11

Additional Topics in Algebra

CHAPTER OUTLINE

CHAPTER CONNECTIONS

For a corporation of any size, decisions made by upper management often depend on a large number of factors, with the desired outcome attainable in many different ways. For instance, consider a legal firm that specializes in family law, with a support staff of 15 employees—6 paralegals and 9 legal assistants. Due to recent changes in the law, the firm wants to send some combination of five support staff to a conference dedicated to the new changes. In Chapter 11, we'll see how counting techniques and probability can be used to determine the various ways such a group can be randomly formed, even if certain constraints are imposed. This application appears as Exercise 34 in Section 11.6.

Check out these other real-world connections:

- Determining the Effects of Inflation (Section 11.1, Exercise 86)
- Counting the Number of Possible Area Codes and Phone Numbers (Section 11.5, Exercise 84)
- Calculating Possible Movements of a Computer Animation (Section 11.2, Exercise 73)
- Tracking and Improving Customer Service Using Probability (Section 11.6, Exercise 53)

11.1 Sequences and Series

Learning Objectives

In Section 11.1 you will learn how to:

- ☐ **A.** Write out the terms of a sequence given the general or *n*th term
- ☐ **B.** Work with recursive sequences and sequences involving a factorial
- ☐ **C.** Find the partial sum of a series
- ☐ **D.** Use summation notation to write and evaluate series
- ☐ **E.** Use sequences to solve applied problems

A *sequence* can be thought of as a pattern of numbers listed in a prescribed order. A *series* is the sum of the numbers in a sequence. Sequences and series come in countless varieties, and we'll introduce some general forms here. In following sections we'll focus on two special types: arithmetic and geometric sequences. These are used in a number of different fields, with a wide variety of significant applications.

A. Finding the Terms of a Sequence Given the General Term

Suppose a person had \$10,000 to invest, and decided to place the money in government bonds that guarantee an annual return of 7%. From our work in Chapter 4, we know the amount of money in the account after x years can be modeled by the function $f(x) = 10{,}000(1.07)^x$. If you reinvest your earnings each year, the amount in the account would be (rounded to the nearest dollar):

Year:	$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)\ldots$
	↓	↓	↓	↓	↓
Value:	\$10,700	\$11,449	\$12,250	\$13,108	\$14,026 . . .

Note the relationship (year, value) is a function that pairs 1 with \$10,700, 2 with \$11,449, 3 with \$12,250 and so on. This is an example of a **sequence.** To distinguish sequences from other algebraic functions, we commonly name the functions a instead of f, use the variable n instead of x, and employ a subscript notation. The function $f(x) = 10{,}000(1.07)^x$ would then be written $a_n = 10{,}000(1.07)^n$. Using this notation $a_1 = 10{,}700$, $a_2 = 11{,}449$, and so on.

The values $a_1, a_2, a_3, a_4, \ldots$ are called the **terms** of the sequence. If the account were closed after a certain number of years (for example, after the fifth year) we have a **finite sequence.** If we let the investment grow indefinitely, the result is called an **infinite sequence.** The expression a_n that defines the sequence is called the **general** or ***n*th term** and the terms immediately preceding it are called the $(n - 1)$st term, the $(n - 2)$nd term, and so on.

WORTHY OF NOTE

Sequences can actually start with any natural number. For instance, the sequence $a_n = \frac{2}{n-1}$ must start at $n = 2$ to avoid division by zero. In addition, we will sometimes use a_0 to indicate a preliminary or inaugural element, as in $a_0 = \$10{,}000$ for the amount of money initially held, prior to investing it.

Sequences

A *finite sequence* is a function a_n whose domain is the set of natural numbers from 1 to n. The terms of the sequence are labeled

$$a_1, a_2, a_3, \ldots, a_k, a_{k+1}, \ldots, a_{n-1}, a_n$$

where a_k represents an arbitrary "interior" term and a_n also represents the last term of the sequence.

An *infinite sequence* is a function a_n whose domain is the set of all natural numbers.

EXAMPLE 1A ► **Computing Specified Terms of a Sequence**

For $a_n = \frac{n+1}{n^2}$, find a_1, a_3, a_6, and a_7.

Solution ► $a_1 = \frac{1+1}{1^2} = 2$ $\qquad a_3 = \frac{3+1}{3^2} = \frac{4}{9}$

$a_6 = \frac{6+1}{6^2} = \frac{7}{36}$ $\qquad a_7 = \frac{7+1}{7^2} = \frac{8}{49}$

EXAMPLE 1B ► **Computing the First *k* Terms of a Sequence**

Find the first four terms of the sequence $a_n = (-1)^n 2^n$. Write the terms of the sequence as a list.

Solution ▶ $a_1 = (-1)^1 2^1 = -2$ $\qquad a_2 = (-1)^2 2^2 = 4$

$a_3 = (-1)^3 2^3 = -8$ $\qquad a_4 = (-1)^4 2^4 = 16$

The sequence can be written $-2, 4, -8, 16, \ldots$, or more generally as $-2, 4, -8, 16, \ldots, (-1)^n 2^n, \ldots$ to show how each term was generated.

Now try Exercises 7 through 32 ▶

WORTHY OF NOTE

When the terms of a sequence *alternate in sign* as in Example 1B, we call it an **alternating sequence.**

☑ **A. You've just learned how to write out the terms of a sequence given the general or *n*th term**

B. Recursive Sequences and Factorial Notation

Sometimes the formula defining a sequence uses the preceding term or terms to generate those that follow. These are called **recursive sequences** and are particularly useful in writing computer programs. Because of how they are defined, recursive sequences must give an inaugural term or **seed element,** to begin the recursion process.

Perhaps the most famous recursive sequence is associated with the work of Leonardo of Pisa (A.D. 1180–1250), better known to history as *Fibonacci.* In fact, it is commonly called the Fibonacci sequence in which each successive term is the sum of the previous two, beginning with 1, 1, ...

EXAMPLE 2 ▶ Computing the Terms of a Recursive Sequence

Write out the first eight terms of the recursive (Fibonacci) sequence defined by $c_1 = 1, c_2 = 1$, and $c_n = c_{n-1} + c_{n-2}$.

Solution ▶ The first two terms are given, so we begin with $n = 3$.

$$\begin{aligned} c_3 &= c_{3-1} + c_{3-2} & c_4 &= c_{4-1} + c_{4-2} & c_5 &= c_{5-1} + c_{5-2}\\ &= c_2 + c_1 & &= c_3 + c_2 & &= c_4 + c_3\\ &= 1 + 1 & &= 2 + 1 & &= 3 + 2\\ &= 2 & &= 3 & &= 5 \end{aligned}$$

At this point we can simply use the fact that each successive term is simply the sum of the preceding two, and find that $c_6 = 3 + 5 = 8$, $c_7 = 5 + 8 = 13$, and $c_8 = 13 + 8 = 21$. The first eight terms are 1, 1, 2, 3, 5, 8, 13, and 21.

Now try Exercises 33 through 38 ▶

WORTHY OF NOTE

One application of the Fibonacci sequence involves the Fibonacci spiral, found in the growth of many ferns and the spiral shell of many mollusks.

Sequences can also be defined using a **factorial,** which is the product of a given natural number with all those that precede it. The expression **5!** is read, "five factorial," and is evaluated as: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Factorials

For any natural number n,

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

Rewriting a factorial in equivalent forms often makes it easier to simplify certain expressions. For example, we can rewrite $5! = 5 \cdot 4!$ or $5! = 5 \cdot 4 \cdot 3!$. Consider Example 3.

EXAMPLE 3 ▶ Simplifying Expressions Using Factorial Notation

Simplify by writing the numerator in an equivalent form.

a. $\dfrac{9!}{7!}$ **b.** $\dfrac{11!}{8!2!}$ **c.** $\dfrac{6!}{3!5!}$

Solution ►

a. $\frac{9!}{7!} = \frac{9 \cdot 8 \cdot \cancel{7}!}{\cancel{7}!}$
$= 9 \cdot 8$
$= 72$

b. $\frac{11!}{8!2!} = \frac{11 \cdot 10 \cdot 9 \cdot \cancel{8}!}{\cancel{8}!2!}$
$= \frac{990}{2}$
$= 495$

c. $\frac{6!}{3!5!} = \frac{6 \cdot \cancel{5}!}{3!\cancel{5}!}$
$= \frac{6}{6}$
$= 1$

Now try Exercises 39 through 44 ►

WORTHY OF NOTE

Most calculators have a factorial option or key. On the TI-84 Plus it is located on a submenu of the MATH key: MATH **PRB** (option) **4: !**

EXAMPLE 4 ► Computing a Specified Term from a Sequence Defined Using Factorials

Find the third term of each sequence.

a. $a_n = \frac{n!}{2^n}$ **b.** $c_n = \frac{(-1)^n(2n - 1)!}{n!}$

Solution ►

a. $a_3 = \frac{3!}{2^3}$
$= \frac{6}{8} = \frac{3}{4}$

b. $c_3 = \frac{(-1)^3[2(3) - 1]!}{3!}$
$= \frac{(-1)(5)!}{3!} = \frac{(-1)[5 \cdot 4 \cdot \cancel{3!}]}{\cancel{3!}}$
$= -20$

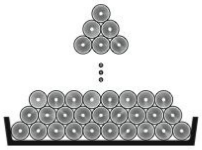
Now try Exercises 45 through 50 ►

☑ **B.** You've just learned how to work with recursive sequences and sequences involving a factorial

C. Series and Partial Sums

Sometimes the terms of a sequence are dictated by context rather than a formula. Consider the stacking of large pipes in a storage yard. If there are 10 pipes in the bottom row, then 9 pipes, then 8 (see Figure 11.1), how many pipes are in the stack if there is a single pipe at the top? The sequence generated is 10, 9, 8, . . . , 3, 2, 1 and to answer the question we would have to *compute the sum of all terms in the sequence.* When the terms of a finite sequence are added, the result is called a **finite series.**

Figure 11.1



Finite Series

Given the sequence $a_1, a_2, a_3, a_4, \ldots, a_n$, the sum of the terms is called a **finite series** or **partial sum** and is denoted S_n:

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

EXAMPLE 5 ► Computing a Partial Sum

Given $a_n = 2n$, find the value of
a. S_4 and **b.** S_7.

Solution ► Since we eventually need the sum of the first seven terms (for Part b), begin by writing out these terms: 2, 4, 6, 8, 10, 12, and 14.

a. $S_4 = a_1 + a_2 + a_3 + a_4$
$= 2 + 4 + 6 + 8$
$= 20$

b. $S_7 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7$
$= 2 + 4 + 6 + 8 + 10 + 12 + 14$
$= 56$

Now try Exercises 51 through 56 ►

☑ **C.** You've just learned how to find the partial sum of a series

D. Summation Notation

When the general term of a sequence is known, the Greek letter *sigma* Σ can be used to write the related series as a formula. For instance, to indicate the sum of the first four terms of $a_n = 3n + 2$, we write $\sum_{i=1}^{4}(3i + 2)$. This result is called **summation** or **sigma notation** and the letter i is called the **index of summation.** The letters j, k, l, and m are also used as index numbers, and the summation need not start at 1.

EXAMPLE 6 ▶ Computing a Partial Sum

Compute each sum:

a. $\sum_{i=1}^{4}(3i + 2)$ **b.** $\sum_{j=1}^{5}\frac{1}{j}$ **c.** $\sum_{k=3}^{6}(-1)^k k^2$

Solution ▶

a. $$\begin{aligned}\sum_{i=1}^{4}(3i + 2) &= (3 \cdot 1 + 2) + (3 \cdot 2 + 2) + (3 \cdot 3 + 2) + (3 \cdot 4 + 2)\\ &= 5 + 8 + 11 + 14 = 38\end{aligned}$$

b. $$\begin{aligned}\sum_{j=1}^{5}\frac{1}{j} &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\\ &= \frac{60}{60} + \frac{30}{60} + \frac{20}{60} + \frac{15}{60} + \frac{12}{60} = \frac{137}{60}\end{aligned}$$

c. $$\begin{aligned}\sum_{k=3}^{6}(-1)^k k^2 &= (-1)^3 \cdot 3^2 + (-1)^4 \cdot 4^2 + (-1)^5 \cdot 5^2 + (-1)^6 \cdot 6^2\\ &= -9 + 16 + -25 + 36 = 18\end{aligned}$$

Now try Exercises 57 through 68 ▶

If a definite pattern is noted in a given series expansion, this process can be reversed, with the expanded form being expressed in summation notation using the nth term.

EXAMPLE 7 ▶ Writing a Sum in Sigma Notation

Write each of the following sums in summation (sigma) notation.

a. $1 + 3 + 5 + 7 + 9$ **b.** $6 + 9 + 12 + \cdots$

Solution ▶

a. The series has five terms and each term is an odd number, or 1 less than a multiple of 2. The general term is $a_n = 2n - 1$, and the series is $\sum_{n=1}^{5}(2n - 1)$.

b. This is an infinite sum whose terms are multiples of 3. The general term is $a_n = 3n$, but the series starts at 2 and not 1. The series is $\sum_{j=2}^{\infty}3j$.

Now try Exercises 69 through 78 ▶

WORTHY OF NOTE

By varying the function given and/or where the sum begins, more than one acceptable form is possible.

For example 7(b) $\sum_{n=1}^{\infty}(3 + 3n)$ also works.

Since the commutative and associative laws hold for the addition of real numbers, summations have the following properties:

Properties of Summation

Given any real number c and natural number n,

(I) $\sum_{i=1}^{n} c = cn$

If you add a constant c "n" times the result is cn.

(II) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

A constant can be factored out of a sum.

(III) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

A summation can be distributed to two (or more) sequences.

(IV) $\sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i = \sum_{i=1}^{n} a_i;\ 1 \le m < n$

A summation is cumulative and can be written as a sum of smaller parts.

The verification of property II depends solely on the distributive property.

Proof: $\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n$ expand sum

$= c(a_1 + a_2 + a_3 + \cdots + a_n)$ factor out c

$= c\sum_{i=1}^{n} a_i$ write series in summation form

The verification of properties III and IV simply uses the commutative and associative properties. You are asked to prove property III in **Exercise 91.**

EXAMPLE 8 ► **Computing a Sum Using Summation Properties**

Recompute the sum $\sum_{i=1}^{4} (3i + 2)$ from Example 6(a) using summation properties.

Solution ► $\sum_{i=1}^{4} (3i + 2) = \sum_{i=1}^{4} 3i + \sum_{i=1}^{4} 2$ property III

$= 3\sum_{i=1}^{4} i + \sum_{i=1}^{4} 2$ property II

$= 3(10) + 2(4)$ $1 + 2 + 3 + 4 = 10$; property I

$= 38$ result

Now try Exercises 79 through 82 ►

☑ **D.** You've just learned how to use summation notation to write and evaluate series

E. Applications of Sequences

To solve applications of sequences, (1) identify where the sequence begins (the initial term) and (2) write out the first few terms to help identify the nth term.

EXAMPLE 9 ▶ Solving an Application — Accumulation of Stock

Hydra already owned 1420 shares of stock when her company began offering employees the opportunity to purchase 175 discounted shares per year. If she made no purchases other than these discounted shares each year, how many shares will she have 9 yr later? If this continued for the 25 yr she will work for the company, how many shares will she have at retirement?

Solution ▶ To begin, it helps to simply write out the first few terms of the sequence. Since she already had 1420 shares before the company made this offer, we let $a_0 = 1420$ be the inaugural element, showing $a_1 = 1595$ (after 1 yr, she owns 1595 shares). The first few terms are 1595, 1770, 1945, 2120, and so on. This supports a general term of $a_n = 1595 + 175(n - 1)$.

After 9 years	After 25 years
$a_9 = 1595 + 175(8)$	$a_{25} = 1595 + 175(24)$
$= 2995$	$= 5795$

After 9 yr she would have 2995 shares. Upon retirement she would own 5795 shares of company stock.

☑ **E. You've just learned to use sequences to solve applied problems**

Now try Exercises 85 through 90 ▶

TECHNOLOGY HIGHLIGHT

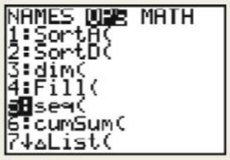
Studying Sequences and Series

To support a study of sequences and series, we can use a graphing calculator to generate the desired terms. This can be done either on the home screen or directly into the LIST feature of the calculator. On the TI-84 Plus this is accomplished using the **"seq("** and **"sum("** commands, which are accessed using the keystrokes 2nd STAT (**LIST**) and the screen shown in Figure 11.2. The **"seq("** feature is option 5 under the **OPS** submenu (press ▶ 5) and the **"sum("** feature is option 5 under the **MATH** submenu (press ◀ 5).

To generate the first four terms of the sequence $a_n = n^2 + 1$, and to find the sum, CLEAR the home screen and press 2nd STAT ▶ 5 to place **"seq("** on the home screen. This command requires four inputs: a_n (the *n*th term), variable used (the calculator can work with any letter), initial term and the last term. For this example the screen reads "**seq**($x^2 + 1$, x, 1, 4)," with the result being the four terms shown in Figure 11.3. To find the *sum* of these terms, we simply precede the **seq**($x^2 + 1$, x, 1, 4) command by **"sum(,"** and two methods are shown in Figure 11.4.

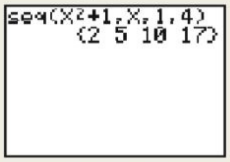
Each of the following sequences have some interesting properties or mathematical connections. Use your graphing calculator to generate the first 10 terms of each sequence and the sum of the first 10 terms. Next, generate the first 20 terms of each sequence and the sum of these terms. What conclusion (if any) can you reach about the sum of each sequence?

