} \tag{18.20}$$

Here the subscript 0 denotes the present time. It is clear from Eq. (20) that for $\Omega > 1$, the universe is closed, for $\Omega \leq 1$, the universe is open.

We note that for nonrelativistic gas (NR)

$$\rho = mn \Rightarrow p \ll \rho\, c^2. \tag{18.21}$$

Then we say that the universe is matter dominated. For extreme relativistic gas (ER)

$$\begin{aligned} p &= \frac{1}{3}\rho\, c^2 \\ \rho\, c^2 &= 3nk_BT \end{aligned} \qquad (18.22)$$

and we say that universe is radiation dominated. Present universe is matter dominated i.e. $p \approx 0$. Thus from Eq. (14), we have

$$d\left(\rho R^3 c^2\right) = 0$$

or

$$\rho R^3 = \text{constant} = \frac{3}{4\pi} M, \qquad (18.23)$$

where M is just the mass of the universe. We define another parameter q, called the deceleration parameter

$$q = -\frac{R\ddot{R}}{\dot{R}^2}. \qquad (18.24)$$

From Eq. (13), using Eq. (23), we get

$$2\dot{R}\ddot{R} = -\frac{2GM}{R^2}\dot{R} = -\frac{8\pi}{3}G\ R\ \rho\dot{R}. \qquad (18.25)$$

Thus

$$\begin{aligned} q &= \frac{4\pi}{3}G\ \rho\ \frac{R^2}{\dot{R}^2} = \frac{1}{2}\frac{\rho}{\rho_c} = \frac{1}{2}\ \Omega \\ q_0 &= \frac{1}{2}\ \Omega_0. \end{aligned} \qquad (18.26)$$

Thus for $q_0 > \frac{1}{2}$, the universe is closed and for $q_0 \leq \frac{1}{2}$, the universe is open.

We now discuss the three cases $k = 0,\ k = 1$ and $k = -1$. We will now put $c = 1$. First we discuss the flat universe ($k = 0$). From Eqs. (13) and (23), we get

$$\dot{R}\sqrt{R} = (2GM)^{1/2}. \qquad (18.27)$$

Integration of Eq. (27) with $R(t \approx 0) \approx 0$ gives

$$R^3(t) = \frac{9GM}{2}\, t^2 = \frac{3M}{4\pi\, \rho(t)}, \tag{18.28}$$

where the last term in Eq. (28) follows from Eq. (23). Hence we have

$$\begin{aligned} \rho^{-1}(t) &= 6\pi G\, t^2 \\ R(t) &= \left(\frac{9GM}{2}\right)^{1/3} t^{2/3}. \end{aligned} \tag{18.29}$$

For the closed universe $k = 1$, we have from Eq. (13) $[\Lambda = 0]$

$$\frac{1}{R^2}\left(\frac{d\,R}{d\,\eta}\right)^2 = \frac{8\pi G}{3}\,\rho\, R^2 - 1, \tag{18.30}$$

where we have put $(dt = R\; d\,\eta)$:

$$\dot{R} = \frac{1}{R}\frac{d\,R}{d\,\eta}. \tag{18.31}$$

Integrating Eq. (30) with the help of Eq. (23), we get

$$\begin{aligned} R &= MG\,(1 - \cos\eta) \\ t &= MG\,(\eta - \sin\eta)\,. \end{aligned} \tag{18.32}$$

Similarly for the open universe $k = -1$, we get

$$\begin{aligned} R &= MG\,(\cosh\eta - 1) \\ t &= MG\,(\sinh\eta - \eta)\,. \end{aligned} \tag{18.33}$$

All the three cases are shown in Fig. 1.

Finally we note that the age of the universe is essentially determined by the matter dominated universe. The radiation era lasts only for a few minutes. Now using [cf. Eq. (29)]

$$\frac{\rho(t)}{\rho_0} = \left(\frac{R_0}{R(t)}\right)^3, \tag{18.34}$$

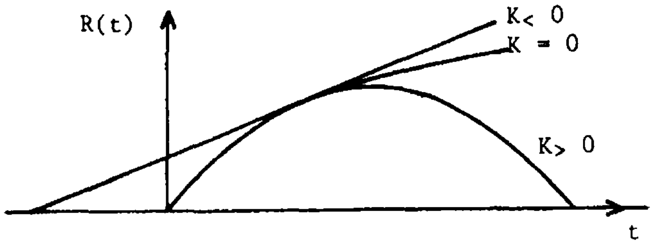

Figure 1 Plot of scale factor $R(t)$ versus time t for closed ($k > 0$), open ($k < 0$) and flat ($k = 0$) universe model.

we have from Eq. (13)

$$\dot{R}^2 - \frac{8\pi\, G}{3}\, \rho_0\, \frac{R_0^3}{R} = -k. \tag{18.35}$$

By using Eq. (20) and Eqs. (18) and (19) for the present time, we obtain

$$\dot{R} = R_0\, H_0 \left[1 - \Omega_0 + \Omega_0 \frac{R_0}{R}\right]^{1/2}. \tag{18.36}$$

The integration of Eq. (36) gives the age of the universe

$$t_u = f\,(\Omega_0)\, H_0^{-1}, \tag{18.37a}$$

where

$$f\,(\Omega_0) = \int_0^1 \left(1 - \Omega_0 + \frac{\Omega_0}{x}\right)^{-1/2} dx \tag{18.37b}$$

with $x = R/R_0$ and $R(t \simeq 0) \simeq 0$. The function $f\,(\Omega_0)$ for $\Omega_0 > 1$, $= 1$ and < 1 is respectively given by

$$f\,(\Omega_0) \;=\; \frac{\Omega_0}{2}\,(\Omega_0 - 1)^{-3/2} \cos^{-1}\left(\frac{2}{\Omega_0} - 1\right) - (\Omega_0 - 1)^{-1}$$

$$\simeq \frac{\pi}{2}\Omega_0^{-1/2}, \text{ for } \Omega_0 \gg 1 \tag{18.38a}$$

$$= \frac{2}{3}, \text{ for } \Omega_0 = 1 \tag{18.38b}$$

$$= (1-\Omega_0)^{-1} - \frac{\Omega_0}{2}(1-\Omega_0)^{-3/2}\cosh^{-1}\left(\frac{2}{\Omega_0}-1\right)$$

$$\simeq 1 + \Omega_0 \ln \Omega_0, \text{ for } \Omega_0 \ll 1. \tag{18.38c}$$

Thus constraints on t_u give constraints on $\Omega_0^{-1/2}\ H_0^{-1}$.

We close this section by summarizing the essential features of the standard model of cosmology. The universe started with a big bang and has been undergoing expansion ever since. This picture is based on two observed facts:

1. The Hubble expansion. The recession of distant cosmological objects was discovered by Hubble in 1920s. They were found to be moving from us with velocities proportional to their distances $v = Hl$.

2. The observation of black body radiation with temperature $T_0 \sim 2.7$ K. This is supposed to be relic of the early universe. The observed isotropy of the background radiation ($\Delta T/T \sim 10^{-5}$) provides the strongest direct support of the cosmological principle.

The model is characterized by four parameters:

(i) The present value of the Hubble parameter

$$H_0 = 100\ h_0 \text{ km s}^{-1} \text{ Mpc}^{-1}. \tag{18.39a}$$

Since Hubble parameter is not very well known, it is written with the ignorance factor h_0. The present estimate for h_0 is

$$0.4 < h_0 < 1. \tag{18.39b}$$

Note that Mpc : Megaparsec $= 3 \times 10^{19}$ km. Thus

$$H_0 = 3.33 \times 10^{-18}\ h_0\ s^{-1} = h_0(1 \times 10^{10} \text{ yr})^{-1}. \tag{18.40}$$

The present age of the universe from Eq. (37a) is given by

$$t_u = f(\Omega_0)\, H_0^{-1} = f(\Omega_0)\, h_0^{-1} \times 10^{10} \text{ yrs}. \tag{18.41}$$

(ii) The present temperature of the cosmic microwave background radiation (CMBR)

$$T_0 = 2.728 \pm 0.002 \text{ K}.$$

(iii) The average mass density

$$\rho_0 = \Omega_0 \, \rho_{c0}$$

$$\rho_{c0} = \frac{3H_0^2}{8\pi G} \approx \left\{ \begin{array}{l} 1.88 \times 10^{-29} \, h_0^2 \text{ gm cm}^{-3} \\ 1.05 \times 10^{-5} \, h_0^2 \text{ GeV cm}^{-3} \end{array} \right\}. \tag{18.42}$$

Accurate estimate of the cosmological density parameter Ω_0 is difficult. Present experimental data is consistent with

$$0.1 \leq \Omega_0 \leq 2. \tag{18.43}$$

Correspondingly Eqs. (41) and (38) give $[f(\Omega_0) = 0.9, 0.67$ and 1.11 for $\Omega_0 = 0.1, 1$ and $2]$.

(iv) The measurement of deceleration parameter $q_0 = \frac{1}{2}\Omega_0$ can also give an estimate of Ω_0, but it is difficult to measure q_0. The present estimates for q_0 are

$$0 \pm 0.5 \qquad \text{to} \qquad 1.5 \pm 0.5.$$

$$t_u \simeq (6.5 \text{ to } 10) \times 10^9 h_0^{-1} \text{ yrs}. \tag{18.44}$$

The best bet on the age of the universe is $(16 \pm 3) \times 10^9$ yrs. This result put constraints on $\Omega_0 h_0^2$.

18.3 Thermal Equilibrium

Consider an arbitrary volume V in thermal equilibrium with a heat bath at temperature T. The particle density n_i (i, particle index) at temperature T is given by

$$n_i = \frac{N_i}{V} = \frac{g_i}{2\pi^2} \left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty \left[\exp\left(\frac{E}{k_B T}\right) \pm 1\right]^{-1} z^2 dz. \tag{18.45}$$

The energy density is given by

$$\rho_i \, c^2 = \frac{g_i}{2\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 (k_B \; T) \int_0^\infty \left[\exp\left(\frac{E}{k_B T}\right) \pm 1\right]^{-1} \left(\frac{E}{k_B T}\right) z^2 dz^2, \tag{18.46}$$

where

$$z = \frac{qc}{k_B T}, \qquad E = \left[(qc)^2 + \left(m_i c^2\right)^2\right]^{1/2} \tag{18.47}$$

and g_i are the number of spin states, q is the momentum of the particle and m_i is its mass. The $+$ sign is for the fermions (F) and $-$ sign is for the bosons (B). In particular for i = photon, $m = 0$, $g = 2$. In writing Eqs. (45) and (46), we have put the chemical potential $\mu_i = 0$. For photon $\mu = 0$. Since particles and antiparticles are in equilibrium with photons $\mu_i = -\mu_{\bar{i}}$. If there is no asymmetry between the number of particles and antiparticles, $\mu_i = \mu_{\bar{i}} = 0$. If the difference between the number of particles and antiparticles is small compared with the number of photons,

$$\left|\frac{\mu_i}{k_B T}\right| = \left|\frac{\mu_{\bar{i}}}{k_B T}\right| \ll 1 \tag{18.48}$$

and the chemical potential can be neglected. For the photon gas, we get from Eqs. (45) and (46)

$$n_\gamma = 2\frac{\zeta\,(3)}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 = 2\frac{1.2}{\pi^2}\left(\frac{1}{\hbar c}\right)^3 (k_B T)^3 \tag{18.49}$$

$$\begin{aligned} \rho_\gamma \; c^2 &= 6\frac{\zeta\,(4)}{\pi^2}\left(\frac{1}{\hbar c}\right)^3 (k_B T)^4 \\ &= \frac{\pi^2}{15}\left(\frac{1}{\hbar c}\right)^3 (k_B T)^4 \approx 2.7 \; n_\gamma \,(k_B T)\,. \end{aligned} \tag{18.50}$$

In Eqs. (49) and (50) $\zeta\,(r)$, $r = 3, 4$ is the Riemann zeta function. For a gas of extreme relativistic particles (ER), $k_B T \gg m_i c^2$, $qc \gg$

$m_i c^2$, we thus get

$$\begin{aligned} n_B &= \left(\frac{g_B}{2}\right) n_\gamma, \qquad \rho_B = \left(\frac{g_B}{2}\right) \rho_\gamma && (18.51a) \\ n_F &= \frac{3}{4}\left(\frac{g_F}{2}\right) n_\gamma, \qquad \rho_F = \frac{7}{8}\left(\frac{g_F}{2}\right) \rho_\gamma . && (18.51b) \end{aligned}$$

The entropy S for the photon gas is given by

$$S = \frac{R^3}{T}\frac{4}{3}\,\rho_\gamma(T)\,. \tag{18.52}$$

For any relativistic gas

$$S = \frac{R^3}{T}\frac{4}{3}\,\rho(T)\,. \tag{18.53}$$

Thus for a gas consisting of extreme relativistic particles (bosons and fermions): $(\hbar = c = 1)$

$$\begin{aligned} n(T) &= \frac{1}{2}\, g'(T)\, n_\gamma(T) \\ &= \frac{1.2}{\pi^2}\, g'(T)\ (k_B T)^3 \end{aligned} \tag{18.54}$$

$$\begin{aligned} \rho(T) &= \frac{1}{2}\, g_*(T)\, \rho_\gamma(T) \\ &= \frac{\pi^2}{30}\, g_*(T)\ (k_B T)^4 \end{aligned} \tag{18.55}$$

$$S = \frac{R^3}{T}\frac{2}{3}\, g_*(T)\, \rho_\gamma(T), \tag{18.56}$$

where

$$\begin{aligned} g'(T) &= \sum_B g_B + \frac{3}{4}\sum_F g_F && (18.57a) \\ g_*(T) &= \sum_B g_B + \frac{7}{8}\sum_F g_F && (18.57b) \end{aligned}$$

are called the "effective" degrees of freedom. We note that entropy per unit volume is given by

$$\frac{1}{k_B}\frac{S}{R^3} = \frac{s}{k_B} = \frac{2\pi^2}{45}\, g_*(T)\ \left(k_B T\right)^3 . \tag{18.58}$$

For non-relativistic gas $k_B T \ll m_i c^2$, we use the Boltzmann distribution

$$n_i = \frac{g_i}{2\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty \exp\left(-\frac{E}{k_B T}\right) z^2\, dz \tag{18.59}$$

$$E \approx m_i c^2 \left[1 + \frac{1}{2}\frac{q^2 c^2}{\left(m_i c^2\right)^2}\right]. \tag{18.60}$$

From Eq. (55), we get

$$n_i = \left[\frac{g_i}{(2\pi)^{3/2}}\right]\left(\frac{k_B T}{\hbar c}\right)^3 \left[\left(\frac{m_i c^2}{k_B T}\right)^{3/2} e^{-m_i c^2/k_B T}\right] \tag{18.61}$$

$$\rho_i = n_i\ m_i . \tag{18.62}$$

18.4 The Radiation Era

For extreme relativistic gas, $p = \frac{1}{3}\ \rho\ c^2$, we get from Eq. (14)

$$R^3\frac{d\ \rho}{d\ R} + 4\rho R^2 = 0. \tag{18.63}$$

Thus, we have

$$\rho = A^2 R^{-4} \qquad (A\text{: constant}). \tag{18.64}$$

Hence

$$\frac{\rho_{N\cdot R}}{\rho_{E\cdot R}} \propto R \rightarrow 0 \qquad \text{as} \quad R \rightarrow 0.$$

Therefore, we have the important result. Early universe is dominated by extreme relativistic particles, i.e. the universe is radiation dominated in early stages. Since $\rho \rightarrow \frac{1}{R^4}$ for the early universe we can neglect the second and third terms on the left-hand side of Eq. (13) as compared with the first term. Thus we get

$$\dot{R} = \sqrt{\frac{8\pi G}{3}\rho}\, R. \tag{18.65}$$

Therefore, the expansion rate is given by

$$H = \frac{\dot{R}}{R} = \sqrt{\frac{8\pi}{3}}\; (G\,\rho)^{1/2}. \tag{18.66}$$

Now using Eq. (55), we get

$$H = \sqrt{\frac{4\pi^3}{45}}\,[g_*(T)]^{1/2}\,\frac{(k_BT)^2}{\hbar M_P} \approx 0.21\; g_*^{1/2}\left(\frac{k_BT}{\text{MeV}}\right)^2\;\text{s}^{-1}. \tag{18.67}$$

Also we have

$$R\dot{R} = A\sqrt{\frac{8\pi G}{3}}. \tag{18.68}$$

Hence we get

$$\begin{aligned} t &= \sqrt{\frac{3}{32\pi G}}\frac{R^2}{A} = \sqrt{\frac{3}{32\pi}}\frac{1}{\sqrt{G\,\rho}} \\ &= \sqrt{\frac{45}{16\pi^3}}\; g_*^{-1/2}\;\frac{\hbar M_P}{(k_BT)^2} \\ &= 2.42\; g_*^{-1/2}\left(\frac{\text{MeV}}{k_BT}\right)^2\;\text{s}. \end{aligned} \tag{18.69}$$

Thus as $t \rightarrow 0$

$$Ht \approx 0.5. \tag{18.70}$$

We consider two examples:

(i) For $g_* = g_\gamma = 2$, we get

$$t \approx 1.7\left(\frac{\text{MeV}}{k_BT}\right)^2\;\text{s}. \tag{18.71}$$

Thus for $k_BT = m_ec^2 \approx 0.51$ MeV, $t = 6.5$ s and $H \approx 0.08$ s^{-1}.

(ii) For $m_\mu > k_BT > m_e$,

$$\begin{aligned} g_* &= g_\gamma + \frac{7}{8}(g_e + 3g_\nu) \\ &= 2 + \frac{7}{8}(4+6) = \frac{43}{4} \end{aligned} \tag{18.72a}$$

and for $m_\pi > k_BT > m_\mu$,

$$\begin{aligned} g_* &= 2 + \frac{7}{8}\ (g_e + g_\mu + 3g_\nu) \\ &= \frac{57}{4} \end{aligned} \tag{18.72b}$$

Here we have taken the number of neutrinos $N_\nu = 3$. Now we get from Eqs. (67) and (69) at $k_BT = 1$ MeV

$$H \approx 0.67\ \left(\frac{k_BT}{\text{MeV}}\right)^2\ \text{s}^{-1} \approx 0.67\ \text{s}^{-1} \tag{18.73a}$$

and

$$t \approx 0.74\ \left(\frac{\text{MeV}}{k_BT}\right)^2\ \text{s} \approx 0.74\ \text{s}. \tag{18.73b}$$

Now Eq. (69) gives the time evolution of the universe in radiation era. From Eq. (66), we have the important result that $H \propto \sqrt{\rho}$ i.e. the higher the energy density in the early universe, the faster will be the expansion rate.

