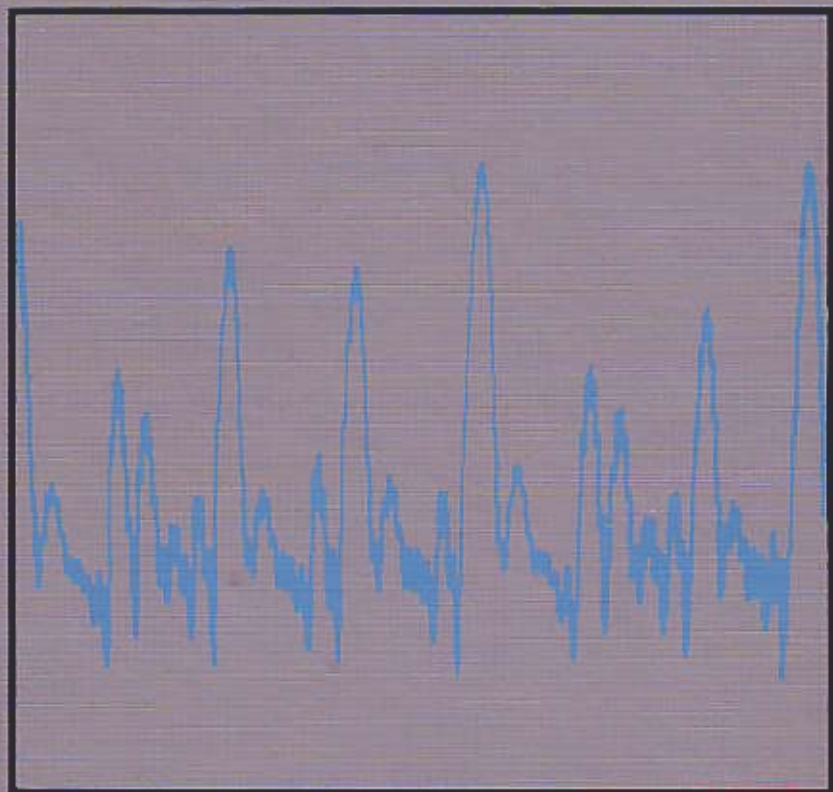


H. HAKEN

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# light

VOLUME 2



**LASER LIGHT DYNAMICS**

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NORTH-HOLLAND

# LIGHT

Volume 2

## LASER LIGHT DYNAMICS

H. HAKEN

*Institut für Theoretische Physik, Stuttgart*



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# Preface to the Preface

Dear Reader,

Before you read this book, and even its preface, the following remarks might be useful to you. Since this book is "Volume 2" you may be inclined to believe that you must know all the contents of "Volume 1" before you can start reading (and, of course, understanding) "Volume 2". But this is not the case. The present "Volume 2" again starts at a rather elementary level, and then proceeds step by step to more difficult matters. Only at these later stages some more advanced theoretical background is required which then can be taken from "Volume 1". I have chosen this way of presentation to make the theory of laser light accessible to a broad audience—ranging from students at the beginning of their graduate studies to professors and scientists interested in recent developments. For details on the relations between the chapters of these books consult the list at the end of the introduction.

**H. Haken**

# Preface

This book is a text which applies to students and professors of physics. Because it offers a broad view on laser physics and presents most recent results on the dynamics of laser light, such as self-pulsing and chaos, it will be of interest also to scientists and engineers engaged in laser research or development. This text starts at a rather elementary level and will smoothly lead the reader into the more difficult problems of laser physics, including the basic features of the coherence and noise properties of laser light.

In the introductory chapters, typical experimental set-ups and laser materials will be discussed, but the main part of this book will be devoted to a *theoretical treatment* of a great variety of laser processes. The laser, or the optical maser, as it was originally called, is one of the most important inventions of this century and has found a great number of important applications in physics, chemistry, medicine, engineering, telecommunications, and other fields. It bears great promises for further applications, e.g. in computers. But also from the point of view of basic research, a study of the physical processes which produce the unique properties of laser light are equally fascinating. The laser is a beautiful example of a system far from thermal equilibrium which can achieve a macroscopically ordered state through "self-organization". It was the first example for a nonequilibrium phase transition, and its study eventually gave birth to synergetics, a new interdisciplinary field of research.

I got involved in laser physics at a rather early stage and under most fortunate circumstances. In 1960 I was working as visiting scientist at the Bell Telephone Laboratories, Murray Hill. There I soon learned that these laboratories were searching for a revolutionary new light source. Two years earlier, in 1958, this source had been proposed by Schawlow and Townes, who derived in particular the laser condition and thus demonstrated the feasibility of this new device. At Bell Telephone Laboratories I soon got involved in a theoretical study of the laser processes and continued it at Stuttgart University. I developed a laser theory whose basic features I published in 1962 and which I then applied to various concrete problems,

jointly with my coworkers. At about the same time, in 1964, Willis Lamb published his theory, which he and his coworkers applied to numerous problems. It is by now well known that these two theories, which are called semiclassical and which were developed independently, are equivalent. The next step consisted in the development of the laser quantum theory which allows one to predict the coherence and noise properties of laser light (and that of light from lamps). This theory which I published in 1964 showed for the first time that the statistical properties of laser light change dramatically at laser threshold. In the following years my group in Stuttgart carried this work further, e.g. to predict the photon statistics close to laser threshold.

From 1965 on, Scully and Lamb started publishing their results on the quantum theory of the laser, using a different approach, and Lax and Louisell presented their theory. Again, all of these theories eventually turned out to be more or less equivalent. In those years experimental laser physics developed (and is still developing) at an enormous pace, but because I shall mainly deal with laser *theory* in this book, I have to cut out a representation of the history of that field.

From my above personal reminiscences it may transpire that laser theory and, perhaps still more, laser physics in general have been highly competitive fields of research. But, what counts much more, laser physics has been for us all a fascinating field of research. When one looks around nowadays, one can safely say that it has lost nothing of its original fascination. Again and again new laser materials are found, new experimental set-ups invented and new effects predicted and discovered. Undoubtedly, for many years to come, laser physics will remain a highly attractive and important field of research, in which fundamental problems are intimately interwoven with applications of great practical importance. I hope that this book will let transpire the fascination of this field.

Over the past nearly 25 years I greatly profited from the cooperation or discussion with numerous scientists and I use this opportunity to thank all of them. There is Wolfgang Kaiser, who was the first at BTL with whom I had discussions on the laser problem. Then there are the members of my group at Stuttgart who in the sixties, worked on laser theory and who gave important contributions. I wish to mention in particular R. Graham, H. Geffers, H. Risken, H. Sauermann, Chr. Schmid, H.D. Vollmer, and W. Weidlich. Most of them now have their own chairs at various universities. Among my coworkers who, in later years, contributed to laser theory and its applications are in particular J. Goll, A. Schenzle, H. Ohno, A. Wunderlin and J. Zorell. Over the years I enjoyed many friendly and stimulating discussions with F.T. Arecchi, W.R. Bennett, Jr., N. Bloembergen, R. Bonifacio, J.H. Eberly, C.G.B. Garret, R.J. Glauber, F. Haake, Yu.

Klimontovich, W. Lamb, M. Lax, W. Louisell, L. Lugiato, L. Mandel, L. Narducci, E.R. Pike, M. Sargent, M. Scully, S. Shimoda, S. Stenholm, Z.C. Wang, E. Wolf, J. Zhang, and many other scientists.

I wish to thank my coworker, Dr. H. Ohno, for his continuous and valuable assistance in the preparation of the manuscript. In particular, he carefully checked the formulas and exercises, contributed some in addition, and drew the figures. My particular thanks go to my secretary, Mrs. U. Funke, who in spite of her heavy administrative work assisted me in many ways in writing the manuscript and typed various versions of it both rapidly and perfectly. Her indefatigable zeal constantly spurred me on to bring it to a finish.

The writing of this book was greatly helped by a program of the Deutsche Forschungsgemeinschaft. This program was initiated by Prof. Dr. Maier-Leibnitz, whom I wish to thank cordially for his support for this project.

H. Haken

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# List of symbols

$A$	constant, edge length of mirror
$A_i$	aperture of mirror $i$
$A_{ij,kl}$	constant in density matrix equation
$A_\lambda$	complex amplitude
$A_\lambda(t)$	complex amplitude, slowly varying part
$A_\mu$	time independent complex amplitude, atomic dipole moment
$A_{\mu\lambda}(t)$	slowly varying mode amplitude, dipole moment
$a_j$	annihilation operator for electron in state $j$
$a_j^+$	creation operator for electron in state $j$
$a_{j,\mu}^+$	creation operator for electron in state $j$ , atom $\mu$
$a_{j,\mu}$	annihilation operator for electron in state $j$ , atom $\mu$
$a_{kk'k''jj'j''}$	coefficient in nonlinear terms
$B$	time independent complex amplitude of the electric field, constant, magnetic induction
$B_\lambda(t)$	slowly varying mode amplitude of electric field
$B(t)$	slowly varying complex amplitude of electric field
$b_\lambda^+$	creation operator of mode A
$b_\lambda$	annihilation operator of mode A, dimensionless complex electric field amplitude
$C$	constant
$C_{ijk}$	coupling coefficient for mode amplitudes $\xi_i, \xi_j, \xi_k$
$c$	speed of light in vacuum
$c'$	speed of light in media
$c_j$	constant coefficient
$c_j(t)$	expansion coefficient in perturbation theory
$c_i$	transmissivity of mirror

$D$	dielectric displacement, normalized inversion, distance of mirrors
$\bar{D}$	normalized inversion
$D_i$	initial inversion in Q-switching
$D_0$	unsaturated inversion
$\mathcal{D}_0$	total unsaturated inversion in externally driven laser
$D(t)$	total saturated inversion in externally driven laser
$D(\mathbf{x}, t)$	atomic inversion density
$D_{\mu ij}$	abbreviation for constants
$D_{\mu\lambda\lambda'}$	abbreviation for constants
$d$	small deviation from normalized inversion
$d_0$	unsaturated inversion of a single atom
$d_\mu$	inversion of atom $\mu$
$d(\omega)$	spectral inversion density
$E$	normalized electric field
$\mathbf{E}$	electric field strength
$E_0$	amplitude factor of electric field strength
$\mathcal{E}_{\text{ext}}$	externally driving electric field
$\hat{E}$	normalized electric field
$\mathbf{E}(t)$	time dependent electric field strength
$\mathcal{E}(t)$	electric field amplitude in externally driven laser
$E_x$	x-component of electric field strength
$E(x)$	electric field strength
$E(\mathbf{x}, t)$	electric field strength
$\mathbf{E}^{(-)}(\mathbf{x}, t)$	electric field (negative frequency part)
$\mathbf{E}^{(+)}(\mathbf{x}, t)$	electric field (positive frequency part)
$E_S(x)$	electric field strength at mirror, S
$E_T$	transmitted electric field amplitude
$E_\lambda(t)$	time dependent amplitude in mode expansion of electric field
$E_\lambda^{(-)}(t)$	negative frequency part of mode amplitude of electric field
$E_\lambda^{(+)}(t)$	positive frequency part of mode amplitude of electric field
$e$	elementary charge, small deviation from normalized electric field
$\mathbf{e}_\lambda$	polarization vector of mode $\lambda$
$F$	normalized electric field strength, cavity cross section

$F(t), F^+(t)$	quantum mechanical fluctuating force (field mode)
$F_{\text{tot}}(t)$	total fluctuating force (field mode)
$f$	Hooke's constant
$f(\mathbf{x})$	spatial dependence of cavity mode
$f_{\lambda}(\mathbf{x})$	space dependent amplitudes
$f(q, t)$	probability distribution
$f(r, \varphi)$	probability distribution function in polar coordinates
$\tilde{f}(\mathbf{x}, y; t)$	probability distribution function
$G$	unsaturated net gain
$G_{\tilde{a},kl}$	correlation coefficient for quantum mechanical fluctuations
$g$	coupling constant
$g_{\mu}$	coupling constants of single mode to atom $\mu$
	coupling coefficient of atom $\mu$ to mode $\lambda$
$g(\mathbf{x})$	spatial dependence of cavity mode
$H$	magnetic field strength
$H$	Hamiltonian
$H_0$	unperturbed Hamiltonian
$H_A$	Hamiltonian of atoms
$H_{\text{af}}$	coupling Hamiltonian single atom-field
$H_{\text{Af}}$	coupling Hamiltonian atoms-field
$H_{A-F}$	coupling Hamiltonian atoms-multimode field
$H_{B_j}$	Hamiltonian of heatbath $j$
$H_{B_j-A}$	coupling Hamiltonian heatbath-atoms
$H_{B_j-f}$	coupling Hamiltonian heatbath-field
$H_f$	Hamiltonian of free field
$H_F$	Hamiltonian for multimode field
$H_n(x)$	Hermitean polynomial
$H^S$	perturbation Hamiltonian
$H_{12}^S$	matrix element of perturbation Hamiltonian
$H_y$	y-component of magnetic field strength
$h, \hbar$	Planck's constant, $A = h/2\pi$
$I$	intensity
$I_i$	incident intensity
$I_{\lambda}$	intensity of mode A
$I_s$	saturation intensity

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$I_T$	transmitted intensity
$i$	imaginary unit
$J_{kk'k''}$	convolution of spatial modes
$\mathbf{j}$	current density
$K(\Lambda, \partial x)$	linear matrix
$K(q)$	force acting on fictitious particle
$K_2(t, t')$	mutual coherence function
$\mathbf{k}$	wave vector
$\mathbf{k}_\lambda$	wave vector of mode $\lambda$
$\mathcal{L}$	optical path in ring cavity
$L$	generalized total Fokker-Planck operator
$L$	cavity length
$L_A$	generalized Fokker-Planck operator for atoms
$L_{A-f}$	generalized Fokker-Planck operator for field mode-atoms
$L_f$	generalized Fokker-Planck operator for field mode
$l_j$	abbreviation
$M$	number of coexisting modes
$M_j$	expectation values of powers of photon number
$M_{\mu\lambda\lambda'\lambda''}$	abbreviation
$m$	electron mass
$N$	Fresnel number
$N$	number of atoms, number of locked modes
$\mathcal{N}$	normalization factor
$N_i$	occupation number of level $i$
$N_{i0}$	stationary occupation number of level $i$
$N_{i,s}$	stationary occupation number of level $i$
$N_{i,\text{thr}}$	occupation number of level $i$ at threshold
$N_{j,\mu}$	occupation number of level $j$ at atom $\mu$
$N(\Lambda, q)$	nonlinear part of equation of motion
$n$	photon number
$\bar{n}$	average photon number
$n_0$	stationary photon number
$n_i$	initial photon number in Q-switching
$n_j$	occupation number
$n_{\text{max}}$	maximal photon number in Q-switching

$n_{\text{sp}}$	number of spontaneously emitted photons
$n(t)$	temporal dependence of photon number in Q-switching
$n_{\text{th}}$	thermal photon number
$n_{\lambda}$	photon number of mode $\lambda$
$\bar{\mathbf{O}}, \bar{\mathbf{O}}^j$	l.h.s. eigenvector
$\mathbf{O}, \mathbf{O}^j$	r.h.s. eigenvector
$O(\beta, \beta^*)$	exponential operator
$P, P(a)$	emission intensity of a laser (at pump strength $a$ )
$\mathbf{P}$	atomic polarization density
$\hat{\mathbf{P}}$	normalized polarization
$P(t)$	total dipole moment in externally driven laser
$P(\mathbf{u}, \mathbf{u}^*)$	Glauber-Sudarshan distribution function
$\mathbf{P}(\mathbf{x}, t)$	atomic polarization density
$\mathbf{P}^{(+)}(\mathbf{x}, t)$	atomic polarization density, positive frequency part
$\mathbf{P}^{(-)}(\mathbf{x}, t)$	atomic polarization density, negative frequency part
$P_{\lambda}(t)$	time dependent amplitude in mode expansion of polarization
$P_{\lambda}^{(-)}(t)$	negative frequency part of mode amplitude (polarization)
$P_{\lambda}^{(+)}(t)$	positive frequency part of mode amplitude (polarization)
	small deviation from normalized polarization, density of photon states, normalized polarization
$\mathbf{p}$	atomic dipole moment
$\mathbf{p}^{(+)}$	positive frequency part of dipole moment
$\mathbf{p}^{(-)}$	negative frequency part of dipole moment
$\mathbf{p}_{\mu}$	dipole moment of atom $\mu$
$p(n, T_0)$	distribution for photon counting
$Q$	quality factor
$Q, Q_{ij}$	diffusion coefficient in Fokker-Planck equation
$q(t)$	coordinate of fictitious particle
$\mathbf{q}(\mathbf{x}, t)$	deviation from stationary state space vector
$R$	reflection coefficient of mirror
$R(t)$	modulus of order parameter
$r(t)$	modulus of complex electric field amplitude
$S$	total dipole moment
$S_-, S_+, S_z$	spin operators

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$T$	longitudinal relaxation time, transmission coefficient of mirror
$T_1$	longitudinal relaxation time
$t$	time variable
$U(\mathbf{x})$	cavity mode function
$U(\mathbf{x}, t)$	state space vector
$U_0(\mathbf{x}, t)$	stationary state space vector
$u$	classical variable corresponding to field mode
$u_\lambda(\mathbf{x})$	spatial dependence of cavity mode
$\mathbf{u}_\lambda(\mathbf{x})$	cavity mode eigenfunction
$V$	cavity volume
$V(q)$	potential of fictitious particle
$v$	velocity of gas atoms, classical variable corresponding to dipole moment
$W$	spontaneous emission rate per atom and time
$W(\bar{n})$	probability distribution for discrete photon numbers
$W_i$	energy of level $i$ , eigenvalue
$W_j(k)$	energy band
$W_{\lambda\mu}$	emission probability of atom $\mu$ to mode A
$W_\lambda(\omega)$	spectral emission probability into mode A
$\bar{W}_\lambda$	abbreviation for constants
$w(\mathbf{x}, t)$	eigenfunction of linear equation
$w_{ji}$	transition probability $i \rightarrow j$
$w_s$	spot radius (size)
$X$	normalized transmitted field, general case
$X, Y$	spatial coordinates, normalized
$x$	normalized transmitted field
$x, y, z$	spatial coordinates
$\mathbf{x}$	space point (vector)
$\mathbf{x}_\mu$	position of atom $\mu$
$Y$	normalized incident field, general case
$y$	normalized incident field
$y(\omega)$	spectral intensity
$z$	axial coordinate



$\alpha$	inhomogeneous line-width, atomic polarizability, absorption coefficient, eigenvalue
$\alpha_l$	critical control parameter
$\alpha_\mu(t)$	complex dipole moment of atom $\mu$
$\alpha(t)$	expectation value of normalized complex dipole moment
$\beta, \beta_j$	eigenvalue
$\Gamma$	real part of eigenvalue
$\Gamma_d(t)$	quantum mechanical fluctuating force (inversion)
$\Gamma_+, \Gamma_-, \Gamma_{ij}(t)$	quantum mechanical fluctuating force (dipoles)
$\Gamma_{\mu+}, \Gamma_{\mu-}$	quantum mechanical fluctuating force (single dipole)
$\gamma$	atomic (natural) line-width
$\gamma_{\parallel}$	longitudinal relaxation constant
$\gamma_{\perp}$	homogeneous line-width
$\Delta$	Laplace operator, normalized detuning parameter
$\Delta V$	volume element
$\Delta\nu$	line-width of laser light, atomic line-width
$\Delta\omega$	band width
$\delta$	constant in density matrix equation, normalized detuning
$\delta(\mathbf{x})$	Dirac's $\delta$ -function
$\delta_{kp}, \delta_{\lambda\lambda'}$	Kronecker's symbol
$\delta_n$	small deviation from stationary photon number
$\delta N_i$	small deviation from stationary occupation number
$\delta\omega$	normalized detuning parameter
$\nabla$	gradient operator (Nabla operator) dielectric constant
$\varepsilon_0$	dielectric constant of vacuum
$\Theta$	angular beam width
$\vartheta$	atomic dipole moment
$\vartheta_{jk}$	dipole moment matrix element
$\kappa$	cavity damping constant
$\kappa(I)$	intensity dependent cavity loss
$\kappa_\lambda$	damping constant of mode A
$\Lambda$	normalized pump parameter
$\lambda$	wavelength, mode index
$\mu$	index of atom, magnetic susceptibility

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$\mu_0$	magnetic susceptibility of vacuum
$\nu$	atomic transition frequency
$\xi$	constant in density matrix equation
$\xi$	electronic coordinate
$\xi_\mu$	spatial displacement of electron at atom $\mu$
$\xi_{k,j}(t)$	time dependent amplitude
$\xi_s(t)$	stable mode amplitude
$\xi_u(t)$	unstable mode amplitude (order parameter)
$\rho$	reduced density matrix, spatial distance
$\rho_0$	spatial density of atoms
$\rho_f$	reduced density matrix of field mode
$\rho_{f,\text{tot}}$	density matrix of field mode coupled to heat bath
$\rho(\omega)$	spectral density of modes
$\rho_{\text{tot}}$	density matrix of total system
$\sigma$	electric conductivity
$\tau$	normalized time variable
$\Phi$	wave function
$\Phi(t)$	phase of complex electric field amplitude
$\Phi(\delta)$	error integral
$\varphi$	phase factor
$\varphi_j$	eigenfunction of unperturbed Hamiltonian
$\varphi_\lambda$	constant phase in mode amplitude
$\chi$	susceptibility
$\chi$	complex absorption coefficient
$\chi(\xi)$	characteristic function
$\chi(\beta, \beta^*)$	characteristic function of Wigner distribution function
$\chi_p(\beta, \beta^*)$	characteristic function of Glauber–Sudarshan function
$\chi_Q(\beta, \beta^*)$	characteristic function of Q-distribution function
$\Psi$	wave function
$\psi$	relative phase of locked modes
$\Omega$	laser frequency in loaded cavity, imaginary part of eigenvalue, general quantum mechanical operator
$\omega$	cavity mode circular frequency
$\omega'$	frequency spacing of cavity modes
$\bar{\omega}$	circular frequency of atomic transition

$\bar{\omega}_0$	central frequency
$\bar{\omega}_\mu$	circular frequency of atom $\mu$
$\omega_\lambda$	frequency of mode $\lambda$ in unloaded cavity
$\omega_M$	modulation frequency
$\omega_{nm}$	transition frequency
$\omega_r$	imaginary part of eigenvalue
$\omega_u$	imaginary part of unstable eigenvalue

## Chapter 1

# Introduction

### 1.1. The maser and laser principle

The word "laser" is an acronym composed of the initial letters of "light amplification by stimulated emission of radiation". The laser principle emerged from the maser principle. The word "maser" is again an acronym standing for "microwave amplification by stimulated emission of radiation". The concept of stimulated emission stems from Einstein when in 1917 he derived Planck's law of radiation. It took nearly 40 years until it was recognized that this process can be used in a device producing coherent microwaves and – in particular – a new type of light – laser light.

The maser was proposed by Basov and Prokhorov (1954–1955) and by Townes (1954), who performed also experiments on that new device. We owe the extension of this principle to the optical region Schawlow and Townes (1958).

One of the first proposals to use stimulated emission was contained in a patent granted in 1951 to V.A. Fabrikant, but being published in the official Soviet patent organ, it became available only in 1959.

In 1977 patents on aspects of the laser principle were granted to Gould. Since his work had not been published it remained unknown to the scientific community.

Because the laser principle is an extension of the maser principle, first the word "optical maser" had been proposed by Schawlow and Townes. However, nowadays the word "laser" is widely used because it is shorter.

In order to understand the laser principle it is useful to first consider the maser principle. The device realizing this principle, which is again called maser, essentially consists of two components. On the one hand a cavity, on the other hand molecules which are in the cavity or which are injected into it. A cavity is practically a metal box of certain shape and dimension. In it specific electromagnetic waves with discrete wave-lengths can be formed (figs. 1.1 and 1.2). The corresponding standing waves shall be denoted in the following as "modes". They possess a discrete sequence of **eigen-**

## 2 1. Introduction

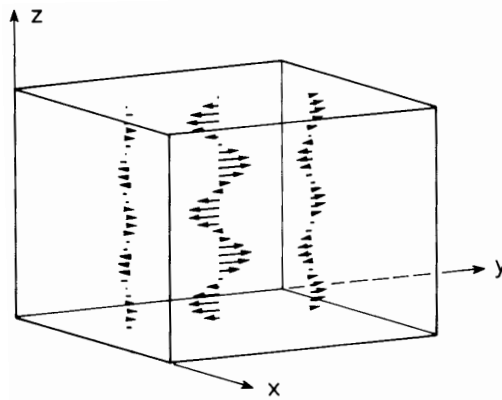


Fig. 1.1. Electro-magnetic field mode in a cavity. Local directions and sizes of the electric field strength are indicated by the corresponding arrows.

frequencies. These modes, which can exist in the cavity in principle, are now to be excited. To this end energetically excited molecules, e.g. ammonia molecules, are injected into the cavity. In order to understand the maser process, for the moment being it is only important to know that a transition between the excited state of the  $\text{NH}_3$  molecule and its ground state can take place which is accompanied by the emission of an electro-magnetic wave with quantum energy  $h\nu = W_i - W_f$ , where  $\nu$  is the frequency of the emitted wave, whereas  $W_i$  and  $W_f$  are the energies of the initial and final

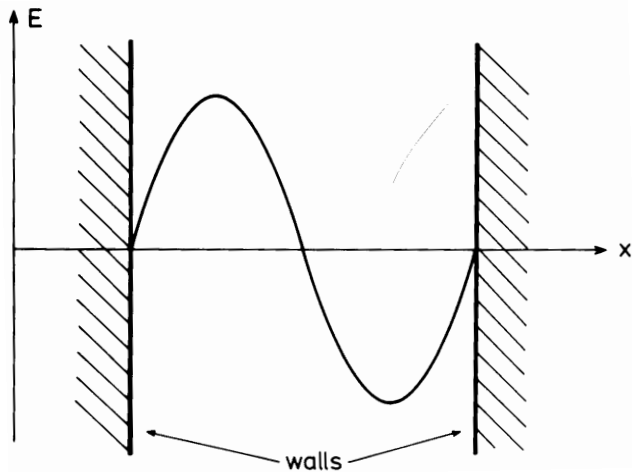


Fig. 1.2. Standing electric wave between two ideally conducting walls.

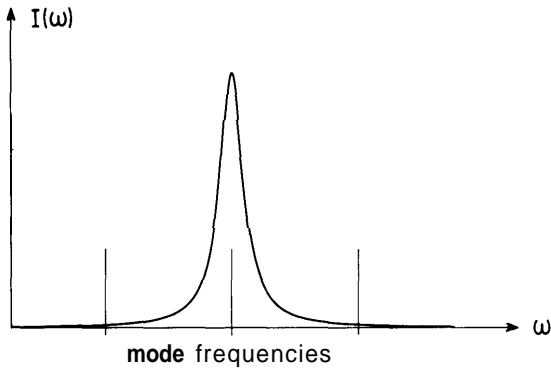


Fig. 1.3. Emission intensity of a molecule versus circular frequency. In most cases, in the microwave region the mode frequencies are so far apart that only one frequency comes to lie within the emission line.

state of the molecule, respectively. As we know (cf. Vol. 1), excited atoms or molecules can be stimulated to emit light quanta if one or several quanta of the electro-magnetic field are already present, and the whole process is called stimulated emission. By means of excited molecules in the cavity, a specific mode can be amplified more and more by stimulated emission. In order to achieve an efficient energy transfer from the molecules to the electro-magnetic field, the frequency of the molecular transition must coincide with the frequency of the mode to be amplified. More precisely speaking, it is necessary that the mode frequency lies within the line-width of the molecular transition. With respect to the molecules used in the maser we can achieve the amplification of a specific mode by choosing the dimensions of the microwave cavity correspondingly. In this way only one frequency falls into the line-width whereas all other mode frequencies lie outside of it (fig. 1.3).

Schawlow and Townes suggested to extend the maser principle to the optical region by using optical transitions between electronic levels of atoms. When one tries to realize the laser principle, fundamental new problems arise as compared to the maser. These problems stem from the fact that the light wave-length is small compared to a cavity of any reasonable dimension. Therefore in general the distance between different mode frequencies becomes very small so that very many modes come to lie within the frequency range of the atomic transition (fig. 1.4). Therefore a suitable mode selection must be made. One possibility consists in omitting the side walls of the resonator and to use only two mirrors mounted in parallel at two opposite sides. The thus resulting Fabry-Perot resonator, which was suggested by

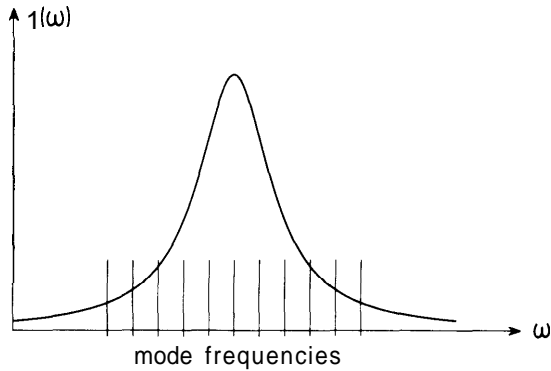


Fig. 1.4. Example for the positions of mode frequencies in the optical region. In general many frequencies come to lie within an emission line.

Schawlow and Townes, and Prokhorov and Dicke, makes a mode selection possible in two ways. Let us consider figs. 1.5 and 1.6. Before the laser process starts, the excited atoms emit light spontaneously into all possible directions. On account of the special arrangement of the mirrors only those light waves can stay long enough in the resonator to cause stimulated emission of atoms, which are sufficiently close to the laser axis, whereas other modes cannot be amplified. This mechanism is particularly efficient because only waves of the same direction, wave-length, and polarization are amplified by the stimulated emission process. In this way the Fabry–Perot interferometer gives rise to a strong discrimination of the modes with respect

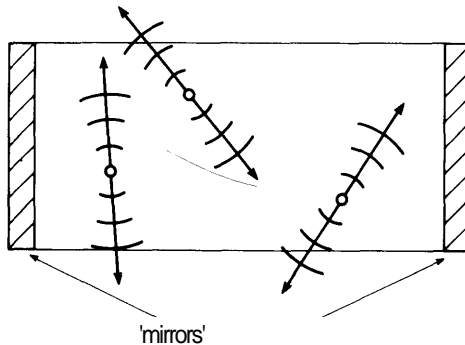


Fig. 1.5. The excited atoms in the laser resonator can radiate light into all directions. Waves which do not run in parallel to the laser axis, leave the resonator quickly and do not contribute to the laser process.

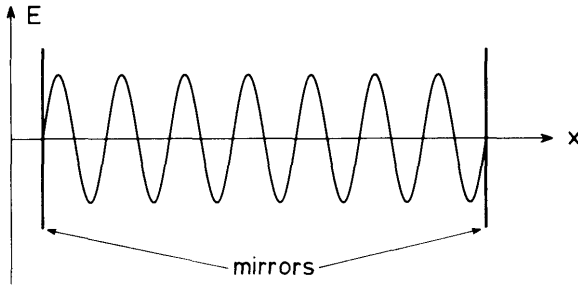


Fig. 1.6. The electric field strength of a standing axial wave in the laser resonator.

to their lifetimes. Furthermore the mirror arrangement can support only those axial modes for which

$$n \frac{\lambda}{2} = L,$$

where  $\lambda$  is the wave-length,  $L$  the distance between the mirrors, and  $n$  an integer. Even under these circumstances quite often still many frequencies may exist within an atomic line-width. The final mode selection, often the selection of a single mode, is achieved by the laser process itself as we shall demonstrate in this book.

The first experimental verification of the laser principle in 1960 is due to Maiman, who used ruby, a red gem. Since then laser physics has been mushrooming and it is still progressing at a rapid pace. Practically each year new materials or laser systems are discovered and still important tasks are ahead of us, for instance the extension of the laser principle into the X-ray and y-ray region. Today a great many laser materials are known and we shall briefly discuss some typical of them in section 2.3.

## 1.2. The problems of laser theory

In this book we shall focus our attention on the *theoretical treatment of the laser process*. As we shall see, a wealth of highly interesting processes are going on in the laser and we shall treat them in detail. But what are the physically interesting aspects and problems of a laser theory? To this end we have to realize that within a laser very many laser-active atoms, say  $10^{14}$  or more, are present which interact with many laser modes. Thus we have to deal with a many-particle problem. Furthermore the laser is an open system. On the one hand the laser emits all the time light through one of its mirrors which has some transmissivity, and on the other hand energy



must be continuously pumped into the laser in order to maintain the laser process. Thus the system is open with respect to an energy exchange with its surrounding. Because the atoms are continuously excited and emit light, the atomic system is kept far from thermal equilibrium. Over the past years it has become evident that the laser represents a prototype of systems which are open and far from thermal equilibrium. Clearly the optical transitions between the atomic levels must be treated according to quantum theory. Indeed, the discrete structure of spectral lines is a direct consequence of quantum theory. Quite evidently we have to deal here with a highly complicated problem whose solution required new ways of physical thinking. This task has been solved in several steps.

### 1.2.1. *Rate equations*

The simplest description which still has the character of a model rests on equations for the temporal change of the numbers of *photons* with which the individual "cavity" modes are occupied. A typical equation for the photon number  $n$  is of the form

$$\frac{dn}{dt} = \text{generation rate} - \text{annihilation rate}.$$

These equations are quite similar to those with which Einstein derived Planck's formula (cf. Vol. I). Such kind of description, which has been used by Tang and Statz and DeMars and many others for laser processes, is still used today when global phenomena, such as the intensity distribution of laser light, are studied. On the other hand such a model-like description based on photon numbers is insufficient for the treatment of many important processes in modern laser physics. This is in particular so if phase relations between laser light waves are important. A theory which describes most laser processes adequately is the semiclassical laser theory.

### 1.2.2. *Semiclassical theory*

This theory deals with the interaction between the *electromagnetic field* of the "cavity" modes and the *laser active atoms* in solids or gases. The field is treated as a classical quantity, obeying Maxwell's equations, whereas the motion of the electrons of the atoms is treated by means of quantum theory.

The source terms in Maxwell's equations, which in a classical treatment stem from oscillating dipoles, are represented by quantum mechanical averages. Furthermore, pumping and decay processes of the atoms are taken into account. The resulting coupled equations are nonlinear and require

specific methods of solution. Such a theory was developed in 1962 by myself and was further developed by my coworkers and myself in the subsequent years. This theory, which we shall present in this book in detail, allows us to treat the multimode problem both in solid state as well as in gas lasers. In this way we shall understand under which conditions only a single mode can be selected by the laser process or, when several modes can coexist. Furthermore we shall find that by means of the laser process the frequencies of the emitted laser light are shifted with respect to the atomic and cavity frequencies. Under well defined approximations, in particular that there are no phase relations between the individual mode amplitudes, the rate equations can be derived from the semiclassical equations and thus given a sound basis. A theory equivalent to our theory was developed independently by Lamb and published by him in 1964, whereby Lamb treated the gas laser. A number of important new phenomena, such as ultrashort pulses occur, when phase locking between modes takes place. The semiclassical equations are still used by numerous scientists as a basis for the study of various laser phenomena and we shall present a number of explicit examples. In this way, the semiclassical theory will form the central part of this book, dealing with the dynamics of laser light.

### *1.2.3. Quantum theory of the laser*

The semiclassical theory, which describes the behavior of the atoms by means of certain quantum mechanical averages and treats the light field as a classical quantity, has a strange consequence. Whereas above a critical pump power, by which the atoms are continuously excited, laser light is created in the form of a completely coherent wave, below that critical pump strength no light emission should take place at all. Of course, a satisfactory laser theory must contain the emission of usual lamps as a special case also, and it must be capable of explaining the difference between the light from lamps, i.e. from thermal sources, and laser light. As we know, light of conventional lamps is produced by spontaneous emission. Spontaneous emission of light is a typical quantum mechanical process. Quite evidently the semiclassical theory cannot treat this process. Thus it becomes necessary to develop a completely quantum mechanical theory of the laser. The previously known quantum mechanical theory, in particular the detailed theory of Weisskopf and Wigner, could explain this spontaneous emission of an individual atom in detail, but this theory was insufficient to describe the laser process.

Thus we were confronted with the task of developing a laser theory which is both quantum mechanical and contains the nonlinearities known from

semiclassical theories. This theory, which I published in 1964, showed that laser light differs basically from light from conventional lamps. Whereas light from conventional lamps consists of individual incoherent wave tracks, laser light essentially consists of a single wave whose phase and amplitude are subject to small fluctuations. Subsequent measurements of the intensity fluctuations of laser light below and above threshold by Armstrong and Smith (1965), and Freed and Haus (1965) fully substantiated my predictions. My approach required the exclusion of the immediate vicinity of the laser threshold. This gap was closed in 1965 by Risken (and subsequently by Hempstead and Lax). Risken interpreted my quantum mechanical laser equation as a classical Langevin equation and established the corresponding Fokker–Planck equation. The stationary solution of the Fokker–Planck equation describes the photon statistics in the laser. We shall deal with the coherence and noise properties of laser light as well as with its photon statistics in chapters 10 and 11. In order to treat these questions, besides the Langevin and Fokker–Planck equations the density matrix equation was used also. Density matrix equations, which describe both the atoms and the light field quantum mechanically, were derived by Haake and Weidlich (1965), and by Scully and Lamb (1966). Solutions of laser density matrix equations in different kinds of representation were given by Scully and Lamb (1966), and by Weidlich, Risken and Haken (1967). This work was carried further by a number of authors, who used still other representations and included higher order terms.

#### *1.2.4. Quantum classical correspondence*

In this section we are abandoning the main stream of this book, to which we shall return in the next section, 1.2.5, and make some technical remarks of interest to theoreticians.

An interesting question arose why a quantum mechanical process can be described by a classical Fokker–Planck equation. This led to a further development of the principle of quantum classical correspondence which allows us to establish a connection between a quantum mechanical description and a classical formulation without loss of quantum mechanical information. Such a transcription had been initiated by Wigner (1932) who treated quantum systems described by the position and momentum operator. A further important step was done by Glauber and Sudarshan (1963) who treated Bose-field operators. In particular, Glauber's careful study of quantum mechanical correlation functions provided a general frame for the description of the coherence properties of light. But, of course, being a general frame, it did not make any predictions on the coherence properties

of laser light. For that purpose, the quantum theory of the laser had to be developed (cf. section 1.2.3). In it the inclusion of the atomic system is indispensable and required a considerable extension of the principle of quantum classical correspondence which was done by Gordon (1967), and Haken, Risken and Weidlich (1967) along different though equivalent lines. Because the principle of quantum classical correspondence has important applications not only in laser physics but also in nonlinear optics, we shall present it in section 11.2.

### *1.2.5. The laser – trailblazer of synergetics*

New vistas on laser theory were opened in 1968 when it was recognized that the transition from light from thermal sources to laser light within an individual laser bears a striking resemblance to phase transitions of systems in thermal equilibrium. Thus the laser became the first example in which the analogy between a phase transition of a system far from thermal equilibrium and one of a system in thermal equilibrium could be established in all details (Graham and Haken, 1968 and 1970; DeGiorgio and Scully, 1970; Kasanvez et al. 1968). It soon turned out that there is a whole class of systems which can produce macroscopic ordered states when driven far from thermal equilibrium. This gave birth to a new branch of scientific study, called "synergetics". In this way deep rooted analogies between quite different systems in physics, chemistry, biology and even in the soft sciences could be established. In this new development the laser played the role of a trailblazer. Within the frame of synergetics it became possible to make further predictions on the behavior of laser light. For instance, on account of analogies between fluid dynamics and laser light the phenomenon of laser light *chaos* was predicted (Haken, 1975). Various routes to chaotic laser light could be discovered experimentally. We shall come to these fascinating questions in chapter 8.

### *1.2.6. Optical bistability*

In this book we shall include other aspects of laser theory also, for instance that of optical bistability. While in conventional lasers the laser is pumped incoherently, devices leading to optical bistability can be viewed as lasers which are driven coherently by an external field. For this reason a good deal of the theoretical methods developed for the laser can be applied to optical bistability. A thorough theoretical treatment is due to Lugiato and others. The name "optical bistability" stems from the fact that under suitable conditions the transmission of light through a resonator filled with atoms

can acquire two different states. The "optical bistability" device bears great promises for the construction of an optical transistor.

### 1.2.7. Two-photon laser

The main part of this book deals with laser processes in which an optical atomic transition generates one photon. As we know, in optical transitions also two or several photons can be absorbed or emitted simultaneously. This has led to the idea of a two-photon laser to which substantial contributions have been given by Walls, Wang and others. We shall include a short description of its theoretical treatment in chapter 12.

## 1.3. The structure of laser theory and its representation in this book

Let us finally discuss the structure of laser theory and its representation in this book. In a strict logical sense the structure of laser theory is as follows. At its beginning we have a fully quantum theoretical treatment of atoms and the light field as we presented it in chapter 7 of Vol. 1. The corresponding equations describe the interaction between atoms and light field. But in addition, the atoms as well as the light field are coupled to their surroundings, for instance the field is coupled to loss mechanisms in the mirrors, or the laser atoms are coupled to their host lattice (fig. 1.7). The coupling of field and atoms to their corresponding surroundings leads to damping and fluctuations which we treated in Vol. I. In this way the basic quantum mechanical equations for the laser result, which is treated as an open system. If we average these basic equations over the fluctuations of the heatbaths representing the surroundings and form adequate quantum mechanical averages, we arrive at the semiclassical laser equations. When we eliminate from these equations the dipole moments of the atoms and average over phases we obtain the rate equations. The rate equations have a much simpler

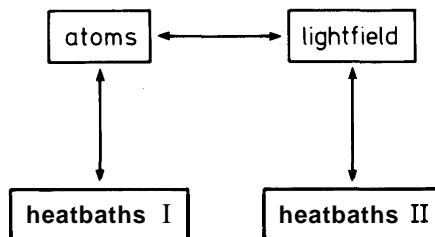
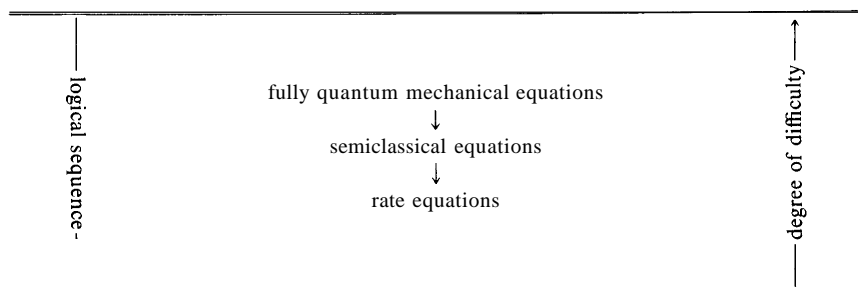


Fig. 1.7. Scheme of the coupling between atoms, light field and heatbaths.

Table 1.1



structure than the fully quantum mechanical equations, at least what the degree of difficulty of interpretation and solution is concerned. For this reason a conflict results with respect to the logical sequence and the pedagogical requirement.

In the present book I prefer the pedagogical aspect in order to keep my promise I gave in the preface, namely to present the whole field in a manner as simple as possible. For this reason I start with the rate equations which I derive heuristically. They will allow us to treat a number of important phenomena (compare table 1.2). After that we shall treat the semiclassical equations which we derive in detail but where we do not need to make use of the fully quantum mechanical equations. The semiclassical equations form the basis for the central part of this book in which we treat a variety of different phenomena such as single and multimode operation and in particular mode locking phenomena, which for instance give rise to ultra-short pulses. Furthermore we shall be concerned with a detailed description of chaotic laser light.

Finally we shall turn to a fully quantum mechanical treatment in which we shall give an outline of the method of quantum mechanical Langevin equations which have the advantage of being tractable in close analogy to the semiclassical equations. We shall include in our representation the density matrix equation and the method of quantum classical correspondence which will allow us to derive a classical Fokker–Planck equation for the quantum mechanical laser process. In this way we shall give a detailed account of the coherence and noise properties of laser light and its photon statistics. The structure of the laser theory is explained in table 1.2.

In conclusion of this introduction I should like to give the reader a hint how to read this book depending on his requirements.

If a reader wants a survey over the whole field without the necessity of going into all the details the following reading can be suggested:

Table 1.2. The structure of laser theory

---

1. <i>Rate equations</i> for photon numbers and atomic occupation numbers
These equations allow the treatment of the following problems: laser condition, intensity distribution over the modes, single mode laser action, multi-mode laser action (coexistence and competition of modes), laser cascades, Q-switching, relaxation oscillations.
2. <i>Semiclassical equations</i>
These rest on Maxwell's equations and the Schrodinger equation of electrons taking into account coupling to heatbaths.
These equations allow a treatment of the following problems (among others): frequency shifts, frequency locking, population pulsations, active and passive mode locking, undamped oscillations, ultrashort pulses, laser light chaos and routes to it, photon echo, wave propagation in an "inverted" material, optical bistability, two-photon laser, and all problems quoted under 1.
3. <i>Quantum mechanical equations</i>
They rest on a fully quantum mechanical treatment of the light field and the atoms by means of the Schrodinger equation or equations equivalent to it, in particular the Heisenberg equations. These equations allow a treatment of the following problems (among others): line-widths of laser light, phase, amplitude and intensity fluctuations (noise), coherence, photon statistics, and all problems quoted under 1 and 2.

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*List of sections for a first reading*

2.1–2.3	Basic properties and types of lasers
3.1	Laser resonators
4.2	Photon model of single mode laser
4.4	Q-switching
5.1–5.6, 5.8–5.9	Semiclassical equations
6.1–6.3	Single mode laser action including transients
6.8	Single mode gas lasers (perhaps)

*The further reading depends on the reader's interest.*

Readers interested in the quantum theoretical foundation of the basic equations and their applications:

Chapter 10 Coherence, noise and photon statistics. Quantum theory of the laser and perhaps chapter 11.

Readers interested in further "macroscopic properties", frequency locking, ultrashort pulses, chaos, etc.:

6.4–6.5	Multimode laser
6.6	Frequency locking

6.7	Laser gyro (perhaps)
7.1	Ultrashort pulses. Some basic mechanisms
8.1	Laser light chaos
8.2	(now needed 7.2)
8.3	
9.1	Optical bistability
9.2	

Readers can also proceed by reading the chapters individually if they want to get to know specific applications of the rate equations, semiclassical equations or the fully quantum mechanical equations. The most advisable way will be to get a survey along the lines indicated above and then to penetrate deeper by reading more sections of the corresponding chapters.

In conclusion of this chapter I present a table showing which knowledge of Volume 1 is required for an understanding of the chapters of the present book.

*Chapters of Volume 1 needed* (if not known otherwise):

Present Vol. 2 number of chapter	Vol. 1 needed chapters
1	1. What is light?
2	2. The nature of light
3	—
4	2. The nature of light
5–9	2+3. The nature of matter
10	6, 7.1–7.6, 8.1, 9.1–9.4. Quantization of field and electron-wave field, coupling to heatbaths
11	5, 6, 7.1–7.6, 8.1, 9.1–9.5
12	Chapters 5 and 6 of Vol. 2
13	



## Chapter 2

# Basic Properties and Types of Lasers

### 2.1. The laser condition

Let us consider the laser depicted in fig. 2.1 more closely, and let us discuss the tasks of its individual parts. The two mirrors mounted at the endfaces fulfil the following functions. When we treat light as a wave, between the two mirrors only standing waves can be formed. Their wave-lengths,  $\lambda$ , are connected with the distance between the mirrors,  $L$ , by the relation  $n\lambda/2 = L$  where  $n$  is an integer. In section 3 we shall briefly discuss the influence of the finite size of mirrors on the formation of these standing waves. On the other hand, when we consider light as consisting of photons, the two mirrors reflect photons running in axial direction again and again. Therefore these photons can stay relatively long in the laser, whereas photons which run in other directions leave the laser quickly. Thus the mirrors serve for a selection of photons with respect to their lifetimes in the laser.

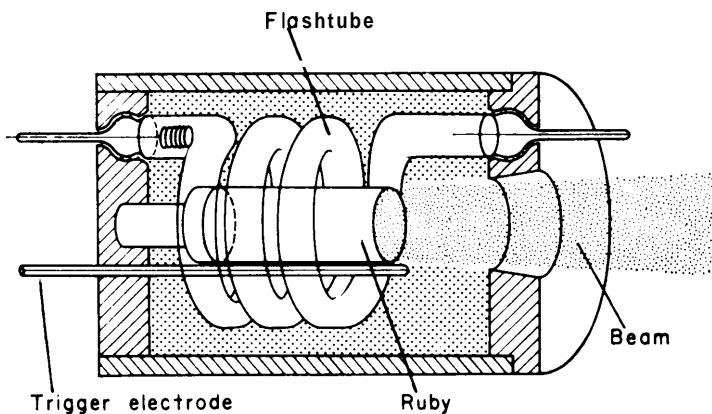


Fig. 2.1. The first experimental set-up of the ruby laser according to Maiman. The ruby rod in the middle is surrounded by a flashtube in the form of a spiral.

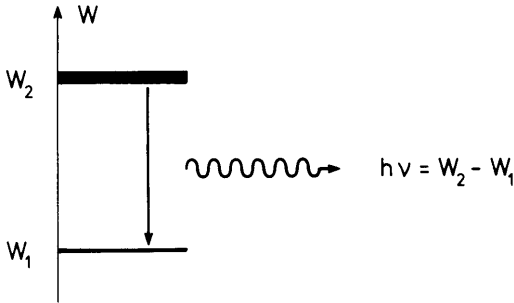
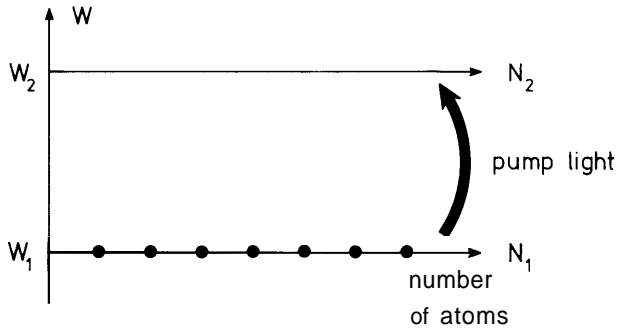


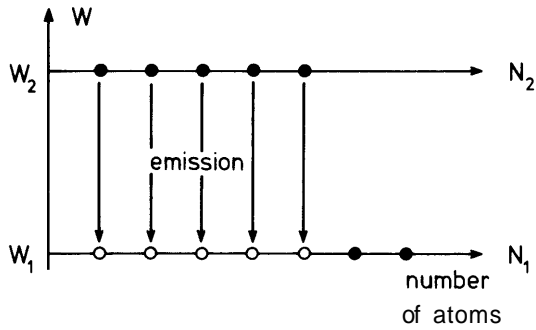
Fig. 2.2. The energy  $W$  of a two-level atom with the energy levels  $W_1$  and  $W_2$  of which the upper one is occupied. During the transition from level 2 to level 1 a photon of quantum energy  $h\nu = W_2 - W_1$ , is emitted.

Let us consider a single kind of photons, for instance those which run in axial direction and which belong to a certain wave-length  $\lambda$ , and let us study how their number  $n$  changes on account of the processes within the laser. To this end we have to make some assumptions on the atoms participating in laser action. We assume that each of the laser atoms has two energy levels between which the optical transition which leads to laser action takes place (fig. 2.2). The external pump light serves the purpose to bring a sufficiently large number of atoms into the excited states of the atoms, whose number we denote by  $N_2$ . The rest of the atoms with number  $N_1$  remains in the ground state (fig. 2.3). The excited atoms emit photons spontaneously with a rate proportional to the number of excited atoms,  $N_2$ . Denoting the rate with which a single excited atom generates a photon per second by  $W$ , the total spontaneous emission rate of photons reads  $WN_2$ . As we know, in addition photons can be generated by stimulated emission (cf. Vol. 1). The corresponding generation rate can be simply obtained from the spontaneous emission rate by a multiplication by  $n$ , i.e. for stimulated emission the generation rate is  $N_2 Wn$ . On the other hand, atoms in their ground states, absorb photons with the absorption rate  $-N_1 Wn$ . Finally we must take into account that the photons may leave the laser, for instance by passing through one of the mirrors or by scattering by impurities in the laser, etc. We denote the inverse of the corresponding lifetime,  $t_1$ , of the photons by  $2\kappa$ . The loss rate is then given by  $-2\kappa n$ . Adding up the contributions which stem from the individual processes just mentioned we obtain the fundamental laser equation

$$\frac{dn}{dt} = (N_2 - N_1) Wn + WN_2 - 2\kappa n. \quad (2.1)$$



(a)



(b)

Fig. 2.3. (a) Through the pump mechanism a number of atoms are lifted from their levels 1 into their levels 2. Thus the number of atoms in their ground states,  $N_1$ , is lowered and those of the atoms in their excited states increased. (b) The excited atoms can make transitions into their ground states by light emission.

The explicit expression for  $W$  was derived in Vol. 1 (eq. (2.96)). Let us rederive that result by some plausibility arguments. The spontaneous emission rate of an atom with respect to all possible kinds of photons is connected with the lifetime  $\tau$  of the atom with respect to spontaneous emission by  $W = 1/\tau$ . In the present context we are interested in the transition of the atoms leading to spontaneous emission of a specific kind of photons only. Therefore we have to divide the transition rate per second,  $1/\tau$ , by the number of all kinds of photons possible. Therefore we have to form  $W = 1/(\tau p)$ , where according to Vol. 1, eq. (2.56) the number  $p$  is given by

$$p = V 8 \pi \nu^2 \Delta \nu / c^3. \quad (2.2)$$

In it  $V$  is the volume of the laser,  $\nu$  the laser light frequency,  $\Delta\nu$  the atomic line-width and  $c$  the velocity of light in the laser medium.

By means of the formulas derived above we may immediately present the laser condition. Laser action sets in if  $n$  increases exponentially. This is guaranteed if the r.h.s. of eq. (2.1) is positive, where we have neglected the spontaneous emission rate  $WN_2$  which is then negligible. In a detailed quantum theoretical treatment of the laser in chapter 10 we shall see that in addition to the argument just presented the light spontaneously emitted (which is described by the term  $WN_2$ ) is incoherent whereas stimulated emission gives rise to coherent light. Using the abbreviations for  $W$  and  $\kappa$  we immediately obtain the laser condition

$$\frac{N_2 - N_1}{V \frac{8\pi\nu^2 \Delta\nu}{c^3} \tau} > \frac{1}{t_1}. \quad (2.3)$$

This condition tells us which laser materials we have to use and how we have to construct a laser. First of all we have to take care that the lifetime  $t_1$  of photons within the laser is big enough. As we shall see below this can be reached by making the distance between the mirrors sufficiently large. In order to find an estimate of  $t_1$  we imagine that the photons run in axial direction and that they quit the laser with a certain probability each time they hit one of the mirrors. This probability can be expressed in a simple way by the reflectivity,  $R$ , of the mirrors. As one readily sees, the lifetime of a photon is proportional to the distance between the mirrors, inversely proportional to the velocity of light, and inversely proportional to  $1 - R$ . We thus obtain the relation

$$t_1 = \frac{L}{(1 - R)c}. \quad (2.4)$$

In order to treat a concrete example let us put

$$R = 90\%, \quad L = 30 \text{ cm}. \quad (2.5)$$

We thus obtain

$$t_1 = 10^{-8} \text{ s}. \quad (2.6)$$

Now let us discuss the left hand side of the inequality (2.3). In order to fulfil (2.3) we must make  $N_2 - N_1$ , i.e. the inversion, as big as possible. The volume  $V$  should be as small as possible or, if we form the ratio between the inversion and the volume, the inversion per volume or, in other words, the inversion density, must be sufficiently large. The factor  $u^2$  should be as

small as possible but because in each case one wants to generate light of a specific wave-length the size of  $v^2$  is fixed and cannot be circumvented. But we see that with increasing frequency it becomes increasingly more difficult to fulfil the laser condition which makes it so difficult to build an X-ray laser. Both the atomic line-width  $\Delta\nu$  and the lifetime  $\tau$  of an atom (with respect to light emission) should be chosen as small as possible. But here fundamental limits exist. As is known from quantum mechanics, the uncertainty relation  $\Delta\nu\tau \geq 1$  holds.

Inserting some typical data such as

$$\nu = 10^{15} \text{ s}^{-1}, \quad c = 3 \times 10^{10} \text{ cm s}^{-1}, \quad \tau = 10^{-8} \text{ s}, \quad \Delta\nu = 10^{10} \text{ s}^{-1}, \quad (2.7)$$

we obtain the inversion density which is necessary for laser action

$$\frac{(N_2 - N_1)}{V} > 10^{10} [\text{cm}^{-3}]. \quad (2.8)$$

In section 2.3 we shall get to know a number of pump mechanisms by means of which we may achieve the necessary inversion.

### Exercises on section 2.1

(1) Calculate  $W$  for the following laser data (ruby):

$$V = 62.8 \text{ cm}^3,$$

$$\nu = 4.32 \times 10^{14} \text{ Hz},$$

$$\Delta\nu = 2.49 \times 10^{13} \text{ Hz},$$

$$c = 2.9979 \times 10^8 \text{ m/s},$$

$$\tau = 3.0 \text{ ms}.$$

(2) Calculate the number of modes of a closed resonator whose edges have the length  $L = 1 \text{ cm}, 10 \text{ cm}, 100 \text{ cm}$ , which are present within the line-width  $\Delta\nu = 6.22 \times 10^9 \text{ Hz}$ . How big is the number of modes in these cases if the modes are axial modes ( $E = \sin kx$ ) in a Fabry-Perot interferometer?

(3) Calculate the quality factor  $2\kappa = 1/t$ , using formula (2.4) for the following cases: length of the laser resonator (=distance between the mirrors)  $L = 1 \text{ cm}, 10 \text{ cm}, 100 \text{ cm}$ , reflectivity  $R = 99\%, 90\%, 10\%$ . How do the results change if the index of refraction is  $n = 2, n = 3$ ? Compare the resonator line-width  $\kappa = 1/(2t_1)$  with the distance between the mode frequencies and the optical line-width of ruby (compare exercise 2).

(4) Calculate the critical inversion density of ruby by means of the laser condition where the following data may be used:

$$V = 62.8 \text{ cm}^3,$$

$$\nu = 4.32 \times 10^{14} \text{ Hz},$$

$$A\nu = 2.49 \times 10^{13} \text{ Hz},$$

$$\tau = 3.0 \text{ ms},$$

$$c' = 1.70 \times 10^8 \text{ m/s} \text{ (} c' = \text{light velocity in ruby)},$$

$$R = 99\%.$$

Hint: Neglect the degeneracies of the levels.

## 2.2. Typical properties of laser light

The typical properties of laser light make the laser an ideal device for many physical and technical applications. Let us quote some of its most important properties.

(1) Laser light can have high intensities. Within laser light pulses, powers far greater than  $10^{10}$  W can be achieved. In order to visualize this power just think that  $10^8$  light bulbs, each with 100 W, are needed to produce the same power. It is more than the power of all American power stations taken together. For applications in laser fusion, lasers with the power of more than  $10^{13}$  W are built or tested experimentally at present. High cw emission can also be achieved. It reaches an order of magnitude of about  $10^5$  W. The achieved top powers are not published (for obvious reasons).

(2) Laser light possesses a high directionality. This stems from the fact that the light within the laser hits the mirrors at its endfaces in form of a plane wave, whereby the mirrors act as a hole giving rise to diffraction (fig. 2.4). In this way the ideal divergence of a plane wave diffracted by a slit is closely approached. A laser with a diameter of a few centimeters can give rise to a laser beam which, when directed to the moon, gives rise to a spot of a few hundred meters in diameter. The strict parallelism of the emerging light results in an excellent focusability which jointly with the high laser light intensity allows a production of very high light intensities in very small volume elements. When one calculates the electric field strength belonging to the corresponding light intensity, field strengths result which are far bigger than  $10^8$  V/cm. These are field strengths to which otherwise electrons in atoms are subjected. In this way ionization of atoms by means of laser light becomes possible.

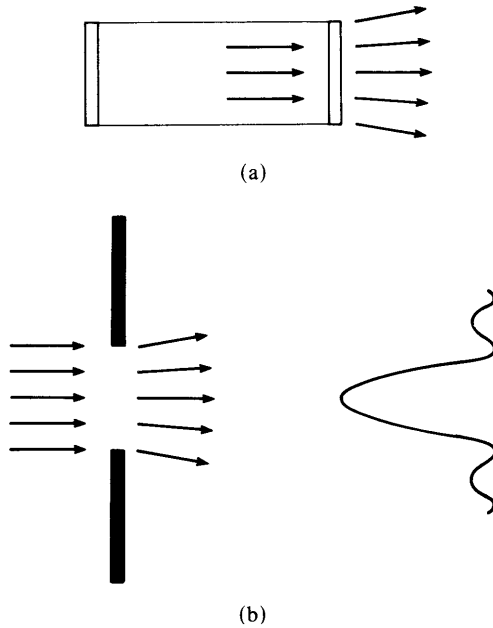


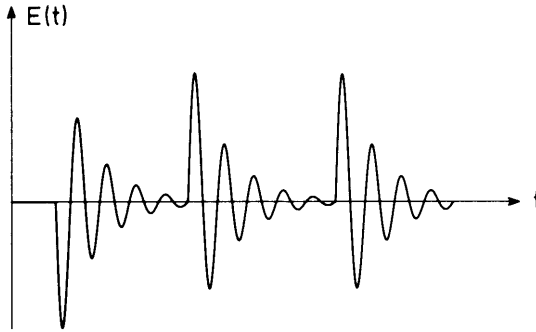
Fig. 2.4. By means of the laser process a plane parallel wave is produced in the laser (a). The divergence of the emitted beam corresponds to that of a plane wave diffracted by a slit (b).

(3) The spectral purity of laser light can be extremely high. The frequency width which is inversely proportional to the emitted power can be  $\delta\nu = 1 \text{ Hz}$  for 1 W emitted power in the ideal case. Experimentally  $\delta\nu = 100 \text{ Hz}$  has been realized. Taking  $\delta\nu = 1$ , the relative frequency width for visible light is  $\delta\nu/\nu = 10^{-15}$  which is of the same order of magnitude as that of the Mossbauer effect. It is important to note that this frequency purity is achieved jointly with a high intensity of the emitted line quite in contrast to spectrographs where high frequency purity is achieved at the expense of intensity. The frequency purity of laser light is closely connected with its coherence (see point (4)).

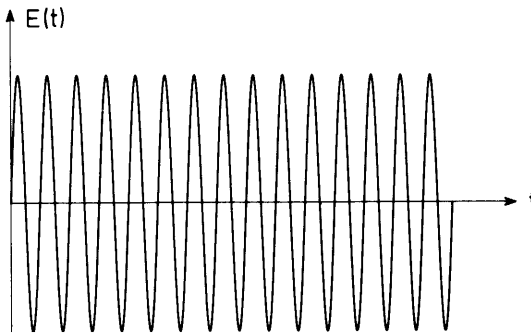
(4) Coherence. While light of usual lamps consists of individual random wave tracks of a few meters length, laser light wave tracks may have a length of 300,000 km.

(5) Laser light can be produced in form of ultrashort pulses of  $10^{-12}$  s duration (picosecond) or still shorter, e.g. 30 femtoseconds (1 femtosecond =  $10^{-15}$  s).

Quite evidently the properties of laser light just mentioned make the laser an ideal device for many purposes which we shall explore in the present



(a)



(b)

**Fig. 2.5.** (a) The electric field strength  $E(t)$  of light of a lamp consists of uncorrelated individual wave tracks. (b) Laser light consists of a single coherent very long wave track.

and the subsequent volume. A most interesting question which we shall study later in great detail consists in the problem how the transition from the emission of a lamp to that of the laser takes place. If we pump the laser only weakly and plot its electric field strength  $E$  versus time we obtain the picture shown in fig. 2.5. The light field consists of entirely uncorrelated individual wave tracks. The whole light field looks like spaghetti. When we increase the pump power beyond a certain threshold, an entirely new behavior of laser light emerges. It becomes an extremely long wave track. This sudden transition which transforms light from one quality into that of another quality becomes apparent also when we plot the emitted power (of a single mode) versus pump power (fig. 2.6). While below laser threshold, i.e. in the range of thermal light, the emitted intensity increases only slowly,



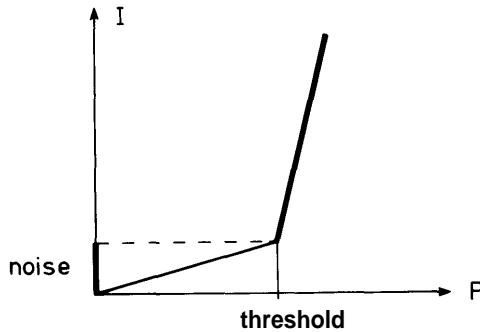


Fig. 2.6. The emitted power  $I$  versus pump power. Within the region of operation of the laser as a lamp the field consists of noise only and increases only slowly with increasing pump power. Above threshold the emitted intensity increases much more strongly with pump power. The intensity is taken with respect to a specific mode.

above threshold it quickly increases. This is a hint that the internal state of order of the laser changes abruptly at laser threshold. This is a process which is strongly reminiscent of phase transitions of superconductors or ferromagnets. Indeed we shall show in chapter 13 that this analogy is very close. Among the more recently discovered properties of laser light are the following. Under suitable conditions, namely high pumping and bad cavity quality, laser light can exhibit chaotic behavior. Laser light chaos is an entirely new type of light which must not be mixed up with so-called "chaotic light from thermal sources". As we have just seen, light from thermal sources consists of very many individual wave tracks. Chaotic light on the other hand still consists of a giant wave track which, however, may show specific fluctuations which we shall explore later in this book. Indeed the study of chaotic laser light has become a new chapter in laser physics.

### 2.3. Examples of laser systems (types of lasers and laser processes)

As we have seen above, a typical laser consists of the following parts: the laser-active material, the pump source, and the resonator. In this section we wish to get to know a number of examples of laser materials. Today there is a great variety of materials which can produce laser action and new materials are still developed. The list of our examples is by no means complete and we wish rather to discuss some laser materials which are of particular importance. Readers who are interested in the basic principles of laser physics only can skip this section totally or may consider only our first example, the ruby laser.

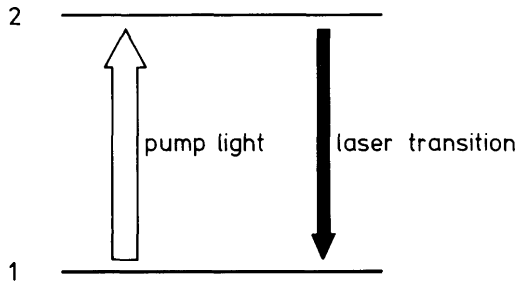


Fig. 2.7. Pumping process and laser light emission of a two-level atom.

2.3.1. *Energy and pump schemes; kinds of line broadening*

Before we discuss individual laser materials it will be useful to give a survey on different energy schemes used between which pump and laser processes go on. Indeed we shall find only a few basic types. We got to know the simplest type already above. In this case the laser material consists of individual atoms each having only two levels. The optical transition which leads to laser action takes place between these two levels. Because sufficiently many atoms must be excited to obtain laser action the atoms must be pumped energetically from their level 1 into their level 2 from the outside (compare fig. 2.7).

This model is entirely sufficient for a theoretical derivation of most properties of laser light. When we wish to build an actual laser, the energy level scheme becomes somewhat more complicated. We may distinguish between three basic types. The first type is represented in fig. 2.8. The

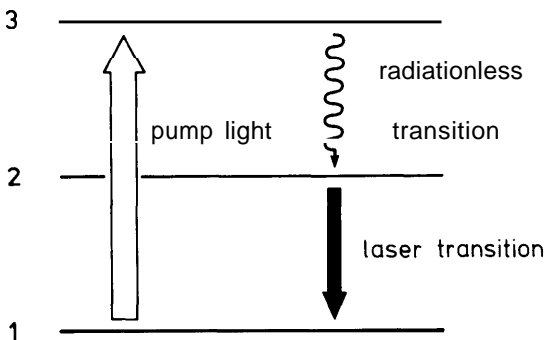


Fig. 2.8. Pumping process, radiationless transition, and laser transition of a three-level atom, where the lower transition leads to laser action.

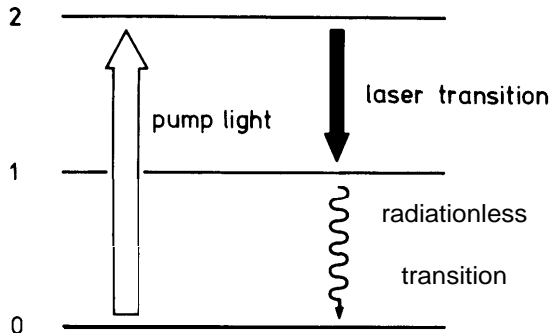
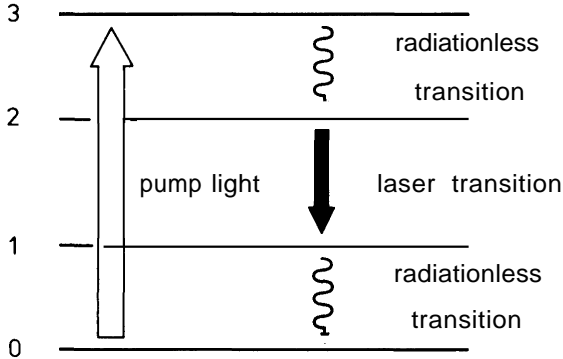


Fig. 2.9. Pumping process, radiationless transition, and laser transition of a three-level atom, where the upper transition leads to laser action.

electron of the atom in the ground state 1 is excited into a state 3. This excitation can be done by an irradiation of the atom by pump light with a frequency corresponding to the transition frequency from level 1 to level 3 (optical pumping according to Kastler). The electron can make a radiationless or radiative transition from level 3 into level 2, from where it makes an optical transition to level 1. This optical transition forms the basis of the laser process. A further pump scheme is represented in fig. 2.9. The ground state of the atom is denoted by 0. Out of this ground state the system is brought into level 2 by optical pumping. From this level 2 the optical transition into the level 1 can take place. The electron in level 1 can recombine to its ground state by a radiationless or radiative transition. The radiationless transitions can be caused by several mechanisms, for instance collisions between gas atoms among each other, or collision of gas atoms with the walls, interaction of atoms in lattices with lattice vibrations, etc. As we have seen when deriving the laser condition, a sufficiently high inversion  $N_2 - N_1$  must be achieved. Because according to the scheme of fig. 2.8 initially practically all atoms are in their ground states, the production of a sufficiently high inversion requires a much higher pump power than that corresponding to the scheme of fig. 2.9. If the recombination from level 1 to level 0 occurs sufficiently rapidly, level 1 will remain occupied only weakly and the inversion can be established by the number of excited atoms,  $N_2$ , alone.

Another pump scheme which is used quite often is that of fig. 2.10. Here the optical pumping occurs from level 0 into level 3. From there a radiationless or radiative recombination of the electron into level 2 occurs. Level 2 serves as initial level for the optical transition to level 1 thus serving as



**Fig. 2.10.** Four-level atom with pumping process, radiationless transitions, and laser transition, where the middle transition leads to laser action.

basis of the laser process. A radiationless or radiative recombination then happens between level 1 to level 0.

It is well known that not only individual electrons in atoms may possess discrete energy levels but that also complex quantum systems possess energy levels. As we shall see below, practically all laser materials can be subsumed under the system of transition processes indicated above. In practical cases, however, a more detailed consideration may be necessary. Quite often the pumping takes place using a whole set of different levels. The reason for that rests in the fact that one wishes to pump the system as strongly as possible. Furthermore the optically active levels 2 and 1 are broadened.

We briefly remind the reader (cf. also Vol. 1) that we have to distinguish between different kinds of line broadening. On the one hand there are level broadenings which are common to all atoms of the system in the same way. This kind of broadening is called "homogeneous line broadening". A broadening which is always present is the "natural line broadening" (fig. 2.11). It results from the finite lifetime of the electron which is leaving the excited state in order to make its optical transition. The linewidth  $\Delta\nu$  is connected with the lifetime  $\tau$  by the relation  $\Delta\nu = 1/\tau$ . We shall meet other kinds of broadening when we consider concrete cases of laser materials. For instance laser active atoms in solids experience different external perturbations, in particular local electric fields, depending on their individual positions. In this way the atomic energy levels are shifted depending on the individual atomic position. This leads to inhomogeneous line broadening. When we consider the ensemble of atoms, due to the individual energy shifts of the atoms the total line appears as a superposition of (homogeneously broadened) lines (fig. 2.12). Another important

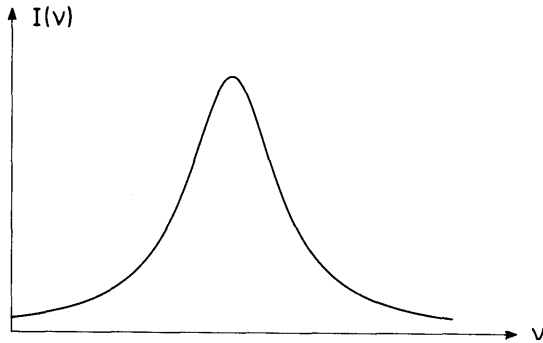


Fig. 2.11. The Lorentzian line.

inhomogeneous broadening is caused by the Doppler effect of moving atoms. This effect is well known from acoustics. When a car, which sounds its horn, passes, the horn seems to have a higher frequency when the car approaches us and to have a lower frequency when the car has passed. This effect occurs also in optics. When an atom moves with the velocity  $v$  towards an observer, the frequency of the light emitted by the atom seems enhanced according to the formula  $\nu' = \nu(1 + v/c)$ , where  $\nu$  is the transition frequency of the atom at rest. When an atom flies in opposite direction the opposite sign applies,  $\nu' = \nu(1 - v/c)$ . If a gas contains atoms moving with different velocities, for instance according to the thermal velocity distribution (Maxwell's distribution), this velocity distribution leads to a corresponding

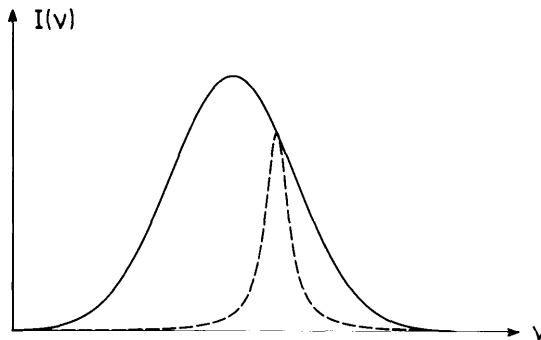


Fig. 2.12. An inhomogeneously broadened line of Gaussian shape (solid line). For comparison the Lorentzian line of the transition with a homogeneously broadened width is indicated also (dashed line).

frequency distribution. In such a case we shall speak of a Doppler broadened line (fig. 2.13).

**2.3.2. Laser materials**

In this section we wish to consider a number of examples. This list is by no means exhaustive but may serve rather the purpose to give the reader an idea how varied laser materials can be. Let us first consider *transitions of electrons in atoms*. Such atoms can be built in as impurities in solids. This leads us to our first class of laser systems, namely:

**Solid state lasers**

(a) Ruby. Ruby was the first material in which laser action was found. Ruby, a well known gem, is a crystal consisting of aluminumoxide,  $Al_2O_3$ .

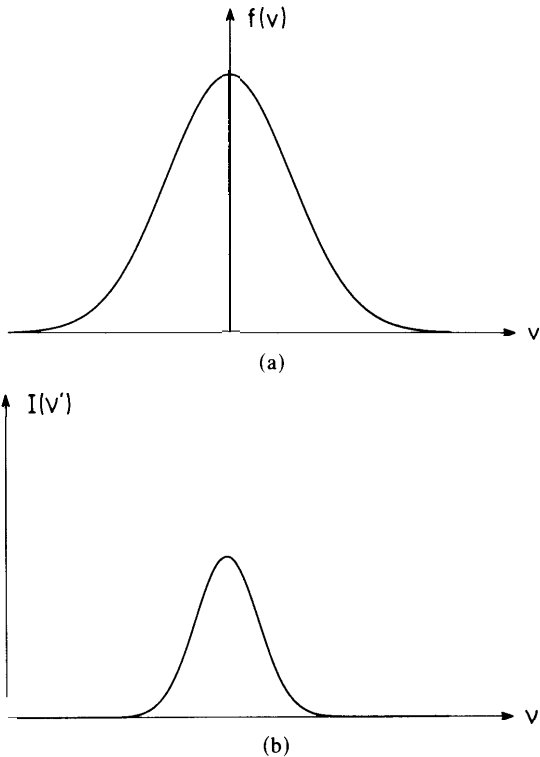


Fig. 2.13. (a) Maxwellian velocity distribution function  $f(v)$  of gas atoms which move at a velocity  $v$  in parallel to the laser axis. (b) The Doppler broadened emission line of gas atoms due to the Maxwellian velocity distribution according to fig. 2.13a.

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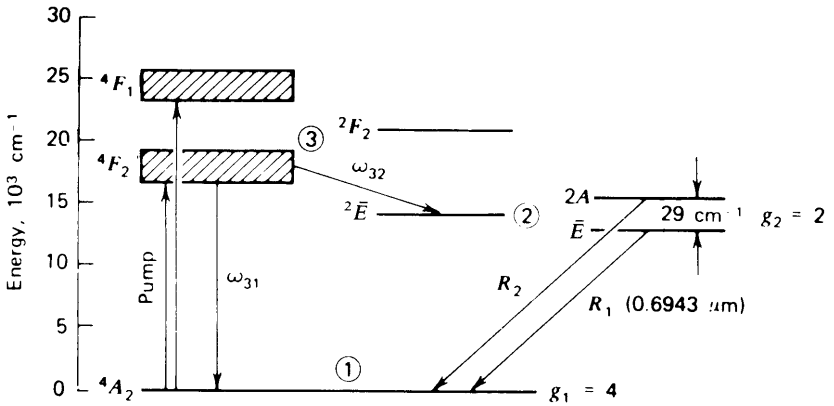


Fig. 2.14. Energy level scheme of ruby: (1) ground state; (2) excited state from which the transition occurs; (3) pump levels. [T.H. Maiman, Phys. Rev. Lett. 4, 564 (1960).]

The lattice is doped with Cr<sup>3+</sup> ions, i.e. triply ionized chromium, typically with a concentration of 0.05% (in weight). The chromium ions lend ruby its red color. Laser action takes place between levels of Cr<sup>3+</sup> whose corresponding energy level scheme is shown in fig. 2.14. Thus basically we are dealing with the scheme corresponding to fig. 2.8, where the participating levels 2 and 3 are split. Optical pumping takes place into the levels denoted by 4F<sub>2</sub> and 4F<sub>1</sub> in fig. 2.14. From these levels the chromium ion relaxes into the level 2 of fig. 2.8 which in fig. 2.14 is denoted by 2F. In fact this level is split into two further sublevels. The lower one of these two levels, which is denoted by E, serves as the initial state for the optical transition, i.e. for the laser transition into level 1 of fig. 2.8 (which actually is fourfold degenerate). The optical transition denoted by R<sub>1</sub> takes place at 0.6943 μm. The lifetime of the upper laser level E-bar is about 3 × 10<sup>-3</sup> s. The linewidth Δν strongly depends on temperature. At 300 K, Δν = 2 × 10<sup>11</sup> Hz. Usually ruby is excited by intense flash lamps but other light sources are used also in order to generate cw emission.

(b) The neodymium glass laser. In this case glass serves as the basic substance which is doped with laser active neodymium ions (Nd<sup>3+</sup>). The pump scheme is that of fig. 2.10, but instead of the single level 3 of fig. 2.10 a whole set of levels is used. It is important to note that the lower laser level 1 is separated from the ground state 0 energetically so far that even at room temperature the occupation number of level 1 differs from that of the level 0 by a factor e<sup>-10</sup>. Thus we may assume that the level 1 is initially practically unoccupied. The optical transition between levels 2 and 1 takes place at λ = 1.06 μm.

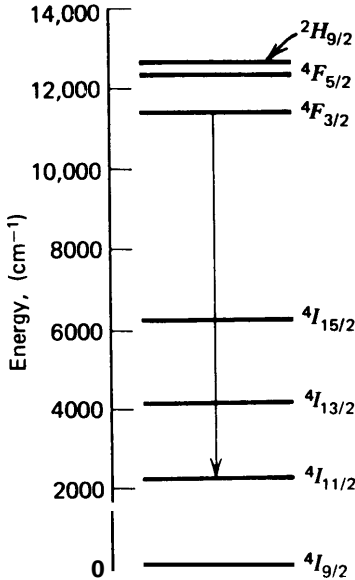


Fig. 2.15. Compare text. [A. Yariv, *Quantum Electronics*, 2nd ed. Wiley, New York 1976.1

(c) The neodymium YAG laser. In this case the neodymium ions are embedded in yttrium–aluminum garnets which consist of  $Y_3Al_5O_{12}$ . The level scheme corresponds to that of fig. 2.15. Laser action takes place at  $\lambda = 1.0641 \mu\text{m}$  at room temperature.

(d) Neodymium pentoxide crystals. Neodymium can be built in at regular lattice sites in the crystals mentioned above and can show laser action.

(e) As a last example of solid state lasers we mention calcium wolframate doped with ions of the rare earths.

### Gas laser

In this case the laser active atoms form a gas. The first example found experimentally was the He–Ne laser in which a gas mixture of helium and neon atoms is used (fig. 2.16). The laser transitions take place in Ne especially at  $\lambda = 0.6328 \mu\text{m}$ ,  $\lambda = 1.15 \mu\text{m}$  and  $\lambda = 3.39 \mu\text{m}$ . The pumping of the Ne atoms is particularly interesting. In the gas mixture which typically contains 1.0 mm Hg of He and 0.1 mm Hg of Ne a dc or ac discharge takes place. By it electrons of sufficiently high energy are liberated which can excite the He atoms by collisions. The electrons of the He atoms recombine by means of a cascade and preferably accumulate in the long living metastable states



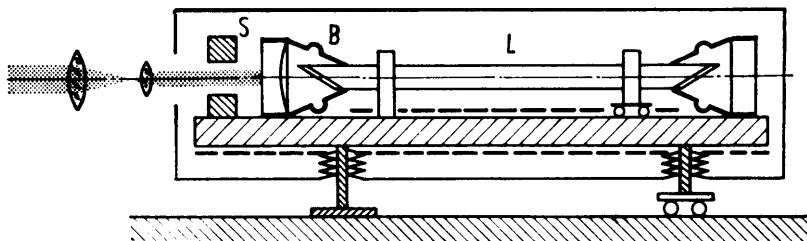


Fig. 2.16. Example of an experimental set-up of the He-Ne laser. The gas discharge tube of the laser is denoted by L. At the endfaces the mirrors are mounted under the Brewster angle. The laser is mounted in the way indicated in order to avoid vibrations.

$2^3S$  and  $2^1S$  (fig. 2.17). Because these long living levels practically coincide with the  $2S$  and  $3S$  levels of Ne, by means of collisions the excited He atoms can transfer their energy to the Ne atoms which are thereby brought into excited states. These states serve as initial states for the laser transition or even for a cascade of laser transitions.

A further important class of gas lasers is represented by ion lasers. Laser active transitions occur in ions of the gases of He or Ar. Ionization and excitation is achieved by electron impact. The wave-length of the emitted light lies in the ultraviolet.

### *Electronic transitions in molecules*

#### *Excimer lasers*

In order to understand the concept of excimers let us consider two atoms or molecules with closed electronic shells, e.g. two He atoms. Being in their ground states they **repell** each other, therefore no He molecule can exist. But if an electron of one atom is excited this atom can form with the other atom a molecule which is called an excimer. If the excited electron recombines, the molecule decays. In this way the laser condition can be fulfilled in an ideal manner because the ground state of the molecule does not exist so that  $N_1=0$ . Laser action of excimer systems was first found in liquid xenon which was energetically pumped by an electronic beam. In the meantime laser action of excimers was found in gaseous  $Xe_2$ ,  $Kr_2$ ,  $Ar_2$  as well as in gaseous compounds of noble gases and halogenides such as  $XeBr$ ,  $XeF$ ,  $XeCl$ ,  $KrF$ ,  $ArF$ , and  $KrCl$ . The atoms are excited by electron beams of high energy or by fast discharges. These lasers can emit light in the ultraviolet and vacuum-ultraviolet.

#### *Chemical lasers*

Here the excited state of an electron in a molecule is generated by a chemical process. An example is provided by the reaction between fluor

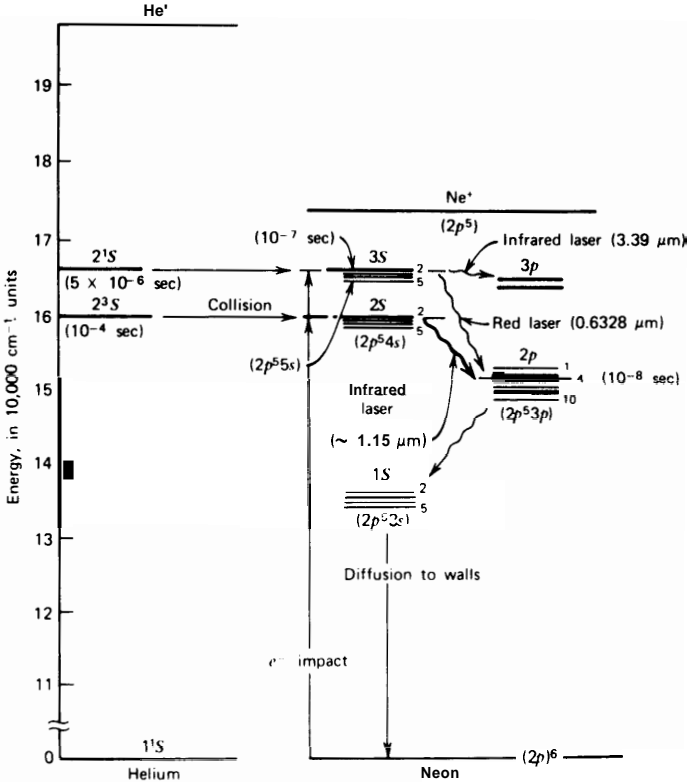
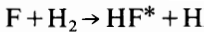


Fig. 2.17. The energy level scheme of helium and neon. The energy of the excited helium atom is transferred by means of collisions to neon. [W.R. Bennett, Appl. Optics, Suppl. 1, Optical Masers, p. 24 (1962).]

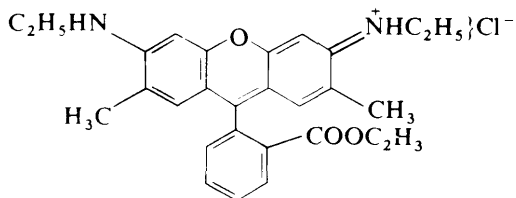
and hydrogen



in which the fluor atom is excited.

**Dye lasers**

Many organic dyes can exhibit a pronounced luminescence which covers a large range of wave-lengths in the visible range of the spectrum. A dye molecule which is quite often used for lasers is Rhodamine 6G. Its molecular structure is shown in fig. 2.18. The optical transitions are caused by electrons. We have to distinguish between two kinds of excitations. In one case the spin of the electron in the excited level is opposite to that of the remaining

Fig. 2.18. Molecular structure of Rhodamine **6G**.

molecule so that a total spin equal 0 results. We call these states S-states (singlet states). In the second case the spin of the excited electron is parallel to that of the remaining molecule. The total spin equals 1 and we are speaking of triplet states. Both kinds of states are further split due to molecular vibrations. The levels are represented in fig. 2.19 by heavy lines. Finally a still finer splitting exists. It stems from the rotation of the molecules which, according to quantum theory, is quantized. In usual experimental setups the dye molecules are in solution. The dye molecules are excited by

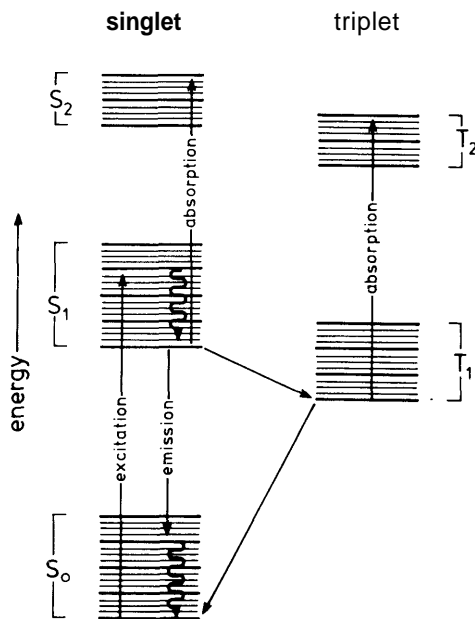


Fig. 2.19. Energy level scheme of a dye. On the left-hand side the groups of the singlet states are plotted which are further split due to oscillations and rotations (compare text). On the right-hand side the triplet states are plotted.

other lasers, e.g. the argon laser, whereby a transition from the group  $S_0$  into the group  $S_1$  occurs. This excitation is followed by a fast recombination to the lowest level of group  $S_1$ . From there the optical transition in one of the states of group  $S_0$  occurs. Besides this optical recombination a transition from  $S_1$  to  $T_1$  happens with a relatively small transition rate. Because the optical transition from  $T_1$  into the ground state of  $S_0$  is forbidden, the states  $T_1$  are long living. Unfortunately the absorption frequency of the transition from  $T_1$  to  $T_2$  coincides with the emission frequency from  $S_1$  to  $S_0$ . As a consequence the emitted laser light is strongly reabsorbed so that laser action is quickly suppressed. Therefore in such a case only laser light pulses can be emitted. However, by adding new substances to the solution a quick recombination of the states  $T_1$  can be achieved so that reabsorption is suppressed. Besides Rhodamine 6G there are a number of further organic dyes showing laser action. By a combination of different kinds of dye molecules a range of wave-lengths from 430 till 800 nm can be covered. On account of their broad luminescence lines, organic dyes are particularly well tunable. Tuning can be achieved, for instance, by reflection gratings.

#### *Laser action caused by molecular oscillations*

The most important example is provided by the  $\text{CO}_2$  gas laser. In these molecules the individual atoms can perform oscillations. The three fundamental kinds of oscillations are shown in fig. 2.20. According to quantum theory the different kinds of oscillations must be quantized so that discrete energy levels result. The energy level diagram belonging to some low lying oscillation levels of  $\text{CO}_2$  is represented in fig. 2.21. One of the laser processes rests on the optical transition between the levels which are denoted in fig. 2.21 by 001 and 100. The excitation of the uppermost level is usually achieved in a plasma discharge in which  $\text{N}_2$  and He participate in addition to  $\text{CO}_2$ . In the plasma discharge a large fraction of the two-atomic  $\text{N}_2$  molecules is excited to make vibrations, whereby the molecules accumulate in the excited state with the vibration quantum number  $n = 1$  of the harmonic oscillator. Collisions with  $\text{CO}_2$  molecules in their ground states make a transfer of the energy from the excited state of  $\text{N}_2$  to an excited state of  $\text{CO}_2$  possible. The remaining small energy difference is transformed into kinetic energy of the molecules after their collisions. The efficiency of  $\text{CO}_2$  lasers is very high and lies at about 30%. In order to achieve high power emission, lasers with a length up to several hundred meters have been built. According to quantum theory, besides the vibrational levels of  $\text{CO}_2$  molecules also discrete rotational levels are possible which may also participate in the laser process. If the gas pressure is increased above 5 Torr, due to the numerous collisions a line-broadening occurs which exceeds the usual

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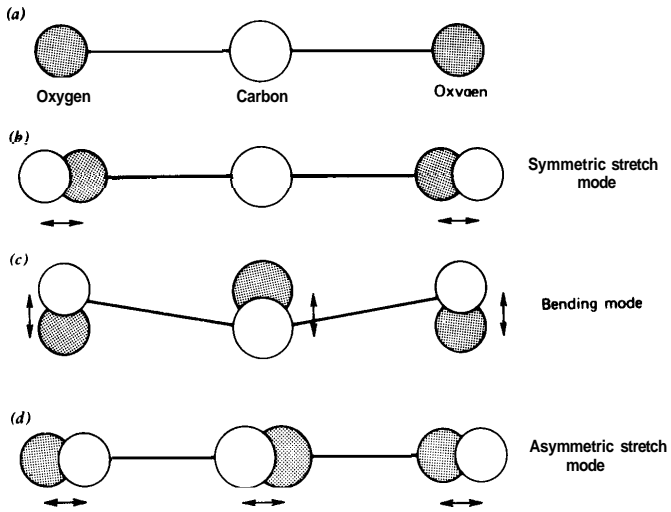


Fig. 2.20. Oscillatory states of the CO<sub>2</sub> molecule.

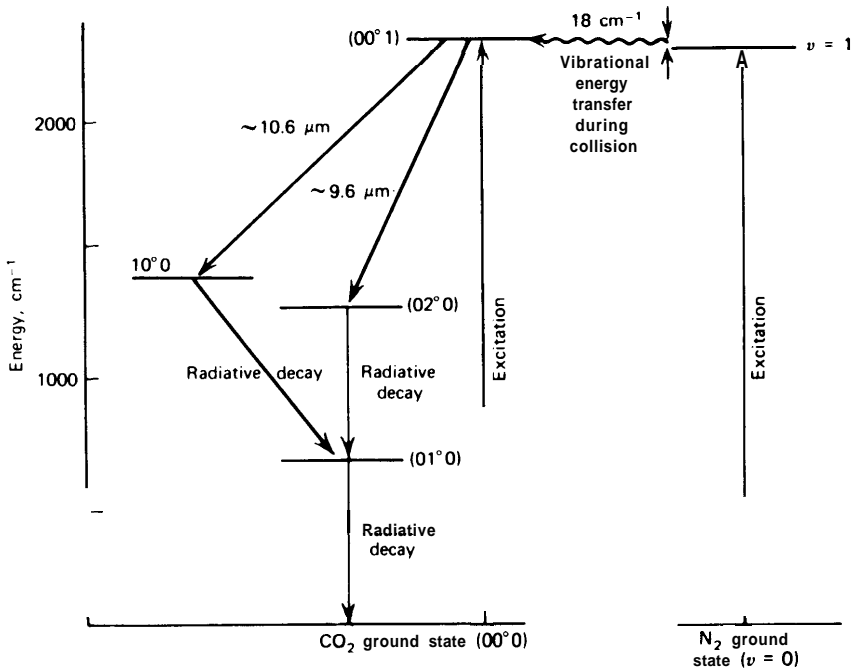


Fig. 2.21. Oscillatory states of CO<sub>2</sub>. [C.K.N. Patel, Phys. Rev. Lett. 12, 588 (1964).]

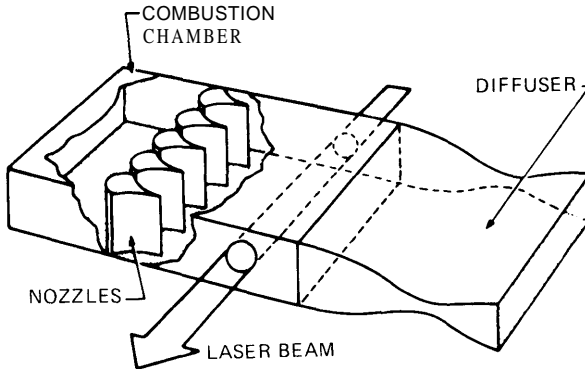


Fig. 2.22. Typical arrangement of a gas dynamic laser. [J.D. Anderson, Jr., *Gas Dynamic Lasers. An Introduction*. Academic Press, New York 1976.1

Doppler broadening. This gives rise to a second laser regime of the  $\text{CO}_2$  laser which is of particular interest for applications.

Particularly high emission powers can be achieved by gas dynamic lasers. Here a mixture of  $\text{CO}_2$ , N.,  $\text{H}_2\text{O}$  or He is used. This gas mixture, which is initially held under high pressure and is very hot, can expand through supersonic jets. During expansion an inversion of the gas atoms is reached so that a laser active medium results. The supersonic gas passes through an arrangement of mirrors whereby a laser light beam is generated (cf. fig. 2.22).

### *Electronic transitions in semiconductors*

Here we are dealing with a further class of solid state lasers. But the electronic states between which the laser transitions take place do not belong to individual impurity atoms but rather to the total crystalline lattice which forms the semiconductor.

A semiconductor is usually a crystal in which the individual atoms form a periodic lattice. In such a periodic structure electrons may propagate like periodically modulated waves with a wave-vector  $k$  (cf. Vol. 1). To a definite  $k$ -vector there belongs a whole set of energies  $W_j(k)$ ,  $j=1, 2, \dots$  (fig. 2.23). When we consider  $W_j(k)$  as a function of  $k$ , the energies form continuous bands which are separated by gaps (compare fig. 2.24 which presents an example of two energy-bands with a single gap). In the electronic ground state of the total crystal the individual energy levels, which we can visualize as being discrete but very dense, are filled up from the bottom with electrons. More precisely speaking, each level is filled with two electrons having

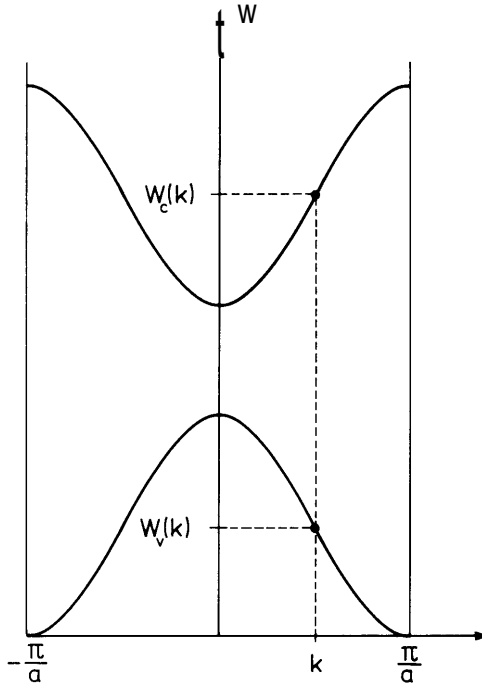


Fig. 2.23. Scheme of the energy bands of an electron in a solid. The energy  $W$  of an electron in the valence band (lower part) and of an electron in the conduction band (upper part) are plotted versus the  $k$ -vector.  $a$  is the lattice constant.

opposite spins. In an insulator the valence band is filled up entirely with electrons. The subsequent band, which is called the conduction band, is empty. As is shown in solid state physics, optical transitions can occur in a periodic lattice only under conservation of the  $k$ -vector, i.e. in the energy level scheme of fig. 2.25 the transitions must take place in vertical direction.