Figure 11.2



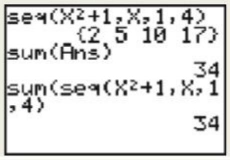
```
NAMES OPS MATH
1:SortA(
2:SortD(
3:dim(
4:Fill(
5:seq(
6:cumSum(
7↓ΔList(
```

Figure 11.3



```
seq(X²+1,X,1,4)
      {2 5 10 17}
```

Figure 11.4



```
seq(X²+1,X,1,4)
      {2 5 10 17}
sum(Ans)
                34
sum(seq(X²+1,X,1
,4)
                34
```

Exercise 1: $a_n = \frac{1}{3^n}$ **Exercise 2:** $a_n = \frac{2}{n(n + 1)}$ **Exercise 3:** $a_n = \frac{1}{(2n - 1)(2n + 1)}$

11.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A sequence is a(n) ________ of numbers listed in a specific ________.

2. A series is the ________ of the numbers from a given sequence.

3. When each term of a sequence is larger than the preceding term, the sequence is said to be ________.

4. When each term of a sequence is smaller than the preceding term, the sequence is said to be ________.

5. Describe the characteristics of a recursive sequence and give one example.

6. Describe the characteristics of an alternating sequence and give one example.

▶ DEVELOPING YOUR SKILLS

Find the first four terms, then find the 8th and 12th term for each *n*th term given.

7. $a_n = 2n - 1$

8. $a_n = 2n + 3$

9. $a_n = 3n^2 - 3$

10. $a_n = 2n^3 - 12$

11. $a_n = (-1)^n n$

12. $a_n = \dfrac{(-1)^n}{n}$

13. $a_n = \dfrac{n}{n + 1}$

14. $a_n = \left(1 + \dfrac{1}{n}\right)^n$

15. $a_n = \left(\dfrac{1}{2}\right)^n$

16. $a_n = \left(\dfrac{2}{3}\right)^n$

17. $a_n = \dfrac{1}{n}$

18. $a_n = \dfrac{1}{n^2}$

19. $a_n = \dfrac{(-1)^n}{n(n + 1)}$

20. $a_n = \dfrac{(-1)^{n+1}}{2n^2 - 1}$

21. $a_n = (-1)^n 2^n$

22. $a_n = (-1)^n 2^{-n}$

Find the indicated term for each sequence.

23. $a_n = n^2 - 2;\ a_9$

24. $a_n = (n - 2)^2;\ a_9$

25. $a_n = \dfrac{(-1)^{n+1}}{n};\ a_5$

26. $a_n = \dfrac{(-1)^{n+1}}{2n - 1};\ a_5$

27. $a_n = 2\left(\dfrac{1}{2}\right)^{n-1};\ a_7$

28. $a_n = 3\left(\dfrac{1}{3}\right)^{n-1};\ a_7$

29. $a_n = \left(1 + \dfrac{1}{n}\right)^n;\ a_{10}$

30. $a_n = \left(n + \dfrac{1}{n}\right)^n;\ a_9$

31. $a_n = \dfrac{1}{n(2n + 1)};\ a_4$

32. $a_n = \dfrac{1}{(2n - 1)(2n + 1)};\ a_5$

Find the first five terms of each recursive sequence.

33. $\begin{cases} a_1 = 2 \\ a_n = 5a_{n-1} - 3 \end{cases}$

34. $\begin{cases} a_1 = 3 \\ a_n = 2a_{n-1} - 3 \end{cases}$

35. $\begin{cases} a_1 = -1 \\ a_n = (a_{n-1})^2 + 3 \end{cases}$

36. $\begin{cases} a_1 = -2 \\ a_n = a_{n-1} - 16 \end{cases}$

37. $\begin{cases} c_1 = 64,\ c_2 = 32 \\ c_n = \dfrac{c_{n-2} - c_{n-1}}{2} \end{cases}$

38. $\begin{cases} c_1 = 1,\ c_2 = 2 \\ c_n = c_{n-1} + (c_{n-2})^2 \end{cases}$

Simplify each factorial expression.

39. $\frac{8!}{5!}$

40. $\frac{12!}{10!}$

41. $\frac{9!}{7!2!}$

42. $\frac{6!}{3!3!}$

43. $\frac{8!}{2!6!}$

44. $\frac{10!}{3!7!}$

Write out the first four terms in each sequence.

45. $a_n = \frac{n!}{(n+1)!}$

46. $a_n = \frac{n!}{(n+3)!}$

47. $a_n = \frac{(n+1)!}{(3n)!}$

48. $a_n = \frac{(n+3)!}{(2n)!}$

49. $a_n = \frac{n^n}{n!}$

50. $a_n = \frac{2^n}{n!}$

Find the indicated partial sum for each sequence.

51. $a_n = n; S_5$

52. $a_n = n^2; S_7$

53. $a_n = 2n - 1; S_8$

54. $a_n = 3n - 1; S_6$

55. $a_n = \frac{1}{n}; S_5$

56. $a_n = \frac{n}{n+1}; S_4$

Expand and evaluate each series.

57. $\sum_{i=1}^{4}(3i - 5)$

58. $\sum_{i=1}^{5}(2i - 3)$

59. $\sum_{k=1}^{5}(2k^2 - 3)$

60. $\sum_{k=1}^{5}(k^2 + 1)$

61. $\sum_{k=1}^{7}(-1)^k k$

62. $\sum_{k=1}^{5}(-1)^k 2^k$

63. $\sum_{i=1}^{4}\frac{i^2}{2}$

64. $\sum_{i=2}^{4} i^2$

65. $\sum_{j=3}^{7} 2j$

66. $\sum_{j=3}^{7}\frac{j}{2^j}$

67. $\sum_{k=3}^{8}\frac{(-1)^k}{k(k-2)}$

68. $\sum_{k=2}^{6}\frac{(-1)^{k+1}}{k^2 - 1}$

Write each sum using sigma notation. Answers are not necessarily unique.

69. $4 + 8 + 12 + 16 + 20$

70. $5 + 10 + 15 + 20 + 25$

71. $-1 + 4 - 9 + 16 - 25 + 36$

72. $1 - 8 + 27 - 64 + 125 - 216$

For the given general term a_n, write the indicated sum using sigma notation.

73. $a_n = n + 3; S_5$

74. $a_n = \frac{n^2 + 1}{n + 1}; S_4$

75. $a_n = \frac{n^2}{3}$; third partial sum

76. $a_n = 2n - 1$; sixth partial sum

77. $a_n = \frac{n}{2^n}$; sum for $n = 3$ to 7

78. $a_n = n^2$; sum for $n = 2$ to 6

Compute each sum by applying properties of summation.

79. $\sum_{i=1}^{5}(4i - 5)$

80. $\sum_{i=1}^{6}(3 + 2i)$

81. $\sum_{k=1}^{4}(3k^2 + k)$

82. $\sum_{k=1}^{4}(2k^3 + 5)$

▶ WORKING WITH FORMULAS

83. Sum of $a_n = 3n - 2$: $S_n = \frac{n(3n - 1)}{2}$

The sum of the first n terms of the sequence defined by $a_n = 3n - 2 = 1, 4, 7, 10, \ldots, (3n - 2), \ldots$ is given by the formula shown. Find S_5 using the formula, then verify by direct calculation.

84. Sum of $a_n = 3n - 1$: $S_n = \frac{n(3n + 1)}{2}$

The sum of the first n terms of the sequence defined by $a_n = 3n - 1 = 2, 5, 8, 11, \ldots, (3n - 1), \ldots$ is given by the formula shown. Find S_8 using the formula, then verify by direct calculation. Observing the results of Exercises 83 and 84, can you now state the sum formula for $a_n = 3n - 0$?

▶ APPLICATIONS

Use the information given in each exercise to determine the *n*th term a_n for the sequence described. Then use the *n*th term to list the specified number of terms.

85. Blue-book value: Steve's car has a blue-book value of \$6000. Each year it loses 20% of its value (its value each year is 80% of the year before). List the value of Steve's car for the next 5 yr. (*Hint:* For $a_1 = 6000$, we need the *next* five terms.)

86. Effects of inflation: Suppose inflation (an increase in value) will average 4% for the next 5 yr. List the growing cost (year by year) of a DVD that costs \$15 right now. (*Hint:* For $a_1 = 15$, we need the *next* five terms.)

87. Wage increases: Latisha gets \$5.20 an hour for filling candy machines for Archtown Vending. Each year she receives a \$0.50 hourly raise. List Latisha's wage for the first 5 yr. How much will she make in the fifth year if she works 8 hr per day for 240 working days?

88. Average birth weight: The average birth weight of a certain animal species is 900 g, with the baby gaining 125 g each day for the first 10 days. List the infant's weight for the first 10 days. How much does the infant weigh on the 10th day?

89. Stocking a lake: A local fishery stocks a large lake with 1500 bass and then adds an additional 100 mature bass per month until the lake nears maximum capacity. If the bass population grows at a rate of 5% per month through natural reproduction, the number of bass in the pond after *n* months is given by the recursive sequence $b_0 = 1500$, $b_n = 1.05b_{n-1} + 100$. How many bass will be in the lake after 6 months?

90. Species preservation: The Interior Department introduces 50 wolves (male and female) into a large wildlife area in an effort to preserve the species. Each year about 12 additional adult wolves are added from capture and relocation programs. If the wolf population grows at a rate of 10% per year through natural reproduction, the number of wolves in the area after *n* years is given by the recursive sequence $w_0 = 50$, $w_n = 1.10w_{n-1} + 12$. How many wolves are in the wildlife area after 6 years?

▶ EXTENDING THE CONCEPT

91. Verify that a summation may be distributed to two (or more) sequences. That is, verify that the following statement is true:

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i.$$

Surprisingly, some of the most celebrated numbers in mathematics can be represented or approximated by a series expansion. Use your calculator to find the partial sums for $n = 4$, $n = 8$, and $n = 12$ for the summations given, and attempt to name the number the summation approximates:

92. $\displaystyle\sum_{k=0}^{n} \frac{1}{k!}$

93. $\displaystyle\sum_{k=1}^{n} \frac{1}{3^k}$

94. $\displaystyle\sum_{k=1}^{n} \frac{1}{2^k}$

▶ MAINTAINING YOUR SKILLS

95. (6.7) Solve $\csc x \sin\left(\frac{\pi}{2} - x\right) = -1$

96. (2.5) Set up the difference quotient for $f(x) = \sqrt{x}$, then rationalize the numerator.

97. (7.2) Given a triangle where $a = 0.4$ m, $b = 0.3$ m, and $c = 0.5$ m, find the three corresponding angles.

98. (6.3) Solve the system using a matrix equation. $\begin{cases} 25x + y - 2z = -14 \\ 2x - y + z = 40 \\ -7x + 3y - z = -13 \end{cases}$

11.2 Arithmetic Sequences

Learning Objectives

In Section 11.2 you will learn how to:

- ☐ **A.** Identify an arithmetic sequence and its common difference
- ☐ **B.** Find the *n*th term of an arithmetic sequence
- ☐ **C.** Find the *n*th partial sum of an arithmetic sequence
- ☐ **D.** Solve applications involving arithmetic sequences

Similar to the way polynomials fall into certain groups or families (linear, quadratic, cubic, etc.), sequences and series with common characteristics are likewise grouped. In this section, we focus on sequences where each successive term is generated by adding a constant value, as in the sequence 1, 8, 15, 22, 29, . . . , where 7 is added to a given term in order to produce the next term.

A. Identifying an Arithmetic Sequence and Finding the Common Difference

An **arithmetic sequence** is one where each successive term is found by adding a fixed constant to the preceding term. For instance 3, 7, 11, 15, . . . is an arithmetic sequence, since adding 4 to any given term produces the next term. This also means if you take the difference of any two consecutive terms, the result will be 4 and in fact, 4 is called the **common difference *d*** for this sequence. Using the notation developed earlier, we can write $d = a_{k+1} - a_k$, where a_k represents any term of the sequence and a_{k+1} represents the term that follows a_k.

Arithmetic Sequences

Given a sequence $a_1, a_2, a_3, \ldots, a_k, a_{k+1}, \ldots, a_n$, where $k, n \in \mathbb{N}$ and $k < n$,
if there exists a common difference d such that $a_{k+1} - a_k = d$,
then the sequence is an *arithmetic sequence.*

The difference of successive terms can be rewritten as $a_{k+1} = a_k + d$ (for $k \geq 1$) to highlight that each following term is found by adding d to the previous term.

EXAMPLE 1 ► **Identifying an Arithmetic Sequence**

Determine if the given sequence is arithmetic.

a. 2, 5, 8, 11, . . . **b.** $\frac{1}{2}, \frac{5}{6}, \frac{4}{3}, \frac{7}{6}, \ldots$

Solution ► **a.** Begin by looking for a common difference $d = a_{k+1} - a_k$. Checking each pair of consecutive terms we have

$$5 - 2 = 3 \qquad 8 - 5 = 3 \qquad 11 - 8 = 3 \quad \text{and so on.}$$

This is an arithmetic sequence with common difference $d = 3$.

b. Checking each pair of consecutive terms yields

$$\frac{5}{6} - \frac{1}{2} = \frac{5}{6} - \frac{3}{6} \qquad\qquad \frac{4}{3} - \frac{5}{6} = \frac{8}{6} - \frac{5}{6}$$
$$= \frac{2}{6} = \frac{1}{3} \qquad\qquad = \frac{3}{6} = \frac{1}{2}$$

Since the difference is not constant, this is not an arithmetic sequence.

Now try Exercises 7 through 18 ►

EXAMPLE 2 ► **Writing the First *k* Terms of an Arithmetic Sequence**

Write the first five terms of the arithmetic sequence, given the first term a_1 and the common difference d.

a. $a_1 = 12$ and $d = -4$ **b.** $a_1 = \frac{1}{2}$ and $d = \frac{1}{3}$

Solution ► **a.** $a_1 = 12$ and $d = -4$. Starting at $a_1 = 12$, add -4 to each new term to generate the sequence: 12, 8, 4, 0, -4

b. $a_1 = \frac{1}{2}$ and $d = \frac{1}{3}$. Starting at $a_1 = \frac{1}{2}$ and adding $\frac{1}{3}$ to each new term will generate the sequence: $\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \frac{11}{6}$

Now try Exercises 19 through 30 ►

☑ **A.** You've just learned how to identify an arithmetic sequence and its common difference

B. Finding the *n*th Term of an Arithmetic Sequence

If the values a_1 and d from an arithmetic sequence are known, we could generate the terms of the sequence by adding *multiples of d to the first term,* instead of adding d to each new term. For example, we can generate the sequence 3, 8, 13, 18, 23 by adding multiples of 5 to the first term $a_1 = 3$:

$$\begin{array}{rcl} 3 = 3 + (0)5 & \qquad & a_1 = a_1 + 0d \\ 8 = 3 + (1)5 & & a_2 = a_1 + 1d \\ 13 = 3 + (2)5 & & a_3 = a_1 + 2d \\ 18 = 3 + (3)5 & & a_4 = a_1 + 3d \\ 23 = 3 + (4)5 & & \underset{\text{current term}}{a_5} = \underset{\text{initial term}}{a_1} + \underset{\text{coefficient of common difference}}{4}d \end{array}$$

It's helpful to note the coefficient of d is 1 less than the subscript of the current term (as shown): $5 - 1 = 4$. This observation leads us to a formula for the nth term.

The *n*th Term of an Arithmetic Sequence

The nth term of an *arithmetic sequence* is given by

$$a_n = a_1 + (n - 1)d$$

where d is the common difference.

EXAMPLE 3 ► **Finding a Specified Term in an Arithmetic Sequence**

Find the 24th term of the sequence 0.1, 0.4, 0.7, 1, ...

Solution ► Instead of creating all terms up to the 24th, we determine the constant d and use the nth term formula. By inspection we note $a_1 = 0.1$ and $d = 0.3$.

$$\begin{aligned} a_n &= a_1 + (n - 1)d && n\text{th term formula} \\ &= 0.1 + (n - 1)0.3 && \text{substitute 0.1 for } a_1 \text{ and 0.3 for } d \\ &= 0.1 + 0.3n - 0.3 && \text{eliminate parentheses} \\ &= 0.3n - 0.2 && \text{simplify} \end{aligned}$$

To find the 24th term we substitute 24 for n:

$$\begin{aligned} a_{24} &= 0.3(24) - 0.2 && \text{substitute 24 for } n \\ &= 7.0 && \text{result} \end{aligned}$$

Now try Exercises 31 through 42 ►

EXAMPLE 4 ► **Finding the Number of Terms in an Arithmetic Sequence**

Find the number of terms in the arithmetic sequence 2, -5, -12, -19, . . . , -411.

Solution ▶ By inspection we see that $a_1 = 2$ and $d = -7$. As before,

$$\begin{aligned} a_n &= a_1 + (n-1)d && \text{nth term formula} \\ &= 2 + (n-1)(-7) && \text{substitute 2 for } a_1 \text{ and } -7 \text{ for } d \\ &= -7n + 9 && \text{simplify} \end{aligned}$$

Although we don't know the number of terms in the sequence, we *do* know the last or nth term is -411. Substituting -411 for a_n gives

$$\begin{aligned} -411 &= -7n + 9 && \text{substitute } -411 \text{ for } a_n \\ 60 &= n && \text{solve for } n \end{aligned}$$

There are 60 terms in this sequence.

Now try Exercises 43 through 50 ▶

If the term a_1 is unknown but a term a_k is given, the nth term can be written

$$a_n = a_k + (n-k)d \text{ since } n = k + (n-k)$$

(the subscript of the term a_k and coefficient of d sum to n).

EXAMPLE 5 ▶ Finding the First Term of an Arithmetic Sequence

Given an arithmetic sequence where $a_6 = 0.55$ and $a_{13} = 0.9$, find the common difference d and the value of a_1.