As we have seen, the radiation density falls off as R^{-4} and the energy density in nonrelativistic matter falls of as R^{-3}. The universe eventually becomes matter dominated. At $t = t_{eq}$, matter density becomes equal to radiation density i.e.

$$\rho_m = \rho_r, \tag{18.74}$$

where from Eqs. (34) we get

$$\rho_m\left(t_{eq}\right)c^2=\Omega_0\;\rho_{c0}c^2\left(\frac{R_0}{R_{eq}}\right)^3 \tag{18.75}$$

and from Eqs. (50), (51) and (64) we get

$$\begin{aligned}\rho_r\left(t_{eq}\right)c^2 &= \frac{\pi^2}{30}\left[g_\gamma\;\left(k_BT_{\gamma 0}\right)^4+\frac{7}{8}3\;g_\nu\;\left(k_BT_{\nu 0}\right)^4\right]\\ &\quad\times\left(\frac{1}{\hbar c}\right)^3\left(\frac{R_0}{R_{eq}}\right)^4\\ &= \frac{\pi^2}{30}\left[2\;+\frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right]\left(k_BT_0\right)^4\left(\frac{R_0}{R_{eq}}\right)^4\left(\frac{1}{\hbar c}\right)^3\end{aligned} \tag{18.76}$$

where we have used $\left(\frac{T_{\nu 0}}{T_{\gamma 0}}\right)^3=\frac{4}{11}$, (cf. Eq. (113)) and $T_{\gamma_0}=T_0$. Hence from Eqs. (75) and (76), we obtain for $T_0=2.728$ K,

$$\begin{aligned}1+z_{eq} &= \frac{R_0}{R_{eq}}=\Omega_0\;\rho_{c0}\;\left(2.27\times 10^6\;\mathrm{MeV}^{-1}\mathrm{cm}^3\right)\\ &= 2.4\times 10^4\;\Omega_0\;h_0^2.\end{aligned} \tag{18.77}$$

From Eqs. (75)-(77), we get

$$k_BT_{eq}=\left(k_BT_0\right)\frac{R_0}{R_{eq}}=5.6\times 10^{-6}\;\Omega_0\;h_0^2\;\mathrm{MeV} \tag{18.78}$$

and from Eqs. (69) and (78), we obtain

$$t_{eq}\approx 3.0\times 10^{10}\;\left(\Omega_0\;h_0^2\right)^{-2}\;\mathrm{s}. \tag{18.79}$$

In the dense early universe, the radiation would have been held in thermal equilibrium with matter and would have scattered repeatedly off free electrons. But when the expansion had cooled the matter below 3000 K ($k_BT\simeq 0.26$ eV), so that from Eq. (69)

with $g_* = 2 + \frac{7}{8}(6) = \frac{29}{4}$, $t \simeq 1.3 \times 10^{13}$ sec $\simeq 4 \times 10^5$ yrs, the primordial plasma would have recombined to atoms, the universe thereafter becoming transparent to light. The experimentally detected microwave photons are therefore direct messengers from an era when the universe had an age of about 4×10^5 yrs. But photons are still around–they fill the universe and no where else to go. The thermal radiation last scattered at this epoch is now detected as the cosmic background radiation. This epic defines a "surface" known as "surface of last scattering".

Some important dates in the evolution of the universe are given in Table 1. These are only estimates.

We end this section by writing some useful numbers. From Eqs. (49) and (50), using the present temperature $T_0 = 2.728$ K, we get

$$n_{\gamma 0} \approx \frac{2.4}{\pi^2}\left(\frac{k_B\, T_0}{\hbar c}\right)^3 \approx 411 \text{ cm}^{-3} \tag{18.80}$$

$$\rho_{\gamma 0} \approx 2.6 \times 10^{-10} \text{ GeV cm}^{-3}. \tag{18.81}$$

Thus n_γ at temperature T is given by

$$n_\gamma \approx 411\left(\frac{T}{2.728}\right)^3 \text{ cm}^{-3}. \tag{18.82}$$

In addition we write down the following estimates. There are about 10^{57} nucleons in a typical star. There are about 10^{11} galaxies in the universe, each galaxy has about 10^{11} stars. Thus there are about 10^{79} baryons in the universe. This is to be compared with 10^{89} photons within the part of the universe we can observe; this number is obtained by thermodynamical arguments. Thus number of baryons/number of photons $\approx 10^{-10}$. The present size of the observable universe is 10^{28} cm. Further the baryon number density n_b is given by $n_b \sim \left(\frac{10^{79}}{\frac{3}{4\pi}(10^{28})^3}\right) \sim 10^{-6}$ cm^{-3}.

Another quantity of interest is baryon density in the universe. First we note from Eq. (23), that it scales as R^{-3}. It is

Table 18.1 Cosmic History (some critical phases)

Era	Age (in seconds)	Temperature K	Remarks
	0		Vacuum to matter transition
Planck	10^{-44}	10^{32}	All forces unify
GUT	10^{-36}	10^{28}	GUT transition, Strong and electroweak forces unify, Baryon number creation
Electro-weak	10^{-10}	10^{18}	$W^{\pm}, Z^0$: Salam–Weinberg transition
Quark	10^{-5}	3×10^{12}	Hadronization
Lepton	8×10^{-5} 8×10^{-3} 0.7 6	1.2×10^{12} 1.2×10^{11} 10^{10} 6×10^{9}	$\mu^{\pm}$ annihilation ν_{μ}'s decouple ν_e's decouple e^+e^- annihilation
Particle	$60 - 80$	$[1.3 - 0.8] \times 10^9$	Nucleosynthesis
Photon	6×10^{12} to 10^{14}	4×10^{12} to 10^3	Radiation era ends Plasma to atom;
	2×10^{17}	2.7	Present

convenient to define the baryon density in terms of parameter η viz.

$$\eta \equiv \frac{n_B}{n_\gamma} \tag{18.83}$$

where $n_B = n_b - n_{\bar{b}}$. The baryon number density is given by

$$\begin{aligned} n_B &= \frac{\rho_B}{m_B} = \frac{\rho_B}{\rho_c}\frac{\rho_c}{m_B} \\ &= \frac{\Omega_B}{m_B}\,\rho_c \end{aligned} \tag{18.84}$$

where ρ_B is the baryon energy density and $\Omega_B \equiv \rho_B/\rho_c$. Now using [cf. Eq. (42)]

$$\rho_{c0} \approx 1.1 \times 10^{-5}\ h_0^2 \text{ GeV cm}^{-3} \tag{18.85}$$

and taking $m_B = 1$ GeV, we obtain

$$n_{B0} \approx \Omega_{B0}\left(1.1 \times 10^{-5}\ h_0^2\right)\ \text{cm}^{-3} \tag{18.86}$$

$$\eta = \frac{n_{B0}}{n_\gamma} = 2.65 \times 10^{-8}\ \Omega_{B0}\ h_0^2. \tag{18.87a}$$

This relation is sometimes written as

$$\Omega_{B0} h_0^2 = 3.78 \times 10^7\ \ \eta \tag{18.87b}$$

and

$$\begin{aligned} \rho_{B0} &= \eta\, n_{\gamma_0}\ (1 \text{ GeV}) = \eta\ (411)\ (1 \text{ GeV})\ \text{cm}^{-3} \\ &= 7.0 \times 10^{-22} \eta \text{ gm cm}^{-3}. \end{aligned} \tag{18.87c}$$

Big Bang nucleosynthesis limit η to [see Sec. 7]

$$2.4 \times 10^{-10} \le \eta \le 4.2 \times 10^{-10}. \tag{18.88}$$

18.5 Freeze Out

At high temperatures ($k_B T \gg m$), thermodynamic equilibrium is maintained through the processes of decays, inverse decays and scatterings. As the universe cools and expands, the reaction rates will fail to keep up with the expansion rate and there will come a time when equilibrium will no longer be maintained. At various stages then, depending on masses and interaction strengths, different particles will decouple with a "freeze out" surviving abundance. We now determine conditions under which the statistical equilibrium is established.

From dimensional analysis, the reaction rate for a typical process can be written as follows. For the decay of a X-particle, the decay rate is given by

$$\Gamma_X = g_d \, \alpha_X \, m_X \, \frac{m_X}{\left[(k_B T)^2 + m_X^2\right]^{1/2}}, \tag{18.89}$$

where m_X is the mass of the X-particle, $\alpha_X = \frac{f_X^2}{4\pi}$ is the measure of coupling strength of X-particle to the decay products, and g_d are number of spin states for the decay channels. Note that

$$\Gamma_X \approx \begin{cases} g_d \, \alpha_X \, m_X & k_B T \ll m_X \\ g_d \, \alpha_X \, \frac{m_X^2}{k_B T} & k_B T \gg m_X . \end{cases} \tag{18.90}$$

The reaction rate for the scattering processes is given by

$$\Gamma = \langle \sigma \, v \rangle \ [\text{number of target particles per unit volume}]. \tag{18.91}$$

For a weak scattering process

$$\langle \sigma \, v \rangle = g_W^4 \, \frac{(k_B T)^2}{\left[(k_B T)^2 + m_W^2\right]^2}. \tag{18.92}$$

Since the number of target particles per unit volume $n \sim (k_BT)^3$, we can write the reaction rate for a weak process

$$\Gamma \sim g_W^4 \, \frac{(k_BT)^5}{\left[(k_BT)^2 + m_W^2\right]^2}. \tag{18.93}$$

For $k_BT \ll m_W$, we get

$$\Gamma \sim \frac{g_W^4}{m_W^4} \, (k_BT)^5 \approx G_F^2 \, (k_BT)^5. \tag{18.94}$$

The condition for thermal equilibrium is

$$\Gamma \geq H \tag{18.95}$$

i.e. the reaction rate Γ must be greater than the expansion rate to maintain the thermodynamic equilibrium.

We now consider a specific example. At about a temperature of 10 MeV, the universe is made up of neutrons, protons, ν's, $\bar{\nu}$'s , $e^{\pm}$ and γ's in thermodynamic equilibrium. At about a few MeV, the neutrinos decouple. To see this consider the processes

$$\nu_e e^- \leftrightarrow \nu_e e^-, \qquad \nu_e \bar{\nu}_e \leftrightarrow e^+ e^-.$$

For these processes $\langle \sigma \ v \rangle = G_F^2/\pi$ s and $(2/3\pi)\, G_F^2$ s respectively. Thus the reaction rate

$$\Gamma \approx \frac{2}{3\pi} G_F^2 \, (k_BT)^5. \tag{18.96}$$

Now using Eq. (73), we find from Eq. (96) [$G_F = 1.166 \times 10^{-5}$ GeV^{-2}]

$$\begin{aligned} & 0.7 \left(\frac{k_BT}{\text{MeV}}\right)^2 s^{-1} \\ = \ & \frac{2}{3\pi} G_F^2 \, (1 \text{ GeV})^4 \, 10^{-12} \left(\frac{k_BT}{\text{MeV}}\right)^5 \frac{1}{\hbar} \, s^{-1} \\ = \ & 0.04 \left(\frac{k_BT}{\text{MeV}}\right)^5 s^{-1}. \end{aligned} \tag{18.97}$$

Thus the decoupling temperature for neutrinos is given by

$$k_B T_D = 2.6 \text{ MeV}. \tag{18.98}$$

Hence for $k_B T < 2.6$ MeV, neutrinos are decoupled. The neutrinos are extreme relativistic particles. For (ER) particles

$$N_{\text{Eq}}^{\text{ER}} = n_{\text{Eq}}^{\text{ER}}\, V \propto T^3\, R^3. \tag{18.99}$$

Now the entropy for ER gas is given by [cf. Eq. (53)]

$$S = \frac{R^3}{T}\frac{4}{3}\, \rho\,(T)\,. \tag{18.100}$$

But $\rho\,(T) \propto R^{-4}$, therefore, $S \sim (1/RT)$. Thus for the entropy to remain constant $T \propto R^{-1}$. Hence from Eq. (99), we have the important result: In equilibrium ER particles are conserved. This can also be seen as follows:

As we have discussed in the beginning of this section, at high temperatures all interacting species i, j, l, m are in thermodynamic equilibrium through the reactions of the type

$$i\; j \leftrightarrow l\; m.$$

As the reaction rate $\Gamma < H$ (the expansion rate), the species involved decouple and their abundance is frozen out. Consider an arbitrary volume V and let N_i be the number of particles of type i in this volume. Thus for the reaction $i\; j \leftrightarrow l\; m$, we have

$$\frac{d\, N_i}{dt} = \Gamma_{\text{prod}}\; N_l - \Gamma_{\text{ann}}\; N_i \tag{18.101}$$

where

$$\Gamma_{\text{prod}} = \langle v\; \sigma_{lm\to ij}\rangle\; n_m \tag{18.102a}$$

$$\Gamma_{\text{ann}} = \langle v\; \sigma_{ij\to lm}\rangle\; n_j. \tag{18.102b}$$

Thus

$$\frac{d\, N_i}{dt} = \langle v\; \sigma_{lm\to ij}\rangle\; n_m\; N_l - \langle v\; \sigma_{ij\to lm}\rangle\; n_j\; N_i. \tag{18.103}$$

Now $N_i = n_i V$, therefore

$$\frac{d\,N_i}{dt} = V\,\frac{d\,n_i}{dt} + n_i\,\frac{d\,V}{dt} \propto \left[R^3 \frac{d\,n_i}{dt} + 3\,n_i\,R^2 \frac{d\,R}{dt}\right]. \quad (18.104)$$

Hence we have from Eq. (103)

$$\frac{d\,n_i}{dt} = -3\,n_i\,\frac{\dot{R}}{R} + \langle v\,\sigma_{lm\to ij}\rangle\;n_m\,n_l - \langle v\,\sigma_{ij\to lm}\rangle\;n_j\,n_i. \quad (18.105)$$

The principle of detailed balance gives

$$\frac{\langle v\,\sigma_{lm\to ij}\rangle}{\langle v\,\sigma_{ij\to lm}\rangle} = \frac{n_l^{Eq}\,n_m^{Eq}}{n_i^{Eq}\,n_j^{Eq}}. \quad (18.106)$$

Thus from Eqs. (103) and (105), we have

$$\frac{d\,N_i^{Eq}}{dt} = \frac{V\;\langle v\,\sigma_{ij\to lm}\rangle\,Eq}{n_i^{Eq}\,n_j^{Eq}}\left[\left(n_l^{Eq}\,n_m^{Eq}\right)^2 - \left(n_i^{Eq}\,n_j^{Eq}\right)^2\right] \quad (18.107)$$

and

$$\frac{d\,n_i^{Eq}}{dt} = -3\,n_i^{Eq}\,\frac{\dot{R}}{R} + \frac{\langle v\,\sigma_{ij\to lm}\rangle\,Eq}{n_i^{Eq}\,n_j^{Eq}}\left[\left(n_l^{Eq}\,n_m^{Eq}\right)^2 - \left(n_i^{Eq}\,n_j^{Eq}\right)^2\right]. \quad (18.108)$$

First we note that n_i^{Eq} etc. are given by Eq. (45). For

$$\frac{d\,N_i^{Eq}}{dt} = 0 \quad (18.109)$$

i.e. for the conservation of N_i^{Eq}, we must have

$$\frac{d\,n_i^{Eq}}{dt} = -3\,n_i^{Eq}\,\frac{\dot{R}}{R}. \quad (18.110)$$

From this equation, we get

$$n_i^{Eq} \propto \frac{1}{R^3}, \quad (18.111)$$

and from Eq. (107), we get

$$(n_i\, n_j)_{\text{Eq}} = (n_l\, n_m)_{\text{Eq}}$$

i.e. in equilibrium extreme relativistic particles are conserved. The condition (111) is always satisfied for the extreme relativistic particle [cf. Eq. (64)].

Weakly interacting particles may decouple when they are ER, massless particles are always ER. For massive particles whose interactions are sufficiently strong to be capable of maintaining equilibrium when $k_B\, T < m$: [cf. Eq. (61)].

$$N_{\text{Eq}}^{\text{NR}} = n_{\text{Eq}}^{\text{NR}}\, V \propto (m\, k_B\, T)^{3/2} \exp\left(-\frac{m}{k_B\, T}\right) \frac{1}{T^3}. \qquad (18.112)$$

18.6 Limit on Neutrino Mass

We now use the result that neutrinos decouple at a temperature of a few MeV (i.e. they go out of equilibrium before $e^- e^+$ annihilation heated up the photon background radiation). Thus T_{ν_0} will be less than T_{γ_0}. Using Eq. (51), we get

$$\frac{n_{\nu_0}}{n_{\gamma_0}} = \frac{3}{4}\left(\frac{T_{\nu 0}}{T_{\gamma 0}}\right)^3. \qquad (18.113)$$

Now using Eq. (53) or (100), the entropy before $e^- e^+$ annihilation is given by

$$S = \frac{4}{3}\left(\rho_{e^-} + \rho_{e^+} + \rho_\gamma\right) \frac{R^3}{T_{\text{before}}}, \qquad (18.114)$$

and the entropy after $e^- e^+$ annihilation is given by

$$S = \frac{4}{3} \rho_\gamma \frac{R^3}{T_{\text{after}}}. \qquad (18.115)$$

Thus we have from Eqs. (114), (115) and (50)

$$\left(\frac{7}{8} \times 2 + \frac{7}{8} \times 2 + 2\right)\, T_{\text{before}}^3 = 2\, T_{\text{after}}^3. \qquad (18.116)$$

Noting that $T_{\text{before}} = T_{\nu 0}$ and $T_{\text{after}} = T_{\gamma 0}$, we get

$$\left(\frac{T_{\nu 0}}{T_{\gamma 0}}\right)^3 = \frac{4}{11}. \tag{18.117}$$

Hence we have from Eq. (113)

$$\frac{n_{\nu 0}}{n_{\gamma 0}} = \frac{3}{11} \tag{18.118}$$

and [cf. Eq. (82)]

$$n_{\nu 0} = \frac{3}{11}(411)\ \text{cm}^{-3}. \tag{18.119}$$

The present neutrinos density in GeV cm^{-3} can be written as

$$\rho_{\nu 0} \approx n_{\nu 0} \left(\sum_i m_{\nu i} \text{ in eV}\right) \times 10^{-9}\ \text{GeV cm}^{-3}. \tag{18.120}$$

Using Eq. (119), we get

$$\rho_{\nu 0} \approx (112) \left(\sum_i m_{\nu i}\ \text{eV}\right) \times 10^{-9}\ \text{GeV cm}^{-3}. \tag{18.121}$$

