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Introduction to Relativistic Continuum Mechanics

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To Liliana and Daniela

Preface

In standard university courses, the teaching of *special relativity* is often limited to show the absolute (i.e. four-dimensional) formulation of relativistic kinematics, mechanics and electromagnetism, whereas the equally interesting long chapters of geometrical continua and fluid dynamics are left for the *general relativity* investigation only. Actually, students become familiar with relativistic kinematics and the relativistic formulation of electromagnetism even from their first-year courses, and this is a really important step in their formation. However, they are given very little information about the so-called *relative observer point of view* in relativity and the different ways in which one can reintroduce the classical concepts of space and time, 3-momentum and energy, etc. from their space-time counterparts, namely the space-time itself, the 4-momentum, etc. The formalisms underlying this decomposition process, or $3+1$ *splitting techniques*, have been widely developed starting from the 1950s but can be found only in General Relativity textbooks.

We consider strongly important to make students familiar with Special Relativity either from the absolute (space-time, four-dimensional) point of view or from the relative (space plus time, $3+1$-dimensional) point of view, in order to give a central role to the measurement problem and to the observer, even studying Special Relativity.

This book, which apart from standard topics, includes also a geometrical introduction to continuous media and fluid dynamics, aims at pursuing an effort in this direction, summarizing the Italian school contribution with the pioneering works of Carlo Cattaneo since 1950 at the University of Rome, Italy.

It is therefore a pleasure to acknowledge Prof. Wolfgang Rindler, an "old friend" of Cattaneo and his coworkers since about 50 years, for many stimulating discussions and useful suggestions in addition to his special effort in improving the content as well as the English of this manuscript. Moreover, we are also grateful to Prof. L. Stazi for his comments and the careful reading of the hundreds of formulas in the manuscript.

A final remark concerns the reader: he is supposed to be familiar enough with ordinary tensor calculus on Riemannian manifolds and endowed with a good amount of patience!

Rome, *Giorgio Ferrarese*
February 2007 *Donato Bini*

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Classical Physics: Axiomatic Formulation

1.1 Methodology

Mathematical sciences contribute to physics either at an instrumental level (algebra and geometry give in fact the descriptional methods, whereas analysis gives the calculus methods for a qualitative and quantitative control of solutions corresponding to a certain schematization) or at a methodological level.

Nowadays, physical theories can be regarded as true axiomatic constructions which make them very close to mathematical theories. However, such a relation between mathematical and physical theories should not be pushed over the limit in which their different roles come to be confused. In fact, both mathematical and physical theories are obviously related through the "hypothetical-deductive method", but they are substantially different. As concerns their similarities, both of them assume a certain set of objects (primary quantities) and relations among them given a priori (axioms or postulates). Axioms must only be compatible. Then, from the primary quantities one obtains, by definition, some other objects (secondary quantities) and the relations between primary and secondary quantities are logically derived by means of theorems.

In any mathematical theory, the choice of objects and axioms is completely free, apart from their logical consistency: to the primary and secondary quantities, the theory cannot give any concrete meaning and what is really important are the relations existing among such objects only; as a consequence of this universality, the theory can be applied to many different situations.

For a physical theory instead, the objects must have a precise meaning in terms of the physical reality they represent, in the sense that they necessarily must have a counterpart in real objects. Moreover, all the observable relations among the real objects must be in agreement with the relations postulated or deduced from the theory. In other words, together with the "internal coherence" of the theory, a "perfect correspondence" between the theory and the reality is required.

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This principle directly inspired both Mach and Einstein. According to such a point of view, a physical theory, differently from a mathematical one, comes out not only as a consequence of a single intuition, but it is the result of a series of intuitions and comparing with the reality, with possible adjustments of the initial theory as well as of the control procedures. Moreover, any physical theory, even in the case of the long-awaited unified model, is never complete or definitive because it must include all phenomena and it has to be in agreement with all the observable relations among them (in the past as well as in the future). Apart from the discovery of new phenomena, the agreement is conditioned to the sensitivity of the instruments, to the refining of the experimental techniques, etc.

Actually, the assessment of the physics is the result of many theories (mechanics, electromagnetism, heat theory, thermodynamics, etc.), each with its own postulates, in addition to the general axioms common to all the theories. If the methods which control the theory reveal a discrepancy between the theoretically expected results and the experimental ones, first to be modified will be the specific axioms of a theory; then, if this is not enough, the general postulates will be changed.

In the history of modern physics, there existed two different moments in which the general postulates have been modified. The first concerns the black-body spectrum: the disagreement between theory and experiments was drastically solved renouncing the continuity hypothesis of the energy exchanges between radiation and matter (Planck, 1900). This led to the birth of quantum mechanics, associated with the names of Bohr (1913), Born, Schrödinger, Heisenberg and Dirac (1915).

The second concerns relativity: the disagreement between theory and experiments (especially after the Michelson–Morley experiment, 1887–1891) was drastically solved renouncing to the traditional ideas of space and time. This led to the Theory of Special Relativity [1], thanks to great contribution of A. Einstein (1905).

Examining the main ideas of that theory is the aim of this book.

1.2 General Axioms

In pre-relativistic physics, there exist two fundamental schemes only: the *material* scheme and the *electromagnetic* one; in correspondence, there exist two different theories: *mechanics* and *electromagnetism*. Each natural phenomenon was framed, in a simple way, in one of these two theories. Therefore, in the assessment of pre-relativistic physics, we find a set of general axioms and two more sets of specific axioms; the first one concerning mechanical phenomena and the second the electromagnetic ones.

There are three general axioms:

A. *Existence of an absolute space, E_3*

This axiom aims at specifying the ambient, i.e. the most natural descriptional context, for all the physical phenomena. The postulated absolute space, E_3, is a strictly Euclidean three-dimensional space. Its points (primary quantities) can be referred, for instance, to orthogonal Cartesian coordinates x^i $(i = 1, 2, 3)$, associated with an orthonormal frame $\mathcal{T} = (\mathrm{O}, \mathbf{c}_i)$, having its origin in O and unit vectors $\mathbf{c}_i$. To denote the distance between the points O and P, we will write $\mathrm{OP} = x^i \mathbf{c}_i$ (the summation with respect to the dummy index i is assumed). Once the orthonormal frame is fixed, the coordinates x^i of P are uniquely determined.

Passing to another orthonormal frame $\mathcal{T}' = (\mathrm{O}', \mathbf{c}_{i'})$, the same point P will have coordinates x'^i: $\mathrm{O'P} = x^{i'} \mathbf{c}_{i'}$.[1] The relation between x^i and $x^{i'}$ is obtained expressing the vectors $\mathbf{c}_{i'}$ as a linear combination of the vectors $\mathbf{c}_i$:

$$\mathbf{c}_{i'} = \mathcal{R}^i{}_{i'} \mathbf{c}_i \quad \sim \quad \mathbf{c}_i = \mathcal{R}^{i'}{}_i \mathbf{c}_{i'} \ , \tag{1.1}$$

where the matrices $||\mathcal{R}^i{}_{i'}||$ and $||\mathcal{R}^{i'}{}_i||$ are inverses of each other (we will follow the convention that the upper index is the row index and the lower index is the column one). In fact, the vectors $\{\mathbf{c}_i\}$ and $\{\mathbf{c}_{i'}\}$, as defined in (1.1), are orthonormal:

$$\mathbf{c}_{i'} \cdot \mathbf{c}_{k'} = \delta_{i'k'} \qquad \sim \qquad \mathbf{c}_i \cdot \mathbf{c}_k = \delta_{ik} \ , \tag{1.2}$$

where δ_{ik} is the Kronecker symbol

$$\delta_{ik} = \begin{cases} 0 \ \text{if} \ i \neq k \ , \\ 1 \ \text{if} \ i = k \ . \end{cases} \tag{1.3}$$

By using (1.1) in (1.2), one obtains the *orthogonality conditions*:

$$\mathcal{R}^i{}_{i'} \mathcal{R}^k{}_{k'} \delta_{ik} = \delta_{i'k'} \quad \sim \quad \mathcal{R}^{i'}{}_i \mathcal{R}^{k'}{}_k \delta_{i'k'} = \delta_{ik} \ . \tag{1.4}$$

Moreover, the triangular relation $\mathrm{OP} = \mathrm{OO'} + \mathrm{O'P}$, assuming

$$\mathrm{OP} = x^i \mathbf{c}_i \ , \quad \mathrm{OO'} = T^i \mathbf{c}_i \ , \quad \mathrm{O'P} = x^{i'} \mathbf{c}_{i'} = x^{i'} \mathcal{R}^i{}_{i'} \mathbf{c}_i \ , \tag{1.5}$$

gives the linear and invertible relations:

$$x^i = \mathcal{R}^i{}_{i'} x^{i'} + T^i \ . \tag{1.6}$$

The invertibility of (1.6), implicit in the interchangeability of the coordinates x^i and $x^{i'}$ (the exchange is equivalent to moving the prime to those letters which are without), is also a consequence of (1.4), which gives the condition

$$\left(\det||\mathcal{R}^i{}_{i'}||\right)^2 = 1 \ ; \tag{1.7}$$

[1] Following a standard convention for tensorial calculus, the prime is put on the index and not on the kernel.

that is, the (orthogonal) matrix $\mathcal{R}^i{}_{i'}$ has its determinant equal to ± 1 and represents a rotation $(+1)$ or an antirotation (-1). Furthermore, (1.6) shows that the coefficients $\mathcal{R}^i{}_{i'}$, besides the meaning of relating vector components in (1.1), coincide with the derivatives of x^i with respect to the $x^{i'}$:

$$\mathcal{R}^i{}_{i'} = \frac{\partial x^i}{\partial x^{i'}} \quad \sim \quad \mathcal{R}^{i'}{}_i = \frac{\partial x^{i'}}{\partial x^i} \, . \tag{1.8}$$

It is implicit in axiom A that the whole physical reality can be represented in terms of the geometric ingredients of E_3: points, curves, surfaces, tensor fields, etc., which have an intrinsic meaning in E_3, that is they are independent of the choice of the triad; this latter always has an accessory role, being determined up to translations and rotations (*equivalence of orthonormal Cartesian frames*).

The use of orthogonal Cartesian coordinates (the most simple in E_3 and with a global meaning), of course, does not prevent the use of other coordinates (polar, cylindrical or curvilinear in general), often with a local meaning only.

B. *Existence of an absolute time*

This axiom postulates the existence of a *universal time*, that is, a well-determined sequence of instants, independent of the space E_3 and hence of the observer and his motion (*absolute clock*). This axiom allows any observer not only to order phenomena happening in a given place (*local calendar*) but also to compare elementary phenomena which occur at different points in E_3, as well as to specify if they are simultaneous (*notion of contemporaneity*) or not (*notion of temporal ordering*). Everything is done objectively, i.e. independently of the observer and his motion.

It is also implicit, in axiom B, that every phenomenon has a well-determined *temporal duration*,[2] the evaluation of which depends on the choice of a temporal coordinate t:

$$t = at' + b \, , \qquad a > 0 \, , \tag{1.9}$$

defined up to a linear transformation, in the sense that it is still possible to choose both the origin (b) and the unit (a).

From the mathematical point of view, the absolute time is represented by an ordered set of instants, say T (not ordered, when inverting the time is allowed), homeomorphic to an oriented straight line R. The Cartesian product $\mathrm{R} \times E_3$ defines the space-time of classical physics, i.e. a new absolute: E_4, which is a four-dimensional manifold, with a foliation structure, and spatially homogeneous and isotropic. From the physical point of view, axiom B implies the existence of synchronizable *standard clocks*, working absolutely, i.e. independently of position, velocity and physical phenomena.

[2] "Tempus absolutum, verum et mathematicum, in se et natura sua, sine relatione ad externum quodvis, aequabiliter fluit, alioque nomine dicitur duratio" [2].

However, *primary quantities* are the points $P \in E_3$ and the instants of time; *derived quantities* are the event E, characterized by a point and an instant, with a physical correspondence in elementary phenomena (e.g. the lighting of a lamp, the ring of a bell, etc.), and the material points, after the introduction of the concept of mass. The latter have their physical counterpart in the material elements, say the molecules,[3] which aggregate to form the various natural bodies or the physical world.

The study of physical reality and its laws obviously presupposes the presence of the observer or the concept of a global reference frame, with its (absolute) measurement instruments for lengths and times.

Generally, one identifies frames with coordinate systems; actually, the concept of frame is more general and related to the so-called *natural undeformable bodies* (rigid frames or solids, for brevity). In fact, a physical reference frame is any natural body on which one can put a system of Cartesian coordinates. So a reference solid has to be necessarily an indeformable body.

In this sense, any Cartesian triad $\mathcal{T}$ in E_3 defines a unique solid frame, made up by the space of all the points in $\mathcal{T}$. Conversely, the same solid frame is characterized not only by $\mathcal{T}$ but by any other triad which can be obtained from $\mathcal{T}$ by time-independent translations and rotations.

In other words, one can attach ∞^6 systems of orthogonal Cartesian coordinates to the same solid reference frame, and all are equivalent. Such coordinates are then related by transformation laws as in (1.6), with $\mathcal{R}^i{}_{i'}$ and T^i independent of time, and defined as *spatial internal coordinates.*

The notion of time allows to examine, relative to of a given solid S, the motion of a point or that of a system of points (that is the motion of natural bodies), and hence to develop ordinary kinematics. In particular, one can develop *rigid kinematics*, or the motion of material systems S' analogous to S, defining other solid frames. From this, finally, *relative kinematics* follows, allowing to compare locally in E_3 and at every instant, the motion of a material element with respect to two different frames. Such a comparison is summarized by the two general laws of addition of velocities (*theorem of relative motions*) and accelerations (*Coriolis theorem*).

It is clear that, from a kinematical point of view, the reference solids are all equivalent or indistinguishable; however, from a physical point of view, i.e. for the formulation of the physics laws, they can be distinguished: this motivates the necessity of a third general axiom.

[3] At least in the cases in which the molecules, due to their internal symmetry, can be represented by their center of mass only. In absence of such internal symmetries, the scheme *material point* must be completed by introducing other geometrical quantities (applied vectors, tensors, etc.) which specify the internal structure.

C. *Existence of a preferred rigid frame, S^**

To this preferred rigid frame, all the physics laws are directly subordered. S^*, which must be still operationally determined, will be identified with the *fixed stars* in mechanics, and with the *cosmic Ether* in electromagnetism.

1.3 Axioms of Newtonian Mechanics

To the primary notions of absolute space and time, Newtonian mechanics adds the notions, also absolute (i.e. independent of the reference frame), of *mass* and *force*. The mass is a *constitutive property of matter*, a scalar quantity denoted by m; the force is the result of the physical action on a test (pointlike) body, due to the presence of other natural bodies or possibly to direct connection with them; an effect which can be schematized with an *applied vector* $\mathbf{f}$.

The fundamental axioms are only two:

1. *The law of motion* ($m\mathbf{a} = \mathbf{f}$, $m > 0$);
2. *The principle of action and reaction.*

Their validity is *limited* only to the preferred frame S^*; this should also be stressed because, differently from m and $\mathbf{f}$, the acceleration $\mathbf{a}$ has not an absolute character but depends on the chosen reference frame for its measurement.

From axiom 1 for the special case of a free material point ($\mathbf{f} = 0$) follows the *law of inertia*, which constrains, from the physical point of view, either the special frame S^* or the absolute scale of times. Let us consider, in fact, particles in inertial motion in empty space, i.e. far enough from other material bodies, in order to not feel any physical influence by them. In S^*, they describe linear trajectories with a uniform velocity or they are at rest; and this cannot be valid in any frame, or for any time scale.

However, independently of the Newtonian formulation, which starts from the two static notions of mass and force, classical mechanics can be structured by a set of axioms (see [3], pp. 192–196), which can all be expressed in terms of the acceleration with respect to S^* only. They allow the introduction of the (*dynamical*) notion of mass and hence that of force, namely

$$\mathbf{f} = m\mathbf{a} \; . \tag{1.10}$$

Definition (1.10) then becomes the *LEX II*, as soon as the *force law*, i.e. the dependence of the force on its effective parameters, is specified. In all those problems which can be framed in the scheme of the "material point" (*restricted problems*), the force law is necessarily of the kind

$$\mathbf{f} = \mathbf{f}(P, \mathbf{v}, t) \; . \tag{1.11}$$

The force law is assumed not to be related to the choice of the reference frame (*principle of force law invariance*), and hence it must be expressible in terms of absolute parameters (like the distance $r = |\mathrm{PQ}|$ of moving points or its temporal derivative $\dot{r}$), as it is in the case of the law of Newtonian gravitation.

The introduced axioms allow to develop the dynamics of the material point (*free or constrained*) relative to the frame S^*, and then that of the material systems, either with a finite number of degrees of freedom or continuous systems. Thereafter, the formulation is extended to any solid reference frame, using the Coriolis theorem. In such an extension, which implies the occurrence of apparent forces, a fact appears which is directly related to the expression of the general postulates of the theory, in terms of acceleration: namely, *the acceleration of a point is invariant, passing from a fixed reference frame S to any other in linear uniform motion with respect to S*: $\mathbf{a} = \mathbf{a}'$. Thus, Newtonian mechanics satisfies an invariantive and fundamental property: the preferred frame S^* can be replaced by any other reference frame in linear uniform motion with respect to it.

In other words, classical mechanics not only admits a single preferred frame but also the whole set $\{S_\mathrm{g}\}$ of the ∞^3 equivalent frames: the *Galilean or inertial frames*, characterized by the law of inertia only (see [4] p. 157). Newtonian mechanics is thus governed by the *Galilean principle of relativity*: *the (differential) laws of the motion are the same in every Galilean frame.*

Let us consider now two Galilean frames, S_g and S'_g. The *Galilean general transformation law* is the change of coordinates associated with the two triads $\mathcal{T}$ and $\mathcal{T}'$, arbitrarily chosen as concerns position and orientation in S_g and S'_g, respectively. These transformations are of the same type as in (1.6), with $\mathcal{R}^i{}_{i'}$ constant and T^i a linear function of t (in fact the T^i are the coordinates of the origin O' of the triad $\mathcal{T}'$, which undergoes a linear uniform motion with respect to $\mathcal{T}$).

Including the time dependence, one then finds the *Galilei group* G_{10}:

$$x^i = \mathcal{R}^i{}_{i'} x^{i'} + u^i t + s^i \,, \quad t = at' + b \quad (i = 1, 2, 3) \,. \tag{1.12}$$

If the two triads $\mathcal{T}$ and $\mathcal{T}'$, in the corresponding solids, are chosen so that

i) they are superposed at $t = 0$: $\mathcal{R}^i{}_{i'} = \delta^i_{i'}$ and $s^i = 0$;
ii) the x^1- and $x^{1'}$- axes have the same orientation of the relative velocity of S'_g with respect to S_g: $\mathbf{u} = u\mathbf{c}_1 = u\mathbf{c}_{1'}$,

then they are said to be in x^1-standard relation. In this case, (1.12) reduce to the *Galilei spatial transformation laws*:

$$x^1 = x^{1'} + ut \,, \qquad x^{2,3} = x^{2',3'} \,. \tag{1.13}$$

Clearly, the Galilean principle of relativity can also be expressed in the following way: *the differential laws of mechanics are formally invariant with respect to the transformations* (1.12), or, in more physical terms, *no mechanical experiment, performed in a given Galilean frame, can show the motion of this frame with respect to another Galilean frame.*

Let us stress that the Galilean principle of relativity (GPR), as we have already seen, depends on the absolute character of the acceleration and the dynamical definition of mass and force. Thus, once it is assumed that the force $\mathbf{f}$ maintains its absolute meaning (*principle of force law invariance*),

$$f^i(x,\dot{x},t)\,\mathbf{c}_i = f^{i'}(x',\dot{x}',t)\ \mathbf{c}_{i'}\ , \quad \forall\ \text{Galilean transf.} \tag{1.14}$$

the GPR appears no longer as a postulate, but as a theorem. Its logical role will be different in relativity theory, where such a principle is assumed a priori to be valid, in an extended form, for all the physics.

1.4 Axioms of Maxwell's Electromagnetism

As in the case of classical mechanics, electromagnetism introduces proper primary quantities. They are the *electric charge* e, the *electric field* $\mathbf{E}$ and the *magnetic field* $\mathbf{H}$. Starting from these essential quantities, one obtains, by definition, the derived quantities, like the *charge density* $\rho \overset{\text{def}}{=} \frac{\mathrm{d}e}{\mathrm{d}C}$, i.e. the charge contained in the element $\mathrm{d}C$ of volume and the *current density* $\mathbf{J} \overset{\text{def}}{=} \rho\mathbf{v}$. The axioms of the theory are two: first of all, Maxwell's equations

$$\begin{cases} \operatorname{div}\mathbf{H} = 0\ , & \operatorname{curl}\mathbf{E} + \dfrac{1}{c}\partial_t\mathbf{H} = 0\ , \\ \operatorname{div}\mathbf{E} = 4\pi\rho\ , & \operatorname{curl}\mathbf{H} - \dfrac{1}{c}\partial_t\mathbf{E} = \dfrac{4\pi}{c}\,\mathbf{J}\ , \end{cases} \tag{1.15}$$

which specify the (differential) relations between the electromagnetic field and the continuous distribution of charges and currents that generate it.[4]

Equation (1.15) form a partial derivatives system, with eight scalar equations (four of which are homogeneous); it is linear in the components of $\mathbf{E}$ or $\mathbf{H}$, and c is a *universal constant*, with the dimensions of a velocity. These equations, valid in S^*, are of said of Eulerian type because the coordinates involved are the t, x^i internal to S^*.

Another axiom concerns the mechanical action $\mathbf{F}$ on a test charge e (*Lorentz force*), namely:

$$\mathbf{F} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right)\ , \tag{1.16}$$

where $\mathbf{v}$ is the velocity of the charge in S^*. This action is proportional to the charge; in the special case of particle at rest ($\mathbf{v} = 0$), it is also proportional to the electric field $\mathbf{F} = e\mathbf{E}$.

However, as stated above, in any physical theory, the choice of the axioms is the synthesis of experimental results and, from this point of view, Maxwell's

[4] System (1.15) is the so-called Heaviside form of Maxwell's equations, in the Gauss unrationalized system of units.

equations do not constitute an exception. They are a summary of many previous experiments, which besides the formalization of electromagnetism, helped in foreseeing future developments of the theory. One of the most significant among such experiments (Oersted) gave the value of the universal constant c, by the invariant relation:

$$\frac{I\Sigma}{M} = \text{inv.} = c \;, \tag{1.17}$$

where M is the dipole moment of a circular current loop, of area Σ, and I is the intensity of the current in the loop. The value of c obtained in this way, namely $c \simeq 3,00,000$ km/s, was practically coincident with that of the light velocity in vacuum, as coming from different experiments.

From (1.15), some general consequences follow.

(i) From $(1.15)_4$, taking the divergence of both sides, and using the identity $\operatorname{div}\operatorname{curl} = 0$, as well as $(1.15)_3$, the *charge conservation equation* follows:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0 \;. \tag{1.18}$$

Equation (1.18) expresses, in Eulerian form, the condition that, in the evolution of the charged continuum in S^*, each region $\mathcal{C} \in C$ maintains its charge, a property similar to the mass conservation law; it can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}} \rho \,\mathrm{d}C = 0 \qquad \sim \qquad \int_{\mathcal{C}} \rho \,\mathrm{d}C = \int_{\mathcal{C}'} \rho' \mathrm{d}C' \;. \tag{1.19}$$

(ii) In the regions where there are no charges ($\rho = 0$) and currents ($\mathbf{J} = \mathbf{0}$), the electromagnetic field satisfies the relations:

$$\begin{cases} \operatorname{div}\mathbf{H} = 0 \;, & \operatorname{curl}\mathbf{E} + \frac{1}{c}\partial_t \mathbf{H} = 0 \;, \\ \operatorname{div}\mathbf{E} = 0 \;, & \operatorname{curl}\mathbf{H} - \frac{1}{c}\partial_t \mathbf{E} = 0 \;. \end{cases} \tag{1.20}$$

Differentiating $(1.20)_2$ with respect to time, one gets

$$\operatorname{curl}\partial_t \mathbf{E} + \frac{1}{c}\partial_{tt}\mathbf{H} = 0 \;, \tag{1.21}$$

so that, using $(1.20)_4$,

$$\operatorname{curl}\operatorname{curl}\mathbf{H} = -\frac{1}{c^2}\partial_{tt}\mathbf{H} \;; \tag{1.22}$$

similarly, from $(1.20)_{2,4}$ one gets

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = -\frac{1}{c}\partial_t \operatorname{curl}\mathbf{H} = -\frac{1}{c^2}\partial_{tt}\mathbf{E} \;. \tag{1.23}$$

Furthermore, from the identity, valid for any vector field $\mathbf{v}$[5]:

[5] The curl of a vector field $\mathbf{v}$ can be expressed as

$$\operatorname{curl}\operatorname{curl}\mathbf{v} = \operatorname{grad}(\operatorname{div}\mathbf{v}) - \Delta_2\mathbf{v} \,, \tag{1.24}$$

and using $(1.20)_{1,3}$, the preceding relations (1.22) and (1.23) become

$$\Box\,\mathbf{H} \equiv \Delta_2\mathbf{H} - \frac{1}{c^2}\partial_{tt}\mathbf{H} = 0 \,, \qquad \Box\,\mathbf{E} \equiv \Delta_2\mathbf{E} - \frac{1}{c^2}\partial_{tt}\mathbf{E} = 0 \,; \tag{1.25}$$

that is, in vacuum, $\mathbf{E}$ and $\mathbf{H}$ both satisfy the d'Alembert equation. It follows that c has a third important meaning. In fact, in analogy with elasticity theory (e.g. the one-dimensional case of a vibrating string), one gets for c the meaning of *propagation velocity* (in vacuum) of $\mathbf{E}$ and $\mathbf{H}$; this is called an electromagnetic wave.[6]

Here, as in elasticity theory, the word "wave" has the meaning of *solution of the field equations.* Such a meaning for c suggested to Maxwell the hypothesis, later confirmed in experiments, that light could be an electromagnetic phenomenon, and this allowed him to make predictions of more general phenomena, related to wave propagation. In fact, from (1.15) to (1.16), one obtains the so-called *physical optics*, and hence, with a limiting procedure, *geometrical optics.*

The meaning of c becomes more clear when one studies the propagation of discontinuity waves for the Maxwell's equations. To see this, let us assume that, at time $t = t_0$, a certain perturbation is introduced in a given electromagnetic field, for instance by means of an electric discharge. This implies that at $t = t_0$, one has $\delta\mathbf{E}$, $\delta\mathbf{H} \neq 0$ in a certain region, limited by a surface σ_0. The region initially perturbed will evolve into a moving surface σ, representing the boundary between the region affected (at that time) by the perturbation and the unperturbed space. In other words, σ is the *instantaneous wave front* of the perturbation, and it is also a *discontinuity surface* for both the field components and their derivatives. If the discontinuities occur only in the maximum order of derivatives appearing in the field equations, σ represents an *ordinary discontinuity wave*; otherwise it is a *shock wave.*

$$\operatorname{curl}\mathbf{v} = \mathbf{c}^i \times \partial_i\mathbf{v} \qquad (\partial_i = \partial/\partial x^i) \,.$$

From this definition,

$$\begin{aligned}\operatorname{curl}\operatorname{curl}\mathbf{v} &= \mathbf{c}^i \times \partial_i(\mathbf{c}^k \times \partial_k\mathbf{v}) = \mathbf{c}^i \times (\mathbf{c}^k \times \partial_{ik}\mathbf{v}) \\ &= \mathbf{c}^k(\mathbf{c}^i \cdot \partial_{ik}\mathbf{v}) - \partial_{ik}\mathbf{v}(\mathbf{c}^i \cdot \mathbf{c}^k) \\ &= \mathbf{c}^k\partial_k(\mathbf{c}^i \cdot \partial_i\mathbf{v}) - \delta^{ik}\partial_{ik}\mathbf{v} \,,\end{aligned}$$

and finally

$$\operatorname{curl}\operatorname{curl}\mathbf{v} = \operatorname{grad}(\operatorname{div}\mathbf{v}) - \Delta_2\mathbf{v} \,.$$

[6] In presence of electromagnetic sources one has instead

$$\Box\,\mathbf{H} = -\frac{4\pi}{c}\operatorname{curl}\mathbf{J} \,, \qquad \Box\,\mathbf{E} = 4\pi\left(\operatorname{grad}\rho + \frac{1}{c^2}\partial_t\mathbf{J}\right) \,.$$

The fact is that Maxwell's equations imply that all the ordinary discontinuity waves, in vacuum or with regular sources, move with velocity c (with respect to S^*), independently of the causes generating the perturbation itself.[7]

In other words, if $f(t, x^i) = 0$ is the Cartesian equation of the wave front σ in S^*, at each point of σ the propagation velocity is always c:

$$\left| \frac{\partial_t f}{\operatorname{grad} f} \right| = \text{const} = c \; . \tag{1.26}$$

Equivalently, every electromagnetic wave front satisfies the (*Eikonal*) differential equation:

$$\delta^{ik} \partial_i f \partial_j f - \frac{1}{c^2} \left(\partial_t f \right)^2 = 0 \; . \tag{1.27}$$

Equation (1.26) represents the simplest propagation law for a surface: constant speed. However, as in the general case, the evolution of the wave front is uniquely determined, as soon as the initial configuration σ_0 and the initial direction of propagation are assigned: σ is parallel to σ_0. In the special case in which σ_0 is reduced to a single point (epicentral waves), the wave fronts are spheres with a common centre; if σ_0 is a plane, one has plane waves etc.

Summarizing: the universal constant c, coming from electromagnetic considerations in Maxwell's theory, assumes the meaning of propagation speed of the electromagnetic waves. From this follows the necessity of selecting a preferred reference frame in which (1.15) hold that is S^*. In any other frame, the propagation velocity would necessarily be different.

Thus, even if (1.15), as well as the differential equations for mechanics, are invariant with respect to coordinate transformations internal to S^*, they do not satisfy the GRP. This latter consequence can be directly confirmed, using (1.12); in fact, neither Maxwell's equations, the force law (1.16) nor the propagation law (1.27) are formally invariant with respect to the Galilei transformations (1.12).

Hence there is a fundamental difference between mechanics and electromagnetism. In the latter case, S^* is the only preferred frame (it was called *cosmic Ether*, being an imponderable medium which allowed the light propagation). In the case of mechanics, instead, S^* cannot be distinguished from all ∞^3 equivalent frames. Assuming the validity of Maxwell's theory in the preferred frame S^*, clearly, the same theory can be extended to any other solid frame (Galilean or not, or even deformable). However, as for the case of the mechanics, because of the fictitious forces, such extension requires explicitly the characterization of the motion of S with respect to S^*.

At this point, once the laws of classical physics are accepted, one has the fundamental problem to localize the preferred frame S^*, taking into account

[7] In other words, such velocity does not depend either on the kinematical status of the charges at the beginning of the propagation or on the propagation direction. Experimentally, this effect can be checked by looking at the process (spontaneous or not) of photon emission, by collision of relativistic particles.

that this should be conditioned only by Maxwell's equations, so that it cannot be physically localized by means of mechanical experiments, but only through electromagnetic experiments. Such a localization was specifically sought for by the classical experiment of Michelson and Morley (1887), as we will see in the next section.

1.5 Optical Experiments and Classical Physics

In this section, we briefly review also few optical experiments which lead to the idea that the cosmic Ether should be identified with the fixed star space (see also [4]).

Astronomical Aberration

If two different observers, in relative motion along a line r, measure the inclination with respect to r of the light emitted by a fixed star A, the two measurements θ and θ' are slightly different and depend on the star position. It follows that by observing the same phenomenon after 6 months (to make the effect bigger), the same part of the sky appears to be deformed.

Such a phenomenon was found by J. Bradley in 1728. He observed that the stars, as seen from the Earth, seem to describe on the sky, in 1 year, a small ellipse, with the semimajor axis $a = 20'', 47$ of the ellipse parallel to the ecliptic, and the semiminor axis $b = a \ \sin \ \lambda$, where λ is the latitude of the star. He called this effect *aberration*, as if it were a sort of optical illusion, related to the motion of the Earth around the Sun. Classical mechanics allows the explanation of the effect, by using the addition of velocities law. In fact, let us assume that, in the reference frame S^*, both the star A and the observer O are at rest; let $\mathbf{c}$ be the (absolute) velocity of the light ray, as emitted from A, so that, for the observer O, the direction of the ray is OA. For the observer O′, in motion with relative velocity $\mathbf{u}$ with respect to O, the apparent direction of the light ray is that of the relative velocity $\mathbf{c}'$, given by the law of addition of velocities:

$$\mathbf{c}' = \mathbf{c} - \mathbf{u} \,. \tag{1.28}$$

Thus, from Fig. 1.1, we find the relation

$$\frac{\sin(\Delta\theta)}{\sin(\pi - \theta')} = \frac{u}{c} \,,$$

from which the approximate formula (to first order in $\Delta\theta$) holds:

$$\Delta\theta = \frac{u}{c} \sin\theta \,. \tag{1.29}$$

Classical mechanics shows, thus, a theoretical effect depending on θ, i.e. on the position of the star, which is of the first order in the ratio u/c. In

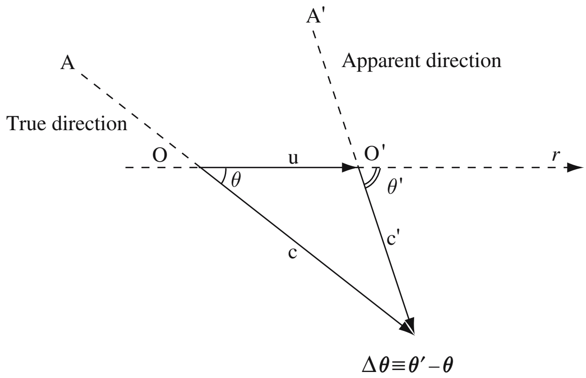

Fig. 1.1. Astronomical aberration

particular, let O′ be an observer on the Earth, and hence in (approximately) uniform linear motion with respect to S^*: $u \simeq 30$ km/s, the orbital velocity of the Earth. With this choice of u, the experimental data are in agreement with (1.29) and with the identification of the cosmic Ether with the space of fixed stars. However, the interpretation of the such a formula is limited because of the existence of big experimental errors.

Luminal Doppler Effect

A second experiment, which confirms the identification of the cosmic Ether with the space of the fixed stars, is a simple application of the Doppler effect. Let us recall here the classical relation which gives the frequency variation law of a monochromatic plane light wave when the observer is in motion with respect to the source. If the observer and the source are both at rest with respect to the Ether S^*, one has

$$\nu = \frac{c}{\lambda} \,,$$

where ν is the *frequency* and λ the *wavelength.* For an observer S′, in motion with respect to the Ether with (constant) velocity $\mathbf{u}$, parallel to the propagation direction of the wave front $\mathbf{n}$ ($\mathbf{n} \cdot \mathbf{n} = 1$), the situation is different. Assuming, for instance, that the observer S′ moves away from the source, the wave relative velocity will be $c - u$, and it will contain a number $(c - u)/\lambda$ of waves. Hence, with respect to S′, the wave frequency will be

$$\nu' = \frac{c-u}{\lambda} = \nu \left(1 - \frac{u}{c}\right) \,,$$

or

$$\nu' = \nu \left(1 - \frac{\mathbf{u} \cdot \mathbf{n}}{c}\right) \,, \tag{1.30}$$

where $\mathbf{u} \cdot \mathbf{n} = u$. Actually (1.30) is also valid when $\mathbf{u}$ is not parallel to $\mathbf{n}$ and represents the frequency variation law of a light wave, passing from the Ether to a frame S′ moving with respect to the Ether with a constant arbitrary velocity $\mathbf{u}$.

In an equivalent form, defining $\Delta\nu = \nu' - \nu$, one has (*luminal Doppler effect*)

$$\frac{\Delta\nu}{\nu} = -\frac{\mathbf{u} \cdot \mathbf{n}}{c} . \tag{1.31}$$

From this relation, one finds that, at a classical level, a transversal Doppler effect does not exists: $\mathbf{u}\cdot\mathbf{n} = 0 \rightarrow \nu' = \nu$. The maximum effect is longitudinal, and it is of the first order in the ratio u/c, as for the stellar aberration.

Let us note that in (1.30), together with the absolute notion of time, also the wave length λ (distance between two successive crests) has been considered as invariant, passing from the Ether to S′, in accordance with the classical invariance of space distances. Now one faces with the problem of how it is possible to operationally control (1.31), without performing measurements in the Ether, which is still to be localized.

Let us assume once again that the Ether will coincide with the space of the fixed stars, and let us consider, as a source, a star in the ecliptical plane. Let S′ be the frame of the Earth and assume we perform frequency measurements at a temporal difference of 6 months. Together with (1.30), one will also have

$$\nu'' = \nu\left(1 + \frac{u}{c}\right) ,$$

where u is the speed associated to Earth orbital motion ($u = 30\,\mathrm{km/s}$). Thus, we have

$$\nu' + \nu'' = 2\nu , \qquad \nu' - \nu'' = -2\frac{u}{c}\nu ,$$

and (1.31) reduces to

$$\frac{\nu' - \nu''}{\nu' + \nu''} = -\frac{u}{c} . \tag{1.32}$$

Equation (1.32) solves our problem because both ν' and ν'' are observable. If the Ether coincides with the fixed star space, one has to find agreement with the experimental data, assuming $u = 30\,\mathrm{km/s}$. This is exactly what happens, independent of the chosen source and within the experimental errors.

Fresnel–Fizeau Effect

An experiment, not so simple in its interpretation, if compared with the previous two, has been done by Fizeau in 1851, to verify an effect of light propagation in a moving medium foreseen by Fresnel in 1818.

In a transparent, homogeneous and isotropic medium, at rest with respect to the Ether, it is known that the speed of light becomes

$$v_0 = \frac{c}{n} < c\,, \tag{1.33}$$

where $n > 1$ characterizes the medium and is called its *refraction index*. What happens if the medium S′, where the light propagates, is in uniform translational motion with respect to the Ether? Figure 1.2 shows the experimental device created by Fizeau to answer this question. A half-silvered plate, placed in A, divides the light ray coming from the source S. The rays follow the rectangular path ABCD, by means of proper mirrors, having an inclination of 45^0, and then reach O, where a screen is placed to show interference fringes. Along their path, the rays pass trough two transparent tubes (arranged longitudinally along AD and BC, respectively), in which a homogeneous and isotropic liquid (water for instance) flows with constant velocity, but in opposite direction in the two tubes.

When the liquid is at rest in the two tubes, in O there are no interference fringes. These appear instead when the liquid is moving, and the fringe's amplitude varies with the fluid velocity u.

Hence, the light velocity is modified by the motion of the transparent medium, either by means of its speed u or by its orientation because this affects differently the two rays, depending on whether the motions in the tubes agree or not.

To see this in detail, let us denote by $v(u)$ the light speed in the tube (with respect to the laboratory), as a function of the speed of the liquid u, in the case in which the two motions agree. If l is the common length of the tubes, the temporal phase-displacement of the two rays Δt (due to the fact that the ray in motion in opposite direction with respect to fluid in the tube will take a longer time to reach O) is given by

$$\Delta t = \frac{2l}{v(-u)} - \frac{2l}{v(u)}\,. \tag{1.34}$$

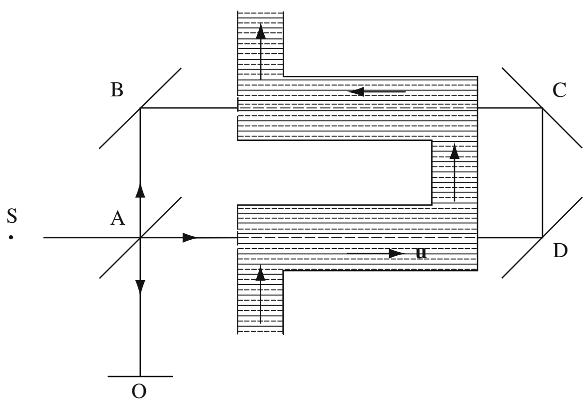

Fig. 1.2. The Fresnel–Fizeau apparatus

Moreover, a first-order Mac-Laurin expansion of the function $v(u)$ gives

$$v(u) \simeq v_0 + ku\ , \qquad k = \left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)_{u=0}\ ; \tag{1.35}$$

thus, (1.33) becomes

$$\Delta t = \frac{4lku}{v_0^2 - k^2u^2} = \frac{4lku}{c^2\left(\frac{1}{n^2} - k^2\frac{u^2}{c^2}\right)}\ ,$$

where the expression (1.34) for v_0 has been used. At the considered first order in u/c, we have

$$\Delta t = \frac{1}{c^2} 4lkn^2u\ . \tag{1.36}$$

Obviously, the temporal displacement Δt depends on the parameters l, n, u, which must be considered as fixed in the course of an experiment, and on the coefficient k, which uniquely determines the position of the fringes. Fizeau found that (1.36) was in agreement with the experimental data if k (assumed to be depending on the used liquid, i.e. on n) was given the form:

$$k = 1 - \frac{1}{n^2}\ . \tag{1.37}$$

From this and from (1.33) and (1.35), the approximated expression for $v(u)$ is

$$v(u) = v_0 + \left(1 - \frac{v_0^2}{c^2}\right)u\ . \tag{1.38}$$

We will see (1.38) again in the next chapter, as a direct consequence of the relativistic addition of velocities theorem. However, in what is stated above, there exists an incongruence. Equation (1.36) has been deduced relative to a frame at rest with respect to the Ether (in fact this is the condition of the observer O and the experimental device, except for the water), and in the same frame, one must give an interpretation to (1.38). Vice versa, (1.38) has been obtained by using (1.37), and comparing (1.36) with the experimental data corresponding to a terrestrial laboratory. Thus, the procedure is not consistent. The incongruence can be avoided, by repeating the calculation of Δt in an Earth laboratory. In doing so, the new value of Δt will be different from what gives (1.36), either for terms of higher order in u/c or because of the presence of the velocity of the laboratory with respect to the Ether. Therefore, (1.36) is still valid in an Earth laboratory, if it is in *slow motion* with respect to the Ether; thus, comparing with experimental data, the validity of (1.38) follows too.

On the contrary, one can also assume that the empirical formula (1.38), which holds to the first order in an Earth laboratory, is still valid to first order also in a laboratory at rest in the Ether. In any case, the value (1.37) of the coefficient k, obtained experimentally assuming the validity of (1.36), cannot, on the basis of a pure logical argumentation, be considered the (experimental) proof of the same (1.36) and, hence, as a proof of the slow motion of the Earth with respect to the Ether.

Fortunately, the theory of electromagnetism helps. In fact, (1.38) can be proved by using Maxwell's equations, for a dielectric nonmagnetizable medium, in slow motion with respect to the Ether. From these equations, one finds exactly the value (1.38) proposed by Fresnel and Fizeau, for the propagation velocity of a plane wave.

Alternatively, the same value can be obtained, assuming that a primary wave moving in the Ether with velocity c, because of electric polarization phenomena, creates dipoles which become, in turn, centres of secondary spherical waves (Huygens theorem). The electromagnetic field, resulting from the superposition of the two kinds of waves, is equivalent to a single wave, propagating with the velocity (1.38).

In conclusion, the Fresnel–Fizeau effect confirms that the Earth is moving slowly with respect to the Ether, but its speed through the Ether is certainly different from the value 30 km/s of the Earth velocity with respect to the fixed stars. Hence, the Ether cannot be considered at rest with respect to the fixed stars.

The Michelson–Morley Experiment

If the Ether were not at rest with respect to the Earth, as follows from the optical experiments described above, the velocity of light with respect to the Earth—because of the addition of velocities—should be different, in different directions and in different seasons. To demonstrate this *light anisotropy on the Earth*, Michelson and Morley in 1887 performed the following experiment, which was improved and repeated later, always with a negative result.

The apparatus, assumed to be at rest with respect to the Earth (laboratory), is shown in Fig. 1.3. A ray of monochromatic light arrives on a half-silvered plate L, placed at 45°. Out come two rays, which follow different paths, for going and coming back, by means of the mirrors S_1 and S_2, and which are combined on a screen O, to show inference fringes. One of the two paths is in the direction of the velocity $\mathbf{u}$ of the Earth with respect to the Ether, and the other has the perpendicular direction. Let T_1 and T_2 denote the flight times of the two rays. For the first ray, we have

$$T_1 = \frac{l_1}{c-u} + \frac{l_1}{c+u} = 2\frac{l_1 c}{c^2-u^2} = 2\frac{l_1}{c\left(1-\dfrac{u^2}{c^2}\right)} \,. \tag{1.39}$$

For the second ray, instead, if c' denotes its velocity with respect to the Earth, we have

$$\mathbf{c} = \mathbf{u} + \mathbf{c}' \,;$$

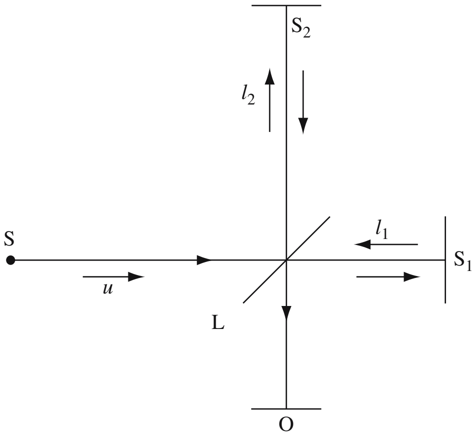

Fig. 1.3. The Michelson–Morley apparatus

from this, as the trajectory is orthogonal to $\mathbf{u}$, the value of c' follows:

$$c' = \sqrt{c^2 - u^2} \ .$$

Therefore, the time used by the second ray is given by

$$T_2 = \frac{2l_2}{c'} = 2\frac{l_2}{c\sqrt{1 - u^2/c^2}} \ . \tag{1.40}$$

The phase-displacement between the two rays, according to classical physics, is then

$$\Delta T = T_2 - T_1 = \frac{2}{c\sqrt{1-\beta^2}}\left(l_2 - \frac{l_1}{\sqrt{1-\beta^2}}\right) , \qquad \beta = u/c \ ,$$

and it is generically nonzero, except for special length of the two arms and special values of u. This implies the existence of interference fringes, actually observed by O.

Let us now rotate the whole apparatus by 90°, so that the two arms, and hence the two paths, are exchanged. The phase-displacement then becomes

$$\Delta T' = \frac{2}{c\sqrt{1-\beta^2}}\left(l_1 - \frac{l_2}{\sqrt{1-\beta^2}}\right) ,$$

differing from the previous one, also in its sign. In fact, assuming $\Delta T > 0$, i.e. $l_2 > l_1/\sqrt{1-\beta^2}$, we get $\Delta T' < 0$, because

$$\frac{l_2}{\sqrt{1-\beta^2}} > \frac{l_1}{1-\beta^2} > l_1 \ ;$$

analogously, when $\Delta T < 0$. Finally, one has

$$\Delta = \Delta T + \Delta T' = \frac{2(l_1 + l_2)}{c\sqrt{1-\beta^2}} \left(1 - \frac{1}{\sqrt{1-\beta^2}} \right) ,$$

or

$$\Delta = \frac{2(l_1 + l_2)}{c(1-\beta^2)} \left(\sqrt{1-\beta^2} - 1 \right) < 0 . \tag{1.41}$$

Assuming $\beta \ll 1$, we have the approximate result

$$\Delta \simeq -\frac{l_1 + l_2}{c} \frac{u^2}{c^2} \neq 0 . \tag{1.42}$$

The theoretical condition $\Delta \neq 0$ implies, after exchanging of the two arms, a displacement of fringes. However, the effect of the second order in u/c could not be seen experimentally, in spite of the precision accuracy of the experimental device, in the different periods of the year and the different ways in which the experiment was performed. Systematically, the rotation of the two arms (even when the platform was placed on a mercury liquid basis) did not show any variation of fringes: $\Delta = 0$.

Within classical physics, assuming $c \neq \infty$, the experimental result admitted the only possible interpretation $u = 0$, i.e. *Ether at rest with respect to the Earth*, in contradiction to the optical experiments described above.

A subsequent hypothesis due to the same Michelson, and then also to Stokes, according to which the Ether was dragged by the Earth, in the vicinity of its surface, was abandoned because no fringes were observed repeating the experiments at mountain level. The Lorentz–Fitzgerald hypothesis of length contraction for moving bodies with respect to the Ether was then introduced (in fact, it implies $|\Delta T| = |\Delta T'|$ and hence $\Delta = 0$, if one supposes that the length of the arm displaced along the velocity of the Earth with respect to the Ether was not l, but $l\sqrt{1-\beta^2}$, differently from the orthogonal one).

According to an epistemological point of view, widely accepted today (*fallibilism*) and not by chance developed, in consequence of the scientific fact we are briefly dealing with, one should suppose that *it is not possible to prove the truth of a theory, but only the falsity.* Therefore, one should conclude that on the basis of the negative result of the Michelson and Morley experiment, all the classical physics should be put in crisis. Because of the impossibility to solve the problem of localizing the Ether (apart from the use of an ad hoc hypothesis), following the logic, one arrived to deny its physical existence as a unique privileged frame. A trace of this way of reasoning can be found already in Poincaré, earlier than in the celebrated essay of Einstein in 1905.

It was immediately clear that the Ether problem should be related to the difference between classical mechanics and electromagnetism, the first admitting an infinity of preferred frames and the second only a single one. Such a difference, in agreement with the idea of a unified physics, was suggested by the effective mixing of natural phenomena and in particular, by the impossibility of separating such related phenomena, like mechanical and electromagnetic ones (see e.g. the particle theory of light).

This demand for unification required, first of all, the extension of the GRP also to the electromagnetic phenomena, and from this point of view, it was the same Einstein to understand concretely this fundamental necessity and to be aware that it had to imply the *invariance of the light velocity* (through the formal invariance contained in Maxwell's equations), hence to think of a fundamental revision of the notions of space and time, i.e. the only notions really common to all the physical phenomena. With his words, in the 1905 essay [4]: "We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body..." Einstein was creating the axiomatic body of the new physics: *special relativity*.

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2

Space-Time Geometry and Relativistic Kinematics

2.1 Introduction to Special Relativity

In the history of physics, Einstein occupies a position analogous, in certain respects, to that of Galilei, so big being the revolution they both determined in the scientific thought. Galilei, after 2000 years, eliminates the Aristotelian ideas on the dynamics (force directly related to velocity) with the help of the well-established methods of experimental control of a theory, and introduces the foundation of classical mechanics; Einstein, on the other hand, on the basis of the recent progress of particle physics, eliminates the distinct notions of space and time and introduces the new relativistic mechanics. In Einstein's ideas, there exist two "corner stones", on which special relativity formulation is based:

i) *Extended relativity principle*, made up by two parts because it posits (a) the *existence*, in Nature, of a class of ∞^3 preferred solid frames S_{g}, *inertial or Galilean frames*, just as in classical mechanics; (b) the *formal invariance*, with respect to these frames, *of all the physics laws* and not only of those of mechanics.
ii) *Light speed axiom*, according to which the light speed, in vacuum, has the same value c in all the Galilean frames, irrespective of the emission properties of the source. This axiom is clearly related to the validity of Maxwell's equations, which are considered the general laws of electromagnetism. According to point (i), these are formally invariant in all the inertial frames. In turn, such a validity, extended from the Ether to all the Galilean frames, implies two facts. On one side, it gives to the absolute quantities of classical electromagnetism (e.g. ρ, $\mathbf{E}$ and $\mathbf{H}$) a *relative meaning*, and, hence, creates the problem to specify their transformation laws, passing from one frame to another; on the other side, it marks the appearance of a *universal constant*: the light speed in vacuum, which constrains the new mechanics to admit a velocity (i.e. c) which has to be the same in each Galilean frame:

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$$c = \begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}t} & \text{in} \quad S_\mathrm{g} \\ \dfrac{\mathrm{d}s'}{\mathrm{d}t'} & \text{in} \quad S'_\mathrm{g}, \end{cases} \tag{2.1}$$

for any light signal in vacuum.

In the classical situation $\mathrm{d}t = \mathrm{d}t'$, (2.1) is not compatible with the addition of velocity law (a linear and uniform motion, with velocity $\mathbf{c}$ with respect to S_g, appears linear and uniform in S'_g too, but with velocity $\mathbf{c}' = \mathbf{c} - \mathbf{u}$, where $\mathbf{u}$ is the relative velocity of S'_g with respect to S_g). In particular, it follows that in any frame, there is a different value of the light velocity, according to its direction (*optical anisotropy*). Hence, it is necessary to assume $\mathrm{d}t \neq \mathrm{d}t'$ together with $\mathrm{d}s \neq \mathrm{d}s'$: the validity of (2.1) implies the necessity to renounce not only a universal time but also the idea of an absolute space. From here it follows Einstein's criticism of the traditional idea of an absolute time, which he rightly considers a conventional quantity, without any operational meaning. But up to what point is it correct to speak about simultaneity, in terms of absolute quantities? Within a given Galilean frame S_g, an operational criterion, to establish if two events, E and F, occurring at two different points A and B, are simultaneous, is the following. Let us assume that the light speed be the same, in each direction (*optical isotropy*), and let us imagine that when the two events occur, two light signals were emitted from A and B, respectively. If these were simultaneously recoiled on a screen placed in the middle point of AB, M, then the two events can be considered as simultaneous. Otherwise, their arrival order will specify the corresponding temporal sequence. This criterion allows us to operationally synchronize the ∞^3 (one for each space point) standard clocks of a given Galilean frame. But, is it possible to transport—as classical mechanics does—the notions of simultaneity and arrival order, from one inertial frame to another? The answer is no. In fact, let us consider, together with S_g, another Galilean frame S'_g, in linear uniform translational motion with respect to S_g, with velocity $\mathbf{u}$.

Let us suppose that the events E and F be simultaneous in S_g, occurring at A and B at the time $t = 0$. Let then A′ and B′ be the points of S'_g superposed, at $t = 0$, to the points A and B, and M′ be the mid-point of A′B′. Initially (i.e. at $t = 0$), M′ (in S'_g) coincides with M (in S_g); however, repeating the previous experiment, because of the motion of M′, the ray emanating from B≡ B′ will meet M′ before the one coming from A≡ A′. Hence, to the observer M′, the switching a light in B will seem to arrive before that the one coming from A: this is the *relativity of simultaneity* or the fact that two events, simultaneous in one frame, are not simultaneous in another Galilean frame.

The situation would have been different if the light speed were infinite: in fact, classical mechanics and its notion of absolute time are both consistent with $c = \infty$; in spite of the fact that from 1675 (Römer), the *finite value* of the light speed was known.

However, assuming the existence of a class of ∞^3 preferred frames, in each of these frames, one can still introduce a universal time t, but this t is, a priori, independent of the time t' of another frame: $t' \neq at + b$. Furthermore, according to Einstein, not only the time but also the lengths have a relative meaning; that is, the ordinary distance between two points, in the same Galilean solid, is not invariant: two events that, in a given Galilean frame occur at distinct points, in another Galilean frame may occur at the same point, and vice versa. In other words, even if all the Galilean solids, assumed to be equivalent, have the same geometric structure (strictly Euclidean), they are no more superposed to the same three-dimensional space E_3, as it is the case for the classical physics. For example, the condition of uniform translational motion (for rods or clocks) in a given S_{g} determines a variation for lengths and times, due to the relative nature of spatial and temporal measurements, and not to a deformation of clocks and rods.

2.2 General Axioms

Let us summarize the general axioms of Einstein physics.

A. *Existence of an absolute space-time*
   This axiom implies that the only primary quantity is the event E, or "elementary phenomenon". The set of all the events $\{E\}$ form the universe in its becoming, and this is the four-dimensional unification of space and time in a unique absolute: the space-time.

   It is implicitly assumed in A that all the physical reality can be represented in terms of the geometrical objects of the space-time (points, curves, hypersurfaces, 4-vectors, etc.). In particular, the Galilean or inertial solid frames can be considered (in a new form, as will be specified in the next postulate) and identified as solids in uniform translational motion with respect to the fixed stars.
B. *Existence of Galilean frames*
   In the space-time, it is possible to identify a class of ∞^3 preferred frames (the *Galilean frames*), each of them characterized by a three-dimensional space, Σ, endowed with a universal time t and with all the ordinary properties; that is the spatial isotropy, the spatial and temporal homogeneity, the validity, in Σ, of the strict Euclidean geometry, and the completeness, in the sense that, in each S_{g}, all the various phenomena can be coordinated (in a relative form). In other words, the generic event E appears, in S_{g}, as occurring at a certain point P of the solid, and at a certain instant t. Thus, once one has selected a Cartesian triad $\mathcal{T}$ in Σ, E is represented by a numerical quadruple (t, x^i) $(i = 1, 2, 3)$: the *space-time coordinates*

of the event E. In another Galilean frame, S'_g, the same event E will be associated with another point $P' \in S'_g$ and by an instant t' of the temporal scale associated with S'_g; hence, it will be represented by a numerical quadruple $(t', x^{i'})$, different from the previous one but determined by this (*Lorentz transformations*).

However, together with $t \neq at' + b$, one will also have $x^i \neq \mathcal{R}^i{}_{i'} x^{i'} + u^i t + s^i$ because, as stated above, the Galilean solids S_g and S'_g are not superposed to a common E_3, as is the case classically.

In any case, because of the completeness of the physical description in S_g, the history of S'_g can be followed in S_g too, even if only in relative terms; that is, the particles of S'_g will appear in uniform translational motion. Thus, even with the unique new absolute: the space-time, the ordinary notions of space and time are not eliminated, in the Einsteinian conception, because these are the fundamental terms of our experience; they, clearly, can be found in the Galilean frames, but with a relative meaning. The experimental physicist, operating in a Galilean frame S_g (laboratory), does his measurements in terms of lengths and time intervals, knowing that the values he can find, in the study of a given phenomenon, will be different from those found by another observer, at rest in another Galilean frame. The additional fact is that, not only the measurements but also the properties that one measures, have, a priori, a relative meaning (velocity, acceleration, mass, charge, electric and magnetic fields, etc.). Hence is the necessity to know the transformation laws of these quantities, passing from one to another Galilean frame. In turns, this requires the link of the relative quantities to the absolute quantities, from which they come and can be seen as a renaissance of the platonic philosophy: the world shows itself by means of shadows, the only things accessible by men; even if these are coherent shadows, with an objective content, as emanating from the "absolute".

However, the physicist especially looks for the fundamental relations between observables; that is, for the physical laws, which, because of their universal character, must satisfy invariance requirements. More precisely, in spite of the relativity of the geometric terms used to formulate physics laws in S_g, the following axiom holds.

C. *Extended relativity principle (ERP)*

All the physics laws are *formally invariant*, passing from one reference frame S_g to another S'_g. That is, no physical experiment may allow to distinguish between S_g and S'_g. This is an extension of the classical *Galilean relativity principle* due to the extension of the absolute space and absolute time axioms. However, the invariance is only formal, in the sense that all the ingredients entering the formulation of the physics laws have a relative meaning. The problem of studying their transformation laws is then naturally posed.

In any case, special relativity substitutes the existence of a universal constant for the indetermination of the physical space and the time, that is

D. *Independence of the emission for the light speed in vacuum*
In each S_g, the light speed in vacuum does not depend on the source motion, and it has the same value in each direction. Because of the ERP, such a speed cannot depend on the chosen Galilean frame S_g, and it should have the value c which it takes in the rest frame of the source: $c_g = c'_g = c$. Let us note that the latter postulate becomes a theorem if Maxwell's equations are accepted as laws of electromagnetism, obviously subordinated to the ERP; that is, every electromagnetic perturbation propagates in vacuum with velocity c, independent of the initial disturbance.

Finally, c is a limit velocity, in the sense that

E. *No material particle, in S_g, can move at (or faster than) the light speed in vacuum*

$$v^2 < c^2 \qquad \text{for any material particle.} \tag{2.2}$$

This axiom, as will be elucidated in what follows, can be obtained by adding the *causality principle* to the preceding axioms. It is widely confirmed, both in the macroscopic and the microscopic range, in high-energy physics experiments. Here we prefer to assume it from the beginning.

In any case, (2.2), valid in every S_g because of the ERP, represents a nonholonomic, unilateral and quadratic constraint, which does not introduce in S_g horizons (i.e. limitation for the particles trajectories), but restricts only the motion laws.

2.3 The Minkowski Space-time

The problem is now that of geometrizing the chosen axioms in order to build up a model for the Universe: the *Minkowski space* M_4, where the Galilean frames should be localized, the Lorentz transformation be derived, i.e. where all the physics theories can be developed. In fact, even if the relative point of view is allowed, and it is close to the phenomenological reality as it appears to the observer, the absolute point of view in M_4 is primary, either for developing the general procedures or to define the various physical quantities.

In classical mechanics, the problem of finding representative spaces for material systems was considered too. As an example, for the motion of a holonomic system, two different formulations exist: one of geometrical–kinematics content (configuration space and phase space) and another of pure geometric content (event space, or the space of the world lines). A material point is a particular holonomic system and hence for it both the possibilities are allowed. However, in the relativistic framework, the second formulation should be favoured, not only for its absolute content (the first has necessarily a relative meaning because such is the time; similarly for the configuration space, which can be identified with a Galilean space) but also because it allows the conceptual vicinity between the time and the space coordinates.

There is, clearly, a fundamental difference: while in a classical context there exist a $1-1$ correspondence (up to a weak condition $\mathrm{d}t/\mathrm{d}\lambda > 0$, see below) between all the physically allowed motions and the curves, in the relativistic case, i.e. in the event space, such a condition does not exist because of the constraint (2.2).

To be more specific, let us consider a Galilean frame S_{g} referred to internal (orthogonal Cartesian) coordinates x^i. From axiom D, let us denote $x^0 = ct$, so that x^0 has the dimensions of a length. Then, we can interpret x^α ($\alpha = 0, 1, 2, 3$) as the Cartesian coordinates of a four-dimensional *affine space*[1] $\mathcal{E}_4$. Such interpretation obviously requires that an affine frame be arbitrarily fixed in $\mathcal{E}_4$, namely an origin Ω and a basis $\{\mathbf{c}_\alpha\}$.

The generic point $E \in \mathcal{E}_4$: $\Omega E = x^\alpha \mathbf{c}_\alpha$, represents an event, i.e. the special event that, in the considered Galilean frame, occurs at $P \equiv x^i$ at the instant $t = x^0/c$. Analogously, a generic motion in S_{g},

$$x^i = x^i(t) \ , \qquad t \in (t_0, t_1), \tag{2.3}$$

has two characteristics: a geometric one (the *relative trajectory*, $\ell \equiv \mathrm{P_0 P_1}$) and a kinematic one (the *law of motion* along the trajectory $s = s(t)$). These are summarized, in $\mathcal{E}_4$, by an oriented curve segment ℓ^+:

$$x^\alpha = x^\alpha(t) \ , \qquad \frac{\mathrm{d}x^0}{\mathrm{d}t} = c > 0 \ , \tag{2.4}$$

where the time of the reference frame is chosen as parameter along the curve.

The latter is not a special parameter because it can be replaced by any other parameter λ, chosen with only the condition of saving the orientation of ℓ^+:

$$x^\alpha = x^\alpha(\lambda) \ , \qquad \frac{\mathrm{d}x^0}{\mathrm{d}\lambda} > 0 \quad \forall \lambda \in (\lambda_0, \lambda_1) \ . \tag{2.5}$$

---

[1] It is well known that an affine space $\mathcal{E}_4$ can be defined as a set of elements (points) in a $1-1$ correspondence with the ordered 4-tuple of real numbers x^α ($\alpha = 0, 1, 2, 3$), defined up to linear and invertible transformations: $x^\alpha = A^\alpha{}_{\alpha'} x^{\alpha'} + A^\alpha$, $A^\alpha{}_{\alpha'} = \partial x^\alpha / \partial x^{\alpha'}$ being a regular matrix, constant as the A^α. Alternatively, $\mathcal{E}_4$ can be thought as a set $\mathcal{E}$ of elements (or points O,P,Q,...) associated with a linear space T_4 such that there exists a surjective application $\alpha : \mathcal{E} \times \mathcal{E} \quad \rightarrow \quad T_4$ between ordered pairs of $\mathcal{E}$ and elements of T_4, that is $(\mathrm{O}, \mathrm{P}) \rightarrow \mathbf{v} = \alpha(\mathrm{O}, \mathrm{P}) = \mathrm{OP}$, satisfying the conditions:

1. $\forall \mathrm{O} \in \mathcal{E}$ and $\mathbf{v} \in T_4 \exists ! P \in \mathcal{E} : \mathrm{OP} = \mathbf{v}$;
2. $\mathrm{OP} + \mathrm{PQ} = \mathrm{OQ} \ , \quad \forall \mathrm{O}, \mathrm{P}, \mathrm{Q} \in \mathcal{E}$ (triangular relation).

Then, five-ordered points of $\mathcal{E}$: Ω and U_α (the origin and the "unit points"), such that the vectors $c_\alpha = \Omega \mathrm{U}_\alpha$ are linearly independent in T_4 and define an affine frame.

Thus, every pointlike motion $\mathcal{M}$ in $S_{\rm g}$ ($x^i = x^i(s)$, $s = s(t)$), in the event space $\mathcal{E}_4$, is represented by a well-determined arc $\ell^+ : x^\alpha = x^\alpha(\lambda)$. In particular,

(i) to the uniform rectilinear motions in $S_{\rm g}$: $\dot{x}^i = \text{const.}\,(i = 1, 2, 3)$ correspond, in $\mathcal{E}_4$, straight lines $\dot{x}^\alpha = \text{const.}$, and among these

(ii) the points of the Galilean frame (at rest in $S_{\rm g}$: $x^i = \text{const.}$) are represented by straight lines, parallel to the x^0-axis.

Conversely, an oriented arc $\ell^+ \in \mathcal{E}_4$ does not define, in general, a physically allowed motion. In fact, even if λ is eliminated: $x^0 = x^0(\lambda), x^i = x^i(\lambda), \quad \to$ $x^i = x^i(t)$, the condition (2.2) should be satisfied, i.e. $\delta_{ik}\dot{x}^i\dot{x}^k - (\dot{x}^0)^2 < 0$, or, after multiplication by $(\mathrm{d}t/\mathrm{d}\lambda)^2 > 0$:

$$\delta_{ik}\left(\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\right)\left(\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}\right) - \left(\frac{\mathrm{d}x^0}{\mathrm{d}\lambda}\right)^2 < 0\,, \qquad \lambda \in (\lambda_0, \lambda_1)\,,$$

the latter being a form invariant with respect to the choice of the parameter λ. Therefore, a necessary and sufficient condition for $\ell^+ \in \mathcal{E}_4$ to represent a physically possible motion in $S_{\rm g}$, is that the following limitation:

$$m_{\alpha\beta}\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\right)\left(\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right) < 0 \quad \forall\lambda \in (\lambda_0, \lambda_1) \tag{2.6}$$

be satisfied, with

$$m_{\alpha\beta} \stackrel{\text{def}}{=} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

The limitation (2.6) can be geometrically interpreted. To this end, let us introduce, in $\mathcal{E}_4$, the scalar product associated to the symmetric matrix (2.7):

$$\mathbf{V}\cdot\mathbf{W} \stackrel{\text{def}}{=} m_{\alpha\beta}V^\alpha W^\beta \qquad \forall\,\mathbf{V},\,\mathbf{W} \in \mathcal{E}_4. \tag{2.8}$$

Such an operation satisfies all the ordinary scalar product properties, namely

1. *commutative*: $\mathbf{V}\cdot\mathbf{W} = \mathbf{W}\cdot\mathbf{V}$;
2. *bilinear*: $(\mathbf{U}+\mathbf{V})\cdot\mathbf{W} = \mathbf{U}\cdot\mathbf{W} + \mathbf{V}\cdot\mathbf{W}$; $(a\mathbf{U})\cdot\mathbf{V} = a(\mathbf{U}\cdot\mathbf{V})$;
3. *nonsingular*: $\mathbf{V}\cdot\mathbf{W} = 0 \quad \forall \quad \mathbf{V} \quad \to \quad \mathbf{W} = \mathbf{0}$,

as it can be directly verified.

As concerns the property 3, the condition $m_{\alpha\beta}V^\alpha W^\beta = 0 \quad \forall V^\alpha$ is equivalent to $m_{\alpha\beta}W^\beta = 0$, that is $W^\beta = 0$, since the matrix $m_{\alpha\beta}$ is regular: $\det||m_{\alpha\beta}|| = -1$.

As in the ordinary case, the condition $\mathbf{V}\cdot\mathbf{W} = 0$ can be geometrically interpreted, in terms of orthogonality of the two vectors; the vectors orthogonal to $\mathbf{V}$ form the orthogonal hyperplane to $\mathbf{V}$, etc.

The operation (2.8) gives, in particular, the meaning of the coefficients $m_{\alpha\beta}$, identifying them as scalar products of the (affine) frame vectors:

$$m_{\alpha\beta} = \mathbf{c}_\alpha \cdot \mathbf{c}_\beta \qquad (\alpha, \beta = 0, 1, 2, 3) \; ; \tag{2.9}$$

moreover, it associates a scalar quantity to any vector $\mathbf{V}$, the norm $||\mathbf{V}||$:

$$||\mathbf{V}|| = m_{\alpha\beta} V^\alpha V^\beta \; , \tag{2.10}$$

which is not necessarily positive; for example, $||\mathbf{c}_0|| = -1$.

The affine space $\mathcal{E}_4$, endowed with the scalar product (2.8), assumes the structure of Euclidean space (not in a strict sense[2]): the *Minkowski space* M_4. Such a space is characterized also by the distance $\delta(E, F) \geq 0$ between the two events E and F

$$\delta^2 \stackrel{\text{def}}{=} |m_{\alpha\beta}(x_F^\alpha - x_E^\alpha)(x_F^\beta - x_E^\beta)| = |\delta_{ik}(x_F^i - x_E^i)(x_F^k - x_E^k) - (x_F^0 - x_E^0)^2| \; , \tag{2.11}$$

built up with the relative spatial distance

$$\Delta\ell = \sqrt{\delta_{ik}(x_F^i - x_E^i)(x_F^j - x_E^j)} \; , \tag{2.12}$$

and the relative time interval: $\Delta t = |t_F - t_E|$, so that

$$\delta^2 = |\Delta\ell^2 - c^2 \Delta t^2| \; . \tag{2.13}$$

This is a geometrical model for the special relativistic absolute: the four-dimensional Minkowski space-time. First of all, its points are the events of the natural world; the history of all the physically allowed motions (sequence of events in causal relation) is given by particular oriented arcs ℓ^+ or *world lines* of M_4. They must satisfy the condition (2.6) which, after introducing the tangent vector $\boldsymbol{\lambda} = \lambda^\alpha \mathbf{c}_\alpha$

$$\lambda^\alpha \stackrel{\text{def}}{=} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \qquad (\alpha = 0, 1, 2, 3) \; , \tag{2.14}$$

is equivalent to the condition

$$||\boldsymbol{\lambda}|| = m_{\alpha\beta} \lambda^\alpha \lambda^\beta < 0 \qquad \forall \lambda \in (\lambda_0, \lambda_1) \; . \tag{2.15}$$

This is a typical property of the world lines which has an intrinsic meaning and does not depend on the choice of the parameter λ. In fact, if $\lambda = \lambda(\lambda')$, then

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}' \frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda} \; ,$$

and hence

$$\boldsymbol{\lambda} \cdot \boldsymbol{\lambda} = \boldsymbol{\lambda}' \cdot \boldsymbol{\lambda}' \left(\frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda} \right)^2 \; ,$$

so that, from (2.15), $||\boldsymbol{\lambda}'|| < 0$.

[2] An Euclidean space is strictly Euclidean if one assumes instead that $\forall\, \mathbf{V} \neq 0$, then $||\mathbf{V}|| > 0$ or $||\mathbf{V}|| < 0$.

At this point, we have considered the mathematical translation of the axioms A and E. We still have to give a representation, in M_4, of the Galilean frames, and also to show how M_4 deals with the relativity principle. However, strengthening the structure of affine space by introducing a scalar product, the various geometrical quantities of M_4 (points, vectors, tensors, etc.) and their properties should be invariant with respect to more general transformation than those (linear and invertible) of an affine space (Lorentz transformations).

Finally, as concerns the definition of a Galilean frame S_{g}, we have that in M_4, as in $\mathcal{E}_4$, its history is represented by straight lines, parallel to $\mathbf{c}_0$; any of the ∞^1 hyperplanes orthogonal to $\mathbf{c}_0$ can represent the physical space Σ, associated with S_{g}. Changing the Galilean frame S_{g} will be then equivalent to exchanging (in M_4) $\mathbf{c}_0$ with another vector of the same kind: $\mathbf{c}'_0$: $\mathbf{c}'_0 \cdot \mathbf{c}'_0 = -1$. The result will be that the equivalence of all the Galilean frames will correspond to the geometrical indistinguishability of the vectors $\mathbf{c}_0$ and $\mathbf{c}'_0$ which are used to represent them.

2.4 The Minkowski Metric

It is now convenient to consider some formal aspects of Minkowski geometry, which, being improperly Euclidean, is quite different from properly Euclidean geometry. First of all, by using the fundamental products,

$$\mathbf{c}_\alpha \cdot \mathbf{c}_\beta = m_{\alpha\beta} = \mathrm{diag}(-1,1,1,1) \ , \tag{2.16}$$

the selected affine frame is constrained being necessarily orthonormal the basis $\{\mathbf{c}_\alpha\}$: the vectors $\mathbf{c}_\alpha$ have *unitary magnitude* (the magnitude or modulus of a vector $\mathbf{V}$ being defined as $V = \sqrt{|\mathbf{V} \cdot \mathbf{V}|}$), and they are *mutually orthogonal*. Orthonormal bases are typical for (properly or not) Euclidean spaces. It can be shown (see [1], p. 82) that, *in any nonsingular Euclidean space, there exist infinite orthonormal bases; moreover, in the same case, the number of vectors having positive (or negative) norm is invariant*: such integer number defines the *signature* of the Euclidean space. From this point of view, the Minkowski space M_4 is an (improper) Euclidean space of signature $+2$, or $- + ++$, because *all the orthonormal bases contain a vector with norm* -1*, and three vectors with norm* $+1$.

With the choice (2.16), the basis $\{\mathbf{c}_\alpha\}$ is orthonormal; however, it is not unique as occurs for the ordinary space. If $\{\mathbf{c}_{\alpha'}\}$ is a basis of the same kind,

$$\mathbf{c}_{\alpha'} \cdot \mathbf{c}_{\beta'} = m_{\alpha'\beta'} = \mathrm{diag}(-1,1,1,1) \qquad (\alpha',\beta' = 0,1,2,3) \ , \tag{2.17}$$

the transformation matrix which defines the change of the basis, $L^\alpha{}_{\alpha'}$ (to distinguish it from the generic transformation matrix $A^\alpha{}_{\alpha'}$),

$$\mathbf{c}_{\alpha'} = L^\alpha{}_{\alpha'} \ \mathbf{c}_\alpha \ , \tag{2.18}$$

should be a *rotation* (also said a 4-rotation), in the sense that it satisfies the properties analogous to (2.4):

$$L^{\alpha}{}_{\alpha'}\ L^{\beta}{}_{\beta'} m_{\alpha\beta} = m_{\alpha'\beta'} \qquad (\alpha, \beta = 0, 1, 2, 3)\ ; \tag{2.19}$$

Equation (2.19) constrains the choice of the $L^{\alpha}{}_{\alpha'}$ (arbitrary, a priori), as was the case for the components of a rotation matrix $\mathcal{R}^{i}{}_{i'}$ in the ordinary space

$$\mathcal{R}^{i}{}_{i'}\ \mathcal{R}^{j}{}_{j'} \delta_{ij} = \delta_{i'j'} \qquad (i, j = 1, 2, 3)\ . \tag{2.20}$$

Here, the conditions obtained from (2.19) are 10 (in fact, because of the symmetry, it is enough to assume $\alpha \leq \beta$) and, hence, the *rotations* in M_4 form a group with $16 - 10 = 6$ parameters.

From (2.19) we also have

$$(\det||L^{\alpha}{}_{\alpha'}||)^2 = 1\ ; \tag{2.21}$$

thus, such a group contains both *rotations* ($\det||L^{\alpha}{}_{\alpha'}|| = 1$) and *antirotations* ($\det||L^{\alpha}{}_{\alpha'}|| = -1$).

In any case, *the coefficients* $m_{\alpha\beta}$ *are necessarily invariant:* $m_{\alpha\beta} = m_{\alpha'\beta'}$ *for all possible orthonormal bases* related by (2.19); they are no longer invariant if the basis $\{\mathbf{c}_{\alpha'}\}$ is generic: $\mathbf{c}_{\alpha'} = A^{\alpha}{}_{\alpha'}\, \mathbf{c}_{\alpha}$ and, in this case, one has

$$m_{\alpha'\beta'} = A^{\alpha}{}_{\alpha'}\ A^{\beta}{}_{\beta'} m_{\alpha\beta} \neq m_{\alpha\beta} \qquad (\alpha, \beta = 0, 1, 2, 3)\ . \tag{2.22}$$

The transformation law (2.22) characterizes, as we will see in the following, a 2-*tensor*: the *metric tensor* of M_4, whose main role is that of defining, in M_4, the scalar product; in other words, it defines the space-time *lengths* and the *angles*, with an abuse of language because here a definition of the angle between two vectors is different from the one valid in the ordinary space.

Another property, equally important, of the metric, is that of *raising* or *lowering* of indices. More precisely, together with the matrix $||m_{\alpha\beta}||$, given in (2.7), let us consider the inverse matrix $||m^{\alpha\beta}||$ such that

$$m^{\alpha\beta} m_{\beta\rho} = \delta^{\alpha}_{\rho}\ ; \tag{2.23}$$

in our case (orthonormal basis), one has $m^{\alpha\beta} = m_{\alpha\beta}$.

By means of the reciprocal elements $m^{\alpha\beta}$, one can construct the *dual basis* $\mathbf{c}^{\alpha}$ of the basis $\mathbf{c}_{\alpha}$, defined as follows:

$$\mathbf{c}^{\alpha} = m^{\alpha\beta}\mathbf{c}_{\beta} \qquad \sim \qquad \mathbf{c}_{\alpha} = m_{\alpha\beta}\mathbf{c}^{\beta}\ , \tag{2.24}$$

or, explicitly,

$$\mathbf{c}^{0} = -\mathbf{c}_0\ , \qquad \mathbf{c}^{i} = \mathbf{c}_i \quad (i = 1, 2, 3)\ . \tag{2.25}$$

From (2.23) and using (2.16) the following property holds:

$$\mathbf{c}^{\alpha} \cdot \mathbf{c}_{\alpha} = \delta^{\alpha}_{\beta}\ . \tag{2.26}$$

With this assumption, one can decompose each vector with respect to the basis $\{\mathbf{c}_\alpha\}$ or its dual $\{\mathbf{c}^\alpha\}$

$$\mathbf{V} = V^\alpha \mathbf{c}_\alpha = V_\alpha \mathbf{c}^\alpha \ . \tag{2.27}$$

The components V^α (along $\mathbf{c}_\alpha$) are termed *contravariant*, while the V_α (along $\mathbf{c}^\alpha$) are said to be *covariant*; from (2.24) follow the invertible relations:

$$V_\alpha = m_{\alpha\beta} V^\beta \qquad \sim \qquad V^\alpha = m^{\alpha\beta} V_\beta \ , \tag{2.28}$$

or, explicitly $V_0 = -V^0$, $V_i = V^i \quad (i = 1, 2, 3)$. It is worth noticing the similar role played in (2.28) by the matrices $||m_{\alpha\beta}||$ and $||m^{\alpha\beta}||$: the first is used to lower an index, the second to raise it. They also play a similar role with respect to the metric: in fact, exchanging of them is equivalent to exchanging of the basis $\{\mathbf{c}_\alpha\}$ with its dual basis $\{\mathbf{c}^\alpha\}$. In other words, from (2.26), one has the following two representations of the metric, covariant and contravariant:

$$m_{\alpha\beta} = \mathbf{c}_\alpha \cdot \mathbf{c}_\beta \ , \qquad m^{\alpha\beta} = \mathbf{c}^\alpha \cdot \mathbf{c}^\beta \ ; \tag{2.29}$$

in addition, there are the two symmetric relations for the components of a vector:

$$V_\alpha = \mathbf{V} \cdot \mathbf{c}_\alpha \ , \qquad V^\alpha = \mathbf{V} \cdot \mathbf{c}^\alpha \ . \tag{2.30}$$

2.5 Vectors and Their Classification. The Lightcone

Let us proceed, now, with the classification of vectors in M_4. We have the following definitions: (1) vectors $\mathbf{u}$ with vanishing norm: $\mathbf{u} \cdot \mathbf{u} = 0$, are *null vectors* (or lightlike or isotropic); (2) vectors $\mathbf{s}$ with positive norm, $\mathbf{s} \cdot \mathbf{s} > 0$, are *spacelike vectors* (like $\mathbf{c}_i$); (3) vectors $\boldsymbol{\gamma}$ with negative norm, $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} < 0$, are *timelike vectors* (like $\mathbf{c}_0$).

Let us consider the whole set of null vectors, at an arbitrary point $\Omega \in M_4$, to be chosen as the origin of the coordinates, for simplicity. Thus, the components of a generic null vector $\mathbf{u} = \Omega\mathrm{U}$ are identified with the coordinates x^α of its end point U. When $\mathbf{u}$ varies within such a family, U describes a hypersurface, defined by the following homogeneous and quadratic relation:

$$m_{\alpha\beta} x^\alpha x^\beta = 0 \ ; \tag{2.31}$$

this is a *three-dimensional cone*, $\mathcal{C}_3$: the *absolute feature of* M_4, or the *lightcone.*

$\mathcal{C}_3$ separates, as we will see later, the vectors with positive norm (external) from those with negative norm (internal). In a properly Euclidean space, like the ordinary one, obviously, the lightcone $\mathcal{C}_3$ degenerates to a single point; differently from what happens in a generic (i.e. nonproperly) Euclidean space, like M_4, where *a nontrivial null vector does exist.*

Let us consider, in fact, the subspace of M_4 generated by the vectors $\lambda\boldsymbol{\gamma}+\mu\mathbf{s}$ ($\boldsymbol{\gamma}$ being a timelike vector, $\mathbf{s}$ a spacelike vector and $\lambda, \mu \in \Re$). The associated norm,

$$\begin{aligned}||\lambda\boldsymbol{\gamma}+\mu\mathbf{s}|| &= (\lambda\boldsymbol{\gamma}+\mu\mathbf{s})\cdot(\lambda\boldsymbol{\gamma}+\mu\mathbf{s}) \\ &= \lambda^2\,||\boldsymbol{\gamma}|| + 2\lambda\mu\boldsymbol{\gamma}\cdot\mathbf{s} + \mu^2||\mathbf{s}||\ ,\end{aligned}$$

is a continuous function of λ and μ, in all the real plane. Once evaluated at the point $\mathrm{P} \equiv (\lambda \neq 0, \mu = 0)$, it is negative, while at the point $\mathrm{Q} \equiv (\lambda = 0, \mu \neq 0)$, it is positive. Thus, there exists a point $\mathrm{R} \equiv (\bar{\lambda}, \bar{\mu})$, internal to the segment PQ (and hence different from Ω) such that $||\bar{\lambda}\boldsymbol{\gamma}+\bar{\mu}\mathbf{s}|| = 0$.

For example, if $\{\mathbf{c}_\alpha\}$ is an orthonormal basis, in the subspace $E_2 \equiv \{\lambda\mathbf{c}_0 + \mu\mathbf{c}_1\}_{\lambda,\mu\in\Re}$, there exist two isotropic straight lines: $-\lambda^2+\mu^2 = 0 \rightarrow \lambda = \pm\mu$, which are defined by the two null vectors: $\mathbf{u}_{1,2} = \mathbf{c}_0 \pm \mathbf{c}_1$.

It is worth noticing that

1. $\mathbf{u}_1 + \mathbf{u}_2 = 2\mathbf{c}_0$ is a timelike vector; that is, the lightcone $\mathcal{C}_3$ is not a vector subspace of M_4.
2. An orthonormal basis cannot contain a null vector, by definition; however, in M_4, there exist bases of null vectors; for example, $\mathbf{c}_0+\mathbf{c}_1$, $\mathbf{c}_0-\mathbf{c}_1$, $\mathbf{c}_0+\mathbf{c}_2$, $\mathbf{c}_0+\mathbf{c}_3$ are four, linearly independent, null vectors. In fact, the condition

$$\alpha^0(\mathbf{c}_0+\mathbf{c}_1)+\alpha^1(\mathbf{c}_0-\mathbf{c}_1)+\alpha^2(\mathbf{c}_0+\mathbf{c}_2)+\alpha^3(\mathbf{c}_0+\mathbf{c}_3) = 0$$

is equivalent to

$$(\alpha^0+\alpha^1+\alpha^2+\alpha^3)\mathbf{c}_0 + (\alpha^0-\alpha^1)\mathbf{c}_1 + \alpha^2\mathbf{c}_2 + \alpha^3\mathbf{c}_3 = 0\ ,$$

or

$$\alpha^0+\alpha^1+\alpha^2+\alpha^3 = 0\ , \quad \alpha^0-\alpha^1 = 0\ , \quad \alpha^2 = 0\ , \quad \alpha^3 = 0\ ,$$

that is $\alpha^\beta = 0$ ($\beta = 0,1,2,3$) and, hence, they form a basis in M_4.
3. The two null vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are not orthogonal

$$(\mathbf{c}_0+\mathbf{c}_1)\cdot(\mathbf{c}_0-\mathbf{c}_1) = -2\ .$$

This is a *general property*, that is *for any two null vectors* $\mathbf{u}$ *and* $\mathbf{u}'$ *not aligned*, one has

$$\mathbf{u}\cdot\mathbf{u}' > 0 \quad \text{or} \quad \mathbf{u}\cdot\mathbf{u}' < 0\ ,$$

the orthogonality being thus excluded.[3]

[3] Let us assume, without loss of generality, a basis $\{\mathbf{c}_\alpha\}$ such that the null vector $\mathbf{u}$ is represented by $\mathbf{u} = u(\mathbf{c}_0+\mathbf{c}_1)$. If $\mathbf{u}'$ is a null vector too, then $\mathbf{u}' = a\mathbf{c}_0 + b^i\mathbf{c}_i$, with $a^2 = \delta_{ik}b^ib^k$. The scalar product with $\mathbf{u}$ is given by $\mathbf{u}'\cdot\mathbf{u} = u(b^1-a)$; it follows that $\mathbf{u}'\cdot\mathbf{u} = 0 \leftrightarrow b^1 = a$, and hence $b^2 = b^3 = 0$, that is $\mathbf{u}'$ parallel to $\mathbf{u}$.

In scalar terms, for any event-origin E, the lightcone is defined by the Cartesian equation (2.31), or, explicitly:

$$\delta_{ik} x^i x^k - (x^0)^2 = 0 \, , \tag{2.32}$$

which represents a circular cone. The intersection with the generic hyperplane $x^0 = ct$, orthogonal to the x^0-axis, is an ordinary sphere: $\delta_{ik} x^i x^k = c^2 t^2$, centred at $\mathrm{O} = (ct, 0, 0, 0)$ and with radius ct.[4] Such a sphere can be seen as a projection of $\mathcal{C}_3$ on Σ, where it represents, for each t, the wave front of a light wave, emitted from $E = \Omega$ at $t = 0$. Furthermore, $\mathcal{C}_3$ separates the external (connected) part from the internal (not connected) one, made up by the two branches of the cone.

Let us consider, now, the set of all the timelike vectors $T \in M_4$. It can be easily shown that,[5] as for null vectors which were not aligned, *two timelike vectors cannot be orthogonal*; thus, for the timelike vectors $\boldsymbol{\gamma}$, $\boldsymbol{\gamma}'$, only the following two cases are allowed:

$$\text{(a) } \boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' > 0 \, , \qquad \text{(b) } \boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' < 0 \, .$$

As a consequence, once $\boldsymbol{\gamma}$ is fixed, the product $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' = m_{\alpha\beta} \gamma^\alpha \gamma'^\beta$ is a *continuous function* of the γ'^β, that is of $\boldsymbol{\gamma}'$. If $\boldsymbol{\gamma}'$ varies in a connected region of T, because of the theorem on the zeros of continuous functions, the sign of $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}'$ should remain unchanged; in other words, it is not possible to have, in such a domain, $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' > 0$ and $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' < 0$: this, in fact, would imply $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' = 0$ for a certain $\boldsymbol{\gamma}'$, which is impossible. Therefore, all the vectors $\boldsymbol{\gamma}'$ such that $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' > 0$, or $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' < 0$, should belong to not connected regions of T, i.e. the two (internal) branches of the lightcone. On the other hand, if $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ belong to the *same branch of the lightcone*, one has $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' < 0$ because $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} < 0$. In different words, *two vectors in the internal part of a branch always have negative product.*

Assuming $\boldsymbol{\gamma} = \mathbf{c}_0$ and $\boldsymbol{\gamma}' = \boldsymbol{\gamma}$, we have, by definition:

$$\begin{cases} \mathcal{C}_3^+ \equiv \{\boldsymbol{\gamma} \, : \, \mathbf{c}_0 \cdot \boldsymbol{\gamma} < 0\} & \text{positive half-cone.} \\ \\ \mathcal{C}_3^- \equiv \{\boldsymbol{\gamma} \, : \, \mathbf{c}_0 \cdot \boldsymbol{\gamma} > 0\} & \text{negative half-cone.} \end{cases}$$

The terminology positive or negative, of course, has not any intrinsic meaning and is introduced only to distinguish between the branches of the lightcone

[4] We notice that vectors on the hyperplane have components $(0, s^1, s^2, s^3)$, while vectors aligned with the x^0-axis have components $(\gamma^0, 0, 0, 0)$.

[5] Otherwise, an orthonormal basis containing two timelike vectors would exist, and the signature will be no more than +2. Moreover, the following property holds that *a vector* $\mathbf{v}$ *orthogonal to a timelike unit (without loss of generality) vector* $\boldsymbol{\gamma}$ is necessarily spacelike. In fact, assuming $\boldsymbol{\gamma} = \mathbf{c}_0$ and $\mathbf{v} = v^\alpha \mathbf{c}_\alpha$, it follows $\mathbf{v} \cdot \boldsymbol{\gamma} = -v^0$. Thus, the hypothesis $\mathbf{v} \perp \boldsymbol{\gamma}$ implies $v^0 = 0$ and hence $\mathbf{v} = v^i \mathbf{c}_i$, so that $||\mathbf{v}|| = \delta_{ik} v^i v^k > 0$. Finally, it should be noted that *a timelike vector and a spacelike one need not be orthogonal.* For example, this is the case for $\boldsymbol{\gamma} = \mathbf{c}_0$ and $\mathbf{s} = -\boldsymbol{\gamma} + 2\mathbf{c}_1$ for which $\mathbf{s} \cdot \boldsymbol{\gamma} = 1$.

which form the two connected parts of T; alternatively, the definition of the branches does not depend on the choice of $\mathbf{c}_0$.

The set of all the spacelike vectors, $S \equiv \{\mathbf{s} : \mathbf{s}\cdot\mathbf{s} > 0\}$, is instead *connected*; in fact, all the cases

$$\mathbf{s}\cdot\mathbf{s}' > 0 \,, \quad \mathbf{s}\cdot\mathbf{s}' = 0 \,, \quad \mathbf{s}\cdot\mathbf{s}' < 0$$

are possible.

A picture of all the vectors in M_4 is obtained by considering, in Σ, a sphere σ centred at Ω and of unit radius, as well as all the vectors $\mathbf{s} \in \Sigma$. Each $\mathbf{s}$ can be associated to a vector $\mathbf{V} = \boldsymbol{\gamma} + \mathbf{s}$, which is timelike, null or spacelike, when $s < 1$, $s = 1$, $s > 1$, respectively. From this follows the meaning of the sphere σ, which characterizes (through $\boldsymbol{\gamma}$) all the vectors of M_4: internal, on the boundary, or external to σ.

Therefore, it follows that M_4 *is a homogeneous Euclidean space, but it is not isotropic* even if there exist no privileged points in M_4, there are, clearly, preferred directions; this is different from all the three-dimensional sections Σ, associated with inertial frames, which are all homogeneous and isotropic (see axiom 2 of B).

In each of the three different classes of vectors, the directions (timelike, null and spacelike) are equivalent, that is geometrically not distinguishable, in the sense that one can pass from one to another with a 4-rotation. In other words, 4-rotations do not change the type of vectors and leave unchanged the lightcone $\mathcal{C}_3$, as well as its two branches: $\mathcal{C}_3^+$ and $\mathcal{C}_3^-$.

2.6 Elements of the Geometry of Minkowski Space-time

After this discussion of the one-dimensional subspaces of M_4, let us consider two-dimensional subspaces. A two-dimensional subspace is defined by the linear combinations of two independent vectors:

$$E_2 \equiv \{\mathbf{v} : \mathbf{v} = \lambda\mathbf{v}_1 + \mu\mathbf{v}_2\}_{\lambda,\mu\in\Re},$$

or, briefly $E_2 \equiv <\mathbf{v}_1, \mathbf{v}_2>$. There exist three different types:

1 *Elliptic subspaces.* They are formed by spacelike vectors only.
For example: $<\mathbf{c}_1, \mathbf{c}_2>$; in fact, for any vector $\mathbf{v} \in <\mathbf{c}_1, \mathbf{c}_2>$, one has

$$(\lambda\mathbf{c}_1 + \mu\mathbf{c}_2)\cdot(\lambda\mathbf{c}_1 + \mu\mathbf{c}_2) = \lambda^2 + \mu^2 \geq 0 \,,$$

with the equality valid only for the trivial case $\lambda = 0$, $\mu = 0$.

2 *Hyperbolic subspaces.* They are formed by vectors of all the three kinds and contain two null (or isotropic) directions.
An example is given by $<\mathbf{c}_0, \mathbf{c}_1>$, and it has already been discussed.

3 *Parabolic subspaces.* They are formed by a null direction $\mathbf{u}$ and by spatial vectors only, which are all orthogonal to $\mathbf{u}$; in fact, if the subspace is $< \mathbf{u}, \mathbf{w} >$, one can always assume $\mathbf{w}$ orthogonal to $\mathbf{u}$: $\mathbf{w} \cdot \mathbf{u} = 0$. Thus, for any vector $\mathbf{v}$ in the subspace, $\mathbf{v} = \lambda \mathbf{u} + \mu \mathbf{w}$, one gets

$$\mathbf{v} \cdot \mathbf{v} = \mu^2 w^2 \geq 0 \,, \quad \mathbf{v} \cdot \mathbf{u} = \mu \mathbf{w} \cdot \mathbf{u} = 0 \,,$$

with the equality valid only if $\mu = 0$, or $\mathbf{v} = \lambda \mathbf{u}$; from this follows the existence of a null direction, with all the other directions being spacelike and orthogonal to the null one.

An example is given by $< \mathbf{c}_0 + \mathbf{c}_1, \mathbf{c}_2 >$, with $\mathbf{c}_0 + \mathbf{c}_1$ a null vector.

Intuitively, one can image a parabolic subspace as the limit of a hyperbolic one, when the two isotropic directions collapse into a single one. More precisely, a hyperbolic subspace is divided, by the two isotropic straight lines, r_1 and r_2, into four not connected parts, two of which contain timelike vectors and the other two contain spacelike vectors; each timelike direction, in turn, admits an orthogonal spacelike direction. When $r_2 \to r_1$, by varying the 2-plane containing Ω, each spacelike vector will become orthogonal to the null direction in which both r_1 and r_2 (as well as the two regions of negative norm vectors) will collapse.

Finally, it is possible to geometrically classify the two-dimensional subspaces (2-planes) of M_4, according to their intersections with the lightcone: these are two real distinct straight lines, two real and coinciding straight lines, or two complex directions, if the subspaces are hyperbolic, parabolic or elliptic, respectively.

The same classification occurs by considering the induced metric from the Minkowskian one, in each subspace, that is the 2-metric associated with the scalar product between vectors of a basis (even not orthogonal) in the same subspace. In fact, by a theorem on completion, every basis in a subspace is a part of a basis of M_4; hence, practically, one has to consider the minors g_{ik} $(i, k = 1, 2)$ of the full metric. From this point of view,

(i) *Elliptical and hyperbolic subspaces are regular* ($\det ||g_{ik}|| \neq 0$), *and with signature* $++$ *or* $-+$, *respectively.*

(ii) *Parabolic subspaces are singular* ($\det ||g_{ik}|| = 0$).

Thus, for the subspace $< \mathbf{c}_1, \mathbf{c}_2 >$ one has

$$\det ||g_{ik}|| = \det ||\mathbf{c}_i \cdot \mathbf{c}_k|| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \,,$$

and the signature is $++$; analogously, for the subspace $< \mathbf{c}_0, \mathbf{c}_1 >$ one has

$$\det ||g_{ik}|| = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \neq 0 \,,$$

and the signature is $-+$; finally, for the subspace $< \mathbf{c}_0 + \mathbf{c}_1, \mathbf{c}_2 >$ one has

$$\det ||g_{ik}|| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \ ,$$

and the induced metric is singular.

Let us consider, now, the three-dimensional linear subspaces of M_4, or 3-planes, or hyperplanes. They are spanned by three linear independent vectors, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$:

$$E_3 \equiv \{\mathbf{v} \, : \, \mathbf{v} = \lambda\mathbf{v}_1 + \mu\mathbf{v}_2 + \nu\mathbf{v}_3\}_{\lambda,\mu,\nu\in\Re} \equiv < \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 > \ .$$

From what has been stated above, it is clear that M_4 will admit the following:

1) *Elliptical hyperplanes* (with signature $+++$); they are formed by spatial vectors, apart for the vector zero.
   An example is given by $\Sigma \equiv < \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 >$.
2) *Hyperbolic hyperplanes* (with signature $-++$); they are formed by vectors of any kind, and they include a two-dimensional cone of null straight lines (see [1], p. 129).
   An example is given by $\Sigma \equiv < \mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2 >$. Here we have

   $$||\lambda\mathbf{c}_0 + \mu\mathbf{c}_1 + \nu\mathbf{c}_2|| = -\lambda^2 + \mu^2 + \nu^2 \ ,$$

   from which, when $\lambda^2 = \mu^2 + \nu^2$, one has ∞^1 null straight lines, that is an ordinary cone.
3) *Parabolic hyperplanes* (singular); for these, as for the parabolic 2-planes, the concept of signature is meaningless because they do not admit orthonormal bases (such bases will necessarily contain the null vector of the hyperplane). Actually, they contain *a null direction* $\mathbf{u}$ *and all the remaining vectors are spacelike.* Moreover, as in the two-dimensional case, *all the vectors are orthogonal to the null direction itself*; thus, this is the *hyperplane orthogonal to* $\mathbf{u}$.

In other words, *to any vector* $\mathbf{v} \in M_4$*, there corresponds, without exceptions, an orthogonal three-dimensional subspace.* This is disjoint from $\mathbf{v}$ (and regular: hyperbolic or elliptic) only when $||\mathbf{v}|| \neq 0$[6]; *when* $\mathbf{v}$ *is null, the normal subspace contains* $\mathbf{v}$ *and it is necessarily parabolic.*

For example, the hyperplane $< \mathbf{c}_0 + \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 >$ is parabolic; thus,

$$\det ||g_{ik}|| = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 \ ;$$

moreover, for any vector of the hyperplane: $\mathbf{v} = \lambda(\mathbf{c}_0 + \mathbf{c}_1) + \mu\mathbf{c}_2 + \nu\mathbf{c}_3$, whence

$$\mathbf{v}\cdot\mathbf{v} = \mu^2 + \nu^2 \geq 0 \ , \quad (\mathbf{c}_0 + \mathbf{c}_1)\cdot\mathbf{v} = 0 \ , \quad \forall\lambda,\mu,\nu \ .$$

[6] In any Euclidean space (proper or not), a linear subspace admits a unique supplemental and orthogonal subspace which is also nonsingular (see [1], p. 82).

2.7 Proper Time

The classification of vectors in M_4, and the lightcone, allows us to geometrically specify, first of all, the whole class of *Galilean frames*, and, in particular, all those frames having the same time orientation: the set of *orthochronous frames*. More precisely, for a generic frame, the unit timelike vector $\boldsymbol{\gamma} = \mathbf{c}_0$ (and, hence, the associated congruence of equi-oriented straight lines) can belong to either of the light half-cones. Orthochronous frames, instead, are characterized by vectors $\boldsymbol{\gamma}$ belonging to the *same branch* of the lightcone:

$$\forall \quad \boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \{\boldsymbol{\gamma}\} \quad \rightarrow \quad \boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' < 0 \,.$$

In the following, we will limit ourselves to orthochronous Galilean frames, the only ones for which the notions of present, past and future are meaningful; that is, we will assume M_4 endowed with only one of the two light half-cones, say $\mathcal{C}_3^+$. We will say, briefly, that M_4 is time oriented, and we will use the notation: $M_4(\mathcal{C}_3^+)$, or, equivalently, M_4^+.

The impossibility to distinguish, from a geometrical point of view, among the unit vectors $\boldsymbol{\gamma}$, clearly has its physical counterpart in the *equivalence of all the associated Galilean frames*, as has been postulated above.

Together with the orthochronous Galilean frames (each with its proper representative physical space, Σ, orthogonal to $\boldsymbol{\gamma}$, and hence elliptical), the light half-cone $\mathcal{C}_3^+$ gives geometrical consistence to the world lines $\ell^+ \equiv E_0E_1$. As we have already seen, they must have, at any point E, the tangent vector contained in $\mathcal{C}_3^+$: thus, necessarily, the whole world line belongs to the half-cone $\mathcal{C}_3^+$, having the vertex at E_0. $\mathcal{C}_3^+$ therefore characterizes all the events which can be connected with E_0 by means of a world line (a straight line or a curve): it is the *future* of E_0, or the geometrical horizon. The latter, as already stated, gives no rise to any physical horizon. In fact, if the event E_0 is characterized in S_g by the pair (P_0, t_0), all that happens at P_0, at the instant t_0, may influence what happens at each $\mathrm{P} \in S_\mathrm{g}$. In particular, a particle emitted suitably at P_0 can reach any $\mathrm{P} \in S_\mathrm{g}$.

Each world line can be parametrized by an *intrinsic parameter* τ, analogous (apart from the dimensions) to the ordinary curvilinear abscissa. It can be defined, indirectly, through the tangent vector condition of norm $-c^2$, namely,

$$\mathbf{V} = \frac{\mathrm{d}\Omega E}{\mathrm{d}\tau} \equiv \left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \right) , \tag{2.33}$$

with

$$\mathbf{V} \cdot \mathbf{V} = -c^2 < 0 \,. \tag{2.34}$$

Comparing with a generic parametrization λ of the same world line,

$$\mathbf{V} = \boldsymbol{\mathcal{V}} \frac{\mathrm{d}\lambda}{\mathrm{d}\tau} \,, \quad \text{with} \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}\tau} > 0 \quad \text{and} \quad \boldsymbol{\mathcal{V}} \stackrel{\text{def}}{=} \frac{\mathrm{d}\Omega E}{\mathrm{d}\lambda} \,,$$

it follows that the condition (2.34) becomes

$$\boldsymbol{\mathcal{V}}\cdot\boldsymbol{\mathcal{V}}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2=-c^2 \quad\rightarrow\quad \frac{\mathrm{d}\tau}{\mathrm{d}\lambda}=\pm\frac{1}{c}\sqrt{-\boldsymbol{\mathcal{V}}\cdot\boldsymbol{\mathcal{V}}}\,.$$

As τ is one of the admissible parameters for the oriented world line ℓ^+, we have $\mathrm{d}\tau/\mathrm{d}\lambda>0$. So (2.34) is equivalent to the first-order differential condition:

$$\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}=\frac{1}{c}\sqrt{-m_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}}\,. \tag{2.35}$$

This condition defines the parameter τ up to an additive constant, as soon as the parametric equations $x^\alpha=x^\alpha(\lambda)$ of the curve ℓ^+ are known:

$$\tau=\tau_0+\frac{1}{c}\int_{E_0}^{E}\sqrt{-m_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda\,. \tag{2.36}$$

τ(E) is invariant with respect to the (completely free) choice of the parameter λ on ℓ^+, as is clear from (2.35):

$$\mathrm{d}\tau=\frac{1}{c}\sqrt{-m_{\alpha\beta}\;\mathrm{d}x^\alpha\;\mathrm{d}x^\beta}\,. \tag{2.37}$$

This is an *absolute quantity*, defined on the world line, and with the dimensions of a time. It is called *proper time* of the particle associated with the world line ℓ^+, and it is proportional to the curvilinear abscissa, by a factor of c: $\mathrm{d}s=c\,\mathrm{d}\tau$. In particular, if one assumes that a Galilean frame is fixed in M_4, and t is the associated relative time, one can put $\lambda=t$, so that (2.35) gives a relation between τ and the ordinary relative velocity of the particle in the Galilean frame under consideration:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t}=\sqrt{1-\frac{v(t)^2}{c^2}}\,,\qquad v(t)^2=\delta_{ik}v^iv^k\,. \tag{2.38}$$

In order to find the physical meaning of the proper time let us distinguish between the case in which the particle world line is a straight line or has curvature. In the first case, $\mathbf{V}$=const., there exists a unique Galilean frame which is the particle's rest frame. This frame is defined by the vector $\boldsymbol{\gamma}=\mathbf{V}/c$ and hence by the timelike congruence of straight lines with the same orientation of ℓ^+. In such a frame, $\mathbf{v}=\mathbf{0}\;\forall t$, so that (2.38) gives $t=\tau$, up to an unessential additive constant: *the particle's proper time coincides with that measured by a standard clock of the Galilean rest frame.*[7]

In the second case, instead, a Galilean rest frame for the particle does not exist, and all that has been said for the rectilinear cases loses its global character. That is, $\forall E\in\ell^+$, $\mathbf{V}(E)$ still defines a Galilean frame, characterized by the unit timelike vector $\boldsymbol{\gamma}=\mathbf{V}(E)/c$ and, even in this case, $(\mathrm{d}\tau/\mathrm{d}t)_E=1$, or $\mathrm{d}\tau=\mathrm{d}t$, but only at the event E. In other words, this is the *instantaneous*

[7] On the platform Σ orthogonal to $\boldsymbol{\gamma}$, the particle's position is always the same.

rest frame of the particle, which depends on the point E considered on the particle's world line.

The introduction of the proper time illustrates the well-known *twin paradox.* Let us assume that a pair of twins (i.e. two material points), move away from a common position P_0 and at the same instant t_0, and hence, from the same event E_0; besides, let us assume that the first will maintain its initial velocity, and that the other will accelerate, until a re-meeting event E_1 is reached. In E_1, i.e. when their world lines intersect again, for the second twin, there has elapsed a lesser quantity of proper time; that is, he is younger than the other. In fact, for the first twin, there exists a Galilean frame S_g such that

$$v_1 = 0 \qquad \forall t \in (t_0, t_1), \rightarrow \; \tau_1 = t_1 - t_0 \; .$$

For the second twin, *in the same Galilean frame*, we have

$$\tau_2 = \int_{\ell_2} \mathrm{d}\tau_2 = \int_{t_0}^{t_1} \sqrt{1 - \frac{v_2^2}{c^2}}\, \mathrm{d}t < \int_{t_0}^{t_1} \mathrm{d}t = t_1 - t_0 \; ,$$

i.e. $\tau_2 < \tau_1$, which completes the proof.

2.8 Test Particle Kinematics (Absolute and Relative)

To the absolute parameter τ corresponds the absolute kinematics of the material point, through the fundamental notions of 4-*velocity* $\mathbf{V}$ and 4-*acceleration* $\mathbf{A}$:

$$\mathbf{V} = \frac{\mathrm{d}\Omega E}{\mathrm{d}\tau} \equiv \left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\right) , \qquad \mathbf{A} = \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}\tau} = \frac{\mathrm{d}^2\Omega E}{\mathrm{d}\tau^2} \equiv \left(\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2}\right) . \tag{2.39}$$

Apart from their names, which are clear in the relative context, $\mathbf{V}$ and $\mathbf{A}$ have also a clear geometrical meaning. The first one, $\mathbf{V}$, is a tangent vector to the world line ℓ^+, and the second one, $\mathbf{A}$, is orthogonal to the world line, as it follows from differentiating (2.34):

$$\mathbf{V} \cdot \mathbf{A} = 0 \; . \tag{2.40}$$

As a consequence, $\mathbf{A}$ belongs to the spacelike platform of the instantaneous rest frame; differently from $\mathbf{V}$ which is timelike, the 4-acceleration is a spacelike vector, i.e. with positive norm:

$$\mathbf{A} \cdot \mathbf{A} > 0 \; . \tag{2.41}$$

The ratio $\mathbf{T} = \mathbf{V}/c$, because of (2.34), is a unit timelike vector, that is the *unit tangent vector to* ℓ^+. Thus, if s denotes the curvilinear abscissa on ℓ^+

$$\mathbf{T} \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}\Omega E}{\mathrm{d}s} \equiv \left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\right) , \tag{2.42}$$

then the relation between $\mathbf{V}$ and $\mathbf{T}$ becomes

$$\mathbf{V} = c\mathbf{T} , \tag{2.43}$$

which is also equivalent to (2.35):

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = c \quad \rightarrow \quad s = c\tau + \mathrm{const} \tag{2.44}$$

Analogously, by differentiating (2.43) with respect to τ, one has

$$\mathbf{A} = c^2\mathbf{C} , \tag{2.45}$$

where $\mathbf{C}$ is the *spacelike curvature vector* of the world line ℓ^+ in E:

$$\mathbf{C} \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} . \tag{2.46}$$

Equations (2.43) and (2.45) specify the geometrical meaning of the vectors $\mathbf{V}$ and $\mathbf{A}$ with respect to the world line ℓ^+. The kinematical meaning of these vectors will be evident once, in M_4, a Galilean frame S_g is fixed by the characteristic vector $\boldsymbol{\gamma}$ and the associated spacelike platform Σ (the physical space).

Let t be the coordinate time of the Galilean frame (with $x^0 = ct$), and let P be the point orthogonal projection of $E \in \ell^+$ on Σ. The following decomposition holds: $\Omega E = \Omega \mathrm{P} + x^0\boldsymbol{\gamma}$; by differentiating this relation with respect to the proper time, one has

$$\mathbf{V} = \left(\frac{\mathrm{d}\,\Omega\mathrm{P}}{\mathrm{d}t} + \frac{\mathrm{d}x^0}{\mathrm{d}t}\boldsymbol{\gamma} \right) \frac{\mathrm{d}t}{\mathrm{d}\tau} . \tag{2.47}$$

Thus, taking into account (2.38), and introducing the notation (*Lorentz factor*):

$$\eta = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} , \tag{2.48}$$

one gets

$$\mathbf{V} = \eta(\mathbf{v} + c\boldsymbol{\gamma}) , \tag{2.49}$$

with $\mathbf{v}$ the particle's *relative velocity* (in S_g):

$$\mathbf{v} = \frac{\mathrm{d}\Omega\mathrm{P}}{\mathrm{d}t} . \tag{2.50}$$

Similarly to (2.49), after a further differentiation, and using the relation:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \eta^3 \frac{\mathbf{v}\cdot\mathbf{a}}{c^2} ,$$

it follows that

$$\mathbf{A} = \eta^2 \left[\mathbf{a} + \frac{\eta^2}{c^2}(\mathbf{v}\cdot\mathbf{a})\,(\mathbf{v}+c\boldsymbol{\gamma}) \right] , \tag{2.51}$$

with $\mathbf{a}$ the *relative acceleration* (in S_{g}):

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\Omega \mathrm{P}}{\mathrm{d}t^2} . \tag{2.52}$$

The relations (2.49) and (2.51) give the relative form, in S_{g}, of the 4-*velocity* and the 4-*acceleration*, and the decomposition has an invariant meaning with respect to the choice of S_{g} due to the absolute character of $\mathbf{V}$ and $\mathbf{A}$:

$$\eta(\mathbf{v}+c\boldsymbol{\gamma}) = \eta'(\mathbf{v}'+c\boldsymbol{\gamma}') = \mathrm{inv.} \tag{2.53}$$

$$\eta^2 \left[\mathbf{a} + \frac{\eta^2}{c^2}(\mathbf{v}\cdot\mathbf{a})\,(\mathbf{v}+c\boldsymbol{\gamma}) \right] = \eta'^2 \left[\mathbf{a}' + \frac{\eta'^2}{c^2}(\mathbf{v}'\cdot\mathbf{a}')\,(\mathbf{v}'+c\boldsymbol{\gamma}') \right] = \mathrm{inv.}$$

Conversely, from (2.49), by using the orthogonality between $\mathbf{v}$ and $\boldsymbol{\gamma}$ and the fact that $\boldsymbol{\gamma}$ is a unit timelike vector, one finds $\mathbf{V}\cdot\boldsymbol{\gamma} = -\eta c$, or

$$\eta = -\frac{1}{c}\mathbf{V}\cdot\boldsymbol{\gamma} , \tag{2.54}$$

so that (2.49) allows us to obtain the expression of $\mathbf{v}$ in terms of $\mathbf{V}$ and $\boldsymbol{\gamma}$, namely,

$$\mathbf{v} = -c\left(\boldsymbol{\gamma} + \frac{\mathbf{V}}{(\mathbf{V}\cdot\boldsymbol{\gamma})}\right) . \tag{2.55}$$

Analogously, from (2.51), one can obtain the relative acceleration $\mathbf{a}$. First of all one has

$$\mathbf{A}\cdot\boldsymbol{\gamma} = -\frac{\eta^4}{c}(\mathbf{v}\cdot\mathbf{a}),$$

and then, from (2.51), one gets

$$\mathbf{a} = \frac{\mathbf{A}}{\eta^2} - \frac{\eta}{c^2}(\mathbf{v}\cdot\mathbf{a})\mathbf{V} = \frac{1}{\eta^2}\left(\mathbf{A} + \frac{1}{\eta c}\right)(\mathbf{A}\cdot\boldsymbol{\gamma})\mathbf{V} ,$$

or, explicitly, as from (2.54),

$$\mathbf{a} = \left(\frac{c}{\mathbf{V}\cdot\boldsymbol{\gamma}}\right)^2 \left(\mathbf{A} - \frac{\mathbf{A}\cdot\boldsymbol{\gamma}}{\mathbf{V}\cdot\boldsymbol{\gamma}}\mathbf{V}\right) . \tag{2.56}$$

The same result should, obviously, be obtained by differentiating (2.55) with respect to t.

The relations (2.49) and (2.51), as well as their inverses (2.55) and (2.56), have a *general character*, either as concerns the pointlike motion or for the Galilean frame. In particular, in the proper Galilean frame S_{g}^0, defined by $\boldsymbol{\gamma} = \mathbf{V}/c$, (2.55) and (2.56) give $\mathbf{v}^0 = 0$ and $\mathbf{a}^0 = \mathbf{A}$; this specifies the kinematical meaning of the 4-acceleration (confirming also its spatial character).

2.9 Lorentz Transformations

We have already stated that the essential features of a Galilean frame, in M_4, are summarized by a unit timelike vector field $\boldsymbol{\gamma}$:

$$\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} = m_{\alpha\beta}\gamma^\alpha \gamma^\beta = -1 \,, \tag{2.57}$$

or by the product $c\boldsymbol{\gamma}$, which represents the 4-velocity of all the particles of the associated reference solid. The congruence of the ∞^3 straight lines (covering the whole Minkowski space), aligned and oriented according to $\boldsymbol{\gamma}$, represents the history of the Galilean frame. Once the origin of the frame Ω is fixed, the straight line passing through Ω, and directed along $\boldsymbol{\gamma}$, is the *temporal axis* of the frame, while the orthogonal subspace through Ω defines the *physical space* (at $t = 0$) of the frame itself; in other words, the space platform of the frame, denoted by Σ.

In the three-dimensional space Σ (signature $+ + +$), one can obviously introduce Cartesian coordinates, or more general internal coordinate systems (polar, cylindrical, etc.).

The passage from *absolute quantities* (in M_4) to their *relative* counterparts (in $S_{\rm g}$) is obtained by spatial (orthogonal) projection on Σ, using the projection operator

$$P_\Sigma = \mathbb{I} + \boldsymbol{\gamma} \otimes \boldsymbol{\gamma} \qquad \sim \qquad {P_\Sigma}^\alpha_\beta = \delta^\alpha_\beta + \gamma^\alpha \gamma_\beta \,. \tag{2.58}$$

In this sense, if ℓ^+ is the world line of a material particle, the relative trajectory on Σ is the projection of ℓ^+ orthogonally to $\boldsymbol{\gamma}$. Such a trajectory clearly depends on the selected Galilean frame.

Analogously, (2.49) and (2.55) and their inverses, (2.51) and (2.56), represent the relations between the local (absolute and relative) kinematical characteristics: velocity and acceleration, respectively, once decomposed along $\boldsymbol{\gamma}$ and Σ. These relations are intrinsic (i.e. only $\boldsymbol{\gamma}$ is needed), and the coordinate system is still at disposal.

Let us study, now, the passage from one Galilean frame to another. To this end let us consider, in M_4, two Galilean frames: $S_{\rm g}$ and $S'_{\rm g}$, characterized by unit timelike vector fields $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$, respectively ($\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \mathcal{C}_3^+$):

$$\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' < 0 \,, \quad \boldsymbol{\gamma} \cdot \boldsymbol{\gamma} = -1 \,, \quad \boldsymbol{\gamma}' \cdot \boldsymbol{\gamma}' = -1 \,. \tag{2.59}$$

The vector $c\boldsymbol{\gamma}'$ is the 4-velocity of any of the particles of $S'_{\rm g}$; in $S_{\rm g}$ this has the following decomposition, according to (2.49):

$$c\boldsymbol{\gamma}' = \rho(\mathbf{u} + c\boldsymbol{\gamma}) \,, \tag{2.60}$$

where $\mathbf{u} \in \Sigma$ is the translational velocity of $S'_{\rm g}$ with respect to $S_{\rm g}$ and, from (2.48) and (2.54),

$$\rho = \frac{1}{\sqrt{1 - u^2/c^2}} = -\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' \,. \tag{2.61}$$

Let us change, now, the role of the two Galilean frames, in (2.60) and (2.61). This is equivalent to change unprimed quantities with primed ones and vice versa; so, by using the fundamental property of the light velocity $c' = c$; we find

$$c\boldsymbol{\gamma} = \rho'(\mathbf{u}' + c\boldsymbol{\gamma}') , \tag{2.62}$$

and hence

$$\rho' = \frac{1}{\sqrt{1 - u'^2/c^2}} = -\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' , \tag{2.63}$$

where now $\mathbf{u}' \in \Sigma'$ is the translational velocity of S_g with respect to S'_g. By comparing (2.61) and (2.63), one has $\rho' = \rho$, so that the so-called *reciprocity lemma* follows:

$$u' = u ; \tag{2.64}$$

that is, *the relative speed of* S_g *with respect to* S'_g *coincides with that of* S_g *with respect to* S_g, and it does not depend on the order in which the two frames are considered.

Moreover, (2.62) allows us to obtain the decomposition of $\mathbf{u}'$ along $\boldsymbol{\gamma}$ and Σ, similar to (2.60) for $\boldsymbol{\gamma}'$. One has

$$c\boldsymbol{\gamma} = \rho'(\mathbf{u}' + c\boldsymbol{\gamma}') = \rho\mathbf{u}' + \rho^2(\mathbf{u} + c\boldsymbol{\gamma}) ,$$

from which, using $1 - \rho^2 = -\rho^2 u^2/c^2$, one finds

$$\mathbf{u}' = -\rho\left(\mathbf{u} + \frac{u^2}{c}\boldsymbol{\gamma}\right) .$$

Summarizing, in M_4, the following transformation laws, associated with the two Galilean frames S_g and S'_g, hold:

$$c\boldsymbol{\gamma}' = \rho(\mathbf{u} + c\boldsymbol{\gamma}), \qquad -\mathbf{u}' = \rho\left(\mathbf{u} + \frac{u^2}{c}\boldsymbol{\gamma}\right) . \tag{2.65}$$

In (2.65), we see either the absolute character of the two Galilean frames ($\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$) or the relative character of the apparent translational velocities $\mathbf{u}$ and $\mathbf{u}'$. These last constant vectors represent two well-determined directions in Σ and Σ' (corresponding to apparent motions).

Thus, as in the case of the special Galilean transformations, the presence of $\mathbf{u}$ and $\mathbf{u}'$ in (2.65) suggests the introduction, in both the platforms Σ and Σ', of (congruent) Cartesian *triads, in standard direction*. We then assume $\mathbf{c}_1 = \mathrm{vers}\,\mathbf{u}$ so that S'_g will appear (in S_g) as moving along the x^1 direction:

$$\mathbf{u} = u\mathbf{c}_1 ; \tag{2.66}$$

$\mathbf{c}_2$ and $\mathbf{c}_3$ will be chosen so that they form, with $\mathbf{c}_1$, an orthogonal left-handed triad. Similarly, in Σ', we can use the triad[8]:

[8] According to a standard notation, we will denote indifferently $\mathbf{c}'_\alpha$ or $\mathbf{c}_{\alpha'}$.

$$\mathbf{c}'_1 = -\mathrm{vers}\,\mathbf{u}' \,, \quad \rightarrow \quad \mathbf{u}' = -u\mathbf{c}'_1 \,, \quad \mathbf{c}'_{2,3} = \mathbf{c}_{2,3} \,. \tag{2.67}$$

The choice is consistent: in fact, from (2.65), $\mathbf{c}_2$ and $\mathbf{c}_3$ are orthogonal to both $\mathbf{c}_1$ and $\boldsymbol{\gamma}$ (and hence to the 2-plane $< \boldsymbol{\gamma}, \mathbf{u} >$), and $\mathbf{c}'_2 = \mathbf{c}_2$ and $\mathbf{c}'_3 = \mathbf{c}_3$ are orthogonal to both $\mathbf{c}'_1$ and $\boldsymbol{\gamma}'$, which belong to the previous 2-plane due to (2.65).

Moreover, the two orthonormal bases $\{\boldsymbol{\gamma}, \mathbf{c}_i\}$ and $\{\boldsymbol{\gamma}', \mathbf{c}'_i\}$ are equi-oriented and can be superposed by using a rotation because the same property holds for the pairs $(\boldsymbol{\gamma}, \mathbf{u})$ and $(\boldsymbol{\gamma}', -\mathbf{u}')$. In fact, the determinant of the transformation (2.65) is +1:

$$\det \begin{Vmatrix} \rho & \dfrac{\rho}{c} \\ \dfrac{\rho u^2}{c} & \rho \end{Vmatrix} = \rho^2(1 - u^2/c^2) = 1 \,.$$

Thus, (2.65) gives the following transformation laws:

$$\begin{cases} \boldsymbol{\gamma}' = \rho(\boldsymbol{\gamma} + \beta\mathbf{c}_1) \,, \quad \beta = u/c \,, \qquad \rho = 1/\sqrt{1-\beta^2}, \\ \mathbf{c}'_1 = \rho(\mathbf{c}_1 + \beta\boldsymbol{\gamma}), \quad \mathbf{c}'_{2,3} = \mathbf{c}_{2,3}, \end{cases} \tag{2.68}$$

where, even if the two Galilean frames are completely arbitrary, the unit vectors $\mathbf{c}_1$ and $\mathbf{c}'_1$ have a precise kinematical meaning, because they characterize the relative motion directions of the two frames:

$$\mathbf{u} = u\mathbf{c}_1 \,, \qquad \mathbf{u}' = -u\mathbf{c}'_1 \,. \tag{2.69}$$

In other words, (2.68) implies the following transformation matrix $||L^\alpha{}_\beta||$:

$$\begin{Vmatrix} \rho & \rho\beta & 0 & 0 \\ \rho\beta & \rho & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \,, \tag{2.70}$$

and they correspond to the special Galilean transformations. For the generic event $E \in M_4$, (2.68) gives rise to the two coordinate representations: $\Omega E = x^\alpha \mathbf{c}_\alpha = x'^\alpha \mathbf{c}'_\alpha$, that is, explicitly,

$$x^0\boldsymbol{\gamma} + x^1\mathbf{c}_1 + x^2\mathbf{c}_2 + x^3\mathbf{c}_3 = x'^0\rho(\boldsymbol{\gamma} + \beta\mathbf{c}_1) + x'^1\rho(\mathbf{c}_1 + \beta\boldsymbol{\gamma}) + x'^2\mathbf{c}_2 + x'^3\mathbf{c}_3 \,.$$

From this, one gets the *direct* relations:

$$x^0 = \rho(x'^0 + \beta x'^1) \,, \quad x^1 = \rho(x'^1 + \beta x'^0), \quad x^{2,3} = x'^{2,3} \,, \tag{2.71}$$

and the inverse:

$$x'^0 = \rho(x^0 - \beta x^1) \,, \quad x'^1 = \rho(x^1 - \beta x^0), \quad x'^{2,3} = x^{2,3} \,; \tag{2.72}$$

because of the reciprocity lemma, the two sets of relations can be obtained, one from the other, by exchanging primed and unprimed quantities and β with $-\beta$.

Equations (2.71) and (2.72) are the x^1-*standard* (*homogeneous*) *special Lorentz transformations*. The inhomogeneous transformations correspond to a choice of the origin Ω' different from Ω, and thus, they differ by a constant translation in M_4 only.

As $\rho = \rho(\beta)$, the transformations (2.71) depend on the single parameter β: $0 \leq \beta < 1$, and they form a *connected group*, $\mathcal{L}_1$. In fact, they contain the identical transformation (for $\beta = 0$), the inverse (for $\beta \to -\beta$), as well as the product of any two transformation, with

$$\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'} \,. \tag{2.73}$$

These are special rotations (of M_4) around Ω: they leave unchanged the 2-plane $<\boldsymbol{\gamma}, \boldsymbol{\gamma}'>$, as well as all the vectors in the orthogonal 2-plane $\Sigma \cap \Sigma'$, which plays the role of a rotation axis.