Solution ▶ At first it seems that not enough information is given, but recall we can express a_{13} as the sum of any earlier term and the appropriate multiple of d. Since a_6 is known, we write $a_{13} = a_6 + 7d$ (note $13 = 6 + 7$ as required).

$$\begin{aligned} a_{13} &= a_6 + 7d && a_1 \text{ is unknown} \\ 0.9 &= 0.55 + 7d && \text{substitute 0.9 for } a_{13} \text{ and 0.55 for } a_6 \\ 0.35 &= 7d && \text{subtract 0.55} \\ d &= 0.05 && \text{solve for } d \end{aligned}$$

Having found d, we can now solve for a_1.

$$\begin{aligned} a_{13} &= a_1 + 12d && \text{nth term formula for } n = 13 \\ 0.9 &= a_1 + 12(0.05) && \text{substitute 0.9 for } a_{13} \text{ and 0.05 for } d \\ 0.9 &= a_1 + 0.6 && \text{simplify} \\ a_1 &= 0.3 && \text{solve for } a_1 \end{aligned}$$

The first term is $a_1 = 0.3$ and the common difference is $d = 0.05$.

Now try Exercises 51 through 56 ▶

☑ **B.** You've just learned how to find the *n*th term of an arithmetic sequence

C. Finding the *n*th Partial Sum of an Arithmetic Sequence

Using sequences and series to solve applications often requires computing the sum of a given number of terms. Consider the sequence $a_1, a_2, a_3, a_4, \ldots, a_n$ with common difference d. Use S_n to represent the sum of the first n terms and write the original series, then the series in reverse order underneath. Since one row increases at the same rate the other decreases, the sum of each column remains constant, and for simplicity's sake we choose $a_1 + a_n$ to represent this sum.

$$\begin{array}{rcccccccccccccl} S_n = & a_1 & + & a_2 & + & a_3 & + \cdots + & a_{n-2} & + & a_{n-1} & + & a_n & \\ S_n = & a_n & + & a_{n-1} & + & a_{n-2} & + \cdots + & a_3 & + & a_2 & + & a_1 & \Big\downarrow \text{add columns vertically} \\ \hline 2S_n = & (a_1 + a_n) & + & (a_1 + a_n) & + & (a_1 + a_n) & + \cdots + & (a_1 + a_n) & + & (a_1 + a_n) & + & (a_1 + a_n) & \end{array}$$

To understand why each column adds to $a_1 + a_n$, consider the sum in the second column: $a_2 + a_{n-1}$. From $a_2 = a_1 + d$ and $a_{n-1} = a_n - d$, we obtain $a_2 + a_{n-1} = (a_1 + d) + (a_n - d)$ by direct substitution, which gives a result of $a_1 + a_n$. Since there are n columns, we end up with $2S_n = n(a_1 + a_n)$, and solving for S_n gives the formula for the first n terms of an arithmetic sequence.

The *n*th Partial Sum of an Arithmetic Sequence

Given an arithmetic sequence with first term a_1, the nth partial sum is given by

$$S_n = n\left(\frac{a_1 + a_n}{2}\right).$$

In words: The sum of an arithmetic sequence is the number of terms times the average of the first and last term.

EXAMPLE 6 ► **Computing the Sum of an Arithmetic Sequence**

Find the sum of the first 75 positive, odd integers: $\sum_{k=1}^{75}(2k - 1)$.

Solution ► The initial terms of the sequence are 1, 3, 5, . . . and we note $a_1 = 1$, $d = 2$, and $n = 75$. To use the sum formula, we need the value of $a_n = a_{75}$. The nth term formula shows $a_{75} = a_1 + 74d = 1 + 74(2)$, so $a_{75} = 149$.

$$\begin{aligned} S_n &= \frac{n(a_1 + a_n)}{2} && \text{sum formula} \\ S_{75} &= \frac{75(a_1 + a_{75})}{2} && \text{substitute 75 for } n \\ &= \frac{75(1 + 149)}{2} && \text{substitute 1 for } a_1\text{, 149 for } a_{75} \\ &= 5625 && \text{result} \end{aligned}$$

The sum of the first 75 positive, odd integers is 5625.

Now try Exercises 57 through 62 ►

☑ **C.** You've just learned how to find the nth partial sum of an arithmetic sequence

By substituting the nth term formula directly into the formula for partial sums, we're able to find a partial sum without actually having to find the nth term:

$$\begin{aligned} S_n &= \frac{n(a_1 + a_n)}{2} && \text{sum formula} \\ &= \frac{n(a_1 + [a_1 + (n - 1)d])}{2} && \text{substitute } a_1 + (n - 1)d \text{ for } a_n \\ &= \frac{n}{2}[2a_1 + (n - 1)d] && \textit{alternative formula for the nth partial sum} \end{aligned}$$

See Exercises 63 through 68 for more on this alternative formula.

Figure 11.5

spiral fern

Figure 11.6

nautilus

D. Applications

In the evolution of certain plants and shelled animals, sequences and series seem to have been one of nature's favorite tools (see Figures 11.5 and 11.6). Sequences and series also provide a good mathematical model for a variety of other situations as well.

EXAMPLE 7 ▶ Solving an Application of Arithmetic Sequences: Seating Capacity

Cox Auditorium is an amphitheater that has 40 seats in the first row, 42 seats in the second row, 44 in the third, and so on. If there are 75 rows in the auditorium, what is the auditorium's seating capacity?

Solution ▶ The number of seats in each row gives the terms of an arithmetic sequence with $a_1 = 40$, $d = 2$, and $n = 75$. To find the seating capacity, we need to find the total number of seats, which is the sum of this arithmetic sequence. Since the value of a_{75} is unknown, we opt for the alternative formula $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$.

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d] \qquad \text{sum formula}$$

$$S_{75} = \frac{75}{2}[2(40) + (75 - 1)(2)] \qquad \text{substitute 40 for } a_1\text{, 2 for } d \text{ and 75 for } n$$

$$= \frac{75}{2}(228) \qquad \text{simplify}$$

$$= 8550 \qquad \text{result}$$

The seating capacity for Cox Auditorium is 8550.

Now try Exercises 71 through 76 ▶

☑ **D.** You've just learned how to solve applications involving arithmetic sequences

11.2 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Consecutive terms in an arithmetic sequence differ by a constant called the ______ ______.

2. The sum of the first n terms of an arithmetic sequence is called the nth ______ ______.

3. The formula for the nth partial sum of an arithmetic sequence is $s_n =$ ________, where a_n is the ____ term.

4. The nth term formula for an arithmetic sequence is $a_n =$ ________, where a_1 is the ______ term and d is the ______ ______.

5. Discuss how the terms of an arithmetic sequence can be written in various ways using the relationship $a_n = a_k + (n - k)d$.

6. Describe how the formula for the nth partial sum was derived, and illustrate its application using a sequence from the exercise set.

▶ DEVELOPING YOUR SKILLS

Determine if the sequence given is arithmetic. If yes, name the common difference. If not, try to determine the pattern that forms the sequence.

7. $-5, -2, 1, 4, 7, 10, \ldots$

8. $1, -2, -5, -8, -11, -14, \ldots$

9. $0.5, 3, 5.5, 8, 10.5, \ldots$

10. $1.2, 3.5, 5.8, 8.1, 10.4, \ldots$

11. $2, 3, 5, 7, 11, 13, 17, \ldots$

12. $1, 4, 8, 13, 19, 26, 34, \ldots$

13. $\frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}, \ldots$

14. $\frac{1}{12}, \frac{1}{15}, \frac{1}{20}, \frac{1}{30}, \frac{1}{60}, \ldots$

15. $1, 4, 9, 16, 25, 36, \ldots$

16. $-125, -64, -27, -8, -1, \ldots$

17. $\pi, \frac{5\pi}{6}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}, \ldots$

18. $\pi, \frac{7\pi}{8}, \frac{3\pi}{4}, \frac{5\pi}{8}, \frac{\pi}{2}, \ldots$

Write the first four terms of the arithmetic sequence with the given first term and common difference.

19. $a_1 = 2, d = 3$

20. $a_1 = 8, d = 3$

21. $a_1 = 7, d = -2$

22. $a_1 = 60, d = -12$

23. $a_1 = 0.3, d = 0.03$

24. $a_1 = 0.5, d = 0.25$

25. $a_1 = \frac{3}{2}, d = \frac{1}{2}$

26. $a_1 = \frac{1}{5}, d = \frac{1}{10}$

27. $a_1 = \frac{3}{4}, d = -\frac{1}{8}$

28. $a_1 = \frac{1}{6}, d = -\frac{1}{3}$

29. $a_1 = -2, d = -3$

30. $a_1 = -4, d = -4$

Identify the first term and the common difference, then write the expression for the general term a_n and use it to find the 6th, 10th, and 12th terms of the sequence.

31. $2, 7, 12, 17, \ldots$

32. $7, 4, 1, -2, -5, \ldots$

33. $5.10, 5.25, 5.40, \ldots$

34. $9.75, 9.40, 9.05, \ldots$

35. $\frac{3}{2}, \frac{9}{4}, 3, \frac{15}{4}, \ldots$

36. $\frac{5}{7}, \frac{3}{14}, -\frac{2}{7}, -\frac{11}{14}, \ldots$

Find the indicated term using the information given.

37. $a_1 = 5, d = 4$; find a_{15}

38. $a_1 = 9, d = -2$; find a_{17}

39. $a_1 = \frac{3}{2}, d = -\frac{1}{12}$; find a_7

40. $a_1 = \frac{12}{25}, d = -\frac{1}{10}$; find a_9

41. $a_1 = -0.025, d = 0.05$; find a_{50}

42. $a_1 = 3.125, d = -0.25$; find a_{20}

Find the number of terms in each sequence.

43. $a_1 = 2, a_n = -22, d = -3$

44. $a_1 = 4, a_n = 42, d = 2$

45. $a_1 = 0.4, a_n = 10.9, d = 0.25$

46. $a_1 = -0.3, a_n = -36, d = -2.1$

47. $-3, -0.5, 2, 4.5, 7, \ldots, 47$

48. $-3.4, -1.1, 1.2, 3.5, \ldots, 38$

49. $\frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}, \frac{1}{4}, \ldots, \frac{9}{8}$

50. $\frac{1}{12}, \frac{1}{15}, \frac{1}{20}, \frac{1}{30}, \ldots, -\frac{1}{4}$

Find the common difference d and the value of a_1 using the information given.

51. $a_3 = 7, a_7 = 19$

52. $a_5 = -17, a_{11} = -2$

53. $a_2 = 1.025, a_{26} = 10.025$

54. $a_6 = -12.9, a_{30} = 1.5$

55. $a_{10} = \frac{13}{18}, a_{24} = \frac{27}{2}$

56. $a_4 = \frac{5}{4}, a_8 = \frac{9}{4}$

Evaluate each sum. For Exercises 61 and 62, use the summation properties from Section 11.1.

57. $\sum_{n=1}^{30}(3n - 4)$

58. $\sum_{n=1}^{29}(4n - 1)$

59. $\sum_{n=1}^{37}\left(\frac{3}{4}n + 2\right)$

60. $\sum_{n=1}^{20}\left(\frac{5}{2}n - 3\right)$

61. $\sum_{n=4}^{15}(3 - 5n)$

62. $\sum_{n=7}^{20}(7 - 2n)$

Use the alternative formula for the nth partial sum to compute the sums indicated.

63. The sum S_{15} for the sequence $-12 + (-9.5) + (-7) + (-4.5) + \cdots$

64. The sum S_{20} for the sequence $\frac{9}{2} + \frac{7}{2} + \frac{5}{2} + \frac{3}{2} + \cdots$

65. The sum S_{30} for the sequence $0.003 + 0.173 + 0.343 + 0.513 + \cdots$

66. The sum S_{50} for the sequence $(-2) + (-7) + (-12) + (-17) + \cdots$

67. The sum S_{20} for the sequence $\sqrt{2} + 2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + \cdots$

68. The sum S_{10} for the sequence $12\sqrt{3} + 10\sqrt{3} + 8\sqrt{3} + 6\sqrt{3} + \cdots$

▶ WORKING WITH FORMULAS

69. Sum of the first n natural numbers: $S_n = \frac{n(n+1)}{2}$

The sum of the first n natural numbers can be found using the formula shown, where n represents the number of terms in the sum. Verify the formula by adding the first six natural numbers by hand, and then evaluating S_6. Then find the sum of the first 75 natural numbers.

70. Sum of the squares of the first n natural numbers: $S_n = \frac{n(n+1)(2n+1)}{6}$

If the first n natural numbers are squared, the sum of these squares can be found using the formula shown, where n represents the number of terms in the sum. Verify the formula by computing the sum of the squares of the first six natural numbers by hand, and then evaluating S_6. Then find the sum of the squares of the first 20 natural numbers: $(1^2 + 2^2 + 3^2 + \cdots + 20^2)$.

▶ APPLICATIONS

71. Temperature fluctuation: At 5 P.M. in Coldwater, the temperature was a chilly 36°F. If the temperature decreased by 3°F every half-hour for the next 7 hr, at what time did the temperature hit 0°F?

72. Arc of a baby swing: When Mackenzie's baby swing is started, the first swing (one way) is a 30-in. arc. As the swing slows down, each successive arc is $\frac{3}{2}$ in. less than the previous one. Find (a) the length of the tenth swing and (b) how far Mackenzie has traveled during the 10 swings.

73. Computer animations: The animation on a new computer game initially allows the hero of the game to jump a (screen) distance of 10 in. over booby traps and obstacles. Each successive jump is limited to $\frac{3}{4}$ in. less than the previous one. Find (a) the length of the seventh jump and (b) the total distance covered after seven jumps.

74. Seating capacity: The Fox Theater creates a "theater in the round" when it shows any of Shakespeare's plays. The first row has 80 seats, the second row has 88, the third row has 96, and so on. How many seats are in the 10th row? If there is room for 25 rows, how many chairs will be needed to set up the theater?

75. Sales goals: *At the time that I was newly hired, 100 sales per month was what I required. Each following month—the last plus 20 more, as I work for the goal of top sales award. When 2500 sales are thusly made, it's Tahiti, Hawaii, and pina coladas in the shade.* How many sales were made by this person in the seventh month? What were the total sales after the 12th month? Was the goal of 2500 total sales met after the 12th month?

76. Bequests to charity: *At the time our mother left this Earth, she gave $9000 to her children of birth. This we kept and each year added $3000 more, as a lasting memorial from the children she bore. When $42,000 is thusly attained, all goes to charity that her memory be maintained.* What was the balance in the sixth year? In what year was the goal of $42,000 met?

▶ EXTENDING THE THOUGHT

77. From a study of numerical analysis, a function is known to be linear if its "first differences" (differences between each output) are constant. Likewise, a function is known to be quadratic if its "first differences" form an *arithmetic sequence.* Use this information to determine if the following sets of output come from a linear or quadratic function:

a. 19, 11.8, 4.6, −2.6, −9.8, −17, −24.2, . . .

b. −10.31, −10.94, −11.99, −13.46, −15.35, . . .

78. From elementary geometry it is known that the interior angles of a triangle sum to 180°, the interior angles of a quadrilateral sum to 360°, the interior angles of a pentagon sum to 540°, and so on. Use the pattern created by the relationship between the number of sides to the number of angles to develop a formula for the sum of the interior angles of an n-sided polygon. The interior angles of a decagon (10 sides) sum to how many degrees?

▶ MAINTAINING YOUR SKILLS

79. (5.7) Identify the amplitude (A), period (P), horizontal shift (HS), vertical shift (VS) and endpoints of the primary interval (PI) for $f(t) = 7\sin\left(\frac{\pi}{3}t - \frac{\pi}{6}\right) + 10$.

80. (3.1) Graph by completing the square. Label all important features: $y = x^2 - 2x - 3$.

81. (2.3) In 2000, the deer population was 972. By 2005 it had grown to 1217. Assuming the growth is linear, find the function that models this data and use it to estimate the deer population in 2008.

82. (6.1) Verify $\frac{\tan x}{\csc x} - \frac{\sin x}{\cos x} = \frac{\sin x - 1}{\cot x}$ is an identity.

11.3 Geometric Sequences

Learning Objectives

In Section 11.3 you will learn how to:

- ☐ **A.** Identify a geometric sequence and its common ratio
- ☐ **B.** Find the *n*th term of a geometric sequence
- ☐ **C.** Find the *n*th partial sum of a geometric sequence
- ☐ **D.** Find the sum of an infinite geometric series
- ☐ **E.** Solve application problems involving geometric sequences and series

Recall that arithmetic sequences are those where each term is found by *adding* a constant value to the preceding term. In this section, we consider **geometric sequences,** where each term is found by *multiplying* the preceding term by a constant value. Geometric sequences have many interesting applications, as do **geometric series.**

A. Geometric Sequences

A geometric sequence is one where each successive term is found by multiplying the preceding term by a fixed constant. Consider growth of a bacteria population, where a single cell splits in two every hour over a 24-hr period. Beginning with a single bacterium ($a_0 = 1$), after 1 hr there are 2, after 2 hr there are 4, and so on. Writing the number of bacteria as a sequence we have:

hours:	a_1	a_2	a_3	a_4	a_5	. . .
	↓	↓	↓	↓	↓	
bacteria:	2	4	8	16	32	. . .

The sequence 2, 4, 8, 16, 32, . . . is a geometric sequence since each term is found by multiplying the previous term by the constant factor 2. This also means that the ratio of any two consecutive terms must be 2 and in fact, 2 is called the **common ratio *r*** for this sequence. Using the notation from Section 11.1 we can write $r = \frac{a_{k+1}}{a_k}$, where a_k represents any term of the sequence and a_{k+1} represents the term that follows a_k.

EXAMPLE 1 ► **Testing a Sequence for a Common Ratio**

Determine if the given sequence is geometric.

a. 1, 0.5, 0.25, 0.125, . . . **b.** $\frac{1}{8}, \frac{1}{4}, \frac{3}{4}, 3, 15, \ldots$

Solution ► Apply the definition to check for a common ratio $r = \frac{a_{k+1}}{a_k}$.

a. For 1, 0.5, 0.25, 0.125, . . . , the ratio of consecutive terms gives

$$\frac{0.5}{1} = 0.5, \quad \frac{0.25}{0.5} = 0.5, \quad \frac{0.125}{0.25} = 0.5, \quad \text{and so on.}$$

This is a geometric sequence with common ratio $r = 0.5$.

b. For $\frac{1}{8}, \frac{1}{4}, \frac{3}{4}, 3, 15, \ldots$, we have:

$$\frac{1}{4} \div \frac{1}{8} = \frac{1}{4} \cdot \frac{8}{1} = 2 \qquad \frac{3}{4} \div \frac{1}{4} = \frac{3}{4} \cdot \frac{4}{1} = 3 \qquad 3 \div \frac{3}{4} = \frac{3}{1} \cdot \frac{4}{3} = 4 \qquad \text{and so on.}$$

Since the ratio is not constant, this is not a geometric sequence.