Now $\rho_{\nu 0}$ must be less than the average density of the universe ρ_0. Thus [cf. Eq. (42)]:

$$\rho_{\nu 0} < \rho_0 \approx 1.1 \times 10^{-5}\ \Omega_0\ h_0^2\ \text{GeV cm}^{-3}. \tag{18.122}$$

Hence we have from Eq. (121)

$$\sum_i m_{\nu i} < 100\ \Omega_0\ h_0^2\ \text{eV}. \tag{18.123}$$

Here the sum runs over all neutrino species with $m_\nu < 1$ MeV. Now if the age of the universe $t_u \geq 13 \times 10^9$ yrs, then, Eqs. (37) and (38) imply that $\Omega_0\ h_0^2 \leq 0.45$ for $h_0 \geq 0.4$, while $t_u \geq 10 \times 10^9$ yrs implies $\Omega_0\ h_0^2 \leq 1$ for $h_0 \geq 0.4$. The above constraints on $\Omega_0\ h_0^2$ give respectively

$$\sum_i m_{\nu i} < 45\ \text{eV} \tag{18.124a}$$

or

$$\sum_i m\nu_i < 100 \text{ eV}. \tag{18.124b}$$

There is a wide consensus among the astrophysicists that at least 90% of the mass in the universe does not shine (i.e. not visible by optical or radio means). It is only detected through its gravitational interaction. Now from Eq. (87c),

$$\rho_{B0} \approx 7.0\ \eta \times 10^{-22} \text{ gm/cm}^3 \tag{18.125}$$

where as discussed in the next section [cf. Eq. (135)] $\eta \leq 4 \times 10^{-10}$ giving

$$\rho_{B0} \leq 2.8 \times 10^{-31} \text{ gm/cm}^3 \tag{18.126}$$

to be compared with the critical density $\rho_{c0} \approx 1.88 \times 10^{-29}\ h_0^2$ gm/cm^3. Furthermore from Eqs. (87b) and (88)

$$0.009 \leq \Omega_{B0} h_0^2 \leq 0.016 \tag{18.127a}$$

or, since $0.4 \leq h_0 \leq 1$,

$$0.009 \leq \Omega_{B0} \leq 0.1. \tag{18.127b}$$

Since $\Omega_0 > 0.1$, the dark matter is mostly of nonbaryonic origin and the universe is not closed by baryons. It must certainly have contributed to the formation of galaxies. Light relic neutrinos are typical candidates for hot dark matter because their velocity was relativistic at the time of decoupling. Eqs. (42) and (121)

$$\Omega_\nu^{HDM} h_0^2 \equiv \frac{\rho_{\nu 0}}{\rho_{c0}} = \frac{\sum_i m_{\nu_i} \text{ in eV}}{94 \text{ eV}}. \tag{18.128}$$

Unfortunately there is no direct particle physics evidence on $\sum_i m_{\nu_i}$. However, if $\Omega_\nu^{HDM} \simeq 0.2$ and $h_0^2 \simeq 0.30$, then $\sum_i m_{\nu_i} \simeq 5 - 6$ eV.

18.7 Primordial Nucleosynthesis

At temperatures ≥ 1 MeV, the weak reactions such as

$$\begin{aligned} \bar{\nu}_e + p &\leftrightarrow e^+ + n \\ e^- + p &\leftrightarrow \nu_e + n \end{aligned} \tag{18.129}$$

are still fast compared with the expansion rate of the universe to maintain thermodynamic equilibrium between p and n. The abundance ratio at equilibrium is given by

$$\frac{n}{p} \sim e^{-\Delta m/(k_B T)}, \qquad k_B T > k_B T_D \sim 1 \text{ MeV}. \tag{18.130}$$

Using $\Delta m = (m_n - m_p) = 1.3$ MeV and $k_B T = k_B T_D = 1$ MeV, we find $n/p = 0.27$. The decoupling temperature T_D is estimated as follows. The rough estimate for the reaction rate in Eq. (129) is given by Eq. (94). A more accurate calculation gives

$$\begin{aligned} \Gamma &= \frac{7\pi}{30} G_F^2 \left(1 + 3g_A^2\right) (k_B T)^5 \\ &\approx 4.22\, G_F^2 \, (k_B T)^5 = 0.8 \left(\frac{k_B T}{\text{MeV}}\right)^5 \text{ s}^{-1}. \end{aligned} \tag{18.131}$$

The decoupling temperature is given by $\Gamma = H$ viz. [cf. Eq. (73), where we have taken $N_\nu = 3$]

$$0.8 \left(\frac{k_B T}{\text{MeV}}\right)^5 = 0.7 \left(\frac{k_B T}{\text{MeV}}\right)^2. \tag{18.132}$$

Thus

$$k_B T = k_B T_D \simeq 1 \text{ MeV}. \tag{18.133}$$

As the temperature cools past the decoupling temperature $k_B T_D \approx 1$ MeV, it is no longer possible to maintain the thermal equilibrium. The ratio n/p thereafter is frozen out and is approximately constant (it decreases slowly due to weak decay of neutron). The freeze out n/p ratio is given by

$$\frac{n}{p} \approx e^{-Q/k_B T_D} \approx 0.16, \tag{18.134}$$

where we have used the $Q-$value $Q = (m_n - m_p) + m_e = 1.8$ MeV. For $T > T_S$, the deuteron formed is knocked out by photo dissociation

$$\gamma + D \to p + n,$$

since the binding energy ΔB for the deuteron is only 2.2 MeV. The formation of deuteron actually starts after $k_B T_S \approx 0.1$ MeV; T_S is called nucleosynthesis temperature. The estimate that $k_B T_S \approx 0.1$ MeV can be obtained as follows:

$$\frac{n_\gamma^{\text{diss}}}{n_B} \sim \frac{1}{\eta} e^{-\Delta B / k_B T} \leq 1. \tag{18.135}$$

Thus

$$-\frac{\Delta B}{k_B T_S} \approx \ln \eta. \tag{18.136}$$

Using $\Delta B \approx 2.2$ MeV and $\eta \approx 10^{-10}$, we find $k_B T_S \approx 0.1$ MeV.

For $T > T_S$, photodissociation is so rapid that deuteron abundance is negligibly small and this provides a bottleneck to further nucleosynthesis. The deuteron "bottleneck" thus delay nucleosynthesis till $k_B T \leq 0.1$ MeV. But once the bottleneck is passed, nucleosynthesis proceeds rapidly and essentially all neutrons are incorporated into ^4He :

$$\begin{aligned}
n + p &\longrightarrow D + \gamma \\
D + D &\longrightarrow {}^3H + p,\ {}^3H_e + n \\
{}^3H + D &\longrightarrow {}^4H_e + n \\
{}^3H + {}^4H_e &\longrightarrow {}^7Li
\end{aligned}$$

It is clear from the above reactions that ^4He abundance is given by

$$Y = \frac{2\,(n\ /\ p)}{1 + n\ /\ p} = \frac{0.32}{1.16} = 0.27. \tag{18.137a}$$

The ratio Y changes from T_D to T_S due to the neutron decay $n \to p + e^- + \bar{\nu}_e$. During this time n/p changes from 0.16 to 0.14. Thus at $T = T_S$,

$$Y = \frac{0.28}{1.14} = 0.25. \tag{18.137b}$$

We conclude that n/p ratio or Y depends on three parameters:

(i) decoupling temperature T_D, which in turn depends on the number of light particles, e.g. number of neutrino flavors N_ν.

(ii) neutron decay in between T_D and T_S i.e. on the decay rate of neutron or neutron half-life $\tau_{1/2}$.

(iii) $\eta = n_B/n_\gamma$.

In fact Y is most sensitive function of Γ/H. Now T_D depends on the expansion rate H; the expansion rate depends upon the effective degrees of freedom g_*, the higher the g_*, the faster the expansion rate. This implies higher T_D and hence higher n/p freeze out abundance. Thus the higher the g_*, the higher will be Y. But $g_* = 2 + \frac{7}{8}(4 + 2N_\nu)$, where N_ν are the number of neutrino species. For $N_\nu = 3$, $g_* = \frac{43}{4}$ and we obtained $T_D \approx 1$ MeV and $Y \approx 0.25$. The observed primordial abundance of ^4He gives $Y = 0.234 \pm 0.002\,(\pm 0.005)$. The half life for neutron decay, the parameter needed in the above analysis, is $\tau_{1/2} = 887 \pm 2\ s$.

Taking $N\nu = 3$ as given by LEP data [cf. Sec. 13]: $N\nu = 2.999 \pm 0.016$, one can use the observed primordial abundances of D, ^{4}He and ^{7}Li, to get a limit on η :

$$2.4 \times 10^{-10} \le \eta \le 4.2 \times 10^{-10}. \tag{18.138}$$

As already remarked this value of η, implies [cf. Eq. (127b)]

$$0.009 \le \Omega_{B0} \le 0.1. \tag{18.139}$$

If $\Omega_0 > 0.1$, then as remarked in the previous section, some other non-baryonic form of matter must account for the difference between Ω_0 and Ω_{B0} . There are some dynamical models which suggest $\Omega_0 = 1$, this requires a large amount of non-baryonic matter.

18.8 Baryon Asymmetry of the Universe: Baryogenesis

There is no evidence for the existence of antibaryons in the universe. The baryons to photons ratio $\eta = n_B/n_\gamma \simeq 3 \times 10^{-10}$. The asymmetry between baryons and antibaryons can be explained as follows: The universe started with a complete matter-antimatter symmetry in a standard big bang picture. In the subsequent evolution of the universe, a net baryon number was generated. This is possible if the following three conditions are satisfied:

(i) There exists a baryon number violating interaction.

(ii) There exist C and CP violation to introduce the asymmetry between particle and antiparticle processes.

(iii) Departure from thermal equilibrium of $X-$particles which mediate the baryon number violating interactions.

The condition (iii) is necessary because if the baryon – violating interactions were always in equilibrium, the number of particles and antiparticles would be given by e^{-m/k_BT} and $e^{-\bar{m}/k_BT}$ and thus would be equal since $\bar{m} = m$ by CPT theorem. The condition (iii) is supplied by the expansion of the universe. The condition (i) is supplied by the $X-$particles (vector and scalar bosons) predicted by grand unified models. At $T = T_D$ (the decoupling temperature i.e. the temperature at which $X-$particles go out of equilibrium), the number density of $X-$particles is given by [cf. Eqs. (49) and (54)]:

$$n_{XD} = \frac{2.4}{\pi^2}\frac{g_X}{2}\ (k_B\, T_D)^3, \tag{18.140}$$

where g_X is the total number of X (and $\bar{X}$) spin states. Now the entropy density at T_D is given by

$$s = \frac{S}{R^3} = k_B\ \frac{\pi^2}{15}\left(\frac{4}{3}\right)\ (k_B\, T_D)^3\, g_*/2, \tag{18.141}$$

where g_* is the effective number of degrees of freedom. The number of baryons at T_D are given by

$$n_B = n_{XD}\,\Delta B. \tag{18.142}$$

But

$$\begin{aligned} k_B\left(\frac{n_B}{s}\right)_D &= (2.4)\,\frac{45}{4\pi^4}\left(\frac{g_X}{g_*}\right)\Delta B \\ &= 0.28\left(\frac{g_X}{g_*}\right)\Delta B. \end{aligned} \tag{18.143}$$

Now g_* is over 100 in a typical GUT. [In SU(5): $\gamma, W^{\pm}, Z^0$, 8G's, 34 Higgs, 6 quarks, 3 leptons, 3 neutrinos, 12 X's. Thus $g_* = (24 \times 2) + 34 + \frac{7}{8}(18 \times 4 + 3 \times 4 + 3 \times 2) = 160.8$.] We, therefore, expect $g_X/g_* \approx 10^{-2}$ to 10^{-1}. Thus we have

$$k_B\left(\frac{n_B}{s}\right)_D \approx 0.28\times\left(10^{-2} - 10^{-1}\right)\Delta B \approx 3\times\left(10^{-3} - 10^{-2}\right)\Delta B. \tag{18.144}$$

But $(n_B/s)_D = (n_B/s)_0$, where 0 denotes the present time. Thus

$$k_B\left(\frac{n_B}{s}\right)_0 \approx 3\times\left(10^{-3} - 10^{-2}\right)\Delta B. \tag{18.145}$$

Now [cf. Eqs. (58) and (54)]

$$\begin{aligned} \left(\frac{s}{k_B}\right) &= \left(\frac{s}{k_B}\right)_{\gamma 0} + \left(\frac{s}{k_B}\right)_{\nu 0} \\ &= \frac{2\pi^4}{45\times 1.2}\left[\frac{1}{2}\,g_\gamma\,n_{\gamma 0} + \frac{1}{2}\sum_i g_{\nu i}\,n_{\nu 0}\right] \\ &= 3.6\left[1 + \frac{21}{4}\frac{1}{2}\frac{4}{11}\right] n_{\gamma 0} \approx 7\,n_{\gamma 0}. \end{aligned} \tag{18.146}$$

Hence from Eq. (145), we get

$$\left(\frac{n_B}{n_\gamma}\right)_0 \approx 21\times\left(10^{-3} \text{ to } 10^{-2}\right)\Delta B \approx 2\times\left(10^{-2} \text{ to } 10^{-1}\right)\Delta B. \tag{18.147}$$

The X−particles can generate ΔB, by the processes of the following type

$$\begin{array}{lll} X & \rightarrow & ql : r \qquad B_1 = 1/3 \\ X & \rightarrow & \bar{q}\bar{q} : 1-r \qquad \bar{B}_2 = -2/3 \\ \bar{X} & \rightarrow & \bar{q}\bar{l} : \bar{r} \qquad \bar{B}_1 = -1/3 \\ \bar{X} & \rightarrow & qq : 1-\bar{r} \qquad B_2 = 2/3. \end{array}$$

The mean baryon number per decay

$$\begin{aligned} B_X &= r\, B_1 + (1-r)\, \bar{B}_2 \\ B_{\bar{X}} &= \bar{r}\, \bar{B}_1 + (1-\bar{r})\, B_2. \end{aligned} \tag{18.148}$$

Thus

$$\begin{aligned} \Delta B &= \frac{1}{2}\left[r\, B_1 + (1-r)\, \bar{B}_2 + \bar{r}\, \bar{B}_1 + (1-\bar{r})\, B_2\right] \\ &= \frac{1}{2}\left[r\, \left(B_1 - \bar{B}_2\right) + \bar{r}\, \left(\bar{B}_1 - B_2\right) + \left(\bar{B}_2 + B_2\right)\right] \\ &= \frac{1}{2}\left(r\, - \bar{r}\right). \end{aligned} \tag{18.149}$$

From Eqs. (149) and (147), we see that we can explain the baryon number generation if $r \neq \bar{r}$, i.e. X−interactions violate C and CP. Also we require $\Delta B \sim 10^{-8}$ in order to explain the present baryon number $\eta = n_B/n_\gamma \approx 10^{-10}$.

Let us now obtain an estimate for T_D. If $k_B\, T_D > m_X$, the thermal equilibrium can be maintained by inverse decays. Thus the condition for departure from equilibrium is [cf. Eqs. (67) and (90)]:

$$\frac{1}{3}\alpha_X\, g_d\, \left(k_B\, T_D\right) \approx 1.66\, g_*^{1/2}\, \frac{\left(k_B\, T_D\right)^2}{M_P}. \tag{18.150}$$

Now using $g_d \approx 12 \times 2 = 24$ and $g_* \approx 160$, we get

$$k_B\, T_D \approx \alpha_X\, (4.0)\, 10^{18}\ \text{GeV}. \tag{18.151}$$

Using $\alpha_X \approx 1/40$ [SU(5) value], we get

$$k_B\, T_D \approx 10^{17}\ \text{GeV}. \tag{18.152}$$

Thus if $X-$boson are vector bosons, $k_B\, T_D >$ mass of vector bosons of SU(5) and therefore vector bosons of SU(5) cannot give rise to baryon asymmetry.

However, for Higgs scalar $\alpha_X \sim 10^{-4}$, and $k_B\, T_D \sim 10^{13}$ GeV. If $k_B T_D <$ the mass of Higgs scalars, inverse decays are not energetically allowed and baryon asymmetry may arise as the scalar bosons go out of thermal equilibrium at $k_B\, T_D \sim 10^{13}$ GeV. However, in SU(5), the scalar bosons give $\Delta B \sim 10^{-15}$, but we require $\Delta B \sim 10^{-8}$.

18.9 Inflation

There are several problems in the standard model of cosmology. We now discuss two of these problems and how to resolve them in an inflationary universe.

18.9.1 Horizon problem

Horizon of the universe $r_H(t)$ (called the particle horizon) at time t is defined as the size which can be causally related during the evolution of the universe. Since signals cannot travel with speed greater than c,

$$r_H(t) = ct$$

$$r_H(t_0) = ct_0 \approx 10^{28}\ \text{cm}, \ \left[t_0 \sim 4.5 \times 10^{17}\ \text{s}\right]. \tag{18.153}$$

This gives the maximum size of the observable universe. We call the present size of the universe $r_0(t_0)$. Let us extrapolate it backward in time:

$$r_0\,(t) = \frac{R(t)}{R(t_0)} r_0\,(t_0) = \frac{R(t)}{R(t_0)}\,(ct_0)\,. \tag{18.154}$$

Thus

$$\frac{r_0\,(t)}{r_H\,(t)} = \frac{R(t)}{R(t_0)}\frac{t_0}{t}. \tag{18.155}$$

Let us take $t = t_s = 1.3 \times 10^{13}$ sec $(4 \times 10^5$ yrs$)$, which corresponds to the epic of the "last scattering surface". This is the time when the universe just started to be matter dominated. Thus [cf. Eqs. (69) and (29)]

$$R(t) \sim \begin{cases} t^{1/2} & t < t_s \\ t^{2/3} & t > t_s . \end{cases} \tag{18.156}$$

Hence, rewriting Eq. (155),

$$\begin{aligned} \frac{r_0\,(t)}{r_H\,(t)} &= \frac{R(t)}{R(t_s)}\frac{R(t_s)}{R(t_0)}\frac{t_0}{t} \\ &= \left(\frac{t_0}{t_s}\right)^{-1/6}\left(\frac{t_0}{t}\right)^{1/2} . \end{aligned} \tag{18.157}$$

Then using $t_0 \approx 4.5 \times 10^{17}$ s $(1.5 \times 10^{10}$ yrs$)$, we have at the last scattering surface

$$\frac{r_0\,(t_s)}{r_H\,(t_s)} \approx 33$$

while

$$\begin{aligned} \frac{r_0\,(t_{\text{Planck}})}{r_H\,(t_{\text{Planck}})} &\approx 10^{30}, \qquad t_{\text{Planck}} \approx 10^{-44}\ \text{s} \\ \frac{r_0\,(t_{Gut})}{r_H\,(t_{Gut})} &\approx 10^{26}, \qquad t_{Gut} \approx 10^{-36}\ \text{s}. \end{aligned} \tag{18.158}$$

The uniformity of the temperature of the background microwave radiation $\left(\frac{\Delta T}{T} \leq 10^{-4}\right)$ provides a strong evidence that universe is isotropic to a high degree of precision. But the present size of the universe extrapolated backward in time to t_s is 33 times the particle horizon. Therefore, there would not have time for transport processes to equalize the temperature over the last scattering surface. Thus it is difficult to explain the high degree of isotropy in the present universe.

