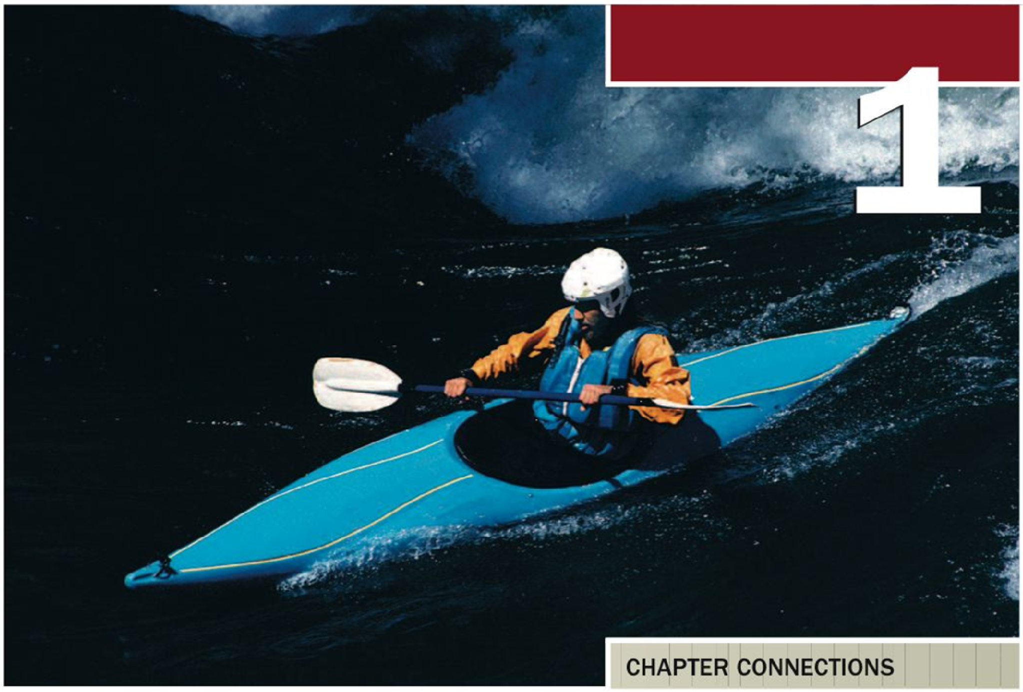


1

Equations and Inequalities

CHAPTER OUTLINE

CHAPTER CONNECTIONS

The more you understand equations, the better you can apply them in context. Impressed by a friend's 3-hr time in a 10-mi kayaking event (5-mi up river, 5-mi down river), you wish to determine the speed of the kayak in still water knowing only that the river current runs at 4 mph. The techniques illustrated in this chapter will assist you in answering this question. This application appears as Exercise 95 in Section 1.6.

Check out these other real-world connections:

- Cradle of Civilization (Section 1.1, Exercise 64)
- Heating and Cooling Subsidies (Section 1.2, Exercise 85)
- Cell Phone Subscribers (Section 1.5, Exercise 141)
- Mountain-Man Triathlon (Section 1.6, Exercise 95)

1.1 Linear Equations, Formulas, and Problem Solving

Learning Objectives

In Section 1.1 you will learn how to:

- **A.** Solve linear equations using properties of equality
- **B.** Recognize equations that are identities or contradictions
- **C.** Solve for a specified variable in a formula or literal equation
- **D.** Use the problem-solving guide to solve various problem types

In a study of algebra, you will encounter many **families of equations,** or groups of equations that share common characteristics. Of interest to us here is the family of **linear equations in one variable,** a study that lays the foundation for understanding more advanced families. In addition to *solving* linear equations, we'll use the skills we develop to *solve for a specified variable* in a formula, a practice widely used in science, business, industry, and research.

A. Solving Linear Equations Using Properties of Equality

An **equation** is *a statement that two expressions are equal*. From the expressions $3(x - 1) + x$ and $-x + 7$, we can form the equation

$$3(x - 1) + x = -x + 7,$$

which is a **linear equation in one variable.** To solve an equation, we attempt to find a specific input or x-value that will make the equation true, meaning the left-hand expression will be equal to the right. Using Table 1.1, we find that $3(x - 1) + x = -x + 7$ is a true equation when x is replaced by 2, and is a false equation otherwise. Replacement values that make the equation true are called **solutions** or **roots** of the equation.

Table 1.1

x	$3(x - 1) + x$	$-x + 7$
−2	−11	9
−1	−7	8
0	−3	7
1	1	6
2	**5**	**5**
3	9	4
4	13	3

CAUTION ► From Section R.6, an **algebraic *expression*** is a sum or difference of algebraic terms. Algebraic expressions can be simplified, evaluated or written in an equivalent form, but cannot be *"solved,"* since we're not seeking a specific value of the unknown.

Solving equations using a table is too time consuming to be practical. Instead we attempt to write a sequence of **equivalent equations,** each one simpler than the one before, until we reach a point where the solution is obvious. Equivalent equations are those that have the same solution set, and are obtained by using the distributive property to simplify the expressions on each side of the equation, and the additive and multiplicative properties of equality to obtain an equation of the form $x =$ constant.

The Additive Property of Equality	The Multiplicative Property of Equality
If A, B, and C represent algebraic expressions and $A = B$, then $A + C = B + C$	If A, B, and C represent algebraic expressions and $A = B$, then $AC = BC$ and $\frac{A}{C} = \frac{B}{C}$, $(C \neq 0)$

In words, the additive property says that like quantities, numbers or terms can be added to both sides of an equation. A similar statement can be made for the multiplicative property. These properties are combined into a general guide for solving linear equations, which you've likely encountered in your previous studies. Note that not all steps in the guide are required to solve every equation.

Guide to Solving Linear Equations in One Variable

- Eliminate parentheses using the distributive property, then combine any like terms.
- Use the additive property of equality to write the equation with all variable terms on one side, and all constants on the other. Simplify each side.
- Use the multiplicative property of equality to obtain an equation of the form $x =$ constant.
- For applications, answer in a complete sentence and include any units of measure indicated.

For our first example, we'll use the equation $3(x - 1) + x = -x + 7$ from our initial discussion.

EXAMPLE 1 ► Solving a Linear Equation Using Properties of Equality

Solve for x: $3(x - 1) + x = -x + 7$.

Solution ►

$3(x - 1) + x = -x + 7$	original equation
$3x - 3 + x = -x + 7$	distributive property
$4x - 3 = -x + 7$	combine like terms
$5x - 3 = 7$	add x to both sides (additive property of equality)
$5x = 10$	add 3 to both sides (additive property of equality)
$x = 2$	multiply both sides by $\frac{1}{5}$ or divide both sides by 5 (multiplicative property of equality)

As we noted in Table 1.1, the solution is $x = 2$.

Now try Exercises 7 through 12 ►

To check a solution by substitution means we substitute the solution back into the original equation (this is sometimes called **back-substitution**), and verify the left-hand side is equal to the right. For Example 1 we have:

$3(x - 1) + x = -x + 7$	original equation
$3(2 - 1) + 2 = -2 + 7$	substitute 2 for x
$3(1) + 2 = 5$	simplify
$5 = 5$✓	solution checks

If any coefficients in an equation are fractional, multiply both sides by the least common denominator (LCD) to *clear the fractions.* Since any decimal number can be written in fraction form, the same idea can be applied to decimal coefficients.

EXAMPLE 2 ► Solving a Linear Equation with Fractional Coefficients

Solve for n: $\frac{1}{4}(n + 8) - 2 = \frac{1}{2}(n - 6)$.

Solution ►

$\frac{1}{4}(n + 8) - 2 = \frac{1}{2}(n - 6)$	original equation
$\frac{1}{4}n + 2 - 2 = \frac{1}{2}n - 3$	distributive property
$\frac{1}{4}n = \frac{1}{2}n - 3$	combine like terms
$4(\frac{1}{4}n) = 4(\frac{1}{2}n - 3)$	multiply both sides by LCD = 4
$n = 2n - 12$	distributive property
$-n = -12$	subtract $2n$
$n = 12$	multiply by -1

Verify the solution is $n = 12$ using back-substitution.

Now try Exercises 13 through 30 ►

✓ **A.** You've just learned how to solve linear equations using properties of equality

B. Identities and Contradictions

Example 1 illustrates what is called a **conditional equation,** since the equation is true for $x = 2$, but false for all other values of x. The equation in Example 2 is also conditional. An **identity** is an equation that is *always true,* no matter what value is substituted for the variable. For instance, $2(x + 3) = 2x + 6$ is an identity with a solution set of all real numbers, written as $\{x | x \in \mathbb{R}\}$, or $x \in (-\infty, \infty)$ in interval notation. **Contradictions** are equations that are *never true,* no matter what real number is substituted for the variable. The equations $x - 3 = x + 1$ and $-3 = 1$ are contradictions. To state the solution set for a contradiction, we use the symbol "$\varnothing$" (the null set) or "{ }" (the empty set). Recognizing these special equations will prevent some surprise and indecision in later chapters.

EXAMPLE 3 ► **Solving an Equation That Is a Contradiction**

Solve for x: $2(x - 4) + 10x = 8 + 4(3x + 1)$, and state the solution set.

Solution ►

$$2(x - 4) + 10x = 8 + 4(3x + 1) \quad \text{original equation}$$
$$2x - 8 + 10x = 8 + 12x + 4 \quad \text{distributive property}$$
$$12x - 8 = 12x + 12 \quad \text{combine like terms}$$
$$-8 = 12 \quad \text{subtract } 12x$$

Since -8 is never equal to 12, the original equation is a contradiction. The solution is the empty set { }.

Now try Exercises 31 through 36 ►

In Example 3, our attempt to solve for x ended with all variables being eliminated, leaving an equation that is *always false*—a contradiction (-8 is never equal to 12). There is nothing wrong with the solution process, the result is simply telling us the original equation has *no solution.* In other equations, the variables may once again be eliminated, but leave a result that is *always true*—an identity.

B. You've just learned how to recognize equations that are identities or contradictions

C. Solving for a Specified Variable in Literal Equations

A **formula** is an equation that models a known relationship between two or more quantities. A **literal equation** is simply one that has two or more variables. Formulas are a type of literal equation, but not every literal equation is a formula. For example, the formula $A = P + PRT$ models the growth of money in an account earning simple interest, where A represents the total amount accumulated, P is the initial deposit, R is the annual interest rate, and T is the number of years the money is left on deposit. To *describe* $A = P + PRT$, we might say the formula has been "solved for A" or that "A is written in terms of P, R, and T." In some cases, before using a formula it may be convenient to solve for one of the other variables, say P. In this case, P is called the **object variable.**

EXAMPLE 4 ► **Solving for Specified Variable**

Given $A = P + PRT$, write P in terms of A, R, and T (solve for P).

Solution ► Since the object variable occurs in more than one term, we first apply the distributive property.

$$A = \mathbf{P} + \mathbf{P}RT \quad \text{focus on } \mathbf{P}\text{—the object variable}$$
$$A = \mathbf{P}(1 + RT) \quad \text{factor out } \mathbf{P}$$
$$\frac{A}{1 + RT} = \frac{\mathbf{P}(1 + RT)}{(1 + RT)} \quad \text{solve for } \mathbf{P} \text{ [divide by } (1 + RT)\text{]}$$
$$\frac{A}{1 + RT} = \mathbf{P} \quad \text{result}$$

Now try Exercises 37 through 48 ►

We solve literal equations for a specified variable using the same methods we used for other equations and formulas. Remember that it's good practice to *focus on the object variable* to help guide you through the solution process, as again shown in Example 5.

EXAMPLE 5 ► Solving for a Specified Variable

Given $2x + 3y = 15$, write y in terms of x (solve for y).

Solution ►

$$2x + 3\mathbf{y} = 15 \quad \text{focus on the object variable}$$
$$3\mathbf{y} = -2x + 15 \quad \text{subtract } 2x \text{ (isolate term with } y\text{)}$$
$$\tfrac{1}{3}(3\mathbf{y}) = \tfrac{1}{3}(-2x + 15) \quad \text{multiply by } \tfrac{1}{3} \text{ (solve for } y\text{)}$$
$$\mathbf{y} = \tfrac{-2}{3}x + 5 \quad \text{distribute and simplify}$$

Now try Exercises 49 through 54 ►

WORTHY OF NOTE

In Example 5, notice that in the second step we wrote the subtraction of $2x$ as $-2x + 15$ instead of $15 - 2x$. For reasons that become clear later in this chapter, we generally write variable terms before constant terms.

Literal Equations and General Solutions

Solving literal equations for a specified variable can help us develop the general solution for an entire family of equations. This is demonstrated here for the family of linear equations written in the form $ax + b = c$. A side-by-side comparison with a specific linear equation demonstrates that identical ideas are used.

Specific Equation		Literal Equation
$2\mathbf{x} + 3 = 15$	focus on object variable	$a\mathbf{x} + b = c$
$2\mathbf{x} = 15 - 3$	subtract constant	$a\mathbf{x} = c - b$
$\mathbf{x} = \frac{15 - 3}{2}$	divide by coefficient	$\mathbf{x} = \frac{c - b}{a}$

Of course the solution on the left would be written as $x = 6$ and checked in the original equation. On the right we now have a general formula for all equations of the form $ax + b = c$.

EXAMPLE 6 ► Solving Equations of the Form $ax + b = c$ Using the General Formula

Solve $6x - 1 = -25$ using the formula just developed, and check your solution in the original equation.

Solution ► For this equation, $a = 6$, $b = -1$, and $c = -25$, this gives

$$x = \frac{c - b}{a} = \frac{-25 - (-1)}{6} = \frac{-24}{6} = -4$$

Check:
$$6x - 1 = -25$$
$$6(-4) - 1 = -25$$
$$-24 - 1 = -25$$
$$-25 = -25 \checkmark$$

Now try Exercises 55 through 60 ►

WORTHY OF NOTE

Developing a general solution for the linear equation $ax + b = c$ seems to have little practical use. But in Section 1.5 we'll use this idea to develop a general solution for *quadratic equations*, a result with much greater significance.

C. You've just learned how to solve for a specified variable in a formula or literal equation

D. Using the Problem-Solving Guide

Becoming a good problem solver is an evolutionary process. Over time and with continued effort, your problem-solving skills grow, as will your ability to solve a wider range of applications. Most good problem solvers develop the following characteristics:

- A positive attitude
- A mastery of basic facts
- Strong mental arithmetic skills
- Good mental-visual skills
- Good estimation skills
- A willingness to persevere

These characteristics form a solid basis for applying what we call the **Problem-Solving Guide,** which simply organizes the basic elements of good problem solving. Using this guide will help save you from two common stumbling blocks—indecision and not knowing where to start.

Problem-Solving Guide

- **Gather and organize information.**
  Read the problem several times, forming a mental picture as you read. *Highlight key phrases.* List given information, including any related formulas. *Clearly identify what you are asked to find.*
- **Make the problem visual.**
  Draw and label a diagram or create a table of values, as appropriate. This will help you see how different parts of the problem fit together.
- **Develop an equation model.**
  Assign a variable to represent what you are asked to find and build any related expressions referred to in the exercise. Write an equation model from the information given in the exercise. *Carefully reread the exercise to double-check your equation model.*
- **Use the model and given information to solve the problem.**
  Substitute given values, then simplify and solve. State the answer in sentence form, and check that the answer is reasonable. Include any units of measure indicated.

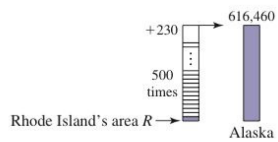
General Modeling Exercises

In Section R.2, we learned to translate word phrases into symbols. This skill is used to build equations from information given in paragraph form. Sometimes the variable *occurs more than once* in the equation, because two different items in the same exercise are related. If the relationship involves a comparison of size, we often use line segments or bar graphs to model the relative sizes.

EXAMPLE 7 ► **Solving an Application Using the Problem-Solving Guide**

The largest state in the United States is Alaska (AK), which covers an area that is 230 square miles (mi^2) more than 500 times that of the smallest state, Rhode Island (RI). If they have a combined area of 616,460 mi^2, how many square miles does each cover?

Solution ► Combined area is 616,460 mi^2, AK covers 230 more than 500 times the area of RI. — *gather and organize information; highlight any key phrases*



make the problem visual

Let R represent the area of Rhode Island. — *assign a variable*

Then $500R + 230$ represents Alaska's area. — *build related expressions*

Rhode Island's area + Alaska's area = Total

$R + (500R + 230) = 616{,}460$ — *write the equation model*

$501R = 616{,}230$ — *combine like terms, subtract 230*

$R = 1230$ — *divide by 501*

Rhode Island covers an area of 1230 mi^2, while Alaska covers an area of $500(1230) + 230 = 615{,}230$ mi^2.

Now try Exercises 63 through 68 ►

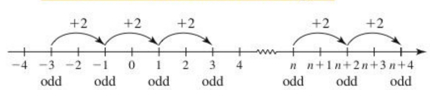
Consecutive Integer Exercises

Exercises involving **consecutive integers** offer excellent practice in assigning variables to unknown quantities, building related expressions, and the problem-solving process in general. We sometimes work with consecutive **odd** integers or consecutive **even** integers as well.

EXAMPLE 8 ► **Solving a Problem Involving Consecutive Odd Integers**

The sum of three consecutive *odd* integers is 69. What are the integers?

Solution ► The sum of three consecutive odd integers . . . — *gather/organize information; highlight any key phrases*



make the problem visual

Let n represent the smallest consecutive odd integer, then $n + 2$ represents the second odd integer and $(n + 2) + 2 = n + 4$ represents the third. — *assign a variable; build related expressions*

In words: first + second + third odd integer = 69 — *write the equation model*

$n + (n + 2) + (n + 4) = 69$ — *equation model*

$3n + 6 = 69$ — *combine like terms*

$3n = 63$ — *subtract 6*

$n = 21$ — *divide by 3*

The odd integers are $n = 21$, $n + 2 = 23$, and $n + 4 = 25$.

$21 + 23 + 25 = 69$ ✓

Now try Exercises 69 through 72 ►

> **WORTHY OF NOTE**
>
> The number line illustration in Example 8 shows that consecutive odd integers are *two units* apart and the related expressions were built accordingly: $n, n + 2, n + 4$, and so on. In particular, *we cannot use* $n, n + 1, n + 3, \ldots$ because n and $n + 1$ are *not two units apart*. If we know the exercise involves *even* integers instead, the same model is used, since even integers are also two units apart. For *consecutive* integers, the labels are $n, n + 1, n + 2$, and so on.

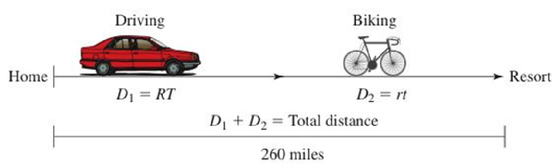
Uniform Motion (Distance, Rate, Time) Exercises

Uniform motion problems have many variations, and it's important to draw a good diagram when you get started. Recall that if speed is constant, the distance traveled is equal to the rate of speed multiplied by the time in motion: $D = RT$.

EXAMPLE 9 ► **Solving a Problem Involving Uniform Motion**

I live 260 mi from a popular mountain retreat. On my way there to do some mountain biking, my car had engine trouble—forcing me to bike the rest of the way. If I drove 2 hr longer than I biked and averaged 60 miles per hour driving and 10 miles per hour biking, how many hours did I spend pedaling to the resort?

Solution ► The sum of the two distances must be 260 mi. The **rates** are given, and the driving time is *2 hr more than biking* time. — gather/organize information; highlight any key phrases; make the problem visual



Let t represent the biking time, — assign a variable
then $T = t + 2$ represents time spent driving. — build related expressions

$$D_1 + D_2 = 260 \quad \text{write the equation model}$$
$$RT + rt = 260 \quad RT = D_1,\ rt = D_2$$
$$60(t + 2) + 10t = 260 \quad \text{substitute } t + 2 \text{ for } T, 60 \text{ for } R, 10 \text{ for } r$$
$$70t + 120 = 260 \quad \text{distribute and combine like terms}$$
$$70t = 140 \quad \text{subtract 120}$$
$$t = 2 \quad \text{divide by 70}$$

I rode my bike for $t = 2$ hr, after driving $t + 2 = 4$ hr.

Now try Exercises 73 through 76 ►

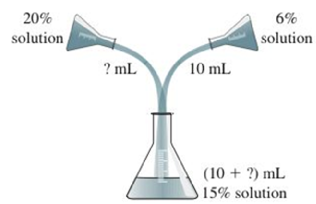
Exercises Involving Mixtures

Mixture problems offer another opportunity to refine our problem-solving skills while using many elements from the problem-solving guide. They also lend themselves to a very useful mental-visual image and have many practical applications.

EXAMPLE 10 ► **Solving an Application Involving Mixtures**

As a nasal decongestant, doctors sometimes prescribe saline solutions with a concentration between 6% and 20%. In "the old days," pharmacists had to create different mixtures, but only needed to stock these concentrations, since any percentage in between could be obtained using a mixture. An order comes in for a 15% solution. How many milliliters (mL) of the 20% solution must be mixed with 10 mL of the 6% solution to obtain the desired 15% solution?

Solution ► Only 6% and 20% concentrations are available; mix a 20% solution with 10 mL of a 6% solution *gather/organize information; highlight any key phrases*



make the problem visual

Let x represent the amount of 20% solution, then $10 + x$ represents the total amount of 15% solution. *assign a variable; build related expressions*

1st quantity times its concentration		2nd quantity times its concentration		1st+2nd quantity times desired concentration	
$10(0.06)$	+	$x(0.2)$	=	$(10 + x)(0.15)$	write equation model
0.6	+	$0.2x$	=	$1.5 + 0.15x$	distribute/simplify

$$0.2x = 0.9 + 0.15x \quad \text{subtract 0.6}$$
$$0.05x = 0.9 \quad \text{subtract } 0.15x$$
$$x = 18 \quad \text{divide by 0.05}$$

To obtain a 15% solution, 18 mL of the 20% solution must be mixed with 10 mL of the 6% solution.

Now try Exercises 77 through 84 ►

WORTHY OF NOTE

For mixture exercises, an estimate assuming equal amounts of each liquid can be helpful. For example, assume we use 10 mL of the 6% solution and 10 mL of the 20% solution. The final concentration would be halfway in between, $\frac{6 + 20}{2} = 13\%$. This is too low a concentration (we need a 15% solution), so we know that more than 10 mL of the stronger (20%) solution must be used.

☑ **D.** You've just learned how to use the problem-solving guide to solve various problem types

TECHNOLOGY HIGHLIGHT

Using a Graphing Calculator as an Investigative Tool

The mixture concept can be applied in a wide variety of ways, including mixing zinc and copper to get bronze, different kinds of nuts for the holidays, diversifying investments, or mixing two acid solutions in order to get a desired concentration. Whether the value of each part in the mix is monetary or a percent of concentration, the general mixture equation has this form:

$$\text{Quantity 1} \cdot \text{Value I} + \text{Quantity 2} \cdot \text{Value II} = \text{Total quantity} \cdot \text{Desired value}$$

Graphing calculators are a great tool for exploring this relationship, because the TABLE feature enables us to test the result of various mixtures in an instant. Suppose 10 oz of an 80% glycerin solution are to be mixed with an unknown amount of a 40% solution. How much of the 40% solution is used if a 56% solution is needed? To begin, we might consider that using equal amounts of the 40% and 80% solutions would result in a 60% concentration (halfway between 40% and 80%). To illustrate, let C represent the final concentration of the mix.

$$\mathbf{10}(0.8) + \mathbf{10}(0.4) = (10 + 10)C \quad \text{equal amounts}$$
$$8 + 4 = 20C \quad \text{simplify}$$
$$12 = 20C \quad \text{add}$$
$$0.6 = C \quad \text{divide by 20}$$

Figure 1.1

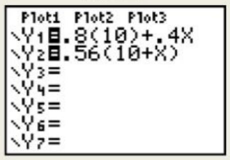
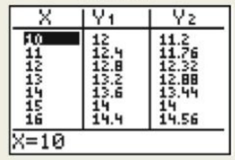


Figure 1.2

Figure 1.3



Since this is too high a concentration (a 56% = 0.56 solution is desired), we know more of the weaker solution should be used. To explore the relationship further, assume *x* oz of the 40% solution are used and enter the resulting equation on the Y= screen as $Y_1 = .8(10) + .4X$. Enter the result of the mix as $Y_2 = .56(10 + X)$ (see Figure 1.1). Next, set up a TABLE using 2nd WINDOW **(TBLSET)** with **TblStart** = 10, ΔTbl = 1, and the calculator set in Indpnt: **AUTO** mode (see Figure 1.2). Finally, access the TABLE results using 2nd GRAPH **(TABLE)**. The resulting screen is shown in Figure 1.3, where we note that 15 oz of the 40% solution should be used (the equation is true when X is 15: $Y_1 = Y_2$).

Exercise 1: Use this idea to solve Exercises 81 and 82 from the Exercises.

1.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. A(n) ______ is an equation that is always true, regardless of the ______ value.

2. A(n) ______ is an equation that is always false, regardless of the ______ value.

3. A(n) ______ equation is an equation having ______ or more unknowns.

4. For the equation $S = 2\pi r^2 + 2\pi rh$, we can say that S is written in terms of ____ and ____.

5. Discuss/Explain the three tests used to identify a linear equation. Give examples and counterexamples in your discussion.

6. Discuss/Explain each of the four basic parts of the *problem-solving guide.* Include a solved example in your discussion.

▶ DEVELOPING YOUR SKILLS

Solve each equation. Check your answer by substitution.

7. $4x + 3(x - 2) = 18 - x$

8. $15 - 2x = -4(x + 1) + 9$

9. $21 - (2v + 17) = -7 - 3v$

10. $-12 - 5w = -9 - (6w + 7)$

11. $8 - (3b + 5) = -5 + 2(b + 1)$

12. $2a + 4(a - 1) = 3 - (2a + 1)$

Solve each equation.

13. $\frac{1}{5}(b + 10) - 7 = \frac{1}{3}(b - 9)$

14. $\frac{1}{6}(n - 12) = \frac{1}{4}(n + 8) - 2$

15. $\frac{2}{3}(m + 6) = \frac{-1}{2}$

16. $\frac{4}{5}(n - 10) = \frac{-8}{9}$

17. $\frac{1}{2}x + 5 = \frac{1}{3}x + 7$

18. $-4 + \frac{2}{3}y = \frac{1}{2}y - 5$

19. $\frac{x+3}{5} + \frac{x}{3} = 7$

20. $\frac{z-4}{6} - 2 = \frac{z}{2}$

21. $15 = -6 - \frac{3p}{8}$

22. $-15 - \frac{2q}{9} = -21$

23. $0.2(24 - 7.5a) - 6.1 = 4.1$

24. $0.4(17 - 4.25b) - 3.15 = 4.16$

25. $6.2v - (2.1v - 5) = 1.1 - 3.7v$

26. $7.9 - 2.6w = 1.5w - (9.1 + 2.1w)$

27. $\frac{n}{2} + \frac{n}{5} = \frac{2}{3}$

28. $\frac{m}{3} - \frac{2}{5} = \frac{m}{4}$

29. $3p - \frac{p}{4} - 5 = \frac{p}{6} - 2p + 6$

30. $\frac{q}{6} + 1 - 3q = 2 - 4q + \frac{q}{8}$

Identify the following equations as an identity, a contradiction, or a conditional equation, then state the solution.

31. $-3(4z + 5) = -15z - 20 + 3z$

32. $5x - 9 - 2 = -5(2 - x) - 1$

33. $8 - 8(3n + 5) = -5 + 6(1 + n)$

34. $2a + 4(a - 1) = 1 + 3(2a + 1)$

35. $-4(4x + 5) = -6 - 2(8x + 7)$

36. $-(5x - 3) + 2x = 11 - 4(x + 2)$

Solve for the specified variable in each formula or literal equation.

37. $P = C + CM$ for C (retail)

38. $S = P - PD$ for P (retail)

39. $C = 2\pi r$ for r (geometry)

40. $V = LWH$ for W (geometry)

41. $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$ for T_2 (science)

42. $\frac{C}{P_2} = \frac{P_1}{d^2}$ for P_2 (communication)

43. $V = \frac{4}{3}\pi r^2 h$ for h (geometry)

44. $V = \frac{1}{3}\pi r^2 h$ for h (geometry)

45. $S_n = n\left(\frac{a_1 + a_n}{2}\right)$ for n (sequences)

46. $A = \frac{h(b_1 + b_2)}{2}$ for h (geometry)

47. $S = B + \frac{1}{2}PS$ for P (geometry)

48. $s = \frac{1}{2}gt^2 + vt$ for g (physics)

49. $Ax + By = C$ for y

50. $2x + 3y = 6$ for y

51. $\frac{5}{6}x + \frac{3}{8}y = 2$ for y

52. $\frac{2}{3}x - \frac{7}{9}y = 12$ for y

53. $y - 3 = \frac{-4}{5}(x + 10)$ for y

54. $y + 4 = \frac{-2}{15}(x + 10)$ for y

The following equations are given in $ax + b = c$ form. Solve by identifying the value of a, b, and c, then using the formula $x = \frac{c - b}{a}$.

55. $3x + 2 = -19$

56. $7x + 5 = 47$

57. $-6x + 1 = 33$

58. $-4x + 9 = 43$

59. $7x - 13 = -27$

60. $3x - 4 = -25$

► WORKING WITH FORMULAS

61. Surface area of a cylinder: $SA = 2\pi r^2 + 2\pi rh$

The surface area of a cylinder is given by the formula shown, where h is the height of the cylinder and r is the radius of the base. Find the height of a cylinder that has a radius of 8 cm and a surface area of 1256 cm^2. Use $\pi \approx 3.14$.

62. Using the equation-solving process for Exercise 61 as a model, solve the formula $SA = 2\pi r^2 + 2\pi rh$ for h.

▶ APPLICATIONS

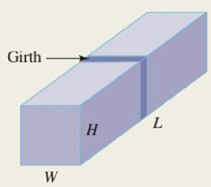
Solve by building an equation model and using the problem-solving guidelines as needed.