Now try Exercises 7 through 24 ►

EXAMPLE 2 ► **Writing the Terms of a Geometric Sequence**

Write the first five terms of the geometric sequence, given the first term $a_1 = -16$ and the common ratio $r = 0.25$.

Solution ► Given $a_1 = -16$ and $r = 0.25$. Starting at $a_1 = -16$, multiply each term by 0.25 to generate the sequence.

$$a_2 = -16 \cdot 0.25 = -4 \qquad a_3 = -4 \cdot 0.25 = -1$$
$$a_4 = -1 \cdot 0.25 = -0.25 \qquad a_5 = -0.25 \cdot 0.25 = -0.0625$$

The first five terms of this sequence are −16, −4, −1, −0.25, and −0.0625.

Now try Exercises 25 through 38 ►

☑ **A.** You've just learned how to identify a geometric sequence and its common ratio

B. Find the *n*th Term of a Geometric Sequence

If the values a_1 and r from a geometric sequence are known, we could generate the terms of the sequence by applying *additional factors of r to the first term,* instead of multiplying each new term by r. If $a_1 = 3$ and $r = 2$, we simply begin at a_1, and continue applying additional factors of r for each successive term.

$$\begin{aligned} 3 &= 3 \cdot 2^0 & a_1 &= a_1r^0 \\ 6 &= 3 \cdot 2^1 & a_2 &= a_1r^1 \\ 12 &= 3 \cdot 2^2 & a_3 &= a_1r^2 \\ 24 &= 3 \cdot 2^3 & a_4 &= a_1r^3 \\ 48 &= 3 \cdot 2^4 & a_5 &= a_1r^4 \end{aligned}$$

current term ↑ initial term ↑ exponent on common ratio

From this pattern, we note the exponent on r is always 1 less than the subscript of the current term: $5 - 1 = 4$, which leads us to the formula for the nth term of a geometric sequence.

The *n*th Term of a Geometric Sequence

The nth term of a *geometric sequence* is given by

$$a_n = a_1r^{n-1}$$

where r is the common ratio.

EXAMPLE 3 ► **Finding a Specific Term in a Sequence**

Find the 10th term of the sequence 3, −6, 12, −24, ...

Solution ► Instead of writing out all 10 terms, we determine the constant ratio r and use the nth term formula. By inspection we note that $a_1 = 3$ and $r = -2$.

$$\begin{aligned} a_n &= a_1r^{n-1} && n\text{th term formula} \\ &= 3(-2)^{n-1} && \text{substitute 3 for } a_1 \text{ and } -2 \text{ for } r \end{aligned}$$

To find the 10th term we substitute $n = 10$:

$$\begin{aligned} a_{10} &= 3(-2)^{10-1} && \text{substitute 10 for } n \\ &= 3(-2)^9 = -1536 && \text{simplify} \end{aligned}$$

Now try Exercises 39 through 46 ►

EXAMPLE 4 ► **Determining the Number of Terms in a Geometric Sequence**

Find the number of terms in the geometric sequence 4, 2, 1, . . . , $\frac{1}{64}$.

Solution ► Observing that $a_1 = 4$ and $r = \frac{1}{2}$, we have

$$\begin{aligned} a_n &= a_1r^{n-1} && n\text{th term formula} \\ &= 4\left(\frac{1}{2}\right)^{n-1} && \text{substitute 4 for } a_1 \text{ and } \frac{1}{2} \text{ for } r \end{aligned}$$

Although we don't know the number of terms in the sequence, we *do* know the last or nth term is $\frac{1}{64}$. Substituting $a_n = \frac{1}{64}$ gives

$$\begin{aligned} \frac{1}{64} &= 4\left(\frac{1}{2}\right)^{n-1} && \text{substitute } \frac{1}{64} \text{ for } a_n \\ \frac{1}{256} &= \left(\frac{1}{2}\right)^{n-1} && \text{divide by 4} \left(\text{multiply by } \frac{1}{4}\right) \end{aligned}$$

From our work in Chapter 4, we attempt to write both sides as exponentials with a like base, or apply logarithms. Since $256 = 2^8$, we equate bases.

$$\left(\frac{1}{2}\right)^8 = \left(\frac{1}{2}\right)^{n-1} \qquad \text{write } \frac{1}{256} \text{ as } \left(\frac{1}{2}\right)^8$$

$$\rightarrow 8 = n - 1 \qquad \text{like bases imply exponents must be equal}$$

$$9 = n \qquad \text{solve for } n$$

This shows there are nine terms in the sequence.

Now try Exercises 47 through 58 ►

If the term a_1 is unknown but a term a_k is given, the nth term can be written

$$a_n = a_k r^{n-k}, \text{ since } n = k + (n - k)$$

(the subscript on the term a_k and the exponent on r sum to n).

EXAMPLE 5 ► **Finding the First Term of a Geometric Sequence**

Given a geometric sequence where $a_4 = 0.075$ and $a_7 = 0.009375$, find the common ratio r and the value of a_1.

Solution ► Since a_1 is not known, we express a_7 as the product of a known term and the appropriate number of common ratios: $a_7 = a_4r^3$ $(7 - 4 = 3$, as required).

$$\begin{aligned} a_7 &= a_4 \cdot r^3 && a_1 \text{ is unknown} \\ 0.009375 &= 0.075r^3 && \text{substitute 0.009375 for } a_7 \text{ and 0.075 for } a_4 \\ 0.125 &= r^3 && \text{divide by 0.075} \\ r &= 0.5 && \text{solve for } r \end{aligned}$$

Having found r, we can now solve for a_1

$$\begin{aligned} a_7 &= a_1r^6 && n\text{th term formula} \\ 0.009375 &= a_1(0.5)^6 && \text{substitute 0.009375 for } a_7 \text{ and 0.5 for } r \\ 0.009375 &= a_1(0.015625) && \text{simplify} \\ a_1 &= 0.6 && \text{solve for } a_1 \end{aligned}$$

The first term is $a_1 = 0.6$ and the common ratio is $r = 0.5$.

☑ **B.** You've just learned how to find the nth term of a geometric sequence

Now try Exercises 59 through 64 ►

C. Find the *n*th Partial Sum of a Geometric Sequence

As with arithmetic series, applications of geometric series often involve computing a sum of consecutive terms. We can adapt the method for finding the sum of an arithmetic sequence to develop a formula for adding the first n terms of a geometric sequence.

For the nth term $a_n = a_1r^{n-1}$, we have $S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1}$. If we multiply S_n by $-r$ then add the original series, the "interior terms" sum to zero.

$$\begin{array}{rcccccccccccc} -\ rS_n = & & -a_1r & + & (-a_1r^2) & + & (-a_1r^3) & + \cdots + & (-a_1r^{n-1}) & + & (-a_1r^n) \\ +\ \ S_n = & a_1 & + & a_1r & + & a_1r^2 & + \cdots + & a_1r^{n-2} & + & a_1r^{n-1} & \\ \hline S_n - rS_n = & a_1 & + & 0 & + & 0 & + 0 + & 0 & + & 0 & + (-a_1r^n) \end{array}$$

We then have $S_n - rS_n = a_1 - a_1r^n$, and can now solve for S_n:

$$\begin{aligned} S_n - rS_n &= a_1 - a_1r^n && \text{difference of } S_n \text{ and } rS_n \\ S_n(1 - r) &= a_1 - a_1r^n && \text{factor out } S_n \\ S_n &= \frac{a_1 - a_1r^n}{1 - r} && \text{solve for } S_n \end{aligned}$$

The result is a formula for the *n*th partial sum of a geometric sequence.

The *n*th Partial Sum of a Geometric Sequence

Given a geometric sequence with first term a_1 and common ratio r, the nth partial sum (the sum of the first n terms) is

$$S_n = \frac{a_1 - a_1r^n}{1 - r} = \frac{a_1(1 - r^n)}{1 - r}, r \neq 1$$

In words: The sum of a geometric sequence is the difference of the first and $(n + 1)$st term, divided by 1 minus the common ratio.

EXAMPLE 6 ► **Computing a Partial Sum**

Find the sum: $\sum_{i=1}^{9} 3^i$ (the first nine powers of 3).

Solution ► The initial terms of this series are $3 + 9 + 27 + \cdots$, and we note $a_1 = 3$, $r = 3$, and $n = 9$. We could find the first nine terms and add, but using the partial sum formula gives

$$\begin{aligned} S_n &= \frac{a_1(1 - r^n)}{1 - r} && \text{sum formula} \\ S_9 &= \frac{3(1 - 3^9)}{1 - 3} && \text{substitute 3 for } a_1\text{, 9 for } n\text{, and 3 for } r \\ &= \frac{3(-19{,}682)}{-2} && \text{simplify} \\ &= 29{,}523 && \text{result} \end{aligned}$$

☑ **C.** You've just learned how to find the *n*th partial sum of a geometric sequence

Now try Exercises 65 through 88 ►

D. The Sum of an Infinite Geometric Series

To this point we've considered only partial sums of a geometric series. While it is impossible to add an infinite number of these terms, some of these "infinite sums" appear to have a limiting value. The sum appears to get ever closer to this value but never exceeds it—much like the asymptotic behavior of some graphs. We will define the sum of this **infinite geometric series** to be this limiting value, if it exists. Consider the illustration in Figure 11.7, where a standard sheet of typing paper is cut in half. One of the halves is again cut in half and the process is continued indefinitely, as shown. Notice the "halves" create an infinite sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \cdots$ with $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$. The corresponding infinite series is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots$.

Figure 11.7

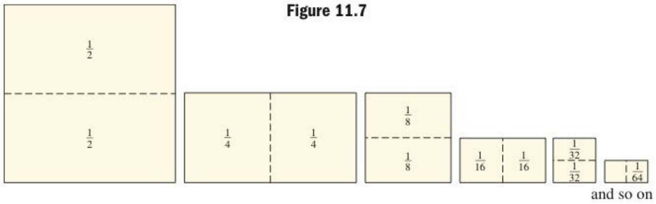
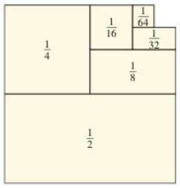


Figure 11.8



If we arrange one of the halves from each stage as shown in Figure 11.8, we would be rebuilding the original sheet of paper. As we add more and more of these halves together, we get closer and closer to the size of the original sheet. We gain an intuitive sense that this series must add to 1, because the *pieces* of the original sheet of paper must add to 1 whole sheet. To explore this idea further, consider what happens to $(\frac{1}{2})^n$ as n becomes large.

$$n = 4: \left(\frac{1}{2}\right)^4 = 0.0625 \qquad n = 8: \left(\frac{1}{2}\right)^8 \approx 0.004 \qquad n = 12: \left(\frac{1}{2}\right)^{12} \approx 0.0002$$

Further exploration with a calculator seems to support the idea that as $n \to \infty$, $(\frac{1}{2})^n \to 0$, although a definitive proof is left for a future course. In fact, it can be shown that for any $|r| < 1$, r^n becomes very close to zero as n becomes large. In symbols: as $n \to \infty$, $r^n \to 0$. For $S_n = \frac{a_1 - a_1r^n}{1 - r} = \frac{a_1}{1 - r} - \frac{a_1r^n}{1 - r}$, note that if $|r| < 1$ and "we sum an infinite number of terms," the second term becomes zero, leaving only the first term. In other words, the limiting value (represented by S_∞) is $S_\infty = \frac{a_1}{1 - r}$.

WORTHY OF NOTE

The formula for the sum of an infinite geometric series can also be derived by noting that $S_\infty = a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots$ can be rewritten as $S_\infty = a_1 + r(a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots) = a_1 + rS_\infty$.

$$S_\infty - rS_\infty = a_1$$
$$S_\infty(1 - r) = a_1$$
$$S_\infty = \frac{a_1}{1 - r}.$$

Infinite Geometric Series

Given a geometric sequence with first term a_1 and $|r| < 1$, the sum of the related infinite series is given by

$$S_\infty = \frac{a_1}{1 - r}; r \neq 1$$

If $|r| > 1$, no finite sum exists.

EXAMPLE 7 ► **Computing an Infinite Sum**

Find the limiting value of each infinite geometric series (if it exists).

a. $1 + 2 + 4 + 8 + \cdots$ **b.** $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$

c. $0.185 + 0.000185 + 0.000000185 + \cdots$

Solution ► Begin by determining if the infinite series is geometric. If so, use $S_\infty = \frac{a_1}{1 - r}$.

a. Since $r = 2$ (by inspection), a finite sum does not exist.

b. Using the ratio of consecutive terms we find $r = \frac{2}{3}$ and the infinite sum exists. With $a_1 = 3$, we have

$$S_\infty = \frac{3}{1 - \frac{2}{3}} = \frac{3}{\frac{1}{3}} = 9$$

c. This series is equivalent to the repeating decimal $0.185185185\ldots = 0.\overline{185}$. The common ratio is $r = \frac{0.000185}{0.185} = 0.001$ and the infinite sum exists:

$$S_\infty = \frac{0.185}{1 - 0.001} = \frac{5}{27}$$

☑ **D.** You've just learned how to find the sum of an infinite geometric series

Now try Exercises 89 through 104 ►

E. Applications Involving Geometric Sequences and Series

Here are a few of the ways these ideas can be put to use.

EXAMPLE 8 ► **Solving an Application of Geometric Sequences: Pendulums**

A pendulum is any object attached to a fixed point and allowed to swing freely under the influence of gravity. Suppose each swing is 0.9 the length of the previous one. Gradually the swings become shorter and shorter and at some point the pendulum will appear to have stopped (although *theoretically* it never does).

a. How far does the pendulum travel on its eighth swing, if the first swing was 2 m?

b. What is the total distance traveled by the pendulum for these eight swings?

c. How many swings until the length of each swing falls below 0.5 m?

d. What total distance does the pendulum travel before coming to rest?

Solution ► **a.** The lengths of each swing form the terms of a geometric sequence with $a_1 = 2$ and $r = 0.9$. The first few terms are 2, 1.8, 1.62, 1.458, and so on. For the 8th term we have:

$$a_n = a_1 r^{n-1} \quad \text{nth term formula}$$
$$a_8 = 2(0.9)^{8-1} \quad \text{substitute 8 for } n\text{, 2 for } a_1\text{, and 0.9 for } r$$
$$\approx 0.956$$

The pendulum travels about 0.956 m on its 8th swing.

b. For the total distance traveled after eight swings, we compute the value of S_8.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad \text{nth partial sum formula}$$
$$S_8 = \frac{2(1 - 0.9^8)}{1 - 0.9} \quad \text{substitute 2 for } a_1\text{, 0.9 for } r\text{, and 8 for } n$$
$$\approx 11.4$$

The pendulum has traveled about 11.4 m by the end of the 8th swing.

c. To find the number of swings until the length of each swing is less than 0.5 m, we solve for n in the equation $0.5 = 2(0.9)^{n-1}$. This yields

$$0.25 = (0.9)^{n-1} \quad \text{divide by 2}$$
$$\ln 0.25 = (n - 1)\ln 0.9 \quad \text{take the natural log, apply power property}$$
$$\frac{\ln 0.25}{\ln 0.9} + 1 = n \quad \text{solve for } n \text{ (exact form)}$$
$$14.16 \approx n \quad \text{solve for } n \text{ (approximate form)}$$

After the 14th swing, each successive swing will be less than 0.5 m.

d. For the total distance traveled before coming to rest, we consider the related infinite geometric series, with $a_1 = 2$ and $r = 0.9$.

$$S_\infty = \frac{a_1}{1 - r} \quad \text{infinite sum formula}$$
$$S_\infty = \frac{2}{1 - 0.9} \quad \text{substitute 2 for } a_1 \text{ and 0.9 for } r$$
$$= 20 \quad \text{result}$$

The pendulum would travel 20 m before coming to rest.

☑ **E. You've just learned how to solve application problems involving geometric sequences and series**

Now try Exercises 107 through 119 ►

11.3 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. In a geometric sequence, each successive term is found by ________ the preceding term by a fixed value *r*.

2. In a geometric sequence, the common ratio *r* can be found by computing the ________ of any two consecutive terms.

3. The *n*th term of a geometric sequence is given by $a_n =$ ______, for any $n \geq 1$.

4. For the general sequence $a_1, a_2, a_3, \ldots, a_k, \ldots$, the fifth partial sum is given by $S_5 =$ ________.

5. Describe/Discuss how the formula for the *n*th partial sum is related to the formula for the sum of an infinite geometric series.

6. Describe the difference(s) between an arithmetic and a geometric sequence. How can a student prevent confusion between the formulas?

▶ DEVELOPING YOUR SKILLS

Determine if the sequence given is geometric. If yes, name the common ratio. If not, try to determine the pattern that forms the sequence.

7. $4, 8, 16, 32, \ldots$

8. $2, 6, 18, 54, 162, \ldots$

9. $3, -6, 12, -24, 48, \ldots$

10. $128, -32, 8, -2, \ldots$

11. $2, 5, 10, 17, 26, \ldots$

12. $-13, -9, -5, -1, 3, \ldots$

13. $3, 0.3, 0.03, 0.003, \ldots$

14. $12, 0.12, 0.0012, 0.000012, \ldots$

15. $-1, 3, -12, 60, -360, \ldots$

16. $-\frac{2}{3}, 2, -8, 40, -240, \ldots$

17. $25, 10, 4, \frac{8}{5}, \ldots$

18. $-36, 24, -16, \frac{32}{3}, \ldots$

19. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

20. $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$

21. $3, \dfrac{12}{x}, \dfrac{48}{x^2}, \dfrac{192}{x^3}, \ldots$

22. $5, \dfrac{10}{a}, \dfrac{20}{a^2}, \dfrac{40}{a^3}, \ldots$

23. $240, 120, 40, 10, 2, \ldots$

24. $-120, -60, -20, -5, -1, \ldots$

Write the first four terms of the sequence, given a_1 and r.