18.9.2 Flatness problem

Why is the universe near the critical density? Stated in other words why the curvature term does not dominate at a certain $R(t)$. To see this problem, we note [cf. Eqs. (20) and (17)]

$$\begin{aligned}(\Omega - 1) &= \frac{\rho - \rho_c}{\rho_c} = \frac{k}{R^2(t)} H^2(t) \\ &= \frac{k}{\dot{R}^2(t)},\end{aligned} \tag{18.159}$$

$$\frac{\Omega - 1}{\Omega_0 - 1} = \frac{\left[\dot{R}(t_0)\right]^2}{\left[\dot{R}(t)\right]^2}. \tag{18.160}$$

Using Eq. (156),

$$\dot{R}(t) \sim \left[\begin{array}{ll} \frac{1}{2}\, t^{-1/2} & t \le t_s \\ \frac{2}{3}\, t^{-1/3} & t \ge t_s. \end{array}\right. \tag{18.161}$$

Thus we get

$$\begin{aligned}\frac{\Omega(t_s) - 1}{\Omega_0 - 1} &= \left(\frac{t_s}{t_0}\right)^{2/3} \\ \frac{\Omega(t) - 1}{\Omega(t_s) - 1} &= \left(\frac{t}{t_s}\right).\end{aligned} \tag{18.162}$$

Taking $t_s \sim 1.3 \times 10^{13}$ s and $t_0 \sim 4.5 \times 10^{17}$ s , we have [for $\Omega_0 = 0.1$ and $\Omega_0 = 2$]

$$\Omega(t_s) \simeq 1 + 10^{-3}\ (\Omega_0 - 1) \approx 1 \mp 10^{-3} \tag{18.163}$$

while for $t_{\text{Planck}} \simeq 10^{-44}$ s

$$\Omega(t_{\text{Planck}}) = 1 \mp 10^{-60}. \tag{18.164}$$

Such a terrible "fine tuning" looks unnatural. The natural solution is either $\rho = \rho_c$, $\Omega = 1$ (for a reason to be discovered) for the whole history of the universe or some nonstandard mechanism intervened to derive $\rho_0 \longrightarrow \rho_{c0}$, $\Omega_0 \longrightarrow 1$.

18.9.3 Inflationary universe

The basic idea of this scenario is that there was an epoch when the vacuum energy density dominated the energy density of the universe. Thus we write

$$\rho = \rho_V + \rho_r = \rho_V + \frac{\pi^2}{30}\, g_*(T)\ (k_B\, T)^4 . \tag{18.165}$$

The radiation era density $\rho_r \sim \frac{1}{R^4}$, but ρ_V is constant independent of R. Suppose $\rho_V \gg \rho_r$ in the early universe. Thus from Eq. (62), we have

$$\frac{\dot{R}}{R} = H = \frac{\sqrt{8\pi\ G\ \rho_V}}{3} = \text{const.} \tag{18.166}$$

Thus ρ_V acts like an effective cosmological constant. We get

$$R\,(t) = e^{Ht} = e^{t/t_{Gut}}, \tag{18.167}$$

where we have put

$$H = \sqrt{\frac{8\pi\ G\ \rho_V}{3}} = t_{Gut}^{-1}. \tag{18.168}$$

The exponential increase of $R(t)$ with t is called the inflation. What can cause this inflation? This scenario may happen in the spontaneously broken grand unified theories (GUT). Consider the phase transition for the symmetric phase $[\langle\phi\rangle = 0]$ $T \gg T_c$ to the broken phase $[\langle\phi\rangle \neq 0\ ]$ $T < T_c$. If this phase transition is of first order, then it is accompanied by latent heat. Note that T_c denotes the critical temperature and ϕ is the Higgs scalar responsible for spontaneous symmetry breaking. For $T \gg T_c$, $\langle\phi\rangle = 0$ is the local and global minimum. At $T = 0$, $\langle\phi\rangle = M_X$ is the local and global minimum. For $T = T_c$, both $\langle\phi\rangle = 0$ and $\langle\phi\rangle \sim M_X$ are minima. Below $T < T_c$, $\langle\phi\rangle = 0$ is a local minimum (false vacuum) (see Fig. 2).

If the universe is trapped in this false vacuum, then it is in a stage of supercooling because to go over to true vacuum it has to cross a potential barrier (see Fig. 2), either by thermal fluctuations or by quantum mechanical tunneling. If V is sufficiently flat, the time required for ϕ to transverse the flat region can be long compared to the expansion time scale t_{Gut}, say $\tau_\phi \approx 65\ t_{Gut}$. During this slow growth phase $\rho_V = V(\phi = 0)$ dominates over ρ_r and we get

$$R(t) = e^{t/t_{Gut}}, \tag{18.169}$$

where

$$t_{Gut}^{-1} = \sqrt{\frac{8\pi}{3}\frac{M_X^4}{M_P^2}} \approx 3 \times 10^{11}\ GeV \approx 10^{36}\ \mathrm{s}^{-1}. \tag{18.170}$$

Here we have put $\rho_V \sim M_X^4 \sim T_c^4$, $M_X \approx 10^{15}$ GeV and $G \sim \frac{1}{M_P^2} \approx 10^{-38}$ GeV^{-2}.

Thus for $t = 65\ t_{Gut} = \tau$, we have

$$\begin{aligned} \frac{r_0(t)}{r_H(t)} &= \frac{R(t)}{R(t_0)}\frac{t_0}{t} \\ &= \frac{R(t)}{R(\tau)}\frac{R(\tau)}{R(t_s)}\frac{R(t_s)}{R(t_0)}\frac{t_0}{t} \\ &= e^{t/t_{Gut}}\ e^{-t/\tau}\frac{\tau^{1/2}}{t_s^{1/2}}\frac{t_s^{2/3}}{t_0^{2/3}}\frac{t_0}{t} \\ &= \left(\frac{t_0}{t_s}\right)^{-1/6}\left(\frac{t_0}{t}\right)^{1/2} e^{t/t_{Gut}}\ e^{-t/\tau}\left(\frac{\tau}{t}\right)^{1/2}. \end{aligned} \tag{18.171}$$

Hence we have

$$\begin{aligned} \frac{r_0(t_{Gut})}{r_H(t_{Gut})} &= 1.75 \times 10^{-1}\ e\ e^{-65}\ \left(10^{53}\right)^{1/2}\sqrt{65}, \\ &\sim 10^{-1} < 1. \end{aligned} \tag{18.172}$$

Note that without the inflation $r_0(t_{Gut})/r_H(t_{Gut}) \sim 10^{26}$. Thus the problem of causal disconnection is solved. We note that $t_{Gut}^{-1} \approx 10^{11} GeV \sim 10^{-25}$ cm. But it exponentially grows to $e^{100}10^{-25} \sim 10^{18}$ cm after $t = 100\ t_{Gut} \approx 10^{-34}$ s.

The flatness problem is also solved in the inflationary universe scenario. Now [cf. Eq. (17)]

$$\Omega\,(t) \equiv \frac{\rho}{\rho_c} = \frac{1}{\left[1 - \frac{k/R^2}{8\pi G\frac{\rho}{3}}\right]}. \tag{18.173}$$

Also $[R(t)]^{-2} \sim e^{-2t/t_{Gut}} \rightarrow 0$ and $\rho = \rho_V$ (constant) for inflationary epoch, we get from Eq. (173)

$$\Omega\,(t) \approx 1. \tag{18.174}$$

Hence we see that Ω is driven to 1 in inflationary scenario.

The latent heat of the phase transition is used to reheat the universe to $T \approx 10^{14} GeV$, thus making baryon synthesis and creation of baryon number possible.

We have not discussed the monopole problem at all. We have only sketched the inflationary universe scenario. A more detailed discussion of inflation is beyond the scope of this book. Figure 3 gives a very rough sketch of the inflationary universe.

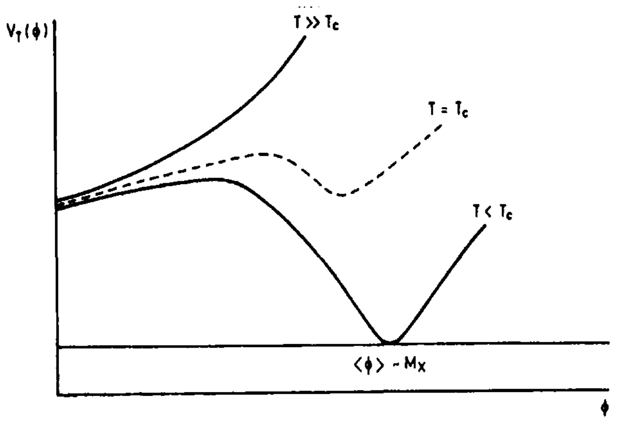

Figure 2 Behavior of the potential term $V_T(\phi)$ for $T < T_c, T = T_c$ and $T \gg T_c$.

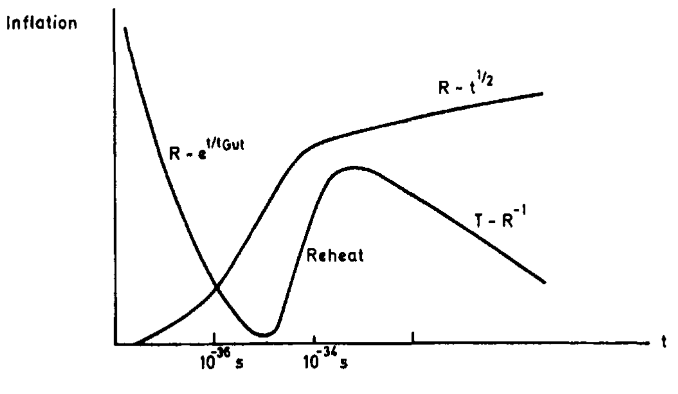

Figure 3 Scale factor $R(t)$ versus t, showing the inflationary phase.

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Appendix A
QUANTUM FIELD THEORY [A SUMMARY]

A.1 Spin 0 Field

Spin zero particle of mass m is described by a field $\phi(x)$ which in the absence of interactions, satisfies the Klein-Gordon equation.

$$\left(m^2 + \Box^2\right)\phi(x) = 0. \tag{A.1}$$

In quantum mechanics, $\phi(x)$ is regarded as a c-number. In quantum field theory, $\phi(x)$ is a field operator which can create and annihilate the field quantum.

The Fourier decomposition of $\phi(x)$ is

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k_0}} \left[a(k)\ e^{-ik.x} + b^\dagger(k)\ e^{ik.x}\right] \tag{A.2a}$$

$$\phi^\dagger(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k_0}} \left[a^\dagger(k)\ e^{ik.x} + b(k)\ e^{-ik.x}\right], \tag{A.2b}$$

where $\phi^\dagger(x)$ is hermitian conjugate of $\phi(x)$ and $k \cdot x = k_0 x_0 - \mathbf{k}.\mathbf{x}$, $k_0 = \sqrt{\mathbf{k}^2 + m^2} > 0$. In Eq. (2), $a(k)$ and $b(k)$ are interpreted as follows:

$a^{\dagger}(k)$: creation operator for the particle (spin 0 and mass m)

$a(k)$: annihilation operator for the particle (spin 0 and mass m)

$b^{\dagger}(k)$: creation operator for the antiparticle (spin 0 and mass m)

$b(k)$: annihilation operator for the antiparticle (spin 0 and mass m).

$a(k)$ and $b(k)$ satisfy the following commutation relations

$$\left[a(k), a^{\dagger}(k')\right] = \delta^3(\mathbf{k}-\mathbf{k}') \tag{A.3a}$$

$$\left[b(k), b^{\dagger}(k')\right] = \delta^3(\mathbf{k}-\mathbf{k}') \tag{A.3b}$$

$$[a(k), b(k')] = \left[a(k), b^{\dagger}(k')\right] = 0. \tag{A.3c}$$

If $|0\rangle$ denotes the vacuum state then one particle state of 4-momentum k is given by

$$|k\rangle = a^{\dagger}(k)|0\rangle. \tag{A.4}$$

Define

$$N_+(k) = a^{\dagger}(k)\ a(k) \tag{A.5a}$$

$$N_-(k) = b^{\dagger}(k)\ b(k). \tag{A.5b}$$

It follows from the commutation relations (3) that $N_+\ (k)$ and $N_-\ (k)$ have the eigenvalues 0, 1, 2, $\cdots$ and are known as number operators for the particles and antiparticles. Then

$$n = \sum_{\mathbf{k}} N_+\ (k) = \text{Total number of particles} \tag{A.6a}$$

$$\overline{n} = \sum_{\mathbf{k}} N_-(k) = \text{Total number of antiparticles.} \tag{A.6b}$$

It may also be noted that for free fields

$$\left[\phi(x),\ \phi^\dagger(x')\right] = i\Delta\ (x - x'), \tag{A.7a}$$

$$\Delta(x) = -\frac{i}{(2\pi)^3}\int d^4k\ \varepsilon(k_0)\ e^{-ik.x}\ \delta\left(k^2 - m^2\right) \tag{A.7b}$$

$$\varepsilon(k_0) \ = \ \begin{cases} +1 & k_0\ > 0 \\ -1 & k_0\ < 0 \end{cases}. \tag{A.7c}$$

We note that

$$\Delta\ (x - x') = 0 \qquad\qquad \text{for}\ (x - x')^2\ <\ 0 \tag{A.8}$$

viz. the space-like distances. Then from (7a), it follows that the commutator is zero for space-like separation. This is the statement of the micro causality. Also

$$\left|\frac{\partial\,\Delta\ (\ x - y)}{\partial\ x_0}\right|_{x_0 = y_0} = -\delta^3\ (\mathbf{x} - \mathbf{y}) \tag{A.9a}$$

and from Eq. (8), we get

$$\Delta\ (0,\ \mathbf{x} - \mathbf{y}) = 0. \tag{A.9b}$$

A.2 Spin 1/2 Particle

Spin 1/2 particle of mass m is described by a field $\Psi(x)$, which is the absence of interactions, satisfies the Dirac equation $\left[\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \boldsymbol{\nabla})\right]$

$$(i\gamma^\mu\partial_\mu\ - m)\,\Psi(x) = 0. \tag{A.10a}$$

The adjoint of $\Psi(x)$, $\bar{\Psi}(x) = \Psi^\dagger(x)\ \gamma^0$ satisfies the equation

$$\overline{\Psi}(x)\left(-i\gamma^\mu\ \overleftarrow{\partial_\mu}\ -m\right) = 0. \tag{A.10b}$$

γ^μ are Dirac matrices. We choose γ^μ :

$$\gamma^{0\dagger} = \gamma^0, \ \gamma^{i\dagger} = -\gamma^i \qquad i = 1, 2, 3, \tag{A.11}$$

γ^μ 's satisfy the anticommutation relation

$$[\gamma^\mu, \gamma^\nu]_+ \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{A.12}$$

There are 16 independent Dirac matrices:

Matrices	Components
1	1
γ^μ	4
$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$	6
$\gamma^5 = i\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$	1
$i\ \gamma^\mu\,\gamma^5$	4

γ^5 is also hermitian

$$\gamma^{5\dagger} = \gamma^5. \tag{A.13}$$

γ^5 anticommutes with γ^μ viz.

$$\gamma^5\,\gamma^\mu = -\gamma^\mu\,\gamma^5. \tag{A.14}$$

In Pauli representation, γ^μ 's can be written as

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_i = -\gamma^i \tag{A.15a}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma_0 \tag{A.15b}$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma_5 = \gamma^5. \tag{A.15c}$$

The Fourier decomposition of $\Psi(x)$ is

$$\Psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3\,p \sqrt{\frac{m}{p_0}}$$

$$\times \sum_{r=1}^{2} \left[a_r(p)\ u_r(p)\, e^{-ip.x} + b_r^\dagger(p)\ v_r(p)\, e^{ip.x}\right] \tag{A.16a}$$

$$\overline{\Psi}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 p \sqrt{\frac{m}{p_0}} \times \sum_{r=1}^{2} \left[a_r^\dagger(p)\ \overline{u}_r(p)\, e^{ip.x} + b_r(p)\ \overline{v}_r(p)\, e^{-ip.x}\right] \tag{A.16b}$$

where

$$\overline{u} = u^\dagger\, \gamma^0, \qquad \overline{v} = v^\dagger\, \gamma^0 \tag{A.17}$$

and u and v satisfy the equations

$$(\gamma.p - m)\, u_r(p) = 0 \tag{A.18a}$$
$$(\gamma.p + m)\, v_r(p) = 0 \tag{A.18b}$$
$$\overline{u}_r(p)\, (\gamma.p - m) = 0 \tag{A.18c}$$
$$\overline{v}_r(p)\, (\gamma.p + m) = 0 \tag{A.18d}$$

$a(p)$ and $b(p)$ are interpreted as follows:

$a_r^\dagger(p)$: creation operator of the particle with momentum $\mathbf{p}$ and spin component r

$a_r(p)$: annihilation operator of the particle with momentum $\mathbf{p}$ and spin component r

$b_r^\dagger(p)$: creation operator of the antiparticle with momentum $\mathbf{p}$ and spin component r

$b_r(p)$: annihilation operator of the particle with momentum $\mathbf{p}$ and spin component r.

The operators a and b satisfy the anticommutation relation

$$\left[a_r(p), a_{r'}^\dagger(p')\right]_+ = \delta_{r\,r'}\, \delta^3\ (\mathbf{p} - \mathbf{p}') \tag{A.19a}$$
$$\left[b_r(p), b_{r'}^\dagger(p)\right]_+ = \delta_{r\,r'}\, \delta^3\ (\mathbf{p} - \mathbf{p}') \tag{A.19b}$$

and all other anticommutation relations give zero. Define number operators:

$$N_r^{(+)}(p) = a_r^\dagger(p)\ a_r(p) \tag{A.20a}$$
$$N_r^{(-)}(p) = b_r^\dagger(p)\ b_r(p)\,. \tag{A.20b}$$

Then from the anticommutation relations (19), we have

$$\left[N_r^{(\pm)}(p)\right]^2 = N_r^{(\pm)}(p)\,. \tag{A.21}$$

Thus, we see that $N_r^{(\pm)}(p)$ have eigenvalue 0 or 1. This means that each state is either empty or has a single particle of definite spin and momentum. Thus the anticommutation relations lead to description of a system of particles which obey the Pauli exclusion principle or in other words obey the Fermi-Dirac statistics.

