APPENDIX I

More on Synthetic Division

As the name implies, synthetic division simulates the long division process, but in a condensed and more efficient form. It's based on a few simple observations of long division, as noted in the division $(x^3 - 2x^2 - 13x - 17) \div (x - 5)$ shown in Figure AI.1.

Figure AI.1

$$\begin{array}{r}
x^2 + 3x + 2 \\
x - 5\overline{)x^3 - 2x^2 - 13x - 17} \\
-\underline{(x^3 - 5x^2)} \qquad\qquad \\
3x^2 - 13x \qquad\quad \\
-\underline{(3x^2 - 15x)} \qquad \\
2x - 17 \\
-\underline{(2x - 10)} \\
-7 \quad \text{remainder}
\end{array}$$

Figure AI.2

$$\begin{array}{r|rrrr}
 & & 1 & 3 & 2 \\
\hline
x - 5 & 1 & -2 & -13 & -17 \\
 & & \underline{5} & & \\
 & & 3 & & \\
 & & & \underline{15} & \\
 & & & 2 & \\
 & & & & \underline{10} \\
 & & & & -7 \quad \text{remainder}
\end{array}$$

A careful observation reveals a great deal of repetition, as any term in red is a duplicate of the term above it. In addition, since the dividend and divisor must be written in decreasing order of degree, the variable part of each term is unnecessary as we can let the *position of each coefficient* indicate the degree of the term. In other words, we'll agree that

$$1 \quad -2 \quad -13 \quad -17 \quad \text{represents the polynomial} \quad 1x^3 - 2x^2 - 13x - 17.$$

Finally, we know in advance that we'll be subtracting each partial product, so we can "distribute the negative," shown at each stage. Removing the repeated terms and variable factors, then distributing the negative to the remaining terms produces Figure AI.2. The entire process can now be condensed by vertically compressing the rows of the division so that a minimum of space is used (Figure AI.3).

Figure AI.3

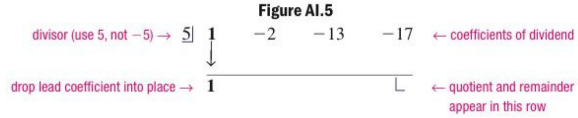
$$\begin{array}{r|rrrrl}
 & & 1 & 3 & 2 & \text{quotient} \\
\hline
x - 5 & 1 & -2 & -13 & -17 & \text{dividend} \\
 & & 5 & 15 & 10 & \text{products} \\
\hline
 & & 3 & 2 & 7 & \text{sums}
\end{array}$$

Figure AI.4

$$\begin{array}{r|rrrrl}
 & & 1 & 3 & 2 & \\
\hline
x - 5 & 1 & -2 & -13 & -17 & \text{dividend} \\
 & \downarrow & 5 & 15 & 10 & \text{products} \\
\hline
 & 1 & 3 & 2 & -7 & \textbf{remainder} \\
 & & \text{quotient} & & &
\end{array}$$

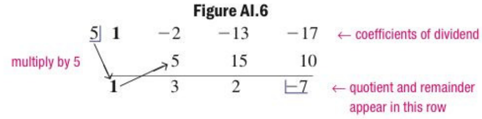
Further, if we include the lead coefficient in the bottom row (Figure AI.4), the coefficients in the top row (in **blue**) are duplicated and no longer necessary, since the quotient and remainder now appear in the last row. Finally, note all entries in the product row (in **red**) are five times the sum from the prior column. There is a direct connection between this multiplication by 5 and the divisor $x - 5$, and in fact, it is the *zero of the divisor* that is used in synthetic division ($x = 5$ from $x - 5 = 0$). A simple change in format makes this method of division easier to use, and highlights the location of the divisor and remainder (the **blue** brackets in Figure AI.5). Note the process begins by "dropping the lead coefficient into place" (shown in **bold**). The full process of synthetic division is shown in Figure AI.6 for the same exercise.