25. $a_1 = 5, r = 2$

26. $a_1 = 2, r = -4$

27. $a_1 = -6, r = -\frac{1}{2}$

28. $a_1 = \frac{2}{3}, r = \frac{1}{5}$

29. $a_1 = 4, r = \sqrt{3}$

30. $a_1 = \sqrt{5}, r = \sqrt{5}$

31. $a_1 = 0.1, r = 0.1$

32. $a_1 = 0.024, r = 0.01$

Find the indicated term for each sequence.

33. $a_1 = -24, r = \frac{1}{2}$; find a_7

34. $a_1 = 48, r = -\frac{1}{3}$; find a_6

35. $a_1 = -\frac{1}{20}, r = -5$; find a_4

36. $a_1 = \frac{3}{20}, r = 4$; find a_5

37. $a_1 = 2, r = \sqrt{2}$; find a_7

38. $a_1 = \sqrt{3}, r = \sqrt{3}$; find a_8

Identify a_1 and r, then write the expression for the nth term $a_n = a_1r^{n-1}$ and use it to find a_6, a_{10}, and a_{12}.

39. $\frac{1}{27}, -\frac{1}{9}, \frac{1}{3}, -1, 3, \ldots$

40. $-\frac{7}{8}, \frac{7}{4}, -\frac{7}{2}, 7, -14, \ldots$

41. $729, 243, 81, 27, 9, \ldots$

42. $625, 125, 25, 5, 1, \ldots$

43. $\frac{1}{2}, \frac{\sqrt{2}}{2}, 1, \sqrt{2}, 2, \ldots$

44. $36\sqrt{3}, 36, 12\sqrt{3}, 12, 4\sqrt{3}, \ldots$

45. $0.2, 0.08, 0.032, 0.0128, \ldots$

46. $0.5, -0.35, 0.245, -0.1715, \ldots$

Find the number of terms in each sequence.

47. $a_1 = 9, a_n = 729, r = 3$

48. $a_1 = 1, a_n = -128, r = -2$

49. $a_1 = 16, a_n = \frac{1}{64}, r = \frac{1}{2}$

50. $a_1 = 4, a_n = \frac{1}{512}, r = \frac{1}{2}$

51. $a_1 = -1, a_n = -1296, r = \sqrt{6}$

52. $a_1 = 2, a_n = 1458, r = -\sqrt{3}$

53. $2, -6, 18, -54, \ldots, -4374$

54. $3, -6, 12, -24, \ldots, -6144$

55. $64, 32\sqrt{2}, 32, 16\sqrt{2}, \ldots, 1$

56. $243, 81\sqrt{3}, 81, 27\sqrt{3}, \ldots, 1$

57. $\frac{3}{8}, -\frac{3}{4}, \frac{3}{2}, -3, \ldots, 96$

58. $-\frac{5}{27}, \frac{5}{9}, -\frac{5}{3}, -5, \ldots, -135$

Find the common ratio *r* and the value of a_1 using the information given (assume $r > 0$).

59. $a_3 = 324, a_7 = 64$ **60.** $a_5 = 6, a_9 = 486$

61. $a_4 = \frac{4}{9}, a_8 = \frac{9}{4}$ **62.** $a_2 = \frac{16}{81}, a_5 = \frac{2}{3}$

63. $a_4 = \frac{32}{3}, a_8 = 54$ **64.** $a_3 = \frac{16}{25}, a_7 = 25$

Find the indicated sum. For Exercises 81 and 82, use the summation properties from Section 11.1.

65. $a_1 = 8, r = -2$; find S_{12}

66. $a_1 = 2, r = -3$; find S_8

67. $a_1 = 96, r = \frac{1}{3}$; find S_5

68. $a_1 = 12, r = \frac{1}{2}$; find S_8

69. $a_1 = 8, r = \frac{3}{2}$; find S_7

70. $a_1 = -1, r = -\frac{3}{2}$; find S_{10}

71. $2 + 6 + 18 + \cdots$; find S_6

72. $2 + 8 + 32 + \cdots$; find S_7

73. $16 - 8 + 4 - \cdots$; find S_8

74. $4 - 12 + 36 - \cdots$; find S_8

75. $\frac{4}{3} + \frac{2}{9} + \frac{1}{27} + \cdots$; find S_9

76. $\frac{1}{18} - \frac{1}{6} + \frac{1}{2} - \cdots$; find S_7

77. $\sum_{j=1}^{5} 4^j$ **78.** $\sum_{k=1}^{10} 2^k$

79. $\sum_{k=1}^{8} 5\left(\frac{2}{3}\right)^{k-1}$ **80.** $\sum_{j=1}^{7} 3\left(\frac{1}{5}\right)^{j-1}$

81. $\sum_{i=4}^{10} 9\left(-\frac{1}{2}\right)^{i-1}$ **82.** $\sum_{i=3}^{8} 5\left(-\frac{1}{4}\right)^{i-1}$

Find the indicated partial sum using the information given. Write all results in simplest form.

83. $a_2 = -5, a_5 = \frac{1}{25}$; find S_5

84. $a_3 = 1, a_6 = -27$; find S_6

85. $a_3 = \frac{4}{9}, a_7 = \frac{9}{64}$; find S_6

86. $a_2 = \frac{16}{81}, a_5 = \frac{2}{3}$; find S_8

87. $a_3 = 2\sqrt{2}, a_6 = 8$; find S_7

88. $a_2 = 3, a_5 = 9\sqrt{3}$; find S_7

Determine whether the infinite geometric series has a finite sum. If so, find the limiting value.

89. $3 + 6 + 12 + 24 + \cdots$

90. $4 + 8 + 16 + 32 + \cdots$

91. $9 + 3 + 1 + \cdots$

92. $36 + 24 + 16 + \cdots$

93. $25 + 10 + 4 + \frac{8}{5} + \cdots$

94. $10 + 2 + \frac{2}{5} + \frac{2}{25} + \cdots$

95. $6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$

96. $-49 + (-7) + (-\frac{1}{7}) + \cdots$

97. $6 - 3 + \frac{3}{2} - \frac{3}{4} + \cdots$

98. $10 - 5 + \frac{5}{2} - \frac{5}{4} + \cdots$

99. $0.3 + 0.03 + 0.003 + \cdots$

100. $0.63 + 0.0063 + 0.000063 + \cdots$

101. $\sum_{k=1}^{\infty} \frac{3}{4}\left(\frac{2}{3}\right)^{k}$ **102.** $\sum_{i=1}^{\infty} 5\left(\frac{1}{2}\right)^{i}$

103. $\sum_{j=1}^{\infty} 9\left(-\frac{2}{3}\right)^{j}$ **104.** $\sum_{k=1}^{\infty} 12\left(\frac{4}{3}\right)^{k}$

▶ WORKING WITH FORMULAS

105. Sum of the cubes of the first n natural numbers: $S_n = \frac{n^2(n+1)^2}{4}$

Compute $1^3 + 2^3 + 3^3 + \cdots + 8^3$ using the formula given. Then confirm the result by direct calculation.

106. Student loan payment: $A_n = P(1 + r)^n$

If P dollars is borrowed at an annual interest rate r with interest compounded annually, the amount of money to be paid back after n years is given by the indicated formula. Find the total amount of money that the student must repay to clear the loan, if \$8000 is borrowed at 4.5% interest and the loan is paid back in 10 yr.

▶ APPLICATIONS

107. Pendulum movement: On each swing, a pendulum travels only 80% as far as it did on the previous swing. If the first swing is 24 ft, how far does the pendulum travel on the 7th swing? What total distance is traveled before the pendulum comes to rest?

108. Pendulum movement: Ernesto is swinging to and fro on his backyard tire swing. Using his legs and body, he pumps each swing until reaching a maximum height, then suddenly relaxes until the swing comes to a stop. With each swing, Ernesto travels 75% as far as he did on the previous swing. If the first arc (or swing) is 30 ft, find the distance Ernesto travels on the 5th arc. What total distance will he travel before coming to rest?

109. Depreciation: A certain new SUV depreciates in value about 20% per year (meaning it holds 80% of its value each year). If the SUV is purchased for \$46,000, how much is it worth 4 yr later? How many years until its value is less than \$5000?

110. Depreciation: A new photocopier under heavy use will depreciate about 25% per year (meaning it holds 75% of its value each year). If the copier is purchased for \$7000, how much is it worth 4 yr later? How many years until its value is less than \$1246?

111. Equipment aging: Tests have shown that the pumping power of a heavy-duty oil pump decreases by 3% per month. If the pump can move 160 gallons per minute (gpm) new, how many gpm can the pump move 8 months later? If the pumping rate falls below 118 gpm, the pump must be replaced. How many months until this pump is replaced?

112. Equipment aging: At the local mill, a certain type of saw blade can saw approximately 2 log-feet/sec when it is new. As time goes on, the blade becomes worn, and loses 6% of its cutting speed each week. How many log-feet/sec can the saw blade cut after 6 weeks? If the cutting speed falls below 1.2 log-feet/sec, the blade must be replaced. During what week of operation will this blade be replaced?

113. Population growth: At the beginning of the year 2000, the population of the United States was approximately 277 million. If the population is growing at a rate of 2.3% per year, what will the population be in 2010, 10 yr later?

114. Population growth: The population of the Zeta Colony on Mars is 1000 people. Determine the population of the Colony 20 yr from now, if the population is growing at a constant rate of 5% per year.

115. Population growth: A biologist finds that the population of a certain type of bacteria doubles *each half-hour.* If an initial culture has 50 bacteria, what is the population after 5 hr? How long will it take for the number of bacteria to reach 204,800?

116. Population growth: Suppose the population of a "boom town" in the old west doubled *every 2 months* after gold was discovered. If the initial population was 219, what was the population 8 months later? How many months until the population exceeds 28,000?

117. **Elastic rebound:** Megan discovers that a rubber ball dropped from a height of 2 m rebounds four-fifths of the distance it has previously fallen. How high does it rebound on the 7th bounce? How far does the ball travel before coming to rest?

118. **Elastic rebound:** The screen saver on my computer is programmed to send a colored ball vertically down the middle of the screen so that it rebounds 95% of the distance it last traversed. If the ball always begins at the top and the screen is 36 cm tall, how high does the ball bounce after its 8th rebound? How far does the ball travel before coming to rest (and a new screen saver starts)?

119. **Creating a vacuum:** To create a vacuum, a hand pump is used to remove the air from an air-tight cube with a volume of 462 in^3. With each stroke of the pump, two-fifths of the air that remains in the cube is removed. How much air remains inside after the 5th stroke? How many strokes are required to remove all but 12.9 in^3 of the air?

▶ EXTENDING THE CONCEPT

120. As part of a science experiment, identical rubber balls are dropped from a certain height on these surfaces: slate, cement, and asphalt. When dropped on slate, the ball rebounds 80% of the height from which it last fell. On cement the figure is 75% and on asphalt the figure is 70%. The ball is dropped from 130 m on the slate, 175 m on the cement, and 200 m on the asphalt. Which ball has traveled the shortest total distance at the time of the fourth bounce? Which ball will travel farthest before coming to rest?

121. Consider the following situation. A person is hired at a salary of $40,000 per year, with a guaranteed raise of $1750 per year. At the same time, inflation is running about 4% per year. How many years until this person's salary is overtaken and eaten up by the actual cost of living?

122. A standard piece of typing paper is approximately 0.001 in. thick. Suppose you were able to fold this piece of paper in half 26 times. How thick would the result be? (a) As tall as a hare, (b) as tall as a hen, (c) as tall as a horse, (d) as tall as a house, or (e) over 1 mi high? Find the actual height by computing the 27th term of a geometric sequence. Discuss what you find.

123. Find an alternative formula for the sum $S_n = \sum_{k=1}^{n} \log k$, that does not use the sigma notation.

124. Verify the following statements:

a. If $a_1, a_2, a_3, \ldots, a_n$ is a geometric sequence with r and a_1 greater than zero, then $\log a_1$, $\log a_2$, $\log a_3, \ldots, \log a_n$ is an arithmetic sequence.

b. If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic sequence, then $10^{a_1}, 10^{a_2}, \ldots, 10^{a_n}$, is a geometric sequence.

▶ MAINTAINING YOUR SKILLS

125. **(1.5)** Find the zeroes of f using the quadratic formula: $f(x) = x^2 + 5x + 9$.

126. **(7.3)** Find a unit vector in the same direction as $3\mathbf{i} - 7\mathbf{j}$.

127. **(4.6)** Graph the rational function:

$$h(x) = \frac{x^2}{x - 1}$$

128. **(5.1)** The cars on the Millenium Ferris Wheel are 100 ft from the center axle. If the top speed of the wheel is 1.5 revolutions per minute, find the linear velocity of a passenger in a car. Round your answer to the nearest whole number. Also, give the velocity in miles per hour.

11.4 Mathematical Induction

Learning Objectives

In Section 11.4 you will learn how to:

- ☐ **A.** Use subscript notation to evaluate and compose functions
- ☐ **B.** Apply the principle of mathematical induction to sum formulas involving natural numbers
- ☐ **C.** Apply the principle of mathematical induction to general statements involving natural numbers

Since middle school (or even before) we have accepted that, "The product of two negative numbers is a positive number." But have you ever been asked to *prove* it? It's not as easy as it seems. We may think of several patterns that yield the result, analogies that indicate its truth, or even number line illustrations that lead us to believe the statement. But most of us have never seen a *proof* (see www.mhhe.com/coburn). In this section, we introduce one of mathematics' most powerful tools for proving a statement, called **proof by induction.**

A. Subscript Notation and Composition of Functions

One of the challenges in understanding a proof by induction is working with the notation. Earlier in the chapter, we introduced subscript notation as an alternative to function notation, since it is more commonly used when the functions are defined by a sequence. But regardless of the notation used, the functions can still be simplified, evaluated, composed, and even graphed. Consider the function $f(x) = 3x^2 - 1$ and the sequence defined by $a_n = 3n^2 - 1$. Both can be evaluated and graphed, with the only difference being that $f(x)$ is continuous with domain $x \in \mathbb{R}$, while a_n is discrete (made up of distinct points) with domain $n \in \mathbb{N}$.

EXAMPLE 1 ► **Using Subscript Notation for a Composition**

For $f(x) = 3x^2 - 1$ and $a_n = 3n^2 - 1$, find $f(k + 1)$ and a_{k+1}.

Solution ►

$$\begin{aligned} f(k+1) &= 3(k+1)^2 - 1 \\ &= 3(k^2 + 2k + 1) - 1 \\ &= 3k^2 + 6k + 2 \end{aligned} \qquad \begin{aligned} a_{k+1} &= 3(k+1)^2 - 1 \\ &= 3(k^2 + 2k + 1) - 1 \\ &= 3k^2 + 6k + 2 \end{aligned}$$

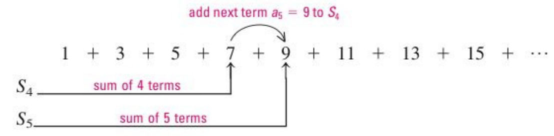
Now try Exercises 7 through 18 ►

☑ **A. You've just learned how to use subscript notation to evaluate and compose functions**

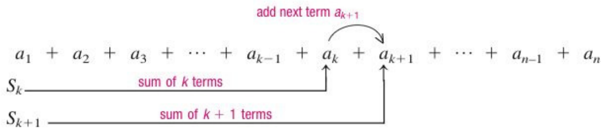
No matter which notation is used, every occurrence of the input variable is replaced by the new value or expression indicated by the composition.

B. Mathematical Induction Applied to Sums

Consider the sum of odd numbers $1 + 3 + 5 + 7 + 9 + 11 + 13 + \cdots$. The sum of the first four terms is $1 + 3 + 5 + 7 = 16$, or $S_4 = 16$. If we now add a_5 (the next term in line), would we get the same answer as if we had simply computed S_5? Common sense would say, "Yes!" since $S_5 = 1 + 3 + 5 + 7 + 9 = 25$ and $S_4 + a_5 = 16 + 9 = 25$✓. In diagram form, we have



Our goal is to develop this same degree of clarity in the *notational scheme* of things. For a given series, if we find the kth partial sum S_k (shown next) and then add the next term a_{k+1}, would we get the same answer if we had simply computed S_{k+1}? In other words, is $S_k + a_{k+1} = S_{k+1}$ true?



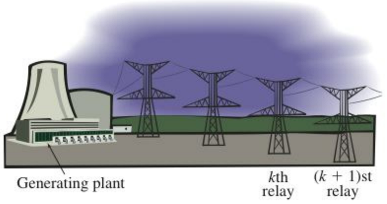
Now, let's return to the sum $1 + 3 + 5 + 7 + \cdots + 2n - 1$. This is an arithmetic series with $a_1 = 1$, $d = 2$, and nth term $a_n = 2n - 1$. Using the sum formula for an arithmetic sequence, an alternative formula for *this sum* can be established.

$$\begin{aligned} S_n &= \frac{n(a_1 + a_n)}{2} && \text{summation formula for an arithmetic sequence} \\ &= \frac{n(1 + \mathbf{2n - 1})}{2} && \text{substitute 1 for } a_1 \text{ and } 2n-1 \text{ for } a_n \\ &= \frac{n(2n)}{2} && \text{simplify} \\ &= n^2 && \text{result} \end{aligned}$$

This shows that the sum of the first n positive odd integers is given by $S_n = n^2$. As a check we compute $S_5 = 1 + 3 + 5 + 7 + 9 = 25$ and compare to $S_5 = 5^2 = 25$✓. We also note $S_6 = 6^2 = 36$, and $S_5 + a_6 = 25 + 11 = 36$, showing $S_6 = S_5 + a_6$. For more on this relationship, **see Exercises 19 through 24.**

While it may seem simplistic now, showing $S_5 + a_6 = S_6$ and $S_k + a_{k+1} = S_{k+1}$ (in general) is a critical component of a proof by induction. Unfortunately, general summation formulas for many sequences cannot be established from known formulas. In addition, just because a formula works for the first few values of n, we cannot assume that it will hold true for *all* values of n (there are infinitely many). As an illustration, the formula $a_n = n^2 - n + 41$ yields a prime number for *every natural number n from 1 to 40,* but fails to yield a prime for $n = 41$. This helps demonstrate the need for a more conclusive proof, particularly when a relationship appears to be true, and can be "verified" in a finite number of cases, but whether it is true in *all* cases remains in question.