How can we achieve laser action in such a crystal? To this end we have to generate an inversion, i.e. we must excite electrons from the valence band into the conduction band. An example is shown in fig. 2.26 schematically. Because of the just mentioned  $k$ -selection rule the electrons can make their transitions independently of each other so that a sufficiently high inversion can be generated if we only bring enough electrons into the upper band, i.e. the conduction band. Experimentally such an inversion can be achieved by irradiating the crystal by a beam of electrons with sufficiently high energy. In this way electrons of the valence band are kicked into the conduction band where they accumulate at its bottom. In many practical applications,

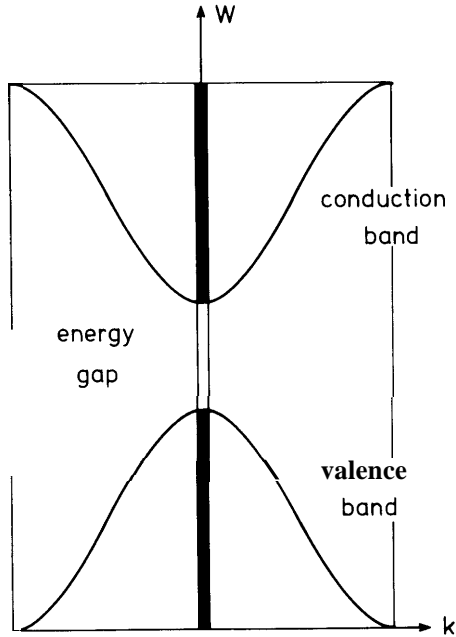


Fig. 2.24. When we project the energy levels onto the  $W$ -axis we clearly see that the conduction band and the valence band (black) are separated by a gap (white).

other kinds of pump mechanisms are used, however. If impurity centers are implanted into a crystal, not only new energy levels are generated but also the conduction and the valence bands are shifted. If different kinds of impurity atoms are implanted into different regions of the crystal, as indicated in fig. 2.27, an energy scheme as indicated in that figure arises. Because in the energetic ground state of the crystal the electrons occupy the lowest electronic energy levels, an occupation scheme as shown in fig. 2.27 results. Because the energetically lowest state is occupied, no optical transitions can take place. In order to generate an inversion, according to fig. 2.26, an electric field is applied to the crystal. This electric field causes an increase of the energy of the electrons at one end of the crystal and a lowering of their energies at the other end. In other words, the energy scheme is tilted. Because the electrons again wish to occupy the energetically lowest states, quite evidently they must make transitions as indicated in fig. 2.28. These transitions are optical transitions from an occupied into an unoccupied energy level and form the basis for laser action. We have been describing the general scheme of a p-n junction, where p and n are abbreviations for



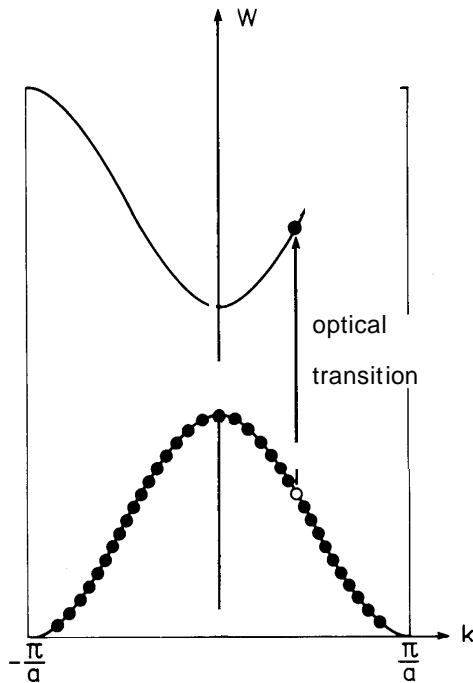


Fig. 2.25. The optical transition of an electron from the valence band into the conduction band takes place under conservation of the  $k$ -wave vector, i.e. vertically.

"positive" and "negative" (charge). An important example of such a semiconductor laser is provided by gallium arsenide (GaAs).

The simple scheme of optical transitions must still be modified because the electronic transitions are strongly influenced by the impurities. In this way the  $k$ -selection rule is violated.

Semiconductor lasers can be very small and may have diameters of less than a fraction of a millimeter. Because these light sources are nevertheless very intense, they can be used in medicine and also in communication networks. For technical reasons the simple  $p$ - $n$  junction we just described has to be modified in various ways. In particular multiple  $p$ - $n$  junctions are used of which fig. 2.29 shows an example.

A further class of semiconductor lasers is formed by exciton lasers. We briefly remind the reader of the concept of an exciton. Let us consider an insulator and let us visualize it as a crystal being built-up of its individual atoms with their localized electrons. If we excite such an insulator, an electron can be removed from its mother atom and transferred to another

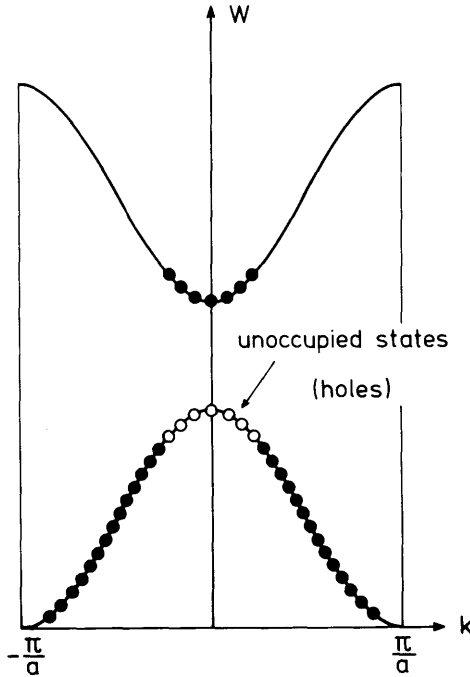


Fig. 2.26. When we excite a number of electrons optically or by electronic collisions, the electrons occupy states in the conduction band to which unoccupied states (holes) correspond in the valence band. In this way an inversion can be generated.

atom. In this way a positively charged hole at the mother atom remains. The negatively charged electron senses the attractive Coulomb force of the remaining hole and can circle around that (cf. Vol. 1). According to quantum theory, the total energy of the system electron + hole is quantized. This new kind of electronic system, consisting of an electron and a hole with quantized energies is called "exciton". If semiconductors are irradiated by high light intensities, high densities of such excitons are generated. When the electron and the hole of the exciton recombine, they can emit their total energy in form of light. When many excitons participate in this process, we might expect laser action to occur. However, a typical difficulty arises with excitons, because the emitted light can again generate new excitons and is thus reabsorbed. In this way the exciton system alone can never produce laser action. However, a new kind of process can occur in crystals. Namely the energy which is liberated by the recombination of an electron and a hole can be split into the energy of a photon and that of a phonon, which is a quantum of the lattice vibrations. In this way not enough *photon* energy is

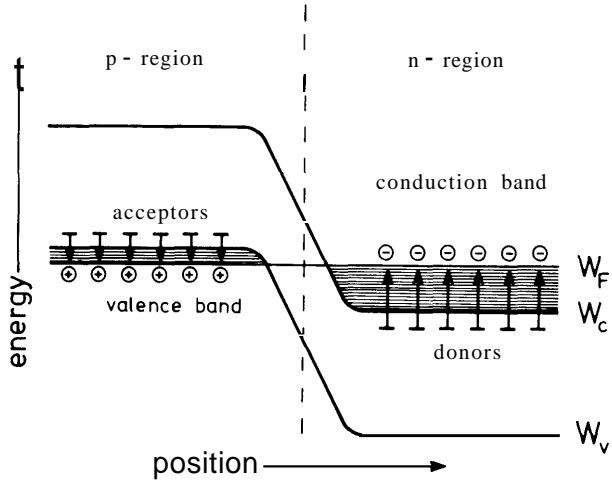


Fig. 2.27. The energy bands can be locally shifted by doping a semiconductor with various impurities. In this figure the energy is plotted as a function of the spatial coordinate. The donors are impurities in the crystalline lattice which can give their electrons to the conduction band. On the other hand, acceptors are impurities which can bind electrons or, in other words, which can generate holes in the valence band.  $F$  is the Fermi energy up to which the electronic levels can be filled up.  $W_c$  is the lower edge of the conduction band,  $W_v$  is the upper edge of the valence band. p and n refer to "positive" and "negative" according to the doping.

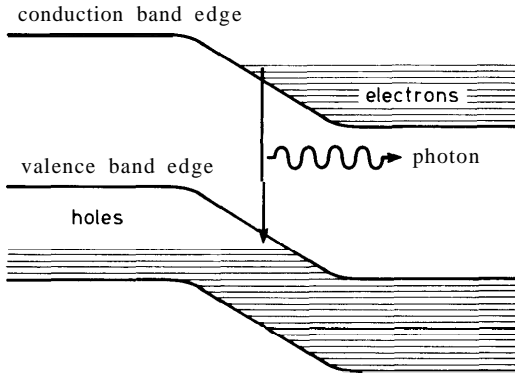


Fig. 2.28. When an external electric field is applied the electrons are shifted to one side (in our figure to the left side), whereas the oppositely charged holes are driven to the opposite side. In this way occupied electronic states come to stand above the unoccupied hole states so that electrons can make a transition as indicated by the vertical arrow and emit photons.

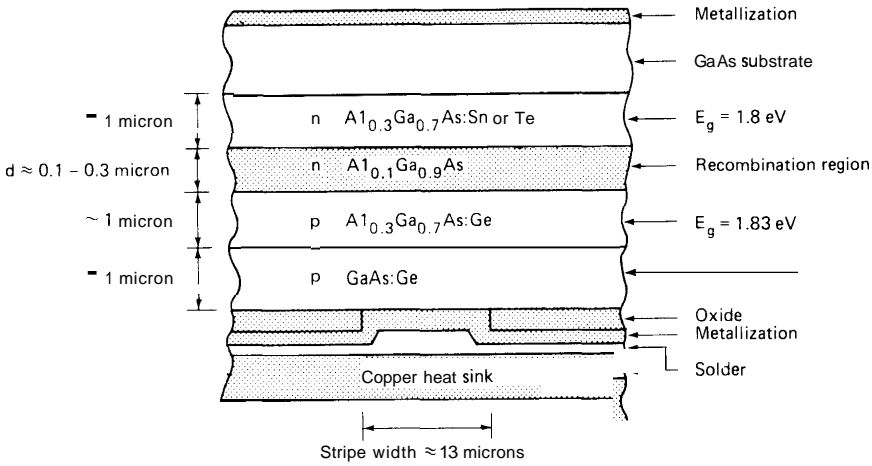


Fig. 2.29. Example for the experimental arrangement of a semiconductor laser. The cross section shows various layers. Laser action takes place in the recombination region. [H. Kressel, I. Ladany, M. Ettenberg and H. Lockwood, *Physics Today*, May 1976, p. 38.]

available for reabsorption and laser light can indeed be generated by exciton recombination (fig. 2.30).

As a final example of laser processes in solids we mention color center lasers. These are ionic crystals of sodium chloride (**NaCl**), potassium bromide (**KBr**), etc. The positively or negatively charged ions of sodium or chloride, respectively, are regularly arranged to form a lattice. In such a lattice defects can be formed in various ways. An important defect consists in the lack of a negatively charged chloride ion at a lattice site. Because the total crystal has been neutral it appears as if the defect possesses a

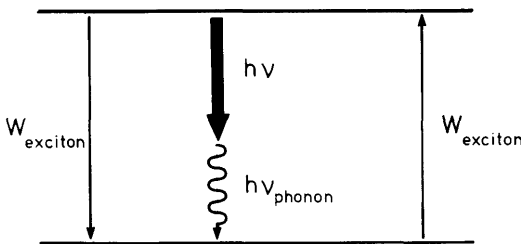


Fig. 2.30. In the exciton laser the total energy liberated by the recombination of an exciton is split into the light quantum energy  $h\nu$  and the phonon energy  $h\nu_{\text{phonon}}$ . In this way not enough light quantum energy is available to regenerate the exciton because reabsorption processes cannot take place.

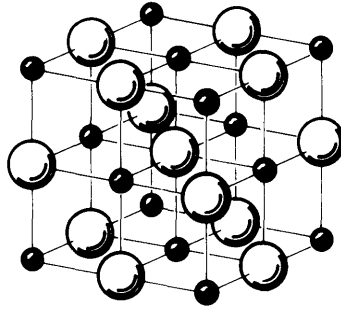


Fig. 2.31. The NaCl lattice. The negatively charged chlorine ions are represented by big white balls, the positively charged sodium ions by small black balls.

positive charge (fig. 2.31). Such a positively charged center can capture an electron which circles around that center. According to quantum theory the energy levels of the electron are quantized. The electron can make optical transitions between them and can emit or absorb light. Because these centers lend the above mentioned crystals their color, they are called color centers. If sufficiently many of these centers are excited, laser action can take place.

An entirely different class of lasers is provided by the free *electron laser*.

In this case the electrons move in vacuum and pass within an electronic beam through a spatially modulated magnetic field (fig. 2.32). By means of the Lorentz force the electrons are periodically deflected. It is well known

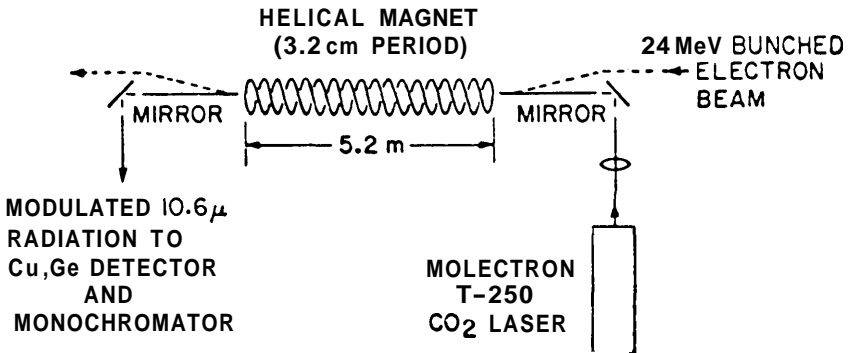


Fig. 2.32. Experimental arrangement of the free electron laser. The pulsed electron beam enters from the right-hand side into a helical magnet. In the experiment indicated, the emitted light is superimposed on that of a CO<sub>2</sub> laser and the modulated radiation is registered. [D.A.G. Deacon, L.R. Elias, J.M.J. Madey, H.A. Schwettman and T.I. Smith, in: Laser Spectroscopy III, eds. J.L. Hall and J.L. Carlesten, Springer, Berlin 1977.]

from classical electrodynamics, that deflected, i.e. accelerated, charges give rise to the emission of electromagnetic waves. By means of the collective emission by many electrons, laser action may become possible. The advantage of this system which has been realized, for instance, at the linear accelerator of Stanford University, is its tunability, which could be reached by a continuous variation of the magnetic field strength.

#### X-ray and $\gamma$ -ray lasers

Due to the many applications of laser light it is, of course, highly desirable to build lasers with very short wave-lengths. With respect to the X-ray and  $\gamma$ -ray region so far only proposals exist. Possible laser-active quantum systems could be excited atomic nuclei which can emit  $\gamma$ -rays. With respect to resonators the principle of the distributed feedback laser has been suggested. Because of the factor  $\nu^2$  in the laser condition (2.3), the realization of such lasers seems to be very difficult, however.

## Chapter 3

# Laser Resonators

### 3.1. Survey

In principle the light field or, more generally speaking, the electro-magnetic wave field contains all possible wave-lengths, all directions of propagation and all directions of polarization. On the other hand the main goal of the laser device consists in the generation of light with definite properties. A first selection, namely with respect to frequency, is achieved by the choice of the laser material. By means of the energy levels  $W$  of the chosen system the frequency  $\nu$  of the emitted light is fixed according to Bohr's formula  $h\nu = W_{\text{initial}} - W_{\text{final}}$ . Of course, the frequencies of the optical transitions are not sharp but they are broadened due to various causes. Such causes may be the finite lifetimes of the levels due to optical transitions or collisions, inhomogeneous crystalline fields, etc. In order to select frequencies further, resonators are used. We met the simplest type of a resonator in Vol. 1 when we studied the modes in a cavity. In a cavity, whose walls have an infinitely high conductivity, standing waves with discrete frequencies can exist. These waves are well defined eigenmodes of the cavity. When scientists tried to extend the maser principle into the optical region it was an open question whether a laser with just two mirrors but otherwise open side walls would allow modes at all (fig. 3.1). Because of the diffraction and transmission losses due to the mirrors, no permanent field could stay in such an open resonator. It turned out, however, that the concept of modes can be well applied to open resonators. The first proof was given by computer calculation. Fox and Li considered an arrangement of two plane parallel mirrors and they prescribed 'an initial field distribution on one of the mirrors. Then they studied the propagation of light and its reflection. After the first steps the initial light field got distorted and its amplitude lowered. However, after say 50 round trips the field mode acquired a final shape and its overall amplitude was decreasing by the same constant factor after each reflection. In this way it was clear how to generalize the concept of modes. One has to look for such field configurations which remain the same in the course

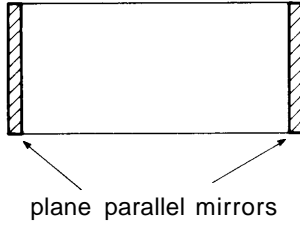


Fig. 3.1. Laser resonator with plane parallel mirrors.

of time with the exception that their amplitudes decrease after each step by the same factor. In this way it becomes possible to calculate the modes of open resonators and we shall give two explicit examples in the second part of this chapter.

In analogy to a closed resonator, in an open resonator a sequence of discrete modes may exist with which a series of discrete frequencies is connected. The emerging mode configurations can be characterized by their specific intensity distribution on the mirrors. Examples are shown in fig. 3.2. By means of the finite lifetime of the modes due to diffraction and especially due to the transmissivity of the mirrors, the amplitude is damped which gives rise to a frequency width. In most lasers this frequency width

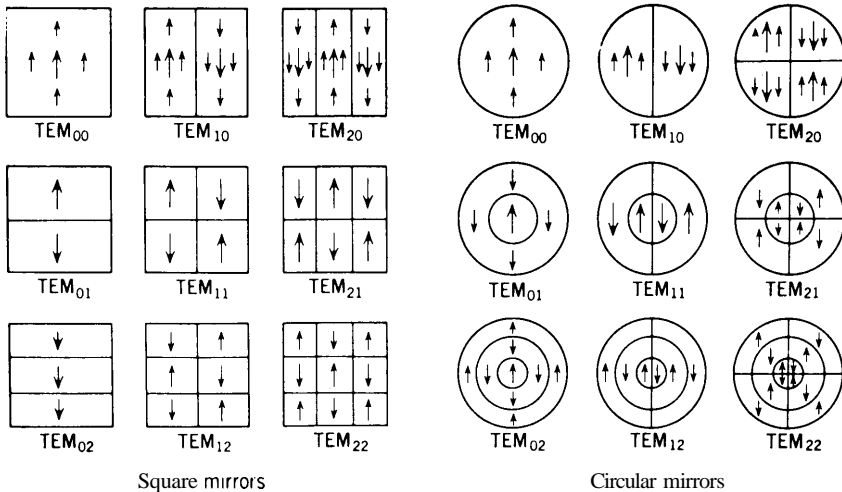


Fig. 3.2. Left part: Distribution of the electric field strength of the laser field over the endfaces of a laser with rectangular cross section. The abbreviation TEM means "transverse electric mode". The indices represent the number of nodes of the field in vertical or horizontal direction. Right part: The same for a laser with circular cross section.



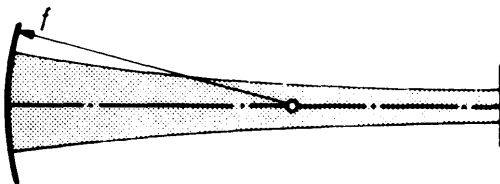


Fig. 3.3. Laser arrangement with a convex and a plane mirror.

is much smaller than the original atomic line-width ("good cavity case"). Important dynamic effects of laser light occur if the frequency width caused by the cavity is bigger than the atomic line-width ("bad cavity case"). A misunderstanding should be avoided at any rate. By the laser process itself, which we shall treat in our book, the effective line-width will be lowered, as compared to cavity and atomic line-width, by many orders of magnitude.

As can be easily visualized, light can sufficiently often be reflected between the mirrors only if these are precisely adjusted in a parallel position. If the laser mirrors are tilted, the light wave track will leave the laser very quickly. Its lifetime in the laser has dropped strongly and the laser condition can no more be fulfilled. For these reasons other arrangements of resonators have been developed where the light modes depend less sensitively on the adjustment of the mirrors. Such an arrangement is shown in fig. 3.3 where one plane mirror is replaced by a mirror with the shape of a section of a sphere.

Another arrangement often used is that of confocal mirrors where the center of one mirror just coincides with the focus of the other mirror. Plane parallel mirrors are often mounted at the end surfaces of the material itself. But also other arrangements are used in which one or both mirrors are separated from the laser active material. Such arrangements may influence the mode selection done by the laser process and can be used correspondingly. As we shall see later in our book, laser action can simultaneously take place in modes which are standing waves. These modes are made possible by experimental setups corresponding to figs. 3.1–3.5.

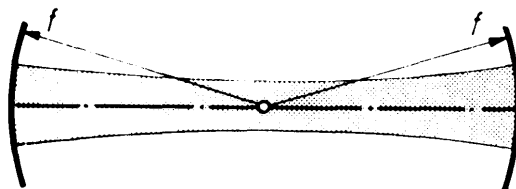


Fig. 3.4. Laser arrangement with two confocal mirrors.

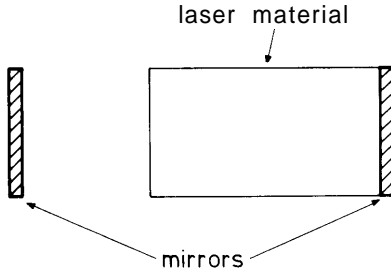


Fig. 3.5. Laser arrangement in which one of the mirrors is mounted in a certain distance from the laser material.

In order to achieve higher reflection coefficients for polarized light, a definite angle between the laser axis and the mirrors is used in order to make use of the Brewster angle. This arrangement is often used in gas lasers, for instance the helium–neon laser. In order to obtain a very sharp frequency, the selection of a single mode becomes necessary. This can be achieved, for instance, by the ring laser shown in fig. 3.6. In such a ring laser running waves are generated. By additional means, e.g. by a Faraday rotator put in-between two mirrors, it becomes even possible to select one direction of propagation. The ring laser in which two waves propagate in opposite direction forms the basis of the laser gyro which will be studied in more detail in section 6.7. The most important property of all these mirror arrangements consists in the fact that in this way the light is fed again and again into the laser (feed-back) whereby the laser wave is more and more amplified until a stationary state is reached. In a way the light is back-scattered coherently into the material. Such a coherent back scattering can be achieved in a quite different way by a method which is well known from

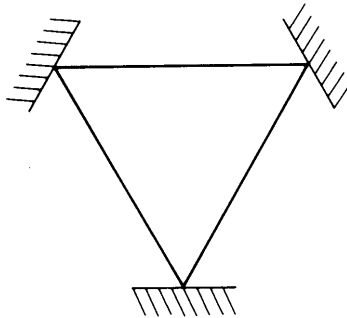


Fig. 3.6. Mirror arrangement of a ring laser. Also four mirrors are in use.

X-ray diffraction. When crystals are irradiated by X-rays each atom acts as a scattering center. If the X-rays impinge on the crystal in specific directions and with specific wave-lengths, the individual back-scattered wave tracks can interfere with the incoming light field, and thus standing waves are formed. These standing waves correspond to the modes we have been discussing before. However, the difference to the former mirror arrangements consists in the fact that the scattering centers (centers of reflection) are distributed all over the crystal in a regular fashion. This principle can be applied to the optical region. By means of a grating the back scattering of the wave tracks is achieved. In this way we arrive at the principle of distributed feed-back lasers.

In a number of cases the laser process can be achieved without the specific feed-back mechanisms we just have studied. For instance in a number of semiconductors the difference between the index of diffraction of the material and that of air is so big that the internal reflection is big enough to achieve the same effect as a mirror. In addition to the arrangement of mirrors we just have mentioned, also more exotic arrangements have been suggested and even verified, e.g. the whispering gallery mode (fig. 3.7).

Before we turn to a more detailed description of methods how the modes in an open resonator can be calculated we mention a few technical terms. The "quality of a resonator",  $Q$ , is defined by  $Q = \omega t_0$  where  $\omega$  is the mode frequency and  $t_0$  its lifetime in the unloaded cavity, i.e. a cavity not yet filled with laser active material.  $t_0$  is the time in which the mode intensity drops down to  $1/e$  of its initial value. In our book we shall use the decay constant  $\kappa = 1/(2t_0)$ . In order to obtain a high  $Q$ , according to physical optics (theory of diffraction) the following criterium must be fulfilled. In the case of two mirrors with apertures  $2A_1$  and  $2A_2$ , respectively, and separated by a distance  $D$ , the inequality

$$\frac{A_1 A_2}{\lambda D} > 1, \quad (3.1)$$

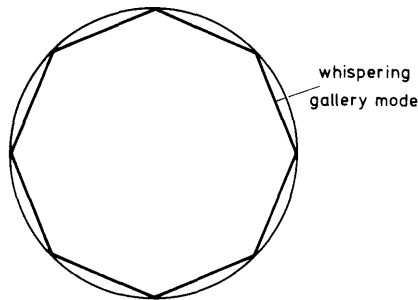


Fig. 3.7. Example of a whispering gallery mode.

where  $A$  is the wave-length, must be fulfilled. The parameter  $N = A^2/\lambda D$  obtained for  $A_1 = A_2 = A$  is called Fresnel number. It is approximately equal to the number of Fresnel zones seen on one mirror from the center of the other mirror. A resonator theory should explain the following points:

- (1) the mode pattern on the mirrors;
- and more generally
- (2) the mode distribution in the interior of the open resonator;
  - (3) the losses due to diffraction, reflection, mirror misalignment, and aberration;
  - (4) the far field pattern.

In sections 3.2 and 3.3 we shall give two examples how the mode patterns can be calculated.

### 3.2. Modes in a confocal resonator\*

This resonator is formed by two spherical mirrors of equal curvature separated by their common radius of curvature. The focal length of a mirror is one half of its radius of curvature, so that the focal points of the reflectors coincide.

The reflectors are assumed to be square with the edge length  $2A$  (compare fig. 3.8), which is small compared to the spacing  $D=R$  where  $R$  is the radius.  $A$  and  $R$  are large compared to the wave-length. Because of the symmetry of the problem we can choose the electric field vector either in  $x$  or  $y$  direction. In the following we shall drop that index  $x$  or  $y$ . Actually

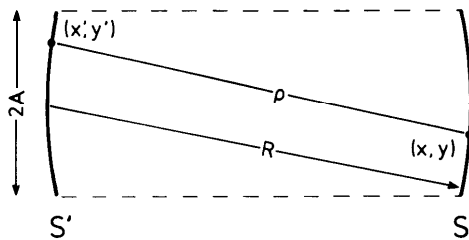


Fig. 3.8. Confocal resonator with mirrors  $S$  and  $S'$ . The coordinates used in the text are indicated.

\*The specific form of the modes we are going to derive in this and the following section is not very important for most of the theoretical conclusions we shall draw in our later chapters. For this reason, readers who are interested mainly in laser processes can skip the reading of sections 3.2 and 3.3 entirely.

all the essential steps of the mathematical analysis can be seen when we deal with a *scalar* field using Huygens' principle.

Let the field strength at point  $x'$  be given by  $E(\mathbf{x}')$ . Then according to Huygens' principle this point is a starting point of a new spherical wave which produces a field strength at point  $x$  which is given by

$$\frac{ik}{2\pi} \frac{\exp[-ik|\mathbf{x}-\mathbf{x}'|]}{|\mathbf{x}-\mathbf{x}'|} E(\mathbf{x}'). \quad (3.2)$$

Here  $k$  is the wave number;  $\mathbf{x}$  and  $\mathbf{x}'$  are vectors defined by

$$\mathbf{x} = (x, y, z), \quad (3.3)$$

$$\mathbf{x}' = (x', y', z'). \quad (3.4)$$

The total field at point  $\mathbf{x}$  is found by integrating over all original points  $\mathbf{x}'$  on a surface. In the following we shall consider the spherical mirrors as such a surface and we shall choose the coordinates  $x$  and  $y$  on them. Corresponding to the two mirrors  $S$  and  $S'$  we write

$$E(\mathbf{x}) = E_S(x, y) \quad (3.5)$$

and

$$E(\mathbf{x}') = E_{S'}(x', y'). \quad (3.6)$$

The field on mirror  $S$  is given by

$$E_S(x, y) = \int_{S'} \frac{ik}{2\pi} \frac{\exp[-ik\rho]}{\rho} E_{S'}(x', y') dS', \quad (3.7)$$

where

$$|\mathbf{x}-\mathbf{x}'| = \rho. \quad (3.8)$$

Because of  $2A < R$  we approximate  $\rho$  by

$$\frac{\rho}{R} \approx 1 - \frac{xx' + yy'}{R^2}. \quad (3.9)$$

In accordance to what we have stated in section 3.1 we define a mode by the property that its field distribution is repeated when going from one mirror to the other one, besides a constant factor  $\sigma$ ,

$$E_S(x, y) = \sigma E_{S'}(x, y). \quad (3.10)$$

Inserting (3.7) into (3.10) we find an equation for  $E_{S'}$ . To solve this equation we proceed in several simple steps. Because it will turn out in a self-consistent

fashion that the spot-size of the field distribution is much smaller than  $R$  we make the following approximation:

$$\frac{\exp[-ik\rho]}{\rho} \approx \frac{\exp[-ikR]}{R} \exp\left[\frac{-ik(xx' + yy')}{R}\right]. \quad (3.11)$$

But when we insert (3.11) into the r.h.s. of (3.7) we may factorize the integral into one referring to  $xx'$  and one referring to  $yy'$ . This suggests to make the following hypothesis in the form of a product:

$$E_S = E_0 f(x) g(y), \quad (3.12)$$

where  $E_0$  is a constant. For formal reasons we decompose also  $\sigma$  into a product:

$$\sigma = \sigma' \sigma'', \quad (3.13)$$

Inserting (3.12) and (3.13) into (3.10), where we have used (3.7) and (3.11), we find

$$\begin{aligned} \sigma' f(x) \sigma'' g(y) &= \frac{ik \exp[-ikR]}{2\pi R} \int_{-A}^A f(x') \exp\left[\frac{ikx'x}{R}\right] dx' \\ &\quad \times \int_{-A}^A g(y') \exp\left[\frac{iky'y}{R}\right] dy'. \end{aligned} \quad (3.14)$$

Because on each of both sides there is a product of a function of  $x$  and one of  $y$  this equation can be fulfilled only if the factors fulfil the following equation:

$$f(x) = \text{const} \int_{-A}^A f(x') \exp[ikxx'/R] dx' \quad (3.15)$$

and a corresponding equation for  $g(x)$ . Eq. (3.15) is called an integral equation and it has been solved exactly. The solutions are given by the angular wave functions in prolate spherical coordinates as defined by Flammer. But probably hardly anybody is familiar with this kind of wave functions. Fortunately, for our purposes the solution of (3.15) can be written in a very familiar form provided the field has not too many nodes in the  $x, y$  plane. In such a case it is well concentrated around the axis so that  $x/A \ll 1$ . If the field is strongly concentrated, the contributions of the integral in the region of  $x' \approx \pm A$  are practically negligible and we can extend the limits of the integral to infinity. Therefore instead of (3.15) we now have to solve

$$f(x) = \text{const} \int_{-\infty}^{+\infty} f(x') \exp[ikxx'/R] dx'. \quad (3.16)$$

After a little bit of guessing one may quite easily find the solution of the equation (3.16). It is nothing but a Gaussian distribution

$$f(x) = \exp[-\frac{1}{2}kx^2/R]. \quad (3.17)$$

We leave it as a little exercise to the reader to check that (3.17) is indeed a solution of (3.16) and to determine the constant in front of the integral. In the following it will be convenient to have a suitable abbreviation, namely

$$X = x\sqrt{k/R}, \quad Y = y\sqrt{k/R}. \quad (3.18)$$

With (3.17) and (3.18) and choosing the same solution for given  $y$  we find the field distribution on the mirrors. It is a Gaussian distribution

$$E(x, y) = E_0 \exp[-\frac{1}{2}(X^2 + Y^2)]. \quad (3.19)$$

We may define the spot radius,  $w_s$ , by that radius where (3.19) has dropped to  $1/e$ , which yields

$$w_s = \sqrt{R\lambda/\pi}. \quad (3.20)$$

Huygens' principle allows one also to calculate the field inside the resonator. Because the derivation of the result is of a more technical nature we just write down the corresponding formulas. We use the abbreviation

$$\xi = 2z/R, \quad (3.21)$$

and find

$$E(x, y, z) = E_0 \sqrt{\frac{2}{1+\xi^2}} \exp\left[\frac{-(X^2 + Y^2)}{(1+\xi^2)}\right] \sin \varphi(X, Y, \xi), \quad (3.22)$$

where

$$\varphi(X, Y, \xi) = k \left( \frac{R}{2} (1+\xi) + \frac{\xi}{(1+\xi^2)} k (X^2 + Y^2) \right) - \left( \frac{\pi}{2} - \varphi_0 \right), \quad (3.23)$$

and

$$\varphi_0 = \frac{1-\xi}{1+\xi}. \quad (3.24)$$

As we may see, in  $x, y$ -direction the field is still Gaussian though the spot size varies along the laser axis. The function  $\sin \varphi$  looks rather complicated but a little analysis reveals that it has the following structure:

$$\sin \varphi \approx \sin(kz + f(z)), \quad (3.25)$$

where  $\sin(kz)$  describes the fast oscillations of the field, whereas  $f(z)$  is a slowly varying function. That means that in  $z$ -direction, i.e. in the direction

of the laser axis, the field mode strongly resembles that of a mode in a closed cavity. The field outside the cavity can also be found by means of Huygens' principle and reads

$$E(x, y, z) = c_t E_0 \sqrt{\frac{2}{1 + \xi^2}} \exp\left[\frac{-(X^2 + Y^2)}{(1 + \xi^2)}\right] \exp[-i\varphi(X, Y, \xi)]. \quad (3.26)$$

where  $c_t$  is the transmissivity of the mirrors. The far field pattern has a spot size which at a distance  $z$  is given by

$$w_s = \left(\frac{R\lambda}{2\pi} (1 + \xi^2)\right)^{1/2}. \quad (3.27)$$

The angular beam width  $\Theta$  can be defined as the ratio  $w_s/z$  for  $z \rightarrow \infty$  which yields

$$\Theta = \sqrt{\frac{R\lambda}{2\pi}} \frac{\xi}{z} = \sqrt{\frac{R\lambda}{2\pi}} \frac{2}{R} = \sqrt{\frac{2\lambda}{\pi R}}. \quad (3.28)$$

The results are represented in figs. 3.9 and 3.10. The field distribution (3.19) represents just the mode which has the lowest losses.

A closer analysis shows that the whole sequence of modes is again of a form quite familiar to physicists. Namely, the general solution of (3.14) can be written as

$$E(x, y) = E_0 H_m(X) H_n(Y) \exp[-\frac{1}{2}(X^2 + Y^2)], \quad (3.29)$$

where  $H_m$  are the Hermitian polynomials,  $m = 0, 1, 2, \dots$  and

$$X = x\sqrt{k/R}, \quad Y = y\sqrt{k/R},$$

at least in the case in which the Fresnel number  $N = (1/2\pi)(A^2 k/R) \rightarrow \infty$ .

For sake of completeness we represent also the field *inside and outside* the resonator in the general case. The *field outside the resonator* is

$$E(x, y, z) = c_t E_0 \sqrt{\frac{2}{1 + \xi^2}} H_m\left(X \sqrt{\frac{2}{1 + \xi^2}}\right) H_n\left(Y \sqrt{\frac{2}{1 + \xi^2}}\right) \times \exp\left[-(X^2 + Y^2) \frac{1}{1 + \xi^2}\right] \exp[-i\varphi(X, Y)]. \quad (3.30)$$

To obtain the *field inside the resonator*, the factor  $c_t$  must be omitted and  $\exp[-i\varphi(X, Y)]$  be replaced by  $\sin(\varphi(X, Y))$ . In order to calculate the losses by diffraction, the finite size of the mirrors is, of course, crucial. The analysis shows, however, that the losses due to diffraction are in



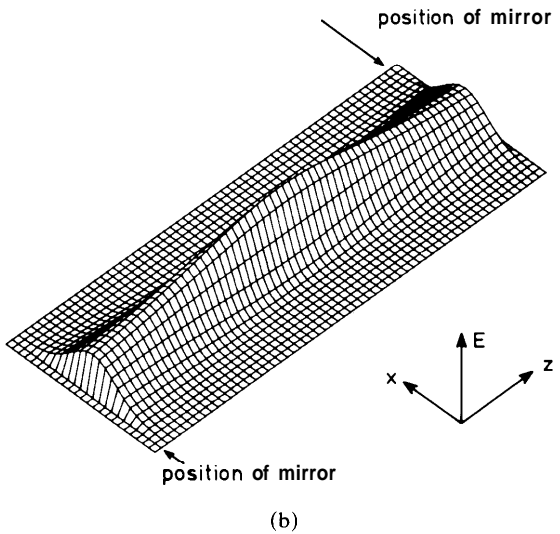
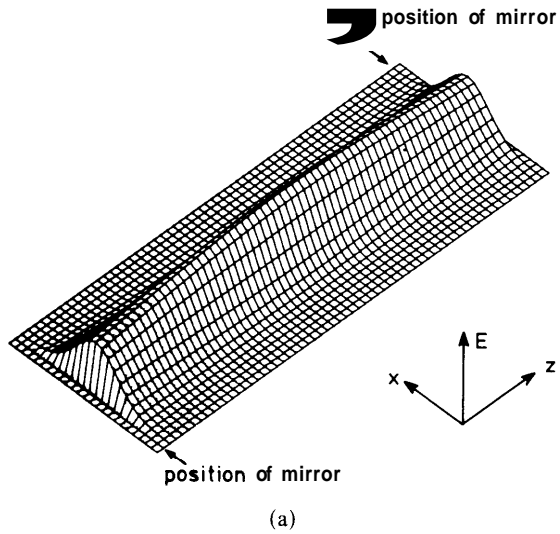


Fig. 3.9. (a) Field distribution between two confocal mirrors, according to eq. (3.22). (a) and (b) show only the envelope. The rapidly oscillating function  $\sin \varphi$  has been omitted. (b) In order to bring out the *narrowing* of the field distribution, in the middle part the field distribution of (a) is somewhat exaggerated.

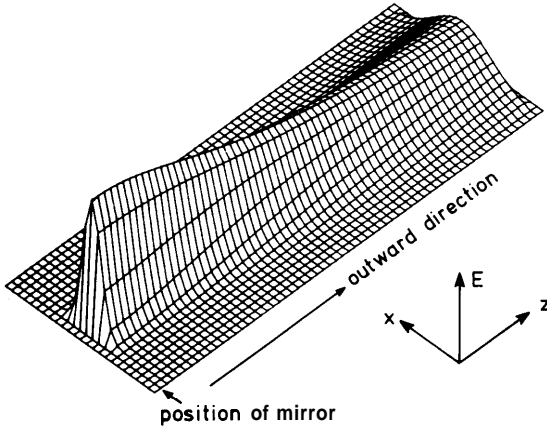


Fig. 3.10. Field distribution outside the confocal resonator, according to eq. (3.26). In this figure only the envelope is shown. The rapidly oscillating function  $\exp[i\varphi]$  has been omitted.

general much smaller, typically by a factor of 100, than losses due to the finite transmissivity of the mirrors. Therefore we shall not be concerned with these kinds of losses. We mention that in  $z$ -direction only discrete values of  $k$  are admitted which are given by

$$2\pi l = 2(\pi/2 - kR + (m+n)\pi/2), \quad (3.31)$$

where  $m$ ,  $n$  and  $l$  are integers.

### 3.3. Modes in a Fabry–Perot resonator

It is not our purpose to present resonator theory in full length. We rather wish to give the reader a feeling how the modes look like. In the foregoing section we have seen how the application of Huygens' principle allows one to determine the field configurations within a confocal resonator in a rather simple fashion. In the present section we want to briefly indicate the results of a model calculation which avoids the approximation on which Huygens' principle is based. For simplicity we consider a two-dimensional model of a Fabry–Perot resonator which consists of two plane strip metal mirrors. In the present approach we assume that the space between the mirrors is filled with active material being described by a complex susceptibility  $\chi = \chi' + i\chi''$ . For a rigorous treatment Maxwell's equations must be used.

Due to the symmetry of the problem we may put either

$$\left. \begin{aligned} E_x &= U(y, z), \\ H_y &= -\frac{i}{\mu\omega} \frac{\partial U}{\partial z}, \quad H_z = \frac{i}{\mu\omega} \frac{\partial U}{\partial y}, \end{aligned} \right\} \quad (3.32)$$

or

$$\left. \begin{aligned} H_x &= U(y, z), \\ E_y &= \frac{i}{(\varepsilon + \chi)\omega} \frac{\partial U}{\partial z}, \quad E_z = -\frac{i}{(\varepsilon + \chi)\omega} \frac{\partial U}{\partial y}. \end{aligned} \right\} \quad (3.33)$$

Inserting the hypothesis (3.32) or (3.33) into Maxwell's equations we readily verify that  $U$  must obey the wave equation

$$\Delta U + k_M^2 U = 0, \quad (3.34)$$

where the wave number is given by

$$k_M^2 = \frac{\omega^2}{c^2} \left( 1 + \frac{\chi}{\varepsilon} \right) \quad (3.35)$$

within the active material and

$$\Delta U + k^2 U = 0, \quad (3.36)$$

with

$$k^2 = \omega^2 / c^2 \quad (3.37)$$

outside the active material.  $\varepsilon$  is the dielectric constant. The mirrors are assumed to have a reflectivity  $r$ , close to one, so that the tangential components of  $\mathbf{E}$  and  $\mathbf{H}$  must satisfy a certain boundary condition, called the Leontovich condition. It reads

$$\mathbf{E}_{\text{tang}} = \sqrt{\frac{\mu}{\varepsilon}} \frac{1-i}{4} (1-r) \mathbf{n} \times \mathbf{H}, \quad (3.38)$$

where  $\mathbf{n}$  is a vector normal to the mirror surface.

The essential results can be summarized as follows. If the Fresnel numbers are sufficiently high, the electric field has the following spatial dependence:

$$\sin \left[ (y+A) \frac{m\pi}{2A} \right] \sin \left[ \left( z + \frac{D}{2} \right) \frac{n\pi}{D} \right]. \quad (3.39)$$

The solution can be readily generalized to a three-dimensional resonator,

$$\sin \left[ (x + A_1) \frac{l\pi}{2A_1} \right] \sin \left[ (y + A_2) \frac{m\pi}{2A_2} \right] \sin \left[ \left( z + \frac{D}{2} \right) \frac{n\pi}{D} \right]. \quad (3.40)$$

$2A_1$  and  $2A_2$  are the edge lengths of the rectangular end mirrors in  $x$  and  $y$ -direction, respectively;  $D$  is their distance.  $l$ ,  $m$  and  $n$  are integers. This result has been derived for axial or nearly axial modes so that  $n$  is a big number ( $\lambda n \approx D$ ) whereas  $l$  and  $m$  are small integers of order unity.

The resonance condition reads approximately

$$\frac{\omega^2}{c^2} \approx \left( \frac{l\pi}{2A_1} \right)^2 + \left( \frac{m\pi}{2A_2} \right)^2 + \left( \frac{n\pi}{D} \right)^2, \quad l, m, n = 0, 1, \dots \quad (3.41)$$

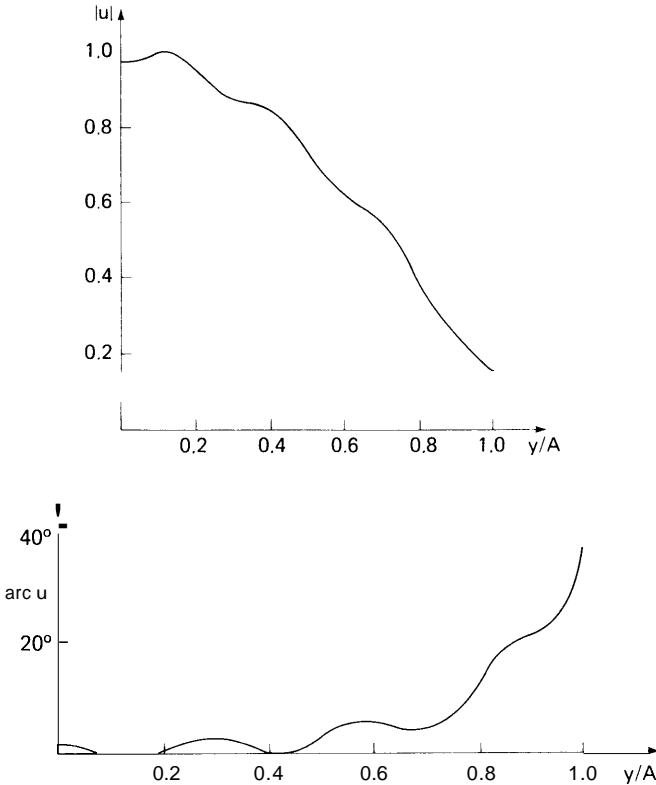
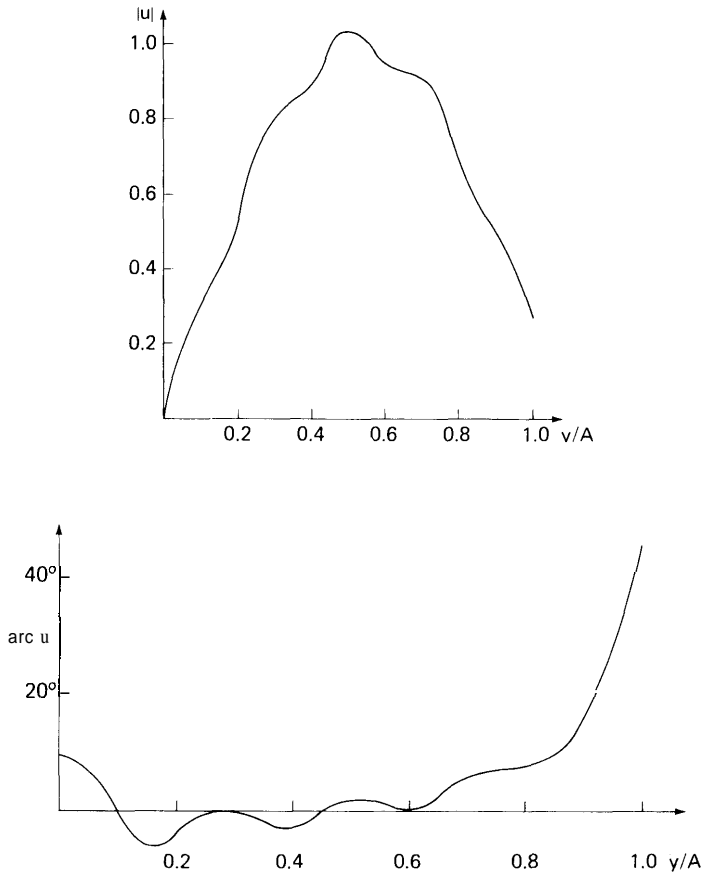


Fig. 3.11. Amplitude (above) and phase (below) distribution of the lowest mode of even symmetry for the two-dimensional resonator ( $N = 6.25$ ). [H. Risken, *Z. Physik* 180, 150 (1964).]



**Fig. 3.12. Amplitude (above) and phase (below) distribution for the lowest mode of odd symmetry for the two-dimensional resonator ( $N=10$ ). [H. Risken, *Z. Physik* 180, 150 (1964).]**

A more detailed analysis reveals that in higher order approximation the functions (3.39) and (3.40) have to be changed in two ways. The  $l$ 's,  $m$ 's and  $n$ 's acquire additional small imaginary parts and the expressions (3.39) and (3.40) have to be supplemented by additional terms. Because the explicit results do not give us much physical insight we rather show the corresponding amplitude and phase distributions for the lowest modes in figs. 3.11 and 3.12. In the remainder of this book we shall be satisfied with the explicit representation of the wave functions in the form (3.39), (3.40) when we deal with standing waves.

## Chapter 4

# The Intensity of Laser Light. Rate Equations

### 4.1. Introduction

In this chapter we shall deal with rate equations. We got to know an example of such equations when we derived the laser condition in section 2.1. As is well known (cf. Vol. 1), when treating the light field quantum mechanically we may attribute the photon number  $n$  to each mode. In this chapter we shall treat  $n$  which, according to quantum theory should be an integer, as a continuous variable. A completely satisfactory derivation of the rate equations can be done only by means of a fully quantum theoretical treatment of the laser or in a rather good approximation by means of the semiclassical laser equations. We postpone this derivation to later chapters and start here right away with the rate equations in order to get a first insight into the physical processes in the laser.

### 4.2. The photon model of a single mode laser

Let us consider a single kind of photons whose number we shall denote by  $n$ . Because in the following we are interested in the genuine laser process we shall neglect the spontaneous emission rate  $WN_2$  (cf. section 2.1). We shall see later anyway that the stimulated emission rate  $(N_2 - N_1)Wn$  plays quite a different role than the rate  $WN_2$ . This becomes manifest in the statistical properties of laser light which we can treat only later, however. Therefore we start from the following equation for the temporal change of the photon number:

$$\frac{dn}{dt} = (N_2 - N_1)Wn - 2\kappa n. \quad (4.1)$$

Due to the laser process not only the photon number changes but also the occupation numbers of the atoms change. Let us consider for simplicity

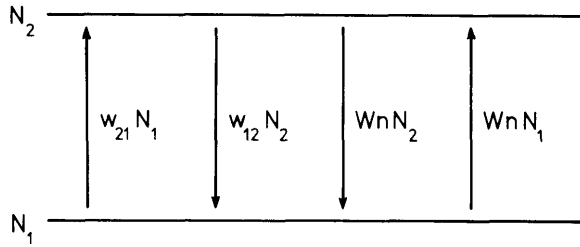


Fig. 4.1. The various optical transitions in a system of two-level atoms with the occupation numbers  $N_1$  and  $N_2$ .

a system of two-level atoms, and let us study the temporal change of the occupation numbers  $N_1$  and  $N_2$ . The number  $N_2$  increases by the excitation of electrons by the pumping process. The transition rate is proportional to the number of electrons available in the ground states of the individual atoms, i.e. it is proportional to  $N_1$ . The corresponding proportionality factor will be denoted by  $w_{21}$  (fig. 4.1). Here and in the following we have to note that we must read the indices of the  $w$ 's from right to left, i.e.  $w_{21}$  refers to the transition from level 1 to level 2. This transition rate depends, of course, on the optical properties of the atoms or, more precisely speaking, on the corresponding optical transition matrix elements. Finally  $w_{21}$  is proportional to the intensity of the pump light. We shall not discuss this in detail but we shall rather consider  $w_{21}$  as a "control parameter" which we can manipulate from the outside.

Atoms being in level 2 can make transitions into the ground state by means of radiationless transitions in which no photons are emitted. Such processes can be caused, for instance, by collisions in gases or by the interaction between atoms with lattice vibrations in solids. The rate of these transitions is, of course, proportional to the number of excited atoms,  $N_2$ . The corresponding proportionality factor will be denoted by  $w_{12}$ .

By means of this description we also take care of radiative processes which do not belong to the emission of the kind of photons under consideration here. Finally the processes of stimulated emission and absorption take place. The number of transitions per second is given by  $(N_2 - N_1) Wn$ . Collecting all the contributions just mentioned we obtain the rate equation for the occupation number of level 2,

$$\frac{dN_2}{dt} = w_{21}N_1 - w_{12}N_2 - (N_2 - N_1)Wn. \quad (4.2)$$

Level 1 can be treated in a completely analogous fashion. We then obtain

$$\frac{dN_1}{dt} = w_{12}N_2 - w_{21}N_1 + (N_2 - N_1)Wn. \quad (4.3)$$

Adding the eqs. (4.2) and (4.3) results in

$$\frac{d(N_1 + N_2)}{dt} = 0. \quad (4.4)$$

This means that the total occupation number of levels 1 and 2 remains constant. It is equal to the total number of all laser atoms

$$N_1 + N_2 = N. \quad (4.5)$$

Because eqs. (4.1) to (4.3) contain the difference of the occupation numbers or, in other words, the inversion  $N_2 - N_1$ , we introduce this quantity as a new variable

$$N_2 - N_1 = D. \quad (4.6)$$

On the other hand  $N_1$  and  $N_2$  can be expressed by means of  $N$  and  $D$ ,

$$N_2 = \frac{1}{2}(N + D), \quad N_1 = \frac{1}{2}(N - D). \quad (4.7)$$

Because  $N$  is a constant we have to deal with a single variable  $D$  only. Therefore it suggests itself to express eqs. (4.1)–(4.3) by means of that variable. Subtracting (4.3) from (4.2) and using (4.7) yields

$$\frac{dD}{dt} = N(w_{21} - w_{12}) - D(w_{21} + w_{12}) - 2WDn. \quad (4.8)$$

This equation can be brought into a form which lets transpire its physical meaning still more when we remember that the  $w$ 's have the meaning of transition rates, i.e. that they are inversely proportional to certain transition times. Therefore we introduce a time constant  $T$  by the relation

$$1/T = w_{21} + w_{12}. \quad (4.9)$$

Furthermore we study which inversion  $D = D_0$  will result if only pump and relaxation processes occur but no laser process. We obtain that quantity by putting the l.h.s. of (4.8) = 0 and by omitting the last term in (4.8). We then readily obtain

$$D_0 = N \frac{w_{21} - w_{12}}{w_{21} + w_{12}} \quad (4.10)$$

For reasons which we shall explain below, this inversion is called the



unsaturated inversion. Inserting  $D_0$  and (4.9) in (4.8) we finally obtain

$$\frac{dD}{dt} = \frac{1}{T} (D_0 - D) - 2WDn. \quad (4.11)$$

When we neglect for the moment being the last term in (4.11), we can readily solve this differential equation and find a solution which tells us that  $D$  relaxes towards the value  $D_0$  within the time  $T$  (compare the exercise).

By use of (4.6) we can write the rate equations of the photons in the form

$$\frac{dn}{dt} = DWn - 2\kappa n. \quad (4.12)$$

Eqs. (4.11) and (4.12) are the fundamental laser equations which we shall discuss in more detail. Because the r.h.s. of these equations contain products of the variables  $D$  and  $n$ , these equations are nonlinear and in general cannot be solved in closed form. Therefore we shall proceed from the simple case to more complicated cases when discussing and solving these equations. In the most simple case  $n$  and  $D$  are time independent, i.e. we consider the stationary state.

### Stationary *solution*

In this case we have

$$\frac{dn}{dt} = 0 \quad \text{and} \quad \frac{dD}{dt} = 0, \quad (4.13)$$

so that (4.11) reduces to

$$(D_0 - D) = 2TDWn. \quad (4.14)$$

Solving this equation with respect to the inversion  $D$  we obtain

$$D = \frac{D_0}{1 + 2TWn}. \quad (4.15)$$

Eq. (4.15) teaches us that the actual inversion  $D$  decreases from the inversion  $D_0$ , which is prescribed by pump and relaxation processes only, when the photon number increases. This fact is expressed in technical terms by saying that a saturation of the inversion occurs. This effect can most easily be represented when the photon number  $n$  is still small. In this case we replace (4.15) by the relation

$$D \approx D_0 - 2D_0TWn. \quad (4.16)$$

As noted above,  $D_0$  is the unsaturated inversion. The term containing  $n$  is called saturation. Inserting (4.15) in (4.12) and using  $dn/dt=0$  we obtain after a slight transformation

$$n\left(\frac{WD_0}{1+2TWn}-2\kappa\right)=0. \tag{4.17}$$

This equation possesses the two solutions

$$n_0=0 \tag{4.18}$$

and

$$n_0=(WD_0-2\kappa)/(4\kappa TW). \tag{4.19}$$

(Because  $n$  is a stationary solution we have supplied  $n$  with the index 0.)  $n_0=0$  means that no photons are produced, i.e. that no laser action takes place. Therefore let us consider (4.19) and in particular the first factor on the r.h.s.,  $(WD_0-2\kappa)$ .

If we pump the laser only weakly,  $D_0$  is small so that

$$(WD_0-2\kappa)<0. \tag{4.20}$$

But because the photon number must not be negative, (4.19) is eliminated as a possible solution and only (4.18) remains. Therefore no laser action takes place. If we increase  $D_0$  such that

$$(WD_0-2\kappa)>0 \tag{4.21}$$

holds, the solution (4.19) with  $n_0>0$  becomes possible and we obtain laser action. It is a simple matter to convince oneself that (4.21) is identical with the laser condition (2.3) (compare exercise). The increase of  $n_0$  with increasing  $D_0$  is represented in fig. 4.2. While below the critical value  $D_0=2\kappa/W$  there is no laser action, it occurs above that value and  $n_0$  increases rapidly. These considerations do not explain why we can exclude the solution  $n_0=0$  in the region (4.21). To this end we have to consider the time dependent equations.

Our result that below laser threshold no photons are present at all stems from our neglect of spontaneous emission. We can take that effect into account only much later when we consider in chapter 10 the quantum theory of the laser.

### Approximate time dependent solutions

Because the time dependent equations cannot be solved in closed form we shall try to solve them approximately. To this end we make two

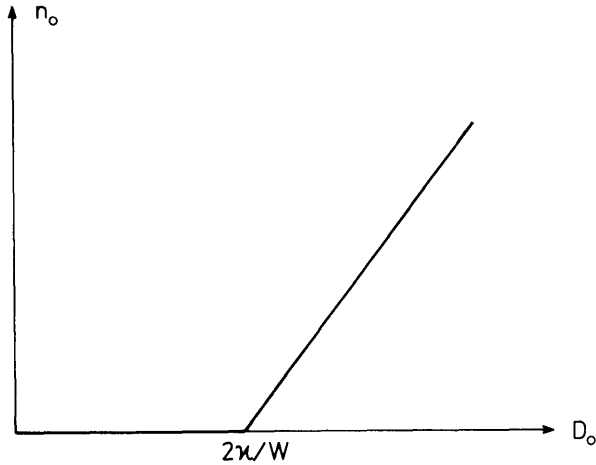


Fig. 4.2. Emitted photon number  $n_0$  versus unsaturated inversion  $D_0$ . In this model, below the critical inversion  $D_0 = 2\kappa/W$  no laser emission takes place. Beyond that inversion the photon number increases linearly.

assumptions. Let us consider the deviation of  $D$  from the unsaturated inversion  $D_0$ , i.e.  $D - D_0$ .

Let us assume that  $D - D_0$  changes only little over times  $T$  (compare exercises). Expressed mathematically this means that

$$\left| \frac{d}{dt} (D - D_0) \right| \ll |D - D_0| / T \quad (4.22)$$

shall hold. Because  $D_0$  is time independent, i.e.  $dD_0/dt = 0$ , we may neglect the l.h.s. of (4.11), i.e.  $dD/dt$  compared to  $(D_0 - D)/T$  which occurs on the r.h.s. or, in other words, we may put  $dD/dt = 0$  in eq. (4.11). This allows us to solve (4.11) immediately by expressing  $D(t)$  by means of the instantaneous value of  $n(t)$  as we did in (4.15) above. But in contrast to that former result  $n$  and  $D$  now depend on time  $t$ . Inserting (4.16) in (4.12) and slightly rearranging the terms we obtain the laser equation

$$\frac{dn}{dt} = (D_0 W - 2\kappa)n - 2D_0 T W^2 n^2. \quad (4.23)$$

This equation can be interpreted as follows. When the laser is pumped only weakly the inversion  $D_0$  is small or even negative. In this case we have  $(D_0 W - 2\kappa) < 0$  and the total r.h.s. of (4.23) is negative, i.e.  $dn/dt$  is negative. Even if at an initial time some photons are present, for instance by spon-

taneous emission, the photon number decreases exponentially. If we increase the pump strength we finally obtain  $(D_0 W - 2\kappa) > 0$ . Because usually in the beginning of the laser process the photon number  $n$  can be assumed small we may first neglect the quadratic term in  $n$  in (4.23) compared to the linear term. In this case an exponential increase of the photon number happens according to (4.23). In other words we have to deal with an "instability". Spontaneously produced photons are multiplied by the process of stimulated emission like an avalanche. Of course, the photon number does not increase exponentially for ever because finally the term quadratic in (4.23) becomes influential and lets the r.h.s. of (4.23) tend to 0. This implies that  $dn/dt$  tends to 0 and eventually the stationary state  $n = n_0$  will be reached. The quadratic term stems from the saturation of the inversion as can be easily derived from (4.16). The saturation makes it possible that a stationary state is reached. If we start with a photon number  $n$  which is bigger than that of the stationary state,  $n_0$ , the second term in (4.23) dominates. In this case the r.h.s. is negative and consequently the photon number decreases and eventually acquires the stationary value,  $n_0$ . These results are represented in figs. 4.3a and 4.3b. The explicit time dependent solution of eq. (4.23) is derived in exercise 3 of this section.

**Exercises on section 4.2**

(1) Solve the equation

$$\frac{dD}{dt} = \frac{1}{T} (D_0 - D).$$

What does it mean that  $D$  relaxes towards  $D_0$  within the time  $T$ ?

(2) Convince yourself that the condition (4.21) is identical with the laser condition (2.3).

*Hint:* Use the definitions of  $\kappa$  and  $W$  as introduced in section 2.1.

(3) This exercise deals with the time dependent solution of eq. (4.23) for the photon number  $n$ ,

$$\frac{dn}{dt} = an - bn^2,$$

where  $a = D_0 W - 2\kappa$ ,  $b = 2D_0 T W^2$ , and at initial time  $t = t_0$  the photon number is given by  $n = n_0 > 0$ . Convince yourself that the solution of (4.23) reads as follows:

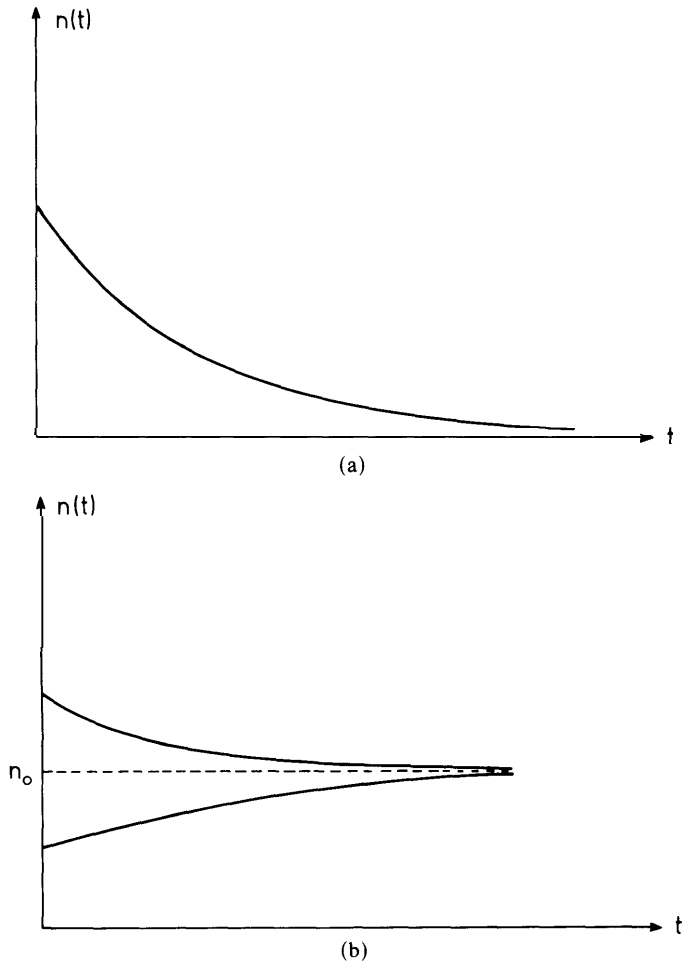


Fig. 4.3. The photon number  $n$  versus time  $t$ . (a) At time  $t=0$  a finite number of photons was given, but the laser was operated below laser threshold. (b) The laser condition is assumed to be fulfilled. The photon number approaches, according to the initial photon number, the stationary value  $n_0$  from above or from below, respectively.

(a) for  $a > 0$

$$n(t) = \frac{ac \exp[a(t - t_0)]}{1 + bc \exp[a(t - t_0)]},$$

where  $c$  is given by

$$c = \frac{n_0}{(a - bn_0)};$$

(b) for  $a < 0$

$$n(t) = \frac{|a|c \exp[a(t-t_0)]}{1 - bc \exp[a(t-t_0)]}.$$

Discuss the time dependence of  $n(t)$ .

(4) Investigate under which conditions the slowness condition (4.22) is fulfilled.

Hint: Use eqs. (4.15) and (4.11), and the results of exercise 3 of this section.

### 4.3. Relaxation oscillations

In this section we wish to further study the time dependent processes in the single mode laser. But we shall no more assume that the inversion follows the photon number instantaneously. While this assumption is well justified if the photon numbers are small (compare exercise 4 of section 4.2), this is no more the case for higher photon numbers. Instead we consider small deviations of  $n$  and the occupation numbers from their corresponding stationary values. To this end we assume that  $n$  or  $N_j$ , initially deviate from their stationary states a little. We wish to show that  $n$  and  $N_j$  perform damped oscillations or, in other words, relaxation oscillations. Let us consider as an example a 3-level laser in which the optical transition occurs between the two upper levels (compare fig. 4.4). The corresponding occupation numbers are again denoted by  $N_1$  and  $N_2$ . Let us assume that the lower transition from level 1 to level 0 takes place very rapidly (compare also the exercise at the end of this section). In this case the laser equations read as follows:

Equation for the photons

$$dn/dt = -2\kappa n + DWn, \quad (4.24)$$

where  $D = N_2 - N_1$ .

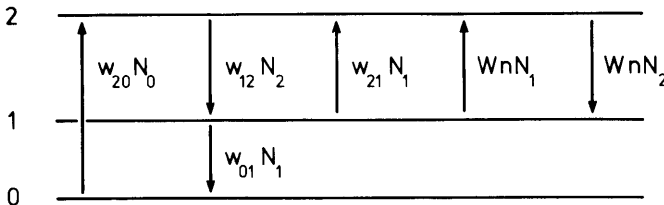


Fig. 4.4. Transition scheme in a system of three-level atoms in which the optical transition takes place from the uppermost to the middle level.

Equation for the occupation numbers

$$dN_2/dt = Nw_{20} - w_{12}N_2 - WN_2n. \quad (4.25)$$

$N$  is the total number of atoms. Because of our assumption that the lower laser level is practically unoccupied we may replace  $D$  by  $N_2$  in the equation for the photons. We now assume that deviations of  $n$  and  $N_2$  from their stationary states  $N_0$  and  $N_2^0$ , respectively, have occurred. We study whether and in which way a stationary state will be reached again. To this end we make the hypothesis

$$D \approx N_2 = N_2^0 + \delta N_2, \quad (4.26)$$

$$n = n_0 + \delta n, \quad (4.27)$$

where the stationary solutions  $n_0$  and  $N_2^0$  are fixed by the conditions

$$-2\kappa + WN_2^0 = 0 \quad (4.28)$$

and

$$Nw_{20} - N_2^0 w_{12} - n_0 N_2^0 W = 0. \quad (4.29)$$

Inserting (4.26) and (4.27) in (4.24) and (4.25) we obtain terms which contain only the stationary solution, terms proportional to  $\delta n$  or  $\delta N_2$  and finally expressions which contain the product  $\delta n \delta N_2$ . We assume that the deviations from the equilibrium values are only small. This allows us to neglect the term  $\delta n \delta N_2$  being of higher order. Because the stationary solution obeys the eqs. (4.28) and (4.29), this solution drops out and we only retain the equations

$$\frac{d}{dt} \delta n = \frac{\delta N_2}{N_2^0} 2\kappa n_0 \quad (4.30)$$

and

$$\frac{d}{dt} \delta N_2 = -\frac{\delta N_2}{N_2^0} Nw_{20} - \delta n N_2^0 W. \quad (4.31)$$

These are two coupled linear differential equations which we can solve as usual by the exponential ansatz

$$\delta n = A \exp(\alpha t) \quad (4.32)$$

and

$$\delta N_2 = B \exp(\alpha t). \quad (4.33)$$

We insert (4.32) and (4.33) in (4.30) and (4.31), perform the differentiation

with respect to time, and divide the resulting equations by the exponential function. We then obtain the following two equations

$$\alpha A = \frac{2\kappa n_0}{N_2^0} B, \quad (4.34)$$

and

$$\alpha B = -\frac{Nw_{20}}{N_2^0} B - N_2^0 WA. \quad (4.35)$$

These are homogeneous equations for the unknowns  $A$  and  $B$ . According to elementary theorems of algebra a non-trivial solution is possible only if the corresponding determinant

$$\begin{vmatrix} \alpha & -2\kappa n_0/N_2^0 \\ N_2^0 W & \alpha + Nw_{20}/N_2^0 \end{vmatrix} = 0 \quad (4.36)$$

vanishes. This determinant can be easily evaluated. Inserting the values for  $N_2^0$  and  $n_0$  according to (4.28) and (4.29) we obtain an equation for  $\alpha$ :

$$\alpha^2 + (w_{20}NW/2\kappa)\alpha + (WNw_{20} - 2w_{12}\kappa) = 0. \quad (4.37)$$

We represent the solution of this quadratic equation in the form

$$\alpha = -\Gamma + i\omega_r \quad (4.38)$$

where  $\Gamma$  is given by

$$\Gamma = w_{20}w_{12}/2w_{\text{thr}} \quad (4.39)$$

and  $\omega_r$  by

$$\omega_r = \left[ -\frac{w_{20}^2 w_{12}^2}{4w_{\text{thr}}^2} + \left( \frac{w_{20}}{w_{\text{thr}}} - 1 \right) 2\kappa w_{12} \right]^{1/2}. \quad (4.40)$$

We have assumed that the second term in the bracket in (4.40) is bigger than the first term, so that  $\omega_r$  is a real frequency. Furthermore we have used the abbreviation

$$w_{\text{thr}} = 2w_{12}\kappa/NW. \quad (4.41)$$

The index "thr" is an abbreviation of "threshold". The most general solution for the photon number may be represented in the form

$$\delta n = A_1 \exp[-(\Gamma - i\omega_r)t] + A_2 \exp[-(\Gamma + i\omega_r)t], \quad (4.42)$$

where  $A_1$  and  $A_2$  are fixed by the initial conditions for the photon number  $n$  and the occupation number  $N_2$ . Obviously the system relaxes towards the stationary state while it performs oscillations (fig. 4.5).



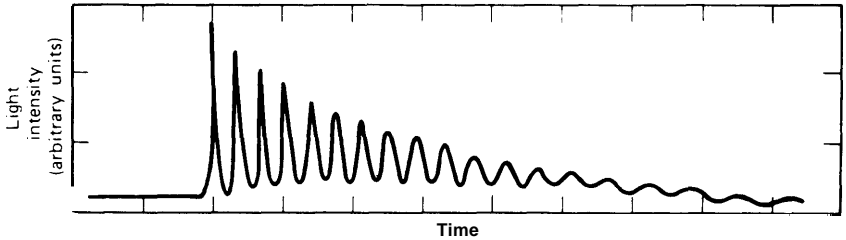


Fig. 4.5. Example of relaxation oscillations. [L.F. Johnson, in: Lasers, ed. A.K. Levine. Dekker, New York 1966.1

### Exercises on section 4.3

(1) Establish the rate equations for the photon number  $n$  and the atomic occupation numbers for the transition scheme of fig. 4.4. Discuss the limiting case that  $w_{12}$  becomes very large. Convince yourself that in this case  $N_1 \approx 0$ . Under which assumptions may one replace the first term on the r.h.s. of (4.25), which in an exact treatment should read  $N_0 w_{20}$ , by  $N w_{20}$ , where  $N_0$  is the number of atoms in the ground state? Why is one allowed to keep for the whole set of equations only those for the photon numbers and for the  $N_2$ ?

(2) Determine  $\delta N_2$  which corresponds to (4.42) and determine  $A_1$  and  $A_2$  by means of the initial condition at time  $t=0$ ,

$$\delta n(0) = \delta n_0, \quad \delta N_2(0) = \delta N_{2,0}.$$

Hint: Express  $B$  by means of  $A$  (for  $A_1$ ,  $a = -\Gamma + i\omega_r$ , and for  $A_2$ ,  $\alpha = -\Gamma - i\omega_r$ ) using eq. (4.34).

### 4.4. Q-switching

The rate equations of the single mode laser (section 4.1) allow us to study the performance of the Q-switched laser. In such a laser the reflectivity of one of the mirrors can be suddenly changed. This change can be achieved by mounting one of the mirrors in a certain distance from one end of the laser material and letting this mirror rotate. In practical cases a rotating prism is used (fig. 4.6). In order to achieve very short switching times, Kerr cells are used also, which are especially practical if the light of the laser active atoms is already polarized for instance in ruby crystals.

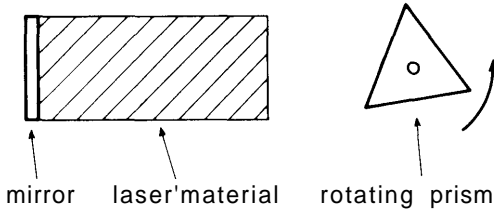


Fig. 4.6. Scheme of an experimental arrangement to achieve Q-switching.

The basic idea of Q-switching is as follows. If we first omit one of the mirrors, the photons in the laser material have a very short lifetime. Even if we pump very highly, the laser condition cannot be fulfilled and no laser action takes place. Because laser action would decrease the inversion, by the absence of laser action we may achieve a very high inversion if a mirror is lacking. If suddenly this mirror is brought into its correct position, laser action can start with a very high initial inversion. Since in eq. (4.12),  $DW - 2\kappa$  is very big we expect an exponential increase of the photon number  $n$  in an avalanche, i.e. a giant pulse to be emitted. The pulse height and width will be limited because according to eq. (4.11) a big photon number  $n$  in the last term of that equation will cause a decrease of the inversion. This in turn will decelerate the photon number production according to eq. (4.12), and if  $D$  becomes negative will even lead to a decrease of the number of photons.

Let us consider these effects in more detail. We start from eq. (4.11) and assume that laser action takes place much more quickly than pump and relaxation processes within the atoms. Furthermore we neglect as usual spontaneous emission. Eq. (4.11) then reduces to

$$dD/dt = -2WnD. \quad (4.43)$$

According to (4.43) the inversion decreases rather slowly for small values of  $n$ , so that we may replace  $D$  on the right hand side of

$$dn/dt = (WD - 2\kappa)n \quad (4.44)$$

and of (4.43) by  $D$ . Furthermore we assume that initially a certain number of photons is present due to spontaneous emission. Their number will be denoted by  $n_i$ . The solutions of (4.44) and (4.43) read

$$n = n_i \exp(\alpha t) \quad (4.45)$$

and

$$D \approx D_i \{1 + 2Wn_i[1 - \exp(\alpha t)]/\alpha\}, \quad (4.46)$$

where

$$\alpha = WD_i - 2\kappa. \quad (4.47)$$

In the initial phase of the pulse,  $n$  increases exponentially with the gain constant  $\alpha$  where upon the inversion decreases on account of (4.43) or (4.46). The increase of  $n$  stops at latest at time  $t_1$  when  $D(t_1) \equiv 0$ . (In reality it ends earlier because of the loss constant  $2\kappa$ .) If we neglect in (4.46) “1” against the exponential function  $\exp(\alpha t)$  it follows from eq. (4.46) that

$$n_i \exp(\alpha t_1) = \alpha/2 W = n_{\max}, \quad (4.48)$$

or equivalently

$$t_1 = (1/\alpha) \ln(\alpha/2 W n_i). \quad (4.49)$$

Eq. (4.48) incidentally represents the number of photons present at time  $t_1$ . After time  $t_1$  we may assume  $D=0$ . The photon number then decreases exponentially according to

$$n(t) = n_{\max} \exp[-2\kappa(t - t_1)]. \quad (4.50)$$

The equations just presented are quite useful at least for an estimate of the quantities  $t_1$ ,  $n_{\max}$  and  $n(t)$ . To solve the equations more accurately computer solutions must be used. Some typical results are presented in fig. 4.7.

#### 4.5. The basic rate equations of the multimode laser

In the preceding sections we have studied a laser, assuming that the atoms emit light only into a single mode. This is, of course, in contrast to the emission of atoms in conventional light sources where light is emitted, for instance, into all possible directions. As we mentioned above, a reason for mode selection is the different lifetimes of different kinds of photons. In this section we wish to study in more detail how mode selection in a laser is achieved. We shall see that simultaneous emission of photons into different modes can happen also.

To this end we have to discuss the structure of the coefficient  $W$  which occurs in the laser equations more closely. So far we have taken this coefficient from Einstein's theory of absorption and emission of photons. As we shall show in detail in later chapters,  $W$  cannot be considered as constant for all kinds of photons. We wish to visualize how  $W$  looks like in reality and to derive its form in a heuristic fashion. (For its derivation from first principles cf. section 6.9.)  $W$  stems of course, from the interaction of the light field with the atoms. If we consider a single standing light wave, e.g. in the form  $\sin kx$ , it is quite clear that this light wave cannot have any

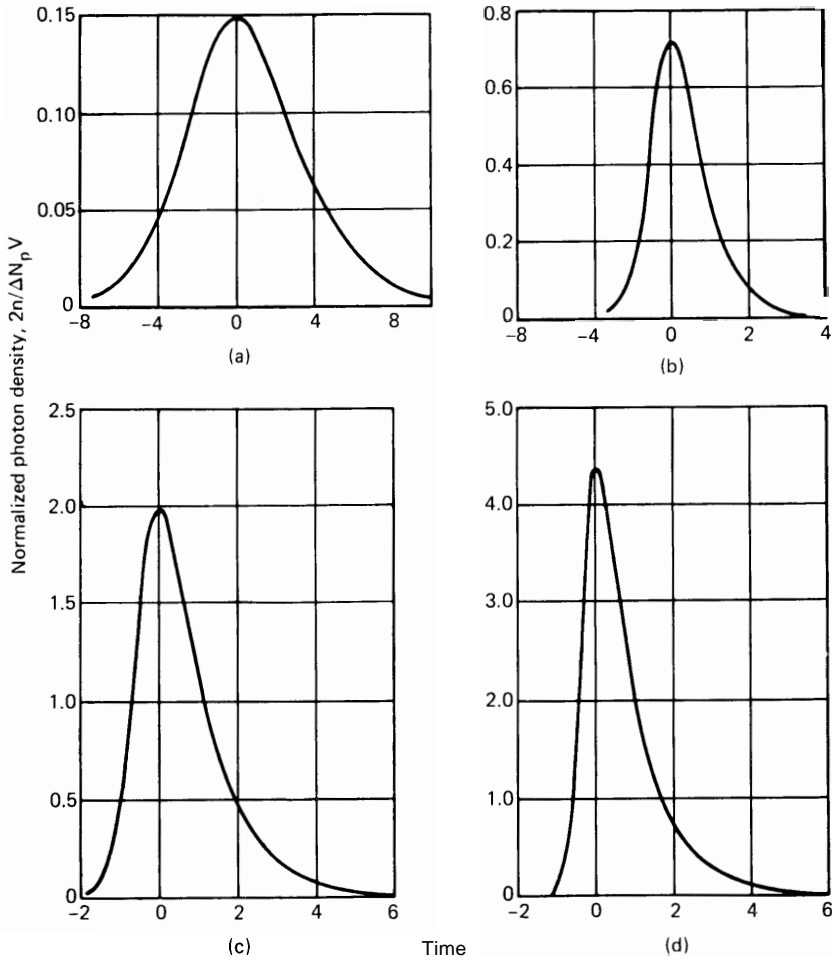


Fig. 4.7. Example of the time dependence of the emitted intensity of a Q-switching laser for various degrees of excitation. [W. Wagner and B.A. Lengyel, *J. Appl. Phys.* 34, 2040 (1963).]

interaction with an atom at the position  $\mathbf{x} = 0$  or at any of the other nodes of the sine wave. On the other hand we may expect a maximal interaction between atom and light wave if the sine function has its maximum. Because energy is exchanged between the atom and the light field we must assume that  $W$  does not depend on the field amplitude but rather on the intensity, i.e. on the absolute square of the field amplitude. Instead of a sine wave also other kinds of wave forms can be generated in the laser resonator (cf.

chapter 3). Denoting the corresponding wave forms by  $u_\lambda(\mathbf{x}_\mu)$  and using the ideas just mentioned we are led to assume the transition probability in the form

$$W \propto |u_\lambda(\mathbf{x}_\mu)|^2. \quad (4.51)$$

Because we shall deal with several wave forms we distinguish the  $u$ 's by the index  $A$ , where  $A$  shall remind the reader of the wave-length. But in our present context it may also denote the various directions of propagation of a wave or its polarization, etc.  $\mathbf{x}_\mu$  denotes the space point where an atom is located. The individual atoms are distinguished by the index  $\mu$ . In this sense  $W$  represents a transition rate caused by the interaction of the wave  $A$  and the atom  $\mu$  (fig. 4.8).

A further dependence of  $W$  on the light field and on the atomic quantities follows when we take into account the polarization of the light field and

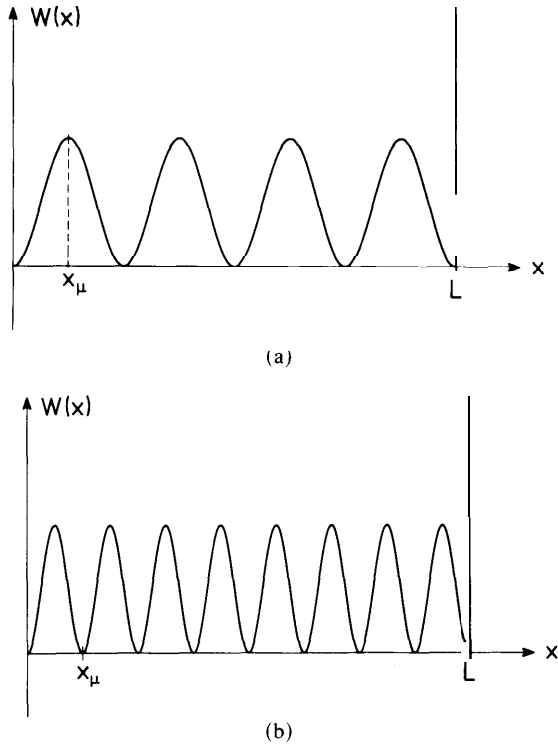


Fig. 4.8. The interaction function  $W$  (eq. (4.51)) versus the spatial coordinate  $x$  along the laser axis.  $L$  is the distance between the mirrors of the laser resonator,  $x_\mu$  the coordinate of atom  $\mu$ . (a)  $x_\mu$  coincides with an antinode; (b)  $x_\mu$  coincides with a node.

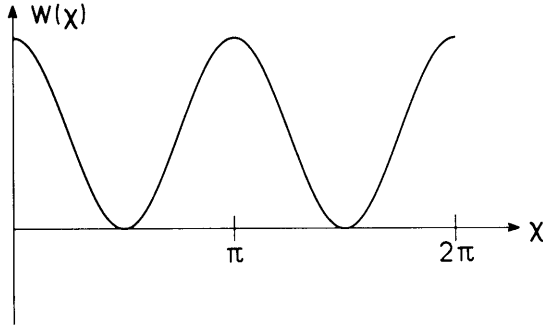


Fig. 4.9. This figure shows the dependence of the interaction function  $W$  on the angle  $\chi$  between atomic dipole moment and field polarization.

of the atoms. The electronic motion within the atoms can be described as that of oscillating dipoles. Here and in the following we shall assume for simplicity that the dipoles in the laser material are oscillating in the same direction.

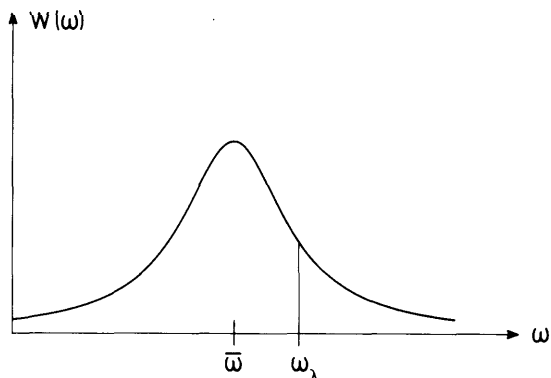
A light wave  $A$  having the direction of polarization  $e_\lambda$  can interact with an atomic dipole moment only to an extent in which the dipole moment of the atom  $\mathfrak{D}$  has its component in the direction of the polarization vector  $e_\lambda$ . Because only intensities play a role we are led to assume the dependence of  $W$  on the polarization in the form (fig. 4.9)

$$W \propto |e_\lambda \mathfrak{D}|^2. \tag{4.52}$$

Finally we have to discuss how the positions of the mode frequencies relative to the frequencies of the optical transitions within the atoms enter into  $W$ . From now on we shall use in this book till its end the following notations for the circular frequencies of atoms and fields: circular frequency of the atomic transition  $\bar{\omega}$ , circular frequency of the light wave in the laser resonator  $\omega_\lambda$ . As is well known from experimental physics, the emission of atoms possesses a certain line shape (cf. fig. 4.10). A single atom therefore does not uniformly radiate light in the region of its line-width into the individual frequencies but rather according to an intensity distribution. In case of a Lorentzian line the intensity of a light wave with frequency  $\omega_\lambda$  and the central atomic transition frequency  $\bar{\omega}$  is given by

$$\frac{2\gamma}{\gamma^2 + (\bar{\omega} - \omega_\lambda)^2}, \tag{4.53}$$

where we omitted a factor,  $I_0$ . Here  $\gamma$  is the line-width or, more precisely speaking, the half width at half intensity. Now let us recall that  $W$  is the

Fig. 4.10. An illustration of the function  $W$  according to eq. (4.53).

rate with which an atom emits light into the mode  $A$ . Therefore we shall require that  $W$  is proportional to the expression (4.53) (fig. 4.10).

In some cases we have to generalize (4.53). For instance, laser-active atoms in a solid may occupy different kinds of positions within the crystalline lattice. Due to their different positions the atoms have different central frequencies  $\bar{\omega}$  which we will have to distinguish by the atomic index  $\mu$ . If there is a distribution of  $\bar{\omega}$ 's over a certain frequency range, an inhomogeneous broadening is present. An inhomogeneous broadening is caused also in gases because of the Doppler shift due to the motion of the gas atoms. In the cases of homogeneous and inhomogeneous broadening we have to assume  $W$  in the form

$$W \propto \frac{2\gamma}{\gamma^2 + (\bar{\omega} - \omega_\lambda)^2}. \quad (4.54)$$

Because  $W$  depends on the mode  $A$  and on the laser atom  $\mu$  under consideration, we attach the indices  $A$  and  $\mu$  to  $W$ .

Let us collect the individual conditions which we impose on  $W$  and let us anticipate the still lacking proportionality factor which we explicitly derive in a later chapter.  $W_{\lambda\mu}$  can then be written in the form

$$W_{\lambda\mu} = \frac{2\gamma}{\gamma^2 + (\bar{\omega}_\mu - \omega_\lambda)^2} |g_{\mu\lambda}|^2, \quad (4.55)$$

where we have used the abbreviation

$$g_{\mu\lambda} = i\mathcal{D}e_\lambda u_\lambda(x_\mu) [\omega_\lambda / (2\hbar\epsilon_0)]^{1/2}. \quad (4.56)$$

Evidently all factors within  $g_{\mu\lambda}$  are known from our above considerations

except for the last one which can be derived by an exact quantum mechanical calculation only.

We are now in a position to formulate the laser equations. We assume that there exists a certain set of modes in the laser resonator and we distinguish them by the index  $\lambda$ . Each mode can be occupied with a certain number of photons  $n_\lambda$ . Because the lifetimes of different modes in the resonator can be different we introduce decay constants  $\kappa_\lambda$  which in general will differ from each other. Because the individual atoms interact with the laser modes in a different way we have to consider the atoms individually. For simplicity we again consider the 2-level scheme leaving its extension to a 3-level scheme as an exercise to the reader.

We denote the occupation numbers of the atom  $\mu$  in the states 1 and 2 by  $N_{1,\mu}$  and  $N_{2,\mu}$ , respectively. The corresponding inversion of atom  $\mu$  is described by  $d_\mu = N_{2,\mu} - N_{1,\mu}$ . Generalizing the laser equation (2.1) we can immediately write down the laser equation for the mode  $\lambda$

$$\frac{dn_\lambda}{dt} = -2\kappa_\lambda n_\lambda + n_\lambda \sum_\mu W_{\lambda\mu} d_\mu + \sum_\mu W_{\lambda\mu} N_{2,\mu}. \quad (4.57)$$

Though this equation was not derived exactly here (what we shall do later) its form is quite plausible. The temporal change of the number of photons of kind  $\lambda$  is given by:

- (1) losses (first term on the r.h.s.);
  - (2) the stimulated emission and absorption processes by the individual atom  $\mu$  (first sum on the r.h.s.);
  - (3) a term representing spontaneous emission (second sum on the r.h.s.).
- We shall omit this last term when we consider the laser process.

A critical reader will quite rightly ask why no phase relations between the modes and the oscillating dipole moments of the electrons of the atoms are taken into account. In fact, (4.57) implies an approximation whose meaning we shall study in a later chapter. Eq. (4.57) can be obtained only if phase relations are neglected which is allowed in many cases but definitely not always. Indeed, very important effects in the laser, such as mode locking, are due to specific phase relations.

The rate equations for the individual atom  $\mu$  must take into account pump and relaxation processes and, in addition, the effect of stimulated emission and absorption. Because not only a single kind of photons but various kinds of photons are emitted, all the photon numbers must be taken into account. On account of these ideas we obtain the rate equations for the atom  $\mu$

$$\frac{dN_{2,\mu}}{dt} = w_{21}N_{1,\mu} - w_{12}N_{2,\mu} - d_\mu \sum_\lambda n_\lambda W_{\lambda\mu}, \quad (4.58)$$



and

$$\frac{dN_{1,\mu}}{dt} = -w_{21}N_{1,\mu} + w_{12}N_{2,\mu} + d_\mu \sum_\lambda n_\lambda W_{\lambda\mu}. \quad (4.59)$$

By adding (4.58) to (4.59) we obtain the conservation law for the total occupation number of the atom  $\mu$

$$\frac{d}{dt}(N_{1,\mu} + N_{2,\mu}) = 0,$$

and thus

$$N_{1,\mu} + N_{2,\mu} \equiv N_\mu = 1. \quad (4.60)$$

Because we are dealing with a single atom, we have put  $N_\mu = 1$ . In a way analogous to the single mode laser (section 4.2), we may derive an equation for the inversion  $d_\mu = N_{2,\mu} - N_{1,\mu}$  from (4.58) and (4.59),

$$\frac{d}{dt}d_\mu = \frac{1}{T}(d_0 - d_\mu) - 2d_\mu \sum_\lambda n_\lambda W_{\lambda\mu}. \quad (4.61)$$

Eqs. (4.57) and (4.61) are the basic laser equations which we wish to discuss now.

### Exercise on section 4.5

(1) Formulate the rate equations corresponding to eqs. (4.57), (4.58) and (4.59) for three-level atoms with the pump scheme of fig. 2.9.

### 4.6. Hole burning. Qualitative discussion

Within the frame of rate equations it will be our goal to calculate the intensity distribution over the individual modes of the laser or, more precisely speaking, we wish to calculate the number of photons belonging to the individual modes  $\lambda$ . Because the rate equations are nonlinear, this task is not quite simple. However, one may get some useful insight into some important mechanisms by a study of the inversion  $d_\mu$  which is crucial for the determination of the photon numbers. To this end we start from eq. (4.57) where we shall ignore the last term describing spontaneous emission as already mentioned above. As is evident from eq. (4.57), the inversion  $d_\mu$  of the individual atoms determines whether the losses described by the term  $-2\kappa_\lambda n_\lambda$  can be compensated. In the following it will be important to note that the inversion  $d_\mu$  is determined in turn by the photon numbers as can

be seen by means of eq. (4.61). We confine our discussion on the stationary state by putting

$$\frac{dd_\mu}{dt} = 0. \quad (4.62)$$

In analogy to the single mode case, in our discussion we may include also processes in which  $d_\mu$  follows the photon number instantaneously in which (4.62) also holds. By means of (4.62) we may resolve (4.61) with respect to  $d_\mu$

$$d_\mu = \frac{d_0}{1 + 2T \sum_\lambda n_\lambda W_{\lambda\mu}}. \quad (4.63)$$

In the following we shall be interested in laser action not too far above threshold. In such a case we may assume that the photon numbers  $n$ , are small so that we may expand the r.h.s. of (4.63) into a power series with respect to the photon numbers. As it will turn out it is sufficient to confine the expansion to the first two terms

$$d_\mu \approx d_0 \left( 1 - 2T \sum_\lambda n_\lambda W_{\lambda\mu} \right). \quad (4.64)$$

We first consider the special case in which only one photon number is unequal 0. In such a case (4.64) reduces to

$$d_\mu \approx d_0 (1 - 2TnW_{\lambda\mu}). \quad (4.65)$$

The mode index  $\lambda$  has been omitted from  $n$  because we are dealing with one mode only. We remind the reader that the index  $\mu$  distinguishes between the different atoms. We now study how the photon number  $n$  influences the inversion  $d_\mu$  of the individual atoms. To this end we recall the definition of  $W_{\lambda\mu}$  which was given in eqs. (4.55) and (4.56). According to that definition,  $W_{\lambda\mu}$  essentially consists of two parts namely one referring to the spatial behavior of the mode  $\lambda$  under consideration and of the central transition frequency of the atom  $\mu$ . We wish to study the influence of these two factors on the behavior of  $d_\mu$  separately. We first consider the case in which we deal with running laser waves which may be realized, e.g. in a ring laser. In such a case the spatial function  $u_\lambda(\mathbf{x}_\mu)$  which occurs in  $W_{\lambda\mu}$  according to (4.55) and (4.56) is given by

$$u_\lambda(\mathbf{x}_\mu) = \frac{1}{\sqrt{V}} \exp[ik_\lambda \mathbf{x}_\mu], \quad (4.66)$$

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where  $V$  is the volume and  $k$ , the wave vector. Because in (4.55) only the absolute value of (4.66) enters,  $W_{\lambda\mu}$  becomes independent of the spatial coordinate. Therefore we may confine our discussion to that part of  $W_{\lambda\mu}$  which has the form

$$W_{\lambda\mu} \propto \frac{2\gamma}{\gamma^2 + (\omega_\lambda - \bar{\omega}_\mu)^2}. \quad (4.67)$$

After these preparatory steps we discuss *hole burning in an inhomogeneously broadened line*. We consider an inhomogeneously broadened atomic line where the transition frequencies  $\bar{\omega}_\mu$  depend on the atomic index  $\mu$ . We first study the behavior of  $d_\mu$  (4.65) when we change  $\beta$ . For sake of simplicity we shall omit the index  $\mu$  and consider fig. 4.11. In it the inversion  $d$  is plotted versus  $\beta$ . If no laser action takes place, i.e. if  $n=0$ , we obtain the unsaturated inversion  $d_0$  which is shown as the upper dashed line. When  $n$  is unequal zero we must subtract the Lorentzian curve (4.67) from  $d_0$ . The maximum of that curve lies at that atomic transition frequency  $\bar{\omega}$  which coincides with the frequency  $\omega_\lambda$  of the laser mode under consideration. The half width of this curve is given by  $\gamma$ . Thus we obtain the solid curve shown in fig. 4.11 which exhibits an incision in the inversion. That means that due to laser action the inversion is reduced close to the resonance line ( $\beta = \omega_\lambda$  or, in other words, that a hole is burned into a line).