General Modeling Exercises

63. Two spelunkers (cave explorers) were exploring different branches of an underground cavern. The first was able to descend 198 ft farther than twice the second. If the first spelunker descended a 1218 ft, how far was the second spelunker able to descend?

64. The area near the joining of the Tigris and Euphrates Rivers (in modern Iraq) has often been called the *Cradle of Civilization*, since the area has evidence of many ancient cultures. The length of the Euphrates River exceeds that of the Tigris by 620 mi. If they have a combined length of 2880 mi, how long is each river?

65. U.S. postal regulations require that a package can have a maximum combined length and girth (distance around) of 108 in. A shipping carton is constructed so that it has a width of 14 in., a height of 12 in., and can be cut or folded to various lengths. What is the maximum length that can be used?



Source: www.USPS.com

66. Hi-Tech Home Improvements buys a fleet of identical trucks that cost \$32,750 each. The company is allowed to depreciate the value of their trucks for tax purposes by \$5250 per year. If company policies dictate that older trucks must be sold once their value declines to \$6500, approximately how many years will they keep these trucks?

67. The longest suspension bridge in the world is the Akashi Kaikyo (Japan) with a length of 6532 feet. Japan is also home to the Shimotsui Straight bridge. The Akashi Kaikyo bridge is 364 ft more than twice the length of the Shimotsui bridge. How long is the Shimotsui bridge?

Source: www.guinnessworldrecords.com

68. The Mars rover *Spirit* landed on January 3, 2004. Just over 1 yr later, on January 14, 2005, the *Huygens* probe landed on Titan (one of Saturn's moons). At their closest approach, the distance from the Earth to Saturn is 29 million mi more than 21 times the distance from the Earth to Mars. If the distance to Saturn is 743 million mi, what is the distance to Mars?

Consecutive Integer Exercises

69. Find two consecutive even integers such that the sum of twice the smaller integer plus the larger integer is one hundred forty-six.

70. When the smaller of two consecutive integers is added to three times the larger, the result is fifty-one. Find the smaller integer.

71. Seven times the first of two consecutive odd integers is equal to five times the second. Find each integer.

72. Find three consecutive even integers where the sum of triple the first and twice the second is eight more than four times the third.

Uniform Motion Exercises

73. At 9:00 A.M., Linda leaves work on a business trip, gets on the interstate, and sets her cruise control at 60 mph. At 9:30 A.M., Bruce notices she's left her briefcase and cell phone, and immediately starts after her driving 75 mph. At what time will Bruce catch up with Linda?

74. A plane flying at 300 mph has a 3-hr head start on a "chase plane," which has a speed of 800 mph. How far from the airport will the chase plane overtake the first plane?

75. Jeff had a job interview in a nearby city 72 mi away. On the first leg of the trip he drove an average of 30 mph through a long construction zone, but was able to drive 60 mph after passing through this zone. If driving time for the trip was $1\frac{1}{2}$ hr, how long was he driving in the construction zone?

76. At a high-school cross-country meet, Jared jogged 8 mph for the first part of the race, then increased his speed to 12 mph for the second part. If the race was 21 mi long and Jared finished in 2 hr, how far did he jog at the faster pace?

Mixture Exercises

Give the total amount of the mix that results and the percent concentration or worth of the mix.

77. Two quarts of 100% orange juice are mixed with 2 quarts of water (0% juice).

78. Ten pints of a 40% acid are combined with 10 pints of an 80% acid.

79. Eight pounds of premium coffee beans worth \$2.50 per pound are mixed with 8 lb of standard beans worth \$1.10 per pound.

80. A rancher mixes 50 lb of a custom feed blend costing \$1.80 per pound, with 50 lb of cheap cottonseed worth \$0.60 per pound.

Solve each application of the mixture concept.

81. To help sell more of a lower grade meat, a butcher mixes some premium ground beef worth \$3.10/lb, with 8 lb of lower grade ground beef worth \$2.05/lb. If the result was an intermediate grade of ground beef worth \$2.68/lb, how much premium ground beef was used?

82. Knowing that the camping/hiking season has arrived, a nutrition outlet is mixing GORP (Good Old Raisins and Peanuts) for the anticipated customers. How many pounds of peanuts worth \$1.29/lb, should be mixed with 20 lb of deluxe raisins worth \$1.89/lb, to obtain a mix that will sell for \$1.49/lb?

83. How many pounds of walnuts at 84¢/lb should be mixed with 20 lb of pecans at \$1.20/lb to give a mixture worth \$1.04/lb?

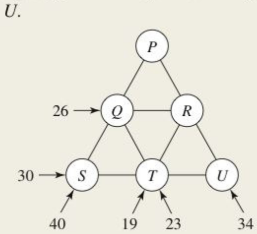
84. How many pounds of cheese worth 81¢/lb must be mixed with 10 lb cheese worth \$1.29/lb to make a mixture worth \$1.11/lb?

▶ EXTENDING THE THOUGHT

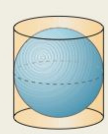
85. Look up and read the following article. Then turn in a one page summary. "Don't Give Up!," William H. Kraus, *Mathematics Teacher,* Volume 86, Number 2, February 1993: pages 110–112.

86. A chemist has four solutions of a very rare and expensive chemical that are 15% acid (cost \$120 per ounce), 20% acid (cost \$180 per ounce), 35% acid (cost \$280 per ounce) and 45% acid (cost \$359 per ounce). She requires 200 oz of a 29% acid solution. Find the combination of any two of these concentrations that will minimize the total cost of the mix.

87. P, Q, R, S, T, and U represent numbers. The arrows in the figure show the sum of the two or three numbers added in the indicated direction (Example: $Q + T = 23$). Find $P + Q + R + S + T + U$.



88. Given a sphere circumscribed by a cylinder, verify the volume of the sphere is $\frac{2}{3}$ that of the cylinder.



▶ MAINTAINING YOUR SKILLS

89. **(R.1)** Simplify the expression using the order of operations.

$-2 - 6^2 \div 4 + 8$

90. **(R.3)** Name the coefficient of each term in the expression:

$-3v^3 + v^2 - \frac{v}{3} + 7$

91. **(R.4)** Factor each expression:

a. $4x^2 - 9$ **b.** $x^3 - 27$

92. **(R.2)** Identify the property illustrated:

$\frac{6}{7} \cdot 5 \cdot 21 = \frac{6}{7} \cdot 21 \cdot 5$

1.2 Linear Inequalities in One Variable

Learning Objectives

In Section 1.2 you will learn how to:

- **A.** Solve inequalities and state solution sets
- **B.** Solve linear inequalities
- **C.** Solve compound inequalities
- **D.** Solve applications of inequalities

There are many real-world situations where the mathematical model leads to a statement of *inequality* rather than equality. Here are a few examples:

Clarice wants to buy a house costing $85,000 or less.
To earn a "B," Shantë must score more than 90% on the final exam.
To escape the Earth's gravity, a rocket must travel 25,000 mph or more.

While conditional linear equations in one variable have a single solution, linear inequalities often have an *infinite number of solutions*—which means we must develop additional methods for writing a solution set.

A. Inequalities and Solution Sets

WORTHY OF NOTE

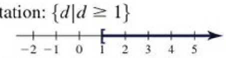
Some texts will use an open dot "∘" to mark the location of an endpoint that is not included, and a closed dot "•" for an included endpoint.

The set of numbers that satisfy an inequality is called the **solution set.** Instead of using a simple inequality to write solution sets, we will often use (1) a form of **set notation,** (2) a **number line** graph, or (3) **interval notation.** Interval notation is a symbolic way of indicating a selected interval of the real number line. When a number acts as the **boundary point** for an interval (also called an **endpoint**), we use a left bracket "[" or a right bracket "]" to indicate **inclusion** of the endpoint. If the boundary point is **not included,** we use a left parenthesis "(" or right parenthesis ")."

EXAMPLE 1 ► **Using Inequalities in Context**

Model the given phrase using the correct inequality symbol. Then state the result in set notation, graphically, and in interval notation: "If the ball had traveled at least one more foot in the air, it would have been a home run."

Solution ► Let d represent additional distance: $d \geq 1$.

- Set notation: $\{d|d \geq 1\}$
- Graph 
- Interval notation: $d \in [1, \infty)$

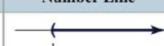
Now try Exercises 7 through 18 ►

WORTHY OF NOTE

Since infinity is really a *concept* and not a number, it is *never included* (using a bracket) as an endpoint for an interval.

The "∈" symbol says the number d is *an element of the set or interval* given. The "∞" symbol represents positive infinity and indicates the interval continues forever to the right. Note that the endpoints of an interval must occur in the same order as on the number line *(smaller value on the left; larger value on the right)*.

A short summary of other possibilities is given here. Many variations are possible.

Conditions ($a < b$)	Set Notation	Number Line	Interval Notation
x is greater than k	$\{x\|x > k\}$		$x \in (k, \infty)$
x is less than or equal to k	$\{x\|x \leq k\}$		$x \in (-\infty, k]$
x is less than b and greater than a	$\{x\|a < x < b\}$		$x \in (a, b)$
x is less than b and greater than or equal to a	$\{x\|a \leq x < b\}$		$x \in [a, b)$
x is less than a or x is greater than b	$\{x\|x < a$ or $x > b\}$		$x \in (-\infty, a) \cup (b, \infty)$

A. You've just learned how to solve inequalities and state solution sets

B. Solving Linear Inequalities

A linear *inequality* resembles a linear *equality* in many respects:

	Linear Inequality	Related Linear Equation
(1)	$x < 3$	$x = 3$
(2)	$\frac{3}{8}p - 2 \geq -12$	$\frac{3}{8}p - 2 = -12$

A linear inequality in one variable is one that can be written in the form $ax + b < c$, where a, b, and $c \in \mathbb{R}$ and $a \neq 0$. This definition and the following properties also apply when other inequality symbols are used. Solutions to simple inequalities are easy to spot. For instance, $x = -2$ is a solution to $x < 3$ since $-2 < 3$. For more involved inequalities we use the **additive property of inequality** and the **multiplicative property of inequality.** Similar to solving equations, we solve inequalities by isolating the variable on one side to obtain a solution form such as *variable* $<$ *number*.

The Additive Property of Inequality

If A, B, and C represent algebraic expressions and $A < B$,

$$\text{then} \quad A + C < B + C$$

Like quantities (numbers or terms) can be added to both sides of an inequality.

While there is little difference between the additive property of *equality* and the additive property of *inequality,* there is an *important difference* between the multiplicative property of *equality* and the multiplicative property of *inequality.* To illustrate, we begin with $-2 < 5$. Multiplying both sides by positive three yields $-6 < 15$, a true inequality. But notice what happens when we **multiply both sides by negative three:**

$$-2 < 5 \quad \text{original inequality}$$
$$-2(-3) < 5(-3) \quad \text{multiply by negative three}$$
$$6 < -15 \quad \text{false}$$

This is a *false* inequality, because 6 is *to the right* of -15 on the number line. Multiplying (or dividing) an inequality by a negative quantity *reverses the order relationship between two quantities* (we say it changes the *sense* of the inequality). We must compensate for this by reversing the inequality symbol.

$$6 > -15 \quad \text{change direction of symbol to maintain a true statement}$$

For this reason, the multiplicative property of inequality is stated in two parts.

The Multiplicative Property of Inequality

If A, B, and C represent algebraic expressions and $A < B$,
then $AC < BC$
if C is a *positive quantity*
(inequality symbol remains the same).

If A, B, and C represent algebraic expressions and $A < B$,
then $AC > BC$
if C is a *negative quantity*
(inequality symbol must be reversed).

EXAMPLE 2 ▶ Solving an Inequality

Solve the inequality, then graph the solution set and write it in interval notation: $\frac{-2}{3}x + \frac{1}{2} \leq \frac{5}{6}$.

Solution ▶

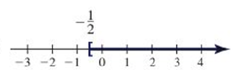
$$\frac{-2}{3}x + \frac{1}{2} \leq \frac{5}{6} \quad \text{original inequality}$$

$$6\left(\frac{-2}{3}x + \frac{1}{2}\right) \leq (6)\frac{5}{6} \quad \text{clear fractions (multiply by LCD)}$$

$$-4x + 3 \leq 5 \quad \text{simplify}$$

$$-4x \leq 2 \quad \text{subtract 3}$$

$$x \geq -\frac{1}{2} \quad \text{divide by } -4\text{, reverse inequality sign}$$

- Graph: 
- Interval notation: $x \in \left[-\frac{1}{2}, \infty\right)$

Now try Exercises 19 through 28 ▶

WORTHY OF NOTE

As an alternative to multiplying or dividing by a negative value, the additive property of inequality can be used to ensure the variable term will be positive. From Example 2, the inequality $-4x \leq 2$ can be written as $-2 \leq 4x$ by adding $4x$ to both sides and subtracting 2 from both sides. This gives the solution $-\frac{1}{2} \leq x$, which is equivalent to $x \geq -\frac{1}{2}$.

To check a linear inequality, you often have an infinite number of choices—any number from the solution set/interval. If a test value from the solution interval results in a true inequality, all numbers in the interval are solutions. For Example 2, using $x = 0$ results in the true statement $\frac{1}{2} \leq \frac{5}{6}$ ✓.

Some inequalities have all real numbers as the solution set: $\{x | x \in \mathbb{R}\}$, while other inequalities have no solutions, with the answer given as the empty set: { }.

EXAMPLE 3 ▶ Solving Inequalities

Solve the inequality and write the solution in set notation:

a. $7 - (3x + 5) \geq 2(x - 4) - 5x$ **b.** $3(x + 4) - 5 < 2(x - 3) + x$

Solution ▶

a.

$$7 - (3x + 5) \geq 2(x - 4) - 5x \quad \text{original inequality}$$

$$7 - 3x - 5 \geq 2x - 8 - 5x \quad \text{distributive property}$$

$$-3x + 2 \geq -3x - 8 \quad \text{combine like terms}$$

$$2 \geq -8 \quad \text{add } 3x$$

Since the resulting statement is always true, the original inequality is true for all real numbers. The solution is $\{x | x \in \mathbb{R}\}$.

b.

$$3(x + 4) - 5 < 2(x - 3) + x \quad \text{original inequality}$$

$$3x + 12 - 5 < 2x - 6 + x \quad \text{distribute}$$

$$3x + 7 < 3x - 6 \quad \text{combine like terms}$$

$$7 < -6 \quad \text{subtract } 3x$$

Since the resulting statement is always false, the original inequality is false for all real numbers. The solution is { }.

☑ **B.** You've just learned how to solve linear inequalities

Now try Exercises 29 through 34 ▶

C. Solving Compound Inequalities

In some applications of inequalities, we must consider more than one solution interval. These are called **compound inequalities,** and require us to take a close look at the operations of **union** "∪" and **intersection** "∩". The intersection of two sets A and B, written $A \cap B$, is the set of all elements *common to both sets.* The union of two sets A and B, written $A \cup B$, is the set of all elements *that are in either set.* When stating the union of two sets, repetitions are unnecessary.

EXAMPLE 4 ► Finding the Union and Intersection of Two Sets

For set $A = \{-2, -1, 0, 1, 2, 3\}$ and set $B = \{1, 2, 3, 4, 5\}$, determine $A \cap B$ and $A \cup B$.

Solution ► $A \cap B$ is the set of all elements in *both A and B:*
$A \cap B = \{1, 2, 3\}$.
$A \cup B$ is the set of all elements in *either A or B:*
$A \cup B = \{-2, -1, 0, 1, 2, 3, 4, 5\}$.

Now try Exercises 35 through 40 ►

WORTHY OF NOTE

For the long term, it may help to rephrase the distinction as follows. The intersection is a *selection* of elements that are common to two sets, while the union is a *collection* of the elements from two sets (with no repetitions).

Notice the intersection of two sets is described using the word "and," while the union of two sets is described using the word "or." When compound inequalities are formed using these words, the solution is modeled after the ideas from Example 4. If "and" is used, the solutions must satisfy *both* inequalities. If "or" is used, the solutions can satisfy *either* inequality.

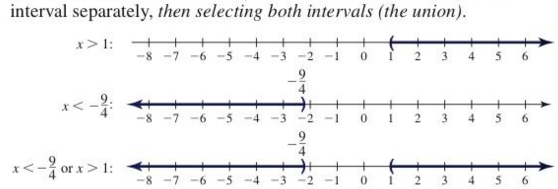
EXAMPLE 5 ► Solving a Compound Inequality

Solve the compound inequality, then write the solution in interval notation:
$-3x - 1 < -4$ **or** $4x + 3 < -6$.

Solution ► Begin with the statement as given:

$-3x - 1 < -4$	or	$4x + 3 < -6$	original statement
$-3x < -3$	or	$4x < -9$	isolate variable term
$x > 1$	or	$x < -\frac{9}{4}$	solve for x, reverse first inequality symbol

The solution $x > 1$ **or** $x < -\frac{9}{4}$ is better understood by graphing each interval separately, *then selecting both intervals (the union).*



Interval notation: $x \in \left(-\infty, -\frac{9}{4}\right) \cup (1, \infty)$.

Now try Exercises 41 and 42 ►

WORTHY OF NOTE

The graphs from Example 5 clearly show the solution consists of two disjoint (disconnected) intervals. This is reflected in the "or" statement: $x < -\frac{9}{4}$ or $x > 1$, and in the interval notation. Also, note the solution $x < -\frac{9}{4}$ or $x > 1$ is not equivalent to $-\frac{9}{4} > x > 1$, as there is no single number that is both greater than 1 and less than $-\frac{9}{4}$ at the same time.

EXAMPLE 6 ► Solving a Compound Inequality

Solve the compound inequality, then write the solution in interval notation:
$3x + 5 > -13$ **and** $3x + 5 < -1$.

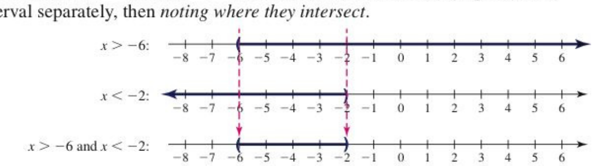
Solution ► Begin with the statement as given:

$3x + 5 > -13$	and	$3x + 5 < -1$	original statement
$3x > -18$	and	$3x < -6$	subtract five
$x > -6$	and	$x < -2$	divide by 3

WORTHY OF NOTE

The inequality $a < b$ (a is less than b) can equivalently be written as $b > a$ (b is greater than a). In Example 6, the solution is read, "$x > -6$ and $x < -2$," but if we rewrite the first inequality as $-6 < x$ (with the "arrowhead" still pointing at -6), we have $-6 < x$ and $x < -2$ and can clearly see that x must be in the single interval between -6 and -2.

The solution $x > -6$ **and** $x < -2$ can best be understood by graphing each interval separately, then *noting where they intersect*.



Interval notation: $x \in (-6, -2)$.

Now try Exercises 43 through 54 ►

The solution from Example 6 consists of the single interval $(-6, -2)$, indicating the original inequality could actually be *joined* and written as $-6 < x < -2$, called a **joint** or **compound inequality** (see Worthy of Note). We solve joint inequalities in much the same way as linear inequalities, but must remember they *have three parts (left, middle, and right)*. This means operations must be applied to *all three parts* in each step of the solution process, to obtain a solution form such as *smaller number* $<$ x $<$ *larger number*. The same ideas apply when other inequality symbols are used.

EXAMPLE 7 ► Solving a Compound Inequality

Solve the compound inequality, then graph the solution set and write it in interval notation: $1 > \frac{2x+5}{-3} \geq -6$.

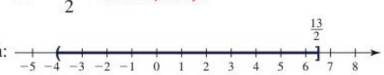
Solution ►

$1 > \frac{2x+5}{-3} \geq -6$ — original inequality

$-3 < 2x + 5 \leq 18$ — multiply all parts by -3; reverse the inequality symbols

$-8 < 2x \leq 13$ — subtract 5 from all parts

$-4 < x \leq \frac{13}{2}$ — divide all parts by 2

- Graph: 
- Interval notation: $x \in (-4, \frac{13}{2}]$

C. You've just learned how to solve compound inequalities

Now try Exercises 55 through 60 ►

D. Applications of Inequalities

Domain and Allowable Values

One application of inequalities involves the concept of allowable values. Consider the expression $\frac{24}{x}$. As Table 1.2 suggests, we can evaluate this expression using any real number *other than zero,* since the expression $\frac{24}{0}$ is undefined. Using set notation the allowable values are written $\{x | x \in \mathbb{R}, x \neq 0\}$. To graph the solution we must be careful to exclude zero, as shown in Figure 1.4.

Table 1.2

x	$\frac{24}{x}$
6	4
−12	−2
$\frac{1}{2}$	48
0	error

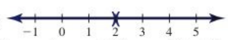
The graph gives us a snapshot of the solution using interval notation, which is written as a union of two **disjoint (disconnected) intervals** so as to exclude zero: $x \in (-\infty, 0) \cup (0, \infty)$. The set of allowable values is referred to as the **domain** of the expression. Allowable values are said to be "*in the domain*" of the expression; values that are not allowed are said to be "*outside the domain.*" When the denominator of a fraction contains a variable expression, values that cause a denominator of zero are outside the domain.

Figure 1.4

EXAMPLE 8 ▶ Determining the Domain of an Expression

Determine the domain of the expression $\frac{6}{x-2}$. State the result in set notation, graphically, and using interval notation.

Solution ▶ Set the denominator equal to zero and solve: $x - 2 = 0$ yields $x = 2$. This means 2 is outside the domain and *must be excluded*.

- Set notation: $\{x | x \in \mathbb{R}, x \neq 2\}$
- Graph: 
- Interval notation: $x \in (-\infty, 2) \cup (2, \infty)$

Now try Exercises 61 through 68 ▶

A second area where allowable values are a concern involves the square root operation. Recall that $\sqrt{49} = 7$ since $7 \cdot 7 = 49$. However, $\sqrt{-49}$ cannot be written as the product of two real numbers since $(-7) \cdot (-7) = 49$ and $7 \cdot 7 = 49$. In other words, $\sqrt{X}$ represents a real number only if the radicand is positive or zero. If X represents an algebraic expression, the domain of $\sqrt{X}$ is $\{X | X \geq 0\}$.

EXAMPLE 9 ▶ Determining the Domain of an Expression

Determine the domain of $\sqrt{x+3}$. State the domain in set notation, graphically, and in interval notation.

Solution ▶ The radicand must represent a nonnegative number. Solving $x + 3 \geq 0$ gives $x \geq -3$.

- Set notation: $\{x | x \geq -3\}$
- Graph: 
- Interval notation: $x \in [-3, \infty)$

Now try Exercises 69 through 76 ▶

Inequalities are widely used to help gather information, and to make comparisons that will lead to informed decisions. Here, the problem-solving guide is once again a valuable tool.

EXAMPLE 10 ▶ Using an Inequality to Compute Desired Test Scores

Justin earned scores of 78, 72, and 86 on the first three out of four exams. What score must he earn on the fourth exam to have an average of at least 80?

Solution ▶ **Gather and organize information;** highlight any key phrases.
First the scores: 78, 72, 86. An average of *at least* 80 means $A \geq 80$.
Make the problem visual.

Test 1	Test 2	Test 3	Test 4	Computed Average	Minimum
78	72	86	x	$\frac{78+72+86+x}{4}$	80

Assign a variable; build related expressions.

Let x represent Justin's score on the fourth exam, then $\frac{78 + 72 + 86 + x}{4}$ represents his average score.

$$\frac{78 + 72 + 86 + x}{4} \geq 80 \quad \text{average must be greater than or equal to 80}$$

Write the equation model and solve.

$$
\begin{aligned}
78 + 72 + 86 + x &\geq 320 && \text{multiply by 4} \\
236 + x &\geq 320 && \text{simplify} \\
x &\geq 84 && \text{solve for } x \text{ (subtract 236)}
\end{aligned}
$$

Justin must score at least an 84 on the last exam to earn an 80 average.

Now try Exercises 79 through 86 ►

As your problem-solving skills improve, the process outlined in the problem-solving guide naturally becomes less formal, as we work more directly toward the equation model. See Example 11.

EXAMPLE 11 ► Using an Inequality to Make a Financial Decision

As Margaret starts her new job, her employer offers two salary options. Plan 1 is base pay of \$1475/mo plus 3% of sales. Plan 2 is base pay of \$500/mo plus 15% of sales. What level of monthly sales is needed for her to earn more under Plan 2?

Solution ► Let x represent her monthly sales in dollars. The equation model for Plan 1 would be $0.03x + 1475$; for Plan 2 we have $0.15x + 500$. To find the sales volume needed for her to earn more under Plan 2, we solve the inequality

$$
\begin{aligned}
0.15x + 500 &> 0.03x + 1475 && \text{Plan 2} > \text{Plan 1} \\
0.12x + 500 &> 1475 && \text{subtract } 0.03x \\
0.12x &> 975 && \text{subtract 500} \\
x &> 8125 && \text{divide by 0.12}
\end{aligned}
$$

If Margaret can generate more than \$8125 in monthly sales, she will earn more under Plan 2.

Now try Exercises 87 and 88 ►

D. You've just learned how to solve applications of inequalities

1.2 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. For inequalities, the three ways of writing a solution set are ________ notation, a number line graph, and ________ notation.

2. The mathematical sentence $3x + 5 < 7$ is a(n) ________ inequality, while $-2 < 3x + 5 < 7$ is a(n) ________ inequality.

3. The ________ of sets A and B is written $A \cap B$. The ________ of sets A and B is written $A \cup B$.

4. The intersection of set A with set B is the set of elements in A ________ B. The union of set A with set B is the set of elements in A ________ B.

5. Discuss/Explain how the concept of domain and allowable values relates to rational and radical expressions. Include a few examples.

6. Discuss/Explain why the inequality symbol must be reversed when multiplying or dividing by a negative quantity. Include a few examples.

▶ DEVELOPING YOUR SKILLS

Use an inequality to write a mathematical model for each statement.

7. To qualify for a secretarial position, a person must type at least 45 words per minute.

8. The balance in a checking account must remain above \$1000 or a fee is charged.

9. To bake properly, a turkey must be kept between the temperatures of 250° and 450°.

10. To fly effectively, the airliner must cruise at or between altitudes of 30,000 and 35,000 ft.

Graph each inequality on a number line.

11. $y < 3$ **12.** $x > -2$

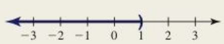
13. $m \le 5$ **14.** $n \ge -4$

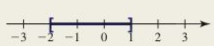
15. $x \ne 1$ **16.** $x \ne -3$

17. $5 > x > 2$ **18.** $-3 < y \le 4$

Write the solution set illustrated on each graph in set notation and interval notation.

19. 

20. 

21. 

22. 

Solve the inequality and write the solution in set notation. Then graph the solution and write it in interval notation.

23. $5a - 11 \ge 2a - 5$

24. $-8n + 5 > -2n - 12$

25. $2(n + 3) - 4 \le 5n - 1$

26. $-5(x + 2) - 3 < 3x + 11$

27. $\frac{3x}{8} + \frac{x}{4} < -4$ **28.** $\frac{2y}{5} + \frac{y}{10} < -2$

Solve each inequality and write the solution in set notation.

29. $7 - 2(x + 3) \ge 4x - 6(x - 3)$

30. $-3 - 6(x - 5) \le 2(7 - 3x) + 1$

31. $4(3x - 5) + 18 < 2(5x + 1) + 2x$

32. $8 - (6 + 5m) > -9m - (3 - 4m)$

33. $-6(p - 1) + 2p \le -2(2p - 3)$

34. $9(w - 1) - 3w \ge -2(5 - 3w) + 1$

Determine the intersection and union of sets A, B, C, and D as indicated, given $A = \{-3, -2, -1, 0, 1, 2, 3\}$, $B = \{2, 4, 6, 8\}$, $C = \{-4, -2, 0, 2, 4\}$, and $D = \{4, 5, 6, 7\}$.

35. $A \cap B$ and $A \cup B$ **36.** $A \cap C$ and $A \cup C$

37. $A \cap D$ and $A \cup D$ **38.** $B \cap C$ and $B \cup C$

39. $B \cap D$ and $B \cup D$ **40.** $C \cap D$ and $C \cup D$

Express the compound inequalities graphically and in interval notation.

41. $x < -2$ or $x > 1$ **42.** $x < -5$ or $x > 5$

43. $x < 5$ and $x \ge -2$ **44.** $x \ge -4$ and $x < 3$

45. $x \ge 3$ and $x \le 1$ **46.** $x \ge -5$ and $x \le -7$

Solve the compound inequalities and graph the solution set.

47. $4(x - 1) \le 20$ or $x + 6 > 9$

48. $-3(x + 2) > 15$ or $x - 3 \le -1$

49. $-2x - 7 \le 3$ and $2x \le 0$

50. $-3x + 5 \le 17$ and $5x \le 0$

51. $\frac{3}{5}x + \frac{1}{2} > \frac{3}{10}$ and $-4x > 1$

52. $\frac{2}{3}x - \frac{5}{6} \le 0$ and $-3x < -2$

53. $\frac{3x}{8} + \frac{x}{4} < -3$ or $x + 1 > -5$

54. $\frac{2x}{5} + \frac{x}{10} < -2$ or $x - 3 > 2$

55. $-3 \le 2x + 5 < 7$ **56.** $2 < 3x - 4 \le 19$

57. $-0.5 \le 0.3 - x \le 1.7$

58. $-8.2 < 1.4 - x < -0.9$

59. $-7 < -\frac{3}{4}x - 1 \le 11$

60. $-21 \le -\frac{2}{3}x + 9 < 7$

Determine the domain of each expression. Write your answer in interval notation.

61. $\frac{12}{m}$ **62.** $\frac{-6}{n}$

63. $\frac{5}{y+7}$ **64.** $\frac{4}{x-3}$

65. $\frac{a+5}{6a-3}$ **66.** $\frac{m+5}{8m+4}$

67. $\frac{15}{3x-12}$ **68.** $\frac{7}{2x+6}$

Determine the domain for each expression. Write your answer in interval notation.