As we have already stated, also the group of the general rotations around Ω, without any special choice of the orthonormal basis vectors, form a group: the six *parameter group of the homogeneous Lorentz transformations*, which will be considered in the following section (see e.g. [2] for a structural analysis of the Lorentz group). We notice here that these transformations will be obtained starting from an orthonormal basis $\mathbf{c}_\alpha$, once there are assigned the vectors $\boldsymbol{\gamma}$ and $\mathbf{u}$, satisfying the constraints $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} = -1$ and $\boldsymbol{\gamma} \cdot \mathbf{u} = 0$. These are only two conditions for the eight variables at disposal, i.e. the components of the two vectors $\boldsymbol{\gamma}$ and $\mathbf{u}$ along the selected orthonormal basis $\{\mathbf{c}_\alpha\}$; therefore in the case $\Omega = \Omega'$ only six more free parameters remain. Otherwise, the free parameters become $6 + 4 = 10$ (*inhomogeneous Lorentz group, or Poincaré group*).

2.10 General Lorentz Transformations: I

In the transformation laws (2.68), general as concerns the Galilean frames S_g and S'_g, the triads $\mathcal{T} \in \Sigma$ and $\mathcal{T}' \in \Sigma'$ are very special because of the x^1-standard relation: $\mathbf{u} = u\mathbf{c}_1$, $\mathbf{u}' = -u\mathbf{c}'_1$. This restriction, clearly, is not essential, because, using (2.68), the general case, in which $\mathcal{T} \in \Sigma$ and $\mathcal{T}' \in \Sigma'$ are arbitrarily chosen, can be obtained. To perform this extension, we proceed as follows:

1. *Choice of an arbitrary triad $\mathcal{T}$ in S_g*

   The problem is to rewrite (2.68) in a symmetric form, cancelling the special property of $\mathbf{c}_1$ (with respect to $\mathbf{c}_{2,3}$) due to the alignment with the relative velocity $\mathbf{u}$: $\mathbf{u} = u\mathbf{c}_1$ ($u^i = u\delta^i_1$). To this end, let us write $\rho\mathbf{c}_1$ in the form $\mathbf{c}_1 + \cdots$, using the following identity:

$$\rho = 1 + (\rho - 1) = 1 + \frac{\rho^2 - 1}{\rho + 1} = 1 + \frac{\rho^2}{1 + \rho}\left(1 - \frac{1}{\rho^2}\right) \,,$$

or

$$\rho = 1 + \frac{u^2}{c^2}\frac{\rho^2}{1+\rho} = 1 + \frac{uu_1}{c^2}\frac{\rho^2}{1+\rho} \ . \tag{2.74}$$

It follows that

$$\rho \mathbf{c}_1 = \mathbf{c}_1 + \frac{u_1}{c^2}\frac{\rho^2}{1+\rho}\mathbf{u} \ ,$$

and, from $(2.68)_2$:

$$\mathbf{c}'_1 = \mathbf{c}_1 + \frac{\rho}{c^2}\left(\frac{\rho}{1+\rho}\mathbf{u} + c\boldsymbol{\gamma}\right)\mathbf{u}\cdot\mathbf{c}_1 \ .$$

Thus, (2.68) assumes the form (general as concerns the choice of $\mathcal{T} \in S_{\mathrm{g}}$):

$$\boldsymbol{\gamma}' = \rho\left(\boldsymbol{\gamma} + \frac{1}{c}\mathbf{u}\right) \ , \qquad \mathbf{c}'_i = \mathbf{c}_i + \frac{\rho}{c^2}\left(\frac{\rho}{1+\rho}\mathbf{u} + c\boldsymbol{\gamma}\right)\mathbf{u}\cdot\mathbf{c}_i \ , \tag{2.75}$$

where now

$$\mathbf{u} = u^i\mathbf{c}_i \ , \qquad \rho = \frac{1}{\sqrt{1 - \frac{1}{c^2}\delta_{ik}u^iu^k}} \ . \tag{2.76}$$

The next step is now

2. *Choice of an arbitrary triad $\mathcal{T}'$ in S'_{g}*

The problem is that of replacing the preferred triad $\mathbf{c}'_i$ in (2.75), with an arbitrary triad in Σ'. For this, it is enough to perform, in Σ', a spatial rotation, using an ordinary orthogonal matrix $\mathcal{R}^i{}_k$. Denoting still by $\mathbf{c}'_i$ the rotated triad, we have, for the most general change of orthonormal bases in M_4:

$$\begin{cases} \boldsymbol{\gamma}' = \rho\left(\boldsymbol{\gamma} + \frac{1}{c}\mathbf{u}\right) , \\ \mathbf{c}'_k = \mathcal{R}^i{}_k\left[\mathbf{c}_i + \frac{\rho}{c^2}\left(\frac{\rho}{1+\rho}\mathbf{u} + c\boldsymbol{\gamma}\right)\mathbf{u}\cdot\mathbf{c}_i\right] \qquad (k = 1,2,3). \end{cases} \tag{2.77}$$

In (2.77), there are (implicitly) six independent parameters: the three components u^i (or $u_i = \mathbf{u}\cdot\mathbf{c}_i = \delta_{ik}u^k$) of the relative velocity of S'_{g} with respect to S_{g}, and three parameters for the matrix $\mathcal{R}^i{}_k$ (e.g. the Euler angles or the Rodriguez parameters, see e.g. [3], p. 113). In terms of coordinates of the generic event $E \in M_4$: $x^0\boldsymbol{\gamma} + x^i\mathbf{c}_i = x'^0\boldsymbol{\gamma}' + x'^i\mathbf{c}'_i$, (2.77) give the *general homogeneous Lorentz transformation*:

$$\begin{cases} x^0 = \rho\left(x'^0 + \frac{1}{c}x'^k\mathcal{R}^i{}_k u_i\right) \\ x^i = \frac{\rho}{c}u^ix'^0 + x'^k\mathcal{R}^{\mathrm{h}}_k\left(\delta^i_k + \frac{1}{c^2}\frac{\rho^2}{1+\rho}u_hu^i\right) \quad (i = 1,2,3). \end{cases} \tag{2.78}$$

In the limit $c \to \infty$, one re-obtains the general homogeneous Galilei transformation (1.12). Equation (2.78) can be inverted easily. Using the notation

$$y^i = \mathcal{R}^i{}_k x'^k \;, \qquad y = y^i u_i \;, \tag{2.79}$$

Equation (2.78) become:

$$x^0 = \rho\left(x'^0 + \frac{1}{c}y\right) \;, \quad x^i = y^i + \frac{\rho}{c}u^i x'^0 + \frac{1}{c^2}\frac{\rho^2}{1+\rho}y u^i \;. \tag{2.80}$$

We will now derive the corresponding expressions for x'^0 and y^i, from which the x'^k will follow immediately. Let us start obtaining y; from $(2.80)_2$, using (2.74), one gets

$$u_i x^i = y + \frac{\rho}{c}u^2 x'^0 + \frac{u^2}{c^2}\frac{\rho^2}{1+\rho}y = \rho y + \frac{\rho}{c}u^2 x'^0 \;,$$

or

$$y = \frac{1}{\rho}u_i x^i - \frac{u^2}{c}x'^0 \;. \tag{2.81}$$

This relation allows us to cast $(2.80)_1$ in the form

$$x^0 = \rho x'^0 + \frac{1}{c}u_i x^i - \frac{u^2}{c^2}\rho x'^0 = \frac{1}{\rho}x'^0 + \frac{1}{c}u_i x^i \;,$$

so that

$$x'^0 = \rho\left(x^0 - \frac{1}{c}u_i x^i\right) \;. \tag{2.82}$$

From (2.81), one obtains the expression for y:

$$y = \frac{u_i x^i}{\rho} - \frac{\rho u^2}{c}\left(x^0 - \frac{u_i x^i}{c}\right) = \rho\left(\frac{1}{\rho^2} + \frac{u^2}{c^2}\right)u_i x^i - \frac{\rho u^2}{c}x^0 \;,$$

that is

$$y = \rho\left(u_i x^i - \frac{u^2}{c}x^0\right) \;. \tag{2.83}$$

At this point we can obtain y^i, from $(2.80)_2$:

$$\begin{aligned} y^i &= x^i - \frac{1}{c}\rho^2 u^i\left(x^0 - \frac{1}{c}u_k x^k\right) - \frac{1}{c^2}\frac{\rho^3}{1+\rho}u^i\left(u_k x^k - \frac{u^2}{c}x^0\right) \\ &= x^i - \frac{\rho^2}{c}u^i\left(1 - \frac{u^2}{c^2}\frac{\rho}{1+\rho}\right)x^0 + \frac{\rho^2}{c^2}u^i\left(1 - \frac{\rho}{1+\rho}\right)u_k x^k \;; \end{aligned}$$

moreover, using the identities,

$$\begin{aligned} &1 - \frac{u^2}{c^2}\frac{\rho}{1+\rho} = \frac{1}{1+\rho}\left(1 + \rho - \frac{u^2}{c^2}\rho\right) = \frac{1}{1+\rho}\left(1 + \frac{1}{\rho}\right) = \frac{1}{\rho} \\ &\quad 1 - \frac{\rho}{1+\rho} = \frac{1}{1+\rho} \;, \end{aligned}$$

one obtains, from (2.82), the inverse form of (2.78)[9]:

$$\begin{cases} x'^0 = \rho(x^0 - \dfrac{1}{c} u_i x^i) \\ \mathcal{R}^i{}_k x'^k = x^i - \dfrac{\rho}{c} u^i x^0 + \dfrac{1}{c^2} \dfrac{\rho^2}{1+\rho} u^i u_k x^k \qquad (i = 1, 2, 3). \end{cases} \tag{2.84}$$

We notice that, in obtaining the Lorentz transformation, we have assumed M_4 to be endowed with one of the two light half-cones. With this assumption, the two Galilean frames $S_{\rm g}$ and $S'_{\rm g}$ are equi-oriented in time (in the future, as well as in the past): $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' < 0$, that is $\rho > 0$. If this were not true, one has $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' > 0$, that is

$$\rho = -\frac{1}{\sqrt{1 - \beta^2}} < 0 \, .$$

Analogously, if the two Galilean frames are equi-oriented in space (i.e. $\mathcal{T}$ and $\mathcal{T}'$ both left-handed, or right-handed), in (2.78), we have $\det||\mathcal{R}^i{}_k|| = +1$.

Thus, the complete (homogeneous) Lorentz group is described by (2.78) and (2.84), through the parameters u_i and $\mathcal{R}^i{}_k$ (the latter not independent of each other), without any sign restriction, for ρ or for the determinant of the matrix $\mathcal{R}^i{}_k$. However, it is not possible to pass continuously from the positive light half-cone to the negative one, as well as, in the same way, it is not possible to change continuously the orientation (left-handed or right-handed) of the spatial triad $\mathcal{T}$ or $\mathcal{T}'$. In other words, the complete homogeneous Lorentz group is not connected, but it is made up of four connected parts. Each part is characterized by an orthonormal basis: $\{\boldsymbol{\gamma}, \mathbf{c}_i\}$ and $\{\boldsymbol{\gamma}', \mathbf{c}'_i\}$, satisfying the following conditions: (i) $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ belong to the same branch of the lightcone: $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' < 0$, and $\rho = 1/\sqrt{1 - u^2/c^2} > 0$; (ii) the bases of M_4 are equi-oriented, in the sense that the determinant of the relative transformation $L^\alpha{}_\beta$ ($\mathbf{c}'_\beta = L^\alpha{}_\beta \mathbf{c}_\alpha$) is 1; this is also equivalent to $\det||\mathcal{R}^i{}_k|| = +1$ because of (2.78) and (2.84). The transformations $L^\alpha{}_\beta$ have then the form:

$$||L^\alpha{}_\beta|| \equiv \left\| \begin{matrix} \rho & \dfrac{\rho}{c} u_h \mathcal{R}^{\rm h}_k \\ \dfrac{\rho u^i}{c} & \left(\delta^i_{\rm h} + \dfrac{1}{c^2} \dfrac{\rho^2}{1+\rho} u^i u_{\rm h} \right) \mathcal{R}^{\rm h}{}_k \end{matrix} \right\| , \tag{2.85}$$

and constitute a connected group (Lorentz proper group), which gives rise to the complete group by adding the space and time reflections. The orthonormal frames $\{\boldsymbol{\gamma}, \mathbf{c}_i\}$, belonging to any of the four connected parts

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[9] See [1], p. 67, taking into account that, from (2.74), one has $\rho^2/(1+\rho) = c^2(\rho - 1)/u^2$.

(with the $\mathbf{c}_i$ defined up to spatial rotations), characterize ∞^3 Galilean frames which are equi-oriented both in space and in time.

Thus, the connected parts are all equivalent, as concerns the relativity principle, and it is not restrictive to assume that M_4 is endowed with orthonormal frames, satisfying conditions (i) and (ii) only. In other words, we will assume M_4 oriented, and endowed with only one of the two light half-cone, let us say $M_4^+(\mathcal{C}_3^+)$. All the relations between the absolute quantities of $M_4^+(\mathcal{C}_3^+)$ will automatically satisfy the relativity principle. So we will proceed by formulating, in $M_4^+(\mathcal{C}_3^+)$, the physical laws, starting from the dynamical ones, passing then to consider the relative formulation in a certain Galilean frame, i.e. defining the relative ingredients starting from the absolute quantities.

2.11 Relativity of Lengths and Times

In the limit $c \to \infty$, (2.65) reduces to $\boldsymbol{\gamma}' = \boldsymbol{\gamma}$: $M_4^+(\mathcal{C}_3^+)$, degenerates to the Cartesian product of an Euclidean 3-space and an oriented straight line, while $\mathbf{u}' = -\mathbf{u}$ is the complete reciprocity theorem. In the same limit, (2.71) reduces to the special Galilei transformation (1.13).

When c is finite, (2.65) states the relative meaning of lengths and times associated with a Galilean frame. Let us write (2.71) and (2.72) in the usual form (i.e. with $t = x^0/c, x^1 = x, x^2 = y, x^3 = z$ and analogously for the primed variables):

$$t = \frac{1}{\sqrt{1-\beta^2}}\left(t' + \frac{\beta}{c}x'\right) , \quad x = \frac{1}{\sqrt{1-\beta^2}}(x' + c\beta t) , \quad y = y' , \quad z = z' , \tag{2.86}$$

as well as

$$t' = \frac{1}{\sqrt{1-\beta^2}}\left(t - \frac{\beta}{c}x\right) , \quad x' = \frac{1}{\sqrt{1-\beta^2}}(x - c\beta t) , \quad y = y' , \quad z = z' , \tag{2.87}$$

where $\beta = u/c$. The typical relativistic mixing of space and time coordinates is evident from these relations; moreover, they allow us to derive some kinematical effects, already contained in the postulates.

1. *Relative meaning of simultaneity*

   Let us consider two events E and F happening, in S'_{g}, at two different points of the x'-axis, say A′ and B′, and here to be simultaneous $t'_E = t'_F = t'_0$, that is

   $$E \equiv (t'_0, x'_E, 0, 0) , \qquad F \equiv (t'_0, x'_F, 0, 0).$$

   How do these events manifest themselves in the frame S_{g}? They still happen on the x-axis ($y_E = z_E = 0$; $y_F = z_F = 0$), but they are no longer simultaneous. In fact, from $(2.86)_1$ one has

$$t_E = \frac{1}{\sqrt{1-\beta^2}}\left(t'_E + \frac{u}{c^2}x'_E\right) , \qquad t_F = \frac{1}{\sqrt{1-\beta^2}}\left(t'_F + \frac{u}{c^2}x'_F\right) , \tag{2.88}$$

so that

$$t_F - t_E = \frac{u}{c^2\sqrt{1-\beta^2}}(x'_F - x'_E) \; ; \tag{2.89}$$

if $x'_F = x'_E$, obviously, the two events coincide, both in S_{g} or in S'_{g}, but, in general, $t_F - t_E \neq 0$.

Also the *time ordering* has a relative meaning: $t'_F > t'_E$ may coexist with $t_F < t_E$, unless the two events are in a causality relation, i.e. E and F are associated with the same timelike world line.

2. *Time dilation*

Let us consider a phenomenon happening at a fixed point of S'_{g}, over a time period T_0. What is its duration in S_{g}? To answer this question, it is enough to specify its duration by means of the initial and final instants and to consider the associated events, E and F.

Let $(x', 0, 0)$ be the point of S'_{g} where the phenomenon happens, and t_E and $t_F = t_E + T_0$ be the initial and final instants. From (2.88) one has

$$T = t_F - t_E = \frac{1}{\sqrt{1-\beta^2}}\left(t'_F + \frac{u}{c^2}x'\right) - \frac{1}{\sqrt{1-\beta^2}}\left(t'_E + \frac{u}{c^2}x'\right) = \frac{t'_F - t'_E}{\sqrt{1-\beta^2}},$$

so that

$$T = \frac{T_0}{\sqrt{1-\beta^2}} > T_0 \; ; \tag{2.90}$$

thus, in S_{g}, the duration of a local phenomenon of S'_{g} appears longer: this result is known as *time dilation*. For example, the life of an observer in S'_{g} appears longer, when measured with the universal time of S_{g}.

3. *Lorentz contraction*

Let a rod A′B′, of length L_0, be at rest along the x'-axis in S'_{g} and let the endpoint A′ be at Ω while B′ is at $x' = L_0$; what is the length of the rod, as measured from S_{g}? Operationally, one has to measure the distance L between the intersections A and B left, on the x-axis, by the world lines of A′ and B′ at a fixed instant t of S_{g}.

Using (2.87) one has

$$x'_{\mathrm{B}} - x'_{\mathrm{A}} = \frac{x_{\mathrm{B}} - ut}{\sqrt{1-\beta^2}} - \frac{x_{\mathrm{A}} - ut}{\sqrt{1-\beta^2}} = \frac{x_{\mathrm{B}} - x_{\mathrm{A}}}{\sqrt{1-\beta^2}} \; ,$$

or $L_0 = L/\sqrt{1-\beta^2}$, so that

$$L = L_0\sqrt{1-\beta^2} < L_0 \; . \tag{2.91}$$

In other words, the rod, in S_{g}, appears contracted by a factor depending on its speed u: this phenomenon is called *Lorentz contraction* of the moving lengths. It is worth noting that the contraction concerns the direction of motion of S'_{g} (that is of the rod) with respect to S_{g}; if the rod were displaced orthogonally to the x-axis, there would be no contraction ($y' = y, z' = z$).

Moreover, as the Lorentz factor should be real, the relative speed u of S'_{g} with respect to S_{g} should always be less than c. This is consistent with postulate E, which prohibits any particle motion at a speed faster than that of light. This axiom could be verified as a whole, assuming—for absurdity—that an electromagnetic signal could travel with a speed greater than c. Then it would be possible to receive the same signal before its emission, and even to send it back to the emitter, before it actually would have been emitted; this would be a clear violation of the causality principle.

It is also interesting that it is still possible to have a weak form of special relativity without assuming the postulate E, accepting the above stated violation of the causality principle.

Note. Equations (2.71)–(2.72) may have a double interpretation:

- Like a Cartesian coordinate change, for the generic event $E \in M_4$, this governs the passage from an affine frame to another and vice versa. It is the most natural interpretation, and it is adapted to the absolute character of the events. In this sense, this should be the primary interpretation.
- Like an endomorphism of M_4, in the sense that the x^α and the x'^α are associated with different points: E and E′ respectively, in the same affine frame $R \equiv (\Omega; \boldsymbol{\gamma}, \mathbf{c}_i)$. From this point of view, it leaves each of the two light half-cones invariant and gives the correct meaning to the notions of past, present and future of an event.

An important consequence of the relativistic speed limit should be noted here. It implies the impossibility of the existence of rigid bodies in relativity. For if any body is pushed at one point, the opposite part of the body cannot immediately start to move, otherwise we would have transmitted a signal at infinite speed. So every body must be deformable, and not rigid.

2.12 Muon Mean Life and the Time Dilation

Let us consider (2.91) for the length contraction. This necessitates a rigid rod for which a *Galilean rest frame* would exist; in this frame, its length is L_0 (*proper length*). In a Galilean frame in which the rod is in linear and uniform motion, with *longitudinal* (i.e. in the direction of the rod) velocity $\mathbf{v}$, its length is no longer L_0, but

$$L = L_0 \sqrt{1 - v^2/c^2}\ . \tag{2.92}$$

In other terms, this is a *purely spatial* phenomenon, i.e. a sequence of events which admits a Galilean frame where they all happen at the same time, but they are localized in points of a segment. In another Galilean frame, the spatial localization of the points is exactly of the same kind, but they are no more simultaneous.

The situation is similar for a *purely temporal* phenomenon, i.e. a sequence of events which admits a Galilean frame where they all are located at the same position, but they correspond to different times and (2.90) holds. We mean that, while in the rest frame, the phenomenon is only characterized by its duration T_0 (*proper duration*), in another Galilean frame S_{g}, the events not only happen in different places but give rise to a uniform motion with velocity $\mathbf{v}$, the duration of which is

$$T = T_0/\sqrt{1 - v^2/c^2}\,. \tag{2.93}$$

In (2.92) and (2.93), the Lorentz factor η appears, which is very close to 1 when $v \ll c$ as for bodies in the solar system. Thus, at least in this regime, the relativistic effects of length contraction and time dilation are not relevant. For instance, if $v/c \sim 10^{-4}$, corresponding to the Earth orbital motion, then $1/\eta = 0.000995$. Hence, from (2.92) and taking into account that the Earth diameter is $D_0 \simeq 10^9$ cm, the Earth, as seen by a Sun observer, would appear contracted (longitudinally) by about 6.5 cm. Analogously, from (2.93), a purely temporal Earth phenomenon, lasting a century, when examined from the Sun, would have a duration of 2.5 min more. The effects become important when the velocities approach that of light, as it happens, microscopically, for elementary particles.

Let us consider the experimental data for the muons which are contained in the cosmic rays. They have the same charge as the electron, but a 200 times heavier mass. Their proper life is very short: $T_0 = 2.15 \times \cdot 10^{-6}$s, and then, after this time (on average), they spontaneously decay into an electron (with the same charge) and two neutrinos.

In a time interval of the amount of their proper life, even if they could move at the speed of light, they could make a very short path: $L = cT_0 = 645$ m. This result would be not acceptable because of the experimental observation of muons in the atmosphere, which were produced at more than 1 km higher. It is explained by the relativistic time dilation, according to which the real life of muons is $T = T_0/\sqrt{1 - v^2/c^2}$ and depends on their velocity. Thus, using the experimental value $v = 0.99\,c$, one gets $\eta = 10$, and the path becomes ~ 6400 m, in agreement with observations (see [5], pp. 33–34).

2.13 Theorem of Relative Motions

Let us start from formulas (2.86) and (2.87), which specify the change of the coordinates, for a space-time event passing from a Galilean frame S_{g} to another frame S'_{g}; the aim is to discuss the problem of *relative motions*.

To this end, let us consider, in S_{g}, a pointlike motion $\mathcal{M}$: $x^i = x^i(t)$, $t \in (t_0, t_1)$. From (2.87) we have, immediately, the parametric equations $x'^{\alpha}(t)$ of the curve corresponding to $\mathcal{M}$ in S'_{g}:

$$t' = \frac{t - \dfrac{\beta}{c} x(t)}{\sqrt{1-\beta^2}} \,, \quad x' = \frac{x(t) - c\beta t}{\sqrt{1-\beta^2}} \,, \quad y' = y(t), \quad z' = z(t) \,. \tag{2.94}$$

From the first relation, in principle, one can deduce $t = t(t')$, and substitute in the others obtaining then $x'^i = x'^i(t')$. From these, by differentiation with respect to t', one gets the components $v'^i = (\mathrm{d}x'^i/\mathrm{d}t')$ of the relative velocity $\mathbf{v}'$ in S'_{g}:

$$v'^i = \frac{\mathrm{d}x'^i}{\mathrm{d}t} \frac{\mathrm{d}t}{\mathrm{d}t'} \,,$$

with

$$\frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{1}{\mathrm{d}t'/\mathrm{d}t} = \frac{\sqrt{1-\beta^2}}{1 - uv^1/c^2} \,. \tag{2.95}$$

The relations between the x-standard components of the relative velocity $\mathbf{v}$ and $\mathbf{v}'$ then follow easily:

$$v'^1 = \frac{v^1 - u}{1 - uv^1/c^2} \,, \quad v'^{2,3} = v^{2,3} \frac{\sqrt{1-\beta^2}}{1 - uv^1/c^2} \,. \tag{2.96}$$

Equation (2.96) represents, though in a scalar form, the relativistic addition of velocity law. It can also be cast in a vectorial form, independent of the choice of the Cartesian triads in S_{g} and S'_{g}. For instance, the denominator of the two fractions in (2.96) is simply $1 - \mathbf{u} \cdot \mathbf{v}/c^2$. Let us introduce, thus, the two quantities, invariant with respect to internal transformations of S_{g}:

$$\alpha = \sqrt{1 - u^2/c^2} \,, \qquad \sigma = 1 - \mathbf{u} \cdot \mathbf{v}/c^2 \,; \tag{2.97}$$

so that (2.96) becomes

$$v'^1 = \frac{v^1 - u}{\sigma} \,, \quad v'^{2,3} = \frac{\alpha}{\sigma} v^{2,3} \,. \tag{2.98}$$

To get the vectorial formula, one needs to remember that this should be an extension of the classical relation $\mathbf{v}' = \mathbf{v} - \mathbf{u}$; that is, it will be a relation between 3-vectors and it should be referred to S_{g} or S'_{g}; a priori, this is not correct: in fact $\mathbf{v}$ and $\mathbf{v}'$ are 3-vectors in the two platforms Σ and Σ', and it is not possible to pass from one to the other, without using $\boldsymbol{\gamma}$ or $\boldsymbol{\gamma}'$. However, an indirect comparison, between the two platforms, can always be done: i.e. it is possible to boost Σ' on Σ by means of a rotation of M_4. This makes the unit vectors $\mathbf{c}'_i$ coincident with the $\mathbf{c}_i$ $(i = 1, 2, 3)$. In other words, between the vectors $\mathbf{s}' \in \Sigma'$ and those $\mathbf{s} \in \Sigma$, there exists an invertible *isometry map*

$\mathcal{R}$ so that $\mathbf{s}' \to \mathcal{R}\mathbf{s}' \in \Sigma$; such a correspondence can be given by *interpreting the components of* $\mathbf{s}'$ *along* $\mathcal{T}'$ *as components along* $\mathcal{T}$:

$$\mathbf{s}' = s'^i \mathbf{c}'_i \qquad \to \qquad \mathcal{R}\mathbf{s}' = s'^i \mathbf{c}_i \ . \tag{2.99}$$

Clearly, this map implies $\mathcal{R}\mathbf{u}' = -\mathbf{u}$ and it leaves invariant those vectors (in Σ') which are orthogonal to $\mathbf{u}'$. Then, one can identify $\mathcal{R}\mathbf{v}' = v'^i \mathbf{c}_i$, or simply, by omitting the symbol $\mathcal{R}$ for brevity, $\mathbf{v}' = v'^i \mathbf{c}_i$.

It is convenient to recast $(2.98)_1$ in the following form:

$$v^1 = \alpha v^1 + (1-\alpha) v^1 = \alpha v^1 + \frac{\beta^2}{(1+\alpha)} v^1 \ ,$$

i.e. using the relation $uv^1 = u\mathbf{v} \cdot \mathbf{c}^1 = \mathbf{u} \cdot \mathbf{v}$:

$$v^1 = \alpha v^1 + \frac{1}{c^2} \frac{\mathbf{u} \cdot \mathbf{v}}{1+\alpha} u \ . \tag{2.100}$$

Thus,

$$v'^1 = \frac{\alpha}{\sigma} v^1 + \frac{1}{\sigma} \left(\frac{1}{c^2} \frac{\mathbf{u} \cdot \mathbf{v}}{1+\alpha} - 1 \right) u \ , \qquad v'^{2,3} = \frac{\alpha}{\sigma} v^{2,3} \ ;$$

hence,

$$\mathbf{v}' = \frac{\alpha}{\sigma} \mathbf{v} + \frac{1}{\sigma} \left(\frac{1}{c^2} \frac{\mathbf{u} \cdot \mathbf{v}}{1+\alpha} - 1 \right) \mathbf{u} \ ,$$

and finally, using $(2.97)_2$,

$$\mathbf{v}' = \frac{1}{\sigma} \left(\alpha \mathbf{v} - \frac{\alpha + \sigma}{1+\alpha} \mathbf{u} \right) \ . \tag{2.101}$$

Equation (2.101) represents the *relativistic theorem of relative motions* (it is also known as the *velocities transformation formula*). It has an intrinsic meaning, i.e. it does not depend on the choice of the two triads $\mathcal{T} \in S_{\mathrm{g}}$ and $\mathcal{T}' \in S'_{\mathrm{g}}$, and it is valid for any choice of the two frames. The Galilean formula, $\mathbf{v}' = \mathbf{v} - \mathbf{u}$, obviously comes from the $c \to \infty$ limit.

Equation (2.101) should be considered together with the link between the relative times t and t'. In differential terms, this link (all along the motion) is expressed by (2.95), which can also be written in intrinsic form, as:

$$\frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{\alpha}{\sigma} \ . \tag{2.102}$$

However, in the case $\mathbf{v}$ =const., we also have σ =const. and hence, from (2.101): $\mathbf{v}'$ =const. Thus, linear and uniform motions have an intrinsic meaning, as in classical mechanics. This was already known: $\mathbf{v}$ =const. implies $\mathbf{V} =$ const. and then $\mathbf{v}'$ =const.

The transformation law of the velocities (2.101) is compatible with the axiom E, in the sense that it implies $v'^2 < c^2$ whenever $v^2 < c^2$. In fact, from (2.101) and (2.97) one has

$$\begin{aligned} v'^2 &= \frac{1}{\sigma^2}\left[\alpha^2 v^2 + \left(\frac{\alpha+\sigma}{1+\alpha}\right)^2 u^2 - 2\frac{\alpha(\alpha+\sigma)}{1+\alpha}\,\mathbf{u}\cdot\mathbf{v}\right] \\ &= \frac{c^2}{\sigma^2}\left[\alpha^2\frac{v^2}{c^2} + \left(\frac{\alpha+\sigma}{1+\alpha}\right)^2 (1-\alpha^2) - 2\frac{\alpha(\alpha+\sigma)}{1+\alpha}(1-\sigma)\right] \\ &= \frac{c^2}{\sigma^2}\left\{\alpha^2\frac{v^2}{c^2} + \frac{\alpha+\sigma}{1+\alpha}[(\alpha+\sigma)(1-\alpha) - 2\alpha(1-\sigma)]\right\} \\ &= \frac{c^2}{\sigma^2}\left[\alpha^2\frac{v^2}{c^2} + \frac{\alpha+\sigma}{1+\alpha}(1+\alpha)(\sigma-\alpha)\right] \\ &= \frac{c^2}{\sigma^2}\left[\alpha^2\left(\frac{v^2}{c^2}-1\right) + \sigma^2\right], \end{aligned}$$

and thus

$$\frac{v'^2}{c^2} - 1 = \left(\frac{\alpha}{\sigma}\right)^2\left(\frac{v^2}{c^2}-1\right) , \tag{2.103}$$

which completes the proof.

Equation (2.103) can be obtained directly, from (2.38), which, because of the absolute meaning of the proper time τ, gives rise to the following invariance property (with respect to the choice of the Galilean frame and along a given world line):

$$\sqrt{1-\frac{v^2}{c^2}}\mathrm{d}t = \sqrt{1-\frac{v'^2}{c^2}}\mathrm{d}t' = \text{inv.} = \mathrm{d}\tau . \tag{2.104}$$

It then follows, using (2.102), that

$$\frac{\alpha}{\sigma} = \sqrt{\frac{1-\dfrac{v'^2}{c^2}}{1-\dfrac{v^2}{c^2}}} = \frac{\eta}{\eta'} ,$$

i.e. (2.103), or, equivalently,

$$\eta' = \eta\frac{\sigma}{\alpha} . \tag{2.105}$$

Let us note that (2.101) can also be derived from the invariance of the 4-*velocity*: $\eta(\mathbf{v}+c\boldsymbol{\gamma}) = \text{inv.} = \eta'(\mathbf{v}'+c\boldsymbol{\gamma}')$; using (2.105), this becomes $\mathbf{v}'+c\boldsymbol{\gamma}' = \alpha/\sigma(\mathbf{v}+c\boldsymbol{\gamma})$, which gives immediately the components $v'^i = \mathbf{v}'\cdot\mathbf{c}'_i$, taking into account (2.68).

Note. The isometric boost of the two spaces Σ and Σ', necessary to compare the relative ingredients associated with the frames S_g and S'_g, can be better geometrically formalized acting directly on vectors, rather than on their Cartesian components with respect to the two triads $\mathcal{T}$ and $\mathcal{T}'$ in x-standard relation. According to this point of view, one interprets the effective components of a vector $\mathbf{v}'\in\Sigma'$ as if they were along the triad $\mathcal{T}$ (that is on Σ), thus verifying that the associated map is an isometry. Therefore, more generally, one can operate *independently on the triads* $\mathcal{T}$ and $\mathcal{T}'$. For the transformed isometric vector $\mathcal{R}\mathbf{v}'$ of a vector $\mathbf{v}' \in \Sigma'$, we have

$$\mathcal{R}\mathbf{v}' = \boldsymbol{\sigma} - \lambda\mathbf{u} \quad \sim \quad \mathcal{R}\mathbf{v}' = \mathbf{s} + \lambda(\rho - 1)\mathbf{u} \,,$$

with $\boldsymbol{\sigma} = \mathbf{s} + \lambda\rho\mathbf{u}$ and

$$\lambda \stackrel{\text{def}}{=} \frac{1}{u^2}\mathbf{v}' \cdot \mathbf{u}' \,, \quad \rho = \frac{1}{\sqrt{1-\beta^2}} \,, \quad \beta = \frac{u}{c} \,.$$

In these relations, there appears $\boldsymbol{\sigma}$, i.e. the orthogonal projection of $\mathbf{v}'$ on the 2-plane $\Sigma \cap \Sigma'$ (the "axis" of the rotation), as well as $\mathbf{s}$, i.e. the orthogonal projection of $\mathbf{v}'$ on Σ, orthogonally to $\boldsymbol{\gamma}$. The two expressions are equivalent. In particular, when $\mathbf{v}' = \mathbf{u}'$, one gets $\boldsymbol{\sigma} = 0$, and hence $\lambda = 1$, $\mathcal{R}\mathbf{u}' = -\mathbf{u}$; if instead $\mathbf{v}' \in \Sigma \cap \Sigma'$ ($\mathbf{v}' \cdot \mathbf{u}' = 0$), one has $\lambda = 0$ and $\mathbf{s} = \boldsymbol{\sigma}$.

2.14 Optical Experiments and Special Relativity

As an application of the theorem (2.101) on relative motions, let us reconsider here the optical experiments, previously discussed in terms of classical physics.

1. *Stellar aberration*
   Vector multiplication, by $\mathbf{u}$, of both sides of (2.101), yields
   $$\mathbf{v}' \times \mathbf{u} = \frac{\alpha}{\sigma}\mathbf{v} \times \mathbf{u} \,,$$
   from which, using $v = v' = c$, it follows that
   $$cu \sin\theta' = \frac{\alpha}{\sigma} cu \sin\theta \,,$$
   or
   $$\sin\theta' = \sin\theta \frac{\sqrt{1 - \dfrac{u^2}{c^2}}}{1 - \dfrac{u}{c}\cos\theta} \,. \tag{2.106}$$
   To first order in $\beta = u/c$:
   $$\frac{\sqrt{1-\beta^2}}{1 - \beta\cos\theta} \simeq 1 + \beta \, \cos\theta \,,$$
   so that $\sin\theta' - \sin\theta \simeq u/c \sin\theta \cos\theta$. Now, using $\theta' = \theta + \Delta\theta$, to first order in $\Delta\theta$ one has $\sin\theta' - \sin\theta \simeq \cos\theta\Delta\theta$, and hence (2.106) assumes the classical form (1.29): $\Delta\theta = \beta \sin\theta$.

2. *Relativistic luminal Doppler effect*
This effect can be obtained by considering, either from an absolute or a relative point of view, the photon as a material particle, taking into account the fundamental property: the *photon frequency* ν satisfies the invariance property

$$\frac{\mathrm{d}t}{\nu} = \frac{\mathrm{d}t'}{\nu'} = \text{inv.} \tag{2.107}$$

with respect to any change of Galilean frames. Using (2.102), one then has the relativistic Doppler effect formula:

$$\frac{\nu' - \nu}{\nu} = \frac{\Delta\nu}{\nu} = \frac{1 - \dfrac{u}{c}\cos\theta}{\sqrt{1 - \dfrac{u^2}{c^2}}} - 1\,, \qquad \cos\theta = \frac{\mathbf{u}\cdot\mathbf{c}}{uc}\,. \tag{2.108}$$

From this follows a *longitudinal effect* ($\mathbf{c}$ parallel to $\mathbf{u}$):

$$\frac{\Delta\nu}{\nu} = \frac{1 - \dfrac{\epsilon u}{c}}{\sqrt{1 - \dfrac{u^2}{c^2}}} - 1\,, \quad \epsilon = \pm 1\,, \tag{2.109}$$

which, to first order in u/c, reduces to the classical effect:

$$\frac{\Delta\nu}{\nu} \simeq \epsilon\frac{u}{c}\,.$$

If, instead, the velocity $\mathbf{c}$ of the light ray is perpendicular to $\mathbf{u}$ (translational velocity of S'_{g} with respect to S_{g}), (2.108) gives the *transverse Doppler effect* formula:

$$\frac{\Delta\nu}{\nu} = \frac{1}{\sqrt{1 - \dfrac{u^2}{c^2}}} - 1 > 0\,. \tag{2.110}$$

One then finds more than a simple relativistic correction: a new phenomenon which is of the second order in u/c:

$$\frac{\Delta\nu}{\nu} \simeq \frac{1}{2}\frac{u^2}{c^2}\,.$$

3. *Fresnel–Fizeau effect*
Let us assume S'_{g} as the water rest frame; the light signal speed is then $v' = c/n$, with n the refraction index of the water. Assuming that the direction of propagation of the light, in S_{g} coincides with $\mathbf{u}$, what is the value of the speed of light in S_{g}? We need to use the inverse formula of (2.101), obtained by exchanging primed and unprimed quantities:

$$\mathbf{v} = \frac{1}{\sigma'}\left(\alpha\mathbf{v}' - \frac{\alpha' + \sigma'}{1+\alpha'}\mathbf{u}'\right) .$$

But $\mathbf{u}' = -\mathbf{u}$, so that

$$\mathbf{v} = \frac{1}{\sigma'}\left(\alpha\mathbf{v}' + \frac{\alpha + \sigma'}{1+\alpha}\mathbf{u}\right) , \tag{2.111}$$

where

$$\alpha = \sqrt{1 - u^2/c^2} , \quad \sigma' = 1 + \mathbf{u}\cdot\mathbf{v}'/c^2 . \tag{2.112}$$

In our case, $\mathbf{v}' = \epsilon c\mathbf{u}/(nu)$, with $\epsilon = \pm 1$; then (2.111) becomes

$$\mathbf{v} = \frac{1}{\sigma'}\left(\alpha\epsilon\frac{c}{n} + \frac{\alpha + \sigma'}{1+\alpha}u\right)\frac{\mathbf{u}}{u} .$$

In other words, the light ray velocity, in S_{g}, is $\mathbf{v} = v\,\mathrm{vers}\,\mathbf{u}$, with

$$v = \frac{1}{\sigma'}\left(\alpha\epsilon\frac{c}{n} + \frac{\alpha + \sigma'}{1+\alpha}u\right), \quad \sigma' = 1 + \frac{\epsilon u}{nc} , \quad \epsilon = \pm 1 . \tag{2.113}$$

This is an exact relativistic formula. To first order in u/c (so that $\alpha \simeq 1$), one gets

$$v = \epsilon\frac{c}{n} + \left(1 - \frac{1}{n^2}\right)u , \quad \epsilon = \pm 1 , \tag{2.114}$$

from which, when $\epsilon = 1$ (propagation along $\mathbf{u}$), one obtains the earlier seen (1.38). Nothing surprising, of course, in this result, because the Lorentz transformations reduce to the Galilei ones, to first order in β.

4. *Michelson–Morley experiment*
   Here a difference appears because the involved effect is of second order in β. Assuming that the Earth is an inertial frame S'_{g}, also in the relativistic context, the light velocity $\mathbf{v}'$ should be the same (c) in each direction (optical isotropy of the Earth), and hence $\Delta T = \Delta T'$, or $\Delta = 0$, according to the previous notation. Thus, the lack of fringe shift is in perfect agreement with special relativity. In other words, the Michelson–Morley experiment in agreement with the special theory of relativity confirms that the Earth is a Galilean frame (to the second order), as in classical mechanics.

2.15 Coriolis Theorem

From the theorem of relative motions (2.101), considered for an arbitrary motion $\mathcal{M}$, after differentiation with respect to t', one gets the *relative acceleration composition law* in the context of Galilean frames, or the *Coriolis theorem*, even if, classically, this theorem concerns any kind of frame and thus has a more general meaning.

More precisely, using (2.102): $\mathrm{d}t/\mathrm{d}t' = \alpha/\sigma$, and (2.97) for the definition of α and σ, from (2.101) one gets

$$\begin{aligned}
\mathbf{a}' &= \frac{\alpha}{\sigma}\left[\frac{1}{\sigma}\left(\alpha\mathbf{a} - \frac{\dot{\sigma}}{1+\alpha}\mathbf{u}\right) - \frac{\dot{\sigma}}{\sigma^2}\left(\alpha\mathbf{v} - \frac{\alpha+\sigma}{1+\alpha}\mathbf{u}\right)\right] \\
&= \frac{\alpha}{\sigma^2}\left[\alpha\mathbf{a} + \frac{(\mathbf{u}\cdot\mathbf{a})\mathbf{u}}{c^2(1+\alpha)} + \frac{\mathbf{u}\cdot\mathbf{a}}{c^2\sigma}\left(\alpha\mathbf{v} - \frac{\alpha+\sigma}{1+\alpha}\mathbf{u}\right)\right] \\
&= \frac{\alpha^2}{\sigma^3}\left[\sigma\mathbf{a} + \frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\left(\mathbf{v} - \frac{1}{1+\alpha}\mathbf{u}\right)\right] \\
&= \frac{\alpha^2}{\sigma^3}\left[\mathbf{a} - \frac{(\mathbf{u}\cdot\mathbf{v})}{c^2}\mathbf{a} + \frac{(\mathbf{u}\cdot\mathbf{a})}{c^2}\mathbf{v} - \frac{(\mathbf{u}\cdot\mathbf{a})}{c^2(1+\alpha)}\mathbf{u}\right] ,
\end{aligned}$$

and finally,

$$\mathbf{a}' = \frac{\alpha^2}{\sigma^3}\left[\mathbf{a} + \frac{1}{c^2}\mathbf{u}\times(\mathbf{v}\times\mathbf{a}) - \frac{(\mathbf{u}\cdot\mathbf{a})}{c^2(1+\alpha)}\mathbf{u}\right] , \tag{2.115}$$

with the inverse relation

$$\mathbf{a} = \frac{\alpha^2}{\sigma'^3}\left[\mathbf{a}' - \frac{1}{c^2}\mathbf{u}\times(\mathbf{v}'\times\mathbf{a}') - \frac{(\mathbf{u}\cdot\mathbf{a}')}{c^2(1+\alpha)}\mathbf{u}\right] , \tag{2.116}$$

where

$$\sigma' = 1 + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{v}' . \tag{2.117}$$

Equation (2.115) represents the relativistic version of the Coriolis theorem in intrinsic form in S_{g}. However, as already stated, it is only a partial generalization, because, in special relativity, only Galilean frames are admitted. Actually, it generalizes the classical theorem: $\mathbf{a}' = \mathbf{a}$ on the acceleration invariance in the context of Galilean frames, to which it reduces in the limit $c \to \infty$. Similarly, also the theorem of relative motion (2.101) generalizes the corresponding classical one: $\mathbf{v}' = \mathbf{v} - \mathbf{v}_\tau$, with $\mathbf{v}_\tau$ the dragging velocity, only in the case $\mathbf{v}_\tau = \mathbf{u}$ =const. Equation (2.115) has been obtained by differentiating with respect to t' the analogous relation (2.101). However, we should have performed two different steps: (1) evaluate, starting from (2.96), the relations among the components a^i and a'^i of the relative accelerations with respect to $\mathcal{T}$ and $\mathcal{T}'$; (2) interpret the result in intrinsic form, boosting the triad of S'_{g} in S_{g}. But the result would have been the same because boosting Σ' on Σ and differentiating with respect to time are two commutable operations.

Equation (2.115) shows that, differently from classical mechanics, the relative acceleration is not invariant passing from one frame to another. The invariance exists only for uniform rectilinear motions: $\mathbf{a} = 0$ implies $\mathbf{a}' = 0$.

Relativistic kinematics determines second-order corrections to the classical relative motion theorems: $\mathbf{v}' = \mathbf{v} - \mathbf{u}$ and $\mathbf{a}' = \mathbf{a}$. In fact, from

$$\frac{\alpha}{\sigma} \simeq 1 + \frac{1}{c^2}\left(\mathbf{u}\cdot\mathbf{v} - \frac{1}{2}u^2\right) ,$$

and

$$\frac{1+\alpha/\sigma}{1+\alpha} \simeq 1 + \frac{1}{2c^2}\left(\mathbf{u}\cdot\mathbf{v} + \frac{1}{2}u^2\right) ,$$

one gets

$$\begin{cases} \mathbf{v}' \simeq \mathbf{v} - \mathbf{u} + \dfrac{1}{c^2}\left[\left(\mathbf{u}\cdot\mathbf{v} - \dfrac{1}{2}u^2\right)\mathbf{v} - \dfrac{1}{2}\left(\mathbf{u}\cdot\mathbf{v} + \dfrac{1}{2}u^2\right)\mathbf{u}\right] \\ \\ \mathbf{a}' \simeq \mathbf{a} + \dfrac{1}{c^2}\left[\left(3\mathbf{u}\cdot\mathbf{v} - \dfrac{1}{2}u^2\right)\mathbf{a} + \mathbf{u}\times(\mathbf{v}\times\mathbf{a}) - \dfrac{1}{2}\mathbf{u}\cdot\mathbf{a}\mathbf{u}\right] . \end{cases}$$

.

Note. As we have already stated, comparison between vectors on Σ and Σ' is related to the isometric boost of Σ' to Σ; it depends only on the two platforms and not on the choice of the origins Ω and Ω'. More precisely, if the two origins do coincide, the boost (rotation around $\Omega = \Omega'$ in the 2-plane $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$) induces a map between points $\mathrm{P}' \in \Sigma'$ and $\mathrm{P} \in \Sigma$. The special Lorentz transformations (2.87) can be then interpreted intrinsically in Σ, associating with x'^i the Cartesian coordinates (in $\mathcal{T}$) of the image of P' in Σ. From this point of view, denoting (with an abuse of notation) with P' this image too, the special Lorentz transformations can be summarized in the following correspondence $(t, \mathrm{P}) \leftrightarrow (t', \mathrm{P}')$ (see [6], p. 41):

$$\begin{cases} t' = \dfrac{1}{\alpha}\left(t - \dfrac{1}{c^2}\Omega\mathrm{P}\cdot\mathbf{u}\right) , \\ \\ \Omega\mathrm{P}' = \Omega\mathrm{P} - \dfrac{1}{\alpha}\left(t - \dfrac{1}{(1+\alpha)c^2}\Omega\mathrm{P}\cdot\mathbf{u}\right)\mathbf{u} . \end{cases}$$

2.16 Vectorial Maps

In order to use the map language, which permits one to treat, in an intrinsic way, mixed tensors of any rank in any vector space, let us briefly summarize here its most important properties (see [1], pp. 24–44).

A *vectorial map* t, defined in a vector space E_n, is a linear map of E_n into E_n (endomorphism):

$$t : \mathbf{v} \quad \to \quad t(\mathbf{v}) \in E_n \qquad \forall \mathbf{v} \in E_n , \tag{2.118}$$

satisfying the *linearity condition*:

$$t(a\mathbf{u} + b\mathbf{v}) = at(\mathbf{u}) + bt(\mathbf{v}) \qquad \forall \mathbf{u}, \mathbf{v} \in E_n, \quad a, b \in R . \tag{2.119}$$

The set Ω of all the maps of E_n has the structure of a linear space. In fact, if $t, t' \in \Omega$, and $a \in R$, we can define

$$t + t' : (t + t')(\mathbf{v}) = t(\mathbf{v}) + t'(\mathbf{v}) \ , \qquad at : (at)(\mathbf{v}) = at(\mathbf{v}) \ , \quad \forall \mathbf{v} \in E_n \ ;$$

the maps $t + t'$ and at are linear (like t and t'), and all the axioms of a vector space are satisfied. It follows that Ω is a vector space isomorphic to the space of (affine) mixed tensors $t^i{}_k$; hence it can be identified with this space, associated with E_n, having dimension n^2.

Such isomorphism (linear and bijective) follows by considering the quantities $t^i{}_k$ obtained by decomposing, with respect to the fixed basis $\{\mathbf{e}_i\} \in E_n$, the transformed vectors of the basis themselves:

$$t(\mathbf{e}_k) = t^i{}_k \mathbf{e}_i \qquad (k = 1, 2, ..., n) \ . \tag{2.120}$$

Thus, the quantities $t^i{}_k$ are the components of the mixed tensor associated with the map t; they are called *coefficients of the map* t, with respect to the basis $\{\mathbf{e}_k\}$. The map t operates by linearity on a generic vector $\mathbf{v} \in E_n$:

$$t(\mathbf{v}) = t(v^k \mathbf{e}_k) = v^k t(\mathbf{e}_k) = v^k t^i{}_k \mathbf{e}_i \ , \tag{2.121}$$

i.e. the transform with respect to a given basis $\{\mathbf{e}_i\}$ of a vector is a vector, obtained by contracting the original components with the coefficients of the map, in that basis:

$$v^i \quad \to \quad v^k t^i{}_k \ . \tag{2.122}$$

If E_n is a Riemannian space, i.e. endowed with a nonsingular metric g_{ik}: $\det||g_{ik}|| \neq 0$, the position of the indices in $t^i{}_k$ is inessential, in the sense that one can pass from $t^i{}_k$ to the other equivalent forms: covariant, contravariant and mixed ($t_i{}^k = g_{ij} g^{hk} t^j{}_h$). In general, instead, the tensor $t^i{}_k$ defines two different map laws, corresponding to the cases in which the contracted index is the first or the second one:

$$v^k \quad \to \quad v^i t^k{}_i \qquad \text{or} \qquad v^k \quad \to \quad v^i t_i{}^k \ , \tag{2.123}$$

and one needs to specify if the transformed vector of $\mathbf{v}$ if *left transformed*, or *right transformed*.

We note that the coefficients of the map are essentially dependent on the choice of the basis $\{\mathbf{e}_i\}$, and they transform according to the tensorial law. However, considering the matrix $||t^i{}_k||$ of the coefficients of the map, it is easy to check that there exist n scalar quantities, related to the matrix, which are independent of the chosen basis $\{\mathbf{e}_i\}$. These are the *fundamental invariants of the map*: I_k $(k = 1, 2, ..., n)$, and coincide with the sum of principal minors of order k of the matrix $||t^i{}_k||$ [10]:

$$I_k = \frac{1}{k!} \delta^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_k} t^{\beta_1}{}_{\alpha_1} \dots t^{\beta_k}{}_{\alpha_k} \ , \qquad k = 1 \dots n \ , \tag{2.124}$$

[10] A minor is principal if its principal diagonal is included in that of the original matrix.

where[11]

$$\delta^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_k} = k!\delta^{\alpha_1}_{[\beta_1}\ldots\delta^{\alpha_k}_{\beta_k]} \ . \tag{2.125}$$

In particular, one has

$$I_1 = t^1{}_1 + t^2{}_2 + \cdots + t^n{}_n = \mathrm{Tr}\, t \ , \qquad I_n = \det||t^i{}_k|| \ .$$

A map is called *proper*, if the invariant of maximum order $I_n = \det||t^i{}_k||$ is nonzero, and this is an absolute property. In this case, t *maintains linear independence*, in the sense that it transforms independent vectors into independent vectors; in particular, the transform of a nonzero vector always is nonzero.

Conversely, if $I_n = 0$, the map is called *degenerate*, and there always exists a nonzero vector $\mathbf{v}$ whose transform is zero. The set of vectors $\mathbf{v} \in E_n$ satisfying the condition $t(\mathbf{v}) = 0$ forms a vector subspace of E_n, which is called the *kernel* of t. For a proper map, the kernel is reduced to the zero vector only.

A vector $\mathbf{v} \in E_n$ such that $t(\mathbf{v})$ is parallel to $\mathbf{v}$,

$$t(\mathbf{v}) = \lambda \mathbf{v} \ , \qquad \lambda \in R \ , \tag{2.126}$$

is called an *eigenvector* of t; λ is the corresponding *eigenvalue* of t, associated with $\mathbf{v}$. Because of the linearity property of t, from (2.126), it is clear that if t admits an *eigenvector* $\mathbf{v}$, then it admits ∞^1 eigenvectors, all parallel to $\mathbf{v}$: $t(a\mathbf{v}) = at(\mathbf{v}) = \lambda a\mathbf{v}$, and forming a one-dimensional subspace, say $< \mathbf{v} >$.

Equation (2.126) can be conveniently written as $t(\mathbf{v}) = \lambda t^0(\mathbf{v})$, where t^0 is the *identity map*: $t^0(\mathbf{v}) = \mathbf{v}$, having as coefficients the Kronecker tensor δ^k_i.[12] Therefore, (2.126) assumes the form:

$$(t - \lambda t^0)(\mathbf{v}) = 0 \ . \tag{2.127}$$

In order that the latter condition to be satisfied, for nonzero vectors (proper eigenvectors), the map $t - \lambda t^0$ must be degenerate, leading to the following condition for λ:

$$\det||t - \lambda t^0|| = 0 \ , \tag{2.128}$$

or, explicitly, the *eigenvalue equation*:

$$\sum_{k=0}^{n} (-1)^k I_{n-k} t^k = 0 \ , \tag{2.129}$$

[11] As standard, indices antisymmetrization is denoted by square brackets while symmetrization by round brackets. For example, for a 2-tensor A we have

$$A_{[\alpha\beta]} = \frac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha}) \ , \qquad A_{(\alpha\beta)} = \frac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha}) \ .$$

[12] The notation $t^0 = \mathbb{I}$ is also used and especially when two or more maps are involved.

or, explicitly

$$(-1)^n\lambda^n + (-1)^{n-1}I_1\lambda^{n-1} + \cdots - I_{n-1}\lambda + I_n = 0 \, . \tag{2.130}$$

The (complex) solutions of (2.130), when substituted back in (2.127), give rise to a *linear system*, which, for each real λ, gives the associated eigenvectors. In scalar terms, one must solve the linear homogeneous system:

$$({t^i}_k - \lambda\delta^i_k)v^k = 0 \, , \tag{2.131}$$

which, because of (2.128), admits at least one real eigensolution v^i (if λ and ${t^i}_k$ are real).

The following general property holds: *eigenvectors associated with distinct eigenvalues are independent.* A case of special interest is when t has n distinct eigenvectors, with which one can form (in an infinite number of ways), a *basis of eigenvectors.* In this case, with a proper selection of the basis vector, the map assumes a diagonal form:

$$\begin{Vmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & 0 & \lambda_n \end{Vmatrix} \, ,$$

where $\lambda_1, ..., \lambda_n$ are the (all distinct) eigenvalues of t. As already stated, the introduction of a nonsingular metric, $g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k$ (with inverse $g^{ik} = \mathbf{e}^i \cdot \mathbf{e}^k$, and $\mathbf{e}^i$ the dual basis of $\mathbf{e}_i$), allows one to identify the mixed tensor ${t^i}_k$ with its covariant counterpart: $t_{ik} = g_{il}{t^l}_k$, or the contravariant one: $t^{ik} = g^{kl}{t^i}_l$ and also the (other) mixed one: ${t_i}^k = g_{il}g^{km}{t^l}_m$.

As a consequence, there exist other equivalent forms to express the parallelism condition (2.126), as well as the condition (2.128). For example, the covariant form of (2.126) is

$$t_{ik}v^k = \lambda v_i = \lambda g_{ik}v^k \, , \tag{2.132}$$

so that (2.128) becomes

$$\det||t_{ik} - \lambda g_{ik}|| = 0 \, . \tag{2.133}$$

In Riemannian spaces, symmetric tensors, $t_{ik} = t_{ki}$ (and $t_{(ik)} = t_{ik}$), have a particular importance as concerns their eigenvectors, satisfying the following property: *eigenvectors corresponding to distinct eigenvalues, are orthogonal to each other, besides being independent.* This does not exclude that a single vector can be isotropic, but this cannot be true for two isotropic vectors because they cannot be orthogonal. In particular, in Riemannian spaces, if the n eigenvalues of a symmetric tensor t are *real and distinct*, the corresponding n eigendirections are orthogonal in pairs, and give rise to a *basis of orthonormal eigenvectors.* Moreover, if E_n is strictly Euclidean, the previous result generalizes so that each symmetric 2-tensor is diagonalizable; that is, as a consequence of the symmetry, one has the reality of the eigenvalues, as well as the existence of orthonormal bases made up of eigenvectors.

Associated with a map t, one has also other maps:

1. the *conjugate map* : Kt (also denoted by t^T), such that $(Kt)^i{}_j = t_j{}^i$;
2. the *complementary map* : Rt, such that $(Rt)^i{}_k\,(Kt)^k{}_j = (\det t)\delta^i_j$;
3. the *inverse map* : t^{-1}, such that $(t^{-1})^i{}_k\,t^k{}_j = t^i{}_k\,(t^{-1})^k{}_j = \delta^i_j$.

A number of relations among the invariants of t and related maps (Kt, Rt, t^{-1}) can be derived. For example, we have

$$\begin{aligned} I_1(Kt) &= I_1(t), \\ I_1(t^2) &= I_1^2(t) - 2I_2(t) \\ I_1(t^3) &= I_1(t)I_1(t^2) - I_2(t)I_1(t) + 3I_3(t)\ , \end{aligned}$$

etc.[13] A used terminology is the following:

1. if $t = Kt$, then t is said a *dilation map*;
2. if $t = -Kt$, then t is said an *axial map*.

Moreover, the Hamilton–Cayley identity [1],

$$\sum_{k=0}^{n}(-1)^k I_{n-k}(t)t^k = 0\ , \tag{2.134}$$

can be used to express the inverse map as a polynom in the map t with coefficients the fundamental invariants:

$$t^{-1} = \frac{1}{I_n(t)}\sum_{j=0}^{n-1}(-1)^j\,I_{n-1-j}(t)t^j \qquad I_0(t) = 1\ . \tag{2.135}$$

For $n = 4$, we have explicitly

$$t^{-1} = \frac{1}{I_4(t)}[I_3(t)\,\mathbb{I} - I_2(t)t + I_1(t)t^2 - t^3]\ , \tag{2.136}$$

which for antisymmetric tensors, having $I_1(t) = 0$ and $I_3(t) = 0$, specializes to the form

[13] Furthermore, if t and τ are two generic vector maps, a number of identities among the associated invariants can be derived, e.g.

$$\begin{aligned} I_2(t+\tau) &= I_2(t) + I_2(\tau) + I_1(t)I_1(\tau) - I_1(t\tau)\ , \\ I_3(t+\tau) &= \frac{1}{3}\Big[I_1(t^3) + I_1(\tau^3) + 3I_1(t^2\tau) + \\ &+ 3I_1(t\tau^2) - I_1^3(t+\tau) + 3I_1(t+\tau)I_2(t+\tau)\Big]\ . \end{aligned}$$

$$t^4 = -I_4(t)\,\mathbb{I} - I_2(t)t^2\ . \tag{2.137}$$

Finally, in special case in which t is a *Rotation* , i.e. $Kt = t$ and $I_n(t) = 1$ the following (Cayley) representation of t holds: there exist a unique skew-symmetric map Q ($KQ = -Q$) so that

$$t = (\mathbb{I} - Q)^{-1}(\mathbb{I} + Q)\ . \tag{2.138}$$

2.17 Levi–Civita Indicator and Ricci Tensor

In any manifold of dimension n (and hence in M_4), one can introduce the affine tensor, also known as *Levi–Civita indicator* $\epsilon^{i_1\cdots i_n}$. This is a n-indices system whose components can only assume values $1, 0, -1$, and precisely

$$\epsilon^{i_1\ldots i_n} = \begin{cases} 0 & \text{if the indices are not all distinct} \\ (-1)^p & \text{if the indices form a } p \text{ class} \\ & \text{permutation (even or odd) of } 12\ldots n. \end{cases} \tag{2.139}$$

$\epsilon^{i_1\cdots i_n}$ can be defined in terms of the generalized Kronecker delta:

$$\epsilon^{i_1\cdots i_n} = \delta^{i_1\ldots i_n}_{1\ldots n}\ , \qquad \epsilon_{i_1\ldots i_n} = \delta^{1\ldots n}_{i_1\ldots i_n}\ . \tag{2.140}$$

In M_4, referred to Cartesian coordinates, we have $\epsilon^{0123} = 1$, and $\epsilon^{\alpha\beta\rho\sigma}$ is an odd-type tensor (changing sign according to the orientation of M_4) whose components transform with the law:

$$\epsilon'^{\alpha\beta\rho\sigma} = \pm \frac{\partial x'^\alpha}{\partial x^\lambda}\frac{\partial x'^\beta}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\nu}\frac{\partial x'^\sigma}{\partial x^\tau}\epsilon^{\lambda\mu\nu\tau}\ , \tag{2.141}$$

with the sign $+$ or $-$ depending on whether

$$\det\left|\left|\frac{\partial x'^\alpha}{\partial x^\lambda}\right|\right| = \pm 1.$$

In the case of Riemannian spaces (associated with a metric g_{ij}, with $g = \det||g_{ij}||$) and general (non-Cartesian) coordinate systems, the role of the Levi-Civita indicator is played by the *Ricci tensor* (see e.g. [1] p. 77) defined by

$$\eta_{i_1\ldots i_n} = [\operatorname{sgn} g]\sqrt{|g|}\epsilon_{i_1\ldots i_n}\ , \tag{2.142}$$

where one has the $[\operatorname{sgn} g] = +1$ when $g > 0$ and $[\operatorname{sgn} g] = -1$ when $g < 0$. $\eta_{i_1\ldots i_n}$ is a true tensor, differently from the Levi–Civita indicator, and it is associated with the unit volume n-form; in M_4, referred to Cartesian coordinates, we have $\eta^{0123} = -\epsilon^{0123} = -1$; when the coordinates are not Cartesian we have instead

$$\eta_{\alpha\beta\mu\nu} = -\sqrt{|g|}\epsilon_{\alpha\beta\mu\nu}\ , \qquad \eta^{\alpha\beta\mu\nu} = \frac{1}{\sqrt{|g|}}\epsilon^{\alpha\beta\mu\nu}\ . \tag{2.143}$$

It is also useful to recall the identities:

$$\eta_{\alpha\beta\rho\sigma}\eta^{\alpha\beta\mu\nu} = -2\delta^{\mu\nu}_{\rho\sigma} \ , \quad \eta_{\alpha\beta\rho\sigma}\eta^{\alpha\lambda\mu\nu} = -\delta^{\lambda\mu\nu}_{\beta\rho\sigma} \ . \tag{2.144}$$

The Ricci tensor is used to define the space-time *dual* of tensors of any rank. In fact, if $t_{i_1\dots i_p}$ is an antisymmetric tensor of rank p (i.e. a p-form), the dual of t is the antisymmetric tensor of rank $4-p$ defined by

$$^*t^{i_{p+1}\dots i_4} = \frac{1}{p!}\eta^{i_1\dots i_p i_{p+1}\dots i_4} t_{i_1\dots i_p} \ . \tag{2.145}$$

In particular, if $t \equiv t_j$ is a vector ($p = 1$), we have

$$^*t^{ijk} = \eta^{mijk} t_m \ ; \tag{2.146}$$

if $t \equiv t_{ij}$ is an antisymmetric tensor of rank 2 ($p = 2$), we have

$$^*t^{ij} = \frac{1}{2}\eta^{lmij} t_{lm} \ , \tag{2.147}$$

so that the dual is also an antisymmetric tensor of rank 2; if $t \equiv t_{ijk}$ is an antisymmetric tensor of rank 3 ($p = 3$), we have

$$^*t^{i} = \frac{1}{6}\eta^{ijkl} t_{jkl} \ , \tag{2.148}$$

so that the dual is a vector; finally, if $t \equiv t_{ijkl}$ is an antisymmetric tensor of rank 4 ($p = 4$), we have

$$^*t = \frac{1}{24}\eta^{ijkl} t_{ijkl} \ , \tag{2.149}$$

so that the dual is a function or a 0-form. The duality operation can be iterated and for an antisymmetric p-tensor we have

$$[^{**}t]^{i_1\dots i_p} = [\mathrm{sgn}\, g]\,(-1)^{p(n-p)} t^{i_1\dots i_p} \ . \tag{2.150}$$

It is worth to note that in the two cases, “space-time” ($n = 4$, $[\mathrm{sgn}\, g] = -1$) and space ($n = 3$, $[\mathrm{sgn}\, g] = 1$), the previous relation imply

$$^{**}t = (-1)^{p-1} t \qquad (\text{space} - \text{time}) \tag{2.151}$$

$$^{**}t = t \qquad (\text{space}) \ . \tag{2.152}$$

A final remark concerns notation for the duality operation. It is conventional to put the * on the left of the symbol denoting the tensor (unless one needs not to specify a right-duality operation different from a left-duality operation, especially when dealing with antisymmetric tensor of rank higher than 2). This notation is but less convenient when one uses index-free notation and products of many vector maps. Therefore, with an abuse of notation, in such cases, we often put the * over the letter denoting the tensor.

2.18 General Lorentz Transformations: II

In this section, we study the geometry of 4-rotations in M_4 using the representation of the general Lorentz group in terms of antisymmetric 2-tensors [6].

As we have already seen in Sect. 2.10, the Lorentz group is the set of all possible coordinate transformations in M_4 associated with two Cartesian (orthonormal) frames $\{\Omega, x^\alpha, \mathbf{c}_\alpha\}$ and $\{\Omega', x'^\alpha, \mathbf{c}'_\alpha\}$. These are linear and inhomogeneous:

$$x^\alpha = L^\alpha{}_\beta x'^\beta + T^\alpha \; ; \tag{2.153}$$

the four coefficients T^α are arbitrary and represent space-time translations while the coefficients $L^\alpha{}_\beta$,

$$L^\alpha{}_\beta = \frac{\partial x^\alpha}{\partial x'^\beta} \; , \tag{2.154}$$

are associated with the orthonormality of the vectors $\mathbf{c}_\alpha$ and $\mathbf{c}'_\alpha$:

$$\mathbf{c}_\alpha \cdot \mathbf{c}_\beta = m_{\alpha\beta} = \mathbf{c}'_\alpha \cdot \mathbf{c}'_\beta \; . \tag{2.155}$$

The coefficients $L^\alpha{}_\beta$ characterize also the transformation laws of the unit vectors of the two considered frames

$$\mathbf{c}'_\beta = L^\alpha{}_\beta \mathbf{c}_\alpha \; , \tag{2.156}$$

so that they can be interpreted as the coefficients of a 4-rotation L which maps the vectors $\mathbf{c}_\alpha$ into the vectors $\mathbf{c}'_\alpha$:

$$\mathbf{c}'_\beta = L\mathbf{c}_\beta \; . \tag{2.157}$$

The role of the two frames can be exchanged by moving simply the position of the prime so that (2.153), (2.154) and (2.156) have their analogous:

$$\begin{cases} x'^\alpha = L'^\alpha{}_\beta x^\beta + T'^\alpha & (T'^\alpha = -L'^\alpha{}_\beta T^\beta) \\ L'^\alpha{}_\beta = \dfrac{\partial x'^\alpha}{\partial x^\beta} \; , & \mathbf{c}_\beta = L'^\alpha{}_\beta \mathbf{c}'_\alpha . \end{cases} \tag{2.158}$$

The matrices $L^\alpha{}_\beta$ and $L'^\alpha{}_\beta$ are inverse of each other because of the identities:

$$\delta^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x'^\beta} = \frac{\partial x'^\alpha}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\beta} \quad \Longleftrightarrow \quad L'^\alpha{}_\rho L^\rho{}_\beta = \delta^\alpha_\beta \; . \tag{2.159}$$

Furthermore, the orthonormality properties (2.155) provide restrictions to the change of basis (2.156), i.e. to the matrix $L^\alpha{}_\beta$; in fact, one has the following ten relations:

$$L^\rho{}_\alpha L^\sigma{}_\beta m_{\rho\sigma} = m_{\alpha\beta} \qquad (\alpha, \beta = 0, 1, 2, 3) \; , \tag{2.160}$$

so that (2.153) specify a *group* of coordinate transformations $\mathcal{L}_{10}$ with 10 parameters: $(16-10) = 6$ for the $L^\alpha{}_\beta$ and 4 for the T^α. This is called *Poincaré* *group* of the Minkowski space M_4. In terms of changes of bases (2.156), (2.160) characterizes the *orthogonal group* $\mathcal{L}_6$ of M_4. Two different points of view are then possible, one associated with the orthogonal Cartesian coordinates of M_4 and the other with the orthonormal frames of M_4.

Let us assume the basis $\mathbf{c}_\alpha$ to be fixed while the basis $\mathbf{c}'_\alpha$ as well as the coefficients $L^\alpha{}_\beta$ of (2.156) to be allowed to vary. Among all numerical matrices satisfying (2.160), there is the *identity matrix*:

$$L^\alpha{}_\beta = \delta^\alpha_\beta \ , \tag{2.161}$$

i.e. the rotation which leaves the initial basis as invariant. Such a rotation has determinant $+1$, whereas a generic Lorentz matrix satisfies the condition (similar to that valid for a rotation in the ordinary space):

$$\det ||L^\alpha{}_\beta|| = \pm 1 \ . \tag{2.162}$$

Therefore, if one considers the $L^\alpha{}_\beta$ as continuous functions of a parameter $t \in (0, T)$, with the initial condition $L^\alpha{}_\beta(0) = \delta^\alpha_\beta$, it is not possible to obtain all the orthonormal bases $\{\mathbf{c}'_\alpha\} \in M_4$: those derived by applying to $\mathbf{c}_\alpha$ a matrix with determinant -1 are excluded. This also happens in the ordinary space where the rotation group (which depends on three parameters) is not connected, but consists in two connected parts, O^+ and O^- (O stands for orthonormal); in fact

1. in both O^+ and O^-, there are only rotations with determinant $+1$ (equi-oriented bases: both left-handed or right-handed);
2. any two bases, one in O^+ and the other in O^-, are related by an anti-rotation: $\det ||\mathcal{R}^i{}_k|| = -1$ (non equi-oriented bases: one left-handed and the other right-handed).

As stated above, in the case of M_4, the connected parts become four: $\mathrm{O}^+(\mathcal{C}_3^\pm)$ and $\mathrm{O}^-(\mathcal{C}_3^\pm)$ because of the presence of the lightcone $\mathcal{C}_3$. In fact, $\mathcal{C}_3$ is a barrier not only between timelike (internal) and spacelike (external) vectors but also for the timelike vectors which cannot pass continuously from a half-cone to the other.

Alternatively, the condition that the orthonormal bases $\{\mathbf{c}_\alpha\}$ and $\{\mathbf{c}'_\alpha\}$ are equi-oriented, i.e. $\det ||L^\alpha{}_\beta|| = 1$, does not imply the possibility to pass from one to the other continuously: it is necessary to add the condition that the two timelike vectors $\mathbf{c}_0$ and $\mathbf{c}'_0$ belong to the same half lightcone. Thus, to remain in one of the four connected parts, $\mathrm{O}^\pm(\mathcal{C}_3^\pm)$, one must consider the set of all *orthonormal bases* O such that for each pair $\{\mathbf{c}_\alpha\}, \{\mathbf{c}'_\alpha\} \in \mathrm{O}$ the following conditions hold:

1. $\mathbf{c}_0$ and $\mathbf{c}'_0$ belong to the same half lightcone;
2. $\det ||L^\alpha{}_\beta|| = 1$.

Assuming the above conditions, each of the four connected parts of M_4 is endowed with equi-oriented orthonormal bases.

2.18.1 Representation of the 4-Rotations with Pairs of Spatial Vectors

Let us consider one of the four connected parts of the homogeneous Lorentz group. In this case, Lorentz transformations generate a continuous group of transformations with six parameters: O_6; in addition, for each pair of orthonormal bases $\{\mathbf{c}_\alpha\}, \{\mathbf{c}'_\alpha\} \in O_6$, the timelike vectors $\mathbf{c}_0 \equiv \boldsymbol{\gamma}$, $\mathbf{c}'_0 \equiv \boldsymbol{\gamma}'$, belong to the same half lightcone and specify *two orthochronous Galilean frames*: S_g and S'_g, having generic spatial triads $\{\mathbf{c}_i\}$ and $\{\mathbf{c}'_i\}$. We note that one can always consider the special case of triads in x^1-standard relation by performing an ordinary rotation in Σ (the space platform associated with $\boldsymbol{\gamma}$ in S_g) and an analogous rotation in Σ' (the space platform associated with $\boldsymbol{\gamma}'$ in S'_g). More precisely, as we have seen in Sect. 2.10, denoting by $\mathbf{u}$ the *relative velocity* of the Galilean frame S'_g with respect to S_g, the following relations hold:

$$\begin{cases} \mathbf{c}'_0 = \rho\left(\mathbf{c}_0 + \dfrac{1}{c}\mathbf{u}\right) \ , \quad \rho = \dfrac{1}{\sqrt{1-\dfrac{u^2}{c^2}}} \\ \mathbf{c}'_k = \mathcal{R}^i{}_k\left[\mathbf{c}_i + \dfrac{\rho}{c}\mathbf{u}\cdot\mathbf{c}_i\left(\mathbf{c}_0 + \dfrac{\rho}{c}\dfrac{\mathbf{u}}{1+\rho}\right)\right] . \end{cases} \tag{2.163}$$

These relations involve six parameters: the three components of $\mathbf{u} \in \Sigma$ and the three parameters corresponding to the spatial rotation $\mathcal{R}$, such that

$$\det ||\mathcal{R}^i{}_k|| = 1 \ . \tag{2.164}$$

Equation (2.163) can be cast into a more familiar form by using the representation of the ordinary rotations in terms of three parameters and, in particular, in terms of a single vector $\mathbf{q}$. In fact for an ordinary rotation $\mathcal{R}$, we have the representation [1, 3]

$$\mathbf{v}' = \mathcal{R}\mathbf{v} = \mathbf{v} + \frac{2}{1+q^2}[\mathbf{q}\times\mathbf{v} + \mathbf{q}\times(\mathbf{q}\times\mathbf{v})] \ , \tag{2.165}$$

implying that the transformed vectors of $\mathbf{c}_i \in \Sigma$ are given by

$$\Delta_k \stackrel{\mathrm{def}}{=} \mathcal{R}^i{}_k\mathbf{c}_i = \mathbf{c}_k + \frac{2}{1+q^2}[\mathbf{q}\times\mathbf{c}_k + \mathbf{q}\cdot\mathbf{c}_k\mathbf{q} - q^2\mathbf{c}_k)] \ . \tag{2.166}$$

Introducing now the components of $\mathbf{q}$ along $\mathbf{c}_k$, $\mathbf{q} = q^k\mathbf{c}_k$, as well as the spatial Levi–Civita alternating symbol

$$\epsilon_{ilk} = \mathbf{c}_i\cdot\mathbf{c}_l\times\mathbf{c}_k \ , \tag{2.167}$$

we find the coefficients $\mathcal{R}^i{}_k$

$$\mathcal{R}^i{}_k = \delta^i_k + \frac{2}{1+q^2}(q^l\epsilon_{lk}{}^i + q^iq_k - q^2\delta^i_k) \ ; \tag{2.168}$$

where $\mathcal{R}^i{}_k q^k = q^i$, i.e. $\mathbf{q}$ is an eigenvector of $\mathcal{R}$; moreover, $\mathbf{q}$ specifies the rotation axis (with its direction) as well as the rotation angle (with its magnitude; in fact $q = \tan(\phi/2)$ where ϕ is the angle of rotation). For example, when $\mathbf{q} = \tan(\phi/2)\mathbf{c}_3$, we find

$$\mathcal{R}^i{}_k = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.169}$$

Similarly, using the decomposition $\mathbf{u} = u^i\mathbf{c}_i$, we also have $\mathcal{R}^i{}_k u_i = \Delta_k \cdot \mathbf{u}$. Adopting such a notation allows us to rewrite (2.163) in a more compact form

$$\begin{cases} \mathbf{c}'_0 = \rho\left(\mathbf{c}_0 + \dfrac{1}{c}\mathbf{u}\right), \\ \mathbf{c}'_k = \Delta_k + \dfrac{\rho}{c}\Delta_k \cdot \mathbf{u}\left(\mathbf{c}_0 + \dfrac{\rho}{c}\dfrac{\mathbf{u}}{1+\rho}\right). \end{cases} \tag{2.170}$$

Note that in the limit $c \to \infty$, (2.163) reduce to

$$\mathbf{c}'_0 = \mathbf{c}_0, \qquad \mathbf{c}'_k = \Delta_k = \mathcal{R}^i{}_k \mathbf{c}_i \ . \tag{2.171}$$

The components of the Lorentz matrix $L^\alpha{}_\beta$ are then given by

$$\begin{cases} L^0{}_0 = \rho\ , \quad L^i{}_0 = \dfrac{\rho}{c}u^i \\ L^0{}_k = \dfrac{\rho}{c}\Delta_k \cdot \mathbf{u}\ , \quad L^i{}_k = \mathbf{c}^i \cdot \Delta_k + \dfrac{1}{c^2}\dfrac{\rho^2}{1+\rho}u^i\Delta_k \cdot \mathbf{u}. \end{cases} \tag{2.172}$$

Let us focus, now, on the *composition law for the product of two rotations*. In fact, in addition to $L : \{\mathbf{c}_\alpha\} \to \{\mathbf{c}'_\alpha\}$ (characterized by the vectors $\mathbf{q}, \mathbf{u} \in \Sigma$) consider a second rotation: $L' : \{\mathbf{c}'_\alpha\} \to \{\mathbf{c}''_\alpha\}$ (characterized by the vectors $\mathbf{q}', \mathbf{u}' \in \Sigma'$) which implies

$$\begin{cases} \mathbf{c}''_0 = \rho'\left(\mathbf{c}'_0 + \dfrac{1}{c}\mathbf{u}'\right)\ , \qquad \rho' = \dfrac{1}{\sqrt{1 - \dfrac{u'^2}{c^2}}} \\ \mathbf{c}''_k = \Delta'_k + \dfrac{\rho'}{c}\Delta'_k \cdot \mathbf{u}'\left(\mathbf{c}'_0 + \dfrac{\rho'}{c}\dfrac{\mathbf{u}'}{1+\rho'}\right)\ , \qquad \Delta'_k = \Delta_k(\mathbf{q}'), \end{cases} \tag{2.173}$$

where

$$\mathbf{u}' = u'^i\mathbf{c}'_i\ , \qquad \mathbf{q}' = q'^i\mathbf{c}'_i\ . \tag{2.174}$$

The transformation

$$L'' = L'L \tag{2.175}$$

directly maps the basis $\{\mathbf{c}_\alpha\}$ into $\{\mathbf{c}''_\alpha\}$ and can be represented by relations analogous to (2.170) with both the vectors $\mathbf{q}'', \mathbf{u}'' \in \Sigma$, namely,

$$\begin{cases} \mathbf{c}_0'' = \rho''\left(\mathbf{c}_0 + \dfrac{1}{c}\mathbf{u}''\right) , & \rho'' = \dfrac{1}{\sqrt{1 - \dfrac{u''^2}{c^2}}} \\ \mathbf{c}_k'' = \Delta_k'' + \dfrac{\rho''}{c}\Delta_k'' \cdot \mathbf{u}''\left(\mathbf{c}_0 + \dfrac{\rho''}{c}\dfrac{\mathbf{u}''}{1+\rho''}\right) , & \Delta_k'' = \Delta_k(\mathbf{q}''). \end{cases} \tag{2.176}$$

The vectors $\mathbf{q}'', \mathbf{u}'' \in \Sigma$ are functions of $\mathbf{q}, \mathbf{u} \in \Sigma$ and $\mathbf{q}', \mathbf{u}' \in \Sigma'$; such a relation thus represents the *composition law* of two 4-rotations. Note that this law can also be obtained directly in terms of the vector $\mathbf{q}$ only, exactly as the composition law of the product of two ordinary rotations (or Rodrigues formula, see [3], p. 113):

$$\mathbf{q}'' = \frac{\mathbf{q} + \mathbf{q}' + \mathbf{q}' \times \mathbf{q}}{1 - \mathbf{q}' \cdot \mathbf{q}} . \tag{2.177}$$

A similar relation (with the only difference of a + sign at the denominator of (2.177)) holds in the three-dimensional hyperbolic case, with signature: $-, +, +$.

In the case of a Minkowski space-time, we find convenient to generalize (2.177) using the representation of 4-rotations L in terms of antisymmetric 2-tensors. This is quite natural since an antisymmetric 2-tensor is equivalent to a pair of spatial vectors, as for the case of the electromagnetic tensor, which can be represented in terms of the electric and magnetic vector fields.

Later we will briefly introduce another method, based on Clifford's algebra [7] and also associated with antisymmetric tensors. We thus proceed now analyzing the general properties of antisymmetric 2-tensors in M_4 [8].

2.18.2 Invariants of an Antisymmetric 2-Tensor

Let us consider a four-dimensional *linear space* E_4, and let $\mathbf{A}$ be a contravariant antisymmetric 2-tensor, i.e. $\mathbf{A} \in E_4 \wedge E_4$

$$\mathbf{A} = \frac{1}{2} A^{\alpha\beta} \mathbf{e}_\alpha \wedge \mathbf{e}_\beta , \tag{2.178}$$

where $\{\mathbf{e}_\alpha\}$ is a generic basis in E_4.[14] Due to a general property (see [1], p. 53), the tensor $\mathbf{A}$

(1) is a bivector: $\mathbf{A} = \mathbf{u} \wedge \mathbf{v}$,

or

(2) it can be expressed (in infinite ways) as the sum of two bivectors:

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_1' , \quad \mathbf{A}_1 = \mathbf{v}_0 \wedge \mathbf{v}_1 , \quad \mathbf{A}_1' = \mathbf{v}_2 \wedge \mathbf{v}_3 , \tag{2.179}$$

[14] As a standard notation, we use bold face letters to denote tensors (noncapital letters for vectors mainly). The vectorial map associated with the mixed representation of a 2-tensor is denoted by the same (capital) letter as for the tensor but is not in bold face.

where $\mathbf{v}_\alpha$ $(\alpha = 0, 1, 2, 3)$ are four linearly independent vectors: $\mathbf{v}_0 \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3 \equiv \mathbf{V} \neq 0$. In case 2), assuming the (ordered) set of vectors $\mathbf{v}_\alpha$ as a basis in E_4, the components of $\mathbf{A}$ with respect to such a basis are

$$A^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} ,$$

and $\det ||A^{\mu\nu}|| = 1$. Moreover, because of the tensorial behaviour of the components $A^{\mu\nu}$, the sign of this determinant is invariant under a change of basis, i.e. it does not depend on the choice of the basis. Hence, it is always positive for every $\mathbf{A}$ which is not a bivector. Therefore, *for every contravariant* (*or covariant*) *antisymmetric* 2*-tensor of the form* (2.179) *we have*

$$\det ||A^{\mu\nu}|| \geq 0 , \tag{2.180}$$

where the equality holds *if and only if* $\mathbf{A}$ is a bivector.

The above property holds, in particular, if $E_4 \equiv M_4$. In this case, denoting the metric tensor as $g_{\alpha\beta}$ (generically), one can consider the various forms of the tensor $\mathbf{A}$: covariant $(A_{\alpha\beta})$, contravariant $(A^{\alpha\beta})$ and mixed $(A^\alpha{}_\beta = g_{\beta\rho}A^{\alpha\rho})$, the latter identifying the vectorial map A. One can then consider the four *invariants* of A:

$$\begin{cases} I_1(A) = A^\alpha{}_\alpha = \mathrm{Tr}\ A = 0 \\ I_2(A) = \dfrac{1}{2}\delta^{\rho\sigma}_{\alpha\beta}A^\alpha{}_\rho A^\beta{}_\sigma = -\dfrac{1}{2}A^\alpha{}_\rho A^\rho{}_\alpha = -\dfrac{1}{2}I_1(A^2) \\ I_3(A) = \dfrac{1}{3!}\delta^{\rho\sigma\nu}_{\alpha\beta\mu}A^\alpha{}_\rho A^\beta{}_\sigma A^\mu{}_\nu = 0 \\ I_4(A) = \det ||A^\alpha{}_\beta|| = g \det ||A^{\alpha\beta}||, \end{cases} \tag{2.181}$$

where $\delta^{\beta_1 \dots \beta_k}_{\alpha_1 \dots \alpha_k}$ is the *generalized Kronecker tensor* introduced in (2.125). Equation $(2.181)_4$ together with (2.180) implies that, if A is nondegenerate, $I_4(A)$ has the sign of the determinant of the space-time metric g; in particular, assuming the basis $\{\mathbf{v}_\alpha\}$ used to represent the bivectors $\mathbf{A}_1$ and $\mathbf{A}'_1$ in (2.179) instead of $\{\mathbf{e}_\alpha\}$, leads to $I_4(A) = g$ implying then $I_4(A) \leq 0$ in M_4. Due to the invariant property of $I_4(A)$, this result specifies its geometrical meaning. In fact, generically $\mathbf{V} = \mathbf{v}_0 \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3$ is associated with its *absolute extension* V_A, defined by the positive invariant:

$$V_A^2 \stackrel{\text{def}}{=} \frac{1}{4!}|V_{\alpha\beta\rho\sigma}V^{\alpha\beta\rho\sigma}| , \qquad V^{\alpha\beta\rho\sigma} = \delta^{\alpha\beta\rho\sigma}_{\mu\nu\tau\epsilon}v_0^\mu v_1^\nu v_2^\tau v_3^\epsilon , \tag{2.182}$$

as well as its *extension relative to a frame* $\{\mathbf{e}_\alpha\}$, V_R:

$$V_R \stackrel{\text{def}}{=} \det ||v^\alpha{}_\mu|| , \qquad \mathbf{v}_\mu = v^\alpha{}_\mu \mathbf{e}_\alpha ; \tag{2.183}$$

V_R then is related to V_A by the property

$$V_A^2 = V_R^2 |g| = \text{inv.} = |\det ||\mathbf{v}_\alpha \cdot \mathbf{v}_\beta|| \,| \,, \tag{2.184}$$

invariant with respect to the choice of the basis $\{\mathbf{e}_\alpha\}$. When $\{\mathbf{e}_\alpha\} \equiv \{\mathbf{v}_\alpha\}$, we find $V_R = 1$ and, using $(2.181)_4$ and (2.184), we see that both scalars $|I_4(A)|$ and V_A^2 assume the same value $|g|$; hence, they coincide: $|I_4(A)| = V_A^2$, and $\mathbf{V}$ can then be written as

$$\mathbf{V} = \mathbf{v}_0 \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3 = \mathbf{A}_1 \wedge \mathbf{A}_1' \equiv \frac{1}{2}\mathbf{A} \wedge \mathbf{A} \,. \tag{2.185}$$

Let us define the (pseudoscalar) invariant $\overset{*}{I}(A)$:

$$[\overset{*}{I}(A)]^2 \overset{\text{def}}{=} -I_4(A) \geq 0 \,. \tag{2.186}$$

We have

$$[\overset{*}{I}(A)]^2 = V_A^2 \,; \tag{2.187}$$

moreover, denoting by ${}^*V$ the dual of V (odd-type scalar):

$${}^*V = \frac{1}{4!}\eta_{\alpha\beta\rho\sigma} V^{\alpha\beta\rho\sigma} \,, \tag{2.188}$$

we have in addition

$${}^*V^2 = -V_A^2 \,, \tag{2.189}$$

so that (2.187) is also equivalent to

$$\overset{*}{I}(A) = {}^*V \,. \tag{2.190}$$

2.18.3 Algebraic Properties of Antisymmetric 2-Tensors

From (2.178) we have

$$\mathbf{V} = \frac{1}{8} A^{\alpha\beta} A^{\rho\sigma} \mathbf{e}_\alpha \wedge \mathbf{e}_\beta \wedge \mathbf{e}_\rho \wedge \mathbf{e}_\sigma \equiv \frac{4!}{8} A^{[\alpha\beta} A^{\rho\sigma]} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\rho \otimes \mathbf{e}_\sigma \,,$$

so that the components of V are given by

$$V^{\alpha\beta\rho\sigma} = 3 A^{[\alpha\beta} A^{\rho\sigma]} \,, \tag{2.191}$$

and we find

$${}^*V \equiv \frac{1}{4!}\eta_{\alpha\beta\rho\sigma} V^{\alpha\beta\rho\sigma} = \frac{1}{8}\eta_{\alpha\beta\rho\sigma} A^{[\alpha\beta} A^{\rho\sigma]} = \frac{1}{8}\eta_{\alpha\beta\rho\sigma} A^{\alpha\beta} A^{\rho\sigma} \,. \tag{2.192}$$

Using then the dual of A

$$\overset{*}{\mathbf{A}} = \frac{1}{2}{}^*A^{\alpha\beta} \mathbf{e}_\alpha \wedge \mathbf{e}_\beta \,, \qquad {}^*A^{\alpha\beta} = \frac{1}{2}\eta^{\alpha\beta}{}_{\rho\sigma} A^{\rho\sigma} \,, \tag{2.193}$$

is easy to prove the additional property of $\overset{*}{I}(A)$:

$$\overset{*}{I}(A) = \frac{1}{4}{}^*A_{\alpha\beta}A^{\alpha\beta} = -\frac{1}{4}I_1(A\overset{*}{A}) \; . \tag{2.194}$$

$\overset{*}{I}(A)$ characterizes the product of A and $\overset{*}{A}$ (see [9], p. 588):

$$\overset{*}{A}A = -\overset{*}{I}(A)A^0 \qquad A^0 \equiv \mathbb{I} \; ; \tag{2.195}$$

moreover, since $\overset{*}{I}(A)$ is a *symmetric function* of A and $\overset{*}{A}$, we also have

$$A\overset{*}{A} = \overset{*}{A}A = -\overset{*}{I}(A)A^0 \; . \tag{2.196}$$

From this equation, taking the determinant of both sides, it follows

$$I_4(A)I_4(\overset{*}{A}) = [\overset{*}{I}(A)]^4 \; ,$$

and (2.186) becomes

$$I_4(A) = I_4(\overset{*}{A}) = -[\overset{*}{I}(A)]^2 \; . \tag{2.197}$$

A second property of $\overset{*}{I}(A)$ follows from (2.196). In fact, replacing A by $A+B$ in (2.196) leads to

$$(A+B)(\overset{*}{A}+\overset{*}{B}) = -\overset{*}{I}(A+B)A^0 \; ;$$

the left-hand side of this equation can be cast in the form

$$A\overset{*}{A} + A\overset{*}{B} + B\overset{*}{A} + B\overset{*}{B} = \frac{1}{4}I_1(A\overset{*}{A})A^0 + A\overset{*}{B} + B\overset{*}{A} + \frac{1}{4}I_1(B\overset{*}{B})A^0$$

while at the right-hand side we have

$$\overset{*}{I}(A+B) = -\frac{1}{4}\left[I_1(A\overset{*}{A}) + I_1(B\overset{*}{B}) + 2I_1(A\overset{*}{B})\right] \; . \tag{2.198}$$

Therefore we find

$$A\overset{*}{B} + B\overset{*}{A} = \frac{1}{2}I_1(A\overset{*}{B})A^0 \; , \tag{2.199}$$

i.e. a relation equivalent to (2.166), to which it reduces when $A = B$. Next, replacing $\overset{*}{B}$ by B and B by $-\overset{*}{B}$ (being ${}^{**} = -1$, as shown in Sect. 2.17), (2.199) becomes

$$AB - \overset{*}{B}\overset{*}{A} = \frac{1}{2}I_1(AB)A^0 \; ; \tag{2.200}$$

multiplying this equation by AB and using (2.196), we can express the square of AB as a linear function of the same AB:

$$(AB)^2 = \overset{*}{I}(A)\overset{*}{I}(B)A^0 + \frac{1}{2}I_1(AB)AB \; . \tag{2.201}$$

We note that when $B = A$, using (2.181) and (2.197), (2.201) gives the Hamilton–Cayley identity for antisymmetric 2-tensors:

$$A^4 = -I_4(A)A^0 - I_2(A)A^2 \ . \tag{2.202}$$

Multiplying (2.200) by A we have

$$ABA = \frac{1}{2}I_1(AB)A - \overset{*}{I}(A)\overset{*}{B} \ ; \tag{2.203}$$

when $B = A$, the latter allows us to obtain A^3 in terms of A and $\overset{*}{A}$:

$$A^3 = -I_2(A)A - \overset{*}{I}(A)\overset{*}{A} \ . \tag{2.204}$$

Equations (2.199) and (2.200) together with (2.201) imply simple properties for the Poisson brackets $[A, B]$:

$$[A, B] \equiv AB - BA \ , \tag{2.205}$$

which give the structure of a Lie algebra to the (Euclidean) space of antisymmetric 2-tensors $\Lambda^2 = M_4 \wedge M_4$. For instance, from (2.199), with $A = B$ and $B = -\overset{*}{A}$, we have: $A\overset{*}{B} + B\overset{*}{A} = \overset{*}{B}A + \overset{*}{A}B$, that is

$$[A, \overset{*}{B}] = [\overset{*}{A}, B] \ . \tag{2.206}$$

Similarly, from (2.200), rewritten also exchanging A with B, one finds

$$[A, B] = -[\overset{*}{A}, \overset{*}{B}] \ . \tag{2.207}$$

Finally, we have

$$^*[A, B] = [A, \overset{*}{B}] = [\overset{*}{A}, B] \ ; \tag{2.208}$$

the parenthesis $[A, \overset{*}{A}]$, instead, vanishes identically.

Therefore, two independent antisymmetric 2-tensors: A and B together with $\overset{*}{A}$, $\overset{*}{B}$, $[A, B]$ and $^*[A, B]$ *form a basis in the linear space* Λ^2.

2.18.4 Bivectors and Their Classification

The properties of antisymmetric 2-tensors outlined in the previous section do not imply any special requirement for A and B. However, it is convenient to distinguish between the *general case*: $I_4(A) \neq 0$ and the *degenerate one*: $I_4(A) = 0$.

The decomposition (2.179) suggests to study the general case starting from bivectors. To this end, let us consider the bivector $\mathbf{A} = \mathbf{u} \wedge \mathbf{v}$, with $\mathbf{u}$ and $\mathbf{v}$ independent and defining a linear subspace $E_2 \equiv <\mathbf{u}, \mathbf{v}>$; let us denote by $A^{\alpha\beta} = u^\alpha v^\beta - u^\beta v^\alpha$ the components of A with respect to the basis $\{\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\}$

of M_4^2; equivalently, $A^{\alpha\beta}$ are the contravariant coefficients of the linear map associated with A which maps each vector $\mathbf{w} \in M_4$ into a vector $A\mathbf{w}$ given by

$$A\mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\mathbf{u} - (\mathbf{u} \cdot \mathbf{w})\mathbf{v} \ , \tag{2.209}$$

belonging to E_2 (principal plane) and orthogonal to $\mathbf{w}$: $A\mathbf{w} \cdot \mathbf{w} = 0$; thus, $A\mathbf{w}$ has the direction of the intersection between E_2 and the hyperplane orthogonal to $\mathbf{w}$. The exception is represented by those vectors orthogonal to E_2, which belong to the kernel of A: $A\mathbf{w} = 0, \forall \mathbf{w} \perp E_2$.

Since $AM_4 = E_2$, all possible (real) eigendirections of A belong to E_2. Furthermore, $I_4(A) = 0$, since A is a bivector. The eigenvalue equation is then given by

$$\det ||A^\alpha{}_\beta - \lambda\delta^\alpha_\beta|| \equiv \lambda^2[I_2(A) + \lambda^2] = 0 \ , \tag{2.210}$$

and admits, besides the double root $\lambda = 0$ (the vectors of the 2-plane normal to E_2 are all in the kernel of A), the roots of the equation

$$I_2(A) + \lambda^2 = 0 \ , \tag{2.211}$$

where $I_2(A)$ is given by

$$I_2(A) = ||\mathbf{u}||\,||\mathbf{v}|| - (\mathbf{u} \cdot \mathbf{v})^2 \ . \tag{2.212}$$

Let us now assume, without loss of generality, that the basis $\{\mathbf{e}_\alpha\}$ is adapted to E_2: $\mathbf{e}_i \in E_2\,(i = 0, 1)$; denoting by $\mathcal{V}$ the extension of $\mathbf{A}$ with respect to $\mathbf{e}_0 \wedge \mathbf{e}_1$, we have

$$I_2(A) = \mathcal{V} g_A \ , \qquad g_A = \det ||\mathbf{e}_i \cdot \mathbf{e}_k|| = ||\mathbf{e}_0||\,||\mathbf{e}_1|| - (\mathbf{e}_0 \cdot \mathbf{e}_1)^2 \ . \tag{2.213}$$

We distinguish then the following three cases:

1. $g_A > 0 \quad \sim \quad I_2(A) > 0$: the signature of E_2 is $(++)$, i.e. E_2 is *elliptic*, and there are not null directions; (2.211) has no real solutions and, consequently, in E_2, there are no real eigendirections.
2. $g_A < 0 \quad \sim \quad I_2(A) < 0$: the signature of E_2 is $(-+)$, i.e. E_2 is *hyperbolic*. Besides the double root $\lambda = 0$, there are two different real eigenvalues: $\lambda = \pm\sqrt{-I_2(A)}$, and hence there exist in E_2 two independent eigendirections: $\mathbf{u}_1$ and $\mathbf{u}_2$. One then has

$$\mathbf{u} \wedge \mathbf{v} = h\mathbf{u}_1 \wedge \mathbf{u}_2 \ , \tag{2.214}$$

where h is the signed extension of the parallelogram $(\mathbf{u}, \mathbf{v})$ with respect to $(\mathbf{u}_1, \mathbf{u}_2)$. Then, from (2.209) follows that

$$A\mathbf{w} = h[(\mathbf{u}_2 \cdot \mathbf{w})\mathbf{u}_1 - (\mathbf{u}_1 \cdot \mathbf{w})\mathbf{u}_2], \qquad \forall \mathbf{w} \in M_4 \ , \tag{2.215}$$

so that

$$\begin{aligned} A\mathbf{u}_1 &= h[(\mathbf{u}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{u}_1 \cdot \mathbf{u}_1)\mathbf{u}_2], \\ A\mathbf{u}_2 &= h[(\mathbf{u}_2 \cdot \mathbf{u}_2)\mathbf{u}_1 - (\mathbf{u}_1 \cdot \mathbf{u}_2)\mathbf{u}_2] \ , \end{aligned} \tag{2.216}$$

and, since $\mathbf{u}_1$ and $\mathbf{u}_2$ are the eigendirections of A ($A\mathbf{u}_1$ parallel to $\mathbf{u}_1$ and $A\mathbf{u}_2$ parallel to $\mathbf{u}_2$), we have that $\mathbf{u}_1 \cdot \mathbf{u}_1 = 0$ and $\mathbf{u}_2 \cdot \mathbf{u}_2 = 0$, i.e. $\mathbf{u}_1$ and $\mathbf{u}_2$ are necessarily the two isotropic directions of E_2; the associated eigenvalues λ are given by $\lambda = \pm(h\mathbf{u}_1 \cdot \mathbf{u}_2)$, in agreement with (2.213). In particular, if $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy the normalization condition $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1$ (which still leaves $\mathbf{u}_1$ and $\mathbf{u}_2$ defined up to a factor), one has $\lambda = \pm h$.

3. $g_A = 0 \sim I_2(A) = 0$: (2.211) has the double root $\lambda = 0$. The vectors of E_2 cannot have positive (negative) norm, because in such a case, g_A would be positive (negative) too. Thus, *there necessarily exists a null vector* $\mathbf{u}$. Assuming that one of the vectors $\mathbf{e}_i$ coincides with $\mathbf{u}$, we see, from (2.213), that $\mathbf{u}$ is orthogonal to all the vectors of E_2. Moreover, it is unique, up to a factor: if there were another one, $\mathbf{u}_1$, not collinear with $\mathbf{u}$, then, from (2.215) with $h = 0$, one would have $A\mathbf{w} = 0, \quad \forall \mathbf{w}$, i.e. $A = 0$, contrarily to the hypothesis.

   Let $\mathbf{s} \in E_2$ be a generic vector orthogonal to $\mathbf{u}$ and, hence, with nonzero norm. The normal (with respect to E_2) not degenerate hyperplane: Π, contains $\mathbf{u}$ and has signature $(-++)$; as a consequence, besides the isotropic direction, E_2 only contains *spatial vectors*: $||\mathbf{s}|| > 0$.

   Actually, one still has a relation similar to (2.215):

$$A\mathbf{w} = h[(\mathbf{s} \cdot \mathbf{w})\mathbf{u} - (\mathbf{u} \cdot \mathbf{w})\mathbf{s}] \ ,$$

   which $\forall \mathbf{w} \in E_2$ becomes

$$A\mathbf{w} = h(\mathbf{s} \cdot \mathbf{w})\mathbf{u} \ , \qquad \forall \mathbf{w} \in E_2 \ .$$

   Thus, each vector $\mathbf{w} \in E_2$ is mapped into a vector parallel to $\mathbf{u}$, i.e. along the only null direction of E_2. In fact, the condition $A\mathbf{w} = h(\mathbf{s} \cdot \mathbf{w})\mathbf{u} = 0$, with $\mathbf{w} \in E_2$, is equivalent to $\mathbf{s} \cdot \mathbf{w} = 0$, that is, $\mathbf{w}$ parallel to $\mathbf{u}$. Note that E_2 cannot have orthonormal bases: otherwise the signature should have been $(++)$; moreover, every orthogonal basis of E_2 necessarily contains the null direction $\mathbf{u}$.

   Therefore E_2 is *parabolic* as well as the orthogonal 2-plane which contains $\mathbf{u}$ (the isotropic vector) and the null directions of A, as in the above cases 1 and 2.

We can now discuss some orthogonality properties of antisymmetric 2-tensors.

Let $\mathbf{A} \equiv (A^{\alpha\beta})$ and $\mathbf{B} \equiv (B^{\alpha\beta})$ be two antisymmetric 2-tensors; the invariant

$$\mathbf{A} \cdot \mathbf{B} \stackrel{\text{def}}{=} A_{\alpha\beta}B^{\alpha\beta} = -I_1(AB) \tag{2.217}$$

is called the *scalar product* of $\mathbf{A}$ and $\mathbf{B}$. When $\mathbf{A} \cdot \mathbf{B} = 0$, the two tensors are said orthogonal. We have that *if* $\mathbf{A}$ *is a bivector, then its dual* $\overset{*}{\mathbf{A}}$ *is a bivector orthogonal to* $\mathbf{A}$:

$$I_4(A) = 0 \quad \to \quad I_4(\overset{*}{A}) = 0 \quad \text{and} \quad A \cdot \overset{*}{A} = 0 \ . \tag{2.218}$$

This follows directly from (2.196) and (2.197). The orthogonality property of the bivectors $\mathbf{A}$ and $\overset{*}{\mathbf{A}}$ is equivalent to the orthogonality of the two subspaces E_2 and E_2^*, to which they belong, respectively. To see this let $\mathbf{A} = \mathbf{u} \wedge \mathbf{v}$; because of the independence of $\mathbf{u}$ and $\mathbf{v}$, we have that a vector $\mathbf{w}$ is orthogonal to the plane $E_2 \equiv < \mathbf{u}, \mathbf{v} >$ associated with A if and only if $A\mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\mathbf{u} - (\mathbf{u} \cdot \mathbf{w})\mathbf{v} = 0$. Moreover, using the antisymmetry of the Ricci tensor η and the symmetry of the tensor $u^\alpha u^\beta$, it follows that

$$ {}^*A^{\alpha\beta} u_\beta = \eta^{\alpha\beta\rho\sigma} u_\beta u_\rho v_\sigma = 0 \;, \quad {}^*A^{\alpha\beta} v_\beta = \eta^{\alpha\beta\rho\sigma} v_\beta u_\rho v_\sigma = 0 \;, $$

that is $\overset{*}{A}\mathbf{u} = 0$ and $\overset{*}{A}\mathbf{v} = 0$. Thus $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to the subspace E_2^* associated with $\overset{*}{A}$; as $\mathbf{u}$ and $\mathbf{v}$ span the subspace E_2 associated with A, we also have the orthogonality of E_2 and E_2^*.

As a consequence of the uniqueness of the orthogonal 2-plane to a given bivector $\mathbf{A} = \mathbf{u} \wedge \mathbf{v}$,[15] it follows that, if $\mathbf{A}$ and $\mathbf{B}$ are bivectors associated with orthogonal subspaces, necessarily one has $\mathbf{B} = \lambda \overset{*}{\mathbf{A}}$, with λ a factor.

Note that (2.218) is contained in (2.196) which for bivectors assumes the form $A\overset{*}{A} = 0$; similarly, from (2.201), the orthogonality of two bivectors $\mathbf{A}$ and $\mathbf{B}$ is equivalent to the condition $AB = 0$.

2.18.5 Canonical Decomposition of Antisymmetric 2-Tensors

Let us consider a generic antisymmetric 2-tensor $\mathbf{A}$, i.e. such that

$$ I_4(A) = I_4(\overset{*}{A}) = -[\overset{*}{I}(A)]^2 < 0 \;. \tag{2.219} $$

In this case, the associated map admits an inverse: A^{-1}, and because of (2.196), for the (left) inversion and (left) duality operation, the following relation holds:

$$ \overset{*}{A} = -\overset{*}{I}(A) A^{-1} \;, \tag{2.220} $$

so that the adjoint (or dual) map differs from the inverse by the factor:

$$ -\overset{*}{I}(A) = \frac{1}{4} I_1(A\overset{*}{A}) \;. \tag{2.221} $$

We find then the following commutation property

$$ \overset{*}{A^{-1}} = [\,\overset{*}{A}\,]^{-1} \;. \tag{2.222} $$

We note that for bivectors (2.222) as well as the original (2.220) is meaningless. The following relation between $\overset{*}{A}$ and the complementary map of A: RA, defined by

[15] When A is parabolic, the uniqueness remains, but the 2-plane orthogonal to E_2 is not supplementary because it contains, like E_2, the isotropic direction of A.

$$(RA)^{\mathrm{T}} \stackrel{\mathrm{def}}{=} \overset{*}{I}(A)\overset{*}{A} \,, \tag{2.223}$$

has, instead, a general validity. Equation (2.223) reduces to (2.220) in the general case $I_4(A) < 0$; for bivectors, instead, the complementary map always vanishes, differently from $\overset{*}{A}$.

Let us turn to the decomposition (2.179) of $\mathbf{A}$ as the sum of two bivectors. We have that $\mathbf{A}$ can always be expressed as the *sum of two orthogonal bivectors*, that is the following canonical decomposition holds:

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}'_1, \qquad \mathbf{A}_1 \cdot \mathbf{A}'_1 = 0 \,, \tag{2.224}$$

and it is unique. To show this let us assume the decomposition (2.224) and use the orthogonality property of $\mathbf{A}_1$ and $\mathbf{A}'_1$; we then have for the associated maps (see Sect. 2.18.4)

$$A'_1 = \lambda \overset{*}{A}_1 \,, \quad A_1 = \lambda' \overset{*}{A}_1{}' \,, \quad \lambda\lambda' = -1 \,;$$

thus the decomposition (2.224) becomes

$$A = A_1 + \lambda \overset{*}{A}_1 = A'_1 + \lambda' \overset{*}{A}_1{}' \,, \tag{2.225}$$

where the pair (A_1, λ) determines A up to the transformation

$$A'_1 = \lambda \overset{*}{A}_1 \,, \quad \lambda' = -1/\lambda \,. \tag{2.226}$$

Consider now the dual of both sides of (2.225), and take into account that, for each antisymmetric 2-tensor in M_4, one has $^{**}A = -A$. Therefore, one has the relation: $\lambda \overset{*}{A} = \lambda \overset{*}{A}_1 - \lambda^2 A_1$ which, once subtracted from (2.225), gives A_1 (as well as A'_1) in terms of A and λ:

$$A_1 = \frac{1}{1+\lambda^2}(A - \lambda \overset{*}{A}) \,, \qquad A'_1 = \frac{\lambda}{1+\lambda^2}(\overset{*}{A} + \lambda A) \,. \tag{2.227}$$

Equation (2.227) is not yet the solution because we must require that A is a bivector. To do this, it is enough to impose $I_1(A_1 \overset{*}{A}_1) = 0$ and use (2.227), so that

$$I_1(A\overset{*}{A} + \lambda A^2 - \lambda \overset{*}{A}{}^2 - \lambda^2 \overset{*}{A} A) = 0 \,.$$

From this relation, using (2.194), $(2.181)_2$ and (2.200) with $B = A$,

$$\begin{cases} I_1(A\overset{*}{A}) = I_1(\overset{*}{A}A) = -4\overset{*}{I}(A), \\ I_1(A^2) = -2I_2(A) \,, \qquad I_1(\overset{*}{A}{}^2) = -I_1(A^2), \end{cases} \tag{2.228}$$

we obtain the equation for λ:

$$\overset{*}{I}(A)\lambda^2 - I_2(A)\lambda - \overset{*}{I}(A) = 0 \,. \tag{2.229}$$

If $\overset{*}{I}(A) = 0$, (2.229) then implies $\lambda = 0$, in agreement with the fact that A is a bivector; if, instead, $\overset{*}{I}(A) \neq 0$, it admits the two roots:

$$\lambda_{1,2} = \frac{1}{2\overset{*}{I}(A)} \left[I_2(A) \pm \sqrt{I_2(A)^2 + 4[\overset{*}{I}(A)]^2} \right] , \quad \lambda_1 \lambda_2 = -1 . \tag{2.230}$$

Apart from the degenerate case, and using for λ any of the values (2.230), (2.227) gives the sought for bivectors $\mathbf{A}_1$ and $\mathbf{A}'_1$ of the canonical decomposition (2.224). We note that exchanging the two roots λ_1 and λ_2 given by (2.230) leaves A invariant, and it is equivalent to exchanging A_1 with A'_1 in (2.227).

Let us study now the relation between the invariants of A and those of A_1 and A'_1. We recall that for 2-tensors B and C we have

$$\begin{aligned} I_2(B+C) &\overset{\text{def}}{=} \frac{1}{2}\delta^{\rho\sigma}_{\alpha\beta}(B^\alpha{}_\rho + C^\alpha{}_\rho)(B^\beta{}_\sigma + C^\beta{}_\sigma) \\ &= I_2(B) + I_2(C) + I_1(B)I_1(C) - I_1(BC) . \end{aligned} \tag{2.231}$$

Using now $B = A_1$ and $C = A'_1$ (antisymmetric orthogonal bivectors) together with (2.224), the previous relation becomes

$$I_2(A) = I_2(A_1) + I_2(A'_1) = -\frac{1}{2}I_1(A_1^2) - \frac{1}{2}I_1(A_1'^2) .$$

In order to evaluate $I_4(A)$, one has to consider instead the odd-type scalar $\overset{*}{I}(A)$ of (2.194):

$$\overset{*}{I}(A) = -\frac{1}{4}I_1(A\overset{*}{A}) = -\frac{1}{2}I_1(A_1\overset{*}{A}{}'_1) = \lambda I_1(A_1^2) ,$$

that is,

$$\overset{*}{I}(A) = -\lambda I_2(A_1) . \tag{2.232}$$

Similarly, exchanging A_1 with A'_1, one finds $\overset{*}{I}(A) = -\lambda' I_2(A'_1)$; thus, since $\lambda\lambda' = -1$, we can write

$$\overset{*}{I}(A)^2 \equiv -I_4(A) = -I_2(A_1)I_2(A'_1) . \tag{2.233}$$

Summarizing, the relations between the invariants of A and those of its orthogonal components A_1 and A'_1 are

$$I_2(A) = I_2(A_1) + I_2(A'_1) , \qquad I_4(A) = I_2(A_1)I_2(A'_1) . \tag{2.234}$$

Equation (2.234) confirms the negativity of $I_4(A)$, since of the two bivectors A_1 and A'_1 one is elliptic ($I_2 > 0$) and the other is hyperbolic ($I_2 < 0$).

Equation (2.234) also shows that the sum of two orthogonal parabolic bivectors is still a parabolic bivector; moreover, using (2.233) and (2.232), one has the following expression for λ^2:

$$\lambda^2 = -\frac{I_2(A_1')}{I_2(A_1)} \, . \tag{2.235}$$

2.18.6 Properties of a 4-Rotation

The spectral analysis of antisymmetric tensors which we have discussed above will be now applied to 4-rotations. We start studying the representation of the Lorentz group in terms of antisymmetric 2-tensors, i.e. the so-called *canonical decomposition*. Thought of as a linear map of vectors, a 4-rotation is a vector valued function: $L \equiv ||L^\alpha{}_\beta||$ satisfying the following properties:

1. it maintains the scalar product of any two vectors: $L\mathbf{u} \cdot L\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$, $\forall\, \mathbf{u}, \mathbf{v} \in M_4$;
2. it leaves unchanged the orientation of M_4, that is it maps each basis $\{\mathbf{e}_\alpha\} \in M_4$ into another one with the same orientation.

From the property (1), using the commutation theorem, one gets the condition $(KL)L = L^0$ where $(KL) = L^{\mathrm{T}}$ is the conjugate map of L. Taking the fourth invariant of both sides, and using the property (2), one then gets $I_4(L) = 1$. Conversely, the latter two conditions imply 1 and 2. Thus,[16] *necessary and sufficient condition for a generic L to be a rotation is*

$$(KL)L = L^0 \, , \qquad I_4(L) = 1 \, . \tag{2.236}$$

Let us now introduce the complementary map of L, (RL):

$$(RL) = I_4(L)(KL)^{-1} \, . \tag{2.237}$$

From (2.236), after left multiplication by $(KL)^{-1}$ and using $(2.236)_2$, we find $L = (RL)$; conversely, such equality which reduces (2.237) to the form $L = I_4(L)(KL)^{-1}$ implies $(KL)L = I_4(L)L^0$, so that $[I_4(L)]^2 = [I_4(L)]^4$. Therefore, *necessary and sufficient condition for L to be a rotation is*

$$(RL) = L \, , \quad I_4(L) > 0 \, . \tag{2.238}$$

It is easy to obtain a relation between the first and the third invariant of a rotation. In fact, using the Hamilton–Cayley identity (2.136) and $(2.236)_2$, one finds

$$L^{-1} = I_3(L)L^0 - I_2(L)L + I_1(L)L^2 - L^3 \, , \tag{2.239}$$

as well as the following expression for (RL):

$$(KL)^{-1} \equiv (RL) = I_3(L)L^0 - I_2(L)(KL) + I_1(L)(KL)^2 - (KL)^3 \, ; \tag{2.240}$$

[16] This is a general property, well known in the ordinary case.

then, evaluating the first invariant of both sides[17]: $I_1(RL) = I_3(L)$ and, from $(2.238)_1$,

$$I_1(L) = I_3(L) \ . \tag{2.241}$$

2.18.7 Expression of a 4-Rotation by Its Antisymmetric Part

In the three-dimensional case, every rotation can be expressed as second degree polynom in its antisymmetric part. We look for an analogous property in the four-dimensional Euclidean space, with signature $(-+++)$. Let us decompose L in the sum of its symmetric part D (i.e. a "dilation map") and its antisymmetric part A (i.e. an "axial map"):

$$L = D + A \ . \tag{2.242}$$

Using $(2.236)_1$ with L given by (2.242) as well as the analogous $L(KL) = L^0$, one has

$$D^2 - AD + DA - A^2 = L^0 \ , \quad D^2 + AD - DA - A^2 = L^0 \ ,$$

from which

$$D^2 = A^0 + A^2 \ , \quad DA = AD \ . \tag{2.243}$$

We have thus expressed D^2 as a function of A; however, our aim[18] is to express D as a function of A. Solving (2.239) with respect to L^3 and using $(2.236)_1$, $L^{-1} = (KL) = D - A$, together with (2.241) leads to

$$L^3 = I_1(L)L^2 - I_2(L)L - KL + I_1(L)L^0 \ ;$$

using the representation of L given by (2.242) and taking into account $(2.243)_2$ then implies

$$\begin{aligned} D^3 + 3D^2A + 3DA^2 + A^3 = {} & I_1(L)(D^2 + 2DA + A^2) + \\ & -I_2(L)(D + A) - D + A + I_1(L)L^0 \ ; \end{aligned}$$

separating the antisymmetric and symmetric parts, we have

$$\begin{cases} D^3 + 3DA^2 = I_1(L)(L^0 + D^2 + A^2) - [1 + I_2(L)]D \\ \\ 3D^2A + A^3 = 2I_1(L)DA + [1 - I_2(L)]A, \end{cases}$$

so that taking into account $(2.243)_1$ leads to

$$\begin{cases} D[(2 + I_2(L))A^0 + 4A^2] = 2I_1(L)(A^0 + A^2) \equiv 2I_1(L)D^2 \\ \\ 3A + 4A^3 = 2I_1(L)DA + [1 - I_2(L)]A. \end{cases}$$

---

[17] As stated in Sect. 2.17, for any nondegenerate map t, we have $I_1(Kt) = I_1(t)$.

[18] Equations (2.243) are necessary condition for L being a 4-rotation; but they are not sufficient because they only represent the first of the two characteristic properties (2.236).

Introducing then the "dilation map" $B \overset{\text{def}}{=} [2 + I_2(L)]A^0 + 4A^2 - 2I_1(L)D$ (satisfying the condition $B = KB$) and using $(2.243)_2$, the above relations can be written as

$$DB = 0\,, \qquad AB = 0\,. \tag{2.244}$$

Moreover, from the definition of B, we have the additional condition

$$D^2B - A^2B = B;$$

the left-hand side of this equation vanishes identically, being $D(DB) - A(AB)$ and using the conditions (2.244). Thus necessarily we have $B = 0$, and this is exactly the sought for relation for D:

$$2I_1(L)D = [2 + I_2(L)]A^0 + 4A^2\,. \tag{2.245}$$

In the general case $I_1(L) \neq 0$, (2.245) gives D in terms of A:

$$D = aA^0 + bA^2\,, \tag{2.246}$$

where the scalars a and b are expressed in terms of invariants of L, instead of A:

$$a = \frac{2 + I_2(L)}{2I_1(L)}\,, \qquad b = \frac{2}{I_1(L)} \neq 0\,. \tag{2.247}$$

2.18.8 Canonical Form of 4-Rotations

Equation (2.246) allows us to write L in the form

$$L = aA^0 + A + bA^2\,, \tag{2.248}$$

with a and b given by (2.247). Our purpose is now to express a and b in terms of the invariants of A. From (2.248) we have

$$I_1(L) \equiv \frac{2}{b} = 4a - 2bI_2(A)\,,$$

that is

$$I_2(A) = 2\frac{a}{b} - \frac{1}{b^2}\,, \tag{2.249}$$

or equivalently

$$I_2(A) = \frac{1}{2}[2 + I_2(L)] - \frac{1}{4}I_1^2(L)\,. \tag{2.250}$$

Moreover, (2.248) gives the product $(KL)L$, using the Hamilton–Cayley identity to express A^4:

$$A^4 = -I_2(A)A^2 - I_4(A)A^0\,, \tag{2.251}$$

that is

$$(KL)L = [a^2 - b^2I_4(A)]A^0 + [2ab - 1 - b^2I_2(A)]A^2\,.$$

Thus, from $(2.236)_1$ and (2.249), we have the relation:

$$L^0 = [a^2 - b^2 I_4(A)]A^0,$$

so that

$$I_4(A) = \frac{a^2 - 1}{b^2} \tag{2.252}$$

or

$$I_4(A) = \frac{1}{16}\left\{[2 + I_2(L)]^2 - 4I_1^2(L)\right\} . \tag{2.253}$$

Moreover, (2.186), $I_4(A) \leq 0$, gives the following restriction to the invariants of L:

$$-1 \leq a \leq 1 \quad \sim \quad [2 + I_2(L)]^2 \leq 4I_1^2(L) , \tag{2.254}$$

with the equality sign holding only if A is a bivector.

Equations (2.249) and (2.252) give, even if not uniquely, the scalars a and b in terms of the invariants of A. In fact, from (2.252) we have

$$\frac{1}{b^2} = \frac{a^2}{b^2} - I_4(A) ;$$

substituting this expression in (2.249) leads to the second degree equation in a/b:

$$\frac{a^2}{b^2} - 2\frac{a}{b} - I_4(A) + I_2(A) = 0 .$$

Consequently,

$$\frac{a}{b} = 1 + \epsilon\sqrt{\mathcal{D}} , \qquad \epsilon = \pm 1 , \tag{2.255}$$

where $\mathcal{D}$ is given by

$$\mathcal{D} \stackrel{\text{def}}{=} 1 - I_2(A) + I_4(A) \equiv \frac{1}{16}[2 - I_2(L)]^2 \geq 0 , \tag{2.256}$$

and it is nonnegative because of (2.250) and (2.253). As we will see later, *this is the only restriction to* A, *in order that the right-hand side of* (2.248) *be a rotation*. Once the ratio a/b is determined, (2.249) gives the expression of b, and then, from (2.255), we get the expression of a:

$$a = \epsilon' \frac{1 + \epsilon\sqrt{\mathcal{D}}}{\sqrt{2(1 + \epsilon\sqrt{\mathcal{D}}) - I_2(A)}} , \qquad b = \frac{\epsilon'}{\sqrt{2(1 + \epsilon\sqrt{\mathcal{D}}) - I_2(A)}} . \tag{2.257}$$

We note that from (2.256) follows the identity

$$2(1 + \epsilon\sqrt{\mathcal{D}}) - I_2(A) \equiv (1 + \epsilon\sqrt{\mathcal{D}})^2 - I_4(A) ;$$

thus, the radicand in (2.257) is nonnegative for each A and vanishes only when $\epsilon = -1$, $I_4(A) = 0$, $I_2(A) = 0$.

2.18.9 Simple Rotations

A 4-rotation L is called *simple* if its antisymmetric part A is a bivector: $\mathbf{A} = \mathbf{u} \wedge \mathbf{v}$, or $I_4(A) = 0$. From (2.252), we have then $a^2 = 1$, or $a = \epsilon'$, where $\epsilon' = \pm 1$ and b is given by (2.255) (or from (2.257), when $I_4(A) = 0$):

$$a = \epsilon' , \qquad b = \frac{\epsilon'}{1 + \epsilon\sqrt{1 - I_2(A)}} , \qquad \epsilon, \epsilon' = \pm 1 .$$

As a consequence of the arbitrariness of A, we have the following representation of simple rotations in M_4:

$$L = \epsilon' \left(A^0 + A + \frac{A^2}{1 + \epsilon\sqrt{1 - I_2(A)}} \right) , \tag{2.258}$$

which recalls the three-dimensional hyperbolic case: $\epsilon' = 1$ (see [1]).

If the principal plane associated with the bivector A: $E_2 =< \mathbf{u}, \mathbf{v} >$ is elliptic: $I_2(A) > 0$ and from the constraint (2.256), one obtains the condition:

$$I_2(A) \leq 1 ; \tag{2.259}$$

the rotation is thus said of *elliptic type* and denoted by L_{e}; it is said of *hyperbolic type*, instead, when $I_2(A) < 0$ and of *parabolic type* when $I_2(A) = 0$. Let us start considering the first two cases: $I_2(A) \neq 0$. We recall that A maps each vector $\mathbf{w} \in M_4$ in a vector of E_2: $A\mathbf{w} = (\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v}$. Thus, each vector belonging to E_2' (the plane orthogonal to E_2) is mapped into the zero element. From (2.258) we also have

$$L\mathbf{w} = \epsilon'\mathbf{w} , \qquad \forall \mathbf{w} \in E_2' , \tag{2.260}$$

that is, *any vector orthogonal to E_2 is fixed, or mapped by L into the opposite*, according to the value of $\epsilon' = \pm 1$.