Figure AI.5



We then multiply this coefficient by the "divisor," place the result in the next column and add. In a sense, we "multiply in the diagonal direction," and "add in the vertical direction." Continue the process until the division is complete.

Figure AI.6



The result is $x^2 + 3x + 2 + \frac{-7}{x - 5}$, read from the last row.

APPENDIX II

More on Matrices

Reduced Row-Echelon Form

A matrix is in reduced row-echelon form if it satisfies the following conditions:

1. All null rows (zeroes for all entries) occur at the bottom of the matrix.
2. The first non-zero entry of any row must be a 1.
3. For any two consecutive, nonzero rows, the leading 1 in the higher row is to the left of the 1 in the lower row.
4. Every column with a leading 1 has zeroes for all other entries in the column.

Matrices A through D are in reduced row-echelon form.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where *Gaussian elimination* places a matrix in *row-echelon form* (satisfying the first three conditions), *Gauss-Jordan elimination* places a matrix in *reduced row-echelon form*. To obtain this form, continue applying row operations to the matrix until the fourth condition above is also satisfied. For a 3×3 system having a unique solution, the diagonal entries of the coefficient matrix will be 1's, with 0's for all other entries. To illustrate, we'll extend Example 4 from Section 9.1 until reduced row-echelon form is obtained.

EXAMPLE 4 ► Solving Systems Using the Augmented Matrix

Solve using Gauss-Jordan elimination: $\begin{cases} 2x + y - 2z = -7 \\ x + y + z = -1 \\ -2y - z = -3 \end{cases}$

$$\begin{cases} 2x + y - 2z = -7 \\ x + y + z = -1 \\ -2y - z = -3 \end{cases} \xrightarrow{R1 \leftrightarrow R2} \begin{cases} x + y + z = -1 \\ 2x + y - 2z = -7 \\ -2y - z = -3 \end{cases} \xrightarrow{\text{matrix form}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & -2 & -7 \\ 0 & -2 & -1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & -2 & -7 \\ 0 & -2 & -1 & -3 \end{array}\right] \xrightarrow{-2R1 + R2 \rightarrow R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -1 & -4 & -5 \\ 0 & -2 & -1 & -3 \end{array}\right] \xrightarrow{-1R2 \rightarrow R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & 5 \\ 0 & -2 & -1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & 5 \\ 0 & -2 & -1 & -3 \end{array}\right] \xrightarrow{2R2 + R3 \rightarrow R3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 7 & 7 \end{array}\right] \xrightarrow{\frac{R3}{7} \rightarrow R3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{-R2 + R1 \rightarrow R1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -6 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[-4R3 + R2 \rightarrow R2]{3R3 + R1 \rightarrow R1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The final matrix is in reduced row-echelon form with solution $(-3, 1, 1)$ just as in Section 9.1.

The Determinant of a General Matrix

To compute the determinant of a general square matrix, we introduce the idea of a **cofactor.** For an $n \times n$ matrix A, $A_{ij} = (-1)^{i+j}|M_{ij}|$ is the cofactor of matrix element a_{ij}, where $|M_{ij}|$ represents the determinant of the corresponding minor matrix. Note that $i + j$ is the sum of the row and column of the entry, and if this sum is even, $(-1)^{i+j} = 1$, while if the sum is odd, $(-1)^{i+j} = -1$ (this is how the sign table for a 3×3 determinant was generated). To compute the determinant of an $n \times n$ matrix, multiply each element in any row or column by its cofactor and add. The result is a tier-like process in which the determinant of a larger matrix requires computing the determinant of smaller matrices. In the case of a 4×4 matrix, each of the minor matrices will be size 3×3, whose determinant then requires the computation of other 2×2 determinants. In the following illustration, two of the entries in the first row are zero for convenience. For

$$A = \begin{bmatrix} -2 & 0 & 3 & 0 \\ 1 & 2 & 0 & -2 \\ 3 & -1 & 4 & 1 \\ 0 & -3 & 2 & 1 \end{bmatrix},$$

$$\text{we have: } \det(A) = -2 \cdot (-1)^{1+1} \begin{vmatrix} 2 & 0 & -2 \\ -1 & 4 & 1 \\ -3 & 2 & 1 \end{vmatrix} + (3) \cdot (-1)^{1+3} \begin{vmatrix} 1 & 2 & -2 \\ 3 & -1 & 1 \\ 0 & -3 & 1 \end{vmatrix}$$