69. $\sqrt{x-2}$ **70.** $\sqrt{y+7}$

71. $\sqrt{3n-12}$ **72.** $\sqrt{2m+5}$

73. $\sqrt{b-\frac{4}{3}}$ **74.** $\sqrt{a+\frac{3}{4}}$

75. $\sqrt{8-4y}$ **76.** $\sqrt{12-2x}$

▶ WORKING WITH FORMULAS

77. Body mass index: $B = \frac{704W}{H^2}$

The U.S. government publishes a body mass index formula to help people consider the risk of heart disease. An index "*B*" of 27 or more means that a person is at risk. Here *W* represents weight in pounds and *H* represents height in inches. (a) Solve the formula for *W*. (b) If your height is 5′8″ what range of weights will help ensure you remain safe from the risk of heart disease?

Source: www.surgeongeneral.gov/topics.

78. Lift capacity: $75S + 125B \le 750$

The capacity in pounds of the lift used by a roofing company to place roofing shingles and buckets of roofing nails on rooftops is modeled by the formula shown, where *S* represents packs of shingles and *B* represents buckets of nails. Use the formula to find (a) the largest number of shingle packs that can be lifted, (b) the largest number of nail buckets that can be lifted, and (c) the largest number of shingle packs that can be lifted along with three nail buckets.

▶ APPLICATIONS

Write an inequality to model the given information and solve.

79. Exam scores: Jacques is going to college on an academic scholarship that requires him to maintain at least a 75% average in all of his classes. So far he has scored 82%, 76%, 65%, and 71% on four exams. What scores are possible on his last exam that will enable him to keep his scholarship?

80. Timed trials: In the first three trials of the 100-m butterfly, Johann had times of 50.2, 49.8, and 50.9 sec. How fast must he swim the final timed trial to have an average time of 50 sec?

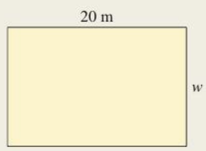
81. Checking account balance: If the average daily balance in a certain checking account drops below $1000, the bank charges the customer a $7.50 service fee. The table gives the daily balance for one customer. What must the daily balance be for Friday to avoid a service charge?

Weekday	Balance
Monday	$1125
Tuesday	$850
Wednesday	$625
Thursday	$400

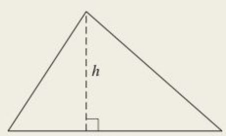
82. Average weight: In the National Football League, many consider an offensive line to be "small" if the average weight of the five down linemen is less than 325 lb. Using the table, what must the weight of the right tackle be so that the line will not be considered too small?

Lineman	Weight
Left tackle	318 lb
Left guard	322 lb
Center	326 lb
Right guard	315 lb
Right tackle	?

83. Area of a rectangle: Given the rectangle shown, what is the range of values for the width, in order to keep the area less than 150 m^2?



84. Area of a triangle: Using the triangle shown, find the height that will guarantee an area equal to or greater than 48 in^2.



85. Heating and cooling subsidies: As long as the outside temperature is over 45°F and less than 85°F ($45 < F < 85$), the city does not issue heating or cooling subsidies for low-income families. What is the corresponding range of Celsius temperatures C? Recall that $F = \frac{9}{5}C + 32$.

86. U.S. and European shoe sizes: To convert a European male shoe size "E" to an American male shoe size "A," the formula $A = 0.76E - 23$ can be used. Lillian has five sons in the U.S. military, with shoe sizes ranging from size 9 to size 14 ($9 \le A \le 14$). What is the corresponding range of European sizes? Round to the nearest half-size.

87. Power tool rentals: Sunshine Equipment Co. rents its power tools for a $20 fee, plus $4.50/hr. Kealoha's Rentals offers the same tools for an $11 fee plus $6.00/hr. How many hours h must a tool be rented to make the cost at Sunshine a better deal?

88. Moving van rentals: Davis Truck Rentals will rent a moving van for $15.75/day plus $0.35 per mile. Bertz Van Rentals will rent the same van for $25/day plus $0.30 per mile. How many miles m must the van be driven to make the cost at Bertz a better deal?

▶ EXTENDING THE CONCEPT

89. Use your local library, the Internet, or another resource to find the highest and lowest point on each of the seven continents. Express the range of altitudes for each continent as a joint inequality. Which continent has the greatest range?

90. The sum of two consecutive even integers is greater than or equal to 12 and less than or equal to 22. List all possible values for the two integers.

Place the correct inequality symbol in the blank to make the statement true.

91. If $m > 0$ and $n < 0$, then mn ________ 0.

92. If $m > n$ and $p > 0$, then mp ________ np.

93. If $m < n$ and $p > 0$, then mp ________ np.

94. If $m \le n$ and $p < 0$, then mp ________ np.

95. If $m > n$, then $-m$ ________ $-n$.

96. If $m < n$, then $\frac{1}{m}$ ________ $\frac{1}{n}$.

97. If $m > 0$ and $n < 0$, then m^2 ________ n.

98. If $m < 0$, then m^3 ________ 0.

▶ MAINTAINING YOUR SKILLS

99. (R.2) Translate into an algebraic expression: eight subtracted from twice a number.

100. (1.1) Solve: $-4(x - 7) - 3 = 2x + 1$

101. (R.3) Simplify the algebraic expression: $2(\frac{5}{6}x - 1) - (\frac{1}{6}x + 3)$.

102. (1.1) Solve: $\frac{4}{5}m + \frac{2}{3} = \frac{1}{2}$

1.3 Absolute Value Equations and Inequalities

Learning Objectives

In Section 1.3 you will learn how to:

- ❑ **A.** Solve absolute value equations
- ❑ **B.** Solve "less than" absolute value inequalities
- ❑ **C.** Solve "greater than" absolute value inequalities
- ❑ **D.** Solve applications involving absolute value

While the equations $x + 1 = 5$ and $|x + 1| = 5$ are similar in many respects, note the first has only the solution $x = 4$, while either $x = 4$ or $x = -6$ will satisfy the second. The fact there are two solutions shouldn't surprise us, as it's a natural result of how absolute value is defined.

A. Solving Absolute Value Equations

The absolute value of a number x can be thought of as its distance from zero on the number line, regardless of direction. This means $|x| = 4$ will have *two solutions,* since there are two numbers that are four units from zero: $x = -4$ and $x = 4$ (see Figure 1.5).

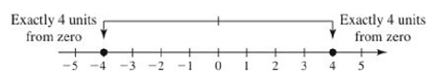


Figure 1.5

WORTHY OF NOTE

Note if $k < 0$, the equation $|X| = k$ has no solutions since the absolute value of any quantity is always positive or zero. On a related note, we can verify that if $k = 0$, the equation $|X| = 0$ has only the solution $X = 0$.

This basic idea can be extended to include situations where the quantity within absolute value bars *is an algebraic expression,* and suggests the following property.

Property of Absolute Value Equations

If X represents an algebraic expression and k is a positive real number,

then $|X| = k$

implies $X = -k$ or $X = k$

As the statement of this property suggests, it can only be applied *after* the absolute value expression has been isolated on one side.

EXAMPLE 1 ► **Solving an Absolute Value Equation**

Solve: $-5|x - 7| + 2 = -13$.

Solution ► Begin by isolating the absolute value expression.

$$-5|x - 7| + 2 = -13 \quad \text{original equation}$$
$$-5|x - 7| = -15 \quad \text{subtract 2}$$
$$|x - 7| = 3 \quad \text{divide by } -5 \text{ (simplified form)}$$

Now consider $x - 7$ as the variable expression "X" in the property of absolute value equations, giving

$$x - 7 = -3 \quad \text{or} \quad x - 7 = 3 \quad \text{apply the property of absolute value equations}$$
$$x = 4 \quad \text{or} \quad x = 10 \quad \text{add 7}$$

Substituting into the original equation verifies the solution set is $\{4, 10\}$.

Now try Exercises 7 through 18 ►

CAUTION ► For equations like those in Example 1, be careful not to treat the absolute value bars as simple grouping symbols. The equation $-5(x - 7) + 2 = -13$ has only the solution $x = 10$, and "misses" the second solution since it yields $x - 7 = 3$ in simplified form. The equation $-5|x - 7| + 2 = -13$ simplifies to $|x - 7| = 3$ and there are actually *two* solutions.

Absolute value equations come in many different forms. Always begin by isolating the absolute value expression, then apply the property of absolute value equations to solve.

EXAMPLE 2 ► **Solving an Absolute Value Equation**

Solve: $\left|5 - \frac{2}{3}x\right| - 9 = 8$

Solution ►

$$\left|5 - \frac{2}{3}x\right| - 9 = 8 \quad \text{original equation}$$

$$\left|5 - \frac{2}{3}x\right| = 17 \quad \text{add 9}$$

$$5 - \frac{2}{3}x = -17 \quad \text{or} \quad 5 - \frac{2}{3}x = 17 \quad \text{apply the property of absolute value equations}$$

$$-\frac{2}{3}x = -22 \quad \text{or} \quad -\frac{2}{3}x = 12 \quad \text{subtract 5}$$

$$x = 33 \quad \text{or} \quad x = -18 \quad \text{multiply by } -\tfrac{3}{2}$$

Check ►

For $x = 33$: $\left|5 - \frac{2}{3}(33)\right| - 9 = 8$

$$|5 - 2(11)| - 9 = 8$$
$$|5 - 22| - 9 = 8$$
$$|-17| - 9 = 8$$
$$17 - 9 = 8$$
$$8 = 8 ✓$$

For $x = -18$: $\left|5 - \frac{2}{3}(-18)\right| - 9 = 8$

$$|5 - 2(-6)| - 9 = 8$$
$$|5 + 12| - 9 = 8$$
$$|17| - 9 = 8$$
$$17 - 9 = 8$$
$$8 = 8 ✓$$

Both solutions check. The solution set is $\{-18, 33\}$.

Now try Exercises 19 through 22 ►

WORTHY OF NOTE

As illustrated in both Examples 1 and 2, the property we use to solve absolute value equations can only be applied *after* the absolute value term has been isolated. As you will see, the same is true for the properties used to solve absolute value inequalities.

For some equations, it's helpful to apply the **multiplicative property of absolute value:**

Multiplicative Property of Absolute Value

If A and B represent algebraic expressions,

then $|AB| = |A||B|$.

Note that if $A = -1$ the property says $|-B| = |-1|\,|B| = |B|$. More generally the property is applied where A is any constant.

EXAMPLE 3 ► **Solving Equations Using the Multiplicative Property of Absolute Value**

Solve: $|-2x| + 5 = 13$.

Solution ►

$$|-2x| + 5 = 13 \quad \text{original equation}$$
$$|-2x| = 8 \quad \text{subtract 5}$$
$$|-2||x| = 8 \quad \text{apply multiplicative property of absolute value}$$
$$2|x| = 8 \quad \text{simplify}$$
$$|x| = 4 \quad \text{divide by 2}$$
$$x = -4 \quad \text{or} \quad x = 4 \quad \text{apply property of absolute value equations}$$

Both solutions check. The solution set is $\{-4, 4\}$.

Now try Exercises 23 and 24 ►

☑ **A.** You've just learned how to solve absolute value equations

B. Solving "Less Than" Absolute Value Inequalities

Absolute value *inequalities* can be solved using the basic concept underlying the property of absolute value equations. Whereas the equation $|x| = 4$ asks for all numbers x whose distance from zero is *equal* to 4, the inequality $|x| < 4$ asks for all numbers x whose distance from zero is *less than* 4.

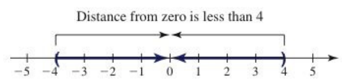


Figure 1.6

As Figure 1.6 illustrates, the solutions are $x > -4$ and $x < 4$, which can be written as the joint inequality $-4 < x < 4$. This idea can likewise be extended to include the absolute value of an algebraic expression X as follows.

WORTHY OF NOTE

Property I can also be applied when the "≤" symbol is used. Also notice that if $k < 0$, the solution is the empty set since the absolute value of any quantity is always positive or zero.

Property I: Absolute Value Inequalities

If X represents an algebraic expression and k is a positive real number,

then $|X| < k$

implies $-k < X < k$

EXAMPLE 4 ► Solving "Less Than" Absolute Value Inequalities

Solve the inequalities:

a. $\dfrac{|3x + 2|}{4} \le 1$ **b.** $|2x - 7| < -5$

Solution ► **a.**

$\dfrac{|3x + 2|}{4} \le 1$ original inequality

$|3x + 2| \le 4$ multiply by 4

$-4 \le 3x + 2 \le 4$ apply Property I

$-6 \le 3x \le 2$ subtract 2 from all three parts

$-2 \le x \le \dfrac{2}{3}$ divide all three parts by 3

The solution interval is $[-2, \frac{2}{3}]$.

b. $|2x - 7| < -5$ original inequality

Since the absolute value of any quantity is always positive or zero, the solution for this inequality is the empty set: { }.

WORTHY OF NOTE

As with the inequalities from Section 1.2, solutions to absolute value inequalities can be checked using a test value. For Example 4(a), substituting $x = 0$ from the solution interval yields:

$\frac{1}{2} \le 1$ ✓

Now try Exercises 25 through 38 ►

☑ **B. You've just learned how to solve less than absolute value inequalities**

C. Solving "Greater Than" Absolute Value Inequalities

For "greater than" inequalities, consider $|x| > 4$. Now we're asked to find all numbers x whose distance from zero is *greater than* 4. As Figure 1.7 shows, solutions are found in the interval to the left of -4, or to the right of 4. The fact the intervals are disjoint (disconnected) is reflected in this graph, in the inequalities $x < -4$ **or** $x > 4$, as well as the interval notation $x \in (-\infty, -4) \cup (4, \infty)$.

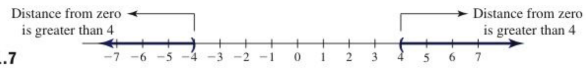


Figure 1.7

As before, we can extend this idea to include algebraic expressions, as follows:

Property II: Absolute Value Inequalities

If X represents an algebraic expression and k is a positive real number,

then $|X| > k$
implies $X < -k$ or $X > k$

EXAMPLE 5 ► **Solving "Greater Than" Absolute Value Inequalities**

Solve the inequalities:

a. $-\frac{1}{3}\left|3 + \frac{x}{2}\right| < -2$ **b.** $|5x + 2| \geq -\frac{3}{2}$

Solution ► **a.** Note the exercise is given as a *less than* inequality, but as we multiply both sides by -3, we must *reverse the inequality symbol*.

$$-\frac{1}{3}\left|3 + \frac{x}{2}\right| < -2 \quad \text{original inequality}$$

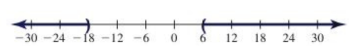
$$\left|3 + \frac{x}{2}\right| > 6 \quad \text{multiply by } -3, \textit{ reverse the symbol}$$

$$3 + \frac{x}{2} < -6 \quad \text{or} \quad 3 + \frac{x}{2} > 6 \quad \text{apply Property II}$$

$$\frac{x}{2} < -9 \quad \text{or} \quad \frac{x}{2} > 3 \quad \text{subtract 3}$$

$$x < -18 \quad \text{or} \quad x > 6 \quad \text{multiply by 2}$$

Property II yields the disjoint intervals $x \in (-\infty, -18) \cup (6, \infty)$ as the solution.



b. $|5x + 2| \geq -\frac{3}{2}$ original inequality

Since the absolute value of any quantity is always positive or zero, the solution for this inequality is all real numbers: $x \in \mathbb{R}$.

C. You've just learned how to solve greater than absolute value inequalities

Now try Exercises 39 through 54 ►

CAUTION ► Be sure you note the difference between the individual solutions of an absolute value equation, and the solution intervals that often result from solving absolute value inequalities. The solution $\{-2, 5\}$ indicates that both $x = -2$ and $x = 5$ are solutions, while the solution $[-2, 5)$ indicates that all numbers between -2 and 5, including -2, are solutions.

D. Applications Involving Absolute Value

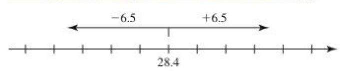
Applications of absolute value often involve finding a range of values for which a given statement is true. Many times, the equation or inequality used must be modeled after a given description or from given information, as in Example 6.

EXAMPLE 6 ▶ Solving Applications Involving Absolute Value Inequalities

For new cars, the number of miles per gallon (mpg) a car will get is heavily dependent on whether it is used mainly for short trips and city driving, or primarily on the highway for longer trips. For a certain car, the number of miles per gallon that a driver can expect varies by no more than 6.5 mpg above or below its field tested average of 28.4 mpg. What range of mileage values can a driver expect for this car?

Solution ▶ Field tested average: 28.4 mpg — *gather information*

mileage varies by no more than 6.5 mpg — *highlight key phrases*



make the problem visual

Let m represent the miles per gallon a driver can expect. Then the difference between m and 28.4 can be no more than 6.5, or $|m - 28.4| \le 6.5$. — *assign a variable*; *write an equation model*

$$|m - 28.4| \le 6.5 \quad \text{equation model}$$
$$-6.5 \le m - 28.4 \le 6.5 \quad \text{apply Property I}$$
$$21.9 \le m \le 34.9 \quad \text{add 28.4 to all three parts}$$

The mileage that a driver can expect ranges from a low of 21.9 mpg to a high of 34.9 mpg.

☑ **D.** You've just learned how to solve applications involving absolute value

Now try Exercises 57 through 64 ▶

TECHNOLOGY HIGHLIGHT

Absolute Value Equations and Inequalities

Graphing calculators can explore and solve inequalities in many different ways. Here we'll use a table of values and a *relational test.* To begin we'll consider the equation $2|x - 3| + 1 = 5$ by entering the left-hand side as Y_1 on the Y= screen. The calculator does not use absolute value bars the way they're written, and the equation is actually entered as $Y_1 = 2$ **abs** $(X - 3) + 1$ (see Figure 1.8). The **"abs("** notation is accessed by pressing MATH, ► **(NUM)** 1 (**option 1** gives only the left parenthesis, you must supply the right). Preset the TABLE as in the previous Highlight (page 81). By scrolling through the table (use the up ▲ and down ▼ arrows), we find $Y_1 = 5$ when $x = 1$ or $x = 5$ (see Figure 1.9).

Although we could also solve the *inequality* $2|x - 3| + 1 \le 5$ using the table (the solution interval is $x \in [1, 5]$), a relational test can help. Relational tests have the calculator return a "1" if a given statement is true, and a "0" otherwise. Enter $Y_2 = Y_1 \le 5$, by accessing Y_1 using VARS ► **(Y-VARS) 1:Function** ENTER, and the "≤" symbol using 2nd MATH **(TEST)** [the "less than or equal to" symbol is option 6]. Returning to the table shows $Y_1 \le 5$ is true for $1 \le x \le 5$ (see Figure 1.9).

Use a table and a relational test to help solve the following inequalities. Verify the result algebraically.

Figure 1.8

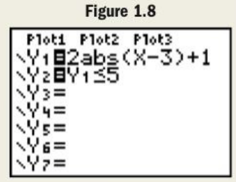
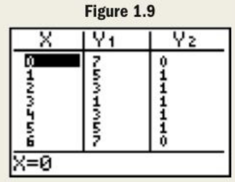


Figure 1.9



Exercise 1: $3|x + 1| - 2 \ge 7$ **Exercise 2:** $-2|x + 2| + 5 \ge -1$ **Exercise 3:** $-1 \le 4|x - 3| - 1$

1.3 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. When multiplying or dividing by a negative quantity, we ______ the inequality to maintain a true statement.

2. To write an absolute value equation or inequality in simplified form, we ______ the absolute value expression on one side.

3. The absolute value equation $|2x + 3| = 7$ is true when $2x + 3 =$ ______ or when $2x + 3 =$ ______.

4. The absolute value inequality $|3x - 6| < 12$ is true when $3x - 6 >$ ______ and $3x - 6 <$ ______.

Describe each solution set (assume $k > 0$). Justify your answer.

5. $|ax + b| < -k$

6. $|ax + b| > -k$

▶ DEVELOPING YOUR SKILLS

Solve each absolute value equation. Write the solution in set notation.

7. $2|m - 1| - 7 = 3$

8. $3|n - 5| - 14 = -2$

9. $-3|x + 5| + 6 = -15$

10. $-2|y + 3| - 4 = -14$

11. $2|4v + 5| - 6.5 = 10.3$

12. $7|2w + 5| + 6.3 = 11.2$

13. $-|7p - 3| + 6 = -5$

14. $-|3q + 4| + 3 = -5$

15. $-2|b| - 3 = -4$

16. $-3|c| - 5 = -6$

17. $-2|3x| - 17 = -5$

18. $-5|2y| - 14 = 6$

19. $-3\left|\frac{w}{2} + 4\right| - 1 = -4$

20. $-2\left|3 - \frac{v}{3}\right| + 1 = -5$

21. $8.7|p - 7.5| - 26.6 = 8.2$

22. $5.3|q + 9.2| + 6.7 = 43.8$

23. $8.7|-2.5x| - 26.6 = 8.2$

24. $5.3|1.25n| + 6.7 = 43.8$

Solve each absolute value inequality. Write solutions in interval notation.

25. $|x - 2| \le 7$

26. $|y + 1| \le 3$

27. $-3|m| - 2 > 4$

28. $-2|n| + 3 > 7$

29. $\frac{|5v + 1|}{4} + 8 < 9$

30. $\frac{|3w - 2|}{2} + 6 < 8$

31. $3|p + 4| + 5 < 8$

32. $5|q - 2| - 7 \le 8$

33. $|3b - 11| + 6 \le 9$

34. $|2c + 3| - 5 < 1$

35. $|4 - 3z| + 12 < 7$

36. $|2 - 7u| + 7 \le 4$

37. $\left|\frac{4x + 5}{3} - \frac{1}{2}\right| \le \frac{7}{6}$

38. $\left|\frac{2y - 3}{4} - \frac{3}{8}\right| < \frac{15}{16}$

39. $|n + 3| > 7$

40. $|m - 1| > 5$

41. $-2|w| - 5 \le -11$

42. $-5|v| - 3 \le -23$

43. $\frac{|q|}{2} - \frac{5}{6} \ge \frac{1}{3}$

44. $\frac{|p|}{5} + \frac{3}{2} \ge \frac{9}{4}$

45. $3|5 - 7d| + 9 \ge 15$

46. $5|2c + 7| + 1 \ge 11$

47. $|4z - 9| + 6 \ge 4$

48. $|5u - 3| + 8 > 6$

49. $4|5 - 2h| - 9 > 11$

50. $3|7 + 2k| - 11 > 10$

51. $-3.9|4q - 5| + 8.7 \le -22.5$

52. $0.9|2p + 7| - 16.11 \ge 10.89$

53. $2 < \left|-3m + \frac{4}{5}\right| - \frac{1}{5}$

54. $4 \le \left|\frac{5}{4} - 2n\right| - \frac{3}{4}$

▶ WORKING WITH FORMULAS

55. Spring Oscillation $|d - x| \le L$

A weight attached to a spring hangs at rest a distance of x in. off the ground. If the weight is pulled down (stretched) a distance of L inches and released, the weight begins to bounce and its distance d off the ground must satisfy the indicated formula. If x equals 4 ft and the spring is stretched 3 in. and released, solve the inequality to find what distances from the ground the weight will oscillate between.

56. A "Fair" Coin $\left|\frac{h - 50}{5}\right| < 1.645$

If we flipped a coin 100 times, we expect "heads" to come up about 50 times if the coin is "fair." In a study of probability, it can be shown that the number of heads h that appears in such an experiment must satisfy the given inequality to be considered "fair." (a) Solve this inequality for h. (b) If you flipped a coin 100 times and obtained 40 heads, is the coin "fair"?

▶ APPLICATIONS

Solve each application of absolute value.

57. Altitude of jet stream: To take advantage of the jet stream, an airplane must fly at a height h (in feet) that satisfies the inequality $|h - 35{,}050| \le 2550$. Solve the inequality and determine if an altitude of 34,000 ft will place the plane in the jet stream.

58. Quality control tests: In order to satisfy quality control, the marble columns a company produces must earn a stress test score S that satisfies the inequality $|S - 17{,}750| \le 275$. Solve the inequality and determine if a score of 17,500 is in the passing range.

59. Submarine depth: The sonar operator on a submarine detects an old World War II submarine net and must decide to detour over or under the net. The computer gives him a depth model $|d - 394| - 20 > 164$, where d is the depth in feet that represents safe passage. At what depth should the submarine travel to go under or over the net? Answer using simple inequalities.

60. Optimal fishing depth: When deep-sea fishing, the optimal depths d (in feet) for catching a certain type of fish satisfy the inequality $28|d - 350| - 1400 < 0$. Find the range of depths that offer the best fishing. Answer using simple inequalities.

For Exercises 61 through 64, (a) develop a model that uses an absolute value inequality, and (b) solve.

61. Stock value: My stock in MMM Corporation fluctuated a great deal in 2009, but never by more than \$3.35 from its current value. If the stock is worth \$37.58 today, what was its range in 2009?

62. Traffic studies: On a given day, the volume of traffic at a busy intersection averages 726 cars per hour (cph). During rush hour the volume is much higher, during "off hours" much lighter. Find the range of this volume if it never varies by more than 235 cph from the average.

63. Physical training for recruits: For all recruits in the 3rd Armored Battalion, the average number of sit-ups is 125. For an individual recruit, the amount varies by no more than 23 sit-ups from the battalion average. Find the range of sit-ups for this battalion.

64. Computer consultant salaries: The national average salary for a computer consultant is \$53,336. For a large computer firm, the salaries offered to their employees varies by no more than \$11,994 from this national average. Find the range of salaries offered by this company.

65. According to the official rules for golf, baseball, pool, and bowling, (a) golf balls must be within 0.03 mm of $d = 42.7$ mm, (b) baseballs must be within 1.01 mm of $d = 73.78$ mm, (c) billiard balls must be within 0.127 mm of $d = 57.150$ mm, and (d) bowling balls must be within 12.05 mm of $d = 2171.05$ mm. Write each statement using an absolute value inequality, then (e) determine which sport gives the least tolerance t $\left(t = \frac{\text{width of interval}}{\textit{average value}}\right)$ for the diameter of the ball.

66. The machines that fill boxes of breakfast cereal are programmed to fill each box within a certain tolerance. If the box is overfilled, the company loses money. If it is underfilled, it is considered unsuitable for sale. Suppose that boxes marked "14 ounces" of cereal must be filled to within 0.1 oz. Write this relationship as an absolute value inequality, then solve the inequality and explain what your answer means. Let W represent weight.

▶ EXTENDING THE CONCEPT

67. Determine the value or values (if any) that will make the equation or inequality true.

a. $|x| + x = 8$ **b.** $|x - 2| \le \frac{x}{2}$

c. $x - |x| = x + |x|$ **d.** $|x + 3| \ge 6x$

e. $|2x + 1| = x - 3$

68. The equation $|5 - 2x| = |3 + 2x|$ has only one solution. Find it and explain why there is only one.

▶ MAINTAINING YOUR SKILLS

69. **(R.4)** Factor the expression completely: $18x^3 + 21x^2 - 60x$.

70. **(1.1)** Solve $V^2 = \frac{2W}{C\rho A}$ for ρ (physics).

71. **(R.6)** Simplify $\frac{-1}{3 + \sqrt{3}}$ by rationalizing the denominator. State the result in exact form and approximate form (to hundredths):

72. **(1.2)** Solve the inequality, then write the solution set in interval notation:

$-3(2x - 5) > 2(x + 1) - 7.$

MID-CHAPTER CHECK

1. Solve each equation. If the equation is an identity or contradiction, so state and name the solution set.

a. $\frac{r}{3} + 5 = 2$

b. $5(2x - 1) + 4 = 9x - 7$

c. $m - 2(m + 3) = 1 - (m + 7)$

d. $\frac{1}{5}y + 3 = \frac{3}{2}y - 2$

e. $\frac{1}{2}(5j - 2) = \frac{3}{2}(j - 4) + j$

f. $0.6(x - 3) + 0.3 = 1.8$

Solve for the variable specified.

2. $H = -16t^2 + v_0t$; for v_0

3. $S = 2\pi x^2 + \pi x^2 y$; for x

4. Solve each inequality and graph the solution set.

a. $-5x + 16 \le 11$ or $3x + 2 \le -4$

b. $\frac{1}{2} < \frac{1}{12}x - \frac{5}{6} \le \frac{3}{4}$

5. Determine the domain of each expression. Write your answer in interval notation.

a. $\frac{3x + 1}{2x - 5}$ **b.** $\sqrt{17 - 6x}$

6. Solve the following absolute value equations. Write the solution in set notation.

a. $\frac{2}{3}|d - 5| + 1 = 7$ **b.** $5 - |s + 3| = \frac{11}{2}$

7. Solve the following absolute value inequalities. Write solutions in interval notation.

a. $3|q + 4| - 2 < 10$

b. $\left|\frac{x}{3} + 2\right| + 5 \leq 5$

8. Solve the following absolute value inequalities. Write solutions in interval notation.

a. $3.1|d - 2| + 1.1 \geq 7.3$

b. $\frac{|1 - y|}{3} + 2 > \frac{11}{2}$

c. $-5|k - 2| + 3 < 4$

9. Motocross: An enduro motocross motorcyclist averages 30 mph through the first part of a 115-mi course, and 50 mph though the second part. If the rider took 2 hr and 50 min to complete the course, how long was she on the first part?

10. Kiteboarding: With the correct sized kite, a person can kiteboard when the wind is blowing at a speed w (in mph) that satisfies the inequality $|w - 17| \leq 9$. Solve the inequality and determine if a person can kiteboard with a windspeed of 9 mph.

REINFORCING BASIC CONCEPTS

Using Distance to Understand Absolute Value Equations and Inequalities

In Section R.1 we noted that for any two numbers a and b on the number line, *the distance between a and b* is written $|a - b|$ or $|b - a|$. In exactly the same way, the equation $|x - 3| = 4$ can be read, "the distance between 3 and an unknown number is equal to 4." The advantage of reading it in this way (instead of *the absolute value of x minus 3 is 4*), is that a much clearer *visualization* is formed, giving a constant reminder there are two solutions. In diagram form we have Figure 1.10.

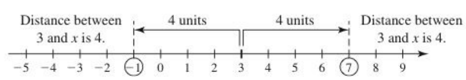


Figure 1.10

From this we note the solution is $x = -1$ or $x = 7$.

In the case of an inequality such as $|x + 2| \leq 3$, we rewrite the inequality as $|x - (-2)| \leq 3$ and read it, "the distance between -2 and an unknown number is less than or equal to 3." With some practice, visualizing this relationship mentally enables a quick statement of the solution: $x \in [-5, 1]$. In diagram form we have Figure 1.11.