Let us see, now, how L acts on the elements of E_2. Since A maps any vector $\mathbf{w} \in E_2$ in a vector orthogonal to this plane, then $A^2\mathbf{w}$ has the original direction. Actually, using (2.204) with $I_4(A) = 0$, for all the vectors of E_2 we have

$$A^2\mathbf{w} = -I_2(A)\mathbf{w} , \qquad \forall \mathbf{w} \in E_2 . \tag{2.261}$$

Thus, substituting in (2.258), we find (only for vectors in E_2):

$$L = \epsilon'(\epsilon\sqrt{1 - I_2(A)}\, A^0 + A) . \tag{2.262}$$

Let us now assume that E_2 is of elliptic type, i.e. with signature $(++)$; in this case, A does not admit (real) eigendirections.[19] Taking into account the orthogonality condition: $A\mathbf{w} \cdot \mathbf{w} = 0$, $\forall \mathbf{w} \in E_2$, and using (2.262) leads to

[19] In order $\mathbf{w}$ be an eigendirection of A, $A\mathbf{w}$ must be aligned with $\mathbf{w}$, but from the properties of A, we have that $A\mathbf{w}$ is orthogonal to $\mathbf{w}$. In an elliptic 2-space, it is not possible to satisfy both these conditions, and hence A does not admit (real) eigendirections.

$$L\mathbf{w} \cdot \mathbf{w} = \epsilon\epsilon' \sqrt{1 - I_2(A)}||\mathbf{w}|| \; ; \tag{2.263}$$

that is, the following invariant property holds:

$$\frac{L\mathbf{w} \cdot \mathbf{w}}{||\mathbf{w}||} = \text{inv.} = \epsilon' \cos\varphi \; , \qquad \forall \mathbf{w} \in E_2 \; , \tag{2.264}$$

where φ, implicitly defined by the relation:

$$\cos\varphi = \epsilon\sqrt{1 - I_2(A)} \; , \qquad 0 < I_2(A) \leq 1 \tag{2.265}$$

is independent of $\mathbf{w}$ and defines the *amplitude of the rotation*: $0 < \varphi < \pi$. This is an effective rotation; in fact, since E_2 is elliptic, $I_2(A) > 0$, both the *null rotation* and the *symmetry*, $\varphi = 0, \pi$, are excluded. Thus, L does not admit eigendirections in E_2, but only in E_2' (*rotation axis*) (Fig. 2.1).

If the subspace E_2 is instead hyperbolic ($I_2(A) < 0$), one should add the two isotropic directions of E_2 (which are now eigendirection of A) to the eigendirections of E_2' (now of elliptic type); the rotation will be said to be hyperbolic of the *first kind*, L_{h}^{+}, or of the *second kind*: L_{h}^{-}, corresponding to the values $\epsilon = -1$ and $\epsilon = 1$ respectively, in the case $\epsilon' = 1$ and conversely in the case $\epsilon' = -1$ (Fig. 2.2).

Moreover, in a hyperbolic subspace E_2, the *Schwartz inequality* gives

$$(\mathbf{a} \cdot \mathbf{b})^2 \geq ||\mathbf{a}|| \, ||\mathbf{b}|| \; , \quad \forall \, \mathbf{a}, \mathbf{b} \in E_2 \; ,$$

with the equality sign holding only when $\mathbf{a}$ and $\mathbf{b}$ are collinear. As a consequence, in each of the four regions in which E_2 is divided by the isotropic

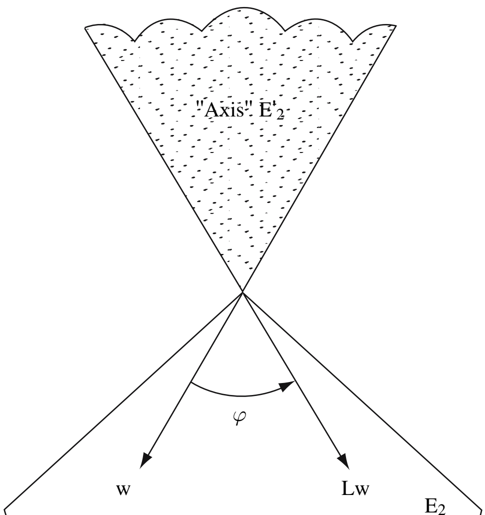

Fig. 2.1. Rotations in an elliptic space

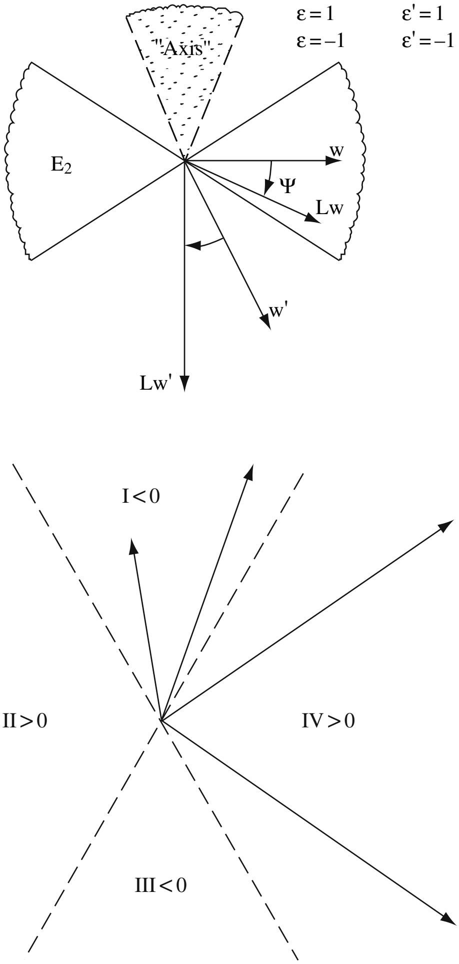

Fig. 2.2. Rotations in a hyperbolic space

directions, it is possible to define the *pseudoangle* $\psi \geq 0$ of the two vectors $\mathbf{a}$ and $\mathbf{b}$. Hence, in each of the four regions, the sign of the scalar product $\mathbf{a} \cdot \mathbf{b}$ is invariant: in the temporal regions I and III (see Fig. 2.2), $\mathbf{a} \cdot \mathbf{b} < 0$, while $\mathbf{a} \cdot \mathbf{b} > 0$ in the spatial regions II and IV. So the pseudoangle ψ is uniquely defined by the relations:

$$\begin{cases} \cosh \psi \stackrel{\text{def}}{=} -\dfrac{\mathbf{a} \cdot \mathbf{b}}{ab}, & \text{in regions I and III} \\ \cosh \psi \stackrel{\text{def}}{=} \dfrac{\mathbf{a} \cdot \mathbf{b}}{ab}, & \text{in regions II and IV} \end{cases} \tag{2.266}$$

and $\psi = 0$ only for $\mathbf{b} = \lambda \mathbf{a}$ with $\lambda > 0$.

Equations (2.262) imply that (2.264) in the hyperbolic case becomes

$$\frac{L\mathbf{w}\cdot\mathbf{w}}{\epsilon\epsilon'||\mathbf{w}||} = \text{inv.} = \cosh\psi \ , \qquad \forall\mathbf{w}\in E_2, \quad \epsilon,\epsilon' = \pm 1 \ , \tag{2.267}$$

where ψ, implicitly defined by the relation

$$\cosh\psi = \sqrt{1 - I_2(A)} \ , \qquad I_2(A) < 0 \ , \tag{2.268}$$

represents the pseudoamplitude of the rotation.

When $\epsilon = 1$ and $\epsilon' = 1$, the four regions of E_2 are invariant for L:

$$L\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = -\cosh\psi < 0 \ , \qquad \forall\boldsymbol{\gamma}\in \text{I, III}, \ \boldsymbol{\gamma} = \text{unit vector},$$

etc. Thus, the half lightcones are conserved; for $\epsilon' = 1$ and $\epsilon = -1$, the regions I and III as well as II and IV (and also the half lightcones) are exchanged, similarly for $\epsilon' = -1$.

We note that a hyperbolic rotation L admits basis of eigenvectors, differently from the elliptic case in which only the vectors in a plane are eigenvectors. These are formed with two isotropic vectors in E_2 and two arbitrary (noncollinear) vectors of E_2'.

$$L = \epsilon'\left(A^0 + A + \frac{1}{2}A^2\right) \ . \tag{2.269}$$

Finally, if $I_2(A) = 0$, the rotation is of parabolic type (Fig. 2.3); it does not admit eigendirections in E_2 besides the isotropic one, say $\mathbf{l}$, belonging to both the orthogonal and parabolic 2-planes E_2 and E_2'. This is a very special case, since the amplitude of the rotation is independent of L. In fact, from (2.269), for each $\mathbf{w}\in E_2$, the transformed vector $L\mathbf{w}$ is given by

$$L\mathbf{w} = \epsilon'\mathbf{w} + \xi\mathbf{l} \ , \qquad \forall\mathbf{w}\in E_2 \ ; \tag{2.270}$$

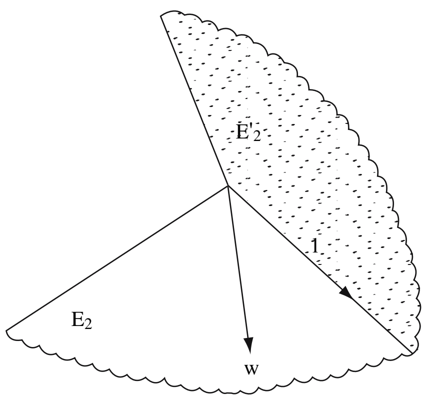

Fig. 2.3. Rotations in a parabolic space

thus the ratio $\frac{L\mathbf{w} \cdot \mathbf{w}}{||\mathbf{w}||}$ is independent of L (and of $\mathbf{w}$):

$$\frac{L\mathbf{w} \cdot \mathbf{w}}{||\mathbf{w}||} = \epsilon' , \qquad \forall \mathbf{w} \in E_2 . \tag{2.271}$$

Hence, apart from the factor $\epsilon' = \pm 1$ and the existence of a whole rotation 2-plane (instead of an axis), the situation for what concerns simple rotations is analogous to that of the three-dimensional case.

2.18.10 4-Rotations as Product of Simple Rotations

Let us turn to the case of a generic rotation: $I_4(A) \neq 0$, decomposing $\mathbf{A}$ as the sum of two orthogonal bivectors: $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}'_1$, with $\mathbf{A}_1 \cdot \mathbf{A}'_1 = 0$. We assume $\mathbf{A}_1$ of elliptic type,[20] so that $\mathbf{A}'_1$ is necessarily hyperbolic. With each of the two bivectors is associated a simple rotation, like in (2.258). We can show now that L is always given by the product of two of such rotations.

Let us start noting that (2.234) imply:

$$\mathcal{D} = (1 - I_2)(1 - I'_2) , \tag{2.272}$$

where we have introduced the notation

$$I_2 = I_2(A_1) < 1 , \quad I'_2 = I_2(A'_1) < 0 . \tag{2.273}$$

We then have

$$2(1 + \epsilon\sqrt{\mathcal{D}}) - I_2(A) = (\epsilon\sqrt{1 - I_2} + \sqrt{1 - I'_2})^2 > 0 , \tag{2.274}$$

and the following expressions for a and b:

$$a = \frac{1 + hh'}{h + h'} , \qquad b = \frac{1}{h + h'} , \tag{2.275}$$

where

$$h = \epsilon\sqrt{1 - I_2} , \qquad h' = \epsilon'\sqrt{1 - I'_2} . \tag{2.276}$$

Rewriting a, using $(2.275)_1$

$$a = 1 + \frac{(1 - h)(1 - h')}{h + h'}$$

we find that (2.248) can be written as:

$$L = A^0 + A + \frac{(1 - h)(1 - h')}{h + h'} A^0 + \frac{A^2}{h + h'} . \tag{2.277}$$

[20] If $\mathbf{A}_1$ is parabolic, then $\mathbf{A}'_1$ is parabolic too, and consequently L is a simple rotation of parabolic type.

At this point, we have to express the right-hand side of (2.277) in terms of A_1 and A'_1. We proceed proving the following identity:

$$\frac{(1-h)(1-h')}{h+h'}A^0 + \frac{A^2}{h+h'} = \frac{A_1^2}{1+h} + \frac{A_1'^{\,2}}{1+h'} \,. \tag{2.278}$$

Starting from (2.200), which holds for each pair of antisymmetric tensors (in particular for $A = B = A_1$) and using $(2.181)_2$, we have

$$\overset{*}{A}_1{}^2 = A_1^2 + I_2(A^2)A^0 \,,$$

so that

$$\frac{1}{\lambda^2}A_1'^2 = A_1^2 + I_2(A^2)A^0 \,;$$

using then (2.235) leads to

$$I_2 A_1'^2 + I_2' A_1^2 + I_2 I_2' A^0 = 0 \,, \tag{2.279}$$

so that

$$A^0 = -\frac{A_1'^2}{1-h'^2} - \frac{A_1^2}{1-h^2} \,,$$

where we have used $I_2 I_2' = (1-h^2)(1-h'^2)$ as a consequence of (2.276). Finally, after substituting this relation on the left-hand side of (2.278) and recalling that $A^2 = A_1^2 + A_1'^2$, the above identity (2.278) is proven.

Summarizing, *each 4-rotation L can be cast in the following form*:

$$L = A^0 + A_1 + A_1' + \frac{A_1^2}{1+h} + \frac{A_1'^2}{1+h'} \,, \tag{2.280}$$

and, due to the orthogonality of A_1 and A'_1, can be written as a *product of two simple rotations*, one of elliptic type ($L_{\rm e}$) and the other hyperbolic ($L_{\rm h}$):

$$L_{\rm e} = A^0 + A_1 + \frac{A_1^2}{1+\epsilon\sqrt{1-I_2}} \,, \qquad L_{\rm h} = A^0 + A_1' + \frac{A_1'^2}{1+\epsilon'\sqrt{1-I_2'}} \,, \tag{2.281}$$

that is

$$L = L_{\rm e} L_{\rm h} = L_{\rm h} L_{\rm e} \,. \tag{2.282}$$

In particular, the above decomposition shows that the isotropic lines of the principal plane of A'_1 are the only eigendirections of L: in fact, these are the only common eigendirections of $L_{\rm h}$ and $L_{\rm e}$.

2.18.11 The Case of Orthogonal Symmetries: $I_1(L) = 0$

Let us now consider the case $I_1(L) = 0$, which also implies $2 + I_2(L) = 0$. This follows from (2.245) which reduces to the form $4A^2 = -[2 + I_2(L)]A^0$, and implies in turn

$$I_2(A) \equiv -\frac{1}{2}I_1(A^2) = 1 + \frac{1}{2}I_2(L) , \tag{2.283}$$

that is,

$$A^2 = -\frac{1}{2}I_2(A)A^0 . \tag{2.284}$$

Using (2.284), we then have $4A^4 = I_2^2(A)A^0$, and the Hamilton–Cayley identity (2.202) provides the condition $[I_2(A)]^2 = 4I_4(A)$. Thus, from (2.197), $I_4(A) \leq 0 \; \forall A$, necessarily follows that $I_2(A) = 0$ and $I_4(A) = 0$, or $I_2(L) = -2$ from (2.283).

Summarizing, the condition $I_1(L) = 0$ implies the two equalities:

$$I_2(L) = -2 , \qquad A = 0;$$

hence, the rotation is only represented by its symmetric part D, satisfying the condition (see (2.243)):

$$L = D , \qquad D^2 = D^0 . \tag{2.285}$$

Such a case is similar to that of the *axial symmetries* in a three-dimensional space; in fact, since

$$I_1(L) = 0 , \quad I_2(L) = -2 , \quad I_3(L) = 0 , \quad I_4(L) = 1 ,$$

L admits two real distinct eigenvalues: $\lambda = \pm 1$ which are both double roots. Hence we have the existence at least of two eigenvectors: $\mathbf{u}$ and $\mathbf{u}'$: $L\mathbf{u} = \mathbf{u}$ and $L\mathbf{u}' = -\mathbf{u}'$, which are necessarily orthogonal, due to the symmetry of L.

Let us denote by $\mathbf{u}_1$ and $\mathbf{u}'_1$ another pair of vectors, forming with $\mathbf{u}$ and $\mathbf{u}'$ a basis in M_4. We have

$$L\mathbf{u}_1 = \lambda\mathbf{u} + \mu\mathbf{u}' + \nu\mathbf{u}_1 + \sigma\mathbf{u}'_1 . \tag{2.286}$$

Applying L to both sides and taking into account (2.285) we find

$$\mathbf{u}_1 = \lambda\mathbf{u} - \mu\mathbf{u}' + L[\nu\mathbf{u}_1 + \sigma\mathbf{u}'_1] , \tag{2.287}$$

and in addition

$$\sigma L\mathbf{u}'_1 = -\lambda(1+\nu)\mathbf{u} + \mu(1-\nu)\mathbf{u}' + (1-\nu^2)\mathbf{u}_1 - \nu\sigma\mathbf{u}'_1 , \tag{2.288}$$

using again (2.286). Adding and subtracting (2.286) and (2.287), we obtain

$$\begin{cases} L[(\nu-1)\mathbf{u}_1 + \sigma\mathbf{u}'_1] = (1-\nu)\mathbf{u}_1 - \sigma\mathbf{u}'_1 - 2\lambda\mathbf{u}, \\ \\ L[(\nu+1)\mathbf{u}_1 + \sigma\mathbf{u}'_1] = (1+\nu)\mathbf{u}_1 + \sigma\mathbf{u}'_1 + 2\mu\mathbf{u}'. \end{cases} \tag{2.289}$$

We now have to distinguish between the two cases: $\sigma \neq 0$ and $\sigma = 0$. In the first case ($\sigma \neq 0$), (2.287) determines $L\mathbf{u}'_1$, starting from $L\mathbf{u}_1$; moreover, (2.289) is equivalent to the conditions

$$L\mathbf{v}' = -\mathbf{v}' \;, \qquad L\mathbf{v} = \mathbf{v} \;,$$

where

$$\mathbf{v} \equiv \mu\mathbf{u}' + (\nu + 1)\mathbf{u}_1 + \sigma\mathbf{u}_1' \;, \qquad \mathbf{v}' \equiv \lambda\mathbf{u} + (\nu - 1)\mathbf{u}_1 + \sigma\mathbf{u}_1' \;; \tag{2.290}$$

therefore the vectors $\mathbf{v}$ and $\mathbf{v}'$ of (2.290), independent of $\mathbf{u}$ and $\mathbf{u}'$, are both eigenvectors of L, corresponding to the eigenvalues $\lambda = -1$ and $\lambda = 1$, respectively, and hence orthogonal.

If instead $\sigma = 0$, from (2.288), we have the following two subcases:

1. $\sigma = 0 \;, \quad \nu = 1 \;, \quad \lambda = 0,$
2. $\sigma = 0 \;, \quad \nu = -1 \;, \quad \mu = 0,$

in which one of the vectors (2.290) reduces to zero. However, in the case 1, we have from (2.286) $L\mathbf{u} = \mu\mathbf{u}' + \mathbf{u}_1$; thus, assuming

$$L\mathbf{u}_1' = \lambda'\mathbf{u} + \mu'\mathbf{u}' + \nu'\mathbf{u}_1 + \sigma'\mathbf{u}_1' \;, \tag{2.291}$$

for the components $L^{\alpha}{}_{\beta}$ along the basis $\{\mathbf{u}, \mathbf{u}', \mathbf{u}_1, \mathbf{u}_1'\}$, we see that $I_1(L) = 1 + \sigma'$, i.e. $\sigma' = -1$. Furthermore, applying L to both sides of (2.291) gives $\mu' = \frac{1}{2}\mu\nu'$; thus, the transformed vector $L\mathbf{u}_1'$ results in $L\mathbf{u}_1' = \lambda'\mathbf{u} + \nu'(\mathbf{u}_1 + \frac{1}{2}\mu\mathbf{u}') - \mathbf{u}_1'$ and the vectors (2.290) are given by

$$\mathbf{v}' \equiv -\frac{1}{2}\lambda'\mathbf{u} - \frac{1}{4}\nu'\mathbf{v} + \mathbf{u}_1' \;, \qquad \mathbf{v} \equiv \mu\mathbf{u}' + 2\mathbf{u}_1 \;; \tag{2.292}$$

similarly, in case 2, we have

$$\mathbf{v}' \equiv \lambda\mathbf{u} - 2\mathbf{u}_1 \;, \qquad \mathbf{v} \equiv \frac{1}{2}\mu'\mathbf{u}' - \frac{1}{4}\nu'\mathbf{v}' + \mathbf{u}_1' \;. \tag{2.293}$$

Finally, in the case $I_1(L) = 0$ (and only in this case), *the rotation L is diagonal* and admits a pair of orthogonal 2-planes generated by eigendirections. In detail, one of such subspaces $< \mathbf{u}, \mathbf{v} >$ is invariant for L, and the other $< \mathbf{u}', \mathbf{v}' >$ is mapped into the opposite. These are *orthogonal symmetries*, say S, with respect to the nondegenerate 2-planes. Again this case can be characterized by using antisymmetric tensors. In fact, if $\mathbf{A} = \mathbf{u} \wedge \mathbf{v}$ is a bivector with an associated subspace E_2 nonparabolic ($I_2(A) \neq 0$), the following representation holds:

$$S = -A^0 - \frac{2}{I_2(A)}A^2 \;, \tag{2.294}$$

where $I_2(A) > 0$ or $I_2(A) < 0$ if E_2 is elliptic or hyperbolic, respectively. After decomposing the generic vector $\mathbf{w} \in M_4$ as the sum of a vector $\mathbf{w}_A \in E_2$ and another $\mathbf{w}_N \in E_2'$, the symmetric of $\mathbf{w}$ with respect to the 2-plane E_2 turns out to be

$$S(\mathbf{w}) = \mathbf{w}_A - \mathbf{w}_N = 2\mathbf{w}_A - \mathbf{w} \;,$$

from which (2.294) follows, using (2.261) and the identity: $A^2\mathbf{w}_A = A^2\mathbf{w}$.

We note that the bivector **A** introduced above is not the antisymmetric part of L (which is instead represented by the symmetric tensor D) but is a bivector associated with the subspace Π that specifies the symmetry and defined up to a multiplicative factor (*Grassmann tensor* of the 2-plane Π). However, (2.294) which characterizes the rotations with $I_1(L) = 0$ in the context of simple rotations completes the previous representation (2.248)–(2.257).

Finally, as concerns the symmetries (2.294), they still generate (by products) the rotation group.

2.18.12 Cayley Representation

For 4-rotations L, we also have the Cayley representation [1]:

$$L = (Q^0 - Q)^{-1}(Q^0 + Q) \equiv (Q^0 + Q)(Q^0 - Q)^{-1} \,, \tag{2.295}$$

where Q is an antisymmetric tensor, satisfying the condition

$$J(Q) \stackrel{\text{def}}{=} I_4(Q_0 - Q) = 1 + I_2(Q) + I_4(Q) \neq 0 \,. \tag{2.296}$$

In analogy with the ordinary case, Q is said to be the *characteristic tensor of the rotation* L. As we will see in the following, Q is defined only for rotations such that

$$I(L) \stackrel{\text{def}}{=} 2 + 2I_1(L) + I_2(L) \neq 0 \,. \tag{2.297}$$

Equation (2.295) can be written explicitly. To this end, let us use the Hamilton-Cayley identity for the map $X = Q^0 - Q$:

$$X^{-1} = \frac{1}{I_4(X)}[I_3(X)\,\mathbb{I} - I_2(X)X + I_1(X)X^2 - X^3] \,. \tag{2.298}$$

Moreover we have $I_1(X) = 4$, $I_2(X) = 6 + I_2(Q)$ and $I_3(X) = 4 + 2I_2(Q)$; using then the Hamilton–Cayley identity also for Q^4,

$$Q^4 = -I_2(Q)Q^2 - I_4(Q)Q^0 \,,$$

one gets[21] the expression of L in terms of Q:

$$\begin{aligned} L = \frac{1}{J(Q)} \{ &[1 + I_2(Q) - I_4(Q)]Q^0 + \\ &2[1 + I_2(Q)]Q + 2Q^2 + 2Q^3 \} \,. \end{aligned} \tag{2.299}$$

This relation represents L as a third-degree polynom in Q instead of the second-degree polynom of the canonical representation (2.248) in terms of the antisymmetric part A. Under the hypothesis (2.296), $J(Q) \neq 0$, (2.299) can be solved for Q

$$Q = (L - L^0)(L + L^0)^{-1} \,; \tag{2.300}$$

[21] See [10], p. 88, taking the contraction $Q = K/k\,, k \neq 0$.

introducing then the notation $Y \overset{\text{def}}{=} L + L^0$, the Hamilton–Cayley identity implies

$$Y^{-1} = \frac{1}{I_4(Y)} \left\{ I_3(Y) L^0 - I_2(Y) Y + I_1(Y) Y^2 - Y^3 \right\} \; .$$

On the other hand, it is easy to show that

$$\begin{cases} I_1(Y) = 4 + I_1(L), \\ I_2(Y) = 6 + 3I_1(L) + I_2(L), \\ I_3(Y) = 4 + 3I_1(L) + 2I_2(L) + I_3(L) \equiv 4 + 4I_1(L) + 2I_2(L). \end{cases} \tag{2.301}$$

Thus, using again the Hamilton–Cayley identity,

$$\begin{aligned} L^4 &= I_1(L)L^3 - I_2(L)L^2 + I_3(L)L - I_4(L)L^0 \\ &= I_1(L)L^3 - I_2(L)L^2 + I_1(L)L - L^0 \; , \end{aligned}$$

Equation (2.300) assumes the form

$$\begin{aligned} Q = \frac{1}{I_4(Y)} &\left\{ -[2I_1(L) + I_2(L)]L^0 + 2[1 + I_1(L) + I_2(L)]L \right. \\ &\left. -2[L^0 + I_1(L)]L^2 + 2L^3 \right\} \; . \end{aligned} \tag{2.302}$$

From (2.299), the relation between Q and A follows immediately:

$$A = \frac{2[1 + I_2(Q)]Q + 2Q^3}{J(Q)} \; . \tag{2.303}$$

Therefore, differently from the ordinary case, A is not a simple function of Q. Similarly, (2.299) gives the fundamental invariants of L, in terms of Q:

$$\begin{cases} I_1(L) \equiv I_3(L) = \dfrac{4}{J(Q)}[1 - I_4(Q)] \\ I_2(L) = 2\left[3 - 4\dfrac{I_2(Q)}{J(Q)}\right] \; , \qquad I_4(L) = 1, \end{cases} \tag{2.304}$$

so that the invariant (2.297) is given by

$$I(L) = \frac{16}{J(Q)} \; . \tag{2.305}$$

Conversely,

$$I_2(Q) = \frac{2}{I(L)}[6 - I_2(L)] \; , \qquad I_4(Q) = 1 - 4\frac{I_1(L)}{I(L)} \; . \tag{2.306}$$

Equation (2.305) shows that the Cayley representation only includes rotations like $I(L) \neq 0$; that is, rotations with $I(L) = 0$, i.e.

$$2 + I_2(L) = -2I_1(L) \quad \rightarrow \quad I_4(A) = 0 \; , \tag{2.307}$$

are excluded. The latters are necessarily simple rotations as the symmetry S (but not all of these, differently from the ordinary case).

2.18.13 Composition Law

We will study now the modifications to the Rodrigues formula (2.177) passing from the three-dimensional case to the four-dimensional one.

Let L and L' be two rotations associated with the antisymmetric tensors Q and Q', respectively

$$L = (Q^0 - Q)^{-1}(Q^0 + Q) \ , \qquad L' = (Q^0 - Q')^{-1}(Q^0 + Q') \ .$$

We look for the map Q'' associated with the rotation product of L' and L: $L'' = L'L$, at least in the generic case: $I(L), I(L'), I(L'') \neq 0$ (see (2.297) for their definition). Let us assume

$$\mathbf{v}' \stackrel{\text{def}}{=} L\mathbf{v} \ , \qquad \mathbf{v}'' = L'\mathbf{v}' \ , \qquad \forall \mathbf{v} \in M_4 \ .$$

The map $\mathbf{v} \to \mathbf{v}''$ associated with the product $L'' = L'L$ is implicitly defined by the relations

$$Q(\mathbf{v}' + \mathbf{v}) = \mathbf{v}' - \mathbf{v} \ , \qquad Q'(\mathbf{v}'' + \mathbf{v}') = \mathbf{v}'' - \mathbf{v}' \ , \tag{2.308}$$

which follow from (2.295) and using the linearity of the map Q; in fact

$$(Q^0 - Q)L(\mathbf{v}) = (Q^0 + Q)(\mathbf{v}) \quad \to \quad \mathbf{v}' - Q(\mathbf{v}') = \mathbf{v} + Q(\mathbf{v}) \ , \tag{2.309}$$

so that

$$Q(\mathbf{v}) + Q(\mathbf{v}') = -\mathbf{v} + \mathbf{v}' \ , \tag{2.310}$$

which immediately reduces to $(2.308)_1$, similarly for the derivation of $(2.308)_2$. We look then for the map Q'' such that

$$Q''(\mathbf{v}'' + \mathbf{v}) = \mathbf{v}'' - \mathbf{v} \ . \tag{2.311}$$

A straightforward calculation shows that

$$\begin{aligned}(Q + Q' + Q'Q - QQ')(\mathbf{v}'' + \mathbf{v}) - Q'QQ'(\mathbf{v}'' + \mathbf{v}') \\ -QQ'Q(\mathbf{v}' + \mathbf{v}) = \mathbf{v}'' - \mathbf{v} \ ;\end{aligned} \tag{2.312}$$

thus, we need to evaluate the product:

$$C \stackrel{\text{def}}{=} -Q'QQ'(\mathbf{v}'' + \mathbf{v}') - QQ'Q(\mathbf{v}' + \mathbf{v}) \ .$$

To this end, we note that from (2.196) and (2.200) we have

$$QQ' = \overset{*}{Q}{}' \overset{*}{Q} + \frac{1}{2} I_1(QQ')Q^0 \ , \qquad Q\overset{*}{Q} = \overset{*}{Q}Q = -\overset{*}{I}(Q)Q^0 \ , \tag{2.313}$$

with

$$\overset{*}{I}(Q) \equiv -\frac{1}{4} I_1(Q\overset{*}{Q}) \ . \tag{2.314}$$

Thus,

$$Q'QQ' = Q'\overset{*}{Q}{}'\overset{*}{Q} + \frac{1}{2}I_1(QQ')Q' \equiv -\overset{*}{I}(Q')\overset{*}{Q} + \frac{1}{2}I_1(QQ')Q' \ ,$$

with

$$\overset{*}{I}(Q') \equiv -\frac{1}{4}I_1(Q'\overset{*}{Q}{}') \ . \tag{2.315}$$

Similarly, exchanging Q and Q' in the above product we have directly

$$QQ'Q = -\overset{*}{I}(Q)\overset{*}{Q}{}' + \frac{1}{2}I_1(QQ')Q \ ;$$

and hence,

$$\begin{aligned} C &= \overset{*}{I}(Q')\overset{*}{Q}(\mathbf{v}'' + \mathbf{v}') - \frac{1}{2}I_1(QQ')Q'(\mathbf{v}'' + \mathbf{v}') \\ &\quad + \overset{*}{I}(Q)\overset{*}{Q}{}'(\mathbf{v}' + \mathbf{v}) - \frac{1}{2}I_1(QQ')Q(\mathbf{v}' + \mathbf{v}) \end{aligned}$$

or, using (2.308):

$$\begin{aligned} C &= \overset{*}{I}(Q')\overset{*}{Q}(\mathbf{v}'' + \mathbf{v}' + \mathbf{v}' - \mathbf{v}) + \overset{*}{I}(Q)\overset{*}{Q}{}'(\mathbf{v}' - \mathbf{v}'' + \mathbf{v}'' + \mathbf{v}) \\ &\quad - \frac{1}{2}I_1(QQ')Q'(\mathbf{v}'' - \mathbf{v}) \\ &= [\overset{*}{I}(Q')\overset{*}{Q} + \overset{*}{I}(Q)\overset{*}{Q}{}'](\mathbf{v}'' + \mathbf{v}) - \frac{1}{2}I_1(QQ')(\mathbf{v}'' - \mathbf{v}) \\ &\quad + \overset{*}{I}(Q')\overset{*}{Q}(\mathbf{v}' - \mathbf{v}) - \overset{*}{I}(Q)\overset{*}{Q}{}'(\mathbf{v}'' - \mathbf{v}') \ . \end{aligned}$$

Moreover, from (2.308) and $(2.313)_2$ we have

$$\begin{aligned} &\overset{*}{I}(Q')\overset{*}{Q}(\mathbf{v}' - \mathbf{v}) - \overset{*}{I}(Q)\overset{*}{Q}{}'(\mathbf{v}'' - \mathbf{v}) \\ &= \overset{*}{I}(Q')\overset{*}{Q}Q(\mathbf{v}' + \mathbf{v}) - \overset{*}{I}(Q)\overset{*}{Q}{}'Q'(\mathbf{v}'' + \mathbf{v}') \\ &= \overset{*}{I}(Q)\overset{*}{I}(Q')(\mathbf{v}'' - \mathbf{v}) \ , \end{aligned}$$

so that C can be written as

$$\begin{aligned} C &= [\overset{*}{I}(Q')\overset{*}{Q} + \overset{*}{I}(Q)\overset{*}{Q}{}'](\mathbf{v}'' + \mathbf{v}) \\ &\quad - \frac{1}{2}I_1(QQ')(\mathbf{v}'' - \mathbf{v}) + \overset{*}{I}(Q)\overset{*}{I}(Q')(\mathbf{v}'' - \mathbf{v}) \ , \end{aligned}$$

and (2.312) becomes

$$\begin{aligned} &[Q + Q' + Q'Q - QQ' + \overset{*}{I}(Q')\overset{*}{Q} + \overset{*}{I}(Q)\overset{*}{Q}{}'](\mathbf{v}'' + \mathbf{v}) + \\ &- \left[\frac{1}{2}I_1(QQ') - \overset{*}{I}(Q)\overset{*}{I}(Q')\right](\mathbf{v}'' - \mathbf{v}) = (\mathbf{v}'' - \mathbf{v}) \ . \end{aligned}$$

Using this result, one finally gets the sought for *composition law* for rotations in terms of Q:

$$Q'' = \frac{Q + Q' + Q'Q - QQ' + \overset{*}{I}(Q')\overset{*}{Q} + \overset{*}{I}(Q)\overset{*}{Q}'}{1 + \frac{1}{2}I_1(QQ') - \overset{*}{I}(Q)\overset{*}{I}(Q')} \,, \tag{2.316}$$

where $\overset{*}{I}(Q)$ is defined by (2.314)–(2.315), and it is related to $I_4(Q)$ by (2.197): $[\overset{*}{I}(Q)]^2 = -I_4(Q)$. In the case of simple rotations, (2.316) reduces to the ordinary law; moreover, as in the three-dimensional case, it includes the limiting case of rotations with $I(L) = 0$, and hence it has a general validity.

2.19 General Lorentz Transformations: III

We have outlined above the fundamental role of antisymmetric 2-tensors (and their associated tensorial space Λ^2) in the representation of rotations in M_4. As in the three-dimensional case, rotations in M_4 are characterized by certain *isotropic tensorial functions* defined in Λ^2. In the preceding sections, the close relation between 4-rotations and antisymmetric 2-tensors has been discussed, using the properties of the associated maps. The same approach can be used in a complex context by means of the technique of null tetrads,[22] taking into account that any 4-rotation admits at least an *isotropic eigenvector*, as well as in the real domain [10], in the context of Clifford's algebra of M_4. We will briefly introduce here this point of view limiting ourselves to the Minkowskian case; the extension to any linear space E_n endowed with a nonsingular metric is straightforward.

Independently of its metric structure, one can associate with M_4 a finite dimensional space Λ. In fact, the various linear spaces Λ^q $(q = 0, 1, 2, 3, 4)$, formed by *antisymmetric tensors of various order* (up to the maximum order 4), are related to M_4 as follows:

- $\Lambda^0 = R$, scalars:
  $a = a\mathbf{e}$,
- $\Lambda^1 = M_4$, vectors:
  $\mathbf{a} = a^\alpha \mathbf{e}_\alpha$,
- $\Lambda^2 = M_4 \wedge M_4$, antisymmetric 2-tensors:
  $\mathbf{A} = \frac{1}{2}A^{\alpha\beta}\mathbf{e}_\alpha \wedge \mathbf{e}_\beta$, with $\mathbf{e}_\alpha \wedge \mathbf{e}_\beta = 2!\mathbf{e}_{[\alpha} \otimes \mathbf{e}_{\beta]}$,
- $\Lambda^3 = M_4 \wedge M_4 \wedge M_4$, antisymmetric 3-tensors:
  $\mathbf{T} = \frac{1}{3!}T^{\alpha\beta\gamma}\mathbf{e}_\alpha \wedge \mathbf{e}_\beta \wedge \mathbf{e}_\gamma$, with $\mathbf{e}_\alpha \wedge \mathbf{e}_\beta \wedge \mathbf{e}_\gamma = 3!\mathbf{e}_{[\alpha} \otimes \mathbf{e}_\beta \otimes \mathbf{e}_{\gamma]}$,
- $\Lambda^4 = M_4 \wedge M_4 \wedge M_4 \wedge M_4$, antisymmetric 4-tensors:
  $\mathbf{Q} = \frac{1}{4!}Q^{\alpha\beta\gamma\rho}\mathbf{e}_\alpha \wedge \mathbf{e}_\beta \wedge \mathbf{e}_\gamma \wedge \mathbf{e}_\rho$, with $\mathbf{e}_\alpha \wedge \mathbf{e}_\beta \wedge \mathbf{e}_\gamma \wedge \mathbf{e}_\rho = 4!\mathbf{e}_{[\alpha} \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma \otimes \mathbf{e}_{\rho]}$,

[22] Such a technique is better included in the more general context of *anholonomic frames*, and it is widely used in general relativity, see e.g. [11].

where $\{\mathbf{e}_\alpha\}$ is an arbitrary basis in M_4 and, to uniform notation, $\mathbf{e} = 1$.

It is convenient to consider the direct sum (Cartesian product) of such spaces Λ:

$$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^3 \oplus \Lambda^4 \; ; \tag{2.317}$$

this is a finite dimensional space with dimension: $1 + 4 + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 16$. We note that both the direct sum of the spaces of symmetric tensors of various order associated with M_4 and the direct sum of all the tensorial spaces associated with M_4 have not a finite dimension.

Up to now the 16-dimensional space Λ has only the structure of linear space. However, it can be endowed with an internal noncommutative product, which we will denote with a $\circ$ (Clifford's product), using the metric of M_4. Let us start defining the $\circ$ product of two antisymmetric tensors, $\mathbf{A}$ and $\mathbf{B}$ *of different order*, say k and h, respectively, with $k < h$. The following representations hold

$$\begin{cases} \mathbf{A} = \dfrac{1}{k!} A^{\alpha_1 \dots \alpha_k} \mathbf{e}_{\alpha_1} \wedge \cdots \wedge \mathbf{e}_{\alpha_k} , \\ \mathbf{B} = \dfrac{1}{h!} B^{\beta_1 \dots \beta_h} \mathbf{e}_{\beta_1} \wedge \cdots \wedge \mathbf{e}_{\beta_h} , \end{cases} \tag{2.318}$$

where $\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_h = 0, 1, 2, 3$.

The exterior product of $\mathbf{A}$ and $\mathbf{B}$ is an antisymmetric tensor of order $(k+h)$ given by

$$\mathbf{A} \wedge \mathbf{B} = \frac{1}{k!h!} A^{\alpha_1 \dots \alpha_k} B^{\beta_1 \cdots \beta_h} \mathbf{e}_{\alpha_1} \wedge \cdots \wedge \mathbf{e}_{\alpha_k} \wedge \mathbf{e}_{\beta_1} \wedge \cdots \wedge \mathbf{e}_{\beta_h} \; . \tag{2.319}$$

Successively, from the tensor product $A^{\alpha_1 \dots \alpha_k} B^{\beta_1 \dots \beta_h}$, by contraction of indices (i.e. using the metric $g_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$ of M_4), we can deduce other antisymmetric tensors of lower rank:

$$A^{\alpha_1 \dots \alpha_{k-1}}{}_{\beta_1} B^{\beta_1 \dots \beta_h} \; , \qquad A^{\alpha_1 \dots \alpha_{k-2}}{}_{\beta_2 \beta_1} B^{\beta_1 \beta_2 \dots \beta_h}, \dots, A_{\beta_k \dots \beta_1} B^{\beta_1 \dots \beta_k \beta_{k+1} \dots \beta_h} \; .$$

It is useful to introduce the notation

$$\mathbf{e}_{\alpha_1 \dots \alpha_k \cdots} = \mathbf{e}_{\alpha_1} \wedge \cdots \wedge \mathbf{e}_{\alpha_k} \wedge \cdots \; ; \tag{2.320}$$

the $\circ$ product (or Clifford's product) of $\mathbf{A}$ and $\mathbf{B}$ is defined by

$$\begin{aligned} \mathbf{A} \circ \mathbf{B} \overset{\text{def}}{=} \; & \mathbf{A} \wedge \mathbf{B} + a_1 A^{\alpha_1 \dots \alpha_{k-1}}{}_{\beta_1} B^{\beta_1 \dots \beta_h} \mathbf{e}_{\alpha_1 \dots \alpha_{k-1} \beta_2 \dots \beta_h} + \\ & a_2 A^{\alpha_1 \dots \alpha_{k-2}}{}_{\beta_2 \beta_1} B^{\beta_1 \beta_2 \dots \beta_h} \mathbf{e}_{\alpha_1 \dots \alpha_{k-2} \beta_3 \dots \beta_h} + \cdots + \\ & a_k A_{\beta_k \dots \beta_1} B^{\beta_1 \dots \beta_k \beta_{k+1} \dots \beta_h} \mathbf{e}_{\beta_{k+1} \dots \beta_h} \; , \end{aligned} \tag{2.321}$$

where

$$a_j = \frac{1}{j!(k-j)!(h-j)!} \; , \qquad j = 1, \dots, k \; .$$

Note that extending the definition of a_j to the value $j = 0$: $a_0 = 1/(h!k!)$ allows writing $\mathbf{A} \wedge \mathbf{B}$ in the form

$$A \wedge B = a_0 A^{\alpha_1 \ldots \alpha_k} B^{\beta_1 \ldots \beta_h} \mathbf{e}_{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_h} .$$

Using the ordinary composition rule of the product, *it* is clear that the product (2.321) can be extended to all the elements of Λ. The $\circ$ *product is noncommutative*, but it can be iterated and hence it results *associative*. Thus, the space Λ, endowed with the linear extension of the product (2.321), becomes an *associative algebra*, and any ordered pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of elements of Λ is associated with a third element $\boldsymbol{\gamma} \in \Lambda$:

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) \quad \to \quad \boldsymbol{\gamma} \stackrel{\text{def}}{=} \boldsymbol{\alpha} \circ \boldsymbol{\beta} \in \Lambda ;$$

this relation defines Clifford's algebra $\mathcal{C}$ of M_4.

As an example, let us consider the $\circ$ product of two vectors $\mathbf{u}$ and $\mathbf{v}$. In this case, $h = k = 1$, and the $\circ$ product reduces to

$$\mathbf{u} \circ \mathbf{v} = \mathbf{u} \wedge \mathbf{v} + \mathbf{u} \cdot \mathbf{v} . \tag{2.322}$$

In fact, (2.321) can be written as

$$\mathbf{u} \circ \mathbf{v} = \frac{1}{1!1!} u^\alpha v^\beta \mathbf{e}_{\alpha\beta} + \frac{1}{0!0!} u_\beta v^\beta \mathbf{e} ,$$

where $\mathbf{e}_{\alpha\beta} = \mathbf{e}_\alpha \wedge \mathbf{e}_\beta$ and $\mathbf{e} = 1$, that is, (2.322). In particular, when $\mathbf{v} = \mathbf{u}$ (2.322) implies

$$\mathbf{u} \circ \mathbf{u} = ||\mathbf{u}|| , \tag{2.323}$$

while, if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal

$$\mathbf{u} \circ \mathbf{v} = \mathbf{u} \wedge \mathbf{v} , \qquad \mathbf{u} \perp \mathbf{v} . \tag{2.324}$$

Thus, if one considers in M_4 an orthonormal basis $\{\mathbf{c}_\alpha\}$, (2.322) implies

$$\mathbf{c}_\alpha \circ \mathbf{c}_\beta = \mathbf{c}_\alpha \wedge \mathbf{c}_\beta + m_{\alpha\beta} \qquad (\alpha, \beta = 0, 1, 2, 3) . \tag{2.325}$$

Clifford's algebra *contains a subalgebra and a group*. In fact, the direct sum operation $\oplus$ is associative and hence the space Λ can also be written as the direct sum of two spaces: Λ^+ and Λ^-, defined by

$$\Lambda^+ \stackrel{\text{def}}{=} \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4 , \qquad \Lambda^- \stackrel{\text{def}}{=} \Lambda^1 \oplus \Lambda^3 , \tag{2.326}$$

both of them with dimension 8; that is

$$\Lambda = \Lambda^+ \oplus \Lambda^- . \tag{2.327}$$

The elements of Λ^+ are said "even", while those of Λ^- are said "odd". On one side, Clifford's algebra induces in Λ^+ a subalgebra: $\mathcal{C}^+$, such that

$$\Lambda^+ \circ \Lambda^+ = \Lambda^+ , \qquad \Lambda^- \circ \Lambda^- \neq \Lambda^- ;$$

hence, the $\circ$ *product is not adapted to the structure* (2.327). On the other side, one can consider the *regular elements of Clifford's algebra*, i.e. those elements $\boldsymbol{\alpha} \in \mathcal{C}$ satisfying the following properties:

1. they are *invertible*. This means that $\boldsymbol{\alpha} \in \mathcal{C}$ has an inverse $\boldsymbol{\alpha}^{-1} \in \mathcal{C}$: $\boldsymbol{\alpha} \circ \boldsymbol{\alpha}^{-1} = \boldsymbol{\alpha}^{-1} \circ \boldsymbol{\alpha} = \mathbf{e} = 1$;
2. they satisfy the property:

$$\boldsymbol{\alpha} \circ \mathbf{v} \circ \boldsymbol{\alpha}^{-1} = \mathbf{w} \in M_4 , \qquad \forall \mathbf{v} \in M_4 . \tag{2.328}$$

Such elements of $\mathcal{C}$ form *Clifford's group*: $\Gamma \in \mathcal{C}$ which is of special importance for the representation of rotations in M_4. In fact Γ contains both even and odd elements, and the quotient $\Gamma/\Lambda^0 - 0$ is isomorphic to the Lorentz group; in other words, each element $\boldsymbol{\alpha} \in \Gamma$ specifies a well-determined Lorentz matrix L:

$$L = L(\boldsymbol{\alpha}) , \qquad \boldsymbol{\alpha} \in \Gamma . \tag{2.329}$$

Conversely, each L specifies an element $\boldsymbol{\alpha} \in \Gamma$, up to a multiplicative factor. In fact, the correspondence (2.329) is associated with the relation:

$$L\mathbf{v} = \boldsymbol{\alpha} \circ \mathbf{v} \circ \boldsymbol{\alpha}^{-1} , \qquad \forall \boldsymbol{\alpha} \in \Gamma, \mathbf{v} \in M_4 , \tag{2.330}$$

or equivalently,

$$L^{-1}\mathbf{v} = \boldsymbol{\alpha}^{-1} \circ \mathbf{v} \circ \boldsymbol{\alpha} , \qquad \forall \boldsymbol{\alpha} \in \Gamma, \mathbf{v} \in M_4 . \tag{2.331}$$

2.19.1 Clifford's Product Composition Law

We can now characterize the regular elements of Clifford's algebra. To this end, we will derive the composition law of Clifford's product. Let us consider then a generic element $\boldsymbol{\alpha} \in \Lambda$:

$$\boldsymbol{\alpha} = a\mathbf{e} + a^\alpha \mathbf{e}_\alpha + \frac{1}{2} A^{\alpha\beta} \mathbf{e}_{\alpha\beta} + \frac{1}{3!} T^{\alpha\beta\gamma} \mathbf{e}_{\alpha\beta\gamma} + \frac{1}{4!} Q^{\alpha\beta\gamma\rho} \mathbf{e}_{\alpha\beta\gamma\rho} ; \tag{2.332}$$

introducing the Ricci tensor $\eta_{\alpha\beta\rho\sigma}$, we have

$$\mathbf{e}_{\alpha\beta\rho} = \eta_{\sigma\alpha\beta\rho} \mathbf{e}^\sigma , \qquad \mathbf{e}_{\alpha\beta\rho\sigma} = \eta_{\alpha\beta\rho\sigma} \Sigma , \tag{2.333}$$

where

$$\Sigma = -\frac{1}{4!} \eta^{\alpha\beta\gamma\delta} \mathbf{e}_{\alpha\beta\gamma\delta} \tag{2.334}$$

is an odd-type antisymmetric 4-tensor, i.e. a basis in Λ^4. Thus, we have the following expression for $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha} = a\mathbf{e} + \mathbf{a} + \mathbf{A} + \tilde{\mathbf{a}} + \tilde{a}\Sigma , \tag{2.335}$$

where $\mathbf{a} = a^\alpha \mathbf{e}_\alpha$, $\mathbf{A} = \frac{1}{2} A^{\alpha\beta} \mathbf{e}_{\alpha\beta}$, and

$$\begin{cases} \tilde{\mathbf{a}} = \frac{1}{3!} T^{\alpha\beta\rho} \eta_{\sigma\alpha\beta\rho} \mathbf{e}^\sigma = \tilde{a}_\sigma \mathbf{e}^\sigma & \text{(odd-type vector)} \\ \tilde{a} = \frac{1}{4!} Q^{\alpha\beta\rho\sigma} \eta_{\alpha\beta\rho\sigma} & \text{(odd-type scalar).} \end{cases} \tag{2.336}$$

Equation (2.335) represents the generic element $\alpha \in \Lambda$ in terms of *a pair of scalars* $(a, \tilde{a})$, a *pair of vectors* $(\mathbf{a}, \tilde{\mathbf{a}})$ and *an antisymmetric* 2-tensor $\mathbf{A}$. Actually, $\tilde{a}_\sigma$ is a pseudovector, and $\tilde{a}$ is a pseudoscalar obtained through the duality operation from an antisymmetric 3-tensor $T^{\alpha\beta\rho}$ and an antisymmetric 4-tensor $Q^{\alpha\beta\rho\sigma}$, respectively:

$$\tilde{a}_\sigma = \frac{1}{3!}\eta_{\sigma\alpha\beta\rho}T^{\alpha\beta\rho} \equiv {}^*T_\sigma, \qquad \tilde{a} = \frac{1}{4!}\eta_{\alpha\beta\rho\sigma}Q^{\alpha\beta\rho\sigma} \equiv {}^*Q \,, \tag{2.337}$$

with the inverse relations,

$$T^{\alpha\beta\rho} = \eta^{\alpha\beta\rho\sigma}\tilde{a}_\sigma \,, \qquad Q^{\alpha\beta\rho\sigma} = -\tilde{a}\eta^{\alpha\beta\rho\sigma} \,. \tag{2.338}$$

Let $\boldsymbol{\beta} \in \Lambda$ be another element of Λ:

$$\boldsymbol{\beta} = \boldsymbol{b}\mathbf{e} + \mathbf{b} + \mathbf{B} + \tilde{\mathbf{b}} + \tilde{\boldsymbol{b}}\boldsymbol{\Sigma} \,, \tag{2.339}$$

and consider the product $\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \gamma$ taking into account (2.321). $\boldsymbol{\gamma}$ is still an element Λ, and it can be written as

$$\boldsymbol{\gamma} = c\mathbf{e} + \mathbf{c} + \mathbf{C} + \tilde{\mathbf{c}} + \tilde{c}\Sigma \,, \tag{2.340}$$

where the various quantities $c, \mathbf{c}, \mathbf{C}, \tilde{\mathbf{c}}, \tilde{c}$ are functions of the analogous quantities of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

For example, multiplying the scalar a by the various elements of β, there arise the following terms:

$$ab \,, \quad a\mathbf{b} \,, \quad a\mathbf{B} \,, \quad a\tilde{\mathbf{b}} \,, \quad a\tilde{b} \,.$$

Multiplying instead the vector $\mathbf{a}$ by $\boldsymbol{\beta}$ and using (2.322) and (2.338) gives rise to the following terms:

$$\begin{aligned}
&\mathbf{a}b \,, \quad \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \cdot \mathbf{b} \,, \qquad \mathbf{a} \wedge \mathbf{B} + a_\alpha B^{\alpha\beta}\mathbf{e}_\beta, \\
&\mathbf{a} \wedge {}^*\tilde{\mathbf{b}} + \frac{1}{2!}a_\alpha({}^*\tilde{\mathbf{b}})^{\alpha\beta\rho}\mathbf{e}_{\beta\rho} \,, \qquad -\frac{1}{3!}a_\alpha\tilde{b}\eta^{\alpha\beta\rho\sigma}\mathbf{e}_{\beta\rho\sigma} \,,
\end{aligned}$$

which are equivalent to

$$\mathbf{a}b \,, \quad \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \cdot \mathbf{b} \,, \quad \overset{*}{B}\mathbf{a} - B\mathbf{a} \,, \quad \mathbf{a} \cdot \tilde{\mathbf{b}}\Sigma + [\mathbf{a} \overset{*}{\wedge} \tilde{\mathbf{b}}] \,, \quad \tilde{b}\mathbf{a} \,,$$

respectively using (2.333). Introducing the compact notation

$$\mathbf{B} = \frac{1}{2}B^{\alpha\beta}\mathbf{e}_{\alpha\beta}, \quad \to \quad B\mathbf{a} = (B^{\alpha\beta}a_\beta)\mathbf{e}_\alpha \,, \tag{2.341}$$

the above terms can be written as

$$\mathbf{a} \cdot \mathbf{b} \,, \quad b\mathbf{a} - B\mathbf{a} \,, \quad \mathbf{a} \wedge \mathbf{b} + [\mathbf{a} \overset{*}{\wedge} \tilde{\mathbf{b}}] \,, \quad \overset{*}{B}\mathbf{a} + \tilde{b}\mathbf{a} \,, \quad \mathbf{a} \cdot \tilde{\mathbf{b}} \,.$$

Summarizing, we have the following relations among the coefficients of the product $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ and those of the single factors:

$$\begin{cases} c = ab - \tilde{a}\tilde{b} + \mathbf{a} \cdot \mathbf{b} + \tilde{\mathbf{a}} \cdot \tilde{\mathbf{b}} + \frac{1}{2}\mathrm{Tr}AB, \\ \mathbf{c} = a\mathbf{b} + b\mathbf{a} - (\tilde{b}\tilde{\mathbf{a}} - \tilde{a}\tilde{\mathbf{b}}) + A\mathbf{b} - B\mathbf{a} - \overset{*}{B}\tilde{\mathbf{a}} - \overset{*}{A}\tilde{\mathbf{b}}, \\ \mathbf{C} = a\mathbf{B} + b\mathbf{A} + [\mathbf{A}, \mathbf{B}] + \mathbf{a} \wedge \mathbf{b} + \tilde{\mathbf{a}} \wedge \tilde{\mathbf{b}} + \overset{*}{[}\, \mathbf{a} \wedge \tilde{\mathbf{b}} + \mathbf{b} \wedge \tilde{\mathbf{a}}] + \tilde{a}\overset{*}{\mathbf{B}} + \tilde{b}\overset{*}{\mathbf{A}}, \\ \tilde{\mathbf{c}} = a\tilde{\mathbf{b}} + b\tilde{\mathbf{a}} + \tilde{b}\mathbf{a} - \tilde{a}\mathbf{b} + A\tilde{\mathbf{b}} - B\tilde{\mathbf{a}} + \overset{*}{A}\mathbf{b} + +\overset{*}{B}\mathbf{a}, \\ \tilde{c} = a\tilde{b} + b\tilde{a} + \mathbf{a} \cdot \tilde{\mathbf{b}} - \mathbf{b} \cdot \tilde{\mathbf{a}} - \frac{1}{2}\mathrm{Tr}A\overset{*}{B}; \end{cases} \tag{2.342}$$

here we have used the standard notation: $AB \equiv (A^{\alpha}{}_{\rho}B^{\rho}{}_{\beta})$ and $[\mathbf{A}, \mathbf{B}]$ is the antisymmetric tensor associated with the commutator $AB - BA$:

$$[\mathbf{A}, \mathbf{B}] \equiv \frac{1}{2}(A^{\alpha}{}_{\rho}B^{\rho\beta} - B^{\alpha}{}_{\rho}A^{\rho\beta})\mathbf{e}_{\alpha\beta} \,. \tag{2.343}$$

As a final remark, we note that (2.342) hold in general and are the corner stones of Clifford's algebra, reducing the product of any two elements of $\mathcal{C}$ to certain tensor algebra operations.

2.19.2 Regular Elements of Clifford's Algebra

Let us study now the regular elements of Clifford's algebra, starting from *even elements*; we have

$$\mathbf{a} = 0 \,, \quad \tilde{\mathbf{a}} = 0 \,. \tag{2.344}$$

The product $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ is still even: $\mathbf{c} = 0 \,, \quad \tilde{\mathbf{c}} = 0$ and (2.342) implies

$$\begin{aligned} c &= ab - \tilde{a}\tilde{b} + \frac{1}{2}\mathrm{Tr}\,(AB), \\ \mathbf{C} &= a\mathbf{B} + b\mathbf{A} + [\mathbf{A}, \mathbf{B}] + \tilde{a}\overset{*}{\mathbf{B}} + \tilde{b}\overset{*}{\mathbf{A}}, \\ \tilde{c} &= a\tilde{b} + b\tilde{a} - \frac{1}{2}\mathrm{Tr}\,(A\overset{*}{B}) \,; \end{aligned} \tag{2.345}$$

furthermore, the products AB, BA, $A\overset{*}{B}$ and $B\overset{*}{A}$ have the same first invariant. In fact, for example, the coefficients of the map $A\overset{*}{B}$ are

$$A^{\alpha}{}_{\rho}\,{}^{*}B^{\rho\beta} = \frac{1}{2}A^{\alpha}{}_{\rho}\eta^{\rho\beta\mu\nu}B_{\mu\nu} \,,$$

so that

$$\mathrm{Tr}(A\overset{*}{B}) = -\frac{1}{2}A_{\beta\rho}\eta^{\beta\rho\mu\nu}B_{\mu\nu} = \mathrm{Tr}(B\overset{*}{A}) \,.$$

In (2.345), c and $\tilde{c}$ are thus symmetric functions of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, differently from $\mathbf{C}$. Hence, *necessary and sufficient condition* to have $\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \boldsymbol{\beta} \circ \boldsymbol{\alpha}$ is

$$AB = BA \qquad \sim \qquad [A, B] = 0 \,; \tag{2.346}$$

that is, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be exchanged if and only if the associated antisymmetric 2-tensors $\mathbf{A}$ and $\mathbf{B}$ commute.

Moreover, necessary and sufficient conditions in order that $\boldsymbol{\alpha} = a\mathbf{e}+\mathbf{A}+\tilde{a}\Sigma$ might have an inverse $\boldsymbol{\beta} = \boldsymbol{b}\mathbf{e} + \mathbf{B} + \tilde{\boldsymbol{b}}\boldsymbol{\Sigma} = \boldsymbol{\alpha}^{-1}$ are

$$\begin{cases} AB = BA \\ ab - \tilde{a}\tilde{b} + \dfrac{1}{2}\mathrm{Tr}\,(AB) = 1 \\ a\mathbf{B} + b\mathbf{A} + \tilde{a}\overset{*}{\mathbf{B}} + \tilde{b}\overset{*}{\mathbf{A}} = 0 \\ a\tilde{b} + b\tilde{a} - \dfrac{1}{2}\mathrm{Tr}\,(A\overset{*}{B}) = 0, \end{cases} \tag{2.347}$$

that is $\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \boldsymbol{\beta} \circ \boldsymbol{\alpha} = e$. To complete the characterization of the even elements, it is necessary to add to (2.347) the conditions

$$\boldsymbol{\alpha} \circ \mathbf{v} \circ \boldsymbol{\beta} \in M_4 \qquad \forall \mathbf{v} \in M_4 \; .$$

To this end, let us evaluate first of all the factor $\boldsymbol{\alpha} \circ \mathbf{v}$ from (2.342):

$$\boldsymbol{\alpha} \circ \mathbf{v} = a\mathbf{v} + A\mathbf{v} - \tilde{a}\mathbf{v} + \overset{*}{A}\mathbf{v} \; ;$$

we can consider then the product $(\boldsymbol{\alpha} \circ \mathbf{v}) \circ \boldsymbol{\beta}$, which is of odd type, so that

$$c = 0 \; , \qquad \mathbf{C} = 0 \; , \qquad \tilde{c} = 0 \; ,$$

and the nonvanishing coefficients are given by

$$\begin{cases} \mathbf{c} = [(ab + \tilde{a}\tilde{b})\,\mathbb{I} + bA - aB + \tilde{a}\overset{*}{B} - \tilde{b}\overset{*}{A} - BA - \overset{*}{B}\overset{*}{A}]\mathbf{v} \\ \tilde{\mathbf{c}} = [(a\tilde{b} - b\tilde{a})\,\mathbb{I} + b\overset{*}{A} + a\overset{*}{B} + \tilde{b}A + \tilde{a}B + \overset{*}{B}A - B\overset{*}{A}]\mathbf{v}. \end{cases} \tag{2.348}$$

The parity condition $\tilde{\mathbf{c}} = 0 \quad \forall \mathbf{v}$ then implies

$$(b\tilde{a} - a\tilde{b})\,\mathbb{I} + B\overset{*}{A} - \overset{*}{B}A = b\overset{*}{A} + a\overset{*}{B} + \tilde{b}A + \tilde{a}B \; ,$$

which, by symmetrization and antisymmetrization gives the two relations:

$$\begin{cases} (b\tilde{a} - a\tilde{b})\,\mathbb{I} + \frac{1}{2}[B\overset{*}{A} + \overset{*}{A}B - \overset{*}{B}A - A\overset{*}{B}] = 0, \\ \frac{1}{2}[B, \overset{*}{A}] - \frac{1}{2}[\overset{*}{B}, A] = b\overset{*}{A} + a\overset{*}{B} + \tilde{b}A + \tilde{a}B. \end{cases} \tag{2.349}$$

Taking into account the identity (2.208), we have that $\boldsymbol{\alpha} \in \Gamma^+$ if one requires additional conditions to (2.347), namely,

$$\begin{cases} (b\tilde{a} - a\tilde{b})\mathbb{I} = \overset{*}{B}A - B\overset{*}{A} \\ \tilde{a}B + \tilde{b}A = -b\overset{*}{A} - a\overset{*}{B}. \end{cases} \tag{2.350}$$

Moreover, by duality, (2.344) gives the relation:

$$\tilde{a}B + \tilde{b}A = a\overset{*}{B} + b\overset{*}{A} \; .$$

Comparing with (2.350), we then obtain the two conditions:

$$\tilde{a}B + \tilde{b}A = 0 \; , \qquad a\overset{*}{B} + b\overset{*}{A} = 0 \; .$$

Next, assuming $\tilde{a} \neq 0$ and $a \neq 0$ (*general case*), we find $B = -\tilde{b}A/\tilde{a} = -bA/a$, and $\tilde{b}/\tilde{a} = b/a = \lambda$. Thus, finally

$$B = -\lambda A \; , \quad \tilde{b} = \lambda\tilde{a} \; , \quad b = \lambda a \; , \tag{2.351}$$

with λ a parameter still unknown . From $(2.347)_2$, with the constraints (2.351), we have

$$(a^2 - \tilde{a}^2)\lambda - \frac{1}{2}\lambda I_1(A^2) = 1 \; ,$$

so that, assuming

$$(a^2 - \tilde{a}^2) + I_2(A) \neq 0 \; , \tag{2.352}$$

one gets the value of λ:

$$\lambda = \frac{1}{(a^2 - \tilde{a}^2) + I_2(A)} \; . \tag{2.353}$$

Equation $(2.347)_4$ in turn becomes

$$2\lambda a\tilde{a} = -\frac{1}{2}\lambda \, \mathrm{Tr}(A\overset{*}{A}) \; ,$$

that is, in agreement with (2.194):

$$a\tilde{a} = \overset{*}{I}(A) \; . \tag{2.354}$$

We have just proven the following theorem:

The even and regular elements of Clifford's algebra in M_4 can be written as $\boldsymbol{\alpha} = a\mathbf{e} + \mathbf{A} + \tilde{a}\Sigma$ and are characterized by the two conditions:

$$(a^2 - \tilde{a}^2) + I_2(A) \neq 0 \; , \qquad a\tilde{a} = \overset{*}{I}(A) \; , \tag{2.355}$$

where $\overset{*}{I}(A)$ is the odd invariant defined by (2.194):

$$\overset{*}{I}(A) = -\frac{1}{4}I_1(A\overset{*}{A}) \equiv \frac{1}{4}A_{\alpha\beta}{}^{*}A^{\alpha\beta} \; , \tag{2.356}$$

also related to $I_4(A)$:

$$[\overset{*}{I}(A)]^2 = -I_4(A) \; . \tag{2.357}$$

The inverse of $\boldsymbol{\alpha}$, because of (2.351)–(2.353), is given by

$$\boldsymbol{\alpha}^{-1} = \frac{\mathbf{1}}{(\boldsymbol{a^2} - \boldsymbol{\tilde{a}^2}) + \boldsymbol{I_1}(\boldsymbol{A^2})}(\boldsymbol{a}\mathbf{e} - \mathbf{A} + \boldsymbol{\tilde{a}}\boldsymbol{\Sigma}) \;, \tag{2.358}$$

that is, apart from the normalization factor, there is only the replacement $\mathbf{A} \to -\mathbf{A}$ in the expression of $\boldsymbol{\alpha}$.

Similarly, assuming $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of *odd type*,

$$\boldsymbol{\alpha} = \mathbf{a} + \tilde{\mathbf{a}} \;, \qquad \boldsymbol{\beta} = \mathbf{b} + \tilde{\mathbf{b}} \;, \tag{2.359}$$

and imposing the conditions,

$$\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \boldsymbol{\beta} \circ \boldsymbol{\alpha} \;, \qquad \boldsymbol{\alpha} \circ \boldsymbol{\beta} = 1 \;, \quad \boldsymbol{\beta} \circ \mathbf{v} \circ \boldsymbol{\alpha} \in M_4 \quad \forall \mathbf{v} \in M_4 \;, \tag{2.360}$$

one can determine the odd-type elements of Clifford's group. We have that *the regular odd-type elements* $\boldsymbol{\alpha} = \mathbf{a} + \tilde{\mathbf{a}}$ *are only those satisfying the conditions:*

$$||\mathbf{a}|| - ||\tilde{\mathbf{a}}|| \neq 0 \;, \qquad \mathbf{a} \cdot \tilde{\mathbf{a}} = 0 \;; \tag{2.361}$$

in this case, the inverse of $\boldsymbol{\alpha}$ is given by:

$$\boldsymbol{\alpha}^{-1} = \frac{1}{||\mathbf{a}|| - ||\tilde{\mathbf{a}}||}(\mathbf{a} - \tilde{\mathbf{a}}) \;. \tag{2.362}$$

In the following, we will consider only the even elements and the associated L-representation (2.330) in terms of antisymmetric 2-tensors. This characterizes the proper Lorentz group, excluding antirotations.

2.19.3 Proper Rotations and Antisymmetric 2-Tensors

We have seen that each regular element of Clifford's subalgebra $\mathcal{C}^+$ (see (2.344)–(2.355)) defines a 4-rotation L. To obtain the expression of L, one has to consider an arbitrary vector $\mathbf{v} \in M_4$ and evaluate the product $\boldsymbol{\alpha}^{-1} \circ \mathbf{v} \circ \boldsymbol{\alpha}$; in fact $L\mathbf{v} = \boldsymbol{\alpha} \circ \mathbf{v} \circ \boldsymbol{\alpha}^{-1}$. In detail, from $(2.348)_1$ and taking into account (2.351)–(2.353), we have the following expression for L:

$$L = \frac{1}{a^2 - \tilde{a}^2 + I_2(A)} \left\{ (a^2 + \tilde{a}^2)\, \mathbb{I} + 2[aA - \tilde{a}\overset{*}{A}] + A^2 + \overset{*}{A}{}^2 \right\} \;, \tag{2.363}$$

where both the antisymmetric 2-tensors A and $\overset{*}{A}$ appear as well as the two scalars a and $\tilde{a}$, which are not independent because of $(2.355)_2$. In the representation (2.363) of L obtained starting from $\alpha \in \Gamma^+$, we see that α does not enter in an essential way, since there always exists an arbitrary factor at disposal. In fact, the right-hand side of (2.363) is invariant under transformations like

$$a \to \lambda a \;, \quad \tilde{a} \to \lambda \tilde{a} \;, \quad A \to \lambda A \;, \tag{2.364}$$

since both the numerator and the denominator in (2.363) are homogeneous and second-degree function of α. We can then use the factor λ to simplify

(2.363). After assuming for instance $a \neq 0$ (that is, A nondegenerate), and defining:

$$q = \tilde{a}/a\ , \quad Q = A/a\ , \tag{2.365}$$

Equation (2.363) can be written as

$$L = \frac{1}{J(Q)}\left\{(1+q^2)\,\mathbb{I} + 2[Q - q\overset{*}{Q}] + Q^2 + \overset{*}{Q}{}^2\right\}\ , \tag{2.366}$$

so that (2.355)–(2.357) assume the form

$$J(Q) = 1 - q^2 + I_2(Q) \neq 0\ , \qquad q = \overset{*}{I}(Q), \qquad q^2 = -I_4(Q)\ . \tag{2.367}$$

Note that (2.366) gives L in terms of the antisymmetric 2-tensors Q and its dual $\overset{*}{Q}$. One can also eliminate the dual using (2.204) and (2.200):

$$\overset{*}{I}(Q)\overset{*}{Q} = -Q^3 - I_2(Q)Q\ , \qquad [\overset{*}{Q}]^2 = Q^2 + I_2(Q)\mathbb{I}\ , \tag{2.368}$$

so that one re-obtains the Cayley representation (2.299):

$$L = (Q^0 + Q)(Q^0 - Q)^{-1}\ . \tag{2.369}$$

Therefore L results in an *isotropic function* of the antisymmetric 2-tensor $Q = (Q^\alpha{}_\beta)$, which is also a third-degree polynom, because of the Hamilton–Cayley identity (2.202).

Moreover, (2.366) corresponds to the vectorial operator $\boldsymbol{\alpha} \circ \mathbf{v} \circ \boldsymbol{\alpha}^{-1}$ associated with the even element $\boldsymbol{\alpha}$, in agreement with (2.330):

$$\boldsymbol{\alpha} = \mathbf{e} \oplus \mathbf{Q} \oplus \boldsymbol{q}\boldsymbol{\Sigma} \in \boldsymbol{\Lambda}\ . \tag{2.370}$$

Hence, if one considers another rotation L':

$$L' = \frac{1}{1 + I_2' + I_4'}\left[(1 + I_2' - I_4')\,\mathbb{I} + 2[1 + I_2']Q' + 2Q'^2 + 2Q'^3\right]\ , \tag{2.371}$$

associated with the antisymmetric tensor $\mathbf{Q}'$, or to the even element $\boldsymbol{\alpha}'$:

$$\boldsymbol{\alpha}' = \mathbf{e} \oplus \mathbf{Q}' \oplus \boldsymbol{q}'\boldsymbol{\Sigma} \in \boldsymbol{\Lambda}\ , \tag{2.372}$$

one can evaluate the product:

$$L'' = L'L\ , \tag{2.373}$$

i.e. the rotation associated with the even element $\boldsymbol{\alpha}''$:

$$\boldsymbol{\alpha}'' = \boldsymbol{\alpha}' \circ \boldsymbol{\alpha}\ , \tag{2.374}$$

clearly defined up to an arbitrary multiplicative factor[23]:

[23] The inverse of the product coincides with the product of the inverses, in inverse order.

$$\boldsymbol{\alpha}' \circ \mathbf{v}' \circ \boldsymbol{\alpha}'^{-1} \equiv \boldsymbol{\alpha}' \circ (\boldsymbol{\alpha} \circ \mathbf{v} \circ \boldsymbol{\alpha}^{-1}) \circ \boldsymbol{\alpha}'^{-1} = (\boldsymbol{\alpha}' \circ \boldsymbol{\alpha}) \circ \mathbf{v} \circ (\boldsymbol{\alpha}' \circ \boldsymbol{\alpha})^{-1} .$$

$\boldsymbol{\alpha}''$ can then be obtained applying the general relations (2.342) with $a = 1$, $\mathbf{a} = 0$, $\mathbf{A} = \mathbf{Q}'$, $\tilde{\mathbf{a}} = 0$, $\tilde{a} = q'$, and $b = 1$, $\mathbf{b} = 0$, $\mathbf{B} = \mathbf{Q}$, $\tilde{\mathbf{b}} = 0$, $\tilde{b} = q$. In fact, assuming $c \neq 0$, that is, assuming the following constraint for Q and Q':

$$1 - qq' + \frac{1}{2}\mathrm{Tr}(QQ') \neq 0 \,, \tag{2.375}$$

the 4-rotation L'' is still of the form (2.366):

$$L' = \frac{1}{1 + I_2'' + I_4''}\left[(1 + I_2'' - I_4'')\,\mathbb{I} + 2[1 + I_2'']Q'' + 2Q''^2 + 2Q''^3\right] \,, \tag{2.376}$$

where Q'' is given by $(2.342)_3$ in terms of Q and Q'. Hence, the following relation (equivalent to C/c) holds:

$$Q'' = \frac{Q + Q' + Q'Q - QQ' + q'\overset{*}{Q} + q\overset{*}{Q}{}'}{1 - qq' + \frac{1}{2}\mathrm{Tr}QQ'} \,, \tag{2.377}$$

which is exactly the composition law (2.316), if one uses $(2.367)_2$.

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3

Test Particle Dynamics

3.1 Fundamental Laws of Test Particle Dynamics

In the relativistic situation, we have seen that the acceleration of a point particle is no longer invariant in the context of Galilean frames. This reflects also into the assessment of the dynamics because it is no more possible to introduce the dynamical concepts of mass and mechanical force following the classical approach, i.e. starting from the postulates on the acceleration.

On the other hand, a theory like the relativistic one cannot be grounded on a static definition of mass and force. The problem can be solved working directly with the *absolute point of view*, i.e. in M_4, and assuming a time orientation. The relative point of view, together with all the associated consequences and modifications, will then follow, a posteriori, from the absolute formulation.

First of all, the *law of inertia* $\mathbf{a} = 0$, as from (2.115), has an absolute meaning and can be expressed by the condition $\mathbf{A} = 0$, $\forall\tau \in (\tau_0, \tau_1)$, or in the rectilinear form of the world line ($\mathbf{V}$ = constant) of the particle itself.

The *second law* of the dynamics, instead, should be adapted to the relativistic case and results no longer in the expression $m\mathbf{a} = \mathbf{f}$ but in its more general version $\dot{\mathbf{p}} = \mathbf{f}$, $\mathbf{p} = m\mathbf{v}$ being the linear momentum which includes also the case of variable mass particles. Such an extension, clearly, requires the preliminary definition (geometrical and dynamical) of the *scheme "material point"* which, because of its simplicity, plays a central role in the new relativistic theory too.

In classical mechanics, such a scheme (geometrical point, endowed with a positive scalar quantity: the *mass* m, invariant by definition), can be used to represent either the material elements (or particles, for brevity), or the finite dimensional material bodies in certain dynamical conditions. A posteriori, in fact, it is justified, in the dynamics of material systems, by the *theorem of the centre of mass motion*. Independent of this theorem, however, a pointlike scheme can be considered also in special relativity, where a material particle can be represented by an oriented world line ℓ^+ (its history in M_4), and by a scalar invariant $m_0 > 0$, locally defined on this line (its *proper mass*). If

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m_0 is *invariant* along all the history of the particle m_0 = constant, one says that the particle has no internal structure. This is a special case because the more general scheme for pointlike particles does not excludes that the mass can vary along ℓ^+, i.e. $m_0 = m_0(E)$, $\forall E \in \ell^+$. In the latter case, one says that the particle has an *internal structure of scalar type*.

We will assume $m_0 > 0$; however, an analogous treatment could be done for the case $m_0 < 0$, associated with "exotic particles".[1] Separately, we will also discuss the case of massless particles $m_0 = 0$, corresponding to *photons*, as well as a unified dynamics of massive and massless particles.

However, the pointlike scheme, which we consider in the following, is not the most general one. In fact, for a more adequate description of matter, it may be necessary to introduce other local quantities, like the spin or other vectors or tensors. Moreover, we will see that the mass will be strictly related to the energy, so that m_0 describes practically internal states of the particle. Thus, the problem of assuming m_0 a continuous function all along ℓ^+, or not (as in quantum mechanics), arises too.

In any case, from a global point of view, for "particle" we will mean the *pair of an oriented world line* (or an arc) ℓ^+ and a function $m_0(E) > 0$, defined $\forall E \in \ell^+$; from a local point of view, instead, the particle will be identified by the event $E \in \ell^+$, the value of m_0 and the 4-velocity $\mathbf{V}$, tangent to the world line at that event [1, 2, 3]. So we define the linear 4-momentum:

$$\mathbf{P} \overset{\text{def}}{=} m_0 \mathbf{V} \qquad \forall E \in \ell^+ \, . \tag{3.1}$$

The applied vector $(E, \mathbf{P})$ summarizes either the kinematical or the material state of the particle because from $\mathbf{P}$ one obtains both m_0 and $\mathbf{V}$. In fact, from (3.1) and the condition $\mathbf{V} \cdot \mathbf{V} = -c^2$, the norm of $\mathbf{P}$ follows: $||\mathbf{P}|| = \mathbf{P} \cdot \mathbf{P} = -m_0^2 c^2$, and hence

$$m_0 = \frac{1}{c}\sqrt{-\mathbf{P}\cdot\mathbf{P}} \, , \qquad \mathbf{V} = \frac{1}{m_0}\mathbf{P} = \frac{c\mathbf{P}}{\sqrt{-\mathbf{P}\cdot\mathbf{P}}} \, . \tag{3.2}$$

As concerns the absolute laws of point mechanics, the law of inertia assumes a quite different form from the ordinary one. More precisely, requiring that in absence of any external action a particle cannot modify either its internal structure or its kinematical state, it can be formulated in the following form:

Law I (*or inertia law*)*: For any isolated particle, both the proper mass and the* 4*-velocity* (*and hence the* 4*-momentum*) *are invariant*:

$$\mathbf{P} = const. \tag{3.3}$$

Due to (3.2), (3.3) summarizes either the condition m_0= constant, or the *geodesic law: the world line of a particle, in the absence of any external action,*

[1] We will see that the canonical formulation of the dynamics will depend on m_0^2.

is a straight timelike line of M_4. From Law I, *Law II* follows easily. In fact, if a particle undergoes the action of an external force, its 4-momentum varies (and this happens either if ℓ^+ has nonvanishing curvature, or if the mass m_0 effectively depends on E). Thus $\mathrm{d}\mathbf{P}/\mathrm{d}\tau \neq 0$ and can be interpreted as a local measure of the 4-force $\mathbf{K}$ acting on the particle and responsible for its deviation from geodesic (inertial) motion. Thus, *Law II* can be formulated in the following form:

Law II: The derivative of the 4*-momentum with respect to proper time equals the* 4*-force:*

$$\mathbf{K}(E) = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\tau} \qquad \forall E \in \ell^+ . \tag{3.4}$$

Equation (3.4), in spite of representing directly a physical law, can be used to define the value of the 4-force $\mathbf{K}(\tau)$ starting from ℓ^+ and $m_0(\tau)$. Conversely, this is the fundamental equation for the absolute dynamics of point particles, once the 4-force characterizing the associated physical action is assigned.

In the so-called *restricted problems* of the material point scheme, the parameters can be at most τ, m_0, E and $\mathbf{V}$, i.e. $\tau, E, \mathbf{P}$:

$$\mathbf{K} = \mathbf{K}(\tau, E, \mathbf{P}) . \tag{3.5}$$

This is a function of nine variables which, in order to represent a real physical action, *should be invariant* with respect to the Lorentz transformations, similarly to the invariance with respect to the Galilean transformation laws of the force: $\mathbf{f} = \mathbf{f}(t, P, \mathbf{v})$ in classical mechanics. Such a property, which is automatically satisfied here because of (3.5) implies that the 4-force K^α be a vectorial function of M_4, depending only on absolute quantities.

Thus, generally, in special relativity, the concept of force can be extended easily either from the absolute or the relative point of view. In general relativity, instead, the absolute mechanism of the gravitational action will be completely modified.

Finally, *Law III*, that is the *action and reaction principle*, completely loses its validity in relativity because here the concept of action at a distance is excluded by principle. And this for two reasons: first of all, the simultaneity of two events has not an absolute meaning; hence, the concept of an instantaneous action is meaningless. As a second reason, there is the fact that every physical action propagates with finite speed, that is, a certain amount of time is necessary for the action to be effective. Hence, the concept of an immediate action is meaningless too.

The action and reaction principle, however, remains valid in collision problems, as well as in all the cases in which there are directed and mediated actions, either in the case of particle–particle interaction or in the case of particle–field or even field–field interaction.

3.2 Cauchy Problem

In the *restricted problems* of the point particle scheme, the fundamental law of absolute dynamics is the following:

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\tau} = \mathbf{K}(\tau, E, \mathbf{P}) \ , \tag{3.6}$$

where $\mathbf{P} = m_0\mathbf{V}$ is the 4-momentum of the particle, and τ is its proper time. Because of the dependence of $\mathbf{K}$ also on E, (3.6) is not a first-order differential equation for the unknown $\mathbf{P}$. Actually, $\mathbf{P}$ should be considered as an auxiliary unknown, depending on E through m_0 and $\mathbf{V}$. Moreover, once the force law is assigned, the evolution of the particle, related to the determination of both m_0 and the world line ℓ^+, is subject to (3.6), but also to the constraint $\mathbf{V} \cdot \mathbf{V} = -c^2$, which takes into account the meaning of the proper time parameter τ. Thus, fundamental equations of point dynamics are

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\tau} = \mathbf{K}(\tau, E, \mathbf{P}), \qquad \mathbf{V} \cdot \mathbf{V} = -c^2 \ . \tag{3.7}$$

These equations, because of the relations $\mathbf{P} = m_0\mathbf{V}$ and $\mathbf{V} = \mathrm{d}\Omega E/\mathrm{d}\tau$, form a differential system in $4 + 1$ equations, for the five unknowns $E(\tau)$ and $m_0(\tau)$, which can be cast into *normal form*, in the following way. Let us use the definition of $\mathbf{P} = m_0\mathbf{V}$ in $(3.7)_1$; we find

$$\frac{\mathrm{d}m_0}{\mathrm{d}\tau}\mathbf{V} + m_0\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}\tau} = \mathbf{K}(\tau, E, m_0, \mathbf{V}) \ .$$

From this, by scalar multiplication by $\mathbf{V}$, and taking into account that $(3.7)_2$ implies $\mathbf{V} \cdot \mathrm{d}\mathbf{V}/\mathrm{d}\tau = 0$, one gets the following *first-order differential system*, normal in the unknowns m_0, $\mathbf{V}$ and ΩE:

$$\frac{\mathrm{d}m_0}{\mathrm{d}\tau} = -\frac{1}{c^2}\mathbf{K} \cdot \mathbf{V}, \quad m_0\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}\tau} = \mathbf{K} + \frac{1}{c^2}(\mathbf{K} \cdot \mathbf{V})\mathbf{V}, \quad \frac{\mathrm{d}\Omega E}{\mathrm{d}\tau} = \mathbf{V} \ ; \tag{3.8}$$

to (3.8), one must add the condition $\mathbf{V} \cdot \mathbf{V} = -c^2$. However, such a condition can be weakened requiring its validity at a certain instant only, let us say initially, at $\tau = \tau_0$. In other words, the system (3.7) is equivalent to (3.8), supplemented by the *initial condition*:

$$\mathbf{V} \cdot \mathbf{V} = -c^2 \qquad \text{at} \qquad \tau = \tau_0 \ . \tag{3.9}$$

To prove this, let us scalar multiply $(3.8)_2$ by $\mathbf{V}$; we find

$$m_0\mathbf{V} \cdot \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}\tau} = \mathbf{K} \cdot \mathbf{V}\left(1 + \frac{1}{c^2}\mathbf{V} \cdot \mathbf{V}\right) \ ,$$

or

$$\frac{m_0 c^2}{2}\frac{\mathrm{d}X}{\mathrm{d}\tau} = (\mathbf{K} \cdot \mathbf{V})X \ , \qquad X = \left(1 + \frac{1}{c^2}\mathbf{V} \cdot \mathbf{V}\right) \ . \tag{3.10}$$

Thus, any solution of system (3.8) gives a unique solution $X(\tau)$ of (3.10). But (3.10) is linear and homogeneous in X and for it a uniqueness theorem is valid. Hence, if $X(\tau)$ vanishes initially, i.e. $X(\tau_0) = 0$ so that (3.9) is satisfied, then $X(\tau) = 0$ or $\mathbf{V} \cdot \mathbf{V} = -c^2$, $\forall \tau > \tau_0$ in a certain neighbourhood of τ_0. In other words, the differential system (3.8) implies the constraint $(3.7)_2$ once this is satisfied initially. Similarly, one may say that $(3.7)_2$ is a conservation equation for the system (3.9), and this is an *involutive constraint* in the terminology of Cartan.

Thus, once the force law is assigned, the absolute dynamics of the point particle is determined by the solution of the normal system (3.8) and, according to regularity properties of the force law, the evolution of the particle is uniquely determined by the initial conditions:

$$m_0 = m_{0,0} \ , \quad E = E_0, \quad \mathbf{V} = \mathbf{V}_0 \quad \text{at} \quad \tau = \tau_0 \ . \tag{3.11}$$

$\mathbf{V}_0$, in turn, is not completely free, but must be chosen compatibly with (3.9). Moreover, neither the orientation of $\mathbf{V}_0$ is free. In fact, in the context of equi-oriented Galilean frames (i.e. assuming M_4 endowed with one of the two half lightcones, say $\mathcal{C}_3^+$), $\mathbf{V}_0$ should belong to this half-cone too.

Hence, other than in the classical situation, the initial data are not free, but they must satisfy the following limitation:

$$m_{0,0} > 0 \ , \quad ||\mathbf{V}_0|| = -c^2 \ , \quad \mathbf{V}_0 \in \mathcal{C}_3^+ \ , \tag{3.12}$$

which, because of the above-mentioned property of conservation for the norm of $\mathbf{V}$, imply that the world line of the particle is timelike and has the same orientation as $\mathbf{V}_0$. Put differently, for any kind of 4-force $\mathbf{K}$, the Cauchy problem (3.8)–(3.11) gives rise to a world line entirely contained in the half-cone $\mathcal{C}_3^+$, with its vertex at E_0. The latter condition gives also a dynamical meaning to such half-cone: it is the *future of* E_0 or all the events which can be influenced by E_0, in the sense that they can be in causal relation with E_0 as a consequence of the presence of $\mathbf{K}$.

It is also useful to distinguish between *dynamical motions* (effectively performed by a material particle, under the action of an external field $\mathbf{K}(\tau, E, \mathbf{P})$ and conditions (3.12)) and *kinematical motions* (a sequence of events not causally connected). These last motions may imply a speed faster than that of light and be represented by world lines external to the lightcone. For example, if a flash light sends signals onto a screen circularly at a distance r, in a certain Galilean frame, then, allowing the flash light to uniformly rotate with period $T < 2\pi r/c$, one has an image which moves uniformly on a circular trajectory, at speed $v > c$. This is not surprising because it is the causality principle which forbids faster than light speed. Different would be the case of postulating the existence of tachionic particles: this would be possible only in a different relativistic theory, because postulate E will be violated [4].

3.3 Classification of 4-Forces. Conservative Forces

Independently of the force law (3.5), in a generic event E, $\mathbf{K}$ can always be decomposed, in a unique way, into the sum of two vectors: one parallel to $\mathbf{V}$ (and hence timelike), and the other orthogonal to it, and hence spacelike and belonging to the hyperplane Σ, orthogonal to $\mathbf{V}$ (see Fig. 3.1):

$$\mathbf{K} = \lambda \mathbf{V} + \boldsymbol{\mathcal{F}} , \qquad \boldsymbol{\mathcal{F}} \in \Sigma : \boldsymbol{\mathcal{F}} \cdot \mathbf{V} = 0 , \quad \lambda = -\frac{1}{c^2} \mathbf{K} \cdot \mathbf{V} . \tag{3.13}$$

This decomposition distinguishes between two kinds of 4-forces, according to whether $\mathbf{K}$ be tangent to the world line of the particle or orthogonal to it:

1. $\mathbf{K} = \lambda \mathbf{V}$ (or $\boldsymbol{\mathcal{F}} = 0$);
2. $\mathbf{K} \cdot \mathbf{V} = 0$ (or $\lambda = 0$).

The first kind of 4-forces are of *thermal type*, whereas the second are of *mechanical type*, and the reason of these names stems from (3.8):

$$\begin{cases} \mathbf{K} = \lambda \mathbf{V} \quad \rightarrow \quad \dfrac{\mathrm{d}}{\mathrm{d}\tau} m_0 = \lambda \neq 0 , \qquad \mathbf{V} = \text{const.} \\ \mathbf{K} \cdot \mathbf{V} = 0 \quad \rightarrow \quad m_0 = \text{const.} , \qquad \mathbf{A} = \mathbf{K}/m_0 . \end{cases} \tag{3.14}$$

The 4-forces of the first kind have no effects on the motion, that is on the curve arc (straight line) in M_4 which represents it, but, through m_0, they influence the internal structure of the material point. Thus, this can be considered an *action of thermal type*.

On the other hand, 4-forces of the second kind have only the effect of accelerating the particle, bending the world line without modifying its internal structure; hence, they correspond to the ordinary mechanical actions. For instance, of such a kind is the effect of an electromagnetic field on a charged particle, i.e. the Lorentz force. As it will be discussed in Chap. 9, electromagnetism specifies this force as

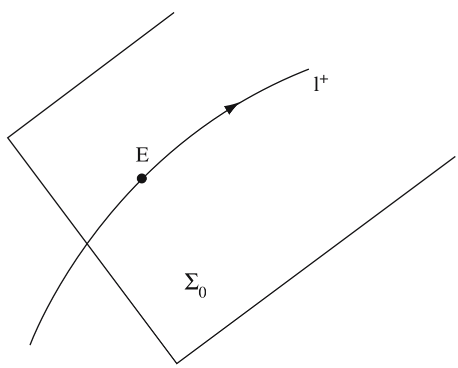

Fig. 3.1. Space-time splitting induced by the world line of a material particle

$$K_\alpha = \frac{e}{c} F_{\alpha\beta} V^\beta \ , \tag{3.15}$$

where e is the (invariant) charge of the particle, V^β its 4-velocity and $F_{\alpha\beta}$ the electromagnetic tensor. This is not, in general, a force law like that specified in (3.5) because $F_{\alpha\beta}$ is subordinated to Maxwell's equations and is not directly given as a function of E. However, $F_{\alpha\beta}$ is antisymmetric: $F_{\alpha\beta} = -F_{\beta\alpha}$, and hence the Lorentz force is always of mechanical type:

$$\mathbf{K} \cdot \mathbf{V} = K_\alpha V^\alpha = \frac{e}{c} F_{\alpha\beta} V^\alpha V^\beta = 0 \ .$$

An important consequence of this is the fact that the electromagnetic field cannot change the proper mass of a particle. Summarizing, the notion of 4-force includes two different physical actions, separated in the classical situation and here strictly related: the thermal action and the mechanical one.

However, (3.5), typical in restricted problems, does not exclude the existence of *positional forces*, i.e. depending only on the point-event E where it is applied: $\mathbf{K} = \mathbf{K}(E)$. Among these, in turn, one can consider *conservative forces*, characterized by the existence of a regular and uniform scalar function $\mathcal{U}(E)$ of E, such that, for all E in the domain of definition and for all world lines passing through E, the mechanical power is an exact derivative, that is

$$\mathbf{K} \cdot \mathbf{V} = \frac{\mathrm{d}\mathcal{U}}{\mathrm{d}\tau} \ , \tag{3.16}$$

or explicitly

$$V^\alpha K_\alpha = V^\alpha \frac{\partial \mathcal{U}}{\partial x^\alpha} = \frac{\mathrm{d}\mathcal{U}}{\mathrm{d}\tau} \qquad \forall E, \mathbf{V};$$

hence:

$$K_\alpha = \frac{\partial \mathcal{U}}{\partial x^\alpha} \ , \qquad \text{or} \qquad \mathbf{K} = \mathrm{Grad}\, \mathcal{U} \ , \tag{3.17}$$

even if $\mathbf{V}$ is not completely arbitrary but subordinated to the condition $\mathbf{V} \cdot \mathbf{V} = -c^2$. In fact, choosing $V^\alpha = (c, 0, 0, 0)$, one gets $K_0 = \partial \mathcal{U}_0 / \partial x^0$ and, in turn, choosing $V^\alpha = (\sqrt{2}c, c, 0, 0)$, $V^\alpha = (\sqrt{2}c, 0, c, 0)$, $V^\alpha = (\sqrt{2}c, 0, 0, c)$, one finds $K_i = \partial \mathcal{U} / \partial x^i$.

Equation (3.17) characterizes $\mathbf{K}$ starting from the potential function $\mathcal{U}$. From this it follows that a conservative 4-force is necessarily positional, like $\mathcal{U}_0$.

The hypersurfaces $\mathcal{U}(x^0, x^1, x^2, x^3) = \text{const.}$ form, in M_4, the field equipotential hypersurfaces. They are three-dimensional and characterize the field itself, through the congruence of the ∞^3 orthogonal curves (flux lines of $\mathbf{K}$). We notice here, from one side, the strict analogy with the classical case (either for their definition or their representation properties) and, from the other side, the big difference between conservative forces in classical mechanics and conservative 4-forces. The former have their conservative meaning with respect to a given Galilean frame, but there is not any absolute notion of conservativity (for instance, the Earth gravitational field is conservative, as well as central, only in the frame in which the Earth is at rest; in a frame in which the Moon is at rest, this field is no longer conservative, nor positional). Definition (3.16) has instead an absolute meaning and leads to the definition of $\mathcal{U}$ as the *intrinsic potential of the* 4*-force* $\mathbf{K}$.

We list few examples of 4-forces:

$$\begin{cases} \mathbf{K} = \text{constant} & \text{uniform field} \\ \\ \mathbf{K} = \phi(|\Omega E|)\text{vers}|\Omega E| & \text{central field, with centre } \Omega \\ \\ \mathbf{K} = \phi(|\Omega E|, \mathbf{V})\text{vers}|\Omega E| & \text{general central field, with centre } \Omega\ , \end{cases}$$

where $|\Omega E| = \sqrt{|\Omega E \cdot \Omega E|}$. The first two fields are conservative, with potential $\mathcal{U} = K_\alpha x^\alpha$ and $\mathcal{U} = \epsilon \int \phi(x)\, \mathrm{d}x$ respectively, with $x = |\Omega E|$ and $\epsilon = \pm 1$, according to the positive/negative sign of the norm of ΩE.

3.4 Constrained Material Point

As in classical mechanics, it is meaningful to consider also in special relativity the scheme of constrained material point. One has to put, a priori, a limitation to the scheme, and this can concern either the world line ℓ^+ (belonging, for instance, to a certain hypersurface) or the material content of the particle. Thus, a relativistic and quite general constraint can be expressed as

$$\phi(E, \mathbf{P}, \tau) \geq 0\ , \tag{3.18}$$

with ϕ a scalar (invariant) function. The constraint (3.18) can be of special kind; for example, *holonomic* (i.e. depending on E and τ only) or *nonholonomic* (i.e. depending on $\mathbf{P}$ too); *unilateral* ($\phi > 0$) or *bilateral* ($\phi = 0$); dependent or independent of τ. Furthermore, through $\mathbf{P}$, the constraint (3.18) could impose limits to the proper mass m_0. In this sense, a very simple bilateral constraint is

$$m_0 = \text{const}\ . \tag{3.19}$$

which refers only to the internal structure of the particle. We define the constraint to be *ideal* if the corresponding 4-reaction is $\mathbf{R} = \lambda \mathbf{V}$, with λ generic.

In the constrained scheme, the fundamental law of relativistic dynamics should be written by distinguishing the active 4-force from the reaction of the constraint, which is a supplementary unknown for the problem; from here the necessity of specifying the dynamical properties of the constraint itself (e.g. ideal constraint or not) follows. Thus, the fundamental equation becomes

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\tau} = \mathbf{K}(E, \mathbf{P}, \tau) + \mathbf{R}\ , \tag{3.20}$$

or, similarly to (3.8):

$$\begin{cases} \dfrac{\mathrm{d}m_0}{\mathrm{d}\tau} = -\dfrac{1}{c^2}(\mathbf{K}+\mathbf{R})\cdot\mathbf{V} \\ m_0\dfrac{\mathrm{d}\mathbf{V}}{\mathrm{d}\tau} = \mathbf{K}+\mathbf{R}+\dfrac{1}{c^2}(\mathbf{K}+\mathbf{R})\cdot\mathbf{V}\,\mathbf{V}\ . \end{cases} \tag{3.21}$$

In particular, when $\mathbf{R} = \lambda\mathbf{V}$, the above equations reduce to

$$\frac{\mathrm{d}m_0}{\mathrm{d}\tau} = -\frac{1}{c^2}\mathbf{K}\cdot\mathbf{V}+\lambda\,, \quad m_0\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}\tau} = \mathbf{K}+\frac{1}{c^2}\mathbf{K}\cdot\mathbf{V}\,\mathbf{V}\,, \tag{3.22}$$

where the unknown λ is specified from the constraint. In particular, if the constraint is that given in (3.19) (particle without any internal structure), $(3.22)_2$ is a "pure" equation for the determination of the motion. Once $(3.22)_2$ is solved, starting from certain initial conditions, then $(3.22)_1$ gives the value of λ:

$$\lambda = \frac{1}{c^2}\mathbf{K}[\tau, E(\tau), \mathbf{P}(\tau)]\cdot\mathbf{V}(\tau)\ ,$$

and hence the value of the reaction of the constraint is also known.

3.5 The Conservative Case. Hamiltonian Formulation

Let us consider now the fundamental system (3.8), which determines, starting from an assigned 4-force law, the point particle dynamics, with variable proper mass (i.e. the system (3.7)). Instead of using the derivative of $\mathbf{P} = m_0\mathbf{V}$, we assume the expression of $\mathbf{V}$ in terms of $\mathbf{P}$, given by $(3.2)_2$, and consider, then, the following first-order differential system in the unknown ΩE and $\mathbf{P}$:

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\tau} = \mathbf{K}(E,\mathbf{P},\tau), \qquad \frac{\mathrm{d}\Omega E}{\mathrm{d}\tau} = \frac{c\mathbf{P}}{\sqrt{-\mathbf{P}\cdot\mathbf{P}}}\ . \tag{3.23}$$

Such a system clearly implies the condition $\mathbf{V}\cdot\mathbf{V} = -c^2$ as a direct consequence of $(3.23)_2$ and is equivalent to (3.7) and hence to (3.8). It is therefore a first-order (normal type) formulation, in eight unknowns: $\mathbf{P} = \mathbf{P}(\tau)$ and $E = E(\tau)$ (in (3.8) there were nine unknowns), which confirms the uniqueness of the motion, once the (regular) force law $\mathbf{K}$ is assigned, together with the initial conditions E_0 and $\mathbf{P}_0$, satisfying the single limitation $\mathbf{P}_0 \in \mathcal{C}_3^+(E_0)$.

In scalar terms, assuming as variables the coordinates of E, namely x^α, and the momenta $P_\alpha = m_{\alpha\beta}P^\beta$, the system (3.23) becomes

$$\frac{\mathrm{d}P_\alpha}{\mathrm{d}\tau} = K_\alpha(E,\mathbf{P},\tau), \qquad \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{cm^{\alpha\beta}P_\beta}{\sqrt{-m^{\rho\sigma}P_\rho P_\sigma}}\ . \tag{3.24}$$

The existence and uniqueness of the motion is obtained by including the initial conditions (*Cauchy problem*):

$$x^\alpha = x_0^\alpha\ , \quad P_\alpha = P_{\alpha,0} \quad \text{at} \quad \tau = \tau_0 \tag{3.25}$$

with the coordinates of the event E_0: x_0^α, belonging to the regularity domain of K_α, and $m^{\alpha\beta}P_{\alpha,0}P_{\beta,0} < 0$, i.e. $\mathbf{P}_0 \in \mathcal{C}_3^+(E_0)$.

System (3.24) is of special interest because when K_α is conservative it gives rise to a Hamiltonian formulation. In fact, if the 4-force is conservative: $K_\alpha = \partial\mathcal{U}/\partial x^\alpha$, by introducing the proper material energy of the particle: $\mathcal{E}_0 = m_0c^2$, as well as its positional energy $\Pi_0 = -\mathcal{U}$, (3.24) are equivalent to the following *canonical system*, with Hamiltonian function $\mathcal{H}(x,P) \equiv \Pi_0 - \mathcal{E}_0$:

$$\frac{\mathrm{d}P_\alpha}{\mathrm{d}\tau} = -\frac{\partial\mathcal{H}}{\partial x^\alpha}, \qquad \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{\partial\mathcal{H}}{\partial P_\alpha} . \tag{3.26}$$

In fact, from $(3.1)_1$, we have

$$\mathcal{H} = -\mathcal{U}(x) - c\sqrt{-m^{\rho\sigma}P_\rho P_\sigma} , \tag{3.27}$$

and hence, using the relation

$$\frac{\partial}{\partial P_\alpha}(-\mathbf{P}\cdot\mathbf{P}) = -2P^\alpha = -2m^{\alpha\rho}P_\rho , \tag{3.28}$$

one gets

$$\frac{\partial\mathcal{H}}{\partial P_\alpha} = \frac{cP^\alpha}{\sqrt{-\mathbf{P}\cdot\mathbf{P}}}, \qquad \frac{\partial\mathcal{H}}{\partial x^\alpha} = -\frac{\partial\mathcal{U}}{\partial x^\alpha} = -K_\alpha; \tag{3.29}$$

thus the differential systems (3.24) and (3.26) coincide.

The dynamical equations (3.26), which summarize the conservative case from an absolute point of view, give an example of Hamiltonian system not equivalent to a Lagrangian one. In fact, the possibility to reduce it to a Lagrangian form is subject to the invertibility of the relations:

$$\dot{q}^\alpha = \frac{\partial\mathcal{H}}{\partial P_\alpha}$$

(in our case $\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{\partial\mathcal{H}}{\partial P_\alpha}$). This corresponds to the requirement that det $||\partial^2\mathcal{H}/\partial P_\alpha\partial P_\beta|| \neq 0$; in our case, instead, we have

$$\left|\left|\frac{\partial^2\mathcal{H}}{\partial P_\alpha\partial P_\beta}\right|\right| = 0 . \tag{3.30}$$

The proof follows easily once the second derivatives of $\mathcal{H}$ are directly evaluated. In fact, from (3.28) and (3.29) we have

$$\frac{\partial^2\mathcal{H}}{\partial P_\alpha\partial P_\beta} = \frac{c}{P}\left(m^{\alpha\beta} + \frac{P^\alpha P^\beta}{P^2}\right) , \qquad P = \sqrt{-\mathbf{P}\cdot\mathbf{P}} , \tag{3.31}$$

from which one gets the result:

$$\frac{\partial^2\mathcal{H}}{\partial P_\alpha\partial P_\beta}P_\beta = \frac{c}{P}\left(P^\alpha + \frac{P^\alpha P^\beta}{P^2}P_\beta\right) = \frac{c}{P}(P^\alpha - P^\alpha) = 0 .$$

Thus, for an arbitrary choice of the P_β variables, the system

$$\frac{\partial^2 \mathcal{H}}{\partial P_\alpha \partial P_\beta} X_\beta = 0 \,, \qquad (\alpha = 0, 1, 2, 3) \,,$$

of four linear and homogeneous equations in the four unknown X_β admits eigensolutions (it is satisfied for $X_\beta = P_\beta \neq 0$). Thus condition (3.30) is satisfied, as it can be shown by using (3.31).

Similarly, the same result can be easily obtained by noticing that the functions at the right-hand side of $(3.24)_2$, i.e. the derivatives $\partial\mathcal{H}/\partial P_\alpha$, besides being regular when $P \neq 0$, are homogeneous of zero order in the P_α. It follows, from one side, the impossibility to obtain the momenta P_α (all independent) as functions of the velocity V^α (subordinated to the condition $m_{\alpha\beta}V^\alpha V^\beta = -c^2$); from the other side, using the Euler theorem:

$$\frac{\partial}{\partial P_\beta}\left(\frac{\partial \mathcal{H}}{\partial P_\alpha}\right) P_\beta = 0,$$

we get again (3.30).

However, being $\mathcal{H}(x, P)$ independent of the parameter τ, the canonical system (3.26) admits the generalized *integral of the energy*:

$$-\mathcal{H} = c\sqrt{-m^{\alpha\beta}P_\alpha P_\beta} + \mathcal{U}(x) = \text{const.} = h \,, \tag{3.32}$$

for all the solutions of (3.26).

This can be seen also from $(3.8)_1$ which, in the conservative case, assumes the form

$$\frac{\mathrm{d}\mathcal{E}_0}{\mathrm{d}\tau} = -\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}\tau} \,, \quad \to \quad \mathcal{E}_0 + \mathcal{U} = \text{const.}$$

We notice that the fact that the (conservative) point particle dynamics can be formulated in Hamiltonian and not in Lagrangian terms can be physically interpreted in the sense that, at least from the absolute point of view, the wave aspect of matter should be preferred, with respect to the particle one. In fact P_α, being a covariant vector, defines a three-dimensional hypersurface (or *wave front*), locally associated to the world line ℓ^+ (*ray*); in turn, the *elementary wave*, is perpendicular to ℓ^+.

The intrinsically conservative case suggests the idea of more general particles described by a Hamiltonian function $H(x, \mathbf{P}, \tau)$, which is not separable like $\mathcal{H}$. In this case, the associated Hamiltonian system, starting from the initial conditions E_0 and $\mathbf{P}_0$, characterizes a world line ℓ^+ (ray) and an elementary wave, not necessarily orthogonal to ℓ^+. For instance, this is the case of spinning test particles for which $\mathbf{P}$ is no more tangent to the ray but also transports the proper material energy of the particle, related to the norm of $\mathbf{P}$.

Note. The first integral (3.32) reduces the rank of the Hamiltonian system (3.29) by a number of two, so that the integration of the system is equivalent to that of an Hamiltonian system of six equations in six unknown, followed by a quadrature. More precisely, one must solve (3.32) with respect to one of the P_α, e.g. P_0:

$$P_0 = \tilde{\mathcal{H}}(x^\alpha, P_i, h) \ , \tag{3.33}$$

and hence consider the *reduced Hamiltonian* $\tilde{\mathcal{H}}$. After this, assuming as integration variables x^0 (in place of τ), we have the reduced system:

$$\frac{\mathrm{d}x^i}{\mathrm{d}x^0} = -\frac{\partial \tilde{\mathcal{H}}}{\partial P_i} \ , \qquad \frac{\mathrm{d}P_i}{\mathrm{d}x^0} = \frac{\partial \tilde{\mathcal{H}}}{\partial x^i} \quad (i = 1, 2, 3) \ , \tag{3.34}$$

from which one can derive the solution: $x^i = x^i(x^0)$, $P_i = P_i(x^0)$. The latter functions, in turn, inserted in (3.33), allow to express P_0 as a function of x^0, too, and hence also the energy $\mathcal{H}$ of (3.27). Thus, performing a quadrature on the original equation $\mathrm{d}x^0/\mathrm{d}\tau = \partial\mathcal{H}/\mathrm{d}P_0$ (where the right-hand side is a known function of x^0), one gets the relation between x^0 and τ.

We notice that the system (3.34) does not admit the energy integral, because $\tilde{\mathcal{H}}$ explicitly depends on x^0; moreover, from the relation $x^0 = ct$, this system represents, practically, the relative dynamics with respect to any Galilean frame, as it will be better discussed in Sect. 3.9. Thus, in the intrinsically conservative case, differently with respect to the absolute dynamics, it is the relative one, described by a regular Hamiltonian system, which is not singular. In any case, differently from the ordinary conservative case, such formulation results invariant with respect to the choice of the frame.

3.6 The Relative Formulation of the Dynamics

We have now the problem to formulate the fundamental equations of the point particle dynamics, in a three-dimensional sense, that is with respect to a Galilean frame. This is in order to obtain relations which can be tested in a Galilean laboratory and also to see the difference with respect to the classical formulation.

Because of the extended relativity principle, these equations should be *formally invariant* passing from one Galilean frame to another; therefore the values of the various involved quantities will change, differently from the absolute formulation.

As already stated, to fix a Galilean frame is equivalent to select, in $\mathcal{C}_3^+$, the timelike unit vector $\boldsymbol{\gamma}$; for a generic motion (ℓ^+, m_0) then the following decomposition of the 4-velocity $\mathbf{V}$ arises:

$$\mathbf{V} = \eta(\mathbf{v} + c\boldsymbol{\gamma}) \ , \qquad \eta = \frac{1}{\sqrt{1 - v^2/c^2}} \ , \tag{3.35}$$

where η is the Lorentz factor; similarly, for the 4-momentum one has $\mathbf{P} = m_0\mathbf{V} = m(\mathbf{v} + c\boldsymbol{\gamma})$, where

$$m = \eta m_0 \tag{3.36}$$

is the *relative mass* of the particle. By defining

$$\begin{cases} \mathbf{p} = m\mathbf{v} & \text{relative 3-momentum} \\ \mathcal{E} = mc^2 & \text{relative material energy,} \end{cases} \tag{3.37}$$

one also gets the following decomposition of the 4-momentum:

$$\mathbf{P} = \mathbf{p} + \frac{\mathcal{E}}{c}\boldsymbol{\gamma} \,, \tag{3.38}$$

which states that the spatial component of the 4-momentum is the *relative 3-momentum*, while the temporal one is the *relative material energy*, a part for the factor of c. Clearly, as $m = \eta m_0$, we have $\mathcal{E} = \eta\mathcal{E}_0$.

In the fixed Galilean frame, the global time coordinate t is given; hence, the world line of any particle can be parametrized by t: $\tau = \tau(t)$. The absolute equation of motion $\mathrm{d}\mathbf{P}/\mathrm{d}\tau = \mathbf{K}$, using (3.38) and the relation $\mathrm{d}t/\mathrm{d}\tau = \eta$, then, becomes

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} + \frac{1}{c}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\boldsymbol{\gamma} = \frac{1}{\eta}\mathbf{K} \,;$$

from here, splitting $\mathbf{K}$ in its components along $\boldsymbol{\gamma}$ and into Σ:

$$\mathbf{K} = \mathbf{K}_\Sigma - (\mathbf{K}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}, \quad \mathbf{K}_\Sigma = P_\Sigma(\mathbf{K}) \tag{3.39}$$

the two equations (one vectorial on Σ, and the other scalar) follow:

$$\dot{\mathbf{p}} = \frac{1}{\eta}\mathbf{K}_\Sigma \,, \qquad \dot{\mathcal{E}} = -\frac{c}{\eta}(\mathbf{K}\cdot\boldsymbol{\gamma}) \,, \tag{3.40}$$

where a dot means differentiation with respect to t. Defining the *relative mechanical force*,

$$\mathbf{F} = \frac{1}{\eta}\mathbf{K}_\Sigma \,, \tag{3.41}$$

Equation (3.40) assumes the Newtonian form of the theorem of linear momentum:

$$\dot{\mathbf{p}} = \mathbf{F} \,. \tag{3.42}$$

In the limit $c \to \infty$, one has $\mathbf{F} = \mathbf{K}_\Sigma$, and the criterion of re-obtaining the classical (vectorial) quantity as the limit of the spatial part of the relativistic (4-vectorial) quantity is satisfied. In the case of $\mathbf{V}$, similarly,

$$\mathbf{v} = \lim_{c\to\infty} \frac{1}{\eta}\mathbf{V}.$$

From $(3.40)_2$, using the relations $c\boldsymbol{\gamma} = \mathbf{V}/\eta - \mathbf{v}$, and $\mathbf{K}\cdot\mathbf{v} = \mathbf{K}_\Sigma\cdot\mathbf{v}$, we have

$$-\frac{c}{\eta}\mathbf{K}\cdot\boldsymbol{\gamma} = \mathbf{F}\cdot\mathbf{v} - \frac{1}{\eta^2}\mathbf{K}\cdot\mathbf{V} \,.$$

Thus, defining the *relative thermal power*:

$$q = -\frac{1}{\eta^2}\mathbf{K} \cdot \mathbf{V} \, , \tag{3.43}$$

Equation $(3.40)_2$ assumes the classical form of the energy theorem:

$$\dot{\mathcal{E}} = \mathbf{F} \cdot \mathbf{v} + q \, . \tag{3.44}$$

We notice that, even in the "Newtonian form", the relativistic point dynamics formulation, summarized by (3.42) and (3.44), is different from the classical one, also because of the widening of the scheme (particles with scalar structure, i.e. with m_0 not a constant): for (3.42), the main difference is that, in the linear momentum $\mathbf{p} = m\mathbf{v}$, the mass m is not a constant, but (explicitly) depends on the relative velocity and on the time t through m_0: $m = m_0(t)/\sqrt{1 - v(t)^2/c^2}$. Note that the dependence on v remains also in the most simple case of particles without internal structure, for which m_0 becomes a characteristic constant.

Equation (3.44) appears deeply modified, for two reasons: (1) the energy of the particle $\mathcal{E}$ does not coincide with the ordinary kinetic energy $mv^2/2$; (2) at the right-hand side appears the thermal power q which should be added to the ordinary mechanical power $\mathbf{F} \cdot \mathbf{v}$. The source of the energy is, then, the whole power:

$$\mathcal{W} = \mathbf{F} \cdot \mathbf{v} + q \, , \tag{3.45}$$

so that the decomposition of the 4-force becomes

$$\mathbf{K} = \eta \left(\mathbf{F} + \frac{\mathcal{W}}{c} \gamma \right) \, . \tag{3.46}$$

We emphasize the fact that $\mathbf{F}$ is the source of linear momentum and not of velocity because of the variability of the relative mass m. In any case, in agreement with the absolute formulation (3.8), also from the relative point of view, the physical action, represented by the 4-force $\mathbf{K}$, has the two effects of (1) accelerating the particle, (2) modifying its material energy:

$$\mathcal{E} = \frac{\mathcal{E}_0}{\sqrt{1 - \frac{v^2}{c^2}}} \, . \tag{3.47}$$

From this point of view, because of the presence of the term q, that is because of the variability of m_0, as from $(3.8)_1$, we have

$$c^2 \frac{\mathrm{d}m_0}{\mathrm{d}\tau} = q_0 \, , \qquad q_0 = \eta^2 q \, . \tag{3.48}$$

Equation (3.44) is *independent of that of the linear momentum*, differently from the classical situation. However, the presence of q, that is a first link between mechanics and heat theory, is not the unique fundamental novelty of the relativistic pointlike scheme. The other conceptual aspect, not less important than the previous one and certainly more general because it is present also for particles without internal structure, is the fact that the energy $\mathcal{E}$ and the mass m of the particle differ by a multiplicative constant, as if they were two different aspects of the same physical quantity (*equivalence between mass and energy*).

Clearly, $\mathcal{E}$ contains either the rest energy $\mathcal{E}_0 = m_0 c^2$ (intrinsic) or the (relative) kinetic energy, due to the relative velocity of the particle: v. The relation with the ordinary kinetic energy is immediately obtained, with an expansion of the right-hand side of (3.47) up to the first order in v^2/c^2. In fact, one has the approximated relation:

$$\mathcal{E} = \mathcal{E}_0 + \frac{1}{2} m_0 v^2 \, , \tag{3.49}$$

which shows, for slow motions $v^2 \ll c^2$, two main terms: the proper material energy $m_0 c^2$ and the relative kinetic energy $T = m_0 v^2/2$. From here it follows the closeness of (3.44) with the classical energy theorem; this is then strengthened in the case m_0 =constant, where $\dot{\mathcal{E}}_0 = 0$, and hence $\dot{\mathcal{E}} = \dot{T}$. The relative material energy as a function of v is plotted in Fig. 3.2.

In particular, one has

$$\lim_{v \to c} \mathcal{E}(v) = +\infty \, .$$

This result confirms that a material point cannot reach the speed of light, unless its relative material energy becomes infinite. Analogously, $m = \mathcal{E}/c^2$ is the inertial mass which measures the increasing difficulty of the point to further accelerate when its speed becomes close to that of the light, in a given Galilean frame. From this point of view, one may consider, as more significant, the absolute parameter m_0, which measures the classical inertia and gives rise to a hierarchy of particles, without internal structure.

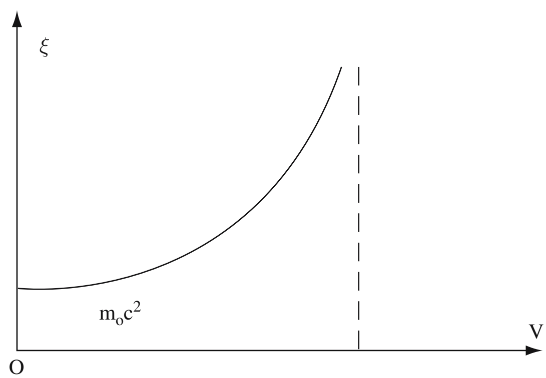

Fig. 3.2. The relative material energy plotted as a function of v

We notice that the various definitions introduced above, all have a real physical content: they establish the physics laws from a relative point of view in a form close to their Newtonian counterparts, in agreement with extended principle of relativity. In particular, the Einsteinian equivalence, between mass and energy, can be easily derived from a general principle of mass and energy conservation, valid for isolated systems.

In other words, if in the ambit of an isolated system, one has increment of material energy $\Delta\mathcal{E}$, in correspondence, a mass defect $\Delta\mathcal{E}/c^2$ is created: this effect has been widely confirmed in experiments.

3.7 Transformation Laws: Unification Between Mechanics and Heat Theory

The relative formulation of the dynamics of material point with scalar structure m_0 is summarized by the two independent equations (one scalar and the other vectorial):

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \mathbf{F}\ , \quad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \mathbf{F}\cdot\mathbf{v} + q = \mathcal{W} \qquad \forall S_{\mathrm{g}}\ . \tag{3.50}$$

They are formally invariant with respect to the choice of any Galilean frame S_{g}, passing to another Galilean frame, S'_{g}, is equivalent to put a prime on all the various quantities in (3.50). They are not substantially invariant because all the various quantities appearing in (3.50), including t and $\mathbf{v}$, have a relative meaning. Thus, it is necessary to specify, as we have already done for t and $\mathbf{v}$, the transformation laws of the fundamental quantities $\mathbf{p}$, $\mathbf{F}$, $\mathcal{E}$, q.

Let us start with transformation law of the mass m. From (3.36), we have

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}\ , \qquad m' = \frac{m_0}{\sqrt{1 - v'^2/c^2}}\ ;$$

thus, by eliminating m_0 (invariant, once specified the event E of the world line):

$$m' = \frac{\sqrt{1 - v^2/c^2}}{\sqrt{1 - v'^2/c^2}}\, m\ .$$

Moreover, from (2.105), it results

$$\frac{\sqrt{1 - v^2/c^2}}{\sqrt{1 - v'^2/c^2}} = \frac{\sigma}{\alpha}\ , \qquad \sigma = 1 - \mathbf{u}\cdot\mathbf{v}/c^2\ , \quad \alpha = \sqrt{1 - u^2/c^2}\ , \tag{3.51}$$

with $\mathbf{u}$ the relative velocity of S'_{g} with respect to S_{g}. Hence,

$$m' = \frac{\sigma}{\alpha}\, m\ . \tag{3.52}$$

From here it follows the introduction of the terminology of *longitudinal mass* or *transversal mass* of use in dynamics, when one passes from a Galilean laboratory to another moving in the direction of $\mathbf{v}$ ($\mathbf{v}$ parallel to $\mathbf{u}$) or in the orthogonal direction ($\mathbf{v}$ orthogonal to $\mathbf{u}$).

The classical invariance of the mass comes from (3.52), in the limit $c \to \infty$:

$$\lim_{c\to\infty} m' = \lim_{c\to\infty} m = m_0 = \text{inv.}$$

From (3.52), after multiplication by c^2, one gets the variation law of the material energy:

$$\mathcal{E}' = \frac{\sigma}{\alpha}\,\mathcal{E}. \tag{3.53}$$

Again, from (3.52), after multiplication by $\mathbf{v}'$, and by using the relativistic addition of velocity law,

$$\mathbf{v}' = \frac{\alpha}{\sigma}\left(\mathbf{v} - \frac{1+\sigma/\alpha}{1+\alpha}\mathbf{u}\right) , \tag{3.54}$$

one gets the variation law of the linear momentum:

$$\mathbf{p}' = \mathbf{p} - \frac{1+\sigma/\alpha}{1+\alpha} m\mathbf{u} . \tag{3.55}$$

Let us pass now to thermal power q, defined by (3.43). We have

$$q = q_0(1 - v^2/c^2) , \qquad q' = q_0(1 - v'^2/c^2) ,$$

where q_0 is a local invariant of the particle, called the *proper thermal power*:

$$q_0 = -\mathbf{K}\cdot\mathbf{V} . \tag{3.56}$$

The variation law of the thermal power follows, from the above relations, by eliminating q_0:

$$q' = \left(\frac{\alpha}{\sigma}\right)^2 q . \tag{3.57}$$

The remaining task is that of finding the transformation law of the relative mechanical force. From the invariance property of $\mathbf{K}$, by (3.46), one gets

$$\eta\left(\mathbf{F} + \frac{\mathcal{W}}{c}\gamma\right) = \eta'\left(\mathbf{F}' + \frac{\mathcal{W}'}{c}\gamma'\right) = \text{inv.} ;$$

thus, from (3.51), it results

$$\mathbf{F}' + \frac{\mathcal{W}'}{c}\gamma' = \frac{\alpha}{\sigma}\left(\mathbf{F} + \frac{\mathcal{W}}{c}\gamma\right) .$$

Finally, projecting on the basis $\{\mathbf{c}'_\alpha\}$:

$$\boldsymbol{\gamma}' = \frac{1}{\alpha}(\boldsymbol{\gamma} + \beta \mathbf{c}_1) , \quad \mathbf{c}_1' = \frac{1}{\alpha}(\mathbf{c}_1 + \beta \boldsymbol{\gamma}), \quad \mathbf{c}'_{2,3} = \mathbf{c}_{2,3} ,$$

the following relations are obtained:

$$\begin{cases} \frac{1}{c}\mathcal{W}' = \frac{1}{\sigma}\left(\frac{1}{c}\mathcal{W} - \beta F_1\right) \\ F_1' = \frac{1}{\sigma}\left(F_1 - \frac{\beta}{c}\mathcal{W}\right) , \qquad F_{2,3}' = \frac{\alpha}{\sigma}F_{2,3}. \end{cases}$$

From here, one gets immediately the variation law of the total power $\mathcal{W}$:

$$\mathcal{W}' = \frac{1}{\sigma}\left(\mathcal{W} - \mathbf{F} \cdot \mathbf{u}\right) . \tag{3.58}$$

For the remaining relation, we can write

$$\begin{aligned} F_1 &= \alpha F_1 + (1-\alpha)F_1 \\ &= \alpha F_1 + \frac{1-\alpha^2}{1+\alpha}F_1 \\ &= \alpha F_1 + \frac{1}{c^2}\frac{u^2}{1+\alpha}F_1 \\ &= \alpha F_1 + \frac{1}{c^2}\frac{\mathbf{u}\cdot\mathbf{F}}{1+\alpha}u . \end{aligned}$$

Interpreting now the components of $\mathbf{F}'$ in S_{g}, instead of S_{g}' (i.e. boosting Σ' on Σ), one gets the relation:

$$\mathbf{F}' = \frac{1}{\sigma}\left[\alpha\mathbf{F} + \frac{1}{c^2}\left(\frac{\mathbf{u}\cdot\mathbf{F}}{1+\alpha} - \mathcal{W}\right)\mathbf{u}\right] . \tag{3.59}$$

We notice that (3.58) can be derived directly from (3.54), (3.57) and (3.59), and taking into account the meaning of $\mathcal{W}$. However, (3.59) shows that, differently from q', the mechanical force $\mathbf{F}'$ depends not only on $\mathbf{F}$ but also on the thermal power q. In fact, we have

$$\mathbf{F}' = \frac{1}{\sigma}\left[\alpha\mathbf{F} - \frac{1}{c^2}\left(\mathbf{F}\cdot\mathbf{w} + q\right)\mathbf{u}\right] , \tag{3.60}$$

where

$$\mathbf{w} \stackrel{\text{def}}{=} \mathbf{v} - \frac{1}{1+\alpha}\mathbf{u} . \tag{3.61}$$

The following fundamental fact arises, as a peculiarity of the relativistic situation: in the framework of special relativity, it is meaningful to formulate a theory for pure mechanics, while *it is a nonsense to formulate a theory for heat, only.* In fact, from (3.57), we have that $q = 0$ implies $q' = 0$, and vice versa: the presence or the absence of a physical action of thermal type is an intrinsic fact. The same is not true for the mechanical force $\mathbf{F}$: if it is absent in a frame, in general it is not into another:

$$\mathbf{F} = 0 \quad \Rightarrow \quad \mathbf{F}' = -\frac{1}{c^2}\frac{q}{\sigma}\mathbf{u} \neq 0 \,,$$

even if a very small, i.e. of the order $1/c^2$. In a relativistic framework, then, from pure mechanics, one is naturally driven into thermomechanics ($q \neq 0$). This situation is somehow similar to the electromagnetism: it is a nonsense, from a relativistic point of view, to develop a theory for the electric field only or the magnetic field only, but the really meaningful theory implies the presence of both fields. This of course will not exclude the possibility of having, in a certain frame of reference, electric or magnetic field only.