How does this hole burning affect the equations for the laser modes? The photons are produced by stimulated emission, i.e. by the second term on the r.h.s. of eq. (4.57). In this term the inversion  $d_\mu$  occurs under a sum

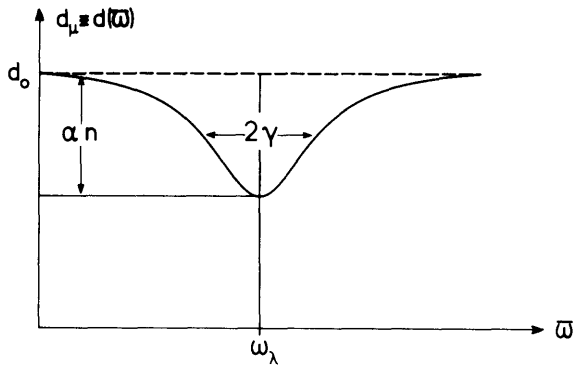


Fig. 4.11. The unsaturated inversion  $d_0$  and the saturated inversion  $d(\bar{\omega})$  versus the atomic transition frequency  $\bar{\omega}$ ;  $\omega_\lambda$  is the mode frequency.

over atomic indices

$$\sum_{\mu} W_{\lambda\mu} d_{\mu}. \tag{4.68}$$

Since we have, at least in general, to deal with very many atoms in a laser (say  $10^{14}$  or more) we may replace the sum by an integral. We first will give a qualitative discussion and then below we shall do the corresponding calculations explicitly. A small region of transition frequencies ( $\omega \dots (\omega + d\bar{\omega})$ ) may contain a certain number of atoms. In general this number is biggest close to the central frequency  $\bar{\omega}_0$  and decreases with increasing frequency difference. In general we have to assume that the number  $dZ$  of atoms within the frequency interval  $d\bar{\omega}$  depends on  $\bar{\omega}$ . Therefore we write

$$dZ = \rho(\bar{\omega}) d\bar{\omega}. \tag{4.69}$$

In many cases of practical interest, for instance in solid state lasers, we may assume that  $\rho(\bar{\omega})$  possesses a Gaussian distribution (cf. fig. 4.12). In order to evaluate the sum over  $\mu$  in (4.68) we have to proceed from the individual atomic indices  $\mu$  to the new variable  $\bar{\omega}$ ,

$$\sum_{\mu} W_{\lambda\mu} d_{\mu} \rightarrow \int W_{\lambda}(\bar{\omega}) d(\bar{\omega}) \rho(\bar{\omega}) d\bar{\omega}. \tag{4.70}$$

We have replaced  $W_{\lambda\mu}$  by  $W_{\lambda}(\bar{\omega})$  and  $d_{\mu}$  by  $d(\bar{\omega})$  because we wanted to replace the variable  $\mu$  by  $\omega$ . In fact the sum over  $\mu$  does not only run over the distribution of atoms over the frequencies  $\bar{\omega}$  but also over the spatial distribution of atoms. But because in the present context we treat space-independent  $W$ 's we shall not discuss the corresponding integration over

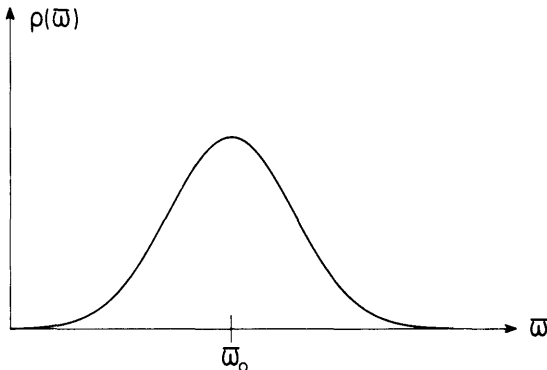


Fig. 4.12. The density of atoms,  $\rho$ , versus their transition frequency  $\bar{\omega}$ ;  $\bar{\omega}_0$  is the center frequency.

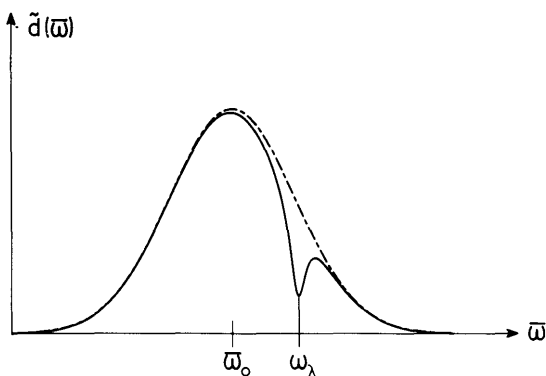


Fig. 4.13. The inversion density  $\tilde{d}(\bar{\omega})$  versus the atomic transition frequency  $\bar{\omega}$ . The hole burned at the position  $\omega_\lambda$  (=mode frequency) can easily be seen.

the atomic positions here. (4.70) suggests to introduce the inversion density  $\tilde{d}$  by the relation

$$\tilde{d}(\bar{\omega}) = d(\bar{\omega}) \rho(\bar{\omega}). \quad (4.71)$$

In order to represent  $\tilde{d}$  we have to multiply the curves of figs. 4.11 and 4.12 with each other. We thus obtain fig. 4.13 showing the dependence of the inversion density on the transition frequency (3. Of course, again a hole is burned into the inversion density at the frequency  $\bar{\omega} = \omega_r$ . How does this kind of hole burning affect the gain which is proportional to (4.68)? Evidently in the sum (4.68) or in the corresponding integral the factor

$$W_{\lambda\mu} \equiv W(\bar{\omega}) = \text{const} \times \frac{2\gamma}{\gamma^2 + (\omega_\lambda - \bar{\omega})^2}, \quad \bar{\omega} \equiv \bar{\omega}_\mu, \quad (4.72)$$

occurs once again (compare also (4.67)). Thus those contributions to the 'sum (4.68) are most important which stem from the surrounding of  $\bar{\omega} = \omega_r$ . Because the depth of the hole is the same for all mode frequencies  $\omega_\lambda$ , the effective gain will become the biggest if the position of the hole coincides with the atomic central frequency  $\bar{\omega}_0$ . In the stationary state the depth of the hole is determined from the condition

$$\frac{dn}{dt} = 0, \quad (4.73)$$

i.e. that the relation

$$2\kappa_\lambda = \sum_{\mu} W_{\lambda\mu} d_{\mu} \quad (4.74)$$

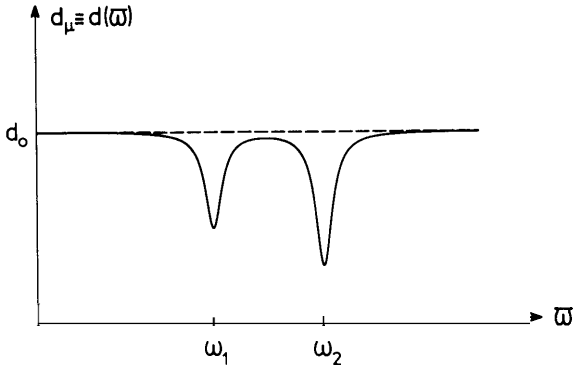


Fig. 4.14. The unsaturated inversion  $d_0$  and the saturated inversion  $d(\omega)$  versus  $\omega$ . The modes with frequencies  $\omega_1$  and  $\omega_2$  have burned two holes into the inversion. If the photon numbers of the two modes differ, the depths of the holes differ also.

is fulfilled. In section 4.7 entitled "quantitative treatment" we shall evaluate in particular the expression (4.74) explicitly.

But here we shall continue our qualitative discussion and consider the behavior of the inversion when *two modes are present*, i.e. when  $n_\lambda \neq 0$  for  $\lambda = 1$  and  $\lambda = 2$ . In order to discuss in which way the inversion is lowered we may proceed in the same way as with one mode but we have to take into consideration that the corresponding Lorentzian lines must be subtracted from the unsaturated inversion  $d_0$  at two positions at  $\omega = \omega_1$  and  $\omega = \omega_2$ . Thus we obtain fig. 4.14 which corresponds to our fig. 4.11 above. Fig. 4.12 can again be used without changes so that we obtain fig. 4.15 as

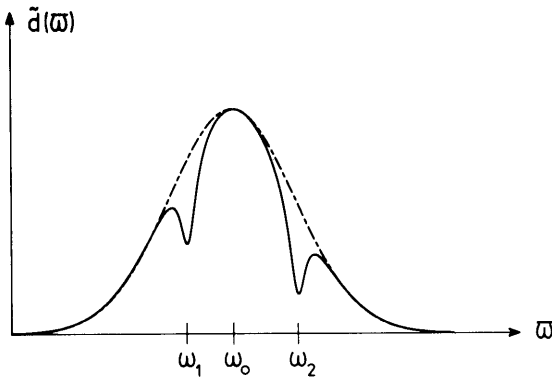


Fig. 4.15. The inversion density belonging to fig. 4.14. The curve results from that of fig. 4.14 by a multiplication with that of fig. 4.12. The two holes are clearly exhibited.

final result for the inversion density. This figure deals with the case in which the frequencies  $\omega_1$  and  $\omega_2$  of the corresponding laser modes have a distance from each other which is large compared to the homogeneous line-width  $\gamma$ . As we have observed above when treating a single mode, the factor  $W_{\lambda\mu}$  which occurs once again in the sum (4.68) essentially picks out the surrounding of  $\bar{\omega} = \omega_1$ . This means, of course, that when two modes are present they do not influence each other with respect to their gains. When we visualize the two laser modes 1 and 2 as two cows on a meadow it means that these cows are grazing on two different parts of the meadow or, in the context of a laser, that the modes draw their energy from two entirely different kinds of atoms. The situation changes drastically when the distance between the two mode frequencies becomes so small that they are lying within the line-width  $2\gamma$ . Then the two cows, so to speak, graze on the same part of the meadow and the question arises how the resulting competition will end. We shall discuss this question in section 4.9.

We now turn to *hole burning in gas lasers* because here especially interesting effects occur. We confine our discussion to a single laser mode. For the case of several modes I have to refer the reader to my book *Laser Theory* (Encyclopedia of Physics). Because gas atoms move, the Doppler effect plays a role which becomes of special importance if laser action takes place. According to the Doppler effect, the transition frequency  $\bar{\omega}_0$  of an atom flying away with the velocity  $v$  from the observer appears shifted according to

$$\bar{\omega} = \bar{\omega}_0(1 - v/c). \quad (4.75)$$

Correspondingly the frequency of an atom flying towards the observer, appears shifted according to

$$\bar{\omega} = \bar{\omega}_0(1 + v/c). \quad (4.76)$$

Because the individual atoms  $\mu$  have different velocities  $v_\mu$ , an effective line broadening is brought about. We may take over our above discussion of the inhomogeneously broadened line of atoms at rest when we use the  $\bar{\omega}_\mu$ 's occurring in (4.75) and (4.76). However, an important difference with respect to the solid state laser occurs because (4.75) and (4.76) contain the velocity components,  $v$ , of the individual atoms in the direction of propagation of the laser waves. When we deal with standing waves, the laser mode consists of two waves running in opposite directions. As a consequence the laser mode resonates with two kinds of atoms, namely those moving in opposite directions with a certain velocity  $|v|$ . Thus each wave burns two holes into the atomic inversion.

In complete analogy to our earlier discussion we still have to take into account the distribution of frequencies. According to (4.75) and (4.76) this

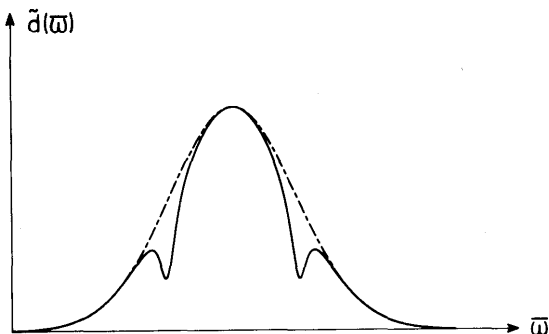


Fig. 4.16. In a gas laser with a standing wave two holes in symmetric positions with respect to the center of the atomic emission line result.

distribution is determined by the distribution of velocities  $v$  which according to the theory of gases is given by a Maxwellian distribution. In this way we obtain fig. 4.16 where we have assumed

$$|\omega_\lambda - \bar{\omega}_0| > 2\gamma. \quad (4.77)$$

By a change of the distance between the mirrors the laser mode frequency may be tuned in such a way that this frequency coincides with  $\bar{\omega}_0$ . As a result both holes coincide and yield a particularly deep incision. Because the laser mode nearly exclusively interacts with the atoms in the region of the incision only, and only here the inversion is strongly decreased, we obtain the following result. If we tune the laser line on the atomic line the gain will become smaller compared to the case in which (4.77) is valid at least for small detuning. We shall treat this effect, which plays an important role in Doppler-free spectroscopy, in section 6.8 quantitatively.

#### 4.7. Quantitative treatment of hole burning. Single mode laser action of an inhomogeneously broadened line

We now return to single mode laser action in a solid state laser with an inhomogeneously broadened line. We start from the mode equation (4.57) which we write down for the special case of a single mode where we drop the index  $A$  of  $n$  but retain it for sake of clarity in  $W_{\lambda\mu}$ . Neglecting as usual spontaneous emission this equation reads

$$\dot{n} = -2\kappa n + n \sum_{\mu} d_{\mu} W_{\lambda\mu}. \quad (4.78)$$



We insert in it the saturated inversion which according to (4.64) reads

$$d_\mu \approx d_0(1 - 2TnW_{\lambda\mu}). \quad (4.79)$$

We adopt running waves (cf. (4.66)) so that  $W_{\lambda\mu}$  does no more depend on the space coordinate  $\mathbf{x}$ . Using the explicit form of  $W_{\lambda\mu}$  as defined in (4.55) and (4.56) and replacing the index  $\mu$  by  $\mathbf{I}$  we may write

$$W_{\lambda\mu} \equiv W_\lambda(\bar{\omega}) = \frac{A}{V} \frac{\gamma}{\gamma^2 + (\bar{\omega} - \omega_\lambda)^2}, \quad (4.80)$$

where we have used the abbreviation

$$A = \bar{\omega}_0 |\mathfrak{D}\mathbf{e}|^2 / \hbar \varepsilon_0. \quad (4.81)$$

By inserting (4.79) into (4.78) we cast eq. (4.78) into the form

$$\dot{n} = n \sum_\mu W_{\lambda\mu} (d_0 - 2TnW_{\lambda\mu}d_0) - 2\kappa n. \quad (4.82)$$

We convert the sum over the atomic indices  $\mu$  into an integral over space and an integral over the frequencies  $\bar{\omega}$ ,

$$\sum_\mu \dots = \int \dots d^3\mathbf{x} \int \dots d\bar{\omega}. \quad (4.83)$$

Because  $W$  does not depend on space the integration over the volume elements  $d^3\mathbf{x}$  yields the volume  $V$  of the resonator. In order to evaluate the integral over  $\bar{\omega}$  we use a Gaussian distribution for the frequency distribution  $\rho(\bar{\omega})$ , i.e.

$$\rho(\bar{\omega}) = \rho_0 (\alpha\sqrt{\pi})^{-1} \exp\left[-\left(\frac{\bar{\omega} - \bar{\omega}_0}{\alpha}\right)^2\right]. \quad (4.84)$$

Here  $\rho_0$  is the density of atoms,  $\rho_0 = N/V$ , where  $N$  is the total number of laser atoms in the resonator and  $V$  the volume of the resonator, and  $\alpha$  is the halfwidth of the Gaussian distribution.

We first treat the first term on the r.h.s. of (4.82),

$$d_0 \sum_\mu W_{\lambda\mu}. \quad (4.85)$$

In order to evaluate this expression we insert (4.80) and (4.84) into it. Observing that integration over the volume in (4.83) cancels against  $V$  in the denominator in (4.80) we have eventually to treat

$$d_0 \rho_0 A \int \frac{\gamma}{\gamma^2 + (\bar{\omega} - \omega_\lambda)^2} (\alpha\sqrt{\pi})^{-1} \exp\left[-\left(\frac{\bar{\omega} - \bar{\omega}_0}{\alpha}\right)^2\right] d\bar{\omega}. \quad (4.86)$$

This integral cannot be evaluated in explicit form. However, it is rather simple to calculate it approximately if we assume  $y \ll a$ . In such a case we obtain in a good approximation

$$(4.86) = d_0 \rho_0 A \frac{\sqrt{\pi}}{a} \exp[-\delta^2], \quad (4.87)$$

where  $\delta$  is defined by

$$\delta = \frac{\omega_\lambda - \bar{\omega}_0}{\alpha}. \quad (4.88)$$

In a similar fashion we can evaluate the second term in (4.82), i.e.

$$-\sum_{\mu} 2Tn W_{\lambda\mu} W_{\lambda\mu} d_0, \quad (4.89)$$

which yields

$$-\rho_0 A d_0 3ATn \frac{\sqrt{\pi} \exp[-\delta^2]}{2\gamma\alpha} \quad (4.90)$$

Taking the corresponding terms together, we may cast (4.82) into the explicit form

$$\dot{n} = \rho_0 A d_0 \frac{\sqrt{\pi}}{\alpha} \exp[-\delta^2] (1 - 3ATn/2\gamma)n - 2\kappa n. \quad (4.91)$$

In the stationary state,  $dn/dt=0$ , this equation can be readily solved for  $n$ . Because  $n$  is the only free parameter in the formula for hole burning, (4.79), we have thus fixed the depth of that hole. We mention that the integrals such as (4.86) can be evaluated exactly and explicitly if the inhomogeneous atomic line is a Lorentzian.

### Exercise on section 4.7

(1) Solve eq. (4.91) for  $n$  (with  $\mathbf{r}\dot{\mathbf{i}}=0$ ) and discuss the dependence of  $n$  on the unsaturated inversion  $d_0$  and on the detuning  $\delta$ .

### 4.8. Spatial hole burning. Qualitative discussion

So far we have studied the case in which  $W_{\lambda\mu}$  does not depend on space. Now we want to study the case in which the spatial dependence of  $W_{\lambda\mu}$  is caused by standing waves. For simplicity we shall assume that the atomic transition line is homogeneously broadened so that  $\bar{\omega}_\mu$  does not depend on

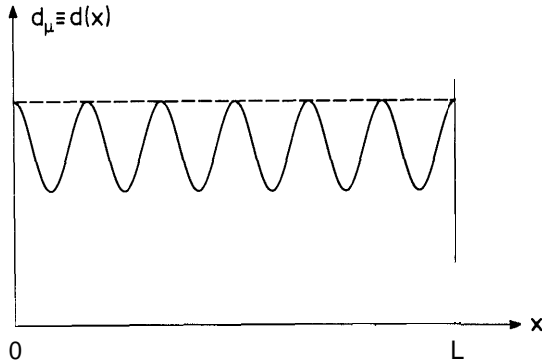


Fig. 4.17. Spatial hole burning. The inversion  $d(x)$  is lowered at the positions  $x$  where the antinodes of the laser modes are.  $L$ : laser length.

$\mu$ . In other words we assume that the central position of all atomic transition frequencies is the same. Since we are primarily interested in the spatial part of  $W_{\lambda\mu}$  we consider only that part. According to our equations (4.55) and (4.56) where  $u_\lambda(x_\mu) = \sqrt{2/V} \sin(k_\lambda x_\mu)$ , this part is given by

$$W_{\lambda\mu} \propto (\sin k_\lambda x_\mu)^2. \quad (4.92)$$

When we plot the inversion  $d_\mu$  as function of the spatial coordinate  $x = x_\mu$  according to (4.65) we obtain the curve of fig. 4.17. Quite evidently holes are periodically burned into the unsaturated inversion precisely at those positions where the standing wave of the laser mode has its maxima. If not only one mode is present, but, for instance, two modes according to (4.64), both modes burn holes into the unsaturated inversion.

In the next sections we shall show how hole burning has an effect on the coexistence or competition between different laser modes. To this end we shall proceed in several steps. We first discuss the special case in which only one mode out of many survives. We then show qualitatively how the coexistence of modes becomes possible and eventually we shall present a detailed mathematical treatment of the effect of hole burning on mode coexistence.

#### 4.9. The multimode laser. Mode competition and Darwin's survival of the fittest

We consider a laser setup which allows running waves only. This may be achieved by a rectangular arrangement of mirrors (compare fig. 3.6) where besides the laser-active medium a cell is present which permits the

propagation of light in only one direction. The plane waves are represented by

$$u \propto \exp(ikx), \quad (4.93)$$

so that

$$|u|^2 = \text{const.} \quad (4.94)$$

This means that all waves may participate at the atomic emission in the same way, at least as far as their spatial behavior is concerned. Thus the transition rate  $W_{\lambda\mu}$  (with  $\bar{\omega}_\mu = \omega_\lambda$ ) simplifies as follows:

$$W_{\lambda\mu} = \frac{2\gamma}{\gamma^2 + (\bar{\omega} - \omega_\lambda)^2} |g|^2 \equiv W_\lambda. \quad (4.95)$$

We are treating a homogeneously broadened atomic line. A closer inspection of eqs. (4.57)–(4.61) shows that we may pull the expressions  $W_\lambda$  in front of the sums over  $\mu$  occurring in eq. (4.57). This means that the photon annihilation and production rate is determined by the total inversion

$$\sum_\mu d_\mu = D \quad (4.96)$$

only. In order to derive an equation for this  $D$  we sum up eq. (4.61) over  $\mu$ . Therefore we need only to take into account the ensemble of atoms but no more their individual occupation numbers. When we confine our analysis to the stationary state we may put  $\dot{n}_\lambda = 0$  and  $D = 0$ . We thus obtain instead of (4.57) the equations

$$0 = -2\kappa_\lambda n_\lambda + n_\lambda W_\lambda D. \quad (4.97)$$

Because we may write (4.97) also in the form

$$(W_\lambda D - 2\kappa_\lambda) n_\lambda = 0, \quad (4.98)$$

this set of equations means that either the photon number  $n_\lambda$ , or its factor vanish,

$$W_\lambda D - 2\kappa_\lambda = 0. \quad (4.99)$$

Let us assume that for a set of certain modes  $n_\lambda \neq 0$ . For these modes which we may numerate  $\lambda = 1, 2, \dots$ , eq. (4.99) must hold. But this implies

$$D = \frac{2\kappa_\lambda}{W_\lambda} = \frac{2\kappa_1}{W_1} = \frac{2\kappa_2}{W_2} = \dots \quad (4.100)$$

Because the inversion  $D$  is uniquely determined, the right hand sides must be equal to each other. Let us consider the following situation. All modes have the same lifetime but they are situated at different frequencies. For

different frequencies the corresponding  $W$ 's differ from each other. Under this hypothesis the expressions  $2\kappa_\lambda / W_\lambda$ ,  $\lambda = 1, 2, \dots$  must be different from each other. Thus we have found a contradiction which stems from the fact that we have assumed a set of modes with nonvanishing photon numbers  $n_\lambda$ . This contradiction can be solved only, if at most only a single  $n_\lambda$  differs from 0, whereas all other  $n_\lambda$  vanish. As one can easily convince oneself that  $n_\lambda$  is different from 0 whose frequency lies closest to the atomic transition frequency. Of course, our considerations can be immediately generalized to the case where a discrimination of the modes is caused by their different lifetimes in the laser resonator. These considerations give rise to the important result that in a laser only one mode is selected if the frequency distribution of the modes and their lifetimes are taken into account. In other words, in the sense of biological selection only one mode can win mode competition and all others have to die out. This is a precise mathematical formulation of Darwin's survival of the fittest. If we visualize the excited atoms as food which is continuously fed into the system and the photons as animals the result has the following meaning: One kind of animal has a better access to the food. It grows more quickly and this kind of animal can eat more food. The other animals cannot compete in eating and eventually perish.

As this example shows, coexistence of animals can be reached if they are enabled to live from separated resources of food so that the animals eating fastest cannot take away the food of the other animals. In biology this is achieved by ecological niches. In the laser, in an abstract sense, a similar situation can be achieved by letting different kinds of atoms generate different kinds of photons. This may happen, for example, in the following cases:

(1) The atomic line is inhomogeneously broadened.

Let us consider the special case in which only two kinds of atoms with different optical transition frequencies  $\bar{\omega}_1$  and  $\bar{\omega}_2$  are present whereby the corresponding emission lines do not overlap. In such a case the photons belonging to the modes with frequencies  $\omega_1 = \bar{\omega}_1$  and  $\omega_2 = \bar{\omega}_2$  are separately supported and can coexist. The general case of continuously distributed central frequencies (inhomogeneously broadened line) has been discussed in section 4.6.

(2) The modes are standing waves.

We find a similar situation if the modes are standing waves and the line is for instance homogeneously broadened. We explain this situation by means of the example of two modes whose spatial distribution is presented in fig. 4.18. If only mode 2 is present it generates an inversion by spatial hole burning as presented in fig. 4.19. Evidently mode 1 experiences an

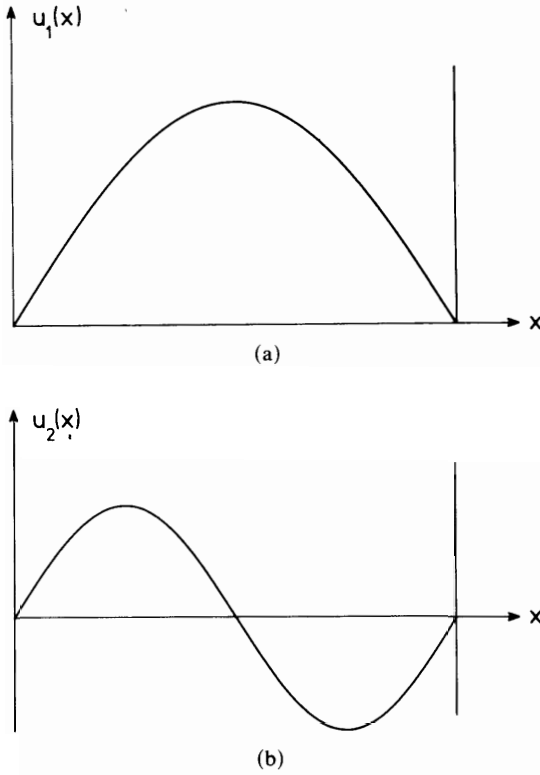


Fig. 4.18. In order to illustrate the possibility of coexistence of standing waves, in (a) and (b) the limiting cases of the two longest wave-lengths are shown. The mode amplitude is plotted versus the spatial coordinate  $x$  in axial direction.

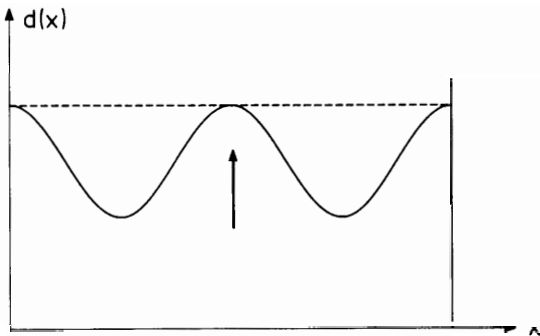


Fig. 4.19. In this figure that inversion is plotted against  $x$  which results from the presence of mode 2 alone.

unsaturated inversion there where the mode 1 itself has its maximum (fig. 4.19). Thus mode 1 can live on that part of the unsaturated inversion. Quite evidently both modes are at least partly supported by two different kinds of atoms which are located at different positions. Thus these modes can coexist provided the unsaturated inversion  $d_0$  is big enough. On account of the figures it might seem as if this effect is important only for large wave-lengths. But one may show that this effect is independent of the mode wave-lengths in the resonator. In this way also several modes can coexist. In the next section we want to prove this rigorously.

#### 4.10. The coexistence of modes due to spatial hole burning. Quantitative treatment

In this section we want to show how the effect of spatial hole burning can be treated quantitatively. To this end we start from our fundamental equations (4.57) for the temporal change of the photon numbers. These equations read

$$\dot{n}_\lambda = -2\kappa_\lambda n_\lambda + n_\lambda \sum_\mu W_{\lambda\mu} d_\mu. \quad (4.101)$$

We assume that the inversion  $d_\mu$  follows the laser light instantaneously as explained in section 4.2. This allows us to put  $\dot{d}_\mu = 0$  in eq. (4.61) and to solve that equation approximately for small enough  $n$ 's, which yields

$$d_\mu \approx d_0 \left( 1 - 2T \sum_\lambda n_\lambda W_{\lambda\mu} \right), \quad (4.102)$$

which we have derived before. By inserting (4.102) into (4.101) we obtain a closed set of equations for the photon numbers  $n_\lambda$  alone,

$$\dot{n}_\lambda = -2\kappa_\lambda n_\lambda + d_0 n_\lambda \sum_\mu W_{\lambda\mu} - d_0 n_\lambda 2T \sum_\mu W_{\lambda\mu} \sum_{\lambda'} n_{\lambda'} W_{\lambda'\mu}. \quad (4.103)$$

In the following we shall focus our attention on the steady state in which

$$\dot{n}_\lambda = 0. \quad (4.104)$$

As a consequence, eqs. (4.103) are transformed into

$$\left( -2\kappa_\lambda + d_0 \sum_\mu W_{\lambda\mu} - d_0 2T \sum_{\lambda'} n_{\lambda'} \sum_\mu W_{\lambda'\mu} \right) n_\lambda = 0. \quad (4.105)$$

This set of equations can be solved if either  $n_\lambda = 0$  or the term in brackets vanishes. Since we are interested in actually lasing modes we shall consider

those equations for which  $n_\lambda \neq 0$ . Therefore in the following we shall discuss the expression in the brackets in more detail. Especially it will be our task to evaluate the sums containing  $W_{\lambda\mu}$ . We treat a special case in which we have standing waves in one direction and we shall assume that the modes are practically constant across the cross section of the laser rod. Actually it is not difficult to generalize the whole treatment to the case where the modes vary spatially across the cross section. For the case under consideration we may write  $W_{\lambda\mu}$  in the form

$$W_{\lambda\mu} = A \frac{\gamma}{\gamma^2 + (\bar{\omega}_\mu - \omega_\lambda)^2} \frac{2}{V} \sin^2(k_\lambda x_\mu), \quad (4.106)$$

where we adopt a homogeneously broadened line, and where  $A$  was defined in (4.81). We mention that sometimes it is more useful to write  $W_{\lambda\mu}$  in a somewhat different form, namely as

$$W_{\lambda\mu} = \frac{2\gamma}{\gamma^2 + (\bar{\omega}_\mu - \omega_\lambda)^2} |g_{\mu\lambda}|^2, \quad (4.107)$$

where the last factor can be written in a formal fashion as

$$|g_{\mu\lambda}|^2 = g^2 V |u_\lambda(\mathbf{x}_\mu)|^2, \quad (4.108)$$

by which the new coupling constant  $g$  is defined. The factor  $2/V$  in front of the sin function is defined in such a way that this function is normalized over the volume of the laser resonator of length  $L$  and total cross section  $F$ ,

$$\int_0^L dx F \frac{2}{V} \sin^2(k_\lambda x) = 1, \quad (4.109)$$

and where

$$FL = V. \quad (4.110)$$

Because the laser atoms are rather closely spaced we may replace the sum over  $\mu$  by an integral over space. In our present case in which  $W_{\lambda\mu}$  does not depend on the spatial coordinates perpendicular to the laser axis we may thus perform the replacement

$$\sum_\mu \rightarrow \frac{N}{V} \int \cdots dx F, \quad (4.111)$$

where  $N$  is the total number of atoms so that  $N/V$  is the density of laser atoms.



#### 94 4. The intensity of laser light. Rate equations

We now apply this procedure to the evaluation of the sum over  $W_{\lambda\mu}$ , i.e.

$$\sum_{\mu} W_{\lambda\mu} \rightarrow \frac{N}{V} \int_0^L dx F \frac{2}{V} \sin^2(k_{\lambda}x) A \frac{\gamma}{\gamma^2 + (\bar{\omega}_0 - \omega_{\lambda})^2}. \quad (4.112)$$

We denote this sum by  $\bar{W}_{\lambda}$ . Its evaluation amounts to performing the integration in (4.112) over the sin-function squared, which immediately yields

$$\sum_{\mu} W_{\lambda\mu} \equiv \bar{W}_{\lambda} = \frac{N}{V} A \frac{\gamma}{\gamma^2 + (\bar{\omega}_0 - \omega_{\lambda})^2}. \quad (4.113)$$

It is a simple matter to evaluate also that sum over  $\mu$  which contains the products of two  $W$ 's again using the replacement of the sum over  $\mu$  by an integral. A brief calculation yields the following result:

$$N \sum_{\mu} W_{\lambda\mu} W_{\lambda'\mu} = \begin{cases} \frac{3}{2} \bar{W}_{\lambda}^2, & \text{for } \lambda = \lambda', \\ \bar{W}_{\lambda} \bar{W}_{\lambda'}, & \text{for } \lambda \neq \lambda'. \end{cases} \quad (4.114)$$

Inserting the results (4.113) and (4.114) into the brackets occurring in eq. (4.105) we are readily led to the following set of equations:

$$-2\kappa_{\lambda} + d_0 \bar{W}_{\lambda} - \frac{d_0 2T}{N} \sum_{\lambda' \neq \lambda} n_{\lambda'} \bar{W}_{\lambda'} \bar{W}_{\lambda} - \frac{d_0 T}{N} 3 \bar{W}_{\lambda}^2 n_{\lambda} = 0. \quad (4.115)$$

After multiplying these equations by

$$N / \bar{W}_{\lambda} d_0 T, \quad (4.116)$$

we obtain

$$3 \bar{W}_{\lambda} n_{\lambda} + 2 \sum_{\lambda' \neq \lambda} n_{\lambda'} \bar{W}_{\lambda'} = l_{\lambda}, \quad (4.117)$$

where we have used the abbreviation

$$l_{\lambda} = \frac{N}{T} \left( 1 - \frac{2\kappa_{\lambda}}{\bar{W}_{\lambda} d_0} \right). \quad (4.118)$$

When we introduce  $\bar{W}_{\lambda} n_{\lambda}$  as a new variable  $y_{\lambda}$ , we readily verify that the set of eqs. (4.117) has a very simple structure. This allows us to find its solution in a straightforward manner.,

$$y_{\lambda} \equiv \bar{W}_{\lambda} n_{\lambda} = l_{\lambda} - \frac{2(l_1 + \dots + l_M)}{2M + 1}. \quad (4.119)$$

Using the definition of  $l_{\lambda}$ , (4.118), we obtain for (4.119)

$$\bar{W}_{\lambda} n_{\lambda} = \frac{N}{T} \left[ \frac{1}{2M + 1} - \frac{2\kappa_{\lambda}}{d_0 \bar{W}_{\lambda}} + \frac{2(2\kappa_1 / \bar{W}_1 + \dots + 2\kappa_M / \bar{W}_M)}{(2M + 1)d_0} \right]. \quad (4.120)$$

In this equation,  $M$  is the total number of modes with nonvanishing  $n$ . In (4.119) and (4.120) we have labeled the modes in such a way that the nonvanishing modes carry the indices  $1, \dots, M$ .

Because  $\bar{W}_\lambda$  is a positive quantity and we are interested in those modes for which  $n$  is positive, we study the conditions under which the term in square brackets in (4.120) is positive,

$$\frac{1}{2M+1} - \frac{2\kappa_\lambda}{d_0\bar{W}_\lambda} + \frac{2(2\kappa_1/\bar{W}_1 + \dots + 2\kappa_M/\bar{W}_M)}{(2M+1)d_0} > 0. \quad (4.121)$$

Two examples for the coexistence of modes

We now want to show under which conditions spatial hole burning, which we have qualitatively discussed in section 4.8, allows the coexistence of several modes. To this end we treat two realistic cases, namely:

(1) All modes have the same decay constants but the distance of their frequencies from the line center increases according to

$$\omega_\lambda = \bar{\omega}_0 + m\delta, \quad (4.122)$$

where  $m$  are positive or negative integers including 0.

(2) The modes have practically all the same frequency but due to their misalignment with respect to the laser axis their decay constants  $\kappa_\lambda$  differ from each other.

(1) Let us turn to the first case. We denote by  $A$ , (or  $m_0$ ) the index of the mode which is farthest away from resonance. Because we assume a symmetric position of the cavity modes with respect to the line center,  $m_0$  is connected with the total number of modes by  $M = 2m_0 + 1$ . After inserting the explicit expression of  $\bar{W}_\lambda$  according to (4.113) and (4.122) into (4.121) it is a simple matter to evaluate the sum over  $m$  occurring in (4.121). After some elementary algebra, and choosing  $m = m_0$  in (4.122) we can cast the condition (4.121) explicitly into the form

$$\frac{8}{3}m_0^3 + m_0^2 - \frac{2}{3}m_0 < \frac{1}{\delta^2} \frac{\gamma}{\kappa} (g^2 D_0 - \kappa\gamma), \quad (4.123)$$

where we have used the usual notation for the total atomic inversion

$$D_0 = Nd_0. \quad (4.124)$$

This formula allows us to calculate the index  $m_0$  of the mode which is farthest away in positive (or negative) direction from the line center.

If we are satisfied with a rough estimate of up to an accuracy of 10% we may replace (4.123) by

$$\frac{8}{3}m_0^3 < \frac{1}{\delta^2} \frac{\gamma}{\kappa} (g^2 D_0 - \kappa \gamma), \quad (4.125)$$

which allows us to quickly determine the number of coexisting modes once  $\delta$  (cf. (4.122)),  $g$  (cf. (4.107), (4.108)),  $D_0$  (cf. (4.124)) and  $\gamma$  and  $\kappa$  are given.

(2) We now turn to the second example in which all modes have practically the same frequency but where the decay constants  $\kappa_\lambda$  are different. We treat this case by way of a model. Our model consists in the assumption that the modes are space independent across the direction perpendicular to the laser axis. A more detailed treatment should take into account that variation also. For simplicity we label the modes in a new way namely by distinguishing them by the index  $m$  instead of  $A$  so that

$$\kappa_m = \kappa(1 + q)^{m-1}. \quad (4.126)$$

$q$  is a quantity which measures how quickly the decay constant increases when the mode index  $m$  increases. We let  $m$  start from  $m = 1$ . We shall further assume

$$Mq \ll 1. \quad (4.127)$$

Inserting this hypothesis into (4.121) we readily obtain

$$\frac{1}{2M + 1} - \frac{2\kappa(1 + q)^{M-1}}{d_0 \bar{W}} + \frac{4\kappa \sum_{m=1}^M (1 + q)^{m-1}}{(2M + 1)d_0 \bar{W}} > 0. \quad (4.128)$$

The geometric sum over  $m$  can be easily evaluated. The result can be considerably simplified by making use of the assumption (4.127). After some trivial algebra we obtain as final result

$$M^2 \leq \frac{1}{q} \left( \frac{d_0 \bar{W}}{2\kappa} - 1 \right) + 1, \quad (4.129)$$

which allows us to determine the maximum number  $M$  of modes which can coexist. If  $q$  is sufficiently small, quite a number of modes can coexist even slightly above laser threshold which is given by

$$d_0 \bar{W} - 2\kappa = 0. \quad (4.130)$$

(Compare exercise 1.)

**Exercises on section 4.10**

(1) Show that (4.130) is the threshold for the first mode to appear.

Hint: Use (4.115).

(2) Calculate  $m_0$  from eq. (4.123) for

$$\gamma = 10^{10} \text{ s}^{-1},$$

$$\kappa = 10^8 \text{ s}^{-1},$$

$$\delta = 10^8 \text{ s}^{-1}, 5 \times 10^8 \text{ s}^{-1},$$

$D_0 = 1\%, 10\%$  above  $D_{0,c}$  (single mode threshold).

Do you need  $g^2$  explicitly?

(3) Calculate  $M$  from eq. (4.129) for

$$q = 0.01,$$

$$\kappa = 10^8 \text{ s}^{-1},$$

$d_0 = 1\%, 10\%$  over the single mode threshold.

Do you need  $W$  and  $\kappa$  explicitly?

(4) What is the relation between  $d_0 W$  and  $g^2 D_0$ ?

## Chapter 5

# The Basic Equations of the Semiclassical Laser Theory

### 5.1. Introduction

In this section we start with the central topic of the present book namely semiclassical laser theory. In the preceding chapter we have described laser action by means of the photon picture where we could motivate the laser equations only heuristically. Therefore it is necessary to derive those equations from first principles. Furthermore we know from classical physics that for a complete description of the light field we not only need its intensity which in a way corresponds to the photon numbers but also the phase of the light field. Because photon numbers do not contain information on phases the rate equations of the preceding chapter are incomplete. These deficiencies can be overcome by the semiclassical laser theory. This theory which we are now going to develop treats the light field as a classical electro-magnetic field which obeys Maxwell's equations. Because laser action is brought about by the interaction between the light field and the atoms we have to treat the motion of the electrons within the atoms adequately. It turns out that we cannot ignore quantum theory entirely and indeed when treating the motions of the electrons we must start from the quantum mechanical treatment. In the following we shall proceed in several steps. First starting from Maxwell's equations we derive a wave equation for the electric field strength. It will turn out that the polarization of the medium acts as a source for the electro-magnetic oscillations. Then we shall study how this polarization is in turn generated by the field. Thus we arrive at equations describing the coupling between field and atoms. Finally we introduce some well founded approximations and thus obtain the fundamental equations for the semiclassical laser theory which we shall treat in the chapters 6 to 9.

## 5.2. Derivation of the wave equation for the electric field strength

We start from the following Maxwell equations:

$$\text{curl } \mathbf{E} = -\dot{\mathbf{B}}, \quad (5.1)$$

$$\text{curl } \mathbf{H} = \mathbf{j} + \dot{\mathbf{D}}. \quad (5.2)$$

The first equation represents the induction law. It describes how vortices of the electric field strength  $\mathbf{E}$  are caused by a temporal change of magnetic induction  $\mathbf{B}$ . Eq. (5.2) describes how the current density  $\mathbf{j}$  causes a vortex of the magnetic field  $\mathbf{H}$  (Oersted's law). The vortex of the magnetic field can be caused by a temporal change of the dielectric displacement  $\mathbf{D}$ , too. As usual we need the connections between  $\mathbf{D}$  and  $\mathbf{E}$  as well as between  $\mathbf{B}$  and  $\mathbf{H}$ . As is shown in electrodynamics, the dielectric displacement  $\mathbf{D}$  depends on the electric field strength  $\mathbf{E}$  via the polarization  $\mathbf{P}$  of the medium in which the dielectric process takes place

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad (5.3)$$

where  $\varepsilon_0$  is the dielectric constant of the vacuum. We briefly remind the reader how to visualize the meaning of the polarization  $\mathbf{P}$ . To this end we assume that the material is composed of individual atoms. When we apply an electric field, the electrons of the individual atoms will be displaced with respect to the atomic nuclei. Because the centers of the charges of the electrons and the nucleus do no more coincide, the applied electric field has induced a dipole at each individual atom. The polarization  $\mathbf{P}$  is defined as the total dipole moment which stems from the individual atomic dipoles and is taken per unit volume. Later on it will be an important task of our theory to calculate the dipole moments of the individual atoms. Furthermore we wish to express the current density  $\mathbf{j}$  by means of the electric field strength  $\mathbf{E}$ . To this end we assume that the material or parts of it possess an electric conductivity  $\sigma$  and use Ohm's law. Therefore we may assume

$$\mathbf{j} = \sigma \mathbf{E}. \quad (5.4)$$

Finally we wish to confine our considerations to non-magnetic materials

$$\mathbf{B} = \mu_0 \mathbf{H}. \quad (5.5)$$

Furthermore we wish to treat wave phenomena and therefore we shall assume that the electric field is transversal which is equivalent to the assumption

$$\text{div } \mathbf{E} = 0. \quad (5.6)$$

It will be our goal to derive a simple equation from the equations mentioned above, namely an equation for the electric field strength. If we know this quantity we may calculate, e.g. the magnetic induction by means of eq. (5.1). In our subsequent treatment we shall deal only with the electric field strength  $\mathbf{E}$  because it contains all the information which we need to develop laser theory. In a first step we differentiate (5.2) with respect to time and obtain

$$\text{curl } \dot{\mathbf{H}} = \dot{\mathbf{j}} + \ddot{\mathbf{D}}. \quad (5.7)$$

In (5.1) we replace the magnetic induction  $\mathbf{B}$  by  $\mu_0 \mathbf{H}$ . Then we express  $\mathbf{H}$  in (5.7) by curl  $\mathbf{E}$  according to (5.1) and thus obtain for the l.h.s. of (5.7)

$$-\text{curl curl } \mathbf{E} \equiv \mathbf{A}\mathbf{E}. \quad (5.8)$$

In deriving (5.8) we have used the following equation, well known from the vector calculus:

$$\text{curl curl } \mathbf{E} = \text{grad div } \mathbf{E} - \mathbf{A}\mathbf{E},$$

where the relation (5.6) has been taken into account. In it  $\mathbf{A}$  is as usual the Laplace operator,  $\mathbf{A} = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ . Finally we replace the quantities  $\dot{\mathbf{j}}$  and  $\ddot{\mathbf{D}}$  on the r.h.s. of (5.7) by means of the relations (5.4) and (5.3), respectively, and bring all expressions containing  $\mathbf{E}$  to the l.h.s. of (5.7). We thus obtain the fundamental wave equation

$$\Delta \mathbf{E} - \frac{1}{c^2} \ddot{\mathbf{E}} + \mu_0 \sigma \dot{\mathbf{E}} = \mu_0 \ddot{\mathbf{P}}, \quad (5.9)$$

where we have put  $\epsilon_0 \mu_0 = 1/c^2$ , with  $c$  the light velocity in vacuum. When we put  $\mathbf{P} = 0$ , (5.9) reduces to the telegraph equation which was derived in the last century. In the exercises we shall discuss some solutions of the telegraph equation. In eq. (5.9) the polarization  $\mathbf{P}$  may be considered as a source term which produces the electric field. On the other hand an electric field in a medium can generate a polarization. Therefore we have to deal with the question how to calculate the polarization  $\mathbf{P}$ .

### Exercises on section 5.2

Solve the telegraph equation (5.9) in one spatial dimension under the following conditions:

- (1) At the points  $x = 0$  and  $x = L$  the field strength  $\mathbf{E}$  must vanish, or
- (2) the medium extends from  $x = 0$  till  $x = \infty$ . At  $x = 0$ ,  $\mathbf{E} = \mathbf{E}_0 \cos(\omega_0 t)$  is prescribed.

What are the most general solutions? Discuss the kind of damping of the solution in the cases (1) or (2).

Hint: In case (1) use the hypothesis  $E = f(t) \sin(kx)$  and determine  $k$  and  $f(t)$ . Note that a linear combination of solutions of the telegraph equation yields a new solution. In case (2) try the hypothesis  $E = \exp[i\omega_0 t] g(x)$  and form an adequate linear combination.

### 5.3. The matter equations

We first remind the reader of classical physics. We consider atoms which we distinguish by an index  $\mu$  and which are localized at the space points  $x_\mu$ . By means of a simple model we treat the motion of an electron within an atom by assuming that it is elastically coupled to the atomic nucleus. The displacement of the electron from its equilibrium position at the nucleus will be denoted by  $\xi_\mu$ . With it a dipole moment

$$p_\mu = (-e)\xi_\mu \tag{5.10}$$

is connected. In classical physics the deviation  $\xi_\mu$  of the electron with charge  $-e$  and mass  $m$  obeys the equation of motion

$$m\ddot{\xi}_\mu + f\xi_\mu = (-e) E(x_\mu, t), \tag{5.11}$$

where  $f$  is Hook's constant. The electric field strength  $E$  is taken at the position  $x_\mu$  of the atom (fig. 5.1). Assuming  $E$  in the form of an harmonic oscillation,

$$E(x, t) = E_0(x) e^{i\omega t} + \text{c.c.}, \tag{5.12}$$

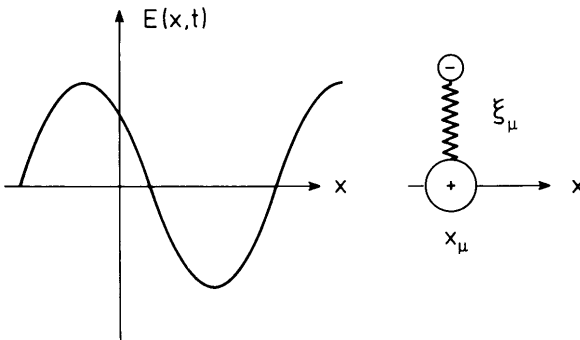


Fig. 5.1. The electric field strength  $E(x, t)$  (left part) hits an atomic dipole at space point  $x_\mu$  and with its elongation  $\xi_\mu$  (right part).



we can solve (5.11) immediately

$$\xi_{\mu} = \frac{-e}{m} \frac{1}{\omega_0^2 - \omega^2} \mathbf{E}(\mathbf{x}_{\mu}, t), \quad (5.13)$$

where we have used the abbreviation

$$\omega_0^2 = f/m. \quad (5.14)$$

Inserting this result into (5.10) we recognize that the dipole moment  $\mathbf{p}_{\mu}$  of the atom  $\mu$  is connected with the electric field strength by a constant factor  $\alpha$ , the polarizability of the atom,

$$\mathbf{p}_{\mu} = \alpha \mathbf{E}(\mathbf{x}_{\mu}, t). \quad (5.15)$$

An inspection of (5.13) reveals that the polarizability is explicitly given by

$$\alpha = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2}. \quad (5.16)$$

Because the polarization of the medium is the sum over the dipole moments per unit-volume it follows from (5.15) that  $\mathbf{P}$  is proportional to  $\mathbf{E}$ . Because in (5.9) the second derivative of  $\mathbf{P}$  with respect to time occurs, the net effect of the polarization is a change of the effective velocity of light in the medium. Of course, what we have presented up to now is nothing but conventional dispersion theory.

This theory is not able to represent laser action adequately, however. Rather we have to deal with the quantum mechanical processes. To this end we assume that only two energy levels within an atom participate at the interaction between the atom and the light field. We denote the electronic coordinate by  $\xi$ . With respect to the electron we treat the problem fully quantum mechanically and therefore start from the Schrodinger equation

$$H\Psi = i\hbar\dot{\Psi}. \quad (5.17)$$

In it the Hamiltonian  $H$  is composed of the unperturbed operator  $H_0$  and the operator of the external perturbation  $H^S$

$$H^S = e\xi \mathbf{E}(t), \quad (5.18)$$

i.e.

$$H = H_0 + e\xi \mathbf{E}(t). \quad (5.19)$$

$H_0$  refers to the unperturbed motion of the electron in the field of its atomic nucleus. In the following we shall assume that the corresponding quantum mechanical problem

$$H_0\varphi_j = W_j\varphi_j, \quad j = 1, 2, \quad (5.20)$$

has been solved already, i.e. that we know the wave functions  $\varphi_j$  and their energies  $W_j$ . In Vol. 1 we got to know how to treat the Schrodinger equation (5.17). To this end we construct the wanted wave function  $\Psi$  as a superposition of the unperturbed wave functions  $\varphi_1$  and  $\varphi_2$ . In other words, we shall assume in the following that the interaction between the electron and the electric field strength  $E$  is only of importance for the two levels under consideration. This may be justified by the fact that the frequency of the electric field strength is in or close to resonance with the frequency of the electronic transition between the corresponding two levels so that the electric field strength stimulates transitions between levels 2 and 1. In order to determine the coefficients  $c_1$  and  $c_2$  we insert as usual the hypothesis

$$\Psi(\boldsymbol{\xi}, t) = c_1(t) \exp[-i W_1 t / \hbar] \varphi_1(\boldsymbol{\xi}) + c_2(t) \exp[-i W_2 t / \hbar] \varphi_2(\boldsymbol{\xi}) \quad (5.21)$$

in (5.17) and multiply the resulting equation by  $\varphi_1^*$  and  $\varphi_2^*$ , respectively. We then integrate over the electron coordinate  $\boldsymbol{\xi}$  and obtain the equations

$$\dot{c}_1 = \frac{1}{i\hbar} \mathbf{E}(t) \mathbf{H}_{12}^S c_2, \quad (5.22)$$

$$\dot{c}_2 = \frac{1}{i\hbar} \mathbf{E}(t) \mathbf{H}_{21}^S c_1. \quad (5.23)$$

In order to simplify these equations we have introduced the matrix elements as follows:

$$\mathbf{H}_{mn}^S = \int \varphi_m^*(\boldsymbol{\xi}) e \boldsymbol{\xi} \varphi_n(\boldsymbol{\xi}) d^3 \boldsymbol{\xi} \exp[i \omega_{mn} t], \quad (5.24)$$

where  $\omega_{mn}$  is given by

$$\omega_{mn} = \frac{1}{\hbar} (W_m - W_n). \quad (5.25)$$

Furthermore we have assumed that the atom does not possess a static dipole moment so that

$$\mathbf{H}_{11}^S = \mathbf{H}_{22}^S = 0. \quad (5.26)$$

When we know the coefficients  $c_1$  and  $c_2$ , which can in principle be obtained by solving eqs. (5.22) and (5.23), we may calculate various important expectation values, for instance that of the atomic dipole moment

$$\mathbf{p} = \int \Psi^*(\boldsymbol{\xi}, t) (-e) \boldsymbol{\xi} \Psi(\boldsymbol{\xi}, t) d^3 \boldsymbol{\xi}. \quad (5.27)$$

In order to simplify the notation we consider only one selected atom so that we neglect the index  $\mu$  for the moment being. We have introduced that index above in order to distinguish the atoms. The expectation value of the dipole moment  $\mathbf{p}$  replaces the classical dipole moment (5.15). Thus, if we can calculate  $\mathbf{p}$ , we may determine the polarization  $\mathbf{P}$  of the medium so that our task to calculate the source term of the wave equation will be solved. Inserting  $\Psi$ , eq. (5.21), in (5.27) and using the relations (5.26) we obtain

$$-\mathbf{p} = c_1^* c_2 \exp[-i\bar{\omega}t] \mathfrak{D}_{12} + c_1 c_2^* \exp[i\bar{\omega}t] \mathfrak{D}_{21}, \quad (5.28)$$

where we have used the abbreviations

$$\bar{\omega} = (W_2 - W_1)/\hbar \quad (5.29)$$

and

$$\mathfrak{D}_{jk} = \int \varphi_j^* e\xi \varphi_k d^3\xi. \quad (5.30)$$

From this result it transpires that we have to know the coefficients  $c_j$  in order to calculate the dipole moment  $\mathbf{p}$ . It has turned out that it is not so useful to solve first eqs. (5.22) and (5.23) and then to determine the dipole moment according to (5.28). Rather it has turned out to be preferable to proceed along lines which we have presented in Vol. 1, for instance when deriving the Bloch equations of spins. There we have seen that we may obtain equations for the expectation values of the spin components directly and in a very simple fashion. We shall follow the same procedure here, i.e. we wish to derive equations for the expectation value of the dipole moment  $\mathbf{p}$ . We note that the dipole moment  $\mathbf{p}$  is known if we know the quantities

$$c_1^* c_2 \exp[-i\bar{\omega}t] = \alpha(t), \quad (5.31)$$

because with their help we may write the dipole moment in the form

$$-\mathbf{p} = \alpha(t) \mathfrak{D}_{12} + \alpha^*(t) \mathfrak{D}_{21}. \quad (5.32)$$

For later purposes we introduce the abbreviations

$$\mathbf{p}^{(+)} = -\alpha(t) \mathfrak{D}_{12}, \quad \mathbf{p}^{(-)} = -\alpha^*(t) \mathfrak{D}_{21}, \quad (5.33)$$

so that the dipole moment of the atoms under consideration can be represented in the form

$$\mathbf{p} = \mathbf{p}^{(+)} + \mathbf{p}^{(-)}. \quad (5.34)$$

We now wish to derive an equation for (5.31). The reader will be well advised if he considers  $\alpha(t)$  as a measure for the dipole moment of the

atoms, i.e. that he connects with the letter  $\alpha$  the meaning of a dipole moment. When we differentiate (5.31) with respect to time we obtain

$$\dot{\alpha} = -i\bar{\omega}\alpha + c_1^* \dot{c}_2 \exp[-i\bar{\omega}t] + \dot{c}_1^* c_2 \exp[-i\bar{\omega}t]. \quad (5.35)$$

We wish to replace the temporal derivatives of  $\dot{c}_2$  and  $\dot{c}_1^*$  by means of the r.h.s. of eqs. (5.22) and (5.23) and their complex conjugates, respectively. To this end we write those equations down in a somewhat different shape

$$\dot{c}_1 = \frac{1}{i\hbar} \mathbf{E}(t) \mathfrak{D}_{12} \exp[-i\bar{\omega}t] c_2, \quad (5.36)$$

$$\dot{c}_2 = \frac{1}{i\hbar} \mathbf{E}(t) \mathfrak{D}_{21} \exp[i\bar{\omega}t] c_1, \quad (5.37)$$

We then obtain for (5.35)

$$\dot{\alpha} = -i\bar{\omega}\alpha - \frac{1}{i\hbar} \mathbf{E}(t) \mathfrak{D}_{21} d, \quad (5.38)$$

where we have introduced the abbreviation

$$d = |c_2|^2 - |c_1|^2. \quad (5.39)$$

It is well known that  $|c_j|^2$  represents the probability of finding an electron in state  $j$ . Equally well,  $|c_j|^2$  can be considered as the occupation number of state  $j$ . Thus (5.39) is a quantum mechanical expression for the occupation number difference or, in other words, for the inversion.

In order to make understandable what follows we remind the reader once again of the Bloch equations of spin. The electron of the atom is not only subjected to the external light field but to other perturbations also. For instance in a gas the atom can collide with other atoms all the time. In a solid the electron can interact with lattice vibrations, etc. As we know from Vol. 1, chapter 7, such effects cause a damping of the dipole moments. We introduce these **dampings** into the theory in a phenomenological manner by adding the damping term  $-\gamma\alpha$  to the r.h.s. of eq. (5.38). The damping constant  $\gamma$  has the same meaning as the inverse of the transverse relaxation time  $T_2$  of a spin (cf. Vol. 1). Thus we find for the atom under consideration the equation

$$\dot{\alpha} = -i\bar{\omega}\alpha - \gamma\alpha - \frac{1}{i\hbar} \mathbf{E}(t) \mathfrak{D}_{21} d. \quad (5.40)$$

Evidently we have been forced to introduce a new unknown variable namely the inversion (5.39). In order to complete the equations of motion we must derive an equation for the inversion. To this end we differentiate

(5.39) with respect to time

$$\dot{d} = c_2^* \dot{c}_2 + \dot{c}_2^* c_2 - c_1^* \dot{c}_1 - \dot{c}_1^* c_1. \quad (5.41)$$

When we replace on its r.h.s. the time derivatives of  $c_j$  according to the relations (5.36) and (5.37) we obtain after a short transformation

$$d = \frac{2}{i\hbar} \mathbf{E}(t) (\mathfrak{D}_{21} a^* - \alpha \mathfrak{D}_{12}). \quad (5.42)$$

This equation describes how the inversion is changed due to the interaction of the electron and the electric field. When we wish to treat laser processes we must not ignore the interaction of the atom with its surrounding, for instance, we have to pump the atom energetically by a pump process. On the other hand the electron may recombine on account of its interaction with its surrounding. For instance, these processes can consist of radiationless transitions. But we also have to take into account transitions in which light is spontaneously emitted without participating at the proper laser process. All these processes taken together will lead to a relaxation of the inversion towards a stationary value  $d_0$  within a characteristic relaxation time  $T$ . This effect can be taken care of by adding the corresponding relaxation term to (5.42) so that we obtain

$$\dot{d} = \frac{2}{i\hbar} \mathbf{E}(t) (\mathfrak{D}_{21} a^* - \alpha \mathfrak{D}_{12}) + \frac{d_0 - d}{T}. \quad (5.43)$$

The time constant  $T$  occurring in this equation corresponds to the longitudinal relaxation time  $T_1$  of the Bloch theory of spins.

Now we have prepared all ingredients in order to formulate the semiclassical laser equations but we have to remind ourselves that we are not dealing with a single atom but with an ensemble of  $N$  atoms in the laser. For each of these atoms we have derived equations for their "dipole moments"  $a$  and inversions  $d$ . We remind the reader that  $\alpha(t)$  is a dimensionless quantity which is proportional to the dipole moment, however. To underline the physical significance of  $\alpha$  we shall call it a dipole moment here and in the following. In order to treat the ensemble of atoms we attach the atomic index  $\mu$  to the corresponding atomic quantities in the equations (5.40)-(5.43) Furthermore we take into account that the electric field strength  $\mathbf{E}$  is a function of the atomic positions  $\mathbf{x}_\mu$ . Thus we obtain the fundamental matter equations of the laser

$$\dot{\alpha}_\mu = (-i\bar{\omega} - \gamma) \alpha_\mu - \frac{1}{i\hbar} \mathbf{E}(\mathbf{x}_\mu, t) \mathfrak{D}_{21} d_\mu, \quad (5.44)$$

$$\dot{d}_\mu = \frac{d_0 - d_\mu}{T} + \frac{2}{i\hbar} \mathbf{E}(\mathbf{x}_\mu, t) (\mathfrak{D}_{21} \alpha_\mu^* - \alpha_\mu \mathfrak{D}_{12}). \quad (5.45)$$

Finally we have to devise a prescription how the macroscopic polarization is connected with the individual dipole moments. To this end we use a mathematical trick by which we write the polarization in the form

$$\mathbf{P}(\mathbf{x}, t) = \sum_\mu \delta(\mathbf{x} - \mathbf{x}_\mu) \mathbf{p}_\mu. \quad (5.46)$$

In it  $\delta$  is Dirac's  $\delta$ -function which can be visualized as having a peak at point  $\mathbf{x} = \mathbf{x}_\mu$  and which vanishes otherwise. The sum runs over all the atoms of the material. When we change the coordinate  $\mathbf{x}$  we palpate the material by means of the  $\delta$ -function and each time when an atomic position  $\mathbf{x}_\mu$  is hit the corresponding dipole moment  $\mathbf{p}$  prevails (compare exercise 1). The individual dipole moments  $\mathbf{p}_\mu$  are connected with the quantities  $\alpha$ , by

$$\mathbf{p}_\mu = \mathbf{p}_\mu^{(+)} + \mathbf{p}_\mu^{(-)} = -(\alpha_\mu(t) \mathfrak{D}_{12} + \alpha_\mu^*(t) \mathfrak{D}_{21}), \quad (5.47)$$

which we have got to know above. Thus we have found a closed system of equations. They consist of the equations for the electric field strength (5.9) and of the matter equations (5.44) and (5.45). They are connected in particular by the polarization  $\mathbf{P}$  according to (5.46) and (5.47). In the following we will transform these equations so that they are more suited for the treatment of the corresponding physical problems.

### Exercises on section 5.3

Dirac's  $\delta$ -function which we introduced in eq. (5.46) is defined by the following property:

(i)  $\delta(\mathbf{x} - \mathbf{x}_0) = 0$ , for  $\mathbf{x} \neq \mathbf{x}_0$ ,

(ii)  $\int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x - x_0) dx = 1$ ,

where  $\varepsilon > 0$  but arbitrary. If  $f(x)$  is a continuous function,

$$\int_a^b f(x) \delta(x - x_0) dx = \begin{cases} f(x_0), & \text{if } a < x_0 < b, \\ 0, & \text{otherwise.} \end{cases} \quad (*)$$

The  $\delta$ -function may be defined in three dimensions also:

(i)  $\delta(\mathbf{x} - \mathbf{x}_0) = 0$ , for  $\mathbf{x} \neq \mathbf{x}_0$ ,

$$(ii) \int_V \delta(\mathbf{x} - \mathbf{x}_0) d^3x = 1,$$

provided  $\mathbf{x}_0$  is contained in  $V$ .

(1) Evaluate  $\int_V f(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}_0) d^3x$  in analogy to the equation marked with (\*).

(2) Average (5.46) over a volume element  $\Delta V$  which contains  $\mathbf{x}_0$ . Let  $\Delta V$  be so small that in it  $\mathbf{p}_\mu$  practically does not change. Write  $\mathbf{p} = \mathbf{p}(\mathbf{x}_0)$  and show  $\bar{\mathbf{P}}(\mathbf{x}_0, t) = \rho \mathbf{p}(\mathbf{x}_0)$ , where the bar above  $\mathbf{P}$  means averaging.  $\rho$  is the density of the atoms, i.e.  $\rho = \Delta N / \Delta V$  where  $\Delta N$  is the number of atoms in the volume  $\Delta V$ .

Hint: The average is defined by

$$\bar{P} = \frac{1}{\Delta V} \int_{\Delta V} P(\mathbf{x}, t) d^3x.$$

#### 5.4. The semiclassical laser equations for the macroscopic quantities electric field strength, polarization, and inversion density

By simple transformations, equations for the macroscopic polarization and the inversion density may be derived from eqs. (5.44) and (5.45). Because we shall not immediately make use of these equations in the subsequent sections, the speedy reader can skip this section. Once again we consider the decomposition (5.46). When we insert the further decomposition (5.47) in it, it suggests itself to introduce the new quantity

$$\mathbf{P}^{(+)}(\mathbf{x}, t) = -\sum_{\mu} \delta(\mathbf{x} - \mathbf{x}_{\mu}) \mathfrak{D}_{1\mu} \alpha_{\mu}(t). \tag{5.48}$$

We denote the quantity conjugate complex to  $\mathbf{P}^{(+)}$  by  $\mathbf{P}^{(-)}$

$$\mathbf{P}^{(+)*} = \mathbf{P}^{(-)}. \tag{5.49}$$

In a way analogous to (5.48) we define the inversion density by summing up over the individual atoms using Dirac's  $\delta$ -function

$$D(\mathbf{x}, t) = \sum_{\mu} \delta(\mathbf{x} - \mathbf{x}_{\mu}) d_{\mu} \tag{5.50}$$

It will be our goal to derive from eqs. (5.44) and (5.45) equations for the macroscopic quantities  $\mathbf{P}$  and  $D$ . To this end we multiply (5.44) on both sides by  $\delta(\mathbf{x} - \mathbf{x}_{\mu}) \mathfrak{D}_{1\mu}$ , and sum up over  $\mu$ . The term  $\delta(\mathbf{x} - \mathbf{x}_{\mu}) \mathbf{E}(\mathbf{x}_{\mu}, t)$

occurring in it may be replaced by

$$\delta(\mathbf{x} - \mathbf{x}_\mu) \mathbf{E}(\mathbf{x}_\mu, t) = \delta(\mathbf{x} - \mathbf{x}_\mu) \mathbf{E}(\mathbf{x}, t), \quad (5.51)$$

(compare exercise). Thus we immediately obtain the relation

$$\frac{d}{dt} \mathbf{P}^{(+)}(\mathbf{x}, t) = (-i\bar{\omega} - \gamma) \mathbf{P}^{(+)}(\mathbf{x}, t) + \frac{1}{i\hbar} [\mathbf{E}(\mathbf{x}, t) \boldsymbol{\vartheta}_{21}] \boldsymbol{\vartheta}_{12} D(\mathbf{x}, t). \quad (5.52)$$

We proceed in a similar way with eq. (5.45) and obtain

$$\frac{d}{dt} D(\mathbf{x}, t) = \frac{D_0 - D(\mathbf{x}, t)}{T} - \frac{2}{i\hbar} \mathbf{E}(\mathbf{x}, t) \cdot (\mathbf{P}^{(-)} - \mathbf{P}^{(+)}) \quad (5.53)$$

where we have introduced the total inversion  $D_0$ , which is produced by the incoherent processes, according to the relation

$$\sum_{\mu} d_0 = N d_0 = D_0. \quad (5.54)$$

Eqs. (5.52) and (5.53) represent the macroscopic matter equations. In order to derive the complete laser equations we must supplement eqs. (5.52) and (5.53) by the field equation (5.9) which we write down once again for sake of completeness

$$\Delta \mathbf{E} - \frac{1}{c^2} \ddot{\mathbf{E}} - \mu_0 \sigma \dot{\mathbf{E}} = \mu_0 (\ddot{\mathbf{P}}^{(+)} + \ddot{\mathbf{P}}^{(-)}). \quad (5.55)$$

Eqs. (5.52), (5.53) and (5.55) represent a very elegant formulation of the equations for the interactions between light and matter and we shall use them later. In these equations a great number of physically most interesting processes is hidden. These equations do not only allow us to describe processes in the laser resonator but also wave propagation phenomena. In the present context, however, we wish to treat the processes within the laser proper. To this end we must take into account the fact that the laser is bounded by mirrors so that we have to deal with standing electric waves.

#### Exercise on section 5.4

Prove eq. (5.51).

Hint: Integrate (5.51) over a small volume on both sides and use the properties of the  $\delta$ -function as listed in exercises on section 5.3.



### 5.5. The laser equations in a resonator

As we know, in general laser processes take place within a resonator. It is defined with respect to the laser by the mirrors mounted at the **endfaces** of the laser material or by externally mounted mirrors. First let us consider the electric or electro-magnetic field in vacuum and the one-dimensional case (compare fig. 5.2). Let us assume that the conductivity of the mirrors is infinite. In such a case the tangential component of the electric field strength must vanish on the mirrors. This boundary condition is fulfilled by the electric field if it has nodes on the mirrors. In this case the electric field strength can be represented by

$$E = E_0 \sin kx. \quad (5.56)$$

Here  $k$  is given by  $m\pi/L$ , where  $L$  is the distance between the mirrors;  $m$  is an arbitrary integer. Thus in the resonator a set of different modes can develop. In the following we shall free ourselves from the specific form of the standing wave (5.56) and we shall denote the spatial part of the wave function by  $\mathbf{u}_\lambda(x)$ . In the special case just considered  $\mathbf{u}_\lambda$  has the form

$$\mathbf{u}_\lambda(x) = \mathbf{e}_\lambda \mathcal{N} \sin k_\lambda x, \quad (5.57)$$

where  $\mathbf{e}_\lambda$  is the unit vector in the direction of the field polarization, i.e. it is parallel to the electric field strength.  $\mathcal{N}$  is a normalization factor and the spatial variation of  $\mathbf{u}_\lambda$  is described by the sine-function as in (5.56). But now we may also assume that  $\mathbf{u}_\lambda$  describes a three-dimensional configuration of the electric field, for instance waves which, so to speak, do not run parallel to the laser axis. The electric field strength which may be a general function of space and time can be represented by a superposition of such

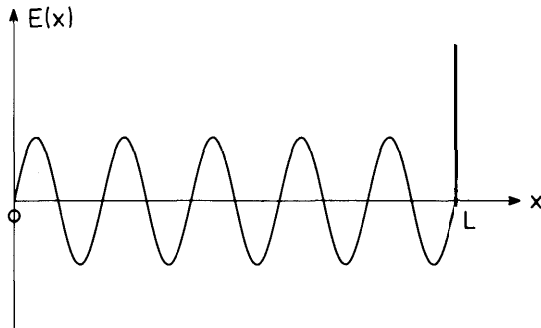


Fig. 5.2. The electric field strength  $E(x)$  between two mirrors separated by the distance  $L$ .

standing waves  $\mathbf{u}_\lambda(\mathbf{x})$ :

$$\mathbf{E}(\mathbf{x}, t) = \sum_{\lambda} E_{\lambda}(t) \mathbf{u}_{\lambda}(\mathbf{x}). \quad (5.58)$$

In it  $E_{\lambda}(t)$  are time dependent amplitudes. In the following we shall assume that the functions  $\mathbf{u}_{\lambda}$  of the individual modes obey a wave equation of the form

$$\Delta \mathbf{u}_{\lambda}(\mathbf{x}) = -k_{\lambda}^2 \mathbf{u}_{\lambda}(\mathbf{x}) = -\frac{\omega_{\lambda}^2}{c^2} \mathbf{u}_{\lambda}(\mathbf{x}). \quad (5.59)$$

As is shown in resonator theory, the functions  $\mathbf{u}_{\lambda}$  obey an orthogonality relation

$$\int \mathbf{u}_{\lambda}(\mathbf{x}) \mathbf{u}_{\lambda'}(\mathbf{x}) d^3x = \delta_{\lambda\lambda'}. \quad (5.60)$$

Because a laser resonator has open sides, the relations (5.60) are valid only approximately. We shall not discuss this question in more detail here, however. Our goal will rather be to derive equations for the field amplitudes  $E_{\lambda}(t)$ . Our starting point is again the wave equation for the electric field strength which we briefly recall

$$\Delta \mathbf{E} - \frac{1}{c^2} \ddot{\mathbf{E}} - \mu_0 \sigma \dot{\mathbf{E}} = \mu_0 \ddot{\mathbf{P}}. \quad (5.61)$$

We insert (5.58) into it and use the relation (5.59) so that we may replace the spatial derivatives by a multiplication by  $-\omega_{\lambda}^2/c^2$ . Then we multiply the resulting equation by  $\mathbf{u}_{\lambda}$  and integrate over the volume of the resonator. By means of the orthogonality relation (5.60) we obtain

$$\omega_{\lambda}^2 E_{\lambda} + \ddot{E}_{\lambda} + \sum_{\lambda'} (\sigma_{\lambda\lambda'} / \epsilon_0) \dot{E}_{\lambda'} = -(1/\epsilon_0) \ddot{P}_{\lambda}. \quad (5.62)$$

In it we have used  $\mu_0 c^2 = 1/\epsilon_0$  and the abbreviation

$$\sigma_{\lambda\lambda'} = \int \mathbf{u}_{\lambda} \sigma \mathbf{u}_{\lambda'} d^3x. \quad (5.63)$$

As we may recall,  $\mathbf{a}$  is the electric conductivity of the material. If the material possesses a homogeneous conductivity we may pull  $\mathbf{a}$  in front of the integral and may use again the orthogonality relation (5.60). If the conductivity varies spatially, for instance if it is concentrated in the mirrors, we must at least in principle be aware of the fact that also such  $\sigma_{\lambda\lambda'}$  may differ from 0 for  $\lambda \neq \lambda'$ . Because this question leads to rather subtle discussions we shall not follow up this problem now but shall come back to

it on later occasions. Finally we have to explain  $P_\lambda$  which occurs on the r.h.s. of (5.62). This is given by

$$P_\lambda = \int \mathbf{u}_\lambda(\mathbf{x}) \mathbf{P}(\mathbf{x}) d^3x. \quad (5.64)$$

If we consider the decomposition (5.58) into the  $u$ 's as Fourier decomposition,  $P_\lambda$  is nothing but a Fourier component of the polarization,  $\mathbf{P}$ . In order to establish the relation of  $P_\lambda$  with the microscopic representation of the polarization, i.e. the individual dipole moments, we recall that the polarization  $\mathbf{P}$  is related to the individual dipole moments  $\mathbf{p}_\mu$  by

$$\mathbf{P}(\mathbf{x}, t) = \sum_\mu \delta(\mathbf{x} - \mathbf{x}_\mu) \mathbf{p}_\mu. \quad (5.65)$$

Inserting (5.65) in (5.64) and recalling the property of the  $\delta$ -function (cf. exercise on section 5.3), we may immediately evaluate (5.64) and obtain

$$P_\lambda = P_\lambda^{(+)} + P_\lambda^{(-)}, \quad P_\lambda = - \underbrace{\sum_\mu \mathbf{u}_\lambda(\mathbf{x}_\mu) \mathfrak{D}_{12} \alpha_\mu(t)}_{P_\lambda^{(+)}} + \underbrace{\text{c.c.}}_{P_\lambda^{(-)}}. \quad (5.66)$$

So far we have transformed the equation for the electric field strength  $\mathbf{E}$  into equations for the individual amplitudes  $\mathbf{E}_\lambda(t)$ . We now do the same for the matter equations. This means only some writing because in eqs. (5.44) and (5.45) we have merely to replace on the r.h.s.  $\mathbf{E}(\mathbf{x}_\mu, t)$  by the corresponding decomposition (5.58). We immediately obtain

$$\dot{\alpha}_\mu = (-i\bar{\omega} - \gamma)\alpha_\mu - \frac{1}{i\hbar} d_\mu \sum_\lambda E_\lambda(t) \mathbf{u}_\lambda(\mathbf{x}_\mu) \mathfrak{D}_{21}, \quad (5.67)$$

$$\dot{d}_\mu = \frac{d_0 - d_\mu}{T} + \frac{2}{i\hbar} (\mathfrak{D}_{21} \alpha_\mu^* - \alpha_\mu \mathfrak{D}_{12}) \sum_\lambda E_\lambda(t) \mathbf{u}_\lambda(\mathbf{x}_\mu). \quad (5.68)$$

Eqs. (5.62), (5.67) and (5.68) represent a good starting point for the treatment of laser processes. In many cases these equations can be still considerably simplified, however, by introducing two approximations which in general are well founded. These approximations will be studied in the next section.

### Exercise on section 5.5

Derive equations which generalize (5.44) and (5.45) or (5.67) and (5.68) to three-level atoms with a pumping scheme of fig. 2.9.

Hint: Start from the Schrodinger equation with three energy levels and derive equations for  $c_1^* c_2$ ,  $c_2^* c_1$ ,  $|c_1|^2$ ,  $|c_2|^2$ ,  $|c_3|^2$ . Add decay terms  $-\gamma(c_1^* c_2)$

and  $-\gamma(c_2^*c_1)$ , respectively, and use in the equations for  $dN_j/dt \equiv d|c_j|^2/dt$  additional transition terms in analogy to the rate equations.

### 5.6. Two important approximations: The rotating wave approximation and the slowly varying amplitude approximation

We first treat the rotating wave approximation. To this end we decompose the mode amplitudes  $E_\lambda(t)$  into their positive and negative frequency parts

$$E_\lambda(t) = E_\lambda^{(+)}(t) + E_\lambda^{(-)}(t), \quad (5.69)$$

where these parts are defined by

$$E_\lambda^{(+)} = A_\lambda \exp[-i\omega_\lambda t], \quad E_\lambda^{(-)} = A_\lambda^* \exp[i\omega_\lambda t]. \quad (5.70)$$

We shall admit that the amplitudes  $A_\lambda$  and  $A_\lambda^*$  are time dependent, but that their time dependence is much slower than that of the exponential functions in (5.70). We further recall that according to its definition (5.31),  $\mathbf{a}_\lambda$  contains a rapidly oscillating factor

$$\alpha_\mu \propto \exp[-i\bar{\omega}t]. \quad (5.71)$$

Let us now consider typical expressions occurring on the r.h.s. of (5.68),

$$\alpha_\mu E_\lambda \propto \exp[i(\omega_\lambda - \bar{\omega})t] \quad \text{and} \quad \exp[-i(\omega_\lambda + \bar{\omega})t]. \quad (5.72)$$

As it will turn out by means of our later calculations, for the laser process only such mode frequencies  $\omega_\lambda$  are important which lie close to the atomic transition frequency  $\bar{\omega}$ . The exponential functions occurring in (5.72) are quite different, because in one of them the *difference* of  $\omega_\lambda$  and  $\bar{\omega}$  occurs, whereas in the other one the sum of  $\omega_\lambda$  and  $\bar{\omega}$  is present. When we integrate over a time interval which is long compared to the time of a single oscillation  $t_0 = 2\pi/\bar{\omega}$  but small compared with times over which the amplitudes  $A_\lambda$  and  $A_\lambda^*$  vary, the following happens. The exponential function occurring in (5.72) which contains the sum of the frequencies,  $\omega_\lambda + \bar{\omega}$ , oscillates very rapidly so that when integrated over a time  $t_0$  this contribution vanishes. On the other hand, during that time the exponential function in (5.72) which depends on the frequency difference has not changed appreciably. As a consequence we may ignore the second term occurring in (5.72) against the first term. In this way (5.68) transforms into

$$\dot{d}_\mu = \frac{d_0 - d_\mu}{T} + \frac{2}{i\hbar} \left( \alpha_\mu^* \sum_\lambda E_\lambda^{(+)}(t) \mathbf{u}_\lambda(\mathbf{x}_\mu) \boldsymbol{\vartheta}_{21} - \alpha_\mu \sum_\lambda E_\lambda^{(-)}(t) \mathbf{u}_\lambda(\mathbf{x}_\mu) \boldsymbol{\vartheta}_{12} \right). \quad (5.73)$$

Eq. (5.67) can be transformed in a similar manner. To this end we recall that  $a_\mu$  contains the rapidly oscillating factor (5.71). When we multiply the equation for  $a_\mu$ , i.e. (5.67), on both sides by  $\exp[-i\bar{\omega}t]$  it hits quantities  $E_\lambda$  thus producing terms of the form (5.72). This allows us to apply the rotating wave approximation to eq. (5.67), too. We thus obtain

$$\dot{\alpha}_\mu = (-i\bar{\omega} - \gamma)\alpha_\mu - \frac{1}{i\hbar} d_\mu \sum_\lambda E_\lambda^{(+)}(t) \mathbf{u}_\lambda(\mathbf{x}_\mu) \mathfrak{D}_{21}. \quad (5.74)$$

Eventually we may decompose the quantities  $E_\lambda$  and  $P_\lambda$  occurring in the wave equation (5.62) into their positive and negative frequency parts (compare for example (5.69)). We thus obtain the field equation

$$\omega_\lambda^2 E_\lambda^{(\pm)} + \ddot{E}_\lambda^{(\pm)} + \sum_{\lambda'} (\sigma_{\lambda\lambda'} / \varepsilon_0) \dot{E}_{\lambda'}^{(\pm)} = -(1/\varepsilon_0) \ddot{P}_\lambda^{(\pm)}. \quad (5.75)$$

Eqs. (5.73) to (5.75) again represent a closed set of field and matter equations.

Our next task will be to further simplify eq. (5.75). This can be achieved by the slowly varying amplitude approximation. To this end we consider for example the positive frequency part  $E_\lambda^{(+)}$

$$E_\lambda^{(+)} = A_\lambda(t) \exp[-i\omega_\lambda t]. \quad (5.76)$$

As we shall see in later chapters, in general it will not be allowed to assume that the amplitudes  $A_\lambda$  are time independent. But it will turn out that  $A_\lambda$  will change much more slowly than its accompanying exponential function. This may be visualized by saying that  $A_\lambda$  performs much fewer oscillations per unit time than its accompanying exponential function. Therefore we may assume that the temporal derivative of  $A_\lambda$  is much smaller than  $\omega_\lambda A_\lambda$ , i.e.

$$\left| \frac{dA_\lambda}{dt} \right| \ll |\omega_\lambda A_\lambda|. \quad (5.77)$$

We use this inequality as follows. Differentiating (5.76) on both sides with respect to time we obtain

$$\frac{d}{dt} E_\lambda^{(+)} = \left( -i\omega_\lambda A_\lambda + \frac{dA_\lambda}{dt} \right) \exp[-i\omega_\lambda t]. \quad (5.78)$$

But according to (5.77) we may replace (5.78) by

$$\frac{d}{dt} E_\lambda^{(+)} \approx -i\omega_\lambda E_\lambda^{(+)}. \quad (5.79)$$

In a similar way we proceed with respect to the second derivatives and

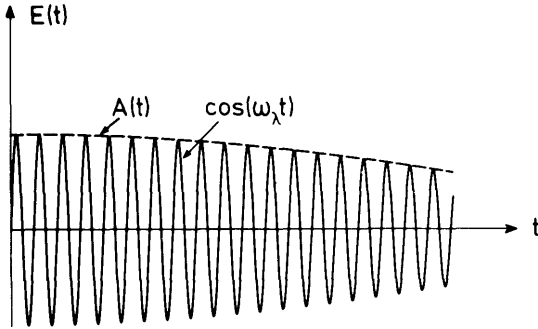


Fig. 5.3. The electric field strength  $E(t)$  consists of an envelope  $A(t)$  and a cosine function.

consider the expression

$$\omega_\lambda^2 E_\lambda^{(+)} + \ddot{E}_\lambda^{(+)} \quad (5.80)$$

Using the decomposition (5.76) we obtain the following expression for (5.80)

$$\exp[-i\omega_\lambda t] (\omega_\lambda^2 A_\lambda - \omega_\lambda^2 \dot{A}_\lambda - 2i\omega_\lambda \dot{A}_\lambda + \ddot{A}_\lambda). \quad (5.81)$$

In analogy to (5.77) we assume that also the inequality

$$|\ddot{A}_\lambda| \ll |\omega_\lambda \dot{A}_\lambda| \quad (5.82)$$

holds. Within this approximation the leading term of (5.81) is obtained as follows:

$$\omega_\lambda^2 E_\lambda^{(+)} + \ddot{E}_\lambda^{(+)} \approx -2i\omega_\lambda \dot{A}_\lambda \exp[-i\omega_\lambda t]. \quad (5.83)$$

In the following we are primarily interested in  $E_\lambda$  instead of  $A_\lambda$ . Therefore we express  $A_\lambda$  by  $E_\lambda$  whereby (5.83) is transformed into

$$-2i\omega_\lambda (\dot{E}_\lambda^{(+)} + i\omega_\lambda E^{(+)}). \quad (5.84)$$

In a similar way we may proceed with the polarization  $P_\lambda$ . Because there is no need to repeat all arguments once again we immediately write down the result

$$\ddot{P}_\lambda^{(\pm)} \approx -\bar{\omega}^2 P_\lambda^{(\pm)}. \quad (5.85)$$

For sake of simplicity we shall assume in the following that

$$\sigma_{\lambda\lambda'} = \delta_{\lambda\lambda'} \sigma_\lambda \quad (5.86)$$

holds, i.e. only those contributions of  $\sigma_{\lambda\lambda'}$  are assumed different from 0 for which  $\lambda = \lambda'$ . Using the approximations (5.79), (5.84) and (5.85) we may

transform the original equation for the electric field strength (5.75) into

$$-2i\omega_\lambda(\dot{E}_\lambda^{(+)} + i\omega_\lambda E^{(+)}) - (\sigma_\lambda / \epsilon_0) i\omega_\lambda E_\lambda^{(+)} = (\bar{\omega}^2 / \epsilon_0) P_\lambda^{(+)}. \quad (5.87)$$

We divide this equation by  $-2i\omega_\lambda$ , introduce the abbreviation

$$\kappa_\lambda = \sigma_\lambda / (2\epsilon_0) \quad (5.88)$$

and put all terms except  $\dot{E}_\lambda^{(+)}$  on the r.h.s. Thus we eventually obtain the rather simple equation

$$\dot{E}_\lambda^{(+)} = (-i\omega_\lambda - \kappa_\lambda) E_\lambda^{(+)} + [i\bar{\omega} / (2\epsilon_0)] P_\lambda^{(+)}. \quad (5.89)$$

We have assumed in addition that we may replace  $\omega_\lambda$  by  $\bar{\omega}$  in the last term.

Let us summarize the results of this section. We have first introduced the rotating wave approximation and then the slowly varying amplitude approximation. The resulting equations are represented in (5.73), (5.74) and (5.89). These equations may serve as a starting point for the laser theory and they are indeed quite often used in that form. On the other hand it has turned out that these equations can be cast in a still simpler and more symmetric form. In sections 5.8 and 5.9 we shall introduce this kind of equations which incidentally will allow us to make close contact with our quantum theoretical treatment of the laser in a later chapter. In that quantum mechanical treatment we shall also quantize the light field. But so far we shall remain in the frame of a semiclassical theory.

### 5.7. The semiclassical laser equations for the macroscopic quantities electric field strength, polarization and inversion density in the rotating wave- and slowly varying amplitude approximation

The rotating wave approximation and the slowly varying amplitude approximation cannot only be applied to the laser equations in a resonator, but they may also serve to simplify the laser equations we derived in section 5.4. Since we shall need the thus resulting equations much later in this book, the reader may skip this section and read it only later when it will be needed.

We start with the field equation (5.55). With respect to its l.h.s. we have several options depending on the specific problem, i.e. standing or running waves. When we use standing waves, we are essentially dealing with the field in a resonator. Since this problem was treated in sections 5.5 and 5.6, we shall deal here with running waves. Incidentally, this allows us to treat the functional dependence of  $E$  on  $\mathbf{x}$  and  $t$  in a symmetric fashion. To this end we consider a plane wave  $\exp[i(\mathbf{k}\mathbf{x} - i\omega t)]$  which is slowly modulated in time *and* space. We therefore write

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}^{(+)}(\mathbf{x}, t) + \mathbf{E}^{(-)}(\mathbf{x}, t), \quad (5.90)$$

where

$$\mathbf{E}^{(\pm)}(\mathbf{x}, t) = \mathbf{E}_0^{(\pm)}(\mathbf{x}, t) \exp[\pm i(\mathbf{k}\mathbf{x} - \omega t)], \quad (5.91)$$

and

$$kc = \omega. \quad (5.92)$$

We form

$$\Delta \mathbf{E}^{(+)} - \frac{1}{c^2} \ddot{\mathbf{E}}^{(+)} \quad (5.93)$$

and insert (5.91) in (5.93). This yields

$$\begin{aligned} \exp[i(\mathbf{k}\mathbf{x} - \omega t)] & [-k^2 \mathbf{E}_0^{(+)} + 2i(\mathbf{k}\nabla) \mathbf{E}_0^{(+)} + \Delta \mathbf{E}_0^{(+)} + (\omega^2/c^2) \mathbf{E}_0^{(+)} \\ & + (2i\omega/c^2) \dot{\mathbf{E}}_0^{(+)} - (1/c^2) \ddot{\mathbf{E}}_0^{(+)}]. \end{aligned} \quad (5.94)$$

In the square brackets, the first and fourth term cancel each other, while the third and sixth term can be neglected due to the slowly varying amplitude approximation. In this way (5.93) reduces to

$$\exp[i(\mathbf{k}\mathbf{x} - \omega t)] [2i(\mathbf{k}\nabla) \mathbf{E}^{(+)} + (2i\omega/c^2) \dot{\mathbf{E}}^{(+)}]. \quad (5.95)$$

In a similar, though simpler fashion, we reduce

$$-\mu_0 \sigma \dot{\mathbf{E}}^{(+)} \quad (5.96)$$

to

$$\exp[i(\mathbf{k}\mathbf{x} - \omega t)] \mu_0 \sigma i \omega \mathbf{E}_0^{(+)}. \quad (5.97)$$

In order to transform the r.h.s. of eq. (5.55) we assume  $\mathbf{P}^{(\pm)}(\mathbf{x}, t)$  in the same form as  $\mathbf{E}^{(\pm)}(\mathbf{x}, t)$ , i.e.

$$\mathbf{P}^{(\pm)}(\mathbf{x}, t) = \mathbf{P}_0^{(\pm)}(\mathbf{x}, t) \exp[\pm i(\mathbf{k}\mathbf{x} - \omega t)], \quad (5.98)$$

where  $\mathbf{P}_0^{(\pm)}$  is a function which varies much more slowly in space and time than  $\exp[\pm i(\mathbf{k}\mathbf{x} - \omega t)]$ . Applying the slowly varying amplitude approximation to  $\mu_0 \ddot{\mathbf{P}}^{(\pm)}(\mathbf{x}, t)$ , we readily obtain

$$-\omega^2 \mu_0 \mathbf{P}_0^{(\pm)}(\mathbf{x}, t) \exp[\pm i(\mathbf{k}\mathbf{x} - \omega t)]. \quad (5.99)$$

We now split the r.h.s. and the l.h.s. of eq. (5.55) into their positive and negative frequency parts, respectively, and collect the corresponding terms (5.95), (5.97) and (5.99). After dividing the resulting equation by  $\exp[i(\mathbf{k}\mathbf{x} - \omega t)]$  we obtain

$$2i(\mathbf{k}\nabla) \mathbf{E}_0^{(+)} + (2i\omega/c^2) \dot{\mathbf{E}}_0^{(+)} + \mu_0 \sigma i \omega \mathbf{E}_0^{(+)} = -\omega^2 \mu_0 \mathbf{P}_0^{(+)}. \quad (5.100)$$



In the last step of our analysis, we multiply this equation by  $c^2/(2i\omega)$ , use the relations

$$\mathbf{k}/k = \mathbf{e}_k, \quad c^2\mu_0 = 1/\epsilon_0, \quad \sigma/(2\epsilon_0) = \kappa \quad (5.101)$$

and on the r.h.s. of (5.100) the approximation  $\omega \approx \bar{\omega}$ . This leaves us with the final result, namely *the field* equation for the slowly varying amplitude

$$\begin{aligned} \nabla \cdot (\mathbf{e}_k \nabla) \mathbf{E}_0^{(+)}(\mathbf{x}, t) + \dot{\mathbf{E}}_0^{(+)}(\mathbf{x}, t) + \kappa \mathbf{E}_0^{(+)}(\mathbf{x}, t) \\ = i\bar{\omega}/(2\epsilon_0) \mathbf{P}_0^{(+)}(\mathbf{x}, t). \end{aligned} \quad (5.102)$$

We wish to reduce the matter equations (5.52) and (5.53) in a similar fashion. To this end we insert (5.90), (5.91) and (5.98) into (5.52) and apply the rotating wave approximation. We immediately obtain *the* equation for the slowly varying amplitude of the polarization

$$\begin{aligned} \dot{\mathbf{P}}_0^{(+)}(\mathbf{x}, t) = (i\omega - i\bar{\omega} - \gamma) \mathbf{P}_0^{(+)}(\mathbf{x}, t) \\ + \frac{1}{i\hbar} (\mathbf{E}_0^{(+)}(\mathbf{x}, t) \mathfrak{D}_{21}) \mathfrak{D}_{12} D(\mathbf{x}, t). \end{aligned} \quad (5.103)$$

Note that  $\bar{\omega}$  need not coincide with  $\omega$ , but that we require only  $|\bar{\omega} - \omega| \ll \omega$ . We insert (5.90), (5.91) and (5.98) into (5.53). When we neglect in

$$\mathbf{E}^{(+)} \mathbf{P}^{(-)} - \mathbf{E}^{(-)} \mathbf{P}^{(+)} - \mathbf{E}^{(+)} \mathbf{P}^{(+)} + \mathbf{E}^{(-)} \mathbf{P}^{(-)} \quad (5.104)$$

the rapidly oscillating terms  $\propto \exp[\pm i(\omega \mp \bar{\omega})t]$  we can transform (5.53) into the equation for the inversion density containing only slowly varying amplitudes

$$\begin{aligned} \frac{d}{dt} D(\mathbf{x}, t) = \frac{D_0 - D(\mathbf{x}, t)}{T} \\ - \frac{2}{i\hbar} (\mathbf{E}_0^{(+)}(\mathbf{x}, t) \mathbf{P}_0^{(-)}(\mathbf{x}, t) - \mathbf{E}_0^{(-)}(\mathbf{x}, t) \mathbf{P}_0^{(+)}(\mathbf{x}, t)). \end{aligned} \quad (5.105)$$

The equations (5.102), (5.103) and (5.105) represent our final result.

### 5.8. Dimensionless quantities for the light field, and introduction of a coupling constant

In this section we continue our outline of section 5.6. In the following we shall introduce dimensionless variables  $b_\lambda$  and  $b_\lambda^*$  instead of the mode amplitudes  $E_\lambda^{(+)}$  and  $E_\lambda^{(-)}$ , respectively. The quantities  $E_\lambda$  and  $b_\lambda$  differ by a simple factor only, namely

$$E_\lambda^{(+)} = i\sqrt{\hbar\omega_\lambda/(2\epsilon_0)} b_\lambda, \quad (5.106)$$

$$E_{\lambda}^{(-)} = -i\sqrt{\hbar\omega_{\lambda}/(2\varepsilon_0)}b_{\lambda}^*. \quad (5.107)$$

As one may show, the energy of an electric field with mode amplitude  $E_{\lambda}$  is proportional to  $|E_{\lambda}|^2$ . On the other hand, in a quantum theoretical treatment  $\hbar\omega_{\lambda}$  is just the energy of a photon. Since  $b_{\lambda}$  is dimensionless we recognize that  $|b_{\lambda}|^2$  must have the meaning of a photon number, may be except for a numerical factor. As it will turn out later,  $|b_{\lambda}|^2$  is precisely the average photon number. We shall elucidate this relation in a later chapter when dealing with the laser equations quantum theoretically. Let us consider the laser equations (5.73), (5.74) and (5.89) more closely. We then recognize that there always the combination  $\mathfrak{D}_{21}\mathbf{u}_{\lambda}(\mathbf{x}_{\mu})$  occurs (or the conjugate complex quantity). Furthermore the factor  $\sqrt{\omega_{\lambda}/(2\hbar\varepsilon_0)}$  occurs. In order to save space it suggests itself to replace this combination by a quantity which we define by

$$\mathbf{g}_{\mu\lambda} = i\mathfrak{D}_{21}\mathbf{u}_{\lambda}(\mathbf{x}_{\mu})\sqrt{\omega_{\lambda}/(2\varepsilon_0\hbar)}. \quad (5.108)$$

It is a rather simple but boring task to rewrite the laser equations by means of the new quantities just introduced. Therefore we shall write down the laser equations in the next section without any intermediate steps.

## 5.9. The basic laser equations

In this section we summarize the basic equations which we have derived in the preceding sections. In this way a reader not interested in their detailed derivation is enabled to start with these equations right from here. We first explain the quantities occurring in the laser equations. Such a quantity is the electric field strength of the light field in the laser. This function which depends on space and time is expanded into a superposition of resonator modes  $\mathbf{u}_{\lambda}(\mathbf{x})$ . The index  $\lambda$  distinguishes the different modes. We assume that the cavity modes are normalized within the volume of the cavity and that they are orthogonal with respect to each other. We allow for an open cavity so that a cavity may consist of two mirrors mounted at the opposite ends of a laser rod. A one-dimensional example for such a mode is provided by a standing wave

$$\mathbf{u}_{\lambda}(x) = \frac{\mathbf{e}_{\lambda}}{\mathcal{J}L} \sin k_{\lambda}x, \quad (5.109)$$

where  $\mathbf{e}_{\lambda}$  is a unit vector in the direction of the polarization of the electric field.  $k_{\lambda}$  is the wave number and  $\omega_{\lambda}$  is the corresponding light frequency in the cavity without laser material. Finally we assume that the light field connected with the mode  $\lambda$  can stay only a finite time in the resonator.