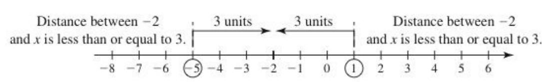


Figure 1.11

Equations and inequalities where the coefficient of x is not 1 still lend themselves to this form of conceptual understanding. For $|2x - 1| \geq 3$ we read, "the distance between 1 and twice an unknown number is greater than or equal to 3." On the number line (Figure 1.12), the number 3 units to the right of 1 is 4, and the number 3 units to the left of 1 is -2.

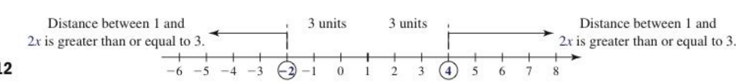


Figure 1.12

For $2x \leq -2$, $x \leq -1$, and for $2x \geq 4$, $x \geq 2$, and the solution is $x \in (-\infty, -1] \cup [2, \infty)$.

Attempt to solve the following equations and inequalities by visualizing a number line. Check all results algebraically.

Exercise 1: $|x - 2| = 5$

Exercise 2: $|x + 1| \leq 4$

Exercise 3: $|2x - 3| \geq 5$

1.4 Complex Numbers

Learning Objectives

In Section 1.4 you will learn how to:

- **A.** Identify and simplify imaginary and complex numbers
- **B.** Add and subtract complex numbers
- **C.** Multiply complex numbers and find powers of *i*
- **D.** Divide complex numbers

For centuries, even the most prominent mathematicians refused to work with equations like $x^2 + 1 = 0$. Using the principal of square roots gave the "solutions" $x = \sqrt{-1}$ and $x = -\sqrt{-1}$, which they found baffling and mysterious, since there is no real number whose square is -1. In this section, we'll see how this "mystery" was finally resolved.

A. Identifying and Simplifying Imaginary and Complex Numbers

The equation $x^2 = -1$ has no real solutions, since the square of any real number is positive. But if we apply the principle of square roots we get $x = \sqrt{-1}$ and $x = -\sqrt{-1}$, which seem to check when substituted into the original equation:

$$x^2 + 1 = 0 \quad \text{original equation}$$

$$(1) \quad (\sqrt{-1})^2 + 1 = 0 \quad \text{substitute } \sqrt{-1} \text{ for } x$$

$$-1 + 1 = 0 \checkmark \quad \text{answer "checks"}$$

$$(2) \quad (-\sqrt{-1})^2 + 1 = 0 \quad \text{substitute } -\sqrt{-1} \text{ for } x$$

$$-1 + 1 = 0 \checkmark \quad \text{answer "checks"}$$

This observation likely played a part in prompting Renaissance mathematicians to study such numbers in greater depth, as they reasoned that while these were not *real number* solutions, they must be *solutions of a new and different kind*. Their study eventually resulted in the introduction of the set of **imaginary numbers** and the **imaginary unit *i*,** as follows.

Imaginary Numbers and the Imaginary Unit

- Imaginary numbers are those of the form $\sqrt{-k}$, where k is a positive real number.
- The imaginary unit i represents the number whose square is -1:

$$i^2 = -1 \text{ and } i = \sqrt{-1}$$

WORTHY OF NOTE

It was René Descartes (in 1637) who first used the term *imaginary* to describe these numbers; Leonhard Euler (in 1777) who introduced the letter *i* to represent $\sqrt{-1}$; and Carl F. Gauss (in 1831) who first used the phrase *complex number* to describe solutions that had both a real number part and an imaginary part. For more on complex numbers and their story, see www.mhhe.com/coburn

As a convenience to understanding and working with imaginary numbers, we rewrite them in terms of i, allowing that the product property of radicals ($\sqrt{AB} = \sqrt{A}\sqrt{B}$) still applies if *only one* of the radicands is negative. For $\sqrt{-3}$, we have $\sqrt{-1 \cdot 3} = \sqrt{-1}\sqrt{3} = i\sqrt{3}$. In general, we simply state the following property.

Rewriting Imaginary Numbers

- For any positive real number k, $\sqrt{-k} = i\sqrt{k}$.

For $\sqrt{-20}$ we have:

$$\begin{aligned}\sqrt{-20} &= i\sqrt{20}\\ &= i\sqrt{4 \cdot 5}\\ &= 2i\sqrt{5},\end{aligned}$$

and we say the expression has been *simplified and written in terms of i*. Note that we've written the result with the unit "*i*" *in front of the radical* to prevent it being interpreted as being *under the radical.* In symbols, $2i\sqrt{5} = 2\sqrt{5}i \neq 2\sqrt{5i}$.

The solutions to $x^2 = -1$ also serve to illustrate that for $k > 0$, there are two solutions to $x^2 = -k$, namely, $i\sqrt{k}$ and $-i\sqrt{k}$. In other words, every negative number has two square roots, one positive and one negative. The first of these, $i\sqrt{k}$, is called the **principal square root** of $-k$.

EXAMPLE 1 ► **Simplifying Imaginary Numbers**

Rewrite the imaginary numbers in terms of i and simplify if possible.

a. $\sqrt{-7}$ **b.** $\sqrt{-81}$ **c.** $\sqrt{-24}$ **d.** $-3\sqrt{-16}$

Solution ►

a. $\sqrt{-7} = i\sqrt{7}$

b. $\sqrt{-81} = i\sqrt{81} = 9i$

c. $\sqrt{-24} = i\sqrt{24} = i\sqrt{4 \cdot 6} = 2i\sqrt{6}$

d. $-3\sqrt{-16} = -3i\sqrt{16} = -3i(4) = -12i$

Now try Exercises 7 through 12 ►

EXAMPLE 2 ► **Writing an Expression in Terms of *i***

The numbers $x = \frac{-6 + \sqrt{-16}}{2}$ and $x = \frac{-6 - \sqrt{-16}}{2}$ are not real, but are known to be solutions of $x^2 + 6x + 13 = 0$. Simplify $\frac{-6 + \sqrt{-16}}{2}$.

Solution ► Using the i notation, we have

$$\frac{-6 + \sqrt{-16}}{2} = \frac{-6 + i\sqrt{16}}{2} \quad \text{write in } i \text{ notation}$$
$$= \frac{-6 + 4i}{2} \quad \text{simplify}$$
$$= \frac{2(-3 + 2i)}{2} \quad \text{factor numerator}$$
$$= -3 + 2i \quad \text{reduce}$$

Now try Exercises 13 through 16 ►

WORTHY OF NOTE

The expression $\frac{-6 + 4i}{2}$ from the solution of Example 2 can also be simplified by rewriting it as two separate terms, then simplifying each term: $\frac{-6 + 4i}{2} = \frac{-6}{2} + \frac{4i}{2} = -3 + 2i$.

The result in Example 2 contains both a **real number part** (-3) and an **imaginary part** $(2i)$. Numbers of this type are called **complex numbers.**

Complex Numbers

Complex numbers are numbers that can be written in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

The expression $a + bi$ is called the **standard form** of a complex number. From this definition we note that all real numbers are also complex numbers, since $a + 0i$ is complex with $b = 0$. In addition, all imaginary numbers are complex numbers, since $0 + bi$ is a complex number with $a = 0$.

EXAMPLE 3 ► **Writing Complex Numbers in Standard Form**

Write each complex number in the form $a + bi$, and identify the values of a and b.

a. $2 + \sqrt{-49}$ **b.** $\sqrt{-12}$ **c.** 7 **d.** $\frac{4 + 3\sqrt{-25}}{20}$

Solution ►

a. $2 + \sqrt{-49} = 2 + i\sqrt{49} = 2 + 7i$; $a = 2, b = 7$

b. $\sqrt{-12} = 0 + i\sqrt{12} = 0 + 2i\sqrt{3}$; $a = 0, b = 2\sqrt{3}$

c. $7 = 7 + 0i$
$a = 7, b = 0$

d. $\dfrac{4 + 3\sqrt{-25}}{20} = \dfrac{4 + 3i\sqrt{25}}{20}$
$= \dfrac{4 + 15i}{20}$
$= \dfrac{1}{5} + \dfrac{3}{4}i$
$a = \dfrac{1}{5}, b = \dfrac{3}{4}$

Now try Exercises 17 through 24 ►

A. You've just learned how to identify and simplify imaginary and complex numbers

Complex numbers complete the development of our "numerical landscape." Sets of numbers and their relationships are represented in Figure 1.13, which shows how some sets of numbers are nested within larger sets and highlights the fact that complex numbers consist of a real number part (any number within the orange rectangle), and an imaginary number part (any number within the yellow rectangle).

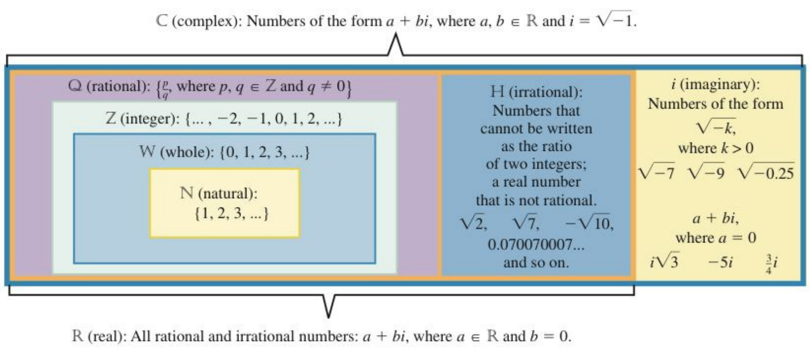


Figure 1.13

B. Adding and Subtracting Complex Numbers

The sum and difference of two polynomials is computed by identifying and combining like terms. The sum or difference of two complex numbers is computed in a similar way, by adding the real number parts from each, and the imaginary parts from each. Notice in Example 4 that the commutative, associative, and distributive properties also apply to complex numbers.

EXAMPLE 4 ► Adding and Subtracting Complex Numbers

Perform the indicated operation and write the result in $a + bi$ form.

a. $(2 + 3i) + (-5 + 2i)$ **b.** $(-5 - 4i) - (-2 - \sqrt{2}i)$

Solution ►

a.
$(2 + 3i) + (-5 + 2i)$ original sum
$= 2 + 3i + (-5) + 2i$ distribute
$= 2 + (-5) + 3i + 2i$ commute terms
$= [2 + (-5)] + (3i + 2i)$ group like terms
$= -3 + 5i$ result

b.
$(-5 - 4i) - (-2 - \sqrt{2}i)$ original difference
$= -5 - 4i + 2 + \sqrt{2}i$ distribute
$= -5 + 2 + (-4i) + \sqrt{2}i$ commute terms
$= (-5 + 2) + [(-4i) + \sqrt{2}i]$ group like terms
$= -3 + (-4 + \sqrt{2})i$ result

Now try Exercises 25 through 30 ►

B. You've just learned how to add and subtract complex numbers

C. Multiplying Complex Numbers; Powers of *i*

The product of two complex numbers is computed using the distributive property and the F-O-I-L process in the same way we apply these to binomials. If any result gives a factor of i^2, remember that $i^2 = -1$.

EXAMPLE 5 ▶ Multiplying Complex Numbers

Find the indicated product and write the answer in $a + bi$ form.

a. $\sqrt{-4}\sqrt{-9}$ **b.** $\sqrt{-6}\,(2 + \sqrt{-3})$
c. $(6 - 5i)(4 + i)$ **d.** $(2 + 3i)(2 - 3i)$

Solution ▶

a.
$$\begin{aligned}\sqrt{-4}\sqrt{-9} &= i\sqrt{4}\cdot i\sqrt{9} && \text{rewrite in terms of } i\\ &= 2i\cdot 3i && \text{simplify}\\ &= 6i^2 && \text{multiply}\\ &= -6 + 0i && \text{result } (i^2 = -1)\end{aligned}$$

b.
$$\begin{aligned}\sqrt{-6}\,(2 + \sqrt{-3}) &= i\sqrt{6}(2 + i\sqrt{3}) && \text{rewrite in terms of } i\\ &= 2i\sqrt{6} + i^2\sqrt{18} && \text{distribute}\\ &= 2i\sqrt{6} + (-1)\sqrt{9}\sqrt{2} && i^2 = -1\\ &= 2i\sqrt{6} - 3\sqrt{2} && \text{simplify}\\ &= -3\sqrt{2} + 2i\sqrt{6} && \text{standard form}\end{aligned}$$

c.
$$\begin{aligned}&(6 - 5i)(4 + i)\\ &= (6)(4) + 6i + (-5i)(4) + (-5i)(i) && \text{F-O-I-L}\\ &= 24 + 6i + (-20i) + (-5)i^2 && i\cdot i = i^2\\ &= 24 + 6i + (-20i) + (-5)(-1) && i^2 = -1\\ &= 29 - 14i && \text{result}\end{aligned}$$

d.
$$\begin{aligned}(2 + 3i)(2 - 3i) &= (2)^2 - (3i)^2 && (A + B)(A - B) = A^2 - B^2\\ &= 4 - 9i^2 && (3i)^2 = 9i^2\\ &= 4 - 9(-1) && i^2 = -1\\ &= 13 + 0i && \text{result}\end{aligned}$$

Now try Exercises 31 through 48 ▶

CAUTION ▶ When computing with imaginary and complex numbers, always write the square root of a negative number in terms of *i* before you begin, as shown in Examples 5(a) and 5(b). Otherwise we get conflicting results, since $\sqrt{-4}\,\sqrt{-9} = \sqrt{36} = 6$ if we multiply the radicands first, which is an incorrect result because the original factors were imaginary. **See Exercise 80.**

Recall that expressions $2x + 5$ and $2x - 5$ are called binomial conjugates. In the same way, $a + bi$ and $a - bi$ are called **complex conjugates.** Note from Example 5(d) that the *product* of the complex number $a + bi$ with its complex conjugate $a - bi$ *is a real number.* This relationship is useful when rationalizing expressions with a complex number in the denominator, and we generalize the result as follows:

WORTHY OF NOTE

Notice that the product of a complex number and its conjugate also gives us a method for *factoring the sum of two squares* using complex numbers! For the expression $x^2 + 4$, the factored form would be $(x + 2i)(x - 2i)$. For more on this idea, **see Exercise 79.**

Product of Complex Conjugates

For a complex number $a + bi$ and its conjugate $a - bi$, their product $(a + bi)(a - bi)$ is the real number $a^2 + b^2$;

$$(a + bi)(a - bi) = a^2 + b^2$$

Showing that $(a + bi)(a - bi) = a^2 + b^2$ is left as an exercise (see Exercise 79), but from here on, when asked to compute the product of complex conjugates, simply refer to the formula as illustrated here: $(-3 + 5i)(-3 - 5i) = (-3)^2 + 5^2$ or 34.

These operations on complex numbers enable us to verify complex solutions by substitution, in the same way we verify solutions for real numbers. In Example 2 we stated that $x = -3 + 2i$ was one solution to $x^2 + 6x + 13 = 0$. This is verified here.

EXAMPLE 6 ► **Checking a Complex Root by Substitution**

Verify that $x = -3 + 2i$ is a solution to $x^2 + 6x + 13 = 0$.

Solution ►

$$\begin{aligned} x^2 + 6x + 13 &= 0 && \text{original equation} \\ (-3 + 2i)^2 + 6(-3 + 2i) + 13 &= 0 && \text{substitute } -3 + 2i \text{ for } x \\ (-3)^2 + 2(-3)(2i) + (2i)^2 - 18 + 12i + 13 &= 0 && \text{square and distribute} \\ 9 - 12i + 4i^2 + 12i - 5 &= 0 && \text{simplify} \\ 9 + (-4) - 5 &= 0 && \text{combine terms } (12i - 12i = 0;\ i^2 = -1) \\ 0 &= 0 \checkmark \end{aligned}$$

Now try Exercises 49 through 56 ►

EXAMPLE 7 ► **Checking a Complex Root by Substitution**

Show that $x = 2 - i\sqrt{3}$ is a solution of $x^2 - 4x = -7$.

Solution ►

$$\begin{aligned} x^2 - 4x &= -7 && \text{original equation} \\ (2 - i\sqrt{3})^2 - 4(2 - i\sqrt{3}) &= -7 && \text{substitute } 2 - i\sqrt{3} \text{ for } x \\ 4 - 4i\sqrt{3} + (i\sqrt{3})^2 - 8 + 4i\sqrt{3} &= -7 && \text{square and distribute} \\ 4 - 4i\sqrt{3} - 3 - 8 + 4i\sqrt{3} &= -7 && (i\sqrt{3})^2 = -3 \\ -7 &= -7 \checkmark && \text{solution checks} \end{aligned}$$

Now try Exercises 57 through 60 ►

The imaginary unit i has another interesting and useful property. Since $i = \sqrt{-1}$ and $i^2 = -1$, we know that $i^3 = i^2 \cdot i = (-1)i = -i$ and $i^4 = (i^2)^2 = 1$. We can now simplify any *higher power of i* by rewriting the expression in terms of i^4.

$$\begin{aligned} i^5 &= i^4 \cdot i = i \\ i^6 &= i^4 \cdot i^2 = -1 \\ i^7 &= i^4 \cdot i^3 = -i \\ i^8 &= (i^4)^2 = 1 \end{aligned}$$

Notice the powers of i "cycle through" the four values $i, -1, -i$ and 1. In more advanced classes, powers of complex numbers play an important role, and next we learn to reduce higher powers using the power property of exponents and $i^4 = 1$. Essentially, we divide the exponent on i by 4, then use the remainder to compute the value of the expression. For i^{35}, $35 \div 4 = 8$ remainder 3, showing $i^{35} = (i^4)^8 \cdot i^3 = -i$.

EXAMPLE 8 ► **Simplifying Higher Powers of *i***

Simplify:

a. i^{22} **b.** i^{28} **c.** i^{57} **d.** i^{75}

Solution ►

a. $i^{22} = (i^4)^5 \cdot (i^2) = (1)^5(-1) = -1$

b. $i^{28} = (i^4)^7 = (1)^7 = 1$

C. You've just learned how to multiply complex numbers and find powers of i

c. $i^{57} = (i^4)^{14} \cdot i = (1)^{14} i = i$

d. $i^{75} = (i^4)^{18} \cdot (i^3) = (1)^{18}(-i) = -i$

Now try Exercises 61 and 62 ►

D. Division of Complex Numbers

Since $i = \sqrt{-1}$, expressions like $\frac{3 - i}{2 + i}$ actually have a radical in the denominator. To divide complex numbers, we simply apply our earlier method of rationalizing denominators (Section R.6), but this time using a *complex* conjugate.

EXAMPLE 9 ► Dividing Complex Numbers

Divide and write each result in $a + bi$ form.

a. $\frac{2}{5 - i}$ b. $\frac{3 - i}{2 + i}$ c. $\frac{6 + \sqrt{-36}}{3 + \sqrt{-9}}$

Solution ►

a. $$\frac{2}{5 - i} = \frac{2}{5 - i} \cdot \frac{5 + i}{5 + i} = \frac{2(5 + i)}{5^2 + 1^2} = \frac{10 + 2i}{26} = \frac{10}{26} + \frac{2}{26}i = \frac{5}{13} + \frac{1}{13}i$$

b. $$\frac{3 - i}{2 + i} = \frac{3 - i}{2 + i} \cdot \frac{2 - i}{2 - i} = \frac{6 - 3i - 2i + i^2}{2^2 + 1^2} = \frac{6 - 5i + (-1)}{5} = \frac{5 - 5i}{5} = \frac{5}{5} - \frac{5i}{5} = 1 - i$$

c. $$\frac{6 + \sqrt{-36}}{3 + \sqrt{-9}} = \frac{6 + i\sqrt{36}}{3 + i\sqrt{9}}$$ convert to i notation

$$= \frac{6 + 6i}{3 + 3i}$$ simplify

The expression can be further simplified by reducing common factors.

$$= \frac{6(1 + i)}{3(1 + i)} = 2$$ factor and reduce

Now try Exercises 63 through 68 ►

Operations on complex numbers can be checked using inverse operations, just as we do for real numbers. To check the answer $1 - i$ from Example 9(b), we multiply it by the divisor:

$$(1 - i)(2 + i) = 2 + i - 2i - i^2 = 2 - i - (-1) = 2 - i + 1 = 3 - i \checkmark$$

D. You've just learned how to divide complex numbers

1.4 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. Given the complex number $3 + 2i$, its complex conjugate is _____.

2. The product $(3 + 2i)(3 - 2i)$ gives the real number _____.

3. If the expression $\frac{4 + 6i\sqrt{2}}{2}$ is written in the standard form $a + bi$, then $a =$ _____ and $b =$ _____.

4. For $i = \sqrt{-1}$, $i^2 =$ __, $i^4 =$ __, $i^6 =$ __, and $i^8 =$ __, $i^3 =$ __, $i^5 =$ __, $i^7 =$ __, and $i^9 =$ __.

5. Discuss/Explain which is correct:
a. $\sqrt{-4} \cdot \sqrt{-9} = \sqrt{(-4)(-9)} = \sqrt{36} = 6$
b. $\sqrt{-4} \cdot \sqrt{-9} = 2i \cdot 3i = 6i^2 = -6$

6. Compare/Contrast the product $(1 + \sqrt{2})(1 - \sqrt{3})$ with the product $(1 + i\sqrt{2})(1 - i\sqrt{3})$. What is the same? What is different?

▶ DEVELOPING YOUR SKILLS

Simplify each radical (if possible). If imaginary, rewrite in terms of *i* and simplify.

7. a. $\sqrt{-16}$ **b.** $\sqrt{-49}$ **c.** $\sqrt{27}$ **d.** $\sqrt{72}$

8. a. $\sqrt{-81}$ **b.** $\sqrt{-169}$ **c.** $\sqrt{64}$ **d.** $\sqrt{98}$

9. a. $-\sqrt{-18}$ **b.** $-\sqrt{-50}$ **c.** $3\sqrt{-25}$ **d.** $2\sqrt{-9}$

10. a. $-\sqrt{-32}$ **b.** $-\sqrt{-75}$ **c.** $3\sqrt{-144}$ **d.** $2\sqrt{-81}$

11. a. $\sqrt{-19}$ **b.** $\sqrt{-31}$ **c.** $\sqrt{\frac{-12}{25}}$ **d.** $\sqrt{\frac{-9}{32}}$

12. a. $\sqrt{-17}$ **b.** $\sqrt{-53}$ **c.** $\sqrt{\frac{-45}{36}}$ **d.** $\sqrt{\frac{-49}{75}}$

Write each complex number in the standard form *a* + *bi* and clearly identify the values of *a* and *b*.

13. a. $\frac{2 + \sqrt{-4}}{2}$ **b.** $\frac{6 + \sqrt{-27}}{3}$

14. a. $\frac{16 - \sqrt{-8}}{2}$ **b.** $\frac{4 + 3\sqrt{-20}}{2}$

15. a. $\frac{8 + \sqrt{-16}}{2}$ **b.** $\frac{10 - \sqrt{-50}}{5}$

16. a. $\frac{6 - \sqrt{-72}}{4}$ **b.** $\frac{12 + \sqrt{-200}}{8}$

17. a. 5 **b.** $3i$

18. a. -2 **b.** $-4i$

19. a. $2\sqrt{-81}$ **b.** $\frac{\sqrt{-32}}{8}$

20. a. $-3\sqrt{-36}$ **b.** $\frac{\sqrt{-75}}{15}$

21. a. $4 + \sqrt{-50}$ **b.** $-5 + \sqrt{-27}$

22. a. $-2 + \sqrt{-48}$ **b.** $7 + \sqrt{-75}$

23. a. $\frac{14 + \sqrt{-98}}{8}$ **b.** $\frac{5 + \sqrt{-250}}{10}$

24. a. $\frac{21 + \sqrt{-63}}{12}$ **b.** $\frac{8 + \sqrt{-27}}{6}$

Perform the addition or subtraction. Write the result in *a* + *bi* form.

25. a. $(12 - \sqrt{-4}) + (7 + \sqrt{-9})$
b. $(3 + \sqrt{-25}) + (-1 - \sqrt{-81})$
c. $(11 + \sqrt{-108}) - (2 - \sqrt{-48})$

26. a. $(-7 - \sqrt{-72}) + (8 + \sqrt{-50})$
b. $(\sqrt{3} + \sqrt{-2}) - (\sqrt{12} + \sqrt{-8})$
c. $(\sqrt{20} - \sqrt{-3}) + (\sqrt{5} - \sqrt{-12})$

27. a. $(2 + 3i) + (-5 - i)$
b. $(5 - 2i) + (3 + 2i)$
c. $(6 - 5i) - (4 + 3i)$

28. a. $(-2 + 5i) + (3 - i)$
b. $(7 - 4i) - (2 - 3i)$
c. $(2.5 - 3.1i) + (4.3 + 2.4i)$

29. **a.** $(3.7 + 6.1i) - (1 + 5.9i)$
b. $\left(8 + \frac{3}{4}i\right) - \left(-7 + \frac{2}{3}i\right)$
c. $\left(-6 - \frac{5}{8}i\right) + \left(4 + \frac{1}{2}i\right)$

30. **a.** $(9.4 - 8.7i) - (6.5 + 4.1i)$
b. $\left(3 + \frac{3}{5}i\right) - \left(-11 + \frac{7}{15}i\right)$
c. $\left(-4 - \frac{5}{6}i\right) + \left(13 + \frac{3}{8}i\right)$

Multiply and write your answer in $a + bi$ form.

31. **a.** $5i \cdot (-3i)$ **b.** $(4i)(-4i)$

32. **a.** $3(2 - 3i)$ **b.** $-7(3 + 5i)$

33. **a.** $-7i(5 - 3i)$ **b.** $6i(-3 + 7i)$

34. **a.** $(-4 - 2i)(3 + 2i)$ **b.** $(2 - 3i)(-5 + i)$

35. **a.** $(-3 + 2i)(2 + 3i)$ **b.** $(3 + 2i)(1 + i)$

36. **a.** $(5 + 2i)(-7 + 3i)$ **b.** $(4 - i)(7 + 2i)$

For each complex number, name the complex conjugate. Then find the product.

37. **a.** $4 + 5i$ **b.** $3 - i\sqrt{2}$

38. **a.** $2 - i$ **b.** $-1 + i\sqrt{5}$

39. **a.** $7i$ **b.** $\frac{1}{2} - \frac{2}{3}i$

40. **a.** $-5i$ **b.** $\frac{3}{4} + \frac{1}{5}i$

Compute the special products and write your answer in $a + bi$ form.

41. **a.** $(4 - 5i)(4 + 5i)$
b. $(7 - 5i)(7 + 5i)$

42. **a.** $(-2 - 7i)(-2 + 7i)$
b. $(2 + i)(2 - i)$

43. **a.** $(3 - i\sqrt{2})(3 + i\sqrt{2})$
b. $(\frac{1}{6} + \frac{2}{3}i)(\frac{1}{6} - \frac{2}{3}i)$

44. **a.** $(5 + i\sqrt{3})(5 - i\sqrt{3})$
b. $(\frac{1}{2} + \frac{3}{4}i)(\frac{1}{2} - \frac{3}{4}i)$

45. **a.** $(2 + 3i)^2$ **b.** $(3 - 4i)^2$

46. **a.** $(2 - i)^2$ **b.** $(3 - i)^2$

47. **a.** $(-2 + 5i)^2$ **b.** $(3 + i\sqrt{2})^2$

48. **a.** $(-2 - 5i)^2$ **b.** $(2 - i\sqrt{3})^2$

Use substitution to determine if the value shown is a solution to the given equation.

49. $x^2 + 36 = 0; x = -6$

50. $x^2 + 16 = 0; x = -4$

51. $x^2 + 49 = 0; x = -7i$

52. $x^2 + 25 = 0; x = -5i$

53. $(x - 3)^2 = -9; x = 3 - 3i$

54. $(x + 1)^2 = -4; x = -1 + 2i$

55. $x^2 - 2x + 5 = 0; x = 1 - 2i$

56. $x^2 + 6x + 13 = 0; x = -3 + 2i$

57. $x^2 - 4x + 9 = 0; x = 2 + i\sqrt{5}$

58. $x^2 - 2x + 4 = 0; x = 1 - \sqrt{3}\,i$

59. Show that $x = 1 + 4i$ is a solution to $x^2 - 2x + 17 = 0$. Then show its complex conjugate $1 - 4i$ is also a solution.

60. Show that $x = 2 - 3\sqrt{2}\,i$ is a solution to $x^2 - 4x + 22 = 0$. Then show its complex conjugate $2 + 3\sqrt{2}\,i$ is also a solution.

Simplify using powers of *i*.

61. **a.** i^{48} **b.** i^{26} **c.** i^{39} **d.** i^{53}

62. **a.** i^{36} **b.** i^{50} **c.** i^{19} **d.** i^{65}

Divide and write your answer in $a + bi$ form. Check your answer using multiplication.

63. **a.** $\frac{-2}{\sqrt{-49}}$ **b.** $\frac{4}{\sqrt{-25}}$

64. **a.** $\frac{2}{1 - \sqrt{-4}}$ **b.** $\frac{3}{2 + \sqrt{-9}}$

65. **a.** $\frac{7}{3 + 2i}$ **b.** $\frac{-5}{2 - 3i}$

66. **a.** $\frac{6}{1 + 3i}$ **b.** $\frac{7}{7 - 2i}$

67. **a.** $\frac{3 + 4i}{4i}$ **b.** $\frac{2 - 3i}{3i}$

68. **a.** $\frac{-4 + 8i}{2 - 4i}$ **b.** $\frac{3 - 2i}{-6 + 4i}$

▶ WORKING WITH FORMULAS

69. Absolute value of a complex number: $|a + bi| = \sqrt{a^2 + b^2}$

The absolute value of any complex number $a + bi$ (sometimes called the *modulus* of the number) is computed by taking the square root of the sum of the squares of a and b. Find the absolute value of the given complex numbers.

a. $|2 + 3i|$ **b.** $|4 - 3i|$
c. $|3 + \sqrt{2}\,i|$

70. Binomial cubes: $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

The cube of any binomial can be found using the formula shown, where A and B are the terms of the binomial. Use the formula to compute $(1 - 2i)^3$ (note $A = 1$ and $B = -2i$).