The spinors u and v satisfy the following orthogonality relations:

$$\overline{u}_r(p)\,u_{r'}(p) = \delta_{r\,r'} = -\overline{v}_r(p)\,v_{r'}(p) \tag{A.22a}$$
$$u_r^\dagger(p)\,u_{r'}(p) = \frac{E_p}{m}\,\delta_{r\,r'} = v_r^\dagger(p)\,v_{r'}(p) \tag{A.22b}$$
$$\overline{v}_r(p)\,u_{r'}(p) = \overline{u}_r(p)\,v_{r'}(p) = 0. \tag{A.22c}$$

They also satisfy the completeness relations

$$\sum_{r=1}^{2}\left[u_\alpha^r(p)\ \overline{u}_\beta^r(p) - v_\alpha^r(p)\ \overline{v}_\beta^r(p)\right] = \delta_{\alpha\,\beta}, \tag{A.23}$$

where α and β are spinor indices; $\alpha,\ \beta = 1,2,3,4$.

$$\sum_{r=1}^{2} u_\alpha^r(p)\ \overline{u}_\beta^r(p) = \left(\frac{\gamma\cdot p + m}{2m}\right)_{\alpha\,\beta} \equiv \left(\Lambda_+(p)\right)_{\alpha\,\beta} \tag{A.24a}$$

$$-\sum_{r=1}^{2} v_\alpha^r(p)\ \overline{v}_\beta^r(p) = \left(\frac{-\gamma\cdot p + m}{2m}\right)_{\alpha\,\beta} \equiv \left(\Lambda_-(p)\right)_{\alpha\,\beta}\,. \tag{A.24b}$$

$\Lambda_+(p)$ and $\Lambda_-(p)$ are the projection operators for particles and antiparticles respectively. One also writes $\gamma^\mu\, p_\mu = \gamma \cdot p = \not{p}$.

Using the Pauli representation of γ-matrices, we can write

$$u_r(p) = R\, w^{(r)} \tag{A.25a}$$

where

$$R = \frac{1}{\sqrt{2m(p_0+m)}} \begin{pmatrix} (p_0+m)\ I \\ \boldsymbol{\sigma}\cdot\mathbf{p} \end{pmatrix} \tag{A.25b}$$

$$p_0 \equiv E_p = \sqrt{p^2+m^2} \tag{A.25c}$$

$$w^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and } w^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.25d}$$

$\overline{u}_r(p)$ is given by

$$\overline{u}_r(p) = w^{(r)\dagger}\, R^\dagger\, \gamma^0 = w^{(r)\dagger}\, \overline{R}, \tag{A.26a}$$

where

$$\overline{R} = \frac{1}{\sqrt{2m(p_0+m)}} \left((p_0+m)\ I,\ -\boldsymbol{\sigma}\cdot\mathbf{p}\right). \tag{A.26b}$$

$v_r(p)$ is given by

$$v_r(p) = -i\gamma^2\, u_r^*(p). \tag{A.27}$$

Finally, we note that for free fields $[\not{\partial}_x = \gamma^\mu \partial_\mu]$

$$\begin{aligned} &\left[\Psi_\alpha(x),\ \overline{\Psi}_\beta(x')\right]_+ \\ =\ & i\,(i\not{\partial}_x + m)_{\alpha\,\beta}\ \Delta(x-x') \\ =\ & -i\ S_{\alpha\,\beta}\ (x-x'), \end{aligned} \tag{A.28a}$$

where

$$\begin{aligned} S\ (x-x') &= (-i\not{\partial}_x - m)\,\Delta(x-x'). \\ -iS\ (x-x')|_{x_0=x_0'} &= -i\left(-i\gamma^0\,\frac{\partial}{\partial\, x^0} - \vec{\gamma}\cdot\ \vec{\nabla} - m\right) \\ &\quad \times \Delta(x-x')|_{x_0\to x_0'} \\ &= \gamma^0\,\delta^3(\mathbf{x}-\mathbf{x}'). \end{aligned} \tag{A.28b}$$

$$\left[\Psi(x),\ \overline{\Psi}(x')\right]_{+_{x_0=x_0'}} = \gamma^0\,\delta^3(\mathbf{x}-\mathbf{x}'). \tag{A.28c}$$

A.3 Trace of γ-Matrices

We note that γ-matrices are traceless

$$\begin{aligned} Tr\ \gamma^\mu &= 0, \qquad \mu = 0,1,2,3 \\ Tr\ \gamma^5 &= 0. \end{aligned} \tag{A.29}$$

Now

$$Tr\ (\gamma^\mu\,\gamma^\nu) = Tr\ (\gamma^\nu\,\gamma^\mu). \tag{A.30}$$

Therefore, from Eq. (12), we have

$$Tr\ (\gamma^\mu\,\gamma^\nu) = g^{\mu\nu}\ Tr\ \left(\hat{1}\right) = 4\ g^{\mu\nu} \tag{A.31}$$

and

$$Tr\ (\gamma\cdot k\ \gamma\cdot p) = 4\ p\cdot k, \tag{A.32}$$

where

$$\gamma\cdot k = \gamma^\mu\ k_\mu. \tag{A.33}$$

Now

$$Tr\ (\gamma^\mu\,\gamma^\nu\,\gamma^\rho) = Tr\ (\gamma^\rho\,\gamma^\mu\,\gamma^\nu) \tag{A.34}$$

$$\gamma^\mu\,\gamma^\nu\,\gamma^\rho = i\varepsilon^{\mu\nu\rho\lambda}\,\gamma_\lambda\,\gamma_5 + g^{\nu\rho}\,\gamma^\mu - g^{\mu\rho}\,\gamma^\nu + g^{\mu\nu}\,\gamma^\rho. \tag{A.35}$$

Therefore,

$$Tr\ (\gamma^\mu\,\gamma^\nu\,\gamma^\rho) = 0 = Tr\ (\gamma^\rho\,\gamma^\mu\,\gamma^\nu). \tag{A.36}$$

From this, we generalize that trace of the product of odd numbers of γ-matrices is zero. Further, we have

$$\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma + \gamma^\sigma\,\gamma^\mu\,\gamma^\nu\,\gamma^\rho = 2\ g^{\rho\sigma}\,\gamma^\mu\,\gamma^\nu - 2\ g^{\nu\sigma}\,\gamma^\mu\,\gamma^\rho + 2\ g^{\mu\sigma}\,\gamma^\nu\,\gamma^\rho. \tag{A.37}$$

Therefore,

$$Tr\ (\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma) = 4\ \left[g^{\sigma\rho}\,g^{\mu\nu} - g^{\nu\sigma}\,g^{\mu\rho} + g^{\mu\sigma}\,g^{\nu\rho}\right]. \tag{A.38}$$

Noting that we can write

$$\gamma^5 = \frac{-i}{4!} \varepsilon_{\alpha\beta\sigma\rho} \gamma^\alpha \gamma^\beta \gamma^\nu \gamma^\rho, \tag{A.39}$$

we have

$$Tr \left(\gamma^5 \gamma^\mu\right) = 0 \tag{A.40}$$

$$Tr \left(\gamma^5 \gamma^\mu \gamma^\nu\right) = 0 \tag{A.41}$$

$$Tr \left(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho\right) = 0 \tag{A.42}$$

and

$$Tr \left(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma\right) = 4i\, \varepsilon_{\mu\nu\rho\sigma}, \tag{A.43}$$

with the definition $\varepsilon_{0123} = 1$ and $\varepsilon_{0ijk} = \varepsilon_{ijk}$ while $\varepsilon^{0123} = -1$. In calculations, we usually come across the matrix elements of the form

$$\begin{aligned} L_{\mu\nu} &= \sum_{\text{spin}} \left[\overline{u}\left(k_2\right) \gamma_\mu \left(1 + a\, \gamma_5\right) u\left(k_1\right)\right] \\ &\quad \times \left[\overline{u}\left(k_1\right) \gamma_\nu \left(1 + a\, \gamma_5\right) u\left(k_1\right)\right]^* && \text{(A.44)} \\ &= \sum_{\text{spin}} \left[\overline{u}\left(k_2\right) \gamma_\mu \left(1 + a\, \gamma_5\right) u\left(k_1\right)\right] && \text{(A.45)} \\ &\quad \times \left[u^\dagger\left(k_1\right) \left(1 + a\, \gamma_5\right) \gamma_\nu^\dagger \gamma^0 u\left(k_2\right)\right]. && \text{(A.46)} \end{aligned}$$

Now $[\gamma^0 \gamma_\nu^\dagger \gamma^0 = \gamma_\nu]$

$$\begin{aligned} \gamma^0 \left(1 + a\, \gamma_5\right) \gamma^\dagger \gamma^0 &= \left(1 - a\, \gamma_5\right) \gamma_\nu \\ &= \gamma_\nu \left(1 + a\, \gamma_5\right). && \text{(A.47)} \end{aligned}$$

Therefore,

$$\begin{aligned} L_{\mu\nu} &= \sum_{\text{spin}} \overline{u}\left(k_2\right) \gamma_\mu \left(1 + a\, \gamma_5\right) u\left(k_1\right) \overline{u}\left(k_1\right) \gamma_\nu \left(1 + a\, \gamma_5\right) u\left(k_2\right) \\ &= \sum_{\text{spin}} \overline{u}_\alpha\left(k_2\right) \left[\gamma_\mu \left(1 + a\, \gamma_5\right)\right]_{\alpha\alpha'} u_{\alpha'}\left(k_1\right) \end{aligned}$$

$$\times\ \overline{u}_{\beta'}(k_1)\ [\gamma_\nu(1+a\ \gamma_5)]_{\beta'\delta}\ u_\delta(k_2)$$
$$= \sum_{\text{spin}}\left(\frac{\gamma\cdot k_2+m_2}{2m_2}\right)_{\delta\alpha}[\gamma_\mu(1+a\gamma_5)]_{\alpha\alpha'}$$
$$\times\left(\frac{\gamma\cdot k_1+m_1}{2m_1}\right)_{\alpha'\beta'}[\gamma_\nu(1+a\gamma_5)]_{\beta'\delta}$$
$$= \frac{1}{4\ m_1 m_2}$$
$$\times \mathrm{Tr}\left[(\not k_2+m_2)\gamma_\mu(1+a\gamma_5)(\not k_1+m_1)\gamma_\nu(1+a\gamma_5)\right]. \tag{A.48}$$

Here

$$\not k_2 = \gamma\cdot k_2, \qquad \not k_1 = \gamma\cdot k_1. \tag{A.49}$$

Using the formulae for the traces of $\gamma-$ matrices given previously, we get

$$L_{\mu\nu} = \frac{4}{4\ m_1 m_2}\left\{\begin{array}{c}(1+a^2)\ (k_{2\mu}\ k_{1\nu}+k_{2\nu}\ k_{1\mu}-k_1\cdot k_2\ g_{\mu\nu}) \\ +m_1 m_2\ (1-a^2)\ g_{\mu\nu}+2ia\varepsilon_{\mu\nu\rho\sigma}\ k_1^\rho\ k_2^\sigma\end{array}\right\}. \tag{A.50}$$

Similarly for

$$L_{\mu\nu} = \sum_{\text{spin}}\left[\overline{v}(k_2)\gamma_\mu(1+a\gamma_5)u(k_1)\right]\left[\overline{v}(k_2)\gamma_\nu(1+a\gamma_5)\ u(k_1)\right]^*, \tag{A.51}$$

we get

$$L_{\mu\nu} = \frac{4}{4\ m_1 m_2}\left\{\begin{array}{c}(1+a^2)\ (k_{2\mu}\ k_{1\nu}+k_{2\nu}\ k_{1\mu}-k_1\cdot k_2\ g_{\mu\nu}) \\ -m_1 m_2\ (1-a^2)\ g_{\mu\nu}+2ia\varepsilon_{\mu\nu\rho\sigma}\ k_1^\rho\ k_2^\sigma\end{array}\right\}. \tag{A.52}$$

For

$$L_{\mu\nu} = \sum\left[\overline{u}(k_2)\ \gamma_\mu(1+a\ \gamma_5)\ v(k_1)\right]$$
$$\times\left[\overline{u}(k_2)\ \gamma_\nu(1+a\ \gamma_5)\ v(k_1)\right]^*, \tag{A.53}$$

we get the same value as given in Eq. (52).

A.4 Spin 1 Field

Electromagnetic field (photon) with mass $m = 0$.

In the absence of interactions, the electromagnetic field $A_\mu(x)$ satisfies the field equation

$$\Box^2 A_\mu(x) = 0. \tag{A.54}$$

There is an additional condition

$$\partial^\mu A_\mu(x) = 0. \tag{A.55}$$

The Fourier decomposition of $A_\mu(x)$:

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{\sqrt{2k_0}} \sum_\lambda \varepsilon_\mu^\lambda(k) \left[a_\lambda(k)\ e^{-ik.x} + a_\lambda^\dagger(k)\ e^{ik.x}\right], \tag{A.56}$$

where $\varepsilon_\mu^\lambda(x)$, are four vectors called polarization vectors. $a_\lambda(k)$ and $a_\lambda^\dagger(k)$ are interpreted respectively as the annihilation and creation operator of the photon with momentum k and polarization $\varepsilon_\mu^\lambda(k)$. They satisfy the following commutation relations

$$\left[a_\lambda(k), a_{\lambda'}^\dagger(k')\right] = \delta_{\lambda\lambda'}\, \delta^3(\mathbf{k} - \mathbf{k}'), \tag{A.57}$$

$$\left[a_\lambda(k), a_{\lambda'}(k')\right] = \left[a_\lambda^\dagger(k), a_{\lambda'}^\dagger(k')\right] = 0. \tag{A.58}$$

The polarization vector $\varepsilon_\mu^\lambda(k)$ satisfies the following relations

$$\varepsilon^\lambda(k) \cdot \varepsilon^{\lambda'}(k) = \delta_{\lambda\lambda'} \tag{A.59}$$

$$k \cdot \varepsilon^\lambda = 0. \tag{A.60}$$

For transverse photon polarization, the four-vector $\varepsilon_\mu^\lambda(k)$ can be chosen as

$$\varepsilon_\mu^\lambda(k) \equiv \left(0,\ \boldsymbol{\varepsilon}^\lambda(k)\right), \tag{A.61}$$

so that we have

$$k \cdot \varepsilon^{\lambda} = \mathbf{k} \cdot \boldsymbol{\varepsilon}^{\lambda} = 0 \tag{A.62}$$

and

$$\begin{aligned}\sum_{\lambda=1}^{2} \varepsilon_{\mu}^{\lambda}(k)\ \varepsilon_{\nu}^{\lambda}(k) &= -g_{\mu\nu} + \frac{k_{\mu}k_{\nu}}{k^2 - (k\cdot\eta)^2} \\ &\quad -\frac{(k\cdot\eta)\, k_{\mu}\eta_{\nu} + k_{\nu}\eta_{\mu}}{k^2 - (k\cdot\eta)^2} + \frac{k^2\eta_{\mu}\eta_{\nu}}{k^2 - (k\cdot\eta)^2}\end{aligned}$$

where $\eta = (1,0,0,0)$.

A.5 Massive Spin 1 Particle

A spin 1 particle of mass m is described by a vector field $\phi_{\mu}(x)$, which in the absence of interactions satisfies the equation

$$\left(m^2 + \Box^2\right)\ \phi_{\mu}(x) = 0 \tag{A.63a}$$

with the subsidiary condition

$$\partial^{\mu}\ \phi_{\mu}(x) = 0. \tag{A.63b}$$

The Fourier decomposition of $\phi_{\mu}(x)$ is given by

$$\begin{aligned}\phi_{\mu}(x) &= \frac{1}{(2\pi)^{3/2}}\int \frac{d^3\ k}{\sqrt{2k_0}} \\ &\quad \times \sum_{\lambda=1}^{3} \varepsilon_{\mu}^{\lambda}(k)\ \left[a_{\lambda}(k)\ e^{-ik.x} + b_{\lambda}^{\dagger}(k)\ e^{ik.x}\right]\end{aligned} \tag{A.64a}$$

$$\begin{aligned}\phi_{\mu}^{\dagger}(x) &= \frac{1}{(2\pi)^{3/2}}\int \frac{d^3\ k}{\sqrt{2k_0}} \\ &\quad \times \sum_{\lambda=1}^{3} \varepsilon_{\mu}^{\lambda^*}(k)\ \left[a_{\lambda}^{\dagger}(k)\ e^{ik.x} + b_{\lambda}(k)\ e^{-ik.x}\right].\end{aligned} \tag{A.64b}$$

$a_\lambda(k)$ and $b_\lambda(k)$ satisfy the following commutation relations:

$$\left[a_\lambda(k),\, a^\dagger_{\lambda'}(k')\right] = \delta_{\lambda\,\lambda'}\,\delta^3\,(\mathbf{k}-\mathbf{k}'), \tag{A.65a}$$

$$\left[b_\lambda(k),\, b^\dagger_{\lambda'}(k')\right] = \delta_{\lambda\,\lambda'}\,\delta^3\,(\mathbf{k}-\mathbf{k}'). \tag{A.65b}$$

$a^\dagger_\lambda(k)$ $(a_\lambda(k))$ are creation (annihilation) operators for the particle with polarization λ and momentum $\mathbf{k}$. $b^\dagger_\lambda(k)$ $(b_\lambda(k))$ are creation (annihilation) operators for the antiparticle with polarization λ and momentum $\mathbf{k}$.