More precisely speaking we assume that the light field amplitude decays in the course of time by means of the damping constant  $\kappa_\lambda$ . Thus the electric field strength is represented in the form

$$\mathbf{E}(\mathbf{x}, t) = \sum_{\lambda} b_{\lambda}(t) \mathbf{u}_{\lambda}(\mathbf{x}) \cdot \mathcal{N}_{\lambda} + \text{conj. compl.} \quad (5.110)$$

where  $b_{\lambda}(t)$  is the amplitude of the field mode A. This amplitude is made dimensionless by means of the factor  $\mathcal{N}_{\lambda}$ ,

$$\mathcal{N}_{\lambda} = i\sqrt{\hbar\omega_{\lambda}/(2\varepsilon_0)}. \quad (5.111)$$

The specific choice of  $\mathcal{N}_{\lambda}$  was made so that we can establish a direct connection with the quantum theory of the light field. The reader is advised to remember in the following that  $b_{\lambda}(t)$  is connected with the electric field strength of the mode A except for a numerical factor. While the light field represents one subsystem of the laser, the other subsystem is provided by matter. We assume that the laser material is built up of individual atoms which we distinguish by the index  $\mu$ . With each atom  $\mu$  a dipole moment  $\mathbf{p}_{\mu}$  is connected which we represent in the form

$$-\mathbf{p}_{\mu} = \mathfrak{D}_{12}\alpha_{\mu}(t) + \mathfrak{D}_{21}\alpha_{\mu}^*(t). \quad (5.112)$$

In it  $\mathfrak{D}_{12}$  is the dipole matrix element which is in quantum theory a prescribed and time independent quantity. Its precise definition is given in eq. (5.30).  $\alpha_{\mu}(t)$  are time dependent functions which determine the temporal behavior of the dipole moments  $\mathbf{p}_{\mu}$ . Because  $\mathbf{p}_{\mu}$  and  $\mathbf{a}_{\mu}$  differ only by the constant vector  $\mathfrak{D}_{12}$ , we shall consider in the following  $\mathbf{a}_{\mu}(t)$  as a dimensionless dipole moment and shall also call it that way. When we consider a system of two-level atoms, the only additional atomic variable still necessary is the inversion  $d_{\mu}$ . It is defined as difference of the occupation numbers of the upper and the lower level of the atom  $\mu$ ;

$$d_{\mu} = N_{2,\mu} - N_{1,\mu}. \quad (5.113)$$

So far we have discussed the subsystems, namely the modes on the one hand and the quantities describing matter on the other hand (dipole moment and inversion). These two subsystems interact with each other by means of the electric interaction between the electrons of the atoms and the electric field. This interaction is described by a coupling constant  $g$  which depends on the indices  $\mu$  and A;

$$g_{\mu\lambda} = i\mathfrak{D}_{21}\mathbf{u}_{\lambda}(\mathbf{x}_{\mu})\sqrt{\omega_{\lambda}/(2\hbar\varepsilon_0)}. \quad (5.114)$$

Evidently this coupling constant contains the dipole matrix element  $\mathfrak{D}_{21}$  as well as the spatial field amplitude  $\mathbf{u}_{\lambda}$  taken at the atomic position  $\mathbf{x}_{\mu}$ . This assumption implies the dipole approximation. As we have shown in the preceding sections, the following equations have been obtained for the

fundamental equations by means of two approximations: namely the rotating wave approximation and the slowly varying amplitude approximation. The corresponding equations read:

(1) Field equations

$$\dot{b}_\lambda = (-i\omega_\lambda - \kappa_\lambda)b_\lambda - i \sum_\mu g_{\mu\lambda}^* \alpha_\mu. \quad (5.115)$$

The physical content of these equations can be explained as follows. The l.h.s. describes the temporal change of the field amplitude. The r.h.s. describes the causes of this change. The first two terms on the r.h.s. describe the oscillation and damping of the field amplitude in the resonator if there is no interaction between the field amplitude and the laser atoms present. The last term describes how the dipole moments act as a driving force on the oscillations of the field. The second group of equations refers to the atoms.

(2) Matter equations

$$\dot{\alpha}_\mu = (-i\bar{\omega}_\mu - \gamma)\alpha_\mu + i \sum_\lambda g_{\mu\lambda} b_\lambda d_\mu, \quad (5.116)$$

$$\dot{d}_\mu = \frac{d_0 - d_\mu}{T} + 2i \sum_\lambda (g_{\mu\lambda}^* \alpha_\mu b_\lambda^* - g_{\mu\lambda} \alpha_\mu^* b_\lambda). \quad (5.117)$$

In analogy to (5.115) the left-hand sides describe the temporal change of the dipole moments and the atomic inversion, respectively. We now discuss the right-hand sides which present the causes of the temporal changes of  $\alpha_\mu$  and  $d_\mu$ . We discuss the first term in (5.116). This contains the transition frequency of the atom  $\mu$ , i.e.  $\bar{\omega}_\mu$ . Because in solids, atoms may occupy sites which are more or less different, the transition frequencies of the individual atoms may differ from each other. We take this fact into account by using the index  $\mu$ . By the interaction of the atom with its surrounding the free oscillation of its dipole moment will be damped. The corresponding damping constant is denoted by  $\gamma$ . The first term on the r.h.s. of (5.116) thus describes the oscillation and damping of the atomic dipole moment if no interaction with the light field takes place. The sum over  $\lambda$  which occurs in (5.116) describes the interaction of all modes  $\lambda$  with the atom under consideration. The factor  $d_\mu$  is of special importance. On account of it the laser equations become nonlinear because here the product between the two quantities  $b_\lambda$  and  $d_\mu$  occurs. This term describes how the electric field represented by its mode amplitudes  $b_\lambda$  drives the dipole moment. But because we are dealing here with a two-level atom, the energy flux between the atom and the field depends on the internal state of the atom. If its electron is in its upper state, energy will be transferred from the atom to the dipole moment. On the other hand if the atom is in its lower state, energy will be transferred from

the field into the atom by absorption. This change of direction is taken care of by the factor  $d_\mu$  whose size depends on the actual occupancy of the two atomic levels.

Let us now turn to a discussion of the r.h.s. of eq. (5.117). The first term describes the relaxation of the inversion caused by the pumping and incoherent relaxation processes.  $d_0$  represents the resulting equilibrium inversion and  $T$  is the corresponding relaxation time. The sum over  $A$  is brought about by the interaction between the field modes  $A$  and the atom  $\mu$ . As may be shown this term is proportional to the energy per second put into the atom or drawn out of it because of the coherent interaction between the atom and the field.

As we shall see in the next chapters, a wealth of phenomena is described by the equations (5.115) till (5.117).

For sake of completeness we now quote a formulation, which is equivalent to eqs. (5.115)–(5.117), but which stresses the continuous distribution of atomic frequencies,  $\bar{\omega}_\mu$ , in an inhomogeneously broadened atomic line. In such a description (compare also sections 4.6 and 4.7) the frequency  $\bar{\omega}_\mu$  is replaced by the continuous (frequency) variable  $\bar{\omega}$ , and correspondingly also the index  $\mu$ . It is further assumed that  $g_{\mu\lambda}$  is space independent (cf. also section 6.1). Therefore the equations (5.116) and (5.117) read in this notation

$$\dot{\alpha}_{\bar{\omega}} = (-i\bar{\omega} - \gamma)\alpha_{\bar{\omega}} + i \sum_{\lambda} g_{\bar{\omega},\lambda} b_{\lambda} d_{\bar{\omega}}, \quad (5.118)$$

$$\dot{d}_{\bar{\omega}} = \frac{d_0 - d_{\bar{\omega}}}{T} + 2i \sum_{\lambda} (g_{\bar{\omega},\lambda}^* \alpha_{\bar{\omega}} b_{\lambda}^* - \text{c.c.}). \quad (5.119)$$

Most important, the sum  $\sum_{\mu}$  in (5.115) is transformed into an integral over  $\bar{\omega}$ , which contains the frequency distribution  $\rho(\bar{\omega})$  (cf. section 4.6), which may be a Gaussian or, in some model calculations, a Lorentzian. In this way, (5.115) is transformed into

$$\dot{b}_{\lambda} = (-i\omega_{\lambda} - \kappa_{\lambda})b_{\lambda} - i \int_{-\infty}^{+\infty} d\bar{\omega} \rho(\bar{\omega}) \alpha_{\bar{\omega}}. \quad (5.120)$$

Note that  $a_{\lambda}$ ,  $d_{\bar{\omega}}$  and  $b_{\lambda}$  are time dependent functions.

### Exercise on section 5.9

Specialize the eqs. (5.115)–(5.117) into those for a single mode laser and assume  $g_{\mu\lambda}$  real. Make the hypothesis  $b = r \exp[i\varphi]$ ,  $a_{\lambda} = P_{\mu} + iQ_{\mu}$ , where  $r$ ,  $\varphi$ ,  $P_{\mu}$  and  $Q_{\mu}$  are assumed real and derive the equations for the new real quantities.

## Chapter 6

# Applications of Semiclassical Theory

### 6.1. The single mode laser. Investigation of stability

As we shall see, the semiclassical laser equations are capable of describing a wealth of laser phenomena. A solution of these equations seems rather difficult, however. Indeed there may be more than  $10^3$  laser modes and perhaps  $10^{18}$  laser atoms which interact with each other. Furthermore the equations are nonlinear because of the terms  $b_\lambda d_\mu$  and  $b_\lambda^* a_\nu$ ,  $b_\lambda \alpha_\mu^*$ . In spite of these difficulties it will be possible to solve the laser equations in an excellent approximation. We shall find a variety of interesting effects. In order to get a first insight how the equations describe laser processes, we shall first consider the special case of a single laser mode. Furthermore we assume exact resonance, i.e. we assume that the frequency  $\omega_\lambda$  of the laser mode under consideration coincides with the atomic transition frequencies which are assumed independent of  $\mu$  (homogeneous line broadening). Therefore we assume  $\omega_\lambda = \bar{\omega}$ . This tuning may be achieved by an appropriate fixation of the distance  $L$  between the mirrors. By way of a model we shall assume that the coupling constants  $g_{\mu\lambda}$  are independent of the mode index  $\lambda$  and of the atomic coordinate  $x_\mu$ :  $g_{\mu\lambda} = g$ .<sup>†</sup> Furthermore we shall assume that  $g$  is real. Because we are dealing with one laser mode only and we need not distinguish between several of them, we shall omit the index  $\lambda$  everywhere. Under these assumptions the laser equations (5.115) to (5.117) reduce to the following equations:

$$\dot{b} = (-i\omega - \kappa)b - ig \sum_{\mu} a_{\mu}, \quad (6.1)$$

$$\dot{\alpha}_{\mu} = (-i\omega - \gamma)\alpha_{\mu} + igbd_{\mu}, \quad (6.2)$$

$$\dot{d}_{\mu} = \frac{1}{T}(d_0 - d_{\mu}) + (2i\alpha_{\mu}gb^* + \text{c.c.}). \quad (6.3)$$

<sup>†</sup>This model can be given a realistic foundation in a ring laser (cf. exercise).

We first study the case in which the pump strength  $d_0$  is so small that we cannot fulfil the laser condition. Under these circumstances we have to deal with a usual lamp. As one may convince oneself, the solutions of eqs. (6.1) to (6.3) read  $b=0$ ,  $a_s=0$ ,  $d_\mu=d_0$ . This result is in so far astonishing as the light field amplitude which is, of course, proportional to  $b$ , vanishes. In fact one should expect that a lamp emits light so that a nonvanishing  $b$  should result. This discrepancy can be resolved in a satisfactory way only in the frame of a quantum theoretical treatment which we shall present in chapter 10. The cause for our present result rests in the fact that  $b$ , which occurs in (6.1), represents only the coherent part of the light field. Light emitted by a conventional lamp is, on the other hand incoherent.

We study what happens when we increase the pump strength  $d_0$ . In order to check whether the solutions  $b=a_s=0$  and  $d_\mu=d_0$  remain stable, we superimpose small deviations  $\delta b$ ,  $\delta\alpha_\mu$  and  $\delta d_\mu$  on  $b$ ,  $a_s$  and  $d_\mu$ , respectively, and linearize the equations (6.1)–(6.3) with respect to these small deviations, i.e. we neglect terms quadratic in these quantities, such as  $\delta d_\mu \delta b$ . While in this approximation the eqs. (6.3) are still fulfilled by  $d_\mu=d_0$ , the eqs. (6.1) and (6.2) acquire the form

$$\delta\dot{b} = (-i\omega - \kappa)\delta b - ig \sum_{\mu} \delta\alpha_{\mu}, \quad (6.4)$$

$$\delta\dot{\alpha}_{\mu} = (-i\omega - \gamma)\delta\alpha_{\mu} + ig\delta b d_0. \quad (6.5)$$

Because according to eq. (6.4) the light field is generated by the sum of all dipole moments, it suggests itself to introduce this sum as a new variable

$$\sum_{\mu} \alpha_{\mu} = S. \quad (6.6)$$

Correspondingly we sum (6.5) up over  $\mu$  and obtain on account of

$$\sum_{\mu} 1 = N = \text{total number of atoms} \quad (6.7)$$

the new equations

$$\delta\dot{b} = (-i\omega - \kappa)\delta b - ig\delta S, \quad (6.8)$$

and

$$\delta\dot{S} = (-i\omega - \gamma)\delta S + igD_0\delta b. \quad (6.9)$$

$D_0 = Nd_0$  is the unsaturated inversion of all atoms. In order to solve these coupled linear equations we make as usual a hypothesis in the form of damped oscillations

$$\delta b = \delta b_0 \exp[(-i\Omega + \Gamma)t], \quad (6.10a)$$

$$\delta S = \delta S_0 \exp[(-i\Omega + \Gamma)t], \quad (6.10b)$$

where  $\delta b_0$  and  $\delta S_0$  are time independent constants.  $\Omega$  is a frequency, whereas  $\Gamma$  is a damping constant, and both are still to be determined. Inserting (6.10) into (6.8) and (6.9) we obtain two homogeneous linear equations for  $\delta b_0$  and  $\delta S_0$ . In order to find a nontrivial solution of these equations, its determinant must vanish. This condition yields the "secular equation"

$$(i\omega + \kappa - i\Omega + \Gamma)(i\omega + \gamma - i\Omega + \Gamma) - g^2 D_0 = 0. \quad (6.11)$$

When we split this equation into its real and imaginary parts, we obtain for the imaginary part  $\omega = \Omega$  and two solutions  $\Gamma_+$  and  $\Gamma_-$ , i.e.

$$\Gamma_{\pm} = -\frac{\kappa + \gamma}{2} \pm \sqrt{\left(\frac{\kappa - \gamma}{2}\right)^2 + g^2 D_0}. \quad (6.12)$$

As long as  $\Gamma_+$  and  $\Gamma_-$  are negative, the deviations (6.10) relax towards 0. The state  $b=0$  in which no coherent emission occurs is stable. When we increase the inversion  $D_0$ , which occurs under the root in (6.12),  $\Gamma_+$  eventually becomes positive. In this case the deviation  $b$  grows exponentially. In other words, the system composed of field mode and atoms becomes unstable. After elementary algebra we obtain from the condition  $\Gamma_+ > 0$ ,

$$D_0 > \kappa\gamma/g^2. \quad (6.13)$$

This is a condition on the inversion  $D_0 = Nd_0$  of the atoms. As can easily be shown (cf. exercise) this condition is just the laser condition which we derived in earlier chapters.

This analysis seems to indicate that the amplitude  $\delta b$  of the laser mode increases exponentially once the condition (6.13) is fulfilled. But this exponential growth does not last forever. Rather a stationary state is eventually reached, which is caused by an equilibrium between the energy input due to pumping and the energy output due to the emission of laser light. In the next section we shall study this stationary state more closely.

### Exercises on section 6.1

(1) Show that in the case of a ring laser with running waves,  $g_{\mu} = g \exp(ik_{\lambda} x_{\mu})$ . Convince yourself that in the single mode laser equations (A fixed),  $\exp(ik_{\lambda} x_{\mu})$  can be transformed away by means of the hypothesis:

$$a_{\mu}(t) = \alpha'_{\mu}(t) \exp(ik_{\lambda} x_{\mu}).$$

If  $g$  is complex, put  $g = |g| e^{i\nu}$  and show that  $e^{i\nu}$  can be transformed away.

(2) Show that the laser condition (6.13) coincides with that of section 2.1.



(3) Derive the instability condition (laser condition) for the case that  $g_{\mu\lambda}$  still depends on  $\mu$  and a homogeneously broadened line.

Hint: Make the hypothesis  $b = \delta b$ ,  $a_\mu = \delta\alpha_\mu$ ,  $d_\mu = d_0 + \delta d_\mu$  and derive linearized equations for  $\delta b$ ,  $\delta\alpha_\mu$ ,  $\delta d_\mu$ . Make the further hypothesis

$$\delta b = \delta b_0 \exp[(-i\Omega + \Gamma)t],$$

$$\delta\alpha_\mu = \delta\alpha_{\mu,0} \exp[(-i\Omega + \Gamma)t],$$

$$\delta d_\mu = 0,$$

$$\delta b_0, \delta\alpha_{\mu,0} \text{ time independent.}$$

Eliminate from the resulting equations  $\delta\alpha_{\mu,0}$  and convince yourself of the following intermediate result:

$$(i\omega + \kappa - i\Omega + \Gamma)(i\omega + \gamma - i\Omega + \Gamma) - d_0 \sum_{\mu} |g_{\mu}|^2 = 0.$$

## 6.2. Single mode laser action. Amplitude and frequency of laser light in the stationary state

We again start from the eqs. (5.115) and (5.117) which we specialize to one mode. Therefore we drop the index A of  $\alpha$  and  $\chi$ . In order to clearly exhibit the connection with the rate equations introduced earlier we keep the index A of the coupling constant  $g_{\mu\lambda}$ . We shall admit that the atomic transition frequencies may differ from each other, i.e. that the line is inhomogeneously broadened. Therefore the laser equations are of the following form:

$$\frac{db}{dt} = (-i\omega - \kappa)b - i \sum_{\mu} g_{\mu\lambda}^* \alpha_{\mu}, \quad (6.14)$$

$$\frac{d\alpha_{\mu}}{dt} = (-i\bar{\omega}_{\mu} - \gamma)\alpha_{\mu} + i g_{\mu\lambda} d_{\mu} b, \quad (6.15)$$

$$\frac{dd_{\mu}}{dt} = \frac{d_0 - d_{\mu}}{\Gamma} + 2i(g_{\mu\lambda}^* \alpha_{\mu} b^* - g_{\mu\lambda} \alpha_{\mu}^* b). \quad (6.16)$$

Because we are dealing with only one mode, the index A is kept fixed and there is no sum over A in the eqs. (6.15)–(6.16). Because we expect that the eqs. (6.14)–(6.16) allow a stationary oscillation of the light field we make the following hypothesis for the field amplitude:

$$b = B e^{-i\Omega t}, \quad (6.17)$$

where the time independent amplitude B and the frequency  $\Omega$  must still be determined. We expect that in the stationary state the inversion acquires

a constant value so that we make the hypothesis

$$d_\mu = \text{const.} \quad (6.18)$$

As it transpires from eq. (6.15), the dipole moments  $a_\mu$  are driven by the oscillating light field. This makes us expect that the individual atoms oscillate at the frequency of the light field. Therefore we make the hypothesis

$$\alpha_\mu = A_\mu e^{-i\Omega t}. \quad (6.19)$$

In (6.19)  $A_\mu$  is a time independent amplitude still to be determined. Inserting (6.17)–(6.19) into eqs. (6.14)–(6.16) we obtain the following equations (after having multiplied them by  $\exp[i\Omega t]$ ):

$$B(i(\omega - \Omega) + \kappa) = -i \sum_\mu g_{\mu\lambda}^* A_\mu, \quad (6.20)$$

$$A_\mu (i(\bar{\omega}_\mu - \Omega) + \gamma) = +i g_{\mu\lambda} d_\mu B, \quad (6.21)$$

$$0 = \frac{d_0 - d_\mu}{T} + 2i(g_{\mu\lambda}^* A_\mu B^* - g_{\mu\lambda} A_\mu^* B). \quad (6.22)$$

Eq. (6.21) can be immediately solved with respect to  $A_\mu$  and we obtain

$$A_\mu = \frac{i g_{\mu\lambda} d_\mu B}{i(\bar{\omega}_\mu - \Omega) + \gamma}. \quad (6.23)$$

Inserting this  $A_\mu$  into (6.22) we obtain the equation

$$0 = \frac{d_0 - d_\mu}{T} - 2d_\mu |B|^2 |g_{\mu\lambda}|^2 \frac{2\gamma}{(\bar{\omega}_\mu - \Omega)^2 + \gamma^2}. \quad (6.24)$$

The last terms in (6.24) are familiar to us from the rate equations. There we introduced terms describing the coupling between the light field and the atoms. The corresponding coupling constants were given by (4.55)

$$|g_{\mu\lambda}|^2 \frac{2\gamma}{(\bar{\omega}_\mu - \Omega)^2 + \gamma^2} = W_{\lambda\mu}. \quad (6.25)$$

Therefore we shall use in the following this abbreviation well known to us. Eq. (6.24) is linear in the still unknown inversion  $d_\mu$ . Therefore we may immediately solve this equation for  $d_\mu$  and thus obtain

$$d_\mu = \frac{d_0}{1 + 2TW_{\lambda\mu}n}. \quad (6.26)$$

Here we have introduced a further abbreviation. Namely, as we shall see immediately the quantity  $|B|^2$  is also well known to us. It just represents

the photon number  $n$  which we have introduced into the rate equations

$$|\mathbf{B}|^2 = n. \quad (6.27)$$

Eq. (6.26) describes how the actual inversion  $d_\mu$  is changed with respect to the unsaturated inversion  $d_0$  when the photon number  $n$  is generated by the laser.  $d_\mu$  is also known as the "saturated inversion". By means of (6.26) we have expressed the inversion by the photon number. When we insert (6.26) into (6.23) we may express the amplitudes  $\mathbf{A}$ , of the dipole moments by the field amplitude  $\mathbf{B}$  alone. The  $\mathbf{A}$ , calculated in this way is finally inserted into (6.20). In this way we obtain the equation

$$B(i(\omega - \Omega) + \kappa) = \sum_{\mu} |g_{\mu\lambda}|^2 B \frac{1}{i(\bar{\omega}_{\mu} - \Omega) + \gamma} \frac{d_0}{1 + 2TW_{\lambda\mu}n}. \quad (6.28)$$

In the following we shall assume, of course, that the laser condition is fulfilled so that a laser amplitude  $B$  unequal zero results. Therefore we may divide eq. (6.28) by  $B$ . In order to discuss this equation further we decompose its left- and right-hand sides into their real and imaginary parts. We obtain for the real parts

$$2\kappa = d_0 \sum_{\mu} \frac{W_{\lambda\mu}}{1 + 2TW_{\lambda\mu}n}, \quad (6.29)$$

where we have used again the abbreviation (6.25).

Precisely the same relation can be derived from the rate equations (4.57), (4.61) if specialized to a single mode. We leave it as an exercise to the reader to convince himself of this fact (cf. exercise). If the photon number  $n$  is sufficiently small, we may expand the ratio in (6.29) with respect to powers of  $n$ . Retaining the first two terms we find the relation

$$2\kappa = d_0 \sum_{\mu} W_{\lambda\mu} - d_0 2Tn \sum_{\mu} W_{\lambda\mu}^2. \quad (6.30)$$

This relation can be considered as an equation for the photon number  $n$  which may be calculated. Because the calculation does not bring us anything new compared to the single mode case treated by rate equations we just refer the reader to our former result (cf. sections 4.7 and 4.10 where we calculated the sums over  $\mu$  explicitly for an inhomogeneously broadened line or standing waves, respectively).

Let us now study the equation which results from the imaginary part of eq. (6.28) (after we have divided it by  $B$ ). We then obtain

$$\omega - \Omega = -\frac{d_0}{2\gamma} \sum_{\mu} (\bar{\omega}_{\mu} - \Omega) \frac{W_{\lambda\mu}}{1 + 2TW_{\lambda\mu}n}. \quad (6.31)$$

Having determined the photon number  $n$  by means of eqs. (6.29) or (6.30) we may insert it into (6.31). This renders (6.31) an equation for the still unknown laser frequency  $\Omega$  alone. Because the photon number  $n$  occurs in (6.31) we shall expect that the frequency  $\Omega$  of the laser light depends on the photon number  $n$ . This is indeed the case if the line is inhomogeneously broadened. We shall present the corresponding results at the end of this section.

Here we first deal with a homogeneously broadened line, where the transition frequencies are all equal,  $\bar{\omega}_\mu = \bar{\omega}$ . We shall show that in this case the frequency is independent of the photon number. In this case we may pull the factor  $\omega - \Omega$  in front of the sum occurring in (6.31). This sum becomes then identical with the one which occurs in (6.29). This leads us directly to the relation

$$\omega - \Omega = -\frac{\kappa}{\gamma} (\bar{\omega} - \Omega), \quad (6.32)$$

from which we may determine the frequency  $\Omega$  by

$$\Omega = \frac{\bar{\omega}\kappa + \omega\gamma}{\gamma + \kappa}. \quad (6.33)$$

This formula tells us that in general the frequency  $\Omega$  of the laser oscillation does not coincide with the frequency of the laser mode in the unloaded resonator. In an unloaded resonator the interaction between light modes and laser atoms is switched off or, more physically speaking, it is a resonator without laser atoms. The meaning of the frequency shift (6.33) can be easily visualized when we recall that the damping constants  $\kappa$  and  $\gamma$  are proportional to the inverse relaxation times of the light field,  $t_1$ , and the atomic dipole moments,  $t_a$ , respectively. Introducing therefore instead of  $\kappa$  and  $\gamma$  the corresponding time constants

$$2\gamma = 1/t_a, \quad 2\kappa = 1/t_1, \quad (6.34)$$

(6.33) can be cast into the form

$$\Omega = \frac{t_a \bar{\omega} + \omega t_1}{t_a + t_1} \quad (6.35)$$

The longer the lifetime of the subsystems composed of the dipole moments or of the field mode are, the bigger is the weight with which we have to attach the atomic or field frequency when we determine  $\Omega$ .

Our above results show that by means of the semiclassical laser equations we may justify the rate equations introduced earlier (at least what the steady

state and the single mode case is concerned). But in addition we may now also determine the frequency of laser light. The phase of laser light remains still undetermined. We shall come back to this problem when we shall deal later with the quantum theory of the laser.

Our results lead to the question whether we may derive the rate equations from the semiclassical equations also in the case of multimode laser action. We shall come back to this question in section 6.9.

In conclusion of this section we want to present the explicit results for the equations for the photon number  $n$  and frequency  $\Omega$  when the sums over  $\mu$  are explicitly evaluated. We quote the important special case of an inhomogeneously broadened line and repeat the results for  $n$  obtained in section 4.6:

Standing wave. According to (4.91), (6.30) can be cast into the form

$$2\kappa = \rho_0 A d_0 \frac{\sqrt{\pi}}{a} e^{-\delta^2} (1 - 3ATn/(2y)), \quad (6.36)$$

where we assumed  $y \ll a$ ;

$\rho_0$  = density of laser atoms,

$A = \bar{\omega}_0 |\mathfrak{D}|^2 / \hbar \epsilon_0$ ,

$\delta = (\Omega - \bar{\omega}_0)/a$  (note the change  $w \rightarrow \Omega!$ ),

$a$  = half width of Gaussian distribution (cf. (4.84)).

The integral occurring in (6.31) can equally well be evaluated under the assumptions  $y \ll a$  and  $(1 + 2TW_{\lambda\mu}n)^{-1} \approx 1 - 2TW_{\lambda\mu}n$ . The result reads

$$\Omega - \omega = -\frac{\rho_0 A d_0}{\alpha} e^{-\delta^2} \Phi(\delta) + nC, \quad (6.37)$$

where

$$\Phi(\delta) = \int_0^\delta e^{u^2} du,$$

$$C = \frac{3}{2} \frac{A^2 \rho_0 d_0 T \sqrt{\pi} \delta e^{-\delta^2}}{2\alpha^2}$$

Precisely speaking, (6.36) and (6.37) are two *coupled* equations for  $n$  and  $\Omega$ . They can be solved iteratively by assuming in a first step  $\Omega = \omega$ .

### Exercise on section 6.2

(1) Show that (6.29) follows from the rate equations (4.57), (4.61) if  $dn/dt = dd_\mu/dt = 0$ .

Hint: Solve (4.61) for  $d_\mu$  and insert the result into (4.57).

(2) Solve (6.36) for  $n$ .

### 6.3. The single mode laser: Transients

In this section we study time dependent solutions of the single mode laser equations. Because the equations are nonlinear, this problem cannot be solved in closed form. Even a computer solution would fail because if a laser contains  $10^{14}$  laser active atoms and only a single mode we ought to solve  $2 \times 10^{14} + 1$  coupled nonlinear differential equations. Therefore we have to devise adequate approximation schemes. In the present and the subsequent section we devise a general approximation scheme which allows us to solve the problem in an excellent approximation. Our scheme requires that the field amplitudes  $|b_\lambda|$  are not too high. This means that our procedure works well close to laser threshold which is, of course, of particular physical interest. Furthermore we shall assume in the following that the cavity width  $\kappa$  is much smaller than the longitudinal and transversal atomic line-widths,  $T^{-1}$ ,  $\gamma$ . In later chapters we shall see that the laser threshold is not the only instability point where the qualitative behavior of light changes dramatically. Indeed we shall see that there is a whole hierarchy of instabilities and we shall represent in our later chapters methods how to cope with these instabilities. In the present section, however, we shall treat the case in which laser action sets in and we start with the single mode laser as an example. To elucidate the decisive steps we assume exact resonance between the mode frequency  $\omega_\lambda \equiv \omega$  and the atomic transition frequencies  $\bar{\omega}_\mu$ , i.e.  $\omega = \bar{\omega}_\mu$ . Furthermore we assume the coupling constant  $g_{\mu\lambda}$  to be independent of  $\mu$  and  $\lambda$ ,  $g_{\mu\lambda} = g$ , where  $g$  is assumed real. In the next section we shall show how this procedure may be extended to the multimode case where we shall drop the specific assumption on resonance and on  $g_{\mu\lambda}$ .

We start from eqs. (6.1)–(6.3). According to eq. (6.1) the dipole moments generate the field mode. Eq. (6.2) in turn tells us that the light mode causes oscillations of the dipoles. According to eq. (6.3) the cooperation of the dipoles and the field mode causes a temporal change of the inversion. Quite evidently these three quantities, field mode, atomic dipole moments and atomic inversion, condition each other. In a certain sense we are dealing here with a vicious circle. In order to escape it, we assume for the moment being that we already know the light field  $b$ . This then fixes the  $\alpha_\mu$ 's and  $d_\mu$ 's because of the matter equations (6.2) and (6.3). Thus in principle we can express  $a_\mu$  by  $b$ . We shall see that this is indeed possible and we shall obtain  $a_\mu$  in the following form:

$$\alpha_\mu = c_1 b + c_2 b |b|^2 + \dots, \quad (6.38)$$

where  $c_1$  and  $c_2$  are certain constant coefficients. If the fields are not too high we may neglect the higher powers of  $b$  which are indicated by dots.

If we retain only the linear term  $\propto b$  in (6.38), the dipole moments of the atoms,  $a_\mu$ , are proportional to the field amplitude  $b$ . Or, in other words, we are back to conventional dispersion theory. Inserting the dipole moments (6.38) in the field equation (6.1) we obtain a closed equation for  $b$ . The circle is now closed again but we can solve this new equation. As the reader may convince himself in the exercises this new equation does not allow any stable solution, at least in general. Indeed the stabilization of laser light can be properly described only if we take into account the nonlinear term in (6.38). It will be our goal to derive (6.38) and then to study the resulting nonlinear equation for  $b$ .

Let us now turn to the nonlinear equations (6.1)–(6.3), where we apply the following iteration procedure. We first assume that the field is given in the form

$$b = B \exp[-i\Omega t]. \quad (6.39)$$

In the following we shall admit that  $B = B(t)$  depends on time. We shall assume, however, that its temporal change is much slower than processes described by the atomic relaxation constants  $\gamma$  and  $1/T$ . As can be shown in detail,  $B$  may be assumed as constant within the individual steps of the iteration procedure. We shall further assume that in lowest approximation a constant inversion

$$d_\mu = d_0 \quad (6.40)$$

has been established by pump and relaxation processes. On the other hand according to eq. (6.2) a field generates nonvanishing dipole moments  $a_\mu$  of the atoms. Because we determine these  $\alpha_\mu$ 's in a first step we call them  $\alpha_\mu^{(1)}$ . According to eq. (6.3) dipole moments and field jointly cause a new inversion which we call  $d_\mu^{(1)}$ . By means of this new inversion which we insert in the r.h.s. of (6.2) we obtain an improved dipole moment  $\alpha_\mu^{(2)}$ . As we shall see in a minute explicitly we shall succeed in expressing the dipole moments  $a_\mu$  by the field amplitude  $b$  alone. When we insert the corresponding expressions into the eq. (6.1) for  $b$  we obtain a closed equation for  $b$  alone. This equation can be considered as a self-consistency condition. The atomic variables are eliminated completely and we have determined the reaction of the field on itself. Our procedure can be described by the following scheme:

$$\text{Start: } \left. \begin{array}{l} \rightarrow b = B \exp[-i\Omega t] \\ d_\mu^{(0)} = d_0 \end{array} \right\} \rightarrow \alpha_\mu^{(1)} \rightarrow d_\mu^{(1)} \rightarrow \alpha_\mu^{(2)} \rightarrow b \quad (6.41)$$

Before we perform the individual steps explicitly we remark that in the case

$\bar{\omega} = \omega$  the frequency  $\Omega$  of the hypothesis (6.39) becomes  $\Omega = \omega$ . In order to simplify the subsequent formulas as much as possible we shall anticipate this relation  $\Omega = \omega$ . We insert  $b = B \exp[-i\omega t]$  and  $d_\mu \equiv d_0$  in (6.2) treating  $B$  as practically time independent so that we neglect its time-derivative. The solution (6.2) reads

$$\alpha_\mu^{(1)}(t) = \frac{igd_0}{\gamma} b(t). \quad (6.42)$$

We now calculate an improved inversion  $d_\mu$  by inserting  $b = B \exp[-i\omega t]$  and (6.42) in (6.3). We assume that  $d_\mu = d_\mu^{(1)}$  changes only little over times of the order of the relaxation time  $T$ , i.e. that we can neglect  $\dot{d}_\mu$  compared to  $(d_0 - d_\mu)/T$ . Therefore we may put in (6.3)

$$\dot{d}_\mu \equiv \dot{d}_\mu^{(1)} = 0, \quad (6.43)$$

and we readily obtain

$$d_\mu^{(1)} = d_0 - \frac{4Tg^2}{\gamma} d_0 |b|^2. \quad (6.44)$$

Now we may repeat the first step namely the calculation of  $a$ , by means of eq. (6.2) where we use the improved  $d_\mu^{(1)}$  instead of  $d_0$ . We thus obtain

$$\alpha_\mu(t) \approx \alpha_\mu^{(2)}(t) = \frac{ig}{\gamma} b(t) \left( d_0 - \frac{4Tg^2}{\gamma} d_0 |b|^2 \right). \quad (6.45)$$

According to (6.45) a given field  $b$  causes the dipoles to oscillate with the same frequency as the field. Of course, the field of a laser is not prescribed but is rather generated by the laser process. This is reflected within our formalism by the fact that we have to insert (6.45) into (6.1). We thus obtain the fundamental self-consistent laser equation

$$\dot{b} = (-i\omega - \kappa)b + \frac{g^2 D_0}{\gamma} b - \frac{4g^4 T D_0}{\gamma^2} |b|^2 b, \quad (6.46)$$

where  $D_0 = Nd_0$  is the unsaturated inversion. This equation describes how the light field of the laser interacts with itself via the atoms. In order to interpret the physical content of eq. (6.46) we use once again the decomposition

$$b(t) = B(t) \exp[-i\omega t], \quad (6.47)$$

i.e. we split off the rapidly oscillating part. We then obtain the following



equation for B:

$$\dot{B}(t) = \underbrace{\left(-\kappa + \frac{g^2 D_0}{\gamma}\right)}_G B(t) - \underbrace{\frac{4g^4 T D_0}{\gamma^2}}_C |B|^2 B(t). \quad (6.48)$$

The first term in the brackets on the r.h.s. stems from the cavity losses, the second positive part stems from the unsaturated inversion. The last term describes the lowering of the effective inversion by the laser process.

This equation tells us that the behavior of the laser system is fundamentally different depending on whether the laser is operated below or above its threshold (6.13). This can be visualized especially nicely when we identify  $B(t)$  with the coordinate  $q(t)$  of a particle (in a purely formal way). By adding an acceleration term  $m\ddot{q}$  to (6.48) and abbreviating the r.h.s. of (6.48) by  $K(q)$  we obtain the equation

$$m\ddot{q} + \dot{q} = K(q). \quad (6.49)$$

This equation is the equation of motion of a damped particle under the action of the force  $K$ . When we let  $m \rightarrow 0$ , we may retain the mechanical interpretation which allows the following conclusions. The force  $K(q)$  may be derived from a potential  $V$

$$K(q) \equiv -\frac{\partial V(q)}{\partial q}, \quad (6.50)$$

where  $V(q)$  is given by

$$V(q) = -\frac{G}{2} q^2 + \frac{C}{4} q^4, \quad (6.51)$$

and  $G$  and  $C$  were defined in (6.48). This potential is represented in fig. 6.1. When the laser is operated below its threshold, i.e.

$$D_0 < \kappa\gamma/g^2, \quad (6.52)$$

the potential is represented by the dashed curve. The only equilibrium point is given by  $q=0$ , i.e. the light field amplitude  $B \equiv q$  is equal to zero. If on the other hand the laser condition is fulfilled, the solid curve applies. Quite evidently the amplitude  $q=0$  is no more stable and there are two new stable positions instead, provided we consider  $q$  as a real quantity. If  $q$  is complex as it is the case with  $B$ , there are equilibrium positions with an arbitrary phase of  $B$ , as we shall demonstrate below. Thus above threshold,  $D_0 > \kappa\gamma/g^2$ , the amplitude of the laser light is unequal zero.  $q$  can be easily calculated by putting  $\dot{q} = \ddot{q} = 0$  in (6.49) and by dividing the resulting

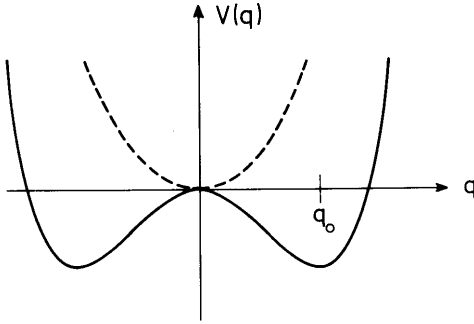


Fig. 6.1. The potential field  $V$  according to eq. (6.49) versus the coordinate  $q$  of the fictitious particle. Dashed line: below laser threshold; solid line: above laser threshold. Above laser threshold two stable values result of which one,  $q_0$ , is indicated.

equation by  $q$  (or  $B$ ). We thus obtain

$$q_0 \equiv |B_0| = \sqrt{G/C}. \quad (6.53)$$

Here the phase of  $B_0 = |B_0| \exp[i\varphi]$  remains undetermined. Indeed it can be chosen freely. Eq. (6.48) is valid provided the field strength does not deviate too much from its stationary value (6.53) and if we do not drive the laser too far above its threshold. Eq. (6.48), which describes the nonlinear relaxation of the laser field amplitude, can be solved exactly by the hypothesis

$$B(t) = r(t) \exp[i\Phi(t)], \quad (6.54)$$

where  $r$  and  $\Phi$  are assumed real. Inserting this hypothesis into (6.48) we obtain the two equations

$$\dot{\Phi} = 0 \quad (6.55)$$

and

$$\dot{r} = Gr - Cr^3. \quad (6.56)$$

It follows from (6.55) that the phase  $\Phi$  is constant but undetermined. The exact solution of (6.56) reads

$$r = \left( \frac{Gh(t)}{1 + h(t)C} \right)^{1/2}. \quad (6.57)$$

In it  $h(t)$  is given by

$$h(t) = \frac{r_0^2}{G - Cr_0^2} \exp[2G(t - t_0)]. \quad (6.58)$$

$r_0$  is the field amplitude at time  $t_0$ . Multiplying eq. (6.56) by  $r$  and putting  $r^2 = n$  we obtain the rate equation

$$\dot{n} = 2Gn - 2Cn^2. \quad (6.59)$$

It agrees with the rate equation of the single mode laser of section 4.1.

### Exercise on section 6.3

Solve the single mode laser equations ( $\omega = \bar{\omega}_\mu = \bar{\omega}$ ,  $g_{\mu\lambda} = g$ , real) in the linear approximation, where in (6.38) only  $c_1 \neq 0$  and all other  $c_k = 0$ .

### 6.4. Multimode action of solid state lasers. Derivation of reduced equations for the mode amplitudes alone

In chapter 4 on rate equations we showed that in a laser several modes can be excited simultaneously. Therefore in this section we wish to treat the question which effects can be expected in multimode laser action. The experimental and theoretical study of such effects has not yet been finished but is going on quite actively and indeed over and over again new types of effects are being discovered. In this and subsequent chapters we shall try to exhibit the most important and interesting effects so far found and we hope that our treatment will enable the reader to explore new effects.

In this section we carry on the line of thought presented in the preceding section, i.e. we confine our analysis to laser modes whose amplitudes are still sufficiently small and to a situation in which laser action starts at the laser threshold introduced before. Later on we shall present further methods and results concerning new kinds of instabilities. Because the laser equations which we derived in chapter 5 are nonlinear, in general it will not be possible to solve them in closed form. In this chapter we shall represent two approximation procedures which will allow us to get a first insight into multimode action. In the present section we shall apply the same method we used when treating the single mode laser and we shall eliminate the atomic variables, i.e. the dipole moments and the inversions. We then obtain equations for the field mode alone which will give rise to new effects especially to phase locking. Furthermore we shall look into the question whether the rate equations which we derived heuristically in chapter 4 can be derived from the original laser equations of chapter 5. As we shall see this is indeed possible provided one may assume that there is no phase locking between different modes, i.e. provided we may average over the phases of the individual modes. But let us start here with the derivation of equations for the laser modes alone.

We start from the fundamental equations (5.115)–(5.117) of section 5.9. We assume that several modes with certain indices, for example  $A_1, \dots, A_n$ , are performing laser action. For each individual mode amplitude we make the hypothesis

$$b_\lambda(t) = B_\lambda \exp[-i\Omega_\lambda t]. \quad (6.60)$$

The frequencies  $\Omega_\lambda$  are still unknown. We shall determine them later on self-consistently. Also the amplitudes  $B_\lambda$  are still unknown quantities. We shall admit that  $B_\lambda$ 's are time dependent but we shall assume that their temporal changes take place on a much longer time scale than the oscillations with the frequencies  $\Omega_\lambda$  and relaxation processes with the constants  $\gamma$  and  $1/T$ . This allows us to use the approximation of slowly varying amplitudes. In a first step of our method of solution we assume that due to pump and relaxation processes the inversion  $d_\mu$  has acquired the value  $d_0$ ,

$$d_\mu^{(0)} = d_0. \quad (6.61)$$

We now insert (6.61) and (6.60) into the equation for the dipole moments (5.116). Because on the r.h.s. of (5.116) a sum over exponential functions of the form (6.60) occurs we write the solution of the equations for  $\alpha_\mu$ , also in the form of a superposition of exponential functions with the corresponding frequencies

$$\alpha_\mu^{(1)} = \sum_\lambda A_{\mu\lambda} \exp[-i\Omega_\lambda t]. \quad (6.62)$$

Using this hypothesis we immediately obtain the relation

$$\begin{aligned} -\sum_\lambda A_{\mu\lambda} i\Omega_\lambda \exp[-i\Omega_\lambda t] &= \sum_\lambda (-i\bar{\omega}_\mu - \gamma) A_{\mu\lambda} \exp[-i\Omega_\lambda t] \\ &+ i \sum_\lambda g_{\mu\lambda} B_\lambda \exp[-i\Omega_\lambda t] d_\mu^{(0)}. \end{aligned} \quad (6.63)$$

On both sides of this equation we compare the factors of the corresponding exponential functions  $\exp[-i\Omega_\lambda t]$ . Thus we can calculate  $A_{\mu\lambda}$  explicitly and insert it in (6.62). Then the following explicit expression for the dipole moments results:

$$\alpha_\mu^{(1)}(t) = d_0 \sum_\lambda g_{\mu\lambda} (-\Omega_\lambda + \bar{\omega}_\mu - i\gamma)^{-1} b_\lambda. \quad (6.64)$$

Within the frame of this first step of our approximation scheme, the dipole moments of the atoms oscillate at the same frequency as the laser modes we assumed to be present. Because the dipole moment  $\alpha_\mu$  is unequal zero and the mode amplitudes  $B_\lambda$  are also assumed unequal zero an additional term occurs in eq. (5.117) for the inversion. Thus we may calculate an

improved inversion  $d_\mu^{(1)}$ . In order to integrate the corresponding equation we make a hypothesis for  $d_\mu^{(1)}$  which just contains the exponential functions  $\exp[(i\Omega_\lambda - i\Omega_{\lambda'})t]$ . We then obtain

$$d_\mu^{(1)} = d_0 \left( 1 - 2 \sum_{\lambda\lambda'} g_{\mu\lambda}^* g_{\mu\lambda'} b_\lambda^* b_{\lambda'} D_{\mu\lambda\lambda'} + \text{conj. compl.} \right). \quad (6.65)$$

The constants  $D_{\mu\lambda\lambda'}$ , occurring in (6.65) are abbreviations for the following expressions:

$$D_{\mu\lambda\lambda'} = -i(\bar{\omega}_\mu - \Omega_{\lambda'} - i\gamma)^{-1} [1/T + i(\Omega_\lambda - \Omega_{\lambda'})]^{-1}. \quad (6.66)$$

We now insert this improved value of the inversion (6.65) and our original hypothesis for the field modes (6.60) into the equations of the dipole moments. The integration can be done in a fashion analogous to the one we just have used so that we may write down the final result immediately,

$$\begin{aligned} \alpha_\lambda^{(2)} = & d_0 \sum_\lambda g_{\mu\lambda} b_\lambda(t) (\bar{\omega}_\mu - \Omega_\lambda - i\gamma)^{-1} \\ & - i2d_0 \sum_{\lambda\lambda'\lambda''} g_{\mu\lambda} g_{\mu\lambda'}^* g_{\mu\lambda''} b_\lambda b_{\lambda'}^* b_{\lambda''} M_{\mu\lambda\lambda'\lambda''}. \end{aligned} \quad (6.67)$$

The constant coefficients  $M_{\mu\lambda\lambda'\lambda''}$  are explicitly given by the expressions

$$\begin{aligned} M_{\mu\lambda\lambda'\lambda''} = & [1/T + i(\Omega_{\lambda'} - \Omega_{\lambda''})]^{-1} (\Omega_{\lambda'} - \Omega_\lambda - \Omega_{\lambda''} + \bar{\omega}_\mu - i\gamma)^{-1} \\ & \times [(\bar{\omega}_\mu - \Omega_{\lambda'} + i\gamma)^{-1} - (\bar{\omega}_\mu - \Omega_{\lambda''} - i\gamma)^{-1}]. \end{aligned} \quad (6.68)$$

This is, of course, a rather lengthy expression and we shall use it later on in this explicit form only at few instances. Much more interesting, however, is the form (6.67). The first sum is already known to us. It means that the dipole moments oscillate coherently with the originally present laser amplitudes  $b_\lambda(t)$ . The additional term in (6.67) stems from the fact that the inversion has been changed by laser action. In contrast to the single mode laser the inversion has become a time dependent function. The inversion performs pulsations with frequencies corresponding to the frequency differences of the individual laser modes. Therefore this effect is called inversion pulsation. By means of the explicit result (6.67) we can now do the last step, namely we may insert the explicit expression for the dipole moments (6.67) into the equations for the laser modes (5.115). We thus obtain our final equations

$$\begin{aligned} \frac{db_\lambda}{dt} = & (-i\omega_\lambda - \kappa_\lambda) b_\lambda - i \sum_{\mu\lambda'} (\bar{\omega}_\mu - \Omega_{\lambda'} - i\gamma)^{-1} g_{\mu\lambda}^* g_{\mu\lambda'} b_{\lambda'} d_0 \\ & - 2d_0 \sum_{\mu\lambda_1\lambda_2\lambda_3} g_{\mu\lambda}^* g_{\mu\lambda_1} g_{\mu\lambda_2}^* g_{\mu\lambda_3} b_{\lambda_1} b_{\lambda_2}^* b_{\lambda_3} M_{\mu\lambda_1\lambda_2\lambda_3}. \end{aligned} \quad (6.69)$$

When we specialize these equations to that of a single mode, exact resonance, i.e.  $\bar{\omega}_\mu = \omega_\lambda = 0$ ,  $g_{\mu\lambda} = g$ , real, we obtain eq. (6.46) which we found in section 6.3.

Eqs. (6.69) represent a result which is quite pleasant for a physicist because we are primarily not interested in the oscillations of the electrons of the laser atoms but rather in the field modes of the laser. But because the field equations (6.69) contain nonlinear terms it is still difficult to solve these equations.

Nevertheless in a number of cases one can rather easily visualize what the equations mean for the  $b$ 's. To this end let us consider the cubic terms. Let us suppose that we insert for the field amplitudes  $b_\lambda$  the hypothesis  $b_\lambda = B_\lambda \exp[-i\Omega_\lambda t]$  and let us assume for the moment being that the amplitudes  $B_\lambda$  are time independent. We then recognize the following. The cubic term can be considered as a driving force for  $b_\lambda$  occurring on the l.h.s. of (6.69). This driving force oscillates at various frequencies depending on which term of the sum over  $A$ ,  $\lambda_2\lambda_3$  is considered. There are special combinations, for instance  $A_1 = A_2$ ,  $A_1 = A_3$  or  $A_2 = A_3$ ,  $\lambda_2 = \lambda_3$ , where the cubic term oscillates in phase with the field mode. In such a case in the steady state the  $B$ 's can be assumed time independent. We shall see below that in this case we can come back to the former rate equations. On the other hand, in eqs. (6.69) additional terms occur in which the frequency of the driving force differs from the originally assumed frequency of the laser mode  $A$ . This causes new phenomena in which phase relations play a crucial role. In the next section we shall treat relatively simple but rather instructive examples of effects which are produced by phase locking. We treat these cases because they clearly demonstrate that phase relations which have been entirely neglected within rate equations can be quite important for lasers.

### 6.5. Simple examples of the multimode case

In the preceding section we succeeded in simplifying the original problem considerably. Whereas our original equations referred not only to the laser modes but also to the numerous atomic variables we finally obtained equations which refer only to the modes. In spite of this fact the resulting equations are still rather complicated, but on the other hand they are capable of describing a great number of phenomena. But let us try to cut a trail through the jungle of these complicated nonlinear equations by focussing our attention on some particularly interesting special cases. These will allow us to get some insight into the structure of these equations and the interaction they describe. Furthermore we can treat a number of effects which are

physically particularly interesting. The simplest case is, of course, that of a single mode in which case we may drop the index A of  $b_\lambda$ . Furthermore we may drop all sums over A. But in contrast to section 6.3 we retain the indices  $\mu$  and may include the nonresonant case. The expression for the inversion (6.65) now reads

$$d_\mu^{(1)} = d_0 \left( 1 - 2T |g_{\mu\lambda}|^2 \frac{2\gamma}{(\Omega_\lambda - \bar{\omega}_\mu)^2 + \gamma^2} n \right). \quad (6.70)$$

When we recall the definition of  $W_{\lambda\mu}$  of eq. (4.55) we immediately recognize that (6.70) agrees with the expression (4.64) which we derived in the frame of rate equations. This expression describes, as we know, hole burning. Correspondingly we obtain from eqs. (6.69) an equation of the form

$$\frac{db}{dt} = (-i\omega - \kappa + G_0 + i\delta\omega_1)b - (s + i\delta\omega_2)|b|^2 b, \quad (6.71)$$

where the constants  $G_0$ ,  $S_0$ ,  $s$ , and  $So$ , are real. We have derived an equation of such a form already in the frame of the iteration procedure of section 6.3 (cf. (6.46)). The additional terms  $S_0$ , and  $So$ , which occur in the case of an incoherently broadened line are of special interest. These terms describe a frequency shift of the laser mode. We have found such terms in section 6.2. But there is a difference between our former result (6.31) and our present one. In section 6.2 we had to confine ourselves to the stationary case but could treat arbitrarily large amplitudes  $b$ . In this section we could find also nonstationary solutions but we had to confine ourselves to  $b$ 's which are not too large.

Let us consider the second simplest case, namely that of two modes. We obtain the following expression for the inversion:

$$\begin{aligned} d_\mu^{(1)} = & \left\{ d_0 \left( 1 - 2T |g_{\mu 1}|^2 \frac{2\gamma}{(\Omega_1 - \bar{\omega}_\mu)^2 + \gamma^2} n_1 \right. \right. \\ & \left. \left. - 2T |g_{\mu 2}|^2 \frac{2\gamma}{(\Omega_2 - \bar{\omega}_\mu)^2 + \gamma^2} n_2 \right) \right\} \\ & + [-2d_0 g_{\mu 1}^* g_{\mu 2} D_{\mu 12} \underbrace{b_1^* b_2}_{\propto \exp(i(\Omega_1 - \Omega_2)t)} \quad - 2d_0 g_{\mu 2}^* g_{\mu 1} D_{\mu 21} \underbrace{b_2^* b_1}_{\propto \exp(i(\Omega_2 - \Omega_1)t)}] \\ & \dagger [\text{conj. complex}]. \end{aligned} \quad (6.72)$$

Though this expression is rather long, it can be easily studied. We derived the expression which stands in the curly bracket already in the context of the rate equations (cf. (4.64)). It again represents hole burning. The

expressions in square brackets, which no longer depend on photon numbers but on the amplitudes of the individual laser modes, e.g. in the form  $b_1^* b_2$ , are new, however. As can be shown, even in such a case the individual laser modes oscillate essentially harmonically, i.e.

$$b_j(t) = B_j \exp[-i\Omega_j t]. \quad (6.73)$$

As we have demonstrated above in eq. (6.72), a time dependent modulation of the inversion results. This effect has been called inversion pulsation. Such effect could not occur in the rate equations because they did not take into account any phase relations. Therefore the question arises in which case such a pulsation can be neglected. To this end we first exhibit the explicit form of  $D_{\mu 21}$  and  $D_{\mu 11}$ :

$$D_{\mu 21} = -\frac{i}{-\Omega_1 + \bar{\omega}_\mu - i\gamma} \left( \frac{1}{T} + i(\Omega_2 - \Omega_1) \right)^{-1}, \quad (6.74)$$

$$D_{\mu 11} = -\frac{1}{-\Omega_1 + \bar{\omega}_\mu - i\gamma} T, \quad (6.75)$$

where the D's were defined in (6.66). We obtain the following relation:

$$\left| \frac{D_{\mu 21}}{D_{\mu 11}} \right| = \frac{1}{[1 + T^2(\Omega_2 - \Omega_1)^2]^{1/2}} \ll 1, \quad (6.76)$$

provided

$$|T(\Omega_2 - \Omega_1)| \gg 1, \quad \text{i.e.} \quad |\Omega_2 - \Omega_1| \gg 1/T. \quad (6.77)$$

It follows that the pulsations are negligible if the frequency distance of the laser modes is large compared to the inverse of the longitudinal relaxation time  $T$  of the inversion. In the opposite case these quantities can acquire the same order of magnitude as the terms occurring in the rate equations so that pulsation processes can play an important role. The equations for the mode amplitudes are becoming rather long. In order to get an overview over the individual contributions we abbreviate the corresponding factors of the mode amplitudes  $b_\lambda$ ,  $\lambda=1,2$ . As an example for the resulting equations we quote that for the mode 1 and discuss the individual contributions subsequently,

$$\begin{aligned} \frac{db_1}{dt} = & \left( -i\omega_1 - \kappa + i \sum_{\mu} |g_{\mu 1}|^2 (-\Omega_1 + \bar{\omega}_\mu + i\gamma)^{-1} \right) b_1 \\ & + (C_{11} n_1 + C_{12} n_2) b_1 + \underline{(F n_1 + F' n_2)} b_2 \\ & + \underline{\underline{(H b f b_2 b_2 + H' b_2^* b_1 b_1)}}. \end{aligned} \quad (6.78)$$



The constants  $C_{ik}$ ,  $F$ ,  $F'$ ,  $H$ ,  $H'$  represent terms which can be easily determined by a comparison with (6.69). The first row and the first term in the second row which contains the factor  $C_{11}$  are known to us from the single mode laser.

The other contributions contain expressions which represent mode coupling. This mode coupling is brought about in various ways. First of all there is the term containing  $C_{12}$ . It corresponds to an expression which we know from the rate equations. It means that the inversion is not only diminished by mode 1 but also by mode 2 (hole burning). The underlined expressions are of special interest. The first kind of underlined expressions is given by

$$\underline{(Fn, +Fn_2)}b_2 \propto \exp[-i\Omega_2 t], \quad (6.79)$$

which means that the mode with index 2 tries to modulate the mode with index 1 via inversion pulsation. Further modulation effects are represented by the further terms (underlined by a wavy line) where frequency dependencies,

$$\underline{\exp[-i(2\Omega_2 - \Omega_1)t]}, \quad (6.80)$$

$$\underline{\exp[-i(2\Omega_1 - \Omega_2)t]}, \quad (6.81)$$

occur. We recognize that an interaction between the individual modes is brought about via the atoms whereby new frequency combinations leading to sidebands are made possible. As we shall see in subsequent chapters such coupling effects, where phase relations occur, play an important role in several aspects. They may bring about phase or frequency locking, and in the case that many modes acquire fixed phase relations ultrashort pulses can be generated.

### Exercise on section 6.5

Determine the constants  $G_0$ ,  $\delta\omega_1$  and  $S_0$ , in the single mode case explicitly by means of (6.69). Compare the expression of  $S_0$ ,  $+|b|^2 S_0$ , with that of eq. (6.31) under the assumption that  $2TW_{\mu\lambda} n \ll 1$ .

### 6.6. Frequency locking of three modes

The case we are going to discuss represents a particularly beautiful example of how the semiclassical laser equations may describe effects which cannot be treated by rate equations. We consider a laser which shows laser action in the modes 1, 2, 3. We assume (as can be verified experimentally) that these three modes belong to the subsequent resonator frequencies  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$ .

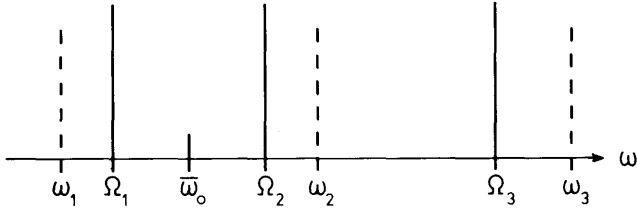


Fig. 6.2. Frequency locking. Scheme of mode frequencies. The abscissa represents the circular frequency.  $\bar{\omega}_0$  is the atomic center frequency.  $\omega_1, \omega_2, \omega_3$  are the mode frequencies in the empty resonator;  $\Omega_1, \Omega_2, \Omega_3$  are the actual mode frequencies when laser action takes place but still without the coupling which causes frequency locking.

In a resonator without laser material, i.e. in the unloaded resonator, these mode frequencies are equidistant. But we know that laser action leads to frequency shifts, some of which we got to know in section 6.2. The new shifted frequencies will be denoted as usual in our book by  $\Omega_\lambda$  (cf. fig. 6.2). When we form the difference from the frequency differences from subsequent modes we obtain the expression

$$(\Omega_2 - \Omega_1) - (\Omega_3 - \Omega_2). \quad (6.82)$$

Experimentally the following results were obtained. The frequencies may be usually shifted by a change of the distance between the mirrors. Usually the three laser modes oscillate with their corresponding frequencies independently of each other. When one changes the distance between the mirrors so that the frequencies are shifted and the expression (6.82) becomes small (typically  $10^3$  Hz) for gas lasers, the frequencies suddenly jump and get locked in a way which we are going to derive. To treat this effect we shall write the mode amplitudes in the form

$$b_\lambda = r_\lambda \exp[-i\Omega_\lambda t - i\varphi_\lambda], \quad (6.83)$$

where  $r_\lambda$  are real amplitudes. We shall admit that the real phases  $\varphi_\lambda$  will be time dependent. Let us consider the cubic terms occurring in the eqs. (6.69) more closely again. When we choose  $\lambda_1 = \lambda_2, \lambda_3 = \lambda_3$ , the r.h.s. oscillates at the same frequency as  $b_\lambda$  on the l.h.s. The same is true for the choice  $\lambda_1 = \lambda_3, \lambda_2 = \lambda_3$ . But now we wish to consider also terms where a frequency combination occurs which is different from the frequency in the mode  $b_\lambda$ . When we consider the equation for mode 1 we obtain such frequency combinations by the choice

$$\lambda_2 = \lambda = 2, \quad \lambda_1 = 1, \quad \lambda_3 = 3, \quad (6.84)$$

or vice versa. Because we shall assume that the expression (6.82) is small the relation

$$\Omega_1 \approx \Omega_{\lambda_2} + \Omega_{\lambda_3} - \Omega_{\lambda_1} \quad (6.85)$$

is approximately (but not exactly) fulfilled. In order to elaborate the essentials we shall assume in the following that the real field amplitudes  $r_\lambda$  are time independent. Inserting the expressions (6.83) into the multimode equations (6.69) and dividing in each case by the exponential function (6.83) we obtain equations of the following form:

$$\begin{aligned} \Omega_1 + \dot{\varphi}_1 &= \tilde{\omega}_1 + \text{Im}(C_1 \exp[i(\Omega_1 + \Omega_3 - 2\Omega_2)t] \\ &\quad \times \exp[-i(2\varphi_2 - \varphi_1 - \varphi_3)]), \end{aligned} \quad (6.86)$$

$$\begin{aligned} \Omega_3 + \dot{\varphi}_3 &= \tilde{\omega}_3 + \text{Im}(C_3 \exp[i(\Omega_1 + \Omega_3 - 2\Omega_2)t] \\ &\quad \times \exp[-i(2\varphi_2 - \varphi_1 - \varphi_3)]), \end{aligned} \quad (6.87)$$

$$\begin{aligned} \Omega_2 + \dot{\varphi}_2 &= \tilde{\omega}_2 + \text{Im}(C_2 \exp[-i(\Omega_1 + \Omega_3 - 2\Omega_2)t] \\ &\quad \times \exp[i(2\varphi_2 - \varphi_1 - \varphi_3)]). \end{aligned} \quad (6.88)$$

The quantities  $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3$  are those frequencies which stem from the original mode frequencies in the unloaded resonator and the various frequency shifts. Of course, in the general case these frequency shifts may depend on the intensity of the laser modes. But because we assume that the intensities are time independent we shall not be concerned with the dependence of  $\tilde{\omega}_\lambda$  on the real amplitudes,  $r_\lambda$ . Similarly we shall assume that the coefficients in front of the exponential functions are time independent constants

$$C_j = C_j(r_1, r_2, r_3). \quad (6.89)$$

This is justified when we neglect temporal changes of  $r_\lambda$ .

We shall try to derive from eqs. (6.86) to (6.88) an equation for the quantity

$$\Psi \equiv (2\Omega_2 - \Omega_1 - \Omega_3)t + (2\varphi_2 - \varphi_1 - \varphi_3), \quad (6.90)$$

which occurs in the exponents on the r.h.s. of (6.86)–(6.88). To this end we multiply (6.88) by 2 and subtract from it the eqs. (6.86) and (6.87). We then obtain a single equation for  $\Psi$  which has the following structure:

$$\dot{\Psi} = \xi + \alpha \sin \Psi + \beta \cos \Psi. \quad (6.91)$$

In it  $\xi$  is an abbreviation

$$\xi = 2\tilde{\omega}_2 - \tilde{\omega}_1 - \tilde{\omega}_3. \quad (6.92)$$

$\alpha$  and  $\beta$  are constants which are composed of the real and imaginary parts

of the C's Eq. (6.91) is a first order differential equation which can be solved by a separation of variables. We then obtain  $\Psi$  as a function of  $t$  in the implicit form

$$t = \int_{\Psi_0}^{\Psi} \frac{d\Psi'}{\xi + \alpha \sin \Psi' + \beta \cos \Psi'}, \quad (6.93)$$

where  $\Psi_0$  is the initial value of  $\Psi$  at time  $t=0$ . It will be our goal to discuss the temporal behavior of  $\Psi(t)$  more closely. The time dependence of  $\Psi(t)$  depends critically on whether

$$(\alpha^2 + \beta^2) < \xi^2 \quad (6.94)$$

or

$$(\alpha^2 + \beta^2) > \xi^2 \quad (6.95)$$

holds. In the first case the integrand of (6.93) does not contain any singularity. We may expand the integrand into a power series of sine and cosine functions. We thus see that the integral behaves as **const.**  $\Psi$  + pulsations. Neglecting these pulsations and small corrections we immediately obtain

$$\Psi = \xi t + \text{const.},$$

where  $\xi$  is given by

$$\xi = 2\tilde{\omega}_2 - \tilde{\omega}_1 - \tilde{\omega}_3 \equiv 2\Omega_2 - \Omega_1 - \Omega_3. \quad (6.96)$$

This is precisely the behavior we expect for normal three-mode laser action.

Let us consider the other case (6.95). Then the integral can diverge. This means that time  $t$  on the l.h.s. of (6.93) tends to infinity while  $\Psi$  acquires a finite value, namely the value

$$\Psi = -\varphi - \arcsin(\xi(\alpha^2 + \beta^2)^{-1/2}), \quad \tan \varphi = \beta/\alpha. \quad (6.97)$$

In this case  $\Psi$  does no more depend on time. As we may see by means of (6.90), this result is only possible if the mode frequencies obey the equation

$$\Omega_2 - \Omega_1 = \Omega_3 - \Omega_2. \quad (6.98)$$

The frequency difference  $\Omega_2 - \Omega_1$  is now locked to the frequency difference  $\Omega_3 - \Omega_2$ . The transition from the unlocked to the locked state is experimentally demonstrated by measuring the beat frequencies of  $\Omega_2 - \Omega_1$  and  $\Omega_3 - \Omega_2$ . One first fixes a distance between the mirrors for which the frequency differences are different from each other (unlocked state). Here the condition (6.94) is fulfilled. When by tuning the resonator the central frequency is shifted closer to the atomic resonance  $\bar{\omega}$ , the distance  $\xi$  of the

frequency differences decreases. When  $|\xi|^2$  reaches the value  $\alpha^2 + \beta^2$  a quick transition into the frequency locked state occurs and instead of the two different frequency differences  $\Omega_1 - \Omega_2$  and  $\Omega_3 - \Omega_2$  only a single one occurs.

### 6.7. The laser gyro

A nice example for the application of lasers is provided by the laser gyro. Incidentally this example shows us how technical applications are linked with a profound understanding of fundamental physical phenomena. By means of the laser gyro it is possible to detect rotations with respect to the inertial system of the cosmos. Thus the laser gyro can replace mechanical gyros. Basically the laser gyro consists of a ring laser which we show in fig. 6.3. If the whole arrangement is rotating with respect to the inertial system of the universe, according to the general theory of relativity the following happens. Observers, in our case the photons, running in the direction of rotation proceed along a path whose length is different from that of observers (photons) going in opposite direction. According to the general theory of relativity, this change of length is proportional to the area  $A$  which is surrounded by the path and proportional to the rotation speed  $\Omega$  in the inertial space. In the case of photons, flying with the speed of light, the

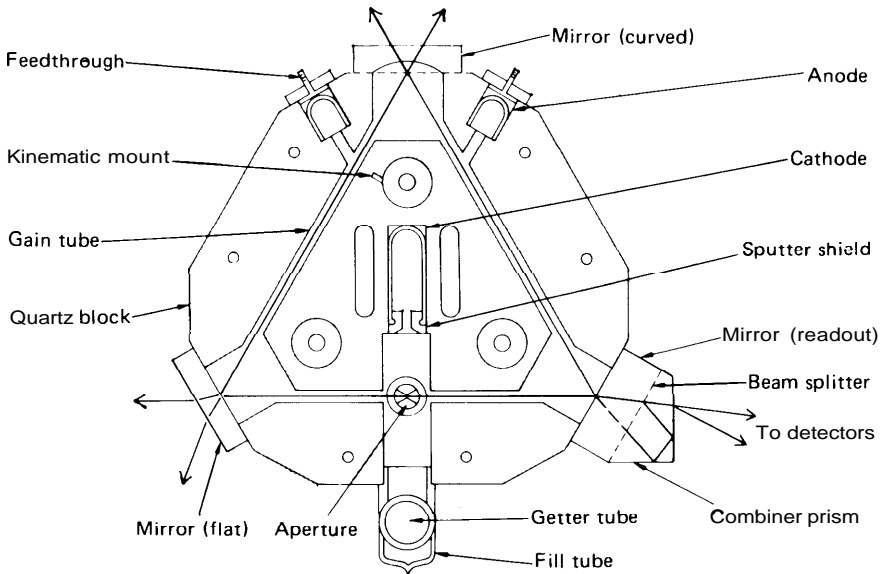


Fig. 6.3. Schematic arrangement of the laser gyro. [J. Killpatrick, IEEE Spectrum, Oct. 1967, p. 44.1

change of length  $\Delta L$  is given by  $4A\Omega/c$  where  $c$  is the light velocity. In 1913 this effect was shown to exist by Sagnac. For a rotation frequency of 2 Hz he could demonstrate changes of lengths of 100 to 200 Å. In order to measure very small changes of lengths, as they are to be expected for small rotation speeds, the laser offers a possibility. In this case the fact will be used that the laser frequency is determined by the length of the ring resonator. An apparent change of length of the resonator leads to a frequency shift. By measuring the frequency shift it becomes possible to measure the rotation speed of the gyro. An experimental arrangement is shown in fig. 6.3. Because the line-width of lasers is very small, a high sensitivity of the gyro can be expected.

For small rotation speeds a difficulty occurs, however, which rests on the following. Because the mirrors can reflect light in the direction of the incident beam, a coupling between waves running in opposite directions may occur and we have to encounter the phenomenon of frequency coupling. This mode coupling is again described by an equation of the form

$$\dot{\Psi} = a + b \sin \Psi. \tag{6.99}$$

$\Psi$  is the relative phase of the modes running in opposite directions,  $a$  is essentially the rotation speed  $\Omega$ , and  $b$  is the back scattering coefficient. As long as  $a > b$  no phase locking occurs. In the opposite case, however, phase locking happens and the rotation can no more be measured (compare fig. 6.4). There are a number of possibilities to do away this latter effect. For instance one may superimpose some trembling or chaotic motion on the system.

### 6.8. The gas laser. Single mode operation

The essential difference between a gas laser and a solid state laser consists in the motion of the gas atoms. Consequently, the coordinate of a single atom is now given by  $\mathbf{x}_\mu + \mathbf{v}_\mu t$ , where  $\mathbf{x}_\mu$  is the coordinate of the atom  $\mu$  at time  $t=0$  and  $\mathbf{v}_\mu$  its velocity. (Of course, due to collisions the atoms are "reshuffled" within their velocity distribution, but the ensemble of atoms remains unaffected by these events.) We shall allow for arbitrary angles,  $\Theta_{\mu\lambda}$ , between the vector of polarization of the light mode  $\lambda$  and the dipole moment of the  $\mu$ th atom. The interaction constant  $g_{\mu\lambda}$  between the mode  $\lambda$  and the atom  $\mu$  thus takes the form

$$g_{\mu\lambda} = -\frac{ig}{2} (\exp[i\mathbf{k}_\lambda \mathbf{x}_\mu + i\mathbf{k}_\lambda \mathbf{v}_\mu t] - \text{c.c.}) \cos \Theta_{\mu\lambda}. \tag{6.100}$$

The summation over  $\mu$  runs in the corresponding equations over the

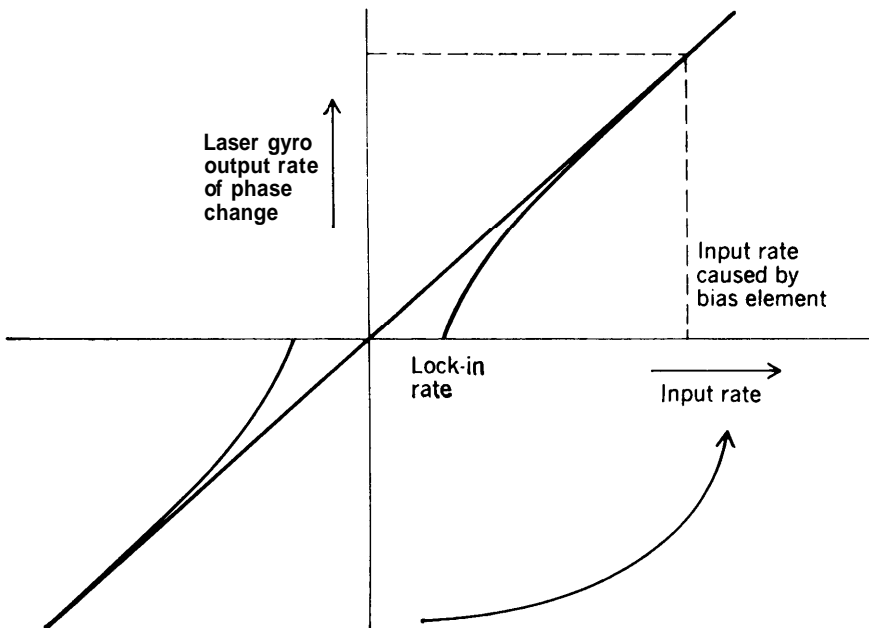


Fig. 6.4. Frequency locking in the laser gyro (compare text).

positions  $\mathbf{x}_\mu$ , the velocities  $\mathbf{v}_\mu$  (for which a Maxwellian distribution is assumed) and over all angles  $\Theta_{\mu\lambda}$ .

The equations of motion can now be taken directly from those referring to fixed atoms, (5.115)–(5.117), if the coupling "constant" (6.100) is used.  $\bar{\omega}_\mu$  is simply to be identified with the center frequency  $\bar{\omega}_0$  of an atom at rest. (The Doppler broadening is automatically taken care of by the explicit representation of the atomic motion in (6.100).)

In a region not too high above laser threshold we may eliminate the atomic variables basically by the same iteration procedure we described in section 6.4. But because of the time dependence of  $g_{\mu\lambda}$  in (6.100), one has to repeat it step by step. We leave the explicit performance of this procedure to the reader as an exercise and merely quote the results for the special case of single-mode operation. Readers interested in two- and multimode operation are referred to my book "Laser Theory".

Making the hypothesis

$$b^*(t) = B^* \exp[i\Omega t], \quad (6.101)$$

we obtain in the same approximation as in section 6.4

$$[\kappa + i(\omega - \Omega)]B^* \exp[i\Omega t] = i \sum_{\mu} g_{\mu}^*(t) \alpha_{\mu}^*(t), \quad (6.102)$$

where  $\alpha_{\mu}^*(t)$  has the following structure:

$$\begin{aligned} \alpha_{\mu}^*(t) = & B^* \exp[i\Omega t] \{ c_+ \exp[ikx_{\mu}(t)] + c_- \exp[-ikx_{\mu}(t)] \\ & + \bar{n} B^* \exp[i\Omega t] \{ d_+ \exp[ikx_{\mu}(t)] + d_- \exp[-ikx_{\mu}(t)] \\ & + f_+ \exp[3ikx_{\mu}(t)] + f_- \exp[-3ikx_{\mu}(t)] \\ & + j_+ \exp[ikx_{\mu}(t)] + j_- \exp[-ikx_{\mu}(t)] \}. \end{aligned} \quad (6.103)$$

$c_{\pm}$ ,  $d_{\pm}$ ,  $f_{\pm}$ ,  $j_{\pm}$  are complex constants, independent of space and time, which are given by

$$c_{\pm} = \pm \frac{id_0 g}{2} \cos \Theta \frac{1}{\Omega - \bar{\omega}_0 - i\gamma \pm \bar{\omega}_{\mu}}, \quad (6.104a)$$

$$\begin{aligned} d_{\pm} = & \pm \gamma T d_0 \frac{g}{2} |g|^2 \cos^3 \Theta \\ & \times \left( \frac{1}{(\Omega - \bar{\omega}_0 + \bar{\omega}_{\mu})^2 + \gamma^2} + \frac{1}{(\Omega - \bar{\omega}_0 - \bar{\omega}_{\mu})^2 + \gamma^2} \right) \frac{1}{i(\Omega - \bar{\omega}_0 - i\gamma \pm \bar{\omega}_{\mu})}, \end{aligned} \quad (6.104b)$$

$$\begin{aligned} f_{\pm} = & \mp \frac{d_0 |g|^2 g \cos^3 \Theta (\gamma \pm i\nu)}{2i(1/T \pm 2i\bar{\omega}_{\mu})} \\ & \times \frac{1}{[(\Omega - \bar{\omega}_0)^2 - (\bar{\omega}_{\mu} \mp i\gamma)^2][\Omega - \bar{\omega}_0 - i\gamma \pm 3\bar{\omega}_{\mu}]}, \end{aligned} \quad (6.104c)$$

$$j_{\pm} = \pm \frac{d_0 (y \pm i\nu) \cos^3 \Theta |g|^2 g}{2i(1/T \pm 2i\bar{\omega}_{\mu})} \frac{1}{[(R - \bar{\omega}_0)^2 - (\bar{\omega}_{\mu} \mp i\gamma)^2][\Omega - \bar{\omega}_0 - i\gamma \pm \bar{\omega}_{\mu}]}. \quad (6.104d)$$

$\bar{\omega}_0$  is the central frequency of the atomic transition,  $x_{\mu}(t) = x_{\mu} + v_{\mu} t$ , and  $\bar{\omega}_{\mu} = kv_{\mu}$ , where  $v_{\mu}$  is the velocity component of atom  $\mu$  in axial direction and  $k = (k, 0, 0)$ . When we multiply (6.103) by  $g_{\mu}^*$  (see (6.102)) and sum over the coordinates, the time dependence with  $\exp[i\bar{\omega}_{\mu} t]$  and  $\exp[i3\bar{\omega}_{\mu} t]$  drops out on account of the orthogonality properties of plane waves so that the r.h.s. has exactly the same time dependence as it is required by the hypothesis (6.101) for  $b^*$ . Performing the integrals over  $x$  and the average over  $\Theta$  explicitly, and assuming a symmetric velocity distribution  $w(\bar{\omega}_{\mu})$  as well as



a homogeneous spatial distribution of the active atoms, we obtain:

$$\begin{aligned} \kappa + i(\Omega - \omega) = & \frac{\rho A d_0}{6} \int \frac{\gamma + i(\bar{\omega} + \bar{\omega}_0 - \Omega)}{(\Omega - \bar{\omega} - \bar{\omega}_0)^2 + \gamma^2} \\ & \times \left( 1 - \frac{3\gamma T \bar{E} A}{5[(\Omega - \bar{\omega}_0 - \bar{\omega})^2 + \gamma^2]} - \frac{3\gamma T \bar{n} A}{5[(\Omega - \bar{\omega}_0 + \bar{\omega})^2 + \gamma^2]} \right) \omega(\bar{\omega}) d\bar{\omega} \\ & + \frac{\rho A^2 d_0 \bar{n}}{10} \int \frac{\nu + i\gamma}{(1/T - 2i\bar{\omega})(\Omega - \bar{\omega}_0 - \bar{\omega} - i\gamma)^2 (\Omega - \bar{\omega}_0 + \bar{\omega} + i\gamma)} \\ & \times \omega(\bar{\omega}) d\bar{\omega}. \end{aligned} \quad (6.105)$$

$\rho$  is the density of the atoms and  $A$  is defined in eq. (4.81). The terms containing  $A$  under the first integral arise from a static depletion of excited atomic states, while the second integral arises from the time dependent response of the atomic system. This can be seen most easily by following up the single steps of our iteration procedure.

Assuming a Gaussian velocity distribution and using constants typical for a He-Ne laser, we can show that the last integral in (6.105) is one order of magnitude smaller than the first one and therefore may be neglected. Keeping terms up to order  $\gamma/\alpha$  and splitting (6.105) into its real and imaginary part, we obtain an equation for the photon density and another one for the frequency shift (a is the half-width of the Gaussian).

(1) Equation for the photon density:

$$\kappa = \frac{\sqrt{\pi} \rho A d_0 e^{-\delta^2}}{6\alpha} \left[ 1 - \frac{3}{5} A \bar{n} T \left( \frac{1}{2\gamma} + \frac{\gamma}{2[\gamma^2 + (\Omega - \bar{\omega}_0)^2]} \right) \right], \quad (6.106)$$

from which we determine the photon density:

$$\bar{n} = \frac{\frac{10\gamma}{3AT} \left( 1 - \frac{3d_{\text{thr}}(\delta)}{d_0} \right)}{1 + \frac{\gamma^2}{\gamma^2 + (\Omega - \bar{\omega}_0)^2}} \quad (6.107)$$

If the photon density is plotted versus frequency it shows a dip. This dip is brought about by the fact that the atoms move in opposite axial directions. As a result, two holes are burnt into the inhomogeneously broadened line at two symmetric points of the line center. If we have fixed atoms instead, no such dip occurs. This dip which was theoretically predicted independently by **Haken** and Sauermann and Lamb, is called saturation dip or Lamb dip. It plays a fundamental role in (nonlinear) saturation spectroscopy.

(2) Equation for the frequency shift:

$$\Omega - \omega = -\frac{\rho A d_0}{3\alpha} e^{-\delta^2} \Phi(\delta) + \bar{n} C', \quad (6.108)$$

where  $A$ ,  $\delta$  and  $\Phi(\delta)$  are defined in eqs. (4.81), (4.88) and (6.37), respectively.  $C'$  is given by

$$C' = \frac{A^2 \rho d_0 T \sqrt{\pi}}{2\alpha} e^{-\delta^2} \left( \frac{1}{10} \frac{\Omega - \bar{\omega}_0}{\gamma^2 + (\Omega - \bar{\omega}_0)^2} + \frac{1}{5} \frac{\delta}{\alpha} \right). \quad (6.109)$$

We compare this result with the expression (6.37) for *fixed* atoms and a single direction of polarization. The first term on the r.h.s. in (6.108) agrees with the corresponding one in (6.37) except for a factor  $\frac{1}{3}$  which stems from the integration over the polarization (only  $\frac{1}{3}$  of the atoms participate in the laser process on the average). This term represents power dependent mode pulling and stems from the Doppler shape of the line.

With respect to the term proportional to  $\bar{n}$ , which comes from the time independent atomic inversion, we observe the following: The part of  $C'$  which stems from the second term in the large brackets on the r.h.s. of (6.109) agrees with the total expression  $C$  (6.37) except for a numerical factor which again stems from the integration over the polarization angles. This second term in  $C'$  is in general, i.e. for not too strong detuning ( $\delta \ll 1$ ), much smaller than the first term, which describes the frequency pushing being due to the existence of two holes burnt into the inversion (note, that a standing wave interacts with atoms whose frequencies are shifted both by  $+\kappa v$ , and  $-\kappa v$ , !). The frequency pushing becomes dominant if the mode is tuned to the center of the line within about a natural line-width. Inserting  $\bar{n}$  according to eq. (6.107) into (6.108) yields as a final result for the frequency shift:

$$\begin{aligned} \Omega = \omega - \kappa \frac{2}{\sqrt{\pi}} \int_0^\delta \exp[u^2] du + \kappa \left( \frac{d_0}{3d_{\text{thr}}(\delta)} - 1 \right) \\ \times \left( -\frac{2}{\sqrt{\pi}} \int_0^\delta \exp[u^2] du + \frac{\gamma(\Omega - \bar{\omega}_0)}{2\gamma^2 + (\Omega - \bar{\omega}_0)^2} \right). \end{aligned} \quad (6.110)$$

## 6.9. Derivation of the rate equations from the semiclassical laser equations

In chapter 4 we derived the rate equations completely heuristically and promised to derive them later on from more fundamental equations. We shall present this derivation here, where it will become clear which assumptions are to be made in order to obtain our previous rate equations. We

assume that the laser equations allow laser modes of the form

$$b_\lambda = B_\lambda \exp[-i\Omega_\lambda t], \quad (6.111)$$

where the amplitudes  $B_\lambda$  may still be slowly varying functions of time. We insert the expression (6.111) into the laser equations (5.116) for the atomic dipole moments and assume that the inversion  $d_\mu$  is time independent or varies only slowly in time. We then obtain quite similarly as in section 6.4 the expression

$$\alpha_\mu = \sum_\lambda \frac{i g_{\mu\lambda} d_\mu b_\lambda}{i(\bar{\omega}_\mu - \Omega_\lambda) + \gamma}. \quad (6.112)$$

We insert this expression immediately into the equations for the field modes (5.115) and multiply the resulting equations by  $b_\lambda^*$ . We now add to these equations their complex conjugates and introduce as usual the photon numbers  $n_\lambda$  by means of the relation

$$n_\lambda = b_\lambda^* b_\lambda. \quad (6.113)$$

The sum of the complex conjugate equations can be written in the following form:

$$\dot{n}_\lambda = -2\kappa_\lambda n_\lambda + \sum_{\mu \lambda'} \left( \frac{g_{\mu\lambda} g_{\mu\lambda'}^* d_\mu b_\lambda b_{\lambda'}^*}{-i(\Omega_{\lambda'} - \bar{\omega}_\mu) + \gamma} + \text{c.c.} \right). \quad (6.114)$$

Let us consider the expressions  $b_\lambda$ ,  $b_\lambda^*$  which occur on the r.h.s. of (6.114) more closely. If we have everywhere  $\lambda' = \lambda$ , we shall use on the r.h.s. the photon numbers (6.113) as in the usual rate equations. We now assume that the oscillations  $\lambda$  are not phase locked. Let us imagine that we perform a phase average on both sides of (6.114) and let us assume that the phases are uncorrelated. Then on the r.h.s. of (6.114) all expressions vanish for which  $\lambda' \neq \lambda$ . In this way the sum over  $\lambda'$  cancels and we may write (6.114) in the form

$$\dot{n}_\lambda = -2\kappa_\lambda n_\lambda + \sum_\mu W_{\lambda\mu} d_\mu n_\lambda. \quad (6.115)$$

Here we have introduced the quantity  $W_{\lambda\mu}$  which we may deduce from the comparison of (6.115) and (6.114) and which is given by

$$W_{\lambda\mu} = \frac{2\gamma |g_{\mu\lambda}|^2}{(\Omega_\lambda - \bar{\omega}_\mu)^2 + \gamma^2}. \quad (6.116)$$

But this is precisely the transition rate we introduced in chapter 4 in a heuristic manner. The only difference rests in the fact that the mode frequencies  $\Omega_\lambda$  are the actual frequencies of the laser modes and no more the mode

frequencies in the unloaded cavity as we had assumed previously in chapter 4. In order to obtain equations for the inversion we proceed in a similar fashion, namely we insert the expression (6.112) in the equations (5.117) of the inversion and again assume that a phase average has been performed. This leads us immediately to the equations

$$\dot{d}_\mu = \frac{d_0 - d_\mu}{T} - 2 \sum_\lambda W_{\lambda\mu} d_\mu n_\lambda. \quad (6.117)$$

We immediately recognize that (6.117) is identical with the previously introduced rate equations. These considerations tell us that we may derive our previous rate equations if we may neglect phase relations between the laser modes and if the changes of inversion and photon numbers are slow compared to the oscillations. This is an assumption which is practically always fulfilled because of the relatively high frequency of the atomic transition. These equations are valid for large photon numbers also, i.e. far above laser threshold. Insofar the equations we just derived go beyond those of sections (6.3) and (6.4) where we had to confine our considerations to a region not too far beyond laser threshold. On the other hand the rate equations rest on the assumption of vanishing phase and frequency correlations so that they do not allow us to treat a number of important phenomena.

## Chapter 7

# Ultrashort Pulses

### 7.1. Some basic mechanisms. Active and passive mode locking

In section 6.6 we saw that locking of laser modes may occur. We wish to study what occurs when many modes are locked together. For the beginning we shall ignore the detailed mechanism which leads to mode locking. Let us first recall how we have introduced the individual modes. When deriving the semiclassical laser equations we decomposed the electric field strength  $E$  into the amplitudes of individual standing waves. When we assume as usual that there is only one direction of polarization we may consider  $E$  as scalar. Thus the decomposition of  $E$  into modes is given by

$$E(t) = \sum_{\lambda} E_{\lambda}(t). \quad (7.1)$$

More precisely speaking  $E_{\lambda}$  is a function of space  $\mathbf{x}$  also,

$$E_{\lambda}(\mathbf{x}, t), \quad \text{for instance} \quad E_{\lambda} \propto \sin k_{\lambda} x. \quad (7.2)$$

For the moment being we are not so much interested in the spatial dependence of the complex amplitude of  $E_{\lambda}$  but primarily in its time dependence. For simplicity we shall assume that all amplitudes have the same modulus, so that we may write

$$E_{\lambda} \propto \exp[i\varphi_{\lambda}]. \quad (7.3)$$

We now consider two typical cases, namely:

- (1) The phases  $\varphi_{\lambda}$  are statistically distributed. As a consequence

$$\overline{E_{\lambda} E_{\lambda'}} = 0 \quad (\lambda \neq \lambda') \quad (7.4)$$

holds (compare also the exercises). We investigate the intensity which results from the average over the phases

$$I = \overline{|E(t)|^2}. \quad (7.5)$$

We may imagine that, for instance, the phases vary statistically in the course of time and that we average the intensity over a certain time.

Inserting (7.1) into (7.5) we obtain

$$\sum_{\lambda, \lambda'} \overline{E_{\lambda}^* E_{\lambda'}}, \tag{7.6}$$

which according to (7.4) may be reduced to

$$\sum_{\lambda} I_{\lambda}(t), \tag{7.7}$$

where we have used the abbreviation

$$I_{\lambda} = |E_{\lambda}|^2. \tag{7.8}$$

If the sum (7.1) comprises N modes having all the same intensity,

$$|E_{\lambda}|^2 = |E_0|^2. \tag{7.9}$$

We obtain as final result

$$I_{\text{uncorrelated}} = N|E_0|^2. \tag{7.10}$$

Thus the total intensity is equal to the intensity of the individual modes multiplied by N. Let us now turn to our second case.

(2) Correlated phases and frequencies. We shall assume that the frequencies of subsequent modes have the same distance from one another, so that we may write  $\varphi_{\lambda} = \omega_{\lambda} t$  where

$$\omega_{\lambda} = \omega + \lambda \omega', \quad \lambda = 0, \pm 1, \pm 2, \dots \tag{7.11}$$

(cf. fig. 7.1). Under the assumption that the spatial factor is constant, (7.1) acquires the form

$$E_{\text{PL}}(t) = \sum_{\lambda} E_0 \exp[i(\omega + \lambda \omega')t]. \tag{7.12}$$

The index "PL" means "phase locked". For simplicity we consider an odd

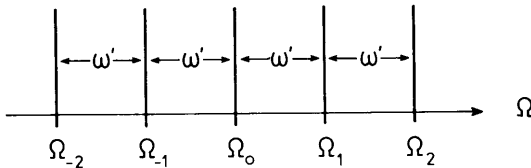


Fig. 7.1. Scheme of mode frequencies with the same distance.

$N$  and let the sum in (7.12) run over the following indices

$$\lambda = -\frac{N-1}{2}, \dots, \lambda = \frac{N-1}{2}. \quad (7.13)$$

The sum in (7.12) is of the geometric type and can be easily evaluated

$$E_{\text{PL}}(t) = E_0 \exp[i\omega t] \frac{\sin(\frac{1}{2}N\omega' t)}{\sin(\frac{1}{2}\omega' t)}. \quad (7.14)$$

When we perform these studies from the very beginning for running waves we have to replace (7.12) by the hypothesis

$$E_{\text{PL}}(t) = \sum_{\lambda} E_0 \exp[i\omega_{\lambda} t - i k_{\lambda} x], \quad k_{\lambda} = \omega_{\lambda}/c. \quad (7.15)$$

In this case the final result reads

$$E_{\text{PL}} = E_0 \exp[i\omega(t-x/c)] \frac{\sin(\frac{1}{2}N\omega'(t-x/c))}{\sin(\frac{1}{2}\omega'(t-x/c))}. \quad (7.16)$$

For the following discussion of (7.16) we choose  $x=0$  for simplicity. The ratio of the sine functions reaches its maximum at the time

$$t = m \frac{2\pi}{\omega'}, \quad m: \text{integer}. \quad (7.17)$$

The maximum intensity belonging to (7.16) is given by

$$I_{\text{PL}} = |E_0|^2 N^2 = N I_{\text{uncorrelated}}, \quad (7.18)$$

i.e. the maximum intensity is proportional to the square of the number of modes. In this way we have found our first important results. If mode coupling is possible, the emitted intensity can be multiplied by a large amount which is the bigger the bigger the number of modes locked to each other is. Fig. 7.2 represents the ratio of the sine functions in (7.16) as a function of  $t-x/c$ . The exponential function in front of the sine functions in (7.16) has been ignored because it represents the carrier wave which we are not interested in for the moment being but rather in its envelope. As we may see, the very intense pulse is followed by several very small ones. The width of the big pulse is given by

$$\tau = \frac{2\pi}{\omega' N}, \quad (7.19)$$

which can easily be deduced from (7.16).  $\omega'$  is the range of frequencies covering the individual mode frequencies (compare (7.11)–(7.13)). It thus

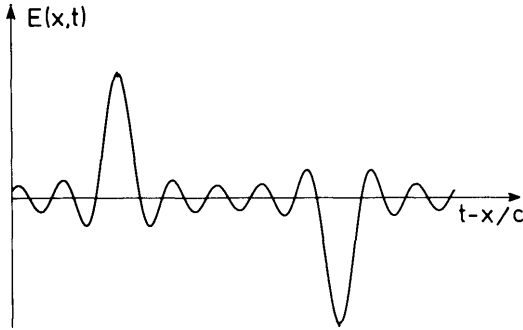


Fig. 7.2. The electric field strength  $E$  according to eq. (7.16) where the factor  $\exp[i\omega(t-x/c)]$  has been omitted.

represents the band width  $\Delta\omega$ ,

$$N_0' = \Delta\omega. \tag{7.20}$$

As may be seen from (7.19), the pulse width which can be reached is inversely proportional to the band width  $\Delta\omega$ . Our results can be summarized as follows. If modes can be locked, high maximum intensities and small pulse widths can be reached if a sufficiently large number of modes can be coupled together.

We now turn to the question how mode locking can be obtained. To this end we proceed in two steps. We first present some more or less qualitative ideas and in subsequent sections we shall present a detailed theory how to cope with "passive" mode locking.

Let us now turn to a qualitative discussion. To this end we must invent mechanisms by which the frequency  $\omega$  and phase  $\varphi$  of a laser mode  $E_0$  can be coupled to the frequency  $\omega_1$  and phase  $\varphi_1$  of a second laser mode  $E_1$ . According to fig. 7.3 we have in particular to bridge the frequency distance  $\omega'$ . This can be reached by modulating the fundamental wave  $E_0$  by a

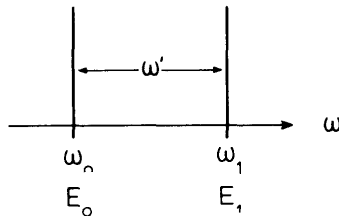


Fig. 7.3. Coupling of a laser mode  $E_0$  with another mode  $E_1$ .



frequency  $\omega'$  so that side bands result, especially with the frequency  $\omega_1 = \omega + \omega'$ . We may expect that side bands created in this manner resonate with mode  $E_1$  and influence in this way the field  $E_1$ . Such a modulation of  $E_0$  becomes possible by a modulation of the losses produced by the mirrors. To this end we make the following hypothesis for the loss rate:

$$\kappa = \kappa_0 + \kappa_1 \sin \omega' t. \quad (7.21)$$

As we know, in the laser equations terms of the form

$$E \cdot \kappa \quad (7.22)$$

occur. When we decompose  $E$  into its individual modes and pick up  $E_0$  as well as the frequency dependent part of  $\kappa$  (cf. 7.21)), we obtain the scheme

$$\begin{array}{ccc} E_0(t) \kappa_1 \sin \omega' t & & \\ \downarrow & \downarrow & \\ \omega & \omega' & \end{array} \quad (7.23)$$

$$\omega'_1 = \omega + \omega'.$$

In this way a new field mode with the frequency  $\omega'_1$  is generated. If there is a resonator mode just at this frequency  $\omega'_1$  it can be excited to a forced oscillation being in phase with  $E_0$ . In a similar way  $E_1$  can excite a further mode  $E_2$  in phase, etc.

Another way to produce mode coupling is provided by a saturable absorber (compare fig. 7.4). In this case a saturable absorber is inserted in-between one of the mirrors and the laser material. The action of such an absorber can be visualized as follows. Let us consider an ensemble of two-level atoms (cf. fig. 7.5). When we irradiate this system by light, two different cases may occur depending on whether the incident intensity is weak or strong. If the intensity is weak, only few atoms will be brought from their ground states to their excited states via absorption and they

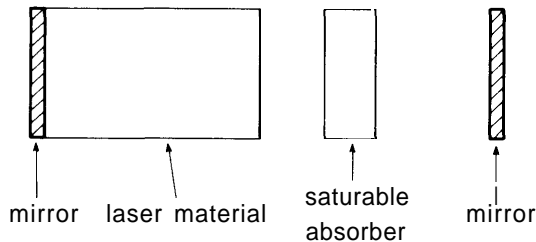


Fig. 7.4. Schematic set-up of a laser with saturable absorber for the generation of ultrashort pulses.



Fig. 7.5. Functioning of a saturable absorber. The wave coming from the left-hand side hits atoms each with two levels (symbolized by horizontal lines) and excites some of them.

recombine again to the ground state. In this way always many atoms are present in the ground state. When this process happens continuously, the incident light wave tracks will always find enough atoms in the ground state being able to absorb the light field.