3.8 The Cauchy Problem in Relative Dynamics

Let us consider now the general equations for point dynamics (3.50), in a generic Galilean frame, assuming that both the frame (i.e. $\boldsymbol{\gamma}$) and the force law (i.e. $\mathbf{K} = \mathbf{K}(\tau, E, \mathbf{P})$) are assigned. Clearly, the component $\mathbf{K}_\Sigma = \mathbf{K} + (\boldsymbol{\gamma} \cdot \mathbf{K})\boldsymbol{\gamma}$ has the same dependence of $\mathbf{K}$. The mechanical force $\mathbf{F} = 1/\eta \mathbf{K}_\Sigma$ seems to have, in addition, the dependence on v, through η; however, if $\mathbf{V}$ and m_0 are known, assigned $\boldsymbol{\gamma}$, also $\mathbf{v}$ (and $m = \eta m_0$ and $\mathcal{E} = mc^2$, as well) is known. Thus, $\mathbf{F}$ has exactly the same dependence as $\mathbf{K}$, and the same is true for $q = -\mathbf{K} \cdot \mathbf{V}/\eta^2$. In relative terms, the variables $\tau, E, \mathbf{P}$ are equivalent to the ordinary quantities P (position), $\mathbf{p}$ (linear 3-momentum), $\mathcal{E}$ (material energy) and t (universal time of the frame); in fact, from E one gets P and t, from $\mathbf{P}$ one has $\mathbf{p}$ and $\mathcal{E}$, and finally from τ one gets t. In this sense, in the restricted problems, the relative force $\mathbf{F}$ and the relative heat power q have the following dependence:

$$\mathbf{F} = \mathbf{F}(x, p, \mathcal{E}, t) \,, \quad q = q(x, p, \mathcal{E}, t) \tag{3.62}$$

or, equivalently

$$\mathbf{F} = \mathbf{F}(x, \dot{x}, \mathcal{E}, t) \,, \quad q = q(x, \dot{x}, \mathcal{E}, t) \,, \tag{3.63}$$

taking into account that $\mathbf{p}$ summarizes $\mathbf{v}$ and m, or $\mathcal{E}$.

From the latter point of view, (3.50) can be cast in scalar terms, in the following second-order system for the unknown x^i ($i = 1, 2, 3$) and $\mathcal{E}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathcal{E}}{c^2}\dot{x}^i\right) = F^i \,, \qquad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = F_i \dot{x}^i + q \,, \qquad F_i = \delta_{ik} F^k \,. \tag{3.64}$$

The similar formulation (first-order ordinary differential system of seven equations for seven unknowns, in the variable t), in terms of position x^i, linear momentum $p_i = \delta_{ik} p^k$ and energy $\mathcal{E}$, appears to have more physical meaning:

$$\dot{x}^i = \frac{c^2}{\mathcal{E}}\delta^{ik} p_k \,, \quad \dot{p}_i = F_i(x, p, \mathcal{E}, t) \,, \quad \dot{\mathcal{E}} = \frac{c^2}{\mathcal{E}} F^i p_i + q(x, p, \mathcal{E}, t) \,. \tag{3.65}$$

For this system, similarly to the corresponding absolute one, the Cauchy problem arises associated with the initial conditions:

$$x^i = x_0^i \ , \quad p_i = p_{i,0} \ , \quad \mathcal{E} = \mathcal{E}_0 \quad \text{at} \quad t = t_0 \ . \tag{3.66}$$

Obviously, because of the meaning of the p_i and $\mathcal{E} = mc^2$, subordinated at the condition $1 - v^2/c^2 > 0$, we have

$$X \stackrel{\text{def}}{=} 1 - \left(\frac{cp}{\mathcal{E}}\right)^2 > 0 \quad (p^2 = \delta_{ik}p^ip^k) \ , \tag{3.67}$$

and the initial data cannot be arbitrary, but they must satisfy the limitations:

$$X_0 = 1 - \left(\frac{cp_0}{\mathcal{E}_0}\right)^2 > 0 \ , \qquad \mathcal{E}_0 > 0 \ , \tag{3.68}$$

and this is a big difference with respect to the classical situation.

Equation (3.68) implies that a discussion on dynamically possible motions can be started on the basis of (3.65), but then at any instant, the constraint (3.67) should be verified. It should be noted that, differently from the problem (3.8) for the absolute dynamics where the conservation equation $1 + \mathbf{V} \cdot \mathbf{V}/c^2 = 0$ was present, in the case under consideration here, (3.67) gives an effective unilateral constraint. The unilateral character of the constraint excludes the possibility of reactions, but it creates a problem completely different from the previous one. In any case, the system (3.65) implies, for the variable $X = 1 - c^2p^2/\mathcal{E}^2$, the following first-order (linear, inhomogeneous) differential condition:

$$\dot{X} = -2\frac{c^2}{\mathcal{E}^2}(XF^ip_i - p^2q/\mathcal{E}) \ ; \tag{3.69}$$

this relation involves either the mechanical force or the thermal power (see (3.62) and shows that the initial value,

$$Y_0 \stackrel{\text{def}}{=} X_0F_0^ip_{i,0} - p_0^2q_0/\mathcal{E}_0 \ , \tag{3.70}$$

plays a central role, discriminating the case $X(t)$ increasing (that is an always positive X), from the decreasing one, in which there can exist critical points $X = 0$ starting from which the solution may be meaningless.

Another form of the system (3.65), which will allow to avoid the constraint (3.67), is obtained assuming as variables x^i, p_i and m_0, in place of x^i, p_i and $\mathcal{E}$. In this case, both the energy $\mathcal{E} = mc^2$ and the mass $m = m_0/\sqrt{1 - v^2/c^2}$ should be expressed in terms of the new variables; otherwise, from

$$m = \frac{m_0}{\sqrt{1 - p^2/(m^2c^2)}} = \frac{m_0m}{\sqrt{m^2 - p^2/c^2}} \ ,$$

it results $m^2 - p^2/c^2 = m_0^2$, so that

$$m = \sqrt{m_0^2 + p^2/c^2} \ , \qquad p^2 = \delta_{ik}p^ip^k \ . \tag{3.71}$$

Taking into account $(3.65)_2$, and by differentiating, one gets the following relation:

$$\dot{m} = \frac{1}{m}\left(m_0\dot{m}_0 + \frac{1}{c^2}p^i F_i\right) ,$$

which maps the energy theorem $(3.65)_3$ into the form: $m_0\dot{m}_0 = qm/c^2$. Thus, system (3.65) transforms as

$$\begin{cases} \dot{x}^i = \dfrac{\delta^{ik}p_k}{\sqrt{m_0^2 + p^2/c^2}} \\ \dot{p}_i = F_i(x, p, m_0^2, t) \\ \left(\frac{1}{2}m_0^2\right)^{\cdot} = \frac{1}{c^2}\sqrt{m_0^2 + p^2/c^2}\, q(x, p, m_0^2, t). \end{cases} \tag{3.72}$$

In this last formulation, m_0 appears through the power m_0^2, allowing the treatment to be valid for both particles ($m_0 > 0$) and exotic particles ($m_0 < 0$). Moreover, the constraint (3.67) rewritten into the form,

$$1 - \frac{p^2}{p^2 + m_0^2c^2} ,$$

is automatically included in $(3.72)_1$, so that *the initial data* x_0^i*,* $p_{i,0}$ *and* $m_{0,0}$ *are completely free*, a part from the condition $m_{0,0} \neq 0$.

3.9 The Intrinsically Conservative Case

In the general formulation (3.72) and (3.65), the functions at the right-hand side are more or less complicated according to the expression of the 4-force law $\mathbf{K}(E, \mathbf{P}, \tau)$. Therefore, they are simplified when $\mathbf{K}$ is special, for instance positional and conservative:

$$K_\alpha = \frac{\partial \mathcal{U}}{\partial x^\alpha} \qquad (\alpha = 0, 1, 2, 3) . \tag{3.73}$$

Let us examine this latter case, assuming that the Cartesian coordinates x^α were internal to the frame: $\boldsymbol{\gamma} = \mathbf{c}_0$. From the general relations given in (3.46)

$$F_i = \frac{1}{\eta}K_i , \quad q = -\frac{1}{\eta^2}\mathbf{K}\cdot\mathbf{V} = -\frac{1}{\eta^2}\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}\tau} ,$$

and using (3.71) to express η, we have

$$\eta = \sqrt{1 + \frac{p^2}{m_0^2c^2}} , \tag{3.74}$$

so that one obtains the following expressions for the mechanical force and the thermal power, respectively:

$$\begin{cases} F_i = (1 + p^2/(m_0^2 c^2))^{-1/2}\, \partial\mathcal{U}/\partial x^i \quad (\text{i} = 1,2,3) \\ q = -(1 + p^2/(m_0^2 c^2))^{-1/2}\, \partial\mathcal{U}/\partial t. \end{cases} \tag{3.75}$$

Equation $(3.75)_1$ shows that the mechanical force $\mathbf{F}$ is neither positional nor conservative. However, it comes from the potential

$$U(x, p, m_0, t) \stackrel{\text{def}}{=} (1 + p^2/(m_0^2 c^2))^{-1/2}\mathcal{U}(x, t)\ , \tag{3.76}$$

which depends on the time t and both p_i and m_0, through the ratio $p^2/(m_0^2 c^2)$. Thus, in any Galilean frame, system (3.72) simplifies as:

$$\dot{x}^i = \frac{\delta^{ik} p_k}{\sqrt{m_0^2 + p^2/c^2}}\ , \quad \dot{p}_i = \partial U/\partial x^i\ , \quad m_0 c^2 + \mathcal{U} = \text{const.}\ , \tag{3.77}$$

while system (3.65) assumes the form:

$$\dot{x}^i = \frac{c^2}{\mathcal{E}}\delta^{ik} p_k\ , \quad \dot{p}_i = \partial U/\partial x^i, \dot{\mathcal{E}} = -\partial U/\partial t\ , \tag{3.78}$$

with the same initial conditions (3.66), as well as the limitations (3.67) and (3.68). Obviously, in order to explicitate the system, one should consider $U = \sqrt{1 - c^2 p^2/\mathcal{E}^2}\,\mathcal{U}(t, x)$. In any case, even if system (3.77) is formally invariant passing from one Galilean frame to another, the characteristic function $U = \mathcal{U}/\eta$ is not invariant (it is instead invariant the absolute potential $\mathcal{U}(E)$). Thus, from (3.76) the invariance property $\eta U = \eta' U' =$ inv. follows; in other words, taking into account (3.51), the transformation law of the relative potential is the following:

$$U' = \frac{\alpha}{\sigma} U\ , \tag{3.79}$$

with the functional identity being subordinated to the Lorentz transformations. Finally, let us note that, by eliminating m_0 using the energy theorem, system $(3.77)_{1,2}$ does not assume a Hamiltonian form.

3.10 Pure Mechanics: Particles with Scalar Structure

As we have already stated, in a relativistic framework, a "pure mechanics" is obtained excluding thermal actions, i.e. assuming

$$q = -\frac{1}{\eta^2}\mathbf{K} \cdot \mathbf{V} = 0\ . \tag{3.80}$$

This is the ordinary pointlike scheme (material point, without internal structure); in fact, (3.80) implies, for $(3.72)_3$, the condition m_0 =const., as from $(3.8)_1$. In other words, (3.80) gives to the energy theorem the classical form, but with a different content for the energy $\mathcal{E}$: $\dot{\mathcal{E}} = F_i v^i$, and this, in turn, is equivalent to the condition $\dot{m}_0 = 0$, that is m_0 =const.

In this case (particles without internal structure: m_0 = const. and $q = 0$), the energy theorem follows from the equations of motion $(3.72)_{1,2}$:

$$\dot{x}^i = \frac{\delta^{ik} p_k}{\sqrt{m_0^2 + p^2/c^2}} \,, \qquad \dot{p}_i = F_i(x, p, t) \quad (i = 1, 2, 3) \,. \tag{3.81}$$

In these last equations, m_0 is a "structural" constant, characteristic of the considered particle, with the material energy $\mathcal{E} = mc^2$ given by the formula:

$$\mathcal{E} = c\sqrt{m_0 c^2 + p^2} \,, \qquad p^2 = \delta^{ik} p_i p_k \,. \tag{3.82}$$

Equation (3.81) ensure the invariance property of the more general (3.65), passing from a Galilean frame to another. This invariance is but, only formal and not substantial. For instance, in S'_{g}, according to (3.60), the mechanical force is given by:

$$\mathbf{F}' = \frac{1}{\sigma}\left[\alpha \mathbf{F} - \frac{1}{c^2}\mathbf{F} \cdot \left(\mathbf{v} - \frac{1}{1+\alpha}\mathbf{u}\right)\right] \,, \tag{3.83}$$

and so the mechanical power is

$$\mathcal{W}' = \frac{1}{\sigma}\left(\mathcal{W} - \mathbf{F} \cdot \mathbf{u}\right) \,, \tag{3.84}$$

differently from the classical situation, where $\mathbf{F}' = \mathbf{F}$ and $\mathcal{W}' = \mathcal{W} - \mathbf{F} \cdot \mathbf{u}$.

3.11 The Conservative Case in a Classical Sense

Let us consider now, in the context of pure mechanics ($q = 0$), the special case in which the mechanical force, relative to S_{g}, comes from a potential U:

$$F_i = \frac{\partial U(x)}{\partial x^i} \,, \qquad (i = 1, 2, 3) \,. \tag{3.85}$$

Obviously, this is not an absolute property because it strictly depends on the choice of the Galilean frame. That is, the hypothesis (3.85) destroys the formal invariance of the dynamical equation (3.81). In fact, from (3.83), as soon as $\mathbf{u} \neq 0$ (i.e. $S'_{\mathrm{g}} \neq S_{\mathrm{g}}$) the mechanical force $\mathbf{F}'$ acquires a dependence on $\mathbf{v}'$, either explicitly or implicitly, through σ, and hence it is no more conservative.

Even with this limitation, which makes S_{g} a preferred frame, the conservative case is still important, as in the classical case, and gives rise to a canonical system. In fact, by introducing the *total relative energy*

$$H = mc^2 - U = c\sqrt{m_0 c^2 + \delta^{ik} p_i p_k} - U(x) \,, \tag{3.86}$$

system (3.81), rewritten in the following form:

$$\dot{x}^i = \frac{c\delta^{ik} p_k}{\sqrt{m_0 c^2 + \delta^{jl} p_j p_l}} \,, \qquad \dot{p}_i = \frac{\partial U}{\partial x^i} \,, \tag{3.87}$$

assumes the *Hamiltonian form*:

$$\dot{x}^i = \frac{\partial H}{\partial p_i} \,, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i} \,. \tag{3.88}$$

Actually, from (3.86):

$$\begin{cases} \dfrac{\partial H}{\partial p_i} = \dfrac{c\delta^{ik} p_k}{\sqrt{m_0 c^2 + \delta^{jl} p_j p_l}} \\ \\ \dfrac{\partial H}{\partial x^i} = -\dfrac{\partial U}{\partial x^i}, \end{cases} \tag{3.89}$$

which completes the proof. In the limit of slow motions, the Hamiltonian function reduces, as from (3.49), to the classical formula for the total energy of a material point in a conservative field (a part for the constant rest energy $m_0 c^2$):

$$H \simeq m_0 c^2 + \frac{1}{2} m_0 v^2 - U \,.$$

However, the conservative case (3.85) is different from the intrinsically conservative one not only as concerns the invariance but also because the function (3.86) has a *nonzero Hessian determinant*, differently from $\mathcal{H}$ defined in (3.27). That is, $(3.87)_1$ is invertible and gives rise to the relations:

$$p_i \equiv m v_i = \frac{m_0 \delta_{ik} \dot{x}^k}{\sqrt{1 - \delta_{ik} \dot{x}^i \dot{x}^k / c^2}} \,; \tag{3.90}$$

moreover, the Hessian of H, with respect to the components of the momentum, has to be different from zero. In fact, assuming $m(p) = \sqrt{m_0^2 + p^2/c^2}$, so that $\partial H / \partial p_i = p^i / m(p)$, it is easy to show that

$$\frac{\partial^2 H}{\partial p_i \partial p_k} = -\frac{1}{m^3 c^2} p^i p^k + \frac{1}{m} \delta^{ik} \,. \tag{3.91}$$

Thus, the matrix $\left|\left| \frac{\partial^2 H}{\partial p_i \partial p_k} \right|\right|$ has the form $||a^{ik} + \lambda \delta^{ik}||$ and, using the general relation

$$\det ||a^{ik} + \lambda \delta^{ik}|| = \lambda^n + I_1 \lambda^{n-1} + \cdots + I_{n-1} \lambda + I_n \,, \tag{3.92}$$

with n the matrix order, and $I_1, \ldots, I_n$ the *principal invariants* of a^{ik} (with respect to the matrix δ_{ik}). In the present case, $\lambda = 1/m$, $n = 3$, $a^{ik} = -p^i p^k / (m^3 c^2)$, and it results in

$$I_1 = -\frac{p^2}{m^3c^2} \,, \quad I_2 = 0 \,, \quad I_3 = 0 \,;$$

so that, from (3.92)

$$\det \left\| \frac{\partial^2 H}{\partial p_i \partial p_k} \right\| = \left(\frac{1}{m}\right)^3 - \frac{p^2}{m^3c^2}\left(\frac{1}{m}\right)^2 = \frac{m_0^2}{\sqrt{(m_0^2 + p^2/c^2)^5}} > 0 \,.$$

Thus, differently from the intrinsically conservative case, the canonical system (3.88) is equivalent to a Lagrangian system, with Lagrangian:

$$\mathcal{L} = [L]_{p_i = p_i(q,\dot{q},t)} \,, \qquad L \stackrel{\text{def}}{=} \frac{\partial H}{\partial p_i} p_i - H \,,$$

where the relations $p_i = p_i(q, \dot{q}, t)$ are obtained by solving the equations $\dot{q}^i = \partial H/\partial p_i$ with respect to the $p+$ when this is possible as it is in the present case. It follows

$$\mathcal{L} = \dot{x}^i p_i(x, \dot{x}, t) - [H]_{p_i = p_i(x,\dot{x},t)} \,;$$

using, then, (3.86) and (3.90), one gets

$$\mathcal{L} = \frac{m_0}{1 - v^2/c^2}(\delta_{ik}\dot{x}^i\dot{x}^k - c^2) + U(x) = -m_0c^2\sqrt{1 - v^2/c^2} + U(x) \,. \quad (3.93)$$

The associated Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \frac{\partial \mathcal{L}}{\partial x^i} = 0 \,, \quad (3.94)$$

give, clearly, the theorem of momentum:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}^i) - \frac{\partial U}{\partial x^i} = 0 \,, \quad (3.95)$$

i.e. give rise to a second-order normal differential formulation of the dynamics in the unknown $x^i(t)$. More precisely, taking into account the energy theorem:

$$\dot{\mathcal{E}} = F_i\dot{x}^i = \frac{\partial U}{\partial x^i}\dot{x}^i \equiv \dot{U} \,, \quad (3.96)$$

one can rewrite (3.95) in the form

$$m\ddot{x}^i = \frac{\partial U}{\partial x^i} - \frac{1}{c^2}\dot{U}\dot{x}^i = \frac{\partial U}{\partial x^i} - \frac{1}{c^2}\frac{\partial U}{\partial x^k}\dot{x}^i\dot{x}^k \,,$$

and thus, they are equivalent to the following second-order, normal system:

$$m_0\ddot{x}^i = \sqrt{1 - \delta_{ik}\dot{x}^i\dot{x}^k/c^2}\left(\delta^{ik} - \frac{1}{c^2}\dot{x}^i\dot{x}^k\right)\frac{\partial U}{\partial x^k} \,. \quad (3.97)$$

Such a system, in the considered Galilean frame, admits the energy integral, as it follows from (3.96): $\mathcal{E} - U = \text{const.}$, that is,

$$\frac{m_0 c^2}{\sqrt{1-\delta_{ik}\dot{x}^i\dot{x}^k/c^2 - U(x)}} = \text{const.} \tag{3.98}$$

for all the solutions of (3.97); furthermore, the Lagrangian function (3.93) does not explicitly depend on the time, so that system (3.94) admits the generalized energy integral:

$$\mathcal{H}(x,\dot{x}) = \frac{\partial \mathcal{L}}{\partial \dot{x}^i}\dot{x}^i - \mathcal{L} = \text{const.}\,, \tag{3.99}$$

for all the solutions of (3.94), and this coincides with (3.98). Similarly, in terms of canonical variables, being t an ignorable coordinate in the Hamiltonian function (3.86), the associated system (3.88) admits as a first integral the same function $H(x,p)$:

$$c\sqrt{m_0c^2 + \delta^{ik}p_ip_k} - U(x) = \text{const.} \tag{3.100}$$

for all the solutions of (3.88). This allows to express, for all dynamical solution, $p^2 = \delta^{ik}p_ip_k$ as a function of the position and the initial data.

3.12 Classical Approximation

As in the conservative case of pure mechanics discussed in (3.97), also in the general case ($\mathbf{F} \neq 0$, $q \neq 0$), the dynamical equations can be written in terms of the four variables x^i and $\mathcal{E}$, or in terms of x^i and m_0. One has to use directly (3.64), which gives rise to the following, normal form, differential system:

$$\begin{cases} \frac{1}{c^2}\mathcal{E}\ddot{x}^i = \left(\delta^{ik} - \mathrm{d}s\frac{1}{c^2}\dot{x}^i\dot{x}^k\right)F_k(x,\dot{x},\mathcal{E},t) - \dfrac{1}{c^2}\dot{x}^i q(x,\dot{x},\mathcal{E},t) \\ \dot{\mathcal{E}} = F_k\dot{x}^k + q, \end{cases} \tag{3.101}$$

or, equivalently,

$$\begin{cases} \dfrac{1}{c^2}\mathcal{E}\mathbf{a} = \mathbf{F} - \dfrac{1}{c^2}(\mathbf{F}\cdot\mathbf{v} + q)\mathbf{v} \\ \dot{\mathcal{E}} = \mathbf{F}\cdot\mathbf{v} + q. \end{cases} \tag{3.102}$$

Assuming instead, as variables x^i and m_0, and using (3.72), the fundamental system (3.101) can be written as

$$\begin{cases} m_0\ddot{x}^i = \sqrt{1-\dfrac{v^2}{c^2}}\left[\left(\delta^{ik} - \dfrac{1}{c^2}\dot{x}^i\dot{x}^k\right)F_k(x,\dot{x},m_0,t) - \dfrac{1}{c^2}q\dot{x}^i\right] \\ \dot{m} = \dfrac{1}{c^2\sqrt{1-\dfrac{v^2}{c^2}}}q(x,\dot{x},m_0,t) \qquad\qquad (v^2 = \delta_{ik}\dot{x}^i\dot{x}^k), \end{cases} \tag{3.103}$$

which shows the meaning of m_0 as the inertial mass for any choice of the Galilean frame. In particular, in the Galilean frame in which the particle is at rest $\boldsymbol{\gamma} = \mathbf{V}/c$, such that $\mathbf{v}_0 = 0$ and $\mathbf{a}_0 = \mathbf{A}$, and using (3.41) and (3.43), one gets

$$\mathbf{F}_0 = \mathbf{K}_\Sigma = \mathbf{K} + \frac{1}{c^2}\mathbf{K}\cdot\mathbf{V}\mathbf{V} \ , \quad q_0 = -\mathbf{K}\cdot\mathbf{V} \ ,$$

and (3.103) become

$$m_0\mathbf{a}_0 = \mathbf{F}_0 \ , \qquad \dot{m}_0 = \frac{1}{c^2}q_0 \ , \tag{3.104}$$

which are equivalent to the original absolute formulation (3.8). We note that $(3.104)_1$ gives again, in the relativistic framework, the classical law $m\mathbf{a} = \mathbf{F}$, even if this result holds in a particular Galilean frame, variable with the particle. Conversely, this may give a criterion to extend, in relativity, the classical physics laws, according to which one can consider the classical laws still valid, but only with respect to the Galilean frame in which the particle is at rest. Then, the formulation can be extended to any Galilean frame, by considering the transformation laws of the various quantities involved.

Finally, it is also worth to note that, in the limit $c \to \infty$, the formulation (3.103) is equivalent to the classical case:

$$m_0\ddot{x}^i = F^i \ , \qquad m_0 = m = \text{const.} \ , \tag{3.105}$$

from which the inertial meaning of m_0 is confirmed and its purely mechanical meaning too, without the thermal coupling.

3.13 Unified Scheme: Particles and Photons

Within the particle scheme, considered up to now, we have excluded both the cases $m_0 < 0$ and $m_0 = 0$. The first case ($m_0 < 0$) can be taken into account, without significative changes in the scheme considered, above and it represents *exotic matter*, or particles with negative material energy (and hence not too much physically relevant). The second case ($m_0 = 0$) must be discussed separately because in this case some fundamental quantities, like the 4-momentum, loose their direct meaning. Allowing to consider also particle with very small masses, it is quite natural to consider the case $m = 0$ as a limiting one of the particle scheme. From this point of view, taking into account the relation

$$m = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} \ ,$$

it is clear that, if $v < c$, the limit $m_0 \to 0$ implies $m \to 0$ too, in every Galilean frame; a particle which would correspond to such a model will not be physically observable. As a consequence, if one would like to consider a physically compatible scheme, also for the case $m_0 = 0$, one should allow $v = c$. This observation suggests the corresponding scheme, for the case $m_0 = 0$, that is a scheme apt to represent particles moving at the speed of light. This is the case of photons, introduced by Einstein in 1905 to explain the photoelectric effect.

They are, then, limiting particles ($m_0 \to 0$ and $v \to c$, so that m will be finite and nonzero), for which the world line is lightlike:

$$\boldsymbol{\lambda} \cdot \boldsymbol{\lambda} = 0 , \qquad \boldsymbol{\lambda} = \frac{\mathrm{d}E}{\mathrm{d}\lambda} . \tag{3.106}$$

Therefore, for these particles, one cannot introduce the notions of proper time τ and 4-velocity $\mathbf{V}$. Similarly, the 4-momentum $\mathbf{P} = m_0\mathbf{V}$ is meaningless because it is derived from m_0 and $\mathbf{V}$. Properly speaking, even if both m_0 and $\mathbf{V}$ have no more meaning, their product may have. In fact, for a generic particle, the 4-momentum $\mathbf{P}$ can be written as

$$m_0\mathbf{V} = m_0\boldsymbol{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} , \qquad \mathrm{d}\tau = \frac{1}{c}\sqrt{-\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}}\mathrm{d}\lambda ,$$

and the scalar quantity, $P^\lambda \equiv m_0\mathrm{d}\lambda/\mathrm{d}\tau$, may have a physical meaning. Then $\mathbf{P} = P^\lambda\boldsymbol{\lambda}$ can be meaningful in the limit $m_0 \to 0$ and $\mathrm{d}\tau \to 0$ if *the latter are infinitesimal of the same order*. From this point of view, which gives to $\mathbf{P}$ a *primitive meaning* with respect to m_0 and $\mathbf{V}$, the following definition appears quite natural for a *particle with scalar structure*: an oriented world line ℓ^+, timelike or lightlike, and a tangent (nonnull) vector field $\mathbf{P}(E)$. When ℓ^+ is timelike, one can introduce the (preferred) proper time parametrization, as well as the 4-velocity $\mathbf{V}$ and the proper mass m_0, defined starting from the decomposition $\mathbf{P} = m_0\mathbf{V}$. When, instead, ℓ^+ is lightlike, i.e. (3.106) holds, for any choice of the parameter λ along ℓ^+, the vector $\mathbf{P}$ remains tangent to ℓ^+, but is lightlike:

$$||\mathbf{P}|| = 0 . \tag{3.107}$$

Summarizing, in the unified scheme (material particles and photons), the fundamental ingredients are two: the *oriented world line* ℓ^+ and the *field of tangent vectors* $\mathbf{P} \neq 0$, with norm

$$||\mathbf{P}|| \leq 0 , \tag{3.108}$$

with the equality holding for photons. Finally, if $\mathbf{P}$ is aligned along ℓ^+ and has the same orientation, one speaks of particles, otherwise of exotic particles.

The condition $\mathbf{P}$ tangent to ℓ^+ represents a strong limitation for the particle scheme. In fact, once introduced on ℓ^+ an *arbitrary parameter* λ, the 4-momentum $\mathbf{P}$ is

$$\mathbf{P} = P^\lambda\boldsymbol{\lambda} , \qquad \boldsymbol{\lambda} = \frac{\mathrm{d}E}{\mathrm{d}\lambda} , \tag{3.109}$$

for all parametric representation, with the scalar P^λ (assumed to be positive) naturally depending on the chosen parametric representation. That is, from the invariance of $\mathbf{P}$: $P^\lambda \boldsymbol{\lambda} = P^{\lambda'} \boldsymbol{\lambda}' = \text{inv.}$, and using the relation

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}' \frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda} , \tag{3.110}$$

one finds the transformation law for P^λ when λ varies:

$$P^{\lambda'} = \frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda} P^\lambda . \tag{3.111}$$

This is the exactly the transformation law for vectors, which motivates the notation for the position of the index λ. Taking then into account the positivity of the quantity $\mathrm{d}\lambda'/\mathrm{d}\lambda$ also follows the *invariance for the sign of* P^λ, which substantiates the two different (but similar) schemes: $P^\lambda > 0$ (particle), and $P^\lambda < 0$ (exotic particle). In the following, in order to avoid an indicator $\epsilon = \pm 1$, we will restrict our attention only to particles: $P^\lambda > 0$.

3.14 Fundamental Invariants

The norm of $\mathbf{P}$ is a first absolute invariant for the unified particles, which we will show to be effective in the relative ambit only. More important is the differential invariant:

$$\frac{\mathrm{d}\lambda}{P^\lambda} = \frac{\mathrm{d}\lambda'}{P^{\lambda'}} = \text{inv.} = I , \tag{3.112}$$

and, by integrating along ℓ^+, the finite invariant $I(E) > 0$, defined up to an additive constant

$$I(E) = \int_{E_0}^{E} \frac{1}{P^\lambda(\lambda)} \mathrm{d}\lambda = \text{inv.} \tag{3.113}$$

Vice versa, by differentiating the invariant (3.113), one gets the component P^λ of $\mathbf{P}$:

$$\frac{\mathrm{d}I}{\mathrm{d}\lambda} = 1/P^\lambda , \tag{3.114}$$

so that the unified scheme can also be characterized by the oriented world line ℓ^+ (timelike or lightlike), endowed with a scalar invariant: $I(E)$, which assimilates the photons to the material particles; in particular, for the latter case, the proper mass $m_0 \stackrel{\text{def}}{=} P^\tau$ is contained in (3.114):

$$\frac{1}{m_0} = \frac{\mathrm{d}I}{\mathrm{d}\tau} . \tag{3.115}$$

Obviously, because of the positiveness of P^λ, I is also an admissible parameter for ℓ^+, and it is such that $\mathbf{P}$ becomes an exact derivative:

$$\mathbf{P} = \mathrm{d}E/\mathrm{d}\lambda , \qquad P^I = 1 . \tag{3.116}$$

In the quantum treatment of photons, the universal constant h (Planck constant) plays a role:

$$h \simeq 6.63\, 10^{-34}\, \mathrm{J} \cdot \mathrm{s} \tag{3.117}$$

with dimensions of an angular momentum (or of the action): $[h] = [ML^2T^{-1}]$. We can include this constant in P^λ, by putting

$$P^\lambda = \frac{h}{c^2}\nu^\lambda \,, \tag{3.118}$$

where the new quantity ν^λ is characterized by the same variation law of P^λ at varying the parameter λ:

$$\nu^{\lambda'} = \nu^\lambda \frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda} \quad \Rightarrow \frac{\mathrm{d}\lambda}{\nu^\lambda} = \text{inv.} \tag{3.119}$$

It is easy to recognize that, *if the parameter λ has the dimensions of a time, then ν^λ is a frequency*. In fact, from one side, $[P] = [MLT^{-1}]$ (being $\mathbf{P}$ a linear momentum) and from the other $[P] = [P^\lambda][L\lambda^{-1}]$, so that, if $[\lambda] = [T]$, it results

$$MLT^{-1} = [P^\lambda]LT^{-1} \,, \qquad \Rightarrow \qquad [P^\lambda] = M \,,$$

and hence,

$$[\nu^\lambda] = [T^{-1}] \,. \tag{3.120}$$

Summarizing, the invariant (3.113) gives rise, by using (3.118), to another invariant for both material particles and photons, that is the proper frequency $\mathcal{V} > 0$:

$$\frac{1}{\mathcal{V}(E)} = \int_{E_0}^{E} \frac{1}{\nu^\lambda(\lambda)} \mathrm{d}\lambda = \text{inv.} \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{\mathcal{V}}\right) = \frac{1}{\nu^\lambda} \,. \tag{3.121}$$

The following relation holds

$$I(E) = \frac{c^2}{h\mathcal{V}(E)} \,, \tag{3.122}$$

so that, in the unified scheme, the invariant I can be replaced by $\mathcal{V}$. Because of the different meaning of such invariants, $\mathbf{P}$ can be obtained starting from (3.109) and using conditions (3.114) and (3.122):

$$P^I = 1 \,, \qquad P^\mathcal{V} = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}I} = -\frac{c^2}{hI^2} \,. \tag{3.123}$$

In the unified scheme, from the absolute point of view, the local invariants I, or $\mathcal{V}$, play the role of the proper mass m_0: hence, the latter has only a partial meaning, like τ and $\mathbf{V}$, for timelike world lines.

3.15 Particle and Photon Dynamics

As concerns the absolute dynamics, in the unified scheme, one cannot use the proper time parametrization of the world line because this has no meaning on null world lines. It is necessary to use an arbitrary parameter λ, and to rewrite (3.6) in the form:

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\lambda} = \mathbf{K}_\lambda(\lambda, E, \mathbf{P}) \;, \tag{3.124}$$

assuming that this equation have absolute character, i.e. not depending on the choice of λ. It follows that the 4-force $\mathbf{K}_\lambda$ should transform like a derivative, or with the covariance law:

$$\mathbf{K}_{\lambda'} = \mathbf{K}_\lambda \frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'} \;, \tag{3.125}$$

complementary to (3.111), so that the parameter λ can be chosen arbitrarily.

In particular, by using the canonical parameter I, the unified dynamics of particles, in the ambit of restrict problems, is governed by the following set of equations:

$$\frac{\mathrm{d}E}{\mathrm{d}I} = \mathbf{P} \;, \qquad \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}I} = \hat{\mathbf{K}}(I, E, \mathbf{P}) \;, \qquad ||\mathbf{P}|| \leq 0 \;, \tag{3.126}$$

where $\hat{\mathbf{K}} \equiv \mathbf{K}_I$ is the generalized 4-force, and $\mathrm{d}E/\mathrm{d}I$ the analogous of the 4-velocity. The absolute parameter I is defined by means of (3.113)–(3.114), both invariant with respect to the choice of λ, vice versa, (3.115) holds only for material particles.

The unified treatment (3.126) includes, obviously, the energy theorem; in fact, assuming

$$||\mathbf{P}|| = -\hat{m}_0^2 c^2 = -\frac{\hat{\mathcal{E}}^2}{c^2} \;, \tag{3.127}$$

from $(3.126)_2$ one obtains the relation

$$\frac{\mathrm{d}}{\mathrm{d}I}\left(\frac{1}{2}\hat{\mathcal{E}}^2\right) = \frac{1}{c^4}\hat{q} \;, \qquad \hat{q} = -\hat{\mathbf{K}} \cdot \mathbf{P} \;, \quad (\mathbf{P} = \frac{\mathrm{d}E}{\mathrm{d}I}) \;, \tag{3.128}$$

where the scalar $\hat{q}$ can be interpreted as *proper thermal power*. From this point of view, which reintroduces the proper mass as a quantity derived from $\mathbf{P}$, the photon is characterized by the condition $\hat{m}_0 = 0$, like a particle without internal structure, which implies also $\hat{q} = 0$ (absence of thermal interaction).

However, formulation (3.128) does not include the limitation $||\mathbf{P}|| \leq 0$, which should be added to the equations. It uses, as a parameter for ℓ^+, the absolute invariant I, which makes the mass as unit: $P^I = 1$, but it does not have the dimensions of a time, and hence it cannot be considered as a substitute for the proper time. In any case, (3.122) defines a second invariant: $\mathcal{V} > 0$ (*absolute frequency*), which also gives the temporal invariant:

$$T \stackrel{\text{def}}{=} \frac{1}{\mathcal{V}} > 0 \,, \tag{3.129}$$

where $\mathcal{V}$ is defined by (3.121), in terms of $\mathbf{P}$:

$$\frac{1}{\mathcal{V}} = \int_{E_0}^{E} \frac{1}{\nu^\lambda} \mathrm{d}\lambda \,, \qquad \nu^\lambda = \frac{c^2}{h} P^\lambda.$$

From here, by differentiating and using (3.159), one gets

$$\frac{\mathrm{d}T}{\mathrm{d}\lambda} = \frac{1}{\nu^\lambda} > 0 \,; \tag{3.130}$$

Thus, together with I and $\mathcal{V}$, also T is admissible along ℓ^+: this is an absolute temporal parameter which, differently than the proper time, is *meaningful also for photons*. To it one can refer all the fundamental quantities for the unified particles, starting from the frequency $\mathcal{V}$, or the equivalent mass $M = h\mathcal{V}/c^2$ which, in terms of T, have the following expressions:

$$\mathcal{V} = \frac{1}{T} \,, \qquad M = \frac{h}{c^2 T}, \tag{3.131}$$

with

$$\mathbf{P} = M\boldsymbol{\mathcal{V}} \,, \qquad \boldsymbol{\mathcal{V}} = \frac{\mathrm{d}E}{\mathrm{d}T} \,. \tag{3.132}$$

Hence, the *absolute dynamics of unified particles* is summarized by the following set of equations, similar to the canonical ones (3.126):

$$M\frac{\mathrm{d}E}{\mathrm{d}T} = \mathbf{P} \,, \quad \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}T} = \mathbf{K}_T \,, \quad ||\mathbf{P}|| \leq 0 \,, \tag{3.133}$$

where M is now a known function of T. The limitation $||\mathbf{P}|| \leq 0$ ensures that the world line of the particle is timelike or lightlike.

Finally, as concerns the relations with the proper mass m_0 and the proper time τ of the material particles, from (3.132), we have $M\boldsymbol{\mathcal{V}} = m_0\mathbf{V}$, so that:

$$M\mathcal{V} = cm_0 \,, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{c}{\mathcal{V}}\frac{\mathrm{d}}{\mathrm{d}T} \,. \tag{3.134}$$

3.16 Unified Relative Dynamics of Particles

Differently from m_0 (absolute quantity), the concept of *relative mass* m can be introduced also in the unified scheme in any Galilean frame, that is also for photons (corpuscular theory of light). In fact, if a Galilean frame is fixed,

$S_{\rm g}(\boldsymbol{\gamma})$, and the ℓ^+ is parametrized by the coordinate time of the frame $\lambda = t$, (3.109), using also (3.118), becomes

$$\mathbf{P} = P^t \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{h\nu}{c^2}(\mathbf{v} + c\gamma) ,$$

with ν the *relative frequency* of the particle (luminal or material) and $\mathbf{v}$ its relative velocity. Thus, in $S_{\rm g}$, also for a photon, the ordinary decomposition (3.38) holds

$$\mathbf{P} = \mathbf{p} + \frac{\mathcal{E}}{c}\gamma , \qquad \mathbf{p} = m\mathbf{v} , \tag{3.135}$$

with the *relative frequency* or *mass*:

$$m = P^t = \frac{\mathrm{d}t}{\mathrm{d}I} = \frac{h\nu}{c^2} , \tag{3.136}$$

together with the *relative material energy*, given by

$$\mathcal{E} = mc^2 = h\nu . \tag{3.137}$$

We notice that the component $P^t = \mathrm{d}t/\mathrm{d}\lambda$ has either the meaning of relative mass or that or relative frequency, and both these quantities are well defined for the unified particles.

Equation (3.137) shows that the energy of a photon is proportional to its relative frequency. Moreover, from (3.119), which gives the variation of the frequency, with respect to that of the parameter along ℓ^+, one gets the variation law of the frequency (and hence of the energy and of the mass) in the ambit of the Galilean frames. In fact, using $\lambda = t$ and $\lambda = t'$, respectively, it follows: $\nu' = \nu \mathrm{d}t'/\mathrm{d}t$, i.e. using (2.102):

$$\nu' = \nu \frac{1 - \mathbf{u} \cdot \mathbf{v}}{\sqrt{1 - u^2/c^2}} , \tag{3.138}$$

in agreement with (3.53). Equation (3.138) defines the *transversal Doppler effect*.

As concerns the relative dynamics of particles, in the unified scheme, it is clear that, once fixed a Galilean frame, and using the associated time coordinate t, it is enough to put $\lambda = t$ in (3.124), obtaining

$$\dot{\mathbf{P}} = \mathbf{K}_t(t, E, P) , \qquad (\,)^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}t} . \tag{3.139}$$

From here, the *relative equations of motion*, using either (3.135), as well as the decomposition of $\mathbf{K}_t$ along $\boldsymbol{\gamma}$ and onto Σ:

$$\mathbf{K}_t = \mathbf{F} + \frac{\mathcal{W}}{c}\gamma . \tag{3.140}$$

More precisely, (3.139), as for the case of material particles, splits into the two Newtonian equations:

$$\dot{\mathbf{p}} = \mathbf{F} \ , \qquad \dot{\mathcal{E}} = \mathcal{W} \ , \tag{3.141}$$

where the total power is given by

$$\mathcal{W} = \mathbf{F} \cdot \mathbf{v} + q \ , \tag{3.142}$$

which defines the relative thermal power q, in terms of $\mathcal{W}$ and $\mathbf{F}$. Equivalently, as in (3.43), q is also given by:

$$q = -\mathbf{K}_t \cdot \frac{\mathrm{d}E}{\mathrm{d}t} \quad \sim \quad q = \left(\frac{\mathrm{d}I}{\mathrm{d}t}\right)^2 \hat{q} \ ; \tag{3.143}$$

in fact, from (3.140) it follows:

$$\mathbf{K}_t \cdot \frac{\mathrm{d}E}{\mathrm{d}t} = \left(\mathbf{F} + \frac{\mathcal{W}}{c}\boldsymbol{\gamma}\right) \cdot (\mathbf{v} + c\boldsymbol{\gamma}) = \mathbf{F} \cdot \mathbf{v} - \mathcal{W} \ .$$

Thus, also in their generalized form (to include material particles and photons), the fundamental equations (3.141) are invariant with respect to the choice of the Galilean frame; the invariance being formal and not substantial because of the relative meaning of the involved quantities. In particular, as (3.119) gave the variation law for the frequency (3.138) (and hence for the energy and the mass), (3.125), for $\lambda = t$ and $\lambda = t'$, gives the transformation laws for the mechanical force $\mathbf{F}$ and the power $\mathcal{W}$. In fact we have

$$\mathbf{F}' + \frac{\mathcal{W}'}{c}\boldsymbol{\gamma}' = \frac{\alpha}{\sigma}\left(\mathbf{F} + \frac{\mathcal{W}}{c}\boldsymbol{\gamma}\right) \tag{3.144}$$

from which, using the standard procedure, one gets again (3.58) and (3.59):

$$\begin{cases} \mathcal{W}' = \dfrac{1}{\sigma}(\mathcal{W} - \mathbf{F} \cdot \mathbf{u}), \\ \mathbf{F}' = \dfrac{1}{\sigma}\left[\alpha \mathbf{F} + \dfrac{1}{c^2}\left(\dfrac{\mathbf{F} \cdot \mathbf{u}}{1 + \alpha} - \mathcal{W}\right)\mathbf{u}\right] . \end{cases} \tag{3.145}$$

Finally, from (3.142) and (3.143), we have the variation law of the thermal power:

$$q' = \left(\frac{\alpha}{\sigma}\right)^2 q \ , \tag{3.146}$$

already found in the case of material particles. Equation (3.146) again gives, also for photons, the intrinsic meaning of the condition $q = 0$ (absence of thermal actions). In fact, (3.141) can be written in the form: $q = -\hat{\mathbf{K}} \cdot \mathbf{P}/(P^t)^2$, that is

$$q = \hat{q}/m^2 \ , \qquad \hat{q} = -\hat{\mathbf{K}} \cdot \mathbf{P} \ , \tag{3.147}$$

where $\hat{q}$ is the scalar invariant defined in (3.128) which, in the unified scheme, corresponds to the proper thermal power q_0. However, the condition $\hat{q} = 0$

which for material particles is equivalent to the invariability of the proper mass m_0, in the unified scheme, according to $(3.126)_3$, corresponds to the conservation of the norm of $\mathbf{P}$:

$$||\mathbf{P}|| = \text{const.} = ||\mathbf{P}_0|| \leq 0 \,. \tag{3.148}$$

Because for a material particle we have $||\mathbf{P}|| = -m_0^2 c^2$, (3.148) shows that photons behave like particles without internal structure and with proper mass $m_0 = 0$. This is but only qualitative because for photons the notion of proper mass, as that of proper rest frame, have no meaning. We notice that (3.148), taking into account the definition of the characteristic function $I(\lambda)$ and $(3.126)_1$, implies that

$$||\boldsymbol{\lambda}|| = ||\mathbf{P}|| \left(\frac{\mathrm{d}I}{\mathrm{d}\lambda}\right)^2 \,, \tag{3.149}$$

or, in relative terms to S_{g}:

$$v^2 - c^2 = ||\mathbf{P}|| \left(\frac{\mathrm{d}I}{\mathrm{d}\lambda}\right)^2 \,; \tag{3.150}$$

in particular, for material particles, from (3.114) and (3.136) we have

$$\frac{\mathrm{d}t}{\mathrm{d}I} = m = \frac{h\nu}{c^2} \,, \tag{3.151}$$

so that (3.149) assumes the ordinary form: $m = m_0/\sqrt{1 - v^2/c^2}$.

Finally, it is worth to mention that, in the unified scheme, the dynamical source $\mathbf{K}_\lambda$, satisfying the invariance property (3.125): $\mathbf{K}_\lambda \mathrm{d}\lambda = \mathbf{K}_{\lambda'} \mathrm{d}\lambda'$ =inv. cannot be given, a priori, through the law $\mathbf{K}_\lambda = \mathbf{K}_\lambda(\lambda, E, \mathbf{P})$; in fact, the considered scheme is not free, but constrained by the condition (3.108):

$$||\mathbf{P}|| \leq 0 \,.$$

Thus, it has partially the meaning of reaction to the constraint, a property which is also inherited by $\mathbf{F}$ and $\mathbf{q}$. This is specially true for the particles on the border (photons), where the constraint becomes bilateral, and from (3.135) implies $p^2 = \mathcal{E}^2/c^2$, or $v = c$.

3.17 An Alternative to the Unified Dynamics

In unified dynamics too one can consider a formulation similar to (3.81). First of all, (3.141) can be summarized by the following scalar relations, in the variables x^i, p_i and $m = \mathcal{E}/c^2$:

$$\dot{x}^i = \frac{1}{m}\delta^{ik} p_k, \quad \dot{p}_i = F_i(t, x, p, m) \,, \quad \dot{m} = \frac{1}{c^2} W(t, x, p, m) \,, \tag{3.152}$$

with the constraint $||\mathbf{P}|| \leq 0$, that is, from (3.135):

$$p^2 - m^2c^2 \leq 0 \qquad \sim \qquad v^2 \leq c^2 \ ; \tag{3.153}$$

we see that such constraint does not represent a limitation for the relative trajectory but only for the motion law. However, as in the ordinary case, *the constraint* $||\mathbf{P}|| \leq 0$ *can be included in the dynamical equations*, by using as a parameter $||\mathbf{P}||$ in place of m or $\mathcal{E}$:

$$||\mathbf{P}|| = \hat{m}^2c^2 \ . \tag{3.154}$$

One finds

$$p^2 - m^2c^2 = \hat{m}^2c^2 \ , \tag{3.155}$$

from which the value of m follows, having in addition the meaning of frequency: $m = h\nu/c^2$, i.e.

$$m = \sqrt{\hat{m}^2 + \frac{p^2}{c^2}} \ . \tag{3.156}$$

We notice that, at least for the material particles, $\hat{m}$ has the meaning of proper mass, and it vanishes for photons; vice versa, the relative mass m is always positive because for photons, one has $v = c \neq 0$.

By differentiating (3.155), one gets

$$\dot{m} = \frac{1}{2m}[(\hat{m}^2)^{\cdot} + 2\mathbf{p} \cdot \dot{\mathbf{p}}/c^2] \ ,$$

which maps system (3.151) in its equivalent form:

$$\begin{cases} \dot{x}^i = \dfrac{\delta^{ik}p_k}{\sqrt{\hat{m}^2 + p^2/c^2}}, \qquad \dot{p}_i = F_i(t,x,p,\hat{m}^2), \\ \\ (\hat{m}^2)^{\cdot} = \dfrac{2}{c^2}\sqrt{\hat{m}^2 + p^2/c^2}\, q(t,x,p,\hat{m}^2). \end{cases} \tag{3.157}$$

We easily recognize that the variables are changed and, in place of m, appears the absolute parameter $\hat{m}^2$; however, the differential system is still of the first order with mechanical and thermal sources given, separately, by F_i and q.

However, *system* (3.156) *includes the constraint* $||\mathbf{P}|| \leq 0$; in fact, from $(3.156)_1$, it follows that

$$(\hat{m}^2 + p^2/c^2)v^2 = p^2 \quad \sim \quad \hat{m}^2v^2 = (1 - v^2/c^2)p^2 \ ,$$

that is the restriction $v^2 \leq c^2$, for all the solutions of the system. System (3.156), as the analogous (3.151), represents the relative dynamics of unified particles, and the parameter $\hat{m}$ discriminates between material particles and photons. For the latter case, one has $\hat{m} = 0$ and $q = 0$ and the system (3.156) reduces to

$$\dot{x}^i = c\delta^{ik} p_k/p, \quad \dot{p}_i = F_i(t, x, p) , \tag{3.158}$$

which implies $v = c$. It is then clear the role played by the covariant vector p_i, which, with the direction, gives rise to the velocity v^i along the ray (relative trajectory of the photon), while with its norm gives the energy carried by the elementary surface, orthogonal to it, as from (3.155):

$$p^2 = m^2 c^2 . \tag{3.159}$$

Differently, from (3.154), one finds $p^2 = (m^2 - \hat{m}^2)c^2$, and the velocity along the ray is determined by p_i and $\hat{m}$.

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4

Applications

4.1 Principal Tetrad for Nonnull Curves in M_4

One of the most interesting aspects of the mathematical methodology is that of generalizing to any space a result introduced and verified in a particular framework. Such a stimulating procedure can be interpreted as the necessity of making *universal* certain concepts, eliminating eventual limits due to the framework where they have been first introduced.

In relativity, this happens very often; results and geometrical quantities, typical of the classical apparatus, are redefined in a completely different framework (for dimension and geometry), like the Minkowski space M_4.

In a strictly geometrical ambit, a typical example of such extension is given by the so-called *Frenet–Serret formulas*, fundamental for the intrinsic classification of curves in M_4. The extension is not difficult, when the curve is *timelike* (or spacelike): a well-determined tetrad corresponds, locally, to the principal triad of the ordinary case; actually, to the ordinary curvature and torsion correspond generically three "curvatures". Differently, the extension for *lightlike curves*, i.e. tangent to the lightcone, is not so simple. In this case, in fact, it is necessary to introduce *quasi-orthonormal bases*, or more generally, anholonomic bases, which are first attached to the curve and then, more and more specialized, in order to be intimately related to the curve itself.

We will discuss here the nondegenerate case only. Without any loss of generality, let us consider a timelike and future-oriented curve ℓ^+ representing the world line of a material (or exotic) particle.

Let us recall that, in M_4, an orthonormal frame is defined by an event Ω, taken as the origin, a timelike axis x^0 and three mutually orthogonal spatial axes x^i $(i = 1, 2, 3)$; all of them characterized by the unit vectors $\mathbf{c}_\alpha$ $(\alpha = 0, 1, 2, 3)$. A world line, in M_4, can be defined by the parametric equations $x^\alpha = x^\alpha(\lambda)$ $(\alpha = 0, 1, 2, 3)$, with λ a generic parameter. If the world line is not lightlike, i.e.

$$m_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \neq 0 \, , \tag{4.1}$$

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as it is the case for ℓ^+ (timelike and future-oriented), one can consider a special parameter s (*curvilinear abscissa*) invariant with respect to the choice of x^α:

$$s = s_0 + \int_{\lambda_0}^{\lambda} \sqrt{-m_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}}\, \mathrm{d}\lambda \; ; \tag{4.2}$$

the minus sign is necessary for timelike curves. Thus, assuming such a parameter, one has

$$x^\alpha = x^\alpha(s) \qquad (\alpha = 0, 1, 2, 3) \; ; \tag{4.3}$$

i.e., using a shorten notation:

$$\Omega E = \Omega E(s) \; , \tag{4.4}$$

so, with a prime, we will denote differentiation with respect to s. As in the ordinary case, at each $E \in \ell^+$, the unit tangent vector $\mathbf{T}$ and its first derivative, i.e. the curvature vector $\mathbf{C}$, have a direct meaning:

$$\mathbf{T} \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}\Omega E}{\mathrm{d}s} \; , \qquad \mathbf{C} \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \; ; \tag{4.5}$$

the curvature vector $\mathbf{C}$, in turn, gives rise to the unit vector $\mathbf{N}$ (*principal normal of the curve*), as well as to the *geodesic curvature* $C > 0$ (we will not consider here the case of a straight line: $\mathbf{T} = \text{const.}$):

$$\mathbf{C} = C\mathbf{N} \; . \tag{4.6}$$

The 2-plane, in E, containing the tangent vector and the principal normal, is still called *osculating plane.*

Let us consider, now, the hyperplane Π, defined by $\mathbf{T}$ and its first and second derivatives:

$$\Pi =< \mathbf{T}, \mathbf{T}', \mathbf{T}'' > \; ; \tag{4.7}$$

We assume ℓ^+ regular enough and generic, in the sense that the vectors $\mathbf{T}$, $\mathbf{T}'$ and $\mathbf{T}''$ are linearly independent. Furthermore, let us assume that either M_4 or the hyperplane Π is (independently) oriented, so that a convenient notation is M_4^+ or Π^+, respectively. It is possible, then, to consider a unit vector $\mathbf{B}$, uniquely defined by the following three conditions:

1. $\mathbf{B}$ is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$;
2. $\mathbf{B} \in \Pi$;
3. the triad $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is congruent to Π^+, say left-handed.

Finally, let $\mathbf{D}$ be the unit normal to Π, oriented so that the tetrad $(\mathbf{T}, \mathbf{N}, \mathbf{B}, \mathbf{D})$ is coherent with the orientation of M_4. In this way, the four vectors $(\mathbf{T}, \mathbf{N}, \mathbf{B}, \mathbf{D})$ uniquely define a basis, dependent only on the point E and the curve ℓ^+, called the *principal tetrad of the curve* ℓ^+ *in* E. The two vectors $\mathbf{B}$ and $\mathbf{D}$ define, in E, two half lines: the *binormal* and the *threenormal*, respectively. The hyperplane Π_n, spanned by $\mathbf{N}$, $\mathbf{B}$ and $\mathbf{D}$:

$$\Pi_n =< \mathbf{N}, \mathbf{B}, \mathbf{D} > , \tag{4.8}$$

is normal to **T**, and then elliptic.

Our purpose is now that of deriving the expressions for the vectors **N**, **B** and **D**, in order to obtain in M_4 the analogous of the ordinary Frenet formulas.

4.2 Frenet–Serret Formalism

In the ordinary three-dimensional case, the Frenet–Serret formulas give, in intrinsic way, the derivatives (with respect to the curvilinear abscissa) of the vectors of the principal triad: **t**, **n** and **b**, in terms of the same triad. Similarly, for a curve in $\ell^+ \in M_4$, endowed with a principal tetrad, one can evaluate the first derivatives of the corresponding vectors: $\mathbf{T}'$, $\mathbf{N}'$, etc.

The vector **B**, in turn, being contained in Π, can be expressed as

$$\mathbf{B} = \lambda \mathbf{T} + \mu \mathbf{T}' + \nu \mathbf{T}'' ; \tag{4.9}$$

so that, using the relation $\mathbf{T}' = C\mathbf{N}$ and its first derivative,

$$\mathbf{T}'' = C'\mathbf{N} + C\mathbf{N}' , \tag{4.10}$$

one gets the following expression for **B**:

$$\mathbf{B} = \lambda \mathbf{T} + (\mu C + \nu C')\mathbf{N} + \nu C \mathbf{N}' .$$

From this relation, using the orthogonality of **B** and **N**, it follows:

$$\mu C + \nu C' = 0 ; \tag{4.11}$$

hence, **B** should be represented as

$$\mathbf{B} = \lambda \mathbf{T} + \nu C \mathbf{N}' , \tag{4.12}$$

that is $\mathbf{N}'$ belongs to the plane spanned by **T** and **B**; moreover, scalar multiplication of (4.12) by **T** gives $0 = -\lambda - \nu C \mathbf{N} \cdot \mathbf{T}' = -\lambda - \nu C^2$, that is: $\lambda = -\nu C^2$. Thus, assuming $C \neq 0$, both λ and μ can be expressed in terms of the curvature C and the parameter ν:

$$\lambda = -\nu C^2 , \qquad \mu = -\nu \frac{C'}{C} ; \tag{4.13}$$

then, (4.12) becomes

$$\mathbf{N}' = C\mathbf{T} + \tau \mathbf{B} , \tag{4.14}$$

where τ represents the first *torsion* of the curve:

$$\tau = \frac{1}{\nu C} . \tag{4.15}$$

Equation (4.14) gives the derivative of $\mathbf{N}$ and, apart from the signs, is similar to the classical formula. As concerns the derivative of $\mathbf{B}$, because of $(4.5)_2$ and (4.14), we have

$$\mathbf{B}' \cdot \mathbf{T} = -\mathbf{B} \cdot \mathbf{T}' = 0 \,, \qquad \mathbf{B}' \cdot \mathbf{N} = -\mathbf{B} \cdot \mathbf{N}' = -\tau \,;$$

thus, defining

$$\beta = \mathbf{B}' \cdot \mathbf{D} = -\mathbf{B} \cdot \mathbf{D}' \,, \tag{4.16}$$

we obtain

$$\mathbf{B}' = -\tau \mathbf{N} + \beta \mathbf{D} \,. \tag{4.17}$$

Finally, passing to $\mathbf{D}'$, we have

$$\mathbf{D}' \cdot \mathbf{T} = -\mathbf{D} \cdot \mathbf{T}' = 0 \,, \qquad \mathbf{D}' \cdot \mathbf{N} = -\mathbf{D} \cdot \mathbf{N}' = 0 \,,$$

so that

$$\mathbf{D}' = -\beta \, \mathbf{B} \,. \tag{4.18}$$

The Frenet–Serret formulas (see [1], pp. 8–12), then follow:

$$\begin{cases} \mathbf{T}' = C\mathbf{N} \,, \quad C > 0 \,, \\ \mathbf{N}' = C\mathbf{T} + \tau \mathbf{B} \,, \\ \mathbf{B}' = -\tau \mathbf{N} + \beta \mathbf{D} \,, \\ \mathbf{D}' = -\beta \mathbf{B} \,. \end{cases} \tag{4.19}$$

4.3 Curvature and Torsions

The scalar quantities C, τ and β are fundamental in the study of curves in M_4 because (exactly like c and τ in the ordinary case) they allow an *intrinsic characterization* of the curve itself, up to a Lorentz transformation. In fact (4.19), completed with $\Omega E' = \mathbf{T}$, form a well-determined first-order *linear system* in the unknown E, $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ and $\mathbf{D}$.

From this point of view, (4.19) play an important role also in the absolute dynamic, giving the *intrinsic equations of the motion*, particularly important in the presence of constraints (for example, when ℓ^+ belongs to a given hypersurface in M_4, etc.). In any case, how can one express the curvature and torsions of ℓ^+, starting from the parametric equations (4.3) of the curve, and what is their geometrical meaning?

Let us notice, first of all, that the scalars C, τ and β do not depend on the orientation of the curve. In fact, changing the sign of the curvilinear abscissa, the vector $\mathbf{T}$ changes sign but $\mathbf{T}' = \mathbf{C}$ remains unchanged, and with this, both C and $\mathbf{N}$ are invariant because of (4.6). Analogously, $\mathbf{B}$ too changes sign but $\mathbf{B}'$ remains unchanged. $\mathbf{D}$, instead, does not change: this implies, using $(4.19)_3$, the invariance of τ and β.

From $(4.19)_1$, being $\mathbf{T}'$ spacelike (because it is orthogonal to $\mathbf{T}$), one gets immediately

$$C = \sqrt{||\mathbf{T}'||} = \sqrt{m_{\alpha\beta}\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2}\frac{\mathrm{d}^2 x^\beta}{\mathrm{d}s^2}} \; . \tag{4.20}$$

As concerns τ, one can use $(4.19)_2$, taking into account that $\mathbf{N}$, and hence $\mathbf{N}'$, is determined by differentiating $(4.19)_1$:

$$\mathbf{N}' = \frac{1}{C}\mathbf{T}'' - \frac{C'}{C^2}\mathbf{T}' \; . \tag{4.21}$$

In fact, using the property $||\mathbf{T}|| = -1$ and $(4.19)_2$, it follows:

$$\tau = \pm\sqrt{|\,||\mathbf{N}'|| + C^2|} \; , \tag{4.22}$$

where $||\mathbf{N}'||$ can be derived from (4.21):

$$||\mathbf{N}'|| = \frac{1}{C^2}||\mathbf{T}''|| + \frac{C'^2}{C^4}||\mathbf{T}'|| - 2\frac{C'}{C^3}\mathbf{T}'\cdot\mathbf{T}'' \; ,$$

and thus, being

$$\mathbf{T}'\cdot\mathbf{T}'' = \frac{1}{2}(\mathbf{T}'\cdot\mathbf{T}')' = CC' \; ,$$

one gets the result

$$||\mathbf{N}'|| = \frac{1}{C^2}\left(m_{\alpha\beta}\frac{\mathrm{d}^3 x^\alpha}{\mathrm{d}s^3}\frac{\mathrm{d}^3 x^\beta}{\mathrm{d}s^3} - C'^2\right) \; . \tag{4.23}$$

In this way, we have already determined C, τ and $\mathbf{N}'$; thus, from $(4.19)_2$ follows $\mathbf{B}$:

$$\mathbf{B} = \frac{1}{\tau}(\mathbf{N}' - C\mathbf{T}) = \frac{1}{\tau}\left(\frac{1}{C}\mathbf{T}'' - \frac{C'}{C^2}\mathbf{T}' - C\mathbf{T}\right) \; ; \tag{4.24}$$

finally $(4.19)_3$ specifies either β or $\mathbf{D}$:

$$\beta = \pm\sqrt{|\,||\mathbf{B}|| - \tau^2|}; \qquad \mathbf{D} = \frac{1}{\beta}\left(\mathbf{B}' + \tau\mathbf{N}\right) \; . \tag{4.25}$$

Let us pass now to discuss the geometrical meaning of the curvatures. Clearly, for C and τ, we can repeat all that has been said in the ordinary case (see [2], pp. 30–34): $C > 0$ measures the displacement of the curve ℓ^+ from the rectilinear behaviour (C is said the *geodesic curvature*); differently from C, τ can assume both signs: it measures the variation of the osculating plane, i.e. the displacement of the curve from the plane behaviour. In fact, if $\tau = 0$, from $(4.19)_2$, one has $\mathbf{N}' = C\mathbf{T}$, so that $(4.19)_1$ implies that $\mathbf{T}''$ belongs to the osculating plane.

Also β can assume both signs; however, its absolute value has a meaning which can be obtained by proceeding in an analogous way as in the ordinary case, for curvature and torsion, namely

$$|\beta| = \lim_{\Delta s \to 0} \frac{\phi}{|\Delta s|} , \tag{4.26}$$

where ϕ is the angle between the two hyperplanes Π and Π', corresponding to the values s and $s' = s + \Delta s$ of the curvilinear abscissa; that is, ϕ is the angle between $\mathbf{D}$ and $\mathbf{D}'$, both spacelike:

$$\cos\phi = \mathbf{D} \cdot \mathbf{D}' . \tag{4.27}$$

As for the ordinary torsion, the sign of β has a precise geometrical meaning. In fact, let us evaluate the signed distance $\delta(E, E')$ of the generic point $E' \in \ell^+$ from the hyperplane $\Pi(E)$. We will assume such a distance as positive or negative, corresponding to E' placed, with respect to $\Pi(E)$, in the same side of $(E, \mathbf{D})$ or in the opposite side, that is,

$$\delta = EE' \cdot \mathbf{D} . \tag{4.28}$$

The distance δ is a quantity of the fourth order in $(s' - s)$, so that to be evaluated it is necessary to expand EE' up to the fourth order:

$$EE' = \mathbf{T}\Delta s + \frac{1}{2}\mathbf{T}'(\Delta s)^2 + \frac{1}{3!}\mathbf{T}''(\Delta s)^3 + \frac{1}{4!}\mathbf{T}'''(\Delta s)^4 + \boldsymbol{\epsilon}_5 ,$$

where $\boldsymbol{\epsilon}_5$ is the rest in the Taylor series. From (4.19), one has $\mathbf{T}' = C\mathbf{N}$, $\mathbf{T}'' = C'\mathbf{N} + C(C\mathbf{T} + \tau\mathbf{B})$ and $\mathbf{T}''' = 3CC'\mathbf{T} + (C^3 - C\tau^2 + C'')\mathbf{N} + (2\tau C' + C\tau')\mathbf{B} + C\tau\beta\mathbf{D}$; thus, substituting in the previous expression, leads to

$$\begin{aligned} EE' = \mathbf{T}\Delta s &+ \frac{1}{2}C\mathbf{N}(\Delta s)^2 + \frac{1}{3!}(C\mathbf{T} + C'\mathbf{N} + C\tau\mathbf{B})(\Delta s)^3 \\ &+ \frac{1}{4!}[3CC'\mathbf{T} + (C^3 - C\tau^2 + C'')\mathbf{N} \\ &+ (2C'\tau + C\tau')\mathbf{B} + C\tau\beta\mathbf{D}](\Delta s)^4 + \boldsymbol{\epsilon}_5 . \end{aligned} \tag{4.29}$$

Then, scalar multiplication by $\mathbf{D}$ gives

$$\delta = \frac{1}{4!}C\tau\beta(\Delta s)^4 + \boldsymbol{\epsilon}_5 \cdot \mathbf{D} , \tag{4.30}$$

so that from (4.30) we have

$$\beta = \frac{4!}{C\tau} \lim_{\Delta s \to 0} \frac{\delta}{(\Delta s)^4} . \tag{4.31}$$

Thus, in a neighbourhood of the point $E \in \ell^+$, the sign of δ is invariant (positive or negative, according to the sign of $\tau\beta$); that is, the curve is *all* *placed from one side*, with respect to the osculating hyperplane $\Pi(E)$, and more precisely, it stands where $\pm\mathbf{D}$ is placed, if $\tau\beta > 0$ or $\tau\beta < 0$, respectively.

The principal tetrad, as well as the Frenet–Serret formulas (4.19), are no longer valid when the curve ℓ^+ is lightlike because the concept of curvilinear abscissa is lost in this case. Thus, as for the case of the photon dynamics, one must use a generic parameter, and the tangent vector to the null curve is defined up to a multiplicative factor. The intrinsic characterization of a null curve, that is the analogous quantities of C, τ, β, can be obtained using a *quasi-orthogonal* basis, $\mathbf{e}_\alpha$. The latter (in general anholonomic) are built up by considering, for the generic point $E \in \ell^+$, a pair of *orthogonal* 2-planes: Π and Π'; the first, for instance, *hyperbolic* (and containing the tangent vector of the null curve ℓ^+) and the second *elliptic*. Among the infinite *adapted basis* to the 2-planes, i.e. characterized by

$$\mathbf{e}_{0,2} \in \Pi \,, \quad \mathbf{e}_{1,3} \in \Pi' \,, \tag{4.32}$$

recalling that Π (hyperbolic) contains two null straight lines, it is meaningful to consider those having:

1. $\mathbf{e}_{0,2}$ are *null vectors*, satisfying the normalization condition $\mathbf{e}_0 \cdot \mathbf{e}_2 = 1$;
2. $\mathbf{e}_{1,3}$ are *spacelike* orthonormal vectors.

It results in

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta_{\alpha\{2-\beta\}} \,, \tag{4.33}$$

where the symbol $\{2-\beta\}$ denotes $2-\beta$ modulus 4. The bases like $\{\mathbf{e}_\alpha\}$ are not orthonormal: in fact, they contain two null vectors ($\mathbf{e}_0$ and $\mathbf{e}_2$) and are termed *quasi-orthonormal* (see e.g. [3]). In the structure (Π, Π'), the vectors $\mathbf{e}_{0,2}$ are defined each up to a multiplicative factor, while $\mathbf{e}_{1,3}$ can be arbitrarily rotated in the 2-plane Π'; a set of equivalent tetrads $\{\mathbf{e}_\alpha\}$ arises, to which one must add the possibility to select the pair of hyperplanes Π and Π'. By using the Cartan method of the *repére mobile* [4, 5], one recognizes that

1. *along any null curve* parametrized by an arbitrary parameter t, i.e. with equations $x^\alpha(t)$, *one has two local invariants*[1]: $I(t)$ and $J(t)$, built up with the derivatives of $x^\alpha(t)$ (up to the third and fourth order, respectively), and the Kronecker tensor; they are independent on the choice of the coordinates and, as concerns the dependence on the parameter t, the differential forms: $I\mathrm{d}t$ and $J\mathrm{d}t$ are invariant too.
   Thus, one can introduce, on the curve, an *absolute parameter* σ ($\mathrm{d}\sigma = I\mathrm{d}t$), analogous to the curvilinear abscissa;

[1] For a nonnull curve, these invariants are four: one corresponding to the curvilinear abscissa and the three others related to the curvatures: C, τ and β.

2. in each point of the curve, there exists a special *quasi-orthogonal tetrad*: $\{\mathbf{e}_\alpha\} \stackrel{\text{def}}{=} (\mathbf{T}, \mathbf{N}, \mathbf{B}, \mathbf{D})$, such that, by adopting the parameter σ, it satisfies the conditions analogous to the Frenet–Serret formulas (4.19):

$$\begin{cases} \frac{\mathrm{d}\,OE}{\mathrm{d}\sigma} = \mathbf{T}\,, \quad \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\sigma} = \mathbf{N}\,, \quad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}\sigma} = \tau\mathbf{T} - \mathbf{B}, \\ \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\sigma} = \beta\mathbf{D} - \tau\mathbf{N}, \quad \frac{\mathrm{d}\mathbf{D}}{\mathrm{d}\sigma} = -\beta\mathbf{T} \end{cases} \tag{4.34}$$

One recognizes immediately that the curvatures are, now, no more three, but only two: τ and β (the first curvature: C, is unitary); however, being $\mathbf{T}$ and $\mathbf{B}$ null vectors, in order to determine τ and β, one only needs $(4.34)_4$, which gives the *ordinary relation*:

$$\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\sigma}\right)^2 = \tau^2 + \beta^2\,. \tag{4.35}$$

The latter is an equation for τ because β, in turn, is determined by the invariant J above mentioned, that is:

$$|\beta| = \left(\frac{\mathrm{d}J}{\mathrm{d}\sigma}\right)^{10}\,. \tag{4.36}$$

Further details can be found in [5].

4.4 Intrinsic Equations

The Frenet–Serret formulas (4.19) can be conveniently used to discuss absolute properties of the motion, especially in the presence of constraints when, for very special external fields, it is possible to distinguish between the geometrical properties from the kinematical ones. As an example, let us consider the case of a massive particle, with *proper mass* $m_0 > 0$, in the (purely positional) external field:

$$\mathbf{K} = \mathbf{K}(E)\,. \tag{4.37}$$

The absolute equation

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\tau} = \mathbf{K}(E)\,, \tag{4.38}$$

taking into account the expression of the momentum: $\mathbf{P} = m_0\mathbf{V}$, becomes

$$\frac{\mathrm{d}m_0}{\mathrm{d}\tau}\mathbf{V} + m_0\mathbf{A} = \mathbf{K}(E)\,, \tag{4.39}$$

where, because of the relation $s = c\tau + \text{const.}$, between the curvilinear abscissa and the proper time, we have

$$\mathbf{V} = c\mathbf{T}\,, \qquad \mathbf{A} = c^2\mathbf{C}\,. \tag{4.40}$$

Projecting on the principal tetrad, $\mathbf{K}$ can be decomposed as

$$\mathbf{K} = K_T\mathbf{T} + K_N\mathbf{N} + K_B\mathbf{B} + K_D\mathbf{D} \,, \tag{4.41}$$

and the *intrinsic equations of motion* become

$$\frac{\mathrm{d}m_0}{\mathrm{d}s} = \frac{1}{c^2}K_T \,, \quad m_0 C = \frac{1}{c^2}K_N, \quad K_B = 0 \,, \quad K_D = 0 \,. \tag{4.42}$$

Relations (4.42) are *not equivalent* to the original ones (4.38) because the chosen basis is itself unknown. However, $(4.42)_{3,4}$ play a role similar to $(4.42)_2$, in the sense that, as the last equations give C, they express the torsions τ and β in purely geometrical terms. In fact, by differentiating with respect to s and using $(4.19)_{3,4}$, one gets

$$\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}s}\cdot\mathbf{B} - \tau K_N = 0 \,, \quad \frac{\mathrm{d}\mathbf{K}}{\mathrm{d}s}\cdot\mathbf{D} = 0 \,, \tag{4.43}$$

so that another differentiation gives

$$\frac{\mathrm{d}^2\mathbf{K}}{\mathrm{d}s^2}\cdot\mathbf{D} - \beta\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}s}\cdot\mathbf{B} = 0 \,, \tag{4.44}$$

where we have assumed the following:

$$\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}s} = T^\alpha\frac{\partial}{\partial x^\alpha}\mathbf{K} \,, \quad \frac{\mathrm{d}^2\mathbf{K}}{\mathrm{d}s^2} = T^\alpha T^\beta\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\mathbf{K} + CN^\alpha\frac{\partial}{\partial x^\alpha}\mathbf{K} \,. \tag{4.45}$$

Therefore, *once the force law* (4.37) *was assigned*, $(4.42)_2$, $(4.43)_1$ and (4.44) give the expression of the three curvatures of ℓ^+: C, τ, β as functions of m_0, $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, $\mathbf{D}$. In this sense, system (4.19), completed with $(4.42)_1$ and the additional equation $\mathrm{d}\Omega E/\mathrm{d}s = \mathbf{T}$, can be solved, allowing the determination of the world line ℓ^+ as well as the proper mass m_0, once initial conditions were fixed.

The problem is simplified if the external field $\mathbf{K}$ is constant; then, from $(4.43)_1$, it follows $\tau = 0$, so that ℓ^+ is flat, which is also derived directly from (4.38): $\mathbf{P} = m_0\mathbf{V} = \mathbf{K}\tau + \mathbf{P}_0$.

Finally, as concerns the *intrinsic equations of the relative motions with respect to a given Galilean frame*, we recall that the following fundamental relations hold

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \mathbf{F} \,, \qquad m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \,. \tag{4.46}$$

From this, at least in the case of particles without internal structure, $m_0 =$ const., we have the following equation:

$$m_\mathrm{t}\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\mathbf{v} + m_\mathrm{n}\mathbf{a} = \mathbf{F} \,, \tag{4.47}$$

where $m_{\rm t}$ and $m_{\rm n}$ are defined by

$$m_{\rm t} = \frac{m_0}{(1 - v^2/c^2)^{3/2}} \; , \qquad m_{\rm n} = \frac{m_0}{(1 - v^2/c^2)^{1/2}} \; , \tag{4.48}$$

and represent the *longitudinal and transversal mass*, respectively. The latter denomination comes from the intrinsic form of (4.47). In fact, decomposing $\mathbf{v}$ and $\mathbf{a}$ along $\mathbf{t}$ and $\mathbf{n}$, respectively (tangent and principal normal unit vectors), and introducing the curvature radius $r(s)$, (4.47) becomes

$$m_{\rm t} \ddot{s} \, \mathbf{t} + m_{\rm n} \frac{\dot{s}^2}{r} \mathbf{n} = \mathbf{F} \; , \tag{4.49}$$

and gives to $m_{\rm t}$ and $m_{\rm n}$ the meaning of *inertial mass* along the tangent and the principal normal, respectively. As in the classical case, the force $\mathbf{F}$ belongs to the osculating plane, so that, projecting on the principal triad, one gets the intrinsic equations:

$$m_{\rm t} \, \ddot{s} = F_{\rm t} \; , \qquad m_{\rm n} \frac{\dot{s}^2}{r} = F_{\rm n} \; , \qquad 0 = F_b \; . \tag{4.50}$$

Equations (4.50) are especially useful in the case of a fast particle, constrained (in a given Galilean frame) onto a fixed curve or surface; however, similarly to what happens for the absolute formulation, they are also useful for a free test particle. In this case, in fact, if the force $\mathbf{F}$ does not depend explicitly on the time: $\mathbf{F} = \mathbf{F}(P, \mathbf{v})$, (4.50), combined with the Frenet–Serret formulas, reduce the kinematical problem to a pure geometric one (the determination of the motion law, being sub-ordered to the resolution of a first-order differential equation for s).

4.5 Conservative Lorentz-like Forces

Let us consider now, in the special relativistic ambit, a class of Lorentz-like forces, i.e.

$$K_\alpha = F_{\alpha\beta} V^\beta \qquad (\alpha = 0, 1, 2, 3) \; , \tag{4.51}$$

being $F_{\alpha\beta}$ an antisymmetric tensor of rank 2: $F_{\alpha\beta} = -F_{\beta\alpha}$. These 4-forces are of mechanical type: $\mathbf{K} \cdot \mathbf{V} = K_\alpha V^\alpha = 0$, and exclude any possible thermal action: $q_0 = 0$; therefore, they imply the conservation of the proper mass of the particle, m_0.

Let us also assume that the tensor $F_{\alpha\beta}$ admits a potential, i.e. there exists a regular vectorial field $\boldsymbol{\phi}(E)$, such that

$$F_{\alpha\beta} = \partial_\alpha \phi_\beta - \partial_\beta \phi_\alpha \; . \tag{4.52}$$

If $\boldsymbol{\phi}$ is not lightlike (a case excluded here), we can decompose it in its modulus and direction: $\phi_\alpha = \phi \gamma_\alpha$, that is

$$\phi = \phi(E)\gamma(E) \ , \qquad ||\gamma|| = \pm 1 \ ; \tag{4.53}$$

it follows

$$F_{\alpha\beta} = \phi(\partial_\alpha \gamma_\beta - \partial_\beta \gamma_\alpha) + \partial_\alpha \phi \gamma_\beta - \partial_\beta \phi \gamma_\alpha \ , \tag{4.54}$$

and we have

$$K_\alpha = \phi(\partial_\alpha \gamma_\beta - \partial_\beta \gamma_\alpha) V^\beta + \partial_\alpha \phi (\boldsymbol{\gamma} \cdot \mathbf{V}) - \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\gamma_\alpha \qquad \left(\frac{\mathrm{d}}{\mathrm{d}\tau} = V^\alpha \partial_\alpha\right) . \tag{4.55}$$

In particular, let $\boldsymbol{\gamma}$ be uniform and timelike:

$$\boldsymbol{\gamma} = \text{const.}, \qquad ||\boldsymbol{\gamma}|| = -1 \ , \tag{4.56}$$

so that it represents a Galilean frame S_g. In this case, (4.55) simplifies to the following (intrinsic) form:

$$\mathbf{K} = (\boldsymbol{\gamma} \cdot \mathbf{V}) \, \mathrm{Grad}\, \phi \ , \qquad \mathrm{Grad}\, \phi \equiv \partial_\alpha \phi - \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\boldsymbol{\gamma} \ . \tag{4.57}$$

We notice that the force field $\mathbf{K}$ now depends on E (through ϕ) and $\mathbf{V}$ (and also from $\boldsymbol{\gamma}$); it is intrinsically conservative if and only if $\phi =$ constant, and in such a case, one has trivially $\mathbf{K} = 0$.

Let us examine, now, the mechanical force $\mathbf{F}$, relative to the reference frame associated with $\boldsymbol{\gamma}$ (S_g), introducing in M_4—without any loss of generality—Cartesian coordinates with $\boldsymbol{\gamma} = \mathbf{c}_0$, and the other three spatial axis belonging to the 3-space Σ, orthogonal to $\boldsymbol{\gamma}$ (i.e. coordinates adapted to S_g). We have then $\gamma^0 = 1$, $\gamma^i = 0$, $\gamma_0 = -1$, $\gamma_i = 0$ and, from (4.57),

$$\mathbf{K} = V_0 \, \mathrm{Grad}\, \phi - \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\boldsymbol{\gamma} \ , \qquad V_0 = -c\eta \ . \tag{4.58}$$

Thus, in S_g, the relative force, $\mathbf{F} = \mathbf{K}_\Sigma/\eta$ associated with $\mathbf{K}$, is given by

$$\mathbf{F} = -c \, \mathrm{grad}\, \phi \ , \qquad \mathrm{grad}\, \phi \equiv \partial_i \phi \ ; \tag{4.59}$$

it comes from a potential $\phi(E)$, depending on the space-time coordinates and hence, on the time coordinate of S_g too; therefore, $\mathbf{F}$ *is not conservative* in general, neither in the preferred frame S_g. It becomes conservative in S_g if and only if $\phi(E)$ does not depend on t; in this case, which will be considered in detail below, we have

$$\mathbf{F} = \mathrm{grad}\, U \ , \qquad U = -c\, \phi(x^1, x^2, x^3) \quad \longrightarrow \quad \mathcal{W} = \mathbf{F} \cdot \mathbf{v} = \dot{U} \ . \tag{4.60}$$

It is evident that the conservativity of the force $\mathbf{F}$ (and also the more general condition (4.59)) is strictly related to S_g, at least for the following two reasons:

1. passing from S_g to another frame S'_g the (Lorentz) transformation formulas of the coordinates involve the time too;

2. in the dynamical problem relative to S'_{g}, the physical action (4.60), even if it remains of mechanical type, is represented by a more general force $\mathbf{F}'$, as given by (3.60):

$$\mathbf{F}' = \frac{1}{\sigma}\left[\alpha\mathbf{F} - \frac{1}{c^2}\mathbf{F}\cdot\left(\mathbf{v} - \frac{\mathbf{u}}{1+\alpha}\right)\mathbf{u}\right] , \tag{4.61}$$

also *depending on the velocity*.

4.6 Central Forces in a Galilean Frame

In the context of the forces like (4.60), even if confined to a well-determined Galilean frame S_{g}, the central ones are characterized by the condition of being directed towards a fixed point $\mathrm{O} \in S_{\mathrm{g}}$, and with intensity only depending on the distance of the point P from O: $\rho = |\mathrm{OP}|$. In this case, one has $\phi = \phi(\rho)$, so that

$$F_i = \partial_i U = \frac{\partial U}{\partial \rho}\,\frac{x^i - x^i_{\mathrm{O}}}{\rho} ,$$

or, in terms of intrinsic quantities of the Galilean frame S_{g}:

$$\mathbf{F} = \frac{\partial U(\rho)}{\partial \rho}\,\frac{\mathrm{OP}}{\rho} , \quad \rho = \sqrt{\delta_{ik}(x^i - x^i_{\mathrm{O}})(x^k - x^k_{\mathrm{O}})} . \tag{4.62}$$

Such a force will be attractive (repulsive) if $\partial U/\partial\rho$ is negative (positive). Moreover, the equation of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \mathbf{F} , \tag{4.63}$$

implies the existence of the *energy integral*: $\mathcal{E} - U = H = \text{const.}$, that is:

$$mc^2 - U(\rho) = H, \qquad m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad (m_0, H = \text{const.}) ; \tag{4.64}$$

it follows that, once the total energy H is fixed, the accessible region for the motion is determined by $H + U(\rho) > 0$.

In a relativistic context too, the typical property of central motions survives, that is *the trajectory is planar* and *the area integral exists*. In fact, by taking the vector product by OP of (4.63), and using (4.62), one finds

$$\mathrm{OP} \times \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{OP} \times m\mathbf{v}) = 0 ,$$

so that a *first integral of the angular momentum* follows:

$$\mathrm{OP} \times m\mathbf{v} = m_0\mathbf{k} , \tag{4.65}$$

with $\mathbf{k}$ a constant vector:

$$m_0\mathbf{k} = \mathrm{OP}_0 \times (m\mathbf{v})_0 \ . \tag{4.66}$$

If $\mathbf{k} = 0$ (that is OP_0 parallel to $\mathbf{v}_0$ or, in particular, $\mathrm{P}_0 \equiv \mathrm{O}$ or $\mathbf{v}_0 = 0$), the motion is necessarily a straight line passing through O, and the law of motion is given by (4.63), projected on the radius OP. In fact, assuming

$$\mathbf{F} = F(\rho)\mathbf{u} \ , \quad F(\rho) = \frac{\mathrm{d}U}{\mathrm{d}\rho} \ , \quad \mathbf{u} = \text{vers OP} \ ,$$

and making explicit the dependence of m from the velocity, given by $(4.64)_2$, from (4.63) one gets the following scalar equation:

$$\frac{m_0\ddot{\rho}}{(1-\dot{\rho}^2/c^2)^{3/2}} = \frac{\mathrm{d}U}{\mathrm{d}\rho} \ ; \tag{4.67}$$

this equation can be, obviously, also obtained by using the energy integral $(4.64)_1$.

If $\mathbf{k} \neq 0$ (*general case*), from (4.65) it follows that the vector OP is always orthogonal to $\mathbf{k}$, and thus *the motion is planar* (determined by the initial values P_0 and $\mathbf{v}_0$), passing through the centre of the force O. Introducing in this plane a system of polar coordinates (ρ, θ), with origin in O, and assuming $\mathbf{c}_3$ aligned with $\mathbf{k}$, one gets the area first integral:

$$m\rho^2\dot{\theta} = m_0 k \ , \qquad \mathbf{k} = k\mathbf{c}_3 \ ; \tag{4.68}$$

in particular, it follows that θ, as a function of the time, increases monotonically, so that the particle will be *never at rest*. By eliminating m, using $(4.64)_1$ and (4.68), one gets

$$\dot{\theta} = \frac{k\mathcal{E}_0}{\rho^2[H+U(\rho)]} \ , \tag{4.69}$$

where $\mathcal{E}_0 = m_0c^2$ is the proper energy. Thus, once the trajectory $\rho = \rho(\theta)$ is known, the law of motion $\theta = \theta(t)$ follows by quadratures.

In this way, independently on the specification of the potential $U(\rho)$, the dynamical problem is reduced to the determination of the (planar) trajectory only. It is then convenient to use the first integrals $(4.64)_1$ and (4.69), in place of the motion (4.63). From the expression of the velocity in polar coordinates,

$$v^2 = \rho^2\dot{\theta}^2 + \dot{\rho}^2 = (\rho^2 + \rho'^2)\dot{\theta}^2 \ , \qquad \rho' = \frac{\mathrm{d}\rho}{\mathrm{d}\theta} \ ,$$

and using the condition (4.69), the square of the velocity follows:

$$v^2 = (\rho^2 + \rho'^2)\frac{k^2\mathcal{E}_0^2}{\rho^4[H+U(\rho)]^2} \ ; \tag{4.70}$$

thus, $(4.64)_1$ becomes

$$\frac{\mathcal{E}_0}{H+U(\rho)} = \sqrt{1 - \frac{k^2}{c^2}\mathcal{E}_0^2\frac{\rho^2+\rho'^2}{\rho^4[H+U(\rho)]^2}} \ ,$$

or

$$\mathcal{E}_0 = \sqrt{[H + U(\rho)]^2 - \frac{k^2}{c^2}\mathcal{E}_0^2 \left[\left(\frac{1}{\rho}\right)^2 + \left(\frac{1}{\rho}\right)'^2\right]} \,. \tag{4.71}$$

One finds, then, the *resolvent equation* of the trajectory; in fact, introducing the new variable, $\xi = 1/\rho$, and squaring both sides of (4.71), one finds

$$\xi'^2 + \xi^2 - \frac{c^2}{k^2\mathcal{E}_0^2}[H + U(\xi)]^2 + \frac{c^2}{k^2} = 0 \,. \tag{4.72}$$

From here, by differentiating with respect to θ, one gets the second-order differential equation of the dynamical trajectories:

$$\xi'' + \xi - \frac{c^2}{k^2\mathcal{E}_0^2}[H + U(\xi)]U'(\xi) = 0 \,, \tag{4.73}$$

which contains the circular trajectories $\xi = \text{const.}$

4.7 The Keplerian Case

Let us assume now that the potential U be of *Newtonian* type, i.e. with the field proportional to the proper mass of the particle,

$$U = f\frac{m_0 M_0}{\rho} = f m_0 M_0 \xi \,, \tag{4.74}$$

where f denotes the Newtonian gravitational constant. Thus m_0, together with its inertial meaning has also a gravitational significance: the hypothesis, less natural, of proportionality to the relative mass gives a different dynamical problem. Equation (4.74) describes, in a relativistic framework, a central gravitational field of Newtonian type (that is instantaneous), due to an isolated central body (the Sun, for example). In the hypothesis (4.74), the equation of the trajectories (4.73) becomes the classic one:

$$\xi'' + \omega^2\xi = \frac{\omega^2}{p} \,, \tag{4.75}$$

where

$$\omega^2 \stackrel{\text{def}}{=} 1 - \left(\frac{fM_0}{k\,c}\right)^2 , \qquad \frac{1}{p} \stackrel{\text{def}}{=} \frac{fHM_0}{k^2\mathcal{E}_0\,\omega^2} \,, \tag{4.76}$$

with the condition

$$1 - \left(\frac{fM_0}{kc}\right)^2 > 0 \,. \tag{4.77}$$

The solution of (4.75) can be written in the form:

$$\xi = A\cos(\omega\theta + \alpha) + \frac{1}{p} , \tag{4.78}$$

where, because of the arbitrariness of α, the constant A can be assumed positive; moreover, the constant H of the total energy, given in $(4.64)_1$, can also be assumed positive because of the presence of the term mc^2, in general bigger than U:

$$H = \frac{\mathcal{E}_0}{\sqrt{1 - v_0^2/c^2}} - U(\rho_0) > 0 . \tag{4.79}$$

By introducing the notation $A \overset{\text{def}}{=} e/p$, it follows from (4.78) that

$$\rho = \frac{p}{1 + e\cos(\omega\theta + \alpha)} , \tag{4.80}$$

where $e \geq 0$ and $\alpha \in [0, 2\pi)$ are constant, which are determined from the initial conditions. Finally, assuming θ having its zero value along the line joining O with the perihelium, it is $\alpha = 0$, and (4.80) reduces to

$$\rho = \frac{p}{1 + e\,\cos\omega\theta} . \tag{4.81}$$

This equation does not represent a conic because $\omega \neq 1$, and the orbit can be limited or not, according to the value of e: limited for $e < 1$, unlimited for $e \geq 1$. This circumstance is strictly related to the value of the total energy H, which we will write in the form:

$$H = E + \mathcal{E}_0 , \tag{4.82}$$

in analogy with the classical two bodies problem. Thus, (4.73) becomes

$$(\xi')^2 + \omega^2\xi^2 - 2\frac{\omega^2}{p}\xi = \frac{c^2}{k^2}\left(\frac{H^2}{\mathcal{E}_0^2} - 1\right) = \frac{E}{m_0k^2}\left(2 + \frac{E}{\mathcal{E}_0}\right) ,$$

and, by using (4.81), one gets

$$\frac{\omega^2}{p^2}(e^2 - 1) = \frac{E}{m_0k^2}\left(2 + \frac{E}{\mathcal{E}_0}\right) . \tag{4.83}$$

Moreover, using (4.82), $(4.76)_2$ becomes

$$\frac{1}{p} = \frac{fM_0}{k^2\omega^2}\left(1 + \frac{E}{\mathcal{E}_0}\right) , \tag{4.84}$$

so that (4.83) is equivalent to

$$\frac{fm_0M_0}{p}(e^2 - 1) = E\left(1 + \frac{1}{1 + E/\mathcal{E}_0}\right) , \tag{4.85}$$

or, introducing H and using (4.84) and $(4.76)_1$:

$$\frac{1-\omega^2}{\omega^2}(e^2-1) = 1 - \frac{\mathcal{E}_0^2}{E^2} \,. \tag{4.86}$$

Therefore, one has a criterion to discriminate among the orbits, by means of the "eccentricity" e, and the energy constant $H > 0$, taking into account that $1-\omega^2 > 0$:

$$\begin{cases} e < 1 & \text{when} \quad H < \mathcal{E}_0 \quad (-\mathcal{E}_0 < E < 0) \\ e = 1 & \text{when} \quad H = \mathcal{E}_0 \quad (E = 0) \\ e > 1 & \text{when} \quad H > \mathcal{E}_0 \quad (E > 0) \,. \end{cases} \tag{4.87}$$

Summarizing, if $e < 1$, (4.81) represents a curve bounded between the two circles, with centre O and radii $r = 1/(1+e)$ and $R = 1/(1-e)$, respectively (in particular, if $e = 0$, i.e. $H = \omega\mathcal{E}_0$, one has exactly a circle). Moreover, when the orbit intersects one of these circles, at the intersection point, one has $\xi' = 0$ (maximum or minimum distance from the centre O), that is, using (4.69):

$$\dot{\rho} \equiv \rho'\dot{\theta} = -\frac{k\mathcal{E}_0\xi'}{H+U(\xi)} = 0 \,;$$

as a consequence, the velocity is transversal, and the trajectory is tangent to the same circle.

Moreover, the advance of the perihelium, after two successive loops, is not 2π (as it would be for an elliptic orbit), but $2\pi/\omega > 2\pi$ (being $\omega < 1$), and the trajectory has the typical form of a *rosette* (see Fig. 4.1). If ω is rational, then the orbit is closed; in the opposite case, the trajectory is *dense* in the corona bounded by the two circles. This is a result of Minkowskian gravity, which finds its complete confirmation in general relativity in the relative formulation of the so-called exterior Schwarzschild problem, where we have the precession of the perihelium, for every freely gravitation particle. There, however, the trajectory equation is different because of the presence of a term like ξ^3 (see [6], p. 260). In the limit $c \to \infty$, one finds the classical case (see [2], p. 304, with $M = M_0$ and $m_* = m = m_0$).

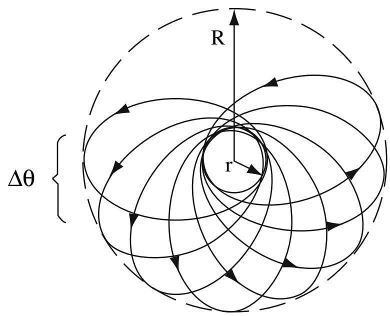

Fig. 4.1. Rosetta-like motion

4.8 Motion of Charged Particles in a Uniform Magnetic Field

Let us now consider another fundamental problem, that is the motion of a charged particle in a given electromagnetic field, as for example that of an accelerated particle in a synchrotron. The motion is considered with respect to a given Galilean frame and, hence, it is described by the fundamental relativistic equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \mathbf{F} \,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(mc^2) = \mathbf{F}\cdot\mathbf{v} + q \,, \tag{4.88}$$

where $m = m_0/\sqrt{1-v^2/c^2}$. The physical action $\mathbf{F}$ is due to the electromagnetic field and, as it is well known, it is purely mechanical: $q = 0$. Thus, the charged test particles we are considering have no any internal structure: $m_0 = \text{const.}$ and, if e denotes the *charge* of the particle, the mechanical force is the Lorentz one:

$$\mathbf{F} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) \,. \tag{4.89}$$

We will study the dynamical problem in the following hypothesis:

1. absence of the electric field: $\mathbf{E} = 0$.[2] Equation (4.88) then becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \frac{e}{c}\mathbf{v}\times\mathbf{H} \,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(mc^2) = \mathbf{F}\cdot\mathbf{v} = 0 \,; \tag{4.90}$$

whatever magnetic field were assigned in the "restricted " problem: $\mathbf{H} = \mathbf{H}(P, \mathbf{v}, t)$, the energy integral holds $mc^2 = \text{const.}$, that is the speed is constant:

$$v = \text{const.} = v_0 < c \,. \tag{4.91}$$

Thus, because of hypothesis 1, the particle's motion is uniform. The trajectory, instead, has no a priori limitations, but it is sub-ordered to the equation:

$$m\mathbf{a} = \frac{e}{c}\mathbf{v}\times\mathbf{H} \,, \tag{4.92}$$

with m constant:

$$m = \frac{m_0}{\sqrt{1-v_0^2/c^2}} \,. \tag{4.93}$$

From (4.92), by differentiation, follows, then, the relation

$$m\dot{\mathbf{a}} = \frac{e}{c}(\mathbf{a}\times\mathbf{H} + \mathbf{v}\times\dot{\mathbf{H}}) \,, \tag{4.94}$$

and we assume another hypothesis:

[2] The case $\mathbf{E} \neq 0$ is discussed in [7], p. 520.

2. the magnetic field is uniform: $\mathbf{H} =$ const. and without any loss of generality, aligned along the z-axis:

$$\mathbf{H} = H\mathbf{c}_3 \ , \qquad H > 0 \, \text{and} \, \text{const.} \tag{4.95}$$

It follows the existence of another first integral, that is the momentum with respect to the z-axis, being $\mathbf{F}$ orthogonal to the z-axis itself: $m\dot{z} =$ const., or

$$\dot{z} = \text{const.} = \dot{z}_0 \ . \tag{4.96}$$

Thus, if one assumes $\dot{z}_0 = 0$, that is $\mathbf{v}_0 \perp \mathbf{z}$, the motion is planar, and it is contained in the plane Π, defined by $(P_0, \mathbf{v}_0)$, and orthogonal to the z-axis. According to the above hypothesis, one has then

3. $\mathbf{v}_0$ is orthogonal to $\mathbf{H}$, and *the trajectory is a circle*. This can be see by projecting (4.92) onto the x–y plane: Π or, directly, from (4.94). In fact, from the hypothesis 2, it assumes the following form:

$$m\dot{\mathbf{a}} = \frac{e}{c}(\mathbf{a} \times \mathbf{H}) \ ,$$

so that the vectors $\mathbf{a}$ and $\dot{\mathbf{a}}$ are orthogonal to each other, and to $\mathbf{H}$, and the condition a =const. follows too.

Finally, in the hypothesis 1–3, the motion is uniform: $\dot{s} =$ const. ($\ddot{s} = 0$), planar, and the magnitude of the acceleration is also constant:

$$a^2 \equiv (\ddot{s})^2 + \left(\frac{v^2}{\rho}\right)^2 = \left(\frac{v^2}{\rho}\right)^2 = \text{const.} \ ;$$

thus, $1/\rho$ is const. and the orbit is a circle. We can determine the centre C and the radius R of the circle, starting from the initial conditions. The centre C is placed in Π, along the line orthogonal to $\mathbf{v}_0$ and passing for P_0. To determine it, let us assume polar coordinates, with origin in C, and the usual notation ρ and θ and conventions for the unit vectors $\mathbf{u}$ and $\boldsymbol{\tau}$. It results $\rho = R$, $\mathbf{v} = R\dot{\theta}\boldsymbol{\tau}$ and $\mathbf{a} = -R\dot{\theta}^2\mathbf{u}$; furthermore, $R\dot{\theta} = \text{const.} = \pm v$, and, from (4.92):

$$-mR\dot{\theta}^2 = \frac{e}{c}R\dot{\theta}H < 0 \ .$$

Thus, one finds that the sign of $\dot{\theta}$ is opposite to that of the charge e:

$$mR\dot{\theta} = -\frac{e}{c}RH \ , \quad v = R|\dot{\theta}| \ , \tag{4.97}$$

and finally,

$$R = \frac{cmv}{|e|H} \ , \tag{4.98}$$

or

$$R = \frac{m_0}{|e|}\frac{cv}{H\sqrt{1 - v^2/c^2}} \ . \tag{4.99}$$

Thus, if v, H and R (radius of the particle accelerator) are known, (4.99) allows to obtain the ratio between the charge and the proper mass of a particle:

$$\frac{|e|}{m_0} = \frac{c}{RH} \frac{v}{\sqrt{1 - v^2/c^2}} \, . \tag{4.100}$$

The analogous result, obtained using classical mechanics, in place of (4.99), corresponds to a lesser value:

$$R_c = \frac{c m_0 v}{|e| H} < R \, ,$$

and, obviously, the difference $R - R_c$ becomes bigger and bigger, as soon as that v approaches the speed of light. Equation (4.99) is in agreement with experiments, so that the use of the relativistic formula is essential in projecting an accelerating machine.

As a final remark, we notice that, because of the presence of c, in (4.99), the magnetic field H should be intense enough, in order that R has a reasonable value for the experimental device (and not of the order of km).

4.9 Extension of Maxwell's Equations to any Galilean Frame

As already stated in Sect. 1.4, Maxwell's theory can be formally extended, *in a classic context*, from the *heter* S^* to an arbitrary Galilean frame S', in motion with respect to S^*. Let $\mathcal{T}_0$ and $\mathcal{T}_\Omega$ be two orthonormal triads, in S^* and S', respectively, with unit vectors $\mathbf{c}_i$ and $\mathbf{c}'_i(t)$, and let $\mathbf{v}_\Omega$ and $\boldsymbol{\omega}$ be the kinematical characteristics of the motion of S' with respect to S^*.

Let us start from the formulation of the classical electromagnetism in vacuo, in the heter frame S^* that is the set of Maxwell's equations (1.15):

$$\begin{cases} \text{div}\, \mathbf{H} = 0 \, , \qquad \text{curl}\, \mathbf{E} + \dfrac{1}{c}\partial_t \mathbf{H} = 0 \, , \\ \text{div}\, \mathbf{E} = 4\pi\rho \, , \qquad \text{curl}\, \mathbf{H} - \dfrac{1}{c}\partial_t \mathbf{E} = \dfrac{4\pi}{c}\mathbf{J} \, . \end{cases} \tag{4.101}$$

Passing from S^* to S', one has to transform these equations using, for the generic point P, in place of the Cartesian coordinates x^i relative to $\mathcal{T}_0 \in S^*$, the analogous coordinates $x^{i'}$ relative to $\mathcal{T}_\Omega \in S'$. To this end, it is necessary to use the transformation formulas $x^i = x^i(t, x')$, obtained from the fundamental relation $\text{OP} = \text{O}\Omega(t) + x^{i'}\mathbf{c}'_i(t)$, with $\text{OP} = x^i \mathbf{c}_i$, that is:

$$x^i = \text{O}\Omega(t) \cdot \mathbf{c}^i + x^{k'} \mathbf{c}'_k(t) \cdot \mathbf{c}^i \, , \tag{4.102}$$

or the inverse relations:

$$x^{i'} = x^k \mathbf{c}_k \cdot \mathbf{c}^{i'}(t) - \mathrm{O}\Omega(t) \cdot \mathbf{c}^{i'} \ . \tag{4.103}$$

We will denote by $\mathbf{E}'$, $\mathbf{H}'$ the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ (both invariant) once expressed in terms of the time t and the variables $x^{k'}$, by using (4.102). The spatial operators div and curl have an invariantive property with respect to the transformations (4.102):

$$\begin{cases} \mathbf{c}^i \times \dfrac{\partial}{\partial x^i} = \mathbf{c}^{i'} \times \dfrac{\partial}{\partial x^{i'}} = \text{inv} \ . \\ \mathbf{c}^i \cdot \dfrac{\partial}{\partial x^i} = \mathbf{c}^{i'} \cdot \dfrac{\partial}{\partial x^{i'}} = \text{inv} \ .; \end{cases} \tag{4.104}$$

the temporal derivative, instead, has not an invariantive meaning:

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{\partial x^{i'}}{\partial t}\frac{\partial}{\partial x^{i'}} \ . \tag{4.105}$$

Moreover, according to (4.103) and using the Poisson formulas for the rigid kinematics, one has

$$\frac{\partial x^{i'}}{\partial t} = -\mathbf{W} \cdot \mathbf{c}^{i'} \quad (i = 1, 2, 3) \ , \tag{4.106}$$

where

$$\mathbf{W} \stackrel{\text{def}}{=} \mathbf{v}_\Omega(t) + \boldsymbol{\omega}(t) \times \Omega P = \mathbf{W}(t, P) \ ; \tag{4.107}$$

in addition, for an arbitrary vector field $\mathbf{v}(t, x)$, the following general decomposition holds

$$\frac{\partial \mathbf{v}}{\partial x^i} = \frac{1}{2}\mathrm{curl}\, \mathbf{v} \times \mathbf{c}_i + \boldsymbol{\sigma}_i^{(v)} \ , \tag{4.108}$$

having assumed

$$\boldsymbol{\sigma}_i^{(v)} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial \mathbf{v}}{\partial x^i} + \mathrm{grad}\, v_i\right) \ . \tag{4.109}$$

Thus, by using (4.105), one gets the following transformation formula:

$$\frac{\partial \mathbf{v}(t, x)}{\partial t} = \frac{\partial \mathbf{v}'(t, x')}{\partial t} - \frac{1}{2}\mathrm{curl}\, \mathbf{v}' \times \mathbf{W} - \boldsymbol{\sigma}_W^{(v')} \ , \tag{4.110}$$

where, following (4.109):

$$\boldsymbol{\sigma}_W^{(v')} \stackrel{\text{def}}{=} W^i \boldsymbol{\sigma}_i^{(v')} = \frac{1}{2}\Big(\partial_W \mathbf{v}' + \mathrm{grad}\mathbf{v}' \cdot \mathbf{W}\Big) \ . \tag{4.111}$$

At this point, the transformation of (4.101) follows immediately, if one takes into account that $\mathbf{E}$, $\mathbf{H}$ and ρ have an absolute meaning, while the current density $\mathbf{J} = \rho\mathbf{v}$, because of the theorem of addition of velocities, becomes

$$\mathbf{J} = \mathbf{J}' + \rho \mathbf{W} \, . \tag{4.112}$$

Therefore, Maxwell's equations in the Galilean frame S' assume the form:

$$\begin{cases} \operatorname{div} \mathbf{H} = 0 \, , & \operatorname{curl} \mathbf{E} + \dfrac{1}{c}\partial_t \mathbf{H} = \Delta_H \, , \\ \operatorname{div} \mathbf{E} = 4\pi\rho \, , & \operatorname{curl} \mathbf{H} - \dfrac{1}{c}\partial_t \mathbf{E} = \dfrac{4\pi}{c}(\mathbf{J}' + \rho\mathbf{W}) - \Delta_E \, , \end{cases} \tag{4.113}$$

where, following (4.111), we have introduced the notation:

$$\Delta_v = \frac{1}{c}\left(\frac{1}{2}\operatorname{curl} \mathbf{v} \times \mathbf{W} + \boldsymbol{\sigma}_W^{(v)}\right) \, , \tag{4.114}$$

for every vector field $\mathbf{v}$, and the vector $\mathbf{W}$ has the kinematical meaning defined in (4.107). Equation (4.113) extends the formulation of classical electromagnetism to an arbitrary frame S', and it shows clearly the noninvariantive content of Maxwell's equations when changing the heter frame S^*: in fact, new terms appear at the right-hand side of $(4.101)_{2,4}$, and these terms identically vanish, whatever the electromagnetic field be, if and only if $\mathbf{W} = 0$, i.e. $\mathbf{v}_\Omega = 0$ and $\omega = 0$, so that $S' \equiv S^*$.

If $S' \neq S^*$, such terms are always present: apparent electromagnetic current densities, analogous to the inertial forces (dragging and Coriolis) of classical mechanics. Clearly, they are really present and can be measured in S', but they disappear in the heter frame S^*. The situation, in M_4, will be completely different (as we will see in Chap. 9) since (4.101) will result instead formally invariant.

It is also clear that, even in the formulation (4.113) of Maxwell's equations, the heter frame still plays a special role; in fact, $\mathbf{W}(t, P)$ is directly connected to the motion of S' with respect to S^*. In particular, if S' is in rectilinear uniform motion with respect to S^*, it results $\mathbf{W} = \mathbf{u} = \text{const.}$

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5

Relativistic Kinematics for a Three-Dimensional Continuum

5.1 Continuum Mechanics. Relative Representation of the Motion

We study here the relativistic mechanics of a continuum,[1] adopting a different point of view from that for a material point. In the latter case the absolute formulation of kinematics and dynamics was considered before the relative one. In the present case, instead, it is convenient to begin with the classical point of view in terms of an arbitrary Galilean frame $S_{\rm g}$ and then pass to the relativistic extension in M_4, that is, to the absolute formulation.

Because of the relative aspect of our treatment, we assume that the underlying reference space is the ordinary Euclidean manifold E_3 (to which a Galilean frame $S_{\rm g}$ is superposed) endowed with the natural topology. The mathematical scheme of the continuum, including both geometrical and kinematical aspects, is obtained by considering connected and bounded subsets C of E_3 (generally variable with the time). Their evolution, in $S_{\rm g}$, is described by the vectorial function

$$\mathrm{OP} = \mathrm{OP}(t, y^1, y^2, y^3)\ , \qquad \mathrm{P} \in C\ , \tag{5.1}$$

where y^i $(i = 1, 2, 3)$ are three curvilinear coordinates which label the generic particle of the continuum at each instant $t \in (t_0, t_1)$. They are often called "label coordinates," or "Lagrangian coordinates," because they label the particle itself.

Once a Cartesian triad $\mathcal{T}$ in $S_{\rm g}$ is fixed (see Fig. 5.1), the Lagrangian coordinates can be interpreted as (curvilinear) coordinates of a point P_* representative of the particle. Varying the particle, i.e. the parameters y^i, P_* describes a three-dimensional configuration field C_* or the *reference configuration* of the

[1] The continuum scheme will not be justified a priori from a statistical point of view, by using probabilistic considerations or limiting processes on the point particle scheme.

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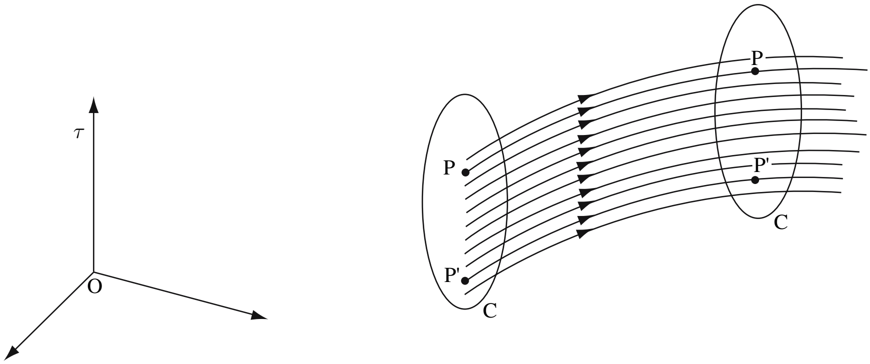

Fig. 5.1. Evolution of a generic continuum in a Galilean frame S_g

system. The configuration field C of the positions of the ∞^3 particles of the system at the generic instant t is called instead the *actual* (or *instantaneous*) *configuration* of the system.

Clearly the Lagrangian coordinates y^i are not uniquely defined, but only up to an invertible transformation

$$y^i = y^i(y') \ , \tag{5.2}$$

which does not involve time. We will assume that

$$A \stackrel{\text{def}}{=} \det \left\| \frac{\partial y^i}{\partial y'^k} \right\| > 0 \ , \tag{5.3}$$

which is more restrictive than $A \neq 0$ and ensures the local invertibility of (5.2). Such a limitation introduces a well-determined orientation in the continuum. For all $t \in (t_0, t_1)$, (5.1) are the (vectorial) parametric equations of the actual configuration C: thus, (5.3) corresponds, in the three-dimensional case, to the condition $\mathrm{d}\lambda/\mathrm{d}\lambda' > 0$, which characterizes the admissible parameters of an oriented curve for the case of a single material point.

The vectorial function (5.1) and the corresponding scalar equations

$$x^i = x^i(t, y^1, y^2, y^3) \ , \qquad (i = 1, 2, 3) \tag{5.4}$$

are assumed to be sufficiently regular (even C^∞), with respect to all four variables and, in particular, invertible for each $t \in (t_0, t_1)$:

$$y^i = y^i(t, x^1, x^2, x^3) \ ; \tag{5.5}$$

this implies that, at each instant, one has a bijective map between the points of the reference configuration and those of the actual one. In other words, two different particles M and M' of the continuum (corresponding to different values of y^i) remain distinct throughout the motion. This does not mean that the spatial trajectories of two particles cannot intersect each other, but only that two particles cannot occupy the same position at the same instant; that is, they must have nonintersecting world lines.

As a consequence, during the evolution, there cannot be collisions or breaks and every loop or closed surface of C_* remains closed at every instant t. This follows from the fundamental requirement that each element of the continuum has its own individuality and cannot be destroyed. Hence, we will not consider any process of matter annihilation or generation.

Equation (5.4) define the position at any instant t of each particle (i.e. for fixed values of y^i). For fixed t, they instead describe the actual configuration of the system parametrically, in terms of y^i. Thus, (5.4) give the motion of the whole system, particle by particle, varying the four parameters y^i and t. This is the so-called *Lagrangian point of view*, which discusses the dynamical characteristics of the system as a function of the particle and time, and hence it assumes $y^i \in C_*$ and $t \in (t_0, t_1)$ to be independent variables.

Conversely, the functions (5.5), which are equivalent to (5.4) but assume $x^i \in C$ and $t \in (t_0, t_1)$, correspond to the *Eulerian point of view*, a different description of motion according to which the kinematical ingredients are the point P and the time t. In fact, (5.5) give, at each instant t, the label of the particle which, at that moment, occupies the point P in C.

We notice that, in the Eulerian description, the variables x^i are defined in a domain (C) which is not fixed in the considered Galilean frame, but varies with t. However, for the points of C, either the Cartesian coordinates x^i or the curvilinear coordinates y^i are admissible. Thus to preserve the orientation of the continuum, from (5.3), the following limitation holds:

$$\mathcal{D} \stackrel{\text{def}}{=} \det \left\| \frac{\partial x^i}{\partial y^k} \right\| > 0 \,. \tag{5.6}$$

We will denote by $\mathbf{e}_i$ the derivatives of the vectorial function (5.1) with respect to the parameters y^i:

$$\mathbf{e}_i \stackrel{\text{def}}{=} \frac{\partial \mathrm{OP}(t,y)}{\partial y^i} \,, \qquad (i = 1,2,3) \,. \tag{5.7}$$

These are three linearly independent vectors because, for each P and t, the relation

$$\alpha^i \mathbf{e}_i = 0 \,, \qquad \mathbf{e}_i = \frac{\partial x^k}{\partial y^i} \mathbf{c}_k \tag{5.8}$$

implies $\alpha^i = 0$. In fact, (5.8) is equivalent to

$$\alpha^i \frac{\partial \mathrm{OP}}{\partial y^i} = \alpha^i \frac{\partial x^k}{\partial y^i} \mathbf{c}_k = 0 \,,$$

i.e. it corresponds to the linear and homogeneous system

$$\alpha^i \frac{\partial x^k}{\partial y^i} = 0 \,, \qquad (k = 1,2,3) \,,$$

whose solution is necessarily $\alpha^i = 0$ because of the limitation (5.6).

The vectors $\mathbf{e}_i$, defined at each point $\mathrm{P} \in C$ by (5.7), form the so-called *natural basis*, depending on both the considered instant and the particle: $\mathbf{e}_i = \mathbf{e}_i(t, y)$. This basis, exactly like the Cartesian one $\mathbf{c}_k$, can be used to decompose a vector applied at $\mathrm{P} \in C$ (P being the position of the particle P_*, at the instant t).

The *Cartesian representation* (in terms of $\mathbf{c}_k$) as well as the *natural representation* (in terms of $\mathbf{e}_k$) are equivalent, and one can consider the transformation laws for the change of basis. From (5.7) one has

$$\mathbf{e}_i = \frac{\partial x^k}{\partial y^i} \mathbf{c}_k \tag{5.9}$$

and the inverse relations

$$\mathbf{c}_k = \frac{\partial y^i}{\partial x^k} \mathbf{e}_i \ . \tag{5.10}$$

The scalar product in the region C represented, in Cartesian terms, by the Kronecker tensor, $\delta_{ik} = \mathbf{c}_i \cdot \mathbf{c}_k$, defines the *Lagrangian metric*:

$$g_{ik} \stackrel{\text{def}}{=} \mathbf{e}_i \cdot \mathbf{e}_k \ , \qquad (i, k = 1, 2, 3) \ . \tag{5.11}$$

In fact, from (5.9) one has the tensorial relation

$$g_{ik} = \frac{\partial x^l}{\partial y^i} \frac{\partial x^m}{\partial y^k} \delta_{lm} \ , \tag{5.12}$$

implying

$$g \stackrel{\text{def}}{=} \det ||g_{ik}|| = \mathcal{D}^2 > 0 \ . \tag{5.13}$$

5.2 Fundamental Kinematical Fields

Differently from the vectors $\mathbf{e}_i$ and the Lagrangian metric g_{ik}, which have a geometrical meaning only in the actual configuration C, the temporal derivatives

$$\mathbf{v} = \frac{\partial \mathrm{OP}}{\partial t} \ , \qquad \mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial^2 \mathrm{OP}}{\partial t^2} \ , \tag{5.14}$$

have a kinematical meaning, i.e. they represent the *velocity* and the *acceleration* of the generic particle of the continuum with respect to S_{g}, in terms of intrinsic quantities. From (5.1) it follows that (5.14) define $\mathbf{v}$ and $\mathbf{a}$ as functions of y^i and t, that is, in Lagrangian form. The corresponding Eulerian form is obtained by substituting the y^i by using (5.5), *once the differentiations are performed.*[2] Decomposing $\mathbf{v}$ and $\mathbf{a}$ with respect to the basis $\mathbf{e}_i$ and

[2] For a generic function f of the coordinates and time we will often use the notation $f(t, y)$ in place of $f(t, y^1, y^2, y^3)$ or $f(t, x)$ in place of $f(t, x^1, x^2, x^3)$.

$\mathbf{c}_k$, one obtains the natural and Cartesian components, respectively. We have the following relations:

$$\mathbf{v} = v^i \mathbf{e}_i \equiv \dot{x}^k \mathbf{c}_k \ , \qquad \left(\dot{x}^k = \partial_t x^k(t, y) \ , \quad \partial_t \stackrel{\text{def}}{=} \frac{\partial}{\partial t} \right) \ ,$$

and, using (5.10) in the last term, one gets

$$v^i \mathbf{e}_i = \dot{x}^k \frac{\partial y^i}{\partial x^k} \mathbf{e}_i \ ,$$

from which, because of the independence of the vectors $\mathbf{e}_i$, we have

$$v^i = \frac{\partial y^i}{\partial x^k} \dot{x}^k \ . \tag{5.15}$$

Clearly, as v^i are functions of y^k and t, the derivatives $\partial y^i / \partial x^k$, as well as $\dot{x}^k$, should be thought of as depending on these variables, too; that is, one must substitute the x^i by using (5.4), *once the derivatives have been performed*.

Similarly, denoting by $\ddot{x}^k$ the Cartesian components of the acceleration, one has the relations

$$\dot{x}^k = \partial_t x^k(t, y) \ , \qquad \ddot{x}^k = \partial_{tt} x^k(t, y) \ , \tag{5.16}$$

so that

$$\mathbf{a} = a^i \mathbf{e}_i \ , \quad a^i = \frac{\partial y^i}{\partial x^k} \ddot{x}^k \ . \tag{5.17}$$

Commonly, the Cartesian components of the velocity $\dot{x}^k$, expressed in terms of x^i and t by means of (5.5), are called *Eulerian velocities* and are denoted by e^k:

$$e^k \stackrel{\text{def}}{=} \left[\partial_t x^k \right]_{y^i = y^i(t,x)} \ . \tag{5.18}$$

Similarly, the Cartesian components of the acceleration, according to (5.16), are the partial derivatives, with respect to time, of the Cartesian components of the velocity: $\ddot{x}^k = \partial_t \dot{x}^k$. To see the relation between the components of these quantities with respect to the natural basis we proceed as follows. On the one hand, one has $\mathbf{a} = a^i \mathbf{e}_i$ and on the other, by definition, $\mathbf{a} = \partial_t \mathbf{v}$; thus

$$a^i \mathbf{e}_i = \partial_t (v^i \mathbf{e}_i) = \partial_t v^i \mathbf{e}_i + v^i \partial_t \mathbf{e}_i = \partial_t v^i \mathbf{e}_i + v^i h_i{}^k \mathbf{e}_k = (\partial_t v^i + v^k h_k{}^i) \mathbf{e}_i \ ,$$

where we have used the notation

$$\partial_t \mathbf{e}_i \stackrel{\text{def}}{=} h_i{}^k \mathbf{e}_k \ ; \tag{5.19}$$

hence we have

$$a^i = \partial_t v^i + h_k{}^i v^k \ , \qquad (i = 1, 2, 3) \ . \tag{5.20}$$

The quantities h_{ik} appearing in (5.20) arise from the decomposition of the vectors $\partial_t \mathbf{e}_k$, according to the natural basis $\mathbf{e}_i$. Since the vectors $\mathbf{e}_i = \mathbf{e}_i(t, y)$

depend on time, it follows that the tensor h_{ik} is, in general, nonvanishing, and the relations (5.20) do not coincide with the analogous Cartesian relations: $a^i \neq \partial_t v^i$. Before specifying better the meaning of h_{ik}, it is convenient to briefly summarize some results already obtained for the metric tensor $m_{\alpha\beta}$ of M_4, in terms of g_{ik}.

First of all, the basis $\mathbf{e}_i$ is not orthonormal, so that the scalar products $g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k$ are different from the Kronecker tensor δ_{ik}. However, the matrix g_{ik} is regular because of (5.13), and it can be inverted; denoting the inverse by g^{ik}, we have

$$g^{ih} g_{hk} = \delta^i_k \ . \tag{5.21}$$

It is therefore, meaningful to consider the dual basis $\mathbf{e}^i$ of the basis $\mathbf{e}_i$, obtained raising the index with the metric g^{ik}:

$$\mathbf{e}^i = g^{ik} \mathbf{e}_k \qquad \sim \qquad \mathbf{e}_i = g_{ik} \mathbf{e}^k \ . \tag{5.22}$$

The following fundamental *duality relation*, obtained using (5.21), holds:

$$\mathbf{e}^i \cdot \mathbf{e}_k = \delta^i_k \ ; \tag{5.23}$$

similar to (5.11), the *contravariant metric* is given by

$$g^{ik} = \mathbf{e}^i \cdot \mathbf{e}^k \ . \tag{5.24}$$

As for $m_{\alpha\beta}$ or $m^{\alpha\beta}$, in S_{g}, the metric g_{ik} or g^{ik} can be used to raise and lower the indices of tensor components, giving rise to different but *equivalent representations*: *covariant* (with lowered indices) or *contravariant* (with raised indices) and also *mixed*. In particular, the scalar product of two vectors $\mathbf{v}$ and $\mathbf{w}$, in S_{g}, can be expressed as

$$\mathbf{v} \cdot \mathbf{w} = g_{ik} v^i w^k = v_k w^k = v^k w_k = g^{ik} v_i w_k \ . \tag{5.25}$$

Let us now consider the fundamental relation (5.19), associated with the decomposition of the vectors $\dot{\mathbf{e}}_i$ (where a dot replaces the partial derivative ∂_t). These vectors, because of the definition (5.7), coincide with the gradient of velocity $\mathbf{v}$ with respect to the coordinates y^i:

$$\dot{\mathbf{e}}_i = \partial_i \mathbf{v}(t, y) \ , \qquad (\partial_i = \partial / \partial y^i) \ . \tag{5.26}$$

It follows immediately that $\dot{\mathbf{e}}_i \cdot \mathbf{e}_k = h_i{}^h \mathbf{e}_h \cdot \mathbf{e}_k = h_i{}^h g_{hk}$, from which we find the meaning of the covariant components $h_{ik} = h_i{}^h g_{hk}$:

$$h_{ik} = \dot{\mathbf{e}}_i \cdot \mathbf{e}_k = \partial_i \mathbf{v} \cdot \mathbf{e}_k \ , \qquad (i, k = 1, 2, 3) \ . \tag{5.27}$$

The last expression shows that h_{ik} is a 2-tensor in the sense that an arbitrary change in the label coordinates $y^i = y^i(y')$ transforms h_{ik} according to the typical law of tensors:

$$h_{ik} = \frac{\partial y'^l}{\partial y^i} \frac{\partial y'^m}{\partial y^k} h'_{lm} \ . \tag{5.28}$$

In fact, from (5.7), one has

$$\mathbf{e}_i = \frac{\partial \mathrm{OP}}{\partial y^i} = \frac{\partial \mathrm{OP}}{\partial y'^k}\frac{\partial y'^k}{\partial y^i} ,$$

and hence the transformation law of the vectors $\mathbf{e}_i$:

$$\mathbf{e}_i = \frac{\partial y'^k}{\partial y^i}\mathbf{e}'_i \qquad \sim \qquad \mathbf{e}'_k = \frac{\partial y^i}{\partial y'^k}\mathbf{e}_i \ . \tag{5.29}$$

Equation (5.27) then gives rise to the relation

$$h_{ik} = \frac{\partial \mathbf{v}}{\partial y'^l}\frac{\partial y'^l}{\partial y^i} \cdot \frac{\partial y'^m}{\partial y^k}\mathbf{e'}_m = \frac{\partial y'^l}{\partial y^i}\frac{\partial y'^m}{\partial y^k}h'_{lm} ,$$

i.e. (5.28). Moreover, (5.29) clarifies the meaning of the limitation (5.3): the two natural basis, $\mathbf{e}_i$ and $\mathbf{e'}_i$, associated with the coordinates y^i and y'^i, respectively, have the same orientation (both left-handed or both right-handed) and this is equivalent to selecting, for the continuum, one of the two possible orientations.