▶ APPLICATIONS

71. Dawn of imaginary numbers: In a day when imaginary numbers were imperfectly understood, Girolamo Cardano (1501–1576) once posed the problem, "Find two numbers that have a sum of 10 and whose product is 40." In other words, $A + B = 10$ and $AB = 40$. Although the solution is routine today, at the time the problem posed an enormous challenge. Verify that $A = 5 + \sqrt{15}i$ and $B = 5 - \sqrt{15}i$ satisfy these conditions.

72. Verifying calculations using *i*: Suppose Cardano had said, "Find two numbers that have a sum of 4 and a product of 7" (see Exercise 71). Verify that $A = 2 + \sqrt{3}i$ and $B = 2 - \sqrt{3}i$ satisfy these conditions.

Although it may seem odd, imaginary numbers have several applications in the real world. Many of these involve a study of electrical circuits, in particular *alternating current* or AC circuits. Briefly, the components of an AC circuit are current *I* (in amperes), voltage *V* (in volts), and the impedance *Z* (in ohms). The impedance of an electrical circuit is a measure of the total opposition to the flow of current through the circuit and is calculated as $Z = R + iX_L - iX_C$ where *R* represents a pure resistance, X_C represents the capacitance, and X_L represents the inductance. Each of these is also measured in ohms (symbolized by Ω).

73. Find the impedance Z if $R = 7\ \Omega$, $X_L = 6\ \Omega$, and $X_C = 11\ \Omega$.

74. Find the impedance Z if $R = 9.2\ \Omega$, $X_L = 5.6\ \Omega$, and $X_C = 8.3\ \Omega$.

The voltage *V* (in volts) across any element in an AC circuit is calculated as a product of the current *I* and the impedance *Z*: $V = IZ$.

75. Find the voltage in a circuit with a current $I = 3 - 2i$ amperes and an impedance of $Z = 5 + 5i\ \Omega$.

76. Find the voltage in a circuit with a current $I = 2 - 3i$ amperes and an impedance of $Z = 4 + 2i\ \Omega$.

In an AC circuit, the total impedance (in ohms) is given by $Z = \frac{Z_1Z_2}{Z_1 + Z_2}$, where *Z* represents the total impedance of a circuit that has Z_1 and Z_2 wired in parallel.

77. Find the total impedance Z if $Z_1 = 1 + 2i$ and $Z_2 = 3 - 2i$.

78. Find the total impedance Z if $Z_1 = 3 - i$ and $Z_2 = 2 + i$.

▶ EXTENDING THE CONCEPT

79. Up to this point, we've said that expressions like $x^2 - 9$ and $p^2 - 7$ are factorable:

$$x^2 - 9 = (x + 3)(x - 3) \quad \text{and}$$
$$p^2 - 7 = (p + \sqrt{7})(p - \sqrt{7}),$$

while $x^2 + 9$ and $p^2 + 7$ are prime. More correctly, we should state that $x^2 + 9$ and $p^2 + 7$ are nonfactorable *using real numbers,* since they actually *can* be factored if complex numbers are used. From $(a + bi)(a - bi) = a^2 + b^2$ we note $a^2 + b^2 = (a + bi)(a - bi)$, showing

$$x^2 + 9 = (x + 3i)(x - 3i) \quad \text{and}$$
$$p^2 + 7 = (p + i\sqrt{7})(p - i\sqrt{7}).$$

Use this idea to factor the following.

a. $x^2 + 36$ **b.** $m^2 + 3$

c. $n^2 + 12$ **d.** $4x^2 + 49$

80. In this section, we noted that the product property of radicals $\sqrt{AB} = \sqrt{A}\sqrt{B}$, can still be applied when at most one of the factors is negative. So what happens if *both* are negative? First consider the expression $\sqrt{-4 \cdot -25}$. What happens if you first multiply in the radicand, then compute the square root? Next consider the product $\sqrt{-4} \cdot \sqrt{-25}$. Rewrite each factor using the *i* notation, then compute the product. Do you get the same result as before? What can you say about $\sqrt{-4 \cdot -25}$ and $\sqrt{-4} \cdot \sqrt{-25}$?

81. Simplify the expression $i^{17}(3 - 4i) - 3i^3(1 + 2i)^2$.

82. While it is a simple concept for real numbers, the square root of a complex number is much more involved due to the interplay between its real and imaginary parts. For $z = a + bi$ the square root of z can be found using the formula: $\sqrt{z} = \frac{\sqrt{2}}{2}(\sqrt{|z| + a} \pm i\sqrt{|z| - a})$, where the sign is chosen to match the sign of b (see Exercise 69). Use the formula to find the square root of each complex number, then check by squaring.

a. $z = -7 + 24i$ **b.** $z = 5 - 12i$

c. $z = 4 + 3i$

▶ MAINTAINING YOUR SKILLS

83. **(R.7)** State the perimeter and area formulas for: (a) squares, (b) rectangles, (c) triangles, and (d) circles.

84. **(R.1)** Write the symbols in words and state True/False.

a. $6 \notin \mathbb{Q}$ **b.** $\mathbb{Q} \subset \mathbb{R}$

c. $103 \in \{3, 4, 5, \ldots\}$ **d.** $\mathbb{R} \not\subset \mathbb{C}$

85. **(1.1)** John can run 10 m/sec, while Rick can only run 9 m/sec. If Rick gets a 2-sec head start, who will hit the 200-m finish line first?

86. **(R.4)** Factor the following expressions completely.

a. $x^4 - 16$ **b.** $n^3 - 27$

c. $x^3 - x^2 - x + 1$ **d.** $4n^2m - 12nm^2 + 9m^3$

1.5 Solving Quadratic Equations

Learning Objectives

In Section 1.5 you will learn how to:

- ❑ **A.** Solve quadratic equations using the zero product property
- ❑ **B.** Solve quadratic equations using the square root property of equality
- ❑ **C.** Solve quadratic equations by completing the square
- ❑ **D.** Solve quadratic equations using the quadratic formula
- ❑ **E.** Use the discriminant to identify solutions
- ❑ **F.** Solve applications of quadratic equations

In Section 1.1 we solved the equation $ax + b = c$ for x to establish a general solution for all linear equations of this form. In this section, we'll establish a general solution for the quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$ using a process known as *completing the square.* Other applications of completing the square include the graphing of parabolas, circles, and other relations from the family of *conic sections.*

A. Quadratic Equations and the Zero Product Property

A **quadratic equation** is one that can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. As shown, the equation is written in **standard form,** meaning the terms are in decreasing order of degree and the equation is set equal to zero.

Quadratic Equations

A quadratic equation can be written in the form

$$ax^2 + bx + c = 0,$$

with $a, b, c \in \mathbb{R}$, and $a \neq 0$.

Notice that a is the leading coefficient, b is the coefficient of the linear (first degree) term, and c is a constant. All quadratic equations have degree two, but can have one, two, or three terms. The equation $n^2 - 81 = 0$ is a quadratic equation with two terms, where $a = 1$, $b = 0$, and $c = -81$.

EXAMPLE 1 ► Determining Whether an Equation Is Quadratic

State whether the given equation is quadratic. If yes, identify coefficients a, b, and c.

a. $2x^2 - 18 = 0$ **b.** $z - 12 - 3z^2 = 0$ **c.** $\frac{-3}{4}x + 5 = 0$

d. $z^3 - 2z^2 + 7z = 8$ **e.** $0.8x^2 = 0$

Solution ►

	Standard Form	Quadratic	Coefficients
a.	$2x^2 - 18 = 0$	yes, deg 2	$a = 2 \quad b = 0 \quad c = -18$
b.	$-3z^2 + z - 12 = 0$	yes, deg 2	$a = -3 \quad b = 1 \quad c = -12$
c.	$\frac{-3}{4}x + 5 = 0$	no, deg 1	(linear equation)
d.	$z^3 - 2z^2 + 7z - 8 = 0$	no, deg 3	(cubic equation)
e.	$0.8x^2 = 0$	yes, deg 2	$a = 0.8 \quad b = 0 \quad c = 0$

Now try Exercises 7 through 18 ►

WORTHY OF NOTE

The word *quadratic* comes from the Latin word *quadratum,* meaning square. The word historically refers to the "four sidedness" of a square, but mathematically to the *area* of a square. Hence its application to polynomials of the form $ax^2 + bx + c$—the variable of the leading term is *squared.*

With quadratic and other polynomial equations, we generally cannot isolate the variable on one side using only properties of equality, because the variable is raised to different powers. Instead we attempt to solve the equation by factoring and applying the **zero product property.**

Zero Product Property

If A and B represent real numbers or real valued expressions

and $A \cdot B = 0$,

then $A = 0$ or $B = 0$.

In words, the property says, *If the product of any two (or more) factors is equal to zero, then at least one of the factors must be equal to zero.* We can use this property to solve higher degree equations after rewriting them in terms of equations with lesser degree. As with linear equations, values that make the original equation true are called *solutions* or *roots* of the equation.

EXAMPLE 2 ► Solving Equations Using the Zero Product Property

Solve by writing the equations in factored form and applying the zero product property.

a. $3x^2 = 5x$ **b.** $-5x + 2x^2 = 3$ **c.** $4x^2 = 12x - 9$

Solution ► **a.**

$3x^2 = 5x$ given equation

$3x^2 - 5x = 0$ standard form

$x(3x - 5) = 0$ factor

$x = 0$ or $3x - 5 = 0$ set factors equal to zero (zero product property)

$x = 0$ or $x = \frac{5}{3}$ result

b.

$-5x + 2x^2 = 3$ given equation

$2x^2 - 5x - 3 = 0$ standard form

$(2x + 1)(x - 3) = 0$ factor

$2x + 1 = 0$ or $x - 3 = 0$ set factors equal to zero (zero product property)

$x = -\frac{1}{2}$ or $x = 3$ result

c.

$$4x^2 = 12x - 9 \quad \text{given equation}$$
$$4x^2 - 12x + 9 = 0 \quad \text{standard form}$$
$$(2x - 3)(2x - 3) = 0 \quad \text{factor}$$
$$2x - 3 = 0 \quad \text{or} \quad 2x - 3 = 0 \quad \text{set factors equal to zero (zero product property)}$$
$$x = \frac{3}{2} \quad \text{or} \quad x = \frac{3}{2} \quad \text{result}$$

This equation has only the solution $x = \frac{3}{2}$, which we call a *repeated root*.

Now try Exercises 19 through 42 ►

CAUTION ► Consider the equation $x^2 - 2x - 3 = 12$. While the left-hand side is factorable, the result is $(x - 3)(x + 1) = 12$ and finding a solution becomes a "guessing game" because the equation is not set equal to zero. If you *misapply* the zero factor property and say that $x - 3 = 12$ or $x + 1 = 12$, the "solutions" are $x = 15$ or $x = 11$, which are both incorrect! After subtracting 12 from both sides $x^2 - 2x - 3 = 12$ becomes $x^2 - 2x - 15 = 0$, giving $(x - 5)(x + 3) = 0$ with solutions $x = 5$ or $x = -3$.

A. You've just learned how to solve quadratic equations using the zero product property

B. Solving Quadratic Equations Using the Square Root Property of Equality

The equation $x^2 = 9$ can be solved by factoring. In standard form we have $x^2 - 9 = 0$ (note $b = 0$), then $(x - 3)(x + 3) = 0$. The solutions are $x = -3$ or $x = 3$, which are simply the *positive and negative square roots of 9*. This result suggests an alternative method for solving equations of the form $X^2 = k$, known as the **square root property of equality.**

Square Root Property of Equality

If X represents an algebraic expression

and $X^2 = k$,

then $X = \sqrt{k}$ or $X = -\sqrt{k}$;

also written as $X = \pm\sqrt{k}$

EXAMPLE 3 ► Solving an Equation Using the Square Root Property of Equality

Use the square root property of equality to solve each equation.

a. $-4x^2 + 3 = -6$ **b.** $x^2 + 12 = 0$ **c.** $(x - 5)^2 = 24$

Solution ► **a.**

$$-4x^2 + 3 = -6 \quad \text{original equation}$$
$$x^2 = \frac{9}{4} \quad \text{subtract 3, divide by } -4$$
$$x = \sqrt{\frac{9}{4}} \quad \text{or} \quad x = -\sqrt{\frac{9}{4}} \quad \text{square root property of equality}$$
$$x = \frac{3}{2} \quad \text{or} \quad x = -\frac{3}{2} \quad \text{simplify radicals}$$

This equation has two rational solutions.

b. $x^2 + 12 = 0$ original equation
$x^2 = -12$ subtract 12
$x = \sqrt{-12}$ or $x = -\sqrt{12}$ square root property of equality
$x = 2i\sqrt{3}$ or $x = -2i\sqrt{3}$ simplify radicals

This equation has two complex solutions.

c. $(x - 5)^2 = 24$ original equation
$x - 5 = \sqrt{24}$ or $x - 5 = -\sqrt{24}$ square root property of equality
$x = 5 + 2\sqrt{6}$ $x = 5 - 2\sqrt{6}$ solve for x and simplify radicals

This equation has two irrational solutions.

B. You've just learned how to solve quadratic equations using the square root property of equality

Now try Exercises 43 through 58 ►

CAUTION ► For equations of the form $(x + d)^2 = k$ [see Example 3(c)], you should resist the temptation to expand the binomial square in an attempt to simplify the equation and solve by factoring—many times the result is nonfactorable. *Any* equation of the form $(x + d)^2 = k$ can quickly be solved using the square root property of equality.

WORTHY OF NOTE

In Section R.6 we noted that for any real number a, $\sqrt{a^2} = |a|$. From Example 3(a), solving the equation by taking the square root of both sides produces $\sqrt{x^2} = \sqrt{\frac{9}{4}}$. This is equivalent to $|x| = \sqrt{\frac{9}{4}}$, again showing this equation must have two solutions, $x = -\sqrt{\frac{9}{4}}$ and $x = \sqrt{\frac{9}{4}}$.

Answers written using radicals are called **exact** or **closed form** solutions. Actually checking the exact solutions is a nice application of fundamental skills. Let's check $x = 5 + 2\sqrt{6}$ from Example 3(c).

check:
$(x - 5)^2 = 24$ original equation
$(5 + 2\sqrt{6} - 5)^2 = 24$ substitute $5 + 2\sqrt{6}$ for x
$(2\sqrt{6})^2 = 24$ simplify
$4(6) = 24$ $(2\sqrt{6})^2 = 4(6)$
$24 = 24$ ✓ result checks ($x = 5 - 2\sqrt{6}$ also checks)

C. Solving Quadratic Equations by Completing the Square

Again consider $(x - 5)^2 = 24$ from Example 3(c). If we had first expanded the binomial square, we would have obtained $x^2 - 10x + 25 = 24$, then $x^2 - 10x + 1 = 0$ in standard form. Note that this equation *cannot be solved by factoring*. Reversing this process leads us to a strategy for solving nonfactorable quadratic equations, by creating a *perfect square trinomial* from the quadratic and linear terms. This process is known as **completing the square.** To transform $x^2 - 10x + 1 = 0$ back into $x^2 - 10x + 25 = 24$ [which we would then rewrite as $(x - 5)^2 = 24$ and solve], we subtract 1 from both sides, then add 25:

$x^2 - 10x + 1 = 0$
$x^2 - 10x = -1$ subtract 1
$x^2 - 10x + 25 = -1 + 25$ add 25
$(x - 5)^2 = 24$ factor, simplify

In general, after subtracting the constant term, the number that "completes the square" is found by squaring $\frac{1}{2}$ the coefficient of the linear term: $\left[\frac{1}{2}(10)\right]^2 = 25$. **See Exercises 59 through 64 for additional practice.**

EXAMPLE 4 ► **Solving a Quadratic Equation by Completing the Square**

Solve by completing the square: $x^2 + 13 = 6x$.

Solution ►

$x^2 + 13 = 6x$	original equation
$x^2 - 6x + 13 = 0$	standard form
$x^2 - 6x + \_\_ = -13 + \_\_$	subtract 13 to make room for new constant
$[(\frac{1}{2})(-6)]^2 = 9$	compute $[(\frac{1}{2})(\textit{linear coefficient})]^2$
$x^2 - 6x + 9 = -13 + 9$	add 9 to both sides (completing the square)
$(x - 3)^2 = -4$	factor and simplify
$x - 3 = \sqrt{-4}$ or $x - 3 = -\sqrt{-4}$	square root property of equality
$x = 3 + 2i$ or $x = 3 - 2i$	simplify radicals and solve for x

Now try Exercises 65 through 74 ►

The process of completing the square can be applied to any quadratic equation with a leading coefficient of 1. If the leading coefficient is not 1, we simply divide through by a before beginning, which brings us to this summary of the process.

WORTHY OF NOTE

It's helpful to note that the number you're squaring in step three, $\left[\frac{1}{2} \cdot \frac{b}{a}\right] = \frac{b}{2a}$, turns out to be the constant term in the factored form. From Example 4, the number we squared was $\left(\frac{1}{2}\right)(-6) = -3$, and the binomial square was $(x - 3)^2$.

Completing the Square to Solve a Quadratic Equation

To solve $ax^2 + bx + c = 0$ by completing the square:

1. Subtract the constant c from both sides.
2. Divide both sides by the leading coefficient a.
3. Compute $\left[\frac{1}{2} \cdot \frac{b}{a}\right]^2$ and add the result to both sides.
4. Factor left-hand side as a binomial square; simplify right-hand side.
5. Solve using the square root property of equality.

EXAMPLE 5 ► Solving a Quadratic Equation by Completing the Square

Solve by completing the square: $-3x^2 + 1 = 4x$.

Solution ►

$-3x^2 + 1 = 4x$	original equation
$-3x^2 - 4x + 1 = 0$	standard form (nonfactorable)
$-3x^2 - 4x = -1$	subtract 1
$x^2 + \frac{4}{3}x + \quad = \frac{1}{3}$	divide by -3
$x^2 + \frac{4}{3}x + \frac{4}{9} = \frac{1}{3} + \frac{4}{9}$	$\left[\frac{1}{2}\frac{b}{a}\right]^2 = \left[\left(\frac{1}{2}\right)\left(\frac{4}{3}\right)\right]^2 = \frac{4}{9}$; add $\frac{4}{9}$
$\left(x + \frac{2}{3}\right)^2 = \frac{7}{9}$	factor and simplify $\left(\frac{1}{3} = \frac{3}{9}\right)$
$x + \frac{2}{3} = \sqrt{\frac{7}{9}}$ or $x + \frac{2}{3} = -\sqrt{\frac{7}{9}}$	square root property of equality
$x = -\frac{2}{3} + \frac{\sqrt{7}}{3}$ or $x = -\frac{2}{3} - \frac{\sqrt{7}}{3}$	solve for x and simplify (exact form)
$x \approx 0.22$ or $x \approx -1.55$	approximate form (to hundredths)

☑ **C. You've just learned how to solve quadratic equations by completing the square**

Now try Exercises 75 through 82 ►

CAUTION ► For many of the skills/processes needed in a study of algebra, it's actually easier to work with the fractional form of a number, rather than the decimal form. For example, computing $\left(\frac{2}{3}\right)^2$ is easier than computing $(0.\overline{6})^2$, and finding $\sqrt{\frac{9}{16}}$ is much easier than finding $\sqrt{0.5625}$.

D. Solving Quadratic Equations Using the Quadratic Formula

In Section 1.1 we found a general solution for the linear equation $ax + b = c$ by comparing it to $2x + 3 = 15$. Here we'll use a similar idea to find a general solution for quadratic equations. In a side-by-side format, we'll solve the equations $2x^2 + 5x + 3 = 0$ and $ax^2 + bx + c = 0$ by completing the square. Note the similarities.

$2x^2 + 5x + 3 = 0$	given equations	$ax^2 + bx + c = 0$
$2x^2 + 5x + \quad = -3$	subtract constant term	$ax^2 + bx + \quad = -c$
$x^2 + \frac{5}{2}x + \_\_\_ = -\frac{3}{2}$	divide by lead coefficient	$x^2 + \frac{b}{a}x + \_\_\_ = -\frac{c}{a}$
$\left[\frac{1}{2}\left(\frac{5}{2}\right)\right]^2 = \frac{25}{16}$	$\left[\frac{1}{2}(\text{linear coefficient})\right]^2$	$\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \frac{b^2}{4a^2}$
$x^2 + \frac{5}{2}x + \frac{25}{16} = \frac{25}{16} - \frac{3}{2}$	add to both sides	$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$
$\left(x + \frac{5}{4}\right)^2 = \frac{25}{16} - \frac{3}{2}$	left side factors as a binomial square	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$
$\left(x + \frac{5}{4}\right)^2 = \frac{25}{16} - \frac{24}{16}$	determine LCDs	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$
$\left(x + \frac{5}{4}\right)^2 = \frac{1}{16}$	simplify right side	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
$x + \frac{5}{4} = \pm\sqrt{\frac{1}{16}}$	square root property of equality	$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$
$x + \frac{5}{4} = \pm\frac{1}{4}$	simplify radicals	$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$
$x = -\frac{5}{4} \pm \frac{1}{4}$	solve for x	$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
$x = \frac{-5 \pm 1}{4}$	combine terms	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$x = \frac{-5 + 1}{4}$ or $x = \frac{-5 - 1}{4}$	solutions	$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

On the left, our final solutions are $x = -1$ or $x = -\frac{3}{2}$. The general solution is called the **quadratic formula,** which can be used to solve *any equation belonging to the quadratic family.*

Quadratic Formula

If $ax^2 + bx + c = 0$, with a, b, and $c \in \mathbb{R}$ and $a \neq 0$, then

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a};$$

also written $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

CAUTION ► It's very important to note the values of a, b, and c come from an equation *written in standard form.* For $3x^2 - 5x = -7$, $a = 3$ and $b = -5$, but $c \neq -7$! In standard form we have $3x^2 - 5x + 7 = 0$, and note the value for use in the formula is actually $c = 7$.

EXAMPLE 6 ► **Solving Quadratic Equations Using the Quadratic Formula**

Solve $4x^2 + 1 = 8x$ using the quadratic formula. State the solution(s) in both exact and approximate form. Check one of the exact solutions in the original equation.

Solution ► Begin by writing the equation in standard form and identifying the values of a, b, and c.

$$4x^2 + 1 = 8x \quad \text{original equation}$$
$$4x^2 - 8x + 1 = 0 \quad \text{standard form}$$
$$a = 4, b = -8, c = 1$$
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(1)}}{2(4)} \quad \text{substitute 4 for } a, -8 \text{ for } b, \text{ and 1 for } c$$
$$x = \frac{8 \pm \sqrt{64 - 16}}{8} = \frac{8 \pm \sqrt{48}}{8} \quad \text{simplify}$$
$$x = \frac{8 \pm 4\sqrt{3}}{8} = \frac{8}{8} \pm \frac{4\sqrt{3}}{8} \quad \text{rationalize the radical (see following Caution)}$$
$$x = 1 + \frac{\sqrt{3}}{2} \quad \text{or} \quad x = 1 - \frac{\sqrt{3}}{2} \quad \text{exact solutions}$$
$$x \approx 1.87 \quad \text{or} \quad x \approx 0.13 \quad \text{approximate solutions}$$

Check ►

$$4x^2 + 1 = 8x \quad \text{original equation}$$
$$4\left(1 + \frac{\sqrt{3}}{2}\right)^2 + 1 = 8\left(1 + \frac{\sqrt{3}}{2}\right) \quad \text{substitute } 1 + \tfrac{\sqrt{3}}{2} \text{ for } x$$
$$4\left[1 + 2\left(\frac{\sqrt{3}}{2}\right) + \frac{3}{4}\right] + 1 = 8 + 4\sqrt{3} \quad \text{square binomial; distribute}$$
$$4 + 4\sqrt{3} + 3 + 1 = 8 + 4\sqrt{3} \quad \text{distribute}$$
$$8 + 4\sqrt{3} = 8 + 4\sqrt{3} \checkmark \quad \text{result checks}$$

D. You've just learned how to solve quadratic equations using the quadratic formula

Now try Exercises 83 through 112 ►

CAUTION ► For $\frac{8 \pm 4\sqrt{3}}{8}$, be careful not to incorrectly "cancel the eights" as in $\frac{\overset{1}{\cancel{8}} \pm 4\sqrt{3}}{\underset{1}{\cancel{8}}} \neq 1 \pm 4\sqrt{3}$.

No! Use a calculator to verify that the results are not equivalent. Both terms in the numerator are divided by 8 and we must either rewrite the expression as separate terms (as above) or factor the numerator to see if the expression simplifies further:
$\frac{8 \pm 4\sqrt{3}}{8} = \frac{\overset{1}{\cancel{4}}(2 \pm \sqrt{3})}{\underset{2}{\cancel{8}}} = \frac{2 \pm \sqrt{3}}{2}$, which is equivalent to $1 \pm \frac{\sqrt{3}}{2}$.

E. The Discriminant of the Quadratic Formula

Recall that $\sqrt{X}$ represents a real number only for $X \geq 0$. Since the quadratic formula contains the radical $\sqrt{b^2 - 4ac}$, the expression $b^2 - 4ac$, called the **discriminant,** will determine the nature (real or complex) and the number of solutions to a given quadratic equation.

The Discriminant of the Quadratic Formula

For $ax^2 + bx + c = 0, a \neq 0$,
1. If $b^2 - 4ac = 0$, the equation has one real root.
2. If $b^2 - 4ac > 0$, the equation has two real roots.
3. If $b^2 - 4ac < 0$, the equation has two complex roots.

Further analysis of the discriminant reveals even more concerning the nature of quadratic solutions. If a, b, and c are rational and the discriminant is a perfect square, there will be two *rational* roots, which means the original equation can be solved by factoring. If the discriminant is not a perfect square, there will be two *irrational* roots that are conjugates. If the discriminant is zero there is one rational root, and the original equation is a perfect square trinomial.

EXAMPLE 7 ▶ Using the Discriminant to Analyze Solutions

Use the discriminant to determine if the equation given has any real root(s). If so, state whether the roots are rational or irrational, and whether the quadratic expression is factorable.

a. $2x^2 + 5x + 2 = 0$ **b.** $x^2 - 4x + 7 = 0$ **c.** $4x^2 - 20x + 25 = 0$

Solution ▶

a. $a = 2, b = 5, c = 2$
$b^2 - 4ac = (5)^2 - 4(2)(2)$
$= 9$

Since $9 > 0$, → two rational roots, factorable

b. $a = 1, b = -4, c = 7$
$b^2 - 4ac = (-4)^2 - 4(1)(7)$
$= -12$

Since $-12 < 0$, → two complex roots, nonfactorable

c. $a = 4, b = -20, c = 25$
$b^2 - 4ac = (-20)^2 - 4(4)(25)$
$= 0$

Since $b^2 - 4ac = 0$, → one rational root, factorable

Now try Exercises 113 through 124 ▶

In Example 7(b), $b^2 - 4ac = -12$ and the quadratic formula shows $x = \frac{4 \pm \sqrt{-12}}{2}$. After simplifying, we find the solutions are the complex conjugates $x = 2 + i\sqrt{3}$ or $x = 2 - i\sqrt{3}$. In general, when $b^2 - 4ac < 0$, the solutions *will be complex conjugates*.

Complex Solutions

The complex solutions of a quadratic equation with real coefficients occur in conjugate pairs.

EXAMPLE 8 ▶ Solving Quadratic Equations Using the Quadratic Formula

Solve: $2x^2 - 6x + 5 = 0$.

Solution ▶ With $a = 2$, $b = -6$, and $c = 5$, the discriminant becomes $(-6)^2 - 4(2)(5) = -4$, showing there will be two complex roots. The quadratic formula then yields

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{quadratic formula}$$

$$x = \frac{-(-6) \pm \sqrt{-4}}{2(2)} \quad b^2 - 4ac = -4,\ \text{substitute 2 for } a, \text{ and } -6 \text{ for } b$$

$$x = \frac{6 \pm 2i}{4} \quad \text{simplify, write in } i \text{ form}$$

$$x = \frac{3}{2} \pm \frac{1}{2}i \quad \text{solutions are complex conjugates}$$

Now try Exercises 125 through 130 ▶

☑ **E.** You've just learned how to use the discriminant to identify solutions

WORTHY OF NOTE

While it's possible to solve by completing the square if $\frac{b}{a}$ is a fraction or an odd number (see Example 5), the process is usually most efficient when $\frac{b}{a}$ is an even number. This is one observation you could use when selecting a solution method.

Summary of Solution Methods for $ax^2 + bx + c = 0$

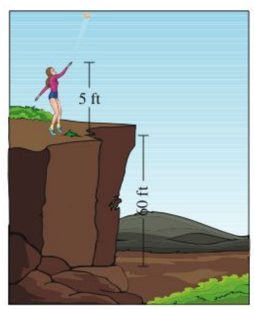
1. If $b = 0$, isolate x and use the square root property of equality.
2. If $c = 0$, factor out the GCF and solve using the zero product property.
3. If no coefficient is zero, you can attempt to solve by
   a. factoring the trinomial
   b. completing the square
   c. using the quadratic formula

F. Applications of the Quadratic Formula

A projectile is any object that is thrown, shot, or *projected* upward with no sustaining source of propulsion. The height of the projectile at time t is modeled by the equation $h = -16t^2 + vt + k$, where h is the height of the object in feet, t is the elapsed time in seconds, and v is the initial velocity in feet per second. The constant k represents the initial height of the object above ground level, as when a person releases an object 5 ft above the ground in a throwing motion. If the person were on a cliff 60 ft high, k would be 65 ft.

EXAMPLE 9 ► Solving an Application of Quadratic Equations

A person standing on a cliff 60 ft high, throws a ball upward with an initial velocity of 102 ft/sec (assume the ball is released 5 ft above where the person is standing). Find (a) the height of the object after 3 sec and (b) how many seconds until the ball hits the ground at the base of the cliff.