Computing the first 3×3 determinant gives -16, the second 3×3 determinant is 14. This gives:

$$\begin{aligned} \det(A) &= -2(-16) + 3(14) \\ &= 74 \end{aligned}$$

APPENDIX III

Deriving the Equation of a Conic

The Equation of an Ellipse

In Section 10.2, the equation $\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$ was developed using the distance formula and the definition of an ellipse. To find the standard form of the equation, we treat this result as a radical equation, isolating one of the radicals and squaring both sides.

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2} \quad \text{isolate one radical}$$
$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 \quad \text{square both sides}$$

We continue by simplifying the equation, isolating the remaining radical, and squaring again.

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2 \quad \text{expand binomials}$$
$$4cx=4a^2-4a\sqrt{(x-c)^2+y^2} \quad \text{simplify}$$
$$a\sqrt{(x-c)^2+y^2}=a^2-cx \quad \text{isolate radical; divide by 4}$$
$$a^2[(x-c)^2+y^2]=a^4-2a^2cx+c^2x^2 \quad \text{square both sides}$$
$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2 \quad \text{expand and distribute } a^2 \text{ on left}$$
$$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2 \quad \text{add } 2a^2cx \text{ and rewrite equation}$$
$$x^2(a^2-c^2)+a^2y^2=a^2(a^2-c^2) \quad \text{factor}$$
$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1 \quad \text{divide by } a^2(a^2-c^2)$$

Since $a > c$, we know $a^2 > c^2$ and $a^2 - c^2 > 0$. For convenience, we let $b^2 = a^2 - c^2$ and it also follows that $a^2 > b^2$ and $a > b$ (since $c > 0$). Substituting b^2 for $a^2 - c^2$ we obtain the standard form of the equation of an ellipse (major axis horizontal, since we stipulated $a > b$): $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Note once again the x-intercepts are $(\pm a, 0)$, while the y-intercepts are $(0, \pm b)$.

The Equation of a Hyperbola

Similar to the development of the equation of an ellipse, the equation $\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a$ could have been developed using the distance formula and the definition of a hyperbola. To find the standard form of this equation, we apply the same procedures as before.

$$\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2} \quad \text{isolate one radical}$$
$$(x+c)^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 \quad \text{square both sides}$$
$$x^2+2cx+c^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2 \quad \text{expand binomials}$$
$$4cx=4a^2+4a\sqrt{(x-c)^2+y^2} \quad \text{simplify}$$
$$cx-a^2=a\sqrt{(x-c)^2+y^2} \quad \text{isolate radical; divide by 4}$$
$$c^2x^2-2a^2cx+a^4=a^2[(x-c)^2+y^2] \quad \text{square both sides}$$
$$c^2x^2-2a^2cx+a^4=a^2x^2-2a^2cx+a^2c^2+a^2y^2 \quad \text{expand and distribute } a^2 \text{ on the right}$$
$$c^2x^2-a^2x^2-a^2y^2=a^2c^2-a^4 \quad \text{add } 2a^2cx \text{ and rewrite equation}$$
$$x^2(c^2-a^2)-a^2y^2=a^2(c^2-a^2) \quad \text{factor}$$
$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1 \quad \text{divide by } a^2(c^2-a^2)$$

From the definition of a hyperbola we have $0 < a < c$, showing $c^2 > a^2$ and $c^2 - a^2 > 0$. For convenience, we let $b^2 = c^2 - a^2$ and substitute to obtain the standard form of the equation of a hyperbola (transverse axis horizontal): $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Note the x-intercepts are $(0, \pm a)$ and there are no y-intercepts.