Things are quite different in the case of a high light intensity, however. In this case so many atoms make transitions into the excited state that for the absorption process not enough atoms are available. In addition induced emission may set in reinforcing the incident light field. Our considerations show that the kind of absorption of the atomic system of the saturable absorber changes when we proceed from low to high light intensities. The precise behavior of the loss rate  $\kappa(I)$ ,  $I = |E|^2$ , may easily be deduced from the laser theory of two-level atoms. To this end we have only to study how the occupation number difference depends on the incident light intensity  $I$ . One easily finds

$$\kappa(I) = \frac{\kappa_0}{1 + \beta I}. \tag{7.24}$$

Thus the curve reproduced in fig. 7.6 results, according to which the loss

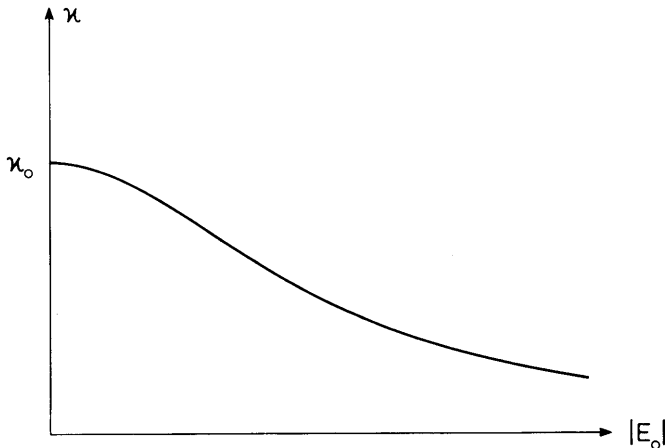


Fig. 7.6. The loss rate  $\kappa$  versus the modulus of the field strength  $E_0$  according to eq. (7.24).

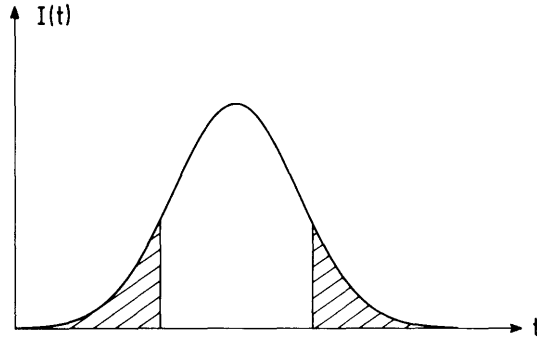


Fig. 7.7. Visualization of the action of a saturable absorber. The tails of the pulse with too low an intensity are cut away.

rate decreases beyond a critical field strength  $\beta^{-1/2}$ . Such saturable absorbers can be realized by organic dyes.

We now consider what happens in the laser of fig. 7.4 due to the saturable absorber. We imagine that by a fluctuation of the light intensity a wave track with a spatially inhomogeneously high intensity has resulted. If such a wave track hits the saturable absorber, those parts of it whose intensity is not high enough will be absorbed, but the other parts having sufficiently high intensity will be let through. In this way the wings of the laser pulses are again and again cut away (fig. 7.7), and the light pulse becomes shorter and higher. In order to recognize how this picture can be connected with loss modulation we consider more closely the pulse running back and forth in the laser. Denoting by  $c'$  the effective speed of light in the total experimental setup and by  $L$  the distance between the mirrors we may write the pulse repetition time as

$$t_1 = 2L/c'. \quad (7.25)$$

The modulation frequency for cutting away the wings of the pulse is thus given by

$$\omega_M = 2\pi/t_1. \quad (7.26)$$

Now let us assume that we decompose the pulse into its individual stationary laser modes with frequencies  $\omega_\lambda$ . Then the interaction of the modes within the saturable absorber leads to new side bands which differ from the fundamental wave frequency  $\omega_\lambda$  just by a multiple of (7.26). On the other hand, the frequency distance of axial modes in the laser is given by (7.26) with (7.25) when we recall the condition that half integers of the laser

wave-length must fit in-between the two mirrors. Thus the sidebands resonate with neighboring modes so that a very efficient frequency locking becomes possible (this resonance is opposed, however, by the different frequency shifts caused by laser action, see also below).

The third example of mode locking is known to us from the special case treated in section 6.6. There mode locking was caused by the nonlinear polarization. The individual contributions of the polarization occur as nonlinear terms in the equations (6.69) and have quite generally the form

$$P \propto E_i E_j E_k^* \tag{7.27}$$

By means of an example we again want to convince ourselves how mode locking is achieved (cf. fig. 7.8). Let us consider the expression (7.27) as a force exciting a mode. Then we obtain the following relation:

$$\begin{aligned}
 & E_0(t) \times E_{-1}^*(t) E_0(t) \\
 & \quad \downarrow \\
 & \omega_1 \approx \omega + \underbrace{(\omega_{-1} - \omega)}_{\omega'} \\
 & \quad \downarrow \\
 & E_1
 \end{aligned}
 \tag{7.28}$$

In practical cases not only three modes as in our case interact but a whole set of them. This leads to the possibility of self-pulsing lasers where mode locking results if they are pumped sufficiently highly.

The three cases, namely external loss modulation, saturable absorber, and gain modulation via nonlinear polarization can be represented in a unique way. To this end we consider two modes with the amplitudes  $b_0$  and  $b_1$ , where the mode  $b_0$  is considered as the fundamental mode to which mode 1 is coupled. The equations of mode 1 can be written in the form

$$\dot{b}_1 = (-i\omega_1 - \kappa_0)b_1 + G_1 b_0 + Z \tag{7.29}$$

In it  $\omega_1$  is the frequency of mode 1 and  $\kappa_0$  its loss (without loss modulation).  $G_1$  describes the saturated gain. The additional term  $Z$  describes the mode

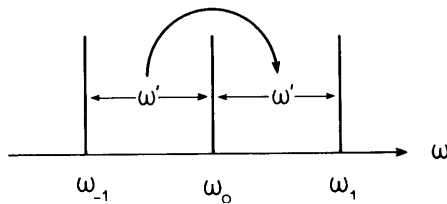


Fig. 7.8. How a coupling between the modes  $\omega_{-1}$ ,  $\omega_0$  and  $\omega_1$  is brought about.

locking effects and depends on the mechanisms under consideration. According to the different mechanisms we have the following expressions:

(a) loss modulation (active mode locking)

$$Z = \kappa_1 \sin \omega' t b_0; \quad (7.30)$$

(b) nonlinear polarization (passive mode locking)

$$Z = \alpha (b_{-1}^* b_0) b_0; \quad (7.31)$$

(c) internal modulation (saturable absorber) (passive mode locking)

$$Z = \left( \frac{\kappa_0}{1 + \beta I} - \kappa_0 \right) b_1 \approx -\kappa_1 I b_1. \quad (7.32)$$

The intensity  $I$  of all modes is taken in the form

$$I = \left| \sum b_\lambda \mathcal{N}_\lambda \exp[ik_\lambda x] \right|^2. \quad (7.33)$$

### Exercise on section 7.1

Prove (7.4).

Hint: The phase average  $\overline{E_\lambda E_{\lambda'}}$  is defined by

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} d\varphi \int_0^{2\pi'} d\varphi' E_\lambda(\varphi) E_{\lambda'}(\varphi').$$

### 7.2. The basic equations of self-pulsing lasers

In this chapter we continue the theme of the preceding section, where we discussed how the mechanisms of phase locking can produce ultrashort laser pulses. We shall focus our attention on a laser which becomes spontaneously self-pulsing, i.e. without an external modulation, e.g. of its mirrors. Ultrashort pulses produced by a ring laser with a homogeneously broadened atomic line were first predicted theoretically. Self-pulsing of laser has indeed been found experimentally, but with an inhomogeneously broadened atomic line. In order to elucidate the basic mechanisms and theoretical treatment we shall present an analytical treatment of the onset of ultrashort laser pulses caused by laser atoms with a homogeneously broadened line. We shall then point out recent developments.

The main results of the following analysis are presented in section 7.6 and the reader not interested in mathematical details can directly proceed to that section. We base our analysis on the equations (5.102), (5.103) and (5.105) for the slowly varying amplitudes  $\mathbf{E}_0^{(+)}$  and  $\mathbf{P}_0^{(+)}$  of the electric field

strength  $E$  and polarization  $P$ , respectively, describing modulated running waves in a ring cavity, and for the inversion density  $D$ . We assume that the pumping is so strong that at least the first laser threshold is reached. The corresponding cw-solution is denoted by  $E_{0,cw}^{(+)}$ ,  $P_{0,cw}^{(+)}$  and  $D_{cw}$ . For simplicity we assume that  $E$ ,  $P$  and  $\mathfrak{D}_{21}$  are polarized in one direction, perpendicular to the direction of propagation along the  $x$ -axis.

It has turned out that some writing within formulas can be saved if we use normalized quantities, i.e.

$$\hat{E} = E_0^{(+)}(x, t) / E_{0,cw}^{(+)} \quad (7.34)$$

$$\hat{P} = P_0^{(+)}(x, t) / P_{0,cw}^{(+)} \quad (7.35)$$

$$\hat{D} = D / D_{cw}, \quad (7.36)$$

and the new pump parameter

$$\Lambda = (D_0 - D_{cw}) / D_{cw}. \quad (7.37)$$

$D_{cw}$  is the inversion at threshold. We further put  $1/T = y$ . A little algebra (cf. exercise) transforms (5.102), (5.103) and (5.105) into the following set of equations:

$$\left\{ \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} + \kappa \right\} \hat{E} = \kappa \hat{P}, \quad (7.38)$$

$$\left\{ \frac{\partial}{\partial t} + \gamma \right\} \hat{P} = \gamma \hat{E} \hat{D}, \quad (7.39)$$

$$\left\{ \frac{\partial}{\partial t} + \gamma_{\parallel} \right\} \hat{D} = \gamma_{\parallel} (1 + \Lambda) - \gamma_{\parallel} \Lambda (\hat{E}^* \hat{P} + \hat{E} \hat{P}^*) / 2. \quad (7.40)$$

The cw-solution reads, of course,

$$\hat{E} = \hat{P} = \hat{D} = 1. \quad (7.41)$$

Because the method by which we obtain pulse solutions of these equations is of a more general interest also with respect to other applications in laser physics, we first give an outline of the general method. Then in the next step we shall apply this method to the concrete case of a ring laser with a homogeneously broadened atomic line.

### Exercise on section 7.2

Solve eqs. (5.102), (5.103) and (5.105) with  $\partial E_0^{(+)} / \partial t = \partial P_0^{(+)} / \partial t = \partial D / \partial t = 0$ . Transform (5.102), (5.103) and (5.105) into (7.38)–(7.40) by use of (7.34)–(7.37) using the explicit expressions for  $E_{0,cw}^{(+)}$ , etc. just determined.

### 7.3. A general method for calculating evolving patterns close to instability points

Let us consider a system which we describe by a space- and time-dependent state vector  $\mathbf{U}(x, t)$ . For simplicity we treat the one-dimensional case. Since in the following we shall have in mind to solve the eqs. (7.38)–(7.40), an explicit example for such a state space vector is provided by

$$\mathbf{U}(x, t) = \begin{pmatrix} \hat{E}(x, t) \\ \hat{P}(x, t) \\ \hat{D}(x, t) \end{pmatrix}. \quad (7.42)$$

For sake of simplicity  $\hat{E}$  and  $\hat{P}$  are treated as scalars but the method can easily cope with the case in which they are vectors. We shall assume that  $\mathbf{U}$  obeys a set of differential equations which we write in the general form

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{G}(\mathbf{U}, \partial_x, A). \quad (7.43)$$

$\mathbf{G}$  is a nonlinear function of  $\mathbf{U}$  which may also contain derivatives of  $\mathbf{U}$  with respect to space.  $A$  is a parameter which may be controlled from the outside, e.g. by the influx of energy. An explicit example for the set of differential equations (7.43) is provided by (7.38)–(7.40). We further use the definition

$$\partial_x = \partial / \partial x. \quad (7.44)$$

We now consider the following situation. We assume that for a given  $A$  we have found a time- and space-independent solution of (7.43). We call this solution  $\mathbf{U}_0$ . Again the system (7.38)–(7.40) provides us with an example because the corresponding solution reads

$$\mathbf{U}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (7.45)$$

The method we are going to present here can be extended to the case where  $\mathbf{U}_0$  is space- and time-dependent and  $x$  is a three-dimensional vector. We have to refer the reader for that case to the book *Advanced Synergetics* (see references). As we have seen in a number of instances of the present book, a solution can become unstable if a control parameter  $A$  (e.g. the pump power) is changed. In order to check the stability we make the hypothesis (cf. section 6.1 for an explicit example)

$$\mathbf{U}(x, t) = \mathbf{U}_0 + \mathbf{q}(x, t). \quad (7.46)$$

Inserting (7.46) into (7.43) we obtain an equation for  $\mathbf{q}$  of the form

$$\frac{\partial \mathbf{q}(\mathbf{x}, t)}{\partial t} = \mathbf{K}(\Lambda, \partial_x) \mathbf{q}(\mathbf{x}, t) + \mathbf{N}(\Lambda, \mathbf{q}), \quad (7.47)$$

where we have split the resulting r.h.s. into a linear term,  $\mathbf{Kq}$ , and a nonlinear term,  $\mathbf{N}$ . If in the original equation  $\mathbf{G}$  contains powers of  $\mathbf{U}$  just up to second order,  $\mathbf{N}$  is also just of second order. We write the individual components of  $\mathbf{N}$  in the form

$$N_l(\Lambda, \mathbf{q}) = \sum_{\mu, \sigma} g_{l\mu\sigma} q_\mu q_\sigma. \quad (7.48)$$

We first wish to study the stability of  $U_0$ . To this end we consider the linearized part of eq. (7.47). Denoting the solutions of the linearization by  $\mathbf{w}$ , we then have to study equations of the form

$$\frac{\partial \mathbf{w}(\mathbf{x}, t)}{\partial t} = \mathbf{K}(\Lambda, \partial_x) \mathbf{w}(\mathbf{x}, t). \quad (7.49)$$

It may be shown quite generally that this equation can be solved by the hypothesis

$$\mathbf{w}(\mathbf{x}, t) = \exp[\beta t] \mathbf{v}(\mathbf{x}). \quad (7.50)$$

In the following we shall confine our analysis to a problem with periodic boundary conditions, so that

$$\mathbf{w}(\mathbf{x} + \mathbf{L}, t) = \mathbf{w}(\mathbf{x}, t) \quad (7.51)$$

must hold. Then  $\mathbf{v}(\mathbf{x})$  can be chosen in the form

$$\mathbf{v}(\mathbf{x}) = \mathbf{O}(1/\sqrt{L}) \exp[i\mathbf{k}\mathbf{x}], \quad (7.52)$$

where  $\mathbf{k}$  must be chosen such that

$$\exp[i\mathbf{k}\mathbf{L}] = 1. \quad (7.53)$$

The factor  $1/\sqrt{L}$  serves for the normalization of this function over the periodicity length  $L$ , while  $\mathbf{O}$  is a constant vector still to be determined. Inserting (7.50) with (7.52) into (7.49) we may readily perform the differentiations indicated by  $\partial_x$ . Because of

$$\partial_x \exp[i\mathbf{k}\mathbf{x}] = i\mathbf{k} \exp[i\mathbf{k}\mathbf{x}], \quad (7.54)$$

we obtain

$$\mathbf{K}(\Lambda, \partial_x) \exp[i\mathbf{k}\mathbf{x}] = \mathbf{K}(\Lambda, i\mathbf{k}) \exp[i\mathbf{k}\mathbf{x}]. \quad (7.55)$$

Dividing the equation, which results from (7.49), by the exponential



functions  $\exp[\beta t]$  and  $\exp[ikx]$ , we obtain a set of ordinary algebraic equations where  $\mathbf{K}$  is a constant matrix and  $\beta$  plays the role of an eigenvalue. In order to find nontrivial solutions of

$$\beta \mathbf{O} = \mathbf{K}(\Lambda, ik) \mathbf{O}, \tag{7.56}$$

the eigenvalues  $\beta$  must be chosen properly. The matrix  $\mathbf{K}$  is parametrized by  $k$  so that the eigenvalues depend on  $k$ . A set of algebraic equations (7.56) possesses a set of eigenvalues  $\beta_j, j=1, \dots, m$ . We therefore denote an eigenvalue  $\beta$  more precisely by  $\beta_j(k)$ . Also  $\mathbf{O}$  depends on the same indices so that we write

$$\beta = \beta_j(k), \quad \mathbf{O} = \mathbf{O}^j(k). \tag{7.57}$$

Using these indices,  $k$  and  $j$ , the solution (7.50) is written in the form

$$w(x, t) = \exp[\beta_j(k)t] v^{kj}(x). \tag{7.58}$$

We further shall assume that also left-hand eigenvectors of (7.56) exist, i.e. that

$$\bar{\mathbf{O}}^j(k) \mathbf{K} = \beta_j(k) \bar{\mathbf{O}}^j(k) \tag{7.59}$$

holds. We assume that the l.h. eigenvectors and the r.h. eigenvectors form an orthonormal set, i.e. that

$$\bar{\mathbf{O}}^j(k) \mathbf{O}^l(k) \equiv \sum_{\nu} \bar{\mathbf{O}}^j_{\nu}(k) \mathbf{O}^l_{\nu}(k) = \delta_{jl}. \tag{7.60}$$

In order to solve the fully nonlinear equations (7.47), we expand the wanted solution  $q$  into a superposition of the complete set of eigenvectors in  $x$ -space spanned by (7.58), i.e. we write

$$q(x, t) = \sum_{k,j} \xi_{k,j}(t) v^{kj}(x), \tag{7.61}$$

where the amplitudes  $\xi(t)$  are still unknown quantities. Inserting (7.61) into (7.47) we readily obtain for the l.h.s. of (7.47)

$$\sum_{k,j} \dot{\xi}_{k,j}(t) v_i^{kj}(x). \tag{7.62}$$

For the linear term on the r.h.s. of (7.47) we obtain

$$\sum_{k,j} \xi_{k,j} \sum_{l'} \mathbf{K}_{ll'}(\Lambda, ik) v_l^{kj}(x), \tag{7.63}$$

which because of (7.56), (7.57) can be transformed into

$$\sum_{kj} \beta_j(k) \xi_{k,j} v_l^{kj}(x). \tag{7.64}$$

The nonlinear terms (7.48) are transformed into

$$\sum_{\mu, \sigma} g_{l\mu\sigma} \sum_{k', j'} \xi_{k', j'} v_{\mu}^{k', j'}(x) \sum_{k'', j''} v_{\sigma}^{k'', j''}(x). \quad (7.65)$$

Equating (7.62) to the sum of (7.64) and (7.65) we obtain equations for the unknowns  $\xi_{k, j}(t)$ . To cast these equations into a more convenient form we multiply them from the left by

$$\bar{O}_l^j(k) \exp[-ikx]. \quad (7.66)$$

Then we sum up over the indices  $l$  and integrate over space from  $x = 0$  till  $x = L$ ,

$$\sum_l \int_0^L \dots dx. \quad (7.67)$$

Because the exponential functions are orthonormal, i.e.

$$\int_0^L \exp[-ikx + ik'x] dx = \delta_{kk'} = \begin{cases} 1 & \text{for } k = k', \\ 0 & \text{for } k \neq k', \end{cases} \quad (7.68)$$

and we may apply (7.60), we readily obtain

$$\dot{\xi}_{k, j} = \beta_j(k) \xi_{k, j} + \sum_{\substack{k', k'' \\ j', j''}} \xi_{k', j'} \xi_{k'', j''} a_{kk'k'', jj'j''} J_{kk'k''}, \quad (7.69)$$

where we have used the abbreviations

$$a_{kk'k'', jj'j''} = \sum_{l\mu\sigma} g_{l\mu\sigma} \bar{O}_l^j(k) O_{\mu}^{j'}(k') O_{\sigma}^{j''}(k''), \quad (7.70)$$

$$J_{kk'k''} = L^{-3/2} \int_0^L \exp[i(-k + k' + k'')x] dx = \delta_{k, k' + k''}. \quad (7.71)$$

In general a transformation from one set of variables to another one does not solve a problem. However, in the present case a considerable simplification can be achieved. To this end we have to distinguish between the unstable and stable modes. We shall call a mode,  $j, k$ , stable if its eigenvalue  $\beta_j(k)$  has a negative real part. In such a case a small perturbation will be damped out and the original state is reached again, for instance in the laser case the solution

$$U_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}.$$

Thus a new state, for instance a pulsation can take place only if at least

one of the eigenvalues becomes positive. A mode,  $j, k$ , is called unstable if the real part of  $\beta_j(k)$  is positive (or vanishes).

As we shall see below, in the case of a ring laser,  $\beta_j(k)$  may indeed acquire a positive real part provided the pump is high enough. This instability point is reached at a specific  $k$  wave number (see below fig. 7.9). We shall call the wave number  $k$ , where this instability is reached first,  $k_c$ . We label the  $\beta$ 's in such a way that this instability occurs for  $j=1$ . We shall denote the pair ( $k=k_c, j=1$ ) by  $u$  indicating "unstable". When we further assume that all other  $\beta$ 's have negative real parts, all the other modes are still stable and we shall use the following abbreviations:

$$(k = k_c, j = 2, 3) \rightarrow s, \quad (k \neq k_c, j = 1, 2, 3, \dots) \rightarrow s.$$

For our general outline we need not restrict the analysis to a single unstable mode but we may admit a set of them distinguishing them by the index  $u$ . With the new notation of  $u$  and  $s$  we may rewrite the set of equations (7.69) in a new self-explanatory fashion

$$\begin{aligned} d\xi_u/dt = \beta_u \xi_u + \sum_{u_1, u_2} C_{uu_1 u_2} \xi_{u_1} \xi_{u_2} \\ + \sum_{s, u} C_{uu_1 s} \xi_{u_1} \xi_s + \sum_{s, s_1} C_{uss_1} \xi_s \xi_{s_1}, \end{aligned} \quad (7.72)$$

where  $a_{kk'k'', jj'j''} J_{kk'k''}$  is written as  $C_{uu_1 u_2}$ ,  $C_{uu_1 s}$  or  $C_{uss_1}$  depending on the index combinations  $k', j'$ , etc. Thus the unstable modes are coupled to the stable modes. In many practical applications the terms  $C_{uu_1 u_2}$  vanish on account of selection rules. For sake of simplicity we shall drop the corresponding sum in (7.72). For the general case consult the references.

The stable modes in turn obey equations of the form

$$\begin{aligned} d\xi_s/dt = \beta_s \xi_s + \sum_{u, u_1} C_{suu_1} \xi_u \xi_{u_1} \\ + \sum_{u, s_1} C_{sus_1} \xi_u \xi_{s_1} \\ + \sum_{s_1, s_2} C_{ss_1 s_2} \xi_{s_1} \xi_{s_2}, \end{aligned} \quad (7.73)$$

where again the  $C$ 's stand for  $a \dots J \dots$  corresponding to the various combinations of indices  $k', j', \dots$ . An important aspect should be mentioned here. Note that "unstable" and "stable" refer to the linear stability analysis. Here we deal with a nonlinear analysis, however, and we shall see that the "unstable" modes become stabilized because of their coupling to the "stable" modes which in turn are coupled to the "unstable" modes. In the following we shall indicate a method by which we can eliminate the

"stable" modes so that we arrive at equations for the "unstable" modes alone. In a way this procedure is similar to the one we explained in sections 6.3 and 6.4, where we eliminated the atomic variables and were led to equations which contain the field modes alone. The main difference between the present procedure and the former one consists in the fact that in the former case the damping constants  $\kappa$ ,  $\gamma$ ,  $1/T$  were directly evident, whereas here the damping constants are obtained as eigenvalues  $\beta$  of the linearized problem. The mode amplitudes  $\xi_u$  will be called *order parameters*. As one may show, the order parameters may serve as a smallness parameter, if we are not too far above the instability point. One may convince oneself rather quickly that the stable mode amplitudes are of at least one order smaller than  $\xi_u$ . This then suggests the following iteration procedure. We wish to express the  $\xi_s$  by  $\xi_u$ . In lowest approximation eqs. (7.73) reduce to

$$\{d/dt - \beta_s\} \xi_s^0 = \sum_{u, u_1} C_{suu_1} \xi_u \xi_{u_1}. \quad (7.74)$$

The formal solution of (7.74) is given by

$$\xi_s^0 = \sum_{u, u_1} \{d/dt - \beta_s\}^{-1} C_{suu_1} \xi_u \xi_{u_1}. \quad (7.75)$$

In order to evaluate the inverse operator in brackets we decompose the order parameters according to

$$\xi_u = R(t) \exp(i\omega t), \quad (7.76)$$

where we anticipate that the order parameters oscillate at a frequency which is approximately given by

$$\omega_u = \text{Im}(\beta_u). \quad (7.77)$$

We assume that close to the transition point,  $R(t)$  can be considered as constant compared to the oscillatory exponential functions. This allows us to replace the operator  $d/dt$  in (7.75) by

$$d/dt \rightarrow \text{Im}(\beta_u + \beta_{u_1}). \quad (7.78)$$

In this way we may replace (7.75) by

$$\xi_s^0 = \sum_{u, u_1} [i(\omega_u + \omega_{u_1}) - \beta_s]^{-1} C_{suu_1} \xi_u \xi_{u_1}. \quad (7.79)$$

We shall call this procedure "adiabatic approximation". In our present treatment we use the "unrenormalized" frequencies  $\omega$  according to (7.77). The procedure can be extended, however, to the case in which frequency shifts are taken into account (cf. Advanced Synergetics, quoted in the references). We mention that the form (7.75) with (7.78) implies that

transient processes which occur on time scales shorter than  $1/\omega_u$  are not covered by this procedure. When we insert the expression (7.79) into the r.h.s. of (7.72), we obtain terms of third order in  $\xi_u$ ,

$$\dot{\xi}_u = \beta_u \xi_u - b \xi_u^3. \quad (7.80)$$

In many cases of practical interest, e.g. at the first laser threshold, were "usual" laser action sets in, the sign of the cubic term is negative so that a stabilization of the total state is achieved. Because it was found that this need not always be the case when ultrashort pulses occur, one has to go two orders of magnitude further and we briefly describe the corresponding steps. In lowest approximation we treated eq. (7.74) whose solution is given in the adiabatic approximation by (7.79).

In the next step of our approximation we insert this solution in eq. (7.73) which in first order then reads

$$[d/dt - \beta_s] \xi_s^{(1)} = \sum_{u, u_1} C_{suu_1} \xi_u \xi_{u_1} + \sum_{u, s_1} C_{sus_1} \xi_u \xi_{s_1}^{(0)}. \quad (7.81)$$

In the adiabatic approximation the solution of (7.81) is given by

$$\xi_s^{(1)} = \xi_s^{(0)} + \sum_{uu_1u_2, s_1} \Gamma_{uu_1u_2}^{ss_1} \xi_u \xi_{u_1} \xi_{u_2}. \quad (7.82)$$

The constants  $\Gamma$  are defined by

$$\Gamma_{uu_1u_2}^{ss_1} = [\omega_{uu_1u_2} - \beta_s]^{-1} C_{sus_1} [\omega_{u_1u_2} - \beta_{s_1}]^{-1} C_{s_1u_1u_2}, \quad (7.83)$$

where we use the general abbreviation

$$\omega_{uu_1u_2\dots u_n} = \text{Im}(\beta_u + \beta_{u_1} + \beta_{u_2} + \dots + \beta_{u_n}). \quad (7.84)$$

Inserting (7.82) into (7.72) we obtain a closed set of equations for the order parameters  $\xi_u$  alone. Because the explicit result becomes somewhat lengthy, we leave this step as an exercise to the reader. In the case that only one complex order parameter is present the (typically) resulting equation reads

$$[d/dt - \beta_u] \xi_u = B |\xi_u|^2 \xi_u + C |\xi_u|^4 \xi_u. \quad (7.85)$$

The reader may easily convince himself that the solution of this nonlinear equation can be discussed in terms of a potential in analogy to our discussion in section 6.3. We shall come back to this question explicitly when dealing with the laser.

### 7.4. Onset of ultrashort laser pulses: linear stability analysis

Our starting point is the equations (7.38)–(7.40). In order to simplify these equations further, we introduce a new scaling of space and time:

$$(x, t) = (\hat{x}, \hat{t})/\gamma, \quad \hat{\gamma} = \gamma_{\parallel}/\gamma, \quad \kappa/\gamma = \hat{\kappa}.$$

Dropping the "hat" ^ everywhere, the equations to be studied read

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} + \kappa\right)E = \kappa P, \tag{7.86}$$

$$\left(\frac{\partial}{\partial t} + \gamma\right)D = \gamma(1 + \Lambda) - \gamma\Lambda EP, \tag{7.87}$$

$$\left(\frac{\partial}{\partial t} + 1\right)P = ED. \tag{7.88}$$

We remind the reader that these equations refer to normalized quantities and that we assume that the pump strength is at least so high that normal laser action can occur. The cw-solution is given by

$$E = P = D = 1. \tag{7.89}$$

In the following we shall study those pulse solutions in which the phases of  $E$  and  $P$  are fixed so that we may use  $E$  and  $P$  as real quantities. According to the previous section we first have to study stability. To this end we make the hypothesis

$$(E, D, P) = (1 + e, 1 + d, 1 + p), \tag{7.90}$$

where the vector  $\begin{pmatrix} e \\ d \\ p \end{pmatrix}$  plays the role of  $q$ . The coefficients of the nonlinearity

(cf. (7.48)) then read

$$g_{\nu\mu\sigma} = -\gamma\Lambda\delta_{\nu 2}(\delta_{\mu 1}\delta_{\sigma 3} + \delta_{\mu 3}\delta_{\sigma 1})/2 + \delta_{\nu 3}(\delta_{\mu 1}\delta_{\sigma 2} + \delta_{\mu 1}\delta_{\sigma 2})/2. \tag{7.91}$$

Furthermore we have

$$K(\Lambda, \partial_x) = \begin{pmatrix} c\partial/\partial x + \kappa & 0 & -\kappa \\ \gamma\Lambda & \gamma & \gamma\Lambda \\ -1 & -1 & 1 \end{pmatrix}. \tag{7.92}$$

The hypothesis

$$(e, d, p) \sim \exp[\beta t + ikx/c] \tag{7.93}$$

transforms the differential equation of the linearized problem into an algebraic equation whose determinant must vanish,

$$\begin{vmatrix} \beta + ik + \kappa & 0 & -\kappa \\ \gamma A & \beta + \gamma & \gamma A \\ -1 & -1 & \beta + 1 \end{vmatrix} = 0. \quad (7.94)$$

The characteristic equation belonging to (7.94) reads

$$\begin{aligned} \beta^3 + \beta^2(1 + \gamma + \kappa + ik) + \beta[ik(1 + \gamma) + \gamma(1 + A + \kappa)] \\ + ik\gamma(1 + A) + 2\kappa\gamma A = 0. \end{aligned} \quad (7.95)$$

For fixed  $k$  this equation has in general three different solutions  $\beta$  which can be distinguished by the index  $j=1, 2, 3$ . A detailed discussion of the solutions of (7.95) is, of course, rather boring. Therefore we present only the essential results. If the cavity losses are small,

$$\kappa < 1 + \gamma, \quad (7.96)$$

an instability occurs if the pump "power"  $A$  exceeds  $A_c$ . The critical value,  $A_c$ , is given by

$$A_c = 4 + 3\gamma + 2[2(1 + \gamma)(2 + \gamma)]^{1/2}. \quad (7.97)$$

Then a range of wave number vectors  $k$  exists in which one of the three eigenvalues  $\beta$  acquires a positive real part. We shall define the critical point as that value of  $k$  for which the first eigenvalue touches the imaginary axis. The corresponding critical eigenvalue reads

$$\beta_c = i[\gamma(3A_c - \gamma)/2]^{1/2}. \quad (7.98)$$

A plot of the different eigenvalues in the complex plane is given by fig. 7.9. The individual points on each curve are parametrized by the value  $k$ . Note that the  $k$ -values must obey the condition  $k = n\pi/L$ , where  $n$  is an integer and  $L$  is the length of the ring cavity, so that  $\exp[ikx]$  fits into the cavity. The branches 1 and 2 correspond to the stable modes, branch 3 shows a region where the modes can become unstable because  $\beta$  acquires a positive real part.  $k=0$  corresponds to the cross section of this branch with the real axis. For  $k \rightarrow -\infty$  the eigenvalues move in the direction of positive imaginary parts, for  $k \rightarrow +\infty$  in the direction of negative imaginary parts. In order to reach the instability of the cw-solution, the wave vector must lie in the unstable region. The stable branches 1 and 2 do not show essential dependence on  $k$ , their real parts are evidently much smaller than the negative real parts of the unstable eigenvalues. If the fields  $E$  and  $P$  are complex and if their stability is studied, we obtain the branches 4 and 5 for their

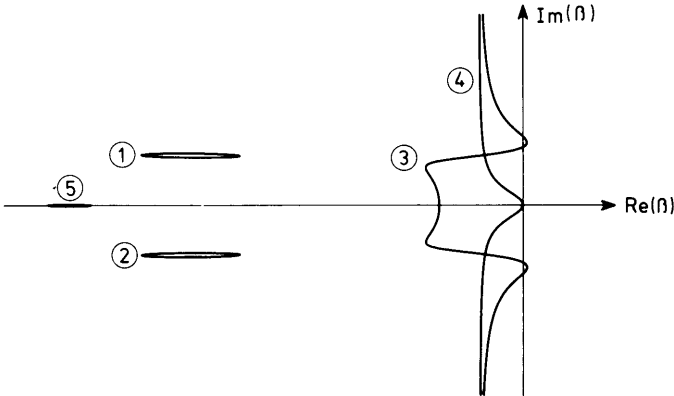


Fig. 7.9. The eigenvalue  $\beta$  in the complex plane in the good cavity case ( $\kappa < 1 + \gamma$  in the notation of this section). The points along each curve correspond to different values of  $k$  which range from  $-\infty$  to  $+\infty$ . The branches 1 and 2 refer to modes with damped amplitudes (stable modes). The branches 4 and 5 represent damped phase modes. Branch 3 contains modes which can become unstable when the imaginary axis is crossed. Since the individual points on each curve are parametrized by  $k$ , the instability occurs at a finite value of  $k$ .

phases. While the latter is strongly damped, branch 4 passes for  $k=0$  through the origin. The corresponding mode is marginal. These branches are independent of the pumping  $A$ . Because one may restrict the discussion on real fields the branches 4 and 5 can be neglected in the following.

The eigenvalues for  $\kappa > 1 + \gamma$ . For sake of completeness we mention that for  $\kappa > 1 + \gamma$  the stable and unstable branches of fig. 7.9 exchange their roles (fig. 7.10). While branch 3 goes to the left, the previously stable branches 1 and 2 go to the imaginary axis and can acquire regions with positive real parts provided the pump is big enough. The branches 4 and 5 show an analogous behavior but are also irrelevant for  $\kappa > 1 + \gamma$ . The instability occurs first at a value  $k \neq 0$ , but by a suitable choice of the resonator length the realization of this value can be prohibited so that an instability of the mode  $k=0$  can be forced to occur. The fields then show spatially homogeneous but temporally chaotic oscillations in time. We shall treat this phenomenon in chapter 8.

## 7.5. Onset of ultrashort laser pulses: nonlinear analysis

In the foregoing section we have shown that for a sufficiently high pump strength the cw-solution becomes unstable and we expect a new type of solution. In order to find this new solution we perform a nonlinear analysis



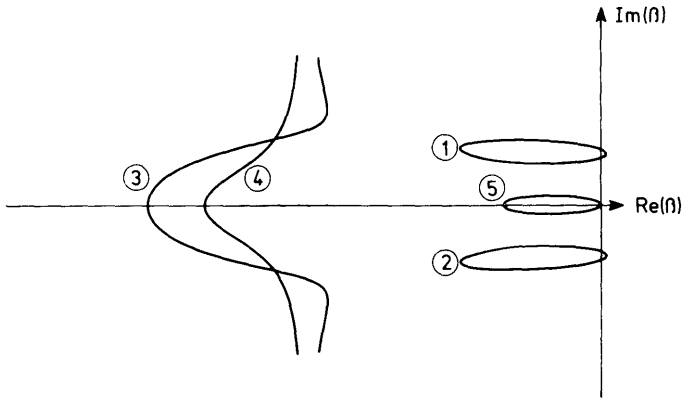


Fig. 7.10. The eigenvalue  $\beta$  in the complex plane in the bad cavity case ( $\kappa > 1 + \gamma$  in the notation of this section). The points on each curve correspond to different  $k$ -values which range from  $-\infty$  to  $+\infty$ . Branches 1 and 2 refer to unstable modes where the imaginary axis is crossed. As can be shown, the first crossing occurs at  $k=0$ . Branches 1 and 2 contain, of course, also damped modes at  $k$ -values unequal 0. Branch 5 stems from the phase mode which can be omitted, however, in the nonlinear analysis because the phase remains constant. The branches 3 and 4 refer to modes with damped amplitude (stable modes).

by means of the method we presented in section 7.3. We take as an example the case of a good cavity, i.e.  $\kappa < 1 + \gamma$ . We further assume that only the eigenvalues  $\beta_j(k)$ ,  $j = u$  ("unstable"),  $k = \pm k_c$ , cross the imaginary axis (cf. fig. 7.9), and that just one pulse fits into the resonator. Furthermore we shall assume for sake of simplicity that the cavity length is chosen such that the instability occurs at the smallest possible value of the pump parameter as given by (7.97).

We denote the order parameters ( $\equiv$  amplitudes of unstable modes) by

$$\xi_{k_c, u} \quad \text{and} \quad \xi_{-k_c, u}.$$

The corresponding eigenvectors  $\mathbf{O}$  are to be written as

$$\mathbf{O}_u(k_c) \quad \text{and} \quad \mathbf{O}_u(-k_c),$$

respectively. Because the fields are real, the relations

$$\xi_{-k_c, u} = \xi_{k_c, u}^* \tag{7.99}$$

and

$$\mathbf{O}_u(-k_c) = \mathbf{O}_u^*(k_c)$$

must hold. Because of  $J_{k, k', k''}$  occurring in (7.69) the amplitudes  $\xi_{k_c, s}$  of the slaved modes have wave vectors  $k$  which are multiple integers of  $k_c$ ,  $k = nk$ ,

The index  $s$  runs over the stable branches (cf. fig. 7.9). With these notations in mind we may write the wanted solution for  $E, P, D$ ,

$$\begin{pmatrix} E \\ D \\ P \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} e \\ d \\ p \end{pmatrix}, \tag{7.100}$$

in the form

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \xi_{k_c, u}(t) \mathbf{O}_u(k_c) \exp[ik_c x] \frac{1}{\sqrt{L}} + \text{c.c.} \\ + \sum_{s, n} \xi_{s, n}(t) \mathbf{O}_s(nk_c) \exp[in k_c x], \tag{7.100a}$$

where we have used the general hypothesis (7.61). The index  $s$  runs over all stable modes belonging to a fixed  $nk_c$ . Once  $\mathbf{O}, \xi_u$  and  $\xi_s$  are determined, we may immediately calculate the spatio-temporal functions  $E, P, D$ . As we know from section 7.3 we can express  $\xi_{k, s}$  by  $\xi_{k_c, u}$ . Therefore we are interested in the equation for  $\xi_{k_c, u}$  alone. We shall see that we can derive an equation of the form

$$\dot{\xi} = \beta \xi + A \xi |\xi|^2 + B \xi |\xi|^4, \tag{7.101}$$

where  $\xi_{k_c, u} \equiv \xi$ . This equation will allow us to determine both the steady state,  $\xi = \xi_0$ , and even transients,  $\xi = \xi(t)$ , so that we can calculate both steady state pulses as well as their transients. The speedy reader can from here on proceed to section 7.6. Readers wanting to see how eq. (7.101) is derived in detail can just continue reading this section. From now on we shall simplify the notation by making the following replacements:

$$\xi_{k_c, u} \rightarrow \xi, \tag{7.102}$$

$$k_c \rightarrow k, \tag{7.102}$$

$$\beta_u(k_c) \rightarrow \beta_u. \tag{7.103}$$

The coefficients in (7.72) and (7.73) which are defined according to (7.70) and (7.71) will be supplemented by indices indicating the corresponding wave vectors. It will be our goal to eliminate the stable modes by means of the iteration procedure explained in section 7.3. In lowest order of our approximation we need to retain only the stable modes belonging to  $k_s = 0$  and to  $k_s = 2k$ , because only these two kinds of modes are driven by the

order parameter,

$$\xi_{0,s} = [-\beta_s(0)]^{-1} 2C_{-k,k}^{suu} |\xi|^2, \quad (7.104)$$

$$\xi_{2k,s} = [2\beta_u - \beta_s(2k)]^{-1} C_{k,k}^{suu} \xi \xi. \quad (7.105)$$

Eqs. (7.104) and (7.105) have been obtained by means of the adiabatic elimination as explained in section 7.3. The expressions (7.104) and (7.105) can now be inserted into the equation for the order parameter  $\xi$  where they give rise to nonlinearities of third order. Because under certain conditions the coefficient of  $\xi \cdot |\xi|^2$  is positive no stabilization is reached and we have to consider the next nonvanishing higher order term. Before we go over to that approximation we introduce a further abbreviation, namely

$$\mathbf{O}^u(k_c) \rightarrow \mathbf{O}. \quad (7.106)$$

We further note that

$$\mathbf{O}^u(-k_c) \rightarrow \mathbf{O}^*. \quad (7.107)$$

An inspection of the formulas for  $\xi$  and  $\xi_s$  of section 7.3 reveals that the eigenvectors  $\mathbf{O}$  occur always in the combination  $\mathbf{O} \cdot \xi$ , whereas in the iterated terms, e.g. (7.104), we always find  $\xi_s = \dots \mathbf{O}$ . This suggests to introduce the following abbreviation

$$\mathbf{K}_{\rho\nu}(m) = \sum_s [m\beta_u - \beta_s(mk)]^{-1} \mathbf{O}_\rho^s(mk) \bar{\mathbf{O}}_\nu^s(mk). \quad (7.108)$$

The first factor under the sum results from the adiabatic approximation. This abbreviation does not only simplify the whole formalism because we have less to write down, but allows us to calculate  $\kappa$  in an easier way (see below). The r.h.s. of (7.108) represents a sum of projection operators  $\mathbf{O}\mathbf{O}$ . By means of the completeness of the eigenvectors  $\mathbf{O}$ ,  $\mathbf{O}$ , i.e. in the notation of section 7.3,

$$\sum_j \dot{\mathbf{O}}_\nu^j(k) \bar{\mathbf{O}}_\mu^j(k) = \delta_{\mu\nu}, \quad (7.109)$$

we may write for  $m \neq 1$

$$\mathbf{K}(m) = [m\beta_u \mathbf{I} + \mathbf{K}(A, mk)]^{-1}, \quad (7.110)$$

where  $\mathbf{I}$  is the unity matrix

$$I_{\sigma\mu} = \delta_{\sigma\mu}. \quad (7.111)$$

$\mathbf{K}_\lambda(mk)$  is defined by (7.92) with the replacement  $\partial/\partial x \rightarrow imk$ . Because of eq. (7.110) we need not solve the complete eigenvalue problem for each wave vector. Rather it is sufficient to determine the inverse (7.110) of a  $3 \times 3$  matrix. Only for  $m = 1$  the eigenvectors and eigenvalues must be calculated

explicitly and the sum (7.108) over the stable modes must be performed. Finally we define the abbreviations

$$G_\rho(0) = \sum_{\nu\mu\sigma} 2K_{\rho\nu}(0) g_{\nu\mu\sigma} O_\mu^* O_\sigma \quad (7.112)$$

and

$$G_\rho(2) = \sum_{\nu\mu\sigma} K_{\rho\nu}(2) g_{\nu\mu\sigma} O_\mu O_\sigma. \quad (7.113)$$

After this little excursion we return to our original problem namely to go one step further in our approximation. To this end we assume that we have inserted the lowest approximation for the stable modes into the eqs. (7.73). As can be immediately seen, two further sets of modes come into play, namely

$$\begin{aligned} [d/dt - \beta_s(k)]\xi_{k,s} &= 2C_{k,0}^{sus_1} \xi\xi_{0,s_1} + 2C_{-k,2k}^{sus} \xi^* \xi_{2k,s_1} \\ &= \sum_{\nu\mu\sigma} 2\bar{O}_\nu^s(k) g_{\nu\mu\sigma} [O_\mu G_\sigma(0) + O_\mu^* G_\sigma(2)] |\xi|^2 \xi, \end{aligned} \quad (7.114)$$

and

$$\begin{aligned} [d/dt - \beta_s(3k)]\xi_{3k,s} &= 2C_{k,2k}^{sus_1} \xi\xi_{2k,s} \\ &= \sum_{\nu\mu\sigma} 2\bar{O}_\nu^s(3k) g_{\nu\mu\sigma} O_\mu G_\sigma(2) \xi\xi\xi. \end{aligned} \quad (7.115)$$

The modes belonging to  $k_s=0$  and  $k_s=2k$  remain unchanged within this iteration step because of the k-selection rule. The corresponding formulas must be inserted in the above eqs. (7.114) and (7.115) and then we must resolve (7.114) and (7.115) with respect to the stable modes.

In analogy to (7.112) and (7.113) we define

$$G_\rho(1) = 2 \sum_{\nu\mu\sigma} K_{\rho\nu}(1) g_{\nu\mu\sigma} [O_\mu G_\sigma(0) + O_\mu^* G_\sigma(2)], \quad (7.116)$$

$$G_\rho(3) = 2 \sum_{\nu\mu\sigma} K_{\rho\nu}(3) g_{\nu\mu\sigma} O_\mu G_\sigma(2). \quad (7.117)$$

Taking into account the stable modes up to the iteration step of first order means that we can confine our analysis to modes up to  $k_s=3k$ . Then the order parameter equation reads

$$\begin{aligned} [d/dt - \beta_u]\xi &= \sum_{\nu\mu\sigma} 2\bar{O}_\nu g_{\nu\mu\sigma} \sum_s [O_\mu O_\sigma^s(0) \xi\xi_{0,s} + O_\mu^* O_\sigma^s(2k) \xi^* \xi_{2k,s} \\ &\quad + \sum_{s_1} (O_\mu^s(k) O_{\sigma^1}^{s_1}(0) \xi_{k,s} \xi_{0,s_1} \\ &\quad + O_\mu^{s*}(k) O_{\sigma^1}^{s_1}(2k) \xi_{k,s}^* \xi_{2k,s_1} \\ &\quad + O_\mu^{s*}(2k) O_{\sigma^1}^{s_1}(3k) \xi_{2k,s}^* \xi_{3k,s_1}]. \end{aligned} \quad (7.118)$$

On the r.h.s. we have now to insert the expression for the stable modes (7.104)–(7.105) and (7.114)–(7.115). We then obtain our final order parameter equation

$$[d/dt - \beta_u]\xi = B|\xi|^2\xi + C|\xi|^4\xi. \quad (7.119)$$

The coefficients are given by the expressions

$$B = \sum_{\nu\mu\sigma} 2\bar{O}_\nu g_{\nu\mu\sigma} [O_\mu G_\sigma(0) + O_\mu^* G_\sigma(2)] \quad (7.120)$$

and

$$C = \sum_{\nu\mu\sigma} 2\bar{O}_\nu g_{\nu\mu\sigma} [G_\mu(1) G_\sigma(0) + G_\mu^*(1) G_\sigma(2) + G_\mu^*(2) G_\sigma(3)]. \quad (7.121)$$

As the reader may note the whole procedure is rather simple. It requires only the evaluation of some sums which can easily be performed on a computer.

## 7.6. Solution of the order parameter equation

The behavior of the ring laser close to the second laser threshold is entirely described by the equation

$$[d/dt - \beta_u]\xi = B|\xi|^2\xi + C|\xi|^4\xi, \quad (7.122)$$

which we derived in the previous section. This equation has a form strongly reminiscent of the form of the single mode laser equation. We decompose the parameters  $\beta_u$ , B, C, which occur in (7.122) into their real and imaginary parts,

$$\beta_u = b + i\omega, \quad (7.123)$$

$$B = d + i\phi, \quad (7.124)$$

$$C = -f + i\psi. \quad (7.125)$$

In (7.125) we have explicitly exhibited the negative sign of the real part which guarantees the stabilization of the system. We approximate the imaginary part of the critical eigenvalue  $\beta_u$  by its value at the critical point (7.98). This means an insignificant simplification and modifies the velocity of the pulse only slightly. We shall decompose the complex amplitude  $\xi$  into its modulus and phase,

$$\xi(t) = R(t) \exp[i\eta(t)], \quad (7.126)$$

where  $R(t)$  and  $\eta(t)$  depend on time explicitly. Inserting (7.126) into (7.122)

and taking the real part of the resulting equation we obtain

$$dR/dt = bR + dR^3 - fR^5. \quad (7.127)$$

This equation can be integrated in closed form but for our purpose it is sufficient to discuss the behavior of the equations qualitatively. To this end we interpret (7.127) as the equation of the overdamped motion of a particle in the potential field  $V(R)$ , where

$$dR/dt = -\partial V(R)/\partial R, \quad (7.128)$$

$$V(R) = -\frac{1}{12}(6bR^2 + 3dR^4 - 2fR^6). \quad (7.129)$$

Quite evidently the behavior of  $R$  is that of a particle which tends towards the closest minimal value of  $V(R)$  where it eventually acquires a stationary state. The phase  $\eta(t)$  obeys the equation

$$d\eta/dt = \omega + \phi R^2 + \psi R^4. \quad (7.130)$$

Obviously the frequency contains amplitude dependent corrections. In the stationary state the order parameter acquires the form

$$\xi = R_0 \exp[i\omega t], \quad (7.131)$$

if we neglect the nonlinear corrections in (7.130).

### 7.6.1. The potential at the critical point

In order to discuss how the shape of the potential changes when the pump parameter  $A$  changes we have to study the individual coefficients. As a detailed analysis reveals, the factor  $C$  changes but little if the pump strength or the critical wave number  $k$  change (the critical wave number  $k \equiv k_c$  is changed if the length  $L$  is changed). However, the real part of the coefficient  $B$  of the cubic term in (7.122) strongly depends on  $k_c$  or, in other words, on the resonator length. Fig. 7.11 shows the real part of  $B$  for the pump parameter  $A = 12$  as a function of the resonator length. For more details we refer the reader to the legend. We mention as an important result that the real part of  $B$  changes its sign. Therefore we obtain two types of potential curves which define two different kinds of behavior, namely an abrupt transition from the cw-state to the pulse state and a smooth transition (compare figs. 7.12 and 7.13).

### 7.6.2. The *fields* of the pulse solutions

Let us briefly summarize what we have achieved so far. We have established an explicit order parameter equation which can easily be solved in the

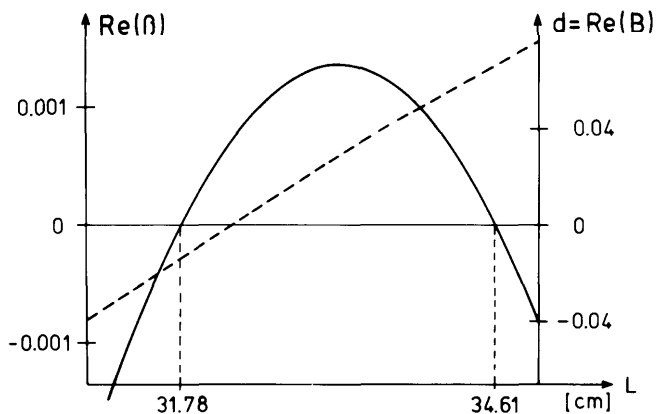


Fig. 7.11. This figure shows the behavior of the real part of  $\beta$  (left ordinate and solid line) and  $d \equiv \text{Re}(B)$  (right coordinate and dashed line) versus resonator length. The vertical dashed lines indicate the region of  $L$  in which the mode becomes unstable. Because the dashed line crosses the 0-axis, the coefficient of the cubic term of the amplitude equation changes its sign within an allowed resonator length.

stationary state but which allows us also to calculate transients. Once we know the order parameter we can calculate the amplitudes of the stable modes. In a last step we can calculate the field, the polarization and the inversion, namely by (7.100) and (7.100a). The normalization factor of the plane waves can be taken care of by a proper normalization of the eigenvec-

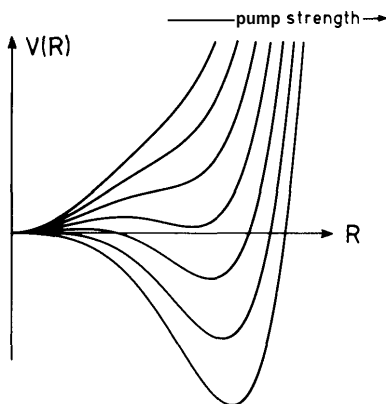


Fig. 7.12. The potential  $V(R)$  of eq. (7.129) as a function of the amplitude  $R$  for various pump strengths, for  $d = \text{Re}(B) > 0$  (cf. fig. 7.11).

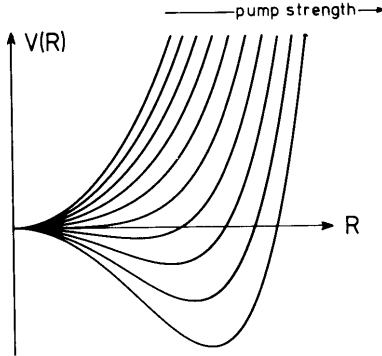


Fig. 7.13. Same as fig. 7.12 but with  $d = \text{Re}(B) < 0$  (cf. fig. 7.11).

tors  $O$  and  $O$  and has been omitted in (7.100a). A number of typical explicit results including transients are shown in figs. 7.14 and 7.15.

Eqs. (7.86)–(7.88), which form the basis of this chapter, have also been solved by direct integration. A typical result is shown in fig. 7.16. In a region not too high above laser threshold excellent agreement between the numerical and the analytical approach is found. The advantage of the analytic

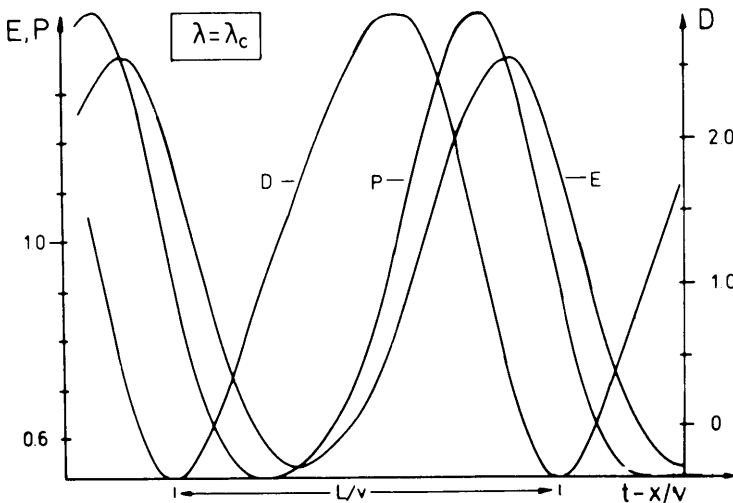


Fig. 7.14. For fixed time  $t$  the pulse shape of  $E$ ,  $P$  and  $D$  is presented as a function of the coordinate along the laser axis according to eq. (7.100a) for  $d > 0$ , at the second threshold. Note the finite amplitude. [H.Haken and H. Ohno, Opt. Commun. 16, 205 (1976).]



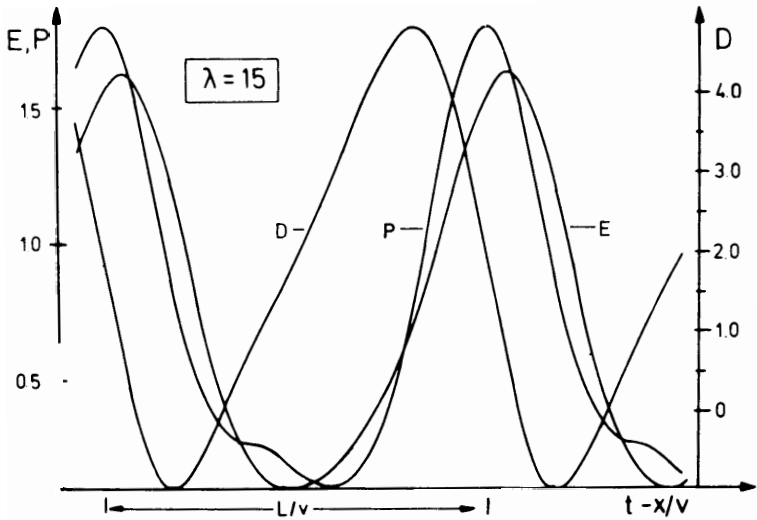


Fig. 7.15. Same as fig. 7.14, but somewhat above the second threshold. [H. Haken and H. Ohno, *Opt. Commun.* 16, 205 (1976).]

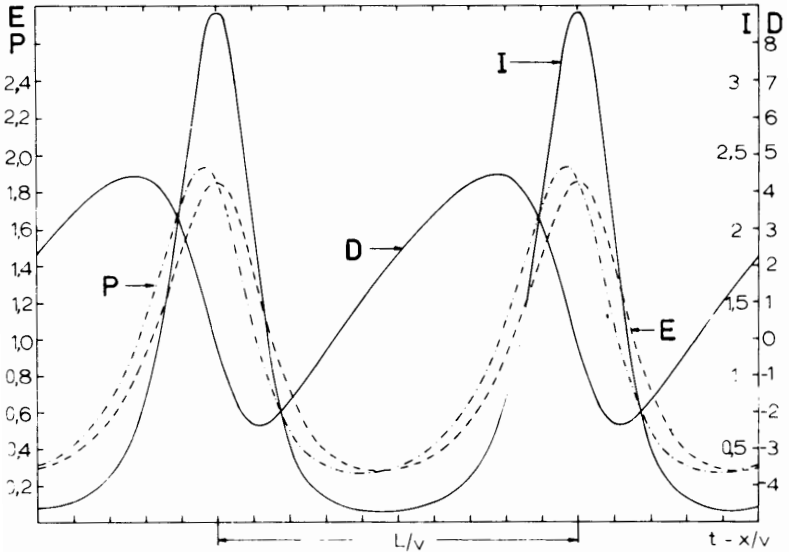


Fig. 7.16. Results obtained by a direct numerical integration of the laser equations (7.86)–(7.88) somewhat above threshold. The ordinates refer to E (left) and D (right). The field E is represented by the dashed line, the polarization by the dash-dotted line, the inversion by the solid line, and the intensity  $I$  by a solid line also. These quantities are plotted versus  $t - x/v$ , where  $t$  = time,  $x$  = coordinate along the axis of the ring laser,  $v$  = pulse velocity and  $L$  = length of the ring laser. [H. Risken and K. Nummedal, *J. Appl. Physics* 39, 466 (1968).]

approach consists in giving us a rather detailed insight into the kind of transition which takes place at the second laser threshold.

### **7.6.3. Recent results**

To make closer contact with reality a number of more detailed calculations are required. For instance, in our treatment we have assumed that there is no spatial variation of the modes other than in the direction of propagation. In reality, however, the mode intensity varies over the cross section perpendicular to the direction of propagation, essentially in the form of a Gaussian distribution. If such a variation is taken into account, it can be shown that the pulse instability disappears. On the other hand a detailed analysis reveals that the coupling of the laser to a saturated absorber may decrease the threshold for laser pulses so that the negative effect of the spatial mode dependence can be more than compensated. Because in the bad cavity case the threshold for onset of pulses or chaos can be considerably lowered if an inhomogeneous atomic line-width is involved, we may speculate that a similar effect holds also for the good cavity case treated in this chapter.

The general method we have outlined above allows a number of further applications, for instance to lasers with saturable absorbers as just mentioned, and to optical bistability (cf. chapter 9).

## **7.7. Models of lasers with saturable absorbers**

In section 7.1 we briefly described saturable absorbers and some of their effects on laser light emission. In this section we wish to formulate the basic equations of a laser with saturable absorber more precisely and indicate the main results which have been achieved so far. As it will transpire a good deal of work has still to be done to calculate the evolving laser pulses. On the other hand we shall see that even under simplifying assumptions interesting features of a laser with saturable absorber can be derived. From an experimental point of view two main arrangements can be considered. In fig. 7.17a the active laser material and the saturable absorber are situated at different positions in-between two mirrors. This is the more common experimental set-up. Another possibility is provided by a laser in which the laser active atoms and the atoms (or molecules) of the saturated absorber are more or less homogeneously distributed over the whole material.

We shall model the laser by a set of two-level atoms (the analysis can easily be extended to a system of three-level atoms). The saturable absorber is modelled also by a set of two-level atoms. But while the laser atoms are incoherently pumped from the outside, the atoms of the saturable absorber

are in their ground states as long as they are not coupled to the laser fields. The formulation of the basic equations is quite simple because it is just a straightforward extension of the basic equations (5.115)–(5.117). For simplicity let us adopt the mode picture and let us consider a single mode only. Because the amplitude  $b$  of the single mode interacts with the set of dipole moments of the lasing atoms and with those of the absorber, eq. (5.115) now acquires the form

$$\dot{b} = (-i\omega - \kappa)b - i \sum_{\mu} g_{\mu} \alpha_{\mu} - i \sum_{\mu'} g'_{\mu'} \alpha'_{\mu'}. \quad (7.132)$$

Here and in the following the prime will indicate quantities belonging to the absorber. The coupling coefficients  $g$  were defined in (5.114). In general the size of the dipole moments of the laser atoms and of the saturable absorber atoms will be different. Depending on the models depicted in figs. 7.17a and 7.17b, the following conditions must be imposed on the coupling coefficients  $g$  and  $g'$ :

- (a)  $g_{\mu} \neq 0$  in region L,  
 $= 0$  outside of region L;  
 $g'_{\mu'} \neq 0$  in region S,  
 $= 0$  outside of region S;

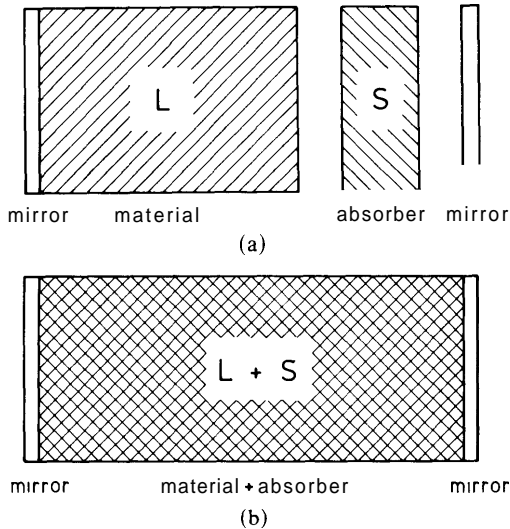


Fig. 7.17. (a) Laser arrangement in which the saturable absorber S is separated from the material L (schematic). (b) Same as (a), but the material and absorber fill the same volume.

$$(b) \left. \begin{array}{l} g_{\mu} \neq 0 \\ g'_{\mu'} \neq 0 \end{array} \right\} \text{ in total region within the cavity.} \quad (7.133)$$

Because of the laser atoms and the atoms of the saturable absorber we have now two groups of matter equations. The first group refers to the laser atoms:

$$\dot{\alpha}_{\mu} = (-i\bar{\omega}_{\mu} - \gamma)\alpha_{\mu} + ig_{\mu}d_{\mu}b, \quad (7.134)$$

and

$$\dot{d}_{\mu} = \frac{d_0 - d_{\mu}}{T} + 2i(g_{\mu}^* \alpha_{\mu} b - g_{\mu} \alpha_{\mu}^* b). \quad (7.135)$$

The next group refers to the atoms of the saturable absorber:

$$\dot{\alpha}'_{\mu'} = (-i\bar{\omega}'_{\mu'} - \gamma')\alpha'_{\mu'} + ig'_{\mu'}d'_{\mu'}b, \quad (7.136)$$

and

$$\dot{d}'_{\mu'} = \frac{d'_0 - d'_{\mu'}}{T'} + 2i(g'^*_{\mu'} \alpha'_{\mu'} b^* - g'_{\mu'} \alpha'_{\mu'} b), \quad (7.137)$$

where it is assumed that the inversion  $d'_0$  without laser action is given by

$$d'_0 = -1, \quad (7.138)$$

whereas  $d_0$  will be chosen positive. In general the atomic line-widths  $\gamma$  and  $\gamma'$  are different from each other and the same holds for the relaxation times  $T$  and  $T'$ . These equations can be easily extended to the multimode case. We leave it as an exercise to the interested reader to write down the corresponding equations, in which  $b$ ,  $\omega$ ,  $\kappa$ ,  $g_{\mu}$  and  $g'_{\mu'}$  must be supplemented with an index  $\lambda$  and the corresponding sums over  $\lambda$  must be performed in eqs. (7.134)–(7.137). In the literature so far only the single mode case has been treated and we briefly indicate the main results.

One first looks for the stationary solution. In this case one makes the hypothesis

$$\left. \begin{array}{l} b(t) = B e^{-i\Omega t}, \\ \alpha_{\mu} = A_{\mu} e^{-i\Omega t}, \\ \alpha'_{\mu'} = A'_{\mu'} e^{-i\Omega t}, \end{array} \right\} \quad (7.139)$$

where  $B$ ,  $A_{\mu}$ ,  $A'_{\mu'}$ ,  $d_{\mu}$  and  $d'_{\mu'}$  are time independent constants. This hypothesis transforms the differential equations (7.132), (7.134)–(7.137) into a set of algebraic equations quite in analogy to those of section 6.2 where we treated the stationary state of the single mode laser. By means of these equations

we may express  $A$ , and  $d_\mu$  by  $B$ , and similarly the quantities  $A'_{\mu'}$  and  $d'_{\mu'}$  by  $B$  in an explicit fashion, so that

$$A_\mu = A_\mu(B), \quad A'_{\mu'} = A'_{\mu'}(B), \quad (7.140a)$$

$$d_\mu = d_\mu(B), \quad d'_{\mu'} = d'_{\mu'}(B). \quad (7.140b)$$

Inserting these functions of  $B$  into the first equation (7.132) we find a single nonlinear equation for  $B$  alone,

$$B[i(\omega - \Omega) + \kappa] = -i \sum_{\mu} g_{\mu} A_{\mu}(B) - i \sum_{\mu'} g'_{\mu'} A'_{\mu'}(B). \quad (7.141)$$

This allows us to determine  $B$  as a function of  $d_0$ . As can be shown for small enough but positive inversion  $d_0$  a stable stationary solution exists. The models treated use a running laser mode. For certain parameter ranges three constant amplitude solutions may coexist and a hysteresis cycle may occur.

When the pump parameter is increased, the onset of oscillations could be shown. Such oscillations can be modelled only by the semiclassical equations (7.132), (7.134)–(7.137), but not by rate equations.

The coexistence of two limit cycles under certain assumptions on the systems parameters could be demonstrated also. Under certain parameter conditions even the emissionless state ( $B = A, = A'_{\mu'} = 0$ ) can be unstable against oscillations.

The problem of a laser with a saturable absorber based on the configuration b (fig. 7.17b) has been formulated also in the *multimode case* based on space- and time-dependent electric fields, atomic polarizations, and inversions in a straightforward generalization of eqs. (7.38)–(7.40). This model exhibits a pulse instability, but so far no pulse-like solutions have been constructed explicitly. An exception is provided by a simplified model in which the action of the saturable absorber is described by an intensity dependent cavity loss as explained in section 7.1. As it seems, still a good deal of work has to be done in this field. The problem is rather difficult because of the many variables and especially of the many possible free parameters which allow one to operate the system in quite different regimes. In view of the results of the single mode laser without saturable absorber, a rich variety of phenomena can be expected and still awaits its exploration.

the "rationality condition" (8.2), the actual laser frequencies  $\Omega_\lambda$  of a free running laser are continuously shifted against their corresponding  $\omega_\lambda$ 's and may be expected to fulfil the "irrationality" condition. Quasiperiodicity can be easily verified by taking the temporal Fourier components of (8.1) which can be done experimentally by a spectrograph. On the other hand, when (8.1) is considered as a function of time, an irregular time dependence is observed.

In chapter 7 we saw that qualitatively new effects may appear when phase and frequency locking takes place. Here ultrashort pulses which are still regular arise. In this chapter we want to study whether still other kinds of behavior can be expected or have been found. One of the most surprising findings is that of "chaotic laser light". The search for such a new type of light was motivated by certain analogies between laser light dynamics and fluid dynamics which we shall describe below. Unfortunately, the word "chaos" or "chaotic light" is used with two entirely different meanings and we shall first discuss this basic differences in order to avoid confusion. In traditional optics light from thermal sources, i.e. from thermally excited atoms, is sometimes called chaotic. In this case no laser action takes place. The atoms are pumped only weakly. After each excitation of an individual atom it starts emitting a wave track spontaneously. Because the acts of spontaneous emission are completely uncorrelated, an entirely random light field results. Neither the rate equations nor the semiclassical equations we have introduced before allow us to treat spontaneous emission adequately. This indeed requires a fully quantum mechanical treatment and we shall come back to it in a later chapter. The only important thing to keep in mind for the moment being is the following. The randomness or chaos of light in this case is produced by fluctuations which are of a quantum mechanical origin which is the cause for spontaneous emission.

In this chapter we will exclusively deal with a second, entirely different type of chaos. We start from the semiclassical laser equations which are obviously deterministic and do not contain any fluctuations a priori. Nevertheless it will turn out that the solutions mean that the emitted light behaves randomly. But the kind of randomness is different from that which we discussed with respect to thermal light, because still many atoms cooperate coherently in order to produce chaotic laser light. This chapter will be devoted to this new kind of chaotic light. We shall first give an example. Then we shall discuss criteria which can tell us whether light is chaotic or, e.g., only quasiperiodic, and then we shall discuss various simple mechanisms which may cause chaotic laser light emission. Finally to conclude our discussion on chaos we shall show that there are various routes to chaos when we start from conventional single mode laser action.

## 8.2. The basic equations

In order to make contact with what is known on chaos in fluid dynamics we shall choose a specific form of the basic laser equations which we derived in section 7.2 and which we repeat for the reader's convenience.

$$\left(\frac{\partial}{\partial t} + \gamma\right)\hat{P} = \gamma\hat{E}\hat{D}, \quad (8.3)$$

$$\left(\frac{\partial}{\partial t} + \gamma_{\parallel}\right)\hat{D} = \gamma_{\parallel}(\Lambda + 1) - \gamma_{\parallel}\Lambda\hat{E}\hat{P}, \quad (8.4)$$

$$\left(\frac{\partial}{\partial t} + \kappa + c\frac{\partial}{\partial x}\right)\hat{E} = \kappa\hat{P}, \quad (8.5)$$

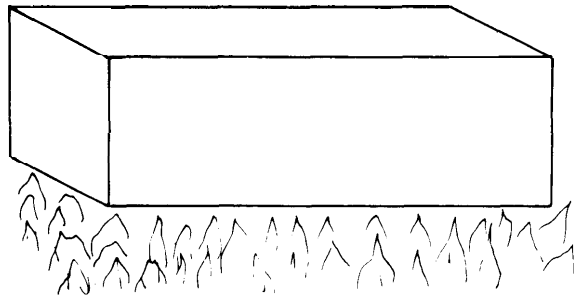
where we have assumed that  $\hat{E}$  and  $\hat{P}$  are real quantities. We briefly remind the reader of the meaning of the individual quantities.  $\gamma$ ,  $\gamma_{\parallel} = 1/T$ ,  $\kappa$  are the usual decay constants used everywhere in this book. It is assumed that the pump strength is so high that it is beyond the first threshold, at which cw laser action occurs.  $\hat{E}$  and  $\hat{P}$  are in our present notation the slowly varying amplitudes of the running waves of the field and the polarization and are normalized, jointly with the inversion density  $\hat{D}$ , with respect to their cw values. Therefore,  $\hat{E} = \hat{P} = \hat{D} = 1$  represents the cw-solution in these normalized quantities.  $\Lambda$  is a normalized pump parameter. In the following we seek a solution of eqs. (8.3)–(8.5), which is space independent (which can be achieved by a proper choice of the length of the ring cavity). This means that we seek a single mode solution. While eqs. (8.3) and (8.4) remain unchanged, (8.5) simplifies to

$$\left(\frac{\partial}{\partial t} + \kappa\right)\hat{E} = \kappa\hat{P}. \quad (8.6)$$

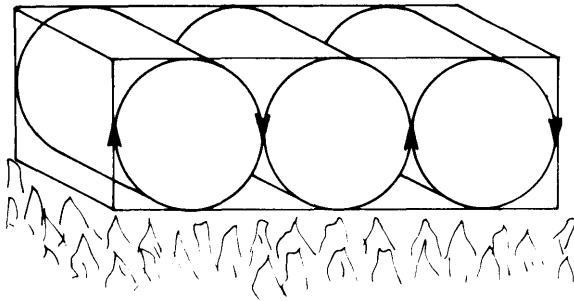
After these preparatory steps let us turn to a specific model of fluid dynamics.

## 8.3. The single mode laser equations and their equivalence with the Lorenz model of turbulence

Let us first take a quick glance at a different field in physics, namely fluid dynamics. There, a long standing and still unsolved problem is the explanation of turbulence. The original purpose of the Lorenz equation is to provide a simple model for turbulence. Lorenz considered a rather typical problem of fluid dynamics which is called the convection instability or Bénard instability. To achieve it, a fluid layer is heated from below (fig. 8.1). The



(a)



(b)

**Fig. 8.1.** (a) Scheme of experiment on the Bénard instability. A fluid layer in a vessel is heated from below. (b) Formation of rolls by the liquid beyond a critical temperature difference between upper and lower surface (schematic).

motion of the fluid is described by the Navier–Stokes equations which we are not going to write down here because they are not important for us. We only mention that they are nonlinear, partial differential equations. In order to cut down the complexity of the problem of solving these equations Lorenz introduced a Fourier decomposition. He expanded the velocity and temperature fields of the fluid into spatial Fourier series. The Fourier coefficients were still time dependent variables. From the infinite series, Lorenz retained only three terms altogether. In this way he derived three coupled differential equations for three variables. Because their physical meaning does not matter in the present context, we shall call these variables  $X$ ,  $Y$ ,  $Z$ . The Lorenz equations have the following form:

$$\dot{X} = \sigma Y - \sigma X, \quad (8.7)$$



$$\dot{Y} = -XZ + rX - Y, \quad (8.8)$$

$$\dot{Z} = XY - bZ. \quad (8.9)$$

For those interested in fluid dynamics we mention that  $\alpha$  is the Prandtl number, and  $r = R/R_c$ , where  $R$  is the Rayleigh number and  $R_c$  the critical Rayleigh number (for onset of convection).  $b = 4\pi^2/(\pi^2 + k_1^2)$  where  $k_1$  is a dimensionless wave number. The equations (8.7)–(8.9) are of quite a simple structure. They are ordinary differential equations and contain only two nonlinearities in the form  $XZ$  and  $XY$ . To the great surprise of many physicists and mathematicians these equations can have solutions which are quite irregular. These solutions were found by means of computer calculations. Fig. 8.2 shows the temporal evolution of  $X(t)$  which is evidently quite irregular. A plot of  $X$ ,  $Y$  and  $Z$  in various planes reveals the following behavior (fig. 8.3). The point  $X(t)$ ,  $Y(t)$ ,  $Z(t)$  circles in one region for a while, but then suddenly jumps into another region, where it moves for a while until it jumps, seemingly randomly, back into the first region, and so on. Quite evidently, deterministic equations of a rather simple form can give rise to an entirely irregular motion which is called "chaotic". In order to produce chaotic motion we need at least three variables obeying first order differential equations. If only two variables are present obeying equations of the form  $\dot{X} = F(X, Y)$  and  $\dot{Y} = G(X, Y)$ , one can show mathematically rigorously that no irregular motion can occur.

But why are the Lorenz equations so important for laser physics? The answer can easily be found through the following steps. Let us first make the rather simple transformations

$$X = \xi, \quad Y = \eta, \quad Z = r - \zeta,$$

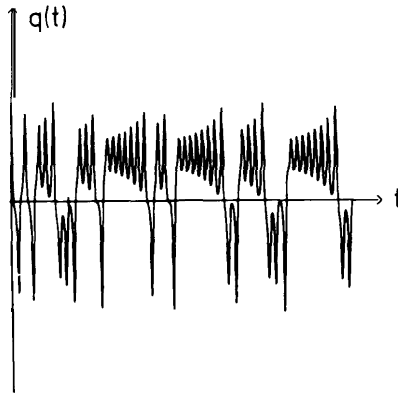


Fig. 8.2. A typical coordinate  $q$  versus time of the Lorenz attractor (arbitrary units).

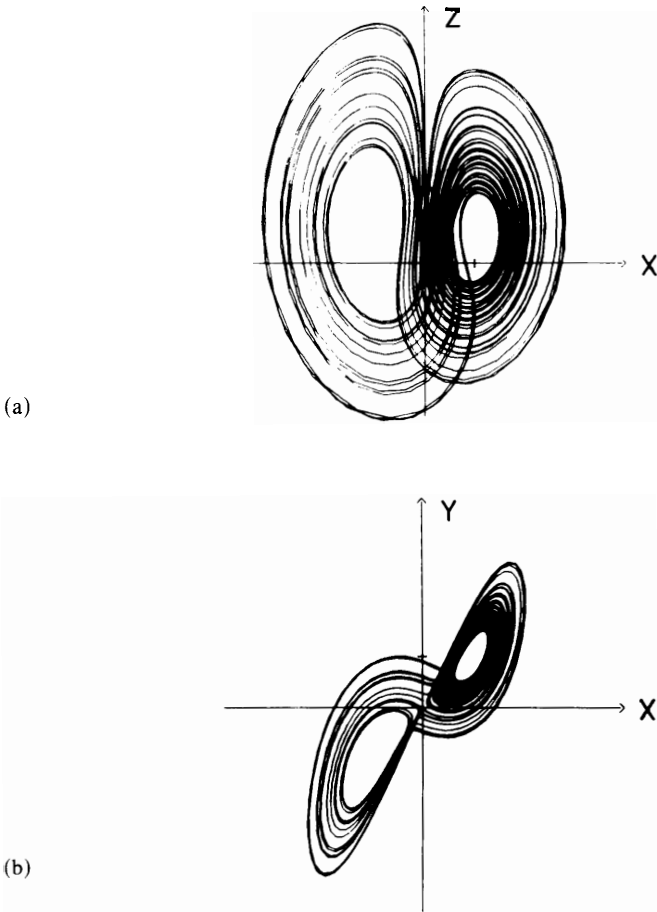


Fig. 8.3. (a) Trajectories of the Lorenz attractor shown as their projection on the Z-X plane. (b) Same as (a) but with the projection on the Y-X plane. (First results of this kind were obtained by E.N. Lorenz.)

through which eqs. (8.7)-(8.9) acquire the new form

$$\dot{\xi} = \sigma\eta - \sigma\xi, \quad \dot{\eta} = \zeta\xi - \eta, \quad \dot{\zeta} = b(r - \zeta) - \xi\eta. \quad (8.10)$$

But these equations are identical with the laser equations (8.3), (8.4) and (8.6), as one may easily verify by using the following substitutions:

$$t \rightarrow t'\sigma/\kappa, \quad \hat{E} \rightarrow \alpha\xi \quad \text{where} \quad \alpha = [b(r-1)]^{-1/2}, \quad r > 1, \\ \hat{P} \rightarrow \alpha\eta, \quad \hat{D} \rightarrow \zeta, \quad \gamma_{\parallel} = \kappa b/\sigma, \quad \gamma = \kappa/\sigma, \quad \Lambda = r-1.$$

In particular the following correspondence holds:

<i>Bénard</i> problem	Laser
$a$ : Prandtl number	$\sigma = \kappa / \gamma$
$r = R/R_c$ ( $R$ , Rayleigh number)	$r = A + 1$
$b = \frac{4\pi^2}{\pi^2 + k_1^2}, \quad k_1^2 = \frac{\pi^2}{2}$	$b = \gamma_{\parallel} / \gamma$

Eqs. (8.10) describe at least two instabilities which have been found independently in lasers and in fluid dynamics. For  $A < 0$  ( $r < 1$ ) there is no laser action (the fluid is at rest), for  $A \geq 0$  ( $r \geq 1$ ) laser action (convective motion) with stable, time independent solutions  $\xi, \eta, \zeta$  occurs. Besides this well known instability a new one occurs provided

laser:

$$\kappa > \gamma + \gamma_{\parallel} \quad \text{and} \quad A > (\gamma + \gamma_{\parallel} + \kappa)(\gamma + \kappa) / \gamma(\kappa - \gamma - \gamma_{\parallel}), \quad (8.11)$$

fluid:

$$\sigma > b + 1 \quad \text{and} \quad r > \sigma(\sigma + b + 3) / (\sigma - 1 - b). \quad (8.12)$$

This instability leads to the irregular motion, an example of which we have shown in fig. 8.2. When numerical values are used in the condition (8.12) it turns out that the Prandtl number must be so high that it cannot be realized by realistic fluids.

From a historical point of view it is interesting to note that this second instability was found independently in laser physics and in fluid dynamics, but the meaning of the second laser instability was recognized rather late.

A numerical analysis reveals that in lasers the pump power must be very high in order to fulfil the condition (8.11). Therefore, after the possibility of chaotic laser light had been demonstrated in principle, other mechanisms were searched for in order to realize chaotic laser light at lower pump power. The fundamental idea is to condense the laser equations (8.3)–(8.6) into still simpler ones by, e.g., eliminating the polarization adiabatically, but to introduce at the same time some modulation effect so that the number of variables remains (at least) three. In the theoretical literature the following modulation effects have been treated:

- (a) time dependent modulation of the cavity loss;
- (b) time dependent modulation of the inversion  $d_0$ ;

(c) injection of a modulated coherent electrical field. Before we present these examples we discuss in the next section criteria upon which one may decide whether chaos is present.

#### 8.4. Criteria for the presence of chaos

In order to decide whether or not chaos is present one might naively be inclined just to look at a plot of the time dependence of a variable of the system. If the plot looks irregular one would call the process "chaotic". Such a procedure leads to the following difficulty, however. Quasiperiodic motion as described, e.g., by (8.1) can also look irregular. Therefore a more detailed analysis is necessary. To this end it has been suggested to study the Fourier transform of (8.1), or in general, of a typical variable  $q(t)$  of the physical system under consideration. Periodic and quasiperiodic motion (or light emission) can be characterized by a set of discrete lines, while broad band emission could indicate chaos. However, here again an intrinsic difficulty arises, because light from thermal sources would have also a broad frequency band. But in this case light emission stems from uncorrelated spontaneous emission acts and has nothing to do with the chaos we are studying here. Another criterion for chaos, which was suggested in the literature, is based on correlation functions of the type  $\langle q(t+\tau)q(t) \rangle$ . According to this criterion, an exponential decay of this correlation function should indicate chaos. But in the case of light from thermal sources the correlation function decays exponentially also, though we are not dealing here with deterministic chaos. Though both "criteria", namely a broad band of the Fourier spectrum and exponential decay of the correlation function, are rather often used in the literature on chaos, these criteria are certainly not sufficient. Therefore other criteria have to be developed.

A criterion which has been coming into the focus of research over the past years is provided by the concept of Lyapunov exponents. As numerical solutions of the Lorenz equations or of similar equations reveal, the time evolution of the variables is very sensitive to initial conditions. Or, in other words, when we change the initial condition even a little bit, in the course of time the two trajectories will increase their distance more and more. More precisely speaking, their distance increases exponentially with elapsing time.

To cast this concept into a mathematical form we consider a general set of nonlinear equations for a state vector  $q$ ,

$$\dot{q} = N(q). \quad (8.13)$$

Then at each time  $t$  we study how a neighboring trajectory evolves by putting

$$q' = q + u, \quad (8.14)$$

where  $u$  is assumed to be a small quantity. Inserting (8.14) into (8.13) and linearizing the resulting equations with respect to  $u$  we obtain equations of

the form

$$\dot{\mathbf{u}} = L(\mathbf{q}(t)) \mathbf{u}, \quad (8.15)$$

where  $L$  is a matrix whose coefficients are still time dependent because they depend on the trajectory  $\mathbf{q}(t)$ . Because of  $\mathbf{u} = \mathbf{q}' - \mathbf{q}$ ,  $|\mathbf{u}|$  measures the distance between two trajectories  $\mathbf{q}'$  and  $\mathbf{q}$ . We expect that in the case of chaos  $\mathbf{u}$  behaves as

$$\mathbf{u} = e^{\lambda t} \mathbf{v}, \quad (8.16)$$

where  $\lambda$  is positive and  $\mathbf{v}$  is a function which changes less rapidly than an exponential function. As it turns out, the form (8.16) is an **over-simplification**, but it can be shown in mathematics that one may define a quantity which corresponds to  $\lambda$  in the following way:

$$\lambda = \lim_{t \rightarrow \infty} \ln \frac{1}{t} |\mathbf{u}|. \quad (8.17)$$

This thus defined  $\lambda$  is called Lyapunov exponent. If  $\mathbf{q}$  spans an  $n$ -dimensional space, there exist at maximum  $n$  different Lyapunov exponents. If at least one of them is positive the criterion for chaos is fulfilled provided some "pathological" cases are excluded. For more details and for an entirely rigorous definition of  $\lambda$  compare the references.

## 8.5. Routes to chaos

According to the interdisciplinary field of synergetics (cf. chapter 13), far reaching analogies in the behavior of quite different systems can be expected irrespective of the nature of the individual parts of a system. These analogies become apparent especially in situations where the qualitative macroscopic behavior of a system changes. Within laser physics, examples for such qualitative changes are provided by the onset of laser action with increasing pump parameter, or by the onset of deterministic chaos. Within fluid dynamics not only the transition to turbulence as described by the Lorenz model is known, but both theoretical and experimental studies show that there may be a hierarchy of different instabilities before the chaotic state is reached.

Among the routes explored are the following:

(1) With the increase of a typical "control parameter" (e.g. the pump power of a laser), more and more oscillations at frequencies  $\omega_1, \omega_2, \dots$  set in. In fluid dynamics this specific route is called the Landau–Hopf picture. It can be observed in lasers where more and more free running modes start

laser action and no frequency locking occurs. According to the original Landau-Hopf picture, in fluid dynamics the turbulent state can be characterized by an infinite number of oscillations at frequencies irrational with respect to each other. This idea has been abandoned due to the experimental results according to which after the occurrence of oscillations at two or three frequencies chaos has been observed in fluid dynamics. In laser physics still more free running modes have been observed, however. A technical term should be mentioned in this context.

Provided the dimension of a vector

$$\mathbf{q} = \sum_n \mathbf{c}_{n_1, n_2, \dots, n_M} \exp[i(n_1 \omega_1 + n_2 \omega_2 + \dots + n_M \omega_M) t] \quad (8.18)$$

is bigger than the number of basic frequencies  $\omega_1, \omega_2, \dots$ , the endpoints of the vector  $\mathbf{q}(t)$  can be represented as trajectory lying on a torus. Therefore the Landau-Hopf picture consists in the idea that with increasing control parameter, tori of higher and higher dimensions are formed.

(2) Another picture based on mathematical arguments on "generic properties" was produced by Newhouse, Ruelle and Takens. According to this picture, after a system has reached an oscillatory state at two basic frequencies, chaos should set in. Such a route is observed in various cases in fluid dynamics ("motion on a two-dimensional torus") though also motion on three-dimensional tori was found. We do not want to bore the reader with mathematical subtleties and therefore interpret the above mentioned term "generic" simply as "typical". The reader should be warned, however, that there are some doubts whether such term, which stems from certain mathematical properties of systems, can be immediately applied to concrete physical situations. Clearly a laser freely running at 4 modes with irrational frequencies contradicts that theorem.

(3) A third route to chaos, which has become rather "popular" now, consists of a sequence of period doublings (fig. 8.4). According to this picture, with increasing control parameter the period of oscillations undergoes a doubling at specific values of that parameter. In a large class of systems the values of the control parameter  $a$ , at which such doublings occur obeys the law

$$\lim_{l \rightarrow \infty} \frac{\alpha_{l+1} - \alpha_l}{\alpha_{l+2} - \alpha_{l+1}} = \delta = 4.6692016, \quad (8.19)$$

where  $\delta$  is called the Feigenbaum number. In particular cases the sequence of period doublings could be followed up to  $n = 5$  or  $6$  and then chaos is observed. The observation of higher numbers  $n$  is impeded by the noise

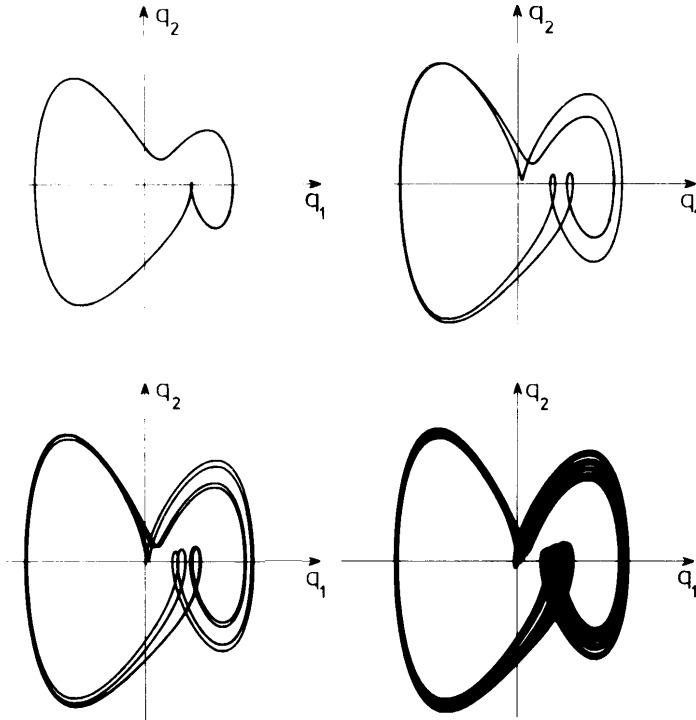


Fig. 8.4. This figure shows trajectories calculated for the Duffing equation which represents a harmonically driven nonlinear oscillator with linear and cubic restoring forces. The trajectories are presented in the plane  $q_1, q_2 = \dot{q}_1$ . The driven amplitude is increased in the order: upper left figure, upper right figure, lower left and lower right figure. At critical values of the driver amplitude, the trajectories split and it takes double as long for a point on the trajectory to return to its original position.

level. Though it seems that period doubling sequences are a widespread phenomenon when nonlinear oscillations are involved, other sequences can be also observed, for instance period triplings and also mixed sequences between doublings and triplings. More generally, other types of subharmonic generation are also found. Therefore a warning should be added namely that in spite of the great enthusiasm among physicists on the universality of the period doubling sequence quite other kinds of subharmonic generation must also be expected and experimentalists would be misled in just finding what they are prepared to see, namely merely period doubling.

(4) Finally we mention the phenomenon of intermittency as a possible route to chaos. Here a typical physical quantity, e.g. the velocity field of a

fluid, remains quiescent for a while, then a chaotic outburst occurs, then a quiescent state reappears, etc.

(5) Finally it may be worthwhile that for instance in chemical reactions alternating sequences between periodic and chaotic oscillations were found when a typical control parameter is continuously increased.

Some of the routes to chaos just mentioned could be verified in the meantime experimentally in laser physics and the study of further routes to chaos in laser physics offers a wide field of future research. In the subsequent sections we shall discuss some situations where laser light chaos can be expected or has been observed. It is beyond the scope of this present book, however, to discuss the mathematical details of these various routes to chaos and I refer the interested reader to my books *Synergetics* and *Advanced Synergetics* where these problems are treated.

## 8.6. How to produce laser light chaos. Some theoretical models

Using the analogy between the single mode laser equations and that of the Lorenz model of turbulence we were able to reveal a possibility of creating chaotic laser light. As we have seen (compare (8.11)), the cavity losses must be particularly high. Therefore this case is referred to as the "bad cavity case". We now wish to study other means of generating chaotic laser light.

We start from the single mode laser equations (6.1)–(6.3) making the same simplifications as in section 6.1. For the reader's convenience we repeat these equations,

$$b = (-i\omega - \kappa)b - ig \sum a, \quad (8.20)$$

$$\dot{\alpha}_\mu = (-i\omega - \gamma)\alpha_\mu + igbd_\mu, \quad (8.21)$$

$$\dot{d}_\mu = \frac{1}{T}(d_0 - d_\mu) + 2ig(\alpha_\mu b^* - \alpha_\mu^* b). \quad (8.22)$$

To remove the terms containing  $i\omega$  we make the substitutions

$$b = \exp[-i\omega t] \mathcal{E}(t) \quad (8.23)$$

and

$$a_\mu = \exp[-i\omega t] \bar{\alpha}_\mu(t). \quad (8.24)$$

Because a sum over  $\mu$  occurs on the r.h.s. of (8.20) it suggests itself to introduce the whole sum as a new variable

$$\sum_\mu \bar{\alpha}_\mu = \mathcal{P}(t). \quad (8.25)$$



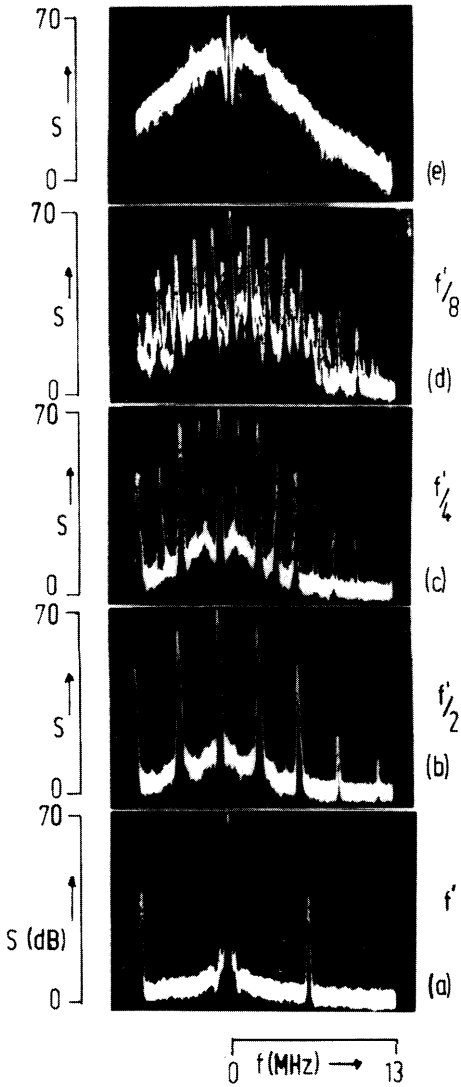


Fig. 8.5. (a)–(e) Period doubling approach to chaos observed in a cw-He-Ne laser. The sequence is caused by tilting of one resonator mirror away from the perfect alignment condition. [C.O. Weiss, A. Godone and A. Olafsson, Phys. Rev. 28, 892 (1983).]

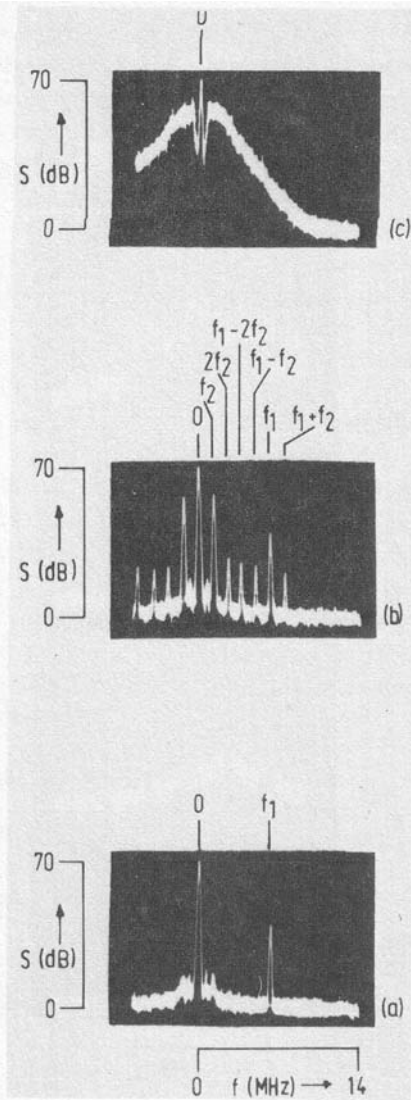


Fig. 8.6. (a)–(c) Observation of the Ruelle–Takens sequence in a cw-He–Ne laser caused by tilting one of the resonator mirrors. An oscillation  $\omega_1$  (a) at frequency  $\omega_1$  is followed by two periodic states at frequency  $\omega_1$ ,  $\omega_2$  (b), followed by chaos (c). [C.O. Weiss, A. Godone and A. Olafsson, Phys. Rev. 28, 892 (1983).]