Solution ► Using the given information, we have $h = -16t^2 + 102t + 65$. To find the height after 3 sec, substitute $t = 3$.

a.
$$\begin{aligned} h &= -16t^2 + 102t + 65 && \text{original equation} \\ &= -16(3)^2 + 102(3) + 65 && \text{substitute 3 for } t \\ &= 227 && \text{result} \end{aligned}$$

After 3 sec, the ball is 227 ft above the ground.

b. When the ball hits the ground at the base of the cliff, it has a height of zero. Substitute $h = 0$ and solve using the quadratic formula.

$$0 = -16t^2 + 102t + 65 \qquad a = -16,\ b = 102,\ c = 65$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{quadratic formula}$$

$$t = \frac{-(102) \pm \sqrt{(102)^2 - 4(-16)(65)}}{2(-16)} \qquad \text{substitute } -16 \text{ for } a, 102 \text{ for } b, 65 \text{ for } c$$

$$t = \frac{-102 \pm \sqrt{14{,}564}}{-32} \qquad \text{simplify}$$

Since we're trying to find the time in seconds, we go directly to the approximate form of the answer.

$$t \approx -0.58 \quad \text{or} \quad t \approx 6.96 \qquad \text{approximate solutions}$$

The ball will strike the base of the cliff about 7 sec later. Since t represents time, the solution $t \approx -0.58$ does not apply.

Now try Exercises 133 through 140 ►

EXAMPLE 10 ► Solving Applications Using the Quadratic Formula

For the years 1995 to 2002, the amount A of annual international telephone traffic (in billions of minutes) can be modeled by $A = 0.3x^2 + 8.9x + 61.8$, where $x = 0$ represents the year 1995 [*Source:* Data from the *2005 Statistical Abstract of the United States,* Table 1372, page 870]. If this trend continues, in what year will the annual number of minutes reach or surpass 275 billion minutes?

Solution ► We are essentially asked to solve $A = 0.3x^2 + 8.9x + 61.8$, when $A = 275$.

$$275 = 0.3x^2 + 8.9x + 61.8 \quad \text{given equation}$$
$$0 = 0.3x^2 + 8.9x - 213.2 \quad \text{subtract 275}$$

For $a = 0.3$, $b = 8.9$, and $c = -213.2$, the quadratic formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{quadratic formula}$$
$$x = \frac{-8.9 \pm \sqrt{(8.9)^2 - 4(0.3)(-213.2)}}{2(0.3)} \quad \text{substitute known values}$$
$$x = \frac{-8.9 \pm \sqrt{335.05}}{0.6} \quad \text{simplify}$$
$$x \approx 15.7 \quad \text{or} \quad x \approx -45.3 \quad \text{result}$$

We disregard the negative solution (since x represents time), and find the annual number of international telephone minutes will reach or surpass 275 billion 15.7 years after 1995, or in the year 2010.

F. You've just learned how to solve applications of quadratic equations

Now try Exercises 141 and 142 ►

TECHNOLOGY HIGHLIGHT

The Discriminant

Quadratic equations play an important role in a study of College Algebra, forming a bridge between our previous and current studies, and the more advanced equations to come. As seen in this section, the discriminant of the quadratic formula ($b^2 - 4ac$) reveals the type and number of solutions, and whether the original equation can be solved by factoring (the discriminant is a perfect square). It will often be helpful to have this information in advance of trying to solve or graph the equation. Since this will be done for each new equation, the discriminant is a prime candidate for a short program. To begin a new program press PRGM ► ► (NEW) ENTER. The calculator will prompt you to name the program using the green ALPHA letters (eight letters max), then allow you to start entering program lines. In PRGM mode, pressing PRGM once again will bring up menus that contain all needed commands. For very basic programs, these commands will be in the **I/O** (Input/Output) submenu, with the most common options being **2:Prompt, 3:Disp,** and **8:CLRHOME.** As you can see, we have named our program *DISCRMNT.*

PROGRAM:DISCRMNT

:CLRHOME	Clears the home screen, places cursor in upper left corner
:DISP "DISCRIMINANT"	Displays the word *DISCRIMINANT* as user information
:DISP "B²−4AC"	Displays $B^2 - 4AC$ as user information
:DISP ""	Displays a blank line (for formatting)
:Prompt A, B, C	Prompts the user to enter the values of *A*, *B*, and *C*
:B²−4AC → D	Computes $B^2 - 4AC$ using given values and stores result in memory location *D*

—*continued*

:CLRHOME	Clears the home screen, places cursor in upper left corner
:DISP "DISCRIMINANT IS:"	Displays the words *DISCRIMINANT IS* as user information
:DISP D	Displays the computed value of *D*

Exercise 1: Run the program for $x^2 - 3x - 10 = 0$ and $x^2 + 5x - 14 = 0$ to verify that both can be solved by factoring. What do you notice?

Exercise 2: Run the program for $25x^2 - 90x + 81 = 0$ and $4x^2 + 20x + 25 = 0$, then check to see if each is a perfect square trinomial. What do you notice?

Exercise 3: Run the program for $y = x^2 + 2x + 10$ and $y = x^2 - 2x + 5$. Do these equations have real number solutions? Why or why not?

Exercise 4: Once the discriminant *D* is known, the quadratic formula becomes $x = \frac{-b \pm \sqrt{D}}{2a}$ and solutions can quickly be found. Solve the equations in Exercises 1–3 above.

1.5 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. A polynomial equation is in standard form when written in ______ order of degree and set equal to ______.
2. The solution $x = 2 + \sqrt{3}$ is called an ______ form of the solution. Using a calculator, we find the ______ form is $x \approx 3.732$.
3. To solve a quadratic equation by completing the square, the coefficient of the ______ term must be a ______.
4. The quantity $b^2 - 4ac$ is called the ______ of the quadratic equation. If $b^2 - 4ac > 0$, there are ______ real roots.
5. According to the summary on page 122, what method should be used to solve $4x^2 - 5x = 0$? What are the solutions?
6. Discuss/Explain why this version of the quadratic formula is incorrect:

$$x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

▶ DEVELOPING YOUR SKILLS

Determine whether each equation is quadratic. If so, identify the coefficients *a*, *b*, and *c*. If not, discuss why.

7. $2x - 15 - x^2 = 0$
8. $21 + x^2 - 4x = 0$
9. $\frac{2}{3}x - 7 = 0$
10. $12 - 4x = 9$
11. $\frac{1}{4}x^2 = 6x$
12. $0.5x = 0.25x^2$
13. $2x^2 + 7 = 0$
14. $5 = -4x^2$
15. $-3x^2 + 9x - 5 + 2x^3 = 0$
16. $z^2 - 6z + 9 - z^3 = 0$
17. $(x - 1)^2 + (x - 1) + 4 = 9$
18. $(x + 5)^2 - (x + 5) + 4 = 17$

Solve using the zero factor property. Be sure each equation is in standard form and factor out any common factors before attempting to solve. Check all answers in the original equation.

19. $x^2 - 15 = 2x$
20. $z^2 - 10z = -21$
21. $m^2 = 8m - 16$
22. $-10n = n^2 + 25$

23. $5p^2 - 10p = 0$ **24.** $6q^2 - 18q = 0$

25. $-14h^2 = 7h$ **26.** $9w = -6w^2$

27. $a^2 - 17 = -8$ **28.** $b^2 + 8 = 12$

29. $g^2 + 18g + 70 = -11$

30. $h^2 + 14h - 2 = -51$

31. $m^3 + 5m^2 - 9m - 45 = 0$

32. $n^3 - 3n^2 - 4n + 12 = 0$

33. $(c - 12)c - 15 = 30$

34. $(d - 10)d + 10 = -6$

35. $9 + (r - 5)r = 33$

36. $7 + (s - 4)s = 28$

37. $(t + 4)(t + 7) = 54$

38. $(g + 17)(g - 2) = 20$

39. $2x^2 - 4x - 30 = 0$

40. $-3z^2 + 12z + 36 = 0$

41. $2w^2 - 5w = 3$

42. $-3v^2 = -v - 2$

Solve the following equations using the square root property of equality. Write answers in exact form and approximate form rounded to hundredths. If there are no real solutions, so state.

43. $m^2 = 16$ **44.** $p^2 = 49$

45. $y^2 - 28 = 0$ **46.** $m^2 - 20 = 0$

47. $p^2 + 36 = 0$ **48.** $n^2 + 5 = 0$

49. $x^2 = \frac{21}{16}$ **50.** $y^2 = \frac{13}{9}$

51. $(n - 3)^2 = 36$ **52.** $(p + 5)^2 = 49$

53. $(w + 5)^2 = 3$ **54.** $(m - 4)^2 = 5$

55. $(x - 3)^2 + 7 = 2$ **56.** $(m + 11)^2 + 5 = 3$

57. $(m - 2)^2 = \frac{18}{49}$ **58.** $(x - 5)^2 = \frac{12}{25}$

Fill in the blank so the result is a perfect square trinomial, then factor into a binomial square.

59. $x^2 + 6x +$ ______ **60.** $y^2 + 10y +$ ______

61. $n^2 + 3n +$ ______ **62.** $x^2 - 5x +$ ______

63. $p^2 + \frac{2}{3}p +$ ______ **64.** $x^2 - \frac{3}{2}x +$ ______

Solve by completing the square. Write your answers in both exact form and approximate form rounded to the hundredths place. If there are no real solutions, so state.

65. $x^2 + 6x = -5$ **66.** $m^2 + 8m = -12$

67. $p^2 - 6p + 3 = 0$ **68.** $n^2 = 4n + 10$

69. $p^2 + 6p = -4$ **70.** $x^2 - 8x - 1 = 0$

71. $m^2 + 3m = 1$ **72.** $n^2 + 5n - 2 = 0$

73. $n^2 = 5n + 5$ **74.** $w^2 - 7w + 3 = 0$

75. $2x^2 = -7x + 4$ **76.** $3w^2 - 8w + 4 = 0$

77. $2n^2 - 3n - 9 = 0$ **78.** $2p^2 - 5p = 1$

79. $4p^2 - 3p - 2 = 0$ **80.** $3x^2 + 5x - 6 = 0$

81. $m^2 = 7m - 4$ **82.** $a^2 - 15 = 4a$

Solve each equation using the most efficient method: factoring, square root property of equality, or the quadratic formula. Write your answer in both exact and approximate form (rounded to hundredths). Check one of the exact solutions in the original equation.

83. $x^2 - 3x = 18$ **84.** $w^2 + 6w - 1 = 0$

85. $4m^2 - 25 = 0$ **86.** $4a^2 - 4a = 1$

87. $4n^2 - 8n - 1 = 0$ **88.** $2x^2 - 4x + 5 = 0$

89. $6w^2 - w = 2$ **90.** $3a^2 - 5a + 6 = 0$

91. $4m^2 = 12m - 15$ **92.** $3p^2 + p = 0$

93. $4n^2 - 9 = 0$ **94.** $4x^2 - x = 3$

95. $5w^2 = 6w + 8$ **96.** $3m^2 - 7m - 6 = 0$

97. $3a^2 - a + 2 = 0$ **98.** $3n^2 - 2n - 3 = 0$

99. $5p^2 = 6p + 3$ **100.** $2x^2 + x + 3 = 0$

101. $5w^2 - w = 1$ **102.** $3m^2 - 2 = 5m$

103. $2a^2 + 5 = 3a$ **104.** $n^2 + 4n - 8 = 0$

105. $2p^2 - 4p + 11 = 0$ **106.** $8x^2 - 5x - 1 = 0$

107. $w^2 + \frac{2}{3}w = \frac{1}{9}$ **108.** $\frac{5}{4}m^2 - \frac{8}{3}m + \frac{1}{6} = 0$

109. $0.2a^2 + 1.2a + 0.9 = 0$

110. $-5.4n^2 + 8.1n + 9 = 0$

111. $\frac{2}{7}p^2 - 3 = \frac{8}{21}p$

112. $\frac{5}{9}x^2 - \frac{16}{15}x = \frac{3}{2}$

Use the discriminant to determine whether the given equation has irrational, rational, repeated, or complex roots. Also state whether the original equation is factorable using integers, but do not solve for *x*.

113. $-3x^2 + 2x + 1 = 0$ **114.** $2x^2 - 5x - 3 = 0$

115. $-4x + x^2 + 13 = 0$ **116.** $-10x + x^2 + 41 = 0$

117. $15x^2 - x - 6 = 0$ **118.** $10x^2 - 11x - 35 = 0$

119. $-4x^2 + 6x - 5 = 0$ **120.** $-5x^2 - 3 = 2x$

121. $2x^2 + 8 = -9x$ **122.** $x^2 + 4 = -7x$

123. $4x^2 + 12x = -9$ **124.** $9x^2 + 4 = 12x$

Solve the quadratic equations given. Simplify each result.

125. $-6x + 2x^2 + 5 = 0$ **126.** $17 + 2x^2 = 10x$

127. $5x^2 + 5 = -5x$ **128.** $x^2 = -2x - 19$

129. $-2x^2 = -5x + 11$ **130.** $4x - 3 = 5x^2$

▶ WORKING WITH FORMULAS

131. Height of a projectile: $h = -16t^2 + vt$

If an object is projected vertically upward from ground level with no continuing source of propulsion, the height of the object (in feet) is modeled by the equation shown, where v is the initial velocity, and t is the time in seconds. Use the quadratic formula to solve for t in terms of v and h. (*Hint:* Set the equation equal to zero and identify the coefficients as before.)

132. Surface area of a cylinder: $A = 2\pi r^2 + 2\pi rh$

The surface area of a cylinder is given by the formula shown, where h is the height and r is the radius of the base. The equation can be considered a quadratic in the variable r. Use the quadratic formula to solve for r in terms of h and A. (*Hint:* Rewrite the equation in standard form and identify the coefficients as before.)

▶ APPLICATIONS

133. Height of a projectile: The height of an object thrown upward from the roof of a building 408 ft tall, with an initial velocity of 96 ft/sec, is given by the equation $h = -16t^2 + 96t + 408$, where h represents the height of the object after t seconds. How long will it take the object to hit the ground? Answer in exact form and decimal form rounded to the nearest hundredth.

134. Height of a projectile: The height of an object thrown upward from the floor of a canyon 106 ft deep, with an initial velocity of 120 ft/sec, is given by the equation $h = -16t^2 + 120t - 106$, where h represents the height of the object after t seconds. How long will it take the object to rise to the height of the canyon wall? Answer in exact form and decimal form rounded to hundredths.

135. Cost, revenue, and profit: The revenue for a manufacturer of microwave ovens is given by the equation $R = x(40 - \frac{1}{3}x)$, where revenue is in thousands of dollars and x thousand ovens are manufactured and sold. What is the minimum number of microwave ovens that must be sold to bring in a revenue of $900,000?

136. Cost, revenue, and profit: The revenue for a manufacturer of computer printers is given by the equation $R = x(30 - 0.4x)$, where revenue is in thousands of dollars and x thousand printers are manufactured and sold. What is the minimum number of printers that must be sold to bring in a revenue of $440,000?

137. Cost, revenue, and profit: The cost of raw materials to produce plastic toys is given by the cost equation $C = 2x + 35$, where x is the number of toys in hundreds. The total income (revenue) from the sale of these toys is given by $R = -x^2 + 122x - 1965$. (a) Determine the profit equation (profit = revenue − cost). During the Christmas season, the owners of the company decide to manufacture and donate as many toys as they can, without taking a loss (i.e., they break even: profit or $P = 0$). (b) How many toys will they produce for charity?

138. Cost, revenue, and profit: The cost to produce bottled spring water is given by the cost equation $C = 16x + 63$, where x is the number of bottles in thousands. The total revenue from the sale of these bottles is given by the equation $R = -x^2 + 326x - 18{,}463$. (a) Determine the profit equation (profit = revenue − cost). (b) After a bad flood contaminates the drinking water of a nearby community, the owners decide to bottle and donate as many bottles of water as they can, without taking a loss (i.e., they break even: profit or $P = 0$). How many bottles will they produce for the flood victims?

139. Height of an arrow: If an object is projected vertically upward from ground level with no continuing source of propulsion, its height (in feet) is modeled by the equation $h = -16t^2 + vt$, where v is the initial velocity and t is the time in seconds. Use the quadratic formula to solve for t, given an arrow is shot into the air with $v = 144$ ft/sec and $h = 260$ ft. See Exercise 131.

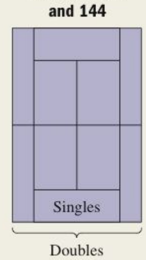
140. Surface area of a cylinder: The surface area of a cylinder is given by $A = 2\pi r^2 + 2\pi rh$, where h is the height and r is the radius of the base. The equation can be considered a quadratic in the variable r. Use the quadratic formula to solve for r, given $A = 4710$ cm^2 and $h = 35$ cm. See Exercise 132.

141. Cell phone subscribers: For the years 1995 to 2002, the number N of cellular phone subscribers (in millions) can be modeled by the equation $N = 17.4x^2 + 36.1x + 83.3$, where $x = 0$ represents the year 1995 [*Source:* Data from the *2005 Statistical Abstract of the United States,* Table 1372, page 870]. If this trend continued, in what year did the number of subscribers reach or surpass 3750 million?

142. U.S. international trade balance: For the years 1995 to 2003, the international trade balance B (in millions of dollars) can be approximated by the equation $B = -3.1x^2 + 4.5x - 19.9$, where $x = 0$ represents the year 1995 [*Source:* Data from the *2005 Statistical Abstract of the United States*, Table 1278, page 799]. If this trend continues, in what year will the trade balance reach a deficit of $750 million dollars or more?

143. Tennis court dimensions: A regulation tennis court for a doubles match is laid out so that its length is 6 ft more than two times its width. The area of the doubles court is 2808 ft^2. What is the length and width of the doubles court?

Exercises 143 and 144



144. Tennis court dimensions: A regulation tennis court for a singles match is laid out so that its length is 3 ft less than three times its width. The area of the singles court is 2106 ft^2. What is the length and width of the singles court?

► EXTENDING THE CONCEPT

145. Using the discriminant: Each of the following equations can easily be solved by factoring, since $a = 1$. Using the discriminant, we can create factorable equations with identical values for b and c, but where $a \neq 1$. For instance, $x^2 - 3x - 10 = 0$ and $4x^2 - 3x - 10 = 0$ can both be solved by factoring. Find similar equations ($a \neq 1$) for the quadratics given here. (*Hint:* The discriminant $b^2 - 4ac$ must be a perfect square.)

a. $x^2 + 6x - 16 = 0$

b. $x^2 + 5x - 14 = 0$

c. $x^2 - x - 6 = 0$

146. Using the discriminant: For what values of c will the equation $9x^2 - 12x + c = 0$ have

a. no real roots **b.** one rational root

c. two real roots **d.** two integer roots

Complex polynomials: Many techniques applied to solve polynomial equations with real coefficients can be applied to solve polynomial equations with *complex coefficients*. Here we apply the idea to carefully chosen quadratic equations, as a more general application must wait until a future course, when the square root of a complex number is fully developed. Solve each equation using the quadratic formula, noting that $\frac{1}{i} = -i$.

147. $z^2 - 3iz = -10$

148. $z^2 - 9iz = -22$

149. $4iz^2 + 5z + 6i = 0$

150. $2iz^2 - 9z + 26i = 0$

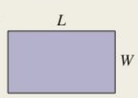
151. $0.5z^2 + (7 + i)z + (6 + 7i) = 0$

152. $0.5z^2 + (4 - 3i)z + (-9 - 12i) = 0$

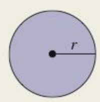
▶ MAINTAINING YOUR SKILLS

153. **(R.7)** State the formula for the perimeter and area of each figure illustrated.

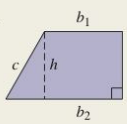
a.



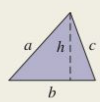
b.



c.



d.



154. **(1.3)** Factor and solve the following equations:

a. $x^2 - 5x - 36 = 0$ **b.** $4x^2 - 25 = 0$

c. $x^3 + 6x^2 - 4x - 24 = 0$

155. **(1.1)** A total of 900 tickets were sold for a recent concert and \$25,000 was collected. If good seats were \$30 and cheap seats were \$20, how many of each type were sold?

156. **(1.1)** Solve for C: $P = C + Ct$.

1.6 Solving Other Types of Equations

Learning Objectives

In Section 1.6 you will learn how to:

- ❑ **A.** Solve polynomial equations of higher degree
- ❑ **B.** Solve rational equations
- ❑ **C.** Solve radical equations and equations with rational exponents
- ❑ **D.** Solve equations in quadratic form
- ❑ **E.** Solve applications of various equation types

The ability to solve linear and quadratic equations is the foundation on which a large percentage of our future studies are built. Both are closely linked to the solution of other equation types, as well as to the graphs of these equations. In this section, we get our first glimpse of these connections, as we learn to solve certain polynomial, rational, radical, and other equations.

A. Polynomial Equations of Higher Degree

In standard form, linear and quadratic equations have a known number of terms, so we commonly represent their coefficients using the early letters of the alphabet, as in $\boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c} = 0$. However, these equations belong to the larger family of **polynomial equations.** To write a general polynomial, where the number of terms is unknown, we often represent the coefficients using subscripts on a single variable, such as a_1, a_2, a_3, and so on. A *polynomial equation of degree n* has the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x^1 + a_0 = 0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and $a_n \neq 0$. Factorable polynomials of degree 3 and higher can also be solved using the zero product property and fundamental algebra skills. As with linear equations, values that make an equation true are called *solutions* or *roots* to the equation.

EXAMPLE 1 ► **Solving Polynomials by Factoring**

Solve by factoring: $2x^3 - 20x = 3x^2$.

Solution ►

$2x^3 - 20x = 3x^2$ — given equation

$2x^3 - 3x^2 - 20x = 0$ — standard form

$x(2x^2 - 3x - 20) = 0$ — common factor is x

$x(2x + 5)(x - 4) = 0$ — factored form

$x = 0$ or $2x + 5 = 0$ or $x - 4 = 0$ — zero product property

$x = 0$ or $x = \frac{-5}{2}$ or $x = 4$ — result

Substituting these values into the original equation verifies they are solutions.

Now try Exercises 7 through 14 ►

EXAMPLE 2 ► Solving Higher Degree Equations

Solve each equation by factoring:

a. $x^3 - 7x + 21 = 3x^2$ **b.** $x^4 - 16 = 0$

Solution ► **a.**

$x^3 - 7x + 21 = 3x^2$ given equation

$x^3 - 3x^2 - 7x + 21 = 0$ standard form; factor by grouping

$x^2(x - 3) - 7(x - 3) = 0$ remove common factors from each group

$(x - 3)(x^2 - 7) = 0$ factored form

$x - 3 = 0$ or $x^2 - 7 = 0$ zero product property

$x = 3$ or $x^2 = 7$ isolate variables

$x = \pm\sqrt{7}$ square root property of equality

The solutions are $x = 3$, $x = \sqrt{7}$, and $x = -\sqrt{7}$.

b.

$x^4 - 16 = 0$ given equation

$(x^2 + 4)(x^2 - 4) = 0$ factor as a difference of squares

$(x^2 + 4)(x + 2)(x - 2) = 0$ factor $x^2 - 4$

$x^2 + 4 = 0$ or $x + 2 = 0$ or $x - 2 = 0$ zero product property

$x^2 = -4$ or $x = -2$ or $x = 2$ isolate variables

$x = \pm\sqrt{-4}$ square root property of equality

Since $\pm\sqrt{-4} = \pm 2i$, the solutions are $x = 2i$, $x = -2i$, $x = 2$, and $x = -2$.

Now try Exercises 15 through 32 ►

In Examples 1 and 2, we were able to solve higher degree polynomials by "breaking them down" into linear and quadratic forms. This basic idea can be applied to other kinds of equations as well, by rewriting them as equivalent linear and/or quadratic equations. For future use, it will be helpful to note that for a third-degree equation in the standard form $ax^3 + bx^2 + cx + d = 0$, a solution using factoring by grouping is always possible when $ad = bc$.

☑ **A.** You've just learned how to solve polynomial equations of higher degree

B. Rational Equations

In Section 1.1 we solved linear equations using basic properties of equality. If any equation contained fractional terms, we "cleared the fractions" using the least common denominator (LCD). We can also use this idea to solve **rational equations,** or equations that contain rational *expressions*.

Solving Rational Equations

1. Identify and exclude any values that cause a zero denominator.
2. Multiply both sides by the LCD and simplify (this will eliminate all denominators).
3. Solve the resulting equation.
4. Check all solutions in the original equation.

EXAMPLE 3 ► Solving a Rational Equation

Solve for m: $\frac{2}{m} - \frac{1}{m - 1} = \frac{4}{m^2 - m}$.

Solution ► Since $m^2 - m = m(m - 1)$, the LCD is $m(m - 1)$, where $m \neq 0$ and $m \neq 1$.

$$m(m-1)\left(\frac{2}{m} - \frac{1}{m-1}\right) = m(m-1)\left[\frac{4}{m(m-1)}\right] \quad \text{multiply by LCD}$$
$$2(m-1) - m = 4 \quad \text{simplify—denominators are eliminated}$$
$$2m - 2 - m = 4 \quad \text{distribute}$$
$$m = 6 \quad \text{solve for } m$$

Checking by substitution we have:

$$\frac{2}{m} - \frac{1}{m-1} = \frac{4}{m^2 - m} \quad \text{original equation}$$
$$\frac{2}{(6)} - \frac{1}{(6)-1} = \frac{4}{(6)^2 - (6)} \quad \text{substitute 6 for } m$$
$$\frac{1}{3} - \frac{1}{5} = \frac{4}{30} \quad \text{simplify}$$
$$\frac{5}{15} - \frac{3}{15} = \frac{2}{15} \quad \text{common denominator}$$
$$\frac{2}{15} = \frac{2}{15} \checkmark \quad \text{result}$$

Now try Exercises 33 through 38 ►

Multiplying both sides of an equation by a variable sometimes introduces a solution that satisfies the *resulting equation,* but not the original equation—the one we're trying to solve. Such "solutions" are called **extraneous roots** and illustrate the need to check all apparent solutions in the original equation. In the case of rational equations, we are particularly aware that any value that causes a zero denominator is outside the domain and cannot be a solution.

EXAMPLE 4 ► Solving a Rational Equation

Solve: $x + \frac{12}{x-3} = 1 + \frac{4x}{x-3}$.

Solution ► The LCD is $x - 3$, where $x \neq 3$.

$$(x-3)\left(x + \frac{12}{x-3}\right) = (x-3)\left(1 + \frac{4x}{x-3}\right) \quad \text{multiply both sides by LCD}$$
$$x^2 - 3x + 12 = x - 3 + 4x \quad \text{simplify—denominators are eliminated}$$
$$x^2 - 8x + 15 = 0 \quad \text{set equation equal to zero}$$
$$(x-3)(x-5) = 0 \quad \text{factor}$$
$$x = 3 \quad \text{or} \quad x = 5 \quad \text{zero factor property}$$

Checking shows $x = 3$ is an extraneous root, and $x = 5$ is the only valid solution.

Now try Exercises 39 through 44 ►

In many fields of study, formulas involving rational expressions are used as equation models. Frequently, we need to solve these equations for one variable in terms of others, a skill closely related to our work in Section 1.1.

EXAMPLE 5 ▶ Solving for a Specified Variable in a Formula

Solve for the indicated variable: $S = \dfrac{a}{1 - r}$ for r.

Solution ▶

$S = \dfrac{a}{1 - r}$ — LCD is $1 - r$

$(1 - r)S = (1 - r)\left(\dfrac{a}{1 - r}\right)$ — multiply both sides by $(1 - r)$

$S - Sr = a$ — simplify—denominator is eliminated

$-Sr = a - S$ — isolate term with r

$r = \dfrac{a - S}{-S}$ — solve for r (divide both sides by $-S$)

$r = \dfrac{S - a}{S}; S \neq 0$ — multiply numerator/denominator by -1

WORTHY OF NOTE

Generally, we should try to write rational answers with the fewest number of negative signs possible. Multiplying the numerator and denominator in Example 5 by -1 gave $r = \frac{S - a}{S}$, a more acceptable answer.

Now try Exercises 45 through 52 ▶

B. You've just learned how to solve rational equations

C. Radical Equations and Equations with Rational Exponents

A **radical equation** is any equation that contains terms with a variable in the radicand. To solve a radical equation, we attempt to isolate a radical term on one side, then apply the appropriate nth power to free up the radicand and solve for the unknown. This is an application of the **power property of equality.**

The Power Property of Equality

If $\sqrt[n]{u}$ and v are real-valued expressions and $\sqrt[n]{u} = v$,

then $(\sqrt[n]{u})^n = v^n$

$u = v^n$

for n an integer, $n \geq 2$.

Raising both sides of an equation to an *even* power can also introduce a false solution (extraneous root). Note that by inspection, the equation $x - 2 = \sqrt{x}$ has only the solution $x = 4$. But the equation $(x - 2)^2 = x$ (obtained by squaring both sides) has both $x = 4$ *and* $x = 1$ as solutions, yet $x = 1$ does not satisfy the original equation. This means we should *check all solutions of an equation where an even power is applied.*

EXAMPLE 6 ▶ Solving Radical Equations

Solve each radical equation:

a. $\sqrt{3x - 2} + 12 = x + 10$ **b.** $2\sqrt[3]{x - 5} + 4 = 0$

Solution ▶ **a.**

$\sqrt{3x - 2} + 12 = x + 10$ — original equation

$\sqrt{3x - 2} = x - 2$ — isolate radical term (subtract 12)

$(\sqrt{3x - 2})^2 = (x - 2)^2$ — apply power property, power is even

$3x - 2 = x^2 - 4x + 4$ — simplify; square binomial

$0 = x^2 - 7x + 6$ — set equal to zero

$0 = (x - 6)(x - 1)$ — factor

$x - 6 = 0$ or $x - 1 = 0$ — apply zero product property

$x = 6$ or $x = 1$ — result, check for extraneous roots

Check ► $x = 6$: $\sqrt{3(6) - 2} + 12 = (6) + 10$

$$\sqrt{16} + 12 = 16$$
$$16 = 16 ✓$$

Check ► $x = 1$: $\sqrt{3(1) - 2} + 12 = (1) + 10$

$$\sqrt{1} + 12 = 11$$
$$13 = 11 ✗$$

The only solution is $x = 6$; $x = 1$ is extraneous.