The Asymptotes of a Central Hyperbola

From our work in Section 10.3, a central hyperbola with a horizontal axis will have asymptotes at $y = \pm\frac{b}{a}x$. To understand why, recall that for asymptotic behavior we investigate what happens to the relation for large values of x, meaning as $|x| \to \infty$. Starting with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

$$b^2x^2 - a^2y^2 = a^2b^2 \quad \text{clear denominators}$$

$$a^2y^2 = b^2x^2 - a^2b^2 \quad \text{isolate term with } y$$

$$a^2y^2 = b^2x^2\left(1 - \frac{a^2}{x^2}\right) \quad \text{factor out } b^2x^2 \text{ from right side}$$

$$y^2 = \frac{b^2}{a^2}x^2\left(1 - \frac{a^2}{x^2}\right) \quad \text{divide by } a^2$$

$$y = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}} \quad \text{square root both sides}$$

As $|x| \to \infty$, $\frac{a^2}{x^2} \to 0$, and we find that for large values of x, $y \approx \pm\frac{b}{a}x$.

APPENDIX IV

Selected Proofs

Proofs from Chapter 3

The Remainder Theorem

If a polynomial $p(x)$ is divided by $(x - c)$ using synthetic division, the remainder is equal to $p(c)$.

Proof of the Remainder Theorem

From our previous work, any number c used in synthetic division will occur as the factor $(x - c)$ when written as (quotient)(divisor) + remainder: $p(x) = (x - c)q(x) + r$. Here, $q(x)$ represents the quotient polynomial and r is a constant. Evaluating $p(c)$ gives

$$\begin{aligned} p(x) &= (x - c)q(x) + r \\ p(c) &= (c - c)q(c) + r \\ &= 0 \cdot q(c) + r \\ &= r \checkmark \end{aligned}$$

The Factor Theorem

Given a polynomial $p(x)$,
(1) if $p(c) = 0$, then $x - c$ is a factor of $p(x)$, and
(2) if $x - c$ is a factor of $p(x)$, then $p(c) = 0$.

Proof of the Factor Theorem

1. Consider a polynomial p written in the form $p(x) = (x - c)q(x) + r$. From the remainder theorem we know $p(c) = r$, and substituting $p(c)$ for r in the equation shown gives:

$$p(x) = (x - c)q(x) + p(c)$$

and $x - c$ is a factor of $p(x)$, if $p(c) = 0$

$$p(x) = (x - c)q(x) \checkmark$$

2. The steps from part 1 can be reversed, since any factor $(x - c)$ of $p(x)$, can be written in the form $p(x) = (x - c)q(x)$. Evaluating at $x = c$ produces a result of zero:

$$\begin{aligned} p(c) &= (c - c)q(x) \\ &= 0 \checkmark \end{aligned}$$

Complex Conjugates Corollary

Given $p(x)$ is a polynomial with real number coefficients, complex solutions must occur in conjugate pairs. If $a + bi$, $b \neq 0$ is a solution, then $a - bi$ must also be a solution.

To prove this for polynomials of degree $n > 2$, we let $z_1 = a + bi$ and $z_2 = c + di$ be complex numbers, and let $\bar{z}_1 = a - bi$, and $\bar{z}_2 = c - di$ represent their conjugates, and observe the following properties:

1. The conjugate of a sum is equal to the sum of the conjugates.

sum: $z_1 + z_2$		sum of conjugates: $\bar{z}_1 + \bar{z}_2$
$(a + bi) + (c + di)$		$(a - bi) + (c - di)$
$(a + c) + (b + d)i$	→ conjugate of sum →	$(a + c) - (b + d)i$ ✓

2. The conjugate of a product is equal to the product of the conjugates.

product: $z_1 \cdot z_2$		product of conjugates: $\bar{z}_1 \cdot \bar{z}_2$
$(a + bi) \cdot (c + di)$		$(a - bi) \cdot (c - di)$
$ac + adi + bci + bdi^2$		$ac - adi - bci + bdi^2$
$(ac - bd) + (ad + bc)i$	→ conjugate of product →	$(ac - bd) - (ad + bc)i$ ✓

Since polynomials involve only sums and products, and the complex conjugate of any real number is the number itself, we have the following:

Proof of the Complex Conjugates Corollary

Given polynomial $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x^1 + a_0,$ where $a_n, a_{n-1}, \cdots, a_1, a_0$ are real numbers and $z = a + bi$ is a zero of p, we must show that $\bar{z} = a - bi$ is also a zero.