5.3 Shear and Vorticity

The tensor $h_i{}^k$ or $h_{ik} = h_i{}^j g_{jk}$ defined in (5.19):

$$\dot{\mathbf{e}}_i = h_{ik}\mathbf{e}^k \equiv \partial_i \mathbf{v} , \tag{5.30}$$

summarizes the two fundamental kinematical elements of a continuum: the *deformation velocity* or *shear* and the *angular velocity*. More precisely, let us denote the symmetric part of h_{ik} by k_{ik} and the antisymmetric part by ω_{ik}, that is:

$$h_{ik} = k_{ik} + \omega_{ik} , \tag{5.31}$$

where

$$\begin{cases} k_{ik} \stackrel{\text{def}}{=} h_{(ik)} \equiv \frac{1}{2}(h_{ik} + h_{ki}), \\ \omega_{ik} \stackrel{\text{def}}{=} h_{[ik]} \equiv \frac{1}{2}(h_{ik} - h_{ki}). \end{cases} \tag{5.32}$$

For the tensor k_{ik}, by using (5.27), one gets

$$k_{ik} = \frac{1}{2}(\dot{\mathbf{e}}_i \cdot \mathbf{e}_k + \dot{\mathbf{e}}_k \cdot \mathbf{e}_i) = \frac{1}{2}\partial_t(\mathbf{e}_i \cdot \mathbf{e}_k) ,$$

that is:

$$k_{ik} = \frac{1}{2}\partial_t g_{ik} . \tag{5.33}$$

This tensor takes into account the temporal variation of the metric $g_{ik}(t, y)$, for each particle, and it is called *deformation velocity tensor*. The tensor ω_{ik}, also termed *vorticity tensor*, has instead the meaning of *angular velocity*. To justify this meaning, let us consider the vector

$$\boldsymbol{\omega} = \frac{1}{2}\omega_{ik}\mathbf{e}^i \times \mathbf{e}^k \ . \tag{5.34}$$

Now since $k_{ik}\mathbf{e}^i \times \mathbf{e}^k = 0$ (k_{ik} is symmetric, and the product $\mathbf{e}^i \times \mathbf{e}^k$ is antisymmetric), one has, using (5.31),

$$\boldsymbol{\omega} = \frac{1}{2}(k_{ik} + \omega_{ik})\mathbf{e}^i \times \mathbf{e}^k = \frac{1}{2}h_{ik}\mathbf{e}^i \times \mathbf{e}^k \ .$$

Therefore, from (5.30), the vector $\boldsymbol{\omega}$ can also be expressed in the form

$$\boldsymbol{\omega} = \frac{1}{2}\mathbf{e}^i \times \partial_t \mathbf{e}_i \ . \tag{5.35}$$

Equation (5.35) is similar to the formula for angular velocity in rigid motion, expressed in terms of an orthonormal triad $\{i_a\}$ (at rest with respect to the moving rigid body) and its derivatives, with respect to time (see [1], p. 125):

$$\boldsymbol{\omega} = \frac{1}{2}i^a \times \frac{\mathrm{d}i_a}{\mathrm{d}t} \ .$$

The fundamental difference between the two relations is that, in the case of a continuum, the natural basis depends on the coordinates y^i (which also leads to the use of partial derivatives in place of the total derivative with respect to time); in other words, at least as concerns changes in the direction, the continuous system behaves as if each particle were a *rigid microsystem*, with respect to a generic (not orthonormal) triad.

Furthermore, because of the identity $\dot{\mathbf{e}}_i = \partial_i \mathbf{v}$, the vector $\boldsymbol{\omega}(t, y)$, defined in (5.35), also has a direct meaning in terms of the velocity field:

$$\boldsymbol{\omega} = \frac{1}{2}\mathbf{e}^i \times \partial_i \mathbf{v} \equiv \frac{1}{2}\mathrm{curl}\, \mathbf{v} \ . \tag{5.36}$$

Using a terminology common in fluid mechanics, $\boldsymbol{\omega}$ represents the *local vortex* of the continuum.

The vector $\boldsymbol{\omega}$ is of course *invariant*, in the sense that it depends only on the considered particle and time, but not on the choice of Lagrangian coordinates:

$$\boldsymbol{\omega} = \frac{1}{2}\omega_{ik}\mathbf{e}^i \times \mathbf{e}^k = \mathrm{inv} = \frac{1}{2}\omega'_{ik}\mathbf{e}'^i \times \mathbf{e}'^k \ . \tag{5.37}$$

In particular, in terms of the Cartesian components $\overset{(c)}{\omega}_{ik}$, one has

$$\boldsymbol{\omega} = \frac{1}{2}\overset{(c)}{\omega}_{ik}\, \mathbf{c}^i \times \mathbf{c}^k \ , \tag{5.38}$$

where

$$\overset{(c)}{\omega}_{ik} = \frac{\partial y^l}{\partial x^i}\frac{\partial y^m}{\partial x^k}\omega_{lm} \ . \tag{5.39}$$

Note that in the last relation it is implicitly assumed that the natural components ω_{lm} are expressed in terms of the coordinates t and x, by means of (5.5). Moreover, from (5.36), the expression for $\boldsymbol{\omega}$, as function of the Eulerian velocity $\mathbf{e} = e^k\mathbf{c}_k$ defined in (5.18), is the following:

$$\boldsymbol{\omega} = \frac{1}{2}\mathrm{curl}\,\mathbf{e} = \frac{1}{2}\mathbf{c}^i \times \frac{\partial \mathbf{e}}{\partial x^i} \ . \tag{5.40}$$

In contrast to the antisymmetric tensor ω_{ik}, which can be represented by the vector $\boldsymbol{\omega}$ (the dual of ω_{ik}, i.e. $\omega_k = 1/2\eta^{kij}\omega_{ij}$, see Chap. 2), the deformation velocity tensor has six (and not three) independent components. It is directly related to the Lagrangian metric g_{ik} by (5.33) and, like $\boldsymbol{\omega}$, it can be expressed in terms of the velocity field $\mathbf{v}$:

$$k_{ik} = \frac{1}{2}(\partial_i\mathbf{v}\cdot\mathbf{e}_k + \partial_k\mathbf{v}\cdot\mathbf{e}_i) \ . \tag{5.41}$$

Equation (5.41) is also valid for the Cartesian components $\overset{(c)}{k}_{ik}$:

$$\overset{(c)}{k}_{ik} = \frac{1}{2}\left(\frac{\partial e_k}{\partial x^i} + \frac{\partial e_i}{\partial x^k}\right), \qquad e_i = \delta_{ik}e^k \ , \tag{5.42}$$

which, in turn, are related to the natural components by the tensorial relation

$$k_{ik} = \frac{\partial x^l}{\partial y^i}\frac{\partial x^m}{\partial y^k}\overset{(c)}{k}_{lm} \ . \tag{5.43}$$

Finally, it is worth mentioning that there are no algebraic relations between the two kinematical quantities ω_{ik} (*angular velocity*) and k_{ik} (*deformation velocity*); however, these quantities are not independent from a differential point of view. More precisely, the gradient of $\boldsymbol{\omega}$ is a function of the first spatial derivatives of the deformation velocity. In fact,

$$\partial_i\boldsymbol{\omega} = \frac{\partial\boldsymbol{\omega}}{\partial x^k}\frac{\partial x^k}{\partial y^i} \ , \tag{5.44}$$

and by introducing, from (5.40), the components of the Eulerian velocity with respect to the Cartesian basis $\mathbf{e} = e_h\mathbf{c}^h$, we have

$$\frac{\partial\boldsymbol{\omega}}{\partial x^k} = \frac{1}{2}\mathbf{c}^i \times \frac{\partial^2\mathbf{e}}{\partial x^i\partial x^k} = \frac{1}{2}\frac{\partial^2 e_h}{\partial x^i\partial x^k}\mathbf{c}^i \times \mathbf{c}^h \ .$$

Thus, taking into account the identity

$$\frac{\partial^2 e_k}{\partial x^i\partial x^h}\mathbf{c}^i \times \mathbf{c}^h = 0$$

then leads to

$$\frac{\partial \boldsymbol{\omega}}{\partial x^k} = \frac{1}{2}\left(\frac{\partial^2 e_h}{\partial x^i \partial x^k} + \frac{\partial^2 e_k}{\partial x^i \partial x^h}\right) \mathbf{c}^i \times \mathbf{c}^h = \frac{1}{2}\frac{\partial}{\partial x^i}\left(\frac{\partial e_h}{\partial x^k} + \frac{\partial e_k}{\partial x^h}\right) \mathbf{c}^i \times \mathbf{c}^h \ ,$$

and using (5.42), we have

$$\frac{\partial \boldsymbol{\omega}}{\partial x^k} = \frac{\partial}{\partial x^i} \overset{(c)}{k}_{hk} \ \mathbf{c}^i \times \mathbf{c}^h \ . \tag{5.45}$$

From this relation it immediately follows that if at a certain instant t the deformation velocity vanishes everywhere: $\overset{(c)}{k}_{hk} = 0$, $\forall \mathrm{P} \in C$, *at that instant, the angular velocity is constant in* C[3]:

$$\overset{(c)}{k}_{hk} = 0, \qquad \forall \mathrm{P} \in C_t \quad \Rightarrow \quad \boldsymbol{\omega} = \text{const.}, \qquad \forall \mathrm{P} \in C_t \ . \tag{5.46}$$

If the condition is satisfied everywhere at any instant, the angular velocity depends only on t: $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$, and the motion is necessarily rigid. This justifies the name of deformation velocity for the tensor k_{ik}.

We will see, in the next section, how (5.45) is modified passing from the Cartesian to the curvilinear coordinates y^i.

5.4 Christoffel Symbols and Covariant Derivative

Let us start from the tensor h_{ik}, defined in (5.27):

$$h_{ik} = \partial_i \mathbf{v} \cdot \mathbf{e}_k \ , \qquad (i, k = 1, 2, 3) \ , \tag{5.47}$$

which, with the symmetric and antisymmetric parts, respectively, gives rise to the deformation velocity and the angular velocity:

$$\begin{cases} k_{ik} \overset{\text{def}}{=} h_{(ik)} = \frac{1}{2}(\partial_i \mathbf{v} \cdot \mathbf{e}_k + \partial_k \mathbf{v} \cdot \mathbf{e}_i), \\ \omega_{ik} \overset{\text{def}}{=} h_{[ik]} = \frac{1}{2}(\partial_i \mathbf{v} \cdot \mathbf{e}_k - \partial_k \mathbf{v} \cdot \mathbf{e}_i). \end{cases} \tag{5.48}$$

In terms of Cartesian coordinates, $\mathbf{v}$ becomes the Eulerian velocity $\mathbf{e}(t, x) = e^k \mathbf{c}_k$, so that (5.48) assumes the form

$$\overset{(c)}{k}_{ik} = \frac{1}{2}\left(\frac{\partial e_k}{\partial x^i} + \frac{\partial e_i}{\partial x^k}\right) , \qquad \overset{(c)}{\omega}_{ik} = \frac{1}{2}\left(\frac{\partial e_k}{\partial x^i} - \frac{\partial e_i}{\partial x^k}\right) . \tag{5.49}$$

To see how the relations (5.49) are modified passing from the Cartesian to the Lagrangian coordinates y^i, we consider the gradient

[3] This is the case if the deformation is *homogeneous*, that is, the components $\overset{(c)}{k}_{hk}$ do not depend on the position in C.

$$\partial_i \mathbf{v} = \partial_i (v^k \mathbf{e}_k) = \partial_i v^k \mathbf{e}_k + v^k \partial_i \mathbf{e}_k \tag{5.50}$$

and, hence, the second derivatives

$$\partial_i \mathbf{e}_k = \frac{\partial^2 \mathrm{OP}}{\partial y^i \partial y^k} \; . \tag{5.51}$$

These are linear combinations of $\mathbf{e}_h$:

$$\partial_i \mathbf{e}_k = \Gamma^h{}_{ik} \mathbf{e}_h = \Gamma_{ik,h} \mathbf{e}^h \; , \tag{5.52}$$

where the three index coefficients $\Gamma^h{}_{ik}$, or their alternatives $\Gamma_{ik,h}$, obtained by lowering the index h:

$$\Gamma_{ik,l} = g_{hl} \Gamma^h{}_{ik} \; , \tag{5.53}$$

denote the contravariant and covariant components of the gradient $\partial_i \mathbf{e}_k$, respectively:

$$\Gamma^h{}_{ik} = \partial_i \mathbf{e}_k \cdot \mathbf{e}^h \; , \qquad \Gamma_{ik,h} = \partial_i \mathbf{e}_k \cdot \mathbf{e}_h \; . \tag{5.54}$$

These coefficients can be expressed by means of the metric g_{ik} and its first derivatives as follows:

$$\Gamma_{ik,h} = \frac{1}{2} (\partial_i g_{kh} + \partial_k g_{hi} - \partial_h g_{ik}) \; , \tag{5.55}$$

which identifies them as the ordinary *first-type Christoffel symbols*. In fact, one has

$$\partial_i g_{hk} = \partial_i (\mathbf{e}_k \cdot \mathbf{e}_h) = \partial_i \mathbf{e}_k \cdot \mathbf{e}_h + \partial_i \mathbf{e}_h \cdot \mathbf{e}_k \; ,$$

and hence

$$\partial_i g_{hk} = \Gamma_{ik,h} + \Gamma_{ih,k} \; , \tag{5.56}$$

giving the derivatives of g_{ik} as functions of the coefficients $\Gamma_{ik,h}$. Conversely, one can invert the relations (5.56), obtaining (5.55). In fact, cyclic permutation of the indices i, h, k, in (5.56), leads to

$$\partial_k g_{hi} = \Gamma_{kh,i} + \Gamma_{ki,h}, \quad \partial_h g_{ik} = \Gamma_{hi,k} + \Gamma_{hk,i} \; ;$$

next, adding the first of these to (5.56) and subtracting the second, (5.55) follows immediately.

The Christoffel symbols, especially the *second-type Christoffel symbols*

$$\Gamma^h{}_{ik} = g^{hl} \Gamma_{ik,l} = \frac{1}{2} g^{hl} (\partial_i g_{kl} + \partial_k g_{li} - \partial_l g_{ik}) \; , \tag{5.57}$$

play an important role in the Lagrangian differentiation of tensorial functions. Taking into account (5.52), the gradient of $\mathbf{v}$ (5.50) assumes the following form:

$$\partial_i \mathbf{v} \stackrel{\text{def}}{=} (\nabla_i v^k)\mathbf{e}_k \ , \tag{5.58}$$

where

$$\nabla_i v^k \stackrel{\text{def}}{=} \partial_i v^k + \Gamma^k{}_{ih} v^h \ . \tag{5.59}$$

In contrast to the case of Cartesian coordinates, $\partial_i \mathbf{v}$ is no longer represented by the simple partial derivative $\partial_i v^k$, but it is necessary to introduce a *differential operator* ∇_i which depends on the Christoffel symbols. The latter operator has an *absolute meaning*, in the sense that, as follows from (5.58), the 2-index quantity $\nabla_i v^k$ has tensorial behaviour under a change of the Lagrangian coordinates y^i:

$$\nabla_i v^k = \frac{\partial y'^h}{\partial y^i} \frac{\partial y^k}{\partial y'^l} \nabla'_h v'^l \ . \tag{5.60}$$

It is called *covariant derivative* of $\mathbf{v}$.

If for the vector $\mathbf{v}$ one considers the decomposition $\mathbf{v} = v_k \mathbf{e}^k$, one needs—in place of (5.52)—the derivatives of the dual basis $\mathbf{e}^k$, which, because of (5.23), can be written as:

$$\partial_i \mathbf{e}^k = -\Gamma^k{}_{ih} \mathbf{e}^h \ . \tag{5.61}$$

This leads to an analog of (5.58):

$$\partial_i \mathbf{v} = (\nabla_i v_k)\mathbf{e}^k \ , \tag{5.62}$$

where

$$\nabla_i v_k \stackrel{\text{def}}{=} \partial_i v_k - \Gamma^h{}_{ik} v_h \ . \tag{5.63}$$

By comparing (5.58) and (5.62), one gets

$$(\nabla_i v^k)\mathbf{e}_k = (\nabla_i v_h)\mathbf{e}^h \ ,$$

that is, using (5.22), the relations

$$(\nabla_i v^k) = g^{hk} \nabla_i v_h \qquad \sim \qquad (\nabla_i v_h) = g_{hk} \nabla_i v^k \ . \tag{5.64}$$

Importantly, *the metric behaves like a constant under covariant differentiation*:

$$\nabla_i g^{hk} = 0 \quad \sim \quad \nabla_i g_{hk} = 0 \quad \sim \quad \nabla_i \delta^k_h = 0 \ , \tag{5.65}$$

as follows easily from the definitions

$$\begin{cases} \nabla_i g^{hk} = \partial_i g^{hk} + \Gamma^h{}_{il} g^{lk} + \Gamma^k{}_{il} g^{hl}, \\[2ex] \nabla_i g_{hk} = \partial_i g_{hk} - \Gamma^l{}_{ih} g_{lk} - \Gamma^l{}_{ik} g_{hl}, \\[2ex] \nabla_i \delta^k_h = -\Gamma^l{}_{ih} \delta^k_l + \Gamma^k{}_{il} \delta^l_h, \end{cases} \tag{5.66}$$

which are *extensions* of (5.59) and (5.63) to the case of tensors with several indices. Thus, we can formulate the following *general rule*. Passing from Cartesian (x^i) to Lagrangian coordinates (y^i), *for any tensorial object, the partial derivative must be replaced with the covariant derivative*:

$$\partial_i \to \nabla_i \ . \tag{5.67}$$

In this way (5.49) become

$$k_{ik} = \frac{1}{2}(\nabla_i v_k + \nabla_k v_k) \ , \quad \omega_{ik} = \frac{1}{2}(\nabla_i v_k - \nabla_k v_i) \ . \tag{5.68}$$

Furthermore, because of the symmetry of the Christoffel symbols, with respect to the lower indices:

$$\Gamma^h{}_{ik} = \Gamma^h{}_{ki} \ , \qquad \Gamma_{ik,h} = \Gamma_{ki,h} \ , \tag{5.69}$$

Equation $(5.68)_2$ can be cast in the equivalent form:

$$\omega_{ik} = \frac{1}{2}(\partial_i v_k - \partial_k v_i) \ . \tag{5.70}$$

Similarly, the differential tensorial relation (5.45) becomes

$$\partial_k \boldsymbol{\omega} = \nabla_i k_{hk} \mathbf{e}^i \times \mathbf{e}^k \ , \tag{5.71}$$

where

$$\nabla_i k_{hk} = \partial_i k_{hk} - \Gamma^l{}_{ih} k_{lk} - \Gamma^l{}_{ik} k_{hl} \ . \tag{5.72}$$

Note.

- The deformation velocity can also be expressed by using (5.33): $k_{ik} = 1/2 \partial_t g_{ik}$, which clearly has a tensorial meaning for arbitrary transformations of the coordinates y^i that do not involve time. In fact, g_{ik} transforms as

$$g_{ik} = \frac{\partial y'^l}{\partial y^i} \frac{\partial y'^m}{\partial y^k} g'_{lm} \ , \tag{5.73}$$

  which implies

$$k_{ik} = \frac{\partial y'^l}{\partial y^i} \frac{\partial y'^m}{\partial y^k} k'_{lm} \ , \qquad k'_{lm} = \frac{1}{2} \partial_t g'_{lm} \ .$$

  However, the above discussion cannot be repeated when passing from the Cartesian coordinates to the Lagrangian ones, which involves time. In other words, to (5.33) does not correspond the Cartesian analog

$$\overset{(c)}{k}{}_{ik} = 1/2 \partial_t \delta_{ik} \equiv 0,$$

  but (5.49) instead. In fact, from (5.33), by using (5.12) in place of (5.73), we have

$$\begin{aligned} k_{ik} &= \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial x^l}{\partial y^i} \frac{\partial x^m}{\partial y^k} \delta_{lm} \right) \\ &= \frac{1}{2} \left[\frac{\partial}{\partial y^i} \left(\frac{\partial x^l}{\partial t} \right) \frac{\partial x^m}{\partial y^k} \delta_{lm} + \frac{\partial x^l}{\partial y^i} \frac{\partial}{\partial y^k} \left(\frac{\partial x^m}{\partial t} \right) \delta_{lm} \right] \ . \end{aligned}$$

At this point, to get the Cartesian components, it is enough to identify the y^i with the x^i, using (5.5) when necessary; from (5.18) we find

$$\overset{(c)}{k}_{ik} = \frac{1}{2}\left[\frac{\partial}{\partial x^i}\left(\frac{\partial x^l}{\partial t}\right)_{y=y(t,x)} \delta^m_k \delta_{lm} + \delta^l_i \frac{\partial}{\partial x^k}\left(\frac{\partial x^m}{\partial t}\right)_{y=y(t,x)} \delta_{lm}\right]$$
$$= \frac{1}{2}\left(\frac{\partial e^l}{\partial x^i}\delta_{kl} + \frac{\partial e^m}{\partial x^k}\delta_{im}\right) ,$$

that is, $(5.49)_1$.

- The second-type Christoffel symbols $\Gamma^i{}_{jk}$ associated with the coordinates y^i through the metric g_{ik} and its first derivatives *do not transform as tensorial quantities*; in fact, as it can be easy checked from (5.73), we have

$$\Gamma^h{}_{ik} = \frac{\partial y'^l}{\partial y^i}\frac{\partial y'^m}{\partial y^k}\frac{\partial y^h}{\partial y'^n}\Gamma'^n{}_{lm} + \frac{\partial^2 y'^m}{\partial y^i \partial y^k}\frac{\partial y^h}{\partial y'^l} , \tag{5.74}$$

where $\Gamma'^n{}_{lm}$ are the coefficients analogous to $\Gamma^n{}_{lm}$:

$$\Gamma'^n{}_{lm} = \frac{1}{2} g'^{np}\left(\frac{\partial g'_{mp}}{\partial y'^l} + \frac{\partial g'_{pl}}{\partial y'^m} - \frac{\partial g'_{lm}}{\partial y'^p}\right) . \tag{5.75}$$

For linear transformations of the Lagrangian coordinates y^i, (5.74) reduces to a tensorial law: $\partial^2 y'^l/\partial y^i \partial y^k \equiv 0$ and the coefficients $\Gamma^j{}_{ik}$ are said to be an *affine tensor*. Contrasting to the first-type Christoffel symbols, the coefficients $\Gamma^j{}_{ik}$ form a well-determined geometrical entity Γ, in the sense that the transformation laws (5.74):

1. allow one to determine the components $\Gamma^j{}_{ik}$ of Γ, relative to the coordinates y^i, once the analogous $\Gamma'^j{}_{ik}$ relative to the coordinates y'^i are known, together with the map of the coordinate change.
2. form a group, that is, the following property holds: *for three admissible coordinate systems y^i, y'^i and y''^i, the transformation $\Gamma'' \to \Gamma$ resulting from* (5.74) *coincides with the product of the two transformations: $\Gamma'' \to \Gamma'$ and $\Gamma' \to \Gamma$ and each inverse transformation is a transformation of the same type.*

Finally, the Cartesian form of the Christoffel symbols follows from (5.57), with $y^i = x^i$ and $g_{ik} = \delta_{ik}$:

$$\overset{(c)}{\Gamma}{}^h{}_{ik} \equiv 0 ; \tag{5.76}$$

hence, from (5.74), with $y'^i = x^i$, one gets

$$\Gamma^h{}_{ik} = \frac{\partial^2 x^l}{\partial y^i \partial y^k}\frac{\partial y^h}{\partial x^l} . \tag{5.77}$$

Equation (5.77) can be checked in two ways, either starting from (5.51) or by using (5.57). In fact, from (5.51) and by using $\mathrm{OP} = x^l \mathbf{c}_l$, we find

$$\partial_i \mathbf{e}_k = \Gamma^h{}_{ik}\mathbf{e}_h = \frac{\partial^2 x^l}{\partial y^i \partial y^k}\mathbf{c}_l \ , \tag{5.78}$$

which is equivalent to (5.51) because of (5.10); analogously, from (5.57), by using (5.12) and the dual relation

$$g^{hl} = \frac{\partial y^h}{\partial x^p}\frac{\partial y^l}{\partial x^q}\delta^{pq} \ ,$$

one has

$$\begin{aligned}\Gamma^h{}_{ik} = \frac{1}{2}\frac{\partial y^h}{\partial x^p}\frac{\partial y^l}{\partial x^q}\delta^{pq}\Bigg[&\frac{\partial^2 x^r}{\partial y^i \partial y^k}\frac{\partial x^s}{\partial y^l}\delta_{rs} + \frac{\partial x^r}{\partial y^k}\frac{\partial^2 x^s}{\partial y^l \partial y^i}\delta_{rs} \\ &+\frac{\partial^2 x^r}{\partial y^k \partial y^l}\frac{\partial x^s}{\partial y^l}\delta_{rs} + \frac{\partial^2 x^s}{\partial y^i \partial y^k}\frac{\partial x^r}{\partial y^l}\delta_{rs} - \frac{\partial^2 x^r}{\partial y^l \partial y^i}\frac{\partial x^s}{\partial y^k}\delta_{rs} \\ &-\frac{\partial^2 x^s}{\partial y^k \partial y^l}\frac{\partial x^r}{\partial y^i}\delta_{rs}\Bigg] = \frac{\partial y^h}{\partial x^q}\delta^s_q\frac{\partial^2 x^s}{\partial y^i \partial y^k} \ ,\end{aligned}$$

which coincides with (5.77).

5.5 Local Analysis of the Motion of a Continuum

The meaning of the kinematical quantities introduced in Sect. 5.3 easily follows by analysing the velocity field of the continuum in the instantaneous configuration C. In fact, let us consider, first, the vectorial function (5.1) and assume that the time t is fixed. A first-order Taylor expansion gives

$$\mathrm{OQ} = \mathrm{OP} + \frac{\partial \mathrm{OP}}{\partial y^i}\Delta y^i + \mathrm{O}(2), \qquad \Delta y^i \overset{\text{def}}{=} y^i_{\mathrm{Q}} - y^i \ ,$$

that is, by using the notation of (5.7):

$$\mathrm{PQ} = \mathbf{e}_i \Delta y^i + \mathrm{O}(2) \ ; \tag{5.79}$$

by applying the same procedure to the velocity function $(5.14)_1$, with fixed t, we have

$$\mathbf{v}_{\mathrm{Q}} = \mathbf{v}_{\mathrm{P}} + \partial_i \mathbf{v}\Delta y^i + \mathrm{O}(2) \ . \tag{5.80}$$

Let us now use the following decomposition of $\partial_i \mathbf{v}$, obtained from (5.30) and (5.31):

$$\partial_i \mathbf{v} = k_{ik}\mathbf{e}^k + \omega_{ik}\mathbf{e}^k = \mathbf{k}_i + \boldsymbol{\omega}_i \ , \tag{5.81}$$

where

$$\mathbf{k}_i \overset{\text{def}}{=} k_{ik}\mathbf{e}^k \tag{5.82}$$

and

$$\boldsymbol{\omega}_i \overset{\text{def}}{=} \omega_{ik}\mathbf{e}^k \equiv \boldsymbol{\omega} \times \mathbf{e}_i . \tag{5.83}$$

From the definition (5.37), the duality relations (5.23) and by using the anti-symmetry of ω_{ik} we then find

$$\begin{aligned}\boldsymbol{\omega} \times \mathbf{e}_i &= \frac{1}{2}\omega_{hk}(\mathbf{e}^h \times \mathbf{e}^k) \times \mathbf{e}_i = \frac{1}{2}\omega_{hk}\left[(\mathbf{e}^h \cdot \mathbf{e}_i)\mathbf{e}^k - (\mathbf{e}^k \cdot \mathbf{e}_i)\mathbf{e}^h\right] \\ &= \frac{1}{2}\omega_{hk}(\delta_i^h \mathbf{e}_k - \delta_i^k \mathbf{e}^h) = \frac{1}{2}(\omega_{ik}\mathbf{e}^k - \omega_{hi}\mathbf{e}^h) = \omega_{ik}\mathbf{e}^k \,.\end{aligned}$$

Thus (5.81) becomes

$$\partial_i \mathbf{v} = \mathbf{k}_i + \boldsymbol{\omega} \times \mathbf{e}_i \,, \tag{5.84}$$

and the approximate formula (5.80), using (5.79), gives the following first-order relation:

$$\mathbf{v}_{\mathrm{Q}} = \mathbf{v}_{\mathrm{P}} + \boldsymbol{\omega} \times \mathrm{PQ} + \mathbf{k}_i \Delta y^i + \mathrm{O}(2) \,. \tag{5.85}$$

The relation (5.85), although only a first approximation,[4] is *invariant with respect to the choice of the Lagrangian coordinates.* This is obvious for all the terms except the last; however, it is easy to show that

$$\mathbf{k}_i \Delta y^i = \mathbf{k}'_i \Delta y'^i = \text{inv.} \tag{5.86}$$

In fact, $\mathbf{k}_i$, as follows from (5.82), form a set of vectors labelled by the index i following a covariant transformation law (exactly as $\mathbf{e}_i$). To show this let us start from the transformation law

$$k_{ik} = \frac{\partial y'^l}{\partial y^i}\frac{\partial y'^m}{\partial y^k} k'_{lm} \,; \tag{5.87}$$

contracting both sides by $\mathbf{e}^k$, one gets

$$k_{ik}\mathbf{e}^k = \frac{\partial y'^l}{\partial y^i}\frac{\partial y'^m}{\partial y^k} k'_{lm}\mathbf{e}^k = \frac{\partial y'^l}{\partial y^i} k'_{lm}\mathbf{e}'^m \,,$$

and hence

$$\mathbf{k}_i = \frac{\partial y'^l}{\partial y^i}\mathbf{k}'_l \,, \tag{5.88}$$

which proves the relation (5.86). Comparing (5.85) with the fundamental formula for rigid kinematics

$$\mathbf{v}_{\mathrm{Q}} = \mathbf{v}_{\mathrm{P}} + \boldsymbol{\omega} \times \mathrm{PQ} \,, \tag{5.89}$$

one immediately recognizes some substantial differences. Equation (5.89) is exact, while (5.85) is only approximate; in (5.89), $\boldsymbol{\omega}$ has a global meaning for the entire rigid body, because it only depends on time. Conversely, in (5.85), $\boldsymbol{\omega}$ has a local meaning, because it also depends on the particle: $\boldsymbol{\omega} = \boldsymbol{\omega}(t, y)$;

[4] The second approximation requires only differentiations and the use of the relation (5.71).

finally, in (5.85), there is an extra term, due to deformations which, like $\boldsymbol{\omega}$, has a local meaning and is absent for the case of the rigid body.

The condition $k_{ik} = 0$, $\forall \mathrm{P} \in C$, not only reduces (5.85) to the form (5.89), but, as we have already seen, it implies the constancy of $\boldsymbol{\omega}$ in C. Moreover, the last term in (5.85) can be seen as the gradient of a quadratic form. More precisely, writing, for the sake of brevity, $\Delta y^i = y^i_{\mathrm{Q}} - y^i \equiv \xi^i$ and introducing the (homogeneous) function

$$K(\xi) = \frac{1}{2} k_{ik} \xi^i \xi^k \ ,$$

we find

$$\mathbf{k}_i \Delta y^i = k_{ik} \mathbf{e}^k \xi^i \equiv \mathrm{grad}_\xi K(\xi) \ ,$$

with k_{ik} depending only on y^i and t, and not on ξ^i. Thus, (5.85) can also be written in the form

$$\mathbf{v}_{\mathrm{Q}} = \mathbf{v}_{\mathrm{P}} + \boldsymbol{\omega} \times \mathrm{PQ} + \mathrm{grad}_\xi K(\xi) + \mathrm{O}(2) \ , \tag{5.90}$$

and has a direct Lagrangian meaning, because the parameters y^i and $y^i + \Delta y^i$ denote two distinct particles of the continuum, the positions of which are the points P and Q, respectively. Clearly, (5.85) characterizes the *instantaneous velocity distribution* of the continuum, in the neighbourhood of the arbitrary point $\mathrm{P} \in C$. The Eulerian form of (5.85) is instead given by

$$\mathbf{e}_{\mathrm{Q}} = \mathbf{e}_{\mathrm{P}} + \boldsymbol{\omega} \times \mathrm{PQ} + \overset{(c)}{\mathbf{k}}_i \ \Delta x^i + \mathrm{O}(2) \ , \tag{5.91}$$

where now $\mathrm{PQ} = \Delta x^i \mathbf{c}_i$ and $\mathbf{e}_{\mathrm{P}}(t, x)$ replaces $\mathbf{v}_{\mathrm{P}}(t, y)$ after eliminating y with the aid of (5.5).

Finally, as concerns the term $\mathbf{k}_i \Delta y^i$, for any fixed $i = 1, 2, 3$, the vector $\mathbf{k}_i$ can be interpreted as the deformation velocity, at $\mathrm{P} \in C$, along the coordinate line $y^i =$ var. because it can be obtained from the sum $\mathbf{k}_j \Delta y^j$, assuming all the $\Delta y = 0$, except for $\Delta y^i = 1$. The vectors $\mathbf{k}_i$ can clearly replace completely the tensor k_{ik}. For instance, the compatibility conditions for the angular velocity, given in (5.71), can be written as

$$\partial_k \boldsymbol{\omega} = \mathbf{e}^i \times (\partial_i \mathbf{k}_k - \partial_k \mathbf{k}_i) \ . \tag{5.92}$$

Moreover, from (5.61), one has $\partial_i \mathbf{k}_k = (\partial_i k_{hk} - k_{kl} \Gamma^l{}_{ih}) \mathbf{e}^h$, so that (5.92) becomes (by using the symmetry properties of the Christoffel symbols)

$$\begin{aligned} \mathbf{e}^i \times (\partial_i \mathbf{k}_k - \partial_k \mathbf{k}_i) &= (\partial_i k_{kh} - \Gamma^l{}_{ih} k_{kl} - \partial_k k_{ih} + \Gamma^l{}_{kh} k_{il}) \mathbf{e}^i \times \mathbf{e}^h \\ &= (\partial_i k_{kh} - \Gamma^l{}_{ik} k_{hl}) \, \mathbf{e}^i \times \mathbf{e}^h = \nabla_i k_{kh} \, \mathbf{e}^i \times \mathbf{e}^h \ . \end{aligned}$$

Similarly, the temporal derivative of the angular velocity, because of (5.30), becomes

$$\partial_t \mathbf{e}^i = -h_k{}^i \mathbf{e}^k \ , \tag{5.93}$$

and, from (5.36), one gets

$$\partial_t \boldsymbol{\omega} = \frac{1}{2}\operatorname{curl}\mathbf{a} - \frac{1}{2}h_k{}^i\mathbf{e}^k \times h_{ij}\mathbf{e}^j \ .$$

Hence, since $h_{ij} = h_{ji} - 2\omega_{ji}$ and $\omega_{ij}\mathbf{e}^j = \boldsymbol{\omega} \times \mathbf{e}_i$, we have

$$\frac{1}{2}h_k{}^i h_{ij}\mathbf{e}^k \times \mathbf{e}^j = -2h_k{}^i\omega_{ji}\mathbf{e}^k \times \mathbf{e}^j = -2k_k{}^i\omega_{ji}\mathbf{e}^k \times \mathbf{e}^j = 2\mathbf{k}^i \times (\boldsymbol{\omega} \times \mathbf{e}_i) \ ;$$

thus one obtains the general formula

$$\partial_t \boldsymbol{\omega} = \frac{1}{2}\operatorname{curl}\mathbf{a} - k\boldsymbol{\omega} + \mathbf{k}_\omega \ , \tag{5.94}$$

where k is the *cubic deformation velocity* and $\mathbf{k}_\omega$ the deformation velocity along $\boldsymbol{\omega}$:

$$k = \mathbf{k}^i \cdot \mathbf{e}_i = g^{ik}k_{ik} \ , \qquad \mathbf{k}_\omega = \omega^i \mathbf{k}_i \ . \tag{5.95}$$

Equation (5.94) shows that the temporal derivative of $\boldsymbol{\omega}$ is uniquely determined by the acceleration $\mathbf{a}$ and the characteristics of the continuum.

5.6 Passing from One Galilean Frame to Another

The above description of the motion of a continuum is valid, either in the classical framework or in the relativistic one, as long as the discussion is limited to a single Galilean frame. In fact, as concerns the "spatial aspect", the relativistic geometry in a given Galilean frame coincides with the classical one; furthermore, within a single Galilean frame, the time is an absolute quantity in special relativity also. As a consequence, *if no more than one frame is involved*, one would not expect differences between classical and relativistic kinematics. But in the relativistic context there are differences in the transformation laws of the various relative quantities (of kinematics or dynamics), when passing from one reference frame to another. This is true for the single material point (as we have already seen) and also for the continuum (as we will see presently). The reason for such a different behaviour is that, while in the classical situation the passage from one Galilean frame S_g to another S'_g (assumed to be in x-standard relation, without any loss of generality), is governed by the *Galilei transformations*

$$x' = x - ut \ , \quad y' = y \ , \quad z' = z \ , \quad t' = t \ , \tag{5.96}$$

in the relativistic context, one has instead the *Lorentz transformations*

$$x' = \frac{1}{\alpha}(x - ut) \ , \quad y' = y \ , \quad z' = z \ , \quad t' = \frac{1}{\alpha}\left(t - \frac{u}{c^2}x\right) \ , \tag{5.97}$$

where $\alpha = \sqrt{1 - u^2/c^2}$. In both (5.96) and (5.97), once the Lagrangian coordinates y^i are fixed, one can pass from the motion relative to S_g: $x^i = x^i(t, y)$ to that relative to S'_g:

$$x'^i = x'^i(t', y) \ . \tag{5.98}$$

In the relativistic case (5.97) the time t' is not invariant but depends on the considered particle instead:

$$t' = \frac{1}{\alpha}\left(t - \frac{u}{c^2}x(t,y)\right) = t'(t,y) \ , \tag{5.99}$$

and conversely,

$$t = t(t', y) \ . \tag{5.100}$$

In the case of a single material point we have already seen the differential relation:

$$\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{\sigma}{\alpha} \ , \tag{5.101}$$

where

$$\sigma = 1 - \frac{1}{c^2}\mathbf{u}\cdot\mathbf{v} \ . \tag{5.102}$$

Here the same relation holds with the ordinary derivatives with respect to time replaced by partial derivatives and recalling that now $\mathbf{v} = \mathbf{v}(t, y)$.

Let us start considering the transformation law for the quantity (5.6):

$$\mathcal{D} = \det\left\|\frac{\partial x^i}{\partial y^k}\right\| = \mathcal{D}(t,y) \ ; \tag{5.103}$$

classically $\mathcal{D}$ is invariant with respect to the choice of Galilean frame, but not with respect to Lagrangian coordinates y^i. Let us evaluate then the derivatives of the x'^i (given by (5.98)) with respect to y^k; using $(5.97)_{1,2,3}$ with t expressed by (5.100), (5.99) reduces to an identity: $t' = t'$; hence $\partial t'/\partial y^k = 0$:

$$0 = \frac{1}{\alpha}\left[\frac{\partial t}{\partial y^k} - \frac{u}{c^2}\left(\frac{\partial x}{\partial y^k} + \frac{\partial x}{\partial t}\frac{\partial t}{\partial y^k}\right)\right] \ . \tag{5.104}$$

It follows that

$$\begin{aligned}
\frac{\partial x'^1}{\partial y^k} &= \frac{1}{\alpha}\left(\frac{\partial x^1}{\partial y^k} + \frac{\partial x^1}{\partial t}\frac{\partial t}{\partial y^k} - u\frac{\partial t}{\partial y^k}\right) , \\
\frac{\partial x'^{2,3}}{\partial y^k} &= \frac{\partial x^{2,3}}{\partial y^k} + \frac{\partial x^{2,3}}{\partial t}\frac{\partial t}{\partial y^k} \ ,
\end{aligned} \tag{5.105}$$

with $\partial t/\partial y^k$ derived from (5.104). Introducing the Cartesian velocity with respect to S_g,

$$\dot{x}^i = \frac{\partial x^i}{\partial t} \quad \rightarrow \quad \mathbf{v} = \dot{x}^i\mathbf{c}_i \ , \tag{5.106}$$

as well as the natural basis,

$$\mathbf{e}_i = \frac{\partial x^l}{\partial y^i}\mathbf{c}_l \ , \tag{5.107}$$

one gets the relation

$$\sigma \frac{\partial t}{\partial y^k} = \frac{u}{c^2}\frac{\partial x^1}{\partial y^k} = \frac{1}{c^2}\mathbf{u}\cdot\mathbf{e}_k \ . \tag{5.108}$$

Thus, (5.105) become

$$\begin{cases} \dfrac{\partial x'^1}{\partial y^k} = \dfrac{1}{\alpha\sigma}\dfrac{\partial x^1}{\partial y^k}\left(\sigma + \dfrac{1}{c^2}u\dot{x}^1 - \dfrac{u^2}{c^2}\right) = \dfrac{\alpha}{\sigma}\dfrac{\partial x^1}{\partial y^k}, \\[2ex] \dfrac{\partial x'^{2,3}}{\partial y^k} = \dfrac{\partial x^{2,3}}{\partial y^k} + \dfrac{1}{c^2\sigma}u\dfrac{\partial x^1}{\partial y^k}\dot{x}^{2,3}. \end{cases} \tag{5.109}$$

Hence, rewriting α/σ in the form

$$\frac{\alpha}{\sigma} = 1 + \frac{1}{c^2\sigma}u\dot{x}^1 + \nu = 1 + \frac{1-\sigma}{\sigma} + \nu \ ,$$

implying

$$\nu = \frac{\alpha - 1}{\sigma} = -\frac{1}{c^2}\frac{u^2}{\sigma(1+\alpha)} \ , \tag{5.110}$$

Equation (5.109) can be made more compact as follows:

$$\frac{\partial x'^i}{\partial y^k} = \frac{\partial x^i}{\partial y^k} + \frac{1}{c^2\sigma}\mathbf{u}\cdot\mathbf{e}_k\dot{x}^i + \nu\delta^i_1\frac{\partial x^1}{\partial y^k} \ ;$$

after contracting by $\mathbf{c}_i$ and using the relation $\delta^i_1\mathbf{c}_i = \mathbf{c}_1 = \mathbf{u}/u$, one has the corresponding vectorial relation valid in S_{g}:

$$\mathbf{e}'_k = \mathbf{e}_k + \mathbf{u}\cdot\mathbf{e}_k\left(\frac{1}{c^2\sigma}\mathbf{v} + \frac{\nu}{u^2}\mathbf{u}\right) \ ,$$

where $\mathbf{e}'_k$ is the vector of the natural basis in S'_{g} *boosted* to S_{g}. Finally, by introducing the components of $\mathbf{u}$ along $\mathbf{e}_k$

$$u_k = \mathbf{u}\cdot\mathbf{e}_k \ , \tag{5.111}$$

and using (5.110), we get the general formula

$$\mathbf{e}'_k = \mathbf{e}_k + \frac{1}{c^2\sigma}u_k\mathbf{w} \ , \tag{5.112}$$

with

$$\mathbf{w} \overset{\mathrm{def}}{=} \mathbf{v} - \frac{\mathbf{u}}{1+\alpha} \ . \tag{5.113}$$

Equation (5.112) represents the fundamental formula for the kinematics of deformation of a continuum and in the classical limit, $c \to \infty$, is consistent with the invariance property: $\mathbf{e}'_k = \mathbf{e}_k$. From this, one can obtain the relation between the two metrics g_{ik} and g'_{ik}, locally associated with the continuum with respect to the two different Galilean frames considered. Moreover, by differentiation with respect to time of both sides of (5.112) and by using (5.101), one gets the quantities $\partial_{t'}\mathbf{e}'_k$, which, like the corresponding quantities in S_g, summarize the (local) angular and deformation velocities of the continuum, with respect to the Galilean frame S'_g. Hence, the associated transformation laws can be computed (see [2]).

As we will see, the classical invariance properties

$$k'_{ik} = k_{ik} \ , \quad \omega'_{ik} = \omega_{ik} \ , \qquad \forall S_\mathrm{g}, S'_\mathrm{g} \ ,$$

are not conserved in the relativistic case. It follows that the ordinary notion of rigidity loses its meaning in relativity, in the sense that $k_{ik} \equiv 0 \nRightarrow k'_{ik} \equiv 0$.

5.7 Kinematical Invariants

Equation (5.112) represents the starting point for obtaining the transformation laws for the main geometrical and kinematical quantities of the continuum: g_{ik}, g^{ik}, k_{ik}, ω_{ik}, $\Gamma^h{}_{ik}$, etc. Before proceeding to derive these laws, we examine some *fundamental relativistic invariants*, which we will compare with the corresponding classical analog. We start by deriving the variation law of the determinant (5.6), which coincides with

$$\mathcal{D} = \mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_3 \qquad (5.114)$$

or

$$\mathcal{D} = \sqrt{\det||g_{ik}||} \ . \qquad (5.115)$$

From (5.109) one has this determinant relative to S'_g:

$$\mathcal{D}' = \frac{\alpha}{\sigma}\det \left\| \begin{matrix} \partial_1 x^1 & \partial_2 x^1 & \partial_3 x^1 \\ \partial_1 x^2 + X^2\partial_1 x^1 & \partial_2 x^2 + X^2\partial_2 x^1 & \partial_3 x^2 + X^2\partial_3 x^1 \\ \partial_1 x^3 + X^3\partial_1 x^1 & \partial_2 x^3 + X^3\partial_2 x^1 & \partial_3 x^3 + X^3\partial_3 x^1 \end{matrix} \right\| \ ,$$

where, for the sake of brevity, we have used the notation $X^{2,3} = \dot{x}^{2,3}u/(c^2\sigma)$; it is easy to verify the relation

$$\mathcal{D}' = \frac{\alpha}{\sigma}\mathcal{D} \ . \qquad (5.116)$$

The left-hand side of (5.116) is a function of the variables y^i and t', while the right-hand side depends on y^i and t. Therefore we have to use (5.99) for the left-hand side and (5.100) for the right-hand side. When $c \to \infty$, one has $\mathcal{D}' = \mathcal{D}$; in other words, $\mathcal{D}$ is not a relativistic invariant with respect to the choice of the Galilean frame, contrary to what happens in classical kinematics. We note that (5.116) has been obtained by means of (5.109), that is in x^1-standard coordinates. Actually it has a general validity, because of the intrinsic meaning of both $\mathcal{D}$ and $\mathcal{D}'$, as concerns the choice of coordinates in the respective Galilean frame. The same result can be derived by using (5.114), that is, starting from (5.112). In fact, using the notation

$$\mathbf{W} = \frac{1}{c^2\sigma}\left(\mathbf{v} - \frac{\mathbf{u}}{1+\alpha}\right) = \frac{1}{c^2\sigma}\mathbf{w} \qquad \Longrightarrow \qquad \mathbf{e}'_k = \mathbf{e}_k + u_k\mathbf{W} \ , \tag{5.117}$$

one finds the following expression for the product $\mathbf{e}'_1 \times \mathbf{e}'_2$:

$$\mathbf{e}'_1 \times \mathbf{e}'_2 = \mathbf{e}_1 \times \mathbf{e}_2 + (u_2\mathbf{e}_1 - u_1\mathbf{e}_2) \times \mathbf{W} \ ,$$

so that

$$\mathbf{e}'_1 \times \mathbf{e}'_2 \cdot \mathbf{e}'_3 = \mathbf{e}_1 \times \mathbf{e}_2 \cdot (\mathbf{e}_3 + u_3\mathbf{W}) + \mathbf{e}_3 \times (u_2\mathbf{e}_1 - u_1\mathbf{e}_2) \cdot \mathbf{W} \ .$$

Hence

$$\mathcal{D}' = \mathcal{D} + (u_3\mathbf{e}_1 \times \mathbf{e}_2 + u_2\mathbf{e}_3 \times \mathbf{e}_1 + u_1\mathbf{e}_2 \times \mathbf{e}_3) \cdot \mathbf{W} \ ;$$

furthermore, because of the duality relations (5.23), we have

$$\mathbf{e}^i = \frac{1}{\mathcal{D}}\mathbf{e}_{i+1} \times \mathbf{e}_{i+2} \qquad \Longrightarrow \qquad \mathbf{e}_i = \mathcal{D}\mathbf{e}^{i+1} \times \mathbf{e}^{i+2} \ , \tag{5.118}$$

and thus

$$\mathcal{D}' = \mathcal{D}(1 + \mathbf{u} \cdot \mathbf{W}) \ . \tag{5.119}$$

Finally, because of (5.117), one has $(1+\mathbf{u}\cdot\mathbf{W}) = \alpha/\sigma$, so that (5.119) coincides with (5.116). Moreover, (5.116) is associated with a relativistic invariant. In fact, using the relation

$$\frac{\partial t'}{\partial t} = \frac{\sigma}{\alpha} = \frac{\eta'}{\eta} \ , \tag{5.120}$$

holds not only for a single material point (as we have already seen) but also for a continuum, in the case in which the generic particle is fixed by its Lagrangian coordinates. Then (5.116) becomes $\mathcal{D}' = \mathcal{D}\eta/\eta'$, giving rise to the following invariance property:

$$\eta'\mathcal{D}' = \eta\mathcal{D} = \text{inv.} \ , \tag{5.121}$$

with the usual substitution of (5.99) or its inverse (5.100). The quantity $\eta\mathcal{D}$ is invariant with respect to any change of Galilean frame, it has a local meaning and, of course, it can be expressed either in the Lagrangian or in the Eulerian form. It generates, in turn, a differential invariant; in fact, differentiating with respect to t' both sides of (5.121) and taking into account (5.120) leads to

$$\frac{\partial \eta'}{\partial t'}\mathcal{D}' + \eta' \frac{\partial \mathcal{D}'}{\partial t'} = \frac{\alpha}{\sigma}\left(\frac{\partial \eta}{\partial t}\mathcal{D} + \eta \frac{\partial \mathcal{D}}{\partial t}\right) ,$$

so that, dividing both sides by $\mathcal{D}' = \mathcal{D}\alpha/\sigma > 0$, one gets the invariant relation

$$\frac{\partial \eta'}{\partial t'} + \frac{\eta'}{\mathcal{D}'}\frac{\partial \mathcal{D}'}{\partial t'} = \frac{\partial \eta}{\partial t} + \frac{\eta}{\mathcal{D}}\frac{\partial \mathcal{D}}{\partial t} = \text{inv.} \tag{5.122}$$

To obtain the classical limit, we note that

$$\partial_t \eta = \partial_t (1 - v^2/c^2)^{-1/2} = \frac{\eta^3}{c^2}\mathbf{v}\cdot\mathbf{a} \,, \tag{5.123}$$

so that $\eta \to 1$ and $\partial_t \eta \to 0$ in the limit $c \to \infty$, and (5.122) becomes

$$\frac{1}{\mathcal{D}'}\frac{\partial \mathcal{D}'}{\partial t'} = \frac{1}{\mathcal{D}}\frac{\partial \mathcal{D}}{\partial t} = \text{inv.} \;;$$

because of the absolute meaning of time, from these equations follows the invariance property of $\mathcal{D}$.

Equation (5.122) states the invariance of the quantity

$$\mathcal{B} \overset{\text{def}}{=} \frac{\partial \eta}{\partial t} + \frac{\eta}{\mathcal{D}}\frac{\partial \mathcal{D}}{\partial t} \,, \tag{5.124}$$

with respect to the choice of the Galilean frame; but the kinematical meaning of $\mathcal{B}$ should still be elucidated.

Equation (5.123) clarifies the dynamical meaning of the first term; thus we have to interpret the ratio $\partial_t\mathcal{D}/\mathcal{D}$, which is clearly independent of the choice of the Lagrangian coordinates y^i (like η, $\partial_t \eta$ and hence $\mathcal{B}$). The following Lagrangian relation holds (see [3], p. 511):

$$\frac{1}{\mathcal{D}}\frac{\partial \mathcal{D}}{\partial t} = \operatorname{div}\mathbf{v} \equiv \mathbf{e}^i \cdot \partial_i \mathbf{v} = k \,, \tag{5.125}$$

where k is the *cubic deformation velocity*, already introduced in (5.95). To prove this we start from the decomposition (5.81)

$$\operatorname{div}\mathbf{v} \equiv \mathbf{e}^i \cdot (\mathbf{k}_i + \boldsymbol{\omega} \times \mathbf{e}_i) = \mathbf{e}^i \cdot k_{ik}\mathbf{e}^k \,,$$

so that

$$\operatorname{div}\mathbf{v} = g^{ik}k_{ik} = \frac{1}{2}g^{ik}\partial_t g_{ik} \,, \tag{5.126}$$

where the last term follows from (5.33). Moreover, $\operatorname{div}\mathbf{v}$ is related to the determinant of g_{ik}:

$$g = \det ||g_{ik}|| \,. \tag{5.127}$$

In fact, denoting by c^{ik} the algebraic complement of g_{ik}, one has $\partial_t g = c^{ik}\partial_t g_{ik} = g g^{ik}\partial_t g_{ik}$, so that

$$g^{ik}\partial_t g_{ik} = \frac{1}{g}\partial_t g = \frac{2}{\sqrt{g}}\partial_t\sqrt{g} \; ,$$

and hence

$$\frac{1}{\sqrt{g}}\partial_t\sqrt{g} = \frac{1}{2}g^{ik}\partial_t g_{ik} \; . \tag{5.128}$$

Equation (5.126) then becomes

$$\operatorname{div}\mathbf{v} = \frac{1}{\sqrt{g}}\partial_t\sqrt{g} \; . \tag{5.129}$$

Since, from (5.13), $\sqrt{g} = \mathcal{D}$ (5.125) is now proved. Furthermore, the invariant $\mathcal{B}$, introduced in its Lagrangian form

$$\mathcal{B} = \partial_t\eta + \eta\operatorname{div}\mathbf{v} = \mathcal{B}(t,y) \; , \tag{5.130}$$

can also be cast in the Eulerian form

$$B(t,x) = \mathcal{B}(t,y(x)) = \frac{\partial\eta}{\partial t} + e^i\frac{\partial\eta}{\partial x^i} + \eta\frac{\partial e^i}{\partial x^i} \equiv \frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x^i}(\eta e^i) \; , \tag{5.131}$$

where $e^i(t,x) = v^i(t,y(x))$ are the Eulerian components of the velocity. This is achieved by replacing η with $\eta(t,x)$ and $\partial_t\eta(t,y)$ with the *substantial derivative*

$$\frac{\partial\eta}{\partial t} + e^i\frac{\partial\eta}{\partial x^i} \; , \tag{5.132}$$

in (5.130), besides the obvious replacement of $\mathbf{v}(t,y)$ with $\mathbf{e}(t,x)$. Finally, taking into account the ordinary decomposition of the 4-velocity $\mathbf{V} = \eta(\mathbf{e} + c\boldsymbol{\gamma})$, that is, $V^0 = c\eta$ and $V^i = \eta e^i$, it follows that B is the four-dimensional divergence of $\mathbf{V}$:

$$B(t,x) = \operatorname{Div}\mathbf{V} = \partial_\alpha V^\alpha \; . \tag{5.133}$$

Equation (5.133) confirms the absolute meaning of B, because the divergence of a vector is a scalar, invariant under linear transformations (and, in particular, under Lorentz transformations). In fact, under a linear transformation $x^\alpha \to x'^\alpha = A'^\alpha{}_\beta x^\beta + A'^\alpha$, we have

$$A'^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\beta} \; ; \tag{5.134}$$

using then the transformation law of the components of a vector $V'^\alpha = A'^\alpha{}_\beta V^\beta$, one obtains

$$\partial'_\alpha V'^\alpha = A'^\alpha{}_\beta\partial_\rho V^\beta\frac{\partial x^\rho}{\partial x'^\alpha} = \frac{\partial x^\rho}{\partial x'^\alpha}\frac{\partial x'^\alpha}{\partial x^\beta}\partial_\rho V^\beta = \delta^\rho_\beta\partial_\rho V^\beta \; ,$$

and hence the invariance property

$$\partial'_\alpha V'^\alpha = \partial_\beta V^\beta = \text{inv.} \tag{5.135}$$

5.8 Dilation Coefficients

Consider the transformation law of the natural basis $\{\mathbf{e}_i\}$ associated with the (arbitrarily fixed) Lagrangian coordinates y^i, passing from one Galilean frame S_g to another S'_g, i.e. (5.112). The dual of the basis $\{\mathbf{e}'_i\}$ is given by

$$\mathbf{e}'^i = \mathbf{e}^i - \frac{1}{c^2\alpha} w^i \mathbf{u} \; ; \tag{5.136}$$

from (5.113) one has

$$\frac{1}{c^2}\mathbf{u}\cdot\mathbf{w} = \alpha - \sigma \; , \tag{5.137}$$

so that it is easy to check the reciprocity relations

$$\mathbf{e}'^i \cdot \mathbf{e}'_k = \delta^i_k \; . \tag{5.138}$$

Equation (5.112),

$$\mathbf{e}'_k = \left(\delta^i_k + \frac{1}{c^2\sigma} w^i u_k\right)\mathbf{e}_i \; ,$$

gives the vectors $\mathbf{e}'_k$ as the transform of the vectors $\mathbf{e}_k$, by means of the *displacement map* $A \equiv (A^i{}_k)$:

$$\mathbf{e}'_k = A\mathbf{e}_k = A^i{}_k \mathbf{e}_i \; , \tag{5.139}$$

with

$$A^i{}_k \stackrel{\text{def}}{=} \delta^i_k + \frac{1}{c^2\sigma} w^i u_k \; . \tag{5.140}$$

The inverse $A^{-1} = B$ of the map A, such that

$$A^i{}_j B^j{}_k = \delta^i_k \; , \tag{5.141}$$

can be obtained from (5.136):

$$\begin{aligned} \mathbf{e}'^i = B^i{}_k \mathbf{e}^k &\equiv \left(\delta^i_k - \frac{1}{c^2\alpha} w^i u_k\right)\mathbf{e}^k \\ &= B^\mathrm{T}\mathbf{e}^i \equiv (B^\mathrm{T})_k{}^i \mathbf{e}^k \; ; \end{aligned} \tag{5.142}$$

in fact, the coefficients of B: $B^i{}_k \equiv \mathbf{e}^i \cdot (B\mathbf{e}_k) = (B^\mathrm{T}\mathbf{e}^i)\cdot\mathbf{e}_k$ are given by

$$B^i{}_k = \delta^i_k - \frac{1}{c^2\alpha}\, w^i u_k \; . \tag{5.143}$$

The presence, in C, of the metric tensor $g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k$ allows one to introduce, besides the mixed form, the completely covariant and contravariant forms of the tensors A and B:

$$\begin{cases} A_{ik} = g_{ik} + \dfrac{1}{c^2\sigma} w_i u_k \; , & A^{ik} = g^{ik} + \dfrac{1}{c^2\sigma} w^i u^k, \\[2ex] B_{ik} = g_{ik} - \dfrac{1}{c^2\alpha} w_i u_k \; , & B^{ik} = g^{ik} - \dfrac{1}{c^2\alpha} w^i u^k. \end{cases} \tag{5.144}$$

The representation (5.139) allows one to compare linear, surface and volume elements, relative to the configurations C and C' of the continuum, associated with two different (arbitrary) Galilean frames S_g and S'_g. From this comparison, the relative dilation coefficients can be defined.

1. *Linear dilation coefficients*
   Let us compare, first of all, the metric tensors. From (5.112) we have

$$g'_{ik} = g_{ik} + 2\epsilon_{ik} \; , \tag{5.145}$$

ϵ_{ik} being the relative *deformation tensor*[5]

$$\epsilon_{ik} = \frac{1}{2c^2\sigma}\left(u_i w_k + u_k w_i + \frac{1}{c^2\sigma} w^2 u_i u_k\right) \; , \tag{5.146}$$

or by using (5.113)

$$\epsilon_{ik} = \frac{1}{2c^2\sigma}\left(u_i v_k + u_k v_i - \frac{1}{\sigma\eta^2} u_i u_k\right) \; . \tag{5.147}$$

Next consider a *linear element* $\mathrm{dP} = \mathrm{d}y^i \mathbf{e}_i$, emanating from $\mathrm{P} \in C$; let dP' be the corresponding element in $\mathrm{P}' \in C'$:

$$\mathrm{dP}' = A\,\mathrm{dP} \equiv \mathrm{d}y^i \mathbf{e}'_i \; ,$$

and decompose dP and dP′ into magnitude and unit vector:

$$\mathrm{dP} = |\mathrm{dP}|\mathbf{a} \; , \qquad \mathrm{dP}' = |\mathrm{dP}'|\mathbf{a}' \; .$$

It is quite natural to define as *linear dilation coefficient*, at P and in the direction $\mathbf{a}$, the ratio

$$\delta_a \overset{\mathrm{def}}{=} \frac{|\mathrm{dP}'| - |\mathrm{dP}|}{|\mathrm{dP}|} = |A\mathbf{a}| - 1 \; . \tag{5.148}$$

We have $|A\mathbf{a}| = \sqrt{(A\mathbf{a}) \cdot (A\mathbf{a})} = \sqrt{g'_{ik} a^i a^k}$, so that from (5.145) it follows that

$$\delta_a = \sqrt{1 + 2\epsilon_{ik} a^i a^k} - 1 \tag{5.149}$$

or explicitly

$$\delta_a = \sqrt{1 + \frac{1}{c^2\sigma}\left[2(\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{a}) - \frac{1}{\eta^2\sigma}(\mathbf{u}\cdot\mathbf{a})^2\right]} - 1 \; . \tag{5.150}$$

In particular, for $a = \mathbf{e}_i/|\mathbf{e}_i| = \mathbf{e}_i/\sqrt{g_{ii}}$ (no sum over repeated indices is needed here), one has the linear dilation coefficients, in the directions of the vectors $\mathbf{e}_i$:

[5] Here the deformation does not have the usual meaning as for a single continuum in a fixed frame of reference, but it is relative to two different frames.

$$\delta_i = \sqrt{1 + \frac{1}{c^2 \sigma g_{ii}} \left(2 u_i v_i - \frac{1}{\eta^2 \sigma} u_i^2 \right)} - 1 \ . \tag{5.151}$$

Let us now consider the angle Θ_{ab} formed by the directions $\mathbf{a}$ and $\mathbf{b}$ emanating from $\mathrm{P} \in C$: $\cos \Theta_{ab} = \mathbf{a} \cdot \mathbf{b}$; the angle formed by the corresponding directions in $\mathrm{P}' \in C'$ is given by

$$\cos {\Theta'}_{ab} \equiv \frac{A\mathbf{a}}{|A\mathbf{a}|} \cdot \frac{A\mathbf{b}}{|A\mathbf{b}|} = \frac{a^i \mathbf{e}'_i \cdot b^k \mathbf{e}'_k}{\sqrt{{g'}_{ik} a^i a^k} \sqrt{{g'}_{lm} b^l b^m}} = \frac{{g'}_{ik} a^i b^k}{(1+\delta_a)(1+\delta_b)} \ ,$$

that is, by using (5.145) and (5.146):

$$\begin{aligned} \cos {\Theta'}_{ab} = \frac{1}{(1+\delta_a)(1+\delta_b)} \Bigg\{ \cos \Theta_{ab} + \frac{1}{c^2 \sigma} \Bigg[(\mathbf{u} \cdot \mathbf{a})(\mathbf{v} \cdot \mathbf{b}) \\ + (\mathbf{u} \cdot \mathbf{b})(\mathbf{v} \cdot \mathbf{a}) - \frac{1}{\eta^2 \sigma} (\mathbf{u} \cdot \mathbf{a})(\mathbf{u} \cdot \mathbf{b}) \Bigg] \Bigg\} \ . \end{aligned} \tag{5.152}$$

In particular, for $a = \mathbf{e}_i / |\mathbf{e}_i|$ and $b = \mathbf{e}_k / |\mathbf{e}_k|$ (no sum over repeated indices is needed here), one has the *angular transformations*, corresponding to the vectors $\mathbf{e}_i$ and $\mathbf{e}_k$ (*shear*):

$$\begin{aligned} \cos {\Theta'}_{ik} = \frac{1}{(1+\delta_i)(1+\delta_k)} \Bigg[\cos \Theta_{ik} \\ + \frac{1}{c^2 \sigma \sqrt{g_{ii} g_{kk}}} \left(u_i v_k + u_k v_i - \frac{1}{\eta^2 \sigma} u_i u_k \right) \Bigg] \ . \end{aligned} \tag{5.153}$$

2. *Surface dilation coefficients*

Let us consider an oriented surface element: $\mathbf{n} \mathrm{d}\, \sigma$, in $\mathrm{P} \in C$, and let $\mathbf{n}' \mathrm{d}\sigma'$ be the corresponding element in $\mathrm{P}' \in C'$; we can determine the relation between the two surface elements, with the usual assumption of boosting $\mathbf{n}' \ \mathrm{d}\sigma'$ onto S_{g}. We have

$$\mathbf{n} \ \mathrm{d}\sigma = \mathrm{dP} \times \mathrm{dQ} = \mathrm{d}y^i \ \mathbf{e}_i \times \mathrm{d}z^k \ \mathbf{e}_k = \sqrt{g} \epsilon_{ikh} \ \mathrm{d}y^i \ \mathrm{d}z^k \ \mathbf{e}^h$$

where ϵ_{ikh} is the Levi-Civita alternating symbol; similarly, one must assume:

$$\begin{aligned} \mathbf{n}' \ \mathrm{d}\sigma' &= (A\mathrm{dP}) \times (A\mathrm{dQ}) = \mathrm{d}y^i \ \mathbf{e}'_i \times \mathrm{d}z^k \ \mathbf{e}'_k = \sqrt{g'} \epsilon_{ikh} \ \mathrm{d}y^i \ \mathrm{d}z^k \ \mathbf{e}'^h \\ &= \eta_{ikh} \ \mathrm{d}y^i \ \mathrm{d}z^k \ \mathbf{e}'^h \ . \end{aligned}$$

It follows, from comparison, that

$$\mathbf{n}' \ \mathrm{d}\sigma' = \sqrt{\frac{g'}{g}} (\mathbf{n} \ \mathrm{d}\sigma \cdot \mathbf{e}_h) \mathbf{e}'^h$$

or by using (5.136)

$$\mathbf{n}' \, \mathrm{d}\sigma' = \sqrt{\frac{g'}{g}}(\mathbf{n} \, \mathrm{d}\sigma - \frac{1}{c^2\alpha}\mathbf{n} \, \mathrm{d}\sigma \cdot \mathbf{wu}) \; .$$

Furthermore, the Jacobian determinants

$$\mathcal{D} = \det \left\| \frac{\partial x^i}{\partial y^k} \right\| \equiv \sqrt{g} \; , \quad \mathcal{D}' = \det \left\| \frac{\partial x'^i}{\partial y^k} \right\| \equiv \sqrt{g'} \; ,$$

satisfy the invariance property (5.121):

$$\eta\mathcal{D} = \eta'\mathcal{D}' = \text{inv.} \quad \Rightarrow \quad \frac{\mathcal{D}'}{\mathcal{D}} = \frac{\eta}{\eta'} = \frac{\alpha}{\sigma} \; ;$$

thus the previous relation becomes

$$\mathbf{n}' \, \mathrm{d}\sigma' = \frac{\alpha}{\sigma}(\mathbf{n} \, \mathrm{d}\sigma - \frac{1}{c^2\alpha}\mathbf{n} \, \mathrm{d}\sigma \cdot \mathbf{wu}) \; . \tag{5.154}$$

From this equation we get the *surface dilation coefficient* in the direction $\mathbf{n}$:

$$\delta_\sigma \stackrel{\text{def}}{=} \frac{\mathrm{d}\sigma' - \mathrm{d}\sigma}{\mathrm{d}\sigma} \; , \tag{5.155}$$

or explicitly

$$\delta_\sigma = \frac{\alpha}{\sigma}\sqrt{1 - \frac{2}{c^2\alpha}(\mathbf{n}\cdot\mathbf{w})(\mathbf{n}\cdot\mathbf{u}) + \frac{1-\alpha^2}{c^2\alpha}(\mathbf{n}\cdot\mathbf{w})^2} - 1. \tag{5.156}$$

3. *Volume dilation coefficients*
   The *cubic dilation coefficient* δ_c is defined by

$$\delta_\mathrm{c} \stackrel{\text{def}}{=} \frac{\mathrm{d}C' - \mathrm{d}C}{\mathrm{d}C} = \frac{\mathcal{D}'}{\mathcal{D}} - 1 = \frac{\alpha}{\sigma} - 1 \; , \tag{5.157}$$

   using (5.121).

In the limit $c \to \infty$ all the dilation coefficients vanish. For $\mathbf{v} = 0$, instead, S_g is the local rest frame of the continuum and the above relations refer to the way in which the continuum (in relative uniform translational motion, with respect to S'_g) differs from the rigidity condition.

Having defined the local deformation coefficients, we can start studying the transformation laws for the angular and deformation velocities.

5.9 Transformation Laws for Angular and Deformation Velocities

Our purpose now is to obtain the gradient of the continuum velocity $\mathbf{v}'(t', y)$, with respect to S'_g. To this end it is enough to differentiate the relation (5.112) with respect to t', using (5.101):

$$\partial_{t'} = \frac{\alpha}{\sigma}\partial_t \, . \tag{5.158}$$

This gives

$$\partial_{t'}\mathbf{e}'_i = \frac{\alpha}{\sigma}\left[\partial_t\mathbf{e}_i + \frac{u_i}{c^2\sigma}\partial_t\mathbf{w} + \frac{1}{c^2\sigma}\left(\mathbf{u}\cdot\partial_t\mathbf{e}_i - \frac{1}{\sigma}\partial_t\sigma u_i\right)\mathbf{w}\right] \, ,$$

with

$$\partial_t\mathbf{w} = \partial_t\mathbf{v} = \mathbf{a}, \quad \partial_t\sigma = -\frac{1}{c^2}\mathbf{u}\cdot\mathbf{a} \, . \tag{5.159}$$

Thus, by using the identities

$$\partial_t\mathbf{e}_i = \partial_i\mathbf{v} \, , \qquad \partial_{t'}\mathbf{e}'_i = \partial_i\mathbf{v}' \, , \tag{5.160}$$

and introducing the map A defined by (5.139), we have

$$\partial_i\mathbf{v}' = \frac{\alpha}{\sigma}A\left(\partial_i\mathbf{v} + \frac{1}{c^2\sigma}u_i\mathbf{a}\right) \, . \tag{5.161}$$

The same result can, obviously, be obtained from the theorem of relative motions:

$$\mathbf{v}' = \frac{1}{\sigma}\left(\alpha\mathbf{v} - \frac{\alpha+\sigma}{1+\alpha}\mathbf{u}\right) \, , \tag{5.162}$$

by differentiating both sides with respect to y^i, which appear only in $\mathbf{v}$ and σ.

In scalar terms, we can set $\partial_i\mathbf{v}' = h'{}_{ik}\mathbf{e}'^k$, analogously to $\partial_i\mathbf{v} = h_{ik}\mathbf{e}^k$; using (5.139), $\mathbf{e}^h\cdot\mathbf{e}'_k = A^h{}_k$, (5.161) becomes

$$h'{}_{ik} = \frac{\alpha}{\sigma}A^h{}_k\left[h_{ih} + \frac{1}{c^2\sigma}u_i a_h + \frac{1}{c^2\sigma}u^j\left(h_{ij} + \frac{1}{c^2\sigma}u_i a_j\right)w_h\right]$$

or

$$h'{}_{ik} = \frac{\alpha}{\sigma}A^h{}_k A_h{}^j\left(h_{ij} + \frac{1}{c^2\sigma}u_i a_j\right) \, , \tag{5.163}$$

as from (5.140). The tensor $h'{}_{ik}$ summarizes (locally) the angular and deformation velocities of the continuum with respect to S'_{g}, analogously to h_{ik} in S_{g}:

$$h'{}_{ik} = k'{}_{ik} + \omega'{}_{ik} \, , \qquad h_{ik} = k_{ik} + \omega_{ik} \, , \tag{5.164}$$

so that (5.163) contains the sought-for relations. Here, clearly, $k'{}_{ik}$ depends either on the deformation velocity k_{ik} and the angular velocity ω_{ik} or on $\mathbf{a}$ and $\mathbf{u}$; an analogous dependence has $\omega'{}_{ik}$. In the classical situation ($c \to \infty$), we have $k'{}_{ik} = k_{ik}$ and $\omega'{}_{ik} = \omega_{ik}$, which give an *absolute meaning* to both the deformation and the angular velocities, in contrast to the relativistic situation. Now, from (5.161), one has the expression for the local angular velocity $\boldsymbol{\omega}'$:

$$\boldsymbol{\omega}' = \frac{1}{2}\mathbf{e}'^i \times \partial_i\mathbf{v}' \, ; \tag{5.165}$$

by using (5.136) and (5.137), we then have

$$\begin{aligned}\boldsymbol{\omega}' &= \frac{1}{2}\frac{\alpha}{\sigma}\mathbf{e}^i \times A\left(\partial_i \mathbf{v} + \frac{1}{c^2\sigma}u_i \mathbf{a}\right) \\ &\quad - \frac{1}{2c^2\alpha}\mathbf{u} \times \frac{\alpha}{\sigma}A\left[w^i \partial_i \mathbf{v} + \left(\frac{\alpha}{\sigma} - 1\right)\mathbf{a}\right] \\ &= \frac{1}{2}\frac{\alpha}{\sigma}\mathbf{e}^i \times A(\partial_i \mathbf{v}) - \frac{1}{2c^2\alpha}\mathbf{u} \times \frac{\alpha}{\sigma}A(w^i \partial_i \mathbf{v} - \mathbf{a}) \ .\end{aligned}$$

Otherwise, from the form (5.139) of A, it follows that

$$\mathbf{e}^i \times A(\partial_i \mathbf{v}) = 2\boldsymbol{\omega} + \frac{1}{c^2\sigma}\mathbf{e}^i \times (\mathbf{u} \cdot \partial_i \mathbf{v})\mathbf{w} \ ,$$

and, by using the decomposition (5.84)

$$\partial_i \mathbf{v} = \mathbf{k}_i + \boldsymbol{\omega} \times \mathbf{e}_i, \tag{5.166}$$

we have

$$\mathbf{u} \cdot \partial_i \mathbf{v} = (\mathbf{k}_u - \boldsymbol{\omega} \times \mathbf{u}) \cdot \mathbf{e}_i \ , \tag{5.167}$$

where $\mathbf{k}_u$ is the deformation velocity along $\mathbf{u}$, already introduced:

$$\mathbf{k}_u = u^i \mathbf{k}_i \ . \tag{5.168}$$

Moreover, (5.137) implies

$$\begin{aligned}\mathbf{e}^i \times A(\partial_i \mathbf{v}) &= 2\boldsymbol{\omega} + \frac{1}{c^2\sigma}(\mathbf{k}_u - \boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{w} \\ &= \left(1 + \frac{\alpha}{\sigma}\right)\boldsymbol{\omega} + \frac{1}{c^2\sigma}(\mathbf{k}_u \times \mathbf{w} - \boldsymbol{\omega} \cdot \mathbf{w}\mathbf{u}) \ ;\end{aligned}$$

the expression for $\boldsymbol{\omega}'$ is then

$$\begin{aligned}\boldsymbol{\omega}' = \frac{1}{2}\frac{\alpha}{\sigma}\left(1 + \frac{\alpha}{\sigma}\right)\boldsymbol{\omega} + \frac{1}{2c^2\sigma}\bigg[\frac{\alpha}{\sigma}(\mathbf{k}_u \times \mathbf{w} - \boldsymbol{\omega} \cdot \mathbf{w}\mathbf{u}) \\ + \mathbf{u} \times A(\mathbf{a} + \mathbf{k}_w - \boldsymbol{\omega} \times \mathbf{w})\bigg] \ ,\end{aligned} \tag{5.169}$$

where $\mathbf{k}_w = w^i \mathbf{k}_i$ is the deformation velocity along $\mathbf{w}$. Equation (5.169) can be further developed by expanding the vector product $\mathbf{u} \times A(\boldsymbol{\omega} \times \mathbf{w})$, using (5.137) and (5.139):

$$\begin{aligned}\mathbf{u} \times A(\boldsymbol{\omega} \times \mathbf{w}) &= \mathbf{u} \times \left(\boldsymbol{\omega} \times \mathbf{w} + \frac{1}{c^2\sigma}\boldsymbol{\omega} \times \mathbf{w} \cdot \mathbf{u}\mathbf{w}\right) \\ &= c^2(\alpha - \sigma)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{u})\mathbf{w} - \frac{1}{c^2\sigma}(\mathbf{u} \times \mathbf{w} \cdot \boldsymbol{\omega})\mathbf{u} \times \mathbf{w} \ ;\end{aligned}$$

the general expression then follows:

$$\boldsymbol{\omega}' = \frac{1}{2}\left(1+\frac{\alpha^2}{\sigma^2}\right)\boldsymbol{\omega} + \frac{1}{2c^2\sigma}\left[\mathbf{H} + \mathbf{u}\times A(\mathbf{K})\right] , \tag{5.170}$$

where

$$\begin{cases} \mathbf{H} = \dfrac{\alpha}{\sigma}\left(\mathbf{k}_u \times \mathbf{w} - \boldsymbol{\omega}\cdot\mathbf{w}\mathbf{u}\right) + \boldsymbol{\omega}\cdot\mathbf{u}\mathbf{w} + \dfrac{1}{c^2\sigma}(\boldsymbol{\omega}\cdot\mathbf{u}\times\mathbf{w})\mathbf{u}\times\mathbf{w}, \\ \mathbf{K} = \mathbf{a} - \mathbf{k}_w. \end{cases} \tag{5.171}$$

In the classical case ($c \to \infty$), (5.170) reduces to the invariance property $\boldsymbol{\omega}' = \boldsymbol{\omega}$, as expected; in the relativistic case, instead, one has a typical mixing of the various kinematical quantities: the angular velocity $\boldsymbol{\omega}'$ has no longer an invariant meaning, being a function of $\mathbf{u}$ and $\mathbf{v}$ (through σ and $\mathbf{w}$), $\mathbf{a}$, $\boldsymbol{\omega}$ and also of the deformation velocity (through $\mathbf{k}_u$ and $\mathbf{k}_w$).

Let us now determine the transformation law of the deformation velocities: $\mathbf{k}_i = \partial_i\mathbf{v} - \boldsymbol{\omega}\times\mathbf{e}_i$.

From (5.170) and (5.139), we have

$$\begin{aligned}\boldsymbol{\omega}'\times\mathbf{e}'_i = \frac{1}{2}\left(1+\frac{\alpha^2}{\sigma^2}\right)\boldsymbol{\omega}\times A\mathbf{e}_i + \frac{1}{2c^2\sigma}\Big[&\mathbf{H}\times A\mathbf{e}_i \\ &+\mathbf{u}\cdot A(\mathbf{e}_i)A(\mathbf{K}) - A(\mathbf{e}_i)\cdot A(\mathbf{K})\mathbf{u}\Big] .\end{aligned}$$

Using then the identity

$$A^{\mathrm{T}}(\mathbf{u}) = \frac{\alpha}{\sigma}\mathbf{u} , \tag{5.172}$$

and the commutation property of the scalar product leads to

$$\begin{aligned}\boldsymbol{\omega}'\times\mathbf{e}'_i = \frac{1}{2}&\left(1+\frac{\alpha^2}{\sigma^2}\right)\boldsymbol{\omega}\times A(\mathbf{e}_i) \\ &+\frac{1}{2c^2\sigma}\left(\mathbf{H}\times A(\mathbf{e}_i) + \frac{\alpha}{\sigma}u_i A(\mathbf{K}) - A^{\mathrm{T}}A(\mathbf{e}_i)\cdot\mathbf{K}\mathbf{u}\right) .\end{aligned}$$

Next, taking into account (5.161), we have

$$\begin{aligned}\mathbf{k}'_i &= \partial_i\mathbf{v}' - \boldsymbol{\omega}'\times\mathbf{e}'_i \\ &= A(\partial_i\mathbf{v}) - \frac{1}{2}\left(1+\frac{\alpha^2}{\sigma^2}\right)\boldsymbol{\omega}\times A\mathbf{e}_i + \frac{1}{2c^2\sigma}\Big[\frac{\alpha}{\sigma}u_i A(2\mathbf{a}-\mathbf{K}) \\ &\quad -\mathbf{H}\times A\mathbf{e}_i + A^{\mathrm{T}}A\mathbf{e}_i\cdot\mathbf{K}\mathbf{u}\Big] ;\end{aligned} \tag{5.173}$$

moreover, from (5.167), it follows that

$$\begin{aligned}A(\partial_i\mathbf{v}) &= \partial_i\mathbf{v} + \frac{1}{c^2\sigma}(\partial_i\mathbf{v}\cdot\mathbf{u})\mathbf{w} \\ &= \mathbf{k}_i + \boldsymbol{\omega}\times\mathbf{e}_i + \frac{1}{2c^2\sigma}(\mathbf{k}_u - \boldsymbol{\omega}\times\mathbf{u})\cdot\mathbf{e}_i\mathbf{w} ,\end{aligned}$$

and, from $(5.171)_1$ and the (already used) identity (5.172),

$$-\mathbf{H} \times A\mathbf{e}_i = \frac{\alpha}{\sigma}\left(\mathbf{w} \cdot A\mathbf{e}_i \mathbf{k}_u - \mathbf{k}_u \cdot A\mathbf{e}_i \mathbf{w} + \boldsymbol{\omega} \cdot \mathbf{w}\mathbf{u} \times A\mathbf{e}_i\right) \\ -\boldsymbol{\omega} \cdot \mathbf{u}\mathbf{w} \times \mathbf{e}_i - \frac{1}{c^2\sigma}(\boldsymbol{\omega} \cdot \mathbf{u} \times \mathbf{w})\left(\frac{\alpha}{\sigma} u_i \mathbf{w} - \mathbf{w} \cdot A\mathbf{e}_i \mathbf{u}\right) .$$

Equation (5.173) thus becomes

$$\mathbf{k}'_i = \frac{\alpha}{\sigma}\mathbf{k}_i - \frac{1}{2}\left(1 - \frac{\alpha}{\sigma}\right)^2 \boldsymbol{\omega} \times \mathbf{e}_i + \frac{\alpha}{2c^2\sigma^2}\left[-u_i\left(\frac{\alpha}{\sigma} + \frac{\sigma}{\alpha}\right)\boldsymbol{\omega} \times \mathbf{w}\right. \\ -2(\boldsymbol{\omega} \times \mathbf{u})_i \mathbf{w} + \boldsymbol{\omega} \cdot \mathbf{w}\mathbf{u} \times A\mathbf{e}_i - \frac{\sigma}{\alpha}\boldsymbol{\omega} \cdot \mathbf{u}\mathbf{w} \times \mathbf{e}_i \\ -\frac{1}{c^2\sigma}(\boldsymbol{\omega} \cdot \mathbf{u} \times \mathbf{w})\left(u_i \mathbf{w} - \frac{\sigma}{\alpha}\mathbf{w} \cdot A\mathbf{e}_i \mathbf{u}\right) + \mathbf{k}_u \cdot \mathbf{e}_i \\ \left. +\mathbf{w} \cdot A\mathbf{e}_i \mathbf{k}_u + u_i(\mathbf{k}_w + A\mathbf{a}) + \frac{\sigma}{\alpha} A^{\mathrm{T}} A\mathbf{e}_i \cdot (\mathbf{a} - \mathbf{k}_w)\mathbf{u}\right] , \qquad (5.174)$$

where the identity $\mathbf{k}_u \cdot \mathbf{w} = \mathbf{k}_w \cdot \mathbf{u}$ has been used and the product $A^{\mathrm{T}}A$ is such that

$$A^{\mathrm{T}}A(\mathbf{e}_i) = \mathbf{e}_i + \frac{1}{c^2\sigma}\left(u_i \mathbf{w} + w_i \mathbf{u} + \frac{1}{c^2\sigma} w^2 u_i \mathbf{u}\right) . \qquad (5.175)$$

In the classical case ($c \to \infty$), (5.170) reduces to the invariance $\mathbf{k}'_i = \mathbf{k}_i$, as expected; in the relativistic case, instead, as for the angular velocity, one has a mixing of the various kinematical quantities, $\mathbf{k}'_i$ being a function of $\mathbf{k}_i$, $\mathbf{u}$ and $\mathbf{v}$ (through σ and $\mathbf{w}$), $\mathbf{a}$ and also of the angular velocity $\boldsymbol{\omega}$.

Equations (5.170) and (5.174) have a *general validity*, because they refer to any continuously deformable system; they show that the classical notion of rigidity is meaningless in a relativistic context: in fact, the absence of deformations in S_{g}: $\mathbf{k}_i = 0$, has no absolute meaning, since in general $\mathbf{k}'_i \neq 0$:

$$\mathbf{k}'_i = -\frac{1}{2}\left(1 - \frac{\alpha}{\sigma}\right)^2 \boldsymbol{\omega} \times \mathbf{e}_i + \frac{\alpha}{2c^2\sigma^2}\left[-u_i\left(\frac{\alpha}{\sigma} + \frac{\sigma}{\alpha}\right)\boldsymbol{\omega} \times \mathbf{w}\right. \\ -2(\boldsymbol{\omega} \times \mathbf{u})_i \mathbf{w} + \boldsymbol{\omega} \cdot \mathbf{w}\mathbf{u} \times A\mathbf{e}_i - \frac{\sigma}{\alpha}\boldsymbol{\omega} \cdot \mathbf{u}\mathbf{w} \times \mathbf{e}_i \\ -\frac{1}{c^2\sigma}(\boldsymbol{\omega} \cdot \mathbf{u} \times \mathbf{w})\left(u_i \mathbf{w} - \frac{\sigma}{\alpha}\mathbf{w} \cdot A\mathbf{e}_i \mathbf{u}\right) \\ \left. +u_i A\mathbf{a} + \frac{\sigma}{\alpha} A^{\mathrm{T}} A\mathbf{e}_i \cdot \mathbf{a}\mathbf{u}\right] \neq 0 . \qquad (5.176)$$

Similar to the deformation velocity (5.176), in the rigid case (in S_{g}) where $\mathbf{k}_i = 0$, the transformation for the angular velocity reduces to the form

$$\boldsymbol{\omega}' = \frac{1}{2}\left(1 + \frac{\alpha^2}{\sigma^2}\right)\boldsymbol{\omega} + \frac{1}{2c^2\sigma}\left[-\frac{\alpha}{\sigma}\boldsymbol{\omega} \cdot \mathbf{w}\mathbf{u}\right. \\ \left. +\boldsymbol{\omega} \cdot \mathbf{u}\mathbf{w} + \frac{1}{c^2\sigma}(\boldsymbol{\omega} \cdot \mathbf{u} \times \mathbf{w})\mathbf{u} \times \mathbf{w} + \mathbf{u} \times A\mathbf{a}\right] . \qquad (5.177)$$

5.10 Born Rigidity. Thomas Precession

Equations (5.170) and (5.174) are completely general as concerns the motion of the continuum and the choice of the two Galilean frames S_g and S'_g. Let us assume that S_g be the rest frame for some particular element P of the continuum; this means that at P:

$$\mathbf{v} = 0\,, \quad \Rightarrow \sigma = 1, \quad \mathbf{u} = -\mathbf{v}'\,, \quad \alpha = 1/\eta'\,, \tag{5.178}$$

and hence, from (5.113)

$$\mathbf{w} = \frac{\eta'}{1+\eta'}\mathbf{v}'\,. \tag{5.179}$$

Thus, we have the following expressions for $\boldsymbol{\omega}'$ and $\mathbf{k}'_i$:

$$\boldsymbol{\omega}' = \frac{1}{2}\left(1+\frac{1}{\eta'^2}\right)\boldsymbol{\omega}^0 + \frac{1}{2c^2\eta'}(\mathbf{H}^0 - \eta'\mathbf{v}'\times A^0\mathbf{K}^0), \tag{5.180}$$

$$\begin{aligned}
\mathbf{k}'_i = {} & \frac{1}{\eta'}\mathbf{k}_i^0 + \frac{1}{2}\left(1-\frac{1}{\eta'}\right)^2 \boldsymbol{\omega}^0\times\mathbf{e}_i^0 \\
& + \frac{1}{2c^2(1+\eta')}\cdot\left[\left(\eta'+\frac{1}{\eta'}\right)v'_i\boldsymbol{\omega}^0\times\mathbf{v}' + 2(\boldsymbol{\omega}^0\times\mathbf{v}')\cdot\mathbf{e}_i^0\mathbf{v}'\right. \\
& + \boldsymbol{\omega}^0\cdot\mathbf{v}'\mathbf{v}'\times(\eta'\mathbf{e}_i^0 - A^0\mathbf{e}_i^0) - \mathbf{k}_{v'}^0\cdot\mathbf{e}_i^0\mathbf{v}' - \mathbf{v}'\cdot A^0\mathbf{e}_i^0\mathbf{k}_{v'}^0 \\
& \left. - v'_i\left(\mathbf{k}_{v'}^0 + \frac{1+\eta'}{\eta'}A^0\mathbf{a}^0\right) - \eta' A^{0\mathrm{T}}A^0\mathbf{e}_i^0\cdot\left(\frac{1+\eta'}{\eta'}\mathbf{a}^0 - \mathbf{k}_{v'}^0\right)\mathbf{v}'\right],
\end{aligned}$$

where $\boldsymbol{\omega}^0$ and $\mathbf{k}_i^0$ are the *proper angular and deformation velocities*, and (5.136), (5.139), (5.171) and (5.175) reduce to

$$\mathbf{H}^0 = \frac{\eta'}{1+\eta'}[(1-\eta')\boldsymbol{\omega}^0\cdot\mathbf{v}'\mathbf{v}' - \mathbf{k}_{v'}^0\times\mathbf{v}']\,, \qquad \mathbf{K}^0 = \mathbf{a}^0 - \frac{\eta'}{1+\eta'}\mathbf{k}_{v'}^0,$$

$$\mathbf{e}'_i = \mathbf{e}_i^0 - \frac{1}{c^2}\frac{\eta'}{1+\eta'}v'_i\mathbf{v}'\,, \qquad A^0(\,) = (\,) - \frac{1}{c^2}\frac{\eta'}{1+\eta'}(\,)\cdot\mathbf{v}'\mathbf{v}',$$

$$A^{0\mathrm{T}}A^0\mathbf{e}_i^0 = \mathbf{e}_i^0 - \frac{1}{c^2}v'_i\mathbf{v}'\,,$$

and, finally, the *proper acceleration* $\mathbf{a}^0$ is related to $\mathbf{a}'$ by the composition law (2.115)

$$\mathbf{a}' = \frac{1}{\eta'^2}\left(\mathbf{a}^0 - \frac{1}{c^2}\frac{\eta'}{1+\eta'}\mathbf{v}'\cdot\mathbf{a}^0\mathbf{v}'\right)\,.$$

Thus, omitting the prime, the following relations hold in any Galilean frame:

$$\boldsymbol{\omega} = \frac{1}{2}\left(1+\frac{1}{\eta^2}\right)\boldsymbol{\omega}^0 + \frac{1}{2c^2(1+\eta)}\left[(1-\eta)\boldsymbol{\omega}^0\cdot\mathbf{v}\mathbf{v} - (1+\eta)(\mathbf{k}_v^0\times\mathbf{v} + \mathbf{v}\times\mathbf{a}^0)\right],$$

$$\mathbf{k}_i = \frac{1}{\eta}\mathbf{k}_i^0 + \frac{1}{2}\left(1-\frac{1}{\eta}\right)^2\boldsymbol{\omega}^0\times\mathbf{e}_i^0 \tag{5.181}$$

$$+\frac{1}{2c^2(1+\eta)} \cdot \tag{5.182}$$
$$\left\{\left(\eta+\frac{1}{\eta}\right) v_i^0 \boldsymbol{\omega}^0 \times \mathbf{v} + 2(\boldsymbol{\omega}^0 \times \mathbf{v}) \cdot \mathbf{e}_i^0 \mathbf{v}\right.$$
$$+(\eta-1)\boldsymbol{\omega}^0 \cdot \mathbf{vv} \times \mathbf{e}_i^0 - \frac{1+\eta}{\eta} v_i^0 \mathbf{k}_v^0 + \left[(\eta-1)\mathbf{e}_i^0 - \frac{1}{c^2}\eta v_i^0 \mathbf{v}\right] \cdot \mathbf{k}_v^0 \mathbf{v}$$
$$\left. -\frac{1+\eta}{\eta}(v_i^0 \mathbf{a}^0 + \eta \mathbf{a}^0 \cdot \mathbf{e}_i^0 \mathbf{v}) + \frac{1}{c^2}(2+\eta) v_i^0 \mathbf{a}^0 \cdot \mathbf{vv}\right\},$$

where

$$\begin{cases} \mathbf{a}^0 = \eta^2 \left(\mathbf{a} + \dfrac{1}{c^2}\dfrac{\eta^2}{1+\eta}\mathbf{a} \cdot \mathbf{vv}\right) , \quad \mathbf{e}_i^0 = \mathbf{e}_i + \dfrac{1}{c^2}\dfrac{\eta^2}{1+\eta} v_i \mathbf{v}, \\ v_i^0 = \mathbf{v} \cdot \mathbf{e}_i^0 = \eta v_i. \end{cases} \tag{5.183}$$

So far, we find that *in the motion of a continuum, with respect to an arbitrary Galilean reference frame* S_{g}*, the values of the angular and deformation velocities* $\boldsymbol{\omega}$ *and* $\mathbf{k}_i$ *are related to the proper values* $\boldsymbol{\omega}^0$ *and* $\mathbf{k}_i^0$ *by the following equations:*

$$\boldsymbol{\omega} = \frac{1}{2}\left(1+\frac{1}{\eta^2}\right)\boldsymbol{\omega}^0 + \frac{1}{2c^2\eta}\left[\frac{1-\eta}{1+\eta}\boldsymbol{\omega}^0 \cdot \mathbf{vv} - \mathbf{v} \times (\eta^2 \mathbf{a} - \mathbf{k}_v^0)\right], \tag{5.184}$$

$$\mathbf{k}_i = \frac{1}{\eta}\mathbf{k}_i^0 + \frac{1}{2}\left(1-\frac{1}{\eta}\right)^2 \boldsymbol{\omega}^0 \times \mathbf{e}_i \tag{5.185}$$
$$+\frac{1}{2c^2(1+\eta)} \cdot$$
$$\left\{2\eta v_i \boldsymbol{\omega}^0 \times \mathbf{v} + 2(\boldsymbol{\omega}^0 \times \mathbf{v}) \cdot \mathbf{e}_i \mathbf{v}\right.$$
$$+(\eta-1)\boldsymbol{\omega}^0 \cdot \mathbf{vv} \times \mathbf{e}_i - \eta^2(1+\eta)(v_i \mathbf{a} + a_i \mathbf{v}) - (1+\eta) v_i \mathbf{k}_v^0$$
$$\left. + \left[(\eta-1)\mathbf{e}_i - \frac{2\eta^2}{c^2(1+\eta)} v_i \mathbf{v}\right] \cdot \mathbf{k}_v^0 \mathbf{v}\right\}.$$

In a relativistic context, the following *definition of rigid motion* is very useful: *a continuum is said to move rigidly in the sense of Born* [4] *if the proper deformation velocity vanishes identically*:

$$\mathbf{k}_i^0 = 0 , \qquad \forall t, \mathrm{P} \in C , \qquad (i = 1,2,3) . \tag{5.186}$$

This is obviously an *absolute property* of the motion, which must not be confused with rigidity in the classical sense. Then (5.186) implies that, in every Galilean frame, there is deformation; more precisely, from (5.184) it follows that

$$\boldsymbol{\omega} = \frac{1}{2}\left(1+\frac{1}{\eta^2}\right)\boldsymbol{\omega}^0 + \frac{1}{2c^2\eta}\left[\frac{1-\eta}{1+\eta}\boldsymbol{\omega}^0 \cdot \mathbf{vv} - \eta^2 \mathbf{v} \times \mathbf{a}\right], \tag{5.187}$$

$$\mathbf{k}_i = \frac{1}{2}\left(1 - \frac{1}{\eta}\right)^2 \boldsymbol{\omega}^0 \times \mathbf{e}_i \qquad (5.188)$$
$$+\frac{1}{2c^2(1+\eta)} \cdot$$
$$\left[2\eta v_i \boldsymbol{\omega}^0 \times \mathbf{v} + 2(\boldsymbol{\omega}^0 \times \mathbf{v}) \cdot \mathbf{e}_i \mathbf{v}\right.$$
$$\left. +(\eta - 1)\boldsymbol{\omega}^0 \cdot \mathbf{v}\mathbf{v} \times \mathbf{e}_i - \eta^2(1+\eta)(v_i \mathbf{a} + a_i \mathbf{v})\right] \neq 0.$$

Moreover, for each continuum motion, the local angular velocity is related to the spatial deformation gradient by means of (5.92); thus, in the case of a Born-rigid motion of the continuum the proper angular velocity $\boldsymbol{\omega}^0$ has a *global meaning* in C, in the sense that at each instant it is independent of the particle:

$$\partial_i \boldsymbol{\omega}^0 = 0 \,. \qquad (5.189)$$

Clearly, $\boldsymbol{\omega}^0$ is not an absolute constant; in fact, according to (5.94):

$$\partial_t \boldsymbol{\omega}^0 = \frac{1}{2}\mathrm{curl}\, \mathbf{a}^0 \neq 0 \,. \qquad (5.190)$$

On the other hand the angular velocity, relative to a generic S_{g}, *has no global meaning*, like $\boldsymbol{\omega}^0$, because it depends on both the velocity $\mathbf{v}$ of the continuum and the acceleration $\mathbf{a}$. More precisely, from $(5.187)_1$ we have

$$\boldsymbol{\omega} = \frac{1}{2}\left(1 + \frac{1}{\eta^2}\right)\boldsymbol{\omega}^0 + \boldsymbol{\psi} \,, \qquad (5.191)$$

with

$$\boldsymbol{\psi} \stackrel{\mathrm{def}}{=} \frac{1}{2c^2\eta}\left[\frac{1-\eta}{1+\eta}\boldsymbol{\omega}^0 \cdot \mathbf{v}\mathbf{v} - \eta^2 \mathbf{v} \times \mathbf{a}\right] \,. \qquad (5.192)$$

Thus, besides the deformation one has an angular precession $\boldsymbol{\psi}$, dependent on $\mathbf{v}$ and $\mathbf{a}$ as well as on $\boldsymbol{\omega}^0$ (it is independent of $\boldsymbol{\omega}^0$ only if $\boldsymbol{\omega}^0$ is perpendicular to $\mathbf{v}$). In the first approximation, we re-obtain the *Thomas precession* [5]:

$$\boldsymbol{\psi} \simeq -\frac{1}{2c^2}\mathbf{v} \times \mathbf{a} \,. \qquad (5.193)$$

5.11 Material Continuum. Number and Matter Density

Given a geometrical continuum C in S_{g}, one can pass to a material one by defining in C a function $\mu = \mu(t, y)$ representing the *relative material density*; with the generic element of the continuum in an initial volume $\mathrm{d}\mathcal{C} = \mathrm{d}y^1\, \mathrm{d}y^2\, \mathrm{d}y^3$ in C, is then associated the *elementary mass*:

$$\mathrm{d}m = \mu \mathcal{D}\, \mathrm{d}\mathcal{C} \,. \qquad (5.194)$$

This can be done in any Galilean frame; thus, in S'_g, the mass of the element is given by $\mathrm{d}m' = \mu' \mathcal{D}' \, \mathrm{d}\mathcal{C}$, and this must satisfy the transformation law (3.52)

$$\mathrm{d}m' = \frac{\sigma}{\alpha} \mathrm{d}m \,. \tag{5.195}$$

Equation (5.195) induces a transformation law for the density μ. In fact, from (5.116), we have

$$\mathrm{d}m' = \frac{\alpha}{\sigma} \frac{\mu'}{\mu} \mathrm{d}m \,,$$

so that (5.195) gives rise to the *transformation law of the density*

$$\mu' = \mu \left(\frac{\sigma}{\alpha} \right)^2 \,, \tag{5.196}$$

as well as to a *finite invariant*

$$\frac{\mu'}{\eta'^2} = \frac{\mu}{\eta^2} = \text{inv.} \tag{5.197}$$

Condition (5.194) can be easily explained in terms of the substitution theorem for multiple integrals, according to which the measure element in C is expressed, in Cartesian coordinates, by the product $\mathrm{d}x^1 \; \mathrm{d}x^2 \; \mathrm{d}x^3$, and in generic curvilinear coordinates y^i by $|\mathcal{D}(y^1, y^2, y^3)| \mathrm{d}y^1 \; \mathrm{d}y^2 \; \mathrm{d}y^3$, with $\mathcal{D}$ the Jacobian determinant of the transformation $x^i = x^i(y^1, y^2, y^3)$. In our case, since $\mathcal{D} > 0$, the measure element is given by $\mathcal{D} \; \mathrm{d}\mathcal{C}$. We also note that the relative scalar $1/\mathcal{D}$ has the meaning of *particle number density.* Thus a material continuum is a geometrical one endowed with a matter density μ. For each of its elements, because of (5.194) and (5.195), we can apply the above analysis of the material point.

When $c \to \infty$, we recover the classical invariance of the density, with respect to changes of Galilean frames: $\mu' = \mu$. In relativity, instead, the ratio μ/η^2 is invariant, and the density, once known in a certain frame, is also known in any other frame through (5.197); it will depend on the relative velocity $\mathbf{u}$ of S'_g with respect to S_g as well as on the velocity $\mathbf{v}(t, y)$ of the element of the continuum in S_g; that is, we have now a *dynamical* notion of mass.

5.12 Absolute Kinematics. Proper Quantities

From the absolute point of view a three-dimensional continuous system is represented by the ∞^3 world lines of the single particles, all future-oriented, and not intersecting each other since the continuum should be thought of as the set of ∞^3 distinct material points (at each instant and in every Galilean frame). In other words, the evolution of a continuum is geometrically defined by a *unit timelike vector field*, whose flow lines are the histories of the particles of the continuum itself. In this sense, the family Γ of such lines represents a generalization of those special (linear) congruences, characterized by constant vector fields, which represent the Galilean frames. In that case, they are global congruences, because the straight lines cover all M_4; for a generic continuum, instead, the flow lines cover only a certain *world tube* $\mathcal{T} \in M_4$.

Once a Galilean frame (that is a temporal direction $\boldsymbol{\gamma}$ and the associated spatial platform Σ) is fixed, the configurations of the continuum C, relative to the various instants t, coincide with the plane sections of the tube $\mathcal{T}$, orthogonal to $\boldsymbol{\gamma}$, or with their spatial projections onto the reference spatial platform Σ. In the interior of $\mathcal{T}$, besides the world lines of the particles, is defined the 4-velocity $\mathbf{V}$. This is determined from the equations of the world lines

$$x^0 = ct\ , \qquad x^i = x^i(t, y)\ , \tag{5.198}$$

by differentiation with respect to the proper time τ of each particle, given by

$$\begin{aligned} \tau(t_0, t, y) &= \frac{1}{c}\int_{t_0}^{t} \sqrt{-m_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}\mathrm{d}t \\ &= \int_{t_0}^{t} \sqrt{1 - \frac{v^2(t, y)}{c^2}}\mathrm{d}t\ , \end{aligned} \tag{5.199}$$

depending on y^i that is on the considered particle. The 4-velocity $\mathbf{V}$ can be obviously expressed either in Lagrangian or Eulerian terms, because of the invertibility of $(5.198)_2$. The 4-velocity corresponding to the generic point $E \in \mathcal{T}$: $\mathbf{V}(E)$ defines the tangent vector, at E, to the corresponding particle world line or the 4-velocity in its rest frame S_0. In S_0, the relative velocity of the particle, at E, vanishes: $\mathbf{v}_0 = 0$, and the invariant relations, in the passage from one Galilean frame to another, assume a precise meaning. They are the proper quantities, i.e. relative to S_0; thus, (5.121) gives rise to the invariant

$$\mathcal{D}_0(t, y) = \eta\mathcal{D} \equiv \frac{\mathcal{D}(t, y)}{\sqrt{1 - \dfrac{v^2(t, y)}{c^2}}} > 0\ , \tag{5.200}$$

which characterizes the *proper volume element* $\mathrm{d}C_0 \equiv \mathcal{D}_0\mathrm{d}\mathcal{C}$ with respect to fixed Lagrangian coordinates. Hence, the *proper numerical density* of the particles $1/\mathcal{D}_0$ follows. Similarly, from (5.197) one gets the *proper density of proper mass*:

$$\mu_0(E) = \frac{\mu}{\eta^2} \equiv \mu(t, y)\left[1 - \frac{v^2(t, y)}{c^2}\right] \quad \sim \quad \mu_0 = \frac{\mathrm{d}m_0}{\mathrm{d}C_0}\ . \tag{5.201}$$

Finally, we have seen that (5.122) is closely related to a first-order differential invariant, namely the divergence of the 4-velocity: $\partial_\alpha V^\alpha$. However, from (5.199)

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \eta\partial_t = \eta(\,)^{\cdot}\ , \tag{5.202}$$

we have the following transformation of (5.122):

$$\mathcal{B} = \partial_t \eta + \frac{\eta}{\mathcal{D}} \partial_t \mathcal{D} = \frac{1}{\mathcal{D}} \partial_t (\eta \mathcal{D}) = \frac{1}{\eta \mathcal{D}} \frac{\mathrm{d}}{\mathrm{d}\tau} (\eta \mathcal{D}) = \frac{1}{\mathcal{D}_0} \frac{\mathrm{d}\mathcal{D}_0}{\mathrm{d}\tau} .$$

Thus the invariant $\mathcal{B}$, defined by (5.130), turns out to be related either to the 4-velocity $\mathbf{V}$, because of (5.133), or to the invariant $\mathcal{D}_0(E) = D_0(x)$:

$$\mathcal{B}(t, y(x)) = B(t, x) = \partial_\alpha V^\alpha . \tag{5.203}$$

From this follows a differential relation between D_0 and $\mathbf{V}$ (as a consequence of (5.125), in the instantaneous rest frame):

$$\frac{1}{D_0} \frac{\mathrm{d}}{\mathrm{d}\tau} D_0 = \partial_\alpha V^\alpha . \tag{5.204}$$

This equation, since $D_0 = D_0(x)$ and $\mathrm{d}D_0/\mathrm{d}\tau = V^\alpha \partial_\alpha D_0$, implies the *conservation of proper numerical density* :

$$\partial_\alpha \left(\frac{1}{\mathcal{D}_0} V^\alpha \right) = 0 , \qquad \forall E \in \mathcal{T} . \tag{5.205}$$

Note. From the various examples of scalar quantities that we have studied in the preceding sections, one sees clearly the difference between scalar functions of the event E and scalar invariants with respect to the Lorentz transformation: the latter depend only on the event E, while the former depend on other variables also.

For example, for a given continuous system in motion, the quantities η, $\partial_t(\eta\mathcal{D})$ are all scalar functions, but not scalar invariants, because they depend on the considered Galilean frame besides the chosen event $E \equiv (x^\alpha) \equiv (t, x)$; more precisely, they are functions of the surface element, containing E and $\gamma \in \mathcal{C}_3^+$. The products $\eta\mathcal{D}$, $\partial_t(\eta\mathcal{D})/\mathcal{D}$ are scalar invariants, because they do not depend on the chosen Galilean frame, but only on the event $E \equiv (x^\alpha) \equiv (x'^\alpha)$.

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6

Elements of Classical Dynamics of a Continuum

6.1 Introduction to Continuum Classical Dynamics

Once the essential geometrical–kinematical quantities (relative or absolute) necessary for the description of a three-dimensional continuum have been introduced, we can move on to the fundamental dynamical aspects of the relativistic theory. First of all, let us examine the *classical framework of the equations of the continuum dynamics* in the context of the Galilean frames where such equations are invariant. The passage to an arbitrary rigid frame is obtained with the usual procedure adding to the equation of motion the inertial forces (dragging and Coriolis forces). As for the kinematical case, here we limit ourselves to the essential elements of the dynamics.[1]

Following the notation already introduced, let $S_{\rm g}$ be a Galilean frame, associated with an arbitrary Cartesian orthogonal triad: $\mathcal{T} \equiv {\rm O}\,\mathbf{c}_1\,\mathbf{c}_2\,\mathbf{c}_3$; let C be the actual configuration (in $S_{\rm g}$) of the continuum system S, x^i $(i = 1, 2, 3)$ the Cartesian coordinates of the generic point ${\rm P} \in C$ and c an arbitrary portion of C with boundary σ.

In the continuum scheme, *the mechanical action is represented by two kinds of force*:

1. *mass or volume forces*, specified by a characteristic vectorial function $\mathbf{F}({\rm P}, ...)$, defined in C, in the sense that, for any portion $c \in C$, the resultant force and the resultant moment (with respect to O) of such forces are expressed, respectively, by the following volume integrals:

$$\mathbf{r}[c] = \int_c \mu \mathbf{F} \, {\rm d}C \, , \qquad \mathbf{m}_{\rm O}[c] = \int_c {\rm OP} \times \mu \mathbf{F} \, {\rm d}C \, ; \tag{6.1}$$

2. *contact or surface forces*, specified by the vectorial function $\boldsymbol{\phi}_n({\rm P})$, defined on the boundary of C and for each direction $\mathbf{n}$: *specific stress at* P *relative to the direction* $\mathbf{n}$. This is a vectorial function depending on P as well as

[1] For a more detailed discussion the reader may refer to [1], p. 525 and [2], p. 181.

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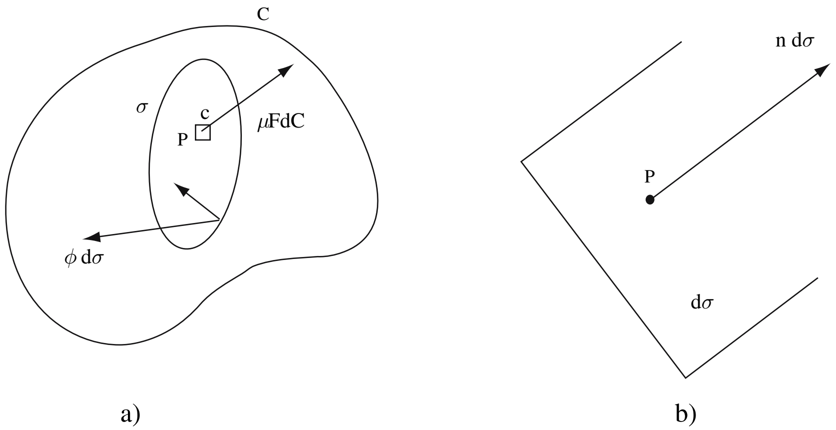

Fig. 6.1. Distribution of forces and stresses for a continuum

on the direction $\mathbf{n}$, i.e. on the considered 3-plane in P (see Fig. 6.1), such that, for any surface $\sigma \in \partial C$, the resultant of force and moment (with respect to O) of such forces are expressed, respectively, by the following surface integrals:

$$\mathbf{r}^{(\mathrm{i})}[\sigma] = \int_\sigma \boldsymbol{\phi}_n \, \mathrm{d}\sigma \,, \qquad \mathbf{m}_{\mathrm{O}}^{(\mathrm{i})}[\sigma] = \int_\sigma \mathrm{OQ} \times \boldsymbol{\phi}_n \, \mathrm{d}\sigma \,. \tag{6.2}$$

Adapting, now, the *fundamental equations of the mechanics* to the arbitrary portion c of the continuum and passing to the limit $c \to P$, with the necessary regularity hypothesis, one gets the following (local) Eulerian conditions, which are no longer depending on the limit itself:

I *Cauchy theorem*, which specifies the dependence of the specific stresses on $\mathbf{n}$:

$$\boldsymbol{\phi}_n = n_i \boldsymbol{\phi}^i \,, \qquad \boldsymbol{\phi}^i = \boldsymbol{\phi}_n \Big|_{n=c^i} \,; \tag{6.3}$$

hence, $\boldsymbol{\phi}_n$ is a linear and homogeneous function of the director cosines of $\mathbf{n}$ in each point of the boundary of C: ∂C.

II *First indefinite equation*:

$$\mu \dot{\mathbf{e}} = \mu \mathbf{F} - \partial_i \boldsymbol{\phi}^i \,, \tag{6.4}$$

where the dot denotes the *substantial derivative*

$$(\,)^{\cdot} \stackrel{\text{def}}{=} \partial_t(\,) + e^i \partial_i(\,) \tag{6.5}$$

III *Second indefinite equation*, that is reciprocity relations for the stresses:

$$\boldsymbol{\phi}_n \cdot \mathbf{n}' = \boldsymbol{\phi}_{n'} \cdot \mathbf{n} \,, \qquad \forall \mathbf{n}, \mathbf{n}' \tag{6.6}$$

To these (Eulerian) equations one must add the *continuity equation*, which represents the principle of mass conservation in local form. More precisely, in the classical situation, for each particle of the continuum the mass element

$$\mathrm{d}m = \mu\mathcal{D}\,\mathrm{d}\mathcal{C}\ , \qquad \mathrm{d}\mathcal{C} = \mathrm{d}y^1\ \mathrm{d}y^2\ \mathrm{d}y^3\ , \tag{6.7}$$

being invariant with respect to the choice of the Galilean frame S_g, is independent on t (in S_g):

$$\mu\mathcal{D} = \text{const.}\ , \qquad \forall y^i \quad \text{fixed}\ . \tag{6.8}$$

In *Lagrangian form*, (6.8) is equivalent to the condition

$$\partial_t(\mu\mathcal{D}) = 0\ , \quad \forall\, t, y^i\ , \tag{6.9}$$

and, from here, one has the *Eulerian form*

$$(\mu D)^{\cdot} = 0\ , \quad \forall\, t, x^i\ , \tag{6.10}$$

where $D(t,x) \stackrel{\mathrm{def}}{=} \mathcal{D}(t, y(x))$. Thus, using (5.125) and (6.9), this last relation can be cast in the form

$$\frac{\dot{D}}{D} = \mathrm{div}\mathbf{e} \equiv \partial_i e^i \tag{6.11}$$

and gives the ordinary continuity equation: $\dot{\mu} + \mu\ \mathrm{div}\,\mathbf{e} = 0$. This equation, in turn, using (6.5), can be transformed to obtain the following:

IV *principle of mass conservation:*

$$\partial_t\mu + \partial_i(\mu e^i) = 0\ . \tag{6.12}$$

The scheme of the continuum is somehow incomplete. In fact, the evaluation of the kinetic energy, defined, for each part $c \in C$ of the continuum, by the integral

$$T[c] = \frac{1}{2}\int_c \mu e^2 \mathrm{d}C\ , \tag{6.13}$$

implies that the kinetic energy of each element can be confused as that of the *centre of mass* only, neglecting the motion relative to the centre of mass itself. In other words, the continuum scheme ignores the thermal energy. Because of this *evaluation defect*, in the continuum scheme the energy theorem is a direct consequence of the equations of motion, as for the case of the single particle. It can be written in the usual form:

$$\dot{T}[c] = W[c]\ , \qquad W[c] = W^{(\mathrm{e})}[c] + W^{(\mathrm{i})}[c]\ , \tag{6.14}$$

with $W^{(\mathrm{e})}[c]$ the power of the external forces (mass forces in c and contact forces on the surface σ) and $W^{(\mathrm{i})}[c]$ the power of the internal (contact) forces, that is:

$$W^{(\mathrm{i})}[c] = \int_c \boldsymbol{\phi}^i \cdot \partial_i \mathbf{e} \, \mathrm{d}C \ . \tag{6.15}$$

The above-mentioned defect of the scheme can be avoided by correcting the energy theorem (6.14) with the aid of the *first law of thermodynamics*, summarized by the following three axioms:

(a) each portion c of the continuum has an *internal energy* $\mathcal{E}[c]$:

$$\mathcal{E}[c] = \int_c \mu \, \epsilon \, \mathrm{d}C \ , \tag{6.16}$$

where ϵ is the specific (i.e. for unit mass) internal energy;

(b) the heat is also energy (apart from a conversion factor) which enters the energy conservation law through its power $Q[c]$ (heat absorbed, in algebraic sense, by c per unit time):

$$Q[c] = \int_c \mu \, q \, \mathrm{d}C \ , \tag{6.17}$$

where q is the specific (i.e. per unit mass) *thermal power*;

(c) for any portion $c \in C$, the following balance relation holds:

$$\dot{T} + \dot{\mathcal{E}} = W^{(\mathrm{e})} + Q \ , \tag{6.18}$$

which corrects (6.14). Hence, using the mass conservation (6.12), (6.18) assumes the form $\dot{\mathcal{E}} = Q - W^{(\mathrm{i})}$, which can be put in the local form:

V *First law of thermodynamics*:

$$\dot{\epsilon} = q - \frac{1}{\mu} w^{(\mathrm{i})} \ , \tag{6.19}$$

with $w^{(\mathrm{i})}$ *the specific power of the internal (contact) forces*, i.e. per unit volume:

$$w^{(\mathrm{i})} = \boldsymbol{\phi}^i \cdot \partial_i \mathbf{e} \ . \tag{6.20}$$

Equation (6.20) can be rewritten in its Eulerian form using the relation

$$\partial_i \mathbf{e} = \overset{(\mathrm{c})}{k}{}_{ik} \, \mathbf{c}^k + \boldsymbol{\omega} \times \mathbf{c}_i . \tag{6.21}$$

For ordinary continua, such a power is independent of the angular velocity. This follows from (6.6); in fact, if one introduces the *Eulerian stress tensor* X^{ik} $(i, k = 1, 2, 3)$, by means of the decomposition

$$\boldsymbol{\phi}^i = X^{ik} \mathbf{c}_k \ , \tag{6.22}$$

then, using (6.3), (6.6) becomes $(X^{ik} - X^{ki}) n_i n'_k = 0$; thus, the symmetry property

$$X^{ik} = X^{ki}, \quad (i, k = 1, 2, 3) \tag{6.23}$$

holds, and (6.20) assumes the form

$$w^{(\mathrm{i})} = X^{ik} \overset{(\mathrm{c})}{k}_{ik} , \tag{6.24}$$

with $\overset{(\mathrm{c})}{k}_{ik}$ denoting the deformation velocity, in Eulerian form:

$$\overset{(\mathrm{c})}{k}_{ik} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} e_k + \frac{\partial}{\partial x^k} e_i\right) . \tag{6.25}$$

6.2 Lagrangian Form of the Fundamental Equations

We now briefly recall some concepts of the Lagrangian mechanics of continua. The relativistic Cauchy equation (6.4) is of Eulerian kind and can be cast into the corresponding Lagrangian form without changing the (arbitrarily chosen) Galilean frame S_{g}.

Let y^i $(i = 1, 2, 3)$ be a set of Lagrangian variables in S_{g}, which we will interpret as the curvilinear coordinates of the points in the actual configuration C of the continuum. Let $\{\mathbf{e}_i\}$ be the *natural basis* relative to the coordinates y^i, $g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k$ the Lagrangian metric with associated Christoffel symbols of the second-type $\Gamma^h{}_{ik}$, and the covariant derivative be denoted by ∇_i. The partial derivatives of the basis vectors give the following geometrical–kinematical relations:

$$\partial_i \mathbf{e}_k = \Gamma^h{}_{ik} \mathbf{e}_h , \quad \partial_t \mathbf{e}_i = \partial_i \mathbf{v} \equiv (\nabla_i v^k) \mathbf{e}_k , \tag{6.26}$$

where $\mathbf{v} = v^k \mathbf{e}_k$ is the Lagrangian velocity, $\partial_i = \partial/\partial y^i$ and $\partial_t = \partial/\partial t$.

The velocity gradient summarizes the two fundamental tensors: $\omega_i{}^k$ (angular velocity) and $k_i{}^k$ (deformation velocity):

$$\nabla_i v^k = \omega_i{}^k + k_i{}^k . \tag{6.27}$$

Introducing the dual basis $\{\mathbf{e}^i\}$ of $\{\mathbf{e}_k\}$ with $\mathbf{e}^i \cdot \mathbf{e}_k = \delta^i_k$, the symmetric tensor $k_{ik} = g_{ij} k^j{}_k$ assumes the form

$$k_{ik} = \frac{1}{2}(\nabla_i v_k + \nabla_k v_i) = \frac{1}{2}\partial_t g_{ik} ; \tag{6.28}$$

similarly, the antisymmetric tensor $\omega_{ik} = g_{ij}\omega^j{}_k = (\nabla_i v_k - \nabla_k v_i)/2$ is equivalent to the vector

$$\boldsymbol{\omega} = \mathbf{e}^i \times \partial_t \mathbf{e}_i = \frac{1}{2}\omega_{ik} \mathbf{e}^i \times \mathbf{e}^k , \qquad \omega_{ik} = \boldsymbol{\omega} \cdot \mathbf{e}_i \times \mathbf{e}_k . \tag{6.29}$$

The Lagrangian form of the continuum dynamical equations is obtained by transforming $\mathbf{F}$ and q, defined by (6.4) and (6.19); this requires the introduction of the *Lagrangian stresses* $\mathbf{Y}^i$:

$$\mathbf{Y}^i = \frac{\partial y^i}{\partial x^k}\boldsymbol{\phi}^k = \frac{\partial y^i}{\partial x^k}X^{kj}\mathbf{c}_j \ . \tag{6.30}$$

In detail, as concerns (6.4), one should take into account the following:

1. on the left-hand side one has the substantial derivative, which, in Lagrangian terms, becomes the partial derivative ∂_t;
2. the divergence of the vectors $\boldsymbol{\phi}^i$ becomes

$$\frac{\partial}{\partial x^i}\boldsymbol{\phi}^i = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\mathbf{Y}^i) \ . \tag{6.31}$$

In fact, from (6.30), one has

$$\begin{aligned}\frac{\partial}{\partial x^i}\boldsymbol{\phi}^i &= \frac{\partial}{\partial x^i}\left(\frac{\partial x^i}{\partial y^k}\mathbf{Y}^k\right) = \frac{\partial y^h}{\partial x^i}\frac{\partial}{\partial y^h}\left(\frac{\partial x^i}{\partial y^k}\mathbf{Y}^k\right)\\ &= \partial_h\mathbf{Y}^k\frac{\partial y^h}{\partial x^i}\frac{\partial x^i}{\partial y^k} + \mathbf{Y}^k\frac{\partial^2 x^i}{\partial y^h\partial y^k}\frac{\partial y^h}{\partial x^i} \ ;\end{aligned}$$

next, using

$$\frac{\partial y^h}{\partial x^i}\frac{\partial x^i}{\partial y^k} = \delta^h_k,$$

and the relations

$$\frac{\partial^2 x^i}{\partial y^h\partial y^k}\frac{\partial y^l}{\partial x^i} = \Gamma^l{}_{hk}, \qquad \Gamma^h{}_{hk} = \frac{1}{\sqrt{g}}\partial_k\sqrt{g} \ , \tag{6.32}$$

one gets (6.31):

$$\frac{\partial}{\partial x^k}\boldsymbol{\phi}^k = \partial_k\mathbf{Y}^k + \Gamma^h{}_{hk}\mathbf{Y}^k = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\mathbf{Y}^i) \ .$$

Equation (6.4) then becomes

$$\mu\partial_t\mathbf{v} = \mu\mathbf{F} - \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\mathbf{Y}^i) \ ; \tag{6.33}$$

introducing the *Lagrangian stress characteristics* Y^{ik} which are symmetric, like the corresponding Eulerian quantities X^{ik}, and defined by

$$\mathbf{Y}^i = Y^{ik}\mathbf{e}_k \ , \tag{6.34}$$

Equation (6.33), in scalar terms, then becomes

$$\mu a^k = \mu F^k - \nabla_i Y^{ik} \ , \tag{6.35}$$

and using the Lagrangian form of (6.12) (*mass conservation*)

$$\partial_t \mu + \mu \nabla_i v^i = 0 \tag{6.36}$$

leads to

$$\mu \partial_t v^k = \mu F^k - \nabla_i Y^{ik} - \mu k v^k \ , \tag{6.37}$$

with $k = g^{ik} k_{ik}$ the *cubic dilation coefficient.* Equation (6.33) and its equivalent (6.37) give the acceleration in terms of the sources and will be used in the *intrinsic formulation* of continuum mechanics [3].

Finally, the power of the internal forces (6.24), in Lagrangian terms, still has the same structure

$$w^{(\mathrm{i})} = Y^{ik} k_{ik}; \tag{6.38}$$

k_{ik} can now be expressed either by means of the velocity v_k or the Lagrangian metric g_{ik}:

$$k_{ik} = \frac{1}{2} \partial_t g_{ik} \ . \tag{6.39}$$

Equation (6.38) then assumes the form

$$w^{(\mathrm{i})} = \frac{1}{2} Y^{ik} \partial_t g_{ik} \ , \tag{6.40}$$

so that for each particle (i.e. for fixed y^i) $w^{(\mathrm{i})} \mathrm{d}t$ is a differential form in the variables $\mathrm{d}g_{ik}$:

$$w^{(\mathrm{i})} \ \mathrm{d}t = \frac{1}{2} Y^{ik} \, \mathrm{d}g_{ik}, \qquad (y^i = \text{fixed}) \ . \tag{6.41}$$

6.3 Isotropic Systems

In Lagrangian terms the fundamental equations are (6.8) and (6.35), that is

$$\begin{cases} \mu a^k - \mu F^k + \nabla_i Y^{ik} = 0 \ , \qquad (k = 1, 2, 3) \\ \\ \mu \mathcal{D} - \mu_* = 0 \ , \qquad \mathcal{D} = \sqrt{g}, \end{cases} \tag{6.42}$$

which involve the mass density μ, the particle number density $1/\mathcal{D}$, the acceleration a^k and the stresses Y^{ik}; the mass force F^k and the reference density μ_* are assumed to be assigned; however, comparing with the Eulerian framework, we now have the *presence of the metric* g_{ik}, either through the Christoffel symbols or, explicitly, in $\mathcal{D}$; furthermore, all the components are referred to the basis $\{\mathbf{e}_i\}$ related to the continuum itself and, hence, the variable with it. In other words, the Lagrangian dynamics is not exhausted in the relations (6.42), involving the metric g_{ik} and the same basis $\{\mathbf{e}_i\}$; from the point of view of the variables the evolutionary problem is enlarged and an *intrinsic Cauchy problem* can be formulated too (see below).