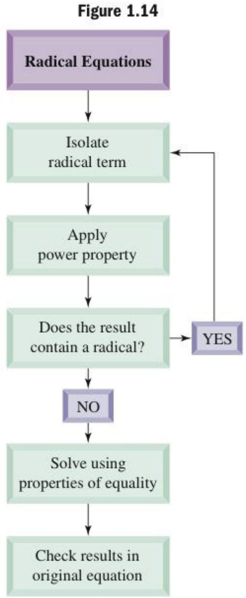
b.

$2\sqrt[3]{x - 5} + 4 = 0$	original equation
$\sqrt[3]{x - 5} = -2$	isolate radical term (subtract 4, divide by 2)
$(\sqrt[3]{x - 5})^3 = (-2)^3$	apply power property, power is odd
$x - 5 = -8$	simplify: $(\sqrt[3]{x - 5})^3 = x - 5$
$x = -3$	solve

Substituting -3 for x in the original equation verifies it is a solution.

Now try Exercises 53 through 56 ►

Figure 1.14



Sometimes squaring both sides of an equation still results in an equation with a radical term, but often there is *one fewer* than before. In this case, we simply repeat the process, as indicated by the flowchart in Figure 1.14.

EXAMPLE 7 ► **Solving Radical Equations**

Solve the equation: $\sqrt{x + 15} - \sqrt{x + 3} = 2$.

Solution ►

$\sqrt{x + 15} - \sqrt{x + 3} = 2$	original equation
$\sqrt{x + 15} = \sqrt{x + 3} + 2$	isolate one radical
$(\sqrt{x + 15})^2 = (\sqrt{x + 3} + 2)^2$	power property
$x + 15 = (x + 3) + 4\sqrt{x + 3} + 4$	$(A + B)^2$; $A = \sqrt{x + 3}, B = 2$
$x + 15 = x + 4\sqrt{x + 3} + 7$	simplify
$8 = 4\sqrt{x + 3}$	isolate radical
$2 = \sqrt{x + 3}$	divide by four
$4 = x + 3$	power property
$1 = x$	possible solution

Check ►

$\sqrt{x + 15} - \sqrt{x + 3} = 2$	original equation
$\sqrt{(1) + 15} - \sqrt{(1) + 3} = 2$	substitute 1 for x
$\sqrt{16} - \sqrt{4} = 2$	simplify
$4 - 2 = 2$	solution checks
$2 = 2 ✓$	

Now try Exercises 57 and 58 ►

Since rational exponents are so closely related to radicals, the solution process for each is very similar. The goal is still to "undo" the radical (rational exponent) and solve for the unknown.

Power Property of Equality

For real-valued expression u and v, with positive integers m, n, and $\frac{m}{n}$ in lowest terms:

If m is odd	If m is even
and $u^{\frac{m}{n}} = v$,	and $u^{\frac{m}{n}} = v(v > 0)$,
then $\left(u^{\frac{m}{n}}\right)^{\frac{n}{m}} = v^{\frac{n}{m}}$	then $\left(u^{\frac{m}{n}}\right)^{\frac{n}{m}} = \pm v^{\frac{n}{m}}$
$u = v^{\frac{n}{m}}$	$u = \pm v^{\frac{n}{m}}$

EXAMPLE 8 ► Solving Equations with Rational Exponents

Solve each equation:

a. $3(x + 1)^{\frac{3}{4}} - 9 = 15$ **b.** $(x - 3)^{\frac{2}{3}} = 4$

Solution ► **a.**

$3(x + 1)^{\frac{3}{4}} - 9 = 15$ original equation; $\frac{m}{n} = \frac{3}{4}$

$(x + 1)^{\frac{3}{4}} = 8$ isolate variable term (add 9, divide by 3)

$\left[(x + 1)^{\frac{3}{4}}\right]^{\frac{4}{3}} = 8^{\frac{4}{3}}$ apply power property, note m is odd

$x + 1 = 16$ simplify $\left[8^{\frac{4}{3}} = \left(8^{\frac{1}{3}}\right)^4 = 16\right]$

$x = 15$ result

Check ►

$3(15 + 1)^{\frac{3}{4}} - 9 = 15$ substitute 15 for x in the original equation

$3\left(16^{\frac{1}{4}}\right)^3 - 9 = 15$ simplify, rewrite exponent

$3(2)^3 - 9 = 15$ $\sqrt[4]{16} = 2$

$3(8) - 9 = 15$ $2^3 = 8$

$15 = 15$ ✓ solution checks

b.

$(x - 3)^{\frac{2}{3}} = 4$ original equation; $\frac{m}{n} = \frac{2}{3}$

$\left[(x - 3)^{\frac{2}{3}}\right]^{\frac{3}{2}} = \pm 4^{\frac{3}{2}}$ apply power property, note m is even

$x - 3 = \pm 8$ simplify $\left[4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = \pm 8\right]$

$x = 3 \pm 8$ result

The solutions are $3 + 8 = 11$ and $3 - 8 = -5$. Verify by checking both in the original equation.

☑ **C.** You've just learned how to solve radical equations and equations with rational exponents

Now try Exercises 59 through 64 ►

CAUTION ► As you continue solving equations with radicals and rational exponents, be careful not to arbitrarily place the "±" sign in front of terms *given* in radical form. The expression $\sqrt{18}$ indicates the positive square root of 18, where $\sqrt{18} = 3\sqrt{2}$. The equation $x^2 = 18$ becomes $x = \pm\sqrt{18}$ after applying the power property, with solutions $x = \pm 3\sqrt{2}$ ($x = -3\sqrt{2}, x = 3\sqrt{2}$), since the square of either number produces 18.

D. Equations in Quadratic Form

In Section R.4 we used a technique called *u-substitution* to factor expressions in quadratic form. The following equations are in quadratic form since the degree of the leading term is twice the degree of the middle term: $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 10 = 0$, $(x^2 + x)^2 - 8(x^2 + x) + 12 = 0$ and $x - 3\sqrt{x + 4} + 4 = 0$ [*Note:* The last equation can be rewritten as $(x + 4) - 3(x + 4)^{\frac{1}{2}} = 0$]. A u-substitution will help to solve these equations by factoring. The first equation appears in Example 9, the other two are in Exercises 70 and 74, respectively.

EXAMPLE 9 ► **Solving Equations in Quadratic Form**

Solve using a u-substitution:

a. $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 10 = 0$ **b.** $x^4 - 36 = 5x^2$

Solution ► **a.** This equation is in quadratic form since it can be rewritten as: $\left(x^{\frac{1}{3}}\right)^{\mathbf{2}} - 3\left(x^{\frac{1}{3}}\right)^{\mathbf{1}} - 10 = 0$, where the degree of leading term is twice that of second term. If we let $u = x^{\frac{1}{3}}$, then $u^2 = x^{\frac{2}{3}}$ and the equation becomes $u^{\mathbf{2}} - 3u^{\mathbf{1}} - 10 = 0$ which is factorable.

$(u - 5)(u + 2) = 0$			factor
$u = 5$	or	$u = -2$	solution in terms of u
$x^{\frac{1}{3}} = 5$	or	$x^{\frac{1}{3}} = -2$	resubstitute $x^{\frac{1}{3}}$ for u
$\left(x^{\frac{1}{3}}\right)^3 = 5^3$	or	$\left(x^{\frac{1}{3}}\right)^3 = (-2)^3$	cube both sides: $\frac{1}{3}(3) = 1$
$x = 125$	or	$x = -8$	solve for x

Both solutions check.

b. In the standard form $x^4 - 5x^2 - 36 = 0$, we note the equation is also in quadratic form, since it can be written as $(x^2)^2 - 5(x^2)^1 - 36 = 0$. If we let $u = x^2$, then $u^2 = x^4$ and the equation becomes $u^2 - 5u - 36 = 0$, which is factorable.

$(u - 9)(u + 4) = 0$			factor
$u = 9$	or	$u = -4$	solution in terms of u
$x^2 = 9$	or	$x^2 = -4$	resubstitute x^2 for u
$x = \pm\sqrt{9}$	or	$x = \pm\sqrt{-4}$	square root property
$x = \pm 3$	or	$x = \pm 2i$	simplify

The solutions are $x = -3$, $x = 3$, $x = -2i$, and $x = 2i$. Verify that all solutions check.

☑ **D.** You've just learned how to solve equations in quadratic form

Now try Exercises 65 through 78 ►

E. Applications

Applications of the skills from this section come in many forms. **Number puzzles** and **consecutive integer** exercises help develop the ability to translate written information into algebraic forms **(see Exercises 81 through 84).** Applications involving **geometry** or a stated relationship between two quantities often depend on these skills, and in many scientific fields, equation models involving radicals and rational exponents are commonplace **(see Exercises 99 and 100).**

EXAMPLE 10 ► **Solving a Geometry Application**

A legal size sheet of typing paper has a length equal to 3 in. less than twice its width. If the area of the paper is 119 in^2, find the length and width.

Solution ► Let W represent the width of the paper. Then $2W$ represents twice the width, and $2W - 3$ represents three less than twice the width: $L = 2W - 3$:

$(\text{length})(\text{width}) = \text{area}$	verbal model
$(2W - 3)(W) = 119$	substitute $2W - 3$ for length

Since the equation is not set equal to zero, multiply and write the equation in standard form.

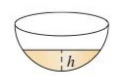
$$2W^2 - 3W = 119 \quad \text{distribute}$$
$$2W^2 - 3W - 119 = 0 \quad \text{subtract 119}$$
$$(2W - 17)(W + 7) = 0 \quad \text{factor}$$
$$W = \tfrac{17}{2} \text{ or } W = -7 \quad \text{solve}$$

We ignore $W = -7$, since the width cannot be negative. The width of the paper is $\frac{17}{2} = 8\frac{1}{2}$ in. and the length is $L = 2\left(\frac{17}{2}\right) - 3$ or 14 in.

Now try Exercises 85 and 86 ►

EXAMPLE 11 ► Solving a Geometry Application

A hemispherical wash basin has a radius of 6 in. The volume of water in the basin can be modeled by $V = 6\pi h^2 - \frac{\pi}{3}h^3$, where h is the height of the water (see diagram). At what height h is the volume of water numerically equal to 15π times the height h?



Solution ► We are essentially asked to solve $V = 6\pi h^2 - \frac{\pi}{3}h^3$ when $V = 15\pi h$.

The equation becomes

$$15\pi h = 6\pi h^2 - \frac{\pi}{3}h^3 \quad \text{original equation, substitute } 15\pi h \text{ for } V$$
$$\frac{\pi}{3}h^3 - 6\pi h^2 + 15\pi h = 0 \quad \text{standard form}$$
$$h^3 - 18h^2 + 45h = 0 \quad \text{multiply by } \tfrac{3}{\pi}$$
$$h(h^2 - 18h + 45) = 0 \quad \text{factor out } h$$
$$h(h - 3)(h - 15) = 0 \quad \text{factored form}$$
$$h = 0 \quad \text{or} \quad h = 3 \quad \text{or} \quad h = 15 \quad \text{result}$$

The "solution" $h = 0$ can be discounted since there would be no water in the basin, and $h = 15$ is too large for this context (the radius is only 6 in.). The only solution that fits this context is $h = 3$.

Check ►

$$15\pi h = 6\pi h^2 - \frac{\pi}{3}h^3 \quad \text{resulting equation}$$
$$15\pi(3) = 6\pi(3)^2 - \frac{\pi}{3}(3)^3 \quad \text{substitute 3 for } h$$
$$45\pi = 6\pi(9) - \frac{\pi}{3}(27) \quad \text{apply exponents}$$
$$45\pi = 54\pi - 9\pi \quad \text{simplify}$$
$$45\pi = 45\pi \checkmark \quad \text{result checks}$$

Now try Exercises 87 and 88 ►

In this section, we noted that extraneous roots can occur when (1) both sides of an equation are multiplied by a variable term (as when solving rational equations) and (2) when both sides of an equation are raised to an even power (as when solving certain radical equations or equations with rational exponents). Example 11 illustrates a third way that extraneous roots can occur, as when a solution checks out fine algebraically, but does not fit the context or physical constraints of the situation.

Revenue Models

In a free-market economy, we know that if the price of an item is decreased, more people will buy it. This is why stores have sales and bargain days. But if the item is sold too cheaply, revenue starts to decline because less money is coming in—even though more sales are made. This phenomenon is analyzed in Example 12, where we use the revenue formula *revenue* = *price* · *number of sales* or $R = P \cdot S$.

EXAMPLE 12 ► **Solving a Revenue Application**

When a popular printer is priced at \$300, Compu-Store will sell 15 printers per week. Using a survey, they find that for each decrease of \$8, two additional sales will be made. What price will result in weekly revenue of \$6500?

Solution ► Let x represent the number of times the price is decreased by \$8. Then $300 - 8x$ represents the new price. Since sales increase by 2 each time the price is decreased, $15 + 2x$ represents the total sales.

$$\begin{aligned} R &= P \cdot S && \text{revenue model} \\ 6500 &= (300 - 8x)(15 + 2x) && R = 6500,\ P = 300 - 8x,\ S = 15 + 2x \\ 6500 &= 4500 + 600x - 120x - 16x^2 && \text{multiply binomials} \\ 0 &= -16x^2 + 480x - 2000 && \text{simplify and write in standard form} \\ 0 &= x^2 - 30x + 125 && \text{divide by } -16 \\ 0 &= (x - 5)(x - 25) && \text{factor} \\ x &= 5 \quad \text{or} \quad x = 25 && \text{result} \end{aligned}$$

Surprisingly, the store's weekly revenue will be \$6500 after 5 decreases of \$8 each (\$40 total), or 25 price decreases of \$8 each (\$200 total). The related selling prices are $300 - 5(8) = \$260$ and $300 - 25(8) = \$100$. To maximize profit, the manager of Compu-Store decides to go with the \$260 selling price.

Now try Exercises 89 and 90 ►

Applications of rational equations can also take many forms. Work and uniform motion exercises help us develop important skills that can be used with more complex equation models. A work example follows here. For more on uniform motion, see **Exercises 95 and 96.**

EXAMPLE 13 ► **Solving a Work Application**

Lyf can clean a client's house in 5 hr, while it takes his partner Angie 4 hr to clean the same house. Both of them want to go to the Cubs' game today, which starts in $2\frac{1}{2}$ hr. If they work together, will they see the first pitch?

Solution ► After 1 hr, Lyf has cleaned $\frac{1}{5}$ and Angie has cleaned $\frac{1}{4}$ of the house, so together $\frac{1}{5} + \frac{1}{4} = \frac{9}{20}$ or 45% of the house has been cleaned. After 2 hr, $2\left(\frac{1}{5}\right) + 2\left(\frac{1}{4}\right)$ or $\frac{2}{5} + \frac{1}{2} = \frac{9}{10}$ or 90% of the house is clean. We can use these two illustrations to form an equation model where H represents hours worked:

$$H\left(\frac{1}{5}\right) + H\left(\frac{1}{4}\right) = 1 \text{ clean house } (1 = 100\%).$$

$$H\left(\frac{1}{5}\right) + H\left(\frac{1}{4}\right) = 1 \quad \text{equation model}$$

$$20H\left(\frac{1}{5}\right) + 20H\left(\frac{1}{4}\right) = 1(20) \quad \text{multiply by LCD of 20}$$

$$4H + 5H = 20 \quad \text{simplify, denominators are eliminated}$$

$$9H = 20 \quad \text{combine like terms}$$

$$H = \frac{20}{9} \quad \text{solve for } H$$

It will take Lyf and Angie $2\frac{2}{9}$ hr (about 2 hr and 13 min) to clean the house. Yes! They will make the first pitch, since Wrigley Field is only 10 min away.

Now try Exercises 93 and 94 ▶

EXAMPLE 14 ▶ Solving an Application Involving a Rational Equation

In Verano City, the cost C to remove industrial waste from drinking water is given by the equation $C = \frac{80P}{100 - P}$, where P is the percent of total pollutants removed and C is the cost in thousands of dollars. If the City Council budgets \$1,520,000 for the removal of these pollutants, what percentage of the waste will be removed?

Solution ▶

$$C = \frac{80P}{100 - P} \quad \text{equation model}$$

$$1520 = \frac{80P}{100 - P} \quad \text{substitute 1520 for } C$$

$$1520(100 - P) = 80P \quad \text{multiply by LCD of } (100 - P)$$

$$152{,}000 = 1600P \quad \text{distribute and simplify}$$

$$95 = P \quad \text{result}$$

On a budget of \$1,520,000, 95% of the pollutants will be removed.

E. You've just learned how to solve applications of various equation types

Now try Exercises 97 and 98 ▶

1.6 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. For rational equations, values that cause a zero denominator must be ______.

2. The equation or formula for revenue models is revenue = ______.

3. "False solutions" to a rational or radical equation are also called ______ roots.

4. Factorable polynomial equations can be solved using the ______ ______ property.

5. Discuss/Explain the power property of equality as it relates to rational exponents and properties of reciprocals. Use the equation $(x - 2)^{\frac{2}{3}} = 9$ for your discussion.

6. One factored form of an equation is shown. Discuss/Explain why $x = -8$ and $x = 1$ are not solutions to the equation, and what must be done to find the actual solutions: $2(x + 8)(x - 1) = -16$.

▶ DEVELOPING YOUR SKILLS

Solve using the zero product property. Be sure each equation is in standard form and factor out any common factors before attempting to solve. Check all answers in the original equation.

7. $22x = x^3 - 9x^2$
8. $x^3 = 13x^2 - 42x$
9. $3x^3 = -7x^2 + 6x$
10. $7x^2 + 15x = 2x^3$
11. $2x^4 - 3x^3 = 9x^2$
12. $-7x^2 = 2x^4 - 9x^3$
13. $2x^4 - 16x = 0$
14. $x^4 + 64x = 0$
15. $x^3 - 4x = 5x^2 - 20$
16. $x^3 - 18 = 9x - 2x^2$
17. $4x - 12 = 3x^2 - x^3$
18. $x - 7 = 7x^2 - x^3$
19. $2x^3 - 12x^2 = 10x - 60$
20. $9x + 81 = 27x^2 + 3x^3$
21. $x^4 - 7x^3 + 4x^2 = 28x$
22. $x^4 + 3x^3 + 9x^2 = -27x$
23. $x^4 - 81 = 0$
24. $x^4 - 1 = 0$
25. $x^4 - 256 = 0$
26. $x^4 - 625 = 0$
27. $x^6 - 2x^4 - x^2 + 2 = 0$
28. $x^6 - 3x^4 - 16x^2 + 48 = 0$
29. $x^5 - x^3 - 8x^2 + 8 = 0$
30. $x^5 - 9x^3 - x^2 + 9 = 0$
31. $x^6 - 1 = 0$
32. $x^6 - 64 = 0$

Solve each equation. Identify any extraneous roots.

33. $\frac{2}{x} + \frac{1}{x+1} = \frac{5}{x^2 + x}$

34. $\frac{3}{m+3} - \frac{5}{m^2 + 3m} = \frac{1}{m}$

35. $\frac{21}{a+2} = \frac{3}{a-1}$

36. $\frac{4}{2y-3} = \frac{7}{3y-5}$

37. $\frac{1}{3y} - \frac{1}{4y} = \frac{1}{y^2}$

38. $\frac{3}{5x} - \frac{1}{2x} = \frac{1}{x^2}$

39. $x + \frac{14}{x-7} = 1 + \frac{2x}{x-7}$

40. $\frac{10}{x-5} + x = 1 + \frac{2x}{x-5}$

41. $\frac{6}{n+3} + \frac{20}{n^2 + n - 6} = \frac{5}{n-2}$

42. $\frac{7}{p+2} - \frac{1}{p^2 + 5p + 6} = -\frac{2}{p+3}$

43. $\frac{a}{2a+1} - \frac{2a^2 + 5}{2a^2 - 5a - 3} = \frac{3}{a-3}$

44. $\frac{-18}{6n^2 - n - 1} + \frac{3n}{2n-1} = \frac{4n}{3n+1}$

Solve for the variable indicated.

45. $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$; for f
46. $\frac{1}{x} - \frac{1}{y} = \frac{1}{z}$; for z
47. $I = \frac{E}{R + r}$; for r
48. $q = \frac{pf}{p - f}$; for p
49. $V = \frac{1}{3}\pi r^2 h$; for h
50. $s = \frac{1}{2}gt^2$; for g
51. $V = \frac{4}{3}\pi r^3$; for r^3
52. $V = \frac{1}{3}\pi r^2 h$; for r^2

Solve each equation and check your solutions by *substitution.* Identify any extraneous roots.

53. a. $-3\sqrt{3x - 5} = -9$ **b.** $x = \sqrt{3x + 1} + 3$
54. a. $-2\sqrt{4x - 1} = -10$ **b.** $-5 = \sqrt{5x - 1} - x$
55. a. $2 = \sqrt[3]{3m - 1}$ **b.** $2\sqrt[3]{7 - 3x} - 3 = -7$
c. $\frac{\sqrt[3]{2m + 3}}{-5} + 2 = 3$ **d.** $\sqrt[3]{2x - 9} = \sqrt[3]{3x + 7}$
56. a. $-3 = \sqrt[3]{5p + 2}$ **b.** $3\sqrt[3]{3 - 4x} - 7 = -4$
c. $\frac{\sqrt[3]{6x - 7}}{4} - 5 = -6$
d. $3\sqrt[3]{x + 3} = 2\sqrt[3]{2x + 17}$
57. a. $\sqrt{x - 9} + \sqrt{x} = 9$
b. $x = 3 + \sqrt{23 - x}$
c. $\sqrt{x - 2} - \sqrt{2x} = -2$
d. $\sqrt{12x + 9} - \sqrt{24x} = -3$
58. a. $\sqrt{x + 7} - \sqrt{x} = 1$
b. $\sqrt{2x + 31} + x = 2$
c. $\sqrt{3x} = \sqrt{x - 3} + 3$
d. $\sqrt{3x + 4} - \sqrt{7x} = -2$

Write the equation in simplified form, then solve. Check all answers by substitution.

59. $x^{\frac{3}{5}} + 17 = 9$

60. $-2x^{\frac{3}{4}} + 47 = -7$

61. $0.\overline{3}x^{\frac{5}{2}} - 39 = 42$

62. $0.\overline{5}x^{\frac{5}{3}} + 92 = -43$

63. $2(x + 5)^{\frac{2}{3}} - 11 = 7$

64. $-3(x - 2)^{\frac{4}{5}} + 29 = -19$

Solve each equation using a *u*-substitution. Check all answers.

65. $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 15 = 0$

66. $x^3 - 9x^{\frac{3}{2}} + 8 = 0$

67. $x^4 - 24x^2 - 25 = 0$

68. $x^4 - 37x^2 + 36 = 0$

69. $(x^2 - 3)^2 + (x^2 - 3) - 2 = 0$

70. $(x^2 + x)^2 - 8(x^2 + x) + 12 = 0$

71. $x^{-2} - 3x^{-1} - 4 = 0$

72. $x^{-2} - 2x^{-1} - 35 = 0$

73. $x^{-4} - 13x^{-2} + 36 = 0$

Use a *u*-substitution to solve each radical equation.

74. $x - 3\sqrt{x + 4} + 4 = 0$

75. $x + 4 = 7\sqrt{x + 4}$

76. $2(x + 1) = 5\sqrt{x + 1} - 2$

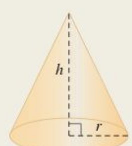
77. $2\sqrt{x + 10} + 8 = 3(x + 10)$

78. $4\sqrt{x - 3} = 3(x - 3) - 4$

▶ WORKING WITH FORMULAS

79. Lateral surface area of a cone: $S = \pi r\sqrt{r^2 + h^2}$

The lateral surface area (surface area excluding the base) S of a cone is given by the formula shown, where r is the radius of the base and h is the height of the cone. (a) Solve the equation for h. (b) Find the surface area of a cone that has a radius of 6 m and a height of 10 m. Answer in simplest form.



80. Painted area on a canvas: $A = \dfrac{4x^2 + 60x + 104}{x}$

A rectangular canvas is to contain a small painting with an area of 52 in^2, and requires 2-in. margins on the left and right, with 1-in. margins on the top and bottom for framing. The total area of such a canvas is given by the formula shown, where x is the height of the *painted* area.

a. What is the area A of the canvas if the height of the painting is $x = 10$ in.?

b. If the area of the canvas is $A = 120\ \text{in}^2$, what are the dimensions of the painted area?

▶ APPLICATIONS

Find all real numbers that satisfy the following descriptions.

81. When the cube of a number is added to twice its square, the result is equal to 18 more than 9 times the number.

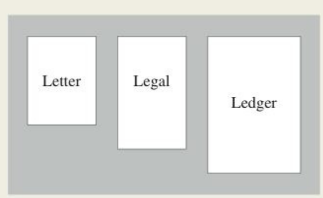
82. Four times a number decreased by 20 is equal to the cube of the number decreased by 5 times its square.

83. Find three consecutive even integers such that 4 times the largest plus the fourth power of the smallest is equal to the square of the remaining even integer increased by 24.

84. Find three consecutive integers such that the sum of twice the largest and the fourth power of the smallest is equal to the square of the remaining integer increased by 75.

85. Envelope sizes: Large mailing envelopes often come in standard sizes, with 5- by 7-in. and 9- by 12-in. envelopes being the most common. The next larger size envelope has an area of 143 in^2, with a length that is 2 in. longer than the width. What are the dimensions of the larger envelope?

86. Paper sizes: Letter size paper is 8.5 in. by 11 in. Legal size paper is $8\frac{1}{2}$ in. by 14 in. The next larger (common) size of paper has an area of 187 in^2, with a length that is 6 in. longer than the width. What are the dimensions of the Ledger size paper?



87. Composite figures—grain silos: Grain silos can be described as a hemisphere sitting atop a cylinder. The interior volume V of the silo can be modeled by $V = \frac{2}{3}\pi r^3 + \pi r^2 h$, where h is the height of a cylinder with radius r. For a cylinder 6 m tall, what radius would give the silo a volume that is numerically equal to 24π times this radius?

88. Composite figures—gelatin capsules: The gelatin capsules manufactured for cold and flu medications are shaped like a cylinder with a hemisphere on each end. The interior volume V of each capsule can be modeled by $V = \frac{4}{3}\pi r^3 + \pi r^2 h$, where h is the height of the cylindrical portion and r is its radius. If the cylindrical portion of the capsule is 8 mm long ($h = 8$ mm), what radius would give the capsule a volume that is numerically equal to 15π times this radius?

89. Running shoes: When a popular running shoe is priced at \$70, The Shoe House will sell 15 pairs each week. Using a survey, they have determined that for each decrease of \$2 in price, 3 additional pairs will be sold each week. What selling price will give a weekly revenue of \$2250?

90. Cell phone charges: A cell phone service sells 48 subscriptions each month if their monthly fee is \$30. Using a survey, they find that for each decrease of \$1, 6 additional subscribers will join. What charge(s) will result in a monthly revenue of \$2160?

Projectile height: In the absence of resistance, the height of an object that is projected upward can be modeled by the equation $h = -16t^2 + vt + k$, where h represents the height of the object (in feet) t sec after it has been thrown, v represents the initial velocity (in feet per second), and k represents the height of the object when $t = 0$ (before it has been thrown). Use this information to complete the following problems.

91. From the base of a canyon that is 480 feet deep (*below* ground level $\rightarrow -480$), a slingshot is used to shoot a pebble upward toward the canyon's rim. If the initial velocity is 176 ft per second:

a. How far is the pebble below the rim after 4 sec?

b. How long until the pebble returns to the bottom of the canyon?

c. What happens at $t = 5$ and $t = 6$ sec? Discuss and explain.

92. A model rocket blasts off. A short time later, at a velocity of 160 ft/sec and a height of 240 ft, it runs out of fuel and becomes a projectile.

a. How high is the rocket three seconds later? Four seconds later?

b. How long will it take the rocket to attain a height of 640 ft?

c. How many times is a height of 384 ft attained? When do these occur?

d. How many seconds until the rocket returns to the ground?

93. Printing newspapers: The editor of the school newspaper notes the college's new copier can complete the required print run in 20 min, while the back-up copier took 30 min to do the same amount of work. How long would it take if both copiers are used?

94. Filling a sink: The cold water faucet can fill a sink in 2 min. The drain can empty a full sink in 3 min. If the faucet were left on and the drain was left open, how long would it take to fill the sink?

95. Triathalon competition: As one part of a Mountain-Man triathalon, participants must row a canoe 5 mi down river (with the current), circle a buoy and row 5 mi back up river (against the current) to the starting point. If the current is flowing at a steady rate of 4 mph and Tom Chaney made the round-trip in 3 hr, how fast can he row in still water? (*Hint:* The time rowing down river and the time rowing up river must add up to 3 hr.)

96. Flight time: The flight distance from Cincinnati, Ohio, to Chicago, Illinois, is approximately 300 mi. On a recent round-trip between these cities in my private plane, I encountered a steady 25 mph headwind on the way to Chicago, with a 25 mph tailwind on the return trip. If my total flying time came to exactly 5 hr, what was my flying time to Chicago? What was my flying time back to Cincinnati? (*Hint:* The flight time between the two cities must add up to 5 hr.)

97. Pollution removal: For a steel mill, the cost C (in millions of dollars) to remove toxins from the resulting sludge is given by $C = \frac{92P}{100 - P}$, where P is the percent of the toxins removed. What percent can be removed if the mill spends \$100,000,000 on the cleanup? Round to tenths of a percent.

98. Wildlife populations: The Department of Wildlife introduces 60 elk into a new game reserve. It is projected that the size of the herd will grow according to the equation $N = \frac{10(6 + 3t)}{1 + 0.05t}$, where N is the number of elk and t is the time in years. If recent counts find 225 elk, approximately how many years have passed? (See Section R.5, Exercise 82.)

99. Planetary motion: The time T (in days) for a planet to make one revolution around the sun is modeled by $T = 0.407R^{\frac{3}{2}}$, where R is the maximum radius of the planet's orbit in millions of miles (*Kepler's third law of planetary motion*). Use the equation to approximate the maximum radius of each orbit, given the number of days it takes for one revolution. (See Section R.6, Exercises 53 and 54.)

a. Mercury: 88 days
b. Venus: 225 days
c. Earth: 365 days
d. Mars: 687 days
e. Jupiter: 4,333 days
f. Saturn: 10,759 days

100. Wind-powered energy: If a wind-powered generator is delivering P units of power, the velocity V of the wind (in miles per hour) can be determined using $V = \sqrt[3]{\frac{P}{k}}$, where k is a constant that depends on the size and efficiency of the generator. Given $k = 0.004$, approximately how many units of power are being delivered if the wind is blowing at 27 miles per hour? (See Section R.6, Exercise 56.)