$a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z^1 + a_0 = p(z)$	evaluate $p(x)$ at z
$a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z^1 + a_0 = 0$	$p(z) = 0$ given
$\overline{a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z^1 + a_0} = \bar{0}$	conjugate both sides
$\overline{a_nz^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1z^1} + \overline{a_0} = \bar{0}$	property 1
$\bar{a}_n(\bar{z}^n) + \bar{a}_{n-1}(\bar{z}^{n-1}) + \cdots + \bar{a}_1(\bar{z}^1) + \bar{a}_0 = \bar{0}$	property 2
$a_n(\bar{z}^n) + a_{n-1}(\bar{z}^{n-1}) + \cdots + a_1(\bar{z}^1) + a_0 = 0$	conjugate of a real number is the number
$p(\bar{z}) = 0$ ✓	result

An immediate and useful result of this theorem is that any polynomial of odd degree must have at least one real root.

Linear Factorization Theorem

If $p(x)$ is a complex polynomial of degree $n \geq 1$, then p has exactly n linear factors and can be written in the form $p(x) = a_n(x - c_1)(x - c_2) \cdot \ldots \cdot (x - c_n)$, where $a_n \neq 0$ and $c_1, c_2, \ldots, c_n$ are complex numbers. Some factors may have multiplicities greater than 1 ($c_1, c_2, \ldots, c_n$ are not necessarily distinct).

Proof of the Linear Factorization Theorem

Given $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is a complex polynomial, the Fundamental Theorem of Algebra establishes that $p(x)$ has a least one complex zero, call it c_1. The factor theorem stipulates $(x - c_1)$ must be a factor of P, giving

$$p(x) = (x - c_1)q_1(x)$$

where $q_1(x)$ is a complex polynomial of degree $n - 1$.

Since $q_1(x)$ is a complex polynomial in its own right, it too must also have a complex zero, call it c_2. Then $(x - c_2)$ must be a factor of $q_1(x)$, giving

$$p(x) = (x - c_1)(x - c_2)q_2(x)$$

where $q_2(x)$ is a complex polynomial of degree $n - 2$.

Repeating this rationale n times will cause $p(x)$ to be rewritten in the form

$$p(x) = (x - c_1)(x - c_2) \cdot \cdots \cdot (x - c_n)q_n(x)$$

where $q_n(x)$ has a degree of $n - n = 0$, a nonzero constant typically called a_n.

The result is $p(x) = a_n(x - c_1)(x - c_2) \cdot \cdots \cdot (x - c_n)$, and the proof is complete.

Proofs from Chapter 4

The Product Property of Logarithms

Given M, N, and $b \neq 1$ are positive real numbers,
$\log_b(MN) = \log_b M + \log_b N$.

Proof of the Product Property

For $P = \log_b M$ and $Q = \log_b N$, we have $b^P = M$ and $b^Q = N$ in exponential form. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^P b^Q) && \text{substitute } b^P \text{ for } M \text{ and } b^Q \text{ for } N \\ &= \log_b(b^{P+Q}) && \text{properties of exponents} \\ &= P + Q && \text{log property 3} \\ &= \log_b M + \log_b N && \text{substitute } \log_b M \text{ for } P \text{ and } \log_b N \text{ for } Q \end{aligned}$$

The Quotient Property of Logarithms

Given M, N, and $b \neq 1$ are positive real numbers,

$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$.