By using the identity $\partial_t \mathbf{e}_i \equiv \partial_i \mathbf{v}$, we have

$$\mathbf{a} = \partial_t (v^i \mathbf{e}_i) = (\partial_t v^i)\mathbf{e}_i + v^i \partial_i \mathbf{v} \ ,$$

and thus the system (6.42) can be written, in terms of the velocity, in the following form:

$$\begin{cases} \mu(\partial_t v^k + v^i \nabla_i v^k) - \mu F^k + \nabla_i Y^{ik} = 0 \ , \qquad (k = 1, 2, 3), \\ \dfrac{1}{\mu}\partial_t \mu + \operatorname{div} \mathbf{v} = 0 \ . \end{cases} \tag{6.43}$$

In either form, Lagrangian or Eulerian, however, the number of equations (four) is different from the number of the unknowns (ten, without considering the metric). This should not be surprising, because the equations obtained do not yet take into account the physical properties of matter which is schematized by the continuum; in this sense, they are valid for a fluid, a solid, an elastic system as well as a plastic one, for reversible or irreversible transformations.

To get the number of equations equal to the number of unknowns, one must introduce the characteristic properties of the considered material. For instance, the *isotropic property*, which we are going to discuss, is enough to reduce to seven the number of the unknowns.

Let us note that, once the reference configuration C_* and the corresponding metric g_{*ik} are fixed, the stress tensor Y^{ik} and the deformation tensor $\epsilon_{*ik} = (g_{ik} - g_{*ik})/2$, pulled back to C_*, are both symmetric; hence, they each have a triad of eigenvectors with respect to g_{*ik}: the *stress and deformation principal triads*, respectively.

We will call the continuum *isotropic*[2] if it admits a configuration C_* such that *the principal deformation triad*, relative to the displacement $C_* \to C$, *is also a principal tension triad with respect to the metric* g_{*ik} of C_* for each transformation of the system and for each C.

This definition uses the concepts of deformation and stress only; the preferred configuration C_*, whose existence is postulated, enters only through the Lagrangian metric g_{*ik}; thus, the invariance with respect to the choice of the Lagrangian coordinates y^i is complete.

The property that the Lagrangian stress tensor Y^{ik} should admit, with respect to the reference metric g_{*ik}, the same eigendirections of the deformation tensor ϵ_{ik}, has an important consequence: the stress tensor Y^{ik} is necessarily a *polynomial function* of the second degree in ϵ_{*ik} (see [4], p. 42):

$$Y_i^{\ k} = p\delta_i^k + q{\epsilon_*}_i^{\ k} + r{\epsilon_*}_i^{\ j}{\epsilon_*}_j^{\ k} \ , \tag{6.44}$$

with

$$Y_i^{\ k} \stackrel{\text{def}}{=} g_{*ij} Y^{jk} \ , \qquad {\epsilon_*}_i^{\ k} \stackrel{\text{def}}{=} \frac{1}{2} {g_*}^{kj} (g_{ij} - g_{*ij}) \ , \tag{6.45}$$

[2] This definition is due to A. Signorini, see [4] p. 136.

where the metric ${g_*}^{ik}$ is the reciprocal of g_{*ik}. Equation (6.44) shows that the stress tensor can be expressed in terms of the three scalars p, q and r, reducing to three its components; obviously, the metrics g_{ik} of C and that g_{*ik} of C_* are both involved.

A *nonviscous fluid*, that is, a continuum which in any configuration C has the specific stress reduced to a single pressure: $\boldsymbol{\phi}_u = p_u\mathbf{u}$, with $p_u > 0$, is a particular isotropic system. Moreover, because of the Cauchy theorem (6.3), p_u is independent of $\mathbf{u}$: $p_u = p$ (pressure). Equivalently, we have

$$Y_i{}^k = p\delta_i^k \,, \tag{6.46}$$

which is a particular case of (6.44).

6.4 Symbolic Relation of Continuum Mechanics

With (6.33) one must associate the *boundary conditions* on ∂C, which give the local identity between the surface force $\mathbf{f}$ and the *normal stress*: $\mathbf{Y}_N = N_i\mathbf{Y}^i$. We thus have the following system

$$\begin{cases} \mathbf{V}(\mathrm{P}) \equiv \mu(\mathbf{F} - \mathbf{a}) - \dfrac{1}{\sqrt{g}}\partial_i(\sqrt{g}\mathbf{Y}^i) = 0 \qquad \text{in}\, C, \\ \mathbf{W}(\mathrm{Q}) \equiv \mathbf{f} - N_i\mathbf{Y}^i = 0 \qquad \text{in}\, \partial C, \end{cases} \tag{6.47}$$

which can be summarized by a single scalar relation: the *symbolic relation of the continuum systems.*

Let us consider an arbitrary vector function $\boldsymbol{\xi}(\mathrm{P})$, defined on $C + \partial C$, and having a regularity class C^n (i.e. a continuous function, with the partial derivative also continuous up to the *nth* order); when $\mathbf{V} = 0$ and $\mathbf{W} = 0$ the integral

$$I[\boldsymbol{\xi}] \equiv \int_C \mathbf{V}\cdot\boldsymbol{\xi}\;\mathrm{d}C + \int_{\partial C} \mathbf{W}\cdot\boldsymbol{\xi}\;\mathrm{d}\Sigma \tag{6.48}$$

clearly vanishes for any choice of the function $\boldsymbol{\xi}$: $I[\boldsymbol{\xi}] = 0$. The converse is less trivial: if the integral (6.48) is zero for any choice of $\boldsymbol{\xi}$, then $\mathbf{V} = 0$ in C and $\mathbf{W} = 0$ in ∂C, simultaneously. To show this, let us assume that, for any choice of $\boldsymbol{\xi}(\mathrm{P})$, it is $I[\boldsymbol{\xi}] = 0$

$$\int_C \mathbf{V}\cdot\boldsymbol{\xi}\;\mathrm{d}C + \int_{\partial C} \mathbf{W}\cdot\boldsymbol{\xi}\;\mathrm{d}\Sigma = 0, \qquad \forall\boldsymbol{\xi}\,, \tag{6.49}$$

but $\mathbf{V}$ is not identically zero; for instance, $V^1 > 0$ for a certain point $\mathrm{P}_0 \in C$. Because of the continuity of $\mathbf{V}(\mathrm{P})$, there exists a neighbourhood of P_0 belonging to C, say a sphere $S_\epsilon(\mathrm{P}_0)$, with centre in P_0 and radius ϵ, in which $V^1 > 0$. Let us assume, then, for $\boldsymbol{\xi}(\mathrm{P})$, the following choice:

$$\boldsymbol{\xi} = \begin{cases} (\epsilon^2 - |\mathrm{P}_0\mathrm{P}|^2)^{n+1}\mathbf{e}^1, & \forall\,\mathrm{P} \in S_\epsilon(\mathrm{P}_0), \\ 0, \quad \forall\,\mathrm{P} \notin S_\epsilon, \end{cases} \tag{6.50}$$

which satisfies the required regularity conditions.[3] With such a choice of $\boldsymbol{\xi}$, the second term of (6.49) vanishes, while the first reduces to

$$\int_{S_\epsilon} V^1(\epsilon^2 - |\mathrm{P}_0\mathrm{P}|^2)^{n+1}\,\mathrm{d}C\,.$$

As $V^1(\epsilon^2 - |\mathrm{P}_0\mathrm{P}|^2)^{n+1} > 0$ in the whole of S_ϵ, the above integral never vanishes, contrary to the hypothesis. Hence (6.49) implies that $\mathbf{V} = 0$ in C necessarily, and similarly one can show that $\mathbf{W} = 0$ in ∂C.

We have thus proven the equivalence between the system (6.47) and the scalar relation (6.49) (*first fundamental lemma of variational calculus*).

Moreover, taking into account the expression (6.47) for $\mathbf{V}$ and $\mathbf{W}$, the functional (6.48) can be written as

$$I[\boldsymbol{\xi}] = \int_C \mu(\mathbf{F} - \mathbf{a}) \cdot \boldsymbol{\xi}\,\mathrm{d}C - \int_C \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\mathbf{Y}^i) \cdot \boldsymbol{\xi}\,\mathrm{d}C + \int_{\partial C} (\mathbf{f} - N_i\mathbf{Y}^i) \cdot \boldsymbol{\xi}\,\mathrm{d}\Sigma\,,$$

and, transforming the second integral by means of the divergence theorem,[4] (6.49) assumes the following form (*symbolic relation of continuum mechanics*):

$$\int_C \mu(\mathbf{F} - \mathbf{a}) \cdot \boldsymbol{\xi}\,\mathrm{d}C + \int_C \mathbf{Y}^i \cdot \partial_i\boldsymbol{\xi} + \int_{\partial C} \mathbf{f} \cdot \boldsymbol{\xi}\,\mathrm{d}\Sigma = 0\,. \tag{6.51}$$

Interpreting the vectorial function $\boldsymbol{\xi}(\mathrm{P})$ as a *nominal velocity field* over the points of C, the latter equation implies that *at any instant, the full nominal power of all the forces acting on the system: mass, inertial, internal and surface, vanishes.*

Using (6.51) one confirms the Lagrangian expression of the specific power of the internal forces

$$w^{(\mathrm{i})} = \mathbf{Y}^i \cdot \partial_i\mathbf{v}\,. \tag{6.52}$$

Clearly, if one pulls back the stresses $\mathbf{Y}^i$ to the reference configuration C_* and uses the decomposition with respect to the basis $\{\mathbf{e}_i^*\}$ corresponding to $\{\mathbf{e}_i\}$[5] then one finds

$$\mathbf{Y}^i = \chi^{ik}\mathbf{e}_k^*\,, \tag{6.53}$$

where the vectors $\mathbf{e}_k^*$ do not depend on time and the coefficients χ^{ik} are termed *Piola–Kirchhoff characteristics.* They are *nonsymmetric* and related by the condition

[3] $\boldsymbol{\xi}$, as defined by (6.50), is C^n in $C + \partial C$.

[4] The product $\mathbf{Y}^i \cdot \boldsymbol{\xi}$, under a Lagrangian coordinate change, transforms as the components of a contravariant vector.

[5] The bijective mapping $C \leftrightarrow C_*$ induces a pointlike correspondence, which is naturally extended to the associated vectors as well as tensors.

$$\chi^{ik}\mathbf{e}_i^* \times \mathbf{e}_k = 0\ . \tag{6.54}$$

Equation (6.38) becomes

$$w^{(\mathrm{i})} = \chi^{ik}\mathbf{e}_k^* \cdot \partial_i \mathbf{v} = \chi^{ik}(k_{ij} + \omega_{ij})\, \mathbf{e}_k^* \cdot \mathbf{e}^j\ . \tag{6.55}$$

and the *angular velocity* ω_{ij} now appears.

6.5 Reversible Transformation Systems. Free Energy

A continuous system is said to undergo a *reversible transformation* if, besides the internal energy ϵ, it admits a second characteristic function: the *entropy* S, additive as well and expressible by the *specific entropy* s,

$$S = \int_C \mu s\ \mathrm{d}C\ ; \tag{6.56}$$

such a function is defined so that for each transformation of the system, and using the *absolute temperature scale*, the ratio between the thermal power q and the temperature θ coincides with the temporal derivative of the function s:

$$\frac{q}{\theta} = \frac{\mathrm{d}s}{\mathrm{d}t}\ . \tag{6.57}$$

Clearly, the specific entropy, like the internal energy, is defined up to an additive constant, and its form is suggested by the physical properties of the material body schematized by the continuum. Again, like ϵ, s should be thought of as a function of the state parameters, say η_1, η_2,..., besides the Lagrangian coordinates y^i and time.

From an analytic point of view, the reversible transformation systems are characterized by the property that the ratio $(q/\theta)\mathrm{d}t$ *is an exact* 1-*form*; that is, for each closed cycle and for each element of the continuum[6]:

$$\oint \frac{q}{\theta}\mathrm{d}t = 0\ .$$

For irreversible transformations instead, this integral is negative, because of the second law of thermodynamics.[7]

[6] That is, any transformation which, starting from a certain state, takes the system again to the same state.

[7] The specific entropy can also be defined for general thermodynamical systems but in place of (6.57) one has the restriction

$$\frac{\mathrm{d}s}{\mathrm{d}t} \geq \frac{q}{\theta}\ ,$$

where the equality holds for reversible transformations only.

The property (6.57) identifies a class of continuous systems which, in a sense, correspond, in the context of thermomechanical phenomena, to the holonomic frictionless systems of analytical mechanics. The latter are characterized by a *Lagrangian function* (in the case of conservative forces),[8] whereas the reversible transformation systems are characterized by the *thermodynamical potential, or free energy*:

$$\mathcal{F} \stackrel{\text{def}}{=} \epsilon - s\theta \,. \tag{6.58}$$

When the function $\mathcal{F}$ is known, at least for isothermal or adiabatic transformations, one gets the same number of equations and unknowns for the evolutionary problem. In fact, from the definition (6.58) and using (6.57), for each transformation of the system one has

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \frac{\mathrm{d}\epsilon}{\mathrm{d}t} - \frac{\mathrm{d}s}{\mathrm{d}t}\theta - s\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}\epsilon}{\mathrm{d}t} - q - s\frac{\mathrm{d}\theta}{\mathrm{d}t} \,,$$

and the first law of thermodynamics gives rise to the following condition:

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = -\frac{1}{\mu}w^{(\mathrm{i})} - s\frac{\mathrm{d}\theta}{\mathrm{d}t} \,. \tag{6.59}$$

Using then the Lagrangian characteristics of tension (introduced in (6.40)) to express $w^{(\mathrm{i})}$ we have

$$\mathrm{d}\mathcal{F} = -\frac{1}{2\mu}Y^{ik}\mathrm{d}g_{ik} - s\mathrm{d}\theta \,. \tag{6.60}$$

Thus the thermodynamical potential $\mathcal{F}$ can only depend on the metric g_{ik} of the actual configuration and on the temperature θ, besides on the Lagrangian coordinates y^i:

$$\mathcal{F} = \mathcal{F}(y, g_{ik}, \theta) \,, \tag{6.61}$$

or on equivalent variables. For instance, once the reference configuration C_* is fixed ($\mathrm{d}g_{*ik} = 0$), the metric g_{ik} can be replaced by the deformation characteristics:

$$\mathcal{F} = \mathcal{F}(y, \epsilon_{*ik}, \theta) \,. \tag{6.62}$$

Equation (6.60) then becomes

$$\mathrm{d}\mathcal{F} = -\frac{1}{\mu}Y^{ik}\mathrm{d}\epsilon_{*ik} - s\mathrm{d}\theta \,, \tag{6.63}$$

leading to the conditions

$$Y^{ik} = -\mu\frac{\partial\mathcal{F}}{\partial\epsilon_{*ik}} \,, \qquad s = -\frac{\partial\mathcal{F}}{\partial\theta}, \quad (i,k = 1,2,3) \,. \tag{6.64}$$

Hence, once the thermodynamical potential is assigned, $(6.64)_1$ gives six more equations to be added to the four equations of (6.49), yielding, at least for

[8] In the case of nonconservative forces one must think of the kinetic energy function.

isothermal transformations (θ = const.), the same number of equations as unknowns. Equation $(6.64)_2$ can then be used to determine the entropy s as a function of the deformation characteristics, the temperature and the internal energy ϵ, as follows from (6.58) and using (6.64):

$$\epsilon = \mathcal{F} - \theta \frac{\partial \mathcal{F}}{\partial \theta} . \tag{6.65}$$

For reversible transformation systems the same number of equations and unknowns is also obtained in the adiabatic case $q = 0$, i.e. s = const. from $(6.57)_1$. To show this, let us start by noting that the Helmholtz postulate[9] implies that

$$\frac{\partial^2 \mathcal{F}}{\partial \theta^2} < 0 \; ; \tag{6.66}$$

$(6.64)_2$ can thus be solved with respect to θ: $\theta = \theta(y, \epsilon, s)$, and the internal energy can be expressed, using (6.68), in terms of the deformation characteristics and entropy:

$$\epsilon = \epsilon(y, \epsilon_*, s) . \tag{6.67}$$

Moreover, for reversible transformation systems, the first law of thermodynamics (6.19) gives the following expression for $\mathrm{d}\epsilon$:

$$\mathrm{d}\epsilon = -\frac{1}{2\mu} Y^{ik} \; \mathrm{d}g_{ik} + \theta \; \mathrm{d}s \equiv -\frac{1}{\mu} Y^{ik} \; \mathrm{d}\epsilon_{ik} + \theta \; \mathrm{d}s \; ,$$

so that

$$Y^{ik} = -\mu \frac{\partial \epsilon}{\partial \epsilon_{*ik}} \; , \qquad \theta = \frac{\partial \epsilon}{\partial s}, \quad (i, k = 1, 2, 3) \; . \tag{6.68}$$

It is easy to see that, when s = const., $(6.68)_1$ gives six relations between stress and deformation, which is what is needed to get the same number of equations as unknowns. Equation $(6.68)_2$ gives instead the absolute temperature in terms of deformation characteristics and the entropy, determining in turn the thermodynamical potential (6.58)

$$\mathcal{F} \equiv \epsilon - s\theta = \epsilon - s \frac{\partial \epsilon}{\partial s} \; . \tag{6.69}$$

We note that $(6.68)_2$ is equivalent to $(6.64)_2$ once the latter is solved with respect to θ. It can also be solved with respect to the entropy s. In fact, from the identity

$$-\frac{\partial \mathcal{F}}{\partial \theta}\bigg|_{\theta = \partial \epsilon / \partial s} \equiv s \; ,$$

[9] This postulate is usually expressed as follows: the specific heat at constant volume, c_v, must always be positive (see [4], p. 110). It is also equivalent to the condition, following from (6.65), that the internal energy ϵ is an increasing function of the absolute temperature $\theta > 0$: $\dfrac{\partial \epsilon}{\partial \theta} = -\theta \, \dfrac{\partial^2 \mathcal{F}}{\partial \theta^2} > 0$.

after differentiating with respect to s one gets

$$-\frac{\partial^2 \mathcal{F}}{\partial \theta^2}\bigg|_{\theta=\partial\epsilon/\partial s} \frac{\partial^2 \epsilon}{\partial s^2} \equiv 1 \; ;$$

hence, because of (6.66), the energy must satisfy the restriction

$$\frac{\partial^2 \epsilon}{\partial s^2} > 0 \; , \tag{6.70}$$

which ensures the solvability of $(6.68)_2$ with respect to s.

So, at least for isothermal or adiabatic transformations, the thermodynamical potential (directly, or indirectly through the energy) allows a correct formulation of the dynamical problem.[10] However, the problem of determining the characteristic function $\mathcal{F}$ still remains.

Without developing a systematic treatise of thermomechanics, we will here limit ourselves to show how, in certain concrete situations, the experience can suggest the choice of the thermodynamical potential as well as the stress–deformation relations, which, in turn, give rise to the constitutive equations for the material system under consideration.

6.6 Perfect Fluids. Characteristic Equation and Specific Heat

A *perfect fluid* is a nonviscous fluid undergoing reversible transformations and without internal constraints; this definition excludes the case of perfect liquids, which are nonviscous fluids satisfying also the incompressibility constraint: $\mathcal{D} = 1$. For a perfect fluid, together with (6.46): $Y^{ik} = pg^{ik}$ $(i, k = 1, 2, 3)$, we have (6.63); then taking into account the identity

$$\frac{1}{\mathcal{D}} \partial_t \mathcal{D} = \frac{1}{2} g^{ik} \partial_t g_{ik} \; , \tag{6.71}$$

(6.60) still holds in the form

$$\mathrm{d}\mathcal{F} = -\frac{p}{\mu \mathcal{D}} \mathrm{d}\mathcal{D} - s \mathrm{d}\theta \; . \tag{6.72}$$

It follows that the thermodynamical potential will depend on the metric g_{ik} only through the invariant $\mathcal{D} \equiv \sqrt{g}$, or the actual density:

[10] For reversible transformations which are not isothermal or adiabatic, the knowledge of the thermodynamical potential is not sufficient to give an equal number of equations and unknowns: it is thus necessary to use other equations, e.g. the heat equation.

$$\mu = \frac{\mu_*}{\mathcal{D}} \ . \tag{6.73}$$

One then has

$$\mathcal{F} = \mathcal{F}(\mu, \theta) \ , \tag{6.74}$$

and from (6.72) and (6.73),

$$\mathrm{d}\mathcal{F} = \frac{p}{\mu^2}\mathrm{d}\mu - s \ \mathrm{d}\theta \ .$$

Hence

$$p = \mu^2 \frac{\partial \mathcal{F}}{\partial \mu}, \qquad s = -\frac{\partial \mathcal{F}}{\partial \theta} \ . \tag{6.75}$$

Equation $(6.75)_1$ shows that for a perfect fluid there exists a well-determined relation between pressure, density and temperature (*characteristic equation of the fluid*):

$$p = f(\mu, \theta) \ , \qquad f \stackrel{\mathrm{def}}{=} \mu^2 \frac{\partial \mathcal{F}}{\partial \mu} \ . \tag{6.76}$$

Knowing f (often from experience) allows one to add to (6.42) and (6.46) one more relation, which is enough (at least for isothermal transformations) to make the dynamical problem determined.[11] However, from (6.76), it follows that the function f alone is not sufficient to fully determine the thermodynamical potential, but only up to an arbitrary function of the temperature; for $\mathcal{F}$, instead, the definition (6.58) allows its determination only up to a linear function of θ.[12]

The function f is often deduced integrating the specific heat at constant volume; in fact, since q is the heat absorbed per unit time, the specific heat is given by

$$c \stackrel{\mathrm{def}}{=} q \frac{\mathrm{d}t}{\mathrm{d}\theta} \ . \tag{6.77}$$

In particular, for reversible transformation systems, because of (6.57), one has $q = \theta \ \mathrm{d}s/\mathrm{d}t$, so that (6.77) becomes

$$c = \theta \frac{\mathrm{d}s}{\mathrm{d}\theta} \ . \tag{6.78}$$

Introducing now the free energy from $(6.64)_2$, one finds

$$c = -\theta \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial \mathcal{F}}{\partial \theta}\right) = -\theta \left(\frac{\partial^2 \mathcal{F}}{\partial \theta^2} + \frac{\partial^2 \mathcal{F}}{\partial \epsilon_{*ik} \partial \theta} \frac{\mathrm{d}\epsilon_{*ik}}{\mathrm{d}\theta}\right) \ . \tag{6.79}$$

[11] One should think of the Eulerian case; the Lagrangian formulation, as already noted, implies the presence of another quantity: the metric g_{ik}.

[12] This is a consequence of the fact that both the internal energy and the entropy are defined up to an additive constant.

If the transformation is not generic, but leaves the metric unchanged: $\mathrm{d}\epsilon_{*ik} = 0$, one then gets the specific heat at constant volume:

$$c_v = -\theta \frac{\partial^2 \mathcal{F}}{\partial \theta^2} > 0 \,, \tag{6.80}$$

or because of (6.65):

$$c_v = \frac{\partial \epsilon}{\partial \theta} > 0 \,. \tag{6.81}$$

From (6.76) and (6.80) it is clear that, at least for perfect fluids, the knowledge of the two functions, f and c_v (both depending on μ and θ), is equivalent to that of the free energy *defined up to a linear function of the temperature.*

6.7 Perfect Gas

Among perfect fluids, *perfect gases* are characterized by the following conditions:

1. the product of pressure and specific volume[13] is a function of temperature only:
$$pV = g(\theta) > 0 \qquad \sim \qquad p = \mu g(\theta) \,; \tag{6.82}$$
2. the internal energy ϵ is independent of the density:
$$\epsilon = \epsilon(\theta) \,. \tag{6.83}$$

Under these hypotheses $(6.75)_1$ becomes $\partial \mathcal{F}/\partial \mu = g(\theta)/\mu$, which, after integration with respect to μ, yields

$$\mathcal{F} = g(\theta) \log \mu + h(\theta) \,, \tag{6.84}$$

where h is an arbitrary function of temperature. For the internal energy ϵ, besides (6.84), one has the expression

$$\epsilon = \mathcal{F} - \theta \frac{\partial \mathcal{F}}{\partial \theta} = \left(g - \theta \frac{\mathrm{d}g}{\mathrm{d}\theta} \right) \log \mu + h - \theta \frac{\mathrm{d}h}{\mathrm{d}\theta} \,, \tag{6.85}$$

compatible with (6.83) only if

$$g - \theta \frac{\mathrm{d}g}{\mathrm{d}\theta} = 0 \qquad \sim \qquad \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\frac{g}{\theta} \right) = 0$$

or

$$g = R\theta \,, \tag{6.86}$$

with R a positive constant. Thus, for a perfect gas the characteristic equation is

[13] That is, the volume per unit mass in the actual configuration $V = 1/\mu$.

$$p = f(\mu, \theta) = R\theta\mu \ , \tag{6.87}$$

while the thermodynamical potential is determined up to an arbitrary function of the temperature:

$$\mathcal{F} = R\theta \log \mu + h(\theta) \ . \tag{6.88}$$

To obtain the function $h(\theta)$ it is sufficient to know the specific heat at constant volume c_v or pressure c_p. In fact, for a perfect fluid (6.79) reduces to the form:

$$c = -\theta \left(\frac{\partial^2 \mathcal{F}}{\partial \theta^2} + \frac{\partial^2 \mathcal{F}}{\partial \mu \partial \theta} \frac{\mathrm{d}\mu}{\mathrm{d}\theta} \right) \ ;$$

moreover, for constant pressure transformations, that is, for $f(\mu, \theta) = \text{const.}$, we have

$$\frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial \mu} \frac{d\mu}{d\theta} = 0 \ ,$$

and one has the following invertible relation between constant volume-and constant pressure-specific heats:

$$c_p = c_v + \theta \frac{\partial^2 \mathcal{F}}{\partial \mu \partial \theta} \frac{\dfrac{\partial f}{\partial \theta}}{\dfrac{\partial f}{\partial \mu}} \ . \tag{6.89}$$

By using (6.87) and (6.88), in the case of a *perfect gas* one sees that the specific heats differ by a constant:

$$c_p = c_v + R \ . \tag{6.90}$$

As a consequence of (6.69) and (6.88) c_v is given by

$$c_v = -\theta \frac{\mathrm{d}^2 h}{\mathrm{d}\theta^2} \ . \tag{6.91}$$

If c_v is assigned, (6.91) specifies $h(\theta)$ and hence $\mathcal{F}$ from (6.88), up to a linear function of temperature. For instance, if $c_v = \text{const.}$, we have $h = -c_v \theta \log \theta$, and from (6.88)

$$\mathcal{F} = \theta (R \log \mu - c_v \log \theta) \ . \tag{6.92}$$

The energy ϵ and the entropy s are then obtained from (6.81) and $(6.75)_2$, respectively:

$$\epsilon = c_v \theta \ , \qquad s = R \log \mu - c_v \log \theta = \log \left(\frac{\mu^R}{\theta^{c_v}} \right) \ . \tag{6.93}$$

Expressing temperature by means of the entropy, it follows that

$$\theta = \mu^{R/c_v} \, e^{-s/c_v} \ ; \tag{6.94}$$

$(6.93)_1$ gives then the internal energy in terms of μ and s:

$$\epsilon = c_v \mu^{R/c_v} \, e^{-s/c_v} \ . \tag{6.95}$$

In the case of adiabatic transformations, introducing a new factor $\gamma \equiv c_p/c_v$ and using (6.90), we have

$$\gamma \equiv \frac{c_p}{c_v} = 1 + \frac{R}{c_v} > 1 \ . \tag{6.96}$$

From (6.94) one then gets the relation

$$\theta = K\mu\gamma - 1 \ , \qquad K = e^{-s/c_v} = \text{const.} \tag{6.97}$$

Thus, for adiabatic as well as isothermal transformations (6.77) reduces to a direct relation between pressure and density (*reduced characteristic equation*):

$$\theta p = c\mu^{\gamma} \ , \qquad c = RK = \text{const.} \tag{6.98}$$

6.8 General Expression for the Power of the Internal Forces

We will pass now from nonviscous fluids to the more general case of isotropic systems. For a generic system undergoing reversible transformations characterized by a *thermodynamical potential* $\mathcal{F} = \mathcal{F}(y, \epsilon, \theta)$, how is the isotropic property represented in terms of $\mathcal{F}$?

To answer this question, it is convenient to derive a general and intrinsic expression for the work of the internal forces, which needs the choice, in C_*, of an arbitrary *anholonomic system* of triads; that is, a *distribution of bases*: $\boldsymbol{\lambda}_{(r)} \equiv \lambda^i_{(r)}$, $r, i = 1, 2, 3,$[14] generally dependent on P_* as well as on time but without any other special meaning, for the moment. With the basis $\{\boldsymbol{\lambda}_{(r)}\}$ is associated the *anholonomic metric* in C_*:

$$g_{(r)(s)} = \boldsymbol{\lambda}_{(r)} \cdot \boldsymbol{\lambda}_{(s)} = g_{*ik} \lambda^i_{(r)} \lambda^k_{(s)} \ , \qquad (r, s = 1, 2, 3) \ , \tag{6.99}$$

with its reciprocal $g^{(r)(s)}$ such that

$$g_{(r)(h)} g^{(s)(h)} = \delta^s_r \qquad \sim \qquad \boldsymbol{\lambda}^{(r)} \cdot \boldsymbol{\lambda}_{(s)} = \lambda^{(r)}_i \lambda^i_{(s)} = \delta^s_r \ , \tag{6.100}$$

where the cobasis vectors $\boldsymbol{\lambda}^{(r)} \equiv (\lambda^{(r)}_i)$ are given by

$$\boldsymbol{\lambda}^{(r)} = \boldsymbol{\lambda}_{(s)} g^{(r)(s)} \qquad \sim \qquad \lambda^{(r)}_i = g^{(r)(s)} \lambda_{(s)i} \equiv g^{(r)(s)} g_{*ik} \lambda^k_{(s)} \ . \tag{6.101}$$

Consider the stress and deformation tensor components along the anholonomic basis introduced above, i.e. $\epsilon_{*(r)(s)}$ and $Y^{(r)(s)}$, respectively. We have the following relations:

[14] In order to distinguish a tensorial index from an ordinal one we will denote the latter with a parenthesis.

$$\epsilon_{*ik} = \lambda_i^{(r)} \lambda_k^{(s)} \epsilon_{*(r)(s)}, \quad Y^{ik} = \lambda^{(h)i} \lambda^{(u)k} Y_{(h)(u)} \ , \tag{6.102}$$

yielding, because of (6.40), the power of the internal forces:

$$w^{(\mathrm{i})} \delta t \equiv Y^{ik} \delta \epsilon_{*ik} = Y^{(r)(s)} \delta \epsilon_{*(r)(s)} + 2 \epsilon_{*(r)(s)} Y^{(h)(s)} \lambda^i_{(h)} \delta \lambda_i^{(r)} \ , \tag{6.103}$$

where $\delta = \mathrm{d}t\ \partial_t$; using then the identity

$$\lambda^i_{(h)} \delta \lambda_i^{(r)} \equiv \delta(\lambda^i_{(h)} \lambda_i^{(r)}) - \lambda_i^{(r)} \delta \lambda^i_{(h)} = -\lambda_i^{(r)} \delta \lambda^i_{(h)} \ ,$$

leads to

$$w^{(\mathrm{i})} \delta t = Y^{(r)(s)} \delta \epsilon_{*(r)(s)} - 2 \epsilon_{*(r)(s)} Y^{(h)(s)} \lambda_i^{(r)} \delta \lambda^i_{(h)} \ . \tag{6.104}$$

Consider next the product $\lambda_{(k)i} \delta \lambda^i_{(h)}$ and specify its symmetric and antisymmetric parts:

$$\lambda_{(k)i} \delta \lambda^i_{(h)} = \frac{1}{2} (\lambda_{(k)i} \delta \lambda^i_{(h)} + \lambda_{(h)i} \delta \lambda^i_{(k)}) + \frac{1}{2} \delta \omega_{(k)(h)} \ , \tag{6.105}$$

where

$$\delta \omega_{(k)(h)} \equiv \lambda_{(k)i} \delta \lambda^i_{(h)} - \lambda_{(h)i} \delta \lambda^i_{(k)} \ . \tag{6.106}$$

The reference metric g_{*ik} is constant with respect to the differentiation δ; from (6.99) the symmetric part can thus be written as

$$\lambda_{(k)i} \delta \lambda^i_{(h)} + \lambda_{(h)i} \delta \lambda^i_{(k)} = \lambda_{(k)}{}^i \delta \lambda_{(h)i} + \lambda_{(h)}{}^i \delta \lambda_{(k)i} \equiv \delta(\lambda^i_{(k)} \lambda_{(h)i}) = \delta g_{(k)(h)} \ ,$$

and (6.105) becomes

$$\lambda_{(k)i} \delta \lambda^i_{(h)} = \frac{1}{2} (\delta g_{(k)(h)} + \delta \omega_{(k)(h)}), \qquad (h,k = 1,2,3) \ . \tag{6.107}$$

Therefore, taking into account (6.101), (6.104) assumes the form:

$$w^{(\mathrm{i})} \delta t = Y^{(r)(s)} \delta \epsilon_{*(r)(s)} - \epsilon_{*(r)(s)} Y^{(h)(s)} g^{(r)(k)} (\delta g_{(k)(h)} + \delta \omega_{(k)(h)}) \ . \tag{6.108}$$

An alternative form to (6.108) is obtained by considering the covariant expressions of the stresses in C:

$$\bar{Y}_{ik} \stackrel{\text{def}}{=} g_{ij} g_{kl} Y^{jl} \qquad \sim \qquad Y^{ik} = g^{ij} g^{kl} \bar{Y}_{jl} \ , \tag{6.109}$$

as well as the deformation tensor ϵ_*^{ik}:

$$\epsilon_*^{ik} \stackrel{\text{def}}{=} \frac{1}{2} (g^{ik} - g_*^{ik}) \ . \tag{6.110}$$

Hence, from (6.38)

$$w^{(\mathrm{i})} \delta t = \frac{1}{2} g^{ij} g^{kl} \bar{Y}_{jl} \delta g_{ik} = -\frac{1}{2} g^{ij} g_{ik} \bar{Y}_{jl} \delta g^{kl} = -\frac{1}{2} \delta^j_k \bar{Y}_{jl} \delta g^{kl} \ ,$$

we get the following form for the power of the internal forces:

$$w^{(\mathrm{i})}\delta t = -\frac{1}{2}\bar{Y}_{ik}\delta g^{ik} \equiv -\bar{Y}_{ik}\delta\epsilon_*^{ik} \ . \tag{6.111}$$

Using now the components along the anholonomic basis $\{\boldsymbol{\lambda}_{(r)}\}$ leads to

$$-w^{(\mathrm{i})}\delta t = \bar{Y}_{(r)(s)}\delta\epsilon_*^{(r)(s)} + 2\epsilon_*^{(r)(s)}\bar{Y}_{(h)(s)}\lambda_i^{(h)}\delta\lambda_{(r)}^i \ ,$$

and thus using (6.107)

$$-\, w^{(\mathrm{i})}\delta t = \bar{Y}_{(r)(s)}\delta\epsilon_*^{(r)(s)} + \epsilon_*^{(r)(s)}\bar{Y}_{(h)(s)}g^{(h)(k)}(\delta g_{(k)(r)} + \delta\omega_{(k)(r)}) \ . \tag{6.112}$$

The expressions (6.108) and (6.112) for the power of the internal forces are quite general, i.e. they do not require any particular choice of the anholonomic frame $\{\boldsymbol{\lambda}_{(r)}\}$. They are clearly simplified if the frame is further specified, for instance, by requiring that, for any t, the $\{\boldsymbol{\lambda}_{(r)}\}$ form an orthonormal triad: $\delta g_{(k)(r)} = 0$. In the following we will explicitly consider this case, with $\{\boldsymbol{\lambda}_{(r)}\}$ coinciding with the principal deformation triad.

By using the exterior product, (6.106) can be written as

$$\delta\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\lambda}^{(r)} \times \delta\boldsymbol{\lambda}_{(r)} = \frac{1}{2}\boldsymbol{\lambda}_{(r)} \times \delta\boldsymbol{\lambda}^{(r)} \tag{6.113}$$

or, equivalently,

$$\delta\boldsymbol{\omega} = -\frac{1}{4}\delta\omega_{(r)(s)}\boldsymbol{\lambda}^{(r)} \times \boldsymbol{\lambda}^{(s)} \ ; \tag{6.114}$$

(6.114) explains the kinematical meaning of the antisymmetric quantities (6.106) [5]. In fact, we have

$$\boldsymbol{\lambda}_{(r)} = \lambda_{(r)}^i\mathbf{e}_i^* \quad \sim \quad \mathbf{e}_i^* = \lambda_{(s)i}\boldsymbol{\lambda}^{(s)} \ , \tag{6.115}$$

so that from (6.113) and using the property $\delta\mathbf{e}_i^* = 0$

$$\delta\boldsymbol{\omega} \equiv \frac{1}{2}\boldsymbol{\lambda}^{(r)} \times (\delta\lambda_{(r)}^i\mathbf{e}_i^* + \lambda_{(r)}^i\delta\mathbf{e}_i^*) = \frac{1}{2}\boldsymbol{\lambda}^{(r)} \times \boldsymbol{\lambda}^{(s)}\lambda_{(s)i}\delta\lambda_{(r)}^i,$$

or, from (6.107),

$$\delta\boldsymbol{\omega} = \frac{1}{4}\boldsymbol{\lambda}^{(r)} \times \boldsymbol{\lambda}^{(s)}(\delta g_{(s)(r)} + \delta\omega_{(s)(r)}) = \frac{1}{4}\boldsymbol{\lambda}^{(r)} \times \boldsymbol{\lambda}^{(s)}\delta\omega_{(s)(r)} \ ,$$

which completes the proof.

6.9 Isotropic Systems: Constitutive Equations

For a system undergoing reversible transformations, the isotropic condition is equivalent to the hypothesis that the thermodynamical potential depends on the deformation characteristics ϵ_{*ik} only, through the associated *invariants* I_k $(k = 1, 2, 3)$, with respect to the reference metric or the equivalent scalars:

$$L = g_*^{ik}\epsilon_{*ik} \ , \quad Q = g_*^{ir}g_*^{ks}\epsilon_{*ik}\epsilon_{*rs} \ , \quad \mathcal{C} = g_*^{ir}g_*^{ls}g_*^{km}\epsilon_{*ik}\epsilon_{*rs}\epsilon_{*lm} \ . \tag{6.116}$$

The isotropic property follows immediately if

$$\mathcal{F} = \mathcal{F}(L, Q, \mathcal{C}) \ , \tag{6.117}$$

(the dependence on the temperature being implicit). The hypothesis that a system undergoes reversible transformations can be cast, from $(6.64)_1$ and (6.116), into the following six equalities:

$$-\frac{1}{\mu}Y^{ik} = \frac{\partial \mathcal{F}}{\partial L}g_*^{ik} + 2\frac{\partial \mathcal{F}}{\partial Q}g_*^{ir}g_*^{ks}\epsilon_{*rs} + 3\frac{\partial \mathcal{F}}{\partial \mathcal{C}}g_*^{ir}g_*^{ls}g_*^{km}\epsilon_{*rs}\epsilon_{*lm} \ . \tag{6.118}$$

Thus, any principal direction $\boldsymbol{\lambda}^{(k)}$ of the deformation tensor (with respect to the metric g_{*ik}), $\epsilon_{*ik}\boldsymbol{\lambda}^{(k)} = Eg_{*ik}\boldsymbol{\lambda}^{(k)}$, is also a principal direction of the tension tensor: $Y^{ik}\boldsymbol{\lambda}_{(k)} = Bg_*^{ik}\boldsymbol{\lambda}_{(k)}$; one then finds the following relation among the corresponding eigenvalues:

$$-\frac{1}{\mu}B = \frac{\partial \mathcal{F}}{\partial L} + 2\frac{\partial \mathcal{F}}{\partial Q}E + 3\frac{\partial \mathcal{F}}{\partial \mathcal{C}}E^2 \ . \tag{6.119}$$

Thus *the deformation and tension tensors admit the same eigenvectors.*

Conversely, let us assume that the triad $\{\boldsymbol{\lambda}_{(r)}\}$ $(r = 1, 2, 3)$ of the eigendirections of ϵ_{*ik} (with respect to g_{*ik}) is also an eigentriad for the tensor Y^{ik}. We can show that using the relation

$$-\frac{1}{\mu}Y^{ik}\delta\epsilon_{*ik} = \delta\mathcal{F}(\epsilon) \ , \tag{6.120}$$

which is valid for any variation of the ϵ_{ik} starting from C_*, there follows a dependence on the ϵ_{*ik} as in (6.117). To see this, let us specialize the anholonomic frame $\{\boldsymbol{\lambda}_{(r)}\}$ $(r = 1, 2, 3)$ in (6.109) to coincide (locally) with the principal deformation triad. This implies the simultaneous reduction of the three tensors ϵ_{*ik}, Y^{ik} and g_{*ik} to the diagonal form:

$$\epsilon_{*(r)(s)} = E_{(r)}\delta_{rs}, \quad Y^{(r)(s)} = B^{(r)}\delta^{rs} \ , \quad g_{(r)(s)} = \delta_{rs} \ , \tag{6.121}$$

with $E_{(r)}$ and $B^{(r)}$ the principal tension and deformation characteristics, respectively. Then, from (6.108), and the antisymmetry of the tensor $\delta\omega_{(r)(s)}$, we have

$$w^{(\mathrm{i})}\delta t = B^{(r)}\delta E_{(r)} - \epsilon_{*(r)(s)}B^{(s)}\delta^{(h)(s)}\delta^{(k)(r)}\delta\omega_{(h)(k)} = B^{(r)}\delta E_{(r)} \ . \tag{6.122}$$

This equation, in turn, specifies the thermodynamical potential using (6.59), namely

$$\mathrm{d}\mathcal{F} = -\frac{1}{\mu} w^{(\mathrm{i})} \mathrm{d}t - s\mathrm{d}\theta \; ,$$

in the sense that, as follows from (6.120), $\mathcal{F}$ reduces to a *differential form* in the three variables, $E_{(r)}$, apart from the dependence on θ:

$$\delta\mathcal{F}(\epsilon) = -\frac{1}{\mu} B^{(r)} \delta E_{(r)} \; ; \tag{6.123}$$

that is, $\delta\mathcal{F}(\epsilon)$ is necessarily of the form (6.117).[15]

For a system undergoing reversible transformations, the free energy (apart from its dependence on the temperature) can be regarded either as a function of the (*direct*) *deformation characteristics*, ϵ_{*ik}, or of the equivalent (*inverse characteristics*) $\epsilon_*^{ik} \stackrel{\text{def}}{=} (g^{ik} - g_*^{ik})/2$. If $\bar{\mathcal{F}}(\epsilon_*)$ denotes the thermodynamical potential expressed in terms of the variables ϵ_*^{ik}, the isotropic property is equivalent to the hypothesis that $\bar{\mathcal{F}}(\epsilon_*)$ depends on ϵ_*^{ik} only through the invariants $\bar{I}_k$ $(k = 1, 2, 3)$ or the equivalent scalars

$$\bar{L} \equiv g_{*ik}\epsilon_*^{ik} \; , \quad \bar{Q} = g_{*ir}g_{*ks}\epsilon_*^{ik}\epsilon_*^{rs}, \quad \bar{C} \equiv g_{*ir}g_{*ls}g_{*km}\epsilon_*^{ik}\epsilon_*^{rs}\epsilon_*^{lm} \; . \tag{6.124}$$

In this case the following equations correspond to (6.118):

$$\frac{1}{\mu}\bar{Y}_{ik} = \frac{\partial\bar{\mathcal{F}}}{\partial\bar{L}} g_{*ik} + 2\frac{\partial\bar{\mathcal{F}}}{\partial\bar{Q}} g_{*ir}g_{*ks}\epsilon_*^{rs} + 3\frac{\partial\bar{\mathcal{F}}}{\partial\bar{C}} g_{*ir}g_{*ls}g_{*km}\epsilon_*^{rs}\epsilon_*^{lm} \; , \tag{6.125}$$

and the eigenvalues satisfy the relations

$$\frac{1}{\mu}\bar{B} = \frac{\partial\bar{\mathcal{F}}}{\partial\bar{L}} + 2\frac{\partial\bar{\mathcal{F}}}{\partial\bar{Q}}\bar{E} + 3\frac{\partial\bar{\mathcal{F}}}{\partial\bar{C}}\bar{E}^2 \; . \tag{6.126}$$

For isotropic systems—even if the fundamental variables are reduced to three—there remains the fundamental problem of the choice of the free energy (6.117): the quadratic relations (6.118)–(6.125) between stress and deformation can only be used to suggest hypotheses for it. For instance, after a suitable choice of the reference configuration, the constitutive equations (6.125) could be selected as linear and homogeneous functions

$$\frac{\partial\bar{\mathcal{F}}}{\partial\bar{L}} = \bar{\lambda}\bar{L} \; , \quad \frac{\partial\bar{\mathcal{F}}}{\partial\bar{Q}} = \bar{\mu} \; , \quad \frac{\partial\bar{\mathcal{F}}}{\partial\bar{C}} = 0 \; ,$$

with $\bar{\lambda}$ and $\bar{\mu}$ (*Lamé constants*) independent of the characteristics ϵ_*^{ik}.

Equation (6.118), as well as the equivalent (6.125), express the stresses in terms of deformations and they are a special example of *constitutive equations*, being characterized by the thermodynamical potential $\mathcal{F}$ of the continuous system. In turn, denoting by $\mathbf{s}$ the displacement $\mathrm{P}_*\mathrm{P}$, one has

$$\mathbf{e}_i = \mathbf{e}_i^* + \partial_i\mathbf{s}, \quad g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k, \tag{6.127}$$

[15] The principal deformation characteristics are functions of the invariants (6.116).

and the deformation characteristics assume the form

$$\epsilon_{ik} = \frac{1}{2}(\partial_i \mathbf{s} \cdot \mathbf{e}^*_k + \partial_k \mathbf{s} \cdot \mathbf{e}^*_i + \partial_i \mathbf{s} \cdot \partial_k \mathbf{s}) \,, \tag{6.128}$$

that is, they can be expressed in terms of $\mathbf{s}$, a variable which can replace the velocity: $\mathbf{v} = \partial_t \mathbf{s}$.

Hence, the constitutive relations (6.118) solve the dynamical problem, because they give the tensions in terms of displacement and, then, allow one to obtain, in the fundamental system (6.43), the same number of equations as unknowns, also taking into account the covariant derivative ∇_i associated with the metric $g_{ik} = g_{*ik} + 2\epsilon_{ik}$.

In the case considered above, up to an arbitrary function of the temperature, the potential is of the form

$$\bar{\mathcal{F}} = \frac{1}{2}(\bar{\lambda}\bar{L}^2 + 2\bar{\mu}\bar{Q}) \,, \tag{6.129}$$

which is physically meaningful, at least for elastic systems, in the case of infinitesimal deformations: $\epsilon_{*ik} \sim -\epsilon_{ik}$. This is, instead, in contrast with the nonlinear elasticity theory, grounded on the hypothesis that the relations (6.125) are quadratic [5], which we will not consider here. We prefer to discuss another, *intrinsic, formulation* of the Lagrangian mechanics of continuous systems, which has a geometrical–kinematical counterpart, both in special and general relativities.

6.10 Dynamical Compatibility of a Continuum

The ordinary formulations of continuum mechanics are generally expressed in terms of displacement components and assume the *choice of a reference configuration*; that is just the point of view used in Sect. 6.9 for an isotropic system undergoing reversible transformations.

We are now going to consider a different point of view which uses as fundamental unknowns the *metric*, the *deformation velocity* and the *angular velocity*, that is, *all variables associated with the actual configuration.* This is a dynamical formulation which, endowed with proper initial data, ensures the compatibility of the evolutionary problem and, in particular, the Euclidean property for the metric.

More precisely such a formulation is of *intrinsic* type, both for the meaning of the variables involved and the choice of the local reference frame, derived from the continuum itself through the Lagrangian variables which are at disposal; in addition one must consider the precise geometrical–kinematical meaning of the chosen variables and their tensorial properties.

Finally, apart from the *choice of the initial data*, *no longer free* but with involutive constraints (in the sense of Cartan), the determination of the motion of the continuum implies the possibility to integrate a well-determined (first-order) Cauchy problem; this is the main problem to which the secondary variables, determined a posteriori by quadratures, are subordinated.

After this short introduction, let us consider *in the Newtonian context* a continuum $\mathcal{C}$ in regular motion with respect to a fixed Galilean frame R_{g}. Let $\{y^i\}$ $(i = 1, 2, 3)$ be an arbitrary set of Lagrangian coordinates, defined up to an invertible transformation

$$\det \left\| \frac{\partial y^i}{\partial y^{i'}} \right\| > 0, \tag{6.130}$$

and let C be the instantaneous configuration of the continuum

$$\mathrm{OP} = \mathrm{OP}(t, y) \qquad \sim \qquad x^i = x^i(t, y) \;, \tag{6.131}$$

where x^i $(i = 1, 2, 3)$ are (global) Cartesian coordinates. Let $\{\mathbf{e}_i\}$ be the natural basis, locally associated at $\mathrm{P} \in C$ with the chosen Lagrangian coordinates, and $\{\mathbf{e}^i\}$ the cobasis, defined by the conditions $\mathbf{e}^i \cdot \mathbf{e}_k = \delta^i_k$. Finally, let g be the Lagrangian metric:

$$g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k \qquad \sim \qquad g_{ik} = \frac{\partial x^r}{\partial y^i}\frac{\partial x^s}{\partial y^k}\delta_{rs} \;, \quad (i, k = 1, 2, 3) \;. \tag{6.132}$$

From the kinematical point of view, the motion of the continuum is determined by the vectorial function (6.131), and it can be reduced to the integration of the following *differential system*:

$$\partial_i \mathbf{e}_k = \Gamma^j{}_{ik}\mathbf{e}_j \;, \qquad \partial_t \mathbf{e}_i = \partial_i \mathbf{v} = h_i{}^k \mathbf{e}_k, \qquad \partial_t \mathbf{v} = \mathbf{a} \;, \tag{6.133}$$

in the four vectorial unknowns: $\mathbf{e}_i$ $(i = 1, 2, 3)$ and $\mathbf{v}$ which are functions of the variables y^i and t. The coefficients on the right-hand side of (6.133) all have a geometrical–kinematical meaning and depend on y^i and t: the second-type *Christoffel symbols*, already defined,

$$\Gamma^j{}_{ik} = \frac{1}{2} g^{hj} (\partial_i g_{hk} + \partial_k g_{hi} - \partial_h g_{ik}) \;, \tag{6.134}$$

and the *tensor* h_{ik}, summarizing the deformation velocity k_{ik} and the angular velocity ω_{ik} of the continuum:

$$h_{ik} = k_{ik} + \omega_{ik} \;, \tag{6.135}$$

being

$$k_{ik} \stackrel{\text{def}}{=} \frac{1}{2}\partial_t g_{ik} \;, \quad \omega_{ik} \stackrel{\text{def}}{=} \boldsymbol{\omega} \cdot \mathbf{e}_i \times \mathbf{e}_k, \quad \boldsymbol{\omega} \stackrel{\text{def}}{=} \frac{1}{2}\mathbf{e}^i \times \partial_t \mathbf{e}_i \;. \tag{6.136}$$

Finally, $\mathbf{v} = \partial_t$ OP and $\mathbf{a} = \partial_t \mathbf{v}$ denote the Lagrangian velocity and acceleration of the generic element of the continuum; the latter, even if with a different role, completes the previous coefficients, so that at the right-hand side of (6.133) appear the characteristics h_{ik} and a^i as well as the metric g_{ik}, through the Christoffel symbols. To ensure the compatibility of the system itself, such coefficients cannot be chosen freely; in fact, they must satisfy the following differential (necessary and sufficient) conditions:

$$R_{ikh}{}^j = 0 \,, \quad A_{ik}{}^h = 0 \,, \quad \partial_t h_{ik} = \nabla_i a_k + h_{ij} h_k{}^j \,, \tag{6.137}$$

where $R_{ikh}{}^j$ is the *curvature tensor* associated with the metric g_{ik}:

$$R_{ikh}{}^j \stackrel{\text{def}}{=} \partial_k \Gamma^j{}_{ih} - \partial_i \Gamma^j{}_{kh} + \Gamma^l{}_{ih} \Gamma^j{}_{kl} - \Gamma^l{}_{kh} \Gamma^j{}_{il} \,, \tag{6.138}$$

while A_{ikh} is related to the gradient of h_{ik}:

$$A_{ikh} \stackrel{\text{def}}{=} \nabla_i \omega_{kh} - \nabla_k k_{hi} + \nabla_h k_{ik} \,, \tag{6.139}$$

with ∇_i being the *covariant derivative.*

The compatibility conditions of the system (6.133) can then be summarized in three groups of equations, corresponding to (6.137).

The first condition has a purely geometrical meaning, i.e. it concerns the metric only (and its first- and second-order derivatives), present through the Christoffel symbols; it is equivalent to six equations (*congruence conditions*), which imply, at each instant, the actual configuration C of the continuum, endowed with an Euclidean flat metric.

Condition $(6.137)_2$, instead, has a geometrical–kinematical meaning, because it contains the metric (through the covariant derivative) as well as the deformation and angular velocities.

Finally, $(6.137)_3$ has an *evolutive meaning* for the tensor h_{ik}, and it is the only equation containing the acceleration a_i.

The system (6.137) *gives rise to a first-order Cauchy problem*, in the variables g_{ik} and h_{ik}:

$$\partial_t g_{ik} = 2h_{(ik)} \,, \qquad \partial_t h_{ik} = \nabla_i a_k + h_i{}^j h_{kj} \,; \tag{6.140}$$

these variables are subjected to the constraints

$$R_{ikh}{}^j = 0 \,, \quad A_{ik}{}^h = 0 \,, \tag{6.141}$$

which must be satisfied at each instant and, in particular, initially. *The constraints* (6.141) *are involutory, in the sense of Cartan.* In fact, from (6.141) (and the Bianchi identities, see e.g. [4]), it follows that the two tensorial fields $R_{ikh}{}^j$ and $A_{ik}{}^h$ must satisfy the following linear homogeneous first-order differential equations:

$$\begin{cases} \partial_t A_{ikh} = R_{khi}{}^j a_j + 2h_{[k}{}^j A_{h]ij} - 2h_i{}^j A_{[kh]j}, \\ \partial_t R_{ikh}{}^j = -R_{ikh}{}^l h_l{}^j + R_{ikl}{}^j h_h{}^l - 2\nabla_{[i} A_{k]h}{}^j, \end{cases} \tag{6.142}$$

whose coefficients contain the characteristics of the continuum a_i and h_{ik}, besides the Christoffel symbols, through ∇_i.

To show this, let us start by noting that the definitions (6.134) and (6.138) imply the identities

$$\partial_t \Gamma^j{}_{ik} = H_{ik}{}^j \ , \quad \partial_t R_{ikh}{}^j = \nabla_i H_{kh}{}^j - \nabla_k H_{ih}{}^j \ , \tag{6.143}$$

where

$$H_{ik,j} \stackrel{\text{def}}{=} \nabla_i k_{kj} + \nabla_k k_{ji} - \nabla_j k_{ik} \tag{6.144}$$

is symmetric with respect to the first pair of indices and depends on the deformation tensor and the metric. In turn, antisymmetrizing $(6.140)_2$, we have $\partial_t \omega_{ik} = \nabla_{[i} a_{k]}$ and, after covariantly differentiating both sides,

$$\partial_t \nabla_h \omega_{ik} = \nabla_h \nabla_{[i} a_{k]} + 2 H_{h[i}{}^j \omega_{k]j} \ ; \tag{6.145}$$

similarly, for the symmetric part $\partial_t k_{ik} = \nabla_{(i} a_{k)} + h_i{}^j h_{kj}$, taking into account (6.144), one gets

$$\begin{aligned} \partial_t(\nabla_h \omega_{ik} - \nabla_i k_{kh} + \nabla_k k_{ih}) = \ & \nabla_h \nabla_{[i} a_{k]} - \nabla_i \nabla_{(k} a_{h)} + \nabla_k \nabla_{(i} a_{h)} \\ & + 2 H_{h[i} j \omega_{k]j} - 2 K_{[ik]}{}^j h_{kj} - 2 g_{hl} K^{lj}_{[i} h_{k]j} \ , \end{aligned}$$

where

$$K_{hik} \stackrel{\text{def}}{=} \nabla_h h_{ik} \ , \tag{6.146}$$

so that

$$A_{hik} = K_{hik} - H_{hik} \ . \tag{6.147}$$

Using now the *Ricci theorem* (see e.g. [4])

$$\nabla_i \nabla_k a_h = \nabla_k \nabla_i a_h + R_{ik}{}^j{}_h a_j \ , \tag{6.148}$$

as well as the antisymmetric properties

$$A_{[hi]k} = K_{[hi]k} \ , \qquad R_{[hik]}{}^j = 0 \ , \tag{6.149}$$

one immediately obtains (after differentiation) $(6.142)_1$. To this linear and homogeneous relation between the tensors $\partial_t A$, A and R there corresponds an analogous relation between $\partial_t R$, A and R, as a differential consequence of $(6.140)_1$. In fact, (6.146) and (6.147) imply $\nabla_j H_{ikh} = \nabla_j \nabla_i h_{hk} - \nabla_j A_{ikh}$, from which, by antisymmetrizing over the first pair of indices and using (6.148), we have

$$2\nabla_{[j} H_{i]kh} = R_{ji}{}^l{}_k h_{lh} + R_{ji}{}^l{}_h h_{kl} - 2\nabla_{[j} A_{i]kh} \ .$$

This confirms $(6.142)_2$ and concludes the proof.

The linearity and homogeneity of the system (6.142) implies the *involutive structure* of the constraints (6.141), in the sense that, because of (6.140), the tensors A_{ihk} and $R_{ijk}{}^h$ vanish at any instant, if they are initially null. It follows that the compatibility conditions of the system (6.133), that is, (6.137), are equivalent to the Cauchy problem (6.140), if the initial data $g_{ik,0}$ and $h_{ik,0}$ satisfy the constraints (6.141) at $t = 0$, i.e. in $C_{|t=0}$.

Apart from the (differential) restrictions on the initial data, the problem of determining the motion of the continuum is thus reduced to the integration of the Cauchy problem (6.140), which, however, requires the knowledge of the acceleration field a_i; this is in agreement with the *Galilei principle*, for the priority of the acceleration in the formulation of the mechanical laws. Apart from this specification, which we will investigate later, the formulation (6.140) represents an evolutionary scheme made up by a first-order Cauchy problem with involutive constraints, naturally giving restrictions to the initial data.

In the case of a continuum, besides the constraint $(6.141)_1$ which concerns the initial metric only, $g_{ik,0}$, the angular and deformation velocities must satisfy the conditions $(6.141)_2$

$$\nabla_i \omega_{kh} = \nabla_k k_{hi} - \nabla_h k_{ki} \; . \tag{6.150}$$

This is a *total differential system* for ω_{ik}, such that the initial angular velocity $\boldsymbol{\omega}_0(y)$ is determined, up to a constant, from the initial deformation velocity $k_{ik,0}$; the latter must then satisfy the *congruence conditions*

$$\nabla_j(\nabla_k k_{hi} - \nabla_h k_{ki}) - \nabla_i(\nabla_k k_{hj} - \nabla_h k_{kj}) = 0 \; . \tag{6.151}$$

So one has an *evolutionary scheme*, independent of the choice of the Galilean frame R_{g}, because the involved variables g_{ik}, h_{ik} and a_i have an intrinsic meaning. Such a scheme is invariant with respect to the choice of the Lagrangian coordinates and unaffected by transformations like

$$t' = t \; , \quad y^i = y^i(y') \; . \tag{6.152}$$

Obviously, in Cartesian coordinates, the metric reduces to the Kronecker tensor δ_i^k, and the covariant derivative ∇_i reduces to the partial one ∂_i. However, the Cauchy problem still has the same structure, with the acceleration law assumed to be known, in agreement with the so-called *restricted problem*:

$$a_i = a_i(t, y^j, g_{ik}, h_{kj}) \; . \tag{6.153}$$

The acceleration plays the role of a *dynamical parameter*, in the sense that the specification of the function (6.153) is related to three groups of equations: *Cauchy, continuity and constitutive equations*. Clearly, the analytic structure of the system (6.140) and (6.141) (Cauchy problem plus constraints for the initial data) remains unchanged, apart from the addition of new dynamical variables. Similarly, the congruence conditions (6.150) and the differential consequence (6.151) remain unchanged.

We will consider the complete dynamical picture later, when studying the relativistic case, which is more transparent and more compact.

6.11 Hyperelastic Continua: Intrinsic Dynamics

In the Cauchy problem (6.140) the acceleration enters into the second group of equations only through its gradient $\nabla_i a_k$; as a concrete and sufficiently general example we will consider the case of a hyperelastic continuum undergoing isothermal or adiabatic transformations, without internal constraints.

The dynamics of such a continuum is governed by the Cauchy and continuity equations (see (6.42))

$$\mu(\mathbf{F} - \mathbf{a}) - \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\mathbf{Y}^i) = 0 \,, \qquad \partial_t(\mu\sqrt{g}) = 0 \,, \tag{6.154}$$

with $g = \det||g_{ik}||$, as well as by the constitutive equations (characteristic of the material), which in the present case are expressed in terms of a *single scalar function* depending on six variables: the isothermal or adiabatic *thermodynamical potential* W. In the context of finite deformations, (6.154) are usually written in the Kirchhoff scalar form, obtained by projecting them onto a fixed triad in the reference configuration C_*, and hence invariant or not, according to the choice of the coordinates y^i in C_*, Cartesian or curvilinear. As concerns the constitutive relations, such a formulation—which requires the introduction of the Piola–Kirchhoff (nonsymmetric) tensor—also requires the definition of a potential function V, built up from W but depending on nine variables instead of six (like W).

Another point of view, directly related to the "moving frame" of stereodynamics (Euler equations, principal and secondary problem), is based on the scalar equation (6.35), obtained from $(6.154)_1$ after projection on the basis $\{\mathbf{e}_i\}$:

$$\mu(F^i - a^i) - \nabla_k Y^{ik} = 0 \,, \qquad (i = 1,2,3) \,. \tag{6.155}$$

These are "intrinsic" Lagrangian equations, in the sense that they are referred to a triad $\{\mathbf{e}_i\}$, moving with the generic particle of the continuum and hence *unknown.* This triad is defined up to transformations like

$$\mathbf{e}_i = \frac{\partial y^{i'}}{\partial y^i}\mathbf{e}_{i'} \,, \tag{6.156}$$

which do not exclude that $\mathbf{e}_i$ may be anholonomic. Thus, (6.155) are not equivalent to $(6.154)_1$, because they presuppose the knowledge of the vectorial functions $\mathbf{e}_i(t, y)$; hence, one is motivated to study their geometrical–kinematical compatibility, as we have done in the previous section.

However, by using the Lagrangian stress tensor Y^{ik} and the metric g_{ik} instead of the deformation ϵ_{ik}, [16] the constitutive equations, because of (6.64) and (6.68), can be written in the following form:

[16] Or the Eulerian one: $X^{rs} = \dfrac{\partial x^r}{\partial y^i}\dfrac{\partial x^s}{\partial y^k}Y^{ik}$, with x^r the Cartesian coordinates of P.

$$Y^{ik} = -2\mu \frac{\partial W}{\partial g_{ik}} , \qquad (i, k = 1, 2, 3) , \tag{6.157}$$

with W depending, as concerns the actual configuration, on y^i and g_{ik}: $W = W(y, g)$. One always assumes the symmetric condition

$$\frac{\partial W}{\partial g_{ik}} = \frac{\partial W}{\partial g_{ki}} , \qquad i \neq k ,$$

which is equivalent to requiring that W depends on six variables only: $g_1 = g_{11}$, $g_2 = g_{22}$, $g_3 = g_{33}$, $g_4 = 2g_{23}$, $g_5 = 2g_{31}$, $g_6 = 2g_{12}$.

The continuity equation $(6.154)_2$, in turn, is equivalent to the scalar condition

$$\mu = \mu_* \sqrt{g_*/g} , \tag{6.158}$$

which gives the mass density in C in terms of the actual metric, starting from the initial data: μ_* and $g_* = \det||g_{*ik}||$.

Finally, using the identity

$$\Gamma^k{}_{ik} = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g}) , \tag{6.159}$$

the constitutive relations (6.157) and (6.158) reduce (6.155) to the form

$$a^i = F^i + 2\frac{\partial W}{\partial g_{ik}} \left(\frac{\partial_k \mu_*}{\mu_*} + \Gamma_k \right) + 2\nabla_k \left(\frac{\partial W}{\partial g_{ik}} \right) , \tag{6.160}$$

where

$$\Gamma_k \stackrel{\text{def}}{=} {}^*\Gamma^i{}_{ik} - \Gamma^i{}_{ik} . \tag{6.161}$$

6.12 Cauchy Problem

Equation (6.160) allows one to give an explicit form for the first-order differential system (6.140) in the variables g_{ik}, k_{ik} and ω_{ik}, at least in the case of hyperelastic continuous systems:

$$\begin{cases} \partial_t g_{ik} = 2k_{ik}, \\ \partial_t k_{ik} = \nabla_{(i} a_{k)} + (k_i{}^l + \omega_i{}^l)(k_{kl} + \omega_{kl}), \\ \partial_t \omega_{ik} = \nabla_{[i} a_{k]}. \end{cases} \tag{6.162}$$

First of all, one must introduce on the right-hand side of (6.162) the acceleration gradient in terms of the fundamental variables, taking into account that, by (6.153), it has the form

$$\nabla_i a_k = \nabla_i F_k + 2\tilde{W}_k{}^l \left[\frac{1}{\mu_*}\nabla_i(\partial_l \mu_*) - \frac{1}{\mu_*^2}\partial_i\mu_*\partial_l\mu_* + \nabla_i \Gamma_l \right] + 2\nabla_i \left(\tilde{W}_k{}^l\right)\left(\frac{\partial_l \mu_*}{\mu_*} + \Gamma_l\right) + 2\nabla_i\nabla_l \tilde{W}_k{}^l \,, \tag{6.163}$$

where the (known) function $\tilde{W}_k{}^l$:

$$\tilde{W}_k{}^l \stackrel{\text{def}}{=} g_{hk}\frac{\partial W}{\partial g_{hl}} \,, \tag{6.164}$$

depends, like W, only on the metric g_{ik} (and on the fixed Lagrangian parameters). For the mass action, instead, the differential system is compatible with a general force law

$$F^i = F^i(t, y, g, h, \partial g) \,, \tag{6.165}$$

which is a priori *independent* of the Lagrangian velocity v^i; however, it is easy to see, from (6.43), that one can obtain a formulation similar to that of (6.47), *in terms of* g_{ik} *and* v^i.

Under the hypothesis (6.165), (6.160) gives the components of the Lagrangian acceleration a^i in terms of the variables t, y^i, g_{ik}, h_{ik}, and derivatives $\partial_j h_{ik}$ and $\partial_{lj} g_{ik}$.[17]

Thus, the dynamics of hyperelastic continua can be summarized in a *well-determined first-order Cauchy problem* for the variables g_{ik}, k_{ik} and ω_{ik} (all having a precise geometrical–kinematical meaning), represented by the system (6.162), with the following initial data: *configuration* C_*, *density* $\mu_*(y)$, *metric* g_{*ik} corresponding to the chosen coordinates y^i, [18] *deformation velocity* $k^*_{ik}(y)$ and *angular velocity* $\omega^*_{ik}(y)$ satisfying in C_* the constraints[19] (6.149).

The mass force $\mathbf{F}$ and the thermodynamical potential W enter the evolution equations (6.162) through the gradient (6.163)[20]; thus, once the function W is assigned, conditions (6.162) and the initial data ensure the geometrical–kinematical compatibility of the scheme.

Moreover, the normal form of the system (6.162) guarantees the uniqueness of the solution, at least in the analytic case (we are thinking of a series expansion in t of the solutions). Once the *principal problem* has solved in this way in order to obtain the motion, one has to integrate the total differential system

[17] One must consider the tensorial meaning of the difference between the Christoffel symbols associated with two different metrics.

[18] If the y^i are Cartesian orthogonal coordinates, one has $g_{*ik} = \delta_{ik}$.

[19] We notice, once again, that $(6.150)_2$ is equivalent to determining the angular velocity $\boldsymbol{\omega}_*(y)$, starting from k^*_{ik}, if ω^*_{ik} are known at a point of C_*. This is a strong limitation, for the choice of the initial data, corresponding to the angular velocity. In turn, the initial deformation velocity is subjected to the congruence conditions (6.151).

[20] The particular case of a perfect fluid presupposes a potential $W = W(\mathcal{D})$ depending on the scalar $\mathcal{D} \stackrel{\text{def}}{=} \sqrt{g/g_*}$, from which $Y^{ik} = pg^{ik}$, being $p = -\mu_* W'$.

(6.133), starting from (6.118) and from the solution of the principal problem, thus obtaining the basis $\mathbf{e}_i$ and configuration C (*secondary problem*).

The intrinsic form considered here can also be obtained in relativity (special or general). In contrast, the approach in terms of displacement which we have examined, for instance, in the case of the hyperelastic continua, as in the isotropic case (but with general validity), can be framed in special relativity, but not in general relativity.

The system (6.162), which presupposes for the Cauchy data the choice of the configuration C_*, satisfies an important requirement: the invariance with respect to the choice of Lagrangian coordinates; that is, (6.162) must have tensorial meaning with respect to transformations like (6.152), since in C_* there are no a priori preferred coordinates.

We will explore in the next chapter how the general picture of continuum mechanics is modified in special relativity.

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7

Elements of Relativistic Dynamics of a Continuum

7.1 Relativistic Extension

Let us focus now on continuous systems in the Minkowski space M_4, assumed to be time-oriented. The admissible frames are not all the orthonormal frames $\mathbf{c}_\alpha$, $(\alpha = 0, 1, 2, 3)$:

$$\mathbf{c}_\alpha \cdot \mathbf{c}_\beta = m_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1) \,, \tag{7.1}$$

but only those having the timelike vector $\mathbf{c}_0 \in \mathcal{C}_3^+$.

In the Galilean frame S_g, associated with the orthonormal frame $\mathbf{c}_\alpha$, we set $\mathbf{c}_0 = \boldsymbol{\gamma}$ $(\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} = -1)$ to stress the different role played by $\mathbf{c}_0$ with respect to the spatial vectors $\mathbf{c}_i$ $(i = 1, 2, 3)$. In fact the three spatial vectors $\mathbf{c}_i$ are defined up to an arbitrary spatial rotation $\mathcal{R}$ in the oriented 3-plane Σ, orthogonal to $\boldsymbol{\gamma}$.

Also for a continuous system in the relativistic context one must distinguish between the *absolute formulation* of the dynamics and that *relative to an arbitrary Galilean frame*. The former is invariant under space-time translations or rotations (Lorentz transformations); the latter is instead invariant under time and space translations as well as spatial rotations.

From a physical point of view, the most significant formulation is the relative formulation, which obeys the relativity principle and hence it is formally invariant with respect to the choice of the Galilean frame. Moreover, the invariance is not *substantial* and we need to specify the *transformation laws* of the various kinematical quantities with respect to a change of the Galilean frame.

Let us start from the absolute point of view with the continuum represented by a congruence of ∞^3 nonintersecting timelike lines, which fill a world tube $\mathcal{T} \in M_4$.

Together with the 4-velocity $\mathbf{V}$ at each point $E \in \mathcal{T}$ are also defined two positive scalar quantities: the *proper numerical density* of particles $1/D_0$ (the proper volume element is proportional to D_0, that is, $\mathrm{d}\mathcal{C}/\mathrm{d}C_0 = 1/D_0$) and the *proper density of proper mass* $\mu_0(E)$.

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The *local tension state* of the continuum, from a relative point of view, is also characterized by the relative specific stresses $\boldsymbol{\phi}_n$; the latter must be substituted by the specific 4-stresses $\mathbf{T}_N$ ($\mathbf{T}$ stands for tension and $\mathbf{N} \in M_4$ is a unit vector), since they must be defined in any Galilean frame, that is, in any spatial section Σ through $E \in \mathcal{T}$. From this point of view, it appears quite natural to require for the $\mathbf{T}_N$ properties similar to the corresponding classical ones. We then have the following *postulates for the relativistic continuous systems* [2, 3, 4]:

I From the absolute point of view, the *tension state* of the continuum is *characterized by a vectorial function* over a 3-surface: $\mathbf{T}_N$, which is defined in the *closure* of $\mathcal{T}$ and for each vector $\mathbf{N}$;

II *The relativistic Cauchy postulate holds*, in the sense that $\mathbf{T}_N$ is a linear and homogeneous function of the components of $\mathbf{N}$:

$$\mathbf{T}_N = N_\alpha \mathbf{T}^\alpha \ , \qquad \mathbf{T}^\alpha = (\mathbf{T}_N)_{\mathbf{N}=\mathbf{c}^\alpha} \ ; \tag{7.2}$$

III *The reciprocity axiom holds*:

$$\mathbf{T}_N \cdot \mathbf{N}' = \mathbf{T}_{N'} \cdot \mathbf{N} \ , \qquad \forall \mathbf{N}, \mathbf{N}' \ , \tag{7.3}$$

that is the Eulerian stress 4-tensor $T^{\alpha\beta}$, given by

$$\mathbf{T}^\alpha = T^{\alpha\beta}\mathbf{c}_\beta, \qquad (\alpha = 0, 1, 2, 3) \ , \tag{7.4}$$

with respect to the Cartesian basis $\mathbf{c}_\alpha$, satisfies the symmetry properties:

$$T^{\alpha\beta} = T^{\beta\alpha}, \qquad (\alpha, \beta = 0, 1, 2, 3) \ . \tag{7.5}$$

Next, one has to specify the relativistic extension of the Cauchy theorem, that is the *evolution equations*. The most natural extension is suggested again by the classical evolution equations, interpreting the latter as the spatial and temporal components of the same space-time equation, respectively; more precisely, from (6.5) and (6.12) one has

$$\mu\dot{\mathbf{e}} = (\mu\,\mathbf{e})^\cdot - \dot{\mu}\,\mathbf{e} = \partial_t(\mu\,\mathbf{e}) + e^i\partial_i(\mu\,\mathbf{e}) + \mu\,\mathbf{e}\partial_i e^i = \partial_t(\mu\,\mathbf{e}) + \partial_i(\mu\,\mathbf{e}e^i) \ ,$$

so that system (6.4)–(6.12) can be cast in the following form:

$$\partial_t\mu + \partial_i(\mu e^i) = 0 \ , \quad \partial_t(\mu\,\mathbf{e}) + \partial_i(\mu\,\mathbf{e}\,e^i) + \partial_i\boldsymbol{\phi}^i = \mu\mathbf{F} \ .$$

Thus, using (5.201)

$$\mu = \eta^2\mu_0 \ , \qquad \eta = 1/\sqrt{1 - e^2/c^2} \ , \tag{7.6}$$

as well as the ordinary decomposition of the 4-velocity (in Eulerian terms):

$$\mathbf{V} = \eta(\mathbf{e} + c\boldsymbol{\gamma}) \qquad \sim \qquad V^0 = \eta c \ , \quad V^i = \eta e^i \ , \tag{7.7}$$

the above equations can be summarized by a single space-time equation:

$$\partial_\alpha(\mu_0 V^\alpha \mathbf{V}) + \partial_i \boldsymbol{\phi}^i = \mu_0 \mathbf{k} , \qquad \mathbf{k} = \eta^2 \mathbf{F} ; \tag{7.8}$$

the latter, in turn, suggests the most natural relativistic extension. In fact, the first term on the left-hand side of (7.8) represents a space-time divergence, and it is therefore invariant under Lorentz transformations. Similarly, the term on the right-hand side has an absolute meaning, since μ_0 is invariant and $\mathbf{k}$ a vector field, not necessarily orthogonal to $\boldsymbol{\gamma}$. The second term on the left-hand side of (7.8), instead, being a spatial divergence, is not invariant. However, by using (7.8), the assumed postulates I and II and the fact that the coordinate stresses $\mathbf{T}^\alpha$ transform like contravariant vectors, one gets their simplest generalization by simply replacing $\partial_i \boldsymbol{\phi}^i$ by $\partial_\alpha \mathbf{T}^\alpha$. Thus, we assume the following dynamical postulate:

IV *Evolution equations*:

$$\partial_\alpha(\mu_0 V^\alpha \mathbf{V}) + \partial_\alpha \mathbf{T}^\alpha = \mu_0 \mathbf{k} , \qquad \forall E \in \mathcal{T} . \tag{7.9}$$

In (7.9) all the fundamental relativistic ingredients of the absolute mechanics of continuous systems appear: the *proper density* of proper mass μ_0, the 4-*velocity* $\mathbf{V}$, the (coordinate) 4-*stresses* $\mathbf{T}^\alpha$ and finally the 4-density of *mass force* $\mathbf{k}$. As in the classical case to the indefinite equations one must add the

V *Boundary conditions*:

$$\mathcal{V}_\alpha \mathbf{T}^\alpha = \mathbf{g} , \tag{7.10}$$

where $\mathcal{V}_\alpha$ is the unit normal, internal to the boundary B of $\mathcal{T}$ (necessarily spacelike). These conditions only require the specification of the surface external 4-forces $\mathbf{g}$, in each point of B.

As in the classical case, in relativity (7.9) and (7.10) can be summarized by a single scalar *symbolic relation*. We will not enter into details here. However, we point out that the classical point of view, which results from adapting the "cardinal equations" of the mechanics to a continuum, has a clear correspondence in the Minkowski space, in the sense that the axioms II–V follow, substantially, from adapting the linear and angular momentum equations of the mechanics to a continuum scheme (integral formulation).

7.2 Proper Mechanical Stresses and Thermal Energy

The richness of the relativistic scheme with respect to the classical one already shown in the case of a point particle also appears in continuum mechanics. Here both the tension and the mass forces do not have in general a purely mechanical character. One must consider that

$$\mathbf{T}_N \cdot \mathbf{V} \neq 0\,, \qquad \mathbf{k} \cdot \mathbf{V} \neq 0\,, \qquad \forall E \in \mathcal{T} \quad \text{and} \quad \mathbf{N} \in M_4\,. \tag{7.11}$$

Thus, the interface between mechanical and thermal actions is not limited to the external forces (including surface forces) but is extended to internal and contact forces too (tension forces). We can then give to (7.9) a more transparent form, decomposing the coordinate 4-stresses into the parts parallel and perpendicular to the 4-velocity $\mathbf{V}$:

$$\mathbf{T}^\alpha = \mathbf{X}^\alpha + Q^\alpha \mathbf{V}\,, \tag{7.12}$$

with

$$\mathbf{X}^\alpha \cdot \mathbf{V} = 0\,. \tag{7.13}$$

The 4-vectors $\mathbf{X}^\alpha$ and $Q^\alpha \mathbf{V}$ are named, respectively, *the purely mechanical and the thermal stresses.* The vectors $\mathbf{X}^\alpha$ satisfy the conditions (7.13) and hence are spacelike vectors. The vectors $Q^\alpha \mathbf{V}$ are parallel to $\mathbf{V}$ and hence are timelike vectors. Note that both $\mathbf{X}^\alpha$ and Q^α belong to the proper frame so that they have an intrinsic meaning.

In the following we will consider only *ordinary continuous systems*, characterized by the additional postulate $\mathbf{X}_N = N_\alpha \mathbf{X}^\alpha$ concerning the mechanical stresses. We note that the vector Q^α, because of the reciprocity axiom, is not independent of the mechanical stresses; in fact, after scalar multiplying (7.12) by $\mathbf{V}$ and using (7.13), we have $\mathbf{T}^\alpha \cdot \mathbf{V} = -c^2 Q^\alpha$; using (7.3) then leads to

$$-c^2 Q^\alpha = \mathbf{T}_V \cdot \mathbf{c}^\alpha = \mathbf{X}_V \cdot \mathbf{c}^\alpha + V_\beta Q^\beta \mathbf{V} \cdot \mathbf{c}^\alpha\,.$$

Thus, one has the following expression for Q^α:

$$Q^\alpha = -\frac{1}{c^2}\left(\mathbf{X}_V \cdot \mathbf{c}^\alpha + \mathbf{V} \cdot \mathbf{Q} V^\alpha\right)\,,$$

that is,

$$\mathbf{Q} = \frac{1}{c^2}\left(\epsilon_{c,0}\,\mathbf{V} - \mathbf{X}_V\right)\,, \tag{7.14}$$

where the proper density of thermal energy conduction

$$\epsilon_{c,0} = -\mathbf{Q} \cdot \mathbf{V} \tag{7.15}$$

has been introduced. From (7.14) we see that the 4-vector Q^α depends on $\mathbf{V}$ and can be expressed using the mechanical stresses and the scalar invariant $\epsilon_{c,0}$. The decomposition (7.12) can thus be written as

$$\mathbf{T}^\alpha = \frac{1}{c^2}\epsilon_{c,0} V^\alpha \mathbf{V} + \mathbf{X}^\alpha - \frac{1}{c^2}\mathbf{X}_V \cdot \mathbf{c}^\alpha \mathbf{V}\,, \tag{7.16}$$

where $\mathbf{V}$, $\epsilon_{c,0}$ and the mechanical (nonsymmetric) stresses $\mathbf{X}^\alpha$ appear.

We define a continuum to be *ordinary* (with symmetric characteristics and without thermal conduction) if the following postulate holds:

VI The proper mechanical stresses satisfy a reciprocity axiom:

$$\mathbf{X}_N \cdot \mathbf{N}' = \mathbf{X}_{N'} \cdot \mathbf{N} , \qquad \forall \mathbf{N}, \mathbf{N}' . \tag{7.17}$$

In this case, $\mathbf{X}_N \cdot \mathbf{V} = 0$, $\quad \forall \mathbf{N}$, implies

$$\mathbf{X}_V = V_\alpha \mathbf{X}^\alpha = 0 , \tag{7.18}$$

that is, absence of mechanical stresses in the direction of $\mathbf{V}(\forall E \in \mathcal{T})$.

For an ordinary relativistic continuum, (7.14) then becomes

$$\mathbf{Q} = \mu_{c,0} \mathbf{V} , \tag{7.19}$$

where $\mu_{c,0}$ is the proper density of thermal conduction:

$$\mu_{c,0} = \frac{\epsilon_{c,0}}{c^2} ; \tag{7.20}$$

Furthermore, (7.16) assumes the reduced form:

$$\mathbf{T}^\alpha = \mathbf{X}^\alpha + \mu_{c,0} V^\alpha \mathbf{V} . \tag{7.21}$$

Using the proper mechanical stresses, the relativistic Cauchy equation (7.9) becomes

$$\partial_\alpha (\mathbf{X}^\alpha + \hat{\mu}_0 V^\alpha \mathbf{V}) = \mu_0 \mathbf{k} , \tag{7.22}$$

where $\hat{\mu}_0$ is the *total proper density*, sum of the pure matter density and thermal conduction:

$$\hat{\mu}_0 = \mu_0 + \mu_{c,0} . \tag{7.23}$$

We stress that the alignment of $\mathbf{Q}$ along $\mathbf{V}$ follows from the symmetry property of the mechanical stresses. For continuous systems having nonsymmetrical tension characteristics one has an enlarged scheme both from a geometrical and physical point of view. Actually, (7.19) will no longer be valid and, from an energetic point of view, the situation will be similar to that of an electromagnetic field.

Equation (7.22) can be given in a scalar (Eulerian, because the independent variables are the x^α) form, by introducing the Cartesian components of the 4-velocities as well as the decomposition

$$\mathbf{X}^\alpha = X^{\alpha\beta} \mathbf{c}_\beta , \qquad (\alpha = 0, 1, 2, 3) , \tag{7.24}$$

where the tension coefficients $X^{\alpha\beta}$ are symmetric because of (7.17): $X^{\alpha\beta} = X^{\beta\alpha}$, and satisfy the conditions $X^{\alpha\beta} V_\beta = 0$. Equation (7.22) then becomes

$$\partial_\alpha M^{\alpha\beta} = \mu_0 k^\beta , \tag{7.25}$$

with $M^{\alpha\beta}$ the *energetic tensor*:

$$M^{\alpha\beta} \stackrel{\text{def}}{=} \hat{\mu}_0 V^\alpha V^\beta + X^{\alpha\beta} \tag{7.26}$$

and the usual conditions:

$$V^{\alpha}V_{\alpha} = -c^2, \quad X^{\alpha\beta} = X^{\beta\alpha}\ , \quad X^{\alpha\beta}V_{\beta} = 0\ , \tag{7.27}$$

so that one has the typical form of *conservative equations with sources.*

Equation (7.25) are four first-order partial differential relations (divergence-like equations) between the mass forces (sources proportional to the proper density of pure matter μ_0) and the energetic tensor $M^{\alpha\beta}$, which summarizes the three fundamental characteristics of a continuous system (see (7.26)): the total proper density $\hat{\mu}_0$,[1] the 4-velocity V^{α} and the purely mechanical proper stresses $X^{\alpha\beta}$.

We will show later that $M^{\alpha\beta}$ (with support in $\mathcal{T}$) summarizes the continuum material scheme under the only condition of admitting a timelike eigenvector. In other words, knowing $M^{\alpha\beta}$ is equivalent to knowing the fundamental ingredients $\hat{\mu}_0$, V^{α} and $X^{\alpha\beta}$, under the limitations (7.27).

From this point of view, for continuous systems the tensor $M^{\alpha\beta}$ plays the same role as the one played by the Lagrangian function for a system with n degrees of freedom: they both describe—in a synthetic way—all the contents of the scheme, in all its generality.

The fundamental equations of the continuum absolute dynamics have the following form:

$$\begin{cases} \partial_{\alpha}(\hat{\mu}_0 V^{\alpha}V^{\beta} + X^{\alpha\beta}) = \mu_0 k^{\beta}, \\ \\ X^{\alpha\beta}V_{\beta} = 0\ , \quad V^{\alpha}V_{\alpha} = -c^2, \end{cases} \tag{7.28}$$

where the source k^{β} must be assigned. These are 9 equations in 16 unknowns: μ_0, $\hat{\mu}_0$, V^{α} and $X^{\alpha\beta}$ (the latter are 10, because of the symmetry of the tensor). The scheme is then compatible, starting from fixed initial and boundary conditions, with infinitely many possible motions of the system, even under regularity hypotheses for the assigned functions k^{α}. Compared with the classical situation, there is one more indetermination due to the presence in $\hat{\mu}_0 = \mu_0 + \mu_{c,0}$ of the thermal inertia term $\mu_{c,0}$. Hence it is necessary to add to (7.28) seven more equations, the so-called *constitutive equations*, concerning the internal structure of the continuum as well as its reaction to external solicitations.

7.3 Space-time Splitting Techniques. The Energy Tensor

The projection of a tensor along a given direction as well as onto the perpendicular 3-space constitutes a very useful and general decomposition method. This method is purely algebraic and is called *natural decomposition.* Here we elucidate it in the case of 2-tensors.

Let $T^{\alpha\beta}$ be an arbitrary 2-tensor and V^{α} a timelike vector, both defined at a given point $E \in M_4$. We have the following *polynomial decomposition* (in V) with tensorial character

[1] From this, once μ_0 is known, one gets the thermal conduction term $\mu_{c,0}$.

$$T^{\alpha\beta} = S^{\alpha\beta} + V^\alpha S^\beta + V^\beta S'^\alpha + SV^\alpha V^\beta \ , \tag{7.29}$$

where the various quantities $S^{\alpha\beta}$, S^β, S'^α and S (respectively, a 2-tensor, two vectors and a scalar) satisfy the orthogonality conditions:

$$S^{\alpha\beta} V_\alpha = 0 \ , \quad S^{\alpha\beta} V_\beta = 0 \ , \quad S^\beta V_\beta = 0 \ , \quad S'^\alpha V_\alpha = 0 \ . \tag{7.30}$$

In fact, from (7.30), multiplication of (7.29) by $V_\alpha V_\beta$ gives

$$T^{\alpha\beta} V_\alpha V_\beta = S(V_\alpha V^\alpha)^2 \ .$$

Thus, after defining

$$V^2 = -(V_\alpha V^\alpha) \ , \tag{7.31}$$

one has the following expression for the scalar S:

$$S = \frac{1}{V^4} T^{\alpha\beta} V_\alpha V_\beta \ . \tag{7.32}$$

Similarly, multiplying (7.29) by V_α and V_β separately gives

$$T^{\alpha\beta} V_\alpha = -V^2 (S^\beta + SV^\beta) \ , \qquad T^{\alpha\beta} V_\beta = -V^2 (S'^\alpha + SV^\alpha) \ ,$$

from which the expressions for the vectors S^β and S'^β follow:

$$S^\beta = -\frac{1}{V^2} T^{\alpha\beta} V_\alpha - SV^\beta \ , \qquad S'^\alpha = -\frac{1}{V^2} T^{\alpha\beta} V_\beta - SV^\alpha \ . \tag{7.33}$$

Finally, the same (7.29) together with (7.32) and (7.33) determines $S^{\alpha\beta}$ as a function of $T^{\alpha\beta}$ and V^α, and this completes the proof.

The decomposition (7.29) is simplified if $T^{\alpha\beta}$ is special. In particular,

- If $T^{\alpha\beta}$ is *symmetric*, from (7.33) it follows that the two vectors S and S' coincide: $S^\alpha = S'^\alpha$ and (7.29) becomes

$$T^{\alpha\beta} = S^{\alpha\beta} + V^\alpha S^\beta + V^\beta S^\alpha + SV^\alpha V^\beta \ , \tag{7.34}$$

  $S^{\alpha\beta}$ being symmetric too;
- If $T^{\alpha\beta}$ is *antisymmetric*, (7.33) imply $S = 0$ and $S^\alpha = -S'^\alpha$, so that (7.29) becomes

$$T^{\alpha\beta} = S^{\alpha\beta} + V^\alpha S^\beta - V^\beta S^\alpha \ , \tag{7.35}$$

  with $S^{\alpha\beta}$ now antisymmetric.

Let us go back to the energy tensor $M^{\alpha\beta}$ of the continuum, satisfying (7.26) and (7.27), and show that it characterizes the continuum scheme itself, giving all the necessary descriptional elements. To this end, let us assume the field $M^{\alpha\beta}$ to be assigned a priori as a symmetric tensor defined in the world tube

$\mathcal{T} \in M_4$, and admitting a future-oriented timelike eigenvector $\mathbf{V}$, $\forall E \in \mathcal{T}$. From $M^{\alpha\beta}$ one can obtain the total proper density and the mechanical stress tensor. In fact, $M^{\alpha\beta}$ can be interpreted as a vectorial map from an arbitrary vector $\mathbf{v}$ into a vector $\mathbf{w} = M(\mathbf{v})$, i.e. $v^\alpha \rightarrow w^\alpha \equiv M^{\alpha\beta} v_\beta$. One can then derive the principal directions of $M^{\alpha\beta}$:

$$M^{\alpha\beta} v_\beta = \lambda v^\alpha \ . \tag{7.36}$$

Moreover, $M^{\alpha\beta}$ can be cast in a *diagonal form*, since it admits an orthonormal basis of eigenvectors. In fact, by hypothesis, it has a timelike eigendirection which defines the world lines of the continuum as well as their 4-velocity $\mathbf{V}$. Thus, in (7.31) one must consider $V^2 = c^2$ and from (7.36) one has

$$M^{\alpha\beta} V_\beta = \lambda V^\alpha \ . \tag{7.37}$$

Equation (7.37) simplifies the natural decomposition of $M^{\alpha\beta}$ along $\mathbf{V}$, which is of the type (7.34) because of the symmetry of $M^{\alpha\beta}$; hence, from (7.32) we have

$$S = \frac{1}{c^4} M^{\alpha\beta} V_\alpha V_\beta = \frac{1}{c^4} \lambda V^\alpha V_\alpha = -\frac{1}{c^2} \lambda \ . \tag{7.38}$$

Equation $(7.33)_1$ then implies

$$S^\beta = -\frac{\lambda}{c^2} V^\beta - S V^\beta = 0 \ ,$$

so that (7.34) can be written as

$$M^{\alpha\beta} = S^{\alpha\beta} - \frac{\lambda}{c^2} V^\alpha V^\beta \ , \qquad S^{\alpha\beta} V_\beta = 0 \ . \tag{7.39}$$

Comparing now the decomposition (7.39) with (7.26) shows that $S^{\alpha\beta}$ can be interpreted as the proper mechanical stress tensor $X^{\alpha\beta}$ and, in turn, $-\lambda/c^2$ is the proper total density $\hat{\mu}_0$.

We also note that $X^{\alpha\beta}$, as a tensor in M_4, defines a degenerate map, admitting $\mathbf{V}$ as a null eigendirection; on the other hand, as a symmetric tensor in Σ_0 (the spacelike platform orthogonal to $\mathbf{V}$) it can be put in a diagonal form. Its (three) eigendirections are called principal directions of proper tension.

Hence a symmetric tensor field $M^{\alpha\beta}$, with a timelike eigenvector, characterizes a symmetric material scheme (ordinary continuum) with mass density μ_0.

Electromagnetism too admits a symmetric energy tensor $E^{\alpha\beta}$, built up from the electromagnetic field. However, such a field is also defined outside the charged matter, and it does not admit a preferred timelike eigenvector. This is a fundamental difference between the two schemes [1].

7.4 Dust Matter and Perfect Fluids

Let us consider the limiting case of material systems, exemplified by dust or sand, for which the 4-tensions can be neglected. This is the *dust matter* scheme, characterized by the twofold condition that both the contact and thermal actions vanish identically:

$$X^{\alpha\beta} = 0 , \qquad \mu_{c,0} = 0 , \qquad \forall E \in \mathcal{T} . \tag{7.40}$$

The energy tensor reduces to the form

$$M^{\alpha\beta} = \mu_0 V^\alpha V^\beta , \tag{7.41}$$

where μ_0 is the pure matter density, and V^α is the 4-velocity.

One can study the special case in which the volume forces vanish too: $k^\alpha = 0$. Equation (7.22) then becomes $\partial_\alpha(\mu_0 V^\alpha \mathbf{V}) = 0$, so that

$$\partial_\alpha(\mu_0 V^\alpha)\mathbf{V} + \mu_0 V^\alpha \partial_\alpha \mathbf{V} = \partial_\alpha(\mu_0 V^\alpha)\mathbf{V} + \mu_0 \mathbf{A} = 0 , \tag{7.42}$$

where $\mathbf{A} = V^\alpha \partial_\alpha \mathbf{V} = \mathrm{d}\mathbf{V}/\mathrm{d}\tau$ is the 4-acceleration. After contracting (7.42) with $\mathbf{V}$ and using $\mathbf{V} \cdot \mathbf{A} = 0$, one obtains the proper mass conservation equation:

$$\partial_\alpha(\mu_0 V^\alpha) = 0, \qquad \forall E \in \mathcal{T} . \tag{7.43}$$

The latter equation implies, in Eulerian terms, the property that along the generic world line of the continuum the elementary proper mass $\mu_0 \mathrm{d}C_0 \equiv \mathrm{d}m_0$ is constant:

$$\mu_0 D_0 = \text{const.} \qquad \sim \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}(\mu_0 D_0) = 0 , \qquad \forall E \in \mathcal{T} . \tag{7.44}$$

In fact, by using the identity

$$\frac{1}{D_0}\frac{\mathrm{d}D_0}{\mathrm{d}\tau} = \partial_\alpha V^\alpha , \tag{7.45}$$

(7.43) becomes

$$V^\alpha \partial_\alpha \mu_0 + \mu_0 \partial_\alpha V^\alpha \equiv \frac{\mathrm{d}\mu_0}{\mathrm{d}\tau} + \frac{\mu_0}{D_0}\frac{\mathrm{d}D_0}{\mathrm{d}\tau} = 0 ,$$

which coincides with (7.44). Thus the world lines of each element are timelike straight lines and the proper mass is conserved just as for a single point-mass in the absence of external forces.

A less extreme case is that of *nonviscous fluids*, namely material systems for which the mechanical stress $\mathbf{X}_n$ is parallel to $\mathbf{n}$, $\forall E \in \mathcal{T}$ and $\forall \mathbf{n} \in \Sigma_0$ (spatial platform in the proper frame, i.e. orthogonal to $\mathbf{V}$):

$$\mathbf{X}_n = p_0 \mathbf{n}, \quad \forall E \in \mathcal{T}, \quad \forall \mathbf{n} \in \Sigma_0 . \tag{7.46}$$

The proportionality factor p_0 is independent of $\mathbf{n}$ and is called *proper pressure* of the fluid at the considered point. We can now evaluate $X^{\alpha\beta}$ from (7.46); using the equality $\mathbf{X}_n = n_\alpha \mathbf{X}^\alpha = n_\alpha X^{\alpha\beta} \mathbf{c}_\beta$, one has

$$n_\alpha X^{\alpha\beta} = p_0 n^\beta \equiv p_0 n_\alpha m^{\alpha\beta} \,,$$

so that

$$(X^{\alpha\beta} - p_0 m^{\alpha\beta}) n_\alpha = 0 \,, \quad \forall \mathbf{n} \in \Sigma_0 \,. \tag{7.47}$$

Because of the arbitrariness of $\mathbf{n} \in \Sigma_0$, (7.47) is equivalent to the condition that, for any fixed value of $\beta = 0, 1, 2, 3$, the vector $X^{\alpha\beta} - p_0 m^{\alpha\beta}$ is orthogonal to Σ_0 and hence parallel to $\mathbf{V}$:

$$X^{\alpha\beta} - p_0 m^{\alpha\beta} = \lambda^\beta V^\alpha \,. \tag{7.48}$$

Using the restriction $(7.27)_3$: $X^{\alpha\beta} V_\alpha = 0$, one can now determine λ^β:

$$-p_0 m^{\alpha\beta} V_\alpha = -c^2 \lambda^\beta \qquad \rightarrow \qquad \lambda^\alpha = \frac{1}{c^2} p_0 V^\alpha \,,$$

that is

$$X^{\alpha\beta} = p_0 \left(m^{\alpha\beta} + \frac{1}{c^2} V^\alpha V^\beta \right) \,, \tag{7.49}$$

and also

$$M^{\alpha\beta} = p_0 m^{\alpha\beta} + \left(\mu_0 + \mu_{c,0} + \frac{p_0}{c^2} \right) V^\alpha V^\beta \,. \tag{7.50}$$

As concerns the number of equations and unknowns, in the case of dust matter there are five equations for five unknown: μ_0 and V^α ($(7.28)_2$ vanishes identically). In the case of nonviscous fluids there are seven unknowns: μ_0, $\hat{\mu}_0$, V^α and p_0 but still only five equations. In fact, $(7.28)_2$ identically vanishes even in this case because of (7.49):

$$p_0 \left(m^{\alpha\beta} + \frac{1}{c^2} V^\alpha V^\beta \right) V_\alpha = p_0 (V^\beta - V^\beta) = 0 \,.$$

To have the same number of unknowns and equations one therefore needs *two constitutive equations*. It is sufficient, for example, that the thermal inertia is absent and that a reduced state equation (i.e. an explicit relation between pressure and proper density) holds; this is exactly the perfect fluid scheme, characterized by (7.50) plus the conditions

$$\mu_{c,0} = 0 \,, \quad \hat{\mu}_0 = \mu_0 = \mu_0(p_0) \,. \tag{7.51}$$

From here one can directly approach *relativistic perfect fluids*.

Let us return to the general case in order to compare the classical and the relativistic situations as well as to show the fundamental role of the first law of thermodynamics in giving us a fully determined set of equations.

7.5 Absolute Dynamics of Ordinary Relativistic Continua

Let us consider the evolution equations (7.22) for an ordinary relativistic continuum in a given Galilean frame:

$$\partial_\alpha(\hat{\mu}_0 V^\alpha \mathbf{V} + \mathbf{X}^\alpha) = \mu_0 \mathbf{k} \ .$$

Expanding the derivative, one has

$$\partial_\alpha(\hat{\mu}_0 V^\alpha)\mathbf{V} + \hat{\mu}_0 V^\alpha \partial_\alpha \mathbf{V} = \mu_0 \mathbf{k} - \partial_\alpha \mathbf{X}^\alpha \ ,$$

where $V^\alpha \partial_\alpha \mathbf{V} = \mathrm{d}\mathbf{V}/\mathrm{d}\tau = \mathbf{A}$. Thus

$$\partial_\alpha(\hat{\mu}_0 V^\alpha)\mathbf{V} + \hat{\mu}_0 \mathbf{A} = \mu_0 \mathbf{k} - \partial_\alpha \mathbf{X}^\alpha \ . \tag{7.52}$$

Multiplying by $\mathbf{V}$ and using $\mathbf{V} \cdot \mathbf{A} = 0$, one has

$$-c^2 \partial_\alpha(\hat{\mu}_0 V^\alpha) = \mu_0 \mathbf{k} \cdot \mathbf{V} - \partial_\alpha \mathbf{X}^\alpha \cdot \mathbf{V} \ ;$$

but $\mathbf{X}^\alpha \cdot \mathbf{V} = 0$ identically, so that

$$\partial_\alpha \mathbf{X}^\alpha \cdot \mathbf{V} = \partial_\alpha(\mathbf{X}^\alpha \cdot \mathbf{V}) - \mathbf{X}^\alpha \cdot \partial_\alpha \mathbf{V} = -\mathbf{X}^\alpha \cdot \partial_\alpha \mathbf{V} \ ,$$

and one finds

$$c^2 \partial_\alpha(\hat{\mu}_0 V^\alpha) = -\mu_0 \mathbf{k} \cdot \mathbf{V} - \mathbf{X}^\alpha \cdot \partial_\alpha \mathbf{V} \ ,$$

or

$$c^2 \partial_\alpha(\hat{\mu}_0 V^\alpha) = \mu_0 r_0 - w_0^{(\mathrm{i})} \ , \tag{7.53}$$

where we have introduced the following *proper quantities*:

$$\begin{cases} r_0 \stackrel{\text{def}}{=} -\mathbf{k} \cdot \mathbf{V} & \text{thermal radiation power density,} \\ w_0^{(\mathrm{i})} \stackrel{\text{def}}{=} \mathbf{X}^\alpha \cdot \partial_\alpha \mathbf{V} & \text{internal forces power density;} \end{cases} \tag{7.54}$$

the thermal radiation power density is meant per unit of proper mass while the internal forces power density is considered per unit of proper volume. Substituting (7.53) in (7.52) leads to the *Newtonian equation*:

$$\hat{\mu}_0 \mathbf{A} = \boldsymbol{\mathcal{F}}_0 \ , \tag{7.55}$$

where $\boldsymbol{\mathcal{F}}_0$ represents the *proper mechanical force per unit proper volume*:

$$\boldsymbol{\mathcal{F}}_0 \stackrel{\text{def}}{=} \mu_0 \mathbf{k} - \partial_\alpha \mathbf{X}^\alpha - \frac{1}{c^2}(\mu_0 r_0 - w_0^{(\mathrm{i})})\mathbf{V} \ . \tag{7.56}$$

The Newtonian equation (7.55) shows in what sense the element of a continuum can be compared to a point-mass (apart from the multiplication by the proper volume element $\mathrm{d}C_0$): as concerns the acceleration, the proportionality factor is now $\hat{\mu}_0 = \mu_0 + \mu_{c,0}$ (total inertial mass per unit proper volume), and it includes both the pure matter inertia μ_0 and the thermal inertia $\mu_{c,0}$. As concerns the proper mechanical force $\boldsymbol{\mathcal{F}}_0$, instead, one must add to the material point term

$$\mu_0\mathbf{f} = \mu_0\left(\mathbf{k} - \frac{1}{c^2}r_0\mathbf{V}\right) \equiv \mu_0\mathbf{k}_{\Sigma_0}\ ,$$

(in place of the 4-force $\mathbf{k}$, one has the proper density of 4-force: $\mathrm{d}\mathbf{K}/\mathrm{d}C_0 = \mu_0\mathbf{k}$) the contribution of the internal (contact) forces

$$\mu_0\mathbf{f}^{(\mathrm{i})} \equiv -\partial_\alpha\mathbf{X}^\alpha + \frac{1}{c^2}w_0^{(\mathrm{i})}\mathbf{V}\ ; \tag{7.57}$$

these terms are both orthogonal to $\mathbf{V}$, because of $(7.54)_2$ and (7.13), that is

$$\mathbf{X}^\alpha \cdot \mathbf{V} = 0 \quad \sim \quad X^{\alpha\beta}V_\beta = 0\ . \tag{7.58}$$

Equation (7.53) represents the *energy theorem* of the relativistic continuum in the proper frame S_0. In fact, after evaluating the derivative and using the identity (7.45), (7.53) becomes

$$\frac{c^2}{D_0}\frac{\mathrm{d}}{\mathrm{d}\tau}(\hat{\mu}_0 D_0) = W_0 \stackrel{\mathrm{def}}{=} \mu_0 r_0 - w_0^{(\mathrm{i})}\ . \tag{7.59}$$

Thus, up to multiplication by the proper volume element $\mathrm{d}C_0 = D_0\mathrm{d}\mathcal{C}$, (7.59) corresponds to the (proper) energy theorem of the material point (even if with a different meaning):

$$\frac{\mathrm{d}\mathcal{E}_0}{\mathrm{d}\tau} = q_0, \qquad (\mathcal{E}_0 \to \hat{\mu}_0 c^2\ \mathrm{d}C_0,\ q_0 \to W_0\ \mathrm{d}C_0)\ .$$

Clearly, passing to the continuum implies a twofold modification, because of the presence of the density of proper thermal energy $\mu_{c,0}c^2$ in addition to the pure matter term $\mu_0 c^2$ and because of the thermal power, which includes the 4-force (mass) term, as well as the contribution due to internal stresses.

The proper power of internal forces $w_0^{(\mathrm{i})}$ can be cast in the classical form (6.24) differently from the relative power. In fact, transforming $(7.54)_2$ through the tension characteristics $X^{\alpha\beta} = X^{\beta\alpha}$ leads to

$$w_0^{(\mathrm{i})} = X^{\alpha\beta}\partial_\alpha V_\beta = \frac{1}{2}X^{\alpha\beta}(\partial_\alpha V_\beta + \partial_\beta V_\alpha)\ .$$

Next introducing the proper deformation 4-velocity

$$\kappa_{\alpha\beta} \stackrel{\mathrm{def}}{=} \frac{1}{2}(\partial_\alpha V_\beta + \partial_\beta V_\alpha)\ , \tag{7.60}$$

one has the (quasi) classical expression (because it is four-dimensional):

$$w_0^{(\mathrm{i})} = X^{\alpha\beta}\ \kappa_{\alpha\beta}\ . \tag{7.61}$$

The latter, in turn, can be reduced to the classical expression if, according to (7.34), we consider the natural decomposition of the tensor $\partial_\alpha V_\beta$:

$$\partial_\alpha V_\beta = H_{\alpha\beta} + V_\alpha S_\beta + V_\beta S'_\alpha + S V_\alpha V_\beta \ , \tag{7.62}$$

with the conditions

$$H_{\alpha\beta} V^\beta = 0, \quad S_{\alpha\beta} V^\alpha = 0, \quad S^\beta V_\beta = 0, \quad S'^\alpha V_\alpha = 0. \tag{7.63}$$

In fact, since $(\partial_\alpha V_\beta) V^\beta = 0$, one has

$$S = 0 \ , \qquad S'^\alpha = 0, \qquad S_\beta = -\frac{1}{c^2} A_\beta . \tag{7.64}$$

Moreover, the 4-acceleration is proportional to the curvature vector of the world line by a factor c^2:

$$A_\alpha = c^2 C_\alpha , \tag{7.65}$$

so that (7.64) imply for (7.62) the form

$$\partial_\alpha V_\beta = H_{\alpha\beta} - V_\alpha C_\beta \ , \tag{7.66}$$

subjected to the conditions given in (7.63):

$$H_{\alpha\beta} V^\beta = 0 \ , \quad H_{\alpha\beta} V^\alpha = 0 \ , \quad C_\alpha V^\alpha = 0 \ . \tag{7.67}$$

Using the restriction (7.58) one then gets the following expression for (7.61):

$$w_0^{(\mathrm{i})} = X^{\alpha\beta} \kappa_{\alpha\beta} = X^{\alpha\beta} H_{(\alpha\beta)} \ , \tag{7.68}$$

which, in terms of proper quantities, has the same meaning as the classical formula (6.24). In fact, as will be more clear later, the spatial tensor $H_{(\alpha\beta)}$ has the meaning of ordinary deformation velocity, even though with respect to the Galilean rest frame.

Finally, as concerns (7.59), besides expressing the energy theorem it is strictly related with the classical mechanics of continuous systems. In fact, by introducing the *density of proper internal energy* $\hat{\epsilon}_0$

$$\mu_0 \hat{\epsilon}_0 \stackrel{\mathrm{def}}{=} \hat{\mu}_0 c^2 \ , \qquad \hat{\epsilon}_0 = c^2 + \frac{\mu_{c,0}}{\mu_0} c^2 \ , \tag{7.69}$$

so that (7.59) becomes

$$\frac{1}{\mu_0 D_0} \frac{\mathrm{d}}{\mathrm{d}\tau} (\mu_0 D_0 \hat{\epsilon}_0) = r_0 - \frac{1}{\mu_0} w_0^{(\mathrm{i})} \ , \tag{7.70}$$

thus representing (in the Galilean proper rest frame S_0) the *first law of thermodynamics* (6.19).

Comparing the relativistic situation with the classical one, (7.70) is formally identical to the expression (6.19) of the first law of thermodynamics, apart from the replacement of the classical quantities by the corresponding proper relativistic quantities. However, there is a relevant difference: in classical mechanics q and ϵ represent the internal energy of the continuum and the exchanged heat (either through radiation or conduction), respectively; in relativity, instead, q_0 and $\hat{\epsilon}_0$ come directly from the adopted scheme, clearly richer than the classical one.

One then finds, as for the material point, the *relativistic unification* between mechanics and thermodynamics. In addition, the relativistic scheme contains another fundamental element for the description of the thermal field: the *proper vector of thermal conduction* $\mathbf{q}_0$. We will come back to this point in the next chapter.

7.6 Relative Dynamics of Ordinary Relativistic Continua

As we have already done for the material point, we are now ready to consider, besides the *absolute formulation*, the *"relative" formulation* with respect to a given Galilean frame. To develop such an approach which is fundamental from a physical point of view one needs to:

1. *specify the temporal direction* $\boldsymbol{\gamma}$, which characterizes the frame and (locally) decompose along $\boldsymbol{\gamma}$ and orthogonally to it, in Σ, all the various tensorial quantities;
2. *correctly* (from both the mathematical and physical point of view) *define the relative quantities*;
3. *obtain the transformation laws* of the relative quantities for an arbitrary change of the frame.

Let us consider then a general Galilean frame S_{g} and let $\boldsymbol{\gamma} = \mathbf{c}_0$ be the temporal direction and $\{\mathbf{c}_i\}$ $(i = 1, 2, 3)$ a spatial triad in Σ (associated with *internal coordinates*). From (7.18) we have the following conditions for the mechanical stresses $\mathbf{X}^\alpha$:

$$V_\alpha \mathbf{X}^\alpha = 0 \qquad \sim \qquad \mathbf{X}^\alpha \cdot \mathbf{V} = 0 \, ,$$

and using the relations $V_0 = -\eta c$ and $V_i = \eta e_i$ one gets the following expression for $\mathbf{X}^0$:

$$\mathbf{X}^0 = \frac{1}{c} e_i \mathbf{X}^i \, . \tag{7.71}$$

Moreover, the natural decomposition of $\mathbf{X}^i$: $\mathbf{X}^i = \boldsymbol{\phi}^i - (\mathbf{X}^i \cdot \boldsymbol{\gamma})\,\boldsymbol{\gamma}$, where

$$\boldsymbol{\phi}^i = X^{ik}\mathbf{c}_k \in \Sigma \, , \qquad (i = 1, 2, 3) \, , \tag{7.72}$$

assumes here a simple form, because of the condition

$$\mathbf{X}^i \cdot \mathbf{V} = 0 \quad \longleftrightarrow \quad \mathbf{X}^i \cdot (\mathbf{e} + c\boldsymbol{\gamma}) = 0 \quad \longleftrightarrow \quad \mathbf{X}^i \cdot \boldsymbol{\gamma} = -\frac{1}{c}\mathbf{X}^i \cdot \mathbf{e} \, ,$$

or

$$\mathbf{X}^i \cdot \boldsymbol{\gamma} = -\frac{1}{c}\boldsymbol{\phi}^i \cdot \mathbf{e} \, .$$

In fact, the latter becomes

$$\mathbf{X}^i = \boldsymbol{\phi}^i + \frac{1}{c}\boldsymbol{\phi}^i \cdot \mathbf{e}\boldsymbol{\gamma} , \qquad (i = 1, 2, 3) , \tag{7.73}$$

and the *proper mechanical stresses* $\mathbf{X}^\alpha$ *are all functions of the vectors* $\boldsymbol{\phi}^i$—given by (7.72)—*and of* $\mathbf{e}$ *and* $\boldsymbol{\gamma}$.

It is quite natural to call the vectors $\boldsymbol{\phi}^i$ the *specific coordinate stresses, relative to the chosen Galilean frame* $S_{\rm g}$. They also satisfy the ordinary Cauchy theorem, so that the specific stress relative to the normal $\mathbf{n} \in \Sigma$ is

$$\boldsymbol{\phi}_n = n_i \boldsymbol{\phi}^i , \qquad \forall \mathbf{n} \in \Sigma, ; , \qquad \forall P \in C . \tag{7.74}$$

The natural decomposition (7.71)–(7.73), even if intrinsic in $S_{\rm g}$, essentially depends on the choice of $S_{\rm g}$. In particular, in the proper frame S_0, one has $\mathbf{X}^0 = 0$ and $\mathbf{X}^i = \boldsymbol{\phi}_0^i$, which specifies the physical meaning of the vectors $\mathbf{X}^\alpha$.

Let us project now the evolution equations (7.22) onto Σ and along $\boldsymbol{\gamma}$, starting from the term $\partial_\alpha(\hat{\mu}_0 V^\alpha \mathbf{V})$. By using the (Eulerian) decomposition

$$\mathbf{V} = \eta(\mathbf{e} + c\boldsymbol{\gamma}) \tag{7.75}$$

and defining the total density of relative mass

$$\hat{\mu} \stackrel{\rm def}{=} \hat{\mu}_0 \eta^2 , \tag{7.76}$$

one has

$$\begin{aligned}\partial_\alpha(\hat{\mu}_0 V^\alpha \mathbf{V}) &= \partial_i[\hat{\mu} e^i(\mathbf{e} + c\boldsymbol{\gamma})] + \frac{1}{c}\partial_t[\hat{\mu} c(\mathbf{e} + c\boldsymbol{\gamma})] \\ &= \partial_i(\hat{\mu} e^i \mathbf{e}) + \partial_t(\hat{\mu}\mathbf{e}) + [\partial_i(\hat{\mu} e^i) + \partial_t \hat{\mu}]c\boldsymbol{\gamma} .\end{aligned}$$

Hence, expanding the partial derivatives and using (6.5) together with the kinematical identity (6.11), we have

$$\partial_\alpha(\hat{\mu}_0 V^\alpha \mathbf{V}) = \frac{1}{D}[(\hat{\mu} D\mathbf{e})^\cdot + (\hat{\mu} D)^\cdot c\boldsymbol{\gamma}] .$$

Similarly, (7.71) and (7.73) imply

$$\partial_\alpha \mathbf{X}^\alpha = \partial_i \boldsymbol{\phi}^i + \frac{1}{c}\partial_i(\boldsymbol{\phi}^i \cdot \mathbf{e})\boldsymbol{\gamma} + \frac{1}{c^2}\partial_t(e_i \boldsymbol{\phi}^i) + \frac{1}{c^3}\partial_t(e_i \boldsymbol{\phi}^i \cdot \mathbf{e})\boldsymbol{\gamma} .$$

Introducing now the *relative mass force per unit relative mass*, $\mathbf{F}$:

$$\mu\mathbf{F} \stackrel{\rm def}{=} \mu_0 \mathbf{k}_\Sigma \equiv \mu_0(\mathbf{k} + \mathbf{k} \cdot \boldsymbol{\gamma}\boldsymbol{\gamma}), \qquad \mu \stackrel{\rm def}{=} \mu_0 \eta^2 , \tag{7.77}$$

gives rise to the *linear momentum theorem*

$$\frac{1}{D}(\hat{\mu} D\mathbf{e})^\cdot = \mu\boldsymbol{\mathcal{F}} \stackrel{\rm def}{=} \mu\mathbf{F} - \partial_i \boldsymbol{\phi}^i - \frac{1}{c^2}\partial_t(e_i \boldsymbol{\phi}^i) , \tag{7.78}$$

as well as to the *energy theorem*

$$\frac{1}{D}(\hat{\mu}Dc^2)^{\cdot} = \mu W \overset{\text{def}}{=} -\mu_0 c\mathbf{k}\cdot\boldsymbol{\gamma} - \partial_i(\boldsymbol{\phi}^i\cdot\mathbf{e}) - \frac{1}{c^2}\partial_t(e_i\boldsymbol{\phi}^i\cdot\mathbf{e})\,. \tag{7.79}$$

In contrast to (7.70), the latter equation is not in its final form, since the right-hand side should correspond to the *total power* W. However, by using (7.75): $-c\boldsymbol{\gamma} = \mathbf{e} - \mathbf{V}/\eta$, and introducing the *specific relative quantities* (per unit relative volume)

$$\begin{cases} \mu\hat{\epsilon} = \hat{\mu}c^2 & \text{internal energy,} \\ \mu r = -\mu_0\mathbf{k}\cdot\mathbf{V}/\eta & \text{radiation thermal power,} \\ w^{(\mathrm{i})} = \boldsymbol{\phi}^i\cdot\left(\partial_i\mathbf{e} + \dfrac{1}{c^2}e_i\partial_t\mathbf{e}\right) & \text{internal forces power,} \end{cases} \tag{7.80}$$

(7.78) can be cast in the typical form:

$$\frac{1}{\mu D}(\mu D\hat{\epsilon})^{\cdot} = W \overset{\text{def}}{=} \boldsymbol{\mathcal{F}}\cdot\mathbf{e} + r - \frac{1}{\mu}w^{(\mathrm{i})}\,. \tag{7.81}$$

Comparing this relation with the material point scheme, we have the additional term $-1/\mu w^{(\mathrm{i})}$, which is related to the internal structure of the continuum. In the relativistic Cauchy equation (7.78) there appears, instead, the term $-1/c^2\partial_t(e_i\boldsymbol{\phi}^i)$, absent classically, but also contained in the expression of the power $w^{(\mathrm{i})}$.

7.7 Transformation Laws of the Fundamental Relative Quantities

Equations (7.78) and (7.80) obey the relativity principle, since *they are formally invariant* with respect to the choice of the Galilean frame S_{g}. However, *they are not substantially invariant* because of the relative meaning of the various quantities. All the relative quantities introduced above (in particular $\hat{\epsilon}$, r and $w^{(\mathrm{i})}$) have instead a real physical content; that is, they cannot be made as vanishing by a simple change of Galilean frame. In fact, the following properties of absolute invariance hold (the index 0 denoting the proper quantities):

$$\begin{cases} \hat{\epsilon} = \hat{\mu}_0c^2/\mu_0 = \text{inv.} = \hat{\epsilon}_0, \\ \eta^3 r = -\mathbf{k}\cdot\mathbf{V} = \text{inv.} = r_0, \\ \eta w^{(\mathrm{i})} = \mathbf{X}^\alpha\cdot\partial_\alpha\mathbf{V} = \text{inv.} = w_0^{(\mathrm{i})}. \end{cases} \tag{7.82}$$

Equations $(7.82)_{1,2}$ directly follow from the definitions $(7.80)_{1,2}$, using (7.76) and (7.69) as well as the relation $\mu = \mu_0\eta^2$.

Equation $(7.82)_3$ follows, instead, from $(7.54)_2$ using (7.58), (7.71) and (7.73), that is:

$$\begin{aligned} w_0^{(\mathrm{i})} &= \mathbf{X}^\alpha \cdot \partial_\alpha \mathbf{V} = \mathbf{X}^\alpha \cdot \partial_\alpha [\eta(\mathbf{e} + c\boldsymbol{\gamma})] \\ &= \partial_\alpha \eta \mathbf{X}^\alpha \cdot \frac{\mathbf{V}}{\eta} + \eta \mathbf{X}^\alpha \cdot \partial_\alpha \mathbf{e} \\ &= \eta \left(\mathbf{X}^i \cdot \partial_i \mathbf{e} + \frac{1}{c} \mathbf{X}^0 \cdot \partial_t \mathbf{e} \right) \\ &= \eta \left(\boldsymbol{\phi}^i \cdot \partial_i \mathbf{e} + \frac{1}{c^2} e_i \boldsymbol{\phi}^i \cdot \partial_t \mathbf{e} \right) = \eta w^{(\mathrm{i})} \ . \end{aligned}$$

From (7.82), using the relations

$$\frac{\eta'}{\eta} = \frac{\sigma}{\alpha} \ , \qquad \alpha = \sqrt{1 - \frac{u^2}{c^2}} \ , \quad \sigma = 1 - \frac{\mathbf{u} \cdot \mathbf{e}}{c^2} \ , \tag{7.83}$$

one immediately gets the transformation laws of r and $w^{(\mathrm{i})}$, when passing from S_g to S'_g:

$$r' = \left(\frac{\alpha}{\sigma} \right)^3 r \ , \qquad w'^{(\mathrm{i})} = \left(\frac{\alpha}{\sigma} \right) w^{(\mathrm{i})} \ , \tag{7.84}$$

which in the classical situation ($c \to \infty$) reduce to invariance laws.

As concerns the mass force of (7.77), the corresponding transformation law can be derived from the one valid for the material point, apart from the fact that $\mathbf{F}$ and q now refer to the unit of relative mass. Thus, from (3.60), after multiplying both sides by $\mathrm{d}m/\mathrm{d}m' = \alpha/\sigma$, one gets the transformation law for $\mathbf{F}$:

$$\mathbf{F}' = \frac{\alpha}{\sigma^2} \left[\alpha \mathbf{F} - \frac{1}{c^2} (\mathbf{F} \cdot \mathbf{w} + r) \mathbf{u} \right] \ , \tag{7.85}$$

involving the thermal power r given by $(7.79)_2$ as well as the vector

$$\mathbf{w} \stackrel{\mathrm{def}}{=} \mathbf{e} - \frac{1}{1+\alpha} \mathbf{u} \ . \tag{7.86}$$

Also in this case, as for the material point, the condition $\mathbf{F} = 0$ does not imply $\mathbf{F}' = 0$, and hence has no absolute meaning. On the other hand, both the conditions $r = 0$ and $w^{(\mathrm{i})} = 0$ are absolute.

We next consider the transformation laws of the relative mechanical stresses $\boldsymbol{\phi}^i = X^{ik} \mathbf{c}_k$ or, equivalently, that of $\boldsymbol{\phi}_n = n_i \boldsymbol{\phi}^i$. To this end we proceed as follows:

1. start from the proper mechanical stresses $\mathbf{X}^\alpha$:

$$\mathbf{X}^\alpha = X^{\alpha\beta} \mathbf{c}_\beta \ , \tag{7.87}$$

   that is, from the symmetric tensor $X^{\alpha\beta}$;

2. use the Lorentz transformation in standard x^1-direction:

$$x'^0 = \frac{1}{\alpha}(x^0 - \beta x^1) , \quad x'^1 = \frac{1}{\alpha}(x^1 - \beta x^0), \quad x'^{2,3} = x^{2,3}; \tag{7.88}$$

3. evaluate the components $X'^{ik} = (\partial x'^i/\partial x^\alpha)(\partial x'^k/\partial x^\beta)X^{\alpha\beta}$, where

$$\frac{\partial x'^i}{\partial x^\alpha} = \begin{pmatrix} -\dfrac{\beta}{\alpha} & \dfrac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{7.89}$$

From (7.89) we thus have the relations

$$\begin{cases} X'^{11} = \dfrac{1}{\alpha^2}(\beta^2 X^{00} - \beta X^{10} - \beta X^{01} + X^{11}), \\ X'^{1a} = \dfrac{1}{\alpha}(X^{1a} - \beta X^{0a}), \\ X'^{2a} = X^{2a} , \quad X'^{33} = X^{33}, \end{cases}$$

where $a = 2, 3$; from the latter, in turn, up to a boost on Σ, we obtain the relative stresses to S'_{g}:

$$\boldsymbol{\phi}'^i = X'^{ik}\mathbf{c}'_k , \tag{7.90}$$

that is,

$$\begin{cases} \boldsymbol{\phi}'^1 = \dfrac{1}{\alpha}\left(\dfrac{1}{\alpha}X^{11}\mathbf{c}_1 + X^{12}\mathbf{c}_2 + X^{13}\mathbf{c}_3 - \dfrac{\beta}{\alpha}X^{10}\mathbf{c}_1\right) \\ \qquad - \dfrac{\beta}{\alpha}\left(\dfrac{1}{\alpha}X^{01}\mathbf{c}_1 + X^{02}\mathbf{c}_2 + X^{03}\mathbf{c}_3\right) + \dfrac{\beta^2}{\alpha^2}X^{00}\mathbf{c}_1, \\ \boldsymbol{\phi}'^a = \dfrac{1}{\alpha}X^{a1}\mathbf{c}_1 + X^{a2}\mathbf{c}_2 + X^{a3}\mathbf{c}_3 - \dfrac{\beta}{\alpha}X^{0a}\mathbf{c}_1 , \qquad (a = 2, 3). \end{cases} \tag{7.91}$$

We must now transform the right sides using the relation

$$\boldsymbol{\phi}^i = X^{ik}\mathbf{c}_k, \qquad \mathbf{X}^0 = X^{0\alpha}\mathbf{c}_\alpha = \frac{1}{c}e_i\left(\boldsymbol{\phi}^i + \frac{1}{c}\boldsymbol{\phi}^i \cdot \mathbf{e}\mathbf{c}_0\right) , \tag{7.92}$$

so that

$$X^{0k}\mathbf{c}_k = \frac{1}{c}e_i\boldsymbol{\phi}^i, \qquad X^{00} = \frac{1}{c^2}e_i\boldsymbol{\phi}^i \cdot \mathbf{e} . \tag{7.93}$$

Moreover, by using the identity

$$\frac{1}{\alpha} = 1 + x , \qquad x = \frac{1}{c^2}\frac{u^2}{\alpha(1+\alpha)} , \tag{7.94}$$

the sum

$$\mathbf{S}^i = \frac{1}{\alpha}X^{i1}\mathbf{c}_1 + X^{i2}\mathbf{c}_2 + X^{i3}\mathbf{c}_3 , \qquad (i = 1, 2, 3) ,$$

reduces to the form

$$\mathbf{S}^i = \boldsymbol{\phi}^i + xX^{i1}\mathbf{c}_1 \ ,$$

that is,

$$\mathbf{S}^i = \boldsymbol{\phi}^i + \frac{1}{c^2}\frac{\boldsymbol{\phi}^i \cdot \mathbf{u}}{\alpha(1+\alpha)}\mathbf{u} \ . \tag{7.95}$$

Similarly, from (7.93) and using the symmetry of X^{ik}, we have

$$X^{0i} = \frac{1}{c} e_k \boldsymbol{\phi}^k \cdot \mathbf{c}^i = \frac{1}{c} e_k \boldsymbol{\phi}^i \cdot \mathbf{c}^k$$

or

$$X^{0i} = \frac{1}{c}\boldsymbol{\phi}^i \cdot \mathbf{e}, \qquad (i = 1, 2, 3) \ ; \tag{7.96}$$

thus, $(7.91)_2$ become

$$\boldsymbol{\phi}'^a = \boldsymbol{\phi}^a - \frac{1}{c^2\alpha}\boldsymbol{\phi}^a \cdot \mathbf{wu}, \qquad (a = 2, 3) \ , \tag{7.97}$$

with $\mathbf{w}$ defined by (7.86).