► EXTENDING THE CONCEPT

101. To solve the equation $3 - \frac{8}{x + 3} = \frac{1}{x}$, a student multiplied by the LCD $x(x + 3)$, simplified, and got this result: $3 - 8x = (x + 3)$. Identify and fix the mistake, then find the correct solution(s).

102. The expression $x^2 - 7$ is not factorable using *integer values.* But the expression *can be written* in the form $x^2 - (\sqrt{7})^2$, enabling us to factor it as a binomial and its conjugate: $(x + \sqrt{7})(x - \sqrt{7})$. Use this idea to solve the following equations:

a. $x^2 - 5 = 0$ **b.** $n^2 - 19 = 0$
c. $4v^2 - 11 = 0$ **d.** $9w^2 - 11 = 0$

Determine the values of x for which each expression represents a real number.

103. $\frac{\sqrt{x - 1}}{x^2 - 4}$ **104.** $\frac{x^2 - 4}{\sqrt{x - 1}}$

105. As an extension of working with absolute values, try the following exercises.

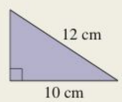
Recall that for $|X| = k$, $X = -k$ or $X = k$.

a. $|x^2 - 2x - 25| = 10$
b. $|x^2 - 5x - 10| = 4$
c. $|x^2 - 4| = x + 2$
d. $|x^2 - 9| = -x + 3$
e. $|x^2 - 7x| = -x + 7$
f. $|x^2 - 5x - 2| = x + 5$

▶ MAINTAINING YOUR SKILLS

106. **(1.1)** Two jets take off on parallel runways going in opposite directions. The first travels at a rate of 250 mph and the second at 325 mph. How long until they are 980 miles apart?

107. **(R.6)** Find the missing side.



108. **(R.3)** Simplify using properties of exponents:

$2^{-1} + (2x)^0 + 2x^0$

109. **(1.2)** Graph the relation given:

$2x - 3 < 7$ and $x + 2 > 1$

SUMMARY AND CONCEPT REVIEW

SECTION 1.1 Linear Equations, Formulas, and Problem Solving

KEY CONCEPTS

- An equation is a statement that two expressions are equal.
- Replacement values that make an equation true are called solutions or roots.
- Equivalent equations are those that have the same solution set.
- To solve an equation we use the distributive property and the properties of equality to write a sequence of simpler, equivalent equations until the solution is obvious. A guide for solving linear equations appears on page 75.
- If an equation contains fractions, multiply both sides by the LCD of all denominators, then solve.
- Solutions to an equation can be checked using back-substitution, by replacing the variable with the proposed solution and verifying the left-hand expression is equal to the right.
- An equation can be:
  1. an identity, one that is always true, with a solution set of all real numbers.
  2. a contradiction, one that is never true, with the empty set as the solution set.
  3. conditional, or one that is true/false depending on the value(s) input.
- To solve formulas for a specified variable, focus on the object variable and apply properties of equality to write this variable in terms of all others.
- The basic elements of good problem solving include:
  1. Gathering and organizing information
  2. Making the problem visual
  3. Developing an equation model
  4. Using the model to solve the application

For a complete review, see the problem-solving guide on page 78.

EXERCISES

1. Use substitution to determine if the indicated value is a solution to the equation given.

   a. $6x - (2 - x) = 4(x - 5), x = -6$ **b.** $\frac{3}{4}b + 2 = \frac{5}{2}b + 16, b = -8$ **c.** $4d - 2 = -\frac{1}{2} + 3d, d = \frac{3}{2}$

Solve each equation.

2. $-2b + 7 = -5$ **3.** $3(2n - 6) + 1 = 7$ **4.** $4m - 5 = 11m + 2$

5. $\frac{1}{2}x + \frac{2}{3} = \frac{3}{4}$ **6.** $6p - (3p + 5) - 9 = 3(p - 3)$ **7.** $-\frac{g}{6} = 3 - \frac{1}{2} - \frac{5g}{12}$

Solve for the specified variable in each formula or literal equation.

8. $V = \pi r^2 h$ for h

9. $P = 2L + 2W$ for L

10. $ax + b = c$ for x

11. $2x - 3y = 6$ for y

Use the problem-solving guidelines (page 78) to solve the following applications.

12. At a large family reunion, two kegs of lemonade are available. One is 2% sugar (too sour) and the second is 7% sugar (too sweet). How many gallons of the 2% keg, must be mixed with 12 gallons of the 7% keg to get a 5% mix?

13. A rectangular window with a width of 3 ft and a height of 4 ft is topped by a semi-circular window. Find the total area of the window.

14. Two cyclists start from the same location and ride in opposite directions, one riding at 15 mph and the other at 18 mph. If their radio phones have a range of 22 mi, how many minutes will they be able to communicate?

SECTION 1.2 Linear Inequalities in One Variable

KEY CONCEPTS

- Inequalities are solved using properties similar to those for solving equalities (see page 87). The one exception is the multiplicative property of inequality, since the truth of the resulting statement depends on whether a positive or negative quantity is used.
- Solutions to an inequality can be graphed on a number line, stated using a simple inequality, or expressed using set or interval notation.
- For two sets A and B: A intersect B ($A \cap B$) is the set of elements in both A **and** B (i.e., *elements common to both sets*). A union B ($A \cup B$) is the set of elements in either A **or** B (i.e., *all elements from either set*).
- Compound inequalities are formed using the conjunctions "and"/"or." These can be either a joint inequality as in $-3 < x \le 5$, or a disjoint inequality, as in $x < -2$ or $x > 7$.

EXERCISES

Use inequality symbols to write a mathematical model for each statement.

15. You must be 35 yr old or older to run for president of the United States.

16. A child must be under 2 yr of age to be admitted free.

17. The speed limit on many interstate highways is 65 mph.

18. Our caloric intake should not be less than 1200 calories per day.

Solve the inequality and write the solution using interval notation.

19. $7x > 35$

20. $-\frac{3}{5}m < 6$

21. $2(3m - 2) \le 8$

22. $-1 < \frac{1}{3}x + 2 \le 5$

23. $-4 < 2b + 8$ and $3b - 5 > -32$

24. $-5(x + 3) > -7$ or $x - 5.2 > -2.9$

25. Find the allowable values for each of the following. Write your answer in interval notation.

a. $\frac{7}{n - 3}$ **b.** $\frac{5}{2x - 3}$ **c.** $\sqrt{x + 5}$ **d.** $\sqrt{-3n + 18}$

26. Latoya has earned grades of 72%, 95%, 83%, and 79% on her first four exams. What grade must she make on her fifth and last exam so that her average is 85% or more?

SECTION 1.3 Absolute Value Equations and Inequalities

KEY CONCEPTS

- To solve absolute value equations and inequalities, begin by writing the equation in simplified form, with the absolute value isolated on one side.
- If X represents an algebraic expression and k is a nonnegative constant:
  - Absolute value equations: $|X| = k$ is equivalent to $X = -k$ or $X = k$
  - "Less than" inequalities: $|X| < k$ is equivalent to $-k < X < k$
  - "Greater than" inequalities: $|X| > k$ is equivalent to $X < -k$ or $X > k$
- These properties also apply when the symbols "$\leq$" or "$\geq$" are used.
- If the absolute value quantity has been isolated on the left, the solution to a less-than inequality will be a single interval, while the solution to a greater-than inequality will consist of two disjoint intervals.
- The multiplicative property states that for algebraic expressions A and B, $|AB| = |A||B|$.

EXERCISES

Solve each equation or inequality. Write solutions to inequalities in interval notation.

27. $7 = |x - 3|$
28. $-2|x + 2| = -10$
29. $|-2x + 3| = 13$

30. $\dfrac{|2x + 5|}{3} + 8 = 9$
31. $-3|x + 2| - 2 < -14$
32. $\left|\dfrac{x}{2} - 9\right| \leq 7$

33. $|3x + 5| = -4$
34. $3|x + 1| < -9$
35. $2|x + 1| > -4$

36. $5|m - 2| - 12 \leq 8$
37. $\dfrac{|3x - 2|}{2} + 6 \geq 10$

38. Monthly rainfall received in Omaha, Nebraska, rarely varies by more than 1.7 in. from an average of 2.5 in. per month. (a) Use this information to write an absolute value inequality model, then (b) solve the inequality to find the highest and lowest amounts of monthly rainfall for this city.

SECTION 1.4 Complex Numbers

KEY CONCEPTS

- The italicized i represents the number whose square is -1. This means $i^2 = -1$ and $i = \sqrt{-1}$.
- Larger powers of i can be simplified using $i^4 = 1$.
- For $k > 0$, $\sqrt{-k} = i\sqrt{k}$ and we say the expression has been *written in terms of i.*
- The standard form of a *complex number* is $a + bi$, where a is the *real number part* and bi is the *imaginary number part.*
- To add or subtract complex numbers, combine the like terms.
- For any complex number $a + bi$, its *complex conjugate* is $a - bi$.
- The *product* of a complex number and its conjugate is a real number.
- The commutative, associative, and distributive properties also apply to complex numbers and are used to perform basic operations.
- To multiply complex numbers, use the F-O-I-L method and simplify.
- To find a *quotient* of complex numbers, multiply the numerator and denominator by the conjugate of the denominator.

EXERCISES

Simplify each expression and write the result in standard form.

39. $\sqrt{-72}$
40. $6\sqrt{-48}$
41. $\dfrac{-10 + \sqrt{-50}}{5}$

42. $\sqrt{3}\sqrt{-6}$
43. i^{57}

Perform the operation indicated and write the result in standard form.

44. $(5 + 2i)^2$ **45.** $\dfrac{5i}{1 - 2i}$ **46.** $(-3 + 5i) - (2 - 2i)$

47. $(2 + 3i)(2 - 3i)$ **48.** $4i(-3 + 5i)$

Use substitution to show the given complex number and its conjugate are solutions to the equation shown.

49. $x^2 - 9 = -34$; $x = 5i$ **50.** $x^2 - 4x + 9 = 0$; $x = 2 + i\sqrt{5}$

SECTION 1.5 Solving Quadratic Equations

KEY CONCEPTS

- The standard form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. In words, we say the equation is written in decreasing order of degree and set equal to zero.
- The coefficient of the squared term a is called the *leading coefficient*, b is called the *linear coefficient*, and c is called the *constant term*.
- The square root property of equality states that if $X^2 = k$, where $k \geq 0$, then $X = \sqrt{k}$ or $X = -\sqrt{k}$.
- Factorable quadratics can be solved using the zero product property, which states that if the product of two factors is zero, then one, the other, or both must be equal to zero. Symbolically, if $A \cdot B = 0$, then $A = 0$ or $B = 0$.
- Quadratic equations can also be solved by *completing the square*, or using the *quadratic formula*.
- If the discriminant $b^2 - 4ac = 0$, the equation has one real (repeated) root. If $b^2 - 4ac > 0$, the equation has two real roots; and if $b^2 - 4ac < 0$, the equation has two complex roots.

EXERCISES

51. Determine whether the given equation is quadratic. If so, write the equation in standard form and identify the values of a, b, and c.

a. $-3 = 2x^2$ **b.** $7 = -2x + 11$ **c.** $99 = x^2 - 8x$ **d.** $20 = 4 - x^2$

52. Solve by factoring.

a. $x^2 - 3x - 10 = 0$ **b.** $2x^2 - 50 = 0$ **c.** $3x^2 - 15 = 4x$ **d.** $x^3 - 3x^2 = 4x - 12$

53. Solve using the square root property of equality.

a. $x^2 - 9 = 0$ **b.** $2(x - 2)^2 + 1 = 11$ **c.** $3x^2 + 15 = 0$ **d.** $-2x^2 + 4 = -46$

54. Solve by completing the square. Give real number solutions in exact and approximate form.

a. $x^2 + 2x = 15$ **b.** $x^2 + 6x = 16$ **c.** $-4x + 2x^2 = 3$ **d.** $3x^2 - 7x = -2$

55. Solve using the quadratic formula. Give solutions in both exact and approximate form.

a. $x^2 - 4x = -9$ **b.** $4x^2 + 7 = 12x$ **c.** $2x^2 - 6x + 5 = 0$

Solve the following quadratic applications. For 56 and 57, recall the height of a projectile is modeled by $h = -16t^2 + v_0t + k$.

56. A projectile is fired upward from ground level with an initial velocity of 96 ft/sec. (a) To the nearest tenth of a second, how long until the object first reaches a height of 100 ft? (b) How long until the object is again at 100 ft? (c) How many seconds until it returns to the ground?

57. A person throws a rock upward from the top of an 80-ft cliff with an initial velocity of 64 ft/sec. (a) To the nearest tenth of a second, how long until the object is 120 ft high? (b) How long until the object is again at 120 ft? (c) How many seconds until the object hits the ground at the base of the cliff?

58. The manager of a large, 14-screen movie theater finds that if he charges \$2.50 per person for the matinee, the average daily attendance is 4000 people. With every increase of 25 cents the attendance drops an average of 200 people. (a) What admission price will bring in a revenue of \$11,250? (b) How many people will purchase tickets at this price?

59. After a storm, the Johnson's basement flooded and the water needed to be pumped out. A cleanup crew is sent out with two powerful pumps to do the job. Working alone (if one of the pumps were needed at another job), the larger pump would be able to clear the basement in 3 hr less time than the smaller pump alone. Working together, the two pumps can clear the basement in 2 hr. How long would it take the smaller pump alone?

SECTION 1.6 Solving Other Types of Equations

KEY CONCEPTS

- Certain equations of higher degree can be solved using factoring skills and the zero product property.
- To solve rational equations, clear denominators using the LCD, noting values that must be excluded.
- Multiplying an equation by a variable quantity sometimes introduces extraneous solutions. Check all results in the original equation.
- To solve radical equations, isolate the radical on one side, then apply the appropriate "nth power" to free up the radicand. Repeat the process if needed. See flowchart on page 132.
- For equations with a rational exponent $\frac{m}{n}$, isolate the variable term and raise both sides to the $\frac{n}{m}$ power. If m is even, there will be two real solutions.
- Any equation that can be written in the form $u^2 + bu + c = 0$, where u represents an algebraic expression, is said to be in quadratic form and can be solved using u-substitution and standard approaches.

EXERCISES

Solve by factoring.

60. $x^3 - 7x^2 = 3x - 21$

61. $3x^3 + 5x^2 = 2x$

62. $x^4 - 8x = 0$

63. $x^4 - \frac{1}{16} = 0$

Solve each equation.

64. $\frac{3}{5x} + \frac{7}{10} = \frac{1}{4x}$

65. $\frac{3h}{h + 3} - \frac{7}{h^2 + 3h} = \frac{1}{h}$

66. $\frac{2n}{n + 2} - \frac{3}{n - 4} = \frac{n^2 + 20}{n^2 - 2n - 8}$

67. $\frac{\sqrt{x^2 + 7}}{2} + 3 = 5$

68. $3\sqrt{x + 4} = x + 4$

69. $\sqrt{3x + 4} = 2 - \sqrt{x + 2}$

70. $3\left(x - \frac{1}{4}\right)^{-\frac{3}{2}} = \frac{8}{9}$

71. $-2(5x + 2)^{\frac{2}{3}} + 17 = -1$

72. $(x^2 - 3x)^2 - 14(x^2 - 3x) + 40 = 0$

73. $x^4 - 7x^2 = 18$

74. The science of *allometry* studies the growth of one aspect of an organism relative to the entire organism or to a set standard. Allometry tells us that the amount of food F (in kilocalories per day) an herbivore must eat to survive is related to its weight W (in grams) and can be approximated by the equation $F \approx 1.5W^{\frac{3}{4}}$.

a. How many kilocalories per day are required by a 160-kg gorilla (160 kg = 160,000 g)?

b. If an herbivore requires 40,500 kilocalories per day, how much does it weigh?

75. The area of a common stenographer's tablet, commonly called a *steno book,* is 54 in^2. The length of the tablet is 3 in. more than the width. Model the situation with a quadratic equation and find the dimensions of the tablet.

76. A batter has just flied out to the catcher, who catches the ball while standing on home plate. If the batter made contact with the ball at a height of 4 ft and the ball left the bat with an initial velocity of 128 ft/sec, how long will it take the ball to reach a height of 116 ft? How high is the ball 5 sec after contact? If the catcher catches the ball at a height of 4 ft, how long was it airborne?

77. Using a survey, a firewood distributor finds that if they charge \$50 per load, they will sell 40 loads each winter month. For each decrease of \$2, five additional loads will be sold. What selling price(s) will result in new monthly revenue of \$2520?

MIXED REVIEW

1. Find the allowable values for each expression. Write your response in interval notation.

a. $\dfrac{10}{\sqrt{x-8}}$ **b.** $\dfrac{-5}{3x+4}$

2. Perform the operations indicated.

a. $\sqrt{-18}+\sqrt{-50}$ **b.** $(1-2i)^2$

c. $\dfrac{3i}{1+i}$ **d.** $(2+i\sqrt{3})(2-i\sqrt{3})$

3. Solve each equation or inequality.

a. $-2x^3+4x^2=50x-100$

b. $-3x^4-375x=0$ **c.** $-2|3x+1|=-12$

d. $-3\left|\dfrac{x}{3}-5\right|\le -12$ **e.** $v^{\frac{4}{3}}=81$

f. $-2(x+1)^{\frac{1}{4}}=-6$

Solve for the variable indicated.

4. $V=\dfrac{1}{3}\pi r^2h+\dfrac{2}{3}\pi r^3$; for h **5.** $3x+4y=-12$; for y

Solve as indicated, using the method of your choice.

6. a. $-20\le 4x+8<56$

b. $-2x+7\le 12$ and $3-4x>-5$

7. a. $5x-(2x-3)+3x=-4(5+x)+3$

b. $\dfrac{n}{5}-2=2-\dfrac{5}{3}-\dfrac{4}{15}n$

8. $5x(x-10)(x+1)=0$

9. $x^2-18x+77=0$ **10.** $3x^2-10=5-x+x^2$

11. $4x^2-5=19$ **12.** $3(x+5)^2-3=30$

13. $25x^2+16=40x$ **14.** $3x^2-7x+3=0$

15. $2x^4-50=0$

16. a. $\dfrac{2}{x}-\dfrac{x}{5x+12}=0$ **b.** $\dfrac{1}{n-1}-\dfrac{2}{n^2-1}=-\dfrac{1}{2}$

c. $\dfrac{2x}{x+3}-\dfrac{36}{x^2-9}=\dfrac{x}{x-3}$

17. a. $\sqrt{2v-3}+3=v$

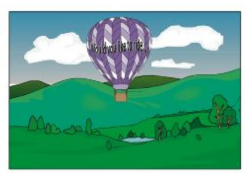
b. $\sqrt[3]{x^2-9}+\sqrt[3]{x-11}=0$

c. $\sqrt{x+7}-\sqrt{2x}=1$

18. The local Lion's Club rents out two banquet halls for large meetings and other events. The records show that when they charge \$250 per day for use of the halls, there are an average of 156 bookings per year. For every increase of \$20 per day, there will be three less bookings. (a) What price per day will bring in \$61,950 for the year? (b) How many bookings will there be at the price from part (a)?

19. The Jefferson College basketball team has two guards who are 6′3″ tall and two forwards who are 6′7″ tall. How tall must their center be to ensure the "starting five" will have an average height of at least 6′6″?

20. The volume of an inflatable hot-air balloon can be approximated using the formulas for a hemisphere and a cone: $V=\frac{2}{3}\pi r^3+\frac{1}{3}\pi r^2h$. Assume the conical portion has height $h=24$ ft. During inflation, what is the radius of the balloon at the moment the volume of air is numerically equal to 126π times this radius?



PRACTICE TEST

1. Solve each equation.

a. $-\frac{2}{3}x - 5 = 7 - (x + 3)$

b. $-5.7 + 3.1x = 14.5 - 4(x + 1.5)$

c. $P = C + kC$; for C

d. $2|2x + 5| - 17 = -11$

2. How much water that is 102°F must be mixed with 25 gal of water at 91°F, so that the resulting temperature of the water will be 97°F?

3. Solve each equation or inequality.

a. $-\frac{2}{5}x + 7 < 19$

b. $-1 < 3 - x \leq 8$

c. $\frac{1}{2}x + 3 < 9$ or $\frac{2}{3}x - 1 \geq 3$

d. $\frac{1}{2}|x - 3| + \frac{5}{4} = \frac{7}{4}$

e. $-\frac{2}{3}|x + 1| - 5 < -7$

4. To make the bowling team, Jacques needs a three-game average of 160. If he bowled 141 and 162 for the first two games, what score S must be obtained in the third game so that his average is at least 160?

Solve each equation.

5. $z^2 - 7z - 30 = 0$ **6.** $x^2 + 25 = 0$

7. $(x - 1)^2 + 3 = 0$ **8.** $x^4 + 16 = 17x^2$

9. $3x^2 - 20x = -12$

10. $4x^3 + 8x^2 - 9x - 18 = 0$

11. $\frac{2}{x - 3} + \frac{2x}{x + 2} = \frac{x^2 + 16}{x^2 - x - 6}$

12. $\frac{4}{x - 3} + 2 = \frac{5x}{x^2 - 9}$

13. $\sqrt{x} + 1 = \sqrt{2x - 7}$

14. $(x + 3)^{\frac{-2}{3}} = \frac{1}{4}$

15. The Spanish Club at Rock Hill Community College has decided to sell tins of gourmet popcorn as a fund-raiser. The suggested selling price is \$3.00 per tin, but Maria, who also belongs to the Math Club, decides to take a survey to see if they can increase "the fruits of their labor." The survey shows it's likely that 120 tins will be sold on campus at the \$3.00 price, and for each price increase of \$0.10, 2 fewer tins will be sold. (a) What price per tin will bring in a revenue of \$405? (b) How many tins will be sold at the price from part (a)?

16. Due to the seasonal nature of the business, the revenue of Wet Willey's Water World can be modeled by the equation $r = -3t^2 + 42t - 135$, where t is the time in months ($t = 1$ corresponds to January) and r is the dollar revenue in thousands. (a) What month does Wet Willey's open? (b) What month does Wet Willey's close? (c) Does Wet Willey's bring in more revenue in July or August? How much more?

Simplify each expression.

17. $\frac{-8 + \sqrt{-20}}{6}$ **18.** i^{39}

19. Given $x = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $y = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ find

a. $x + y$ **b.** $x - y$ **c.** xy

20. Compute the quotient: $\frac{3i}{1 - i}$.

21. Find the product: $(3i + 5)(5 - 3i)$.

22. Show $x = 2 - 3i$ is a solution of $x^2 - 4x + 13 = 0$.

23. Solve by completing the square.

a. $2x^2 - 20x + 49 = 0$

b. $2x^2 - 5x = -4$

24. Solve using the quadratic formula.

a. $3x^2 + 2 = 6x$ **b.** $x^2 = 2x - 10$

25. Allometric studies tell us that the necessary food intake F (in grams per day) of nonpasserine birds (birds other than song birds and other small birds) can be modeled by the equation $F \approx 0.3W^{\frac{3}{4}}$, where W is the bird's weight in grams. (a) If my Green-winged macaw weighs 1296 g, what is her anticipated daily food intake? (b) If my blue-headed pionus consumes 19.2 g per day, what is his estimated weight?

CALCULATOR EXPLORATION AND DISCOVERY

Evaluating Expressions and Looking for Patterns

These "explorations" are designed to explore the full potential of a graphing calculator, as well as to use this potential to investigate patterns and discover connections that might otherwise be overlooked. In this *Exploration and Discovery,* we point out the various ways an expression can be evaluated on a graphing calculator. Some ways seem easier, faster, and/or better than others, but each has advantages and disadvantages depending on the task at hand, and it will help to be aware of them all for future use.

One way to evaluate an expression is to use the TABLE feature of a graphing calculator, with the expression entered as Y_1 on the Y= screen. If you want the calculator to generate inputs, use the 2nd WINDOW (**TBLSET**) screen to indicate a starting value

(**TblStart=**) and an increment value (**ΔTbl=**), and set the calculator in **Indpnt:** AUTO **ASK** mode (to input specific values, the calculator should be in **Indpnt: AUTO** ASK mode). After pressing 2nd GRAPH (**TABLE**), the calculator shows the corresponding input and output values. For help with the basic TABLE feature of the TI-84 Plus, you can visit Section R.7 at www.mhhe.com/coburn.

Expressions can also be evaluated on the home screen for a single value or a series of values. Enter the expression $-\frac{3}{4}x + 5$ on the Y= screen (see Figure 1.15) and use 2nd MODE (**QUIT**) to get back to the home screen. To evaluate this expression, access Y_1 using VARS ► (**Y-VARS**), and use the first option **1:Function** ENTER. This brings us to a submenu where any of the equations Y_1 through Y_0 (actually Y_{10}) can be accessed. Since the default setting is the one we need **1:Y1,** simply press ENTER and Y_1 appears on the home screen. To evaluate a single input, simply enclose it in parentheses. To evaluate more than one input, enter the numbers as a set of values with the set enclosed in parentheses. In Figure 1.16, Y_1 has been evaluated for $x = -4$, then simultaneously for $x = -4, -2, 0$, and 2.

A third way to evaluate expressions is using a list, with the desired inputs entered in List 1 (L1), and List 2 (L2) defined in terms of L1. For example, $L2 = -\frac{3}{4}L1 + 5$ will return the same values for inputs of $-4, -2, 0$, and 2 seen previously on the home screen (remember to clear the lists first). Lists are accessed by pressing STAT **1:Edit.** Enter the numbers $-4, -2, 0$ and 2 in L1, then use the right arrow ► to move to L2. It is important to note that you *next press the up arrow key* ▲ so that the cursor overlies L2. The bottom of the screen now reads **L2=** (see Figure 1.17) and the calculator is waiting for us to define L2. After entering $L2 = -\frac{3}{4}L1 + 5$ and pressing ENTER we obtain the same outputs as before (see Figure 1.18).

The advantage of using the "list" method is that we can *further explore or experiment with the output values* in a search for patterns.

Exercise 1: Evaluate the expression $0.2L1 + 3$ on the list screen, using consecutive integer inputs from -6 to 6 inclusive. What do you notice about the outputs?

Exercise 2: Evaluate the expression $\sqrt{2}L1 - \sqrt{9.1}$ on the list screen, using consecutive integer inputs from -6 to 6 inclusive. We suspect there is a pattern to the output values, but this time the pattern is very difficult to see. Compute the difference between a few successive outputs from L2 [for Example $L2(1) - L2(2)$]. What do you notice?

Figure 1.15

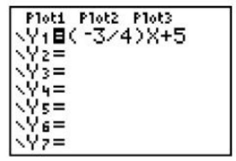


Figure 1.16

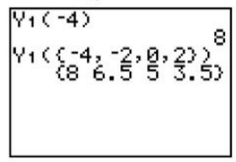


Figure 1.17

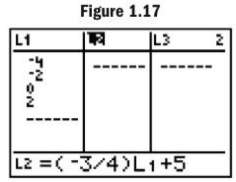
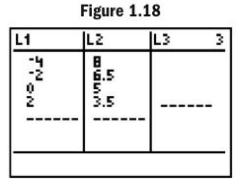


Figure 1.18



STRENGTHENING CORE SKILLS

An Alternative Method for Checking Solutions to Quadratic Equations

To solve $x^2 - 2x - 15 = 0$ by factoring, students will often begin by looking for two numbers whose product is -15 (the constant term) and whose sum is -2 (the linear coefficient). The two numbers are -5 and 3 since $(-5)(3) = -15$ and $-5 + 3 = -2$. In factored form, we have $(x - 5)(x + 3) = 0$ with solutions $x_1 = 5$ and $x_2 = -3$. When these solutions are compared *to the original coefficients,* we can still see the sum/product relationship, but note that while $(5)(-3) = -15$ still gives the constant term, $5 + (-3) = 2$ gives the linear coefficient *with opposite sign.* Although more difficult to accomplish, this method can be applied to *any* factorable quadratic equation $ax^2 + bx + c = 0$ if we divide through by a, giving $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. For $2x^2 - x - 3 = 0$, we divide both sides by 2 and obtain $x^2 - \frac{1}{2}x - \frac{3}{2} = 0$, then look for two numbers whose product is $-\frac{3}{2}$ and whose sum is $-\frac{1}{2}$. The numbers are $-\frac{3}{2}$ and 1 since $\left(-\frac{3}{2}\right)(1) = -\frac{3}{2}$ and $-\frac{3}{2} + 1 = -\frac{1}{2}$, showing the solutions are $x_1 = \frac{3}{2}$ and $x_2 = -1$. We again note the product of the solutions is the constant $-\frac{3}{2} = \frac{c}{a}$, and the sum of the solutions is the linear coefficient *with opposite sign:* $\frac{1}{2} = -\frac{b}{a}$. No one actually promotes this method for solving trinomials where $a \neq 1$, but it does illustrate an important and useful concept:

If x_1 and x_2 are the two roots of $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, then $x_1x_2 = \frac{c}{a}$ and $x_1 + x_2 = -\frac{b}{a}$.

Justification for this can be found by taking the product and sum of the general solutions $x_1 = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$. Although the computation looks impressive, the product can be computed as a binomial times its conjugate, and the radical parts add to zero for the sum, each yielding the results as already stated.

This observation provides a useful technique for checking solutions to a quadratic equation, *even those having irrational or complex roots!* Check the solutions shown in these exercises.

Exercise 1: $2x^2 - 5x - 7 = 0$

$x_1 = \frac{7}{2}$

$x_2 = -1$

Exercise 2: $2x^2 - 4x - 7 = 0$

$x_1 = \frac{2 + 3\sqrt{2}}{2}$

$x_2 = \frac{2 - 3\sqrt{2}}{2}$

Exercise 3: $x^2 - 10x + 37 = 0$

$x_1 = 5 + 2\sqrt{3}\,i$

$x_2 = 5 - 2\sqrt{3}\,i$

Exercise 4: Verify this sum/product check by computing the sum and product of the general solutions.