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Embodied Mathematics

Comments on Lakoff & Núñez[1]

Introduction

A major issue in consciousness research has to do with the nature of first person experience. Experience of mathematics and mathematical objects is an aspect of this. How can people do mathematics? What internal cognitive structures and processes provide a foundation for mathematical thought? What is the ontological status of the objects of such thought?

George Lakoff and Rafael Núñez (henceforth L&N) consider mathematical abilities from the perspective of linguistics (Lakoff's specialty) and cognitive science (Núñez's specialty). In a stimulating, provocative, and ultimately flawed book they seek to establish that mathematical ability is based on metaphors that allow conceptualization of abstract mathematical ideas in terms of the inferential structure of more concrete sensory and motor images. By locating the basis of mathematical ability in metaphors that are neurally embodied, they locate mathematical experiences with all other (supposed) neural correlates of consciousness. Their book is, however, frustrating. There are two reasons for this. The first is the number of mathematical errors. The second is the authors' underlying hostility to views of mathematics other than their own. Similar criticisms have been made by a number of mathematical reviewers of the book (Gold, 2001; Auslander, 2001; Paulos, 2001; Goldin, 2001; Henderson, 2002).

L&N's view of mathematics is consistent with that taken by Ruben Hersh (1997) as well as with the social constructivism of Paul Ernest (1998). The strongest emphasis, however, is placed on the idea of neural embodiment. Indeed, L&N call their view 'embodied mathematics'. They assert, 'Mathematics as we know it is limited and structured by the human brain and human mental

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[1] **George Lakoff & Rafael E. Núñez,** *Where Mathematics Comes From* (New York: Basic Books, 2000, ISBN 0465037704, $23.50)

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capacities. The only mathematics we know or can know is a brain-and-mind-based mathematics.' With their next step, however, they enter slippery philosophical territory by proclaiming that, 'There is no way to know whether theorems proved by human mathematicians have any objective truth, external to human beings or any other beings.'

Summary of the Book

The first two chapters provide necessary background concepts from cognitive science. The basic point is that humans have an innate capacity to recognize small numbers of objects, to remember short lists, to estimate, to use symbols, and to repeat actions.[2] L&N's thesis is that more complex mathematical thought and practice is grounded in these basic cognitive capacities through metaphoric elaboration and conceptual blending.[3]

Most important are 'grounding metaphors', which allow projections 'from everyday experience (like putting things into piles) onto abstract concepts (like addition)' and 'linking metaphors', that make connections between different branches of mathematics. An example of a grounding metaphor is: 'Sets Are Containers', while an example of a linking metaphor is 'Lines Are Collections of Points'. An important linking *and* grounding metaphor is the 'Basic Metaphor of Infinity' (BMI). All the mathematical uses of infinity are supposed to arise from 'a single general conceptual metaphor in which processes that go on indefinitely are conceptualized as having an end and an ultimate result'.

It is important to understand that the word metaphor is used in a technical sense as, 'a grounded inference-preserving cross-domain mapping — a neural mechanism that allows us to use the inferential structure of one conceptual domain (say, geometry) to reason about another (say arithmetic)'.

While it is possible to question some of the metaphors that are proposed,[4] and to criticize the lack of appeal to other facets of cognitive science that may have equal relevance (Goldin, 2001), this book makes an important contribution. Unfortunately, L&N do not restrict themselves to the cognitive basis of mathematical ability and it is at this point that they go astray. With apparent ideological motives, they try to discredit what they call the 'Romance of Mathematics'. As defined by them, this is a straw man, combining an extreme version of mathematical Platonism with an elitist view of mathematics as the ultimate science, and mathematicians as the ultimate experts on rationality.[5]

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[2] Research on this apparently innate 'number sense' is described in detail in an excellent book by Debaene (1997).

[3] For a review of research on conceptual blending see Fauconnier & Turner (2002).

[4] Henderson (2002) suggests an alternate metaphor of infinity, for example.

[5] This criticism is not new. The Islamic theologian Abu Hamid al Ghazali (1058–1111), in his *Confessions*, criticizes the notorious impiety of mathematicians, whose status as experts on rationality, so he asserts, sets a bad example that can lead the true believer astray.