As concerns $(7.91)_1$, there appear terms as in (7.97) and additional terms as in (7.93) and (7.96):

$$\begin{aligned}
\boldsymbol{\phi}'^1 &= \frac{1}{\alpha}\left(\boldsymbol{\phi}^1 - \frac{1}{c^2\alpha}\boldsymbol{\phi}^1 \cdot \mathbf{wu}\right) - \frac{\beta}{\alpha}\left(\frac{1}{c}e_i\boldsymbol{\phi}^i + xX^{01}\mathbf{c}_1\right) + \frac{1}{c^2}\frac{\beta^2}{\alpha^2}e_i\boldsymbol{\phi}^i \cdot \mathbf{ec}_1 \\
&= \boldsymbol{\phi}^1 - \frac{1}{c^2\alpha}\boldsymbol{\phi}^1 \cdot \mathbf{wu} + x\left(\boldsymbol{\phi}^1 - \frac{1}{c^2\alpha}\boldsymbol{\phi}^1 \cdot \mathbf{wu}\right) - \frac{\beta}{\alpha}\left(\frac{1}{c}e_i\boldsymbol{\phi}^i + x\boldsymbol{\phi}^1 \cdot \mathbf{ec}_1\right) \\
&\quad + \frac{1}{c^2}\frac{\beta^2}{\alpha^2}e_i\boldsymbol{\phi}^i \cdot \mathbf{ec}_1 \ .
\end{aligned}$$

Using the relation $\boldsymbol{\phi}^1 = \dfrac{1}{u}\boldsymbol{\phi}_u$ leads to

$$\begin{aligned}
\boldsymbol{\phi}'^1 &= \boldsymbol{\phi}^1 - \frac{1}{c^2\alpha}\boldsymbol{\phi}^1 \cdot \mathbf{wu} + \frac{1}{c}\frac{\beta}{\alpha}\frac{1}{1+\alpha}\left(\boldsymbol{\phi}_u - \frac{1}{c^2\alpha}\boldsymbol{\phi}_u \cdot \mathbf{wu}\right) \\
&\quad - \frac{1}{c}\frac{\beta}{\alpha}\boldsymbol{\phi}_e - \frac{1}{c^3\alpha^2}\frac{\beta}{1+\alpha}\boldsymbol{\phi}_u \cdot \mathbf{eu} + \frac{\beta}{c^3\alpha^2}\boldsymbol{\phi}_e \cdot \mathbf{eu} \ .
\end{aligned}$$

Noting that $\boldsymbol{\phi}_u \cdot \mathbf{e} = \boldsymbol{\phi}_e \cdot \mathbf{u}$, after some algebraic manipulation one gets

$$\begin{aligned}
\boldsymbol{\phi}'^1 &= \boldsymbol{\phi}^1 - \frac{1}{c^2\alpha}\boldsymbol{\phi}^1 \cdot \mathbf{wu} - \frac{1}{c^2\alpha}u\boldsymbol{\phi}_w - \frac{1}{c^4\alpha^2}\frac{u}{1+\alpha}\boldsymbol{\phi}_u \cdot \mathbf{wu} \\
&\quad + \frac{1}{c^4\alpha^2}u\boldsymbol{\phi}_w \cdot \mathbf{eu} \ ,
\end{aligned}$$

and finally

$$\boldsymbol{\phi}'^1 = \boldsymbol{\phi}^1 - \frac{1}{c^2\alpha}\boldsymbol{\phi}^1 \cdot \mathbf{wu} - \frac{u}{c^2\alpha}\left(\boldsymbol{\phi}_w - \frac{1}{c^2\alpha}\boldsymbol{\phi}_w \cdot \mathbf{wu}\right) \ . \tag{7.98}$$

Therefore, taking into account the meaning of u in x^1-standard coordinates, (7.97) and (7.98) are summarized by the following relation, without any restriction on the choice of the triad $\mathcal{T} \in S_{\mathrm{g}}$:

$$\boldsymbol{\phi}'^i = \boldsymbol{\phi}^i - \frac{1}{c^2\alpha}\boldsymbol{\phi}^i \cdot \mathbf{wu} - \frac{u^i}{c^2\alpha}\left(\boldsymbol{\phi}_w - \frac{1}{c^2\alpha}\boldsymbol{\phi}_w \cdot \mathbf{wu}\right) . \tag{7.99}$$

After multiplying by n_i one then also gets the corresponding relation between the mechanical stresses in S_{g} and S'_{g}, with respect to the given direction $\mathbf{n}$:

$$\boldsymbol{\phi}'_n = \boldsymbol{\phi}_n - \frac{1}{c^2\alpha}\boldsymbol{\phi}_n \cdot \mathbf{wu} - \frac{\mathbf{n}\cdot\mathbf{u}}{c^2\alpha}\left(\boldsymbol{\phi}_w - \frac{1}{c^2\alpha}\boldsymbol{\phi}_w \cdot \mathbf{wu}\right) , \tag{7.100}$$

where the dependence on the velocity $\mathbf{e}$ and on the continuum element is through the vector $\mathbf{w}$ given by (7.86).

In the limit $c \to \infty$ one recovers the classical invariance: $\boldsymbol{\phi}'_n = \boldsymbol{\phi}_n$; in the relativistic context, instead, the condition $\boldsymbol{\phi}_n = p\mathbf{n}$ in S_{g} does not have an invariant meaning, since $\boldsymbol{\phi}'_n \neq p'\mathbf{n}$. In other words, the classical concept of nonviscous fluid is meaningless in relativity, and the hypothesis of pure pressure can only be formulated in the proper frame. However, since $\boldsymbol{\phi}_w = w_i\boldsymbol{\phi}^i$, (7.99) can be cast into the form

$$\boldsymbol{\phi}'^i = \left(\delta^i_l - \frac{1}{c^2\alpha}u^i w_l\right)\left(\boldsymbol{\phi}^l - \frac{1}{c^2\alpha}\boldsymbol{\phi}^l \cdot \mathbf{wu}\right) , \tag{7.101}$$

and thus in terms of the tension characteristics:

$$X'^{ik} = B^i{}_l X^{lm} B^k{}_m , \tag{7.102}$$

through the spatial 2-tensor

$$B^i{}_l \stackrel{\mathrm{def}}{=} \delta^i_l - \frac{1}{c^2\alpha}u^i w_l , \tag{7.103}$$

which has already been introduced in Chap. 5.

7.8 Energy Theorem and the First Law of Thermodynamics

In the classical context, the first law of thermodynamics is meant to be a substitute for the energy theorem, i.e. it is introduced to correct the evaluation of the kinetic energy. Equation (7.80) confirms this interpretation, in the sense that it is equivalent to the first law of thermodynamics, once it satisfies the relativistic Cauchy (7.78). To show this, it is enough to eliminate from (7.78) and (7.80) the specific mechanical power: $\boldsymbol{\mathcal{F}} \cdot \mathbf{e}$. After multiplying (7.78) by $\mathbf{e}$ and using the relation

$$e^2 = c^2 \left(1 - \frac{1}{\eta^2}\right) , \tag{7.104}$$

one has

$$\mu \boldsymbol{\mathcal{F}} \cdot \mathbf{e} = \frac{1}{D}(\hat{\mu}D)^{\cdot} e^2 + \frac{1}{2}\hat{\mu}(e^2)^{\cdot} = \frac{1}{D}(\hat{\mu}Dc^2)^{\cdot} \left(1 - \frac{1}{\eta^2}\right) + \frac{1}{\eta^3}\hat{\mu}c^2\dot{\eta} ,$$

so that (7.80) becomes

$$\frac{1}{D\eta^2} \left(\hat{\mu}Dc^2\right)^{\cdot} - \frac{\hat{\mu}c^2}{\eta^3}\dot{\eta} = \mu r - w^{(\mathrm{i})} .$$

Multiplying the latter expression by η finally gives

$$\frac{1}{D} \left(\frac{\hat{\mu}}{\eta} Dc^2\right)^{\cdot} = \eta(\mu r - w^{(\mathrm{i})}) ;$$

introducing then the specific internal energy $\mu\hat{\epsilon} = \hat{\mu}c^2$, as from $(7.80)_1$, leads to

$$\frac{1}{D} \left(\frac{\mu}{\eta} D\hat{\epsilon}\right)^{\cdot} = \eta(\mu r - w^{(\mathrm{i})}) . \tag{7.105}$$

This is exactly the relativistic form of the first law of thermodynamics given by (6.19). Because of the invariance properties $\eta D = \text{inv.} = D_0$ and $\mu/\eta^2 = \text{inv.} = \mu_0$ and using the substantial derivative

$$\eta(\,)^{\cdot} \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}\tau}(\,) = V^\alpha \partial_\alpha(\,) , \tag{7.106}$$

(7.105) is equivalent to the energy theorem (7.70) in the proper Galilean frame of the generic continuum element:

$$\frac{1}{D_0} \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\mu_0 D_0 \hat{\epsilon}_0\right) = \mu_0 r_0 - w_0^{(\mathrm{i})} . \tag{7.107}$$

Furthermore, using the kinematical identity (5.201)

$$\frac{1}{D_0} \frac{\mathrm{d}D_0}{\mathrm{d}\tau} = \partial_\alpha V^\alpha \quad \sim \quad \partial_\alpha \left(\frac{1}{D_0} V^\alpha\right) = 0 , \tag{7.108}$$

allows us to cast (7.107) into a *balance equation*:

$$\partial_\alpha(\mu_0 \hat{\epsilon}_0 V^\alpha) = \mu_0 r_0 - w_0^{(\mathrm{i})} = \eta(\mu r - w^{(\mathrm{i})}) . \tag{7.109}$$

In conclusion, from the relative point of view, the general framework of relativistic continuum mechanics is summarized by the following Eulerian equations:

$$\begin{cases} \dfrac{1}{D}\left(\dfrac{1}{c^2}\mu D\hat{\epsilon}\mathbf{e}\right)^{\cdot} = \dfrac{1}{\eta^2}\mu\mathbf{k}_\Sigma - \partial_i\boldsymbol{\phi}^i - \dfrac{1}{c^2}\partial_t(e_i\boldsymbol{\phi}^i), \\ \\ \dfrac{1}{D}\left(\dfrac{1}{\eta}\mu D\hat{\epsilon}\right)^{\cdot} = \eta(\mu r - w^{(\mathrm{i})}), \end{cases} \tag{7.110}$$

where the dot denotes the molecular derivative (7.106). Obviously, the relative numerical density $1/D$ is not independent of the velocity $\mathbf{e}$ from the differential point of view since we have

$$\frac{\dot{D}}{D} = \operatorname{div}\mathbf{e}\ , \tag{7.111}$$

which takes the place of the classical equation of mass conservation. The general equations are then 5 in 12 unknowns: μ (*mass density*), $1/D$ (*numerical density*), e_i (*Eulerian velocity*), X^{ik} (*tension characteristics*) and $\hat{\epsilon}$ (*specific internal energy*). To have the same number of equations as unknowns, one needs seven more equations. This is not surprising, since one should include into the equations the characteristic properties of the continuum (*thermodynamical state functions*). These are the constitutive equations which, as in the classical case, involve the energy ϵ and the mechanical stress X^{ik}, i.e. the so-called characteristic functions $\hat{\epsilon} = \hat{\epsilon}(\mu, D, e_i)$ and $X^{ik} = X^{ik}(\mu, D, e_i)$, necessary to make the scheme fully determined.

7.9 Continua Without Material Structure

The system (7.110) can be conveniently transformed after separating $\mu_0\hat{\epsilon}_0 = \hat{\mu}_0 c^2$ (with $\hat{\epsilon} = \hat{\epsilon}_0$ from (7.82)) into the pure matter proper energy from the proper thermal energy:

$$\mu_0\hat{\epsilon} = \mu_0 c^2 + \epsilon_{c,0}, \quad \hat{\epsilon} = c^2 + \varepsilon\ , \tag{7.112}$$

where $\varepsilon \overset{\text{def}}{=} \epsilon_{c,0}/\mu_0$ or, in relative terms,

$$\mu\hat{\epsilon} = \mu c^2 + \frac{\mu}{\mu_0}\epsilon_{c,0} = \mu c^2 + \mu\varepsilon\ . \tag{7.113}$$

Equations (7.110) and (7.111) then become

$$\begin{cases} \dfrac{1}{D}\left(\mu D\mathbf{e}\right)^{\cdot} = \mu_0\mathbf{k}_\Sigma - \partial_i\boldsymbol{\phi}^i - \dfrac{1}{c^2}\left[\partial_t(e_i\boldsymbol{\phi}^i) + \dfrac{1}{D}(\mu D\varepsilon\mathbf{e})^{\cdot}\right], \\ \dfrac{1}{D}\left(\dfrac{1}{\eta}\mu D\right)^{\cdot} = -\dfrac{1}{c^2}\left[\dfrac{1}{D}\left(\dfrac{1}{\eta}\mu D\varepsilon\right)^{\cdot} - \eta(\mu r - w^{(\mathrm{i})})\right], \\ \dfrac{\dot{D}}{D} = \operatorname{div}\,\mathbf{e}, \end{cases} \tag{7.114}$$

which shows the sources of linear momentum and energy, respectively, considered in the ordinary sense, that is, in the strictly material context. From the absolute point of view, up to the boundary conditions, such sources were summarized in the volume 4-forces and in the 4-stresses; from the relative point of view, instead, they are identified with $\mathbf{k}_\Sigma$, r, the mechanical stresses $\boldsymbol{\phi}^i$ and with the thermal energy ε.

Thus, in order to have the same number of equations as unknowns, one must specify these sources, which are a priori completely free, apart from the invariance conditions imposed by the relativistic theory.

Apart from the analogy of (7.110) and (7.111) with the material point dynamics, the presence in $\hat{\epsilon}$ of the thermal contribution both to the inertia and to the internal energy represents a completely new feature of relativistic continuum mechanics. This is even more evident if one considers a continuum of classic type, without internal structure, i.e. a continuum such that each particle has the proper mass as a conserved quantity:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\mu_0 D_0) = 0 \quad \sim \quad \mu_0 D_0 = \text{const. for each element of } S\,. \tag{7.115}$$

In each Galilean frame, the following conservation theorem holds:

$$\left(\frac{\mu}{\eta} D\right)^{\cdot} = 0 \quad \sim \quad \frac{\mu}{\eta} D = \text{const. for each element of } S\,, \tag{7.116}$$

and (7.114) reduces to the form

$$\frac{\mu}{\eta}\dot{\varepsilon} = \eta(\mu r - w^{(\mathrm{i})})\,, \tag{7.117}$$

which is the ordinary *first law of thermodynamics*. The presence on the right-hand side of the thermal power shows that it cannot be deduced from the equations of motion. Thus, in the continuum scheme, even without any internal structure, the energy theorem remains independent of the equations of motion. This is a peculiar property of the continuum scheme, because it has no counterpart in the dynamics of particles without internal structure.

In conclusion, for an *ordinary relativistic continuum*, the relative equations are six (one more than in the general case, because of the constraint (7.115)):

$$\begin{cases} \left(\dfrac{\mu}{\eta} D\right)^{\cdot} = 0\,, \quad \dfrac{\dot{D}}{D} = \operatorname{div}\mathbf{e}, \\ \dfrac{1}{D}(\mu D\mathbf{e})^{\cdot} = \dfrac{1}{\eta^2}\mu\mathbf{k}_\Sigma - \partial_i\boldsymbol{\phi}^i - \dfrac{1}{c^2}\left[\partial_t(e_i\boldsymbol{\phi}^i) + \dfrac{1}{D}(\mu D\varepsilon\mathbf{e})^{\cdot}\right], \\ \dfrac{\mu}{\eta}\dot{\varepsilon} = \eta(\mu r - w^{(\mathrm{i})}), \end{cases} \tag{7.118}$$

and there are five unknowns: μ, D and e_i. The last equation is a constraint on the laws of the sources ε, r and X^{ik} (with the last two terms coming from the internal force power). For instance, it can be used to determine r, starting from the constitutive functions $\varepsilon = \varepsilon(...)$ and $X^{ik} = X^{ik}(...)$, which remain free. Obviously, in such a case $\mathbf{k}_\Sigma$ must also be assigned.

If, instead, the thermal power is assigned, the constitutive equations for $\varepsilon = \varepsilon(...)$ and $X^{ik} = X^{ik}(...)$ are necessarily subject to the thermodynamical constraint $(7.118)_3$. We note that $(7.118)_4$ represents the conservation of the particle number all along the motion. In fact, it can be written as

$$\frac{1}{D}\partial_t D = -\frac{1}{D}\partial_i D e^i + \partial_i e^i \equiv D\partial_i \left(\frac{1}{D}e^i\right) ,$$

or

$$\partial_t \left(\frac{1}{D}\right) + \partial_i \left(\frac{1}{D}e^i\right) = 0 . \tag{7.119}$$

This completes the proof if one compares the equation just obtained with the ordinary mass conservation equation, taking into account that $1/D$ represents the number of particles of the continuum per unit volume in S_{g}.

Finally, as concerns the classical situation ($c \to \infty$), the system (7.114) allows us to re-obtain the ordinary Cauchy equation and it also shows that the energy theorem reduces to the mass conservation theorem: $\mu D = \text{const}$. The coupling with thermodynamics is then lost and the first law of thermodynamics must be assumed as an extra postulate representing the *relativistic correction to order* $1/c^2$ of the same $(7.114)_2$.

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8

Elements of Relativistic Thermodynamics of a Continuum

8.1 Introduction

We have already noted that relativity with its fundamental notion of space-time is the most convenient framework for the description of physics, in particular for mechanical and electromagnetic phenomena. In fact, in the mechanical context, it represents the first important step in the process of conceptual unification of physics. This process, starting from ordinary particles (constant proper mass), becomes more and more meaningful in the dynamics of particles endowed with internal material structure, where the energy theorem is independent of the momentum theorem. There are two important aspects concerning such unification: (1) the mass and the kinetic energy combine into the material energy; (2) the mechanical and thermal actions combine into the 4-force.

Passing from the pointlike scheme to the more general one of a continuum, the unifying process is continued in a natural way. In fact,

1. the notion of 4-*stress summarizes the mechanical stress, the internal energy and the thermal conduction vector*;
2. the mass or *volume* 4-*force includes both the mechanical action and the thermal radiation*;
3. the *surface* 4-*force* takes into account both the mechanical action on a surface and the effect of thermal contact with other bodies.

There are clear *advantages* associated with the *absolute formulation* of the thermomechanics of continua. However, besides the elegant and synthetic absolute formulation one has to consider, especially from the physical point of view, the *relative formulation* with corresponding *properties of invariance*, either *formal* with respect to the choice of the Galilean frame or *contentwise* under time translations and space translations and rotations. In order to have a physical content and a general validity a relative theory must be in turn formulated in an arbitrary Galilean frame. In addition, the theory must include

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the transformation laws of the relative ingredients, passing from one frame to another.

The relativistic framework is deeply modified with respect to the classical one (and even simplified) either for the *relativistic corrections* to the traditional physical ingredients (mechanical stress, power of internal forces, internal energy, mechanical power, etc.) or for the changes required to the fundamental equation and, finally, for the relative meaning and dynamical content of such quantities, typical in relativity. In such a framework, the classical result becomes better clarified, especially as concerns the *decoupling mechanism of certain quantities* (see e.g. the role of the first law of thermodynamics).

In this chapter we will limit our attention to the case of *nonpolar continua*, characterized by the condition that the 4-stress tensor be symmetric [1].

The more general scheme of *polar relativistic continua*, not considered here because of its complexity, is related to nonsymmetric energy–momentum tensors. The latter are continuous systems which already in the classical context require the introduction of mass and stress moments [2, 3] from the dynamical point of view; moreover, from the geometrical–kinematical point of view, they imply the enlargement of the pointlike structure to spinning particles, by means of applied vector fields (*directors*) (see e.g. the case of Cosserat continua [4, 5]) or tensor fields (*general microstructures*). The study of polar continua in the relativistic context is motivated by the conjecture that the further unification of thermodynamical and electromagnetic properties of matter will require a relation between the antisymmetric part of the stress tensor and the electromagnetic field.

However, we will not further discuss the fascinating problem of unification, but we will start again from the ordinary scheme, in order to complete some general aspects concerning the mechanical stresses and their associated proper values, e.g. the Lagrangian power of internal forces and the associated notion of isotropy, the class of systems undergoing reversible transformations and the thermodynamics of perfect fluids.

8.2 Nonpolar Continua

Let us consider the decomposition of the coordinate 4-stresses $\mathbf{T}^\alpha$ ($\alpha = 0, 1, 2, 3$):

$$\mathbf{T}^\alpha = \hat{\mathbf{T}}^\alpha + Q^\alpha \mathbf{V} , \qquad \hat{\mathbf{T}}^\alpha \cdot \mathbf{V} = 0 , \quad (\alpha = 0, 1, 2, 3) . \tag{8.1}$$

The total stress $\mathbf{T}_N = N_\alpha \mathbf{T}^\alpha$ ($\forall E \in \mathcal{T}$ and $\forall \mathbf{N} \in M_4$) splits into two parts:

$$\mathbf{T}_N = \hat{\mathbf{T}}_N + Q_N \mathbf{V} , \quad \hat{\mathbf{T}}_N = N_\alpha \hat{\mathbf{T}}^\alpha , \qquad Q_N = N_\alpha Q^\alpha , \tag{8.2}$$

which we will call *proper total mechanical stress* and *proper thermal flux* at the point E and along the direction $\mathbf{N}$, respectively. The former, orthogonal to $\mathbf{V}$ according to $(8.1)_2$, is a spacelike vector. The latter is instead parallel to $\mathbf{V}$ and hence timelike. In the following we will use the notation

$$\mathbf{T}^\alpha = T^{\alpha\beta}\mathbf{c}_\beta \ , \qquad \hat{\mathbf{T}}^\alpha = \hat{T}^{\alpha\beta}\mathbf{c}_\beta \ , \tag{8.3}$$

without assuming a priori the symmetry of $T^{\alpha\beta}$.

Let us decompose $\hat{T}^{\alpha\beta}$ into its symmetric and antisymmetric parts:

$$\hat{T}^{\alpha\beta} = X_0^{\alpha\beta} + F_0^{\alpha\beta} \ , \quad X_0^{\alpha\beta} \stackrel{\text{def}}{=} \hat{T}^{(\alpha\beta)} \ , \quad F_0^{\alpha\beta} \stackrel{\text{def}}{=} \hat{T}^{[\alpha\beta]} \ . \tag{8.4}$$

We will call $X_0^{\alpha\beta} = X_0^{\beta\alpha}$ the *proper mechanical stress tensor* and $F_0^{\alpha\beta} = -F_0^{\beta\alpha}$ the *proper electromagnetic tensor*. These two quantities are unified by $\hat{T}^{\alpha\beta}$. Consider now the natural decomposition along $\mathbf{V}$ of all three tensors introduced above: $X_0^{\alpha\beta}$, $F_0^{\alpha\beta}$ and Q^α, locally associated with the tension tensor $T^{\alpha\beta}$. We have

$$X_0^{\alpha\beta} = X^{\alpha\beta} + X^\alpha V^\beta + X^\beta V^\alpha + XV^\alpha V^\beta \ , \tag{8.5}$$

and similarly

$$F_0^{\alpha\beta} = H^{\alpha\beta} + E^\alpha V^\beta - E^\beta V^\alpha \ , \qquad c^2 Q^\alpha = q_0^\alpha + \epsilon_{c,0} V^\alpha \ , \tag{8.6}$$

where all the newly introduced tensors are spatial, i.e. they belong to the platform Σ_0 orthogonal to $\mathbf{V}$ at $E \in \mathcal{T}$:

$$\begin{cases} X^{\alpha\beta} = X^{\beta\alpha} \ , \quad X^{\alpha\beta}V_\beta = 0 \ , \quad X^\alpha V_\alpha = 0, \\ \\ H^{\alpha\beta} = -H^{\beta\alpha} \ , \quad H^{\alpha\beta}V_\beta = 0 \ , \quad E^\alpha V_\alpha = 0 \ , \quad q_0^\alpha V_\alpha = 0. \end{cases} \tag{8.7}$$

Clearly, they are not all independent; for example $(8.1)_2$, after the decomposition (8.4), becomes

$$X_0^{\alpha\beta}V_\beta + F_0^{\alpha\beta}V_\beta = 0 \ .$$

Because of (8.5), (8.6) and (8.7), this equation is equivalent to the condition $X^\alpha + XV^\alpha + E^\alpha = 0$, that is:

$$X = 0 \ , \quad X^\alpha = -E^\alpha \ .$$

Therefore, (8.1) and (8.5) assume, respectively, the following forms:

$$X_0^{\alpha\beta} = X^{\alpha\beta} - E^\alpha V^\beta - E^\beta V^\alpha \tag{8.8}$$

and

$$T^{\alpha\beta} = X^{\alpha\beta} + H^{\alpha\beta} + \frac{1}{c^2} q_0^\alpha V^\beta - 2E^\beta V^\alpha + \frac{1}{c^2}\epsilon_{c,0} V^\alpha V^\beta \ , \tag{8.9}$$

with $\epsilon_{c,0}$ introduced in (7.20). One can then evaluate the antisymmetric part of the tension tensor:

$$T^{[\alpha\beta]} = H^{\alpha\beta} + \left(E^\alpha + \frac{1}{2c^2} q_0^\alpha\right) V^\beta - \left(E^\beta + \frac{1}{2c^2} q_0^\beta\right) V^\alpha \, .$$

Hence, the necessary and sufficient condition for the tension tensor $T^{\alpha\beta}$ to be symmetric is

$$H^{\alpha\beta} = 0 \, , \qquad E^\alpha = -\frac{1}{2c^2} q_0^\alpha \, . \tag{8.10}$$

In this case (*nonpolar continua*), which is the only one we consider hereafter, $T^{\alpha\beta}$ has the form

$$T^{\alpha\beta} = X^{\alpha\beta} + \frac{1}{c^2}(q_0^\alpha V^\beta + q_0^\beta V^\alpha + \epsilon_{c,0} V^\alpha V^\beta) \tag{8.11}$$

and is summarized by the 4-*velocity* V^α, the (spatial and symmetric) *proper mechanical stress tensor* $X^{\alpha\beta}$, the scalar invariant *proper thermal inertia*[1] $\mu_{c,0} = \epsilon_{c,0}/c^2$ and the *spatial vector* q_0^α *of the proper thermal conduction.*

We note that expression (8.11) is the typical (relative) form of the energy tensor associated with the electromagnetic scheme. Comparing the two fields, *material* and *electromagnetic*, one must assume $\mu_0 = 0$ and interpret $\epsilon_{c,0}$ as the proper electromagnetic energy density, $\mathbf{q}_0$ as the Poynting vector and, finally, $X^{\alpha\beta}$ as Maxwell's stress tensor. However, in this formal analogy, which can be extended to any Galilean frame with $\boldsymbol{\gamma} \neq \boldsymbol{\gamma}_0 \equiv \mathbf{V}/c$, Maxwell's energy tensor has no direct counterpart in the proper mechanical stress tensor $\hat{T}^{\alpha\beta}$ given by (8.4):

$$\hat{T}^{\alpha\beta} = X^{\alpha\beta} + \frac{1}{c^2} q_0^\beta V^\alpha \, , \tag{8.12}$$

rewritten here by using (8.8)–(8.10). Such a tensor, different from $X^{\alpha\beta}$, is also nonsymmetric:

$$\hat{T}^{[\alpha\beta]} = \frac{1}{c^2}(q_0^\beta V^\alpha - q_0^\alpha V^\beta) \, . \tag{8.13}$$

Nonpolar continua are still characterized by the *reciprocity axiom*:

$$\mathbf{T}_N \cdot \mathbf{N}' = \mathbf{T}_{N'} \cdot \mathbf{N} \, , \qquad \forall E \in \mathcal{T} \, , \quad \forall \mathbf{N}, \mathbf{N}' \in M_4 \, . \tag{8.14}$$

If we assume this axiom to be valid for the proper mechanical stress tensor $\hat{\mathbf{T}}_N$ also,

$$\hat{\mathbf{T}}_N \cdot \mathbf{N}' = \hat{\mathbf{T}}_{N'} \cdot \mathbf{N} \, , \qquad \forall E \in \mathcal{T} \, , \quad \forall \mathbf{N}, \mathbf{N}' \in M_4 \, , \tag{8.15}$$

then $\hat{T}^{\alpha\beta}$ *is symmetric too, like* $T^{\alpha\beta}$:

$$X^{\alpha\beta} = X_0^{\alpha\beta} \, , \qquad F_0^{\alpha\beta} = 0,$$

and, as a consequence of (8.6) and (8.10), one has again the scheme of nonpolar continua, without thermal conduction: $\mathbf{q}_0 = 0$ and $\hat{T}^{\alpha\beta} = X^{\alpha\beta}$.

[1] The idea of incorporating in the tension tensor a proper energy term different from that of pure matter $\mu_0 V^\alpha V^\beta$ is due to C. Cattaneo [6].

8.3 Proper Thermodynamics of the Nonpolar Continua

For a nonpolar continuous system (8.11) and $(8.3)_1$ imply that the 4-stresses have the form

$$\mathbf{T}^\alpha = \mathbf{X}^\alpha + \frac{1}{c^2}[V^\alpha \mathbf{q}_0 + (q_0^\alpha + \epsilon_{c,0} V^\alpha)\mathbf{V}] \, , \tag{8.16}$$

where we have used the notation

$$\mathbf{X}^\alpha \stackrel{\text{def}}{=} X^{\alpha\beta}\mathbf{c}_\beta \, . \tag{8.17}$$

Thus the *proper ordinary stresses* $\mathbf{X}_N$ along $\mathbf{N}$

$$\mathbf{X}_N \stackrel{\text{def}}{=} N_\alpha \mathbf{X}^\alpha \tag{8.18}$$

satisfy the *reciprocity property*

$$\mathbf{X}_N \cdot \mathbf{N}' = \mathbf{X}_{N'} \cdot \mathbf{N} \, , \qquad \forall E \in \mathcal{T} \, , \quad \forall \mathbf{N}, \mathbf{N}' \in M_4 \, . \tag{8.19}$$

The coordinate stresses $\mathbf{X}^\alpha$ as well as $\mathbf{q}_0$ are spatial vectors:

$$\mathbf{V} \cdot \mathbf{V} = -c^2 \, , \quad \mathbf{q}_0 \cdot \mathbf{V} = 0 \, , \qquad \mathbf{X}^\alpha \cdot \mathbf{V} = 0 \, . \tag{8.20}$$

The latter condition, because of (8.19), is equivalent to $\mathbf{X}_V \cdot \mathbf{c}^\alpha = 0$ for every possible choice of the Cartesian basis $\mathbf{c}_\alpha$; hence $\mathbf{X}_V = 0$ implying the *linear dependence* of the vectors $\mathbf{X}^\alpha$:

$$\mathbf{X}_V = V_\alpha \mathbf{X}^\alpha = 0 \, . \tag{8.21}$$

Thus, when $V_0 \neq 0$, (8.21) *allows us to express the coordinate stress* $\mathbf{X}^0$ *as a function of the others*:

$$\mathbf{X}^0 = -\frac{V_i}{V_0}\mathbf{X}^i \, . \tag{8.22}$$

Moreover, introducing the *total proper mass density* $\hat{\mu}_0$

$$\hat{\mu}_0 \stackrel{\text{def}}{=} \mu_0 + \mu_{c,0} \, , \qquad \mu_{c,0} \stackrel{\text{def}}{=} \frac{\epsilon_{c,0}}{c^2} \, , \tag{8.23}$$

as a *sum of pure matter inertia and thermal inertia* (see (7.23)), the relativistic Cauchy equation (7.9) can be written in the form

$$\partial_\alpha \left[\left(\hat{\mu}_0 V^\alpha + \frac{1}{c^2} q_0^\alpha \right) \mathbf{V} + \mathbf{X}^\alpha + \frac{1}{c^2} V^\alpha \mathbf{q}_0 \right] = \mu_0 \mathbf{k} \tag{8.24}$$

or, in scalar terms,

$$\partial_\alpha M^{\alpha\beta} = \mu_0 k^\beta \, , \tag{8.25}$$

where the *energy tensor* $M^{\alpha\beta}$ has the form

$$M^{\alpha\beta} = \hat{\mu}_0 V^\alpha V^\beta + X^{\alpha\beta} + \frac{1}{c^2}(V^\alpha q_0^\beta + V^\beta q_0^\alpha) \, , \tag{8.26}$$

with the restrictions

$$V^\alpha V_\alpha = -c^2 \, , \quad X^{\alpha\beta} = X^{\beta\alpha} \, , \quad X^{\alpha\beta} V_\beta = 0 \, , \quad q_0^\alpha V_\alpha = 0 \, . \tag{8.27}$$

To (8.27) one must add the *boundary conditions* on $\partial\mathcal{T}$:

$$\nu_\alpha \mathbf{T}^\alpha = \mathbf{g} \, , \tag{8.28}$$

with $\boldsymbol{\nu}$ being the internal unit normal vector to the hypersurface $\partial\mathcal{T}$, which is the boundary of the world tube $\mathcal{T}$. The field $\mathbf{g}$ should be assigned on $\partial\mathcal{T}$, corresponding to the external thermomechanical contact. However, (8.24) can be written in intrinsic terms as we have already done in (5.46) for the case $\mathbf{q}_0 = 0$ (*absence of thermal conduction*); performing the derivative and scalar multiplying by $\mathbf{V}$ give first of all the scalar equation analogous to (7.53),

$$c^2 \partial_\alpha \left(\hat{\mu}_0 V^\alpha + \frac{1}{c^2} q_0^\alpha \right) = -\mu_0 \mathbf{k} \cdot \mathbf{V} - \left(\mathbf{X}^\alpha + \frac{1}{c^2} V^\alpha \mathbf{q}_0 \right) \cdot \partial_\alpha \mathbf{V} \, ,$$

from which we obtain the *proper energy theorem*

$$c^2 \partial_\alpha (\hat{\mu}_0 V^\alpha) = \mu_0 q_0 - w_0^{(\mathrm{i})} \, , \tag{8.29}$$

where the following proper quantities (per unit proper volume) have been introduced:

$$\begin{cases} \mu_0 q_0 \overset{\mathrm{def}}{=} \mu_0 (r_0 + q_{c,0}) & \text{total thermal power,} \\ \mu_0 r_0 \overset{\mathrm{def}}{=} -\mu_0 \mathbf{k} \cdot \mathbf{V} & \text{radiation thermal power,} \\ \mu_0 q_{c,0} \overset{\mathrm{def}}{=} -\left(\partial_\alpha q_0^\alpha + \dfrac{1}{c^2} \mathbf{q}_0 \cdot \mathbf{A} \right) & \text{conduction thermal power,} \\ w_0^{(\mathrm{i})} \overset{\mathrm{def}}{=} \mathbf{X}^\alpha \cdot \partial_\alpha \mathbf{V} & \text{internal forces power .} \end{cases} \tag{8.30}$$

By substituting (8.29) into (8.24) we finally obtain the equation analogous to (7.55):

$$\rho_0 \mathbf{A} = \hat{\boldsymbol{\mathcal{F}}}_0 \, , \tag{8.31}$$

where $\hat{\boldsymbol{\mathcal{F}}}_0$ is the *total proper mechanical force* per unit proper volume:

$$\hat{\boldsymbol{\mathcal{F}}}_0 \overset{\mathrm{def}}{=} \mu_0 \mathbf{k} - \partial_\alpha \left(\mathbf{X}^\alpha + \frac{1}{c^2} q_0^\alpha \mathbf{V} + \frac{1}{c^2} V^\alpha \mathbf{q}_0 \right) - \frac{1}{c^2} (\mu_0 q_0 - w_0^{(\mathrm{i})}) \mathbf{V} \, ; \tag{8.32}$$

this expression reduces to the previous $\boldsymbol{\mathcal{F}}_0$ when the additional terms due to the thermal flux $\mathbf{q}_0$ are neglected. As concerns the proper power of internal forces $w_0^{(\mathrm{i})}$, one still has the expression (7.68): $w_0^{(\mathrm{i})} = X^{\alpha\beta} \kappa_{\alpha\beta}$; moreover, by introducing the proper density of internal energy $\hat{\epsilon}$ using (7.112)

$$\mu_0 \hat{\epsilon} \overset{\text{def}}{=} \hat{\mu}_0 c^2 \quad \rightarrow \quad \hat{\epsilon} = c^2 + \varepsilon \,, \quad \varepsilon \overset{\text{def}}{=} \frac{\epsilon_{c,0}}{\mu_0} \,, \tag{8.33}$$

the energy theorem (8.29) assumes exactly the form (7.70), valid when $\mathbf{q}_0 = 0$,

$$\frac{1}{\mu_0 D_0} \frac{\mathrm{d}}{\mathrm{d}\tau} (\mu_0 D_0 \hat{\epsilon}) = q_0 - \frac{1}{\mu_0} w_0^{(\mathrm{i})} \,, \tag{8.34}$$

and represents the *first law of thermodynamics in absolute terms and in the proper frame.*

Finally, in order to have the same number of equations as unknowns one can consider (8.31) and (8.34) *as fundamental equations*, so that, using (8.33), the following system of equations (*proper formulation*) arises:

$$\begin{cases} \mu_0 \left(1 + \dfrac{\varepsilon}{c^2}\right) \dfrac{\mathrm{d}\mathbf{V}}{\mathrm{d}\tau} = \hat{\boldsymbol{\mathcal{F}}}_0, \\[2ex] \dfrac{\mathrm{d}\Omega E}{\mathrm{d}\tau} = \mathbf{V} \,, \qquad ||\mathbf{V}|| = -c^2, \\[2ex] \dfrac{1}{\mu_0 D_0} \dfrac{\mathrm{d}}{\mathrm{d}\tau} \left[\mu_0 D_0 \left(1 + \dfrac{\varepsilon}{c^2}\right)\right] = \dfrac{1}{c^2} \left(q_0 - \dfrac{1}{\mu_0} w_0^{(\mathrm{i})}\right) . \end{cases} \tag{8.35}$$

To these equations one must add the *boundary conditions* (8.28), the *initial conditions* and the *equation of conservation of the proper numerical density*:

$$\partial_\alpha \left(\frac{1}{D_0} V^\alpha\right) = 0 \,. \tag{8.36}$$

According to this point of view (i.e. choosing E, $\mathbf{V}$, μ_0 and D_0 as fundamental variables), in order to have the same number of equations as unknowns we must specify all the sources; that is, not only the *external fields* $\mathbf{k}$ and $\mathbf{g}$, with their thermomechanical content, but also the law of the 4-tensions $\mathbf{T}^\alpha$, i.e. $\mathbf{X}^\alpha$, $\mathbf{q}_0$ and ε. Obviously, $\mathbf{X}^\alpha$, $\mathbf{q}_0$ and ε will depend, a priori, on the same variables appearing in the law of $\mathbf{T}^\alpha$: E, $\mathbf{V}$, μ_0 and D_0: this is the so-called *equipresence principle* [7].

8.4 Relative Formulation

The system of absolute equations (8.35) and (8.36) plays a central role because of its invariance property with respect to the choice of the Cartesian coordinates x^α $(\alpha = 0, 1, 2, 3)$. However, the (three-dimensional) physical content of this system is not evident, because all the involved quantities are not directly measurable (unless the observer's frame would coincide with the local rest frame of the continuum). To see such a content one has to consider instead the corresponding formulation relative to an arbitrary Galilean frame. As in the case $\mathbf{q}_0 = 0$ previously examined the following three steps are then necessary:

1. select arbitrarily in $\mathcal{C}_3^+$ the temporal direction $\boldsymbol{\gamma}$ characterizing the frame and decompose, locally, all the various tensorial quantities along $\boldsymbol{\gamma}$ and the normal hyperplane Σ;
2. define the various relative quantities, either from the (formal) mathematical point of view or from a more physical point of view;
3. derive the *transformation laws* of all the involved relative quantities under an arbitrary change of the frame.

Let us start by examining (8.24); for the sake of brevity we assume that the Cartesian coordinates x^α are *adapted to the chosen frame*: $\mathbf{c}_0 = \boldsymbol{\gamma}$. First of all we note that, as for the ordinary case $\mathbf{q}_0 = 0$, the proper stresses $\mathbf{X}^\alpha$ are not all independent, because of (8.22); in fact, using the decomposition $\mathbf{V} = \eta(\mathbf{e} + c\boldsymbol{\gamma})$ with $\mathbf{e} = \mathbf{e}(t, x)$ the *Eulerian velocity*, we have

$$\mathbf{X}^0 = \frac{1}{c} e_i \mathbf{X}^i \, . \tag{8.37}$$

The three vectors $\mathbf{X}^i$ can be further decomposed as follows:

$$\mathbf{X}^i = \boldsymbol{\phi}^i - \mathbf{X}^i \cdot \boldsymbol{\gamma}\boldsymbol{\gamma} \, ,$$

with $\boldsymbol{\phi}^i \in \Sigma$. Hence, using $(8.20)_3$, i.e. $\mathbf{X}^i \cdot \mathbf{e} + c\mathbf{X}^i \cdot \boldsymbol{\gamma} = 0$, one gets

$$\mathbf{X}^i = \boldsymbol{\phi}^i + \frac{1}{c}\boldsymbol{\phi}^i \cdot \mathbf{e}\boldsymbol{\gamma} \, , \tag{8.38}$$

so that, combining (8.37) and (8.38), we have that the mechanical stresses $\mathbf{X}^\alpha$ $(\alpha = 0, 1, 2, 3)$ are well-determined functions of $\mathbf{e}$, $\boldsymbol{\gamma}$ and the three vectors $\boldsymbol{\phi}^i$:

$$\mathbf{X}^\alpha = \left(\delta_i^\alpha + \frac{1}{c} e_i \delta_0^\alpha\right)\left(\boldsymbol{\phi}^i + \frac{1}{c}\boldsymbol{\phi}^i \cdot \mathbf{e}\boldsymbol{\gamma}\right) \, . \tag{8.39}$$

We call $\boldsymbol{\phi}^i$ $(i = 1, 2, 3)$ the *coordinate mechanical stresses relative to* $\boldsymbol{\gamma}$ (so that $\boldsymbol{\phi}_n = n_i \boldsymbol{\phi}^i$ is the stress relative to $\mathbf{n} \in \Sigma$). They belong to Σ and can also be written as

$$\boldsymbol{\phi}^i = X^{ik}\mathbf{c}_k \, . \tag{8.40}$$

The tensor X^{ik}, which is *spatial* (being the complete spatial projection of $X^{\alpha\beta}$) *and symmetric* in the considered frame, gives rise to the *Eulerian tension characteristics*. Similarly, the proper thermal conduction vector $\mathbf{q}_0$ can be decomposed in the form $\mathbf{q}_0 = \mathbf{q} - \mathbf{q}_0 \cdot \boldsymbol{\gamma}\boldsymbol{\gamma}$, so that $(8.20)_2$ implies

$$\mathbf{q}_0 = \mathbf{q} + \frac{1}{c}\mathbf{q} \cdot \mathbf{e}\boldsymbol{\gamma} \, . \tag{8.41}$$

The spatial vector $\mathbf{q} \in \Sigma$ is called *relative thermal conduction vector*; it vanishes if and only if $\mathbf{q}_0 = 0$.

Let us now project (8.24) on Σ and along $\boldsymbol{\gamma}$ starting from the term $\partial_\alpha(\rho_0 V^\alpha \mathbf{V})$. Using the decomposition of $\mathbf{V}$: $\mathbf{V} = \eta(\mathbf{e} + c\boldsymbol{\gamma})$, and defining as in the ordinary case $(\mathbf{q}_0 = 0)$

$$\rho \overset{\text{def}}{=} \eta^2 \rho_0 \ , \tag{8.42}$$

we have the following decomposition:

$$\partial_\alpha(\rho_0 V^\alpha \mathbf{V}) = \frac{1}{D}[(\rho D\mathbf{e})^{\cdot} + (\rho D)^{\cdot} c\boldsymbol{\gamma}] \ ,$$

where $(\,)^{\cdot}$ denotes the substantial derivative:

$$(\,)^{\cdot} = \partial_t(\,) + e^i(t,x)\partial_{x^i}(\,) \ , \tag{8.43}$$

and the Eulerian kinematical identity

$$\frac{\dot{D}}{D} = \text{div}\ \mathbf{e} \ , \tag{8.44}$$

has been used.

Similarly, from (8.39) we have

$$\partial_\alpha \mathbf{X}^\alpha = \partial_i \boldsymbol{\phi}^i + \frac{1}{c}\partial_i(\boldsymbol{\phi}^i \cdot \mathbf{e})\boldsymbol{\gamma} + \frac{1}{c^2}\partial_t(e_i \boldsymbol{\phi}^i) + \frac{1}{c^3}\partial_t(e_i \boldsymbol{\phi}^i \cdot \mathbf{e})\boldsymbol{\gamma} \ .$$

Using then (8.43) and (8.44) leads to

$$\partial_\alpha(V^\alpha \mathbf{q}_0) = (\eta \mathbf{q}_0)^{\cdot} + \eta \mathbf{q}_0 \partial_i e^i \equiv \frac{1}{D}(\eta D \mathbf{q}_0)^{\cdot} \ ;$$

(8.41) implies

$$\partial_\alpha(V^\alpha \mathbf{q}_0) = \frac{1}{D}(\eta D \mathbf{q})^{\cdot} + \frac{1}{cD}(\eta D \mathbf{q} \cdot \mathbf{e})^{\cdot}\, \boldsymbol{\gamma} \ .$$

Finally we have

$$\partial_\alpha(q_0^\alpha \mathbf{V}) = \partial_i(\eta q^i \mathbf{e}) + \frac{1}{c^2}\partial_t(\eta \mathbf{q} \cdot \mathbf{ee}) + c\partial_i(\eta q^i)\boldsymbol{\gamma} + \frac{1}{c}\partial_t(\eta \mathbf{q} \cdot \mathbf{e})\boldsymbol{\gamma} \ .$$

Next, after introducing the *relative mass force* $\mu \mathbf{F}$

$$\mu \mathbf{F} \overset{\text{def}}{=} \mu_0 \mathbf{k}_\Sigma \equiv \mu_0(\mathbf{k} + \mathbf{k} \cdot \boldsymbol{\gamma}\boldsymbol{\gamma}) \ , \tag{8.45}$$

(8.24) gives the *momentum theorem*

$$\frac{1}{D}(\hat{\mu} D \mathbf{e})^{\cdot} = \hat{\boldsymbol{\mathcal{F}}} \overset{\text{def}}{=} \boldsymbol{\mathcal{F}} + \boldsymbol{\mathcal{F}}_c \ , \tag{8.46}$$

where the *total force density* $\hat{\boldsymbol{\mathcal{F}}}$ includes, besides the ordinary term $\boldsymbol{\mathcal{F}}$ (see (7.78)), the contribution due to the thermal flux $\boldsymbol{\mathcal{F}}_c$:

$$\begin{cases} \boldsymbol{\mathcal{F}} \overset{\text{def}}{=} \mu \mathbf{F} - \partial_i \boldsymbol{\phi}^i - \dfrac{1}{c^2}\partial_t(e_i \boldsymbol{\phi}^i), \\[2ex] \boldsymbol{\mathcal{F}}_c \overset{\text{def}}{=} -\dfrac{1}{c^2}\left[\dfrac{1}{D}(\eta D \mathbf{q})^{\cdot} + \partial_i(\eta q^i \mathbf{e}) + \dfrac{1}{c^2}\partial_t(\eta \mathbf{q} \cdot \mathbf{ee})\right] . \end{cases} \tag{8.47}$$

The *energy theorem* is then given by

$$\frac{1}{D}(\hat{\mu}Dc^2)^{\cdot} = -\mu_0 c\mathbf{k}\cdot\boldsymbol{\gamma} - \partial_i(\boldsymbol{\phi}^i\cdot\mathbf{e}) - \frac{1}{c^2}\left[\frac{1}{D}(\eta D\mathbf{q}\cdot\mathbf{e})^{\cdot} + \partial_t(e_i\boldsymbol{\phi}^i\cdot\mathbf{e}) + \partial_t(\eta\mathbf{q}\cdot\mathbf{e})\right] - \partial_i(\eta q^i)\ .$$

In contrast to (8.46), the above equation is not in a physically meaningful form yet; this form can be obtained by using the relation $-c\boldsymbol{\gamma} = \mathbf{e} - \mathbf{V}/\eta$ and expanding the derivatives on the right-hand side. In fact, using the identity

$$e^2 = c^2\left(1 - \frac{1}{\eta^2}\right)\ , \tag{8.48}$$

and introducing the following relative quantities per unit volume:

$$\begin{cases} \mu\hat{\epsilon} \stackrel{\text{def}}{=} \hat{\mu}c^2 & \text{internal energy,} \\ \mu r \stackrel{\text{def}}{=} -\dfrac{1}{\eta}\mu_0\mathbf{k}\cdot\mathbf{V} & \text{radiation power,} \\ \mu q_c \stackrel{\text{def}}{=} -\dfrac{1}{\eta}\partial_i q^i - \dfrac{1}{c^2}\left[\dfrac{1}{\eta}\partial_t(\mathbf{q}\cdot\mathbf{e}) + \eta\mathbf{q}\cdot\dot{\mathbf{e}}\right] & \text{conduction power,} \\ w^{(i)} \stackrel{\text{def}}{=} \boldsymbol{\phi}^i\cdot\left(\partial_i\mathbf{e} + \dfrac{1}{c^2}e_i\partial_t\mathbf{e}\right) & \text{internal forces power ,} \end{cases} \tag{8.49}$$

we can write the energy theorem in its more familiar form:

$$\frac{1}{\mu D}(\mu D\hat{\epsilon})^{\cdot} = \hat{\boldsymbol{\mathcal{F}}}\cdot\mathbf{e} + \hat{Q}\ , \tag{8.50}$$

where $\hat{Q}$ is the *total thermal power*:

$$\hat{Q} \stackrel{\text{def}}{=} q - \frac{1}{\mu}w^{(\mathrm{i})}\ , \qquad q \stackrel{\text{def}}{=} r + q_c\ . \tag{8.51}$$

8.5 Transformation Laws of the Fundamental Relative Quantities

The general equations (8.46) and (8.50) satisfy the relativity principle, since they are *formally invariant* with respect to the choice of the Galilean frame associated with the vector $\boldsymbol{\gamma}$. However, they are *not physically invariant*, because of the relative meaning of the various quantities involved.

All the quantities introduced above, and in particular $\hat{\epsilon}$, r, q_c and $w^{(\mathrm{i})}$, have a real physical meaning. If we denote by an index 0 the proper quantities, the following *properties of invariance* hold:

$$\begin{cases} \hat{\epsilon} = \dfrac{\hat{\mu}_0 c^2}{\mu_0}, \\ \eta^3 r = -\mathbf{k} \cdot \mathbf{V} = \text{inv.} = r_0, \\ \eta^3 q_c = -\dfrac{1}{\mu_0}\left(\partial_\alpha q_0^\alpha + \dfrac{1}{c^2}\mathbf{q}_0 \cdot \mathbf{A}\right) = \text{inv.} = q_{c,0}, \\ \eta w^{(\mathrm{i})} = \mathbf{X}^\alpha \cdot \partial_\alpha \mathbf{V} = w_0^{(\mathrm{i})}, \end{cases} \tag{8.52}$$

which complete (7.82), by the inclusion of the thermal flux. Hence, as in the ordinary case, the transformation laws of the quantities r, q_c and $w^{(\mathrm{i})}$, passing from one Galilean frame S_g to another S'_g, easily follow:

$$r' = \left(\frac{\alpha}{\sigma}\right)^3 r\,, \quad q'_c = \left(\frac{\alpha}{\sigma}\right)^3 q_c\,, \quad w'^{(\mathrm{i})} = \left(\frac{\alpha}{\sigma}\right) w^{(\mathrm{i})}\,. \tag{8.53}$$

Analogously, for the mass forces and the coordinate stresses (see (8.38) and (8.45)) the transformation laws are exactly those of the ordinary case ($\mathbf{q}_0 = 0$), that is, (7.85) and (7.99):

$$\begin{cases} \mathbf{F}' = \dfrac{\alpha}{\sigma^2}\left[\alpha \mathbf{F} - \dfrac{1}{c^2}(\mathbf{F} \cdot \mathbf{w} + r)\mathbf{u}\right], \\ \boldsymbol{\phi}'^i = \left(\delta_k^i - \dfrac{1}{c^2\alpha} u^i w_k\right)\left(\boldsymbol{\phi}^k - \dfrac{1}{c^2\alpha}\boldsymbol{\phi}^k \cdot \mathbf{w}\mathbf{u}\right), \quad (i = 1, 2, 3), \end{cases} \tag{8.54}$$

where the dependence on the Eulerian velocity $\mathbf{e}$ is either through the scalar $\sigma = 1 - \mathbf{e} \cdot \mathbf{u}/c^2$ or through the vector $\mathbf{w}$:

$$\mathbf{w} = \mathbf{e} - \frac{\mathbf{u}}{1 + \alpha}\,. \tag{8.55}$$

Finally, for the thermal conduction vector $\mathbf{q}$, from (8.42) we have

$$\mathbf{q}' = \mathbf{q} - \frac{1}{c^2\alpha}\mathbf{q} \cdot \mathbf{w}\mathbf{u}\,. \tag{8.56}$$

Using then the relativistic theorem of addition of velocities

$$\mathbf{e}' = \frac{1}{\sigma}\left(\alpha \mathbf{e} - \frac{\alpha + \sigma}{1 + \alpha}\mathbf{u}\right), \tag{8.57}$$

we also have the following transformation law for the vector $\mathbf{w}$:

$$\mathbf{w}' = \frac{\alpha}{\sigma}\mathbf{w}\,. \tag{8.58}$$

Equation $(8.54)_1$ shows that, from a relativistic point of view, the cases $\mathbf{q} = 0$ (absence of thermal conduction) and $r = 0$ (absence of thermal radiation) are allowed. On the other hand, a pure heat theory (in the absence of a mechanical interaction) is meaningless; in fact, $\mathbf{F} = 0$ in S_{g} does not imply $\mathbf{F}' = 0$ in S'_{g}, but rather

$$\mathbf{F}' = -\frac{\alpha}{\sigma^2} r\mathbf{u} \neq 0 \, .$$

In the classical case ($c \to \infty$) the situation is obviously different, all the quantities introduced above having an invariant meaning with respect to the choice of the Galilean frame. Furthermore, most of the relations decouple and give rise to the ordinary theories.

8.6 Classical Form of the Relativistic Cauchy Equation

The relativistic equation (8.46) contains the various nonclassical terms, which appear on both sides and generate a thermodynamical coupling, absent in the ordinary theory of continuous systems. However, if we use (8.44), (8.46) then assumes the typical *conservative form*:

$$\partial_t(\hat{\mu}\mathbf{e}) + \partial_i(\hat{\mu}\mathbf{e}e^i) = \hat{\boldsymbol{\mathcal{F}}} \, , \tag{8.59}$$

similar to the absolute equation from which it is derived. To this equation, in general, one should not couple the mass conservation equation, differently from the classical case. Taking into account the meaning (8.42) of $\hat{\mu}$, it is convenient to separate the pure matter term ($\mu = \eta^2\mu_0$) from that related to the thermal conduction ($\mu_c = \eta^2\mu_{c,0}$) by introducing the thermal energy density ε, already used in (8.33):

$$\mu\varepsilon = \mu_c c^2 \, ; \tag{8.60}$$

(8.42) then assumes the form

$$\hat{\mu} = \mu\left(1 + \frac{\varepsilon}{c^2}\right) \, , \tag{8.61}$$

where ε is a scalar invariant. The decomposition (8.61) can also be used in the motion equation (8.59), showing the thermodynamical coupling through the thermal energy ε, which represents a *strictly relativistic result*. Clearly, when $\mathbf{q} \neq 0$ the vector $\boldsymbol{\mathcal{F}}_c$, given by $(8.47)_2$, is a coupling term too.

A first alternative form to (8.59) can be obtained by taking into account $(8.47)_1$, which gives the equivalent expression:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\hat{\mu}\mathbf{e} + \frac{1}{c^2}e_k\boldsymbol{\phi}^k\right) + \frac{\partial}{\partial x^i}\left[\left(\hat{\mu}\mathbf{e} + \frac{1}{c^2}e_k\boldsymbol{\phi}^k\right)e^i\right] \\ &= \mu\mathbf{F} - \frac{\partial}{\partial x^i}\left(\boldsymbol{\phi}^i - \frac{1}{c^2}e^i e_k\boldsymbol{\phi}^k\right) + \boldsymbol{\mathcal{F}}_c \, . \end{aligned}$$

The latter equation suggests the introduction of the coordinate *dynamical stresses*:

$$\hat{\boldsymbol{\phi}}^i \stackrel{\text{def}}{=} \boldsymbol{\phi}^i - \frac{1}{c^2} e^i e_k \boldsymbol{\phi}^k , \tag{8.62}$$

which, both in the classical approximation ($c \to \infty$) and in the proper frame $\mathbf{e} = 0$, coincide with the ordinary stresses $\boldsymbol{\phi}^i$. The vectors $\hat{\boldsymbol{\phi}}^i$ are in 1–1 correspondence with the $\boldsymbol{\phi}^i$; in fact, from (8.62) and using (8.48) one has

$$e_i \hat{\boldsymbol{\phi}}^i = \frac{1}{\eta^2} e_k \boldsymbol{\phi}^k , \tag{8.63}$$

so that

$$\boldsymbol{\phi}^i = \hat{\boldsymbol{\phi}}^i + \frac{\eta^2}{c^2} e^i e_k \hat{\boldsymbol{\phi}}^k . \tag{8.64}$$

Therefore, the previous equation, at least in the case $\mathbf{q} = 0$, expresses the *classical form*

$$\frac{\partial \boldsymbol{\mathcal{P}}}{\partial t} + \frac{\partial}{\partial x^i}(\boldsymbol{\mathcal{P}} e^i) = \mu \mathbf{F} - \frac{\partial \hat{\boldsymbol{\phi}}^i}{\partial x^i} , \tag{8.65}$$

where $\boldsymbol{\mathcal{P}}$ represents the *total linear momentum* per unit volume:

$$\boldsymbol{\mathcal{P}} \stackrel{\text{def}}{=} \mu \left(1 + \frac{\varepsilon}{c^2}\right) \mathbf{e} + \frac{\eta^2}{c^2} e_i \hat{\boldsymbol{\phi}}^i . \tag{8.66}$$

The conservative form (8.65), typical of the classical situation, is still valid in the relativistic case when $\mathbf{q} = 0$, with a larger meaning due to both the mechanical sources (dynamical stresses $\hat{\boldsymbol{\phi}}^i$ in place of ordinary stresses $\boldsymbol{\phi}^i$) and the newly defined linear momentum, with the addition of the two terms related to the internal energy ε and to the dynamical stresses $\hat{\boldsymbol{\phi}}^i$. Moreover, the conservative form (8.65) remains valid also in the presence of thermal flux $\mathbf{q}$. In fact, by introducing the dynamical thermal conduction $\hat{\mathbf{q}}$, through the same law (8.62):

$$\hat{\mathbf{q}} \stackrel{\text{def}}{=} \mathbf{q} - \frac{1}{c^2} \mathbf{q} \cdot \mathbf{e}\,\mathbf{e} \qquad \sim \qquad \mathbf{q} = \hat{\mathbf{q}} - \frac{\eta^2}{c^2} \hat{\mathbf{q}} \cdot \mathbf{e}\,\mathbf{e} , \tag{8.67}$$

the additional mechanical force $\boldsymbol{\mathcal{F}}_c$ given by $(8.47)_2$ assumes the following form:

$$\boldsymbol{\mathcal{F}}_c = -\left[\frac{\partial \boldsymbol{\mathcal{P}}_c}{\partial t} + \frac{\partial (\boldsymbol{\mathcal{P}}_c e^i)}{\partial x^i} + \frac{\partial \hat{\boldsymbol{\phi}}_c^i}{\partial x^i}\right] ,$$

where

$$\boldsymbol{\mathcal{P}}_c \stackrel{\text{def}}{=} \frac{1}{c^2}\left(\eta \hat{\mathbf{q}} + \frac{2\eta^3}{c^2} \hat{\mathbf{q}} \cdot \mathbf{e}\mathbf{e}\right) , \qquad \hat{\boldsymbol{\phi}}_c^i \stackrel{\text{def}}{=} \frac{\eta}{c^2} \hat{q}^i \mathbf{e} . \tag{8.68}$$

Thus, in the general nonpolar case with $\mathbf{q} \neq 0$, the relativistic Cauchy equation retains its classical form (8.65):

$$\frac{\partial \hat{\boldsymbol{\mathcal{P}}}}{\partial t} + \frac{\partial}{\partial x^i}(\hat{\boldsymbol{\mathcal{P}}} e^i) = \mu \mathbf{F} - \frac{\partial \widehat{\boldsymbol{\phi}^i}}{\partial x^i} , \tag{8.69}$$

where the total linear momentum $\hat{\boldsymbol{\mathcal{P}}}$ and the total stresses $\widehat{\boldsymbol{\phi}^i}$ also include, besides the ordinary contributions $\boldsymbol{\mathcal{P}}$ and $\hat{\boldsymbol{\phi}}^i$, the thermal terms $\boldsymbol{\mathcal{P}}_c$ and $\hat{\boldsymbol{\phi}}^i{}_c$:

$$\begin{cases} \hat{\boldsymbol{\mathcal{P}}} \stackrel{\text{def}}{=} \boldsymbol{\mathcal{P}} + \boldsymbol{\mathcal{P}}_c \equiv \mu \mathbf{e} + \dfrac{1}{c^2}\left(\mu\varepsilon\mathbf{e} + \eta^2 e_k \hat{\boldsymbol{\phi}}^k + \eta\hat{\mathbf{q}} + \dfrac{2\eta^3}{c^2}\hat{\mathbf{q}}\cdot\mathbf{ee}\right), \\ \widehat{\boldsymbol{\phi}^i} \stackrel{\text{def}}{=} \hat{\boldsymbol{\phi}}^i + \hat{\boldsymbol{\phi}}^i_c \equiv \left(\delta^i_k - \dfrac{1}{c^2}e^i e_k\right)\left(\boldsymbol{\phi}^k + \dfrac{\eta}{c^2}q^k\mathbf{e}\right). \end{cases} \tag{8.70}$$

In particular, neglecting terms of higher order in $1/c^2$, (8.70) give the approximate expressions:

$$\begin{cases} \hat{\boldsymbol{\mathcal{P}}} \simeq \mu\mathbf{e} + \dfrac{1}{c^2}\left(\mu\varepsilon\mathbf{e} + e_k\hat{\boldsymbol{\phi}}^k + \mathbf{q}\right), \\ \widehat{\boldsymbol{\phi}^i} \simeq \boldsymbol{\phi}^i - \dfrac{1}{c^2}(e^i e_k \boldsymbol{\phi}^k - q^i\mathbf{e}). \end{cases} \tag{8.71}$$

8.7 Transformation Laws of the Dynamical Stresses

Let us consider the relation (8.62) between the ordinary stresses $\boldsymbol{\phi}^i$ and the dynamical ones $\hat{\boldsymbol{\phi}}^i$ rewritten in the form

$$\hat{\boldsymbol{\phi}}^i \stackrel{\text{def}}{=} \left(\delta^i_k - \frac{1}{c^2}e^i e_k\right)\boldsymbol{\phi}^k \equiv \hat{g}^i{}_k\boldsymbol{\phi}^k , \tag{8.72}$$

through the introduction of the *Eulerian tensor*

$$\hat{g}^{ik} \stackrel{\text{def}}{=} \left(\delta^{ik} - \frac{1}{c^2}e^i e^k\right) , \tag{8.73}$$

i.e. a (regular) spatial tensor such that

$$\det||\hat{g}^{ik}|| = \frac{1}{g\eta^2} > 0 , \tag{8.74}$$

with reciprocal tensor $\hat{g}_{ik}$ (denoted by the same symbol):

$$\hat{g}_{ik} = \delta_{ik} + \frac{\eta^2}{c^2}e_i e_k , \qquad \hat{g}^{ik}\hat{g}_{kj} = \delta^i_j . \tag{8.75}$$

The two tensors $\hat{g}^{ik}$ and $\hat{g}_{kj}$ depend on the choice of the Galilean frame S_{g}, and hence have a relative meaning. In the proper frame ($\mathbf{e} = 0$) they coincide with the spatial metric, a fact that will allow us to better specify their geometrical meaning. It is easy to see that, passing from S_{g} to S'_{g}, the following transformation laws hold:

$$\hat{g}'^{ik} = A^i{}_j A^k{}_h \hat{g}^{jh}; \qquad \hat{g}'_{ik} = B^j{}_i B^h{}_k \hat{g}_{jh} , \tag{8.76}$$

where the matrices

$$A^i{}_k \stackrel{\text{def}}{=} \delta^i_k + \frac{1}{c^2\sigma} w^i u_k \ , \qquad B^i{}_k \stackrel{\text{def}}{=} \delta^i_k - \frac{1}{c^2\alpha} w^i u_k \tag{8.77}$$

satisfy the relation $A^i{}_k B^k{}_j = \delta^i_j$ (see Chap. 5). We also note that, because of (8.58) and the reciprocity theorem of the velocities

$$\mathbf{u}' = -\mathbf{u} \ , \tag{8.78}$$

the tensors A and B given by (8.77) *are not only inverse to each other, but can also be transformed from one to the other*:

$$B'^i{}_k = A^i{}_k \ . \tag{8.79}$$

Let us now consider the transformation law of the ordinary stresses $(8.54)_2$:

$$\boldsymbol{\phi}'^i = B_k{}^i \left(\boldsymbol{\phi}^k - \frac{1}{c^2\alpha} \boldsymbol{\phi}^k \cdot \mathbf{wu} \right) \ ; \tag{8.80}$$

using (8.76) leads to

$$\begin{aligned} \hat{\boldsymbol{\phi}}'^i \equiv \hat{g}'^i{}_k \hat{\boldsymbol{\phi}}'^k &= A^i{}_j A_k{}^h \hat{g}^j{}_h B_l{}^k \left(\boldsymbol{\phi}^l - \frac{1}{c^2\alpha} \boldsymbol{\phi}^l \cdot \mathbf{wu} \right) \\ &= A^i{}_j \delta^h_l \hat{g}^j{}_h \left(\boldsymbol{\phi}^l - \frac{1}{c^2\alpha} \boldsymbol{\phi}^l \cdot \mathbf{wu} \right) \ , \end{aligned}$$

so that

$$\hat{\boldsymbol{\phi}}'^i = A^i{}_j \left(\hat{\boldsymbol{\phi}}^j - \frac{1}{c^2\alpha} \hat{\boldsymbol{\phi}}^j \cdot \mathbf{wu} \right) \ . \tag{8.81}$$

If we decompose the vectors $\hat{\boldsymbol{\phi}}^i$ along the Cartesian basis, that is

$$\hat{\boldsymbol{\phi}}^i = \tilde{X}^{ik} \mathbf{c}_k \ , \tag{8.82}$$

it is easy to find the relation with the nonsymmetric tension characteristics:

$$\tilde{X}^{ik} = \hat{g}^i{}_h X^{hk} \equiv X^{ik} - \frac{1}{c^2} e^i e_h X^{hk} \ . \tag{8.83}$$

Moreover, (8.81) implies:

$$\tilde{X}'^{ik} = A^i{}_j \tilde{X}^{hk} B_h{}^k \ , \tag{8.84}$$

that is, *the two (different) nonsymmetric tensors* $\tilde{X}'^{ik}$ and $\tilde{X}^{ik}$ *have* in $S_{\rm g}$ *the same principal invariants.* The same property is no longer true for the ordinary characteristics X^{ik}, given by (8.40); in fact, the latter are symmetric and transform as follows (see $(8.54)_2$):

$$X'^{ik} = B^i{}_j X^{jh} B_h{}^k \ . \tag{8.85}$$

8.8 Nonviscous Fluids and Dynamical Pressure

The introduction of the dynamical stresses (8.62) in a given Galilean frame S_{g} implies a relativistic consistence for the hypothesis of a pure pressure:

$$\hat{\boldsymbol{\phi}}^i = p\mathbf{c}^i \qquad \sim \qquad \tilde{X}^{ik} = p\delta^{ik} \; ; \tag{8.86}$$

in fact, using (8.81), we find $\hat{\boldsymbol{\phi}}'^i = p'\mathbf{c}'^i$, $\forall S'_{\mathrm{g}}$, and hence such hypothesis is invariant with respect to the choice of the Galilean frame. The scalar p has the meaning of proper pressure of the fluid, and this is a different way to define nonviscous fluids. The hypothesis (8.86), using (8.64) and (8.75), is indeed equivalent to the condition that the ordinary stresses $\boldsymbol{\phi}^i$ have the form

$$\boldsymbol{\phi}^i = p\hat{g}^{ik}\mathbf{c}_k \qquad \sim \qquad X^{ik} = p\hat{g}^{ik} \; , \tag{8.87}$$

implying that in the proper frame there is no proper viscosity:

$$\boldsymbol{\phi}_0^i = p_0\delta^{ik}\mathbf{c}_k \; . \tag{8.88}$$

In fact, consider $(8.54)_2$ and specialize S_{g} to be the proper frame S_0. Thus $\mathbf{e} = 0$ and, from (8.57), for a fluid element at rest in S_0:

$$\mathbf{e}' = -\mathbf{u} \; . \tag{8.89}$$

After multiplying by n_i and using the reciprocity axiom, $(8.54)_2$ assumes the form:

$$\boldsymbol{\phi}'_n = \boldsymbol{\phi}_n^0 + \frac{1}{c^2}\frac{1}{\alpha(1+\alpha)}(\mathbf{u}\cdot\mathbf{n}\boldsymbol{\phi}_u^0 + \boldsymbol{\phi}_u^0\cdot\mathbf{n}\mathbf{u}) + \frac{\mathbf{u}\cdot\mathbf{n}}{c^4\alpha^2(1+\alpha)^2}\boldsymbol{\phi}_u^0\cdot\mathbf{u}\mathbf{u} \; .$$

Replacing $\mathbf{u}$ through (8.89) and omitting the prime for the sake of brevity lead to the following dependence between ordinary proper stresses and those relative to an arbitrary Galilean frame S_{g}:

$$\boldsymbol{\phi}_n = \boldsymbol{\phi}_n^0 + \Delta\boldsymbol{\phi}_n \quad , \tag{8.90}$$

where

$$\Delta\boldsymbol{\phi}_n = \frac{1}{c^2}\frac{\eta^2}{1+\eta}\left[\boldsymbol{\phi}_e^0\cdot\mathbf{n}\mathbf{e} + \mathbf{e}\cdot\mathbf{n}\left(\boldsymbol{\phi}_e^0 + \frac{\eta^2}{c^2(1+\eta)}\boldsymbol{\phi}_e^0\cdot\mathbf{e}\mathbf{e}\right)\right] \; . \tag{8.91}$$

Equation (8.90) shows that in a relativistic context *it is necessary to distinguish between static and dynamical stresses*; that is, in any Galilean frame S_{g} the stresses depend on the dynamical state of the fluid element (apart from other conditions) $\boldsymbol{\phi}_n = \boldsymbol{\phi}_n(\mathbf{e}, \boldsymbol{\phi}_n^0)$, different from what happens classically.

We notice that for any direction $\mathbf{n}$ orthogonal to $\mathbf{e}$ the relativistic correction to the ordinary stresses $\Delta\boldsymbol{\phi}_n$ is always parallel to the velocity $\mathbf{e}$ and *it is a linear and homogeneous function of* $\boldsymbol{\phi}_e^0 = e_i\boldsymbol{\phi}^i$.

Let us assume the continuum to be a nonviscous fluid, that is, from (8.87):

$$\boldsymbol{\phi}_n^0 = p_0 \mathbf{n} \,, \qquad \forall \mathbf{n} \in \Sigma_0 \,. \tag{8.92}$$

From (8.90) we then have

$$\boldsymbol{\phi}_n = p_0 \left[\mathbf{n} + \frac{\eta^2}{c^2(1+\eta)} \mathbf{e} \cdot \mathbf{n} \left(2 + \frac{\eta^2}{1+\eta} \frac{e^2}{c^2} \right) \mathbf{e} \right] = p_0 \left(\mathbf{n} + \frac{\eta^2}{c^2} \mathbf{e} \cdot \mathbf{n} \mathbf{e} \right) \,. \tag{8.93}$$

Equation (8.92) *implies the following form of coordinate stresses in any Galilean frame* S_g:

$$\boldsymbol{\phi}^i = p_0 \left(\mathbf{c}^i + \frac{\eta^2}{c^2} e^i \mathbf{e} \right) = p_0 \hat{g}^{ik} \mathbf{c}_k \,, \tag{8.94}$$

which is not compatible with a pure pressure; such a compatibility concerns instead the dynamical stresses $\hat{\boldsymbol{\phi}}^i = p_0 \mathbf{c}^i$ only, confirming the equivalence between (8.86) and (8.92).

8.9 Lagrangian Form of the Relativistic Cauchy Equation

As we have seen above, the relativistic Cauchy equation (8.69) has been derived using Eulerian coordinates. We will proceed now to transform it in its Lagrangian form [4]. To this end, let us assume for the continuum a generic set of *Lagrangian coordinates* y^i $(i = 1, 2, 3)$, i.e. curvilinear coordinates for the points in the actual configuration C. Let $\{\mathbf{e}_i\}$ denote the natural basis associated with the coordinates y^i in $C \in S_\mathrm{g}$ and $g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k$ the Lagrangian metric with associated Christoffel symbols of the second-type ${\Gamma^h}_{ik}$ and the covariant derivative ∇_i.

The partial derivatives (spatial and temporal) of the basis vectors $\{\mathbf{e}_i\}$ (and similarly for the cobasis $\{\mathbf{e}^i\}$) give the following geometrical–kinematical relations:

$$\partial_i \mathbf{e}_k = {\Gamma^h}_{ik} \mathbf{e}_h \,, \quad \partial_t \mathbf{e}_i = \partial_i \mathbf{v} \,, \quad \left(\partial_i = \frac{\partial}{\partial y^i} \,, \quad \partial_t = \frac{\partial}{\partial t} \right) \,, \tag{8.95}$$

where $\mathbf{v} = v^i \mathbf{e}_i$ denotes the Lagrangian velocity. The velocity gradient $\nabla_i v_k$ summarizes, in turn, the two fundamental tensors: ω_{ik} (*angular velocity*) and k_{ik} (*deformation velocity*):

$$\nabla_i v_k = \omega_{ik} + k_{ik} \,, \qquad v_k = g_{ki} v^i \,. \tag{8.96}$$

The *symmetric tensor* k_{ik}:

$$k_{ik} \stackrel{\mathrm{def}}{=} \frac{1}{2} (\nabla_i v_k + \nabla_k v_i) \,, \tag{8.97}$$

can also be written as

$$k_{ik} = \frac{1}{2}\partial_t g_{ik} \; ; \tag{8.98}$$

the *antisymmetric tensor* ω_{ik} is instead equivalent to the vector

$$\boldsymbol{\omega} = \frac{1}{2}\mathbf{e}^i \times \partial_t \mathbf{e}_i = \frac{1}{2}\mathrm{curl}\,\mathbf{v} \; , \tag{8.99}$$

where $\mathbf{e}^i$ is the dual basis of $\mathbf{e}_i$: $\mathbf{e}^i \cdot \mathbf{e}_k = \delta^i_k$ and

$$\boldsymbol{\omega} = \frac{1}{2}\omega_{ik}\mathbf{e}^i \times \mathbf{e}^k \; , \quad \omega_{ik} = \boldsymbol{\omega} \cdot \mathbf{e}_i \times \mathbf{e}_k \; . \tag{8.100}$$

The Lagrangian form of (8.69) requires the introduction of the Lagrangian coordinate *static stresses* $\mathbf{Y}^i$ or the *dynamical ones* $\hat{\mathbf{Y}}^i$:

$$\mathbf{Y}^i = \frac{\partial y^i}{\partial x^k}\boldsymbol{\phi}^k \; , \qquad \hat{\mathbf{Y}}^i = \frac{\partial y^i}{\partial x^k}\hat{\boldsymbol{\phi}}^k \; . \tag{8.101}$$

Moreover, we have the following: (1) the left-hand side of (8.69) can be expressed using the substantial derivative (8.43), which in Lagrangian terms is exactly the temporal derivative ∂_t; (2) for the vectors $\boldsymbol{\phi}^i$, in Lagrangian terms, one still has the ordinary expression for the divergence:

$$\frac{\partial \boldsymbol{\phi}^i}{\partial x^i} = \frac{1}{\mathcal{D}}\partial_i(\mathcal{D}\mathbf{Y}^i) \; , \quad \mathcal{D} = \sqrt{g} \; . \tag{8.102}$$

Using $(8.101)_1$, we find

$$\begin{aligned} \frac{\partial \boldsymbol{\phi}^i}{\partial x^i} &= \frac{\partial}{\partial x^i}\left(\frac{\partial x^i}{\partial y^k}\mathbf{Y}^k\right) \\ &= \frac{\partial}{\partial y^h}\left(\frac{\partial x^i}{\partial y^k}\mathbf{Y}^k\right)\frac{\partial y^h}{\partial x^i} = \partial_h \mathbf{Y}^k \frac{\partial x^i}{\partial y^k}\frac{\partial y^h}{\partial x^i} + \mathbf{Y}^k \frac{\partial^2 x^i}{\partial y^h \partial y^k}\frac{\partial y^h}{\partial x^i} \; . \end{aligned}$$

Next, using the relations

$$\frac{\partial y^h}{\partial x^i}\frac{\partial x^i}{\partial y^k} = \delta^h_k, \quad \frac{\partial^2 x^i}{\partial y^h \partial y^k}\frac{\partial y^h}{\partial x^i} = \Gamma^l{}_{hk}, \quad \Gamma^h{}_{hk} = \frac{1}{\mathcal{D}}\partial_k \mathcal{D} \tag{8.103}$$

leads to

$$\frac{\partial \boldsymbol{\phi}^i}{\partial x^i} = \partial_k \mathbf{Y}^k + \Gamma^h{}_{hk}\mathbf{Y}^k \equiv \frac{1}{\mathcal{D}}\partial_k(\mathcal{D}\mathbf{Y}^k) \; .$$

Thus, (8.69) cast in Lagrangian form becomes

$$\frac{1}{\mathcal{D}}\partial_t(\mathcal{D}\hat{\boldsymbol{\mathcal{P}}}) = \mu\mathbf{F} - \frac{1}{\mathcal{D}}\partial_i(\mathcal{D}\widehat{\mathbf{Y}}^i) \; , \tag{8.104}$$

where we have the *linear momentum* and the *generalized stresses* given by (8.62) and (8.70):

$$\begin{cases} \hat{\boldsymbol{\mathcal{P}}} \stackrel{\text{def}}{=} \mu\mathbf{v} + \dfrac{1}{c^2}\left(\mu\varepsilon\mathbf{v} + \eta^2 v_i \hat{\mathbf{Y}}^i + \eta\hat{\mathbf{q}} + \dfrac{2\eta^3}{c^2}\hat{\mathbf{q}}\cdot\mathbf{v}\mathbf{v}\right), \\ \widehat{\mathbf{Y}}^i \stackrel{\text{def}}{=} \hat{\mathbf{Y}}^i + \dfrac{\eta}{c^2}\hat{q}^i\mathbf{v}, \end{cases} \tag{8.105}$$

where

$$\hat{\mathbf{Y}}^i = \hat{g}^i{}_k\mathbf{Y}^k \ , \quad \hat{q}^i = \hat{g}^i{}_k q^k \ , \quad \hat{g}^i{}_k = \delta^i_k - \frac{1}{c^2}v^i v_k \ . \tag{8.106}$$

As in the classical case, (8.104) can be written in scalar terms in a number of ways, according to the stress characteristics and the bases used. For instance, assuming

$$\hat{\mathbf{Y}}^i = \tilde{Y}^{ik}\mathbf{e}_k \ , \quad \mathbf{Y}^i = Y^{ik}\mathbf{e}_k \ , \tag{8.107}$$

it is possible to introduce the *dynamical tension* characteristics $\tilde{Y}^{ik}$:

$$\tilde{Y}^{ik} = \hat{g}^i{}_j Y^{jk} \ ; \tag{8.108}$$

these are not symmetric, different from the corresponding Eulerian quantities $\tilde{X}^{ik}$ given by (8.82). With this choice and using (8.95), (8.104) becomes

$$\frac{1}{\mathcal{D}}\partial_t(\mathcal{D}\hat{\mathcal{P}}^k) + \hat{\mathcal{P}}^h\nabla_h v^k = \mu F^k - \nabla_h\tilde{Y}^{hk} \ , \tag{8.109}$$

that is

$$\partial_t\hat{\mathcal{P}}^k = \mu F^k - \nabla_h\tilde{Y}^{hk} - k\hat{\mathcal{P}}^k - \hat{\mathcal{P}}^h({\omega_h}^k + {k_h}^k) \ , \tag{8.110}$$

where $k \stackrel{\text{def}}{=} g^{ik}k_{ik}$ is the *cubic deformation velocity*. Equation (8.110), using (8.105), determines the acceleration $\mathbf{a}$ as a function of the sources and paves the way to the intrinsic formulation of the mechanics of relativistic continua, although the nonsymmetric character of $\tilde{Y}^{ik}$ makes (8.109) not familiar.

8.10 Lagrangian Form of the Power of the Internal Forces

In order to complete the Lagrangian formulation, we have to transform the power of the internal force given by $(8.49)_4$. In Eulerian terms, we have

$$w^{(\mathrm{i})} = X^{ik}\hat{k}_{ik} \quad \sim \quad w^{(\mathrm{i})} = \boldsymbol{\phi}^i\cdot\left(\partial_t\mathbf{e} + \frac{1}{c^2}e_i\partial_t\mathbf{e}\right) \ , \tag{8.111}$$

where

$$\hat{k}_{ik} \equiv k_{ik} + \frac{1}{2c^2}\left(e_i\frac{\partial e_k}{\partial t} + e_k\frac{\partial e_i}{\partial t}\right) \tag{8.112}$$

and k_{ik} is the ordinary deformation tensor:

$$k_{ik} = \frac{1}{2}\left(\frac{\partial e_k}{\partial x^i} + \frac{\partial e_i}{\partial x^k}\right) \ . \tag{8.113}$$

The temporal derivative of $\mathbf{e}$ can be cast in Lagrangian form taking into account that, for any Eulerian quantity, (8.43) holds:

$$\partial_t(\,) = (\,)^{\cdot} - e^i \partial_i(\,) \,.$$

Thus, using $(8.101)_1$, $(8.111)_2$ becomes

$$w^{(\mathrm{i})} = \mathbf{Y}^i \cdot \left[\partial_i \mathbf{v} + \frac{1}{c^2} v_i (\partial_t \mathbf{v} - v^k \partial_k \mathbf{v})\right] \,,$$

that is

$$w^{(\mathrm{i})} = Y^{ik} \left[\nabla_i v_k + \frac{1}{c^2} v_i (\partial_t \mathbf{v} \cdot \mathbf{e}_k - v^h \nabla_h v_k)\right] \,.$$

Moreover, because of the identity

$$\partial_t \mathbf{v} \cdot \mathbf{e}_k = \partial_t v_k - \mathbf{v} \cdot \partial_k \mathbf{v} \,,$$

we find

$$w^{(\mathrm{i})} = Y^{ik} \left\{\nabla_i v_k + \frac{1}{c^2} v_i [\partial_t v_k - v^h (\nabla_k v_h + \nabla_h v_k)]\right\} \,.$$

The symmetry of the tension characteristics (either Eulerian, X^{ik} or Lagrangian, Y^{ik}) then implies

$$w^{(\mathrm{i})} = Y^{ik} \left[k_{ik} + \frac{1}{c^2} v_i (\partial_t v_k - 2 v^h k_{hk})\right] \,, \tag{8.114}$$

so that, using (8.98),

$$w^{(\mathrm{i})} = \frac{1}{2} Y^{ik} \left[\partial_t g_{ik} + \frac{2}{c^2} v_i (\partial_t v_k - v^h \partial_t g_{hk})\right] \,. \tag{8.115}$$

This expression immediately gives the classical limit of $w^{(\mathrm{i})}$:

$$\lim_{c \to \infty} w^{(\mathrm{i})} = \frac{1}{2} Y^{ik} \partial_t g_{ik} \,,$$

and it can be further transformed in order to have a single temporal derivative; in fact, the last term can be written as $v^h \partial_t g_{hk} = \partial_t v_k - g_{hk} \partial_t v^h$, so that

$$w^{(\mathrm{i})} = \frac{1}{2} Y^{ik} \left(\partial_t g_{ik} + \frac{2}{c^2} v_i g_{hk} \partial_t v^h\right) \,. \tag{8.116}$$

Introducing now the *covariant form* of the tension tensor

$$Y_{lh} = Y^{ik} g_{il} g_{kh} \tag{8.117}$$

and using the identity

$$Y_{lh}\partial_t g^{lh} = -Y^{ik}\partial_t g_{ik} ,$$

we finally get

$$w^{(\mathrm{i})} = -\frac{1}{2}Y_{ik}\partial_t \hat{g}^{ik} , \tag{8.118}$$

where the tensor $\hat{g}^{ik}$ has been introduced in $(8.106)_3$:

$$\hat{g}^{ik} = g^{ik} - \frac{1}{c^2}v^i v^k . \tag{8.119}$$

Summarizing, while expressions (8.115) and (8.118) for the power of the internal forces (expressed in terms of g_{ik} and g^{ik}, respectively) are equivalent in the classical case, in relativity only (8.118) assumes a fundamental role, giving $w^{(\mathrm{i})}$ as a *differential form in the variables* $\hat{g}^{ik}$ (for each element of the continuum, i.e. for fixed y^i).

Another useful expression for $w^{(\mathrm{i})}$ in terms of the dynamical stress variables $\hat{Y}^{ik}$ can easily be obtained:

$$\hat{Y}^{ik} \stackrel{\mathrm{def}}{=} \hat{g}^i{}_j \hat{g}^k{}_h Y^{jh} \equiv \hat{g}^{ij}\hat{g}^{kh}Y_{jh} . \tag{8.120}$$

$\hat{Y}^{ik}$ are *symmetric, different from the* $\tilde{Y}^{ik}$ given by (8.107) and have a 4-degree polynomial form in the Lagrangian velocities v^i. In fact, using the inverse relations we have

$$Y_{jh} = \hat{g}_{ji}\hat{g}_{hk}\hat{Y}^{ik} , \tag{8.121}$$

where the tensor $\hat{g}_{ik}$ has been defined in (8.75):

$$\hat{g}_{ik} = g_{ik} + \frac{\eta^2}{c^2}v_i v_k ; \tag{8.122}$$

thus, (8.118) can also be written as

$$w^{(\mathrm{i})} = \frac{1}{2}\hat{Y}^{ik}\partial_t \hat{g}_{ik} . \tag{8.123}$$

Obviously, both (8.118) and its counterpart (8.123) are formally invariant with respect to the choice of the Galilean frame and are also in agreement with the transformation law $(8.53)_3$; in fact, in Lagrangian terms, for each element of the continuum we have

$$\partial_{t'} = \frac{\eta}{\eta'}\partial_t \equiv \frac{\alpha}{\sigma}\partial_t . \tag{8.124}$$

We notice that $w^{(\mathrm{i})}$ vanishes in S_{g} (and hence in every S'_{g}) only for motions satisfying the conditions

$$\hat{k}_{ik} \equiv \frac{1}{2}\partial_t \hat{g}_{ik} = 0 ; \tag{8.125}$$

these are the so-called *rigid motions in the sense of Born*, already seen in Chap. 5. More precisely, consider the Minkowski metric $m_{\alpha\beta}$ and the associated natural decomposition in S_{g} along $\boldsymbol{\gamma}$:

$$m_{\alpha\beta} = \tilde{m}_{\alpha\beta} - \gamma_\alpha \gamma_\beta \ , \qquad \tilde{m}_{\alpha\beta}\gamma^\beta = 0 \ .$$

Comparing this form with the corresponding absolute one, along **V**,

$$m_{\alpha\beta} = m^0_{\alpha\beta} - \frac{1}{c^2} V_\alpha V_\beta \ , \qquad m^0_{\alpha\beta} V^\beta = 0 \ ,$$

we have, irrespective of the choice of the coordinates,

$$m^0_{\alpha\beta} = \tilde{m}_{\alpha\beta} + \frac{1}{c^2} V_\alpha V_\beta - \gamma_\alpha \gamma_\beta \ .$$

Thus

$$m^0_{\alpha\beta} = \tilde{m}_{\alpha\beta} + \frac{\eta}{c}\left(e_\alpha \gamma_\beta + e_\beta \gamma_\alpha + \frac{\eta}{c} e_\alpha e_\beta \right) \ , \tag{8.126}$$

and the total spatial part of the proper metric $m^0_{\alpha\beta}$ turns out to be given by

$$\tilde{m}^0_{\alpha\beta} = \tilde{m}_{\alpha\beta} + \frac{\eta^2}{c^2} e_\alpha e_\beta \ ; \tag{8.127}$$

in coordinates adapted to S_g, it coincides with the spatial Eulerian tensor:

$$\tilde{m}^0_{ik} = \delta_{ik} + \frac{\eta^2}{c^2} e_i e_k \equiv \hat{g}_{ik} \ . \tag{8.128}$$

These relations specify the geometrical meaning of the Eulerian tensor $\hat{g}_{ik}$, i.e. the proper spatial metric, induced in the space Σ of S_g; moreover, the condition (8.125) has an absolute meaning and implies (in S_g) the vanishing of the proper deformation velocity: $k^0_{ik} = 0$.

Therefore in relativity the local deformation compatible with the vanishing of the power of the internal forces is not related to the ordinary rigid motion, but to the Born-rigid one.

8.11 Energy Theorem and First Law of Thermodynamics

Equation (8.50) confirms the classical interpretation of the first law of thermodynamics as a substitute for the energy theorem [8]. To see this, it is enough to eliminate from (8.50) the mechanical power $\hat{\boldsymbol{\mathcal{F}}} \cdot \mathbf{e}$ by using the relativistic Cauchy equation (8.46); in fact, after multiplying this equation by **e** and using (8.49), we have

$$\begin{aligned} \mu \hat{\boldsymbol{\mathcal{F}}} \cdot \mathbf{e} &= \frac{1}{D}(\hat{\mu} D)^{\cdot} e^2 + \frac{1}{2}\hat{\mu}(e^2)^{\cdot} \\ &= \frac{1}{D}(\hat{\mu} D c^2)^{\cdot} - \frac{1}{D\eta^2}(\hat{\mu} D c^2)^{\cdot} + \frac{1}{\eta^3}\hat{\mu} c^2 \dot{\eta} \ ; \end{aligned}$$

using $(8.49)_1$, (8.50) then becomes

$$\frac{1}{D\eta^2}(\hat{\mu}Dc^2)^{\cdot} - \frac{1}{\eta^3}\hat{\mu}c^2\dot{\eta} = \mu q - w^{(\mathrm{i})} ,$$

and using $(8.52)_{2,3,4}$ as well as the relation $D\eta = D_0$:

$$\frac{\eta^2}{D_0}(\hat{\mu}Dc^2)^{\cdot} - \hat{\mu}c^2\dot{\eta} = \mu q_0 - \eta^2 w_0^{(\mathrm{i})} .$$

This equation, due to (8.42) and the similar relation $\mu = \eta^2\mu_0$, coincides with

$$\eta\frac{(\hat{\mu}_0 D_0 c^2)^{\cdot}}{D_0} = \mu_0 q_0 - w_0^{(\mathrm{i})} ,$$

which, because of the Eulerian identity

$$\eta(\,)^{\cdot} = V^\alpha\partial_\alpha(\,) \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}(\,) , \tag{8.129}$$

reduces to the first law of thermodynamics (8.34). The latter, in turn, coincides with the energy theorem (8.50), as follows studying it in the proper Galilean frame of the generic element of the continuum. In fact, using the Eulerian identity (8.36)

$$\frac{1}{D_0}\frac{\mathrm{d}D_0}{\mathrm{d}\tau} = \partial_\alpha V^\alpha , \tag{8.130}$$

we find the conservation law of the total proper internal energy (8.29):

$$\partial_\alpha(\mu_0\hat{\epsilon}V^\alpha) = \mu_0 q_0 - w_0^{(\mathrm{i})} . \tag{8.131}$$

We can also write (8.131) in the Eulerian form:

$$\frac{\partial\mathcal{E}}{\partial t} + \frac{\partial}{\partial x^i}(\mathcal{E}e^i) \equiv \frac{1}{D}(D\mathcal{E})^{\cdot} = \eta(\mu q - w^{(\mathrm{i})}) , \tag{8.132}$$

where $\mathcal{E}$ is the *relative internal energy density* for both pure matter and thermal energy density:

$$\mathcal{E} \overset{\mathrm{def}}{=} \frac{1}{\eta}\hat{\mu}c^2 \equiv \frac{1}{\eta}(\mu c^2 + \mu\varepsilon) . \tag{8.133}$$

As we have already seen in the case of the relativistic Cauchy equation, (8.132) can be conveniently rewritten, enlarging the energetic content of $\mathcal{E}$. In fact, because of (8.67), $(8.49)_3$ can be written in the form

$$\begin{aligned}
-\mu q_c = {} & \frac{1}{\eta}\frac{\partial q^i}{\partial x^i} + \frac{1}{c^2\eta}\left[\frac{\partial}{\partial t}(\mathbf{q}\cdot\mathbf{e}) + \frac{\partial}{\partial x^i}(\mathbf{q}\cdot\mathbf{e}e^i)\right. \\
& \left. -\frac{\partial}{\partial x^i}(\mathbf{q}\cdot\mathbf{e}e^i) + \eta^2(\mathbf{q}\cdot\dot{\mathbf{e}})\right] \\
= {} & \frac{1}{\eta}\frac{\partial\hat{q}^i}{\partial x^i} + \frac{1}{c^2\eta}\left[\frac{\partial}{\partial t}(\mathbf{q}\cdot\mathbf{e}) + \frac{\partial}{\partial x^i}(\mathbf{q}\cdot\mathbf{e}e^i) + \eta^2(\mathbf{q}\cdot\dot{\mathbf{e}})\right]
\end{aligned}$$

or

$$-\mu\eta q_c = \frac{\partial \hat{q}^i}{\partial x^i} + \frac{1}{c^2}\left[\frac{\partial}{\partial t}(\eta^2\hat{\mathbf{q}}\cdot\mathbf{e}) + \frac{\partial}{\partial x^i}(\eta^2\hat{\mathbf{q}}\cdot\mathbf{e}e^i) + \eta^2(\hat{\mathbf{q}} + \frac{\eta^2}{c^2}\hat{\mathbf{q}}\cdot\mathbf{ee})\cdot\dot{\mathbf{e}}\right].$$

Now, by using the identity

$$\mathbf{e}\cdot\dot{\mathbf{e}} = \frac{c^2}{\eta^3}\dot{\eta}\,, \tag{8.134}$$

we obtain the following expression for q_c:

$$-\mu\eta q_c = \frac{\partial \hat{q}^i}{\partial x^i} + \frac{1}{c^2}\left[\frac{\partial}{\partial t}(\eta^2\hat{\mathbf{q}}\cdot\mathbf{e}) + \frac{\partial}{\partial x^i}(\eta^2\hat{\mathbf{q}}\cdot\mathbf{e}e^i) + \eta\hat{\mathbf{q}}\cdot(\eta\mathbf{e})^{\cdot}\right]. \tag{8.135}$$

Hence, (8.132) turns out to be equivalent to

$$\frac{\partial\hat{\mathcal{E}}}{\partial t} + \frac{\partial}{\partial x^i}(\hat{\mathcal{E}}e^i) = \hat{Q}\,, \tag{8.136}$$

where the energy density $\hat{\mathcal{E}}$ includes also thermal conduction:

$$\hat{\mathcal{E}} \overset{\text{def}}{=} \mathcal{E} + \frac{1}{c^2}\eta^2\hat{\mathbf{q}}\cdot\mathbf{e} \equiv \frac{\mu c^2}{\eta}\left(1 + \frac{\varepsilon}{c^2}\right) + \frac{1}{c^2}\mathbf{q}\cdot\mathbf{e}\,, \tag{8.137}$$

and the total source $\hat{Q}$ has the form

$$\hat{Q} \overset{\text{def}}{=} \eta(\mu r - w^{(\mathrm{i})}) - \frac{\partial \hat{q}^i}{\partial x^i} - \frac{\eta}{c^2}\hat{\mathbf{q}}\cdot(\eta\mathbf{e})^{\cdot}\,. \tag{8.138}$$

We note that the last term in the expression for $\hat{Q}$ is genuinely relativistic, and depends on the relative acceleration $\dot{\mathbf{e}}$; hence, *the first law of thermodynamics is coupled with the Cauchy equation* and this is a novelty with respect to the classical situation.

Moreover, one immediately finds the Lagrangian form of (8.136) as well as that of the quantities $\hat{\mathcal{E}}$ and $\hat{Q}$, given by (8.137) and (8.138). The Eulerian differential system (8.69) and (8.136):

$$\begin{cases} \dfrac{\partial\hat{\boldsymbol{\mathcal{P}}}}{\partial t} + \dfrac{\partial}{\partial x^i}(\hat{\boldsymbol{\mathcal{P}}}e^i) = \mu\mathbf{F} - \dfrac{\partial\widehat{\boldsymbol{\phi}}^i}{\partial x^i}, \\ \dfrac{\partial\hat{\mathcal{E}}}{\partial t} + \dfrac{\partial}{\partial x^i}(\hat{\mathcal{E}}e^i) = \hat{Q}, \end{cases} \tag{8.139}$$

with the general content specified in (8.70), (8.137) and (8.138), is *formally invariant* with respect to the choice of the Galilean frame; it gives four (conservative) scalar equations in the same number of unknowns: μ and $\mathbf{e}$; these are the *general equations of the thermodynamics of nonpolar relativistic continua* and need the sources as well as the associated initial and boundary conditions to be assigned.

For a single mass point the sources are reduced to the 4-force only, which summarizes the mechanical and thermal action. For a continuum, instead, besides the mass 4-force (with the quantities $\mathbf{F}$ and r) there are the 4-stresses which, from a relative point of view, give rise to the mechanical stress and the internal energy together with the thermal conduction vector. Thus, taking into account (8.62), (8.67), $(8.70)_2$ and (8.138), (8.139) should be completed (apart from initial and boundary conditions) assigning the functional dependence of the various quantities, as in the classical case: $\mathbf{F}$ and r for the external (volume) force, [2] $\boldsymbol{\phi}^i$, ϵ and $\mathbf{q}$[3] as concerns the constitutive behaviour of the system. These sources require the *equipresence principle* as a direct consequence of their absolute nature; moreover, at least in the case considered here, the sources are a priori free. One must then take into account the invariant properties of the sources and specify the state variables on which they depend; these can partially be suggested by the Lagrangian expression (8.118) for the power of internal forces.

8.12 Finite Deformations in Relativity. Isotropic Systems

The theory of finite deformations [9] with the associate typical tensors (Cauchy–Green, pure deformation, local rotation, etc.) can be extended in relativity. Assume a reference kinematical state $(C_*, \mathbf{v}_*)$ characterized by the configuration C_* and the velocity field $\mathbf{v}_*$ to be arbitrarily fixed in the chosen Galilean frame S_g. We will denote by a "$_*$" all the quantities relative to the reference configuration C_* and without the "$_*$" all those associated with the actual configuration C corresponding to the motion of the continuum.

For the generic element of the continuum, besides the tensor $\hat{g}^{ik}$ (given by (8.119) and representing the proper spatial metric at the coordinate time t), we can consider the one associated with the reference configuration C_*:

$$\hat{g}_*^{ik} = g_*^{ik} - \frac{1}{c^2} v_*^i v_*^k \,. \tag{8.140}$$

Then the following definition of *direct and inverse deformation characteristics* is quite natural passing from the configuration C_* to C:

$$\begin{cases} \hat{\epsilon}_*^{ik} \stackrel{\text{def}}{=} \dfrac{1}{2}(\hat{g}^{ik} - \hat{g}_*^{ik}) \equiv \epsilon_*^{ik} - \dfrac{1}{2c^2}(v^i v^k - v_*^i v_*^k), \\ \hat{\epsilon}^{ik} \stackrel{\text{def}}{=} \dfrac{1}{2}(\hat{g}_*^{ik} - \hat{g}^{ik}) \equiv \epsilon^{ik} - \dfrac{1}{2c^2}(v_*^i v_*^k - v^i v^k). \end{cases} \tag{8.141}$$

[2] In the boundary conditions the analogous surface sources also appear.

[3] A priori, in the scheme both the temperature and the heat equation do not appear. A brief discussion concerning these aspects will be outlined when discussing the Cauchy problem.

Equation (8.141) refer to Lagrangian coordinates but it can also be easily adapted to Eulerian coordinates. The *covariant form* of the deformation tensors is naturally defined as follows:

$$\begin{cases} \hat{\epsilon}_{*ik} \overset{\text{def}}{=} \dfrac{1}{2}(\hat{g}_{ik} - \hat{g}_{*ik}) \equiv \epsilon_{*ik} + \dfrac{1}{2c^2}(\eta^2 v^i v^k - \eta_*^2 v_*^i v_*^k), \\ \hat{\epsilon}_{ik} \overset{\text{def}}{=} \dfrac{1}{2}(\hat{g}_{*ik} - \hat{g}_{ik}) \equiv \epsilon_{ik} + \dfrac{1}{2c^2}(\eta_*^2 v_*^i v_*^k - \eta^2 v^i v^k). \end{cases} \tag{8.142}$$

Similar to the classical case [4], *the invariants of the direct deformation, with respect to the metric* $\hat{g}_{*ik}$ *or* ${\hat{g}_*}^{ik}$, *are functions of the invariants of the inverse deformation* with respect to the metric $\hat{g}_{ik}$ or $\hat{g}^{ik}$. But, actually, the mixed forms of the deformation tensor which can be considered are four:

$$ {\hat{\epsilon}_*}^{ij}\hat{g}_{*kj}\ , \quad \hat{\epsilon}^{ij}\hat{g}_{kj}\ , \quad \hat{\epsilon}_{*jk}{\hat{g}_*}^{ij}\ , \quad \hat{\epsilon}_{jk}\hat{g}^{ij}\ , \tag{8.143}$$

and are not all independent:

$$ {\hat{\epsilon}_*}^{ij}\hat{g}_{*kj} = \hat{\epsilon}_{jk}\hat{g}^{kj}\ , \qquad \hat{\epsilon}_*^{jk}{\hat{g}_*}^{ij} = \hat{\epsilon}^{ij}\hat{g}_{jk}\ .$$

We can choose, for instance, the first of the mixed forms in (8.143) and introduce the notation

$$ {\hat{\epsilon}_*}^i{}_k = {\hat{\epsilon}_*}^{ij}\hat{g}_{*kj}\ , \qquad \hat{\epsilon}^i{}_k = \hat{\epsilon}_{jk}\hat{g}^{ij}\ , \tag{8.144}$$

so that ${\hat{\epsilon}_*}^i{}_k = \hat{\epsilon}^i{}_k$.

In order to build up a relativistic theory of finite deformations one has to consider in C and C_* the induced proper metrics $\hat{g}^{ik}$ and $\hat{g}_*^{ik}$ instead of the natural metric g_{ik} or g_{*ik}. The conditions $v^2 < c^2$ and $v_*^2 < c^2$ ensure that both these are proper Euclidean metrics.

The classical definition of isotropy [4, 8] can be extended in relativity as follows. The continuum $\mathcal{C}$ is *isotropic,*[4] *with respect to the kinematical status* $(C_*, \mathbf{v}_*)$ of S_g, if for each motion and at each instant *the symmetric tensors* Y_{ik} *and* $\hat{\epsilon}_{ik}$ *(or* $\hat{\epsilon}^{ik}$*) admit the same principal directions* with respect to the metric $\hat{g}_{ik}$ (or $\hat{g}^{ik}$), for all $E \in \mathcal{T}$.

The above condition is equivalent to the existence of constitutive relations like [10]

$$\hat{Y}^i{}_k \equiv \hat{g}^{ij}Y_{jk} = p\delta^i_k + q\hat{\epsilon}^i{}_k + r\hat{\epsilon}^i{}_j\hat{\epsilon}^j{}_k\ , \tag{8.145}$$

where p, q and r are scalar invariants.[5] Isotropic systems are then characterized by the condition that, once a certain kinematical state is fixed (local or

[4] For the sake of simplicity, we will assume that there always exists an isotropic state corresponding to a planar spatial section of the world tube $\mathcal{T}$. This requires a preferred Galilean frame, in which the status of the system is considered at certain instant. However, the isotropic state for the continuum has a local meaning and it corresponds in general to a *curved section* of $\mathcal{T}$: in this case the meaning of instantaneous configuration is lost.

[5] Products of more that two matrices $\hat{\epsilon}^i{}_j$ can always be expressed in terms of these quantities by using the Hamilton–Cayley identity.

global, necessary to evaluate the finite deformations), the Lagrangian tensor of the dynamical stresses $\hat{Y}^i{}_k$ is a quadratic function of the mixed deformation tensor $\hat{\epsilon}^i{}_k$ in briefly, an *isotropic function.*

Similarly, pulling back $\hat{Y}^i{}_k$ to the isotropic configuration C_* and using the identity $\hat{\epsilon}^i{}_k = \hat{\epsilon}_*{}^i{}_k$ the quadratic relation (8.145) becomes

$$\hat{Y}^i{}_k = p\delta^i_k + q\hat{\epsilon}_*{}^i{}_k + r\hat{\epsilon}_*{}^i{}_j\hat{\epsilon}_*{}^j{}_k \ . \tag{8.146}$$

Lowering then, in (8.146), the index i with the metric $\hat{g}_{ik}$:

$$\hat{g}_{ik} = \hat{g}_{*ik} + 2\hat{\epsilon}_{*ik} \ , \tag{8.147}$$

and using the Hamilton–Cayley identity for the last term of (8.146), one obtains the Lagrangian characteristics $Y_{ik} \equiv \hat{g}_{ij}\hat{Y}^j{}_k$ in terms of direct deformations:

$$Y_{ik} = P\hat{g}_{*ik} + Q\hat{\epsilon}_{*ik} + R\hat{\epsilon}_{*ij}\hat{\epsilon}_{*hk}\hat{g}_*{}^{jh} \ , \tag{8.148}$$

where P, Q and R are functions of p, q and r, and of the direct (or inverse) deformation invariants.

As in the classical case, nonviscous fluids are included in (8.145) in the case of vanishing q and r. We notice that the hypothesis of isotropy (8.145), formulated directly in S_{g}, has an invariant meaning with respect to the choice of the Galilean frame. In fact, the covariant tensors Y_{ik}, $\hat{g}_{ik}$ and $\hat{\epsilon}_{ik}$ have the same transformation laws (see (8.85) and $(8.77)_2$); thus, (8.145), invariant with respect to the choice of the Lagrangian coordinates, becomes

$$Y_{ik} = p\hat{g}_{ik} + q\hat{\epsilon}_{ik} + r\hat{\epsilon}_{ij}\hat{\epsilon}_{hk}\hat{g}^{jh} \ ; \tag{8.149}$$

it has an absolute meaning and can then be examined from a relative point of view.

8.13 Continua Without Material Structure

An interesting reduced scheme is that of a continuum without internal structure, corresponding to the classical scheme of a mass conservation system. As we have already seen, in the case $\mathbf{q} = 0$ we have the *proper mass conservation law*:

$$V^\alpha \partial_\alpha (\mu_0 D_0) = 0 \ , \tag{8.150}$$

i.e. the absolute property

$$\mu_0 D_0 \equiv \frac{\mu D}{\eta} = \text{const.} \ , \tag{8.151}$$

valid for each element of the continuum, with the meaning that the proper density of proper mass μ_0 is proportional to the proper numerical density $1/D_0$.

In this case, the sources are not completely free; in fact, (8.131), because of (8.33), assumes the form

$$(c^2 + \varepsilon)\partial_\alpha(\mu_0 V^\alpha) + \mu_0 \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} = \mu_0 q_0 - w_0^{(\mathrm{i})} \ , \tag{8.152}$$

which, using (8.129), gives rise to the following restriction:

$$\frac{\mu}{\eta}\dot{\varepsilon} = \eta(\mu q - w^{(\mathrm{i})}) \ . \tag{8.153}$$

This is the relativistic form of the first law of thermodynamics; it involves the sources ε, r, q and the Lagrangian tension characteristics Y^{ik} through the power of the internal forces. Equation (8.152) suggests the following definition of *systems undergoing reversible transformations*, motivated by the classical case, as those systems for which there exists a function of state (like ε) called the *proper entropy* s, such that for each transformation of the continuum

$$\frac{r_0}{\theta_0} = \frac{\mathrm{d}s}{\mathrm{d}\tau} \ , \tag{8.154}$$

where θ_0 is the *absolute temperature*.

From the relative point of view, being $r_0 = \eta^3 r$, the identity (8.129) reduces (8.154) to the form

$$\frac{\eta^3 r}{\theta_0} = \eta \dot{s} \ ;$$

thus, for systems undergoing reversible transformations, in each Galilean frame one has

$$\frac{r}{\theta} = \dot{s} \ , \tag{8.155}$$

where θ is the *relative temperature*:

$$\theta = \frac{\theta_0}{\eta^2} \ . \tag{8.156}$$

The converse is also valid: if (8.155) holds in any Galilean frame and with the following invariance properties:

$$s = \text{inv.} \ , \qquad \eta^3 r = \text{inv.} \ , \qquad \eta^2\theta = \text{inv.} \ , \tag{8.157}$$

then (8.154) holds too.

The reversibility condition (8.154) introduces the relative temperature θ as an integrating factor for r and another function of state: the entropy s. The latter directly concerns the thermal radiation r and allows one to write (8.131) in the form

$$\left(\frac{\mu D}{\eta}\right)^{\cdot} = \frac{\mu D}{\eta}\theta_0 \dot{s} - \left(\frac{\mu D}{\eta}\varepsilon\right)^{\cdot} - \eta D w^{(\mathrm{i})} + \mu\eta D q_c \ , \tag{8.158}$$

or

$$\left(\frac{\mu D}{\eta}\right)^{\cdot} = -\left(\frac{\mu D}{\eta}\mathcal{F}\right)^{\cdot} - s\left(\frac{\mu D}{\eta}\theta_0\right)^{\cdot} - \eta D w^{(\mathrm{i})} + \mu\eta D q_c \,, \tag{8.159}$$

where $\mathcal{F}$ is the *free energy* (per unit proper mass):

$$\mathcal{F} \overset{\text{def}}{=} \varepsilon - s\theta_0 \,. \tag{8.160}$$

As a consequence, the sources become $\mathbf{F}$, $\boldsymbol{\phi}^i$ and $\mathbf{q}$ as well as the two functions of state: ϵ and s (or equivalent functions like $\mathcal{F}$ and s).

In the general case too, using (8.158) and (8.159), the reversibility hypothesis can give suggestions about the properties of the sources, taking into account the expression of the internal force power (8.118) or (8.123). However, similar to what happens in the classical case, such hypothesis is particularly effective for a continuum without internal material structure and not in the general case.

8.14 Reversible Systems Without Material Structure

Let us now assume that (8.151) and (8.154) are satisfied; then, from (8.158), at least for $q_c = 0$, we have the following restriction for the sources:

$$\dot{\varepsilon} = \theta_0 \dot{s} - \frac{1}{\mu_0} w^{(\mathrm{i})} \,, \tag{8.161}$$

which is unconditionately valid, i.e. it holds for each reversible transformation of the system. Using (8.118) (or (8.123)), it follows that, for each element of the continuum, the function of state ε depends on the entropy s, as well as the variables $\hat{g}^{ik}$ (or $\hat{g}_{ik}$) given by (8.119):

$$\varepsilon = \varepsilon(y^i, s, \hat{g}^{ik}) \,. \tag{8.162}$$

Moreover, the continuum admits constitutive equations of the form

$$\theta_0 = \frac{\partial \varepsilon}{\partial s} \,, \qquad Y_{ik} = 2\mu_0 \frac{\partial \varepsilon}{\partial \hat{g}_{ik}} \,, \tag{8.163}$$

implying that it is necessarily hyperelastic.

Equations (8.163) allow to deal with the isoentropic case: $s =$ const. in a purely mechanical context (that is, without considering the coupling with the heat equation; the latter is necessary when the temperature is considered as a new variable). The isothermal case, instead, is related to the free energy $\mathcal{F}$. More precisely, (8.159) gives the following restriction:

$$\dot{\mathcal{F}} = -s\dot{\theta}_0 - \frac{1}{\mu_0} w^{(\mathrm{i})} \,; \tag{8.164}$$

thus for each element of the continuum, $\mathcal{F}$ depends on one side on the absolute temperature θ_0 and on the variables $\hat{g}^{ik}$ (or $\hat{g}_{ik}$):

$$\mathcal{F} = \mathcal{F}(y^i, \theta_0, \hat{g}^{ik}) \; ; \tag{8.165}$$

on the other side, the following constitutive relations hold:

$$s = -\frac{\partial \mathcal{F}}{\partial \theta_0} \; , \qquad Y_{ik} = 2\mu_0 \frac{\partial \mathcal{F}}{\partial \hat{g}^{ik}} \; . \tag{8.166}$$

In both cases, isentropic or isothermal, assuming the validity of the Helmholtz postulate[6]

$$\frac{\partial^2 \varepsilon}{\partial s^2} > 0 \quad \sim \quad \frac{\partial^2 \mathcal{F}}{\partial \theta_0^2} < 0 \; , \tag{8.167}$$

the characteristic functions of state reduce to a single one: ε or $\mathcal{F}$. In other words, as in the classical case, systems undergoing reversible transformations are characterized by a single constitutive function, which allows us to specify all the sources, apart from the mechanical action $\mathbf{F}$ and the surface thermo-mechanical one.

At least from the constitutive point of view such relativistic systems are the counterpart to the ordinary Lagrangian systems in the context of continuous systems, both of them described by a single function.

If the mass 4-force is *intrinsically conservative*, that is

$$\mathbf{k} = \mathrm{Grad}\, \mathcal{U}(x) \; , \tag{8.168}$$

with $\mathcal{U}$ a scalar invariant, one finds in each S_{g}[7]

$$\mu \mathbf{F} = \mu_0 (\mathbf{k} + \mathbf{k} \cdot \boldsymbol{\gamma}\boldsymbol{\gamma}) \equiv \widetilde{\mathrm{Grad}}\, \mathcal{U}(x) \tag{8.169}$$

and

$$\mu r = -\frac{\mu_0}{\eta} \mathbf{k} \cdot \mathbf{V} \equiv -\frac{\mu_0}{\eta} \frac{\mathrm{d}\mathcal{U}}{\mathrm{d}\tau} \; ,$$

[6] Such a postulate states that the specific heat is always positive for a constant configuration [8]. It is equivalent to the condition that *the internal energy is an increasing function of the absolute temperature* $\theta_0 > 0$:

$$\frac{\partial \varepsilon}{\partial \theta_0} \equiv -\theta_0 \frac{\partial^2 \mathcal{F}}{\partial \theta_0^2} > 0 \; ,$$

once (8.160) and $(8.166)_1$ for the internal energy

$$\varepsilon = \mathcal{F} - \theta_0 \frac{\partial \mathcal{F}}{\partial \theta_0}$$

are considered.

[7] Note that the symbol $\widetilde{\mathrm{Grad}}$ denotes the projection orthogonal to $\boldsymbol{\gamma}$ of the space-time gradient Grad.

that is, from (8.129):

$$\mu r = -\mu_0 \dot{\mathcal{U}} \,. \tag{8.170}$$

Therefore, each continuum with applied mass solicitation like (8.168) necessarily undergoes reversible transformations too, with

$$s = -\frac{\mathcal{U}}{\theta_0} \,, \qquad \theta = \frac{\theta_0}{\eta^2} \,, \tag{8.171}$$

where θ_0 is now an arbitrary positive constant. This is a very special case, in which all the sources (including mechanical mass forces) are derived by a single function of state: the internal energy or the thermodynamical potential; in fact, $(8.166)_1$ specifies the intrinsic potential $\mathcal{U}$, according to $(8.171)_1$:

$$\mathcal{U} = \theta_0 \frac{\partial \mathcal{F}}{\partial \theta_0} \,, \qquad \theta_0 = \text{const.} \tag{8.172}$$

8.15 Isotropic Reversible Systems Without Material Structure

Let us assume again that the continuum $\mathcal{C}$ has no material structure and is subjected to reversible transformations, so that, together with (8.151), the constitutive relations (8.163)–(8.166) hold.

If one avoids a direct coupling with the heat theory also in the relativistic context, that is, if one only considers isothermal ($\theta_0 = \text{const.}$) or isentropic ($s = \text{const.}$) transformations, it comes out:

$$\mathrm{Y}_{ik} = 2\mu_0 \frac{\partial W}{\partial \hat{g}_{ik}} \,, \tag{8.173}$$

where the potential W is given by

$$W = \begin{cases} \epsilon(y^i, \hat{g}^{ik}, s)|_{s=\text{const.}} & \text{adiabatic internal energy,} \\ \mathcal{F}(y^i, \hat{g}^{ik}, \theta_0)|_{\theta_0=\text{const.}} & \text{isothermal free energy.} \end{cases} \tag{8.174}$$

In this case one has a *completely determined mechanical scheme*, in the sense that, once the potential W and the constitutive law of the thermal flux $\mathbf{q}$ is known (in isothermal conditions one must assume $\mathbf{q} = 0$), one has the same number of equations as unknowns. Apart from initial and boundary conditions (which can easily be derived in relative terms from the absolute formulation), the final set of equations in Lagrangian form is the following:

$$\frac{1}{\mathcal{D}} \partial_t (\mathcal{D} \hat{\boldsymbol{\mathcal{P}}}) = \mu \mathbf{F} - \frac{1}{\mathcal{D}} \partial_i (\mathcal{D} \widehat{\mathbf{Y}}^i) \,, \qquad \partial_t \left(\frac{\mu \mathcal{D}}{\eta} \right) = 0 \,, \tag{8.175}$$

where

$$\begin{cases} \hat{\boldsymbol{\mathcal{P}}} \stackrel{\text{def}}{=} \mu\left(1+\dfrac{\varepsilon}{c^2}\right)\mathbf{v} + \dfrac{1}{c^2}\left(\eta^2 v_i \hat{\mathbf{Y}}^i + \eta\hat{\mathbf{q}} + 2\dfrac{\eta^3}{c^2}\hat{\mathbf{q}}\cdot\mathbf{v}\mathbf{v}\right) \\ \widehat{\mathbf{Y}}^i \stackrel{\text{def}}{=} \hat{\mathbf{Y}}^i + \dfrac{\eta}{c^2}\hat{q}^i\mathbf{v}\,, \quad \hat{\mathbf{Y}}^i = \hat{g}^{ik}Y_{kh}\mathbf{e}^h\,, \quad \hat{\mathbf{q}} \stackrel{\text{def}}{=} \mathbf{q} - \dfrac{1}{c^2}\mathbf{q}\cdot\mathbf{v}\mathbf{v}, \\ Y_{ik} = 2\dfrac{\mu}{\eta^2}\dfrac{\partial W}{\partial \hat{g}^{ik}}\,, \quad \hat{g}^{ik} = g^{ik} - \dfrac{1}{c^2}v^i v^k; \end{cases} \tag{8.176}$$

this requires that the internal energy ϵ in the adiabatic case coincides with W and is related to $\mathcal{F}$ in the isothermal case, taking into account (8.160) and $(8.166)_1$:

$$\epsilon = \begin{cases} W & \text{adiabatic case,} \\ W - \left(\theta_0\dfrac{\partial \mathcal{F}}{\partial \theta_0}\right)_{\theta_0=\text{const.}} & \text{isothermal case.} \end{cases} \tag{8.177}$$

As in the classical case, (8.175) can be written in scalar terms in different ways, but the Lagrangian form of the system introduces one more unknown: the metric g_{ik}. Hence, one is forced to pass to the *intrinsic formulation.*

Equation (8.175) requires that both *the specific mass force* $\mathbf{F}$ and the *potential function* $W(y^i, \hat{g}^{ik})$, characteristic of the material, as well as the constitutive law of $\mathbf{q}$ are all assigned.

The determination of W is related to the experimental study of the response of a material (i.e. of the internal stresses) to the various kind of solicitation: pression or simple flexion, presso-flexion, torsion, etc. Symmetry properties as the existence of preferred configurations (*natural status, isotropic status, etc.*) may eventually reduce the number of variables on which W depends and even suggest the functional form. For instance, for isotropic systems, W becomes a function of three variables, instead of six. In fact, W depends on the direct deformation $\hat{\epsilon}_{*ik}$ only through its fundamental invariants $\hat{I}_k$ (with respect to the metric $\hat{g}_{*ik}$) or equivalent variables. To show this let us recall the isotropy property (8.148) which reduces (8.173) to the following differential form:

$$(P\hat{g}_{*ik} + Q\hat{\epsilon}_{*ik} + R\hat{\epsilon}_{*i}^{\;j}\hat{\epsilon}_{*jk})\dot{\hat{\epsilon}}_*^{\;ik} = \mu_0\dot{W}(\hat{\epsilon}_*^{\;ik})\,, \tag{8.178}$$

deducible from

$$Y_{ik} = 2\mu_0\frac{\partial W}{\partial \hat{g}_{ik}} = \mu_0\frac{\partial W}{\partial \hat{\epsilon}_{*ik}}\,, \tag{8.179}$$

where a dot here denotes that the infinitesimal variation W can thus be considered as a function of the *direct deformation characteristics* $\hat{\epsilon}_*^{\;ik}$, in agreement with $(8.141)_1$:

$$\hat{g}^{ik} = \hat{g}_*^{\;ik} + 2\hat{\epsilon}_*^{\;ik} \tag{8.180}$$

(the metric $\hat{g}_*^{\;ik}$ or $\hat{g}_{*ik}$ should be known).