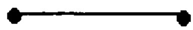
The polarization vector ε^λ_μ satisfies the relation

$$\varepsilon^\lambda \cdot \varepsilon^{\lambda'} = \delta_{\lambda\,\lambda'}, \qquad k\cdot\varepsilon^\lambda = 0 \tag{A.66a}$$

$$\sum_{\lambda=1}^{3} \varepsilon^\lambda_\mu\,\varepsilon^\lambda_\nu = -g_{\mu\,\nu} + \frac{k_\mu\,k_\nu}{m^2}. \tag{A.66b}$$

A.6 Feynman Rules for S-Matrix in Momentum Space

For each internal photon line:  $\frac{-i}{(2\pi)^4}\,\frac{g_{\mu\nu}}{k^2+i\varepsilon}$

For each internal fermion line:  $\frac{i}{(2\pi)^4}\,\frac{m+\gamma\cdot p}{p^2-m^2+i\varepsilon}$

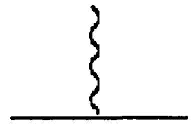
For each internal pion line:  $\frac{i}{(2\pi)^4}\,\frac{1}{k^2-m_\pi^2+i\varepsilon}$

For each external fermion line entering the graph, depending upon whether the line is in the initial or final state  $\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{p_0}}u_r(p)$ or $\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{p_0}}v_r(p)$

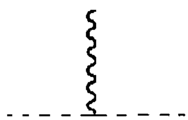
For each external fermion line leaving the graph, depending upon whether the line is in the final or initial state  $\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{p_0}}\overline{u}_r(p)$ or $\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{p_0}}\overline{v}_r(p)$

For each external photon line:  $\frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\, k_0}}\, \varepsilon_\mu^\lambda (k)$

For each external spin 0 meson line:  $\frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\, k_0}}$

For photon-fermion vertex:  $-i\, e\, \gamma^\mu$, $H_I = \ e\, \overline{\Psi}\, \gamma^\mu\, \Psi A_\mu$

For pion-fermion vertex:  $g\, \gamma^5$, $H_I = i\, g\, \overline{\Psi}\, \gamma^5\, \Psi \phi$

For photon-meson vertex:  $-ie\ \ (q + q')_\mu$

A factor $(2\pi)^4\, \delta^4 (p - p' \pm k)$ at each vertex

A factor (-1) for each closed fermion loop

For a massive vector boson of mass m_W  $\frac{i}{(2\pi)^4} \frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}}{k^2 - m_W^2}$

This gives the propagator of the vector boson in unitary gauge.

Further one has $\int d^4\, l$ for each loop integral where the four momentum l is not fixed by energy-mometum conservation. Multiply by $\delta_p = 1\ \ (-1)$ and $-1\ \ (1)$ respectively for the direct and exchange term of fermion (antifermion)-fermion (antifermion) scattering.

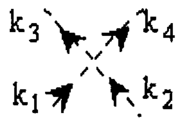
Feynman rules for a hermitian self-interacting spin 0 boson with the Lagrangian

$$L = \frac{1}{2} \left[(\partial_\mu\, \phi)^2 - \mu^2\, \phi^2 \right] - \frac{\lambda}{4!}\, \phi^4$$

are as follows

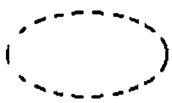
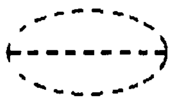
For each external line $\frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\ k_0}}$

For each internal line $\frac{i}{(2\pi)^4} \frac{1}{\sqrt{2\ k_0}} \frac{1}{k^2 - \mu^2 + i\varepsilon}$

For vertex 

$-i\lambda$

A factor
$(2\pi)^4 \delta^4 (k_1 + k_2 - k_3 - k_4)$
at each vertex
For each loop integral
$\int d^4 l$ statistical factors

$\frac{1}{2!}$  $\frac{1}{3!}$  etc.

A.7 An Application of Feynman Rules

As a simple application of Feynman rules, we consider the process

$$
\begin{aligned}
e^- + e^+ &\rightarrow \mu^- + \mu^+ \\
p_1 + p_2 &= p_1' + p_2'
\end{aligned}
$$

$$
\begin{aligned}
S &= \int d^4 k \, \frac{m_\mu}{\sqrt{p_{10}'\, p_{20}'}} \frac{1}{(2\pi)^3}\, \overline{u}(p_1')\,(-i\ e)\ \gamma^\lambda\, v(p_2') \\
&\times \frac{-i}{(2\pi)^4} \frac{g_{\lambda\nu}}{k^2}\, \overline{v}(p_2)\,(-i\ e\ \gamma^\nu)\, u(p_1)\, \frac{1}{(2\pi)^3} \frac{m_e}{\sqrt{p_{10}\, p_{20}}} \\
&\times (2\pi)^4 \delta^4 (p_1 + p_2 - k)\ (2\pi)^4 \delta^4\ (k - p_1' - p_2') \,. \quad \text{(A.67)}
\end{aligned}
$$

Therefore, using the relation $S = 1 + i\ (2\pi)^4 \delta^4 (P_i - P_f)\, T$:

$$
T = \frac{1}{(2\pi)^6} \frac{m_\mu\ m_e g_{\lambda\nu}}{\sqrt{p_{10}\ p_{20}\ \ p_{10}'\ p_{20}'}} \frac{e^2}{k^2} \left[\overline{u}(p_1')\ \ \gamma^\lambda\ v(p_2')\right]\ \left[\overline{v}(p_2)\ \ \gamma^\nu\ u(p_1)\right], \quad \text{(A.68)}
$$

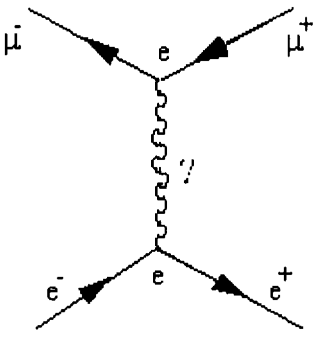


Figure 1 One photon exchange Feynman diagram for the process $e^- \, e^+ \to \mu^- \, \mu^+$.

and we put

$$F = \frac{e^2}{k^2} \left[\overline{u} \left(p_1' \right) \ \gamma^\lambda \ v \left(p_2' \right) \right] \ \left[\overline{v} \left(p_2 \right) \ \gamma_\lambda \ u \left(p_1 \right) \right]. \tag{A.69}$$

Therefore,

$$|F|^2 = \frac{e^4}{k^4} \overline{\sum_{\text{spin}}} \sum_{\text{spin}} \left| \overline{u} \left(p_1' \right) \ \gamma^\lambda \ v \left(p_2' \right) \right|^2 \ \left| \overline{v} \left(p_2 \right) \ \gamma_\lambda \ u \left(p_1 \right) \right|^2. \tag{A.70}$$

Using Eqs. (51)-(53) ($a = 0$), we get

$$|F|^2 = \frac{e^4}{k^4} \frac{1}{4} \frac{1}{m_e^2} \frac{1}{m_\mu^2} \left[\begin{array}{c} p_1' \cdot p_2 \ p_2' \cdot p_1 + p_2' \cdot p_2 \ p_1' \cdot p_1 + m_e^2 \ p_1' \cdot p_2' \\ + m_\mu^2 \ p_2 \cdot p_1 \ + 2 m_e^2 m_\mu^2 \end{array} \right]. \tag{A.71}$$

Now

$$s = \left(p_1 + p_2 \right)^2 = \left(p_1' + p_2' \right)^2 = E_{cm}^2 = k^2 \tag{A.72}$$

and in the center of mass system

$$\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}; \qquad \mathbf{p}_1' = -\mathbf{p}_2' = \mathbf{p}' \tag{A.73}$$

with

$$|\mathbf{p}| = \frac{\sqrt{s - 4\ m_e^2}}{2}, \qquad |\mathbf{p}'| = \frac{\sqrt{s - 4\ m_\mu^2}}{2}. \tag{A.74}$$

Therefore,

$$\begin{aligned}
& p_1' \cdot p_2\ p_2' \cdot p_1 + p_2' \cdot p_2\ p_1' \cdot p_1 \\
&= 2\left[|\mathbf{p}|^2\ |\mathbf{p}'|^2\ \cos^2\theta + \frac{s^2}{16}\right] \\
&= \frac{1}{8}\left[\left(s - 4\ m_e^2\right)\left(s - 4\ m_\mu^2\right)\cos^2\theta + s^2\right] \\
&= \frac{1}{8}\left[s^2\ \left(1 + \cos^2\theta\right) + 16\ m_e^2 m_\mu^2\ \cos^2\theta \right. \\
&\quad \left. -4\ \left(m_e^2 + m_\mu^2\right) s\ \cos^2\theta\right]
\end{aligned} \tag{A.75}$$

and

$$\begin{aligned}
|F|^2 &= \frac{e^4}{s^2}\frac{1}{16\ m_e^2 m_\mu^2}\left[s^2\ \left(1 + \cos^2\theta\right)\right. \\
&\quad \left. +4s\ \left(m_e^2 + m_\mu^2\right)\left(1 - \cos^2\theta\right) + 16\ m_e^2 m_\mu^2\ \cos^2\theta\right].
\end{aligned} \tag{A.76}$$

Hence from Eq. (2.39), we get

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \alpha^2\frac{1}{s}\frac{|\mathbf{p}'|}{|\mathbf{p}|}\frac{1}{4}\left\{\left(1 + \cos^2\theta\right) + \frac{4\ (m_e^2 + m_\mu^2)}{s}\ \left(1 - \cos^2\theta\right)\right. \\
&\quad \left. + \frac{16\ m_e^2 m_\mu^2}{s^2}\ \cos^2\theta\right\} \\
&= \alpha^2\ \frac{\beta_\mu}{\beta_e}\frac{1}{s}\left[\left(1 + \cos^2\theta\right) + \left(2 - \beta_e^2 - \beta_\mu^2\right)\sin^2\theta\right. \\
&\quad \left. + \left(1 - \beta_e^2\right)\left(1 - \beta_\mu^2\right)\cos^2\theta\right],
\end{aligned} \tag{A.77}$$

where

$$\begin{aligned}\beta_\mu &= \frac{2\ |\mathbf{p}'|}{\sqrt{s}} = \frac{\sqrt{s - 4\ m_\mu^2}}{\sqrt{s}}, \\ \beta_e &= \frac{2\ |\mathbf{p}|}{\sqrt{s}} = \frac{\sqrt{s - 4\ m_e^2}}{\sqrt{s}}. \end{aligned} \tag{A.78}$$

Finally, we note that

$$\begin{aligned}\sigma &= \frac{4\pi\alpha^2}{3}\frac{1}{s}\frac{|\mathbf{p}'|}{|\mathbf{p}|}\left[1 + \frac{2\left(m_e^2 + m_\mu^2\right)}{s} + \frac{4\ m_e^2\ m_\mu^2}{s^2}\right] \\ &= \frac{4\pi\alpha^2}{3}\frac{\beta_\mu}{\beta_e}\left\{1 + \frac{2 - \beta_e^2 - \beta_\mu^2}{2} + \frac{\left(1-\beta_e^2\right)\left(1-\beta_\mu^2\right)}{4}\right\}. \end{aligned} \tag{A.79}$$

In the relativistic limit, $\beta_e \approx 1,\ \ \beta_\mu \approx 1\ \left(s \gg m_e^2, m_\mu^2\right)$, we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1 + \cos^2\theta\right) \tag{A.80a}$$

$$\sigma = \frac{4\pi\alpha^2}{3}\frac{1}{s}. \tag{A.80b}$$

A.8 Charge Conjugation

Dirac equation in the presence of electromagnetic field is given by

$$\left[i\gamma^\mu\ \left(\partial_\mu + i\ e\ A_\mu\right) - m\right]\ \Psi(x) = 0. \tag{A.81}$$

For the adjoint field $\overline{\Psi}$, Eq. (1) can be written:

$$\left[-i\left(\gamma^\mu\right)^T\ \left(\partial_\mu - i\ e\ A_\mu\right) - m\right]\ \overline{\Psi}^T(x) = 0. \tag{A.82}$$

Under the charge conjugation

$$\Psi(x) \rightarrow \Psi^c(x) = U_c\ \Psi(x)\ U_c^{-1} \tag{A.83}$$

$$A_\mu(x) \rightarrow A_\mu^c(x) = U_c\ A_\mu\ (x)\ U_c^{-1}. \tag{A.84}$$

If the Dirac equation is invariant under charge conjugation then:

$$\left[i\gamma^{\mu}\ \left(\partial_{\mu}+i\ e\ A_{\mu}^{c}\right)-m\right]\ \Psi^{c}(x)=0. \tag{A.85}$$

Now we can write Eq. (83) as

$$C\left[-i\left(\gamma^{\mu}\right)^{T}\ \left(\partial_{\mu}-i\ e\ A_{\mu}\right)-m\right]\ C^{-1}C\overline{\Psi}^{T}(x)=0, \tag{A.86}$$

where C is a unitary matrix, called the charge conjugation matrix. Equation (87) is identical to Eq. (86), provided that

$$\begin{aligned}\gamma^{\mu} &= -C\left(\gamma^{\mu}\right)^{T}\ C^{-1} && \text{(A.87)}\\ A_{\mu}^{c} &= -A_{\mu} && \text{(A.88)}\\ \Psi^{c}(x) &= C\overline{\Psi}^{T}(x). && \text{(A.89)}\end{aligned}$$

Also one can write

$$\begin{aligned}\Psi^{c}(x) &= C\overline{\Psi}^{T}(x)=-\gamma^{0}C\Psi^{*} && \text{(A.90)}\\ \overline{\Psi}^{c}(x) &= -\overline{\Psi}^{T}(x)\,C^{-1}. && \text{(A.91)}\end{aligned}$$

In Pauli-representation fo γ-matrices

$$\left(\gamma^{\mu}\right)^{T}=\begin{cases}\gamma^{\mu} & \text{for}\quad \mu=0,2\\ -\gamma^{\mu} & \text{for}\quad \mu=1,3.\end{cases} \tag{A.92}$$

Therefore we have from Eq. (88):

$$\begin{aligned}C &= -i\gamma^{2}\gamma^{0} && \text{(A.93)}\\ C^{\dagger} &= -C=C^{T} && \\ C^{2} &= -1. && \text{(A.94)}\end{aligned}$$

Hence in this representation

$$\Psi^{c}(x)=-i\gamma^{2}\Psi^{*}. \tag{A.95}$$

On the other hand, in the Weyl representation of γ-matrices

$$\begin{aligned}\gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ \gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \end{aligned} \tag{A.96}$$

In this representation, one can write

$$\begin{aligned}\Psi &= \left(\frac{1-\gamma^5}{2}\right)\Psi + \left(\frac{1+\gamma^5}{2}\right)\Psi \\ &= \Psi_L + \Psi_R = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} \\ &= \begin{pmatrix} \xi \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \end{aligned} \tag{A.97}$$

where

$$\Psi_L = \left(\frac{1-\gamma^5}{2}\right)\Psi \equiv \xi, \quad \Psi_R = \left(\frac{1+\gamma^5}{2}\right)\Psi \equiv \eta \tag{A.98}$$

are two component left-handed and right-handed spinors. In this representation the relations (93) – (96) are again satisfied. Hence from Eq. (96), we get

$$\begin{aligned}\xi^c &= -i\sigma^2\eta^* \\ \eta^c &= i\sigma^2\xi^*. \end{aligned} \tag{A.99}$$

Sometime it is convenient to write a right-handed field in terms of a left-handed antiparticle field (cf. Eq. (100)):

$$\eta = i\sigma^2\xi^{c^*} \tag{A.100}$$

so that Eq. (98) becomes

$$\Psi = \begin{pmatrix} \xi \\ i\sigma^2\xi^{c^*} \end{pmatrix}. \tag{A.101}$$

The Majorana spinor Ψ_M is defined as

$$\begin{aligned}\Psi_M^C &= \Psi_M = C\overline{\Psi}_M^T \\ \Psi_{M\alpha} &= C_{\alpha\beta}\overline{\Psi}_M^\beta.\end{aligned}$$

Hence in the Weyl representation

$$\Psi_M = \begin{pmatrix} \xi \\ i\sigma^2\xi^* \end{pmatrix}. \tag{A.102}$$

We also note that in the Weyl representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{A.103}$$

where

$$\begin{aligned}\sigma^\mu &= \left(1, \sigma^i\right) = (1, \vec{\sigma}) \\ \bar{\sigma}^\mu &= \left(1, -\sigma^i\right) = (1, -\vec{\sigma}).\end{aligned} \tag{A.104}$$

Now the Dirac Lagrangian

$$L = \bar{\Psi}\left(i\gamma^\mu\partial_\mu - m_D\right)\Psi + \frac{m_M}{2}\left(\Psi^T C^{-1}\Psi - \bar{\Psi}C\bar{\Psi}^T\right) \tag{A.105}$$

(where the second term in Eq. (106) is the Majorana mass term and violates lepton number conservation) can be written in terms of two component chiral fields using Eqs. (99), (100), (104) and (106):

$$\begin{aligned}L &= i\left[\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi + \xi^{c\dagger}\bar{\sigma}^\mu\partial_\mu\xi^c\right] - m_D\left[\xi^{*^T} i\sigma^2\xi^{c^*} + \xi^{c^T}\left(-i\sigma^2\right)\xi\right] \\ &\quad + \frac{m_M}{2}\left[\xi^T i\sigma^2\xi - \xi^{c^{*^T}} i\sigma^2\xi^{c^*} + \xi^{c^T} i\sigma^2\xi^c - \xi^{*^T} i\sigma^2\xi^*\right],\end{aligned} \tag{A.106}$$

where we have used

$$\sigma^2\,\sigma^\mu\,\sigma^2 = \left(\bar{\sigma}^\mu\right)^T$$

$$\begin{aligned}\left(\sigma^2\xi^{c^*}\right)^\dagger \sigma^\mu\partial_\mu\left(\sigma^2\xi^{c^*}\right) &= \xi^{c^T}\left(\bar{\sigma}^\mu\right)^T\partial_\mu\xi^{c^*} \\ &= -\partial_\mu\xi^{c^\dagger}\bar{\sigma}^\mu\xi^c \quad \text{(fermion fields anticommute)} \\ &= \xi^{c^\dagger}\bar{\sigma}^\mu\partial_\mu\xi^c \quad \text{(partial integration).}\end{aligned}$$

Equation (107) can be put in the compact form

$$L = i\xi_i^\dagger\bar{\sigma}^\mu\partial_\mu\xi_i - \frac{1}{2}\left(m_{ij}\xi_i^T\left(-i\sigma^2\right)\xi_j + h.c.\right) \tag{A.107}$$

where i, j =1, 2 and m_{ij} is the symmetric mass matrix

$$\begin{pmatrix} m_M & m_D \\ m_D & m_M \end{pmatrix}$$

and $\xi_1 = \xi = \Psi_L$ and $\xi_2 = \xi^c = -i\sigma^2\eta^* = -i\sigma^2\Psi_R^*$.