Proof of the Quotient Property

For $P = \log_b M$ and $Q = \log_b N$, we have $b^P = M$ and $b^Q = N$ in exponential form. It follows that

$$\begin{aligned} \log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^P}{b^Q}\right) && \text{substitute } b^P \text{ for } M \text{ and } b^Q \text{ for } N \\ &= \log_b(b^{P-Q}) && \text{properties of exponents} \\ &= P - Q && \text{log property 3} \\ &= \log_b M - \log_b N && \text{substitute } \log_b M \text{ for } P \text{ and } \log_b N \text{ for } Q \end{aligned}$$

The Power Property of Logarithms

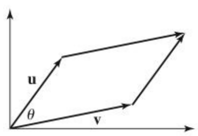
Given M and $b \neq 1$ are positive real numbers and any real number x,
$\log_b M^x = x \log_b M$.

Proof of the Power Property

For $P = \log_b M$, we have $b^P = M$ in exponential form.
It follows that

$$\begin{aligned} \log_b(M)^x &= \log_b(b^P)^x && \text{substitute } b^P \text{ for } M \\ &= \log_b(b^{Px}) && \text{properties of exponents} \\ &= Px && \text{log property 3} \\ &= (\log_b M)x && \text{substitute } \log_b M \text{ for } P \\ &= x \log_b M && \text{rewrite factors} \end{aligned}$$

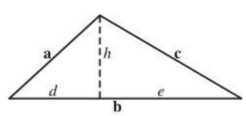
Proof of the Determinant Formula for the Area of a Parallelogram



Since the area of a triangle is $A = \frac{1}{2}ab\sin\theta$, the area of the corresponding parallelogram is twice as large: $A = ab\sin\theta$. In terms of the vectors $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, we have $A = |\mathbf{u}||\mathbf{v}|\sin\theta$, and it follows that

$$\begin{aligned}
\text{Area} &= |\mathbf{u}||\mathbf{v}|\sqrt{1-\cos^2\theta}, && \text{Pythagorean identity}\\
&= |\mathbf{u}||\mathbf{v}|\sqrt{1-\frac{(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{u}|^2|\mathbf{v}|^2}} && \text{substitute for } \cos\theta\\
&= |\mathbf{u}||\mathbf{v}|\sqrt{\frac{|\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{u}|^2|\mathbf{v}|^2}} && \text{common denominator}\\
&= \sqrt{|\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2} && \text{simplify}\\
&= \sqrt{(a^2+b^2)(c^2+d^2)-(ac+bd)^2} && \text{substitute}\\
&= \sqrt{a^2c^2+a^2d^2+b^2c^2+b^2d^2-(a^2c^2+2acbd+b^2d^2)} && \text{expand}\\
&= \sqrt{a^2d^2-2acbd+b^2c^2} && \text{simplify}\\
&= \sqrt{(ad-bc)^2} && \text{factor}\\
&= |ad-bc| && \text{result}
\end{aligned}$$

Proof of Heron's Formula Using Algebra



Note that $\sqrt{a^2-d^2} = h = \sqrt{c^2-e^2}$. It follows that

$$\begin{aligned}
\sqrt{a^2-d^2} &= \sqrt{c^2-e^2}\\
a^2-d^2 &= c^2-e^2\\
a^2-(b-e)^2 &= c^2-e^2\\
a^2-b^2+2be-e^2 &= c^2-e^2\\
a^2-b^2-c^2 &= -2be\\
\frac{b^2+c^2-a^2}{2b} &= e
\end{aligned}$$