An Important Error

It serves no purpose to list all the mathematical errors in the book, but one mistake is worth discussing because it illustrates a significant point. L&N claim to have invented 'granular numbers'. They do this through a construction that uses the BMI to generate a 'first infinitesimal number' that is the basis for the granular numbers. This is where they fall into error. As Gold (2001) points out, no such 'first infinitesimal' exists. L&N (2001) have not responded to this by accepting that something must be wrong with their derivation. Instead, they say that they have 'invented the granulars by applying the BMI. This is overtly *not* a process within formal mathematics. It is the use of a cognitive process for creating mathematical ideas.' They go on to say that their 'invention' of the first infinitesimal 'is not a mathematical result in the classic sense (i.e., it is not a result obtained by proving a mathematical theorem). Therefore it is not a "mathematical error" to say such a thing.'

This shows a profound misunderstanding of mathematics — both of what it is and of how it is developed. Mathematical results are *not* just formal proofs. They include, among other things, the formulation of new definitions — inventing, creating, or discovering new mathematical ideas. L&N seem to think that this creative aspect of mathematics is nothing more than a matter of the clever application of conceptual metaphors. They say: '"The first infinitesimal" is a consequence of the inferential structure of the BMI…. The fact that this infinitesimal is the "first infinitesimal," is an entailment of the metaphor….' As it happens, though, their concept is nonsense. The error lies in the definition. Omitting details, it boils down to defining this purported object as a number having the form 1/H where H is an integer greater than all real numbers. Since all integers *are* real numbers, however, this definition is stating that H is an integer greater than any integer. Use of the BMI has led to contradiction.[6]

Gold notes that application of the BMI requires bridging the gap between the finite steps of an infinite, or infinitely repeated sequence, and the projected endpoint of this sequence. L&N say this is a misunderstanding on her part since the BMI simply assumes that the sequence has a final end or result. They fail to get the point. Her remark is about the transition from metaphoric entailment to mathematical definition. It is not enough to say that something is metaphorically generated and therefore is a legitimate mathematical idea. It must also be demonstrated that the idea generated is free of contradiction. Conceptual metaphors can only be trusted so far — eventually there will be a point at which they break down. Indeed, much of abstract mathematics begins at exactly the point where a metaphor breaks down; where our expectations turn out to have led us astray. When we come to that point, continued reliance on the metaphor leads to illusion. Any further progress depends on pure abstraction.

[6] More precisely, unreflective application of the BMI leads to a conclusion that suffers from the fallacy of continuity — the assumption that the properties of the limit of an infinite sequence must be the same as the properties of points in the sequence itself.

Is There a Transcendent Mathematical Reality?

Plato himself regarded mathematics as a bridge between the illusory world of the senses and the true reality of ideal Forms. Mathematicians may require sensory images, such as geometric diagrams, but the actual content of mathematics is pure form; not a particular imperfect triangle, drawn in sand, but the idea of a triangle. With training, the step from the sensory world of images to the abstract world of pure Forms becomes easier and a person becomes capable of recognizing the higher Forms of Truth, Beauty, Justice, and finally, the Good.

Mathematical Platonism, however, is a modern term coined by Paul Bernays (1935) to describe the folk psychology of most mathematicians. In various forms it involves a belief that mathematics exists in a transcendent, mind-independent reality; that mathematical objects are objectively real; and that mathematical truth is certain, universal, and absolute.

In this Platonic view, mathematics is a pleroma that can never be exhausted by finite human minds. Hence, a distinction is made between the world of transcendental mathematics and the mathematics that is humanly comprehensible; between what Gödel (1995) called objective and subjective mathematics. Human mathematicians are in the position of explorers and mathematical results are discoveries rather than inventions or human creations; reports on territory that mathematicians have explored.[7]

L&N, however, reject the idea of transcendent mathematical reality as a source of mathematical elitism. They mount a two-pronged attack.