A.9 Bibliography

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Appendix B
RENORMALIZATION GROUP AND RUNNING COUPLING CONSTANT

B.1 Feynman Rules for Quantum Chromodynamics

For canonical covariant quantization, the QCD Lagrangian given in Eq. (7.32) is written as [repeated indices imply summation]

$$L = -\frac{1}{4} G_A^{\mu\nu} G_{A\mu\nu} + \bar{q}^a i\, \gamma^\mu \left(\partial_\mu - i g_s T_A\right)_a^b q_b - \frac{1}{2\xi} \left(\partial^\mu G_{A\mu}\right)^2 + \text{Ghosts} \tag{B.1}$$

where

$$G_A^{\mu\nu} = \partial^\mu G_A^\nu - \partial^\nu G_A^\mu + g_s f_{ABC} G_B^\mu G_C^\nu$$

$$[T_A, T_B] = i f_{ABC} T_C, \quad T_A = \frac{1}{2} \lambda_A$$

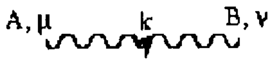
$$Tr\ (T_A, T_B) = \frac{1}{2} \delta_{AB}, \text{ for the fundamental representation.}$$

$$\begin{aligned} \sum f_{ACD} f_{BCD} &= C_2(G)\, \delta_{AB} \\ &= N\, \delta_{AB}, \quad \text{for } SU(N) \text{ gauge group} \end{aligned} \tag{B.2}$$

$$(T_A)_c^a\ (T_A)_b^c = C_F\ \delta_b^a = \frac{N^2-1}{2N}\ \delta_b^a, \quad \text{for } SU(N).$$

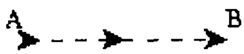
In the Lagrangian (1), $-\frac{1}{2\xi}\left(\partial^\mu G_{A\mu}\right)^2$ is the gauge fixing term, ξ being the fixing parameter. The supplementary nonphysical fields, called ghosts are needed for covariant quantization in order to cancel the probabilities of observing scalar (or time-like) and longitudinal gluons.

Quantizing in a renormalizable gauge leads to the following Feynman rules:

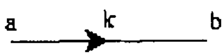


$$\delta_{AB}\frac{-i}{k^2}\left[g_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2}\right],$$

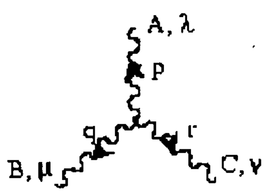
gluon propagator



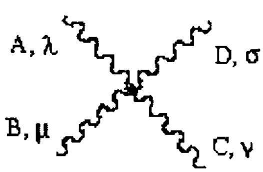
$-i\delta_{AB}/k^2$, ghost propagator



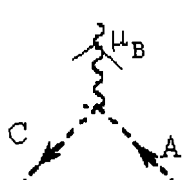
$i\delta^a_b \, \frac{1}{\not{k} - m}$, quark propagator



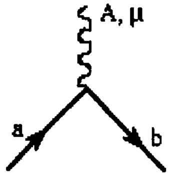
$$-g_s f_{ABC}\left[\begin{array}{c}(p-q)_\nu \; g_{\lambda\mu} + (q-r)_\lambda \, g_{\mu\nu} \\ +(r-p)_\mu \; g_{\nu\lambda}\end{array}\right]$$



$$\begin{array}{l}-ig_s^2 \; f_{ABE} \; f_{CDE}\,(g_{\lambda\nu}\; g_{\mu\sigma} - g_{\lambda\sigma}\; g_{\mu\nu}) \\ -ig_s^2 \; f_{ACE} \; f_{BDE}\,(g_{\lambda\mu}\; g_{\nu\sigma} - g_{\lambda\sigma}\; g_{\mu\nu}) \\ -ig_s^2 \; f_{ADE} \; f_{CBE}\,(g_{\lambda\nu}\; g_{\mu\sigma} - g_{\lambda\mu}\; g_{\sigma\nu})\end{array}$$



$$g_s f_{ABC} \; p_\mu$$



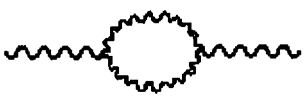
$ig_s \gamma_\mu \, (T_A)_{ab}$

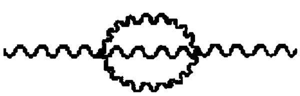
The other factors are

(i) $\int \frac{d^4 l}{(2\pi)^4}$ for each loop integral

(ii) (-1) for closed fermion (ghost) loop

(iii) Statistical factors like

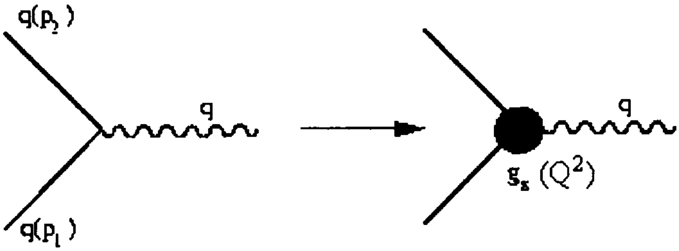
$\frac{1}{2}$  ,

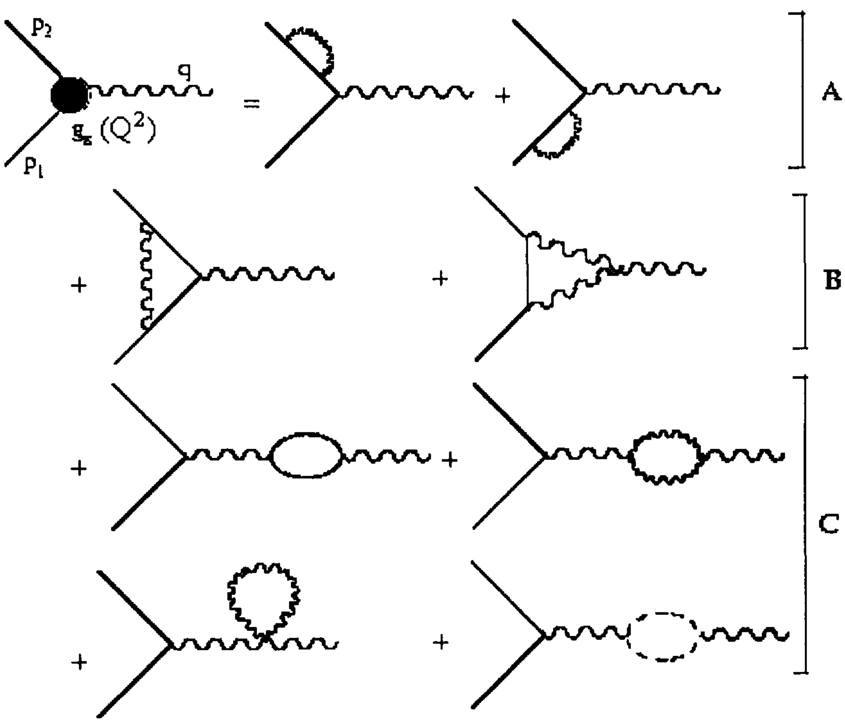
$\frac{1}{3!}$  .

B.2 Renormalization Group, Effective Coupling Constant and Asymptotic Freedom

We now show that the self-coupling of gluons envisaged in the first term of the Lagrangian (1) has the consequences that QCD has a remarkable property of being asymptotically free i.e. the quark - quark force becomes weak at large momentum transfer or short distances, such as probed in deep-inelastic collision [cf. Chap. 14]. In other words, the coupling constant α_s depends on the momentum transfer in such a way that $\alpha_s(Q^2) \to 0$ as $Q^2 \to \infty$.

Consider the radiative corrections to quark - quark - gluon (qqG) vertex, where at one loop level these corrections are shown in fig. 1, $[Q^2 = -q^2]$.





One loop corrections to quark - quark - gluon (qqG) vertex.

The one loop corrections to qqG vertex shown above are infinite. One must define a high l^2- cut off (l being loop momentum) λ^2, so that the loop integrals converge. We have then

$$\Gamma_{A\mu} = -i\, T_A\, \gamma_\mu\, \Gamma_s\, \left(Q^2, \lambda, g_s\right) \tag{B.3a}$$

where

$$\Gamma_s\, \left(Q^2, \lambda, g_s\right) = g_s \left[1 + g_s^2\, \left(a_0 + b_0 \ln \frac{\lambda^2}{Q^2}\right) \cdots \right], \qquad \lambda^2 \gg Q^2 \tag{B.3b}$$

where $\cdots$denotes the corrections from higher order loops. Note here that the cut-off dependent logarithmic contributions from diagrams A [involving quark self-energy diagram] and the first of diagrams B [involving the quark gluon vertex function] cancel due to gauge invariance as is also the case in quantum electrodynamics. Since the theory is renormalizable, we must be able to write it as

$$\Gamma_s\, \left(Q^2/\lambda^2, g_s\right) = Z_s^{1/2}\, \left(\lambda^2/\mu^2, g_s\right) \Gamma_s^R\, \left(Q^2/\mu^2, g_s\right) \tag{B.4}$$

where μ is called the renormalization scale and Z_s is a multiplicative renormalization constant. One may define the renormalization scale through the relation

$$\Gamma_s^R\, \left(Q^2/\mu^2, g_s\right)\Big|_{Q^2=\mu^2} = g_s. \tag{B.5}$$

Then neglecting a_0 in Eq. (3b)

$$Z_s^{1/2}\, \left(\lambda^2/\mu^2, g_s\right) = \left[1 + g_s^2\, b_0 \ln \frac{\lambda^2}{\mu^2}\, \cdots \right] \tag{B.6a}$$

and [cf. Eq. (3b)]

$$\Gamma_s\, \left(Q^2/\lambda^2, g_s\right) = g_s Z_s^{1/2}\, \left(\lambda^2/Q^2, g_s\right). \tag{B.6b}$$

Thus

$$\begin{aligned} g_s\left(Q^2\right) &\equiv \Gamma_s^R\, \left(Q^2/\mu^2, g_s\right) \\ &= g_s\, Z_s^{-1/2}\, \left(\lambda^2/\mu^2, g_s\right) Z_s^{1/2}\, \left(\lambda^2/Q^2, g_s\right). \end{aligned} \tag{B.7}$$

This relation expresses the basic renormalization group property.

It is more conveniently expressed through an equivalent differential equation which follows from the μ-independence of Γ_s so that

$$\frac{d\Gamma_s}{d\mu} = 0, \tag{B.8a}$$

or, using Eq. (4),

$$\frac{1}{\Gamma_s^R(\mu)} \frac{d\Gamma_s^R(\mu)}{d\mu} = -\frac{1}{Z_s^{1/2}} \frac{dZ_s^{1/2}}{d\mu}. \tag{B.8b}$$

This can be rewritten as $[\Gamma_s^R(\mu) = g_s(\mu)]$

$$\frac{dg_s(\mu)}{d\ln\mu} = -g_s \frac{1}{Z_s^{1/2}} \frac{dZ_s^{1/2}}{d\ln\mu} = \beta(g_s) \tag{B.9a}$$

so that

$$\beta(g_s) = -g_s \frac{1}{Z_s^{1/2}} \frac{dZ_s^{1/2}}{d\ln\mu} \tag{B.9b}$$

where $Z_s^{1/2}$ is given in Eq. (6a). Equations (9) are known as the renormalization group equations for the effective coupling constant $g_s(\mu)$. Writing

$$Z_s^{1/2} = 1 + \sum_{k=1}^{\infty} \left(\ln \frac{\lambda^2}{\mu^2} \right)^k Z_{s,k}^{1/2}(g_s), \tag{B.10a}$$

Eq. (9b) gives

$$\begin{aligned} \beta(g_s) &= -2g_s^3 \frac{dZ_{s,1}^{1/2}}{dg_s^2} \\ &= -2g_s^3 \left[b_0 + b_1\, g_s^2 + \cdots \right], \end{aligned} \tag{B.10b}$$

where we have used Eq. (6a).

To integrate Eq. (9a), it is convenient to write it, on using Eq. (10b), as [putting $\mu^2 = Q^2$]

$$\frac{d\ (1/g_s^2)}{d\ \ln Q^2} = 2 \left[b_0 + b_1\, g_s^2 + \cdots \right]$$

or

$$\begin{aligned} d\ \ln Q^2 &= d\ \left(1/g_s^2\right) \frac{1}{2b_0} \left[1 + \frac{b_1}{b_0}\ \left(1/g_s^2\right)^{-1} + \cdots\right]^{-1} \\ &= d\ \left(1/g_s^2\right) \frac{1}{2b_0} \left[1 - \frac{b_1}{b_0}\ \left(1/g_s^2\right)^{-1} + \cdots\right]. \end{aligned} \quad \text{(B.11)}$$

If we keep only the lowest order term, we have

$$\frac{d\left(\alpha_s^{-1}\right)}{d\ln Q^2} = b \quad \text{(B.12a)}$$

where $\alpha_s = \frac{g_s^2}{4\pi}$, $b = 8\pi\ b_0$. Integration of Eq. (12a) gives

$$\alpha_s^{-1}\left(Q^2\right) = \alpha_s^{-1}\left(\mu^2\right) + b\ln\frac{Q^2}{\mu^2}. \quad \text{(B.12b)}$$

Note that what renormalization group does is to relate the coupling constant at two different scales. We may also write Eq. (12b) as

$$\alpha_s^{-1}\left(Q^2\right) = b\ln\frac{Q^2}{\Lambda_{QCD}^2} \quad \text{(B.12c)}$$

where $\left[\alpha_s^{-1}\left(\mu^2\right) \equiv \alpha_\mu^{-1}\right]$

$$\frac{1}{b}\alpha_\mu^{-1} - \ln\mu^2 \ = -\ln\Lambda_{QCD}^2$$

or

$$\frac{\Lambda_{QCD}^2}{\mu^2} = \exp\left(-\frac{1}{b\alpha_\mu}\right). \quad \text{(B.12d)}$$

$\Lambda_{\rm QCD}$ is one parameter which determines the size of $\alpha_s\left(Q^2\right)$. It must be determined from experiment. Thus finally we have from Eq. (12)

$$\begin{aligned} \alpha_s\left(Q^2\right) &= \frac{1}{\alpha_\mu^{-1} + b\ln\frac{Q^2}{\mu^2}} + 0\ \left(\alpha_s^2\left(Q^2\right)\right) \\ &= \frac{1}{b\ln\frac{Q^2}{\Lambda_{QCD}^2}} + 0\ \left(\alpha_s^2\left(Q^2\right)\right). \end{aligned} \quad \text{(B.13)}$$

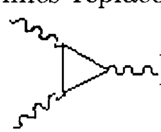
Table B.1 Renormalization Constants

Z_3	$1+\frac{g_s^2}{32\pi^2}\left[\left(\frac{13}{3}-\xi\right)C_2-\frac{4}{3}\,n_f\right]\ln\frac{\lambda^2}{\mu^2}$
Z_1	$1+\frac{g_s^2}{32\pi^2}\left[\left(\frac{17}{6}-\frac{3}{2}\xi\right)C_2-\frac{4}{3}\,n_f\right]\ln\frac{\lambda^2}{\mu^2}$
Z_{3F}	$1-\frac{g_s^2}{16\pi^2}\xi\;C_F\;\ln\frac{\lambda^2}{\mu^2}$
Z_{1F}	$1-\frac{g_s^2}{16\pi^2}\xi\;C_F\;\ln\frac{\lambda^2}{\mu^2}-\frac{g_s^2}{32\pi^2}\left(\frac{3}{2}+\frac{1}{2}\xi\right)C_2\;\ln\frac{\lambda^2}{\mu^2}.$

Note that we have been able to sum the leading logs here [compare (13) with $\alpha_\mu\ \left(1-\alpha_\mu\ b\ \ln\frac{Q^2}{\mu^2}+\cdots\right)$ in the ordinary perturbation theory]. Thus Eq. (13) goes beyond the ordinary perturbation theory. The perturbation now is with respect to $\alpha_s\left(Q^2\right)$.

We now determine b. For this purpose, we need Z_s [cf. Eqs. (8) and (9)]. But we note from Fig. 1 that Z_s is given by

$$Z_s^{1/2}=Z_{3F}\ Z_{1F}^{-1}\ Z_3^{1/2} \tag{B.14}$$

where the renormalization constants Z_{3F}, Z_{1F} and Z_3 arise respectively from diagrams A [self-energy part of the fermions (quarks) propagator], B [vertex part for the fermion] and C [the vacuum polarization or the self-energy part of the gluon propagator]. The values of these constants are summarized in Table 1, which also includes Z_1 which corresponds to the triple gluon vertex [i.e. the first of diagrams (B) with the quark lines replaced by the gluon lines while the second is replaced by ]. In this table C_2 and C_F are defined in Eq. (1) and n_f denote the number of fermion flavors.

From Eq. (14) and Table 1, we have to order g_s^2

$$Z_s^{1/2}=1+\frac{g_s^2}{32\pi^2}\left(\frac{11}{3}C_2-\frac{2}{3}\ n_f\right)\ln\frac{\lambda^2}{\mu^2}. \tag{B.15}$$

Thus from Eq. (10b) [note that the gauge fixing parameter ξ is canceled out]

$$\beta(g_s) = -2g_s^3 \left[b_0 + 0 \ \left(g_s^2\right)\right], \tag{B.16a}$$

so that

$$b_0 = \frac{1}{32\pi^2}\left(\frac{11}{3}C_2 - \frac{2}{3}\ n_f\right) \tag{B.16b}$$

$$b = 8\pi b_0 = \frac{1}{4\pi}\left(\frac{11}{3}C_2 - \frac{2}{3}\ n_f\right). \tag{B.16c}$$

Hence in summary, we have from Eq. (13)

$$\alpha_s\left(Q^2\right) = \frac{1}{\alpha_\mu^{-1} + \frac{1}{4\pi}\left(\frac{11}{3}C_2 - \frac{2}{3}\ n_f\right)\ln\frac{Q^2}{\mu^2}} + O\ \left(\alpha_s^2\left(Q^2\right)\right) \tag{B.17a}$$

$$= \frac{1}{\left(\frac{11}{3}C_2 - \frac{2}{3}\ n_f\right)\ln\frac{Q^2}{\Lambda_{\text{QCD}}^2}} + O\ \left(\alpha_s^2\left(Q^2\right)\right). \tag{B.17b}$$

It is a very useful equation and the single parameter Λ_{QCD} becomes the QCD scale which effectively defines the energy scale at which the running coupling constant attains its maximum. Λ_{QCD} can be determined from experiments and turns out to be [see Chap. 7]

$$\Lambda_{\text{QCD}} = 140 \pm 60 \text{ MeV}. \tag{B.18}$$

Note that for $\text{SU}_c(3)$ $[C_2 = 3]$, $\left(11 - \frac{2}{3}\ n_f\right)$ is positive for $\frac{2}{3}\ n_f \ < 11$ (which is certainly true for known six quark flavors $n_f = 6$) and then $\alpha_s(Q^2)$ decreases as Q^2 increases. This is made possible because of coefficient 11 which comes from the self coupling of gluons, non Abelian nature of QCD. The logarithmic deviation from asysmptotic freedom is a characteristic of QCD and the tests of the theory have to be sought to detect logarithmic scaling violations.