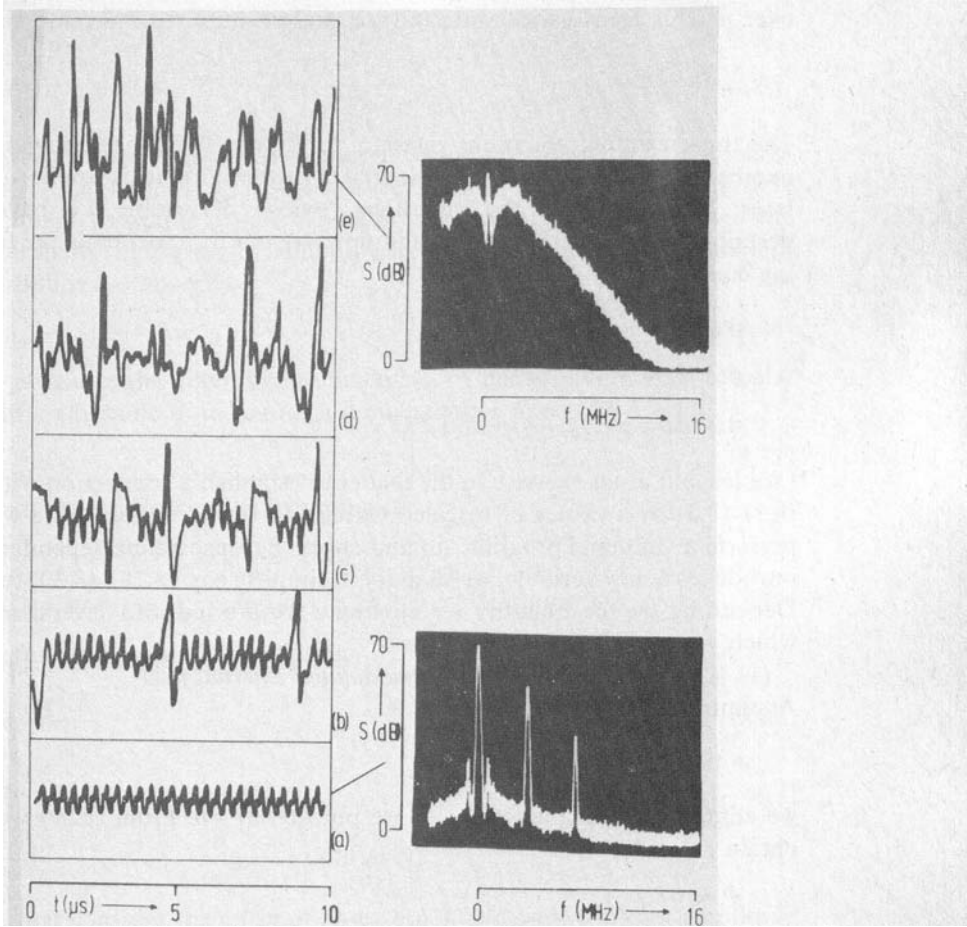


Fig. 8.7. Time dependence of laser output as a control parameter; mirror tilting angle is varied from stable or oscillatory state (a) to chaotic state (e). Spectra corresponding to (a) and (e) are also shown, observed  $-9$  MHz from the main line center. Experimental set-up is in principle the same as in fig. 8.6. The time sequence shows in the cases (b)–(d) clearly the phenomenon of intermittency. [C.O. Weiss, A. Godone and A. Olafsson, *Phys. Rev.* 28, 892 (1983).]

To obtain an equation for this new variable we have to sum up (8.21) over the atomic index  $\mu$ . This leads us to introduce

$$\sum_{\mu} d_{\mu} = \mathcal{D} \tag{8.26}$$

as a new variable. The equation for  $\mathcal{D}$  can be found by summing up (8.22)

over  $\mu$ . This leads us to introduce the total unsaturated inversion via

$$\sum_{\mu} d_0 = \mathcal{D}_0. \quad (8.27)$$

The thus resulting equations refer to  $\mathcal{E}$ ,  $\mathcal{P}$  and  $\mathcal{D}$ . In the following in a number of cases we shall assume that an external field is applied to the laser. Supplementing the equations, whose derivation we have just described, by the corresponding additional term  $\kappa \mathcal{E}_{\text{ext}}$ , we obtain the following equations:

$$d\mathcal{E}/dt = -\kappa(\mathcal{E} - \mathcal{E}_{\text{ext}}) - ig\mathcal{P}, \quad (8.28)$$

$$d\mathcal{P}/dt = -\gamma\mathcal{P} + ig\mathcal{E}\mathcal{D}, \quad (8.29)$$

$$d\mathcal{D}/dt = \gamma_{\parallel}(\mathcal{D}_0 - \mathcal{D}) + 2ig(\mathcal{P}\mathcal{E}^* - \mathcal{P}^*\mathcal{E}). \quad (8.30)$$

(We leave it as an exercise to the reader to establish a connection with eqs. (8.3)–(8.5) by a choice of rescaled variables.) In the following we wish to perform a "minimal program" to find chaos. Because a time dependent  $\mathcal{E}_{\text{ext}}$  introduces a new variable, we shall try to simplify eqs. (8.28)–(8.30) further. Depending on the quantity we eliminate we are led into several models which we are now going to discuss:

(1) Laser chaos produced by a modulated external *field*

Assuming that

$$\kappa \ll \gamma_{\parallel} \ll \gamma, \quad (8.31)$$

we eliminate  $\mathcal{P}$  adiabatically, i.e. we put  $d\mathcal{P}/dt = 0$ . From (8.29) we then obtain

$$\mathcal{P} = ig\mathcal{E}\mathcal{D}/\gamma. \quad (8.32)$$

Making the corresponding approximation for  $\mathcal{D}$  in (8.30) and using (8.32) in it we readily obtain

$$\mathcal{D} = \frac{\mathcal{D}_0}{1 + 4g^2|\mathcal{E}|^2/\gamma\gamma_{\parallel}}. \quad (8.33)$$

Inserting (8.33) in (8.32) we finally express  $\mathcal{P}$  by  $\mathcal{E}$  alone

$$\mathcal{P} = ig\mathcal{E} \frac{\mathcal{D}_0}{\gamma(1 + 4g^2|\mathcal{E}|^2/\gamma\gamma_{\parallel})}. \quad (8.34)$$

Inserting this  $\mathcal{P}$  into (8.28) we find a closed equation for  $\mathcal{E}$ . Our procedure we just made is related to the one we described in section 6.3 but somewhat more rigorous because we do not rely on an expansion of  $\mathcal{P}$  in powers of

$\mathcal{E}$ . The reader is advised to study the relation between

$$d\mathcal{E}/dt = -\kappa(\mathcal{E} - \mathcal{E}_{\text{ext}}) + g^2 \mathcal{E} \frac{\mathcal{D}_0}{\gamma(1 + 4g^2|\mathcal{E}|^2/\gamma\gamma_{\parallel})} \quad (8.35)$$

and (6.46). As one may convince oneself by use of a potential  $V$  in analogy to (6.49)–(6.51) for  $\mathcal{E}_{\text{ext}}=0$ , the solution of eq. (8.35) relaxes to a time independent constant,  $\mathcal{E}=\mathcal{E}_0$ . Therefore, in order to produce chaos the external field is important. Calling the frequency of the external field  $\omega_{\text{ext}}$  we introduce the quantity

$$(\omega_{\text{ext}} - \omega)/\kappa = \delta\omega, \quad (8.36)$$

which measures the detuning in units of  $\kappa$ . To perform numerical calculations it is advisable to introduce a dimensionless time  $\tau$  by

$$t = \tau/\kappa, \quad (8.37)$$

and to rescale the variables according to

$$\mathcal{E} = \hat{\mathcal{E}}(\tau) (\gamma\gamma_{\parallel})^{1/2} \exp[i\delta\omega\tau]/(2g), \quad (8.38)$$

$$\mathcal{E}_{\text{ext}} = A(\tau) (\gamma\gamma_{\parallel})^{1/2} \exp[i\delta\omega\tau]/(2g). \quad (8.39)$$

Using furthermore the abbreviation

$$R = \frac{g^2 \mathcal{D}_0}{\gamma}, \quad (8.40)$$

we arrive at our basic equation

$$\frac{d\hat{\mathcal{E}}}{d\tau} = -i\delta\omega\hat{\mathcal{E}} + \left( \frac{R}{1 + |\hat{\mathcal{E}}|^2} - 1 \right) \hat{\mathcal{E}} + A(\tau). \quad (8.41)$$

Let us first consider the effect of an external field with constant amplitude  $A(\tau)=a$ . By putting the left-hand side equal to zero, we can readily determine the steady state solution  $\hat{\mathcal{E}}_s$  by solving the equation

$$-i\Omega\hat{\mathcal{E}}_s + (z_s - 1)\hat{\mathcal{E}}_s + a = 0, \quad (8.42)$$

where we have used the abbreviations

$$z_s \equiv \frac{R}{1 + \hat{\mathcal{E}}_s^2}, \quad \Omega \equiv \delta\omega. \quad (8.43)$$

Then we may perform a linear stability analysis with which the reader is by now quite familiar. This analysis reveals that the steady state becomes always unstable provided  $R$  is sufficiently large. A slightly more detailed analysis, which we shall not enter here, reveals that this instability causes

an oscillation of  $\hat{\mathcal{E}}_s$  but no chaos. As can be shown, in this region the electric field is modulated, although the incident electric field has a constant amplitude. In order to obtain chaos we consider a modulated external field in the form

$$A(\tau) = a + a' \cos(\Omega' \tau), \quad a > a' \geq 0. \quad (8.44)$$

A numerical solution of eq. (8.41) indeed reveals chaos for a sufficiently high amplitude  $a'$ . When  $a' = 0$ , the system shows a limit cycle\* behavior with the angular velocity  $\Omega_0 = 0.2714$ . When  $a'$  increases, the system acquires a quasiperiodic motion with two characteristic frequencies  $\omega'$  and  $\omega_0$ . By increasing  $a'$  further the limit cycle is entrained by the external force  $A(\tau)$ . As  $\Omega_0/\Omega' \cong 0.6031 \cong \frac{3}{5}$ , the entrainment occurs at a rational frequency of 0, i.e.  $\frac{3}{5}\Omega'$ . Therefore, when we observe the time evolution at time intervals  $2\pi/\Omega'$ , the quintuple cycle (we will use this terminology hereafter) is seen to be realized. This periodic state loses its stability at  $a' = 0.0339$  to lead to a chaotic state. The power spectra of the periodic and the chaotic states are shown in fig. 8.8. A broad peak is clearly seen in the chaotic state. To prove the chaotic behavior we plot the separation distance of two initially adjacent points. The method is the following: After a large number of steps when the phase point can be considered to be trapped in the attractor we take this phase point and choose another point which is separated from this point by a small distance. In the present case the real part of  $\hat{\mathcal{E}}$  is chosen separated by the distance 0.00001. Then the distance,  $D(\tau)$ , between these two points is plotted in fig. 8.9 versus the time  $\tau$ . In the quasiperiodic state ( $a' = 0.01$ ) it can be seen that the two phase points remain close to each other. In the periodic state ( $a' = 0.03$ ) the phase points approach one another as the system evolves. The reason is that the periodic state appears due to the entrainment of the phase point by the external field, and the relative phase of the phase point to the external force  $A(\tau)$  becomes fixed on the attractor. Therefore, the two phase points coincide with each other as  $\tau \rightarrow \infty$ . On the other hand in the chaotic state the two phase points get more separated as time goes on. The saturation behavior appears after  $\tau - 400$ . This is due to the fact that the size of the strange attractor (in the present case it is of the order one) is finite. This behavior of  $D(\tau)$  is quite in line with the other examples of chaos.

At sufficiently large  $a'$  ( $\geq 0.15$ ), the time evolution of the system is periodic with the frequency  $\Omega'$ . Between this completely entrained state and the chaos mentioned above there appear various states. The bifurcation scheme

\*Readers not familiar with the nomenclature such as "limit cycles", "strange attractors", etc. are referred to my book: Advanced Synergetics.

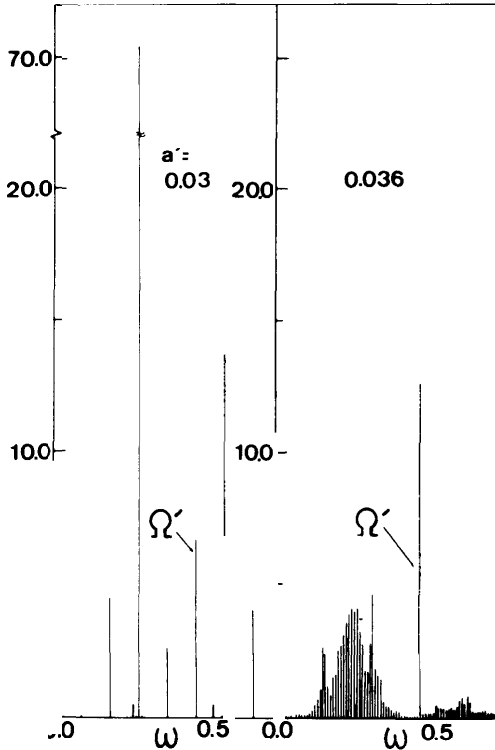


Fig. 8.8. The power spectrum of the periodic ( $a'=0.03$ , left part of figure) and the chaotic ( $a'=0.036$ , right part of figure) states. The sharp peaks at the frequency  $\omega=0.45$  in both figures correspond to the frequency of the external modulated amplitude. The average is taken over the sequence over the spectrum 50 times. [T. Yamada and R. Graham, Phys. Rev. Lett. 45, 1322 (1980).]

shows a window structure. For example, the system has an octuple periodic state at  $a'=0.05$  and a chaotic state at  $a'=0.07$ . The detailed bifurcation scheme with the variation of  $a'$  as well as with that of  $\Omega'$  is interesting, but is beyond the scope of this book.

If the set of parameters,  $R$ ,  $a$  and  $R$ , is chosen such that the system is deep inside the limit cycle region, it becomes harder to find chaos. The reason may be that near the transition region between the steady state and the limit cycle state the orbit of limit cycle is easily affected by the external force, while deep inside the limit cycle region a strong modulation of  $A(\tau)$  is necessary to change the limit cycle orbit and it may violate the inequality  $a > a'$ .

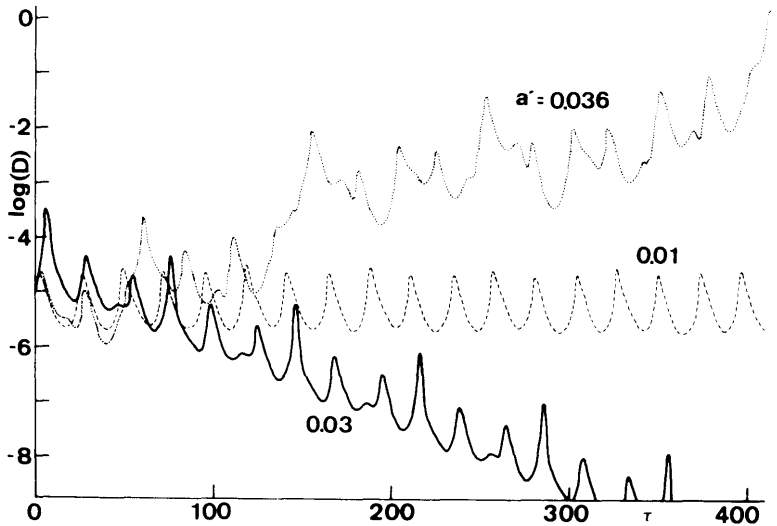


Fig. 8.9. The separation distance  $D(\tau)$  of two initial adjacent points against time  $\tau$ . The logarithmic value of  $D(\tau)$  is plotted. The three curves correspond to the quasiperiodic ( $a' = 0.01$ ), the periodic ( $a' = 0.03$ ), and the chaotic ( $a' = 0.036$ ) states. [T. Yamada and R. Graham, Phys. Rev. Lett. 45, 1322 (1980).]

The existence of chaos studied here seems not to depend critically on the particular approximation (the adiabatic approximation) made at the beginning. Sufficiently close to the transition region between the steady state and the limit-cycle states we can always expect to get a bifurcation scheme leading to chaos if only we choose appropriate values of the parameters to reach the chaotic state. Let us now consider a second approach to laser chaos.

**(2) Laser chaos produced by an external oscillating field with constant amplitude, and a modulated inversion**

Because the inversion  $\mathcal{D}_0$  enters eq. (8.41) via  $R$  (compare (8.40)) we put

$$R = R_0 + R' \cos(\Omega' t). \tag{8.45}$$

Thus the model equation is provided by

$$\frac{d\hat{\mathcal{E}}}{d\tau} = -i\delta\omega\hat{\mathcal{E}} + \left( \frac{R_0 + R' \cos(\Omega' t)}{1 + |\hat{\mathcal{E}}|^2} - 1 \right) \hat{\mathcal{E}} + A_0. \tag{8.46}$$

It has been solved numerically for specific parameter values.

In order to have a three-dimensional phase space of the system, it is necessary that both  $\Omega'$  and  $A$ , in eq. (8.46) are different from zero. For  $\Omega' = 0$  or  $A = 0$ , eq. (8.46) may be reduced to two equations of first order. For  $R' = 0$ , eq. (8.46) has a stationary state  $\hat{\mathcal{E}}_c$ . A limit cycle appears when



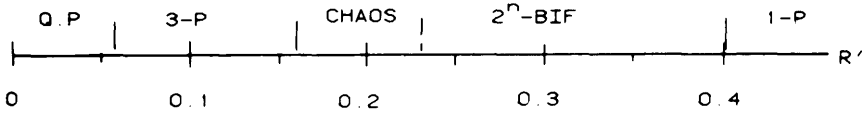


Fig. 8.10. Bifurcation scheme (compare text) with  $R'$  as control parameter; Q.P.=quasiperiodic, 3-P=three-periodic,  $2^n$ -BIF=periodic doubling process. [H.J. Scholz, T. Yamada, H. Brand and R. Graham, Phys. Letters 82, 321 (1981).]

this stationary state becomes unstable for sufficiently large  $R_0$ . Choosing  $R_0=2$ ,  $\omega=0.5$ ,  $A=0.4$ ,  $\Omega'=0.4$  and varying  $R'$ , one finds the bifurcation scheme depicted in fig. 8.10. Slightly above  $R'=0.16$ , the transition from the three-periodic state into the chaotic one takes place via an intermittent mechanism (see fig. 8.11). At  $R'=0.1610$  one observes only a few chaotic bursts interrupting the periodic motion. The number of chaotic bursts increases as the parameter  $R'$  becomes greater. The chaotic regime ranges at least up to  $R' \approx 0.22$  where a complicated subharmonic bifurcation behavior is observed. With increasing  $R'$  the intermittent region is followed by a fully chaotic one. This regime has been investigated by various methods for  $R'=0.18$ . The power spectrum of the real part of the electric field amplitude shows broad peaks. While these calculations have been confined to the single mode case, the results indicate that chaos may be common in

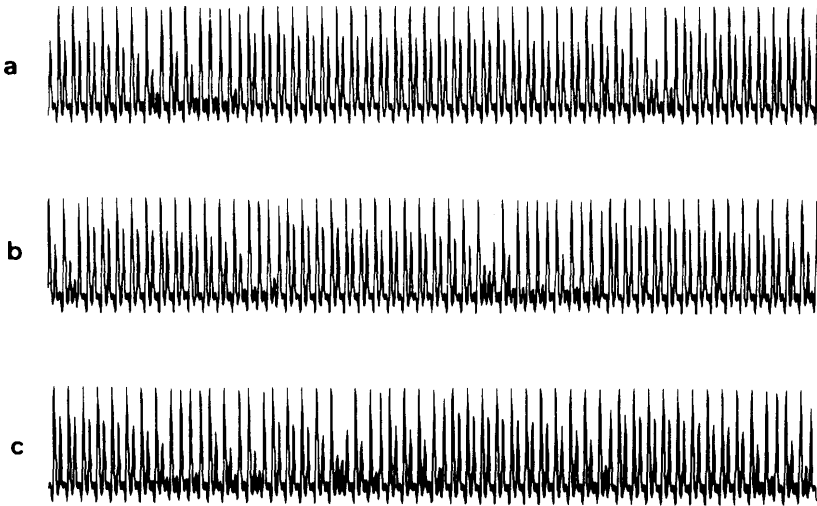


Fig. 8.11. Time evolution of the real part of the electric field for (a)  $R'=0.1610$ , (b)  $R'=0.1615$ , (c)  $R'=0.1620$ . The intermittency is clearly visible.

multimode lasers without external modulation. Indeed, in multimode lasers the atomic inversion felt by each mode is internally modulated at the difference frequency between different modes and the amplitudes of modes provide a driving field for other modes (cf. sections 6.4 and 7.1).

### 8.7. Single mode laser with injected signal. Chaos, breathing, spiking

In the foregoing section we studied the effect of an injected coherent light field. We treated cases where we could eliminate the atomic inversion and polarization so that the laser equations could be reduced to a rather simple equation for  $b$  (or  $\mathcal{E}$ ) alone. In this section we will drop this adiabatic approximation and treat the full set of laser equations. The basic equations are well known to us and read

$$\begin{aligned}\dot{\mathcal{E}} &= -i\omega\mathcal{E} - \kappa(\mathcal{E} - \mathcal{E}_{\text{ext}}) - i\mathbf{g}\mathcal{P}, \\ \dot{\mathcal{P}} &= -i\bar{\omega}\mathcal{P} - \gamma\mathcal{P} + i\mathbf{g}\mathcal{E}\mathcal{D}, \\ \dot{\mathcal{D}} &= \gamma_{\parallel}(\mathcal{D}_0 - \mathcal{D}_{\mu}) + 2i(\mathbf{g}^*\mathcal{P}\mathcal{E}^* - \mathbf{g}\mathcal{P}^*\mathcal{E}).\end{aligned}\tag{8.47}$$

We start from the full set of equations (8.47), where we assume  $\mathcal{E}_{\text{ext}}$  in the form  $\mathcal{E}_{\text{ext}} = A \exp[-i\omega_0 t]$ . In order to remove  $\exp[-i\omega_0 t]$  from eqs. (8.47) we put

$$\mathcal{E}(t) = \tilde{\mathcal{E}}(t) \exp[-i\omega_0 t],\tag{8.48}$$

$$\mathcal{P}(t) = \tilde{\mathcal{P}}(t) \exp[-i\omega_0 t].\tag{8.49}$$

In this way we obtain

$$\dot{\tilde{\mathcal{E}}} = [-i(\omega - \omega_0) - \kappa]\tilde{\mathcal{E}} + \kappa A - i\mathbf{g}\tilde{\mathcal{P}},\tag{8.50}$$

$$\dot{\tilde{\mathcal{P}}} = [-i(\bar{\omega} - \omega_0) - \gamma]\tilde{\mathcal{P}} + i\mathbf{g}\tilde{\mathcal{E}}\tilde{\mathcal{D}},\tag{8.51}$$

$$\dot{\tilde{\mathcal{D}}} = \gamma_{\parallel}(\tilde{\mathcal{D}}_0 - \tilde{\mathcal{D}}_{\mu}) + 2i(\mathbf{g}^*\tilde{\mathcal{P}}\tilde{\mathcal{E}}^* - \mathbf{g}\tilde{\mathcal{P}}^*\tilde{\mathcal{E}}).\tag{8.52}$$

In order to reduce the number of parameters which can be freely chosen and also to make the system (8.50)–(8.52) more apt for a computer calculation, suitably scaled quantities must be introduced. They are defined as follows:

$$C = \alpha L d_0 / (2T),\tag{8.53}$$

where  $a$  is the unsaturated absorption coefficient per unit length,  $L$  the length of the sample, and  $T$  the intensity transmission coefficient of the

mirrors,

$$\tau = yt, \tag{8.54}$$

$$\tilde{\kappa} = \kappa / y = CT / \mathcal{L}\gamma, \tag{8.55}$$

where  $\mathcal{L}$  is the total length of the ring cavity,

$$\tilde{\gamma} = \gamma_{\parallel} / \gamma, \tag{8.56}$$

$$\Delta = (\bar{\omega} - \omega_0) / \gamma, \tag{8.57}$$

$$\Theta = (\omega - \omega_0) / \tau. \tag{8.58}$$

The control parameters are  $C$ ,  $A$ ,  $\Theta$  and  $\tilde{\gamma}$ , and  $y$  to be defined below.

We first note that the steady state solution of (8.50)–(8.52) can be found analytically. We scale the incident field amplitude  $\mathbf{A}$  to the square root of the saturation intensity and call this new quantity  $y$ . Using the same scaling for the laser field amplitude  $\tilde{\mathcal{E}}$  we denote the corresponding quantity by  $\mathbf{x}$ . The relation between the input and output is then given by

$$y = |\mathbf{x}| \left[ \left( 1 - \frac{2C}{1 + \Delta^2 + |\mathbf{x}|^2} \right)^2 + \left( \Theta + \frac{2C\Delta}{1 + \Delta^2 + |\mathbf{x}|^2} \right)^2 \right]^{1/2}. \tag{8.59}$$

A plot of this relation for specific parameter values is given in fig. 8.12. To

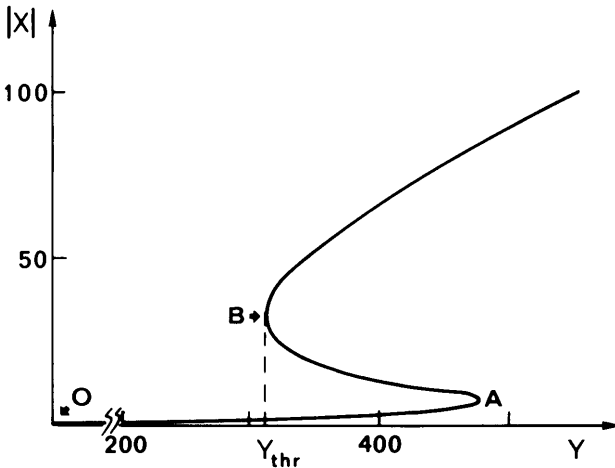


Fig. 8.12.  $|\mathbf{x}|$  versus  $y$  according to eq. (8.59). For  $C = 500$ ,  $A = \Theta = 5$ ,  $\tilde{\kappa} = 1$ , the segment A, B is unstable. [L.A. Lugiato, L.M. Narducci, D.K. Bandy and C.A. Pennise, Opt. Commun. 46, 64 (1983).]

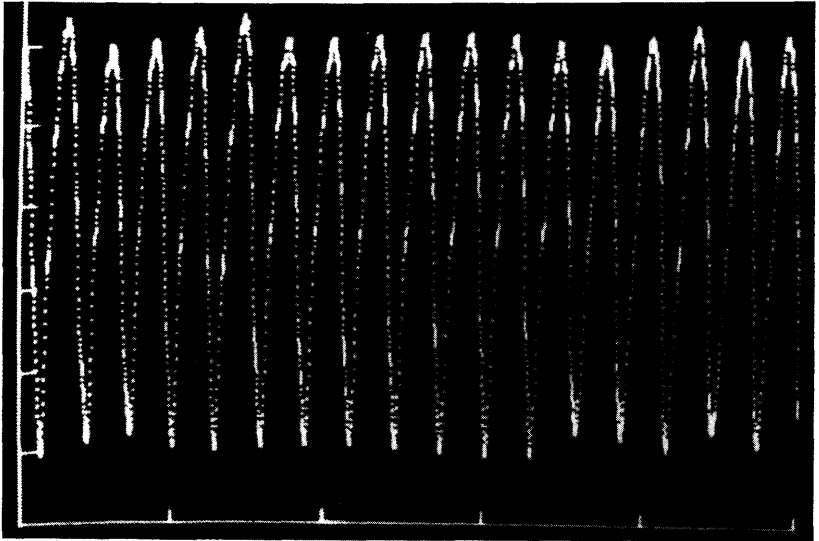


Fig. 8.13. Time evolution of the normalized emitted field  $|x|$  for  $C = 500$ ,  $A = \Theta = 5$ ,  $\tilde{\kappa} = \tilde{\gamma} = 1$ . The abscissa is measured in  $\tau$  units. The curve shows erratic behavior. Driver amplitude  $y = 117$ . [L.A. Lugiato, L.M. Narducci, D.K. Bandy and C.A. Pennise, *Opt. Commun.* 46, 64 (1983).]

describe the time dependent behavior of the system computer calculations are necessary. In these calculations  $\gamma_{\parallel} = y$  was chosen. In the following we shall assume resonance between the cavity mode (with frequency  $\omega$ ) and the frequency  $\bar{\omega}$  of the atomic line center,  $\omega = \bar{\omega}$ . When the external field is switched on, the output intensity begins to oscillate immediately in a regular way with a frequency  $\bar{\omega} - \omega_0$  where  $\omega_0$  is the frequency of the incident light. The average output intensity varies in a continuous way so that the behavior of the system is a direct continuation of the stable lasing state in the absence of the injected signal. With further increased  $A$  the system begins to display an irregular self-pulsing behavior (fig. 8.13). For still higher  $A$  the chaotic pattern is well developed (fig. 8.14). The laser field shows outbursts in which each burst is followed by a number of rapid noisy oscillations. A further increase of  $A$  brings the system out of the chaotic domain through a system of period doubling bifurcations. An inverse order on irregular self-pulsing sets in (figs. 8.15). At this point the system enters a new regime. First the amplitude of the simple oscillation decreases continuously, upon increasing  $y$ , while, at the same time a gradual lengthening of the time scale heralds the appearance of a "breathing" behavior, i.e. a stable modulation of the self-pulsing envelope, over a narrow range of

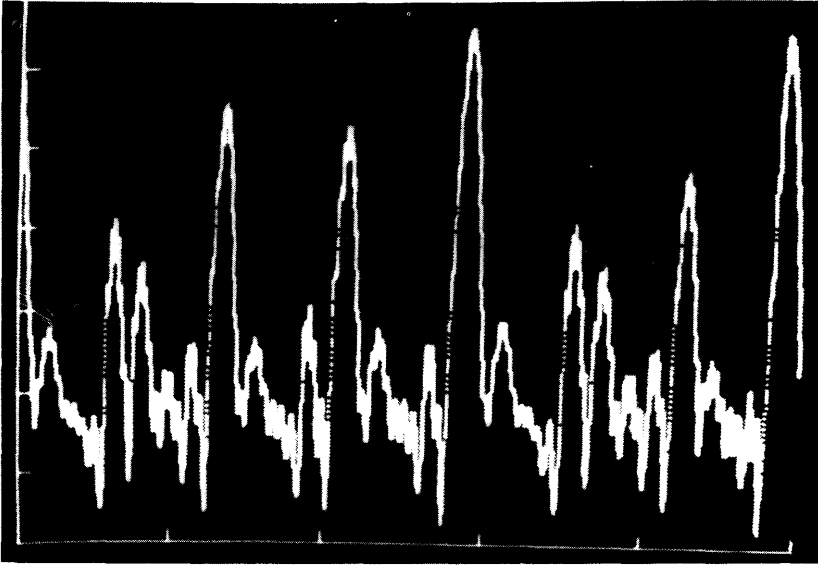


Fig. 8.14. The same as fig. 8.13 but for a driver amplitude  $\gamma = 250$ . The field shows bursting. [L.A. Lugiato, L.M. Narducci, D.K. Bandy and C.A. Pennise, *Opt. Commun.* 46, 64 (1983).]

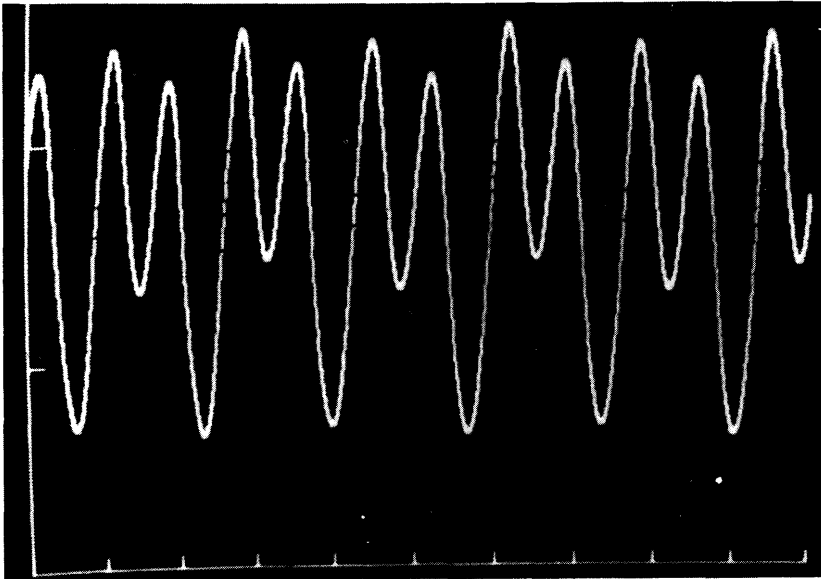


Fig. 8.15. The same as figs. 8.13 and 8.14 but driver amplitude  $\gamma = 279$ . The field exhibits a period 4 behavior. [L.A. Lugiato, L.M. Narducci, D.K. Bandy and C.A. Pennise, *Opt. Commun.* 46, 64 (1983).]

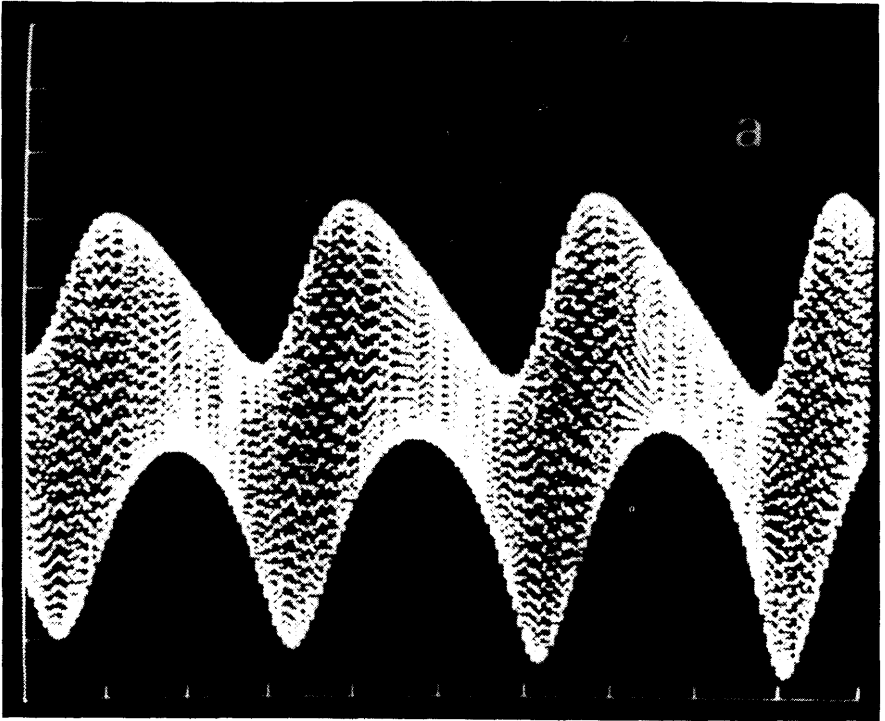


Fig. 8.16. Time evolution of the normalized emitted field  $|x|$  with the same parameters as in fig. 8.13. For  $y = 310.3$  the system displays a marked modulation of the self-pulsing envelop. [L.A. Lugiato, L.M. Narducci, D.K. Bandy and C.A. Pennise, *Opt. Commun.* 46, 64 (1983).]

values of  $y$ . The time scale of the breathing pattern is about 50 times longer than that of simple oscillations. Larger values of the injected field bring about a dramatic increase of the breathing pattern ("heavy breathing", fig. 8.16) and eventually lead to a "spiking" regime (fig. 8.17) in which very narrow spikes are followed by long periods of lethargy. Normally, the spikes have varying peak heights and their temporal separation grows as  $y$  approaches the turning point of the state equation (8.59). Finally, when  $y$  is made larger than  $y_{\text{thr}}$ , the system quickly approaches a steady state (injection locking) in the upper branch

If the parameters are chosen in such a way that the injection locking threshold lies beyond the turning point, breathing and spiking have not been seen. One sees instead a gradual reduction of the self-pulsing amplitude which, eventually, vanishes at the injection threshold. (The period of self-pulsing is about 0.3 unit of  $\tau$  in this case.)

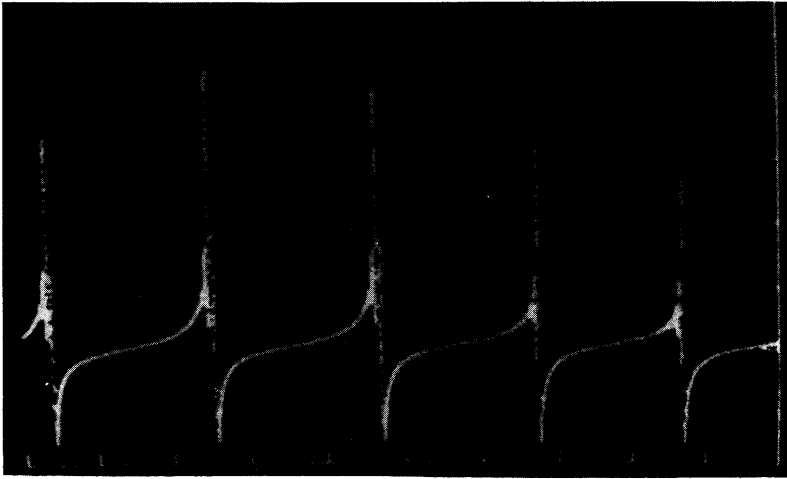


Fig. 8.17. The same as fig. 8.16 but with  $y = 31$ . The field exhibits spiking action [L.A. Lugiato, L.M. Narducci, D.K. Bandy and C.A. Pennise, *Opt. Commun.* **46**, 64 (1983).]

As we have outlined in section 8.4, a criterion for the existence of a strange attractor is given by the exponential divergence of irregular trajectories whose starting points in phase space are arbitrarily close to each other. Such an exponential divergence is indeed found in the chaotic laser regimes.

The chaotic behavior is independent of the Lorenz model, because for  $y=0$ , the stationary state of the laser is stable for our chosen parameters. On the other hand, one may select control parameters that lie within the Lorenz instability domain (this is defined by the condition  $(1 + \tilde{\kappa} + \hat{y})(\tilde{\kappa} + 2C) < 2\tilde{\kappa}(2C - 1)$ ). In this case, one observes large amplitude self-pulsing with evidence of irregular behavior even for small values of the injected field amplitude. Furthermore, unlike the previous case, there appears to be no period doubling cascade upon emerging from the chaotic domain; one finds instead an intermittent behavior of the type shown in fig. 8.14. If  $y$  is increased further, one arrives at simple oscillations followed by nearly the same sequence discussed in the previous case (i.e. breathing and spiking).

The value  $C=500$  is not very interesting from a practical viewpoint, but it is considered here because it presents a rich phenomenology. However, one finds nearly the same sequence of solutions as shown in figs. 8.13–8.15 for values of  $C$  as small as 20, which are, in fact, accessible to high gain lasers. (Breathing, on the other hand, seems to be absent in resonance ( $\omega_A = \omega_c$ ), or hard to find.)

In general, in order to obtain chaotic behavior from the present model, the rates  $\kappa$  and  $\gamma$  must be kept of the same order of magnitude; this is unlike the Lorenz model ( $\gamma=0$ ) where one must insist on  $\kappa > \gamma + \gamma_{\parallel}$ . In addition, chaotic patterns are seen to persist even if  $\gamma$  and  $\gamma_{\parallel}$  are quite different from one another.



## Chapter 9

# Optical Bistability

### 9.1. Survey

The phenomenon of optical bistability can be used in various ways in promising optical devices. Therefore we shall explain this phenomenon and its theoretical treatment in some detail. Let us consider the experimental set-up described in fig. 9.1. An incident coherent light field produced by a laser impinges on a mirror where it is partly reflected and partly transmitted. Then it may propagate in the form of a wave  $E_1$  further within a medium until it hits a second mirror. There it is partly reflected ( $E_2$ ) and partly transmitted. We are interested in the way the transmitted light field  $E_T$  depends on the incident light field  $E_i$ . In the following we shall assume that the Fabry–Perot resonator of fig. 9.1 is tuned or nearly tuned to the incident light. When the cavity is empty, the transmitted power  $I_T$  is proportional to the incident power  $I_i$  and the proportionality constant depends on the detuning and finesse of the cavity. Qualitatively new phenomena may occur when the cavity is filled with material resonant or nearly resonant with the incident field. In contrast to the usual laser case where the material in the cavity is incoherently pumped from the outside, we are considering here a material which is in its ground state if no coherent field  $E_i$  is present.

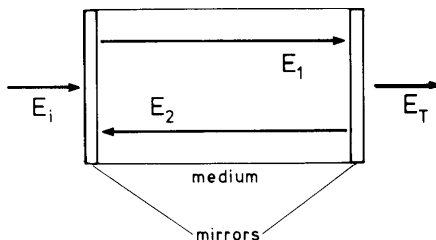


Fig. 9.1. Scheme of experimental set-up (compare text).

Therefore the material will absorb the impinging light. However, this absorption may depend in a nonlinear fashion on the incident light field because saturation effects occur. Therefore  $E_T$  becomes a nonlinear function of  $E_i$ . As we shall see below, the behavior of the system is determined by the ratio of the absorption parameter  $\alpha L$  and the mirror transmissivity  $T$ , where  $a$  is the unsaturated absorption coefficient per unit length on resonance and  $L$  the length of the sample.

Let us discuss what happens when we increase the ratio  $\alpha L/T$  (compare fig. 9.2), where we plot the transmitted intensity versus the incident intensity. As one recognizes, the slope may become greater than unity or, in other words, the differential gain  $dI_T/dI_i$  becomes larger than unity. If under this condition one slowly modulates the incident intensity, the modulation is transferred to the transmitted field via the nonlinear relation  $I_T = I_T(I_i)$  and turns out to be amplified. Thus the system works as an optical transistor. If one further increases the ratio  $\alpha L/T$ , the steady state curve  $I_T = I_T(I_i)$  becomes S-shaped. While the segments with positive slope are stable, the segment with negative slope is unstable. Hence there is a certain range of values of  $I_i$  where the system is bistable. If we slowly sweep the incident power from 0 to a value beyond the bistable region and then sweep it back, we obtain a hysteresis cycle with a low and a high transmission branch. This bistable behavior arises from the interplay of the nonlinearities of the atom-field interaction with the feed-back of the mirrors and will be the main subject of our further study. The threshold value of  $\alpha L/T$  for which one yields bistability depends on several parameters as the cavity mistuning,

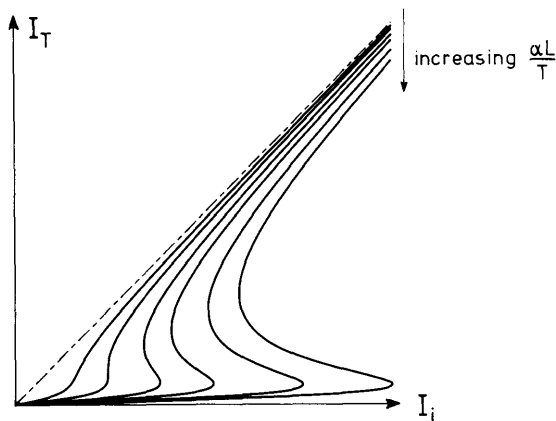


Fig. 9.2. Transmitted intensity  $I_T$  versus input intensity  $I_i$  for various parameter values of  $\alpha L/T$ .

the atomic detuning, the inhomogeneous line-width, etc. When the incident field is in perfect resonance with the atomic line, dispersion does not play any role so that one speaks of purely absorptive bistability. In the general case we have to deal with absorptive and dispersive bistability. When the atomic detuning is so large that absorption becomes negligible one speaks of purely dispersive bistability. Besides the just described all-optical (or intrinsic) bistable systems also hybrid electro-optical systems have been devised in many variants. A typical device of this type is obtained by replacing the absorber by an electro-optical crystal which is monitored by the output field and produces changes in refractive index proportional to the output power. From these remarks it transpires that these systems have a great potential as devices. They can work as optical transistors, memory elements, pulse shapers which eliminate the noisy part of the input light, clippers, discriminators, limitators. In addition to what we shall show below, this system can work as converter of cw-light into pulses in close analogy to the pulses we studied in sections 7.4–7.6. Chaotic states were also found.

## 9.2. A specific model

In order to make our presentation as transparent as possible we shall focus our attention on a specific model though occasionally we shall deal with some more general cases. In the first step of our analysis we replace the arrangement of fig. 9.1 by one which is seemingly more complicated but which allows a simpler theoretical description. Because the response of the medium in the cavity is nonlinear, the interference between the field  $E_1$  running in the right direction and of field  $E_2$  running in the left direction produces nonlinear interference effects which we want to avoid in the theoretical treatment. To this end we consider a device which deals with waves running in one direction only (for details consult fig. 9.3). We adopt a specific model to treat the response of the nonlinear medium. To this end we consider two-level atoms with homogeneous broadening. Furthermore we shall assume that the waves can be described as plane waves.

We adopt the slowly varying amplitude approximation and the rotating wave approximation. The field propagating in the medium is decomposed as usual into its positive and negative frequency parts  $\mathbf{E}^{(+)}$  and  $\mathbf{E}^{(-)}$ . The equations for the slowly varying amplitudes  $\mathbf{E}_0^{(+)}$  and  $\mathbf{P}_0^{(+)}$  of the field  $\mathbf{E}^{(+)}$  and polarization  $\mathbf{P}^{(+)}$ , respectively, were derived in section 5.7: (5.102), (5.103), (5.105). We start from these equations, which we write down again for the reader's convenience. To simplify the notation, we drop the index "0" so that  $\mathbf{E}$  and  $\mathbf{P}$  are now the *slowly varying* amplitudes alone.

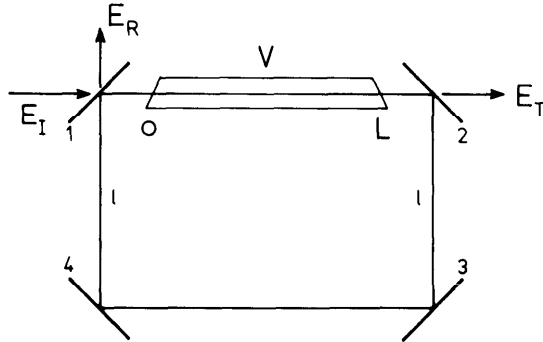


Fig. 9.3. Experimental set-up assumed in the model calculations; 1 and 2 are semi-transparent mirrors, 3 and 4 fully reflecting mirrors; V denotes the medium. [L.A. Lugiato, *Theory of Optical Bistability*, in: *Progress in Optics*, Vol. XXI, ed. E. Wolf, p. 71. North-Holland, Amsterdam 1984.1

We further assume that all vectors are pointing in the same direction and are  $\perp z$ . The field equations read:

$$\frac{\partial E^{(+)}}{\partial t} + c \frac{\partial E^{(+)}}{\partial z} = \frac{i\bar{\omega}}{2\varepsilon_0} P^{(+)} \quad (9.1)$$

The response of the medium to the field is described by the matter equations

$$\frac{\partial P^{(+)}}{\partial t} = [-i(\bar{\omega} - \omega_0) - \gamma] P^{(+)} + \frac{1}{i\hbar} E |\vartheta|^2 D, \quad (9.2)$$

and

$$\frac{\partial D}{\partial t} = \gamma_{\parallel} (D_0 - D) - \frac{2}{i\hbar} (E^{(+)} P^{(-)} - E^{(-)} P^{(+)}). \quad (9.3)$$

We include a detuning between the frequency of the atomic transition,  $\omega_0$ , and the frequency of the incident light,  $\bar{\omega}$  ( $\equiv \omega$  in our previous notation). Due to the boundary conditions at mirror 1 this frequency will then be taken over by the oscillation of the field and polarization within the medium. In contrast to the laser case where  $D_0$  was positive we assume here no pumping so that

$$D_0 = -N/V \quad (N \text{ total number of atoms}). \quad (9.4)$$

We remind the reader that  $D$  is the inversion density

$$D = (N_2 - N_1)/V. \quad (9.5)$$

In order to fix the problem entirely we have to add boundary conditions. Assuming that the distance between the end of the cavity and mirror 2 can be neglected the boundary condition reads

$$E_T(t) = \sqrt{T} E(L, t). \quad (9.6)$$

$T$  is the transmissivity of the mirror. The corresponding boundary condition at mirror 1 reads (where the reflectivity  $R = T - 1$ )

$$E(0, t) = \sqrt{T} E_1 + R \exp[-i\delta_0] E(L, t - \Delta t). \quad (9.7)$$

Here we take into account that the field  $E(L, t)$  is reinjected into the cavity. Relations (9.6) and (9.7) can be easily transformed into identical ones for  $E^{(+)}$

Evidently the second term supplies a feed-back. In it  $R$  is the reflectivity of the mirror,  $\delta_0$  a phase shift given by

$$\delta_0 = \frac{\omega_c - \omega_0}{c/\mathcal{L}}, \quad (9.8)$$

and  $\Delta t$  the time lag given by the time of flight of the light between mirrors 2 and 1 via mirrors 3 and 4 so that

$$\Delta t = (2l + L)/C. \quad (9.9)$$

$\omega_c$  is a frequency of the cavity mode closest to resonance with the incident field and

$$\mathcal{L} = 2(L + l). \quad (9.10)$$

### 9.3. Steady state behavior of the model of section 9.2

The steady state is characterized by

$$\frac{\partial E^{(+)}}{\partial t} = 0. \quad (9.11)$$

One can immediately convince oneself that  $P^{(+)}$  and  $D$  can then be chosen also time independent. Quite in analogy to our procedure in the case of a single mode laser we may express the polarization and the inversion density by  $E^{(+)}$  and  $E^{(-)}$  (cf. section 6.2). Inserting this result as well as (9.11) into (9.1) we readily find an equation of the general form

$$\frac{dE^{(+)}}{dz} = -\chi |E^{(+)}|^2 E^{(+)}, \quad (9.12)$$

where  $\chi$  is explicitly given by

$$\chi = \alpha(1 - i\Delta) \left( 1 + \Delta^2 + \frac{|E^{(+)}|^2}{I_s} \right)^{-1}. \quad (9.13)$$

From electrodynamics we know the significance of  $\chi$ . It is nothing but the complex dielectric susceptibility

$$X = \chi_a + i\chi_d, \quad (9.14)$$

where  $\chi_a$  and  $\chi_d$  are the absorptive and dispersive components, respectively. The quantities in (9.13) are defined as follows.  $A$  represents the detuning between the incident light with frequency  $\omega_0$  ( $\equiv \omega$ ) and the atomic transition frequency ( $\omega$  measured in units of  $\gamma$ ),

$$A = (\omega - \omega_0)/\gamma. \quad (9.15)$$

$I_s$  is the saturation intensity defined by

$$I_s = \frac{\hbar^2 \gamma \gamma_{\parallel}}{4|\vartheta|^2}. \quad (9.16)$$

When we specialize (9.13) taking  $A = 0$  and  $E^{(+)}$  very small we find

$$\chi = \alpha E^{(+)}. \quad (9.17)$$

From this jointly with (9.14) it transpires that  $\alpha$  has the meaning of the absorption constant. Within our specific model (9.1)–(9.3) it can be explicitly calculated and reads

$$\alpha = \frac{\bar{\omega} |\vartheta|^2}{2\varepsilon_0 \hbar V c \gamma} N. \quad (9.18)$$

In order to explore the physical meaning of our results we specialize them to the case of perfect resonance between incident light, atoms, and the cavity. We further introduce the normalized dimensionless electric field  $F$  by putting

$$F = \frac{E^{(+)}}{\sqrt{I_s}} = \frac{2|\vartheta|E^{(+)}}{\hbar\sqrt{\gamma\gamma_{\parallel}}}. \quad (9.19)$$

Then (9.12) acquires the explicit form

$$\frac{dF}{dz} = -\alpha \frac{F}{1 + F^2}, \quad (9.20)$$

where we have assumed that the field is real. We normalize the incident and transmitted amplitudes in a fashion analogous to (9.19) so that we

introduce the corresponding quantities  $y, x$  by

$$y = \frac{E_1^{(+)}}{\sqrt{I_s T}}, \quad (9.21)$$

$$x = \frac{E_T^{(+)}}{\sqrt{I_s T}}. \quad (9.22)$$

In this way our original equations for the boundary conditions (9.6) and (9.7) acquire the form

$$x = F(L), \quad (9.23)$$

$$F(0) = Ty + Rx. \quad (9.24)$$

It is a simple matter to solve the first order differential equation (9.20) and to express  $F$  as a function of  $z$ . We readily obtain

$$\ln\left(\frac{F(0)}{x}\right) + \frac{1}{2}[F^2(0) - x^2] = \alpha L. \quad (9.25)$$

When we combine (9.23), (9.24) and (9.25) we find an exact relation between the transmitted field  $x$  and the incident field  $y$ ,

$$\ln\left[1 + T\left(\frac{y}{x} - 1\right)\right] + \frac{x^2}{2}\left\{\left[1 + T\left(\frac{y}{x} - 1\right)\right]^2 - 1\right\} = \alpha L. \quad (9.26)$$

As can be seen, this equation depends on two parameters,  $\alpha L$  and  $T$ . The meaning of eq. (9.26) can be understood by a graphical representation as is shown in fig. 9.4. The steady state values of  $x$  are the intersections of the straight line (9.24) with the curve (9.25). The first one is the boundary condition of the cavity. The second is the transfer function of the medium which expresses the field at  $z=0$  as a function of the field at  $z=L$  and vice versa. It has neither maxima nor minima but it has an inflection point. The angular coefficient  $R_c$  of the tangent at the inflection point is such that  $0 < R_c < 1$ .  $R_c$  depends only on  $\alpha L$ . For  $R < R_c$  there is only one intersection point for all values of  $y$ . For  $R > R_c$  there is a range of values of  $y$  in correspondence of which one finds three intersection points  $x_a < x_b < x_c$ . Points  $x_a$  and  $x_c$  turn out to be stable while point  $x_b$  is unstable. Thus we are dealing here with the bistable situation. If we plot the steady state solutions  $x$  as a function of the incident field  $y$  we obtain an S-shaped curve (fig. 9.5) which gives rise to a hysteresis cycle. From this analysis it transpires that bistability arises from the combined action of the nonlinear transfer of the medium (eq. (9.25)) and of the feed-back from the mirrors

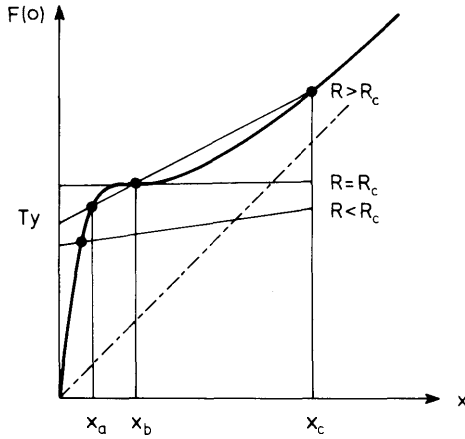


Fig. 9.4. Qualitative graph of the normalized field  $F(0)$  at  $z=0$  as a function of the field  $F(L) \equiv x$  at  $z=L$  (transfer function of the atomic medium at steady state). For  $R=0$  one has  $F(0)=y$ .  $x$  and  $y$  are proportional to the transmitted and incident fields  $E_T$  and  $E_i$ , respectively. For a generic  $R$ , the function  $x=x(y)$  is obtained by intersecting the curve with the straight line  $F(0)=RF(L)+Ty$ . [L.A. Lugiato, Theory of Optical Bistability, in: Progress in Optics, Vol. XXI, ed. E. Wolf, p. 71. North-Holland, Amsterdam 1984.]

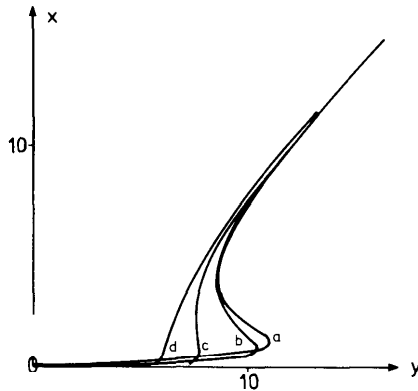


Fig. 9.5. Plot of transmitted light versus incident light at steady state for  $C = \alpha L/2T$  fixed equal to 10 and different values of  $\alpha L$  and  $T$ . For  $\alpha L \rightarrow 0$  one approaches the behavior predicted by the mean field theory. (a)  $\alpha L=20$ ,  $T=1$ ; (b)  $\alpha L=10$ ,  $T=0,5$ ; (c)  $\alpha L=2$ ,  $T=0,1$ ; (d) mean field,  $C=10$ . [L.A. Lugiato, Theory of Optical Bistability, in: Progress in Optics, Vol. XXI, ed. E. Wolf, p. 71. North-Holland, Amsterdam 1984.]



(eq. (9.24)). This feed-back action is essential because as one sees from figs. 9.4 and 9.5 there is no bistability for  $R = 0$ .

#### 9.4. The general case of an arbitrary susceptibility

We now wish to derive a general equation for the transmissivity of the cavity. This quantity is defined as the ratio of the transmitted intensity and the incident intensity, i.e.

$$\mathcal{T} = I_T / I_I. \quad (9.27)$$

We define these intensities by

$$I_I = |E_I^{(+)}|^2, \quad (9.28)$$

$$I_T = |E_T^{(+)}|^2. \quad (9.29)$$

We consider the general case also with respect to the field. Because we now admit that it is complex we write

$$E^{(+)}(t) = \rho(z) \exp[i\varphi(z)] \quad (9.30)$$

and obtain by inserting it into (9.12)

$$\frac{d\rho}{dz} = -\chi_a(\rho^2)\rho, \quad (9.31)$$

$$\frac{d\varphi}{dz} = -\chi_d(\rho^2). \quad (9.32)$$

By use of (9.6) and (9.7) we readily obtain for the transmissivity (9.27)

$$\mathcal{T} = \frac{T^2}{(\eta - R)^2 4R\eta \sin^2 \left\{ \frac{1}{2}[\varphi(L) - \varphi(0) - \delta_0] \right\}}, \quad (9.33)$$

where we have used the abbreviation

$$\eta = \rho(0)/\rho(L), \quad (9.34)$$

where

$$\eta \geq 1. \quad (9.35)$$

Let us now consider some special cases. If the cavity is empty so that  $\chi_a = \chi_d = 0$ , we have  $\eta = 1$  and  $\rho(L) = \rho(0)$ . (9.33) reduces to

$$\mathcal{T} = \frac{1}{1 + 4R \sin^2 \left( \frac{1}{2} \delta_0 / T^2 \right)}, \quad (9.36)$$

which represents the usual expression of the transmissivity as a function of the cavity detuning  $\delta_0$ .

In the special case of two-level atoms which we have been considering in the model treated above, (9.33) can be cast into a slightly more explicit form. To this end we introduce normalized incident and transmitted intensities by

$$Y = \frac{I_I}{I_s T}, \quad (9.37)$$

and

$$X = |F(L)|^2 = \frac{I_T}{I_s T}, \quad (9.38)$$

respectively, where  $I_s$  was defined above (9.16). From eqs. (9.31) and (9.32) with (9.6), (9.7), (9.13), (9.14) and (9.19) we obtain

$$X = \frac{2}{\eta^2 - 1} [\alpha L - (1 + \Delta^2) \ln \eta], \quad (9.39)$$

and

$$\varphi(L) - \varphi(0) = \Delta \ln \eta. \quad (9.40)$$

By solving eq. (9.39) with respect to  $\eta$  one obtains the function  $\eta = \eta(X)$ . Hence by inserting (9.39) into (9.33) one finds the expression of the transmissivity as a function of the normalized transmitted intensity:

$$\mathcal{T} = \frac{X}{Y} = \frac{T^2}{[\eta(X) - R]^2 + 4R\eta(X) \sin^2[\frac{1}{2}(A \ln \eta(X) - \delta_0)]}. \quad (9.41)$$

Therefore in this case the shape of the function  $\mathcal{T}(X)$  is governed by the dependence of  $\eta$  on  $X$ . In particular, for large  $X$ ,  $\eta$  approaches unity so that the transmissivity becomes constant and equal to the empty cavity value (9.36). When dispersion is dominant, under suitable conditions one or few resonances survive. The possibility of multistability (fig. 9.6) depends on the number of oscillations that the function

$$\sin [\frac{1}{2}(\Delta \ln \eta(X) - \delta_0)]$$

undergoes. As one sees from eq. (9.39),  $\eta$  is a monotonically decreasing function of  $X$  which varies from 1 to  $\exp[\alpha L / (1 + \Delta^2)]$ . Hence the quantity  $A \ln \eta - \delta_0$  varies from  $-\delta_0$  to  $\alpha L \Delta / (1 + \Delta^2) - \delta_0$ . Therefore the number of oscillations of the sine function is determined by the parameter  $\alpha L \Delta / (1 + \Delta^2)$ .

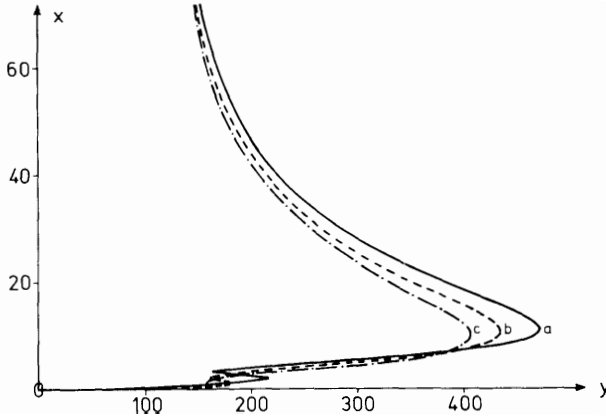


Fig. 9.6. Optical multistability,  $x = \sqrt{X}$  is the normalized transmitted field. In all curves  $C = 900$ ,  $A = 5$ ,  $\Theta = 0,05$ . Curve a corresponds to the case of homogeneous broadening ( $T_2^* = \infty$ ); curve b,  $\gamma T_2^* = 1$ ; curve c,  $\gamma T_2^* = 0.5$ . Clearly, multistability disappears with increased inhomogeneous broadening.

Eq. (9.41) can be rephrased as follows

$$Y = X(\eta) \frac{1}{T^2} \{ (\eta - R)^2 + 4R\eta \sin^2 [\frac{1}{2}(\Delta \ln \eta - \delta_0)] \}. \tag{9.42}$$

Eqs. (9.39) and (9.42) together give a parametric representation  $X = X(\eta)$ ,  $Y = Y(\eta)$  of the function  $X(Y)$  of transmitted versus incident intensity.

*The stationary solution in the "mean field limit"*

From now on, we consider exclusively the case of a two-level atomic system. In general, the susceptibility has the structure

$$\chi(|E|^2) = \alpha \tilde{\chi}(|F|^2), \tag{9.43}$$

where  $a$  and  $F$  are defined in (9.18) and (9.19), respectively. Therefore the steady state solution becomes particularly simple in the case  $\alpha L \ll 1$ , because as one sees from eqs. (9.31) and (9.32) the field becomes practically uniform in space. More precisely, we shall perform the multiple limit

$$\alpha L \rightarrow 0, \quad T \rightarrow 0, \quad \delta_0 \rightarrow 0, \tag{9.44}$$

with

$$C \equiv \alpha L / (2T), \quad \text{constant,}$$

$$@ = -\frac{\delta_0}{T} \frac{\omega_c - \omega_0}{cT/\mathcal{L}}, \quad \text{constant.}$$

It is easy to derive the steady state solution in the limit (9.44). In fact, from (9.31), (9.32) and (9.35), (9.44) we have to first order in  $\alpha L$

$$\begin{aligned}\eta &= 1 + \alpha L \tilde{\chi}_a(|F(L)|^2) = 1 + \alpha L \tilde{\chi}_a(X), \\ \varphi(L) - \varphi(0) &= \alpha L \tilde{\chi}_d(X).\end{aligned}\quad (9.45)$$

By inserting (9.45) into (9.33) we obtain in the limit (9.44)

$$\mathcal{T} = \{[1 + 2C\tilde{\chi}_a(X)]^2 + [\Theta - 2C\tilde{\chi}_d(X)]^2\}^{-1} \quad (9.46)$$

and because  $\mathcal{T} = X/Y$  we have

$$Y = X\{[1 + 2C\tilde{\chi}_a(X)]^2 + [\Theta - 2C\tilde{\chi}_d(X)]^2\}. \quad (9.47)$$

The limit (9.44) is called "mean field limit" in the literature on optical bistability. For a homogeneously broadened system, from (9.13) we have

$$Y = X \left[ \left( 1 + \frac{2C}{1 + \Delta^2 + X} \right)^2 + \left( \Theta - \frac{2C\Delta}{1 + \Delta^2 + X} \right)^2 \right]. \quad (9.48)$$

In the particular case  $\mathbf{A} = \Theta = 0$ , (9.48) can be written in terms of amplitudes instead of intensities as follows:

$$y = x + \frac{2Cx}{1 + x^2}. \quad (9.49)$$

Eqs. (9.48) and (9.49) can also be recovered directly from the exact solutions (9.39), (9.42) and (9.26), respectively, by performing the limit (9.44). Let us briefly comment on the physical meaning of the limit (9.44). First,  $\alpha L \rightarrow 0$  (i.e.  $\alpha \rightarrow 0$ ) is the weak coupling limit in the interaction between the electric field and the atoms. However, if we only let  $\alpha L \rightarrow 0$  but keep  $T$  finite,  $C$  vanishes and therefore we obtain the empty cavity solution  $Y = X(1 + \Theta^2)$ . On the contrary, if we also let  $T \rightarrow 0$ , the parameter  $C$  is arbitrary and we obtain the nonlinear terms in eq. (9.47), which produce all the interesting phenomena. The physical meaning of the limit  $T \rightarrow 0$  is that the mean lifetime  $\alpha\mathcal{L}/cT$  of the photons in the cavity becomes infinite so that the photons can experience the interaction with the atoms even when this becomes vanishingly small. Finally the limit

$$\delta_0 = \frac{\omega_c - \omega_0}{c/\mathcal{L}} \rightarrow 0, \quad \Theta = \frac{\omega_c - \omega_0}{cT/\mathcal{L}}, \text{ finite}$$

means that the cavity detuning must be smaller than the free spectral range, but on the same order of magnitude of the cavity line-width  $k$ , given by

$$k = cT/\mathcal{L}. \quad (9.50)$$

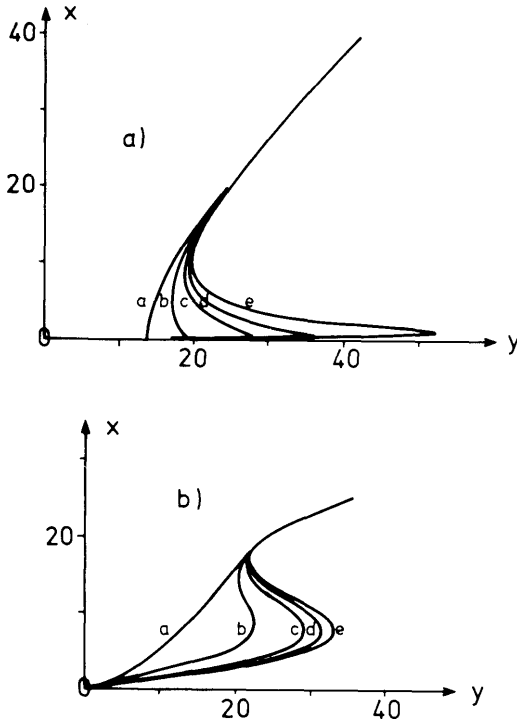


Fig. 9.7. Plot of the transmitted amplitude  $x = \sqrt{X}$  as a function of the incident amplitude  $y$  in the homogeneously broadened case. In both figs. (a) and (b) curves a, b, c, d show the exact stationary solution (eqs. (9.39) and (9.42)), curve e is the mean field result. (a)  $C = 50$ ,  $A = \Theta = 0$ ; (b)  $C = 50$ ,  $A = 10$ ,  $\Theta = 2.25$ . For curves a,  $\alpha L = 100$ ,  $T = 1$ ; for curves b,  $\alpha L = 50$ ,  $T = 0, 5$ ; for curves c,  $\alpha L = 20$ ,  $T = 0, 2$ ; for curves d,  $\alpha L = 10$ ,  $T = 0, 1$ . [L.A. Lugiato, Theory of Optical Bistability, in: Progress in Optics, Vol. **XXI**, ed. E. Wolf, p. 71. North-Holland, Amsterdam 1984.]

Fig. 9.7 shows how the curve (9.48) is approached in the limit (9.44). The part of the curves with negative slope is unstable so that one finds a hysteresis cycle. Curve e in fig. 9.7a is obtained from (9.48) for  $C = 50$ ,  $A = \Theta = 0$  (purely absorptive case); curve e in fig. 9.7b comes from (9.48) for  $C = 50$ ,  $A = 10$ ,  $\Theta = 2.25$  (dispersive case). In both figures 9.7a and 9.7b the curves a, b, c, d show the exact solution (9.39) and (9.42) for different values of  $\alpha L$  and of the transmissivity, chosen in such a way that  $C = \alpha L / (2T)$  is constant equal to 50. For large values of  $\alpha L$  and  $T$ , as in curve a, there is no bistability, whereas the bistable behavior increases by decreasing  $\alpha L$  and  $T$ . In this way one approaches the mean field result (9.48) which is already a good approximation for  $\alpha L \approx 1$ . For  $C$  and  $T$  fixed, the mean

field curve is a better approximation in the dispersive case (fig. 9.7b) than in the absorptive one (fig. 9.7a). This is due to the fact that absorption is reduced in the dispersive case so that the variation of the field in space is not strong even for  $\alpha L$  large.

In the following two subsections we shall analyze the mean field state equation (9.48), which expresses the incident intensity as a function of the transmitted intensity. It depends on three parameters, the cooperativity parameter  $C$ , the atomic detuning  $A$  and the cavity mistuning  $\Theta$ . In comparing (9.48) with experimental data, the definition (9.44) of  $C$  must be changed into

$$C = \alpha LJ / 2\pi, \quad (9.51)$$

where  $J$  is the effective finesse of the cavity. A general property of eq. (9.48) is that, contrary to (9.39) and (9.42), it can never produce multistability, but at most it can lead to bistability.

Bistability condition in the resonant case (mean *field* limit)

The field internal to the cavity is in general quite different from the incident field, because there is a reaction field, co-operatively produced by the atoms, which counteracts the incident one.

In the purely absorptive, resonant case  $A = \Theta = 0$ , the steady state behavior is described by (9.49). The nonlinear term  $2Cx/(1+x^2)$  arises from the reaction field and hence from atomic co-operation, which is measured by the parameter  $C$ . For very large  $x$ , (9.49) reduces to the empty cavity solution  $x = y$  (i.e.  $E_T = E_I$ ). The atomic system is saturated so that the medium is bleached. In this situation each atom interacts with the incident field as if the other atoms were not there; this is the nonco-operative situation, and in fact the quantum statistical treatment shows that atom–atom correlations are negligible. On the other hand, for small  $x$ , (9.49) reduces to  $y = (2C + 1)x$ . Here the linearity arises simply from the fact that for small external field the response of the system is linear. In this situation the atomic system is unsaturated, for large  $C$  the atomic cooperation is dominant and one has strong atom–atom correlations. The curves  $y(x)$  obtained by varying  $C$  are analogous to the Van der Waals curves for the liquid–vapor phase transition, with  $y$ ,  $x$  and  $C$  playing the role of pressure, volume and temperature, respectively. For  $C < 4$ ,  $y$  is a monotonic function of  $x$  so that one has no bistability (fig. 9.8). However, in part of the curve the differential gain  $dx/dy$  is larger than unity so that in this situation one has the possibility of transistor operation. In fact, if the incident intensity is adiabatically modulated around a value of  $I$ , such that  $dI_T/dI_I = (x/y) dx/dy > 1$ , the modulation is amplified in the transmitted light.

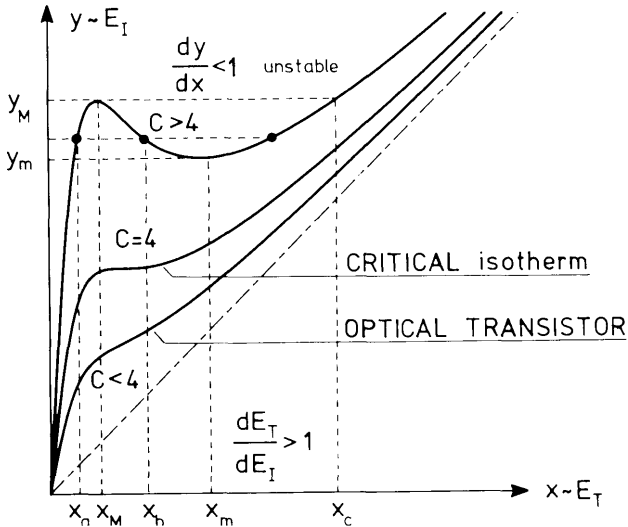


Fig. 9.8. Plot of the mean field state equation (9.49) for purely absorptive bistability with  $\Theta=0$ , for different values of the bistability parameter  $C$ . [L.A. Lugiato, Theory of Optical Bistability, in: Progress in Optics, Vol. XXI, ed. E. Wolf, p. 71. North-Holland, Amsterdam 1984.]

For  $C = 4$  (critical curve) the graph has an inflection point with horizontal tangent. Finally for  $C > 4$  the curve develops a maximum and a minimum, which for  $C \gg 1$  correspond to

$$(x_M \approx 1, y_M \approx C) \quad \text{and} \quad (x_m \approx \sqrt{2C}, y_m \approx \sqrt{8C}).$$

Hence for  $y_m < y < y_M$  one finds three stationary solutions  $x_a < x_b < x_c$ . As a more detailed analysis reveals, solutions  $x_b$  on the part of the curve with negative slope are unstable. Therefore we have a bistable situation and by exchanging the axes  $x$  and  $y$  we immediately obtain the hysteresis cycle of transmitted versus incident light. Since atomic co-operation is dominant in the states  $x_a$  and negligible in the states  $x_c$ , we shall call  $x_a$  "co-operative stationary state" and  $x_c$  "one-atom stationary state", according to the literature.

**Bistability conditions in the general case (mean field limit)**

Let us now consider eq. (9.48) for general values of  $A$  and  $O$ . We assume that  $A, O > 0$  because for  $A, O < 0$  it is more difficult to obtain bistability. For definiteness, we take  $A, O \geq 0$ ; however, the situation is symmetrical with respect to a simultaneous sign change in  $A$  and  $O$ . The function  $Y(X)$

defined by (9.48) has always a single inflection point at

$$X_{\text{inf}} = \frac{2C - \Delta\Theta + 1}{C + \Delta\Theta - 1} (\Delta^2 + 1). \quad (9.52)$$

In order to have bistability the conditions are

$$X_{\text{inf}} > 0, \quad \left. \frac{dY}{dX} \right|_{X_{\text{inf}}} < 0. \quad (9.53)$$

The first condition guarantees that the inflection point is within the physical region  $X > 0$ , while the second one identifies the values of the parameters for which the curve  $Y(X)$  has a maximum and a minimum. For  $\Delta\Theta > 0$  the first of conditions (9.53) reads

$$2C > \Delta\Theta - 1. \quad (9.54)$$

The second condition gives

$$(2C - \Delta\Theta + 1)^2 (C + 4\Delta\Theta - 4) > 27C(\Delta + \Theta)^2. \quad (9.55)$$

The analysis of (9.55) and (9.54) leads to conclude that:

- (1) Bistability is impossible for  $C < 4$ .
- (2) For a fixed value of  $C > 4$ , the largest hysteresis cycle is obtained for  $A = \Theta = 0$  and bistability exists only in a finite domain of the plane  $\{A, \Theta\}$  around the origin.
- (3) If we keep  $C$  and  $A$  fixed and  $C$  satisfies condition (9.55) for  $\Theta = 0$ , by increasing  $\Theta$  the size of the hysteresis cycle increases until it reaches a maximum and then decreases. Finally the cycle vanishes in correspondence to a value of  $\Theta$  smaller than  $(2C + 1)/A$  (see (9.54)).
- (4) If we keep  $C > 4$  fixed and increase  $A$  and  $\Theta$  simultaneously from zero with the ratio  $\Delta/\Theta$  kept fixed, the hysteresis cycle of the curve  $X(Y)$  shifts to the left and decreases in size, until it disappears.

Therefore in homogeneously broadened, two-level systems when absorptive bistability for  $\Theta = 0$  is not possible also dispersive bistability for general values of  $A$  and  $\Theta$  is impossible. This is no longer true in the case of *inhomogeneously broadened* systems ( $T_2^* < \infty$ ). For fixed  $A$ ,  $\Theta$  and inhomogeneous relaxation time  $T_2^*$  one obtains bistability provided  $C$  is larger than a suitable value  $C_{\text{min}}$  which depends on  $A$ ,  $\Theta$  and  $T_2^*$ .  $C_{\text{min}}$  increases rapidly with  $(T_2^*)^{-1}$ . The important point is that for  $(\gamma T_2^*)^{-1} \ll 1$  one finds values of  $C$  such that the system is not bistable for  $A = \Theta = 0$  but becomes bistable for  $A$  and  $\Theta$  large enough. In other words, for these values of  $T_2^*$  and  $C$  one does not find absorptive bistability, but only dispersive bistability.



So far we have only considered hysteresis cycles obtained by varying the incident field intensity and keeping the parameters  $C$ ,  $A$  and  $\Theta$  fixed. Of course, one can also consider cycles obtained by keeping  $Y$  fixed and varying  $C$ , or  $A$ , or  $\Theta$ , or some of these parameters simultaneously. E.g., one can perform an experiment in which one adiabatically sweeps the incident field frequency, thereby varying  $A$  and  $\Theta$  together.

#### Absorptive versus dispersive bistability

As we said, we have purely absorptive optical bistability when the atomic detuning  $A$  vanishes. On the other hand, we have purely dispersive optical bistability when  $A$  is so large that the absorptive part of the nonlinear susceptibility is negligible, hence the state equation (9.47) reduces to

$$Y = X\{1 + [\Theta - 2C\tilde{\chi}_d(X)]^2\}. \quad (9.56)$$

In the case of a homogeneously broadened, two-level system (9.56) is a good approximation of (9.47) when

$$\Delta^2 \gg 1, \quad \Delta\Theta \gg 1, \quad \Delta \gg \Theta. \quad (9.57)$$

Moreover if also the condition

$$\frac{2C}{\Delta\Theta} - 1 \ll 1 \quad (9.58)$$

is satisfied, (9.56) reduces in turn to the relation

$$Y = X \left[ 1 + \left( \Theta - \frac{2C}{\Delta} + \frac{2C}{\Delta^3} X \right)^2 \right]. \quad (9.59)$$

Eq. (9.59) is a particular example of the "cubic model" of purely dispersive optical bistability

$$Y = X[1 + (B - AX)^2], \quad (9.60)$$

which gives bistability for  $B > \sqrt{3}$ . Eq. (9.60) describes optical bistability in several materials, e.g. in a Kerr medium in a certain limit.

Let us now describe the physical mechanisms that give rise to hysteresis in absorptive and dispersive optical bistability. In the absorptive case, let us consider for simplicity the resonant situation  $\Theta = 0$ . In the cooperative (i.e. lower transmission) branch the transmission is small because the presence of the saturable absorber drastically decreases the quality factor  $Q$  of the cavity. Most of the incident light is reflected from the cavity. Increasing the incident field, the absorber begins to saturate, which allows  $Q$  to increase. This in turn increases the internal field which again increases

the saturation and so on, until the absorber is bleached so that  $I_T \approx I_1$ . On the other hand, when the system is in the one-atom (i.e. higher transmission) branch and the incident intensity is decreased, the field internal to the cavity is already strong enough to maintain the absorber saturated and therefore the transmitted light switches "off" at an incident power lower than that necessary to switch "on", thereby producing hysteresis.

In the case of purely dispersive optical bistability the mechanism is quite different. In the empty cavity, the transmission is low because the empty cavity frequency  $\omega_c$  is detuned from the incident frequency  $\omega_0$ . If the atomic and cavity detuning have the same sign, by increasing the incident field the nonlinear refractive index changes the effective optical length of the cavity towards resonance. This in turn increases the internal field which further drives the effective cavity frequency  $\omega'_c = \omega_c - \kappa\chi_d(X)$  towards the incident field frequency and so on, until resonance is reached so that  $I_T \approx I_1$ . On the other hand, when the system is in the higher transmission branch and the incident intensity is decreased, the internal field is already strong enough to maintain resonance, which again produces hysteresis.

In order to complete the discussion of the steady state behavior, let us illustrate the relative advantage of absorptive and dispersive optical bistability. First of all, it is clear that dispersive optical bistability is "easier" mainly for two reasons:

- (a) it does not require saturation of the medium as it appears from the cubic model (9.60);
- (b) in absorptive optical bistability the resonance condition between the incident field and the atoms cannot be easily maintained for a time long enough to allow the system to reach steady state, due to the jitter of the laser frequency.

As we have seen in the previous section, in the case of homogeneous broadening absorptive optical bistability has the advantage of exhibiting the largest hysteresis cycle for fixed  $C$ , when  $A = \Theta = 0$ . However, this is no longer true in the case of inhomogeneous broadening. Furthermore, even in the case of homogeneous broadening the switching from the low transmission to the high transmission branch occurs for lower values of the input field when  $A$  and  $\Theta$  are different from zero. This is an important advantage, also because the presence of a field too intense in the absorber might produce undesirable effects, e.g. excessive heating of the medium.

On the other hand, from the viewpoint of theory and hence of the comparison between experiment and theory, absorptive optical bistability with  $\Theta = 0$  is certainly much easier to deal with, because in eqs. (9.1)–(9.3) all the fields can be assumed real. This is the reason why most theoretical papers treat the absorptive case.

## 9.5. Concluding remarks on chapter 9

In this section I have tried to give a short outline of some basic ideas on optical bistability, closely following the first part of an article by Lugiato. Especially in the limit (9.44) a number of further phenomena could be studied in the literature. Using the decomposition of the field  $E$  into cavity modes, equations strongly reminiscent of the multimode laser equations of the semiclassical theory have been established. Their exact or approximate solution allows one to study relaxation phenomena. In particular, qualitatively new phenomena could be found such as the onset of pulses and chaos under constant incident intensity. Also a detailed quantum theoretical treatment of these phenomena is available. But an inclusion of these phenomena is beyond the scope of our book. It may be worthwhile, however, pointing out that the methods used in these treatments are either identical with the methods we have presented here with respect to the laser or can be considered as certain useful extensions, such as the "dressed mode" approach of Lugiato and Benza. For further details of the above mentioned results, we refer the reader to the references, in particular to Lugiato's article.

# Quantum Theory of the Laser I

## **A first approach via quantum mechanical Langevin equations. Coherence, noise, photon statistics**

### **10.1. Why quantum theory of the laser?**

The semiclassical theory of the laser which we presented in the preceding chapters enabled us to explain or even to predict many properties of laser light. According to these studies, laser action sets in above a certain threshold of the pump, whereas below that threshold no emission occurs at all. But this latter finding is highly unsatisfactory, because even below laser threshold light emission happens – namely the light emission by usual lamps. An adequate laser theory must be able to describe the transition from the light of usual lamps to laser light and it should contain light emission by usual lamps as a special case. Thus quite evidently, we must have left out an important aspect of laser theory. In order to elucidate the problem we consider the emission of light from usual light sources more closely.

As we know (cf. for example Vol. 1) light is spontaneously emitted by excited atoms. This spontaneous emission cannot be dealt with in the frame of a theory which treats the light field classically. We have met this fact already in the first volume when we calculated the Einstein coefficients of absorption and emission. There we could derive the corresponding coefficients of absorption and stimulated emission but not those of spontaneous emission. As we have shown in Vol. 1, spontaneous emission can be treated adequately only when we quantize the light field. In that volume we have also seen that the damping of a classical or quantum mechanical quantity is always accompanied by a corresponding fluctuation. For example, the light field in the resonator is damped due to the transmissivity of the mirrors. According to our studies in Vol. 1 we have to expect fluctuations of the light field amplitude. Both the fluctuations produced by spontaneous emission as well as those accompanying cavity losses are not contained in the semiclassical laser equations. We shall see that a fully

quantum mechanical description of a laser becomes necessary when we wish to understand the difference between a laser and a conventional lamp. As we shall see, laser light fluctuations cannot be neglected but decisively determine the detailed coherence properties of laser light. When we consider the properties of laser light not so much from the point of view of optics but rather from that of electronics we will speak of noise properties rather than of coherence properties.

Finally, when we consider laser light as composed of photons we are led to study photon statistics. A fully quantum theoretical treatment of the laser is not only important with respect to the properties of laser light, its coherence, noise, and photon statistics but it is also of a fundamental importance for laser theory. Namely such a treatment will allow us to derive the laser equations from first principles.

Since the rigorous quantum theory of laser light is rather involved we shall proceed in two steps. In this chapter we shall deal with quantum mechanical Langevin equations. This will allow us to present the most interesting and important properties of laser light, namely its coherence, noise and photon statistics in a way which can fairly easily be understood and which immediately allows a comparison with experimental results. Then in chapter 11 we shall develop a second approach to the quantum theory of laser light, this time based on the density matrix equation. The density matrix equation will be transformed into a generalized Fokker–Planck equation which then will be reduced, under suitable conditions, to the one we shall use directly in section 10.5. Readers, who are not so much interested in the details of these quantum mechanical derivation, can skip chapter 11. For readers, who are not so familiar with quantum theory, especially that dealing with quantized fields, we give an important hint. As the reader will soon find out when reading the following sections, the basic quantum mechanical laser equations are strongly reminiscent of the semiclassical laser equations. Indeed, the quantum mechanical laser equations look the same as the semiclassical laser equations with the exception of an additional term which represents fluctuating forces. Though the corresponding equations are "operator equations" their physical meaning can still be grasped at a classical level.

## 10.2. The laser Hamiltonian

In this section we shall derive the quantum mechanical laser equations from first principles. To this end we consider the laser system more closely. First of all the laser consists of a material containing the laser active atoms (or other quantum systems). We know that there may be a field in the cavity.

Furthermore we know that the atoms and the field interact with each other. In a quantum mechanical formulation one usually starts with writing down a Hamiltonian which in a classical interpretation has the meaning of an energy. In a quantum mechanical treatment the Hamiltonian becomes the Hamilton operator. We shall write down the explicit form of the Hamilton operators below but first we introduce the following abbreviations. The Hamiltonian operator of the field will be denoted by  $H_f$ , that of the atoms by  $H_A$ , and that describing the interaction field-atoms by  $H_{Af}$ . A treatment of the laser with these Hamiltonians is, however, quite insufficient. Namely, the field is coupled to the mirrors which will cause damping and fluctuations of the field. We shall describe the mirrors or other systems to which the field (with the exception of the laser atoms) is coupled by a heatbath (or "reservoir").

Similarly the atoms are coupled to heatbaths. They are pumped from the outside and the excited atoms may decay in various ways. They may decay, for instance, by radiationless transitions, or they may decay by radiative emission into the nonlasing modes of the light field. The motion of electrons in the atoms may be perturbed by their interactions with lattice vibrations or by atom-atom collision. All these effects are taken care of by an appropriate coupling of the atoms to heatbaths.

It will be important for our following discussion that the detailed physical nature of these heatbaths need not be known. Rather for our analysis it will be sufficient to know only a few features which we shall discuss below.

We denote the Hamiltonians of the heatbaths 1 and 2 by  $H_{B_1}$  and  $H_{B_2}$ , respectively. Similarly the Hamiltonian of the interaction between heatbaths 1 and the field is denoted by  $H_{B_1-f}$ , that of the interaction between heatbaths 2 and the atoms by  $H_{B_2-A}$ .

According to the fundamental rules of quantum mechanics the Hamiltonian of the total system is obtained by a sum over the individual contributions so that the total Hamiltonian reads

$$H = H_f + H_A + H_{Af} + H_{B_1} + H_{B_1-f} + H_{B_2} + H_{B_2-A}. \quad (10.1)$$

Here and in the following we shall speak of Hamiltonians, though, more precisely, we should speak of Hamiltonian operators.

We now have to consider the explicit form of these Hamiltonians. To this end we use the results we derived in Vol. 1, chapters 5 to 7. But the reader will note that a good deal of the results to be presented here can be understood directly without a detailed knowledge of their previous derivation. Because the single mode case shows all the important features we will treat only that case. We decompose the electric field strength in the resonator

as follows

$$E_z = i(b - b^+) \sqrt{\hbar\omega / \epsilon_0} (1/\sqrt{L}) \sin kx. \quad (10.2)$$

In a classical treatment,  $b^+$  and  $b$  are time dependent amplitudes. In our present treatment they are operators which obey the commutation relation (cf. Vol. 1)

$$bb^+ - b^+b = 1. \quad (10.3)$$

With their help the quantized field energy can be written in the form

$$H_f = \hbar\omega b^+b, \quad (10.4)$$

so that we have found the explicit form of the field Hamiltonian. The operators  $b^+$  and  $b$  describe the creation or annihilation, respectively, of a photon of the field mode under consideration. As we have seen in Vol. 1, we cannot only describe the creation and annihilation of photons, but also the corresponding processes with respect to electrons. Let us consider a single atom with two levels 1 and 2. The creation of an electron in level 1 is described by the creation operator  $a_1^+$  and in level 2 by  $a_2^+$ . Correspondingly the annihilation of an electron in levels 1 or 2 is described by annihilation operators  $a_1$  or  $a_2$ , respectively. For sake of completeness we mention that these operators obey commutation relations which are as follows:

$$a_j^+ a_k^+ + a_k^+ a_j^+ = 0, \quad (10.5)$$

$$a_j a_k + a_k a_j = 0, \quad (10.6)$$

$$a_j^+ a_k + a_k a_j^+ = \delta_{kj}, \quad (10.7)$$

where  $j$  and  $k$  can acquire the values 1 or 2. With help of these operators, according to Vol. 1, the Hamiltonian of a single atom, which we denote by  $H_a$ , acquires the form

$$H_a = W_1 a_1^+ a_1 + W_2 a_2^+ a_2. \quad (10.8)$$

$W_j$  is the energy of the electron in its level  $j = 1, 2$ . Because we can choose the zero of energy arbitrarily we shall choose

$$W_1 = 0. \quad (10.9)$$

We relate the transition energy to the atomic transition frequency  $\bar{\omega}$  by

$$W_2 = \hbar\bar{\omega}. \quad (10.10)$$

Therefore in the following (10.8) will be used in the form

$$H_a = \hbar\bar{\omega} a_2^+ a_2. \quad (10.11)$$

Because there are many laser active atoms in the cavity we shall distinguish them as previously by the index  $\mu$  so that we have to attach this second index to the operators  $a$ ,

$$a_j^+ \rightarrow a_{i,\mu}^+, \tag{10.12}$$

$$a_j \rightarrow a_{\mu},$$

The energy of the atomic system is a sum over the individual energies,

$$H_A = \hbar\bar{\omega} \sum_{\mu} a_{2,\mu}^+ a_{2,\mu}. \tag{10.13}$$

Again for sake of completeness we quote the corresponding commutation relations for the atomic operators

$$a_{j\mu}^+ a_{k\mu'}^+ + a_{k\mu'}^+ a_{j\mu}^+ = 0, \tag{10.14}$$

$$a_{j\mu} a_{k\mu'} + a_{k\mu'} a_{j\mu} = 0, \tag{10.15}$$

$$a_{j\mu}^+ a_{k\mu'} + a_{k\mu'} a_{j\mu}^+ = \delta_{jk} \delta_{\mu\mu'}. \tag{10.16}$$

Finally we have to deal with the explicit form of the Hamiltonian which describes the interaction between the field mode under consideration and an atom. This form has been derived in Vol. 1 and reads\*

$$H_{af} = \hbar g a_1^+ a_2 (b + b^+) + \hbar g a_2^+ a_1 (b + b^+). \tag{10.17}$$

The meaning of this interaction Hamiltonian can be easily visualized when we recall that  $a_1^+$ ,  $a_2$  and  $b^+$ ,  $b$  are creation or annihilation operators. For instance the term  $a_1^+ a_2 b^+$  describes the creation of a photon while an electron is annihilated in state 2 and created again in state 1 (remember that the sequence of operators is read from the right to the left). A complete survey of the processes described by (10.17) is presented in fig. 10.1.  $g$  is a coupling coefficient which is proportional to the optical dipole matrix element

$$g_{\lambda,12} = -i \sqrt{\frac{\bar{\omega}}{2\hbar\epsilon_0}} \int \varphi_1^*(\mathbf{x}) \mathbf{e} \mathbf{x} \varphi_2(\mathbf{x}) d^3 \mathbf{x} \mathbf{u}_{\lambda}(x_0). \tag{10.18}$$

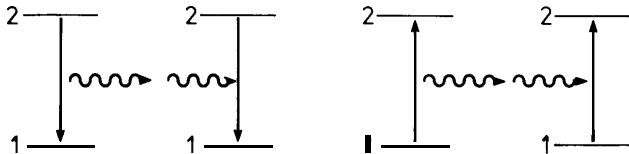


Fig. 10.1 Schematic representation of the processes described by eq. (10.17).

\*Here and in the following we shall assume  $g$  real which can always be achieved by the proper choice of a phase factor of  $a_1^+ a_2$ .



(This form can be deduced from Vol. 1, eq. (7.38) by help of the relation (7.103) of that volume.)  $\mathbf{u}_\lambda$  is the field mode which in the present case is assumed to be polarized in z-direction and to be given by

$$\mathbf{u}(\mathbf{x}_0) = \frac{\mathbf{e}_z}{JL} \sin kx_0. \quad (10.19)$$

$\mathbf{x}_0 = (x_0, y_0, z_0)$  is the position of the atom. In the following we shall adopt the rotating wave approximation which we got to know in section 5.6 and which we can reformulate in a different manner. Namely, it amounts to confining our analysis to real transitions in which a photon is emitted while the atom goes from its upper state to its ground state, or a photon is absorbed while the atom goes from its ground state to its upper state. Under this assumption we may reduce the Hamiltonian (10.17) to the form

$$H_{\text{af}} = \hbar g a_1^+ a_2 b^+ + \hbar g a_2^+ a_1 b. \quad (10.20)$$

When the field mode interacts with a set of atoms distinguished by an index  $\mu$ , we find the total interaction Hamiltonian by summing up over the individual contributions (10.20) provided we distinguish the atomic operators by an additional index  $\mu$  as before. We thus obtain

$$H_{\text{Af}} = \sum_{\mu} (\hbar g_{\mu} a_{1,\mu}^+ a_{2,\mu} b^+ + \hbar g_{\mu}^* a_{2,\mu}^+ a_{1,\mu}), \quad (10.21)$$

where we had to supply  $g$  by the index  $\mu$  because according to (10.18)  $g$  depends on the atomic coordinate  $\mathbf{x}_{0,\mu}$ . For sake of simplicity we shall drop the index  $\mu$  of  $g$ , because at least for running waves the dependence of  $g_{\mu}$  on  $\mathbf{x}_{0,\mu}$  can be removed by a suitable transformation of  $a$ ,  $a_{\mu}^+$  (cf. exercises on page 125).

When we take the sum over Hamiltonians (10.4), (10.13) and (10.21) we find a Hamiltonian which describes the interaction of the field with the set of atoms. But this resulting Hamiltonian is not enough to describe a laser because the field and the atoms are coupled to their corresponding heatbaths (reservoirs). The effect of the heatbaths on the field operators and on the atomic operators can be taken care of by the additional operators  $H_{B_1}$ ,  $H_{B_1-f}$ ,  $H_{B_2}$ ,  $H_{B_2-A}$  in the total Hamiltonian (10.1). In contrast to  $H_f$ ,  $H_A$  and  $H_{\text{Af}}$  we need not know these additional Hamiltonians explicitly. Indeed it was shown in Vol. 1 that only few general properties of these Hamiltonians must be known. The basic idea of the next step consists in eliminating the heatbath variables implicitly contained in  $H_{B_1}, \dots, H_{B_2-A}$ . According to Vol. 1, this may be done in two ways, either in the frame of quantum mechanical Langevin equations or in that of the density matrix equation.

In sections 10.3 and 10.4 we shall follow up the first approach, while section 11.1 will be devoted to the second.

### 10.3. Quantum mechanical Langevin equations

#### 10.3.1. A *field* mode coupled to a *heatbath*

To derive quantum mechanical Langevin equations we use the Heisenberg picture. In this picture the operators are treated as time dependent quantities whereas the wave functions are time independent. The time dependence of the operators is determined by the Heisenberg equations of motion which can be obtained as follows. Let  $\Omega$  be an operator whose time dependence we wish to study. Then its time derivative is given by the equation

$$\dot{\Omega} = \frac{i}{\hbar} [H, \Omega] \equiv \frac{i}{\hbar} (H\Omega - \Omega H), \quad (10.22)$$

i.e. the time derivative of  $\Omega$  is given by the commutator between the Hamiltonian  $H$  and the operator  $\Omega$ . Let us briefly remind the reader how this formalism works by means of a simple example.

Let us consider the field operator  $b$  and its corresponding Hamiltonian  $H_f$  alone so that we consider a freely oscillating field without any couplings to atoms or reservoirs. Using  $\Omega = b$  and inserting  $H_f$  (eq. (10.4)) instead of  $H$  into (10.22) we obtain

$$\dot{b} = \frac{i}{\hbar} \hbar\omega (b^+bb - bb^+b). \quad (10.23)$$

Because of the commutation relation (10.3) the r.h.s. can be transformed into

$$\dot{b} = -i\omega b. \quad (10.24)$$

We leave the performance of this transformation as an exercise to the reader.