Proof by induction is based on a relatively simple idea. To help understand how it works, consider n relay stations that are used to transport electricity from a generating plant to a distant city. If we know the generating plant is operating, and if we assume that the kth relay station (any station in the series) is making the transfer to the $(k + 1)$st station (the next station in the series), then we're sure the city will have electricity.



WORTHY OF NOTE

No matter how distant the city or how many relay stations are involved, if the generating plant is working and the kth station relays to the $(k + 1)$st station, the city will get its power.

This idea can be applied mathematically as follows. Consider the statement, "The sum of the first n positive, even integers is $n^2 + n$." In other words, $2 + 4 + 6 + 8 + \cdots + 2n = n^2 + n$. We can certainly verify the statement for the first few even numbers:

The first even number is 2 and . . .	$(1)^2 + 1 = 2$
The sum of the first *two* even numbers is $2 + 4 = 6$ and . . .	$(2)^2 + 2 = 6$
The sum of the first *three* even numbers is $2 + 4 + 6 = 12$ and . . .	$(3)^2 + 3 = 12$
The sum of the first *four* even numbers is $2 + 4 + 6 + 8 = 20$ and . . .	$(4)^2 + 4 = 20$

While we could continue this process for a very long time (or even use a computer), *no finite number of checks can prove a statement is universally true.* To prove the statement true for *all* positive integers, we use a reasoning similar to that applied in the relay stations example. If we are sure the formula works for $n = 1$ (the generating station is operating), and assume that if the formula is true for $n = k$, it must also be true for $n = k + 1$ [the kth relay station is transferring electricity to the $(k + 1)$st station], then the statement is true for all n (the city will get its electricity). The case where $n = 1$ is called the **base case** of an inductive proof, and the assumption that the formula is true for $n = k$ is called the **induction hypothesis.** When the induction hypothesis is applied to a sum formula, we attempt to show that $S_k + a_{k+1} = S_{k+1}$. Since k and $k + 1$ are arbitrary, the statement must be true for all n.

Mathematical Induction Applied to Sums

Let S_n be a sum formula involving positive integers.

If **1.** S_1 is true, and

2. the truth of S_k implies that S_{k+1} is true,

then S_n must be true for all positive integers n.

WORTHY OF NOTE

To satisfy our finite minds, it might help to show that S_n is true for the first few cases, prior to extending the ideas to the infinite case.

Both parts 1 and 2 must be verified for the proof to be complete. Since the process requires the terms S_k, a_{k+1}, and S_{k+1}, we will usually compute these first.

EXAMPLE 2 ► **Proving a Statement Using Mathematical Induction**

Use induction to prove that *the sum of the first n perfect squares is given by*

$$1 + 4 + 9 + 16 + 25 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution ► Given $a_n = n^2$ and $S_n = \frac{n(n+1)(2n+1)}{6}$, the needed components are . . .

$$\text{For } a_n = n^2\text{: } a_k = k^2 \qquad \text{and} \qquad a_{k+1} = (k+1)^2$$

$$\text{For } S_n = \frac{n(n+1)(2n+1)}{6}\text{:} \quad S_k = \frac{k(k+1)(2k+1)}{6} \quad \text{and} \quad S_{k+1} = \frac{(k+1)(k+2)(2k+3)}{6}$$

1. Show S_n is true for $n = 1$.

$$S_n = \frac{n(n+1)(2n+1)}{6} \qquad \text{sum formula}$$

$$S_1 = \frac{1(2)(3)}{6} \qquad \textit{base case: } n = 1$$

$$= 1\checkmark \qquad \text{result checks, the first term is 1}$$

2. Assume S_k is true,

$$1 + 4 + 9 + 16 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \text{induction hypothesis: } S_k \text{ is true}$$

and use it to show the truth of S_{k+1} follows. That is,

$$\underbrace{1 + 4 + 9 + 16 + \cdots + k^2}_{S_k} + \underbrace{(k+1)^2}_{a_{k+1}} = \underset{S_{k+1}}{\frac{(k+1)(k+2)(2k+3)}{6}}$$

Working with the left-hand side, we have

$$\underbrace{1 + 4 + 9 + 16 + \cdots + k^2}_{} + (k + 1)^2$$

$$\begin{aligned}
&= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 && \text{use the induction hypothesis: substitute } \frac{k(k+1)(2k+1)}{6} \text{ for } 1 + 4 + 9 + 16 + 25 + \cdots + k^2 \\
&= \frac{k(k + 1)(2k + 1) + 6(k + 1)^2}{6} && \text{common denominator} \\
&= \frac{(k + 1)[k(2k + 1) + 6(k + 1)]}{6} && \text{factor out } k + 1 \\
&= \frac{(k + 1)[2k^2 + 7k + 6]}{6} && \text{multiply and combine terms} \\
&= \frac{(k + 1)(k + 2)(2k + 3)}{6} && \text{factor the trinomial, result is } S_{k+1}\checkmark
\end{aligned}$$

Since the truth of S_{k+1} follows from S_k, the formula is true for all n.

☑ **B. You've just learned how to apply the principle of mathematical induction to sum formulas involving natural numbers**

Now try Exercises 27 through 38 ►

C. The General Principle of Mathematical Induction

Proof by induction can be used to verify many other kinds of relationships involving a natural number n. In this regard, the basic principles remain the same but are stated more broadly. Rather than having S_n represent a sum, we take it to represent *any statement or relationship* we might wish to verify. This broadens the scope of the proof and makes it more widely applicable, while maintaining its value to the sum formulas verified earlier.

The General Principle of Mathematical Induction

Let S_n be a statement involving natural numbers.

If **1.** S_1 is true, and
2. the truth of S_k implies that S_{k+1} is also true

then S_n must be true for all natural numbers n.

EXAMPLE 3 ► Proving a Statement Using the General Principle of Mathematical Induction

Use the general principle of mathematical induction to show the statement S_n is true for all natural numbers n. S_n: $2^n \geq n + 1$

Solution ► The statement S_n is defined as $2^n \geq n + 1$. This means that S_k is represented by $2^k \geq k + 1$ and S_{k+1} by $2^{k+1} \geq k + 2$.

1. Show S_n is true for $n = 1$:

$$\begin{aligned}
S_n:&\quad 2^n \geq n + 1 && \text{given statement} \\
S_1:&\quad 2^1 \geq 1 + 1 && \text{base case: } n = 1 \\
&\quad 2 \geq 2\checkmark && \text{true}
\end{aligned}$$

Although not a part of the formal proof, a table of values can help to illustrate the relationship we're trying to establish. It *appears* that the statement is true.

n	1	2	3	4	5
2^n	2	4	8	16	32
$n+1$	2	3	4	5	6

2. Assume that S_k is true,

$$S_k\colon\quad 2^k \geq k + 1 \qquad \text{induction hypothesis}$$

and use it to show that the truth of S_{k+1}. That is,

$$S_{k+1}\colon\quad 2^{k+1} \geq k + 2.$$

Begin by working with the left-hand side of the inequality, 2^{k+1}.

$$\begin{aligned} 2^{k+1} &= 2(2^k) && \text{properties of exponents} \\ &\geq 2(\boldsymbol{k+1}) && \textit{induction hypothesis:} \text{ substitute } k+1 \text{ for } 2^k \\ & && \text{(symbol changes since } k+1 \text{ is less than } 2^k\text{)} \\ &\geq 2k + 2 && \text{distribute} \end{aligned}$$

Since k is a positive integer, $2k + 2 \geq k + 2$,

showing $2^{k+1} \geq k + 2$.

Since the truth of S_{k+1} follows from S_k, the formula is true for all n.

Now try Exercises 39 through 42 ►

WORTHY OF NOTE

Note there is no reference to a_n, a_k, or a_{k+1} in the statement of the general principle of mathematical induction.

EXAMPLE 4 ► **Proving Divisibility Using Mathematical Induction**

Let S_n be the statement, "*$4^n - 1$ is divisible by 3 for all positive integers n.*" Use mathematical induction to prove that S_n is true.

Solution ► If a number is evenly divisible by three, it can be written as the product of 3 and some positive integer we will call p.

1. Show S_n is true for $n = 1$:

$$\begin{aligned} S_n\colon\quad 4^n - 1 &= 3p && 4^n - 1 = 3p,\ p \in \mathbb{Z} \\ S_1\colon\quad 4^{(1)} - 1 &= 3p && \text{substitute 1 for } n \\ 3 &= 3p\checkmark && \text{statement is true for } n = 1 \end{aligned}$$

2. Assume that S_k is true . . .

$$\begin{aligned} S_k\colon\quad 4^k - 1 &= 3p && \text{induction hypothesis} \\ 4^k &= 3p + 1 \end{aligned}$$

and use it to show the truth of S_{k+1}. That is,

$$S_{k+1}\colon\quad 4^{k+1} - 1 = 3q \text{ for } q \in \mathbb{Z} \text{ is also true.}$$

Beginning with the left-hand side we have:

$$\begin{aligned} 4^{k+1} - 1 &= 4 \cdot 4^k - 1 && \text{properties of exponents} \\ &= 4 \cdot (\boldsymbol{3p+1}) - 1 && \text{induction hypothesis: substitute } 3p+1 \text{ for } 4^k \\ &= 12p + 3 && \text{distribute and simplify} \\ &= 3(4p + 1) = 3q && \text{factor} \end{aligned}$$

The last step shows $4^{k+1} - 1$ is divisible by 3. Since the original statement is true for $n = 1$, and the truth of S_k implies the truth of S_{k+1}, the statement, "*$4^n - 1$ is divisible by 3*" is true for all positive integers *n*.

Now try Exercises 43 through 47 ▶

We close this section with some final notes. Although the base step of a proof by induction seems trivial, both the base step and the induction hypothesis are necessary parts of the proof. For example, the statement $\frac{1}{3^n} < \frac{1}{3n}$ is false for $n = 1$, but true for all other positive integers. Finally, for a fixed natural number *p*, some statements are false for all $n < p$, but true for all $n \geq p$. By modifying the base case to begin at *p*, we can use the induction hypothesis to prove the statement is true for all *n* greater than *p*. For example, $n < \frac{1}{3}n^2$ is false for $n < 4$, but true for all $n \geq 4$.

☑ **C.** You've just learned how to apply the principle of mathematical induction to general statements involving natural numbers

11.4 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. No ________ number of verifications can prove a statement __________ true.

2. Showing a statement is true for $n = 1$ is called the ________ ________ of an inductive proof.

3. Assuming that a statement/formula is true for $n = k$ is called the ____________ ____________.

4. The graph of a sequence is ________, meaning it is made up of distinct points.

5. Explain the equation $S_k + a_{k+1} = S_{k+1}$. Begin by saying, "Since the *k*th term is arbitrary . . ." (continue from here).

6. Discuss the similarities and differences between mathematical induction applied to sums and the general principle of mathematical induction.

▶ DEVELOPING YOUR SKILLS

For the given *n*th term a_n, find a_4, a_5, a_k, and a_{k+1}.

7. $a_n = 10n - 6$ **8.** $a_n = 6n - 4$

9. $a_n = n$ **10.** $a_n = 7n$

11. $a_n = 2^{n-1}$ **12.** $a_n = 2(3^{n-1})$

For the given sum formula S_n, find S_4, S_5, S_k, and S_{k+1}.

13. $S_n = n(5n - 1)$ **14.** $S_n = n(3n - 1)$

15. $S_n = \frac{n(n+1)}{2}$ **16.** $S_n = \frac{7n(n+1)}{2}$

17. $S_n = 2^n - 1$ **18.** $S_n = 3^n - 1$

Verify that $S_4 + a_5 = S_5$ for each exercise. These are identical to Exercises 13 through 18.

19. $a_n = 10n - 6$; $S_n = n(5n - 1)$

20. $a_n = 6n - 4$; $S_n = n(3n - 1)$

21. $a_n = n$; $S_n = \frac{n(n+1)}{2}$

22. $a_n = 7n$; $S_n = \frac{7n(n+1)}{2}$

23. $a_n = 2^{n-1}$; $S_n = 2^n - 1$

24. $a_n = 2(3^{n-1})$; $S_n = 3^n - 1$

▶ WORKING WITH FORMULAS

25. Sum of the first n cubes (alternative form): $(1 + 2 + 3 + 4 + \cdots + n)^2$

Earlier we noted the formula for the sum of the first n cubes was $\frac{n^2(n + 1)^2}{4}$. An alternative is given by the formula shown.

a. Verify the formula for $n = 1, 5$, and 9.

b. Verify the formula using $1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$.

26. Powers of the imaginary unit: $i^{n+4} = i^n$, where $i = \sqrt{-1}$

Use a proof by induction to prove that powers of the imaginary unit are cyclic. That is, that they cycle through the numbers i, -1, $-i$, and 1 for consecutive powers.

▶ APPLICATIONS

Use mathematical induction to prove the indicated sum formula is true for all natural numbers n.

27. $2 + 4 + 6 + 8 + 10 + \cdots + 2n$;
$a_n = 2n, S_n = n(n + 1)$

28. $3 + 7 + 11 + 15 + 19 + \cdots + (4n - 1)$;
$a_n = 4n - 1, S_n = n(2n + 1)$

29. $5 + 10 + 15 + 20 + 25 + \cdots + 5n$;
$a_n = 5n, S_n = \frac{5n(n + 1)}{2}$

30. $1 + 4 + 7 + 10 + 13 + \cdots + (3n - 2)$;
$a_n = 3n - 2, S_n = \frac{n(3n - 1)}{2}$

31. $5 + 9 + 13 + 17 + \cdots + (4n + 1)$;
$a_n = 4n + 1, S_n = n(2n + 3)$

32. $4 + 12 + 20 + 28 + 36 + \cdots + (8n - 4)$;
$a_n = 8n - 4, S_n = 4n^2$

33. $3 + 9 + 27 + 81 + 243 + \cdots + 3^n$;
$a_n = 3^n, S_n = \frac{3(3^n - 1)}{2}$

34. $5 + 25 + 125 + 625 + \cdots + 5^n$;
$a_n = 5^n, S_n = \frac{5(5^n - 1)}{4}$

35. $2 + 4 + 8 + 16 + 32 + 64 + \cdots + 2^n$;
$a_n = 2^n, S_n = 2^{n+1} - 2$

36. $1 + 8 + 27 + 64 + 125 + 216 + \cdots + n^3$;
$a_n = n^3, S_n = \frac{n^2(n + 1)^2}{4}$

37. $\frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \cdots + \frac{1}{(2n - 1)(2n + 1)}$;
$a_n = \frac{1}{(2n - 1)(2n + 1)}, S_n = \frac{n}{2n + 1}$

38. $\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots + \frac{1}{n(n + 1)}$;
$a_n = \frac{1}{n(n + 1)}, S_n = \frac{n}{n + 1}$

Use the principle of mathematical induction to prove that each statement is true for all natural numbers n.

39. $3^n \geq 2n + 1$

40. $2^n \geq n + 1$

41. $3 \cdot 4^{n-1} \leq 4^n - 1$

42. $4 \cdot 5^{n-1} \leq 5^n - 1$

43. $n^2 - 7n$ is divisible by 2

44. $n^3 - n + 3$ is divisible by 3

45. $n^3 + 3n^2 + 2n$ is divisible by 3

46. $5^n - 1$ is divisible by 4

47. $6^n - 1$ is divisible by 5

▶ EXTENDING THE THOUGHT

48. You may have noticed that the sum formula for the first n integers was *quadratic,* and the formula for the first n integer squares was *cubic.* Is the formula for the first n integer cubes, if it exists, a quartic (degree four) function? Use your calculator to run a quartic regression on the first five perfect cubes (enter 1 through 5 in L_1 and the cumulative sums in L_2). What did you find? Use proof by induction to show that the sum of the first n cubes is:

$$1 + 8 + 27 + \cdots + n^3 = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n + 1)^2}{4}.$$

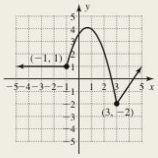
49. Use mathematical induction to prove that $\frac{x^n - 1}{x - 1} = (1 + x + x^2 + x^3 + \cdots + x^{n-1})$.

50. Use mathematical induction to prove that for $1^4 + 2^4 + 3^4 + \cdots + n^4$, where $a_n = n^4$, $S_n = \frac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30}$.

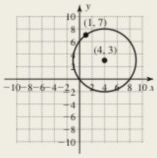
▶ MAINTAINING YOUR SKILLS

51. **(6.2)** Verify the identity $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$.

52. **(2.7)** State the domain and range of the piecewise function shown here.



53. **(2.1)** State the equation of the circle whose graph is shown here.



54. **(7.4)** Given $\mathbf{p} = \langle\sqrt{3}, -1\rangle$ and $\mathbf{q} = \langle 1, 1\rangle$, find

a. the dot product $\mathbf{p} \cdot \mathbf{q}$

b. the angle between the vectors

MID-CHAPTER CHECK

In Exercises 1 to 3, the nth term is given. Write the first three terms of each sequence and find a_9.

1. $a_n = 7n - 4$

2. $a_n = n^2 + 3$

3. $a_n = (-1)^n(2n - 1)$

4. Evaluate the sum $\sum_{n=1}^{4} 3^{n+1}$

5. Rewrite using sigma notation.
$1 + 4 + 7 + 10 + 13 + 16$

Match each formula to its correct description.