B.3 Running Coupling Constant in Quantum Electrodynamics (QED)

For QED, only fermion loops (i.e. the first of diagrams C in Fig. 1 with gluon replaced by photon and g_s^2 by e^2) contribute to electric charge renormalization so that in Table 1 only Z_3 without C_2 is relevant. Note, however, that the contributing charged fermions are $e, u, d, \mu, c, s, \tau, b$ and t so that $e^2\,(n_f/2)$ in the expression for Z_3 is replaced by

$$e^2 \sum_f Q_f^2 = e^2\, \frac{n_f}{2}\, \left[1 + 3\left(\frac{4}{9} + \frac{1}{9}\right)\right]$$

where $(n_f/2)$ are the number of generations [3 in our case] and the factor 3 outside the parenthesis is due to the color. Thus

$$Z_{em}^{1/2} = 1 + \frac{e^2}{32\pi^2}\frac{1}{2}\left[-\frac{4}{3}\left(8\,\frac{n_f}{3}\right)\right]\ln\frac{\lambda^2}{\mu^2} \tag{B.19a}$$

giving

$$\beta_{em} = -\frac{2e^3}{32\pi^2}\,\left(-\frac{16 n_f}{9}\right). \tag{B.19b}$$

The equation analogues to (12) is then

$$\frac{d\alpha_e^{-1}(Q^2)}{dQ^2} = b_{em} = \frac{1}{4\pi}\,\left(-\frac{16 n_f}{9}\right) \tag{B.20a}$$

giving

$$\alpha_e(Q^2) = \frac{1}{\alpha_e^{-1}(Q^2) - \frac{4 n_f}{9\pi}\ln\frac{Q^2}{\mu^2}} \tag{B.20b}$$

and increases with Q^2 in contrast to $\alpha_s(Q^2)$ which decreases with Q^2.

Let us apply Eq.(20b) for $\mu^2 = m_e^2$, where $\alpha_e(m_e^2)$ is determined from Thompson scattering, for example, $\left[\alpha \equiv \alpha_e(m_e^2) = \frac{1}{137}\right]$. No matter how small α one has, one can always increase Q^2 to a point where $\alpha_e(Q^2)$ which was given in Eq.(20b) becomes infinite [Landau ghost]. This, however, occurs [for six flavors] at

$$Q^2 = m_e^2 \exp\left(\frac{3\pi}{8}\alpha^{-1}\right) \approx 10^{63}\ \mathrm{GeV}^2 \tag{B.21}$$

which is even larger than $M_P^2 \approx 10^{38}$ GeV^2 by several orders of magnitude.

Finally, we wish to remark that the formula (20) holds for $m_e^2 \le Q^2 < m_W^2$. For $Q^2 \ge m_W^2$, we have to consider the contribution of charged $W^\pm$ bosons to β_{em}. In this case

$$b_{em} = \frac{1}{4\pi}\left(-\frac{16}{9}n_f + \frac{22}{3}\right) \tag{B.22}$$

and $\alpha_e(Q^2)$ still increases with Q^2 for $n_f = 6$ (or > 6).

B.4 Running Coupling Constant for SU(2) Gauge Group

For SU(2) group from Eq. (2) $C_2 = 2$ and therefore from Eq. (16c)

$$b_{SU(2)} = \frac{1}{4\pi}\left(\frac{22}{3} - \frac{2}{3}n_f\right) \tag{B.23}$$

and correspondingly Eq. (17a) becomes

$$\alpha_2\left(Q^2\right) = \frac{1}{\alpha_\mu^{-1} + \frac{1}{4\pi}\left(\frac{22}{3} - \frac{2}{3}n_f\right)\ln\frac{Q^2}{\mu^2}} \tag{B.24}$$

where $\alpha_2 = \frac{g_2^2}{4\pi}$, g_2 being the coupling constant associated with $qqW^\pm$ vertex, $W^\pm$, W_3 being the gauge bosons associated with SU(2) gauge group. Note that for six quark flavors ($n_f = 6$), $\left(\frac{22}{3} > 4\right)$and $\alpha_2\left(Q^2\right)$ is falling with Q^2, although at a rate less than $\alpha_s\left(Q^2\right)$ for the SU$_C$(3) group.

B.5 Renormalization Group Equation and High Q^2 Behavior of Green's Function

Consider now in general a renormalized Green's function (propagator or vertex function or a related quantity) in QCD denoted by

$$\Gamma_R \ (p_i, \alpha_s, \mu, \xi) = Z^{-1}\left(\lambda^2/\mu^2, \alpha_s, \xi\right) \ \Gamma \ (p_i, \alpha_{s0}, \xi_0), \qquad \text{(B.25)}$$

where Z is a multiplicative renormalization factor and Γ on the right-hand side knows nothing about μ so that $\frac{d\Gamma}{d\mu} = 0$. This implies that Γ_R satisfies the renormalization group equation

$$\frac{1}{\Gamma_R}\frac{d\Gamma_R}{d\mu} = -\frac{1}{Z}\frac{dZ}{d\mu}$$

$$\left[\frac{\partial}{\partial\mu} + \frac{d\alpha_s}{d\mu}\frac{\partial}{\partial\alpha_s} + \frac{d\xi}{d\mu}\frac{\partial}{\partial\xi}\right]\Gamma_R = \left[-\frac{1}{Z}\frac{dZ}{d\mu}\right]\Gamma_R$$

or

$$\left[\mu\frac{\partial}{\partial\mu} + 2\overline{\beta} \ (\alpha_s)\frac{\partial}{\partial\alpha_s} + \delta\left(\alpha_s, \xi\right)\frac{\partial}{\partial\xi} - 2\gamma \ (\alpha_s)\right]\Gamma_R \ (p_i, \alpha_s, \mu, \xi) = 0 \qquad \text{(B.26a)}$$

where [cf. Eq. (9) and (10)]

$$\begin{aligned}
\delta\left(\alpha_s, \xi\right) &= \mu\frac{d\xi}{d\mu} \\
\gamma &= -\frac{1}{2Z}\frac{dZ}{d\ln\mu} = -g_s^2\frac{dZ}{dg_s^2} \\
\overline{\beta} \ (\alpha_s) &= \frac{1}{2}\mu\frac{d\alpha_s}{d\mu} = \frac{1}{4\pi}g_s \ \beta\left(g_s\right).
\end{aligned} \qquad \text{(B.26b)}$$

To simplify matters, let us work in the Landau gauge $\xi = 0$, then

$$\left[\mu\frac{\partial}{\partial\mu} + 2\overline{\beta} \ (\alpha_s)\frac{\partial}{\partial\alpha_s} - 2\gamma \ (\alpha_s)\right]\Gamma_R \ (p_i, \alpha_s, \mu) = 0. \qquad \text{(B.27)}$$

The above equation also determines the high Q^2 behavior of Γ_R. To see this, we first note that there is another constraint on Γ which comes from dimensional analysis. Assume we scale all momenta in $\Gamma_R\ (p_i, \alpha_s, \mu)$, $p_i \to \lambda\ p_i$

$$\Gamma_R\ (\lambda p_i, \alpha_s, \mu) = \mu^D\ F\ \left(\lambda^2\ \frac{p_i \cdot p_j}{\mu^2}, \alpha_s\right), \tag{B.28}$$

D is the dimension of the Green's function (e.g. for inverse gluon propagator $\Gamma \sim p^2$ and we have $D = 2$). F is a dimensionless function of dimensionless variables. From Euler's theorem for homogeneous function

$$\left[\lambda \frac{\partial}{\partial \lambda} + \mu \frac{\partial}{\partial \mu} - D\right] \Gamma_R\ (\lambda\ p_i, \alpha_s, \mu) = 0. \tag{B.29}$$

Put $t = \ln \lambda$ and combine the naive scaling equation (29) with the renormalization group equation (27) [which gives the dynamical constraint] to eliminate $\mu \frac{\partial}{\partial \mu}$ and obtain

$$\left[-\frac{\partial}{\partial t} + 2\overline{\beta}\left(\alpha_s\right) \frac{\partial}{\partial \alpha_s} + D - 2\gamma\ \left(\alpha_s\right)\right] \Gamma_R\ (\lambda\ p_i, \alpha_s, \mu) = 0. \tag{B.30}$$

Its general solution can be obtained by the method of characteristics. First one solves [cf. Eq. (26b) with $t = \ln \mu$]

$$\frac{d\ \overline{\alpha}_s\ (t, \alpha_s)}{dt} = 2\overline{\beta}\ (\overline{\alpha}_s\ (t)) \tag{B.31}$$

with the condition $\overline{\alpha}_s\ (0, \alpha_s) = \alpha_s$. The general solution of Eq. (30) can then be expressed in terms of that of the above differential equation. In this way one obtains

$$\Gamma_R\ (\lambda\ p_i, \alpha_s, \mu) = \lambda^D\ \Gamma_R\ (\ p_i, \overline{\alpha}_s\left(t\right), \mu) \exp\left[-2 \int_0^t dt' \gamma\ \left(\overline{\alpha}_s\left(t'\right)\right)\right]. \tag{B.32}$$

What we learn from this general solution is that the behavior of Green's functions when all momenta are scaled up is governed by $\bar{\alpha}_s\left(t\right)$. Now as already seen in Sec. 2

$$\bar{\beta}\left[\left(\overline{\alpha}_s\left(t\right)\right)^2\right] = -\left(\overline{\alpha}_s\left(t\right)\right)^2 b + \cdots \tag{B.33a}$$

and similarly we can expand

$$\gamma\left(\overline{\alpha}_s(t)\right) = \gamma_0\overline{\alpha}_s(t) + \cdots \tag{B.33b}$$

$$\Gamma_R\ \left(p_i, \overline{\alpha}_s(t), \mu\right) = \Gamma_{R_0}\ \left(p_i, \mu\right) + \Gamma_{R_1}\ \left(p_i, \mu\right)\overline{\alpha}_s(t) + \cdots . \tag{B.33c}$$

Thus to solve Eq. (31) in the lowest order, we make use of Eq. (33a) and rewrite it as

$$dt = -\frac{d\bar{\alpha}_s}{2b\bar{\alpha}_s^2\left[1 + \cdots\right]}$$

giving

$$\bar{\alpha}_s(t) = \frac{1}{\alpha_s^{-1} + 2bt} + 0\left(\bar{\alpha}_s^2\right). \tag{B.34}$$

Remember that $t \sim \ln\lambda$ and this is the same functional dependence for $\bar{\alpha}_s$ as before for $\alpha_s(Q^2)$ in Sec.2. Thus noting that for large t, $\bar{\alpha}_s(t) \sim \frac{1}{2bt}$, the use of Eqs. (33) and (34) in the first order enable us to write Eq. (32) for large t or λ as

$$\begin{aligned} &\Gamma_R\ \left(\lambda\, p_i, \alpha_s, \mu\right) \\ \sim\ &\lambda^D\, \Gamma_{R0}\ \left(p_i, \mu\right)\exp\left(-2\int_0^t \frac{\gamma_0}{2bt'}dt'\right) \\ =\ &\lambda^D\, \Gamma_{R0}\ \left(p_i, \mu\right) t^{-\gamma_0/b} \\ =\ &\lambda^D\, \Gamma_{R0}\ \left(p_i, \mu\right)\left(\ln\lambda\right)^{-\gamma_0/b} \end{aligned} \tag{B.35}$$

where γ or γ_0 is called the anomalous dimension of Γ_R, which can be determined from Eq. (26b). If it were zero, we would have obtained canonical scaling behavior λ^D as in the traditional parton model [cf. Chap. 14]. Noting that $\bar{\alpha}_s(t) \sim \frac{1}{2bt}$ $[t \sim \ln\lambda]$, we can say from Eq. (35) that the large Q^2 behavior of $\Gamma_R\ (Q^2)$ is

$$\frac{\Gamma_R\ (Q^2)}{\Gamma_R\ (Q_0^2)} = \left(\frac{Q^2}{Q_0}\right)^D\left[\frac{\alpha_s\ (Q^2)}{\alpha_s\ (Q_0^2)}\right]^{\gamma_0/b} \tag{B.36}$$

where the second factor can be written as

$$\begin{aligned}
\left[\frac{\alpha_s\ (Q^2)}{\alpha_s\ (Q_0^2)}\right]^{\gamma_0/b} &= \left\{\alpha_s\ \left(Q^2\right)\left[\alpha_\mu^{-1} + b\ln\frac{Q_0^2}{\mu^2}\right]\right\}^{\gamma_0/b} \\
&= \left\{\alpha_s\ \left(Q^2\right)\left[\alpha_\mu^{-1} + b\ln\frac{Q^2}{\mu^2} - b\ln\frac{Q^2}{Q_0^2}\right]\right\}^{\gamma_0/b} \\
&= \left\{1 - b\ \alpha_s\ \left(Q^2\right)\ln\frac{Q^2}{Q_0^2}\right\}^{\gamma_0/b}, \qquad \text{(B.37)}
\end{aligned}$$

with $b = \frac{1}{4\pi}\left[\frac{11}{3}C_2 - \frac{2}{3}n_f\right]$ [cf. Eq. (16c)]. Thus it is clear that the renormalization group equation has enabled us to sum up terms of the form $[\alpha_s\ (Q^2)\ln\ Q^2]^N$ whereas in ordinary perturbation theory we would have to deal with a power series in $\alpha_s\ (Q^2)\ln\ Q^2$.

Analogous logarithmic violation of the scaling will hold in the deep inelastic structure functions and similar physical quantities.

Let us now consider some simple applications:

B.5.1 Gluon propagator

From Table 1,

$$Z_3 = 1 + \frac{2\alpha_s}{8\pi}\left[\left(\frac{13}{3} - \xi\right)C_2 - \frac{4}{3}\ n_f\right]\ln\frac{\lambda}{\mu} + 0\ \left(\alpha_s^2\right) \qquad \text{(B.38)}$$

$$\gamma_v = \frac{\alpha_s}{8\pi}\left[\left(\frac{13}{3} - \xi\right)C_2 - \frac{4}{3}\ n_f\right] + 0\ \left(\alpha_s^2\right)$$

$$\gamma_{v0} = \frac{1}{8\pi}\left[\left(\frac{13}{3} - \xi\right)C_2 - \frac{4}{3}\ n_f\right]. \qquad \text{(B.39)}$$

$$D_{AB\mu\nu} = -i\delta_{AB}\left[\left(g_{\mu\nu} - \frac{k_\mu\ k_\nu}{k^2}\right) + \xi\frac{k_\mu\ k_\nu}{k^2}\right]\frac{1}{k^2}d\ \left(-k^2\right) \quad \text{(B.40a)}$$

where

$$d\ \left(-k^2\right) \sim \left[\alpha_s\ \left(-k^2\right)\right]^{\gamma_{v0}/b}. \qquad \text{(B.40b)}$$

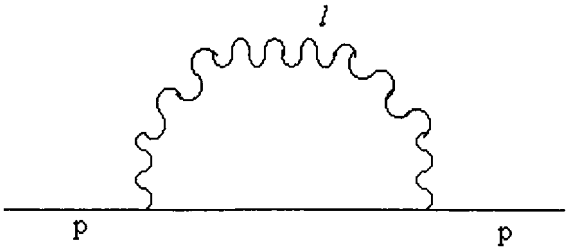


Figure 1 Fermion self-energy at one-loop level.

B.5.2 Fermion propagator

$$S_F^{-1}(p) = \left(\not{p} - m_\mu - \sum(p)\right), \tag{B.41a}$$

where

$$\sum(p) = m_\mu \sum\nolimits_1 \left(p^2\right) + (\not{p} - m_\mu) \sum\nolimits_2 \left(p^2\right). \tag{B.41b}$$

Let us define an effective or running mass through the following equations

$$S_F(p) = s_c \left(p^2\right) \frac{\not{p} + m(p^2)}{p^2 - m^2(p^2)} \tag{B.40c}$$

$$s_c \left(p^2\right) = \left(1 + \sum\nolimits_2 \left(p^2\right)\right)^{-1} \tag{B.40d}$$

$$m\left(p^2\right) = m_\mu \left[1 + \frac{\sum_1(p^2)}{1 + \sum_2(p^2)}\right]. \tag{B.40e}$$

The fermion self-energy diagram is given in Fig. 2 below.

This determines $\sum(p)$ at one-loop level. The renormalization mass m_R is defined by

$$m_R = Z_m \, m_B, \tag{B.42}$$

where Z_m is the multiplicative mass renormalization constant and is given by

$$Z_m = 1 + \sum\nolimits_1$$

$$= 1 + 2\frac{3\alpha_s}{4\pi} C_F \ \ln \frac{\lambda}{\mu} \tag{B.43}$$

while from Table 1:

$$\begin{aligned} Z_{3F} &= \left(1 - \sum\nolimits_2\right) - 1 \\ &= 1 - 2\frac{\alpha_s}{4\pi} \left(C_F \ \xi\right) \ln \frac{\lambda}{\mu}. \end{aligned} \tag{B.44}$$

Thus to the leading order

$$\gamma_{m0} = \frac{3 \ C_F}{4\pi} \tag{B.45}$$

$$\gamma_{F0} = \frac{C_F}{4\pi} \left(-\xi\right), \tag{B.46}$$

where for $SU_c(3)$ $C_F = \frac{4}{3}$ [cf. Eq.(2)]. Hence

$$s_c\left(p^2\right) \sim \left[\alpha_s \ \left(-p^2\right)\right]^{\gamma_{F0}/b} \tag{B.47a}$$

while

$$S_F\left(p^2\right) \sim \frac{1}{\not p} \left[\alpha_s \ \left(-p^2\right)\right]^{\gamma_{F0}/b}. \tag{B.47b}$$

For large p^2 we note from Eq. (36) that

$$\begin{aligned} \frac{m\left(p^2\right)}{m\left(p_0^2\right)} &= \left[\frac{\alpha_s \ \left(-p^2\right)}{\alpha_s \ \left(-p_0^2\right)}\right]^{\gamma_{m0}/b} \\ &= \left[\frac{\alpha_s \ \left(-p^2\right)}{\alpha_s \ \left(-p_0^2\right)}\right]^{3C_F/\left(\frac{11}{3}C_2 - \frac{2}{3} \ n_f\right)} \\ &= \left[\frac{\alpha_s \ \left(-p^2\right)}{\alpha_s \ \left(-p_0^2\right)}\right]^{4/\left(11 - \frac{2}{3} \ n_f\right)}. \end{aligned} \tag{B.48}$$

B.6 Bibliography

See bibliography at the end of Appendix A and Sec. C of the bibliography at the end of Chap. 7.

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