Now let us consider the interaction between the field and the heatbath  
1. The time derivative of  $b$  is then given by the equation

$$\dot{b} = \frac{1}{\hbar} [(H_f + H_{B_1} + H_{B_1-f}), b]. \quad (10.25)$$

Because the Hamiltonian of the heatbath and the interaction Hamiltonian contain the variables of the heatbath, corresponding Heisenberg equations must be established for the heatbath variables. As has been shown in Vol. 1, the heatbath variables can be eliminated from these equations and one finds a closed equation for the field operator  $b$  alone. The thus resulting

equation reads

$$\dot{b} = -i\omega b - \kappa b + F(t), \quad (10.26)$$

where  $\kappa$  describes the damping of the field mode.  $\kappa$  is identical with the quantity we introduced in the semiclassical laser equations in a phenomenological way in section 5.5. As shown in Vol. 1, in the present case  $\kappa$  can be derived from first principles.  $F$  is a fluctuating force which is an operator. The properties of  $F(t)$  and its Hermitian conjugate were derived in Vol. 1 (formulas (9.83)–(9.86)). When we denote the quantum mechanical average over the reservoir variables, which are still implicit in  $F$ , according to Vol. 1 the following relations hold:

$$\langle F(t) \rangle = \langle F^+(t) \rangle = 0, \quad (10.27)$$

$$\langle F^+(t) F^+(t') \rangle = \langle F(t) F(t') \rangle = 0, \quad (10.28)$$

$$\langle F^+(t) F(t') \rangle = \bar{n}(T) 2\kappa \delta(t - t'), \quad (10.29)$$

$$\langle F(t) F^+(t') \rangle = (\bar{n}(T) + 1) 2\kappa \delta(t - t'). \quad (10.30)$$

$\bar{n}(T)$  is the mean photon number of the field mode at temperature  $T$ . This is the temperature of the reservoir to which the field mode is coupled. In the optical case, at room temperature  $\bar{n}$  is much smaller than unity so it can be neglected in practically all cases. For sake of completeness, however, we shall carry  $\bar{n}$  on in our following analysis.

### 10.3.2. Electrons (of atoms) coupled to heatbaths

Now let us turn to the Heisenberg equation of the electron of an individual atom. We start with a free atom which does not interact with any other system. In the following we shall make contact as close as possible with our semiclassical treatment. There we have used the inversion (or, in other words, the population difference) as a variable. Because  $a_1^+ a_1$  and  $a_2^+ a_2$  are the operators which measure the occupation of the atomic levels 1 and 2, respectively, we introduce the operator  $d$  via

$$d = a_2^+ a_2 - a_1^+ a_1. \quad (10.31)$$

The Heisenberg equation for  $d$  reads, of course,

$$\dot{d} = \frac{i}{\hbar} [H_a, d] \quad (10.32)$$

with  $H_a$  given by (10.11). On account of the commutation relations (10.5)–(10.7) one readily derives (compare the exercises) that the r.h.s. of (10.32)

vanishes,

$$d = 0. \quad (10.33)$$

Furthermore we introduce the operators

$$\alpha = a_1^+ a_2, \quad (10.34)$$

$$\alpha^+ = a_2^+ a_1. \quad (10.35)$$

A little algebra with help of the commutation relations just mentioned leads us to the following Heisenberg equations of motion:

$$\dot{\alpha} = -i\bar{\omega}\alpha, \quad (10.36)$$

$$\dot{\alpha}^+ = i\bar{\omega}\alpha^+ \quad (10.37)$$

(compare the exercises).

We now turn to a derivation of the Heisenberg equation of motion of the electron (atom) when it is coupled to heatbaths. To this end we again remind the reader of the main results of Vol. 1, chapter 9.

Let us introduce the mean values of the occupation numbers of the electronic levels  $j = 1, 2$  by means of

$$n_j = \langle \Phi | a_j^+ a_j | \Phi \rangle, \quad j = 1, 2. \quad (10.38)$$

Because of pump and decay processes these occupation numbers may change, the temporal change being described by the well known rate equations

$$\frac{dn_2}{dt} = w_{21}n_1 - w_{12}n_2, \quad (10.39)$$

$$\frac{dn_1}{dt} = -w_{21}n_1 + w_{12}n_2. \quad (10.40)$$

$w_{21}$  and  $w_{12}$  are the transition rates caused by pump and incoherent decay processes, respectively (cf. fig. 10.2). These pump and decay processes are

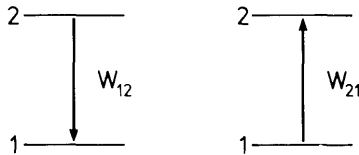


Fig. 10.2. Scheme of radiationless transitions or nonlasing optical transitions (left-hand side) and pump transitions (right-hand side) with the corresponding transition rate constants.

caused by the interaction of the electron with heatbath 2. As was shown in Vol. 1 the effect of heatbaths in the fully quantum mechanical equations for the operators can be taken care of by the following equations:

$$(a_2^+ a_2)' = w_{21} a_1^+ a_1 - w_{12} a_2^+ a_2 + \Gamma_{22}(t), \quad (10.41)$$

$$(a_1^+ a_1)' = -w_{21} a_1^+ a_1 + w_{12} a_2^+ a_2 + \Gamma_{11}(t). \quad (10.42)$$

Because in this section we have introduced the inversion  $d$  (10.31) as a new variable, we derive an equation for that quantity by simply subtracting (10.42) from (10.41). This yields

$$\dot{d} = 2w_{21} a_1^+ a_1 - 2w_{12} a_2^+ a_2 + \Gamma_{22} - \Gamma_{11}. \quad (10.43)$$

Because the electron must be in any of the two states 1 or 2, the relation

$$a_2^+ a_2 + a_1^+ a_1 = 1 \quad (10.44)$$

holds. Writing down the eq. (10.31) again we have

$$a_2^+ a_2 - a_1^+ a_1 = d. \quad (10.45)$$

By means of (10.44) and (10.45) we can express  $a_2^+ a_2$ ,  $a_1^+ a_1$  by  $d$ ,

$$a_2^+ a_2 = \frac{1}{2}(1 + d), \quad (10.46)$$

$$a_1^+ a_1 = \frac{1}{2}(1 - d). \quad (10.47)$$

Introducing (10.46) or (10.47) and using the abbreviations

$$\gamma_{\parallel} = w_{12} + w_{21}, \quad (10.48)$$

$$d_0 = \frac{w_{21} - w_{12}}{w_{21} + w_{12}}, \quad (10.49)$$

$$\Gamma_{22} - \Gamma_{11} = \Gamma_d. \quad (10.50)$$

eq. (10.43) acquires the form

$$\dot{d} = \gamma_{\parallel}(d_0 - d) + \Gamma_d. \quad (10.51)$$

A comparison between this equation and that of the corresponding equation (4.11) or (5.43) of the semiclassical theory reveals that  $d_0$  is the unsaturated inversion whereas  $\gamma_{\parallel} = 1/T$  is the damping constant of the inversion which is the inverse of the longitudinal relaxation time  $T$ . The equations for (10.34) and (10.35) can be immediately taken from the results of Vol. 1, (9.97) and (9.98), so that

$$\dot{\alpha} = -i\bar{\omega}\alpha - \gamma\alpha + \Gamma_-(t), \quad (10.52)$$

$$\dot{\alpha}^+ = i\bar{\omega}\alpha - \gamma\alpha^+ + \Gamma_+(t), \quad (10.53)$$

where

$$\begin{aligned} \Gamma_- &\equiv \Gamma_{12}, \\ \Gamma_+ &\equiv \Gamma_{12}^+ \equiv \Gamma_{21}. \end{aligned} \quad (10.54)$$

For later purposes we need the correlation functions between the fluctuating operator forces  $\Gamma$ . When the quantum mechanical average over the variables of the reservoirs 2 is taken we find relations of the form

$$\langle \Gamma_{11}(t) \Gamma_{11}(t') \rangle \equiv G_{11,11} \delta(t-t'), \dots \quad (10.55)$$

or more generally

$$\langle \Gamma_{jk}(t) \Gamma_{lm}(t') \rangle \equiv G_{jk,lm} \delta(t-t'). \quad (10.56)$$

The  $G$ 's have been derived in Vol. 1, formulas (9.104)–(9.109), and are repeated here

$$G_{11,11} = w_{12}n_2 + w_{21}n_1, \quad (10.57)$$

$$G_{11,22} = -w_{21}n_1 - w_{12}n_2 = G_{22,11}, \quad (10.58)$$

$$G_{22,22} = w_{21}n_1 + w_{12}n_2, \quad (10.59)$$

$$G_{12,12} = G_{21,21} = 0, \quad (10.60)$$

$$G_{12,21} = w_{12}n_2 - w_{21}n_1 + 2\gamma n_1, \quad (10.61)$$

$$G_{21,21} = w_{21}n_1 - w_{12}n_2 + 2\gamma n_2. \quad (10.62)$$

Because in the case of the laser we have to consider many atoms being distinguished by the index  $\mu$  we must say a word how the correlation functions must be generalized to that case. We shall assume as usual that each atom is coupled to its own heatbaths, so that the heatbath variables are independent of each other. In such a case it turns out that quite generally

$$\langle \Gamma_{\mu,jk}(t) \Gamma_{\mu',lm}(t') \rangle \propto \delta_{\mu\mu'}. \quad (10.63)$$

It is not difficult to transform the correlation functions for  $\Gamma_{jk}$  to those referring to the fluctuating forces of (10.50)–(10.54). A little analysis produces the following relations:

$$\langle \Gamma_{\mu-}(t) \rangle = \langle \Gamma_{\mu+}(t) \rangle = \langle \Gamma_{\mu d}(t) \rangle = 0, \quad (10.64)$$

$$\langle \Gamma_{\mu+}(t) \Gamma_{\mu'-}(t') \rangle = \left( \gamma(1 + \langle d_{\mu} \rangle) + \frac{1}{2T} (d_0 - \langle d_{\mu} \rangle) \right) \delta_{\mu\mu'} \delta(t-t'), \quad (10.65)$$

$$\langle \Gamma_{\mu-}(t) \Gamma_{\mu'+}(t') \rangle = \left( \gamma(1 - \langle d_{\mu} \rangle) - \frac{1}{2T} (d_0 - \langle d_{\mu} \rangle) \right) \delta_{\mu\mu'} \delta(t-t'), \quad (10.66)$$

$$\langle \Gamma_{\mu d}(t) \Gamma_{\mu' d}(t') \rangle = (2/T)(1 - d_0 \langle d_{\mu} \rangle) \delta_{\mu\mu'} \delta(t - t'), \quad (10.67)$$

$$\langle \Gamma_{\mu+}(t) \Gamma_{\mu' d}(t') \rangle = -(1/T)(1 + d_0) \langle \alpha_{\mu}^+ \rangle \delta_{\mu\mu'} \delta(t - t'), \quad (10.68)$$

$$\langle \Gamma_{\mu-}(t) \Gamma_{\mu' d}(t') \rangle = (1/T)(1 - d_0) \langle \alpha_{\mu} \rangle \delta_{\mu\mu'} \delta(t - t'). \quad (10.69)$$

10.3.3. Field and atoms coupled to heatbaths. The quantum mechanical Langevin equations of the laser

So far we have derived the Heisenberg equations for the field coupled to its heatbath and for the atoms coupled to their heatbaths. Now we wish to consider the full system in which the field and the atoms interact with each other and each of these subsystems is coupled to its corresponding heatbaths. The field operator  $b$  then obeys the Heisenberg equation

$$\dot{b} = \frac{1}{\hbar} [H, b], \quad (10.70)$$

where  $H$  is now given by (10.1). Because  $b$  commutes with all Hamiltonians which do not contain the field operator  $b^+$ , we need to consider on the r.h.s. of (10.70) only those terms of  $H$  which contain  $b^+$ . Since we have treated the interaction between the field and its heatbath 1 above, the only part of the Hamiltonian,  $H$ , which has not been considered yet, is stemming from the atom–field interaction. Therefore, in the case of the interaction with the single atom we have to study

$$\left. \frac{db}{dt} \right|_{f-a} = \frac{1}{\hbar} [(\hbar g \alpha b^+ + \hbar g \alpha^+ b), b], \quad (10.71)$$

where we made use of (10.20). Because  $b$  commutes with the atomic operators we may immediately apply the commutation relation (10.3) so that we obtain

$$\left. \frac{db}{dt} \right|_{f-a} = -ig\alpha. \quad (10.72)$$

This result can be immediately generalized to the case of many atoms where we find

$$\left. \frac{db}{dt} \right|_{f-A} = -ig \sum_{\mu} \alpha_{\mu}. \quad (10.73)$$

After all these intermediate steps we are now in a position to write down the final equation for  $b$  when it interacts with the atoms and the heatbaths.

Taking the terms (10.26) and (10.73) together we find

$$\dot{b} = (-i\omega - \kappa)b - ig \sum_{\mu} \alpha_{\mu} + F(t). \quad (10.74)$$

For what follows it is important to note that the correlation functions of  $F$  and  $F^+$  (cf. (10.27)–(10.30)) are unaffected by the interaction between the field mode and the atoms.

In a similar fashion we have to derive the Heisenberg equation of motion for the atomic operators if the interaction between the field and the atom is taken into account. The evaluation of the corresponding equation

$$\left. \frac{d\alpha}{dt} \right|_{f-a} = \frac{i}{\hbar} [H_{af}, \alpha] \quad (10.75)$$

requires some elementary algebra using the commutation relations. We leave this algebra as an exercise to the reader and immediately write down the final result

$$\left. \frac{d\alpha}{dt} \right|_{f-a} = igbd. \quad (10.76)$$

It shows that in order to determine the time dependence of  $a$  we need not only to know that of the operator  $b$  but also that of the inversion operator  $d$  defined in (10.31). Therefore we have to derive an equation for  $d$  also. Before we turn to this question we write down the equation for  $a$  if the full Hamiltonian  $H$  (10.1) is taken into account. According to the terms stemming from the "free" motion of  $a$ , from its coupling to heatbaths and from its coupling to the field we obtain the equation

$$\dot{\alpha} = (-i\bar{\omega} - \gamma)\alpha + igbd + \Gamma_{-}. \quad (10.77)$$

The generalization to atoms with index  $\mu$  is straightforward. We just have to supplement the atomic variables  $a$  and  $d$  by the corresponding index  $\mu$ ,

$$\dot{\alpha}_{\mu} = (-i\bar{\omega} - \gamma)\alpha_{\mu} + igbd_{\mu} + \Gamma_{\mu-}. \quad (10.78)$$

The equation for the Hermitian conjugate reads, of course,

$$\dot{\alpha}_{\mu}^{\dagger} = (i\bar{\omega} - \gamma)\alpha_{\mu}^{\dagger} - igb^{\dagger}d_{\mu} + \Gamma_{\mu+}. \quad (10.79)$$

Finally we have to derive an equation for  $d$ . Because the construction of this equation is quite obvious we write it down immediately

$$\dot{d}_{\mu} = \frac{d_0 - d_{\mu}}{T} + 2ig(\alpha_{\mu}b^{\dagger} - \alpha_{\mu}^{\dagger}b) + \Gamma_{\mu d}. \quad (10.80)$$

The first two terms on the r.h.s. stem from the action of the heatbaths on



the electron. The bracket stems from the interaction between the electron and the field, and the last term represents the fluctuating force. The eqs. (10.78)–(10.80) jointly with (10.74) represent the fundamental laser equations. For sake of completeness we mention that the atomic fluctuating forces and the fluctuating forces of the field are uncorrelated so that

$$\langle \Gamma_j(t) F(t') \rangle = \langle \Gamma_j(t) \rangle \langle F(t') \rangle = 0, \tag{10.81}$$

$$\langle \Gamma_j(t) F^+(t') \rangle = 0. \tag{10.82}$$

We note that according to Vol. 1 the correlation functions between the  $F$ 's and between the  $F$ 's and  $F$ 's are not affected by the coherent interaction term  $H_{A,F}$ , so that we may apply the relations (10.27)–(10.30)' (10.64)–(10.69), (10.81) and (10.82). We note however, that in the relations (10.64)–(10.69) the average values on the r.h.s. must be evaluated taking into account the total laser equations including the atom–field interaction. The final laser equations (10.78)–(10.80) and (10.74) exhibit a pronounced analogy to the equations of the single mode laser in the semiclassical theory (6.1)–(6.3). There are two differences, however. The obvious one consists in the occurrence of the additional fluctuating forces acting on field and atomic operators. The other difference consists in the fact that the quantities  $b$ ,  $b^+$ ,  $a$ ,  $\alpha^+$  and  $d$  are now quantum mechanical operators obeying specific commutation relations. But as we shall see, the form of the equations will allow us to proceed in close analogy to the case of the semiclassical equations.

In conclusion we indicate how to proceed from the quantum mechanical laser equations to the semiclassical equations. To this end we take the quantum mechanical average  $\langle \cdot \cdot \rangle$  on both sides of (10.74), (10.78)–(10.80). As we shall demonstrate later in this book, above laser threshold the quantities  $\langle bd \rangle$ ,  $\langle \alpha_\mu b^+ \rangle$  and  $\langle \alpha_\mu^+ b \rangle$  can be factorized in a good approximation, i.e.

$$\langle bd_\mu \rangle \approx \langle b \rangle \langle d_\mu \rangle, \quad \langle \alpha_\mu b^+ \rangle \approx \langle \alpha_\mu \rangle \langle b^+ \rangle \equiv \langle \alpha_\mu \rangle \langle b \rangle^*,$$

$$\langle \alpha_\mu^+ b \rangle \approx \langle \alpha_\mu^+ \rangle \langle b \rangle \equiv \langle \alpha_\mu \rangle^* \langle b \rangle.$$

Because the averages over the fluctuating forces vanish and  $\langle b \rangle$ , etc. are classical quantities, we thus arrive precisely at the semiclassical equations which in this way have been derived from first principles.

### Exercises on section 10.3

- (1) Derive (10.24) by means of (10.22) and (10.3).

(2) Prove (10.33) by means of (10.32), (10.11).

Hint: use (10.5)–(10.7).

What does this result mean physically?

(3) Derive (10.36), (10.37).

#### 10.4. Coherence and noise

It will be our task to solve the basic equations (10.74), (10.78) to (10.80). The striking similarity of the quantum mechanical equations to those of the semiclassical laser theory suggests to formally proceed as with these latter equations. This is in fact possible provided we take care of the sequence of the operators in the nonlinear terms, and by taking into account the fluctuating forces  $\mathbf{F}$  and  $\mathbf{I}$ . The first step consists in an elimination of the atomic "variables"  $a$ ,  $\alpha_\mu^+$  and  $d_\mu$  in analogy to our procedure of section 6.3. Because that procedure was outlined there explicitly we do not repeat it but write down the final result

$$\dot{b} = (-i\omega - \kappa + g^2 D_0 / \gamma) b - C b^+ b b + F_{\text{tot}}. \quad (10.83)$$

$D_0$  is given by

$$D_0 = N d_0, \quad (10.84)$$

and  $C$  is defined by

$$C = 4g^4 T D_0 / \gamma^2. \quad (10.85)$$

$F_{\text{tot}}$  is defined by

$$F_{\text{tot}}(t) = F(t) - (ig/\gamma) \sum_{\mu} \Gamma_{\mu-}(t). \quad (10.86)$$

With respect to the terms which do not contain fluctuations, eq. (10.83) possesses the same degree of accuracy as our former semiclassical equation (6.46). With respect to the additional terms, i.e. to the fluctuations, it must be noted that one term, which stems from the fluctuations of the inversion, has been neglected, because, in general, this causes only small effects. For what follows it is important to know the properties of the fluctuating force  $F_{\text{tot}}$ . Using the definition (10.86) and the correlation functions (10.27)–(10.30), (10.64)–(10.69) and (10.81), (10.82) of the fluctuating forces we may easily derive the following relations:

$$\langle F_{\text{tot}}^+(t) \rangle = 0, \quad \langle F_{\text{tot}}(t) \rangle = 0, \quad (10.87)$$

$$\langle F_{\text{tot}}^+(t) F_{\text{tot}}^+(t') \rangle = \langle F_{\text{tot}}(t) F_{\text{tot}}(t') \rangle = 0, \quad (10.88)$$

$$\langle F_{\text{tot}}^+(t) F_{\text{tot}}(t') \rangle = \left( 2\kappa n_{\text{th}} + \frac{g^2}{\gamma^2} \sum_{\mu} [\gamma(1 + \langle d_{\mu} \rangle)] + \frac{1}{2T} (d_0 - \langle d_{\mu} \rangle) \right) \delta(t - t'), \quad (10.89)$$

$$\approx 2\kappa \left[ n_{\text{th}} + \frac{N_{2,s}}{(N_2 - N_1)_{\text{thr}}} \right] \delta(t - t'). \quad (10.90)$$

When proceeding from (10.89) to (10.90) we have assumed that steady state laser action occurs and that we may put within a surrounding of the laser threshold  $d_0 - \langle d_{\mu} \rangle \approx 0$  in a good approximation. Finally,  $N_{1,\mu} + N_{2,\mu} = l$  jointly with the threshold condition  $D_{0,\text{thr}} = (N_2 - N_1)_{\text{thr}} = \kappa \gamma / g^2$  yields the final form (10.90).  $N_{2,s}$  is the saturated occupation number of level 2 of the atoms, and  $n_{\text{th}}$  the number of photons in thermal equilibrium.

Before we start solving eq. (10.83) let us visualize its meaning by interpreting  $b$  as a classical variable. Our subsequent discussions will show that such a meaning can indeed be attached to that operator. To this end we use the potential model which we introduced in section 6.3. Let us interpret  $b$  as the coordinate  $q$  of a particle. Then eq. (10.83) can be considered as an equation for the overdamped motion of a particle. Thus this equation has the structure

$$m\ddot{q} + \dot{q} = K(q) + F(t) \equiv -\frac{\partial V}{\partial q} + F(t), \quad (10.91)$$

where the acceleration term  $m\ddot{q}$  is assumed very small. The force on the r.h.s. (10.91) is represented as a sum stemming from  $K(q)$  which corresponds to the semiclassical result and  $F(t)$  which represents the additional fluctuating force. As we have seen in section 6.3, there are two different solutions to eq. (10.83) depending on whether the laser is operated below or above its threshold. For this reason we shall perform the discussion of these two cases separately.

#### 10.4.1. Operation below threshold

In this case  $G < 0$ , where

$$-\kappa + g^2 D_0 / \gamma \equiv G \quad (10.92)$$

is the unsaturated net gain.

The potential  $V$  is represented by the dashed curve of fig. 10.3. In this case the fictitious particle has a coordinate  $q$  which remains close to  $q = 0$ , i.e. we expect that  $q$  is a small quantity. Consequently we shall expect that in eq. (10.83) the nonlinear term  $b^+ b b$ , if any suitable expectation value is

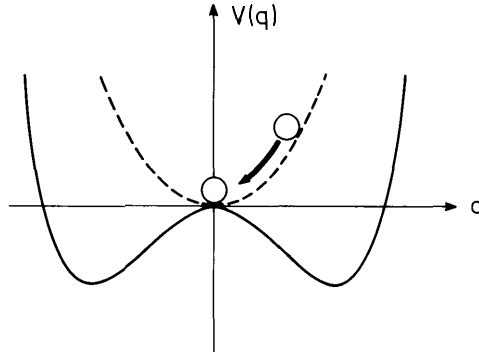


Fig. 10.3. The potential of a fictitious particle versus coordinate  $q$ . Dashed line: below laser threshold. After each excitation act due to the fluctuating force, the particle relaxes towards its equilibrium position. Solid line: potential above laser threshold.

taken, is much smaller than the other linear terms and may be neglected. (This conclusion can be justified also rigorously.) In this way (10.83) reduces to

$$\dot{b} = (-i\omega - |G|)b + F_{\text{tot}}. \quad (10.93)$$

Its solution can be directly read off from our previous treatment of such an equation done in Vol. 1, section 9.1. Because (10.93) is linear, commutation relations do not play any role and we can treat at least in a formal fashion  $b$  as a classical variable. Identifying, in a formal way, the velocity  $v(t)$ , introduced in section 9.1 of Vol. 1, with  $b(t)$ , we obtain

$$b(t) = \int_0^t \exp[-(i\omega + |G|)(t-t')] F_{\text{tot}}(t') dt' + b(0) \exp[-(i\omega + |G|)t]. \quad (10.94)$$

In order to visualize the effect of  $F_{\text{tot}}$  we represent it by way of a model in the form

$$F_{\text{tot}} = \sum f_{\mu} \delta(t - t_{\mu}) \quad \text{with} \quad f_{\mu} = f \exp[i\varphi_{\mu}]. \quad (10.95)$$

Indeed we have seen in Vol. 1, eq. (9.3), that the meaning of the fluctuating forces  $F$  can be visualized best when we consider them as a sequence of individual pushes occurring randomly at times  $t_{\mu}$ . Then our fictitious particle with coordinate  $q$  behaves like a ball in-between two hills, but being kicked by soccer players in a statistical sequence. The term  $-|G|b$  of eq. (10.93) which represents the damping force can be easily visualized as the friction

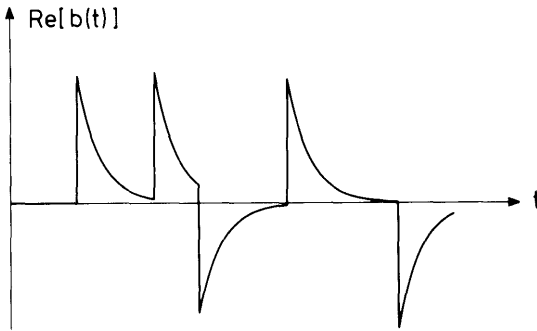


Fig. 10.4. This figure shows the time dependence of the amplitude  $b$ , which is interpreted as real quantity, and of which we have omitted the factor  $\exp[-i\omega t]$ .

force with which the grass acts on the soccer. Let us interpret  $F$  as a real quantity and let us consider  $b$ , after omitting the factor  $\exp[-i\omega t]$ , as a real quantity also. Then  $b(t)$  will exhibit a temporal behavior as indicated by fig. 10.4. When we take the oscillations  $\exp[-i\omega t]$  into account, we find fig. 10.5 instead. We recognize that the light field amplitude  $b$  consists of a sequence of decaying wave tracks. Because the individual kicks are **uncorrelated** with respect to their phases the individual wave tracks of fig. 10.5 possess uncorrelated phases also. When we let the inversion  $D_0$  increase by enhanced pumping, the absolute value of  $G$  decreases. As (10.94) tells us, in this case the individual wave tracks decay more slowly. As we shall see below, a slower decay of the individual wave tracks means that the atomic emission line becomes smaller. Therefore a line narrowing with increasing pumping is expected. Such effects have been found in lasers indeed and scientists have thought for a while that the differences between

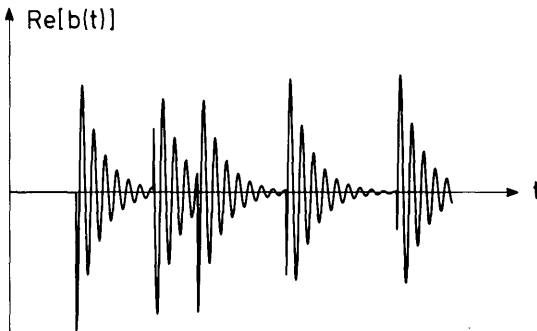


Fig. 10.5. The same as in fig. 10.4 but taking into account the factor  $\exp[-i\omega t]$ .

light from ordinary lamps and from lasers consists only in such a line narrowing. But we shall see that this is not the only and essential difference. In fact laser light behaves above laser threshold in a qualitatively entirely different manner than below threshold.

But let us still discuss the situation below laser threshold and let us study the coherence properties of light as described by (10.94). Because the kicks occur with equal probability in both directions the mean value of (10.94) vanishes, i.e.

$$\langle b \rangle = 0. \quad (10.96)$$

Therefore in a next step we investigate the coherence function of second order which we derived in Vol. 1, sections 2.2 and 8.1. There we derived the coherence functions by means of the field amplitude  $E$  where we decomposed the field amplitude into its positive and negative frequency parts. While the positive frequency part oscillates with  $\exp[-i\omega t]$ , the negative frequency part oscillates with  $\exp[i\omega t]$ . Eq. (10.24) tells us that the positive frequency part corresponds to the operator  $b$ , whereas the negative frequency part corresponds to  $b^+$ . Thus we may use the following replacement (cf. (5.106), (5.107) or Vol. 1):

$$\begin{aligned} E^+ &\rightarrow b i \sqrt{\hbar\omega/2\varepsilon_0} \mathbf{u}(\mathbf{x}), \\ E^- &\rightarrow -b^+ i \sqrt{\hbar\omega/(2\varepsilon_0)} \mathbf{u}(\mathbf{x}). \end{aligned} \quad (10.97)$$

Because the factors of  $b$  and  $b^+$  are fixed quantities which do not fluctuate, we can extract these factors from the averaging procedure.

Our former classical average of section 2.2 of Vol. 1 can be translated into quantum statistical averages as we have shown in section 8.1 of that volume. We shall denote the averaging by brackets. Our present discussion shows that the coherence properties of laser light are represented by expressions of the form

$$\langle b^+(t) b(t') \rangle. \quad (10.98)$$

We have met expressions of the form (10.98) with (10.94) in Vol. 1. Therefore we can immediately use the corresponding result and obtain

$$\langle b^+(t) b(t') \rangle = \exp[i\omega(t-t') - |G| |t-t'|] \langle b^+(t') b(t') \rangle. \quad (10.99)$$

In this formula transients have been neglected, i.e. we study the steady state emission where time  $t$  is large but  $t-t'$  finite. Because we study the steady state, at least in the case of sufficiently large times  $t'$  we may use the relation

$$\langle b^+(t') b(t') \rangle = \bar{n}, \quad (10.100)$$

where  $\mathbf{A}$  is time independent. Because  $b^+b$  is the operator of the photon number (cf. Vol. 1),  $\bar{n}$  has the meaning of an average photon number. It can be directly calculated by means of (10.94). To this end we insert (10.94) in (10.100) and obtain for sufficiently large times

$$\bar{n} = \int_0^t d\tau \int_0^t d\tau' \exp[-|G|(2t - \tau - \tau') + i\omega(\tau - \tau')] \langle F_{\text{tot}}^+(\tau) F_{\text{tot}}(\tau') \rangle. \tag{10.101}$$

The double integral can be immediately evaluated by means of (10.90) (compare exercises), and we obtain

$$\bar{n} = \frac{\kappa}{|G|} (n_{\text{th}} + n_{\text{sp}}), \tag{10.102}$$

where we have used the abbreviation

$$n_{\text{sp}} = \frac{N_{2,s}}{(N_2 - N_1)_{\text{thr}}}. \tag{10.103}$$

The individual expressions in (10.102) have the following meaning:  $n_{\text{th}}$  ( $\equiv n_{\text{thermal}}$ ) stems from the correlation function of  $F^+$  and  $F$  and represents the number of photons belonging to the mode frequency  $\omega$  which are present in thermal equilibrium. In the optical range of the laser this number is negligibly small, whereas it plays the dominant role in the microwave region (maser).  $n_{\text{sp}}$  ( $\equiv n_{\text{spontaneous}}$ ) is the number of spontaneously emitted light quanta.  $n_{\text{sp}}$  stems from the correlation function of

$$F_{\text{atom}} \equiv (-ig/\gamma) \sum_{\mu} \Gamma_{\mu-}(t) \quad \text{and} \quad F_{\text{atom}}^+ \equiv (ig/\gamma) \sum_{\mu} \Gamma_{\mu+}(t).$$

The relation (10.103) is valid for the system of two-level atoms considered here.  $N_{2,s}$  is the saturated occupation number of the upper atomic level. Thus it represents the number which is realized in the laser process.  $(N_2 - N_1)_{\text{thr}}$  is the inversion of the atoms at laser threshold. The factor  $\kappa/|G|$  occurring in (10.102), which causes a multiplication of the photon number ( $n_{\text{th}} + n_{\text{sp}}$ ), is of particular interest as we may see as follows. According to (10.92),  $G$  is the unsaturated net gain which is given below laser threshold by

$$|G| = \kappa - g^2 D_0 / \gamma.$$

If the inversion  $D_0 = 0$ , i.e. if equally many atoms are in the excited and in the ground state, the effects of stimulated emission and absorption compensate each other. In this case  $|G| = \kappa$  and the number of photons actually present is given by  $\mathbf{A} = n_{\text{th}} + n_{\text{sp}}$ . When we increase  $D_0$ , stimulated

emission becomes bigger than absorption, and thus  $|G|$  becomes smaller than  $\kappa$ , i.e.  $\kappa/|G| > 1$ . We then obtain an enhanced number of photons  $\bar{n} > n_{\text{th}} + n_{\text{sp}}$ . The laser system is still operating below laser threshold but within a region of *amplification*. When  $D_0$  approaches its value at laser threshold  $D_{0,\text{thr}} = \kappa\gamma/g^2$ ,  $|G|$  tends to zero.

As a consequence, the amplification factor  $\kappa/|G|$  tends to infinity. It should be noted that this divergence is an artefact caused by our approximation which neglects the nonlinear terms  $-b^+bb$ . A rigorous calculation without linearization will show that the photon number remains finite at laser threshold, as we shall demonstrate in section 10.5. By means of (10.99) we can immediately calculate the complex degree of coherence. According to the corresponding relation of Vol. 1 it can be represented in the present case by

$$\frac{\langle b^+(t) b(t') \rangle}{\langle b^+b \rangle} = \exp[i\omega(t-t') - |G||t-t'|]. \quad (10.104)$$

As we may see, this degree of coherence decreases exponentially which is, of course, caused by the finite duration of the individual wave tracks. The correlation function (10.104) can be measured experimentally either directly by Young's double slit experiment (cf. Vol. 1) or an arrangement described in fig. 10.6. Another possibility is provided by measuring the line-width occurring in the spectral decomposition of light in a spectrometer.

The foundation of this possibility is provided by a theorem due to Wiener–Chintschin which we do not derive here mathematically exactly, but which we can interpret in a rather simple fashion. Let us imagine an ideal spectrometer which decomposes the field amplitude  $b(t)$  into its

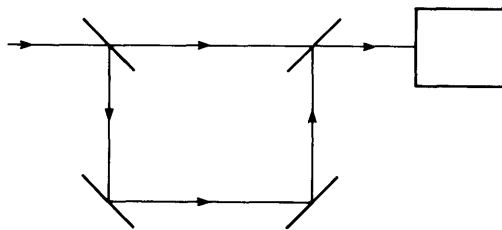


Fig. 10.6. Scheme of an experimental arrangement for the measurement of  $\langle (b^+(t) + b^+(t')) \times (b(t) + b(t')) \rangle$ . The first mirror (upper left) which is semi-transparent splits the incoming beam into two beams. The second mirror (lower left) and the third mirror (lower right) take care of the deflection of the beam which by the mirror upper right is reunited with the original beam. In this way on the original wave an identical but delayed wave is superimposed.



Fourier components

$$b(t) = \int_{-\infty}^{+\infty} d(\omega) \exp(i\omega t) d\omega. \quad (10.105)$$

Because  $b(t)$  and  $b^+(t)$  are composed of fluctuating quantities according to (10.94), the Fourier coefficients  $d(\omega)$  are also fluctuating, i.e. random quantities. As a detailed discussion shows, the properties (10.88) and (10.90) of the fluctuating forces lead to the following relation

$$\langle d^+(\omega) d(\omega') \rangle = \delta(\omega - \omega') \langle d^+(\omega) d(\omega) \rangle. \quad (10.106)$$

When we form the correlation function (10.98) by means of the decomposition (10.105) the resulting double integral over  $\omega$  reduces to a single integral because of (10.106) and we obtain

$$\langle b^+(t) b(t') \rangle = \int_{-\infty}^{+\infty} (d^+(\omega) d(\omega)) \exp[i\omega(t-t')] d\omega. \quad (10.107)$$

But the quantity

$$y(\omega) = \langle d^+(\omega) d(\omega) \rangle \quad (10.108)$$

represents just the intensity of the spectral line at the frequency  $\omega$ . The relation (10.107) represents the Wiener–Chintschin theorem. The correlation function on the l.h.s. can be represented as the Fourier transform of the spectrum. The spectral distribution can be calculated explicitly in the present case. To this end we study the steady state where we put  $t' = 0$ . Due to (10.99) we may write the l.h.s. of (10.107) in the form

$$\langle b^+(t) b(0) \rangle = \exp[i\bar{\omega}t - |G||t|] \bar{n}. \quad (10.109)$$

Because of (10.107), (10.108) and (10.109) we obtain the relation

$$\bar{n} \exp[i\bar{\omega}t - |G||t|] = \int_{-\infty}^{+\infty} y(\omega) \exp[i\omega t] d\omega. \quad (10.110)$$

According to Fourier's theorem we can calculate the spectral distribution (10.110) by

$$y(\omega) = \frac{\bar{n}}{2\pi} \int_{-\infty}^{+\infty} \exp[-i\omega t] \exp[i\bar{\omega}t - |G||t|] dt. \quad (10.111)$$

The integral can be evaluated without any difficulty. We obtain

$$y(\omega) = \frac{1}{2\pi} \frac{2|G|\bar{n}}{(\bar{\omega} - \omega)^2 + G^2}. \quad (10.112)$$

We see that the correlation function which is given by (10.109) belongs to a Lorentzian line with half-width  $|G|$ . If the decay time  $1/|G|$  becomes longer, a line-width becoming smaller is observed. This result has been anticipated intuitively above.

As we shall see below the fundamental difference between the statistical properties of light from conventional lamps and those of light from lasers appears only through correlation functions of fourth order (and higher). In fig. 10.7 we remind the reader of how a correlation function of fourth order can be measured. Let us consider the correlation function

$$\langle b^+(t) b^+(t') b(t') b(t) \rangle. \quad (10.113)$$

Our subsequent calculations are somewhat clumsy. For this reason the speedy reader can skip them and immediately proceed to the final result presented in eqs. (10.124) and (10.125). In order to make our calculation as transparent as possible we assume that  $b$  has been calculated according to (10.94), where we represent the fluctuating forces by means of individual pushes according to (10.95). Strictly speaking we are dealing here with fluctuating operator forces but a detailed analysis shows that all what we are doing below can be done in a strict sense with operators also. But in our present context we rather wish to give the reader a feeling how the final result can be obtained rather than trying here to give a formal but rigorous derivation. We focus our attention on the stationary state so that we can neglect transients. This means that we may drop the term containing  $b(0)$  in (10.94). According to the individual contributions we represent  $b(t)$ ,  $b^+(t)$  in the form

$$b(t) = \sum_{\mu} \beta_{\mu}(t), \quad b^+(t) = \sum_{\mu} \beta_{\mu}^+(t). \quad (10.114)$$

We shall assume that the contributions which stem from different pushes are uncorrelated so that we may use the relation

$$\langle \beta_{\mu}^+ \beta_{\mu'} \rangle = \langle \beta_{\mu}^+ \beta_{\mu} \rangle \delta_{\mu\mu'}. \quad (10.115)$$

Because the evaluation of (10.113) by means of (10.114) is somewhat lengthy, we first treat the case of a simpler correlation function, namely (10.98), as an exercise. Inserting (10.114) and the Hermitian conjugate expression in it we obtain

$$\langle b^+(t) b(t') \rangle = \sum_{\mu\mu'} \langle \beta_{\mu}^+(t) \beta_{\mu'}(t') \rangle. \quad (10.116)$$

On account of the Kronecker symbol  $\delta_{\mu\mu'}$  in (10.115) the double sum over

$\mu, \mu'$  reduces to a single sum over  $\mu$ . In this way (10.116) is transformed into

$$\langle b^+(t) b(t') \rangle = \sum_{\mu} \langle \beta_{\mu}^+(t) \beta_{\mu}(t') \rangle. \quad (10.117)$$

In order to evaluate (10.113) by means of (10.114) we have to consider averages over products of four  $\beta$ 's, namely

$$\langle \beta_{\mu_1}^+ \beta_{\mu_2}^+ \beta_{\mu_3} \beta_{\mu_4} \rangle. \quad (10.118)$$

Again we shall assume that the  $\beta$ 's with different indices  $\mu$  are uncorrelated and that also terms of the form  $\beta_{\mu_1}^+ \beta_{\mu_2}^+$  or  $\beta_{\mu_1} \beta_{\mu_2}$ , with the same indices  $\mu_1 = \mu_2 = \mu$  yield 0. In this case the expression (10.118) can give rise to nonvanishing expressions in the following two cases only:

- (1)  $\mu_1 = \mu_4$  and simultaneously  $\mu_3 = \mu_2$ ,
- (2)  $\mu_1 = \mu_3$  and simultaneously  $\mu_2 = \mu_4$ .

Inserting (10.114) in (10.113) we obtain

$$\sum_{\mu \mu' \mu'' \mu'''} \langle \beta_{\mu}^+(t) \beta_{\mu'}^+(t') \beta_{\mu''}(t') \beta_{\mu'''}(t) \rangle. \quad (10.119)$$

It contains only two types of contributions which do not vanish, namely

$$\mu = \mu''', \quad \mu' = \mu'', \quad (10.120)$$

and

$$\mu = \mu'', \quad \mu' = \mu'''. \quad (10.121)$$

Correspondingly we can split (10.119) into two parts and reduce the four-fold sums into double sums

$$\begin{aligned} (10.119) &= \left( \sum_{\mu} \langle \beta_{\mu}^+(t) \beta_{\mu}(t) \rangle \right) \left( \sum_{\mu'} \langle \beta_{\mu'}^+(t') \beta_{\mu'}(t') \rangle \right) \\ &+ \left( \sum_{\mu} \langle \beta_{\mu}^+(t) \beta_{\mu}(t') \rangle \right) \left( \sum_{\mu'} \langle \beta_{\mu'}^+(t') \beta_{\mu'}(t) \rangle \right). \end{aligned} \quad (10.122)$$

Using the intermediate result (10.116) we arrive at the following final formula:

$$\langle b^+(t) b^+(t') b(t') b(t) \rangle = (\langle b^+(t) b(t) \rangle)^2 + |\langle b^+(t) b(t') \rangle|^2, \quad (10.123)$$

where  $\langle b^+(t') b(t') \rangle = \langle b^+(t) b(t) \rangle$ . Both for theoretical studies and for measurements it is often useful to subtract the photon number from (10.123) so that we introduce as a new quantity

$$K_2(t, t') = \langle b^{\dagger}(t) b^{\dagger}(t') b(t') b(t) \rangle - (\langle b^{\dagger}(t) b(t) \rangle)^2. \quad (10.124)$$

By means of (10.123) we finally obtain

$$K_2(t, t') = |\langle b^+(t) b(t') \rangle|^2. \quad (10.125)$$

In our derivation of this result we have not been very careful with the sequence of the  $\beta$ 's because in our model-like fashion we have treated them as classical quantities. In reality they are operators which in general do not commute with each other. A careful analysis reveals, however, that result (10.125) holds also in the case of operators. The essential result of (10.125) means that the fourth order correlation function can be expressed by a correlation function of lower order. By means of the result (10.99), (10.100) we can express  $K_2$  explicitly

$$K_2 = \bar{n}^2 \exp[-2\gamma_{\text{eff}}|t - t'|], \quad \gamma_{\text{eff}} \equiv |G|. \quad (10.126)$$

The correlation functions can be measured in the Hanbury–Brown–Twiss experiment which is explained in figs. 10.7 and 10.8. As can be shown quite generally, also all higher correlation functions can be expressed by correlation functions of first and second order, i.e. by (10.96), (10.98), provided  $b$  consists of many statistically independent contributions, or, in other words, if  $b$  is Gaussian distributed. Because the field strength  $E$  is directly connected with  $b$  and  $b^+$ , we thus have found that the fluctuating field strength of

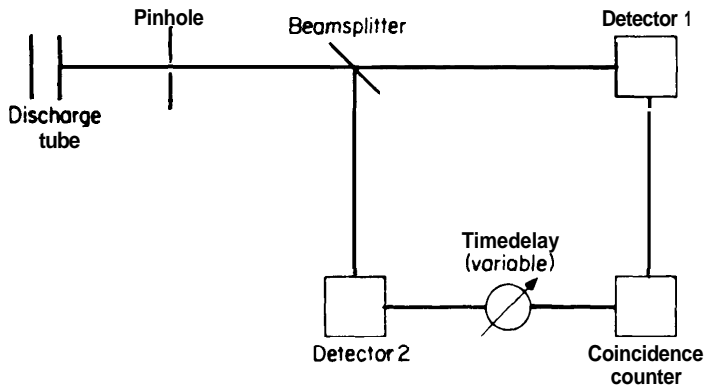


Fig. 10.7. Experimental arrangement for the measurement of a correlation function of fourth order by which the correlation between photon numbers is measured. In experiments made before the discovery of a laser, a gas discharge tube was used as light source. In experiments with laser light it is replaced by the laser and the pinhole is no more necessary. The beam is split by a semi-transparent mirror (beam splitter). In receivers 1 and 2 the incoming photons are registered. The photon numbers registered by the receiver 2 are transmitted with a certain time delay to the coincidence counter where they are processed jointly with the photon numbers measured by the receiver 1.

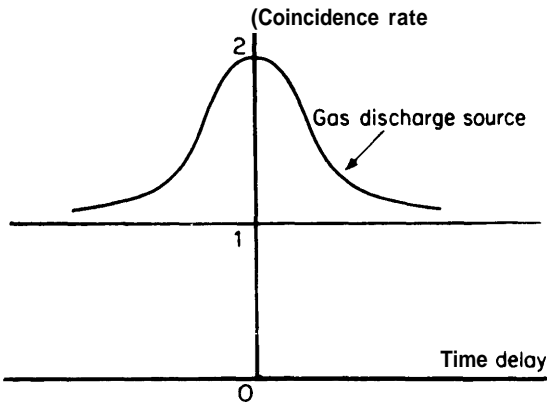


Fig. 10.8. The results of the experiment by Hanbury–Brown–Twiss according to the experimental arrangement of fig. 10.7. The figure shows the results for light from lamps. In the figure the coincidence rate is plotted versus time delay. The curve refers to a gas discharge lamp.

usual lamps obeys a Gaussian distribution. We shall come back to this point in section 10.5.

#### 10.4.2. Behavior above threshold

We now study the properties of laser light above laser threshold. When we interpret  $b$  again as a classical variable we can immediately read off the behavior of  $b$  from fig. 10.3, where the solid curve applies. It is important to note that  $b$  is a complex quantity. In such a case we have to extend eq. (10.91) to equations for the real and imaginary parts of  $b$ . It then transpires that the behavior of  $b$  can be represented by means of the motion of a fictitious particle in two dimensions,  $x$  and  $y$ , where  $b = x + iy$ . The force occurring in the equation of motion can again be derived from the potential which we represent in fig. 10.9. Without fluctuating forces the particle will acquire a resting position with the distance  $r_0$  from the origin, and an arbitrary but fixed phase angle. The fluctuating forces, which we interpret as kicks, have two kinds of effects. In radial direction they try to push the particle up-hill, whereby the distance of the particle coordinate from the bottom of the valley is increased. We shall denote these fluctuating deviations by  $\rho(t)$ . Because the particle, which is pushed up-hill, will try to relax to its equilibrium position we can assume that  $\rho(t)$  remains a small quantity. Furthermore we have to consider the kicks in tangential direction. Because no restoring force acts in this direction, the particle will perform some kind of diffusion process in that direction. As a consequence the phase  $\varphi(t)$  will

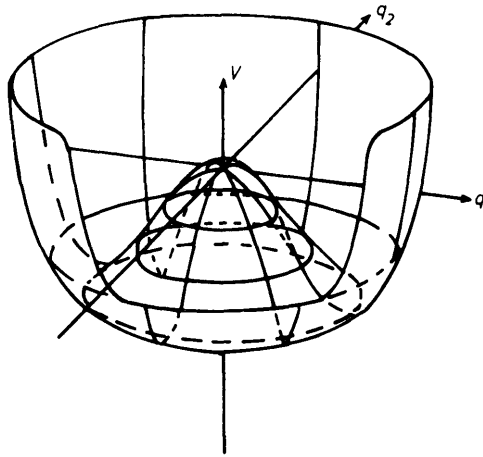


Fig. 10.9. The potential for the motion of the fictitious particle in two dimensions. It represents the behavior of the complex laser amplitude above laser threshold.

fluctuate. Finally taking into account that  $b$  oscillates with the carrier frequency  $\omega$ , our discussion leads us to the following hypothesis for  $b$ :

$$b = (r_0 + \rho(t)) \exp[i\varphi(t)] \exp[-i\omega t]. \tag{10.127}$$

A detailed discussion reveals that our procedure remains valid if  $b$  is an operator. As it turns out in this case,  $r_0$  remains a classical number whereas  $p$  and  $\varphi$  become operators. We shall not discuss the details which result from the properties of  $p$  and  $\varphi$  being operators. As can be shown the laser can be described in a very good approximation by treating  $p$  and  $\varphi$  as if they are classical quantities. In order to determine  $p$  and  $\varphi$  we insert (10.127) into (10.83), which we repeat for the reader's convenience

$$\dot{b} = (-i\omega - \underbrace{\kappa + g^2 D_0 / \gamma}_G) b - C b^+ b b + F_{\text{tot}}. \tag{10.128}$$

Inserting the explicit form (10.127) in (10.128), performing the differentiation with respect to time, and dividing the resulting equation by the exponential functions occurring in (10.127) we obtain

$$i\dot{\varphi} r_0 + \dot{\rho} = G(r_0 + \rho) - C(r_0^3 + 3r_0^2 \rho + \dots) + \underbrace{\exp[i\omega t - i\varphi(t)] F_{\text{tot}}}_{\tilde{F}_{\text{tot}}}. \tag{10.129}$$

We determine  $r_0$  by means of the condition that eq. (10.129) is fulfilled

identically without fluctuations, i.e. for  $\varphi = p = F_{\text{tot}} = 0$ . We thus find

$$G = Cr_0^2. \tag{10.130}$$

By splitting eq. (10.129) into its real and imaginary part we obtain

$$\dot{\varphi} = \frac{1}{ir_0} \text{Im}(\tilde{F}_{\text{tot}}) \tag{10.131}$$

and

$$\dot{\rho} = -\gamma_a \rho + \text{Re}(\tilde{F}_{\text{tot}}), \tag{10.132}$$

where we have used the abbreviation

$$\gamma_a = 2G > 0. \tag{10.133}$$

(10.131) can be immediately integrated. We leave it to the reader as an exercise to convince himself that  $\varphi$  possesses the following correlation function:

$$\langle (\varphi(t) - \varphi(0))^2 \rangle = 2\gamma_a t. \tag{10.134}$$

Under the assumption  $\kappa \ll \gamma$  the effective line-width  $\gamma$ , is given by

$$\gamma_\varphi \equiv \Delta\omega = \frac{\hbar\omega}{P} \kappa^2 (n_{\text{sp}} + n_{\text{th}}), \tag{10.135}$$

where

$$P = 2\kappa\bar{n}\hbar\omega. \tag{10.136}$$

As we shall see  $A\omega$  represents the spectroscopic line-width of laser light. We remind the reader of the meaning of the quantities occurring in (10.135).  $\omega$  is the laser light frequency (=central frequency of the atomic optical transition because we have assumed exact resonance).  $2\kappa$  is the inverse lifetime of a mode in the resonator without laser action.  $n_{\text{sp}}$  and  $n_{\text{th}}$  are the spontaneously emitted and thermally activated photons, respectively (cf. (10.103)).  $P$  is the emission power of the laser, where  $\bar{n}$  is the mean photon number present in the laser. We can easily determine the correlation function between  $b^+$  at time  $t$  and  $b(0)$ . As we shall see, somewhat above threshold  $\rho(t) \ll r_0$ . Therefore, in a good approximation, we can neglect the term  $p$  when we calculate the correlation function between  $b^+$  and  $b$ . The calculation, which we shall discuss more closely in the exercises, shows that the correlation function is given by

$$\langle b^+(t) b(0) \rangle \approx r_0^2 \exp[-\gamma_\varphi t]. \tag{10.137}$$

Evidently a Lorentzian line occurs whose width decreases with increasing laser light power  $P$ .

But now we wish to show that the statistical properties of laser light differ from those of light from usual lamps fundamentally. To this end we consider the correlation function  $K_2$  which we introduced in (10.124) and insert (10.127) in it. Anticipating that  $\mathbf{p}$  is a small quantity we calculate this correlation function only up to its leading terms in  $\mathbf{p}$ . As it turns out, these are terms quadratic in  $\mathbf{p}$  so that we obtain

$$K_2 = 4r_0^2(\langle \rho_i \rho_i \rangle - \langle \rho_i \rangle \langle \rho_i \rangle). \quad (10.138)$$

The quantity  $\rho_i$  obeys eq. (10.132). This equation has a mathematical structure which corresponds entirely to that of eq. (10.93), if we put there  $\omega = 0$  and replace  $|G|$  by  $\gamma_a$  and  $F_{\text{tot}}$  by  $\text{Re}(F_{\text{tot}})$ . Therefore the solution of eq. (10.132) is readily found. In particular we obtain

$$\langle \rho_i \rangle = \langle \rho_i \rangle = 0. \quad (10.139)$$

In this way (10.138) reduces to

$$4r_0^2 \langle \rho_i \rho_i \rangle. \quad (10.140)$$

By means of the solution of (10.132) we can determine the correlation function between  $\mathbf{p}$ , and  $\rho_i$  in analogy to the calculation of (10.99). We then obtain our final result

$$K_2(\tau) = (n_{\text{sp}} + n_{\text{th}}) \frac{\kappa}{C} \exp[-2C\langle n \rangle \tau], \quad (10.141)$$

where  $n_{\text{sp}}$  is the number of spontaneously emitted photons as before and is given by

$$n_{\text{sp}} = \frac{N_{2,s}}{(N_2 - N_1)_{\text{thr}}}. \quad (10.142)$$

$n_{\text{th}}$  is the number of thermally activated photons which in the case of the laser is negligibly small (note, however, that in the case of the maser  $n_{\text{th}}$  dominates over  $n$ ). We have further used that

$$\gamma_a \equiv 2C\langle n \rangle, \quad (10.143)$$

and the abbreviation

$$C = \frac{4\kappa^2 T}{(N_2 - N_1)_{\text{thr}}} \quad (10.144)$$

( $T$  is the relaxation time of the inversion).

In order to compare the behavior of  $K_2$  below and above threshold it is useful to divide  $K_2$  by the photon number  $\bar{n}$  and to plot it against the



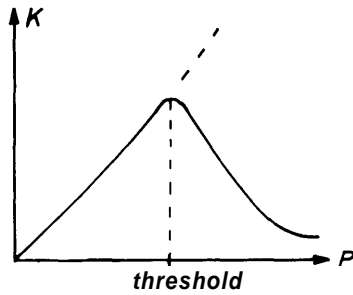


Fig. 10.10. The correlation function  $K = K_2/\bar{n}$  versus pump power

emitted power. For  $\tau = 0$  we obtain the curve of fig. 10.10. As we may see the quantity  $K_2/\bar{n}$  increases linearly below laser threshold according to eq. (10.126), whereas it decreases above laser threshold in a way proportional  $1/\bar{n}$ . In this way the decisive difference between the behavior of laser light and that of light of usual lamps is revealed. The corresponding experimental results are shown in fig. 10.11 and they substantiate the predictions in detail. It should be noted that because of the mathematical methods (linearization below and above threshold) we have to exclude a small region around laser threshold. We shall discuss this region in the next section. Let us summarize our results.

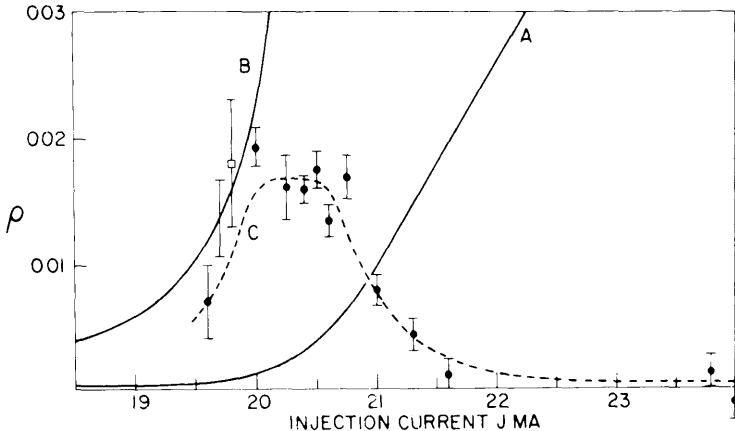


Fig. 10.11. The first experiment by which  $K$  was measured.  $K$  of the strongest mode of a semiconductor laser is plotted versus the injection current  $j$  which corresponds to the pump parameter. The experimental points show error bars indicating the standard deviation of the counting fluctuations. Incidentally the curve A of the emitted intensity of the mode is shown in arbitrary units. [J.A. Armstrong and A.W. Smith, Phys. Rev. Letters 14, 68 (1965).]

We have found that the physical nature of light is fundamentally different depending on whether the laser is operated below or above threshold. Below threshold light consists of exponentially damped wave tracks whose phases are entirely uncorrelated. Because the decay time of the wave tracks increases with increasing pump intensity, an emission line becoming smaller and smaller results. Above laser threshold, light acquires quite different properties. It consists of an amplitude stabilized wave which can perform small fluctuations around its stable amplitude. On this wave amplitude a phase diffusion is superimposed which also gives rise to a finite line-width of laser light. This finite line-width decreases more and more with increasing emission intensity.

### Exercises on section 10.4

(1) Prove (10.87) and (10.90).

Hint: Use (10.27)–(10.30), (10.64)–(10.69) and (10.81), (10.82).

(2) Calculate  $\bar{n}$  which occurs in (10.100).

Hint: Use (10.94) and (10.90).

(3) Show that  $\langle \rho(t)\rho(0) \rangle = \gamma_0 \rho(t)$ .

Hint: Integrate (10.132) and use the fact that  $\tilde{F}_{\text{tot}}(t)$  and  $\tilde{F}_{\text{tot}}^+(t')$  are  $\delta$ -correlated, i.e. that

$$\langle \tilde{F}_{\text{tot}}(t)\tilde{F}_{\text{tot}}^+(t') \rangle = \text{const.} \times \delta(t-t').$$

(4) Prove (10.137).

Hint: Put  $b(t) \approx r_0 \exp[i\varphi(t)]$  as an approximate solution of (10.83).

Assume that  $\varphi(t)$  is decomposed into individual terms  $\varphi_\mu$  so that  $\varphi(t) = \sum_\mu \varphi_\mu(t)$  and assume that these are small and statistically independent of each other, i.e.  $\langle \varphi_\mu \varphi_{\mu'} \rangle = 0$  for  $\mu \neq \mu'$ . Put further

$$\exp[i\varphi(t)] = \exp\left[i \sum_\mu \varphi_\mu(t)\right] = \prod_\mu \exp[i\varphi_\mu(t)] \approx \prod_\mu (1 + i\varphi_\mu - \frac{1}{2}\varphi_\mu^2).$$

### 10.5. The behavior of the laser at its threshold. Photon statistics

In the previous sections we demonstrated that the behavior of laser light above threshold differs dramatically from that below threshold. But our methods did not allow us to treat the rather small, but interesting region around threshold where this change takes place. An elegant way to answer

this question is provided by a study of the distribution function of laser light. This can be done in various ways. One approach is provided by the density matrix equation of the laser, and its direct solution. Another approach consists in using the method of "quantum-classical correspondence" through which we may transform the density matrix equation into a "generalized Fokker-Planck equation". This equation can then be considerably simplified, for instance at and close to the laser threshold, and can eventually be solved, which yields the wanted distribution function. We shall present this approach in chapter 11. Because this approach is mathematically more difficult, in the present section we rather want to continue our line of thought. It is, to some extent, based on intuition and seemingly not so rigorous, but it will provide us with a quick access to the essential features of photon statistics close to and at laser threshold. A rigorous foundation for our present procedure, in which we treat  $b$  as a  $c$ -number, will be given in the subsequent chapter.

As we have seen in the previous sections, we can interpret the quantum mechanical Langevin equation of laser light (10.83) practically as a classical equation. In this section we shall adopt the attitude that this former equation,

$$\frac{db}{dt} = (-i\omega + G)b - C(b^+b)b + F_{\text{tot}}, \quad (10.145)$$

refers to *classical quantities*. In Vol. 1 we have shown how to construct a Fokker-Planck equation for the distribution function belonging to a Langevin equation. We wish to establish a Langevin equation for real variables. To this end we proceed in two steps. By means of the transformation

$$b = e^{-i\omega t} B \quad (10.146)$$

we split off an exponential factor which contains the atomic transition frequency  $\omega$ . Because  $b$  and thus  $B$  are complex quantities we write  $B$  in the form

$$B = x + iy, \quad (10.147)$$

where  $x$  and  $y$  are real time dependent variables. In analogy to (10.147) we decompose the fluctuating force  $F_{\text{tot}}$  into its real and imaginary part by putting

$$e^{i\omega t} F_{\text{tot}} = F_x + iF_y, \quad (10.148)$$

$F_x$  and  $F_y$  are real fluctuating forces which shall be interpreted as  $c$ -number forces. We insert (10.147) and (10.148) into (10.145) and decompose the

thus resulting equation into two equations for the real quantities  $x$  and  $y$ ,

$$\frac{dx}{dt} = Gx - C(x^2 + y^2)x + F_x, \quad (10.149)$$

$$\frac{dy}{dt} = Gy - C(x^2 + y^2)y + F_y. \quad (10.150)$$

We have shown in Vol. 1 how to construct a Fokker–Planck equation which belongs to an equation of the form (10.149). It is not difficult to generalize this procedure to coupled equations such as (10.149) and (10.150). In such a case we have to construct a Fokker–Planck equation for a distribution function which depends on the two independent variables  $x$ ,  $y$ , and on time  $t$ ,  $\tilde{f}(x, y; t)$ .  $\tilde{f}(x, y; t) dx dy$  is the probability to find at time  $t$  the variables  $x$  and  $y$  in the intervals  $x \dots x + dx$ ,  $y \dots y + dy$ . We shall present a concrete example on  $\tilde{f}(x, y; t)$  below. The Fokker–Planck equation for  $\tilde{f}$  acquires the form

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial t} = & -\frac{\partial}{\partial x} \{ [Gx - C(x^2 + y^2)x] \tilde{f} \} \\ & -\frac{\partial}{\partial y} \{ [Gy - C(x^2 + y^2)y] \tilde{f} \} + Q_{xx} \frac{\partial^2 \tilde{f}}{\partial x^2} + Q_{yy} \frac{\partial^2 \tilde{f}}{\partial y^2}. \end{aligned} \quad (10.151)$$

$Q_{xx}$  and  $Q_{yy}$  are the diffusion coefficients. They are determined by means of the correlation functions of the fluctuating forces by

$$\langle F_i(t) F_j(t') \rangle = \delta(t - t') Q_{ij}, \quad i, j = x, y. \quad (10.152)$$

Whereas in a classical treatment, which we present here,  $Q_{ij}$  is uniquely defined by (10.152), a more subtle discussion is required if the  $F$ 's stem from quantum mechanical Langevin equations such as (10.83). In such a case the sequence of the operator forces  $F^+$ ,  $F$  is important. Here we anticipate the result of the rigorous quantum theoretical treatment to be presented in chapter 11. Close to laser threshold one is allowed to calculate (10.152) by means of symmetrized  $F$ 's, i.e. by  $F$  replaced by  $\frac{1}{2}(F^+ + F)$ , or by  $F$ 's used in "normal order", i.e.  $\langle F^+(t) F(t') \rangle$ , with practically the same results. In particular a closer discussion reveals that

$$Q_{xy} = Q_{yx} = 0 \quad (10.153)$$

and

$$Q_{xx} = Q_{yy} = Q, \quad (10.154)$$

where  $Q = \frac{1}{2}\kappa(n_{th} + n_{sp})$ ;  $n_{th}$  and  $n_{sp}$  were introduced above (10.102, 10.103).

For a further treatment of the Fokker–Planck equation (10.151) we transform the coordinates  $x, y$  into polar coordinates

$$x = r \cos \varphi, \quad y = r \sin \varphi. \tag{10.155}$$

In this way  $\tilde{f}$  is transformed into a new distribution function  $f$  which depends on  $r$  and  $\varphi$ ,  $\tilde{f} \rightarrow f(r, \varphi)$ , or more precisely

$$\tilde{f} \, dx \, dy = f r \, dr \, d\varphi. \tag{10.156}$$

Because the explicit performance of this transformation within the Fokker–Planck equation is of no physical interest, we immediately write down the new Fokker–Planck equation

$$\frac{\partial f}{\partial t} = - \left( \frac{1}{r} \frac{\partial}{\partial r} (Gr^2 - Cr^4) f \right) + Q \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2} \right]. \tag{10.157}$$

In general the solutions of (10.157) can be obtained only by means of computers. The stationary solution of (10.157) can be found explicitly, however. In this case

$$\frac{\partial f}{\partial t} = 0. \tag{10.158}$$

We first try to visualize that in the stationary state the distribution function  $f$  does not depend on  $\varphi$ . From our preceding sections it is known that laser light undergoes a phase diffusion. Using again the picture of a particle diffusing in a potential valley we immediately see that after a certain time the probability of finding a specific phase becomes equal for all phases. This means that the probability of finding the particle at any point on the bottom of the valley is of the same size everywhere. As a consequence we have

$$\frac{\partial f}{\partial \varphi} = 0. \tag{10.159}$$

In this way eq. (10.157) reduces to

$$- \frac{\partial}{\partial r} [(Gr^2 - Cr^4) f] = Q \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right). \tag{10.160}$$

We can immediately integrate this equation and obtain

$$(Gr^2 - Cr^4) f = Qr \frac{\partial f}{\partial r} + C'. \tag{10.161}$$

The constant  $C'$  can be determined as follows. Because  $f$  is a distribution

function it must be normalizable in the whole range  $0 \leq r \leq \infty$ . Consequently  $f$  (and its derivative with respect to  $r$ ) must vanish sufficiently strongly for  $r \rightarrow \infty$ . Letting  $r \rightarrow \infty$  in (10.161), we readily find

$$C' = 0, \quad (10.162)$$

so that (10.161) simplifies to

$$\frac{\partial f}{\partial r} = \frac{1}{Q} (Gr - Cr^3)f. \quad (10.163)$$

This first order differential equation can be easily solved. Its solution reads

$$f = \frac{\mathcal{N}}{2\pi} \exp[r^2 G/(2Q) - r^4 C/(4Q)]. \quad (10.164)$$

$\mathcal{N}$  is a normalization factor which is defined by

$$\frac{1}{\mathcal{N}} = \int_0^\infty \exp[\dots] r \, dr. \quad (10.165)$$

By means of the potential function  $V(q)$  of a fictitious particle, which we introduced in (6.51),  $f$  can be written in a particularly simple form

$$f = \frac{\mathcal{N}}{2\pi} \exp[-V(r)/Q], \quad (10.166)$$

where  $q \equiv r$ .

A discussion of the function (10.164) tells us how laser light behaves close to and at laser threshold. In fig. 10.12  $f$  is presented as a function of  $r$  for various parameter values  $G$ . Obviously for  $G < 0$  the maximum of  $f$  lies at  $r = 0$ . This maximum is shifted to higher  $r$  values with increasing  $G > 0$ . But we know that  $r^2$  is identical with  $|b|^2$ . In a classical interpretation  $|b|^2$  is the intensity of light (besides a constant factor), whereas in a quantum mechanical interpretation we may consider  $r^2$  as a qualitative measure of the number of photons. Figs. 10.13 and 10.14 represent  $f$  as a function of  $r$  and  $\varphi$ , where for  $G > 0$  a probability crater can be observed. The potential of the fictitious particle is represented below the distribution function. It is evident how the potential jointly with the fluctuating forces shape the distribution function. In the case  $G < 0$  there is only one minimum of the potential. Correspondingly there is only one maximum of the distribution function. For  $G > 0$  we find a circular valley of the potential. Correspondingly a circular rim of a mountain with a crater in the middle is found representing the probability distribution.

In order to compare the theory, which we have outlined above, with experiments, the following two points must be observed. In order to rep-

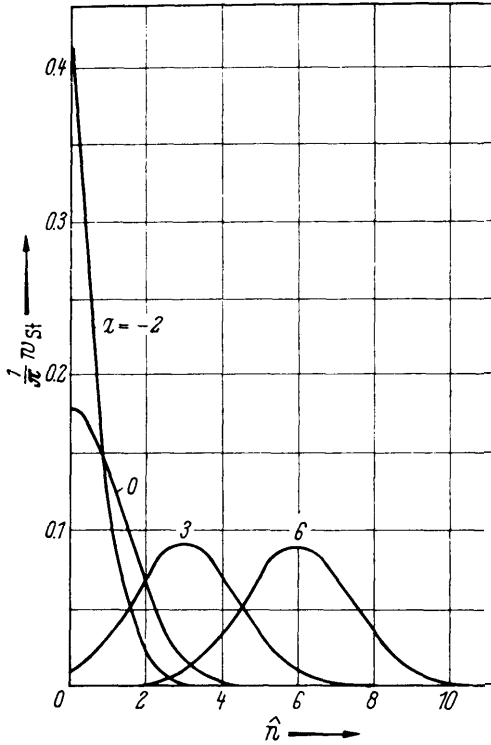


Fig. 10.12. The laser distribution function (10.164) versus  $\hat{n} = r^2$  in reduced units (cf. (10.167), (10.188)).

represent the various laser parameters in a unique fashion it is useful to introduce new dimensionless quantities. Furthermore the variable  $r$  of the distribution function (10.164) is continuous. As just mentioned, in a classical interpretation  $r^2 = |b|^2$  corresponds to an intensity. On the other hand, within a quantum mechanical interpretation,  $r^2$  represents some measure of the discrete photon numbers. Thus we have to present the transformation from continuous to discrete photon numbers. But let us first introduce dimensionless quantities by means of the transformation

$$\hat{r} = \sqrt[4]{C/Q}r, \quad \hat{t} = \sqrt{CQ}t, \quad \hat{n} = \hat{r}^2, \quad a = G/\sqrt{QC} \tag{10.167}$$

(read  $\hat{r}$ :  $r$  "hat"). In this way the distribution function (10.164) can be represented in the form

$$W(\hat{n}) = \mathcal{N} \exp\left(-\frac{\hat{n}^2}{4} + a\frac{\hat{n}}{2}\right). \tag{10.168}$$

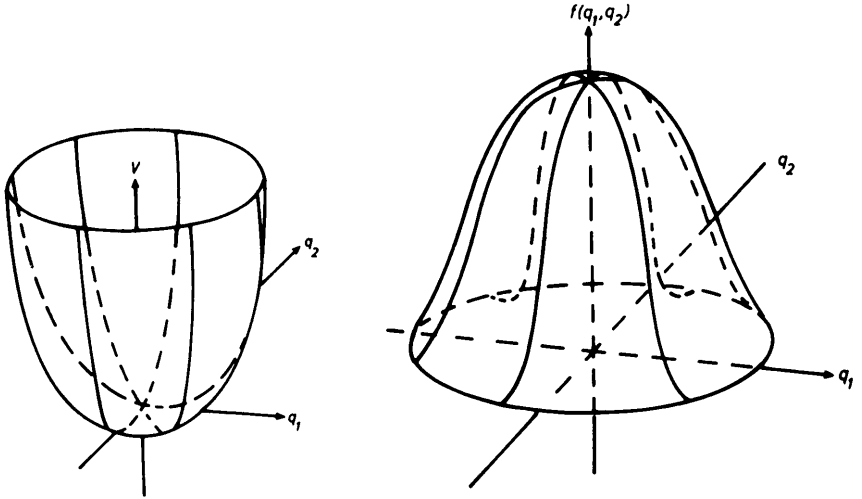


Fig. 10.13. The left-hand side represents the potential of the fictitious particle below threshold. On the right-hand side  $f(q_1, q_2)$  with  $r^2 = q_1^2 + q_2^2$  is plotted versus the coordinates  $q_1, q_2$ :  $f$  is bell shaped.

We now discuss the transformation from continuous to discrete variables. Experimentally, discrete photon numbers  $n$  are counted within given counting intervals of time duration  $T_0$  by means of a photo detector mounted outside of the mirrors of the laser resonator. The thus resulting distribution

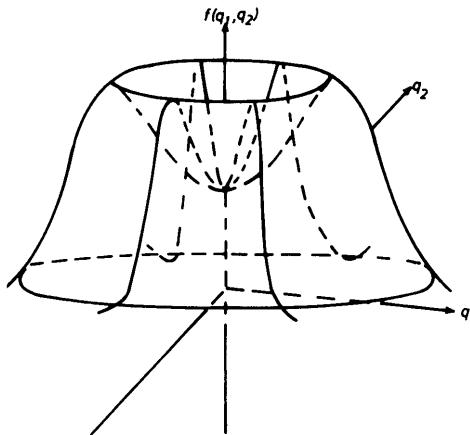


Fig. 10.14. Above threshold the potential function of fig. 10.9 applies. In the present figure the corresponding distribution function  $f(q_1, q_2)$  is shown. The probability "crater" can easily be seen.



function of the counted photon numbers will be denoted by  $p(n, T_0)$ . As a detailed theory reveals,  $p$  can be brought in connection with the distribution function (10.168).<sup>\*</sup> Provided  $T_0$  is small compared to the relaxation times of the nonlinear oscillators (10.145), the relation

$$p(n, T_0) = \int_0^\infty \frac{(s\hat{n})^n}{n!} e^{-s\hat{n}} W(\hat{n}) d\hat{n}, \tag{10.169}$$

holds. The constant  $s$ , which is determined by the photo detector, is proportional to  $T_0$  and depends in particular on the sensitivity of the detector.

When we insert (10.168) in (10.169) and assume a sufficiently large average number of photons, we obtain a practically continuous distribution function

$$p(n, T_0) = \frac{1}{sF_0(a)} \exp\left[-\frac{1}{4}\left(\frac{n}{s}\right)^2 + \frac{1}{2}a\frac{n}{s}\right]. \tag{10.170}$$

In other words, we obtain the same distribution function as before, but with a scaled photon number  $n/s$ .  $F_0(a)$  is given by

$$F_0(a) = \int_0^\infty \exp\left[-\frac{\tilde{I}^2}{4} + a\frac{\tilde{I}}{2}\right] d\tilde{I}, \tag{10.171}$$

where according to (10.167)  $a$  is proportional to the unsaturated net gain  $G$ . Because this latter quantity is proportional to the unsaturated inversion  $d_0$  and thus proportional to the pump power,  $a$  is also called pump parameter.

We shall not discuss the transformation just introduced any further but we rather wish to present some typical examples of the theoretical results and their experimental verification. Let us consider fig. 10.12 once again. Quite evidently the photon statistics changes close to threshold,  $G=0$ , i.e.  $a=0$ . The nature of the photon statistics below and above threshold can be read off already from the mean squared deviation of the photon number,

$$\langle (n - \langle n \rangle)^2 \rangle \equiv \langle n^2 \rangle - \langle n \rangle^2, \tag{10.172}$$

where  $\langle \dots \rangle$  means quantum statistical average.

This quantity can be measured experimentally. In this way the continuous transition from the distribution function below threshold to that above threshold can be studied in detail.

In the quantum statistical average (10.172),  $n$  stands for the photon number operator  $b^+b$ . Whereas so far our simplifying procedure, in which we treated  $b^+$ ,  $b$  as classical quantities, can be justified (see chapter II), the evaluation of (10.172) requires particular care because of the operator

<sup>\*</sup>For a detailed derivation, cf. the literature given in the references.

properties of  $b$ ,  $b$ . Inserting  $n = bb$  in (10.172) we obtain

$$\langle (b^+b)^2 \rangle - \langle b^+b \rangle^2,$$

or more explicitly

$$\langle (b^+bb^+b) \rangle - \langle b^+b \rangle^2. \quad (10.173)$$

As we know,  $b$ ,  $b^+$  obey the commutation relation

$$bb^+ - b^+b = 1. \quad (10.174)$$

By means of this relation, the product  $b^+bb^+b$  in (10.173) can be rewritten as  $b^+b^+bb + b^+b$ . In this way (10.173) transforms into

$$\underbrace{\langle b^+b^+bb \rangle - \langle b^+b \rangle^2}_{K_2(\tau=0)} + \langle b^+b \rangle. \quad (10.175)$$

As we shall prove in chapter 11, the expression  $K_2$  in (10.175) can be evaluated by means of the distribution function  $f$  as if  $b^+$ ,  $b$  were classical quantities. Equivalently,  $\langle b^+b^+bb \rangle$  and  $\langle b^+b \rangle$  can be calculated by use of the solutions of the classical Langevin equations. Therefore we may identify  $K_2$  in (10.175) with the expression  $K_2(\tau=0)$  of section 10.4, where we calculated it explicitly below and above threshold. In this way we obtain:

below threshold

$$K_2(\tau=0) = \langle b^+b \rangle^2; \quad (10.176)$$

above threshold

$$K_2(\tau=0) = \frac{N_{2,s}}{4\kappa T}. \quad (10.177)$$

We can calculate the mean squared deviation (10.172) by inserting (10.176), (10.177) in (10.175), (10.172). We thus obtain:

below threshold

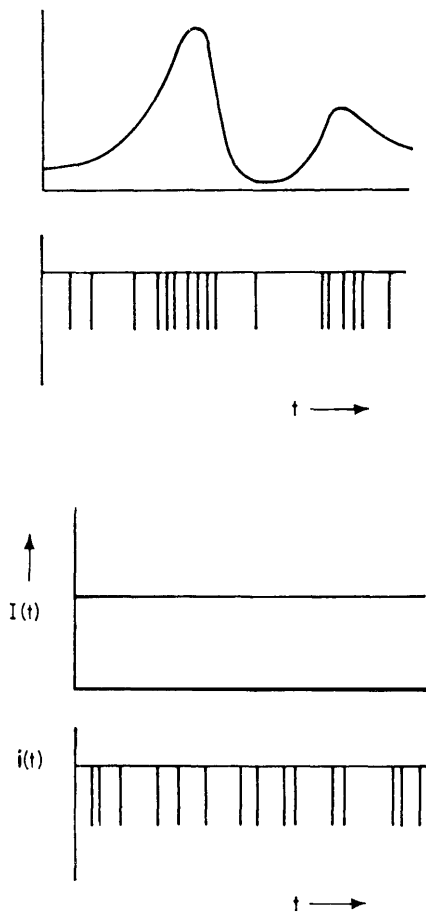
$$\begin{aligned} \langle (n - \langle n \rangle)^2 \rangle &= \langle b^+b \rangle^2 + \langle b^+b \rangle \\ &\equiv \langle n \rangle (\langle n \rangle + 1). \end{aligned} \quad (10.178)$$

This result is typical for Bose–Einstein statistics, where photons tend to form clusters (compare fig. 10.15). On the other hand we obtain:

above threshold ( $n \rightarrow \infty$ )

$$\langle (n - \langle n \rangle)^2 \rangle \approx \langle n \rangle. \quad (10.179)$$

This relation is well known to hold for a Poisson distribution. It means that the photons try to keep a mean distance from each other (compare fig. 10.16). We have been able to find these results by means of the solutions of the laser equation (10.128). But we had to exclude a small range close to and at laser threshold. This gap can now be closed by an explicit use of



Figs. 10.15 and 10.16. These figures show the dependence of the light intensity versus time in the corresponding upper parts of the figures. The corresponding lower parts represent incoming individual photons in the course of time. Fig. 10.15: The case of a lamp. Photon bunching takes place. The light intensity exhibits correspondingly strong fluctuations. Fig. 10.16: The case of the laser. The photons maintain a mean distance between each other. Correspondingly, a smooth light intensity appears. [E.R. Pike, in: Quantum Optics, eds. L. McKay and A. Maitland. Academic Press, New York 1970.1

the distribution functions (10.164) or (10.170), respectively. We remind the reader that we shall justify the use of classical averages instead of the quantum statistical averages in chapter 11. In order to describe the smooth transition from (10.178) to (10.179) we write the l.h.s. of (10.178) (or (10.179)) in the form

$$\langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle (1 + H_2 \langle n \rangle), \quad (10.180)$$

where  $H_2$  is a function of  $\langle n \rangle$  still to be determined. Quite evidently,  $H_2 = 1$  below and  $H_2 = 0$  above threshold. Solving (10.180) with respect to  $H_2$  and using the abbreviations  $M_2 = \langle n^2 \rangle$ ,  $M_1 = \langle n \rangle$  we obtain

$$H_2 = \frac{M_2 - M_1^2}{M_1^2} - \frac{1}{M_1}. \quad (10.181)$$

In this way  $H_2$  is directly expressed by measurable quantities, namely the moments of  $n^2$  and  $n$ . Fig. 10.17 represents a comparison between the theory based on the distribution function (10.164) and experiments. The results show both a good agreement between theory and experiment and reveal also a continuous transition from (10.178) to (10.179). The pump parameter has been replaced by a normalized emission intensity. A similarly good agreement has been found for other quantities also, which can be expressed by higher moments, such as  $M_3 = \langle n^3 \rangle$  and  $M_4 = \langle n^4 \rangle$ .

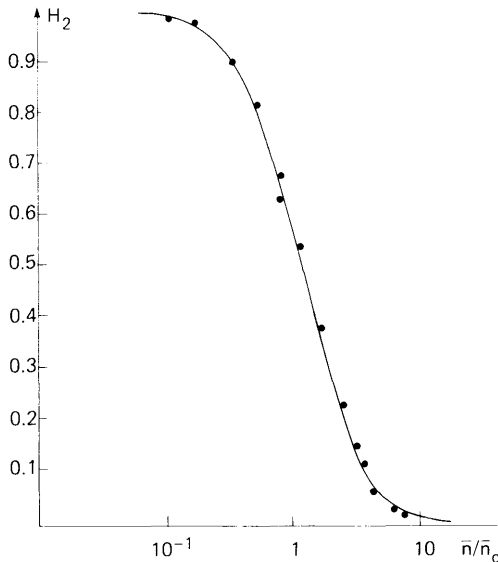


Fig. 10.17.  $H_2$  versus normalized number of emitted photons (experimental, after results of Arecchi et al., theoretical results after Risken, cf. references).

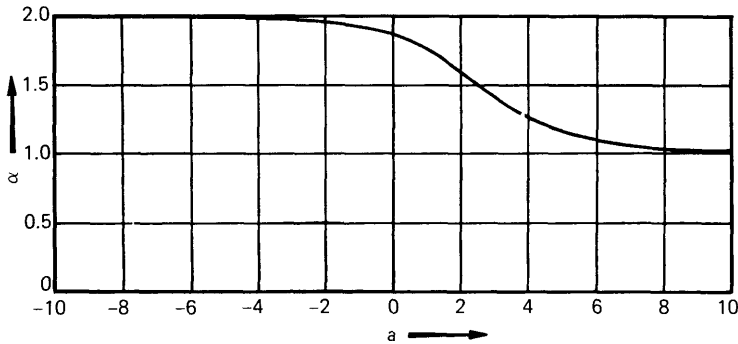


Fig. 10.18. The function  $\alpha(a)$  occurring in (10.183). [H. Risken, Z. Physik 191, 302 (1966).]

We have shown in detail in Vol. 1, section 2.2, that the coherence properties of light are represented by correlation functions of field strengths, e.g. by  $\langle E^{(-)}(t)E^{(+)}(t') \rangle$ . Two particularly important expressions were calculated in the previous sections, namely

$$\langle b^+(t) b(0) \rangle \quad \text{and} \quad K_2(\tau) = \langle b^+(0) b^+(\tau) b(\tau) b(0) \rangle - \langle b^+ b \rangle^2$$

for regions below and above threshold. In the vicinity of the threshold these expressions can be calculated by means of the time dependent solutions of the Fokker–Planck equation (10.157). These solutions were determined by a numerical calculation. Out of these results we present the following which shows how the laser line-width  $\Delta\omega$  changes at laser threshold.  $\Delta\omega \equiv \Delta\omega(a)$  varies continuously as a function of the pump parameter  $a$  and one finds

$$\Delta\omega(a) \cdot P(a) = \Delta\omega_0 P_0 \cdot \alpha(a). \tag{10.182}$$

$P(a)$  is the emission intensity of the laser at the pump strength  $a$ .  $\Delta\omega_0$  and  $P_0$  denote the line-width and the emission intensity, respectively, of the laser above its threshold.  $\Delta\omega_0$  is given by (10.135) and  $P_0$  by (10.136). As we may see,  $\Delta\omega_0 \cdot P_0$  is independent of the emitted power so that (10.182) can be written in the form

$$\Delta\omega(a) \cdot P(a) = \text{const.} \cdot \alpha(a). \tag{10.183}$$

$\alpha(a)$  was calculated numerically and is represented in fig. 10.18. If the emission powers  $P$  below and above threshold were equal to each other we should find a line-width below threshold which is twice that above threshold.

## Chapter 11

# Quantum Theory of the Laser II

### A second approach via the density matrix equation and quantum classical correspondence

#### 11.1. The density matrix equation of the laser

In the foregoing chapter we developed the quantum theory of the laser by means of quantum mechanical Langevin equations. These equations have the advantage that their physical meaning can be quite easily visualized due to their analogy with the semiclassical laser equations. Below and above threshold they can be rather easily solved even in the quantum mechanical case by means of linearization or quasi-linearization. On the other hand the rather small but most interesting region close to and at laser threshold could not be treated directly by means of the quantum mechanical Langevin equations. Not because these equations become invalid, but because no way of their solution is known in that region. Therefore we resorted to a Fokker–Planck equation in section 10.5. In that section we could base the derivation of a classical Fokker–Planck equation from quantum mechanical Langevin equations merely on heuristic arguments. The present chapter serves the purpose of filling that gap. We wish to derive that former Fokker–Planck equation from first principles whereby the complicated quantum mechanical problem is cut down stepwise by well-defined and well justified approximations. In this section we shall make a first step towards that goal by deriving the density matrix equation of the laser. The reader should be familiar with some basic features of a density matrix equation, for instance as presented in Vol. 1.

Again as in the foregoing chapter we start from the total Hamiltonian which reads

$$H = \underbrace{H_f + H_A + H_{Af}}_{H_0} + H_{B_1} + H_{B_1-f} + H_{B_2} + H_{B_2-A}. \quad (11.1)$$

In it  $H_0$  is composed of the Hamiltonian  $H_f$  of the free modes, of the

Hamiltonian  $H_A$  of the laser active atoms, and of the Hamiltonian  $H_{Af}$  describing the interaction between the field and the atoms. We shall adopt the same model as in the foregoing chapter, namely a single mode and a set of two-level atoms being in resonance with the field mode. The explicit form of the Hamiltonians is given by (10.4), (10.13), (10.21). The remaining terms in (11.1) describe the coupling of the field mode and the atoms to their corresponding heatbaths. A possible way of treating the problem defined by the Hamiltonian (11.1) would be to solve the corresponding Schrodinger equation

$$H\Psi = i\hbar\dot{\Psi}, \tag{11.2}$$

which, of course, contains all bath variables. These variables are of no direct interest to us, however, and we wish to eliminate them. To this end we adopt the approach via the density matrix. The density matrix  $\rho_{tot}$ , which refers to the total system described by (11.1), obeys the equation of motion

$$\frac{d\rho_{tot}}{dt} = -\frac{i}{\hbar}[H, \rho_{tot}]. \tag{11.3}$$

On the other hand we are interested in a density matrix which contains the proper laser variables, i.e. the field mode and the atomic variables, but no more the heatbath variables. To this end we introduce a new density matrix  $\rho$  by averaging the original density matrix  $\rho_{tot}$  over the heatbath variables which are assumed to be in their thermal equilibria at their specific temperatures,

$$\rho = \langle \rho_{tot} \rangle. \tag{11.4}$$

Then our obvious goal will be to derive an effective density matrix equation for  $\rho$ . Such a program has been followed up in Vol. 1 where we showed how a field mode coupled to a heatbath or an atom coupled to a heatbath can be treated. For the reader's convenience we briefly remind him of the main steps.

Let us consider a field mode coupled to its heatbath. The density matrix of this total system is denoted by

$$\rho_{f,tot} \tag{11.5}$$

and obeys the equation

$$\frac{d\rho_{f,tot}}{dt} = -\frac{i}{\hbar}[H_f + H_{B_1} + H_{B_1-f}, \rho_{f,tot}]. \tag{11.6}$$

We have shown that we may derive an equation for the density matrix  $\rho_f$

which refers to the field mode variables  $b^+$ ,  $b$  alone. The equation for

$$\rho_f = \langle \rho_{f,\text{tot}} \rangle \quad (11.7)$$

reads

$$\begin{aligned} \dot{\rho}_f = & -\frac{i}{\hbar} [\hbar\omega b^+ b, \rho_f] + \delta \{ [b^+ \rho_f, b] + [b^+, \rho_f b] \} \\ & + \xi \{ [b \rho_f, b^+] + [b, \rho_f b^+] \}, \end{aligned} \quad (11.8)$$

where the constants  $\delta$  and  $\xi$  are connected with the damping constant  $\kappa$  of the field mode amplitudes and the number of thermally present photons,  $n_{\text{th}}$ , by

$$\delta = \kappa n_{\text{th}}, \quad (11.9)$$

$$\xi = \kappa (n_{\text{th}} + 1). \quad (11.10)$$

After this brief reminder let us return to the problem of deriving a reduced equation for the density matrix  $p$  of the total system field + atoms (11.4). The total temporal change of  $p$  consists of three parts:

(1) The change caused by the coherent interaction *between field* and atoms. This change is given as usual by

$$\left( \frac{\partial \rho}{\partial t} \right)_{\text{coh}} = -\frac{i}{\hbar} [H_0, \rho]. \quad (11.11)$$

(2) The two other parts stem from the coupling of the field to its heatbaths and from the coupling of the atoms to their heatbaths, so that

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H_0, \rho] + \left( \frac{\partial \rho}{\partial t} \right)_{B_1-f} + \left( \frac{\partial \rho}{\partial t} \right)_{B_2-A}. \quad (11.12)$$

The change of  $p$  due to the coupling field–heatbath can be directly taken from (11.8), because the coherent motion of  $\rho_f$  is taken care of by (11.11). The terms still to be considered of (11.8) are given by

$$\left( \frac{\partial \rho}{\partial t} \right)_{B_1-f} = \delta \{ [b^+ \rho, b] + [b^+, \rho b] \} + \xi \{ [b \rho, b^+] + [b, \rho b^+] \}. \quad (11.13)$$

The last term in (11.12) describing the effect of the heatbaths on the atoms can again be taken from the general results from Vol. 1. According to that volume quite generally we have

$$\left( \frac{\partial \rho}{\partial t} \right)_{B_2-A} = \sum_{\mu ij} \{ [(a_i^+ a_j)_\mu \rho, (a_j^+ a_i)_\mu A_{ji,ij}] + [(a_i^+ a_j)_\mu, \rho (a_j^+ a_i)_\mu] A_{ji,ij}^* \}. \quad (11.14)$$



Here we have summed up over the individual atoms  $\mu$  where it is assumed that the heatbaths of the individual atoms are statistically independent from each other. Because we deal with two-level atoms only, the indices  $i$  and  $j$  run over 1 and 2 only (here and in the following formulas we have corrected some minor misprints in Vol. 1).

$A$  are constants which we do not need to know explicitly because they are connected with phenomenological quantities, namely:

(1) the transition rate from level  $j$  to level  $m$  under incoherent processes,

$$w_{mj} = A_{jm,mj} + A_{jm,mj}^* \quad (11.15)$$

(2) the line-width connected with transitions between levels  $m$  and  $n$ ,

$$\gamma_{mn} = \sum_i \operatorname{Re}(A_{mi,im} + A_{ni,in}^*) \equiv \frac{1}{2} \sum_i (w_{im} + w_{in}); \quad (11.16)$$

(3) frequency shifts for transitions connecting levels  $m$  and  $n$

$$\Delta\omega_{mn} = -\sum_i \operatorname{Im}(A_{ni,in} + A_{mi,im}^*). \quad (11.17)$$

Because frequency shifts can be absorbed in the original energies of the atomic levels ("renormalization"), we may assume that (11.17) vanishes, and as a more detailed analysis shows, we may even assume  $A$  to be real. In such a case and for the two-level atoms under consideration (11.15) and (11.16) simplify to

$$w_{21} = 2A_{12,21}, \quad (11.18)$$

$$\gamma \equiv \gamma_{12} = \gamma_{21} = \frac{1}{2}(w_{12} + w_{21}) + \Delta, \quad (11.19)$$

where the term

$$\Delta = \frac{1}{2}(w_{11} + w_{22}) \quad (11.20)$$

describes phase fluctuations which are not caused by real transitions. In order not to overload our further analysis we shall ignore the term (11.20) and refer the readers, who are interested in these specific details, to H. Haken, Laser Theory.

Using our former notation  $a_1^\dagger a_2 = a, \dots$  we can cast (11.14) into the form

$$\left( \frac{\partial \rho}{\partial t} \right)_{B_2-A} = \sum_{\mu} \left[ \frac{w_{12}}{2} \{ [\alpha_{\mu}, \rho \alpha_{\mu}^+] + [\alpha_{\mu} \rho, \alpha_{\mu}^+] \} \right. \\ \left. + \frac{w_{21}}{2} \{ [\alpha_{\mu}^+, \rho \alpha_{\mu}] + [\alpha_{\mu}^+ \rho, \alpha_{\mu}] \} \right]. \quad (11.21)$$

(11.12) jointly with (11.11), (11.13) and (11.21) represents the desired density

matrix equation. With respect to the degree of accuracy this density matrix equation is entirely equivalent to that of the quantum mechanical Langevin equations of section 10.3 (provided we include the terms (11.20) in (11.21)). It should be noted that both approaches imply the same kind of approximation, namely it is assumed that the interaction between the field mode and the atoms is not so strong that the interaction of these individual systems with their individual **heatbath** is appreciably disturbed. In the case of a very strong interaction between field and atoms new kinds of effects may occur, for instance a quenching of the interaction between atoms and their heatbaths.

In order to proceed further, one possibility is given by trying to solve the density matrix equation (11.12) directly. Approximate solutions of this density matrix equation were given in the literature. Readers interested in those approaches are referred to the references. In the context of the present book we rather wish to follow up the line indicated at the beginning of this section, namely to derive a classical Fokker–Planck equation starting from the quantum mechanical density matrix equation. To this end we have to provide a link between the quantum mechanical and the classical description by means of the method of quantum classical correspondence.

## 11.2. A short course in quantum classical correspondence. The example of a damped field mode (harmonic oscillator)

### 11.2.1. A formal analogy between quantum statistical and classical statistical averages

The method which we are going to describe briefly is of interest both to laser theory and to nonlinear optics. Let us start from the quantum mechanical Langevin equation of a damped field mode. The equation of the annihilation operator  $b$  is given by (see also Vol. 1)

$$\dot{b} = (-i\omega - \kappa)b + F(t). \quad (11.22)$$

By means of the transformations

$$b = \tilde{b} \exp[-i\omega t], \quad (11.23)$$

$$F(t) = \tilde{F}(t) \exp[-i\omega t], \quad (11.24)$$

we can cast (11.22) into the simpler form

$$\dot{\tilde{b}} = -\kappa\tilde{b} + \tilde{F}(t). \quad (11.25)$$

This equation is analogous to a *classical* Langevin equation which we have

met in Vol. 1, namely

$$\dot{q} = -\kappa q + F. \tag{11.26}$$

This equation can be interpreted as that of the overdamped Brownian motion of a particle. We wish to exploit the analogy between (11.25) and (11.26) to devise a method how to calculate quantum mechanical expectation values by means of c-number procedures ("c" = "classical"). If the quantum mechanical oscillator is coupled to a heatbath, the proper definition of its quantum mechanical and quantum statistical expectation value ( $\langle b \rangle$ ) is given by

$$\langle b \rangle = \text{Tr}\{b\rho\}, \tag{11.27}$$

where  $\rho$  is the density matrix, and Tr means "trace". In order to evaluate (11.27) one has to solve, of course, the density matrix equation for  $\rho$ . As a further step towards the analogy we wish to establish, we consider how the evaluation of an average value corresponding to (11.27) would be done in classical physics. Here the average value of  $q$  is defined by

$$\langle q \rangle = \int q f(q, t) dq, \tag{11.28}$$

where  $f(q, t)$  is the distribution function. As we know (see Vol. 1)  $f(q, t) dq$  gives us the probability of finding the particle at time  $t$  in the interval  $q \dots q + dq$ . The distribution function  $f$  obeys a Fokker–Planck equation which according to Vol. 1, p. 291, is given by

$$\frac{df(q, t)}{dt} = \frac{\partial}{\partial q}(\kappa q f) + \frac{Q}{2} \frac{\partial^2 f}{\partial q^2}. \tag{11.29}$$

In order to make our following procedure understandable we must briefly remind the reader how the Fokker–Planck equation (11.29) is derived from the equation (11.26). (Readers interested in more details are referred to my book Synergetics. An Introduction.)