6. $S_n = \frac{n(a_1 + a_n)}{2}$

7. $a_n = a_1 r^{n-1}$

8. $S_\infty = \frac{a_1}{1 - r}$

9. $a_n = a_1 + (n - 1)d$

10. $S_n = \frac{a_1(1 - r^n)}{1 - r}$

a. sum of an infinite geometric series

b. nth term formula for an arithmetic series

c. sum of a finite geometric series

d. summation formula for an arithmetic series

e. nth term formula for a geometric series

11. Identify a_1 and the common difference d. Then find an expression for the general term a_n.

a. 2, 5, 8, 11, ...

b. $\frac{3}{2}, \frac{9}{4}, 3, \frac{15}{4}, \ldots$

Find the number of terms in each series and then find the sum.

12. $2 + 5 + 8 + 11 + \cdots + 74$

13. $\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \frac{7}{2} + \cdots + \frac{31}{2}$

14. For an arithmetic series, $a_3 = -8$ and $a_7 = 4$. Find S_{10}.

15. For a geometric series, $a_3 = -81$ and $a_7 = -1$. Find S_{10}.

16. Identify a_1 and the common ratio r. Then find an expression for the general term a_n.

a. 2, 6, 18, 54, ...

b. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

17. Find the number of terms in the series then compute the sum.
$\frac{1}{54} + \frac{1}{18} + \frac{1}{6} + \cdots + \frac{81}{2}$

18. Find the infinite sum (if it exists).

$-49 + (-7) + (-1) + (-\frac{1}{7}) + \cdots$

19. Barrels of toxic waste are stacked at a storage facility in pyramid form, with 60 barrels in the first row, 59 in the second row, and so on, until there are 10 barrels in the top row. How many barrels are in the storage facility?

20. As part of a conditioning regimen, a drill sergeant orders her platoon to do 25 continuous standing broad jumps. The best of these recruits was able to jump 96% of the distance from the previous jump, with a first jump distance of 8 ft. Use a sequence/series to determine the distance the recruit jumped on the 15th try, and the total distance traveled by the recruit after all 25 jumps.

REINFORCING BASIC CONCEPTS

Applications of Summation

The properties of summation play a large role in the development of key ideas in a first semester calculus course, and the following summation formulas are an integral part of these ideas. The first three formulas were verified in Section 11.4, while proof of the fourth was part of Exercise 48 on page 714.

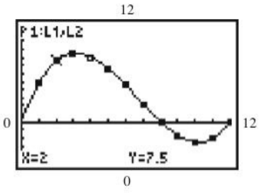
(1) $\sum_{i=1}^{n} c = cn$ (2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

To see the various ways they can be applied consider the following.

Illustration 1 ▶ Over several years, the owner of Morgan's LawnCare has noticed that the company's monthly profits (in thousands) can be approximated by the sequence $a_n = 0.0625n^3 - 1.25n^2 + 6n$, with the points plotted in Figure 11.9 (the continuous graph is shown for effect only). Find the company's approximate annual profit.

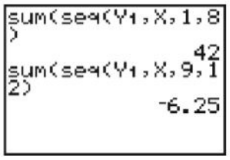
Figure 11.9



Solution ▶ The most obvious approach would be to simply compute terms a_1 through a_{12} (January through December) and find their sum: **sum(seq(Y1, X, 1, 12)** (see Section 11.1 Technology Highlight), which gives a result of 35.75 or $35,750.

As an alternative, we could add the amount of profit earned by the company in the first 8 months, then add the amount the company lost (or broke even) during the last 4 months. In other words, we could apply summation property IV: $\sum_{i=1}^{12} a_n = \sum_{i=1}^{8} a_n + \sum_{i=9}^{12} a_n$ (see Figure 11.10), which gives the same result: $42 + (-6.25) = 35.75$ or $35,750.

Figure 11.10



```
sum(seq(Y1,X,1,8
)
                42
sum(seq(Y1,X,9,1
2)
             -6.25
```

As a third option, we could use summation properties along with the appropriate summation formulas, and compute the result manually. Note the function is now written in terms of "*i*." *Distribute summations and factor out constants* (properties II and III):

$$\sum_{i=1}^{12} (0.0625i^3 - 1.25i^2 + 6i) =$$

$$0.0625\sum_{i=1}^{12} i^3 - 1.25\sum_{i=1}^{12} i^2 + 6\sum_{i=1}^{12} i$$

Replace each summation with the appropriate summation formula, substituting 12 for n:

$$= 0.0625\left[\frac{n^2(n+1)^2}{4}\right] - 1.25\left[\frac{n(n+1)(2n+1)}{6}\right] + 6\left[\frac{n(n+1)}{2}\right]$$

$$= 0.0625\left[\frac{(12)^2(13)^2}{4}\right] - 1.25\left[\frac{(12)(13)(25)}{6}\right] + 6\left[\frac{(12)(13)}{2}\right]$$

$$= 0.0625(6084) - 1.25(650) + 6(78) \text{ or } 35.75$$

As we expected, the result shows profit was $35,750. While some approaches seem "easier" than others, all have great value, are applied in different ways at different times, and are necessary to adequately develop key concepts in future classes.

Exercise 1: Repeat Illustration 1 if the profit sequence is $a_n = 0.125x^3 - 2.5x^2 + 12x$.

11.5 Counting Techniques

Learning Objectives

In Section 11.5 you will learn how to:

- ☐ **A.** Count possibilities using lists and tree diagrams
- ☐ **B.** Count possibilities using the fundamental principle of counting
- ☐ **C.** Quick-count distinguishable permutations
- ☐ **D.** Quick-count nondistinguishable permutations
- ☐ **E.** Quick-count using combinations

How long would it take to estimate the number of fans sitting shoulder-to-shoulder at a sold-out basketball game? Well, it depends. You could actually begin counting 1, 2, 3, 4, 5, . . . , which would take a very long time, or you could try to simplify the process by counting the number of fans in the first row and multiplying by the number of rows. Techniques for "quick-counting" the objects in a set or various subsets of a large set play an important role in a study of probability.

A. Counting by Listing and Tree Diagrams

Consider the simple spinner shown in Figure 11.11, which is divided into three equal parts. What are the different possible outcomes for two spins, spin 1 followed by spin 2? We might begin by organizing the possibilities using a **tree diagram.** As the name implies, each choice or possibility appears as the branch of a tree, with the total possibilities being equal to the number of (unique) paths from the beginning point to the end of a branch. Figure 11.12 shows how the spinner exercise would appear (possibilities for two spins). Moving from top to bottom we can trace nine possible paths: *AA*, *AB*, *AC*, *BA*, *BB*, *BC*, *CA*, *CB*, and *CC*.

Figure 11.11

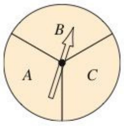
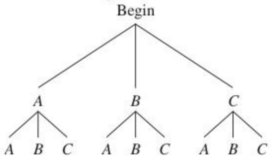


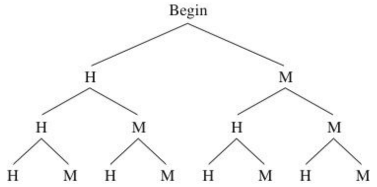
Figure 11.12



EXAMPLE 1 ► **Listing Possibilities Using a Tree Diagram**

A basketball player is fouled and awarded three free throws. Let H represent the possibility of a hit (basket is made), and M the possibility of a miss. Determine the possible outcomes for the three shots using a tree diagram.

Solution ► Each shot has two possibilities, hit (H) or miss (M), so the tree will branch in two directions at each level. As illustrated in the figure, there are a total of eight possibilities: HHH, HHM, HMH, HMM, MHH, MHM, MMH, and MMM.



Now try Exercises 7 through 10 ►

WORTHY OF NOTE

Sample spaces may vary depending on how we define the experiment, and for simplicity's sake we consider only those experiments having outcomes that are equally likely.

To assist our discussion, an **experiment** is any task that can be done repeatedly and has a well-defined set of possible outcomes. Each repetition of the experiment is called a **trial.** A **sample outcome** is any potential outcome of a trial, and a **sample space** is a set of all possible outcomes.

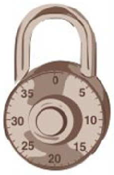
In our first illustration, the *experiment* was spinning a spinner, there were *three sample outcomes* (*A*, *B*, or *C*), the experiment had *two trials* (spin 1 and spin 2), and there were *nine* elements in the *sample space*. Note that after the first trial, each of the three sample outcomes will again have three possibilities (A, B, and C). For two trials we have $3^2 = 9$ possibilities, while three trials would yield a sample space with $3^3 = 27$ possibilities. In general, we have

A "Quick-Counting" Formula for a Sample Space

If an experiment has N sample outcomes that are equally likely and the experiment is repeated t times, the number of elements in the sample space is N^t.

EXAMPLE 2 ► **Counting the Outcomes in a Sample Space**

Many combination locks have the digits 0 through 39 arranged along a circular dial. Opening the lock requires stopping at a sequence of three numbers within this range, going counterclockwise to the first number, clockwise to the second, and counterclockwise to the third. How many three-number combinations are possible?



Solution ► There are 40 sample outcomes ($N = 40$) in this experiment, and three trials ($t = 3$). The number of possible combinations is identical to the number of elements in the sample space. The quick-counting formula gives $40^3 = 64{,}000$ possible combinations.

Now try Exercises 11 and 12 ►

☑ **A.** You've just learned how to count possibilities using lists and tree diagrams

B. Fundamental Principle of Counting

The number of possible outcomes may differ depending on how the event is defined. For example, some security systems, license plates, and telephone numbers exclude certain numbers. For example, phone numbers cannot begin with 0 or 1 because these are reserved for operator assistance, long distance, and international calls. Constructing a three-digit area code is like filling in three blanks $\underset{\text{digit}}{\_\_}\ \underset{\text{digit}}{\_\_}\ \underset{\text{digit}}{\_\_}$ with three digits. Since the area code must start with a number between 2 and 9, there are eight choices for the first blank. Since there are 10 choices for the second digit and 10 choices for the third, there are $8 \cdot 10 \cdot 10 = 800$ possibilities in the sample space.

EXAMPLE 3 ► **Counting Possibilities for a Four-Digit Security Code**

A digital security system requires that you enter a four-digit PIN (personal identification number), using only the digits 1 through 9. How many codes are possible if

a. Repetition of digits is allowed?
b. Repetition is not allowed?
c. The first digit must be even and repetitions are not allowed?

Solution ►

a. Consider filling in the four blanks $\underset{\text{digit}}{\_\_}\ \underset{\text{digit}}{\_\_}\ \underset{\text{digit}}{\_\_}\ \underset{\text{digit}}{\_\_}$ with the number of ways the digit can be chosen. If repetition is allowed, the experiment is similar to that of Example 2 and there are $N^t = 9^4 = 6561$ possible PINs.

b. If repetition is not allowed, there are only eight possible choices for the second digit of the PIN, then seven for the third, and six for the fourth. The number of possible PIN numbers decreases to $9 \cdot 8 \cdot 7 \cdot 6 = 3024$.

c. There are four choices for the first digit (2, 4, 6, 8). Once this choice has been made there are eight choices for the second digit, seven for the third, and six for the last: $4 \cdot 8 \cdot 7 \cdot 6 = 1344$ possible codes.

Now try Exercises 13 through 20 ►

WORTHY OF NOTE

In Example 4, we could also reason that since there are 6! = 720 random seating arrangements and 240 of them consist of Bob and Carol sitting together [Example 4(a)], the remaining 720 − 240 = 480 must consist of Bob and Carol *not* sitting together. More will be said about this type of reasoning in Section 11.6.

Given *any* experiment involving a sequence of tasks, if the first task can be completed in p possible ways, the second task has q possibilities, and the third task has r possibilities, a tree diagram will show that the number of possibilities in the sample space for task_1–task_2–task_3 is $p \cdot q \cdot r$. Even though the examples we've considered to this point have varied a great deal, this idea was fundamental to counting all possibilities in a sample space and is, in fact, known as the **fundamental principle of counting** (FPC).

Fundamental Principle of Counting (Applied to Three Tasks)

Given any experiment with three defined tasks, if there are p possibilities for the first task, q possibilities for the second, and r possibilities for the third, the total number of ways the experiment can be completed is $p \cdot q \cdot r$.

This fundamental principle can be extended to include any number of tasks.

EXAMPLE 4 ► **Counting Possibilities for Seating Arrangements**

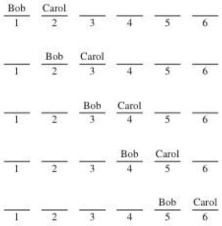
Adrienne, Bob, Carol, Dax, Earlene, and Fabian bought tickets to see *The Marriage of Figaro*. Assuming they sat together in a row of six seats, how many different seating arrangements are possible if

a. Bob and Carol are sweethearts and must sit together?

b. Bob and Carol are enemies and must not sit together?

Solution ► **a.** Since a restriction has been placed on the seating arrangement, it will help to divide the experiment into a sequence of tasks: *task 1:* they sit together; *task 2:* either Bob is on the left or Bob is on the right; and *task 3:* the other four are seated. Bob and Carol can sit together in five different ways, as shown in Figure 11.13, so there are five possibilities for task 1. There are two ways they can be side-by-side: Bob on the left and Carol on the right, as shown, or Carol on the left and Bob on the right. The remaining four people can be seated randomly, so task 3 has 4! = 24 possibilities. Under these conditions they can be seated $5 \cdot 2 \cdot 4! = 240$ ways.

Figure 11.13



b. This is similar to Part (a), but now we have to count the number of ways they can be separated by *at least one seat: task 1:* Bob and Carol are in nonadjacent seats; *task 2:* either Bob is on the left or Bob is on the right; and *task 3:* the other four are seated. For task 1, be careful to note there is no multiplication involved, just a simple counting. If Bob sits in seat 1, there are four nonadjacent seats. If Bob sits in seat 2, there are three nonadjacent seats, and so on. This gives 4 + 3 + 2 + 1 = 10 possibilities for Bob and Carol not sitting together. Task 2 and task 3 have the same number of possibilities as in Part (a), giving $10 \cdot 2 \cdot 4! = 480$ possible seating arrangements.

Now try Exercises 21 through 28 ►

☑ **B. You've just learned how to count possibilities using the fundamental principle of counting**

C. Distinguishable Permutations

In the game of Scrabble® (Milton Bradley), players attempt to form words by rearranging letters. Suppose a player has the letters P, S, T, and O at the end of the game. These letters could be rearranged or *permuted* to form the words POTS, SPOT, TOPS, OPTS, POST, or STOP. These arrangements are called permutations of the four letters. A permutation is any new arrangement, listing, or sequence of objects obtained by changing an existing order. A **distinguishable permutation** is a permutation that produces a result different from the original. For example, a distinguishable permutation of the digits in the number 1989 is 8199.

Example 4 considered six people, six seats, and the various ways they could be seated. But what if there were fewer seats than people? By the FPC, with six people and four seats there could be $6 \cdot 5 \cdot 4 \cdot 3 = 360$ different arrangements, with six people and three seats there are $6 \cdot 5 \cdot 4 = 120$ different arrangements, and so on. These rearrangements are called distinguishable permutations. You may have noticed that for six people and six seats, we used all six factors of 6!, while for six people and four seats we used the first four, six people and three seats required only the first three, and so on. Generally, for n people and r seats, the first r factors of $n!$ will be used. The notation and formula for *distinguishable permutations of n objects taken r at a time* is ${}_nP_r = \frac{n!}{(n-r)!}$. By defining $0! = 1$, the formula includes the case where all n objects are selected, which of course results in ${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$.

Distinguishable Permutations: Unique Elements

If r objects are selected from a set containing n unique elements ($r \le n$) and placed in an ordered arrangement, the number of distinguishable permutations is

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{or} \quad {}_nP_r = n(n-1)(n-2)\cdots(n-r+1)$$

EXAMPLE 5 ► **Computing a Permutation**

Compute each value of ${}_nP_r$ using the methods just described.

a. ${}_7P_4$ **b.** ${}_{10}P_3$

Solution ► Begin by evaluating each expression using the formula ${}_nP_r = \frac{n!}{(n-r)!}$, noting the third line (in bold) gives the first r factors of $n!$.

a.
$$\begin{aligned} {}_7P_4 &= \frac{7!}{(7-4)!} \\ &= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}} \\ &= \mathbf{7 \cdot 6 \cdot 5 \cdot 4} \\ &= 840 \end{aligned}$$

b.
$$\begin{aligned} {}_{10}P_3 &= \frac{10!}{(10-3)!} \\ &= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}} \\ &= \mathbf{10 \cdot 9 \cdot 8} \\ &= 720 \end{aligned}$$

Now try Exercises 29 through 36 ►

EXAMPLE 6 ► **Counting the Possibilities for Finishing a Race**

As part of a sorority's initiation process, the nine new inductees must participate in a 1-mi race. Assuming there are no ties, how many first- through fifth-place finishes are possible if it is well known that Mediocre Mary will finish fifth and Lightning Louise will finish first?

Solution ► To help understand the situation, we can diagram the possibilities for finishing first through fifth. Since Louise will finish first, this slot can be filled in only one way, by Louise herself. The same goes for Mary and her fifth-place finish: $\underset{\text{1st}}{\underline{\text{Louise}}}$ $\underset{\text{2nd}}{\underline{\qquad}}$ $\underset{\text{3rd}}{\underline{\qquad}}$ $\underset{\text{4th}}{\underline{\qquad}}$ $\underset{\text{5th}}{\underline{\text{Mary}}}$. The remaining three slots can be filled in ${}_7P_3 = 7 \cdot 6 \cdot 5$ different ways, indicating that under these conditions, there are $1 \cdot 7 \cdot 6 \cdot 5 \cdot 1 = 210$ different ways to finish.