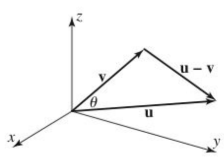
This shows:

$$\begin{aligned}
A &= \frac{1}{2}bh\\
&= \frac{1}{2}b\sqrt{c^2-e^2}\\
&= \frac{1}{2}b\sqrt{c^2-\left(\frac{b^2+c^2-a^2}{2b}\right)^2}\\
&= \frac{1}{2}b\sqrt{c^2-\left(\frac{b^4+2b^2c^2-2a^2b^2-2a^2c^2+a^4+c^4}{4b^2}\right)}\\
&= \frac{1}{2}b\sqrt{\frac{4b^2c^2}{4b^2}-\left(\frac{b^4+2b^2c^2-2a^2b^2-2a^2c^2+a^4+c^4}{4b^2}\right)}\\
&= \frac{1}{2}b\sqrt{\frac{4b^2c^2-b^4-2b^2c^2+2a^2b^2+2a^2c^2-a^4-c^4}{4b^2}}
\end{aligned}$$

$$= \frac{1}{4}\sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}$$
$$= \frac{1}{4}\sqrt{[(a + b)^2 - c^2][c^2 - (a - b)^2]}$$
$$= \frac{1}{4}\sqrt{(a + b + c)(a + b - c)(c + a - b)(c - a + b)}$$

From this point, the conclusion of the proof is the same as the trigonometric development found on page 730.

Proof that $\mathbf{u} \cdot \mathbf{v} = \text{comp}_{\mathbf{v}}\, \mathbf{u} \times |\mathbf{v}|$



Consider the vectors in the figure shown, which form a triangle. Applying the Law of Cosines to this triangle yields:

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos\theta$$

Using properties of the dot product (page 758), we can rewrite the left-hand side as follows:

$$\begin{aligned} |\mathbf{u} - \mathbf{v}|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{u}|^2 - 2\,\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2 \end{aligned}$$

Substituting the last expression for $|\mathbf{u} - \mathbf{v}|^2$ from the Law of Cosines gives

$$\begin{aligned} |\mathbf{u}|^2 - 2\,\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2 &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,|\mathbf{u}||\mathbf{v}| \cos\theta \\ -2\,\mathbf{u} \cdot \mathbf{v} &= -2\,|\mathbf{u}||\mathbf{v}| \cos\theta \\ \mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}||\mathbf{v}| \cos\theta \\ &= |\mathbf{u}| \cos\theta \times |\mathbf{v}|. \end{aligned}$$

Substituting $\text{comp}_{\mathbf{v}}\, \mathbf{u}$ for $|\mathbf{u}| \cos\theta$ completes the proof:

$$\mathbf{u} \cdot \mathbf{v} = \text{comp}_{\mathbf{v}}\, \mathbf{u} \times |\mathbf{v}|$$

Proof of DeMoivre's Theorem: $(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$

For $n > 0$, we proceed using mathematical induction.

1. Show the statement is true for $n = 1$ (base case):

$$(\cos x + i \sin x)^1 = \cos(1x) + i \sin(1x)$$
$$\cos x + i \sin x = \cos x + i \sin x \quad ✓$$

2. Assume the statement is true for $n = k$ (induction hypothesis):

$$(\cos x + i \sin x)^k = \cos(kx) + i \sin(kx)$$

3. Show the statement is true for $n = k + 1$:

$$\begin{aligned} (\cos x + i \sin x)^{k+1} &= (\cos x + i \sin x)^k(\cos x + i \sin x)^1 \\ &= [\cos(kx) + i \sin(kx)](\cos x + i \sin x) && \text{induction hypothesis} \\ &= \cos(kx)\cos x - \sin(kx)\sin x + i[\cos(kx)\sin x + \sin(kx)\cos x] && \text{F-O-I-L} \\ &= \cos[(k + 1)x] + i \sin[(k + 1)x] \quad ✓ && \text{sum/difference identities} \end{aligned}$$

By the principle of mathematical induction, the statement is true for all positive integers. For $n < 0$ (the theorem is obviously true for $n = 0$), consider a positive integer m, where $n = -m$.

$$(\cos x + i \sin x)^n = (\cos x + i \sin x)^{-m}$$

$$= \frac{1}{(\cos x + i \sin x)^m} \quad \text{negative exponent property}$$

$$= \frac{1}{\cos(mx) + i \sin(mx)} \quad \text{DeMoivre's theorem for } n > 0$$

$$= \cos(mx) - i \sin(mx) \quad \text{multiply numerator and denom by } \cos(mx) - i \sin(mx) \text{ and simplify}$$