Their first line of attack goes back to Aristotle, who inquired whether the Ideal Triangle was equilateral or isosceles. The form of the argument is that if there is a transcendent mathematics then the mathematical ideas it contains must be unique. For example, there must be a unique transcendent idea of numbers, something that numbers 'really are'. But, L&N argue, numbers are 'characterized in mathematics in ontologically inconsistent ways'. As an example of this claim they point out that depending on the branch of mathematics, numbers can be thought of as points on the number line, as sets, and as positional values in combinatorial games. Each of these, however, excludes the other two. Sets are not points on the number line, and neither of these are positions in combinatorial games.

This argument fails because it is based on confusion between an ideal and its possible forms of representation — between the container and the content. The various descriptions of numbers are not 'ontologically inconsistent' because the representations used for numbers in each case are epistemological. They are how numbers are known in the given system. It is precisely the fact that numbers appear in such a great variety of contexts, defined in apparently inconsistent ways, that prompts us to abstract the concept of number itself.

The second attack begins with the reasonable claim that all human capacities must 'be accounted for by neural and cognitive mechanisms'. Thus, mathematical ability must be grounded in cognitive capacities, their metaphoric extensions,

[7] An excellent discussion of mathematical Platonism is given in Rucker (1982).

and in the neural mechanisms that underlie these. From this, L&N conclude that even if there were some form of transcendent mathematics, we could never know it and, while this existence cannot be disproved, it is irrelevant except in terms of the negative social and cultural effects that it produces.[8]

This is a version of the only strong argument that has ever been raised against the Platonic view of mathematics: there is no apparent way that the limited and sensory bound human mind could ever have access to a mind-independent world (see, e.g., Benacerraf & Putnam, 1983). The argument fails: the fact that human mathematics is based in human cognitive capacities does not mean that these capacities cannot provide recognition of transcendent mathematical truth. What it does do is point to the well-known issue of qualia, and to the hard problem of consciousness.

Kurt Gödel, an avowed Platonist, maintained that the question of the reality of mathematical objects is no different from the question of the reality of sensory objects. Perceptions of sensory objects are constructed in the mind by cognitive operations on sensory intuitions. Perceptions of mathematical objects are constructed in the mind by cognitive operations on mathematical intuitions. There is no reason, in principle, to privilege one set of perceptions over the other by assigning it 'true' reality. What is more fruitful is to explore the nature of the constructions.

It may seem that the perception of sensory objects is more objective because there is a causal story — objects are assumed to exist and to have properties that cause their perception. Light from an object falls on the retina exciting electrochemical impulses that travel along the optic nerve to visual areas in the brain where the signals are processed into neural patterns of excitation that are the neural correlates of the experience of seeing, for example, a tree. Where is the equivalent causal story in mathematics?

One response is that when a mathematician 'sees' the truth of a theorem, the neural activity that allows physical seeing, coupled to neural activity underlying basic combinatorial operations on visually perceived symbols, allows a perception that a certain sequence of symbol manipulations is valid.[9]

This is not what most mathematicians mean, however, when they speak of 'seeing' a mathematical truth. It is a more direct recognition of something that is experienced as mind-independent. Although they would be loath to admit it, L&N point to a possible answer to the question of how such 'seeing' is possible. We are caused to 'see' a mathematical object or a mathematical truth by the neural activity involved in the employment of the cognitive metaphors used in thinking about it, just as we are caused to 'see' a tree by the neural activity involved in the sensory processing that results in the perceived image of a tree. There is, in other words, a direct analogy between everyday sensory qualia such as colours, and perceptions of abstract mathematical objects.

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[8] This form of argument goes back to the ancient sophist Gorgias (de Romilly, 1992): it doesn't exist; even if it did exist, nobody could know it; even if somebody could know it, they could never communicate their knowledge.

[9] This view characterizes the formalist school of mathematics.

As often formulated, the problem of access to mind-independent mathematical objects is misconceived. The mystery is not in the ability to perceive mathematical objects, but in the ability to perceive any 'object' whatsoever.

Mathematics, as carried out by human beings, *is* embodied. It suffers all of the slings and arrows that go along with that embodiment. In emphasizing this, Lakoff & Núñez perform a valuable service. Ironically, however, their attempt to give mathematics a more human face ignores what is perhaps the most significant human aspect of mathematics. For the Platonist, it is the ability to have intuitive access to what is transcendent, whatever the mode of its existence, that is uniquely human.

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