### 11.2.2. A classical Fokker–Planck equation for the damped quantum mechanical oscillator

Let us first consider the special case of (11.26) in which there is no fluctuating force  $F(t)$ . Once the initial condition of  $q$  is fixed,  $q(t)$  is fixed also. In such a case we know that the particle will be at time  $t$  with certainty in the interval  $q \dots q + dq$ , provided  $q(t)$  lies in that interval, and the probability will be 0 elsewhere. The distribution function describing this property is

simply given by the  $\delta$ -function

$$f_0(q, t) = \delta(q - q(t)). \quad (11.30)$$

Now let us assume that the heatbath producing the fluctuating force  $F(t)$  is acting on the particle coordinate  $q$ . This force causes random pushes so that the particle will follow up different paths for different realizations of the random events. When we now wish to know the probability of finding the particle at time  $t$  in the interval  $q \cdots q + dq$  we have to average the distribution function (11.30) over the different paths caused by the heatbath, i.e. we have to replace (11.30) by

$$f(q, t) = \langle \delta(q - q(t)) \rangle_B. \quad (11.31)$$

The form (11.31) jointly with the properties of the fluctuating forces can be used to derive the Fokker–Planck equation (11.29) which belongs to (11.26) (cf. H. Haken, Synergetics. An Introduction). In the present context another property of (11.31) is still more important, however. Namely let us use the Fourier representation of the  $\delta$ -function, i.e.

$$\delta(q - q(t)) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp[i(q(t) - q)\xi] d\xi. \quad (11.32)$$

Inserting (11.32) in (11.31) and noting that the integration over the variable  $\xi$  has nothing to do with the heatbath average we obtain

$$\langle \delta(q - q(t)) \rangle_B = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \langle \exp[i(q(t) - q)\xi] \rangle_B d\xi, \quad (11.33)$$

or, using the same argument again we obtain

$$(11.33) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp[-iq\xi] \langle \exp[iq(t)\xi] \rangle_B d\xi. \quad (11.34)$$

The statistical average over the exponential function which occurs in (11.34) is well known in statistical physics and is called the characteristic function  $\chi$ ,

$$\langle \exp[iq(t)\xi] \rangle_B = \chi(\xi). \quad (11.35)$$

After these preliminary steps we are in a position to define a distribution function for quantum mechanical variables. We first note that according to the basic principles of quantum theory classical observables, such as  $q(t)$ , are replaced by operators  $b$  in quantum mechanics. As we know (see Vol. 1), in quantum mechanics we have different choices with respect to the time dependence of  $b$ . In the Schrodinger picture the operators  $b, b^+$  are time

independent and all the time dependence of the quantum mechanical system is described by the wave function  $\varphi$  or, in a more elegant fashion, by the time dependent density matrix. Another description is based on the Heisenberg picture where the operators  $b, b^+$  are time dependent, but the wave function is time independent. In the present context we shall use the Schrodinger picture which was already used without being mentioned explicitly in section 11.1. We now establish an analogy between the formation of a statistical average such as in (11.35) and a quantum mechanical average such as in (11.27). First of all we define in an obvious way the quantum mechanical and quantum statistical average over  $\exp[ib\xi]$  by

$$\langle \exp[ib\xi] \rangle \equiv \text{Tr}\{\exp[ib\xi] \rho(t)\}. \quad (11.36)$$

But in this way we have now defined a quantum mechanical characteristic function which occurs in analogy to (11.35). While for the real variables  $q(t)$  the characteristic function (11.35) is sufficient to characterize all statistical properties required, in the case of  $b$  we have to deal in the classical domain with a complex variable, and in the quantum mechanical domain with an operator  $b$  and its Hermitean conjugate. Therefore instead of the one real variable  $\xi$  we have now to use a complex variable  $\beta$  and its complex conjugate  $\beta^*$ . Therefore in generalizing (11.36) we may define a characteristic function  $\chi$  for the harmonic oscillator (field mode) by

$$\chi(\beta, \beta^*) = \text{Tr}\{\exp[i\beta b + i\beta^* b^+] \rho\}. \quad (11.37)$$

At this moment a fundamental difference between characteristic functions of classical stochastic variables, i.e. (11.35), and characteristic functions referring to operators, i.e. (11.37), becomes apparent. Namely because the operators  $b$  and  $b^+$  do not commute, we may define characteristic functions in various ways depending on the way we write down exponential functions containing  $b$  and  $b^+$ . For instance, we would split the exponential function occurring in (11.37) into a product which leads us to the characteristic function

$$\chi_P(\beta, \beta^*) = \text{Tr}\{\exp[i\beta^* b^+] \exp[i\beta b] \rho\}. \quad (11.38)$$

But because the operators  $b$  and  $b^+$  do not commute, (11.38) is a function different from (11.37). As the reader may recall, in quantum mechanics we read operators from the right to the left. Therefore in (11.38) the exponential function containing the operator  $b$  must be applied prior to the exponential function containing  $b^+$ . If on the other hand we exchange the sequence between these two exponential functions we are led to a third characteristic function, namely

$$\chi_Q(\beta, \beta^*) = \text{Tr}\{\exp[i\beta b] \exp[i\beta^* b^+] \rho\}. \quad (11.39)$$

In this way the characteristic functions (11.37), (11.38), (11.39) will give rise to different distribution functions which are called the "Wigner distribution function  $\chi$ ", the "Glauber–Sudarshan distribution function  $\chi_P$ ", and the "Q-distribution function  $\chi_Q$ ", respectively. In the following analysis we will use (11.38) as explicit example.

Let us recall what our goal was. We wanted to find a distribution function corresponding to a quantum mechanical process. Our starting point was eq. (11.31) which defined a classical distribution function  $f$  for a classical variable  $q(t)$ . The translation of this distribution function into one for quantum mechanical processes is provided by (11.34) where we have to replace the classical characteristic function (11.35) by the quantum mechanical characteristic function (11.38). To this end we have merely to take the Fourier transform of the characteristic function (11.38) where we replace the former real variable  $q$  by the complex variables  $u$  and  $u^*$  and the single integration over  $\xi$  by a two-dimensional integration  $d^2\beta$ . We shall not dwell on the mathematical details how to perform the integration in the complex plane. We rather exploit some formal properties and refer the reader to the specific literature for mathematical rigor. In analogy to (11.33), (11.34) we define the Glauber–Sudarshan distribution function by

$$P(u, u^*) = \pi^{-2} \int \exp[-i\beta u - i\beta^* u^*] \chi_P(\beta, \beta^*) d^2\beta, \quad (11.40)$$

or, making explicit use of (11.38) by

$$P(u, u^*) = \pi^{-2} \int \exp[-i\beta u - i\beta^* u^*] \text{Tr}\{\exp[i\beta^* b^+] \times \exp[i\beta b] \rho\} d^2\beta. \quad (11.41)$$

What is known and what is unknown of the right-hand side of (11.41)?  $u$  and  $u^*$  are independent variables, while  $\beta$  and  $\beta^*$  are integration variables.  $b^+$  and  $b$  are operators with given properties. Therefore the only unknown quantity is the density matrix  $\rho$ . Or, in other words, once we know  $\rho$ , we can calculate  $P$ , at least in principle. Thus one way would be to solve the density matrix equation for  $\rho$  first and then to calculate (11.41). Our goal is more ambitious, however, because we wish to derive a Fokker–Planck equation for  $P$ . To this end we shall transform the density matrix equation for  $\rho$  into an equivalent equation for  $P$ . In order to achieve this goal we need some formal tricks. First we introduce the abbreviation

$$O = \exp[i\beta^* b^+] \exp[i\beta b], \quad (11.42)$$

so that the characteristic function  $\chi$  can be written as

$$\chi(\beta, \beta^*) = \text{Tr}\{O\rho\}. \quad (11.43)$$

Note that for sake of simplicity we shall drop the index  $P$  of  $\chi$  in the following.

Let us first derive a differential equation for the characteristic function  $\chi$ . To this end we differentiate both sides of (11.43) with respect to time and obtain

$$d\chi/dt = \text{Tr}\{O(d\rho/dt)\}. \quad (11.44)$$

For  $d\rho/dt$  in (11.44) we insert the r.h.s. of (11.13) where we make use of the explicit form of  $\delta$  and  $\xi$  (cf. (11.9), (11.10)). Furthermore we make use of a theorem on the cyclic property of traces which can be expressed by the relation

$$\text{Tr}\{ABC\} = \text{Tr}\{CAB\}. \quad (11.45)$$

In this way we rewrite (11.44) as

$$d\chi/dt = \kappa \text{Tr}\{[2b^+Ob - Ob^+b - b^+bO]\rho\} + 2\kappa n_{\text{th}} \text{Tr}\{[b^+Ob - bb^+O - Ob^+b + bOb^+]\rho\}. \quad (11.46)$$

It will be our goal to reexpress the r.h.s. of (11.46) by  $\chi$  where we shall admit that  $\chi$  may be differentiated with respect to  $\beta$  or  $\beta^*$  or may be multiplied by these quantities.

Let us consider to this end (11.42) more closely. When we differentiate the r.h.s. of (11.42) with respect to  $i\beta^*$ , we readily find

$$b^+O = \frac{\partial O}{\partial(i\beta^*)}, \quad (11.47)$$

and by differentiating it with respect to  $i\beta^*$  and  $i\beta$  we derive

$$b^+Ob = \frac{\partial^2 O}{\partial(i\beta^*) \partial(i\beta)}. \quad (11.48)$$

Furthermore using the commutation relation  $bb^+ - b^+b = 1$  we can derive, as was shown explicitly in Vol. 1, the relation

$$bO = i\beta^*O + Ob. \quad (11.49)$$

Multiplying (11.49) from the left by  $b^+$  we find

$$b^+bO = i\beta^*b^+O + b^+Ob, \quad (11.50)$$

where the right-hand side can be written again as a derivative with respect

to  $i\beta^*$  and  $i\beta$ ,

$$i\beta^* \frac{\partial O}{\partial(i\beta^*)} + \frac{\partial^2 O}{\partial(i\beta^*) \partial(i\beta)}. \quad (11.51)$$

As we may see from (11.48) and (11.51), the operator expressions of the left-hand sides may be expressed by derivatives of  $O$ . In a similar way we can proceed with the remaining terms of (11.46) and we leave it as an exercise to the reader to derive the corresponding relations.

We now show how relations of the form (11.48) can be used to express the r.h.s. of (11.46) by means of  $\chi$ . To this end we study

$$\text{Tr}\{b^+ O b \rho\}. \quad (11.52)$$

Using (11.48) we may transform it into

$$\text{Tr}\left\{\frac{\partial^2 O}{\partial(i\beta^*) \partial(i\beta)} \rho\right\}. \quad (11.53)$$

Because taking the trace does not refer to the variables  $\beta$  and  $\beta^*$ , we can perform the differentiation in front of the trace so that we obtain

$$\frac{\partial^2}{\partial(i\beta^*) \partial(i\beta)} \text{Tr}\{O \rho\}. \quad (11.54)$$

According to (11.43), the remaining trace is identical with the characteristic function  $\chi$  so that we obtain as final result

$$(11.52) = \frac{\partial^2}{\partial(i\beta^*) \partial(i\beta)} \chi(\beta, \beta^*). \quad (11.55)$$

In quite the same way we can deal with all the other terms also, and we leave that treatment to the reader again as an exercise. Collecting all the terms we find

$$\frac{d\chi}{dt} = -\kappa \left( i\beta \frac{\partial}{\partial(i\beta)} + i\beta^* \frac{\partial}{\partial(i\beta^*)} \right) \chi + 2\kappa n_{\text{th}}(i\beta^*)(i\beta) \chi, \quad (11.56)$$

which is the wanted equation for the characteristic function. In a last step of our analysis we wish to transform eq. (11.56), which refers to the characteristic function, into an equation for the P-distribution function. To this end we differentiate (11.41) on both sides with respect to time, which on the left-hand side yields

$$\frac{dP(u, u^*)}{dt} = \int d^2\beta \exp[-i\beta u - i\beta^* u^*] \frac{d\chi}{dt}. \quad (11.57)$$



We express  $d\chi/dt$  on the r.h.s. of (11.57) by the r.h.s. of (11.56). Let us again treat a specific term of the r.h.s. of (11.56) as an example. Let us consider the expression

$$i\beta \frac{\partial \chi}{\partial (i\beta)}, \quad (11.58)$$

which gives rise, if inserted in (11.57), to

$$\int d^2\beta \exp[-i\beta u - i\beta^* u^*] i\beta \frac{\partial \chi}{\partial (i\beta)}. \quad (11.59)$$

The multiplication of the exponential function in (11.59) by  $i\beta$  can be expressed by the differentiation of that exponential function with respect to  $u$ . Because the integration over  $\beta$  does not effect the differentiation of the exponential function with respect to  $u$  we can write (11.59) in the form

$$-\frac{\partial}{\partial u} \int d^2\beta \exp[-i\beta u - i\beta^* u^*] \frac{\partial \chi}{\partial (i\beta)}. \quad (11.60)$$

We now perform a partial integration with respect to  $i\beta$  which transforms the integral in (11.60) into one where the exponential function is differentiated with respect to  $i\beta$ . We shall assume that  $\chi$  vanishes at infinity so that the partial integration gives rise to the following final result:

$$\frac{\partial}{\partial u} \int d^2\beta (-u) \exp[-i\beta u - i\beta^* u^*] \chi. \quad (11.61)$$

Evidently the r.h.s. of (11.61) can be expressed by

$$-\frac{\partial}{\partial u} (uP(u, u^*)), \quad (11.62)$$

where use of the definition of  $P$  (11.40) has been made.

Let us briefly summarize what we have achieved so far. We have shown how the first term on the r.h.s. of (11.56) gives rise to a term in the equation for  $P$  we are looking for. The second term on the r.h.s. of (11.56) gives rise to

$$-\frac{\partial}{\partial u^*} (u^* P(u, u^*)), \quad (11.63)$$

in quite an analogous fashion. Note that in both terms we have omitted the factor  $\kappa$  for the moment. In a still simpler way one may demonstrate that

the last term in (11.56) (except for the constant factor) gives rise to

$$(i\beta^*)(i\beta)\chi \rightarrow \frac{\partial^2 P}{\partial u \partial u^*}. \quad (11.64)$$

Collecting the terms (11.62)–(11.64) and using them on the r.h.s. of (11.57), we arrive at the fundamental equation for the classical distribution function  $P$ ,

$$dP(u, u^*)/dt = \kappa \left( \frac{\partial}{\partial u} u + \frac{\partial}{\partial u^*} u^* \right) P + 2\kappa n_{\text{th}} \frac{\partial^2}{\partial u \partial u^*} P. \quad (11.65)$$

Quite evidently we have found a Fokker–Planck equation in the classical variables  $u$  and  $u^*$  so that the problem of solving a density matrix equation has been rigorously transformed into that of solving a completely classical Fokker–Planck equation. In the following we shall use a short-hand notation for (11.65), namely

$$\dot{P} = L_f P, \quad (11.66)$$

where  $L_f$  is the differential operator which occurs on the r.h.s. of (11.65). We leave it as an exercise to the reader to solve eq. (11.65).

### 11.2.3 How to calculate quantum mechanical averages by classical averages

At the beginning of this section we asked ourselves whether we can develop a formalism by which we can evaluate a quantum mechanical average, say of the form (11.27), by means of c-number procedures, i.e. by classical averages. We now want to demonstrate that this goal can be achieved and that in the specific problem (11.27) we find

$$\text{Tr}\{b\rho\} = \int u P(u, u^*) d^2u. \quad (11.67)$$

To derive this relation we write the l.h.s. of (11.67) in a specific form, namely

$$\text{Tr}\{b\rho\} = [\text{Tr}\{O(\beta, \beta^*)b\rho\}]_{\beta=\beta^*=0}, \quad (11.68)$$

where  $O$  was defined by

$$O(\beta, \beta^*) = \exp[i\beta^* b^+] \exp[i\beta b]. \quad (11.69)$$

We now remind the reader of the properties of the  $\delta$ -function. Without

searching for mathematical rigor we make use of the following properties:

$$\int \delta(\beta) \delta(\beta^*) f(\beta, \beta^*) d^2\beta = f(0, 0) \tag{11.70}$$

and

$$\delta(\beta) \delta(\beta^*) = \frac{1}{\pi^2} \int \exp[-i\beta u - i\beta^* u^*] d^2u, \tag{11.71}$$

where  $f$  is an arbitrary continuous function of  $\beta$  and  $\beta^*$ . We now insert for  $f$  the expression

$$\frac{d}{d(i\beta)} \text{Tr}\{O(\beta, \beta^*)\rho\} = f(\beta, \beta^*) \tag{11.72}$$

in (11.70) and form

$$\int d^2\beta \frac{1}{\pi^2} \int d^2u \exp[-i\beta u - i\beta^* u^*] \left( \frac{d}{d(i\beta)} \text{Tr}\{O(\beta, \beta^*)\rho\} \right). \tag{11.73}$$

We perform a partial integration over  $i\beta$ . By a change of the sequence of integrations we transform (11.73) into

$$\int d^2u u \frac{1}{\pi^2} \int d^2\beta \exp[-i\beta u - i\beta^* u^*] \text{Tr}\{O(\beta, \beta^*)\rho\}. \tag{11.74}$$

$\underbrace{\hspace{15em}}_{P(u, u^*)}$

We note that by way of construction, (11.73) agrees with the l.h.s. of (11.67), whereas the r.h.s. of (11.74) can be expressed by  $P$ . In this way we indeed find

$$\text{Tr}(b\rho) = \int u P(u, u^*) d^2u, \tag{11.75}$$

which agrees with the assertion (11.67). A basic step in this derivation was provided by (11.72) where we expressed the operator  $Ob$  by the derivative of  $O$  with respect to  $i\beta$ . This procedure may be generalized to the  $n$ th power of  $b$  so that

$$\begin{aligned} \text{Tr}\{b^n\rho\} &= [\text{Tr}\{O(\beta, \beta^*)b^n\rho\}]_{\beta=\beta^*=0} \\ &= \left[ \left( \frac{\partial}{\partial(i\beta)} \right)^n \text{Tr}\{O(\beta, \beta^*)\rho\} \right]_{\beta=\beta^*=0}. \end{aligned} \tag{11.76}$$

Performing the same steps as before we then verify

$$\text{Tr}\{b^n \rho\} = \int u^n P(u, u^*) d^2u. \quad (11.77)$$

Finally we may evaluate the trace

$$\text{Tr}\{(b^+)^m b^n \rho\} \quad (11.78)$$

by means of derivatives, namely by means of

$$(11.78) = \left[ \left( \frac{\partial}{\partial(i\beta^*)} \right)^m \left( \frac{\partial}{\partial(i\beta)} \right)^n \text{Tr}\{O(\beta, \beta^*) \rho\} \right]_{\beta=\beta^*=0}. \quad (11.79)$$

In this way we find again by repeating the steps as before that the following relation holds:

$$\text{Tr}\{(b^+)^m b^n \rho\} = \int (u^*)^m u^n P(u, u^*) d^2u. \quad (11.80)$$

An important remark must be made, namely the nice relation between the left- and the right-hand side of (11.80) holds only if the product of creation and annihilation operators  $b$  and  $b^+$ , respectively, is written in normal order, where the creation operators stand on the left-hand side of all annihilation operators. If we define a function of  $b^+$ ,  $b$  in normal order we have the translation rule

$$\text{Tr}\{f(b^+, b) \rho\} = \int f(u^*, u) P(u, u^*) d^2u. \quad (11.81)$$

This is the final result of this section.

Let us summarize what we have learned in this section. In order to study the stochastic process, which the operator  $b$  (or  $b^+$ ) undergoes, the action of a **heatbath** can be formulated in several ways:

- (1) by means of quantum mechanical Langevin equations;
- (2) by means of the solution of the density matrix equation and jointly by the evaluation of expectation values (quantum mechanical averages) by means of the solution of the density matrix equation;
- (3) we may solve in an entirely equivalent fashion a classical Fokker–Planck equation (11.65) and calculate quantum mechanical expectation values by means of an entirely classical integration via (11.81).

In this way it has become possible to translate a quantum mechanical problem into a classical problem without any loss of generality. We have described this procedure for a special representation, namely the Glauber–Sudarshan representation (11.38). The reader may try it as an exercise to

formulate the corresponding treatment for the two other distribution functions based on (11.37) and (11.39). For the individual steps, which imply the differentiation of exponential functions of operators, we refer the reader to Vol. 1 where all the necessary tricks are presented.

**Exercises on section 11.2**

(1) Solve eq. (11.65) for the steady state, i.e. where  $dP/dt=0$ .

Hint: Try the hypothesis

$$P = \mathcal{N} \exp[-Cu^*u],$$

and determine C by inserting P into (11.65).

(2) Find time dependent solutions of (11.65).

Hint: Try the hypothesis

$$P = \mathcal{N}(t) \exp[-h(t)(u - u(t))(u^* - u^*(t))],$$

and determine the unknown functions  $u(t), u^*(t), \mathcal{N}(t), h(t)$ .

**11.3. Generalized Fokker–Planck equation of the laser**

In the foregoing section we have shown how the density matrix equation of a damped field mode can be transformed into a classical Fokker–Planck equation without any loss of "information". We may therefore ask the question whether a similar procedure can be applied to the density matrix equation of the laser (11.12) which comprises the field and the atomic (or electron) variables. There is, however, an obstacle which consists in the difference between Bose operators,  $b, b^+$ , and Fermi operators of electrons  $a_f, a_k$ . Though at a first sight their corresponding commutation relations differ only by a sign, it is well known to the experts that this causes considerable trouble when one tries to derive operator equations similar to those of the form (11.49). In spite of these difficulties it is possible to derive again a Fokker–Planck-type equation, though due to the specific properties of the Fermi operators this Fokker–Planck equation contains derivatives up to infinitely high order so that we call the corresponding equation "generalized Fokker–Planck equation".

Since the detailed procedure does not give us any more physical insights than that of the preceding section, we shall not present these details here. We rather refer the reader to my book "Laser Theory" in which the detailed steps are presented. Here we rather present the main steps.

In analogy to our considerations when deriving the quantum mechanical Langevin equations we shall use pairs of operators  $a_j^+$ ,  $a_k$  in a form well known to us, namely

$$a_1^+ a_2 = \alpha, \quad (11.82)$$

$$a_2 a_1 = \alpha^+, \quad (11.83)$$

$$a_2^+ a_2 - a_1^+ a_1 = d. \quad (11.84)$$

Because we shall deal with a set of atoms distinguished by the index  $\mu$  we shall supplement the quantities on the right-hand sides of (11.82)–(11.84) by that index,

$$\alpha_\mu, \quad \alpha_\mu^+, \quad d_\mu. \quad (11.85)$$

In order to simplify the calculation we shall adopt the model of a single mode laser with a running wave so that the spatial dependence of the coupling coefficients  $g_\mu$  can be transformed away. Because the field mode then interacts with the operator of the total dipole moment, we shall introduce that operator and its Hermitean conjugate as new operators

$$\sum_\mu \alpha_\mu = S^-, \quad (11.86)$$

$$\sum_\mu \alpha_\mu^+ = S^+. \quad (11.87)$$

Finally we introduce the sum over  $d_\mu$  as a new variable which we call  $2S_z$

$$\sum_\mu d_\mu = 2S_z. \quad (11.88)$$

The notation (11.86)–(11.88) stems from the fact that  $S^+$ ,  $S^-$ , and  $S_z$  can be considered as spin operators as far as their commutation relations are concerned. We shall not use this fact here explicitly, though it plays a role in the detailed derivation of the generalized Fokker–Planck equation we wish to describe.

In the foregoing section we have seen that one may establish a correspondence between the operators  $b$ ,  $b^+$  and  $u$ ,  $u^*$

$$b \leftrightarrow u, \quad b^+ \leftrightarrow u^*. \quad (11.89)$$

In a similar fashion we introduce the correspondence

$$S^- \leftrightarrow v, \quad S^+ \leftrightarrow v^*, \quad 2S_z \leftrightarrow D. \quad (11.90)$$

From the formal point of view we now perform the same steps as above in section 11.2. We first have to define a characteristic function by means of

an exponential function in analogy to (11.43) and (11.42). Because of the operator properties of (11.86)–(11.88) we have several options with respect to an arrangement of the exponential functions. In the following we shall adopt the following choice:

$$O_A = \exp[i\xi^* S^+] \exp[i\zeta S_z] \exp[i\xi S^-], \quad (11.91)$$

which is suggested by the analogy to the choice (11.42) we made before. It should be noted that other choices have been followed up in the literature as well, for instance one with an exchanged sequence of  $S^+$  and  $S^-$ , or one, in which the operators  $S^+$ ,  $S^-$ ,  $S_z$  occur as a linear combination within the same exponential function. Each of these individual choices has advantages and disadvantages, especially concerning the form of the solution of the Fokker–Planck equation and the way in which expectation values and correlation functions are evaluated.

In order to define the characteristic function of the total system field mode + atoms we introduce the operator

$$O = O_A O_f, \quad (11.92)$$

which in particular depends on the parameters  $\xi$ ,  $\xi^*$ ,  $\zeta$ ,  $\beta$  and  $\beta^*$ ,

$$O = O(\xi, \xi^*, \zeta, \beta, \beta^*). \quad (11.93)$$

It is now straightforward to formulate the distribution function of the laser by means of

$$\begin{aligned} & f(u, u^*, v, v^*; \mathcal{D}) \\ &= \mathcal{N} \int \cdots \int \exp[-i(v\xi + v^*\xi^* + \zeta D/2 + u\beta + u^*\beta^*)] \\ & \quad \times \chi(\xi, \xi^*, \zeta, \beta, \beta^*) d^2\xi d\zeta d^2\beta, \end{aligned} \quad (11.94)$$

where the characteristic function is, of course, defined by

$$\chi = \text{Tr}\{O(\xi, \xi^*, \zeta, \beta, \beta^*) \rho(t)\}. \quad (11.95)$$

As mentioned before, the explicit derivation of the final generalized Fokker–Planck equation is rather tedious so that we quote in the present context only the final result. The equation of the distribution function reads

$$df/dt = Lf, \quad (11.96)$$

where the linear operator  $L$  is defined by

$$L = L_f + L_{Af} + L_A. \quad (11.97)$$

Its individual parts are as follows.  $L_f$  stems from the coupling of the field mode to the heatbath. It is given by (11.65) and is repeated here for the reader's convenience,

$$L_f = \kappa \left[ \frac{\partial}{\partial u} u + \frac{\partial}{\partial u^*} u^* \right] + 2\kappa n_{\text{th}} \frac{\partial^2}{\partial u \partial u^*}. \quad (11.98)$$

$L_{Af}$  describes the coherent interaction between the field mode and the set of atoms and is explicitly given by

$$\begin{aligned} L_{Af} = & -ig \left\{ \left[ e^{-2\partial/\partial D} v^* - \frac{\partial^2}{\partial v^2} v + \frac{\partial}{\partial v} D \right] u \right. \\ & - \left[ -\frac{\partial^2}{\partial v^{*2}} v^* + e^{-2\partial/\partial D} v + \frac{\partial}{\partial v^*} D \right] u^* \\ & \left. + \left[ -\frac{\partial}{\partial u} + u^* \right] v - \left[ -\frac{\partial}{\partial u^*} + u \right] v^* \right\}. \quad (11.99) \end{aligned}$$

Finally  $L_A$  stems from the interaction of the atoms with their heatbaths. This term is the most complicated one and has the explicit form

$$\begin{aligned} L_A = & \frac{w_{21}}{2} \left\{ N(e^{-2\partial/\partial D} - 1) + N e^{2\partial/\partial D} \frac{\partial^4}{\partial v^2 \partial v^{*2}} + 2N \frac{\partial^2}{\partial v \partial v^*} \right. \\ & + \frac{\partial}{\partial v} \left[ 2 \frac{\partial^2}{\partial v \partial v^*} + 2 e^{-2\partial/\partial D} - 1 \right] v + \frac{\partial}{\partial v^*} \left[ 2 \frac{\partial^2}{\partial v \partial v^*} + 2 e^{-2\partial/\partial D} - 1 \right] v^* \\ & \left. - 2 \left[ \left( e^{-2\partial/\partial D} - 1 \right) - e^{2\partial/\partial D} \frac{\partial^4}{\partial^2 v \partial^2 v^*} \right] \frac{D}{2} \right\} \\ & + \frac{w_{12}}{2} \left\{ N(e^{2\partial/\partial D} - 1) + \frac{\partial}{\partial v} v + \frac{\partial}{\partial v^*} v^* + 2(e^{2\partial/\partial D} - 1) \frac{D}{2} \right\}. \quad (11.100) \end{aligned}$$

If the phases of the atomic dipole moments are destroyed not only by real transitions but also by virtual transitions, the following term must be added to the r.h.s. of (11.100) (where  $\eta = 2\Delta$ ):

$$\frac{\eta}{2} \left\{ \frac{\partial}{\partial v} v + \frac{\partial}{\partial v^*} v^* + 2 \frac{\partial^2}{\partial v \partial v^*} e^{2\partial/\partial D} \frac{D}{2} + N \frac{\partial^2}{\partial v \partial v^*} e^{2\partial/\partial D} \right\}. \quad (11.100a)$$

In the following considerations we shall not take care of this term explicitly but rather quote its effect at the last stage of our approach only.

Clearly, the Fokker–Planck equation we derived here stems only from those parts of the original density matrix equation (11.12), which contains the coupling of the field and the atoms to their corresponding heatbath,



and from the interaction between the atoms to the field. The Hamiltonian describing the free motions of the field and atoms is not taken care of here. Indeed it can be shown that the free motion can be split off in a trivial way.

(11.96) together with the expressions (11.97)–(11.100) represents the generalized Fokker–Planck equation of the laser. We note that some terms appear in the desired form of a Fokker–Planck equation containing first and second order derivatives only. However, also derivatives up to fourth order with respect to  $v$  and  $v^*$  occur and derivatives with respect to  $D$  up to infinite order via the exponential functions. Therefore the question arises whether we can reduce this still rather complicated generalized Fokker–Planck equation to an ordinary Fokker–Planck equation. We shall discuss the corresponding procedure in the next section.

#### 11.4. Reduction of the generalized Fokker–Planck equation

In this section we wish to show how the generalized Fokker–Planck equation, whose derivation we sketched in the previous section, can be reduced to an ordinary Fokker–Planck equation. To do this we have to anticipate the size of the various quantities  $u$ ,  $v$ ,  $D$  in the laser. Of course, in the original equation of the general form

$$df/dt = Lf, \quad (11.101)$$

in which  $L$  contains  $u$ ,  $v$ ,  $D$  and derivatives with respect to  $u$ ,  $v$ ,  $D$ , these quantities can have any size. Therefore a proper answer to the question what typical size these quantities will acquire in the laser can be given only once the function  $f$  is known. This function has the meaning of a distribution function and just tells us what size  $u$ ,  $v$ ,  $D$  will have when averaged *over*  $f$ . Here a difficulty quite typical for physics arises, because  $f$  is not known. Therefore in one way or another we have to anticipate  $f$ , or at least the order of magnitude of  $u$ ,  $v$ ,  $D$  induced by  $f$ . Because we expect that eventually a Fokker–Planck equation of the type (10.157) will result for a laser close to threshold, in our first step we shall be guided by the solution (10.164) of eq. (10.157). From it a certain order of magnitude for  $u$ ,  $v$ ,  $D$  results so that we can make an appropriate expansion of eq. (11.96) keeping the leading terms provided we have introduced a measure of smallness. As we shall see, in this way we can actually derive an equation of the form (10.157) at threshold so that our whole procedure is self-consistent.

Before we present the main steps we make a general remark. As it will turn out, the choice of a smallness parameter depends on the region in which the laser is operated, i.e. if it is operated close to or away from its threshold. In the following we shall adopt an order of magnitude which is

typical for a laser at threshold. The other case is not so interesting because it leads us back to a treatment equivalent to linearization or quasi-linearization of quantum mechanical Langevin equations. Close to threshold the following relation is assumed to be valid

$$|u|^2 \approx n_{\text{thr}}. \quad (11.102)$$

Note that  $n_{\text{thr}}$  must not be mixed up with  $n_{\text{th}}$ , because  $n_{\text{thr}}$  means "photon number close to threshold", whereas  $n_{\text{th}}$  refers to the photon number present in thermal equilibrium. A relation between the size of the atomic dipole moments proportional to  $v$  and the field mode amplitude  $u$  can be established via the semiclassical theory where we assumed a steady state relation (which can be justified because close to threshold the effective relaxation time tends to zero (critical slowing down)). In this way we obtain

$$|v| = \frac{\kappa}{g} |u|. \quad (11.103)$$

In order to find an appropriate expansion parameter we assume that the density of the laser atoms  $\rho_0$  is kept fixed but that we let the dimension of the laser go to infinity. Denoting the total number of laser atoms by  $N$  and the laser volume by  $V$  we have, of course,

$$\rho_0 = N/V, \quad (11.104)$$

so that with  $N \rightarrow \infty$  we must assume

$$N \propto V. \quad (11.105)$$

The coupling constant  $g$ , which occurs in (11.103) depends on the volume so that

$$g \propto V^{-1/2} \propto N^{-1/2}. \quad (11.106)$$

From the behavior of the photon number close to threshold we conclude

$$n_{\text{thr}} \propto N^{1/2}. \quad (11.107)$$

We further introduce the relations

$$\gamma_{\parallel} = w_{12} + w_{21}, \quad (11.108)$$

$$2\gamma = w_{12} + w_{21} (+\eta). \quad (11.109)$$

where we note that in general an additional term  $\eta$  occurs which is due to phase destroying processes not connected with real transitions. In order to simplify our procedure we shall ignore  $\eta$  here but we note that we insert it again into our final result.

Finally we know from the semiclassical theory that the threshold inversion is given by

$$D_{\text{thr}} = \frac{\kappa\gamma}{g^2}. \quad (11.110)$$

For what follows it is convenient to introduce some abbreviations, namely

$$K = \frac{g^2 N}{\kappa\gamma}, \quad (11.111)$$

$$\delta_1 = \frac{1}{4} \sqrt{\frac{\gamma}{\kappa + \gamma} \frac{\gamma_{\parallel}}{\kappa}}, \quad (11.112)$$

$$\delta_2 = \delta_1 \frac{\kappa}{\gamma}. \quad (11.113)$$

We note that because of (11.106),  $K$  of (11.111) is independent of  $N$ . On the other hand, rewriting (11.110) by means of (11.111), we find that the increase of

$$D_{\text{thr}} = \frac{N}{K} \propto N \quad (11.114)$$

is proportional to  $N$ .

In the next step of our analysis we introduce *normalized quantities* which are of the order of unity close to threshold. Therefore we transform the field amplitude, the dipole moment, and the inversion according to

$$\tilde{u} = \frac{u}{\sqrt{n_{\text{thr}}}}, \quad \tilde{v} = \frac{v}{v_{\text{thr}}}, \quad \tilde{D} = \frac{D}{D_{\text{thr}}}. \quad (11.115)$$

While in the present context  $n_{\text{thr}}$ ,  $v_{\text{thr}}$  and  $D_{\text{thr}}$  merely serve as abbreviations, it will become evident later that these quantities are just the values of the photon number, of the total dipole moment and of the total inversion at threshold, respectively.

Furthermore anticipating the results in a self-consistent fashion we put

$$n_{\text{thr}} = \delta_1 N^{1/2}. \quad (11.116)$$

Because of (11.103) we find

$$v_{\text{thr}}^2 = \frac{\kappa^2}{g^2} n_{\text{thr}} \propto N^{3/2}. \quad (11.117)$$

We now insert (11.115) into the r.h.s. of (11.101) where derivatives with respect to a quantity  $q$  are counted as  $q^{-1}$  as concerns the order of magnitude. The resulting expression can be considered as a function of  $N^{1/2}$ , and the

idea consists in expanding this expression into inverse powers of  $N^{1/2}$ . In order to find an equation which guarantees a bounded distribution function we have to include terms of order  $N^{-1/2}$ . In this way we obtain

$$L_A = (w_{12} - w_{21})K \frac{\partial}{\partial \tilde{D}} + \gamma \left( \frac{\partial}{\partial \tilde{v}} \tilde{v} + \frac{\partial}{\partial \tilde{v}^*} \tilde{v}^* \right) + \gamma_{\parallel} \frac{\partial}{\partial \tilde{D}} \tilde{D} \\ + N^{-1/2} \frac{w_{21}K}{\delta_2} \frac{\partial^2}{\partial \tilde{v} \partial \tilde{v}^*} \quad (11.118)$$

and

$$L_{Af} = -ig \left\{ -2 \frac{\partial}{\partial \tilde{D}} \tilde{v}^* \tilde{u} \sqrt{\delta_1 \delta_2} \sqrt{K} + \frac{\partial}{\partial \tilde{v}} \tilde{D} \tilde{u} \sqrt{\delta_1 / \delta_2} \frac{1}{\sqrt{K}} N^{1/2} \right. \\ \left. - \frac{\partial}{\partial \tilde{u}} \tilde{v} \sqrt{\delta_2 / \delta_1} \frac{1}{\sqrt{K}} N^{1/2} - \text{c.c.} \right\}, \quad (11.119)$$

while  $L$ , retains its original form. We note that it is also possible to retain terms which are important away from threshold but this is not our concern here.

If additional phase destroying processes are included, the factor  $w_{21}$  of  $\partial^2 / \partial \tilde{v} \partial \tilde{v}^*$  must be replaced by

$$w_{21} + \frac{1}{2} \eta (1 + \tilde{D}/K). \quad (11.120)$$

We decompose the complex classical variables  $u$  and  $v$  into their real and imaginary parts and write

$$\mathbf{u} = \begin{pmatrix} \text{Re } u \\ \text{Im } u \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} \text{Re } v \\ \text{Im } v \end{pmatrix}. \quad (11.121)$$

We further introduce the well known abbreviation

$$D_0 = N \frac{w_{21} - w_{12}}{w_{12} + w_{21}} \quad (11.122)$$

Then it is an easy matter to cast the Fokker–Planck equation containing the terms (11.118), (11.119) and (11.98) into the form

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{u}} \{ (-\kappa \mathbf{u} + g \mathbf{v}) f \} + \nabla_{\mathbf{v}} \{ (-\gamma \mathbf{v} + g D \mathbf{u}) f \} \\ + \frac{\partial}{\partial D} \{ [\gamma_{\parallel} (D_0 - D) - 4g \mathbf{u} \mathbf{v}] f \} \\ = \frac{\kappa}{2} n_{\text{th}} \nabla_{\mathbf{u}} f + \frac{1}{4} N w_{21} \Delta_{\mathbf{v}} f + \gamma_{\parallel} N \frac{\partial^2 f}{\partial D^2}. \quad (11.123)$$

In the case of additional phase destroying processes  $Nw_{21}$  must be replaced by

$$Nw_{21} + \frac{N}{2}\eta + D\frac{\eta}{2}. \quad (11.124)$$

This Fokker–Planck equation still refers to the field and atomic variables. On the other hand we wish to treat a laser at threshold where, as we know from the semiclassical approach, we can eliminate the atomic variables. Indeed close to threshold we may eliminate the atomic variables from the Fokker–Planck equation in a simple fashion. This can be done in two ways, either in the Fokker–Planck equation directly or via a Langevin equation. Which way is chosen is partly a way of personal taste, partly one of simplicity. In fact, the seeming round-about via Langevin equations is simpler so that we choose that way. As is shown in classical statistical physics, the Fokker–Planck equation (11.123) is entirely equivalent to the following set of Langevin equations:

$$\left(\frac{d}{dt} + \kappa\right)u + igv = \Gamma_u, \quad (11.125)$$

$$\left(\frac{d}{dt} + \gamma\right)v - iguD = i\Gamma_v, \quad (11.126)$$

$$\left(\frac{d}{dt} + \gamma_{\parallel}\right)(D - D_0) + 2ig(v^*u - vu^*) = \Gamma_D. \quad (11.127)$$

The drift coefficients which occur in the Fokker–Planck equation (11.123) are connected with the fluctuating force by the relations

$$Q_j = \lim_{T \rightarrow \infty} \frac{1}{4\mathbf{T}} \int_0^T \int_0^T \langle \Gamma_j(t_1) \Gamma_j^*(t_2) \rangle d\mathbf{t}_1 d\mathbf{t}_2, \quad j = u, v, D. \quad (11.128)$$

According to the Fokker–Planck equation (11.123), the diffusion coefficients read explicitly

$$Q_u = \frac{\kappa}{2}n_{\text{th}}, \quad (11.129)$$

$$Q_v = \frac{1}{4}Nw_{12}. \quad (11.130)$$

Note that (11.123) does not contain diffusion coefficients stemming from  $\Gamma_D$  because they have been neglected due to their smallness.

We now resort to the method of adiabatic elimination which we have used at various occasions in this book. To this end we assume that

$$\kappa \ll \gamma, \gamma_{\parallel} \quad (11.131)$$

holds. Under this assumption we may eliminate the atomic variables  $v$  and  $D$  adiabatically, whereby we end up with the classical Langevin equation for the field amplitude  $u$  alone,

$$\dot{u} = \beta(\tilde{d} - |u|^2)u + \Gamma. \quad (11.132)$$

In it we have used the abbreviations

$$\beta = \frac{4\kappa^2}{\gamma D_{\text{thr}}}, \quad (11.133)$$

$$\tilde{d} = \frac{\gamma_{\parallel}}{4\kappa}(D_0 - D_{\text{thr}}). \quad (11.134)$$

$D_{\text{thr}}$  was defined above (11.110). Under the assumption (11.131), the fluctuating force  $\Gamma$  is given by

$$\Gamma = \Gamma_u + \frac{g}{\gamma}\Gamma_v. \quad (11.135)$$

By means of (11.128) and using the fact that  $\Gamma_u$  and  $\Gamma_v$  are uncorrelated we readily obtain

$$Q = Q_u + \frac{g^2}{\gamma^2}Q_v, \quad (11.136)$$

where  $Q_u$  and  $Q_v$  are given explicitly by (11.129) and (11.130).

We now observe that the classical Langevin equation (11.132) possesses a classical Fokker–Planck equation which is identical with (10.151) of section 10.5. But in section 10.5 we derived this classical Fokker–Planck equation in a heuristic fashion from the quantum mechanical (Langevin) equations, whereas here we derive it from the quantum mechanical equations via quantum classical correspondence. To complete our derivation we wish to cast  $Q$  into the form we used in section 10.5. Using (11.129), (11.130) we may write  $Q$  in the form

$$Q = \frac{\kappa}{2}n_{\text{th}} + \frac{g^2}{\gamma^2} \frac{1}{4}Nw_{21}. \quad (11.137)$$

By means of the relation

$$\frac{g^2}{\gamma^2} = \frac{\kappa}{\gamma D_{\text{thr}}}, \quad (11.138)$$

Q acquires the form

$$Q = \frac{\kappa}{2} n_{\text{th}} + \frac{1}{4} \frac{\kappa N w_{21}}{\gamma(N_{2,\text{thr}} - N_{1,\text{thr}})}. \quad (11.139)$$

We now remind the reader of the relation (11.109), i.e.

$$2\gamma = w_{12} + w_{21}, \quad (11.140)$$

and of (10.39) and (10.40) from which we obtain for N atoms

$$\frac{N w_{21}}{w_{12} + w_{21}} \approx N_{2,\text{thr}} \approx N_{2,s}. \quad (11.141)$$

By means of (11.140) and (11.141) we transform (11.139) into

$$Q = \frac{\kappa}{2} (n_{\text{th}} + n_{\text{sp}}), \quad (11.142)$$

where  $n_{\text{th}}$  is the number of thermal light quanta and (cf. (10.103))

$$n_{\text{sp}} = \frac{N_{2,s}}{(N_2 - N_1)_{\text{thr}}} \quad (11.143)$$

the number of spontaneously emitted light quanta. (11.142) represents the Q we introduced in section 10.5.

In conclusion we mention that the general Fokker–Planck equation (11.123) can be applied when the laser is operated well above laser threshold (where the adiabatic elimination might no more be applicable).

## 11.5. Concluding remarks

The Fokker–Planck equation we derived in section 11.4 can be readily solved in the stationary state. The method of quantum classical correspondence allows us to calculate expectation values of the field operators  $b^+$ ,  $b$  by means of classical expectation values using the classical distribution function  $\mathbf{f}$  (or  $\mathbf{P}$  in our former notation).

In this way we have a well defined procedure how to evaluate for instance  $K_2$  introduced in section 10.4, at least in the case  $\tau = 0$ . In fact, the method of quantum classical correspondence can be extended also to time dependent correlation functions, provided they are in normal order and in a temporal sequence. Therefore  $K_2$  can be calculated for  $\tau \neq 0$  also by means of classical averages (for more details, cf. H. Haken, Laser Theory). We hope that the reader has seen how the whole justification of our results presented in sections 10.4 and 10.5 works, at least in principle.

# A Theoretical Approach to the Two-Photon Laser

## 12.1. Introduction

In this book we have been concerned with laser action produced by optical transitions where each individual transition is accompanied with the generation (or annihilation) of a single photon. In Vol. 1 we got acquainted with a process in which two photons of quantum energies  $\hbar\omega_1$  and  $\hbar\omega_2$  are absorbed simultaneously, whereby an electron of an atom makes a transition between two levels 1 and 2 and the relation  $\hbar\omega_1 + \hbar\omega_2 = W_2 - W_1$  holds. In this relation,  $W_j$  is the energy of the electron in its state  $j$ . Because on the microscopic level, quantum processes are reversible we must expect that there is also a possibility of the simultaneous emission of two photons by means of a single electronic transition. Provided these transitions are stimulated emission processes, we are led in a natural way to the concept of a two-photon laser. At the same time this problem provides us with a nice example how the methods developed in this book can be applied to a variety of optical processes. Incidentally we shall see how we may derive the various approximations such as the semiclassical approach from the fundamental quantum mechanical equations in a rather elegant fashion.

In a fully quantum theoretical formulation we start from a Hamiltonian. It consists of the Hamiltonian of the field modes, that of the set of laser atoms, and that of the interaction between these two subsystems. As we know from Vol. 1, two-photon absorption can be caused by the virtual absorption of a single photon where the electron is first brought from state 1 to an intermediate state  $i$  and from there by a second virtual absorption of a further photon to the final state 2. Instead of treating these individual virtual transitions explicitly we may start right away from a phenomenological Hamiltonian which describes the process of two-photon absorption (or emission) jointly with the corresponding electronic transition. We shall formulate this Hamiltonian in the subsequent section.



We then may proceed to the Heisenberg equations of motion for suitable creation and annihilation operators of photons. When we imagine that the field and the atoms are coupled to reservoirs, we may introduce into the Heisenberg equations of motion suitable damping terms and fluctuating forces. When we take the average over the quantum fluctuations and the quantum mechanical state of the system, we obtain semiclassical equations for the two-photon laser, which can be considered as the straightforward extension of the equations of the single-photon laser. It remains as a nice exercise for the reader to translate the other methods, like that of the density matrix equation or of the Fokker-Planck equation, to the two-photon laser. In the next section we shall perform the first steps of this program explicitly.

## 12.2. Effective Hamiltonian, quantum mechanical Langevin equations and semiclassical equations

The Hamiltonian  $H$  of the field modes coupled to a set of two-level atoms is given by

$$H = H_F + H_A + H_{AF} \quad (12.1)$$

where the individual terms are defined as follows: Hamiltonian of the field mode

$$H_F = \sum_{\lambda} \hbar \omega_{\lambda} b_{\lambda}^{\dagger} b_{\lambda}; \quad (12.2)$$

Hamiltonian of the atoms

$$H_A = \sum_{\mu} \hbar \bar{\omega}_{\mu} a_{2,\mu}^{\dagger} a_{2,\mu}; \quad (12.3)$$

interaction Hamiltonian

$$H_{AF} = \frac{\hbar}{2} \sum_{\mu} \sum_{\lambda \lambda'} (g_{\mu\lambda\lambda'} \alpha_{\mu} b_{\lambda}^{\dagger} b_{\lambda'}^{\dagger} + g_{\mu\lambda\lambda'}^* \alpha_{\mu}^{\dagger} b_{\lambda} b_{\lambda'}). \quad (12.4)$$

Eq. (12.4) describes the process of two-photon emission or absorption accompanied by a corresponding electronic transition in a phenomenological way.  $a_{\mu}$ ,  $\alpha_{\mu}^{\dagger}$  are the usual dipole moment operators. Their significance becomes obvious in the present context when we recall that they are connected with the creation and annihilation operators of an electron in its individual levels by means of the relations

$$a_{\mu} = a_{1,\mu}^{\dagger} a_{2,\mu} \quad \text{and} \quad \alpha_{\mu}^{\dagger} = a_{2,\mu}^{\dagger} a_{1,\mu}. \quad (12.5)$$

For sake of simplicity we shall assume running waves so that the coupling

coefficients can be written in the form

$$g_{\mu\lambda\lambda'} = \exp[-i(k_\lambda + k_{\lambda'})x_\mu] g_{\lambda\lambda'}. \quad (12.6)$$

As a somewhat more detailed analysis shows the coupling coefficients  $g$  are symmetric with respect to the arguments  $\lambda$  and  $\lambda'$

$$g_{\mu\lambda\lambda'} = g_{\mu\lambda'\lambda}. \quad (12.7)$$

We write the Heisenberg equation of motion of an arbitrary operator  $\Omega$  in the form

$$\frac{d\Omega}{dt} = \frac{i}{\hbar} [H, \Omega] + \left( \frac{\partial \Omega}{\partial t} \right)_{B-S}. \quad (12.8)$$

The second term on the right-hand side of (12.8) stems from the coupling between the system and the heatbaths and gives rise to damping constants and fluctuating forces. It is a simple matter to evaluate the commutator on the right-hand side of (12.8). By means of the commutation relations of the operators  $b^+$ ,  $b$ ,  $a$ , ... we obtain

$$\dot{b}_\lambda = (-i\omega_\lambda - \kappa_\lambda) b_\lambda - i \sum_{\mu \lambda'} g_{\mu\lambda\lambda'} \alpha_\mu b_{\lambda'}^+ + F_\lambda(t), \quad (12.9)$$

$$\dot{\alpha}_\mu = (-i\bar{\omega}_\mu - \gamma) \alpha_\mu + \frac{i}{2} \sum_{\lambda\lambda'} g_{\mu\lambda\lambda'}^* b_\lambda b_{\lambda'} d_\mu + \Gamma_\mu(t), \quad (12.10)$$

$$\dot{d}_\mu = \frac{d_{\mu,0} - d_\mu}{T} + i \sum_{\lambda\lambda'} (g_{\mu\lambda\lambda'} \alpha_\mu b_\lambda^+ b_{\lambda'}^+ - \text{h.c.}) + \Gamma_{\mu d}. \quad (12.11)$$

These are the quantum mechanical Langevin equations which can be considered as a straightforward extension of the single photon case treated in section 10.3. In order to obtain semiclassical equations we replace the operators by their average values and replace for sake of convenience these average values by their arguments, e.g.

$$\langle b_\lambda \rangle \rightarrow b_\lambda. \quad (12.12)$$

We furthermore assume that we may factorize these averages into products of averages. Finally it is assumed that the fluctuating forces have the following property:

$$\langle F_\lambda(t) \rangle = \langle F_\mu(t) \rangle = \langle \Gamma_{\mu d}(t) \rangle = 0. \quad (12.13)$$

In the next section we shall be concerned with the thus resulting equations which we do not need to write down explicitly once more. In the following we just have to interpret the quantities  $b$ , etc. as classical time dependent variables and to omit the fluctuating forces.

### 12.3. Elimination of atomic variables

For sake of completeness we shall derive the main results for the whole set of modes  $\Lambda$ . Readers who are not interested in the rather complicated formulas of this section can skip it and proceed directly to section 12.4, in which we shall be concerned with the much simpler case of single mode operation and a homogeneously broadened atomic line.

We assume that the classical field amplitudes  $b_\lambda$  are still small quantities and that the damping constants of the cavity modes,  $\kappa_\lambda$ , are much smaller than  $\gamma$  and  $1/T$ , where  $1/T \ll \gamma$  is assumed to hold. Under these circumstances we may adopt the same iteration procedure as in section 6.4. In the first step we assume that the inversion is equal to the unsaturated inversion, i.e.

$$d_\mu^0 = d_{\mu,0}. \tag{12.14}$$

Furthermore we assume that coherent modes have developed,

$$b_\lambda = B_\lambda(t) \exp[-i\Omega_\lambda t], \tag{12.15}$$

where  $B_\lambda(t)$  is an amplitude whose time dependence is assumed to be much slower than that of the accompanying exponential function.  $\Omega_\lambda$  is the mode frequency when laser action takes place. In order to obtain the first approximation we insert (12.14) and (12.15) into the semiclassical version of (12.10). Under steady state conditions we obtain the solution

$$\alpha_\mu^{(1)} = \sum_{\lambda\lambda'} A_{\mu\lambda\lambda'} b_\lambda b_{\lambda'}, \tag{12.16}$$

with

$$A_{\mu\lambda\lambda'} = -\frac{1}{2} d_{\mu,0} g_{\mu\lambda\lambda'}^* (\Omega_\lambda + \Omega_{\lambda'} - \bar{\omega}_\mu + i\gamma)^{-1}, \tag{12.17}$$

where the slowly varying amplitude approximation has been made. In the next step of the iteration procedure we insert (12.15) and (12.16) into (12.11). The solution can be written in the form

$$d_\mu^{(1)} = d_{\mu,0} + \sum_{\substack{\lambda\lambda' \\ \lambda_1\lambda_1' \\ \lambda_2\lambda_2'}} C_{\mu\lambda\lambda'\lambda_1\lambda_1'\lambda_2\lambda_2'} b_\lambda^* b_{\lambda'}^* b_{\lambda_1} b_{\lambda_1'}. \tag{12.18}$$

Because the coefficients  $C$  are rather complicated expressions, we do not exhibit them here explicitly but rather leave their explicit determination as an exercise to the reader.

We now insert (12.18) and (12.15) into the classical version of (12.10). The solution is again straightforward and has the general form of

$$\alpha_\mu^{(2)} = \alpha_\mu^{(1)} + \sum_{\substack{\lambda\lambda' \\ \lambda_1\lambda_1' \\ \lambda_2\lambda_2'}} D_{\mu\lambda\lambda'\lambda_1\lambda_1'\lambda_2\lambda_2'} b_\lambda b_{\lambda'} b_{\lambda_1}^* b_{\lambda_1'}^* b_{\lambda_2} b_{\lambda_2'}. \tag{12.19}$$

Again the explicit determination of the coefficients  $\mathbf{D}$  is left as an exercise to the reader. We may now insert (12.19) into (12.9) which closes the circle, i.e. we find a self-consistency equation for the field modes  $b_\lambda$ . These equations have the general form

$$\begin{aligned} \dot{b}_\lambda = & (-i\omega_\lambda - \kappa_\lambda)b_\lambda + \sum_{\substack{\mu\lambda' \\ \lambda_1\lambda_1'}} M_{\mu\lambda'\lambda_1\lambda_1'} b_{\lambda'}^* b_{\lambda_1} b_{\lambda_1'} \\ & + \sum_{\substack{\mu\lambda' \\ \lambda_1\lambda_1'}} \sum_{\substack{\lambda_2\lambda_2' \\ \lambda_3\lambda_3'}} N_{\mu\lambda\lambda'\lambda_1\lambda_1'\lambda_2\lambda_2'\lambda_3\lambda_3'} b_{\lambda'}^* b_{\lambda_1} b_{\lambda_1'} b_{\lambda_2}^* b_{\lambda_2'} b_{\lambda_3} b_{\lambda_3'}. \end{aligned} \quad (12.20)$$

For sake of completeness we have to write down the coefficients  $\mathbf{M}$  and  $\mathbf{N}$  explicitly. They read as follows:

$$M_{\mu\lambda'\lambda_1\lambda_1'} = \frac{1}{2} g_{\mu\lambda\lambda'} g_{\mu\lambda_1\lambda_1'}^* d_{\mu,0} (\Omega_{\lambda_1} + \Omega_{\lambda_1'} - \bar{\omega}_\mu + i\gamma)^{-1}, \quad (12.21)$$

$$N_{\mu\lambda'\dots\lambda_3'} = \frac{1}{4} d_{\mu,0} g_{\mu\lambda\lambda'} g_{\mu\lambda_1\lambda_1'}^* g_{\mu\lambda_2\lambda_2'} g_{\mu\lambda_3\lambda_3'}^* [\cdot \cdot \cdot], \quad (12.22)$$

where we have used the abbreviation

$$\begin{aligned} [\cdot \cdot \cdot] = & (1/T + i(\Omega_{\lambda_2} + \Omega_{\lambda_2'} - \Omega_{\lambda_3} - \Omega_{\lambda_3'}))^{-1} \\ & \times (\Omega_{\lambda_1} + \Omega_{\lambda_1'} - \Omega_{\lambda_2} - \Omega_{\lambda_2'} + \Omega_{\lambda_3} + \Omega_{\lambda_3'} - \bar{\omega}_\mu + i\gamma)^{-1} \\ & \times [(\Omega_{\lambda_3} + \Omega_{\lambda_3'} - \bar{\omega}_\mu - \bar{\omega}_\mu + i\gamma)^{-1} - (\Omega_{\lambda_2} + \Omega_{\lambda_2'} - \bar{\omega}_\mu - i\gamma)^{-1}]. \end{aligned} \quad (12.23)$$

Quite evidently the equations (12.20) are the analogues of the equations (6.69) of the multimode laser which were valid somewhat above laser threshold. In the following we do not intend to present here the complete theory taking into account all modes but we rather wish to treat a particularly simple example, namely that in which only a single mode has a sufficiently long lifetime to support laser action.

#### 12.4. Single mode operation, homogeneously broadened line and running wave

In this case there is only one field mode present so that  $A = A' = \lambda_0$ . In the following we shall drop that index for simplicity so that

$$g_{\mu\lambda\lambda'} \rightarrow g_\mu. \quad (12.24)$$

Because we are dealing with running waves we may assume the form (12.6) and write

$$|g_\mu|^2 = g^2. \quad (12.25)$$

We furthermore assume  $d_{\mu,0} = d_0$  and introduce the sum over the atomic indices,  $\mu$ ,

$$\sum_{\mu} d_{\mu,0} = Nd_0 = D_0, \quad (12.26)$$

where  $D_0$  is the total unsaturated inversion. Finally we use the relation between the photon number  $n$  and  $|b|^2$ ,

$$|b|^2 = n. \quad (12.27)$$

Under these assumptions the equations (12.20) acquire the very simple form

$$\begin{aligned} \dot{b} = & (-i\omega - \kappa)b + \frac{1}{2}D_0[\gamma - i(2\Omega - \bar{\omega})]^{-1}g^2nb \\ & - \frac{1}{2}D_0g^4\gamma T[\gamma - i(2\Omega - \bar{\omega})]^{-1}[(2\Omega - \bar{\omega})^2 + \gamma^2]^{-1}n^3b. \end{aligned} \quad (12.28)$$

In order to bring out the main features we assume exact resonance, i.e. that the mode frequency in the unloaded cavity  $\omega$  is related to the atomic transition frequency of the homogeneously broadened line by

$$2\omega = \omega_0. \quad (12.29)$$

Under this assumption the mode frequency  $\Omega$  in the case that laser action takes place, agrees with  $\omega$ ,

$$\Omega = \omega. \quad (12.30)$$

By means of the hypothesis

$$b = B(t) \exp[-i\omega t], \quad (12.31)$$

where  $B$  is a complex quantity, (12.28) is transformed into

$$\dot{B} = -\kappa B + \frac{1}{2}D_0(g^2/\gamma)|B|^2B - \frac{1}{2}D_0g^4(T/\gamma^2)|B|^6B. \quad (12.32)$$

This equation can be considered as an analogue of the laser equation (6.48). We may write it in the form

$$\dot{B} = -\partial V/\partial B^*, \quad (12.33)$$

where the potential function  $V$  is explicitly given by

$$V = \kappa|B|^2 - \frac{1}{4}D_0(g^2/\gamma)|B|^4 + \frac{1}{8}D_0g^4(T/\gamma^2)|B|^8. \quad (12.34)$$

The potential is plotted for various values of the unsaturated inversion  $D_0$  which serves as pump parameter (fig. 12.1). As can be seen, we are dealing with a first order phase transition. It requires always some energy input in order to reach the minimum of the potential  $V$  by an appropriate field amplitude  $B$ . For this reason it is important to initiate or help laser action by means of an injected signal which is in resonance with the mode frequency

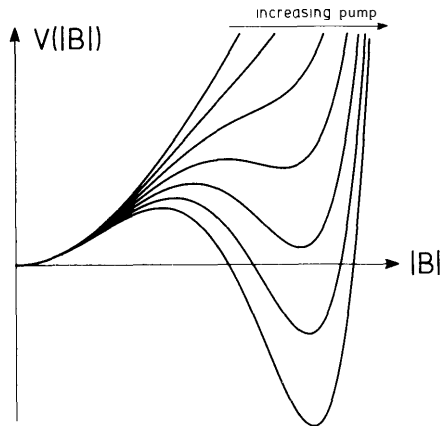


Fig. 12.1. The potential function  $V$  versus  $|B|$  for various values of the unsaturated inversion  $D_0$  according to eq. (12.34).

o. In such a case eq. (12.32) is replaced by

$$\dot{B} = -\kappa B + C_1 |B|^2 B - C_2 |B|^6 B + B_0, \quad (12.35)$$

where  $B_0$  is proportional to the amplitude of the injected signal. It is a simple matter to cast (12.35) again into the form (12.33) where the potential is now given by

$$V = \kappa |B|^2 - \frac{1}{4} D_0 (g^2 / \gamma) |B|^4 + \frac{1}{8} D_0 g^4 (T / \gamma^2) |B|^8 - B_0 B^* - B_0^* B. \quad (12.36)$$

It is a nice exercise for the reader to discuss the form of the potential function  $V$  as a function of the parameters  $D_0$  and  $B_0$ .

In this short chapter we have presented the simplest case of the two-photon laser to show some of the typical features introduced by this new photon emission mechanism. The case we have been treating is called the degenerate case, because the energy of the electronic transition is split into two equal amounts  $\mathbf{k} \omega = \mathbf{k} \omega_1$ .

In the non-degenerate case the emission of two photons with energies  $\hbar \omega_1$  and  $\hbar \omega_2$  must be treated. In this case by a proper choice of the decay constants  $\kappa_1$  and  $\kappa_2$  these two modes may be selected. In such a case two equations for the modes  $A_1$  and  $A_2$  can be derived from the general eqs. (12.20) by means of specialization. Then the task remains to solve the remaining two equations. A detailed discussion will be beyond the scope of this book, however, so that we refer the interested reader to the literature.

## Chapter 13

# The Laser – Trailblazer of Synergetics

### 13.1. What is synergetics about?

At various occasions in this book we alluded to synergetics, and also to analogies between the behavior of a laser at threshold and phase transitions. In this chapter we wish to elaborate somewhat on these analogies and in particular on the significance of the laser as a prototype of systems which produce spatial or temporal structures by self-organization.

Let us first explain the word "synergetics". It consists of two Greek words and means "cooperation", or "science of cooperation". When scientists wish to study the properties of their objects of research, very often they decompose them into individual parts. For instance, a physicist decomposes a crystal into its atoms, or a biologist decomposes an organ into its individual cells. In many cases it turns out that the properties of the total system cannot be explained by a mere superposition of the properties of the individual parts of the system. Rather the individual subsystems cooperate in a well defined fashion which sometimes even appears purposeful. In this way properties of the total system are created which even qualitatively differ from the properties of the individual subsystems. It is a main goal of synergetics to unearth general principles through which the individual subsystems produce macroscopic properties of the total system. The scope of this research program is rather broad because the individual subsystems may be, for instance, atoms, molecules, cells, computers, or even human individuals. The laser has played a fundamental role with respect to the discovery of these general principles. On the other hand, in the spirit of synergetics it has become possible to predict qualitatively new properties of laser light, e.g. chaotic laser light (cf. chapter 8). Therefore within the frame of this book we shall discuss some of the most important aspects of synergetics. We shall show that the transition from light of lamps to laser

light represents an instructive example of self-organization. Furthermore we wish to show that the laser is a typical example of a whole new class of phenomena, namely of nonequilibrium phase transitions.

### 13.2. Self-organization and the slaving principle

We have shown in chapter 10 that the statistical properties of laser light change at laser threshold drastically. Let us consider once again fig. 10.10. In this figure the normalized noise intensity is plotted against the pump power. According to that figure, the statistical properties of laser light change qualitatively at the laser threshold. Below laser threshold noise increases more and more while above threshold it decreases again. We have seen how to visualize this behavior. Below laser threshold, light consists of individual wave tracks which are emitted from the individual atoms independently of each other. Above laser threshold, a practically infinitely long wave track is produced. In order to make contact with other processes of self-organization let us interpret the processes in a lamp or in a laser by means of Bohr's model of the atom (fig. 13.1). A lamp produces its light in such a way that the excited electrons of the atoms make their transitions from the outer orbit to the inner orbit entirely independently of each other. On the other hand, the properties of laser light can be understood only if we assume that the transitions of the individual electrons occur in a correlated fashion. Let us translate these processes into an anthropomorphic picture. Imagine that some men stand at the border of a channel filled with

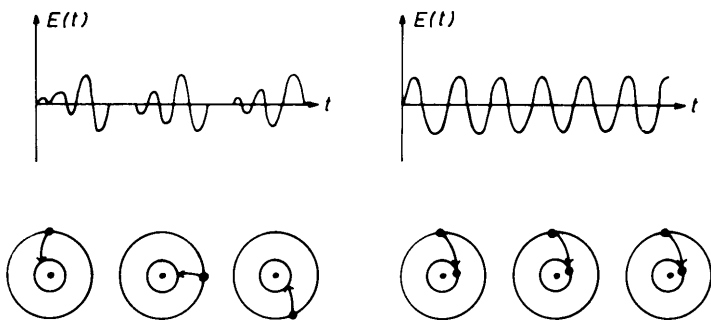


Fig. 13.1. Upper part: The electric field strengths versus time in a lamp (left-hand side) and in a laser (right-hand side). Lower part: In Bohr's model of the atom an electron circles around the nucleus. When it makes its transition from the outer orbit to the inner orbit, it emits a light wave which is plotted in the upper part of this figure. In the lamp the electrons make their transitions to the lower orbit in an uncorrelated fashion. In the laser they make their transitions in phase, i.e. in a correlated fashion.



water. The men are assumed to symbolize the atoms, and the water the light field. By pushing bars into the water the men can excite the water to some motion. In order to describe the light produced by a lamp in a model-like fashion we let the men push their bars independently of each other into the water. In this way a randomly oscillating water surface appears, representing incoherent light. In order to understand the coherence of laser light, we must assume that the men push their bars into the water in a well correlated fashion (fig. 13.2). In our daily life the latter process can easily be understood by assuming that there is a boss who gives the appropriate orders to the men when they have to push their bars into the water. But – and this is the decisive point – in the laser there is nobody who gives such orders to the atoms. Thus the behavior of the atoms is a typical example of self-organization.

As it has turned out over the last one or two decades, there are many other branches in science, such as physics, chemistry, and biology, where similar processes involving self-organization take place. We use the laser example to demonstrate how self-organization is made possible. To this

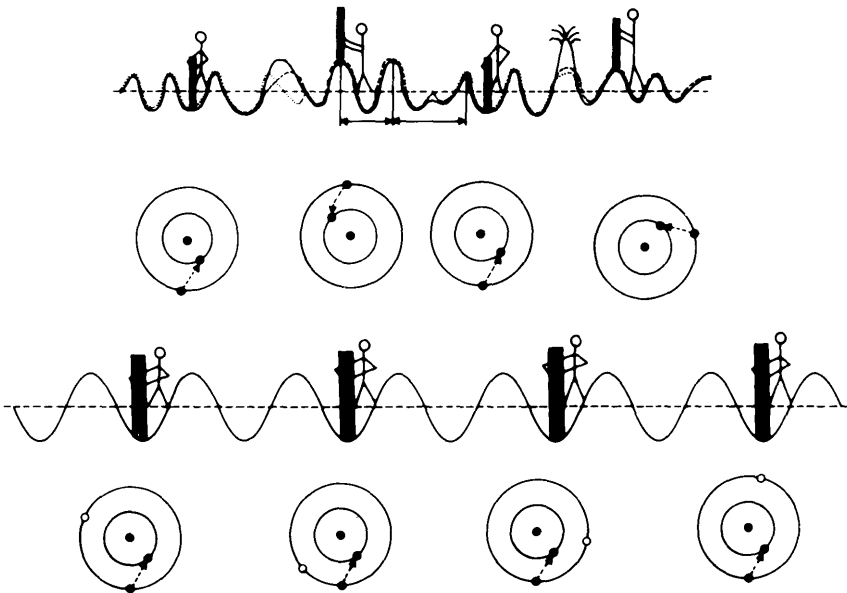


Fig. 13.2. Visualization of the behavior of a lamp or a laser. In both cases the atoms are represented by men and the light field by water in a channel at the border of which the men stand. By pushing bars into the water the men produce a water wave. In the case of a lamp this pushing occurs irregularly and independently. In the case of the laser the bars are pushed periodically and in phase.

end we use the equations of a single mode laser including fluctuating forces, but we shall assume for simplicity that the laser equations including their fluctuating forces refer to classical quantities. We furthermore transform the rapid oscillations with frequency  $\omega$  away so that the laser equations acquire the familiar form

$$dB/dt = -\kappa B - i \sum_{\mu} g A_{\mu} + \tilde{F}, \quad (13.1)$$

$$dA_{\mu}/dt = -\gamma A_{\mu} + ig d_{\mu} B + \tilde{I}_{\mu}, \quad (13.2)$$

$$dd_{\mu}/dt = (d_0 - d_{\mu})/T + 2ig(A_{\mu} B^* - A_{\mu}^* B). \quad (13.3)$$

In the last equation we have dropped the fluctuating forces because they are not so important. In many realistic cases the damping constant  $\kappa$  is much smaller than the damping constant  $\gamma$ . This gives rise to the following idea which we have exploited at several instances in our present book, e.g. in sections 6.3 and 6.4. Because  $\kappa$  is small, we expect that  $B$  decreases only slowly according to eq. (13.1). As we have seen in chapter 10,  $B$  decreases below laser threshold with a damping constant which is even considerably smaller than  $\kappa$ . But also above laser threshold  $B$  relaxes very slowly when we take the laser process into account.

According to (13.2) the temporal change of  $A_{\mu}$  is caused by the field amplitude  $B$  standing on the r.h.s. (provided we neglect for the moment being fluctuations). According to (13.2) we shall expect that also  $A_{\mu}$  changes only slowly. This immediately leads us to the inequality

$$|dA_{\mu}/dt| \approx |\kappa A_{\mu}| \ll |\gamma A_{\mu}|. \quad (13.4)$$

Because  $\kappa$  is much smaller than  $\gamma$  we can put the l.h.s. of (13.2) practically to zero. That means that we can resolve (13.2) with respect to  $A_{\mu}$ ,

$$A_{\mu}(t) = (ig/\gamma) d_{\mu} B(t) + \tilde{I}_{\mu}/\gamma. \quad (13.5)$$

This equation tells us that the amplitude of the dipoles, which is proportional to  $A_{\mu}$ , is instantaneously given by the field amplitude  $B(t)$  (and by the fluctuating force). This is probably the simplest example of a principle which has turned out to be of fundamental importance in synergetics and which is called the *slaving principle*.

In the present context it can be formulated as follows. Fast relaxing quantities (e.g.  $A_{\mu}$ ) adopt their values instantaneously with respect to the values of slowly varying quantities (e.g.  $B(t)$ ) or, to use a terminus technicus, fast relaxing quantities are *slaved* by slowly relaxing quantities. Within synergetic it is shown that this principle can be considerably extended in various ways (cf. for example section 7.3). But it is far beyond the scope of the

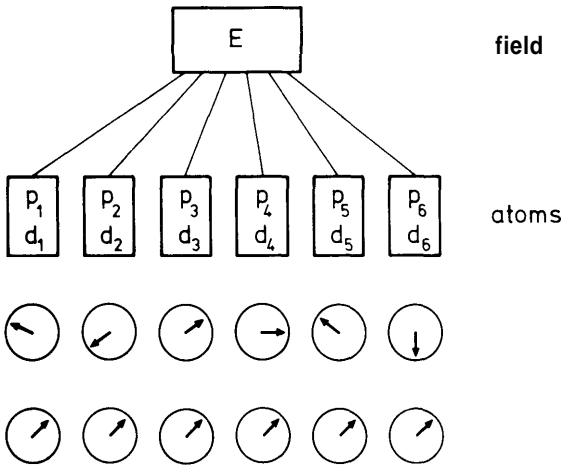


Fig. 13.3. Visualization of the slaving principle. Upper part: The electric field strength  $E$  slaves the atomic dipole moments and inversion. Lower part: First row of circles, below threshold fluctuations dominate and the dipole moments point into random directions; second row of circles, above threshold the dipole moments are slaved by  $E$ .

present book to give here any more details. What is important in the present context is the fact that the slaving principle gives rise to an enormous reduction of the degrees of freedom (cf. fig. 13.3). Because  $A_\mu(t)$  is prescribed by  $B(t)$ , all the atomic dipole moments have to obey the field. In a more detailed treatment of the laser case which we did in sections 6.3 and 6.4 it can be shown that the inversion  $d_\mu$  can be expressed by  $B$  also instantaneously. Because  $A_\mu$  and  $d_\mu$  can be expressed by  $B(t)$  these quantities can be eliminated from eqs. (13.1)–(13.3) and we obtain (for not too big  $B$ 's) the equation

$$\dot{B} = GB - C(B^+B)B + \tilde{F}_{\text{tot}}, \tag{13.6}$$

which we obtained before in this book. It determines the total behavior of the laser and thus the behavior of the individual dipole moments and the inversion of the individual atoms. Below laser threshold  $B$  is small. It transpires from (13.5) that in this case the behavior of the dipoles is essentially described by the fluctuations  $\tilde{F}_\mu$  so that the behavior of the dipoles is entirely uncorrelated. Above laser threshold the coherent field  $B$  grows more and more and it can slave the degrees of freedom of the dipole moments and of the inversion. Within synergetics it has turned out that (13.6) is a quite typical equation describing effects of self-organization. For instance an equation of the type (13.6) describes the onset of a convection pattern in fluid dynamics (cf. section 8.3) or the occurrence of a macroscopic

pattern in chemical reactions. On the other hand in synergetics it is shown that there are also other classes of equations describing macroscopic properties. But (13.6) was the first example of such type of equation in systems far from thermal equilibrium.

### 13.3. Nonequilibrium phase transitions

The laser was the first example in which the occurrence of nonequilibrium phase transitions could be demonstrated. Before we deal with this problem we wish to remind the reader of some important properties of phase transitions of systems in thermal equilibrium. Examples for phase transitions are provided by ferromagnets or superconductors. In both cases we are dealing with systems in thermal equilibrium. When we lower the temperature  $T$  below a critical temperature  $T_c$ , the macroscopic behavior of such a system changes dramatically. In the ferromagnet suddenly a macroscopic magnetization is produced, while in a superconductor the electrical resistance disappears entirely.

In order to treat such transitions, various theories were developed. The most well known theories are the Landau theory of phase transitions and the more recently developed theories based on the renormalization group by Wilson. For our purposes it will be sufficient to remind the reader of the Landau theory. Let us consider a system in thermal equilibrium and let us treat the ferromagnet as example. A ferromagnet can be considered as being composed of elementary magnets each with a magnetic moment  $\mu$ . Let us further assume that the elementary magnets can point in only two directions, up or down. Let us denote the number of elementary magnets showing upwards by  $M_\uparrow$  and of those pointing downwards by  $M_\downarrow$ . The total magnetization of the material is then given by

$$M = (M_\uparrow - M_\downarrow)\mu. \quad (13.7)$$

In the following we replace the variable  $M$  by a variable  $q$  because we wish to treat the problem somewhat more generally. In the following we shall denote  $q$  as "order parameter", because it describes the degree of order of a system (e.g. of the ferromagnet). Let us recall some basic facts of thermodynamics. In the present case the free energy depends on two quantities, namely temperature  $T$  and magnetization  $q$ . If  $q$  is not too big, we may expand the free energy into a Taylor series,

$$\mathcal{F}(q, T) = \mathcal{F}(0, T) + \mathcal{F}'(0, T)q + \dots + \frac{1}{4!} \mathcal{F}^{(4)}(0, T)q^4 + \dots \quad (13.8)$$

In many cases of practical interest the first and third derivative vanish for

symmetry reasons,

$$\mathcal{F}' = \mathcal{F}''' = 0. \tag{13.9}$$

In such a case (13.8) is reduced to

$$\mathcal{F}(q, T) = \mathcal{F}(0, T) + \frac{\alpha}{2} q^2 + \frac{\beta}{4} q^4, \tag{13.10}$$

where we have abbreviated the constant coefficients by  $\alpha/2$  and  $\beta/4$ , respectively.

It is shown in statistical physics that the probability of finding the system at temperature  $T$  and with a specific magnetization  $q$  is given by the formula

$$f = \mathcal{N} \exp(-\mathcal{F}(q, T)/k_B T). \tag{13.11}$$

In it  $k_B$  is Boltzmann's constant and  $\mathcal{N}$  the normalization factor. The most probable order parameter is determined by the condition  $\mathcal{F} = \min!$  Let us investigate the position of the corresponding minimum or minima depending on the coefficient  $a$ . In the Landau theory of phase transitions this coefficient is assumed to be of the form

$$\alpha = a(T - T_c) \quad (a > 0), \tag{13.12}$$

i.e. it changes its sign at the critical temperature  $T = T_c$ . Therefore we shall distinguish between the two domains  $T > T_c$  and  $T < T_c$  (compare table 13.1), i.e. for  $a > 0$  the minimum of  $\mathcal{F}$  lies at  $q = q_0 = 0$ . Let us consider the entropy of the system. According to formulas of thermodynamics,  $S$  is given by

$$S = -\frac{\partial \mathcal{F}(q, T)}{\partial T}. \tag{13.13}$$

In the temperature range we thus obtain

$$S = S_0 = -\frac{\partial \mathcal{F}(0, T)}{\partial T} \tag{13.14}$$

The second derivative of  $\mathcal{F}$  with respect to temperature gives us the specific heat (besides a factor  $T$ )

$$c = T \left( \frac{\partial S}{\partial T} \right). \tag{13.15}$$

Using (13.14) we thus obtain

$$c = T \left( \frac{\partial S_0}{\partial T} \right). \tag{13.16}$$

Let us repeat the same steps for the case  $T < T_c$ , i.e.  $a < 0$ . This yields a new equilibrium value  $q = \pm q_1$  and a new entropy represented in table 13.1.

Table 13.1

	State	
	Disordered	Ordered
Temperature	$T > T_c$	$T < T_c$
Parameter (external)	$\alpha > 0$	$\alpha < 0$
Most probable order parameter $q$ , $f(q) = \max!$	$q_0 = 0$	$q_1 = \pm(-\alpha/\beta)^{1/2}$
$\mathcal{F} = \min!$		Broken symmetry
Entropy, $S = -\partial\mathcal{F}(q_0, T)/\partial T$	$S_0 = -\partial\mathcal{F}(0, T)/\partial T$ $S_0 + (a^2/(2\beta))(T - T_c)$ <span style="display: block; margin: 0 auto; width: 80%; border-top: 1px solid black; padding-top: 2px;">continuous at <math>T = T_c</math></span>	
Specific heat, $c = T(\partial S/\partial T)$	$T(\partial S_0/\partial T)$ $T(\partial S_0/\partial T) + (a^2/(2\beta))T$ <span style="display: block; margin: 0 auto; width: 80%; border-top: 1px solid black; padding-top: 2px;">discontinuous at <math>T = T_c</math></span>	

As one can derive from table 13.1 the entropy  $S$  is continuous at  $T = T_c$ . When we calculate the specific heat, we obtain two different expressions below and above the critical temperature and thus a jump of the specific heat at  $T = T_c$ . This phenomenon is called a phase transition of second order, because the second derivative of the free energy is discontinuous. But because the entropy is continuous, this transition is also called a continuous phase transition. In statistical physics also the temporal change of the order parameter is studied. Quite often in a purely phenomenological manner it is assumed that the temporal change of the order parameter is given by an equation of the form

$$\dot{q} = -\frac{\partial\mathcal{F}}{\partial q}. \tag{13.17}$$

In our concrete example (13.10), (13.17) acquires the form

$$\dot{q} = -\alpha q - \beta q^3, \tag{13.18}$$

which coincides with (13.6) provided we assume  $B$  real and drop the fluctuating forces.

In (13.17) and (13.18) we have omitted a constant factor on the r.h.s. which merely fixes the time scale. (13.18) allows us to discuss some typical phenomena of phase transitions which we came across in the realm of laser

theory in quite a different context. When we let  $a$ , in (13.18) tend to 0, we find a phenomenon which is denoted in phase transition theory as *critical slowing down*. We came across this phenomenon in section 6.3 in quite a different context. At the transition point a *symmetry breaking instability* occurs because for  $T < T_c$  the equilibrium position  $q_0 = 0$  becomes unstable and is replaced by one of the two equilibrium positions  $q_1 = \sqrt{|\alpha|/\beta}$  or  $q_1 = -\sqrt{|\alpha|/\beta}$ . Finally when we add fluctuating forces to (13.18) (cf. (13.6)), these forces are particularly efficient if  $a$ , is close to 0 and the restoring force goes with the third power of  $q$  so that at small values of  $q$  the restoring force is very small. In this case we are dealing with *criticalfluctuations* of  $q$ .

So far we have reminded the reader of some fundamental concepts of phase transition theory of systems in thermal equilibrium. When we consider the individual formulas of the Landau theory of phase transitions we readily recognize a striking analogy with the laser equations. Indeed (13.11) with  $\mathcal{F}$  given by (13.10) precisely corresponds to the laser distribution function (with  $r \equiv q$ ). We thus recognize that the potential  $V$  of the fictitious particle we introduced in laser theory plays the same role as the free energy in phase transition theory of systems in thermal equilibrium. We further see that eq. (13.18) has precisely the same form as the laser equation as mentioned before. The main difference consists in the fact that  $q$  is a real variable whereas the field amplitude  $B$  is complex. But we readily see how to translate the concepts of critical slowing down, critical fluctuations and symmetry breaking into laser theory. From a formal point of view we observe in the case of the laser precisely the same phenomena which occur in phase transitions in thermal equilibrium. The decisive difference rests in the fact that the laser is a system far from thermal equilibrium. This system is open because energy is pumped all the time into it and is going out in form of laser light. We must clearly state that this analogy is purely formal. The pump power, or equivalently the unsaturated inversion, corresponds to temperature. As can be shown, the radiated laser power corresponds to the entropy. The specific heat is now replaced by the differential efficiency, i.e. by the change of emitted power when the pump power is changed. In spite of the fact that this analogy is purely formal, a discussion of laser light phenomena in terms of phase transition theory has proven very fruitful. This is particularly so because besides phase transitions of second order also such of first order are known. In such transitions a hysteresis loop occurs. Such phase transitions can be realized by specific experimental laser set-ups.

In conclusion we wish to mention that nonequilibrium phase transitions have been found in the meantime in many other systems also, such as fluids and chemical reactions.

# References and Further Reading

Since an enormous number of publications has appeared in the field of laser physics, it is impossible to give all the references. Therefore, with a few exceptions, references are given to theoretical papers only (for references with respect to experimental work, consult the books and series below), where again only a small selection can be given. I have included those which pioneered the corresponding approach and those upon which the present book is based. My apologies go to all those research workers whose work could not be quoted here. For the use of graduate students and research workers I have included a number of references of quite recent papers in fields where research is progressing most rapidly.

Comprehensive monographs or texts on laser physics (note that "quantum optics" and "quantum electronics" are often used synonymously for "laser physics"):

Bloom, A.L.: Gas Lasers. Wiley, New York, 1968

*Haken, H.*: Laser Theory, Encyclopedia of Physics Vol. XXV/2c. Springer, Berlin, 1970, 2nd corr. ed. 1984

Laser Handbook Vols. I-II, Eds. Arecchi, F.T. and Schulz-Dubois, O.E. North-Holland, Amsterdam, 1972

Lengyel, B.A.: Lasers. Wiley, New York, 1971

Maitland, A. and Dunn, M.H.: Laser Physics. Elsevier, New York, 1970

*Pantell, R.H.* and *Puthoff, H.E.*: Fundamentals of Quantum Electronics. Wiley, New York, 1969

Sargent III, M., Scully, M.O. and Lamb, Jr, W.E.: Laser Physics. Addison-Wesley, Reading, MA, 1974

*O'Shea, D.C.*, *Callen, W.R.* and Rhodes, W.T.: Introduction to Lasers and their Applications. Addison-Wesley, Reading, MA, 1978

Siegman, A.E.: Introduction to Lasers and Masers. McGraw-Hill, New York, 1971

Smith, W.V. and Sorokin, P.P.: The Laser. McGraw-Hill, New York, 1966

Svelto, O.: Principles of Lasers. Plenum, New York, 1976

Yariv, A.: Quantum Electronics. Wiley, New York, 1975



The following series deal, at least in part, with laser physics:

Springer Topics in Applied Physics, Springer, Berlin  
 Springer Series in Optical Sciences, Springer, Berlin  
 Progress in Optics, Ed. *Wolf*, E., North-Holland, Amsterdam  
 Physics of Quantum Electronics, Eds. Jacob, S., Sargent *III*, M. and *Scully*,  
 M.O. Addison Wesley, New York

More detailed references will be given in the corresponding chapters.

## Chapter 1

### § 1.1. The maser and laser principle

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Basov, N.G. and Prokhorov, A.M.: J. Exp. Theoret. Phys. USSR 27, 431 (1954); 28, 249 (1955)

Gordon, J.P., *Zeiger*, H.J. and Townes, C.H.: Phys. Rev. 95, 282 (1954); 99, 1264 (1954)

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*Maiman*, T.H.: Brit. Comm. Electron. 7, 674 (1960); Nature 187, 493 (1960)

### § 1.2.1. Rate equations

Tang, C.L., Statz, H. and *deMars*, G.A.: J. Appl. Phys. 34, 2289 (1963)

Statz, H. and *deMars*, G.A., in: Quantum Electronics, Ed. Townes, C.H., p. 530. Columbia University Press, New York, 1960

### § 1.2.2. Semiclassical theory

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*Haken*, H. and Sauermann, H.: Z. Phys. 173, 261 (1963); 176, 47 (1963)

*Haken*, H.: Laser Theory, Encyclopedia of Physics Vol. XXV/2c. Springer, Berlin, 1970, 2nd corr. ed. 1984

Lamb, Jr, W.E.: Phys. Rev. 134A, 1429 (1964).

### Self-pulsing lasers:

*Risken*, H. and Nummedal, K.: J. Appl. Phys. 39, 4662 (1968); Phys. Lett 26A, 275 (1968)

Graham, R. and *Haken*, H.: Z. Phys. 213, 420 (1968)

§ 1.2.3. Quantum theory of the laser

*Weisskopf*, V. and Wigner, E. P.: Z. Phys. 63, 54 (1930); 65, 18 (1930)  
(spontaneous-emission theory of lamps)  
*Haken*, H.: Z. Phys. 181, 96 (1964)

Armstrong, J.A. and Smith, A. W.: Phys. Rev. Lett. 14, 68 (1965)  
Freed, C. and *Haus*, H.A.: Appl. Phys. Lett. 6, 85 (1965); Phys. Rev. 141,  
287 (1966)

For detailed and very accurate measurements:

Pike, E.R., in: Quantum Optics, Eds. Kay, M.S. and Maitland, A., Academic  
Press, New York, 1970, where further references are given

Arecchi, F.T., Rodari, G.S. and Sona, A.: Phys. Lett. 25A, 59 (1967)

*Risken*, H.: Z. Phys. 186, 85 (1965)  
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Weidlich, W. and Haake, F.: Z. Phys. 185, 30 (1965); 186, 203 (1965)  
Scully, M.O. and Lamb, Jr, W.E.: Phys. Rev. Lett. 16, 853 (1966); Phys.  
Rev. 166, 246 (1968)  
Weidlich, W., *Risken*, H. and *Haken*, H.: Z. Phys. 201, 396 (1967)  
Lugiato, L.A.: Nuovo Cimento 50B, 89 (1979)  
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§ 1.2.4. Quantum classical correspondence

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Glauber, R.J.: Phys. Rev. Lett. 10, 84 (1963); Phys. Rev. 130, 2529 (1963);  
131, 2766 (1963)  
Sudarshan, E.C.G.: Phys. Rev. Lett. 10, 277 (1963)  
Gordon, J.P.: Phys. Rev. 161, 367 (1967)  
*Haken*, H., *Risken*, H. and Weidlich, W.: Z. Phys. 206, 355 (1967)

§ 1.2.5. The laser - trailblazer of synergetics

Graham, R. and *Haken*, H.: Z. Phys. 213, 240 (1968); 235, 166 (1970); 237,  
31 (1970)  
*DeGiorgio*, V. and Scully, M.O.: Phys. Rev. A2, 1170 (1970)  
Kazantsev, A.P., Rautian, S.Q. and Surdutovich, G.J.: Sov. Phys. JETP 27,  
756 (1968)  
*Haken*, H. and Graham, R.: Umschau 6, 191 (1971)  
*Haken*, H.: Synergetics. An Introduction, 3rd ed. Springer, Berlin, 1983

## Chapter 3

## § 3.1. Survey

Fox, A.G. and Li, T.: Bell Syst. Tech. J. 40, 489 (1961)

## § 3.2. Modes in a confocal resonator

Boyd, G.D. and Kogelnik, H.: Bell Syst. Tech. J. 41, 1347 (1962)

## § 3.3. Modes in a Fabry-Perot resonator

*Risken*, H.: Z. Phys. 180, 150 (1964)

## Chapter 4

## § 4.1. Introduction

For early work cf. references of § 1.2.1.

## § 4.3. Relaxation oscillations

Collins, R.J., Nelson, D.F., Schawlow, A.L., Bond, W., Garrett, C.G.B. and Kaiser, W.: Phys. Rev. Lett. 5, 303 (1960)

Korobkin, V.V. and Uspenski, A.V.: Sov. Phys. JETP 45, 1003 (1963)

Buley, E.R. and Cummings, F.W.: Phys. Rev. 134, A1454 (1964)

## § 4.4. Q-switching

Q-switching was introduced by:

Hellwarth, R. W., in: *Advances in Quantum Electronics*, Ed. Singer, J., p. 334, Columbia University Press, New York, 1961

## § 4.5. The basic rate equations of the multi-mode laser

*Haken*, H.: *Laser Theory*, Encyclopedia of Physics Vol. XXV/2c. Springer, Berlin, 1970, 2nd corr. ed. 1984

## § 4.6. Hole burning. Qualitative discussion

Bennet, Jr, W.R.: Phys. Rev. 126, 580 (1962) (spectral hole burning)

## § 4.7. Quantitative treatment of hole burning. Single mode laser action of an inhomogeneously broadened line

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Lamb, Jr, W.E.: Phys. Rev. 134A, 1429 (1964) (gas lasers)

## § 4.8. Spatial hole burning. Qualitative discussion see § 4.10.

§ 4.9. The multi-mode laser. Mode competition and Darwin's survival of the fittest

*Haken*, H. and Sauer mann, H.: *Z. Phys.* 173, 261 (1963)

§ 4.10. The coexistence of modes due to spatial hole burning. Quantitative treatment

*Haken*, H. and Sauer mann, H.: *Z. Phys.* 173, 261 (1963)

Tang, C.L., Statz, H. and *deMars*, G.A.: *J. Appl. Phys.* 34, 2289 (1963)

## Chapter 5

General remarks on chapters 5 and 6.

Since most of the important features of laser action can be treated by means of the semiclassical laser equations, there exists an enormous literature on them. The formulation of the basic equations (at least for two-level atoms) rests mainly on the analogy between a system of two-level atoms and a system of spins, described by Bloch's equations. For this analogy cf. Vol. 1 of *Light*, section 4. In that book, also the Bloch equations are derived (Bloch, F.: *Phys. Rev.* 70,460 (1946)). The equations for the two-level atoms are combined with Maxwell's equations. For this reason, the resulting set of equations is occasionally called Maxwell-Bloch equations. The methods of quantum statistics allow one to derive similar equations for multi-level systems (see, e.g., *Haken*, H.: *Laser Theory*, *Encyclopedia of Physics* Vol. XXV/2c. Springer, Berlin, 1970, 2nd corr. ed. 1984). The essential element of the semiclassical theory presented in chapters 5 and 6 is its method of solution, by which the nonlinear, multivariable equations can be solved, in particular by an elimination of the atomic variables:

*Haken*, H. and Sauer mann, H.: *Z. Phys.* 173, 261 (1963); 176, 47 (1963)

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§ 5.3. The matter equations

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Wangsness, R.K. and Bloch, F.: *Phys. Rev.* 89,728 (1953) (relaxation of spin)

Feynman, R.P., Vernon, L. and Hellwarth, R. W.: *J. Appl. Phys.* 28, 49 (1957) (elaborate analogy spin 1/2 - two-level system)

### § 5.6. Two important approximations. The rotating wave approximation and the slowly varying amplitude approximation

For the application of the rotating wave approximation in spin-resonance see, e.g.:

Bloch, F. and Siegert, A.J.: *Phys. Rev.* 57, 522 (1940)

Rabi, J.J., Ramey, N.F. and Schwinger, J.: *Rev. Mod. Phys.* 26, 107 (1954)

## Chapter 6

### § 6.6. Frequency locking of laser modes

Lamb, Jr, W.E.: *Phys. Rev.* 134A, 1429 (1964)

### § 6.7. Laser gyro

Sagnac, G.: *Compt. Rend.* 157, 708, 1410 (1913); *J. Phys. Radium* 4, 177 (1914)

Post, E.J.: *Rev. Mod. Phys.* 39, 475 (1967)

Chow, W., Hambenne, J., Hutchings, T., Sanders, V., *Sargent III*, M. and Scully, M.O.; *J. Quant. Electr.* 16, 918 (1980)

For an early review on the laser gyro:

Killpatrick, J.: *IEEE Spectrum*, Oct. 1967, p. 44

Effect of noise:

*Zubairy*, M.S., Scully, M.O. and Just, K.: *Opt. Comm.* 36, 175 (1981)

Scully, M.O., in: *Laser Spectroscopy IV*, Eds. Walther, H. and *Rothe*, K. W., p. 21. Springer, Berlin, 1979

For an early paper on the effect of noise on frequency locking:

*Haken*, H., Saueremann, H., Schmid, C. and *Vollmer*, H.D.: *Z. Phys.* 206,369 (1967)

Cresser, J.D., *Louisell*, W.H., Meystre, P., Schleich, W. and Scully, M.O.: *Phys. Rev.* A25, 2214 (1982)

*Dorschner*, T.A., *Haus*, H.A., Holz, H., Smith, I.M. and Statz, H.: *IEEE J. Quant. El.* QE-16, 1376 (1980)

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### § 6.8. The gas laser. Single mode operation

*Haken*, H. and Saueremann, H.: *Z. Phys.* 176, 47 (1963)

Lamb, Jr, W.E.: *Phys. Rev.* 134A, 1429 (1964)

For the equivalence of these approaches:

*Haken*, H.: *Laser Theory*, *Encyclopedia of Physics* Vol. XXV/2c. Springer, Berlin, 1970, 2nd corr. ed. 1984

§ 6.9. Derivation of the rate equations from the semiclassical laser equations  
See, e.g.:

*Haken*, H.: Laser Theory, Encyclopedia of Physics Vol. XXV/2c. Springer, Berlin, 1970, 2nd corr. ed. 1984

### Chapter 7

Mode-locking is well known in radio engineering, where an oscillator is locked in its frequency to the frequency of an external signal. For a detailed representation see, e.g., the books:

Gardner, F.M.: Phase Lock Techniques. Wiley, New York, 1966

Viterbi, H.J.: Principles of Coherent Communication. McGraw-Hill, New York, 1966

An early experimental verification in lasers where one gas laser was locked to another one was given by:

Stover, H.L. and Steiner, H. W.: Appl. Phys. Lett. 8, 91 (1966)

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Uchida, T.: IEEE J. Quant. El. 3, 7 (1967)

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For further references on early work including analogies to microwave circuits see:

*Haken*, H.: Laser Theory, Encyclopedia of Physics Vol. XXV/2c. Springer, Berlin, 1970, 2nd corr. ed. 1984

Since active mode locking by means of loss modulation plays an important role in practical applications, we quote some early papers. Linear theories were presented by:

Di Domenico, Jr, M.: J. Appl. Phys. 35, 2870 (1964)

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Cromwell, M.H.: IEEE J. Quant. El. QE-1, 12 (1965)

For nonlinear theories see:

*McDuff*, O.P. and Harris, S.E.: IEEE J. Quant. El. QE-3, 101 (1967)  
(inhomogeneously broadened line)

*Haken*, H. and Pauthier, M.: IEEE J. Quant. El. QE-4, 454 (1968)  
(homogeneously broadened line; this paper shows the existence of various pulse shapes)

A similar theoretical treatment is performed by:

Kuizenga, D.J. and Siegmann, A.E.: IEEE J. Quant. El. QE-6, 694 (1970)  
(this paper contains experimental results also)

Another important tool for achieving ultrashort pulses is passive mode-locking by saturable absorbers. For first theoretical treatments see:

*Garmire*, E. and *Yariv*, A.: IEEE J. Quant. El. QE-3, 222 (1967)

*Schwarz*, S.E.: IEEE J. Quant. El. QE-4, 509 (1968)

*Basov*, N.G., *Kriukov*, P.G., *Letokhov*, V.S. and *Sevatzki*, Jr, V.: IEEE J. Quant. El. QE-4, 1968 (1968)

For more recent treatments see, e.g.:

*Lugiato*, L.A., *Mandel*, P., *Dembinski*, S.T. and *Kossakowski*, A.: Phys. Rev. A18, 338 (1978)

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*Arimondo*, E., *Casagrande*, F., *Lugiato*, L.A. and *Glorieux*, P.: Appl. Phys. B30, 57 (1983)

§ 7.3. A general method for calculating evolving patterns close to instability points

*Haken*, H.: Synergetics. An Introduction, 3rd ed. Springer, Berlin, 1983

§ 7.4. Onset of ultrashort laser pulses: linear stability analysis

*Graham*, R. and *Haken*, H.: Z. Phys. 213, 420 (1968)

*Risken*, H. and *Nummedal*, K.: J. Appl. Phys. 39, 4662 (1968); Phys. Lett. 26A, 275 (1968)

§ 7.5. Onset of ultrashort laser pulses: nonlinear analysis

*Ohno*, H.: Thesis, Stuttgart, 1980

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