Condition (8.178) holds for any transformation of the system, that is, for any choice of the variables $\hat{\epsilon}_*^{ik}$ (or $\dot{\hat{\epsilon}}_{*ik}$). Moreover, after introducing the invariants

$\hat{J}_k$ (*linear, quadratic and cubic* of $\hat{\epsilon}_*^{ik}$) in place of the deformation invariants $\hat{I}_k$:

$$\hat{J}_1 = \hat{g}_{*ik}\hat{\epsilon}_*^{ik} , \quad \hat{J}_2 = \hat{\epsilon}_*{}^i{}_j\hat{\epsilon}_*{}^j{}_i , \quad \hat{J}_3 = \hat{\epsilon}_*{}^i{}_j\hat{\epsilon}_*{}^j{}_h\hat{\epsilon}_*{}^h{}_i , \tag{8.181}$$

we find

$$\dot{\hat{J}}_1 = \hat{g}_{*ik}\dot{\hat{\epsilon}}_*{}^{ik} , \quad \dot{\hat{J}}_2 = 2\hat{\epsilon}_*{}^i{}_j\dot{\hat{\epsilon}}_*{}^j{}_i , \quad \dot{\hat{J}}_3 = 3\hat{\epsilon}_*{}^i{}_j\hat{\epsilon}_*{}^j{}_h\dot{\hat{\epsilon}}_*{}^h{}_i , \tag{8.182}$$

where $\dot{\hat{\epsilon}}_*{}^h{}_i = \hat{g}_{*il}\dot{\hat{\epsilon}}_*{}^{jl}$. Equation (8.178) thus becomes

$$P\dot{\hat{J}}_1 + \frac{1}{2}Q\dot{\hat{J}}_2 + \frac{1}{3}R\dot{\hat{J}}_3 = \mu_0\dot{W}(\hat{J}) \tag{8.183}$$

and turns out to be equivalent to the condition that, for each element of the continuum, W depends on the direct deformation only through the three variables $\hat{J}_k$ (in a 1–1 correspondence with the deformation invariants $\hat{I}_k$). Furthermore, in (8.148) one has

$$P = \frac{\mu}{\eta^2}\frac{\partial W(\hat{J})}{\partial \hat{J}_1} , \quad Q = 2\frac{\mu}{\eta^2}\frac{\partial W(\hat{J})}{\partial \hat{J}_2} , \quad R = 3\frac{\mu}{\eta^2}\frac{\partial W(\hat{J})}{\partial \hat{J}_3} , \tag{8.184}$$

with the general relations

$$\hat{J}_1 = \hat{I}_1 , \quad \hat{J}_2 = \hat{I}_1^2 - 2\hat{I}_2 , \quad \hat{J}_3 = \hat{I}_1^3 - 3\hat{I}_1\hat{I}_2 + 3\hat{I}_3 . \tag{8.185}$$

In this way, all the ingredients necessary to develop a relativistic finite elasticity theory are introduced, in particular a second degree theory, similar to the classical one, due to Signorini [11, 12]. One has to require the condition that, according to (8.181), the constitutive relations (8.149) were exactly of the second degree in the inverse deformation or in the direct deformation [13] for an analogous theory.

8.16 Perfect Fluids with Heat Transfer

Consider now the special case of a *perfect fluid*, characterized by the absence of viscosity (8.92) and by a reduced constitutive relation between proper pressure p_0 and proper density of proper mass μ_0. We note that, similar to the classical case [8], in the context of continua without material structure a relativistic perfect fluid can also be defined through the condition that for each element of the continuum the internal forces (a) do not contrast the disjunction of elements of the continuum, (b) do not provide work for any transformation without change of proper volume; and (c) that the system undergoes reversible transformations.

Such hypotheses are summarized by the condition that the thermodynamical potential $\mathcal{F}$ given by (8.160) *depends on the metric* $\hat{g}^{ik}$ *through the proper numerical density of the particles* $1/\mathcal{D}_0$:

$$\frac{1}{\mathcal{D}_0} = \frac{1}{\eta \mathcal{D}} \equiv \sqrt{\det||\hat{g}^{ik}||} \; ; \tag{8.186}$$

moreover, in any configuration C, we find

$$Y_{ik} = p_0 \hat{g}_{ik}, \quad p_0 = -\mu_0 \mathcal{D}_0 \frac{\partial \mathcal{F}(\mathcal{D}_0, \theta_0)}{\partial \mathcal{D}_0} \, . \tag{8.187}$$

Using now the proper mass conservation law (due to the assumed absence of material structure)

$$\mu_0 \mathcal{D}_0 \equiv \frac{\mu \mathcal{D}}{\eta} = \text{const.} > 0 \; , \tag{8.188}$$

we have the *characteristic equation*

$$p_0 = p_0(\mu_0, \theta_0) \; . \tag{8.189}$$

Finally, for a perfect fluid with thermal conduction, the set of evolution equations, in Lagrangian form, is the following:

$$\frac{1}{\mathcal{D}} \partial_t (\mathcal{D} \hat{\boldsymbol{\mathcal{P}}}) = \mu \mathbf{F} - \frac{1}{\mathcal{D}} \partial_i (\mathcal{D} \widehat{\mathbf{Y}}^i) \; , \qquad \partial_t \left(\frac{\mu \mathcal{D}}{\eta} \right) = 0 \; , \tag{8.190}$$

where

$$\begin{cases} \hat{\boldsymbol{\mathcal{P}}} = \left[\mu + \dfrac{1}{c^2} \left(\mu \varepsilon + \eta^2 p_0 + \dfrac{2}{c^2} \eta^3 \hat{\mathbf{q}} \cdot \mathbf{v} \right) \right] \mathbf{v} + \dfrac{1}{c^2} \eta \hat{\mathbf{q}}, \\ \hat{\mathbf{q}} = \mathbf{q} - \dfrac{1}{c^2} \mathbf{q} \cdot \mathbf{v} \mathbf{v}, \\ \widehat{\mathbf{Y}}^i = \left(p_0 \delta^i_k + \dfrac{1}{c^2} \eta \hat{q}^i v_k \right) \mathbf{e}^k, \\ p_0 = -\dfrac{\partial \mathcal{F}(x, \theta_0)}{\partial x} \, , \quad x \overset{\text{def}}{=} \dfrac{1}{\mu_0}. \end{cases} \tag{8.191}$$

When $\mathbf{q} = 0$ and considering isothermal or adiabatic transformations, the system (8.190) gives rise to a purely mechanical scheme, as for the more general case of a continuum without material structure and undergoing reversible transformations. Such a scheme is completely determined starting from the function $\mathcal{F}$; when $\mathbf{q} \neq 0$, at least in the general case, in order to have the same number of equations as unknowns one needs either the relativistic heat equation (not yet formulated in a very satisfactory way) or the evolution equation for the thermal flux $\mathbf{q}$, that is the so-called *entropy principle.*

8.17 Introduction to the Cauchy Problem

The classical approach to physics is very different if compared with the relativistic one. Consider, for example, the case of mechanics; in the classical theory, based on the Galilean principle of relativity, the only possible point of view is the relative one. On the other hand in relativity there are three possible formulations:

1. the *absolute point of view*, which is framed in the four-dimensional space-time (flat in special relativity or curved in general relativity) and expressed in tensorial language. It is simple and elegant because of its geometrical content but, unfortunately, it deals with four-dimensional (absolute) objects, which are but not (directly) observable.
2. the *relative point of view*, which is tied to an arbitrary three-dimensional reference frame, i.e. a Galilean "solid", and is expressed in terms of physical (observable) quantities. This point of view, instead, is more efficient for the applications, because of its three-dimensional content, and uses quantities which are directly observable. Moreover, it is *formally invariant* with respect to the choice of a reference frame (principle of relativity).
3. the *proper Galilean frame point of view*, associated with the world lines of the continuum itself. In this case, once given a reference frame, the proper quantities become the observables. For instance, we have seen how the hypothesis of *pure pressure* for a relativistic fluid is formulated in the proper Galilean frame. Though it represents an absolute property of the continuum, it *assumes a dynamic character in every frame*. Similar to a Born-rigid motion which *appears deformable* in every frame, so a relativistic nonviscous fluid in general appears as viscous in any frame (see Sect. 8.8).

The unifying aspect of relativity is particularly evident in the mechanics of continua, where *the* 4-*stress tensor combines three proper quantities: mechanical stress, thermal flux and internal energy density*. Again, the three kinematic ingredients: *acceleration, angular velocity and deformation velocity* (distinct in classical theory), *are summarized*, in relativity, *by a single* 4-*tensor*: the space-time gradient of the 4-velocity: $\mathbf{V}$ [14, 15].

In this context we will to discuss the *intrinsic Cauchy problem* in special relativity for thermomechanical continua, *intrinsic* in the sense of the rigid Euler dynamics or according to the "rèpere mobile". In fact, let us consider an anholonomic frame distribution and the associated essential ingredients (geometrical and physical); we have a *principal (Cauchy) problem* and then a *secondary problem*, sub-ordered to the first and totally integrable. Here the assumed variables (all spatial) are *metric, angular and deformation velocities, acceleration and mass density.* We will also discuss the corresponding *conditions of compatibility*; the latter, classically, constrain only the deformation velocity, whereas in relativity, constrain both the acceleration and the angular velocity. Consequently, *the initial constraints involve the acceleration as well as the constitutive functions which*, in such a way, *have influence also on the initial data.*

8.17.1 Relativistic Compatibility

Denote by Γ a *timelike congruence* of world lines identifying a kinematic continuum in M_4. *The lines of* Γ *never intersect each other* (conservation of the particles' number) and are characterized by the timelike unit tangent vector field $\boldsymbol{\gamma}$, $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} = -1$, so that $\mathbf{V} = c\boldsymbol{\gamma}$ represents the local 4-velocity of the continuum itself.

Introduce local coordinates y^α with $(\alpha = 0, 1, 2, 3)$ and $y^0 = ct$ *adapted* to Γ; let $\{\mathbf{e}_\alpha\}$ be the *natural basis* associated with y^α and $g_{\alpha\beta}(y)$ the corresponding *metric*, analogous to (5.11) but now in four dimensions.

Γ induces in M_4 an *almost-product orthogonal structure* 1×3, locally defined by the *timelike direction* $\boldsymbol{\gamma}$ and by the *spatial platform* Σ, i.e. the orthogonal complement to $\boldsymbol{\gamma}$ in the tangent space. This structure allows a systematic and *natural decomposition* of all tensor fields in M_4 which can be directly achieved by using an adapted *anholonomic basis*. For example, a convenient *almost-natural basis* $\{\widetilde{\mathbf{e}}_\alpha\}$ is the following[8]:

$$\widetilde{\mathbf{e}}_0 \stackrel{\text{def}}{=} \mathbf{V} \sim V^0\mathbf{e}_0, \quad \widetilde{\mathbf{e}}_i \stackrel{\text{def}}{=} \mathbf{e}_i - \frac{V_i}{V_0}\mathbf{e}_0 \sim \widetilde{\mathbf{e}}_i = \mathbf{e}_i + \frac{1}{c^2}V_i\mathbf{V} \,, \tag{8.192}$$

having tensorial behaviour under transformations of the coordinates y^α *internal* to Γ, that is

$$y^{0'} = y^{0'}(y), \qquad y^{i'} = y^{i'}(y^1, y^2, y^3), \tag{8.193}$$

and giving rise to the (Euclidean) *induced metric* on Σ

$$\gamma_{ik} = \widetilde{\mathbf{e}}_i \cdot \widetilde{\mathbf{e}}_k \,, \tag{8.194}$$

with inverse γ^{ik}.

The following *fundamental relations* are associated with (8.192):

$$\begin{cases} \widetilde{\partial}_i\widetilde{\mathbf{e}}_k = \Gamma^j{}_{ik}\widetilde{\mathbf{e}}_j + \dfrac{1}{c^2}H_{ik}\mathbf{V}, & \partial\widetilde{\mathbf{e}}_i = H_i{}^k\widetilde{\mathbf{e}}_k + \dfrac{1}{c^2}A_i\mathbf{V}, \\[2ex] \widetilde{\partial}_i\mathbf{V} = H_i{}^k\widetilde{\mathbf{e}}_k, & \partial\mathbf{V} = A^j\widetilde{\mathbf{e}}_j, \end{cases} \tag{8.195}$$

where $H_i{}^k = \gamma^{kj}H_{ij}$ and $\widetilde{\partial}_\alpha \equiv (\partial, \widetilde{\partial}_i)$ are the *Pfaffian derivatives* corresponding to the frame vectors of (8.192):

$$\partial \stackrel{\text{def}}{=} V^\alpha\frac{\partial}{\partial y^\alpha}, \qquad \widetilde{\partial}_i \stackrel{\text{def}}{=} \frac{\partial}{\partial y^i} - \frac{V_i}{V_0}\frac{\partial}{\partial y^0} \sim \frac{\partial}{\partial y^i} + \frac{1}{c^2}V_i\partial. \tag{8.196}$$

Equation (8.195) contain all the geometrical–kinematical ingredients for the description of the continuum Γ, namely the *proper deformation velocity* K_{ik}, the *angular velocity* Ω_{ik} (which together form the tensor $H_{ik} = K_{ik} + \Omega_{ik}$), the 4-*acceleration* $A_i = c^2C_i$ (C_i being the curvature vector of the world lines of Γ) and finally the *spatial Christoffel symbols* $\Gamma^j{}_{ik}$. One can evaluate the *anholonomic tensor* associated with the derivatives (8.196),

[8] We have indicated here both the general form and the corresponding one in adapted coordinates to Γ, the latter being specified by a $\sim$.

$$[\widetilde{\partial}_\alpha, \widetilde{\partial}_\beta] = A^\sigma{}_{\alpha\beta}\widetilde{\partial}_\sigma \ , \tag{8.197}$$

whose nonvanishing components are

$$A^0{}_{i0} = \frac{A^i}{c^2} \ , \qquad A^0{}_{ik} = 2\Omega_{ik} \ . \tag{8.198}$$

A direct calculation gives then the following expressions:

$$\begin{cases} C_i = \gamma_0 \left[\widetilde{\partial}_i \gamma^0 + \partial\left(\dfrac{\gamma_i}{\gamma_0}\right)\right], \\ K_{ik} = \dfrac{1}{2}\partial\gamma_{ik}, \\ \Omega_{ik} = \gamma_0 \left[\widetilde{\partial}_i \left(\dfrac{\gamma_k}{\gamma_0}\right) - \widetilde{\partial}_k \left(\dfrac{\gamma_i}{\gamma_0}\right)\right], \\ \Gamma^j{}_{ik} \stackrel{\text{def}}{=} \frac{1}{2}\gamma^{jh}(\widetilde{\partial}_i\gamma_{kh} + \widetilde{\partial}_k\gamma_{hi} - \widetilde{\partial}_h\gamma_{ik}). \end{cases} \tag{8.199}$$

Similarly one can evaluate the *Riemann or curvature tensor* of M_4, defined by

$$\mathbf{R}_{\alpha\beta\rho} = [\widetilde{\partial}_\alpha, \widetilde{\partial}_\beta]\widetilde{\mathbf{e}}_\rho - A^\sigma{}_{\beta\alpha}\widetilde{\partial}_\sigma\widetilde{\mathbf{e}}_\rho \equiv R_{\alpha\beta\rho}{}^\sigma\widetilde{\mathbf{e}}_\sigma \ , \tag{8.200}$$

and identically zero, M_4 being a flat space-time. However, when its components are considered as functions of the tensors H_{ik}, A_i and $\Gamma^j{}_{ik}$, the vanishing condition is equivalent to certain relations among these fields which are just the *compatibility conditions* of the differential system (8.195). More precisely, the curvature tensor, because of its symmetries, has only *three types of independent components*: $R_{ikh}{}^j$, $R_{ikh}{}^0$ and $R_{0ik}{}^0$ (see e.g. [16], (5.65) and (5.67), as well as [17, 18]). A direct evaluation shows the following set of *anholonomic conditions*:

$$\begin{cases} R_{ikh}{}^j \equiv P_{ikh}{}^j + \dfrac{1}{c^2}\left(H_{kh}H_i{}^j - H_{ih}H_k{}^j - 2\Omega_{ik}H_h{}^j\right) = 0, \\ R_{ikh}{}^0 \equiv B_{ikh} = \widetilde{\nabla}_i H_{kh} - \widetilde{\nabla}_k H_{ih} - \dfrac{2}{c^2}\Omega_{ik}A_h = 0, \\ R_{0i0k} \equiv C_{ik} = \partial H_{ik} - \left(\widetilde{\nabla}_i + \dfrac{1}{c^2}A_i\right)A_k - H_{ij}H_k{}^j = 0, \end{cases} \tag{8.201}$$

where $P_{ikh}{}^j$ is the *spatial curvature tensor* associated with the spatial connection $\Gamma^h{}_{ik}$:

$$P_{ikh}{}^j \stackrel{\text{def}}{=} \widetilde{\partial}_i\Gamma^j{}_{kh} - \widetilde{\partial}_k\Gamma^j{}_{ih} + \Gamma^l{}_{kh}\Gamma^j{}_{il} - \Gamma^l{}_{ih}\Gamma^j{}_{kl} \ , \tag{8.202}$$

and $\widetilde{\nabla}_i$ denotes the *Cattaneo's transverse covariant derivative* [19], i.e. the *covariant extension* of the Pfaffian derivatives $\widetilde{\partial}_i$ by means of the connection

$\Gamma^j{}_{ik}$.[9] We note that *the conditions* $(8.201)_{1,2}$ *generalize the Gauss–Mainardi–Codazzi equations to the case of a distribution of* 3*-planes* $\{\Sigma\}$; in fact, if $\{\Sigma\}$ is integrable,[10] i.e. if $\Omega_{ik} = 0$, we have the ordinary form of the equations with H_{ik} symmetric representing the second *quadratic form* of Σ.

The last set of equations $(8.201)_3$, having *evolutive character*, yields the time derivative of the tensor H_{ik}; hence, as in the classical case [20], the dynamical compatibility leads to the following system:

$$\partial\gamma_{ik} = 2H_{(ik)}, \qquad \partial H_{ik} = \left(\widetilde{\nabla}_i + \frac{1}{c^2}A_i\right) A_k - H_{ij}H_k{}^j \,, \tag{8.203}$$

with the supplementary conditions $(8.201)_{1,2}$:

$$R_{ikh}{}^j = 0, \qquad B_{ikh} = 0. \tag{8.204}$$

Compared with the classical situation, here we no longer have the separation of the variables γ_{ik} and H_{ik}. Moreover, in $(8.204)_1$ we have not only the metric, through the spatial Christoffel symbols $\Gamma^j{}_{ik}$, but also the tensor H_{ik}; the acceleration A_i appears instead in $(8.204)_2$.

The constraints (8.204) when expressed in terms of $\Gamma^h{}_{ik}$, H_{ik} and A_i *are still involutive*. In fact, we have the following *spatial identities* (i.e. Bianchi identities, see [18], p. 88):

$$\widetilde{\nabla}_{[l}P_{ik]h}{}^j + \frac{2}{c^2}\Omega_{[ik}H_{l]h}{}^j = 0, \tag{8.205}$$

where the 3-tensor $H_{ik}{}^j$ is related to the deformation velocity by

$$H_{ik}{}^j \stackrel{\text{def}}{=} \gamma^{jh}\left(\widetilde{\nabla}_i K_{kh} + \widetilde{\nabla}_k K_{hi} - \widetilde{\nabla}_h K_{ik}\right); \tag{8.206}$$

moreover, since $C_{ik} = 0$ we also have a first-order differential system, linear and homogeneous in the spatial tensors $R_{ikh}{}^j$ and B_{ikh}:

$$\begin{cases} \partial R_{ikhj} = H_j{}^l R_{ikhl} - H_h{}^l R_{ikjl} - \left(\widetilde{\nabla}_k + \dfrac{1}{c^2}A_k\right) B_{hji} \\ \qquad + \left(\widetilde{\nabla}_i + \dfrac{1}{c^2}A_i\right) B_{hjk} - \dfrac{1}{c^2}\left(A_j B_{ikh} - A_h B_{ikj}\right), \\ \partial B_{ikh} = -R_{ikhl}A^l + H_h{}^l B_{ikl} + H_i{}^l B_{hlk} - H_k{}^l B_{hli}. \end{cases} \tag{8.207}$$

Therefore, as in the classical situation, once the proper acceleration field is assigned

[9] For any spatial vector $\mathbf{X} = X^k\widetilde{\mathbf{e}}_k$ the Cattaneo's transverse covariant derivative is given by

$$\widetilde{\nabla}_i X^k = \widetilde{\partial}_i X^k + \Gamma^k{}_{ij}X^j.$$

[10] In this case Γ is said to be a *normal congruence*.

$$A_i = A_i(y, \gamma_{jk}, H_{jk}, ...) \, , \tag{8.208}$$

the evolution of the continuum can be reduced to the Cauchy problem (8.203), with given *initial values* $\gamma_{ik,0}$ and $H_{ik,0}$ (on an initial hypersurface) and subjected to the conditions (8.204).

The evolution equations still have a precise geometrical meaning (vanishing of the curvature tensor, Bianchi identities, etc.), and the classical ingredients have their direct counterparts in the proper ingredients of the continuum. Moreover, apart from the presence of the tensor H_{ik} in $(8.201)_1$, *there is a new variable in* $(8.201)_2$: *the proper acceleration* A_i. On the other hand, the condition $B_{ikh} = 0$ implies $B_{[ikh]} = 0$, from which the following relations between the tensors Ω_{ik} and A_i hold (*Jacobi identity*):

$$\partial\Omega_{ik} = \frac{1}{c^2}\widetilde{\partial}_{[i}A_{k]}, \quad \widetilde{\nabla}_{[i}\Omega_{kh]} - \frac{1}{c^2}\Omega_{[ik}A_{h]} = 0. \tag{8.209}$$

Consequently, introducing the tensor A_{ikh}:

$$A_{ikh} \stackrel{\text{def}}{=} B_{ikh} - \frac{3}{2}B_{[ikh]}, \tag{8.210}$$

which is in 1–1 correspondence with B_{ikh}:

$$B_{ikh} = A_{ikh} - 3A_{[ikh]}, \tag{8.211}$$

Equation $(8.204)_2$ can be written in the form of a (total) *differential system* for the angular velocity Ω_{ik}:

$$\begin{aligned} A_{ikh} \equiv \widetilde{\nabla}_h\Omega_{ik} + \widetilde{\nabla}_i K_{hk} - \widetilde{\nabla}_k K_{ih} \\ + \frac{1}{c^2}\left(\Omega_{kh}A_i + \Omega_{hi}A_k - \Omega_{ik}A_h\right) = 0 \, ; \end{aligned} \tag{8.212}$$

here, different from the classical case, we have the presence of the acceleration A_i as well as that of the metric, through the spatial Christoffel symbols. However, the system (8.212) *no longer has the unlimited integrability of the classical case* but there are compatibility conditions [18].

8.17.2 Intrinsic Cauchy Problem in Relativity

The relative decomposition of the Riemann tensor (8.201) with $A_i = c^2 C_i$:

$$\begin{cases} R_{ikh}{}^{j} \equiv P_{ikh}{}^{j} + H_{kh}H_i{}^{j} - H_{ih}H_k{}^{j} - 2\Omega_{ik}H_h{}^{j}, \\ \\ R_{ikh}{}^{0} = B_{ikh} \equiv \widetilde{\nabla}_i H_{kh} - \widetilde{\nabla}_k H_{ih} - 2\Omega_{ik}C_h, \\ \\ R_{0i0k} = C_{ik} \equiv \partial H_{ik} - c^2\left(\widetilde{\nabla}_i + C_i\right)C_k - H_{ij}H_k{}^{j}, \end{cases} \tag{8.213}$$

associates with the curvature tensor three independent spatial tensors: R_{ikhj}, B_{ikh} and C_{ik}; R_{ikhj} obviously satisfies all the algebraic properties of a curvature tensor; B_{ikh} is antisymmetric with respect to its first pair of indices and satisfies a cyclic property according to $(8.209)_2$; finally, C_{ik} is a symmetric tensor, because of the condition $(8.209)_1$, with the additional conditions (8.209):

$$\partial\Omega_{ik} = \widetilde{\nabla}_{[i}C_{k]}, \quad \widetilde{\nabla}_{[i}\Omega_{kh]} - C_{[i}\Omega_{kh]} = 0\ . \tag{8.214}$$

If the Cauchy problem is formulated in anholonomic terms, the following *differential system of first order* (in time) for the variables γ_{ik} and $H_{ik} = \Omega_{ik} + K_{ik}$ holds:

$$\begin{cases} \partial\gamma_{ik} = 2H_{(ik)}, \\ \partial H_{ik} = H_i{}^j H_{kj} + 2(\widetilde{\nabla}_i + C_i)C_k. \end{cases} \tag{8.215}$$

Together with (8.215) we should consider, a priori, further conditions (8.214). However, $(8.214)_1$ is a consequence of the system (8.215); in fact, (8.215) can be written in the equivalent form:

$$\begin{cases} \partial\gamma_{ik} = 2K_{ik}, \quad \partial\Omega_{ik} = \widetilde{\nabla}_{[i}C_{k]}, \\ \partial K_{ik} = H_i{}^j H_{kj} + 2\widetilde{\nabla}_{(i}C_{k)} + 2C_iC_k. \end{cases} \tag{8.216}$$

Summarizing, the *effective constraints* for the variables γ_{ik} and H_{ik} (*in involution, because of the Bianchi identities*) are given by $(8.213)_{1,2}$ only:

$$\begin{cases} P_{ikh}{}^j + H_{kh}H_i{}^j - H_{ih}H_k{}^j - 2\Omega_{ik}H_h{}^j = 0, \\ \widetilde{\nabla}_i H_{kh} - \widetilde{\nabla}_k H_{ih} - 2\Omega_{ik}C_h = 0, \end{cases} \tag{8.217}$$

since $(8.214)_2$ is a consequence of $(8.217)_2$, noting that

$$\frac{1}{2}B_{[ikh]} = \widetilde{\nabla}_{[i}\Omega_{kh]} - C_{[i}\Omega_{kh]} = 0\ .$$

Thus, we have $6 + 9 = 15$ *restrictions* (all independent) *to the initial data*: $\gamma_{ik,0}$ and $K_{ik,0}$ with their first and second derivatives.

In the Minkowski case the field equations (8.216) as well the constraints (8.217) both depend on the curvature vector C_i. So, what is the role that the acceleration plays in (8.216) and (8.217)?

Clearly, the acceleration components are *not additional field variables* (besides γ_{ik} and H_{ik}), because they obey Galilei principle; hence, they are functions of the thermodynamic variables of the continuum Γ. This fact loses its meaning in the case of the vacuum, just because of the absence of matter. However, it is worth to note that for any choice of the reference frame Γ the formulations (8.216) and (8.217) of the evolution problem in M_4 have an *invariant meaning* for every coordinate transformation *internal* to Γ:

$$y^{0\prime} = y^{0\prime}(y), \quad y^{i\prime} = y^{i\prime}(y^1, y^2, y^3).$$

Therefore, it has an *intrinsic meaning* in the considered frame, and this is a substantial difference from the formulation of the analogous problem in terms of coordinates.

As a consequence, the case of vacuum is completely different from the case of presence of matter. In other words, in the case of vacuum, Γ cannot have more than a purely geometrical–kinematical meaning; it is completely at disposal and there are *no preferred choices due to physical reasons* (i.e. there is no inertia without matter). Some simplifying choices, as concerns the gravitational equations, can be suggested only by the initial conditions.

8.17.3 Thermodynamical Continuum at Rest in Γ

Let us consider first of all the case of a thermomechanical continuum at rest in Γ. Let the coordinates be adapted to Γ and satisfy the condition:

$$\gamma_0 = -1 \quad \sim \quad g_{00} = -1, \tag{8.218}$$

always compatible and invariant under coordinate transformations like

$$y^{0\prime} = y^0 + \Psi(y^1, y^2, y^3), \quad y^{i\prime} = y^{i\prime}(y^1, y^2, y^3). \tag{8.219}$$

The field equations (8.216) are then combined with the *conservation equations* of the matter: $\nabla_\beta M^{\alpha\beta} = 0$, so that we must determine both the rest congruence of the continuum Γ_0 (coinciding with Γ) and the *proper dynamical variables.* Taking into account the expressions (8.199) for C_i and Ω_{ik}:

$$C_i = \gamma_0 \left(\tilde{\partial}_i \gamma^0 + \partial \frac{\gamma_i}{\gamma_0} \right) , \qquad \Omega_{ik} = \frac{1}{2} \gamma_0 \left[\tilde{\partial}_i \left(\frac{\gamma_k}{\gamma_0} \right) - \tilde{\partial}_k \left(\frac{\gamma_i}{\gamma_0} \right) \right] , \tag{8.220}$$

which reduce in this case ($\gamma_0 = -1$) to

$$C_i = \partial \gamma_i \, , \qquad \Omega_{ik} = \frac{1}{2} \left(\tilde{\partial}_i \gamma_k - \tilde{\partial}_k \gamma_i \right) , \tag{8.221}$$

we have the following Cauchy problem for the (*anholonomic*) variables γ_{ik}, γ_i, $H_{ik}, M_k{}^0$ and $M_0{}^0$:

$$\begin{cases} \partial \gamma_{ik} = 2H_{(ik)}, \quad \partial \gamma_i = C_i, \\ \partial H_{ik} = (\tilde{\nabla}_i + C_i) C_k + H_i{}^j H_{kj}, \\ \partial M_0{}^0 = -K M_0{}^0 - (\tilde{\nabla}_i + C_i) M_0{}^i + H_{ik} M^{ik}, \quad K = H_i{}^i, \\ \partial M_k{}^0 = -K M_k{}^0 + 2H_{[ik]} M^{0i} + C_k M_0{}^0 - (\tilde{\nabla}_i + C_i) M^i{}_k. \end{cases} \tag{8.222}$$

Furthermore, the Cauchy data on a given surface must satisfy the involutive constraints (8.217).

The energy tensor $M^{\alpha\beta}$ is not specified yet, and the form of the Cauchy problem depends on the structure of $M^{\alpha\beta}$, i.e. it is different according to the considered continuum: *dust, perfect fluid, mechanical, thermomechanical, polar, neutral or with electromagnetic field.*

As a concrete example, let us consider a *nonpolar continuum*

$$M^{\alpha\beta} = \mu_0 V^\alpha V^\beta + T^{\alpha\beta}, \tag{8.223}$$

where the proper stress tensor $T^{\alpha\beta}$ (*mechanical and thermal*) is given by

$$T^{\alpha\beta} = X^{\alpha\beta} + Q^{\alpha\beta} + \frac{\epsilon_{c,0}}{c^2} V^\alpha V^\beta, \tag{8.224}$$

with $\epsilon_{c,0} = \mu_{c,0} c^2$ the *conduction thermal energy.* $T^{\alpha\beta}$ includes *the proper mechanical stresses* $X^{\alpha\beta}$:

$$X^{\alpha\beta} = X^{\beta\alpha}, \quad X^{\alpha\beta} V_\beta = 0, \tag{8.225}$$

and *the proper thermal stresses*:

$$Q^{\alpha\beta} = \frac{1}{c^2}\left(q_0^\alpha V^\beta + q_0^\beta V^\alpha\right), \tag{8.226}$$

depending on the *thermal flux* q_0^α:

$$q_0^\alpha V_\alpha = 0. \tag{8.227}$$

The tensor $M^{\alpha\beta}$ then takes the (standard) form

$$M^{\alpha\beta} = \hat{\mu}_0 V^\alpha V^\beta + X^{\alpha\beta} + Q^{\alpha\beta}, \qquad \hat{\mu}_0 \stackrel{\text{def}}{=} \mu_0 + \mu_{c,0}, \tag{8.228}$$

where $\hat{\mu}_0$ is the *total energy density.* Therefore in the proper frame $\boldsymbol{\gamma} = \mathbf{V}/c$ and using the anholonomic basis (8.192) the components of $M^{\alpha\beta}$ are

$$M^{00} = \hat{\mu}_0 c^2, \quad M^{0i} = \frac{1}{c} q_0^i, \quad M^{ik} = X^{ik}\,, \quad (i = 1, 2, 3). \tag{8.229}$$

To determine the vector q_0^i it is necessary to add to (8.222) certain *supplementary equations*,[11] namely the *Fourier equation* (modified in the sense of Cattaneo [21])

$$\partial q_{0i} = -\nu\left(q_{0i} + k_f \widetilde{\partial}_i \theta_0\right) \tag{8.230}$$

and the *heat equation*

[11] We follow here the scheme introduced in [22] and further developed in $[23]_3$ (see p. 118). The modification to the Fourier law introduced by Cattaneo [24] has been cast in covariant form later by Kranis [25]. Different modifications also exist due to Vernotte, Eckart, etc. and they are briefly reviewed by Kranis in the abovementioned paper.

$$\mathtt{C}\,\partial\theta_0 = -\theta_0 \frac{\partial w_0^{(\mathrm{i})}}{\partial\theta_0} + Q \; . \tag{8.231}$$

In (8.230) k_f is the Fourier constant while the coefficient $1/\nu$ has the dimensions of time and represents the thermal inertia; finally $\mathtt{C}$ is the specific thermal capacity of the medium, $w_0^{(\mathrm{i})}$ is the mechanical power:

$$w_0^{(\mathrm{i})} = X^{\alpha\beta}\nabla_\alpha V_\beta = X^{ik} H_{ik} \; , \tag{8.232}$$

and Q the thermal power:

$$Q = -\nabla_\alpha q_0^\alpha = -\left(\widetilde{\nabla}_i + C_i\right) q_0^i \; . \tag{8.233}$$

Thus, $(8.222)_4$ reduces to the form

$$\begin{aligned}\frac{1}{c^2}(\hat{\mu}_0 c^2 \delta^i_k + X^i{}_k)A_i = -\Big[&\widetilde{\nabla}_i X^i{}_k - \frac{\nu}{c}(q_{0k} + k_f \widetilde{\partial}_k \theta_0) \\ &+ \frac{1}{c} K q_{0k} - \frac{2}{c} H_{[ik]} q_0^i \Big]\end{aligned} \tag{8.234}$$

and can be solved with respect to the acceleration $A_i = A_i(\hat{\mu}_0, X_{ik}, H_{ik}, q_{0i}, \theta_0)$, if the condition

$$\det \parallel \hat{\mu}_0 c^2 \delta^i_k + X^i{}_k \parallel \neq 0 \tag{8.235}$$

is satisfied. Thus the *proper formulation* of the Cauchy problem results in the following set of equations:

$$\begin{cases} \partial\gamma_{ik} = 2H_{(ik)} \; , \quad \partial\gamma_i = C_i \; , \\ \partial H_{ik} = (\widetilde{\nabla}_i + C_i)C_k + H_i{}^j H_{kj} + P_{ik} \; , \qquad P_{ik} = P_{jik}{}^j \; , \\ \partial\mu_0 = -K\mu_0 - \dfrac{1}{c^2}\left[\partial\epsilon_{c,0} + K\epsilon_{c,0} + H_{ik}X^{ik} + \dfrac{1}{c}(\widetilde{\nabla}_i + C_i)q_0^i\right] \; , \\ \partial q_{0i} = -\nu(q_{0i} + k_f \widetilde{\partial}_i \theta_0) \; , \\ \partial\theta_0 = -\dfrac{1}{\mathtt{C}}\left[\theta_0 \dfrac{\partial}{\partial\theta_0}(X^{ik}H_{ik}) + (\widetilde{\nabla}_i + C_i)q_0^i\right] \; , \end{cases} \tag{8.236}$$

with the constraints (8.217) for the initial data. Such a differential problem must then be completed with the *constitutive equations*

$$\epsilon_{c,0} = \epsilon_{c,0}(Y), \quad X_{ik} = X_{ik}(Y) \; , \tag{8.237}$$

where Y denotes the *set of the unknowns* of the system (8.236):

$$Y \equiv (\gamma_{ik},\, \gamma_i,\, H_{ik},\, \mu_0,\, q_{0i}, \theta_0) \; .$$

We notice here the typical relativistic link between the initial conditions and the constitutive equations (8.237): both *functions* $\epsilon_{c,0}$ *and* X_{ik} *are not free* but related by the constraint (8.217), because of their dependence on $\hat{\mu}_0 = \mu_0 + \epsilon_{c,0}/c^2$, C_i and H_{ik}. For a continuum without thermal flux (*ordinary continuum*) we have instead a purely mechanical scheme, i.e. the variables are only $\gamma_{ik}, \gamma_i, H_{ik}$ and μ_0.

Finally, the case of a continuum examined in an arbitrary reference frame (i.e. the case $\Gamma \neq \Gamma_0$) can be treated similarly (the whole discussion as well as all the mathematical details can be found in [23]$_3$).

We conclude this chapter noting that we have considered here only the *formulation* of the general continuum relativistic dynamics, with special attention to the *intrinsic aspects* of the associated Cauchy problem. The resulting set of differential equations, completed by assigned constitutive relations and initial data, forms a system of coupled partial differential equations with (polynomial) analytic coefficients but still containing Pfaffian derivatives (essential for the intrinsic formulation outlined above). Putting the system in its normal form and verifying the hypotheses of the Cauchy–Kowalesky theorem [26], i.e. discussing local existence of the solutions, is an open problem.

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9

Relativistic Electromagnetism in Vacuum

9.1 Introduction

After the formulation of the general axioms of special relativity, we have examined the relativistic aspects of mechanics (with the associated specific postulates) in both the schemes of material point and continuous material systems. Actually, since the general axioms of special relativity have been introduced in order to solve the incompatibility between classical mechanics and electromagnetism, it is also interesting to study the modifications induced to electromagnetism. These modifications, however, will be less important than the really revolutionary ones that occurred in the conceptual apparatus of mechanics; in fact the postulate that the light velocity is constant relative to any Galilean frame is related to the idea that Maxwell's equations are formally invariant passing from a Galilean frame to another, as required by the *extended relativity principle*. As a consequence, the electromagnetic phenomena (in vacuum) in any fixed Galilean frame are still governed by the ordinary Maxwell's equations, as we are going to discuss in detail.

Let us recall that, relative to classical physics, electromagnetism is summarized by two sets of axioms. From one side, we have Maxwell's equations in vacuum, which determine the differential relations between the *electric field* $\mathbf{E}$ and the *magnetic field* $\mathbf{H}$ and the associated sources (*charges and currents*):

$$\begin{cases} \operatorname{div}\mathbf{H} = 0\ , & \operatorname{curl}\mathbf{E} + \dfrac{1}{c}\partial_t \mathbf{H} = 0\ , \\ \operatorname{div}\mathbf{E} = 4\pi\rho, & \operatorname{curl}\mathbf{H} - \dfrac{1}{c}\partial_t \mathbf{E} = \dfrac{4\pi}{c}\mathbf{J}\ , \end{cases} \tag{9.1}$$

with ρ and $\mathbf{J}$ the charge and current density, respectively.

From the other side, we have the *Lorentz formula* for the mechanical force acting on charged matter due to the fields $\mathbf{E}$ and $\mathbf{H}$ (classically separated):

$$\mathbf{F} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}\right)\ , \tag{9.2}$$

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where e is the charge in relative motion with velocity $\mathbf{v}$. When $\mathbf{v} = 0$, (9.2) gives the electrostatic or Coulomb force.

In classical physics, (9.1) and (9.2) have no general validity (that is for any Galilean frame), but their validity is postulated in a special frame S_*, the *cosmic Ether*, and hence all the quantities appearing in (9.1) and (9.2), $\mathbf{E}$, $\mathbf{H}$, ρ, $\mathbf{J}$ and e, are invariant.

Even with this limitation, Maxwell's equations have two fundamental consequences. The first concerns the propagation speed of an electromagnetic perturbation (ordinary discontinuity waves) in vacuum: this equals the universal constant c, irrespective of the initial characteristics of the perturbation (e.g. the electromagnetic source as well as its motion with respect to the Ether, etc.). Initial conditions can influence certain properties of the perturbation, like the frequency, but not the speed, which is always (experimentally) coincident with light speed in vacuum, c. This numerical coincidence has just represented the first element in favour of the interpretation of light as an electromagnetic phenomenon. A second element has been the transversality common to both the electromagnetic waves (deduced from Maxwell's equations) and luminal waves (experimental fact).

Another important consequence of (9.1) is the *continuity equation* of the electric charge:

$$\partial_t \rho + \operatorname{div} \mathbf{J} = 0 , \qquad \mathbf{J} = \rho \mathbf{v} , \tag{9.3}$$

which expresses, for the charge, a typical property of the mass, i.e. its conservation in the absolute frame.

Electromagnetism, like mechanics, also had a number of experimental confirmations as well as many theoretical developments, connected with a pure electromagnetic field (in vacuum) or in the presence of matter, at rest with respect to the Ether, or in slow motion with respect to this. More precisely, the agreement between theory and observations is satisfying enough when the following two conditions hold: (1) *velocity* u of the laboratory, with respect to the Ether, as well as velocities v' of particles, with respect to the laboratory, are very small with respect to the light velocity: $u/c \ll 1$, $v'/c \ll 1$; (2) *instrumental precision of the first order, in* u/c *and* v'/c.

If the instrumental precision is higher and allows the evaluation of second-order effects (as the experiment of Michelson and Morley) the disagreement between theory and experiments appears, and both Newtonian mechanics and electromagnetism have to be considered in the fully relativistic context with fundamental modifications, as concerns the mechanical aspects, and less important modifications, as concerns the electromagnetic ones.

In any case, the two classically distinct physical theories: mechanics and electromagnetism, because of the different invariance properties with respect to the choice of the frame, find their proper geometrical and physical unification in the relativistic situation.

9.2 Sources and Electromagnetic Action. Axioms

We pass now to the formulation of the specific axioms of relativistic electromagnetism, by requiring preliminarily:

1. agreement with all the relativistic postulates already introduced (including the extended relativity principle), either in general or in the case of mechanics;
2. extension of classical electromagnetism, in the sense that the (classical) first-order agreement between theory and experiment is maintained.

Furthermore, as the new theory has to be formulated both in absolute and relative terms, all the physical quantities of the classical theory: *electric charge* e*, charge density* ρ*, current density* $\mathbf{J}$*, electric field* $\mathbf{E}$*, magnetic field* $\mathbf{H}$ *and Lorentz force* $\mathbf{F}$, classically defined in the absolute space, should now be introduced in any Galilean frame and, analogously to the mechanical quantities, they would have a *relative meaning*, depending on the considered Galilean frame. Thus, in a relative formulation of electromagnetism (a prioritary point of view, with respect to the absolute one), besides fixing the ingredients and the fundamental relations (field equations), one also has to specify their transformation laws.

We start considering the specific postulates, concerning sources and Lorentz force.

For the sources, that is for the electric charge because currents are derived quantities, we have the following:

Axiom I
The electric charge of a material point (or that of the generic element of charged continuous material system) is invariant (in magnitude and sign), passing from one frame to another:

$$e = e' = \text{inv.} \tag{9.4}$$

In particular we have $e = e_0$, e_0 being the *proper charge* of the particle, evaluated in the local rest frame. Axiom I gives to the electric charge a completely different role if compared with that of mass in relativity. The latter, in fact, is characterized by the law: $m\sqrt{1 - v^2/c^2} = \text{inv.} = m_0$. The validity of (9.4) is obviously sub-ordered to the agreement with experiments of all the possible consequences that can be derived from it (and from the other axioms); for instance, experiments concerning thermal effects support such a validity (see [1], p. 38). In any case, axiom I directly gives the transformation law of the charge density ρ.

To see this, let us consider the generic fluid element of a charged continuous material system. In an arbitrary Galilean frame S_{g}, the charge of such element is expressed by

$$\rho \, \mathrm{d}C = \rho \mathcal{D} \, \mathrm{d}\mathcal{C} \, , \qquad \mathrm{d}\mathcal{C} \stackrel{\text{def}}{=} \mathrm{d}y^1 \, \mathrm{d}y^2 \, \mathrm{d}y^3 \, , \tag{9.5}$$

where ρ is the relative (in S_{g}) charge density, and $\mathcal{D} \equiv \det||\partial x^i/\partial y^j||$ is the reciprocal of the relative numerical density of the particles. From (9.4) we have the following invariance property:

$$\rho \, \mathrm{d}C = \rho' \, \mathrm{d}C' \qquad \sim \qquad \rho\mathcal{D} = \rho'\mathcal{D}' = \text{inv.} \tag{9.6}$$

Furthermore, the product $\eta\mathcal{D}$ (and not $\mathcal{D}$) is also invariant, passing from one frame to another, so that we have

$$\frac{\rho}{\eta} = \frac{\rho'}{\eta'} = \rho_0 = \text{inv.} \,, \tag{9.7}$$

with ρ_0 the *proper charge density*; the latter, for a charged continuous system, plays the same role of μ_0 for neutral material continuous system; thus, like μ_0 allows the introduction of the proper 4-density of linear momentum $\mu_0\mathbf{V}$, it gives rise to the proper 4-density of current:

$$\mathbf{S} \stackrel{\text{def}}{=} \rho_0 \mathbf{V} \,, \tag{9.8}$$

which summarizes the relative current density $\mathbf{J} = \rho\mathbf{v}$ and the charge density ρ. In fact, in a given Galilean frame, the 4-velocity $\mathbf{V}$ has the ordinary decomposition $\mathbf{V} = \eta(\mathbf{v} + c\boldsymbol{\gamma})$, and (9.8) becomes

$$\mathbf{S} = \rho(\mathbf{v} + c\boldsymbol{\gamma}) \,, \qquad \rho = \eta\rho_0 \,. \tag{9.9}$$

From here, one has the relative quantities:

$$\mathbf{J} = \mathbf{S}_\Sigma = \mathbf{S} + \mathbf{S}\cdot\boldsymbol{\gamma}\boldsymbol{\gamma} \equiv P_\Sigma(\mathbf{S}) \,, \qquad \rho = -\frac{1}{c}\mathbf{S}\cdot\boldsymbol{\gamma} \,. \tag{9.10}$$

The formal analogy between (9.9) and the decomposition of the 4-momentum of a particle ($\rho \to m$, $\mathbf{J} \to \mathbf{P}$) immediately gives the transformation laws of the relative quantities ρ and $\mathbf{J}$:

$$\rho' = \frac{\sigma}{\alpha}\rho \,, \qquad \mathbf{J}' = \mathbf{J} - \frac{1 + \dfrac{\sigma}{\alpha}}{1+\alpha}\rho\mathbf{u} \,; \tag{9.11}$$

the first relation, of course, is in agreement with (9.7). From the absolute point of view, the sources are described by a timelike vector field $\mathbf{S}$; in fact, from (9.8), we have $||\mathbf{S}|| = -\rho_0^2 c^2$; a priori, it can have a temporal orientation, coinciding or not with that chosen for the space-time M_4 (and induced on the world lines of test particles). Such an orientation, in fact, specifies the sign of ρ_0 in (9.8), so that the field $\mathbf{S}$ completely describes the sources, and with no ambiguity (world lines and ρ_0):

$$\rho_0 = \frac{\epsilon}{c}\sqrt{-||\mathbf{S}||} \,, \qquad \mathbf{V} = \frac{\epsilon c}{\sqrt{-||\mathbf{S}||}}\mathbf{S} \,, \qquad (\epsilon = \pm 1) \,. \tag{9.12}$$

At this point, after introducing the electromagnetic sources (with axiom I), we should specify the electric (**E**) and magnetic fields (**H**). With an inversion of the logical order, instead, we will assign, first, the axiom concerning the force generated by **E** and **H**, which actually mediates between electromagnetism and mechanics.

Axiom II The electromagnetic action in M_4 is represented by a mechanical 4-force **K**, such that, in any Galilean frame, the associated *relative force* **F** coincides with the Lorentz force. Axiom II gives a purely mechanical meaning to the electromagnetic action ($q = 0$) and allows for **F** the validity of the Lorentz force (9.2) also in the relativistic context, extending thus the validity to any Galilean frame. Explicitly, from the general relation:

$$\mathbf{K} = \eta \left(\mathbf{F} + \frac{\mathcal{W}}{c} \boldsymbol{\gamma} \right) , \qquad \mathcal{W} = \mathbf{F} \cdot \mathbf{v} + q \, , \tag{9.13}$$

we have that axiom II is equivalent to the two conditions:

$$\mathbf{F} = e \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{H} \right) , \qquad \mathcal{W} \stackrel{\text{def}}{=} \mathbf{F} \cdot \mathbf{v} \equiv e \mathbf{E} \cdot \mathbf{v} \, . \tag{9.14}$$

Thus, in any Galilean frame and for any choice of internal coordinates (assuming $\boldsymbol{\gamma} = \mathbf{c}_0$), the components of **K** have the form

$$K_0 = -\frac{\eta}{c} \mathcal{W} = -\frac{e\eta}{c} E_i v^i \, , \qquad K_i = \eta F_i = \eta e \left[E_i + \frac{1}{c} (\mathbf{v} \times \mathbf{H})_i \right] ; \tag{9.15}$$

the problem of summarizing these components in a certain law for K_α then arises. This is suggested from the observation that such components are:

1. *proportional* to the charge e;
2. *linear function* of the velocity **v** (and hence of the 4-velocity).

It then appears quite natural to assume for K_α the following expression:

$$K_\alpha = \frac{e}{c} F_{\alpha\beta} V^\beta \, , \quad (\alpha = 0, 1, 2, 3) \, , \tag{9.16}$$

up to a factor. The quantity $F_{\alpha\beta}$ is a 2-tensor, **K** and **V** being two 4-vectors and e and c two invariant quantities (see e.g. [2], p. 22, for the tensoriality criterion); it is necessarily antisymmetric, because of the condition $K_\alpha V^\alpha = 0$ valid for any V^α. Moreover, in order to summarize the electric and magnetic fields, it should have six independent components, as it is the case for an antisymmetric 2-tensor.

9.3 Electromagnetic Tensor

Assuming that (9.16) summarizes the relations (9.15), independent of the choice of the Cartesian coordinate system, we still have to specify how $F_{\alpha\beta}$

is related to the electric and magnetic fields. Such a relation follows from a direct comparison of (9.15) and (9.16); in fact, for the various components we have $F_{0i} = -E_i$, $F_{12} = H_3$, $F_{13} = -H_2$, etc. However, the relation can be written directly in tensorial form, by using the natural decomposition introduced in Chap. 2. If $\boldsymbol{\gamma} \equiv (\gamma^\alpha)$ denotes the 4-vector along which the natural projection is taken, the decomposition of $F_{\alpha\beta}$ turns out to be

$$F_{\alpha\beta} = H_{\alpha\beta} + \gamma_\alpha E_\beta - \gamma_\beta E_\alpha, \tag{9.17}$$

where the antisymmetric property of $F_{\alpha\beta}$ has been used and the vector E_α and the antisymmetric tensor $H_{\alpha\beta}$ satisfy the conditions:

$$E_\alpha \gamma^\alpha = 0 \,, \qquad H_{\alpha\beta}\gamma^\beta = 0 \,. \tag{9.18}$$

Without any loss of generality, we will assume γ^α as a unit timelike vector: $\gamma_\alpha \gamma^\alpha = -1$, so that it represents the chosen Galilean frame. Independent of the choice of the coordinates x^α and using (9.18), we have

$$E_\alpha = F_{\alpha\beta}\gamma^\beta \,, \qquad H_{\alpha\beta} = F_{\alpha\beta} - \gamma_\alpha E_\beta + \gamma_\beta E_\alpha, \tag{9.19}$$

which represent just the electric and magnetic fields, as we are going to show in detail, a fact that motivates the notation used.

More precisely, let us assume that the Cartesian basis $\{\mathbf{c}_\alpha\}$ is adapted to S_g, in the sense that

$$\boldsymbol{\gamma} = \mathbf{c}_0 \,; \tag{9.20}$$

that is

$$\gamma^0 = 1 \,, \quad \gamma^i = 0; \quad \gamma_0 = -1 \,, \quad \gamma_i = 0 \,, \quad (i = 1, 2, 3) \,. \tag{9.21}$$

From the above conditions, (9.18) become

$$E_0 = 0 \,, \quad H_{\alpha 0} = 0 \quad \Rightarrow \quad E_\alpha = \delta^i_\alpha E_i \,, \quad H_{\alpha\beta} = \delta^i_\alpha \delta^j_\beta H_{ij} \,, \tag{9.22}$$

and the decomposition (9.17) for the sum $F_{\alpha\beta}V^\beta$ of (9.16) implies

$$F_{\alpha\beta}V^\beta = \delta^i_\alpha H_{ik} V^k - \delta^0_\alpha E_i V^i + \delta^i_\alpha E_i V^0 \,. \tag{9.23}$$

Thus, since

$$V^0 = \eta c \,, \qquad V^i = \eta v^i \,, \tag{9.24}$$

it results that, with

$$H_{ik} \equiv \begin{pmatrix} 0 & H_3 & -H_2 \\ -H_3 & 0 & H_1 \\ H_2 & -H_1 & 0 \end{pmatrix} , \tag{9.25}$$

(9.23) assumes the form

$$F_{\alpha\beta}V^{\beta} = -\delta^0_\alpha \eta \mathbf{E}\cdot\mathbf{v} + \delta^i_\alpha \eta c \left[E_i + \frac{1}{c}(\mathbf{v}\times\mathbf{H})_i\right] .$$

Equation (9.15) then follows after multiplication by e/c. Thus we have proven that, due to axiom II, the Lorentz 4-force is necessarily of type (9.16), with $F_{\alpha\beta}$ related to the electric and magnetic fields by (9.17), for any choice of the Galilean frame $S_{\rm g}(\boldsymbol{\gamma})$ and of the coordinates x^α. Using *adapted coordinates* to $S_{\rm g}$ (9.17) simplifies as

$$F_{0i} = E_i \ , \qquad F_{ik} = H_{ik} \tag{9.26}$$

or

$$F_{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & H_3 & -H_2 \\ E_2 & -H_3 & 0 & H_1 \\ E_3 & H_2 & -H_1 & 0 \end{pmatrix} . \tag{9.27}$$

Axiom II is therefore equivalent to postulating the existence of an antisymmetric 2-tensor: the electromagnetic tensor field $F_{\alpha\beta}$ which summarizes in any Galilean frame the electric and magnetic fields, according to (9.17) and the electromagnetic action by means of (9.16). Finally, the tensorial behaviour of $F_{\alpha\beta}$ implies for the components in any other coordinate system the *general transformation law*:

$$F'_{\alpha\beta} = \frac{\partial x^\rho}{\partial x'^\alpha}\frac{\partial x^\sigma}{\partial x'^\beta}F_{\rho\sigma} \ . \tag{9.28}$$

9.4 Absolute Formulation of Maxwell's Equations

We now have to specify the relativistic equations for the electromagnetic field, that is the relations between $\mathbf{E}$ and $\mathbf{H}$ (i.e. the tensor $F_{\alpha\beta}$) and the sources ρ and $\mathbf{J} = \rho\mathbf{v}$ (i.e. the current density vector S^α). We have the following:

Axiom III

The evolution of the electromagnetic field, in vacuum, is governed by the ordinary Maxwell's equations, in any Galilean frame. This is a quite natural axiom which implies, for its compatibility, that Maxwell's equations are formally invariant under Lorentz transformations and hence can have an absolute formulation in M_4. The latter is indeed possible because Maxwell's equations can be written in terms of the two fundamental ingredients: the electromagnetic field $F_{\alpha\beta}$ (and its first-order derivatives, which also have a tensorial meaning) and the current density S^α. More precisely, the ordinary Maxwell's equation (9.1) can be cast in the following form:

$$\begin{cases} \partial_\rho F_{\alpha\beta} + \partial_\alpha F_{\beta\rho} + \partial_\beta F_{\rho\alpha} = 0, \\ \partial_\beta F^{\alpha\beta} = \dfrac{4\pi}{c} S^\alpha \ , \qquad (\alpha,\beta,\rho = 0,1,2,3), \end{cases} \tag{9.29}$$

where

$$F^{\alpha\beta} = m^{\alpha\rho} m^{\beta\sigma} F_{\rho\sigma} \tag{9.30}$$

is the completely contravariant form of F.

To show that the system (9.29) is equivalent to the system (9.1), we notice first of all that system (9.29) also contains eight independent equations: $(9.29)_2$ are four equations like $(9.29)_1$, which apparently are $4^3 = 64$. This number is reduced since in $(9.29)_1$ one cannot choose two coinciding indices; in fact, if $\alpha = \beta$, they become

$$\partial_\rho F_{\alpha\alpha} + \partial_\alpha F_{\alpha\rho} + \partial_\alpha F_{\rho\alpha} = 0 \ ,$$

which is identically zero, because of the antisymmetry of $F_{\alpha\beta}$. Furthermore, the antisymmetric property of F implies that the left-hand side of $(9.25)_1$:

$$T_{\rho\alpha\beta} = \partial_\rho F_{\alpha\beta} + \partial_\alpha F_{\beta\rho} + \partial_\beta F_{\rho\alpha} \ , \tag{9.31}$$

is an *antisymmetric* 3-*tensor* itself, and hence it has only $\binom{4}{3} = 4$ independent components, as for instance those with strictly increasing indices: T_{012}, T_{013}, T_{023}, T_{123}.

With the Galilean frame S_{g} fixed and with the $F_{\alpha\beta}$ given by (9.27), it is easy to see that $(9.29)_1$ summarize the homogeneous Maxwell's equations. In fact, the latter can be obtained as indicated below:

$$\begin{cases} \left(\dfrac{1}{c}\partial_t \mathbf{H} + \mathrm{curl}\ \mathbf{E}\right)_{3,2,1} = 0 & \text{corresponding to indices 012, 013, 023,} \\ \mathrm{div}\ \mathbf{H} = 0 & \text{corresponding to indices 123.} \end{cases}$$

Concerning $(9.29)_2$, from (9.27) and (9.30), we have

$$F^{0i} = -F_{0i} = E_i \ , \qquad F^{ik} = F_{ik} = H_{ik} \ , \tag{9.32}$$

so that the contravariant components of F are given by

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & H_3 & -H_2 \\ -E_2 & -H_3 & 0 & H_1 \\ -E_3 & H_2 & -H_1 & 0 \end{pmatrix} . \tag{9.33}$$

Taking into account (9.9): $S^0 = c\rho$, $S^i = \rho v^i$, we see that $(9.29)_2$ summarize the inhomogeneous Maxwell's equations; more precisely

$$\begin{cases} \mathrm{div}\ \mathbf{E} = 4\pi\rho, & \text{corresponding to the index 0,} \\ \left(\mathrm{curl}\ \mathbf{H} - \dfrac{1}{c}\partial_t \mathbf{E}\right)_{1,2,3} = 0 & \text{corresponding to indices 1, 2, 3.} \end{cases}$$

Equation (9.29), besides representing (9.1) in any Galilean frame, are invariant under linear transformation of the x^α, and hence have a tensorial behaviour in M_4.[1] Moreover, (9.29) contain the continuity equation (9.3) which, being of Eulerian type, can be written as

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{e}) = 0 \,. \tag{9.34}$$

In fact, after differentiating $(9.29)_2$ with respect to the index α and then contracting this index, we have

$$\partial_\alpha \partial_\rho F^{\alpha\rho} = \frac{4\pi}{c} \partial_\alpha S^\alpha \,;$$

using now the symmetric property of the second derivatives $\partial_\alpha \partial_\rho$ (Schwartz theorem), as well as the antisymmetric property of F, leads to $\partial_\alpha \partial_\rho F^{\alpha\rho} = 0$, implying the scalar (invariant) condition

$$\partial_\alpha S^\alpha = 0 \,. \tag{9.35}$$

Using (9.9), the latter equation can be written in the form

$$\frac{1}{c} \partial_t S^0 + \partial_i S^i \equiv \frac{1}{c} \partial_t (\rho c) + \partial_i (\rho e^i) = 0 \,,$$

which is exactly (9.34).

Equation (9.35), having absolute meaning just as (9.34) from which it has been derived, represents the *charge conservation* in any Galilean frame and for all the evolution of a charged continuous system, a property similar to that of mass conservation of material systems. This is a different property with respect to the invariance of the charge assumed by axiom I. Therefore, in M_4, the two separated theories, classical electromagnetism and Newtonian mechanics, are naturally unified in a single theory with the same invariance properties. The most important modifications have concerned mechanics: in fact, the unification of thermal and mechanical action has been obtained through the new idea of space and time (relative, and no more absolute quantities) as well as the identification of the two concepts of mass and energy, previously distinct.

For the electromagnetic field, we have the inclusion of the electric and magnetic fields in the single electromagnetic tensor $F_{\alpha\beta}$ as well as the unification of charge and current density through the vector S^α. Moreover, there is a progress with respect to the classical situation. In fact, in the classical context every electromagnetic problem had to be formulated in the absolute space, or the Ether frame; relativistically, instead, all the Galilean frames are equally valid and indistinguishable, also for

[1] Actually $(9.29)_1$ are also invariant under general coordinate transformation because they can be written as the exterior derivative of $F_{\alpha\beta}$ (see e.g. [2]).

electromagnetic phenomena. Thus, the Galilean frame is completely available and the solution, once obtained in a certain Galilean frame, can then be automatically transferred to any other frame by simply performing a change of coordinates and using the relativistic transformation laws.

9.5 Homogeneous Form of Maxwell's Equations

Let us multiply $(9.29)_1$ (written with respect to a certain Cartesian coordinate system x^α) for the Levi-Civita indicator $\epsilon^{\sigma\rho\alpha\beta}$ and contract the indices ρ, α, β. Using the antisymmetry of both ϵ and F we have

$$3\epsilon^{\sigma\rho\alpha\beta}\partial_\rho F_{\alpha\beta} = 0 \qquad \to \qquad \partial_\rho(\epsilon^{\sigma\rho\alpha\beta}F_{\alpha\beta}) = 0 \; .$$

After introducing the *dual of* F:

$${}^*F^{\sigma\rho} \stackrel{\text{def}}{=} \frac{1}{2}\epsilon^{\sigma\rho\alpha\beta}F_{\alpha\beta} \; , \qquad (\sigma, \rho = 0, 1, 2, 3) \; , \tag{9.36}$$

which is still an antisymmetric tensor (because of the antisymmetry of ϵ), $(9.29)_1$ assume the form

$$\partial_\rho {}^*F^{\sigma\rho} = 0 \; ,$$

similar to the left-hand side of $(9.29)_2$. Thus, the standard form of Maxwell's equations is the following:

$$\partial_\rho {}^*F^{\alpha\rho} = 0, \qquad \partial_\rho F^{\alpha\rho} = \frac{4\pi}{c}S^\alpha \; , \qquad (\alpha = 0, 1, 2, 3) \; , \tag{9.37}$$

where the same differential operator (a divergence) enters, also confirming that they are eight independent equations only. ${}^*F$ is obviously formed with the electric and magnetic fields. In fact, using (9.27) and (2.139), we have

$$\begin{gathered}{}^*F^{01} = \frac{1}{2}(\epsilon^{0123}F_{23} + \epsilon^{0132}F_{32}) = \epsilon^{0123}F_{23} = F_{23}, \\ {}^*F^{02} = \epsilon^{0231}F_{31} = F_{31} \; , \qquad {}^*F^{03} = F_{12} \; ,\end{gathered}$$

that is

$${}^*F^{0i} = H^i \; , \qquad (i = 1, 2, 3) \; ; \tag{9.38}$$

analogously,

$${}^*F^{12} = F_{03} = -E_3 \; , \quad {}^*F^{23} = F_{01} = -E_1 \; , \quad {}^*F^{31} = F_{02} = -E_2,$$

or

$$-\,{}^*F^{ik} = E^{ik} \equiv \begin{pmatrix} 0 & E^3 & -E^2 \\ -E^3 & 0 & E^1 \\ E^2 & -E^1 & 0 \end{pmatrix} . \tag{9.39}$$

Finally, the tensor ${}^*F^{\alpha\beta}$ is

$$
{}^*F^{\alpha\beta} \equiv \begin{pmatrix} 0 & H^1 & H^2 & H^3 \\ -H^1 & 0 & -E^3 & E^2 \\ -H^2 & E^3 & 0 & -E^1 \\ -H^3 & -E^2 & E^1 & 0 \end{pmatrix} , \tag{9.40}
$$

and its completely covariant form ${}^*F_{\alpha\beta}$ is obtained by changing sign to the elements of the first row and the first column in the table (9.40); the components $F^{\alpha\beta}$ follow, instead, by replacing in the table (9.40) $\mathbf{H}$ with $\mathbf{E}$ and E^{ik} with $-H^{ik}$ simultaneously.

The matrix representation (9.40) can also be written in a more compact form, using the definition (9.36). It is necessary to consider the natural decomposition of the Levi-Civita indicator with respect to the vector $\boldsymbol{\gamma}$ characterizing the frame.[2] Such a decomposition, due to the antisymmetric property of $\epsilon^{\rho\sigma\alpha\beta}$, is necessarily *linear* in γ^α:

$$
\epsilon^{\rho\sigma\alpha\beta} = \tilde{\epsilon}^{\rho\sigma\alpha\beta} + \gamma^\sigma \tilde{\epsilon}^{\rho\alpha\beta} - \gamma^\rho \tilde{\epsilon}^{\sigma\alpha\beta} + \gamma^\alpha \tilde{\epsilon}^{\sigma\rho\beta} - \gamma^\beta \tilde{\epsilon}^{\sigma\rho\alpha} , \tag{9.41}
$$

where the tensors $\tilde{\epsilon}^{\rho\sigma\alpha\beta}$ and $\tilde{\epsilon}^{\rho\alpha\beta}$ (with rank 4 and 3, respectively) are antisymmetric and spatial, in the sense that they satisfy the following conditions:

$$
\tilde{\epsilon}^{\rho\sigma\alpha\beta}\gamma_\beta = 0 , \qquad \tilde{\epsilon}^{\rho\alpha\beta}\gamma_\beta = 0 . \tag{9.42}
$$

Furthermore, using adapted coordinates to $S_{\rm g}$ ($\gamma^0 = 1, \gamma^i = 0$), $(9.42)_1$ reduces to $\tilde{\epsilon}^{\rho\sigma\alpha 0} = 0$ and, because of its antisymmetry, it follows that $\tilde{\epsilon}^{\rho\sigma\alpha\beta}$ vanishes identically:

$$
\tilde{\epsilon}^{\rho\sigma\alpha\beta} = 0. \tag{9.43}
$$

We notice that (9.43), directly verified using adapted coordinates, holds in any coordinate system, because $\tilde{\epsilon}^{\rho\sigma\alpha\beta}$ is a tensor. Thus, the decomposition (9.41) becomes

$$
\epsilon^{\rho\sigma\alpha\beta} = \gamma^\sigma \tilde{\epsilon}^{\rho\alpha\beta} - \gamma^\rho \tilde{\epsilon}^{\sigma\alpha\beta} + \gamma^\alpha \tilde{\epsilon}^{\sigma\rho\beta} - \gamma^\beta \tilde{\epsilon}^{\sigma\rho\alpha} , \tag{9.44}
$$

i.e. the only surviving quantity is the spatial and antisymmetric tensor $\tilde{\epsilon}^{\rho\alpha\beta}$. Such a tensor in a system of adapted coordinates has only nonvanishing components of the form $\tilde{\epsilon}^{ijk}$, with $i, j, k = 1, 2, 3$, and it assumes, in Σ, a role similar to that played by the tensor (2.139) in M_4: it is called the *spatial Levi-Civita indicator*.

Therefore, using (9.17) and (9.18), (9.36) implies the following decomposition for ${}^*F^{\alpha\beta}$:

$$
{}^*F^{\alpha\beta} = \frac{1}{2}(\gamma^\sigma \tilde{\epsilon}^{\rho\alpha\beta} H_{\alpha\beta} - \gamma^\rho \tilde{\epsilon}^{\sigma\alpha\beta} H_{\alpha\beta} - \tilde{\epsilon}^{\sigma\rho\beta} E_\beta - \tilde{\epsilon}^{\sigma\rho\alpha} E_\alpha) ,
$$

[2] Actually one should consider the Ricci tensor when the coordinates are not Cartesian. Here we have assumed Cartesian coordinates and the Ricci tensor coincides with the Levi-Civita indicator.

that is

$$ {}^*F^{\sigma\rho} = -E^{\sigma\rho} + \gamma^\sigma H^\rho - \gamma^\rho H^\sigma \ , \tag{9.45} $$

where

$$ E^{\sigma\rho} \stackrel{\text{def}}{=} \tilde{\epsilon}^{\sigma\rho\alpha} E_\alpha \ , \qquad H^\rho \stackrel{\text{def}}{=} \frac{1}{2}\tilde{\epsilon}^{\rho\alpha\beta} H_{\alpha\beta} \ . \tag{9.46} $$

Equation (9.45), which gives the electric and magnetic fields in terms of ${}^*F$ and $\boldsymbol{\gamma}$ (i.e. the frame S_g):

$$ H^\rho = -\gamma_\sigma {}^*F^{\sigma\rho} \ , \qquad E^{\sigma\rho} = \gamma^\sigma H^\rho - \gamma^\rho H^\sigma - {}^*F^{\sigma\rho} \ , \tag{9.47} $$

has a general validity, like (9.17). In a system of adapted coordinates it summarizes (9.40). Finally, we notice that passing from $F^{\alpha\beta}$ to ${}^*F^{\alpha\beta}$ we have the replacements $H_{\alpha\beta} \to -E_{\alpha\beta}$ and $E_\alpha \to H_\alpha$, as already stated.

9.6 Transformation Laws of Electric and Magnetic Fields

Independent of the choice of the coordinates x^α the decomposition (9.17) of the electromagnetic field is invariant with respect to the choice of the Galilean frame S_g (specified by $\boldsymbol{\gamma}$). The decomposition of $F_{\alpha\beta}$ along the unit timelike vector $\boldsymbol{\gamma}'$ of another Galilean frame S'_g is therefore completely similar to (9.17):

$$ F_{\alpha\beta} = H'_{\alpha\beta} + \gamma'_\alpha E'_\beta - \gamma'_\beta E'_\alpha \ , \tag{9.48} $$

with the limitations

$$ E'_\alpha \gamma'^\alpha = 0 \ , \qquad H'_{\alpha\beta}\gamma'^\beta = 0 \ , \qquad H'_{\alpha\beta} = -H'_{\beta\alpha} \ . \tag{9.49} $$

We thus have the following (local) *invariance property*:

$$ H'_{\alpha\beta} + \gamma'_\alpha E'_\beta - \gamma'_\beta E'_\alpha = H_{\alpha\beta} + \gamma_\alpha E_\beta - \gamma_\beta E_\alpha = \text{inv.} \ , \tag{9.50} $$

which gives the relation between the electric and magnetic fields, relative to the two frames. In fact, by using the relations

$$ \boldsymbol{\gamma}' = \frac{1}{\alpha}\left(\boldsymbol{\gamma} + \frac{1}{c}\mathbf{u}\right) \ , \quad \mathbf{c}'_1 = \frac{1}{\alpha}\left(\mathbf{c}_1 + \frac{u}{c}\gamma\right) \ , \quad \alpha = \sqrt{1 - \frac{u^2}{c^2}} \ , \tag{9.51} $$

or in components:

$$ \gamma'^\beta = \frac{1}{\alpha}\left(\gamma^\beta + \frac{u^\beta}{c}\right) \ , \qquad (\beta = 0, 1, 2, 3) \ . \tag{9.52} $$

Multiplying (9.50) by γ'^α, contracting and then using (9.52) as well as (9.18) and (9.49) lead to

$$ -E'_\beta = \frac{1}{\alpha}\left[-E_\beta + \frac{1}{c}(-\gamma_\beta u^\alpha E_\alpha + u^\alpha H_{\alpha\beta})\right] \ , $$

that is

$$E'_\beta = \frac{1}{\alpha}\left(E_\beta + \frac{1}{c}H_{\beta\alpha}u^\alpha + \frac{1}{c}\mathbf{u}\cdot\mathbf{E}\gamma_\beta\right) .$$

This formula can be expressed in S_{g} using a system of adapted coordinates (so that $E_0 = 0$, $H_{0\alpha} = 0$, $\gamma_0 = -1$) and recalling table (9.25); it results in

$$E'_0 = -\frac{1}{\alpha c}\mathbf{u}\cdot\mathbf{E} , \qquad E'_i = \frac{1}{\alpha}\left[E_i + \frac{1}{c}(\mathbf{u}\times\mathbf{H})_i\right] .$$

After contracting these components now with the vectors $\mathbf{c}_\alpha$ ($\mathbf{c}_0 = \boldsymbol{\gamma}$) and using the notation $\eta = 1/\alpha$, we have

$$\mathbf{E}' = \frac{1}{\alpha}\left(\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{H} + \frac{1}{c}\mathbf{u}\cdot\mathbf{E}\boldsymbol{\gamma}\right) . \tag{9.53}$$

This formula holds in M_4 and hence cannot be used directly for measurements performed in different frames. However, we can consider the *isometric boost* of Σ' (to which $\mathbf{E}'$ belongs) on Σ (to which $\mathbf{E}$ and $\mathbf{u}\times\mathbf{H}$ belong); this procedure is equivalent to interpret *the components of* $\mathbf{E}'$ *along the basis* $\mathbf{c}'_i$ *of* Σ' *as components with respect to the basis* $\mathbf{c}_i$ *of* Σ.

We thus proceed to evaluate the components $\mathbf{E}'\cdot\mathbf{c}'_i$, starting from (9.53). Using $(9.51)_2$ and the relations $\mathbf{c}'_{2,3} = \mathbf{c}_{2,3}$ and $\mathbf{u} = u\mathbf{c}_1$, we have

$$\begin{cases} \mathbf{E}'\cdot\mathbf{c}'_1 = \dfrac{1}{\alpha^2}(E_1 - \beta^2 E_1) = E_1, \\ \mathbf{E}'\cdot\mathbf{c}'_{2,3} = \dfrac{1}{\alpha^2}\left[E_{2,3} + \dfrac{1}{c}(\mathbf{u}\times\mathbf{H})_{2,3}\right]. \end{cases} \tag{9.54}$$

Equation $(9.54)_1$ can then be written as

$$E_1 = \frac{1}{\alpha}E_1 + \left(1 - \frac{1}{\alpha}\right)E_1 = \frac{1}{\alpha}E_1 + \frac{\alpha^2 - 1}{\alpha(\alpha+1)}E_1 = \frac{1}{\alpha}E_1 - \frac{1}{c^2}\frac{\mathbf{u}\cdot\mathbf{E}}{\alpha(1+\alpha)}u ;$$

then contracting (9.54) with the basis vectors $\mathbf{c}_i$ of Σ, we have the *representation of* $\mathbf{E}'$ *in* S_{g} (which we still denote by $\mathbf{E}'$):

$$\mathbf{E}' = \frac{1}{\alpha}\left(\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{H} - \frac{1}{c^2}\frac{\mathbf{u}\cdot\mathbf{E}}{1+\alpha}\mathbf{u}\right) . \tag{9.55}$$

A similar relation can be derived for $\mathbf{H}$, using the invariant decomposition (9.45) and repeating the above procedure. This is equivalent to replacing $H_{\alpha\beta}$ by $-E_{\alpha\beta}$ and E_α by H_α. As for (9.55), we have the *general relation*:

$$\mathbf{H}' = \frac{1}{\alpha}\left(\mathbf{H} - \frac{1}{c}\mathbf{u}\times\mathbf{E} - \frac{1}{c^2}\frac{\mathbf{u}\cdot\mathbf{H}}{1+\alpha}\mathbf{u}\right) . \tag{9.56}$$

It is meaningless to perform the limit $c \to \infty$ to obtain the classical relations corresponding to (9.55) and (9.56); in fact, in the framework of ordinary electromagnetism, the electric and magnetic fields only live in the Ether frame. In other words, the classical situation of a Galilean relativity is only compatible with a *static theory of electromagnetism*, with $\mathbf{E}$ and $\mathbf{H}$ having an invariant meaning with respect to the choice of the Galilean frame; in addition, (9.1) and (9.2) do not contain anymore $1/c$ terms.

In the relativistic context, instead, as from (9.55) and (9.56), the fields $\mathbf{E}'$ and $\mathbf{H}'$ are functions of both $\mathbf{E}$ and $\mathbf{H}$, exactly as are the force and the thermal power in mechanics. Equations (9.55) and (9.56) show that, if in a frame only the electric field is present: $\mathbf{H} = 0$, a magnetic field will appear in any other frame: $\mathbf{H}' = -1/(\alpha c)\mathbf{u} \times \mathbf{E}$, even if very small. Equivalently, the condition $\mathbf{H} = 0$ (or $\mathbf{E} = 0$) has no absolute meaning, and hence from a relativistic point of view a pure theory of the electric field or the magnetic field has no meaning at all.

9.7 Invariants

The electromagnetic field $F_{\alpha\beta}$ can be interpreted, in M_4, as a vectorial map, and hence it has a set of determined eigenvalues and eigenvectors. It is convenient to consider the mixed form: $F^{\alpha}{}_{\beta}$, typical for a vectorial map, which has only a sign variation with respect to table (9.27) in the first row:

$$F^{0}{}_{\beta} = -F_{0\beta} \ , \quad F^{i}{}_{\beta} = F_{i\beta} \ .$$

Thus, we have

$$F^{\alpha}{}_{\beta} \equiv \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & H_3 & -H_2 \\ E_2 & -H_3 & 0 & H_1 \\ E_3 & H_2 & -H_1 & 0 \end{pmatrix} ; \tag{9.57}$$

with $F^{\alpha}{}_{\beta}$ are associated the following four invariants:

$$\begin{cases} I_1 = F^{\alpha}{}_{\alpha} = \mathrm{Tr} F, \\ I_2 = \dfrac{1}{2}\delta^{\rho\sigma}_{\alpha\beta} F^{\alpha}{}_{\rho} F^{\beta}{}_{\sigma} = -\dfrac{1}{2} F^{\alpha}{}_{\rho} F^{\rho}{}_{\alpha} = -\dfrac{1}{2} I_1(F^2), \\ I_3 = \dfrac{1}{3!}\delta^{\rho\sigma\nu}_{\alpha\beta\mu} F^{\alpha}{}_{\rho} F^{\beta}{}_{\sigma} F^{\mu}{}_{\nu}, \\ I_4 = \det ||F^{\alpha}{}_{\beta}||, \end{cases} \tag{9.58}$$

where $\delta^{\beta_1 \cdots \beta_k}_{\alpha_1 \cdots \alpha_k}$ is the *generalized Kronecker tensor* already defined in Chap. 2:

$$\delta^{\alpha_1 \cdots \alpha_k}_{\beta_1 \cdots \beta_k} = k! \delta^{\alpha_1}_{[\beta_1} \cdots \delta^{\alpha_k}_{\beta_k]} \ . \tag{9.59}$$

Only two of these invariants are meaningful, since

$$I_1 = 0 \ , \qquad I_3 = 0. \tag{9.60}$$

Equation $(9.60)_1$ is evident; to show $(9.60)_2$ we recall that any third order antisymmetric matrix has always null determinant:

$$I_3 = \begin{vmatrix} 0 & E_1 & E_2 \\ E_1 & 0 & H_3 \\ E_2 & -H_3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & E_1 & E_3 \\ E_1 & 0 & -H_2 \\ E_3 & H_2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & E_2 & E_3 \\ E_2 & 0 & H_1 \\ E_3 & -H_1 & 0 \end{vmatrix}$$
$$+ \begin{vmatrix} 0 & H_3 & -H_2 \\ -H_3 & 0 & H_1 \\ H_2 & -H_1 & 0 \end{vmatrix}$$
$$= E_1(H_3E_2 - H_3E_2) + E_1(-H_2E_3 + H_2E_3) + E_2(H_1E_3 - H_1E_3) = 0 \ .$$

The other invariants I_2 and I_4 are nonzero:

$$I_2 = -E_1^2 - E_2^2 - E_3^2 + H_3^2 + H_2^2 + H_1^2$$

or

$$I_2 = H^2 - E^2 \ ; \tag{9.61}$$

$$I_4 = -E_1 \begin{vmatrix} E_1 & H_3 & -H_2 \\ E_2 & 0 & H_1 \\ E_3 & -H_1 & 0 \end{vmatrix} + E_2 \begin{vmatrix} E_1 & 0 & -H_2 \\ E_2 & -H_3 & H_1 \\ E_3 & H_2 & 0 \end{vmatrix} - E_3 \begin{vmatrix} E_1 & 0 & H_3 \\ E_2 & -H_3 & 0 \\ E_3 & H_2 & -H_1 \end{vmatrix}$$
$$= -E_1H_1\mathbf{E}\cdot\mathbf{H} - E_2H_2\mathbf{E}\cdot\mathbf{H} - E_3H_3\mathbf{E}\cdot\mathbf{H} \ ,$$

that is

$$I_4 = -(\mathbf{E}\cdot\mathbf{H})^2 \le 0 \ . \tag{9.62}$$

We then have the following invariance properties:

$$H^2 - E^2 = H'^2 - E'^2 = \text{inv.} \ , \qquad \mathbf{E}\cdot\mathbf{H} = \mathbf{E}'\cdot\mathbf{H}' = \text{inv.} \ , \tag{9.63}$$

which can also be directly verified using the transformation formulas (9.55) and (9.56).

Thus, for an electromagnetic field, we have to distinguish between the general case: $I_{2,4} \neq 0$, and the special case in which one or both the invariants vanish. When $I_4 = 0$, the electric and magnetic fields are mutually orthogonal in any Galilean frame; if instead $I_2 = 0$, then the two vectors $\mathbf{E}$ and $\mathbf{H}$ have the same magnitude in any frame and hence *they are always both present.*

If there exists a Galilean frame in which the electric field (or the magnetic field) vanishes, it is necessarily $I_4 = 0$, but I_2 should be nonvanishing; otherwise, the whole electromagnetic field vanishes identically. When both the invariants are null, the electromagnetic field is said to be *singular or radiative*; in this case,

$$\mathbf{E}\cdot\mathbf{H} = 0 \ , \qquad H^2 = E^2 \neq 0, \qquad \forall S_{\mathrm{g}} \ . \tag{9.64}$$

The invariants (9.61) and (9.62) can also be obtained by the products of $\mathbf{F}$ with itself or its dual $^*\mathbf{F}$. In fact, from (9.17):

$$F_{\alpha\beta} = H_{\alpha\beta} + \gamma_\alpha E_\beta - \gamma_\beta E_\alpha \ , \qquad F^{\alpha\beta} = H^{\alpha\beta} + \gamma^\alpha E^\beta - \gamma^\beta E^\alpha \ ,$$

one gets $F_{\alpha\beta}F^{\alpha\beta} = H_{\alpha\beta}H^{\alpha\beta} - E_\beta E^\beta - E_\alpha E^\alpha$, and using adapted coordinates (without any loss of generality, because the product $F_{\alpha\beta}F^{\alpha\beta}$ is invariant), the latter becomes

$$F_{\alpha\beta}F^{\alpha\beta} = 2(H^2 - E^2) \equiv 2I_2 \ . \tag{9.65}$$

Analogously, from (9.45): ${}^*F^{\alpha\beta} = -E^{\alpha\beta} + \gamma^\alpha H^\beta - \gamma^\beta H^\alpha$, one has

$$F_{\alpha\beta}{}^*F^{\alpha\beta} = -H_{\alpha\beta}E^{\alpha\beta} - E_\beta H^\beta - E_\alpha H^\alpha,$$

which using (9.25) and (9.39) becomes

$$F_{\alpha\beta}{}^*F^{\alpha\beta} = -4\,\mathbf{E}\cdot\mathbf{H} \equiv -4\sqrt{-I_4} \ . \tag{9.66}$$

9.8 Energy Tensor of the Electromagnetic Field

From Maxwell's equations (9.29)

$$\partial_\rho F_{\alpha\beta} + \partial_\alpha F_{\beta\rho} + \partial_\beta F_{\rho\alpha} = 0 \ , \qquad \partial_\rho F^{\alpha\rho} = \frac{4\pi}{c} S^\alpha \ , \tag{9.67}$$

one can derive evolution equations similar to those of a continuous system. More precisely, consider the evolution problem of a *charged continuous system*, starting from given initial and boundary conditions. In M_4 the continuum follows a world tube $\mathcal{T}$, characterized by the vector field $\mathbf{S} = \rho_0 \mathbf{V}$ describing the distribution of both charges and currents from an absolute point of view. In turn, such a distribution generates in M_4 (in the interior as well as the exterior parts of $\mathcal{T}$) an electromagnetic field $F_{\alpha\beta}$, satisfying (9.67) and constraining the motion of the continuum itself through the Lorentz force. For the generic element of the continuum one then has an autoinduced action represented, in agreement with (9.16), by the elementary force $\mathrm{d}K_\alpha$:

$$\mathrm{d}K_\alpha = \frac{1}{c}\mathrm{d}e F_{\alpha\beta}V^\beta = \frac{1}{c}\rho_0 \mathrm{d}C_0 F_{\alpha\beta}V^\beta \ ,$$

with $\mathrm{d}C_0$ the proper volume element. After introducing the proper density of 4-force

$$\rho_0 f_\alpha \stackrel{\text{def}}{=} \frac{\mathrm{d}K_\alpha}{\mathrm{d}C_0} \ , \tag{9.68}$$

one then gets the following law:

$$\rho_0 f_\alpha = \frac{1}{c} F_{\alpha\beta} S^\beta \ , \tag{9.69}$$

which specifies the dependence of $\rho_0 f_\alpha$ on both the electromagnetic field $F_{\alpha\beta}$ and the sources S^β. Hence, in $\mathcal{T}$ one has new vector fields for the autoinduced mechanical action, in the sense that the continuum (conductor and charges) generates the electromagnetic field through Maxwell's equations, and this, in turn, affects the motion of the continuum itself through the action (9.69); the latter is, of course, sub-ordered to Maxwell's equations (9.67). We can then eliminate the current density (9.69) and *finally express f_α in terms of $F_{\alpha\beta}$ and its first derivatives.* More precisely, such a dependence can be put in a *divergence form*:

$$\rho_0 f_\alpha = -\partial_\rho E_\alpha{}^\rho \ , \tag{9.70}$$

where $E_\alpha{}^\rho$ is a 2-tensor built up by the electromagnetic field and still to be determined. To prove (9.70) let us start from $(9.67)_2$, which reduces (9.69) to the form

$$\rho_0 f_\alpha = \frac{1}{4\pi} F_{\alpha\beta}\partial_\rho F^{\beta\rho} = \frac{1}{4\pi}[\partial_\rho(F_{\alpha\beta}F^{\beta\rho}) - F^{\beta\rho}\partial_\rho F_{\alpha\beta}] \ . \tag{9.71}$$

Transform then the last term in (9.71) using $(9.67)_1$:

$$-F^{\beta\rho}\partial_\rho F_{\alpha\beta} = F^{\beta\rho}(\partial_\alpha F_{\beta\rho} + \partial_\beta F_{\rho\alpha}) = F^{\beta\rho}\partial_\alpha F_{\beta\rho} + F^{\beta\rho}\partial_\beta F_{\rho\alpha} \ ,$$

that is, exchanging the indices ρ and β in the last product

$$-F^{\beta\rho}\partial_\rho F_{\alpha\beta} = F^{\beta\rho}\partial_\alpha F_{\beta\rho} + F^{\rho\beta}\partial_\rho F_{\beta\alpha} \ ,$$

leads to

$$-2F^{\beta\rho}\partial_\rho F_{\alpha\beta} = F^{\beta\rho}\partial_\alpha F_{\beta\rho} \equiv \frac{1}{2}\partial_\alpha(F^{\beta\rho}F_{\beta\rho}) \ .$$

Equation (9.71) thus becomes

$$\rho_0 f_\alpha = \frac{1}{4\pi}\left[\partial_\rho(F_{\alpha\beta}F^{\beta\rho}) + \frac{1}{4}\partial_\alpha(F^{\beta\rho}F_{\beta\rho})\right] \ ,$$

which coincides with (9.70) after defining $E_\alpha{}^\beta$ as

$$E_\alpha{}^\beta \stackrel{\text{def}}{=} \frac{1}{4\pi}\left(F_{\alpha\rho}F^{\rho\beta} - \frac{1}{4}\delta_\alpha^\beta F^{\rho\sigma}F_{\rho\sigma}\right) \equiv \frac{1}{4\pi}\left(F_{\alpha\rho}F^{\rho\beta} - \frac{1}{2}\delta_\alpha^\beta I_2\right) \ . \tag{9.72}$$

The tensor $E_\alpha{}^\beta$ is called the *energy–momentum tensor* of the electromagnetic field $F_{\alpha\beta}$. It has vanishing trace:

$$I_1(E) \equiv E_\alpha{}^\alpha = 0 \ , \tag{9.73}$$

and is symmetric, as one can easily see by considering the contravariant (or covariant) form:

$$4\pi E^{\alpha\rho} = F^\alpha{}_\beta F^{\rho\beta} - \frac{1}{4}m^{\alpha\rho}F^{\beta\sigma}F_{\beta\sigma} \ . \tag{9.74}$$

This is a quadratic homogeneous function of the electromagnetic field $F_{\alpha\beta}$, and hence it is defined either in $\mathcal{T}$, where it satisfies the conditions (9.70), or in the exterior of $\mathcal{T}$, where one has the conservation conditions

$$\partial_\rho E^{\alpha\rho} = 0 \ . \tag{9.75}$$

9.9 Splitting of the Energy Tensor of the Electromagnetic Field

In relative terms, the energetic tensor (9.74) will be expressed, as $F_{\alpha\beta}$, in terms of electric and magnetic fields. In fact, using the decomposition (9.17) of $F^{\alpha}{}_{\beta}$:

$$F^{\alpha}{}_{\beta} = H^{\alpha}{}_{\beta} + \gamma^{\alpha} E_{\beta} - \gamma_{\beta} E^{\alpha},$$

one has, from Eq. (9.18),

$$F^{\alpha}{}_{\beta} F^{\rho\beta} = H^{\alpha}{}_{\beta} H^{\rho\beta} + \gamma^{\rho} H^{\alpha}{}_{\beta} E^{\beta} + \gamma^{\alpha} H^{\rho\beta} E_{\beta} + \gamma^{\alpha}\gamma^{\rho} E_{\beta} E^{\beta} - E^{\alpha} E^{\rho} \ .$$

Taking into account (9.65), we then have

$$\begin{aligned} 4\pi E^{\alpha\rho} = {} & H^{\alpha}{}_{\beta} H^{\rho\beta} + \gamma^{\alpha} H^{\rho\beta} E_{\beta} \\ & + \gamma^{\rho} H^{\alpha}{}_{\beta} E^{\beta} - E^{\alpha} E^{\rho} + E^2 \gamma^{\alpha}\gamma^{\rho} - \frac{1}{2}(H^2 - E^2) m^{\alpha\rho} \ . \end{aligned}$$

Consider now the decomposition of the metric tensor; because of its symmetry, we have

$$m^{\alpha\beta} = \tilde{m}^{\alpha\beta} + \gamma^{\alpha}\tilde{m}^{\beta} + \gamma^{\beta}\tilde{m}^{\alpha} + \tilde{m}\gamma^{\alpha}\gamma^{\beta} \ ,$$

with the conditions $\tilde{m}^{\alpha\beta}\gamma_{\alpha} = 0$, $\tilde{m}^{\alpha}\gamma_{\alpha} = 0$. Contracting by $\gamma^{\alpha}\gamma^{\beta}$ implies $-1 = \tilde{m}$; contracting then by γ_{β} leads to $\gamma^{\alpha} = -\tilde{m}^{\alpha} + \gamma^{\alpha}$, so that $\tilde{m}^{\alpha} = 0$. Finally, the decomposition of the tensor $m^{\alpha\beta}$ is

$$m^{\alpha\beta} = \tilde{m}^{\alpha\beta} - \gamma^{\alpha}\gamma^{\beta} \ , \tag{9.76}$$

and (9.76) becomes

$$4\pi E^{\alpha\rho} = M^{\alpha\rho} + \gamma^{\alpha} P^{\rho} + \gamma^{\rho} P^{\alpha} + \mathcal{W}\gamma^{\alpha}\gamma^{\rho} \ , \tag{9.77}$$

since

$$\begin{cases} M^{\alpha\rho} \stackrel{\text{def}}{=} -E^{\alpha}E^{\rho} + H^{\alpha}{}_{\beta}H^{\rho\beta} - \dfrac{1}{2}(H^2 - E^2)\tilde{m}^{\alpha\rho} & \text{Maxwell's stress tensor,} \\ P^{\rho} \stackrel{\text{def}}{=} H^{\rho\beta}E_{\beta} & \text{Poynting vector,} \\ \mathcal{W} \stackrel{\text{def}}{=} \dfrac{1}{2}(E^2 + H^2) & \textit{electromagnetic energy.} \end{cases} \tag{9.78}$$

Equation (9.77) represents the natural decomposition of the tensor $4\pi E^{\alpha\rho}$ along $\boldsymbol{\gamma}$ and onto Σ, the hypersurface normal to $\boldsymbol{\gamma}$; the tensor $M^{\alpha\rho}$, like $E^{\alpha\rho}$, is symmetric and it is spatial as the vector P^{α}:

$$M^{\alpha\rho}\gamma_{\rho} = 0, \qquad P^{\rho}\gamma_{\rho} = 0 \ .$$

Furthermore, using a system of adapted coordinates, from (9.76) one gets

$$\tilde{m}^{ik} = m^{ik} = \delta^{ik}, \qquad (i,k=1,2,3) \qquad \text{spatial metric} . \tag{9.79}$$

From $(9.78)_{1,2}$ and using (9.25) one obtains the components of the Poynting vector:

$$P^i = (\mathbf{E}\times\mathbf{H})^i \quad \sim \quad \mathbf{P} = \mathbf{E}\times\mathbf{H} , \tag{9.80}$$

as well as those of Maxwell's stress tensor:

$$M^{ik} = -E^i E^k - H^i H^k + \mathcal{W}\delta^{ik} . \tag{9.81}$$

We notice that $\mathbf{P}$ is an eigenvector of M^{ik}, associated with the eigenvalue $\mathcal{W}$: $M^{ik}P_k = \mathcal{W}P^i$. The decomposition (9.77) as well as the various associated spatial quantities all have a relative meaning, that is depending on the chosen Galilean frame $\boldsymbol{\gamma}$. However, by changing the frame one would have a decomposition similar to (9.77) and (9.78), apart from the addition of an overall prime. At this point, given the general formulas (9.55) and (9.56), one should determine the transformation laws of the various quantities (9.78); in particular one can show that *there exist an infinite number of Galilean frames in which the Poynting vector vanishes.*

We notice that the relative law (9.77) for the energetic tensor of an electromagnetic field is formally analogous to those of a polar continuous medium, with heat conduction; this analogy allows to compare the two schemes, even if so different. Moreover, besides the analogies, we recall that the energetic tensor (9.77) is also defined in the exterior of the world tube $\mathcal{T}$ associated with the sources; for a material continuum the energetic tensor has instead no meaning in the exterior region, that is outside matter.

9.10 Spectral Analysis of the Electromagnetic Tensor

Definition (9.72) implies that the spectral analysis of the energetic tensor $E^{\alpha\beta}$ is strictly related to that of the electromagnetic field $F_{\alpha\beta}$, which has only two nonvanishing invariants. In fact, the characteristic equation for $F_{\alpha\beta}$ is biquadratic:

$$\lambda^4 + I_2\lambda^2 + I_4 = 0 , \tag{9.82}$$

with

$$\begin{cases} I_2 = \dfrac{1}{2}F_{\alpha\beta}F^{\alpha\beta} = H^2 - E^2, \\ I_4 = -\dfrac{1}{4}\left(\dfrac{1}{2}F_{\alpha\beta}{}^*F^{\alpha\beta}\right)^2 = -(\mathbf{E}\cdot\mathbf{H})^2 \leq 0. \end{cases} \tag{9.83}$$

Equation (9.82) in the case $I_2^2 - 4I_4 \neq 0$ (that is, excluding the case $I_2 = I_4 = 0$) has two roots for $x = \lambda^2$, one positive and the other negative:

$$x_1 = \frac{1}{2}\left(-I_2 + \sqrt{I_2^2 - 4I_4}\right) > 0,\ x_2 = -\frac{1}{2}\left(I_2 + \sqrt{I_2^2 - 4I_4}\right) < 0 , \tag{9.84}$$

so that the electromagnetic field has two real opposite eigenvalues $\pm\lambda_1$ and two purely imaginary eigenvalues $\pm\,\mathrm{i}\lambda_2$:

$$\begin{cases} \lambda_1 = \sqrt{\dfrac{1}{2}\left(-I_2 + \sqrt{I_2^2 - 4I_4}\right)} > 0, \\ \lambda_2 = \sqrt{\dfrac{1}{2}\left(I_2 + \sqrt{I_2^2 - 4I_4}\right)} > 0. \end{cases} \tag{9.85}$$

Moreover, $\mathbf{F}$ can be decomposed (even if not uniquely, see Chap. 2) in the sum of two orthogonal bivectors $\mathbf{F}_1$ and $\mathbf{F}'_1$:

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}'_1 \ , \tag{9.86}$$

with $\mathbf{F}_1 \in \Pi$, a *hyperbolic* 2-*plane* invariant for $\mathbf{F}_1$, and $\mathbf{F}'_1 \in \Pi'$, an *elliptic* 2-*plane* invariant for $\mathbf{F}'_1$, being Π and Π' orthogonal to each other: $\mathbf{F}_1(\Pi') = 0$ and $\mathbf{F}'_1(\Pi) = 0$.

From the vectorial map point of view, $F_1 \in \Pi$ (hyperbolic) has two real null eigenvectors, while $F'_1 \in \Pi'$ (elliptic) has no real eigenvectors. Equivalently, $F_{\alpha\beta}$ admits, in general, only two real eigenvectors, both of them null; these are associated to the real eigenvalues $\pm\lambda_1$. Such two directions reduce to a single one, when Π is parabolic: $I_2(F_1) = 0$; in this case, Π' also becomes parabolic, i.e. $I_2(F'_1) = 0$.

However, with respect to the orthogonal decomposition (9.86) the following general relations hold:

$$I_2 = I_2(F_1) + I_2(F'_1) \ , \qquad I_4 = I_2(F_1) \cdot I_2(F'_1) \ , \tag{9.87}$$

so that the two isotropic directions of Π reduce to a single one if and only if $I_2 = 0$ and $I_4 = 0$, that is $\lambda_1 = 0$ and $\lambda_2 = 0$. This case is called *radiative*: $F_{\alpha\beta}$ reduces to a bivector ($I_4 = 0$) of parabolic type ($I_2 = 0$). In other words, the above-mentioned property of the electromagnetic field can also be expressed as follows: every electromagnetic field $F_{\alpha\beta}$ admits only two eigendirections, both isotropic, which coincide only in the radiative (singular) case.

Excluding the singular case, the components $F^\alpha{}_\beta$ of $\mathbf{F}$ (i.e. the coefficients of the associated vectorial map) are simplified when referring to an orthonormal basis $\{\mathbf{d}_\alpha\}$ adapted to the two planes Π and Π', in the sense that

$$\mathbf{d}_{0,1} \in \Pi \ , \qquad \mathbf{d}_{2,3} \in \Pi' \ . \tag{9.88}$$

In this case, the transformed vectors $\mathbf{F}_\beta = {}^{(d)}F^\alpha{}_\beta \mathbf{d}_\alpha$ have the form

$$\mathbf{F}_0 = \lambda \mathbf{d}_1 \ , \quad \mathbf{F}_1 = \rho \mathbf{d}_0 \ , \quad \mathbf{F}_2 = \mu \mathbf{d}_3 \ , \quad \mathbf{F}_3 = \nu \mathbf{d}_2 \ ,$$

so that the matrix ${}^{(d)}F^\alpha{}_\beta$ turns out to be

$${}^{(d)}F^\alpha{}_\beta = \begin{pmatrix} 0 & \rho & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu \\ 0 & 0 & \mu & 0 \end{pmatrix} .$$

From this representation, one can write the associated completely covariant form (only the elements of the first row change sign); because of the antisymmetry of the electromagnetic tensor, one finds $\rho = \lambda$ and $\mu = -\nu$. Thus, for any choice of the adapted basis (9.88) the matrix $F^\alpha{}_\beta$ is given by

$$^{(d)}F^\alpha{}_\beta = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu \\ 0 & 0 & \mu & 0 \end{pmatrix} \quad \sim \quad ^{(d)}F^{\alpha\beta} = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu \\ 0 & 0 & \mu & 0 \end{pmatrix} . \tag{9.89}$$

Therefore, $^{(d)}F^\alpha{}_\beta$ results to be expressed in terms of the two scalars λ and μ which are invariants, because of the relations

$$I_2 = -\lambda^2 + \mu^2 , \qquad I_4 = -\lambda^2\mu^2 . \tag{9.90}$$

Comparing (9.90) with (9.85) one gets $\lambda^2 = \lambda_1^2$ and $\mu^2 = \lambda_2^2$.

Equation (9.89) is the *canonical form of the electromagnetic tensor*; the singular case in which the two characteristic 2-planes Π and Π', in spite of being orthogonal, are both of parabolic type and have a common isotropic direction l^α, is excluded. In fact, in this case there are no orthogonal adapted bases and $F_{\alpha\beta}$ is necessarily a parabolic bivector, which can be written as

$$F_{\alpha\beta} = \xi(l_\alpha v_\beta - l_\beta v_\alpha) . \tag{9.91}$$

Here ξ is an arbitrary factor (a multiplicative parameter for the isotropic vector l_α) and v_α is a spatial vector, orthogonal to $\mathbf{l}$ (like all the vectors in Π and Π'), which can be assumed to be normalized to 1:

$$v^2 = v_\beta v^\beta = 1 , \tag{9.92}$$

because in (9.91) $\mathbf{v}$ can always be scaled by an arbitrary factor re-absorbed then in ξ.

9.11 Spectral Analysis of the Energy Tensor of the Electromagnetic Field

We pass now to study the decomposition of the energetic tensor $E_{\alpha\beta}$, a second-degree homogeneous function of $F_{\alpha\beta}$:

$$4\pi E_{\alpha\beta} = F_{\alpha\rho}F_\beta{}^\rho - \frac{1}{4}m_{\alpha\beta}(F_{\rho\sigma}F^{\rho\sigma}) ,$$

that is, as from (9.83):

$$4\pi E_{\alpha\beta} = -F_{\alpha\rho}F^\rho{}_\beta - \frac{1}{2}I_2\, m_{\alpha\beta} . \tag{9.93}$$

In fact, $F_{\alpha\beta}$ considered as a vectorial map, induces either in Π or Π' the orthogonal affinity, and the quadratic form of $E_{\alpha\beta}$ implies that it has eigendirections belonging to Π or to Π' only. Thus, *the tensor* $E_{\alpha\beta}$, different from $F_{\alpha\beta}$, *is diagonal*, and it admits ∞^2 orthogonal tetrads. The latter are the orthonormal bases (9.88) adapted to Π and Π', for which an arbitrary rotation is still possible in both planes.

As concerns the eigenvalues, excluding the singular case, which will be examined later, it is clear that there are *two distinct eigenvalues*, say λ and λ', each of them with multiplicity 2. Furthermore, since $I_1(E^\alpha{}_\beta) = 0$, they are necessarily opposite: $\lambda = -\lambda'$.

Let us determine, first, the eigenvalue of $E_{\alpha\beta}$ for the directions in Π. Denote by $\mathbf{u} \in \Pi$ the null eigenvector of $F_{\alpha\beta}$ associated with the eigenvalue λ_1 given by $(9.85)_1$; from (9.93) it follows that

$$4\pi E_{\alpha\beta} u^\beta = -\lambda_1^2 u_\alpha - \frac{1}{2} I_2 u_\alpha \; ;$$

hence, $\mathbf{u}$ is eigenvector of $4\pi E_{\alpha\beta}$, associated with the eigenvalue

$$\lambda = -\lambda_1^2 - \frac{1}{2} I_2 = -\frac{1}{2}\sqrt{I_2^2 - 4I_4} \; .$$

Next, assuming

$$k = \frac{1}{2}\sqrt{I_2^2 - 4I_4} \geq 0 \; , \tag{9.94}$$

in the *general case* $(k > 0)$ the eigenvalues of $4\pi E_{\alpha\beta}$ are $-k, -k, k, k$, and the matrix ${}^{(d)}E^\alpha{}_\beta$ of the components with respect to the tetrad (9.88) is given by

$$4\pi {}^{(d)}E^\alpha{}_\beta = \begin{pmatrix} -k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix} \quad \sim \quad 4\pi {}^{(d)}E^{\alpha\beta} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix} . \tag{9.95}$$

Clearly, both forms (9.89) and (9.95), having a general meaning, are referred to an orthonormal basis $\{d_{(\rho)}\}$[3] which essentially depends on the point $E \in M_4$ in which the electromagnetic field is evaluated, exactly as the two planes Π and Π'.

If, from (9.89) and (9.95), one needs Cartesian components along a fixed basis $\mathbf{c}_\alpha$, it is enough to decompose the vectors $\mathbf{d}_{(\rho)}$ as

$$\mathbf{d}_{(\rho)} = d^\alpha{}_{(\rho)} \mathbf{c}_\alpha \; , \tag{9.96}$$

and use the transformation laws (both $E_{\alpha\beta}$ and $F_{\alpha\beta}$ are tensors):

$$E^{\alpha\beta} = d^\alpha{}_{(\rho)} d^\beta{}_{(\sigma)} {}^{(d)}E^{(\rho)(\sigma)} \; , \quad F^{\alpha\beta} = d^\alpha{}_{(\rho)} d^\beta{}_{(\sigma)} {}^{(d)}F^{(\rho)(\sigma)} \; . \tag{9.97}$$

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[3] The index in parenthesis is not a tensorial index but only ordinal.

The canonical general forms for F and E also follow:

$$\begin{aligned} F^{\alpha\beta} &= \lambda(d^{\alpha}{}_{(0)}d^{\beta}{}_{(1)} - d^{\beta}{}_{(0)}d^{\alpha}{}_{(1)}) - \mu(d^{\alpha}{}_{(2)}d^{\beta}{}_{(3)} - d^{\beta}{}_{(2)}d^{\alpha}{}_{(3)}) , \\ 4\pi E^{\alpha\beta} &= k(d^{\alpha}{}_{(0)}d^{\beta}{}_{(0)} - d^{\alpha}{}_{(1)}d^{\beta}{}_{(1)} + d^{\alpha}{}_{(2)}d^{\beta}{}_{(2)} + d^{\alpha}{}_{(3)})d^{\beta}{}_{(3)} . \end{aligned} \tag{9.98}$$

Finally, in the singular case, one has

$$4\pi E_{\alpha}{}^{\beta} = F_{\alpha\rho}F^{\beta\rho} = \xi^2(l_{\alpha}v_{\rho} - l_{\rho}v_{\alpha})(l^{\beta}v_{\rho} - l^{\rho}v^{\beta}) = \xi^2 l_{\alpha}l^{\beta}v_{\rho}v^{\rho} ,$$

or, using (9.92),

$$4\pi E^{\alpha\beta} = \xi^2 l^{\alpha}l^{\beta} . \tag{9.99}$$

9.12 Electromagnetic 4-Potential. Gauge Invariance

Let us consider now the homogeneous Maxwell's equations $(9.29)_1$:

$$\partial_{\rho}F_{\alpha\beta} + \partial_{\alpha}F_{\beta\rho} + \partial_{\beta}F_{\rho\alpha} = 0 , \qquad \forall E \in M_4 ; \tag{9.100}$$

they are satisfied identically in M_4, when the electromagnetic field admits a *vector potential* $\phi_{\alpha}(x)$, defined and regular all over M_4; in that case, $F_{\alpha\beta}$ can be expressed by the following relation:

$$F_{\alpha\beta} = \partial_{\alpha}\phi_{\beta} - \partial_{\beta}\phi_{\alpha}. \tag{9.101}$$

In fact, the Schwartz theorem allows to commute partial derivatives, so that $\partial_{\rho}F_{\alpha\beta} = \partial_{\rho}\partial_{\alpha}\phi_{\beta} - \partial_{\beta}\partial_{\rho}\phi_{\alpha}$, and the proof only requires a cyclic permutation of the indices ρ, α and β.

The representation (9.101) has a general validity in the neighbourhood of any point $E \in M_4$, in the sense that (9.100) necessarily imply (9.101). This is a general property of closed differential forms of any order (see e.g. [3], p. 37): the electromagnetic field, being antisymmetric, defines a second-order differential form, which is closed because of (9.100) and from this, the existence of a local potential vector.

The vector field $\phi_{\alpha}(x)$ defined by (9.101) is called the 4-*potential of the electromagnetic field* $F_{\alpha\beta}$; more precisely, it is only one of the 4-potentials of the electromagnetic field $F_{\alpha\beta}$ because it is not uniquely defined by (9.101) (exactly as the scalar potential of a conservative force). In fact, every field like

$$\phi'_{\alpha} = \phi_{\alpha} + \partial_{\alpha}\varphi , \tag{9.102}$$

with $\varphi(x)$ a scalar (invariant) function, still satisfies (9.101):

$$\partial_{\alpha}\phi'_{\beta} - \partial_{\beta}\phi'_{\alpha} = \partial_{\alpha}\phi_{\beta} - \partial_{\beta}\phi_{\alpha} = F_{\alpha\beta},$$

for any choice of the potential $\varphi(x)$. In other words, the 4-potential $\phi_{\alpha}(x)$ defined by (9.101) is not intrinsically related to the electromagnetic field $F_{\alpha\beta}$, but it can undergo a transformation $\phi_\alpha \to \phi'_\alpha$, as in (9.102), with φ an *arbitrary scalar potential.* This is called the *gauge invariance* of the electromagnetic field with respect to its vector potential and shows that, to describe the field $F_{\alpha\beta}$ through (9.101), the components $\phi_\alpha(x)$ are not completely arbitrary: one has at disposal the function $\varphi(x)$; hence the independent components of ϕ_α are only three. One can then impose a priori an additional differential condition to the 4-potential which in no way influences the electromagnetic field. The choice of such a condition is often related to the inhomogeneous Maxwell's equations, in order to simplify their form, for instance.

In fact, writing such equations in terms of ϕ_α gives

$$\partial^\rho(\partial_\alpha\phi_\rho - \partial_\rho\phi_\alpha) = \frac{4\pi}{c} S_\alpha \ ,$$

that is

$$\partial_\alpha(\partial^\rho\phi_\rho) = \partial^\rho\partial_\rho\phi_\alpha + \frac{4\pi}{c} S_\alpha \ .$$

In this form, the inhomogeneous Maxwell's equations show a scalar field, given by the four-dimensional divergence of $\boldsymbol{\phi} = (\phi^\rho)$[4]:

$$\mathrm{Div}\,\boldsymbol{\phi} = \partial_\rho\phi^\rho = m^{\rho\sigma}\partial_\rho\phi_\sigma = \frac{1}{2}m^{\rho\sigma}(\partial_\rho\phi_\sigma + \partial_\sigma\phi_\rho) \ , \tag{9.103}$$

as well as a second-order differential operator: the *D'Alembert operator*, with parameter c (wave equation for light):

$$\Box_c \stackrel{\mathrm{def}}{=} m^{\rho\sigma}\partial_\rho\partial_\sigma = \delta^{ik}\partial_i\partial_k - \frac{1}{c^2}\partial^2_{tt} \ . \tag{9.104}$$

Using such a notation $(9.29)_2$ become

$$\partial_\alpha(\mathrm{Div}\,\boldsymbol{\phi}) = \Box_c\,\phi_\alpha + \frac{4\pi}{c} S_\alpha \ , \qquad (\alpha = 0,1,2,3) \ . \tag{9.105}$$

It is then quite natural to choose the *supplementary condition* for the 4-potential as

$$(\mathrm{Div}\,\boldsymbol{\phi}) = 0 \ , \quad \forall E \in M_4 \ , \tag{9.106}$$

which is known as *Lorentz gauge condition.* The latter can be satisfied in infinite ways, taking into account the transformation (9.102) and with a proper choice of the arbitrary function φ. More precisely, assuming that (9.106) is not directly satisfied by the potential ϕ_α, one has to require such a condition for the transformed function ϕ'_α, imposing that φ is a solution of the differential equation

$$\Box_c\,\varphi = -\frac{1}{2}m^{\rho\sigma}(\partial_\rho\phi_\sigma + \partial_\sigma\phi_\rho) \ ,$$

[4] Note that Div and Grad operations correspond to divergence and gradient in the space-time.

with the right-hand side of this equation a *known function of* x^α. Summarizing, Maxwell's equations (9.29) can also be written as:

$$\Box_c \boldsymbol{\phi} = -\frac{4\pi}{c}\mathbf{S} \,, \qquad \operatorname{Div}\boldsymbol{\phi} = 0 \,, \quad \forall E \in M_4 \,, \tag{9.107}$$

without any restriction on the electromagnetic field, given a posteriori by (9.101). Equations (9.107), like (9.29), imply that the source $\mathbf{S}$ satisfies, in the world tube $\mathcal{T} \in \mathcal{M}_4$, the conservation equation

$$\operatorname{Div}\mathbf{S} = 0 \,, \quad \forall E \in \mathcal{T} \,, \tag{9.108}$$

and, from this point of view, the differential condition (9.108) is a direct consequence of the field equations (9.107). If one assumes, instead, $(9.107)_1$ and (9.108) as field equations then

$$\Box_c \boldsymbol{\phi} = -\frac{4\pi}{c}\mathbf{S} \,, \quad \forall E \in M_4, \qquad \operatorname{Div}\boldsymbol{\phi} = 0 \,, \quad \forall E \in \mathcal{T}, \tag{9.109}$$

implying no longer $(9.107)_2$, but the more general differential condition

$$\Box_c(\operatorname{Div}\boldsymbol{\phi}) = 0 \,, \quad \forall E \in M_4 \,.$$

The latter, in turn, under regularity conditions at the infinity (a fact which should be better specified), is equivalent to the Lorentz condition: $\operatorname{Div}\boldsymbol{\phi} = 0$.

We notice the close analogy between $(9.109)_1$, i.e. the vectorial equation in M_4: $\Box_c \boldsymbol{\phi} = -4\pi/c\mathbf{S}$, and the Poisson equation: apart from the different second-order differential operator $\Box_c$, which reduces to Δ_2 in the limit $c \to \infty$, the analogy between the gravitational field and the electromagnetic one is complete when $\boldsymbol{\phi} \to U$, $\mathbf{S} \to \mu$, $1/c \to f$ (Newtonian gravitational constant). As we see here, to the single gravitational potential, in the electromagnetic analogy, corresponds the four potentials ϕ_α. In general relativity we will have a larger number of potentials, from 1 to 10: $U \to g_{\alpha\beta}$, and in the so-called unified theories (geometrization of the gravitational field and the electromagnetic one), the potentials become 14, at least.

9.13 The Material and the Electromagnetic Schemes

We have already seen that the Lorentz 4-force comes from a *superpotential* $E^{\alpha\beta}$:

$$-\rho_0 f^\alpha = \partial_\rho E^{\alpha\rho} \,, \tag{9.110}$$

which is closely related to the electromagnetic field $F_{\alpha\beta}$:

$$4\pi E^{\alpha\rho} = F^\alpha{}_\beta F^{\rho\beta} - \frac{1}{4} m^{\alpha\rho} F \,, \qquad F = F_{\rho\sigma}F^{\rho\sigma} \,, \tag{9.111}$$

but $F_{\alpha\beta}$ and $E_{\alpha\beta}$ are not in 1–1 correspondence, because (9.111) are not directly invertible. Equations (9.110) are the conservation equations for the electromagnetic field, like the similar equations for material continuous systems:

$$\mu_0 f_m^\alpha = \partial_\rho M^{\alpha\rho} \; ; \tag{9.112}$$

(9.112) have also the meaning of *evolution equations* for the material system, different from (9.110) which only summarize the action autoinduced from the electromagnetic field governed by Maxwell's equations.

Clearly, the analogy between (9.110) and (9.112) is purely formal, because the two tensor fields $E^{\alpha\rho}$ and $M^{\alpha\rho}$ have an algebraic structure completely different, in agreement with the two distinct schemes, the one material and the other electromagnetic. More precisely, while $M^{\alpha\rho}$ is represented by

$$M^{\alpha\rho} = \hat{\mu}_0 V^\alpha V^\rho + X^{\alpha\rho} \tag{9.113}$$

and summarizes the mechanical characteristics of the continuous system: $\hat{\mu}_0$[5]: *total material density*, $\mathbf{V}$: 4-*velocity* and $X^{\alpha\beta}$: *proper mechanical stress tensor*, the field (9.111), instead, does not summarize all the ingredients of the electromagnetic scheme, at least for two reasons: it is only partially related to $F_{\alpha\beta}$ and totally ignores the distribution of charges and currents as described by the function $\mathbf{S} = \rho_0 \mathbf{V}$.

In other words, (9.110) are simple algebraic consequences of Maxwell's equations, considered as evolution equations for the electromagnetic field as well as the charged continuous material, which generates it.

Similarly, the autoinduced field f^α given by (9.69) only partially substitutes the sources S^α: in fact, even if (9.69) are invertible, because of the condition $\det ||F_{\alpha\beta}|| \neq 0$, the vectors f_α and S^α are orthogonal. Moreover, apart from the different role of (9.110) and (9.112) as well as that of the tensor fields $E^{\alpha\beta}$ and $M^{\alpha\beta}$ the (local) algebraic structure of such tensors is different. To see this, we can compare the decompositions of the two tensors, inside the world tube $\mathcal{T}$, described by charges and currents.[6]

Thus, from one side we have (9.113), where $\mathbf{V}$ is an eigendirection of $M^{\alpha\beta}$, since $X^{\alpha\beta} V_\beta = 0$, $(\alpha = 0, 1, 2, 3)$; from the other side, from (9.77), evaluated in the proper frame of the generic charged element, that is for $\boldsymbol{\gamma} = \mathbf{V}/c$, we have

$$4\pi E^{\alpha\rho} = \frac{1}{c^2} \mathcal{W}_0 V^\alpha V^\rho + \frac{1}{c}(P_0^\alpha V^\rho + P_0^\rho V^\alpha) + M_0^{\alpha\rho} \; , \tag{9.114}$$

with an obvious meaning of symbols.

We see that the structure of the tensor $E^{\alpha\rho}$ is more general with respect to that of the energetic tensor associated with an ordinary continuous scheme (i.e. nonpolar). In fact, (9.114) assumes, for each $E \in \mathcal{T}$, two preferred directions: $\mathbf{V}$ (4-velocity of the charge) and $\mathbf{P}_0$ (proper Poynting vector); the former is

[5] We assume here no thermal conduction stresses for simplicity.

[6] Note that for the material continuous system *the energetic tensor has no meaning outside* $\mathcal{T}$.

temporal while the latter spatial, because of the orthogonality condition: $\mathbf{P}_0 \cdot \mathbf{V} = 0$. This then represents the energetic tensor associated with a continuous system, with internal structure of vectorial type, that is with a "director" $\mathbf{P}_0$. In different words, (9.114) implies the following coordinate 4-stresses:

$$\mathbf{T}^\alpha = 4\pi E^{\alpha\rho}\mathbf{c}_\rho \ , \tag{9.115}$$

analogous to (7.12)

$$\mathbf{T}^\alpha = \mathbf{X}^\alpha + Q^\alpha \mathbf{V} \ , \tag{9.116}$$

with

$$\mathbf{X}^\alpha = \mathbf{M}_0^\alpha + \frac{1}{c}\mathbf{P}_0 V^\alpha \ , \qquad Q^\alpha = \frac{1}{c}P_0^\alpha + \frac{1}{c^2}\mathcal{W}_0 V^\alpha \ ; \tag{9.117}$$

hence, the purely mechanical 4-stresses do not satisfy axiom VI:

$$X^{\alpha\beta} = M_0^{\alpha\beta} + \frac{1}{c}V^\alpha P_0^\beta \neq X^{\beta\alpha} \ .$$

Therefore, the problem of unification of the two relativistic schemes, material continuum and electromagnetic field, if really solvable, should be framed in the context of *polar continua*. This implies the loss of the spatial reciprocity axiom VI and hence the enlargement of the continuum scheme from the geometrical–kinematical point of view, introducing as a "director" the proper heat conduction vector, which has the Poynting vector as a counterpart in the electromagnetic field. However, in this case, polarity has a different meaning with respect to the classical situation, because the mechanical 4-stress tensor is assumed to be nonsymmetric, different from the ordinary stress tensor, which is instead symmetric.

Another enlargement of the scheme, even from a dynamical point of view, is obtained by introducing "pairs" (mass or contact pairs, both mechanical or thermal) and rejecting the reciprocity axiom III: these are relativistic polar continua, in the most general sense, that is with nonsymmetric energetic tensor, and consequent asymmetry of the ordinary stress tensor as well as of the momentum of stress tensor.

9.14 Evolution Equations for a Charged Material System

Let us consider now, from a general point of view, the evolution of a charged continuum material system in the absence of thermal stresses; this is a mixed scheme in which, besides the thermodynamical complication, there is a direct coupling between matter and electromagnetic field [4]. More precisely, the energetic tensor of the material system,

$$M^{\alpha\beta} = \hat{\mu}_0 V^\alpha V^\beta + X^{\alpha\beta} \ , \tag{9.118}$$

is sub-ordered to the evolution equations:

$$\partial_\rho M^{\alpha\rho} = \mu_0 f_m^\alpha + \rho_0 f^\alpha \,, \qquad (\alpha = 0, 1, 2, 3)\,, \tag{9.119}$$

where the sources, at the right-hand side, include either the mass force (internal and external, of mechanical and of thermal type) or the mechanical action induced by the electromagnetic field on the continuum, given by (9.69):

$$\rho_0 f^\alpha = \frac{1}{c} F^{\alpha\beta} S_\beta \,. \tag{9.120}$$

This, in turn, is built by the electromagnetic tensor $F_{\alpha\beta}$ and the associated sources S_α, both constrained by Maxwell's equations. Thus, the complete set of equations is the following:

$$\begin{cases} \partial_\rho M^{\alpha\rho} = \mu_0 f_m^\alpha + \rho_0 f^\alpha \,, \qquad \rho_0 f^\alpha = \dfrac{1}{c} F^{\alpha\beta} S_\beta, \\ \partial_\rho F_{\alpha\beta} + \partial_\alpha F_{\beta\rho} + \partial_\beta F_{\rho\alpha} = 0, \\ \partial_\rho F^{\alpha\rho} = \dfrac{4\pi}{c} S^\alpha, \end{cases} \tag{9.121}$$

where, besides the initial and boundary conditions, the energetic tensor $M^{\alpha\rho}$ is the same as in (9.118), and the mass force f_m^α is assigned. It is clear that, from (9.121), the coupling between the two fields (material and electromagnetic) is only through the Lorentz force f^α, also expressed as follows:

$$\rho_0 f^\alpha = -\partial_\rho E^{\alpha\rho} \,,$$

with $E^{\alpha\rho}$ a well-determined function of the electromagnetic field $F_{\alpha\beta}$ as in (9.111). Therefore, the final set of general equations (9.121) for the coupling of matter and charge assumes the form

$$\begin{cases} \partial_\rho (M^{\alpha\rho} + E^{\alpha\rho}) = \mu_0 f_m^\alpha \,, \\ \partial_\rho F_{\alpha\beta} + \partial_\alpha F_{\beta\rho} + \partial_\beta F_{\rho\alpha} = 0, \\ \partial_\rho F^{\alpha\rho} = \dfrac{4\pi}{c} S^\alpha, \end{cases} \tag{9.122}$$

and the interaction between the two fields is governed by the energetic tensor of the electromagnetic field:

$$4\pi E^{\alpha\rho} = F^\alpha{}_\beta F^{\rho\beta} - \frac{1}{4} m^{\alpha\rho} F_{\rho\sigma} F^{\rho\sigma} \,. \tag{9.123}$$

Finally, (9.107) gives rise to the reduced set of equations:

$$\partial_\rho(M^{\alpha\rho}+E^{\alpha\rho}) = \mu_0 f_m^\alpha\,, \qquad \Box_c\,\boldsymbol{\phi} = -\frac{4\pi}{c}\mathbf{S}\,, \quad \mathrm{Div}\,\boldsymbol{\phi} = 0\,, \tag{9.124}$$

in terms of the potential vector $\boldsymbol{\phi}$, which is used to express $F_{\alpha\beta}$ and hence $E_{\alpha\beta}$.

Moreover, (9.122) shows that for the continuum one still has conservation equations as (9.112):

$$\partial_\rho \hat{E}^{\alpha\rho} = \mu_0 f_m^\alpha\,, \tag{9.125}$$

where $\hat{E}^{\alpha\rho}$ is the *total energetic tensor*:

$$\begin{aligned}\hat{E}^{\alpha\rho} &\stackrel{\text{def}}{=} M^{\alpha\rho}+E^{\alpha\rho}\\ &\equiv \hat{\mu}_0 V^\alpha V^\beta + X^{\alpha\beta} + \frac{1}{4\pi}\left(F^\alpha{}_\beta F^{\rho\beta} - \frac{1}{4}m^{\alpha\rho}F_{\rho\sigma}F^{\rho\sigma}\right)\,,\end{aligned} \tag{9.126}$$

that is the sum of the energy–momentum tensors: material and electromagnetic; explicitly, in the matter case:

$$\begin{aligned}\hat{E}^{\alpha\rho} = &\left(\hat{\mu}_0 + \frac{1}{4\pi c^2}\mathcal{W}_0\right)V^\alpha V^\rho + X^{\alpha\rho}\\ &+\frac{1}{4\pi}M_0^{\alpha\rho} + \frac{1}{4\pi c}(V^\alpha P_0^\rho + V^\rho P_0^\alpha)\,.\end{aligned} \tag{9.127}$$

As we have already seen, for a neutral continuous system, (9.125) can be interpreted as conservation equations (with sources, in the region internal to matter, and without sources in the exterior) for the linear momentum and the total energy of the system: matter plus electromagnetic field. More precisely, in any Galilean frame, (9.125) give rise to the conservation of linear momentum and energy for the set of three fundamental fields: *pure matter*, *internal tension* (either of mechanical or thermal type) and *electromagnetic*, with energetic tensors: $\mu_0 V^\alpha V^\rho$, $T^{\alpha\beta} = \mu_{c,0}V^\alpha V^\beta + X^{\alpha\beta}$ and $E^{\alpha\beta}$, respectively; they should be considered in their form, associated with the chosen Galilean frame and from here, the total matter density: $\hat{\mu} = \mu + \mu_c + \mu_e$, the total energy density: $\mu\hat{\epsilon} = \hat{\mu}c^2$, the total coordinate stresses (material and Maxwell's), etc.

Besides the mixing of the various energetic forms and linear momentum, we notice that the coupling which we have been considering here grounds on the hypothesis that in the interior of the material continuum one can assume the vacuum Maxwell's equations as holding. Properly speaking, in the presence of matter in place of **H** and **E** one has to consider the electric and magnetic induction, that is the induced electromagnetic field, which is related to the permeability (electric and magnetic) of the considered matter; similarly, one should specify if the material is a conductor, the ordinary Ohm law, etc.

Thus, the set of equations (9.123) does not represent the general (complicated) coupling of the two fields (matter and charge): one has to specify for example the constitutive equations, so that practically one should look at them as a first approach to a more general problem.

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