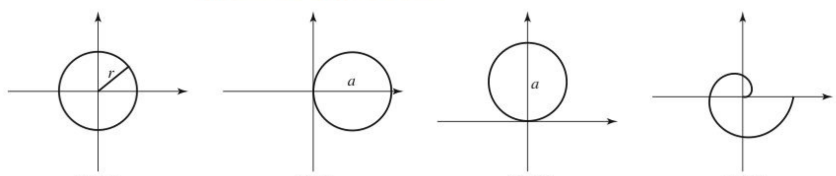
$$= \cos(-mx) + i \sin(-mx) \quad \text{even/odd identities}$$

$$= \cos(nx) + i \sin(nx) \quad n = -m$$

APPENDIX V

Families of Polar Curves

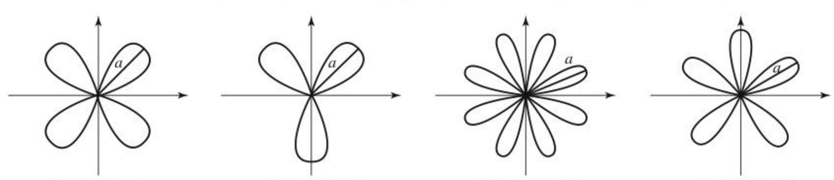
Circles and Spiral Curves



Circle $r = \frac{a}{2}$ | Circle $r = a\cos\theta$ | Circle $r = a\sin\theta$ | Spiral $r = k\theta$

a represents the diameter of the circle

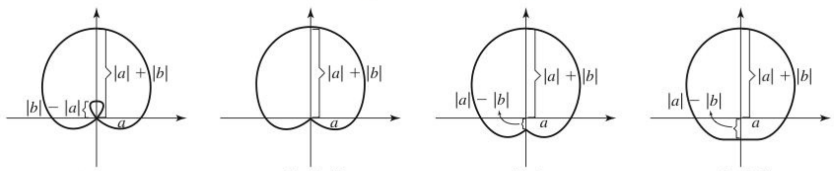
Roses: $r = a\sin(n\theta)$ (illustrated here) and $r = a\cos(n\theta)$



Four-petal rose $r = a\sin(2\theta)$ | Three-petal rose $r = a\sin(3\theta)$ | Eight-petal rose $r = a\sin(4\theta)$ | Five-petal rose $r = a\sin(5\theta)$

If n is odd → there are n petals, if n is even → there are $2n$ petals.
$|a|$ represents the maximum distance from the origin
(the radius of a circumscribed circle)

Limaçons: $r = a + b \sin \theta$ (illustrated here) and $r = a + b \cos \theta$



Limaçon (inner loop if $|a| < |b|$) $r = a + b \sin \theta$

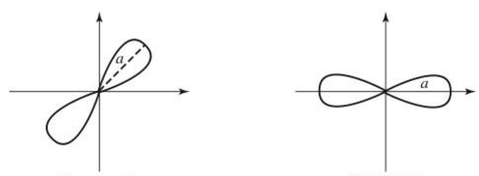
Cardioid (limaçon where $|a| = |b|$) $r = a + b \sin \theta$

Apple (limaçon where $|a| > |b|$) $r = a + b \sin \theta$

Eye-ball (limaçon where $|a| \geq 2|b|$) $r = a + b \sin \theta$

$|a| + |b|$ represents the maximum distance from the origin (along the axis of symmetry)

Lemniscates: $r^2 = a^2\sin(2\theta)$ and $r^2 = a^2\cos(2\theta)$



Lemniscate $r^2 = a^2 \sin(2\theta)$

Lemniscate $r^2 = a^2 \cos(2\theta)$

a represents the maximum distance from the origin (the radius of a circumscribed circle)