☑ **C.** You've just learned how to quick-count distinguishable permutations

Now try Exercises 37 through 42 ►

D. Nondistinguishable Permutations

As the name implies, certain permutations are nondistinguishable, meaning you cannot tell one apart from another. Such is the case when the original set contains elements or sample outcomes that are identical. Consider a family with four children, Lyddell, Morgan, Michael, and Mitchell, who are at the photo studio for a family picture. Michael and Mitchell are identical twins and cannot be told apart. In how many ways can they be lined up for the picture? Since this is an ordered arrangement of four children taken from a group of four, there are ${}_4P_4 = 24$ ways to line them up. A few of them are

Lyddell	Morgan	Michael	Mitchell	Lyddell	Morgan	Mitchell	Michael
Lyddell	Michael	Morgan	Mitchell	Lyddell	Mitchell	Morgan	Michael
Michael	Lyddell	Morgan	Mitchell	Mitchell	Lyddell	Morgan	Michael

But of these six arrangements, half will appear to be the same picture, since the difference between Michael and Mitchell cannot be distinguished. In fact, of the 24 total permutations, every picture where Michael and Mitchell have switched places will be nondistinguishable. To find the distinguishable permutations, we need to take the total permutations (${}_4P_4$) *and divide by* 2!, *the number of ways the twins can be permuted:* $\frac{{}_4P_4}{(2)!} = \frac{24}{2} = 12$ distinguishable pictures.

These ideas can be generalized and stated in the following way.

WORTHY OF NOTE

In Example 7, if a Scrabble player is able to play all seven letters in one turn, he or she "bingos" and is awarded 50 extra points. The player in Example 7 did just that. Can you determine what word was played?

Nondistinguishable Permutations: Nonunique Elements

In a set containing n elements where one element is repeated p times, another is repeated q times, and another is repeated r times $(p + q + r = n)$, the number of nondistinguishable permutations is

$$\frac{{}_nP_n}{p!q!r!} = \frac{n!}{p!q!r!}$$

The idea can be extended to include any number of repeated elements.

EXAMPLE 7 ► **Counting Distinguishable Permutations**

A Scrabble player has the seven letters S, A, O, O, T, T, and T in his rack. How many distinguishable arrangements can be formed as he attempts to play a word?

Solution ► Essentially the exercise asks for the number of distinguishable permutations of the seven letters, given T is repeated three times and O is repeated twice. There are $\frac{{}_7P_7}{3!2!} = 420$ distinguishable permutations.

Now try Exercises 43 through 54 ►

☑ **D.** You've just learned how to quick-count nondistinguishable permutations

E. Combinations

Similar to nondistinguishable permutations, there are other times the total number of permutations must be reduced to quick-count the elements of a desired subset. Consider a vending machine that offers a variety of 40¢ candies. If you have a quarter (Q), dime (D), and nickel (N), the machine wouldn't care about the order the coins were deposited. Even though QDN, QND, DQN, DNQ, NQD, and NDQ give the ${}_3P_3 = 6$ possible permutations, the machine considers them as equal and will vend your snack. Using sets, this is similar to saying the set $A = \{X, Y, Z\}$ has only one subset with three elements, since $\{X, Z, Y\}$, $\{Y, X, Z\}$, $\{Y, Z, X\}$, and so on, all represent the same set. Similarly, there are six, two-letter permutations of X, Y, and Z (${}_3P_2 = 6$): XY, XZ, YX,

YZ, ZX, and ZY, but only three two-letter subsets: $\{X, Y\}$, $\{X, Z\}$ and $\{Y, Z\}$. When permutations having the same elements are considered identical, the result is the number of possible **combinations** and is denoted ${}_nC_r$. Since the r objects can be selected in $r!$ ways, we divide ${}_nP_r$ by $r!$ to "quick-count" the number of possibilities: ${}_nC_r = \frac{{}_nP_r}{r!}$, which can be thought of as *the first r factors of* $n!$, *divided by* $r!$. By substituting $\frac{n!}{(n-r)!}$ for ${}_nP_r$ in this formula, we find an alternative method for computing ${}_nC_r$ is $\frac{n!}{r!(n-r)!}$. Take special note that when r objects are selected from a set with n elements and the order they're listed is unimportant (because you end up with the same subset), the result is a *combination*, not a permutation.

Combinations

The number of combinations of n objects taken r at a time is given by

$$ {}_nC_r = \frac{{}_nP_r}{r!} \qquad \text{or} \qquad {}_nC_r = \frac{n!}{r!(n-r)!} $$

EXAMPLE 8 ► **Computing Combinations Using a Formula**

Compute each value of ${}_nC_r$ given.

a. ${}_7C_4$ **b.** ${}_8C_3$ **c.** ${}_5C_2$

Solution ► **a.** ${}_7C_4 = \frac{7 \cdot 6 \cdot 5 \cdot 4}{4!} = 35$ **b.** ${}_8C_3 = \frac{8 \cdot 7 \cdot 6}{3!} = 56$ **c.** ${}_5C_2 = \frac{5 \cdot 4}{2!} = 10$

Now try Exercises 55 through 64 ►

EXAMPLE 9 ► **Applications of Combinations-Lottery Results**

A small city is getting ready to draw five Ping-Pong balls of the nine they have numbered 1 through 9 to determine the winner(s) for its annual raffle. If a ticket holder has the same five numbers, they win. In how many ways can the winning numbers be drawn?

Solution ► Since the winning numbers can be drawn in any order, we have a combination of 9 things taken 5 at a time. The five numbers can be drawn in ${}_9C_5 = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{5!} = 126$ ways.

Now try Exercises 65 and 66 ►

Somewhat surprisingly, there are many situations where the order things are listed is not important. Such situations include

- The formation of committees, since the order people volunteer is unimportant
- Card games with a standard deck, since the order cards are dealt is unimportant
- Playing BINGO, since the order the numbers are called is unimportant

When the order in which people or objects are selected from a group is unimportant, the number of possibilities is a *combination,* not a permutation.

Another way to tell the difference between permutations and combinations is the following memory device: *P*ermutations have *P*riority or *P*recedence; in other words, the *P*osition of each element matters. By contrast, a *C*ombination is like a *C*ommittee of *C*olleagues or *C*ollection of *C*ommoners; all members have equal rank. For permutations, *a*-*b*-*c* is different from *b*-*a*-*c*. For combinations, *a*-*b*-*c* is the same as *b*-*a*-*c*.

EXAMPLE 10 ► **Applications of Quick-Counting—Committees and Government**

The Sociology Department of Lakeside Community College has 12 dedicated faculty members. (a) In how many ways can a three-member textbook selection committee be formed? (b) If the department is in need of a Department Chair, Curriculum Chair, and Technology Chair, in how many ways can the positions be filled?

Solution ►

a. Since textbook selection depends on a *C*ommittee of *C*olleagues, the order members are chosen is not important. This is a *C*ombination of 12 people taken 3 at a time, and there are ${}_{12}C_3 = 220$ ways the committee can be formed.

b. Since those selected will have *P*osition or *P*riority, this is a *P*ermutation of 12 people taken 3 at a time, giving ${}_{12}P_3 = 1320$ ways the positions can be filled.

Now try Exercises 67 through 78 ►

☑ **E.** You've just learned how to quick-count using combinations

The Exercise Set contains a wide variety of additional applications. See **Exercises 81 through 107.**

TECHNOLOGY HIGHLIGHT

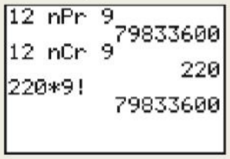
Calculating Permutations and Combinations

Both the ${}_nP_r$ and ${}_nC_r$ functions are accessed using the MATH key and the **PRB** submenu (see Figure 11.14). To compute the permutations of 12 objects taken 9 at a time (${}_{12}P_9$), clear the home screen and enter a 12, then press MATH ◄ **2:${}_n$P${}_r$** to access the ${}_nP_r$ operation, which is automatically pasted on the home screen after the 12. Now enter a 9, press ENTER and a result of 79833600 is displayed (Figure 11.15). Repeat the sequence to compute the value of ${}_{12}C_9$ (MATH ◄ **3:${}_n$C${}_r$).** Note that the value of ${}_{12}P_9$ is much larger than ${}_{12}C_9$ and that they differ by a factor of 9! since ${}_nC_r = \frac{{}_nP_r}{r!}$.

Figure 11.14

```
MATH NUM CPX PRB
1:rand
2:nPr
3:nCr
4:!
5:randInt(
6:randNorm(
7:randBin(
```

Figure 11.15



```
12 nPr 9
        79833600
12 nCr 9
             220
220*9!
        79833600
```

Exercise 1: The Department of Humanities has nine faculty members who must serve on at least one committee per semester. How many different committees can be formed that have (a) two members, (b) three members, (c) four members, and (d) five members?

Exercise 2: A certain state places 45 Ping-Pong balls numbered 1 through 45 in a container, then draws out five to form the winning lottery numbers. How many different ways can the five numbers be picked?

Exercise 3: Dairy King maintains six different toppings at a self-service counter, so that customers can top their ice cream sundaes with as many as they like. How many different sundaes can be created if a customer were to select any three ingredients?

11.5 EXERCISES

CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A(n) ______ is any task that can be repeated and has a(n) ________ set of possible outcomes.

2. If an experiment has N equally likely outcomes and is repeated t times, the number of elements in the sample space is given by ___.

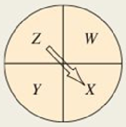
3. When unique elements of a set are rearranged, the result is called a(n) __________ permutation.

4. If some elements of a group are identical, certain rearrangements are identical and the result is a(n) __________ permutation.

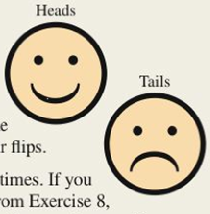
5. A three-digit number is formed from digits 1 to 9. Explain how forming the number with repetition differs from forming it without repetition.

6. Discuss/Explain the difference between a permutation and a combination. Try to think of new ways to help remember the distinction.

DEVELOPING YOUR SKILLS



7. For the spinner shown here, (a) draw a tree diagram illustrating all possible outcomes for two spins and (b) create an ordered list showing all possible outcomes for two spins.



8. For the fair coin shown here, (a) draw a tree diagram illustrating all possible outcomes for four flips and (b) create an ordered list showing the possible outcomes for four flips.

9. A fair coin is flipped five times. If you extend the tree diagram from Exercise 8, how many elements are in the sample space?

10. A spinner has the two equally likely outcomes A or B and is spun four times. How is this experiment related to the one in Exercise 8? How many elements are in the sample space?

11. An inexpensive lock uses the numbers 0 to 24 for a three-number combination. How many different combinations are possible?

12. Grades at a local college consist of A, B, C, D, F, and W. If four classes are taken, how many different report cards are possible?

License plates. In a certain (English-speaking) country, license plates for automobiles consist of two letters followed by one of four symbols (■, ◆, ○, **or** ●), followed by three digits. How many license plates are possible if

13. Repetition is allowed?

14. Repetition is not allowed?

15. A remote access door opener requires a five-digit (1–9) sequence. How many sequences are possible if (a) repetition is allowed? (b) repetition is not allowed?

16. An instructor is qualified to teach Math 020, 030, 140, and 160. How many different four-course schedules are possible if (a) repetition is allowed? (b) repetition is not allowed?

Use the fundamental principle of counting and other quick-counting techniques to respond.

17. Menu items: At Joe's Diner, the manager is offering a dinner special that consists of one choice of entree (chicken, beef, soy meat, or pork), two vegetable servings (corn, carrots, green beans, peas, broccoli, or okra), and one choice of pasta, rice, or potatoes. How many different meals are possible?

18. Getting dressed: A frugal businessman has five shirts, seven ties, four pairs of dress pants, and three pairs of dress shoes. Assuming that all possible arrangements are appealing, how many different shirt-tie-pants-shoes outfits are possible?

19. Number combinations: How many four-digit numbers can be formed using the even digits 0, 2, 4, 6, 8, if (a) no repetitions are allowed; (b) repetitions are allowed; (c) repetitions are not allowed and the number must be less than 6000 and divisible by 10.

20. Number combinations: If I was born in March, April, or May, after the 19th but before the 30th, and after 1949 but before 1981, how many different MM–DD–YYYY dates are possible for my birthday?

Seating arrangements: William, Xayden, York, and Zelda decide to sit together at the movies. How many ways can they be seated if

21. They sit in random order?

22. York must sit next to Zelda?

23. York and Zelda must be on the outside?

24. William must have the aisle seat?

Course schedule: A college student is trying to set her schedule for the next semester and is planning to take five classes: English, art, math, fitness, and science. How many different schedules are possible if

25. The classes can be taken in any order.

26. She wants her science class to immediately follow her math class.

27. She wants her English class to be first and her fitness class to be last.

28. She can't decide on the best order and simply takes the classes in alphabetical order.

Find the value of $_nP_r$ in two ways: (a) compute r factors of $n!$ and (b) use the formula $_nP_r = \frac{n!}{(n-r)!}$.

29. $_{10}P_3$ **30.** $_{12}P_2$ **31.** $_9P_4$

32. $_5P_3$ **33.** $_8P_7$ **34.** $_8P_1$

Determine the number of three-letter permutations of the letters given, then use an organized list to write them all out. How many of them are actually words or common names?

35. T, R, and A **36.** P, M, and A

37. The regional manager for an office supply store needs to replace the manager and assistant manager at the downtown store. In how many ways can this be done if she selects the personnel from a group of 10 qualified applicants?

38. The local chapter of Mu Alpha Theta will soon be electing a president, vice-president, and treasurer. In how many ways can the positions be filled if the chapter has 15 members?

39. The local school board is going to select a principal, vice-principal, and assistant vice-principal from a pool of eight qualified candidates. In how many ways can this be done?

40. From a pool of 32 applicants, a board of directors must select a president, vice-president, labor relations liaison, and a director of personnel for the company's day-to-day operations. Assuming all applicants are qualified and willing to take on any of these positions, how many ways can this be done?

41. A hugely popular chess tournament now has six finalists. Assuming there are no ties, (a) in how many ways can the finalists place in the final round? (b) In how many ways can they finish first, second, and third? (c) In how many ways can they finish if it's sure that Roberta Fischer is going to win the tournament and that Geraldine Kasparov will come in sixth?

42. A field of 10 horses has just left the paddock area and is heading for the gate. Assuming there are no ties in the big race, (a) in how many ways can the horses place in the race? (b) In how many ways can they finish in the win, place, or show positions? (c) In how many ways can they finish if it's sure that John Henry III is going to win, Seattle Slew III will come in second (place), and either Dumb Luck II or Calamity Jane I will come in tenth?

Assuming all multiple births are identical and the children cannot be told apart, how many distinguishable photographs can be taken of a family of six, if they stand in a single row and there is

43. one set of twins

44. one set of triplets

45. one set of twins and one set of triplets

46. one set of quadruplets

47. How many distinguishable numbers can be made by rearranging the digits of 105,001?

48. How many distinguishable numbers can be made by rearranging the digits in the palindrome 1,234,321?

How many distinguishable permutations can be formed from the letters of the given word?

49. logic **50.** leave

51. lotto **52.** levee

A Scrabble player (see Example 7) has the six letters shown remaining in her rack. How many distinguishable, six-letter permutations can be formed? (If all six letters are played, what was the word?)

53. A, A, A, N, N, B

54. D, D, D, N, A, E

Find the value of ${}_nC_r$: (a) using ${}_nC_r = \frac{{}_nP_r}{r!}$ (r factors of $n!$ over $r!$) and (b) using ${}_nC_r = \frac{n!}{r!(n-r)!}$.

55. ${}_9C_4$ **56.** ${}_{10}C_3$ **57.** ${}_8C_5$

58. ${}_6C_3$ **59.** ${}_6C_6$ **60.** ${}_6C_0$

Use a calculator to verify that each pair of combinations is equal.

61. ${}_9C_4$, ${}_9C_5$ **62.** ${}_{10}C_3$, ${}_{10}C_7$

63. ${}_8C_5$, ${}_8C_3$ **64.** ${}_7C_2$, ${}_7C_5$

65. A platoon leader needs to send four soldiers to do some reconnaissance work. There are 12 soldiers in the platoon and each soldier is assigned a number between 1 and 12. The numbers 1 through 12 are placed in a helmet and drawn randomly. If a soldier’s number is drawn, then that soldier goes on the mission. In how many ways can the reconnaissance team be chosen?

66. Seven colored balls (red, indigo, violet, yellow, green, blue, and orange) are placed in a bag and three are then withdrawn. In how many ways can the three colored balls be drawn?

67. When the company’s switchboard operators went on strike, the company president asked for three volunteers from among the managerial ranks to temporarily take their place. In how many ways can the three volunteers “step forward,” if there are 14 managers and assistant managers in all?

68. Becky has identified 12 books she wants to read this year and decides to take four with her to read while on vacation. She chooses *Pastwatch* by Orson Scott Card for sure, then decides to randomly choose any three of the remaining books. In how many ways can she select the four books she’ll end up taking?

69. A new garage band has built up their repertoire to 10 excellent songs that really rock. Next month they’ll be playing in a *Battle of the Bands* contest, with the winner getting some guaranteed gigs at the city’s most popular hot spots. In how many ways can the band select 5 of their 10 songs to play at the contest?

70. Pierre de Guirré is an award-winning chef and has just developed 12 delectable, new main-course recipes for his restaurant. In how many ways can he select three of the recipes to be entered in an international culinary competition?

For each exercise, determine whether a permutation, a combination, counting principles, or a determination of the number of subsets is the most appropriate tool for obtaining a solution, then solve. Some exercises can be completed using more than one method.

71. In how many ways can eight second-grade children line up for lunch?

72. If you flip a fair coin five times, how many different outcomes are possible?

73. Eight sprinters are competing for the gold, silver, and bronze medals. In how many ways can the medals be awarded?

74. Motorcycle license plates are made using two letters followed by three numbers. How many plates can be made if repetition of letters (only) is allowed?

75. A committee of five students is chosen from a class of 20 to attend a seminar. How many different ways can this be done?

76. If onions, cheese, pickles, and tomatoes are available to dress a hamburger, how many different hamburgers can be made?

77. A caterer offers eight kinds of fruit to make various fruit trays. How many different trays can be made using four different fruits?

78. Eighteen females try out for the basketball team, but the coach can only place 15 on her roster. How many different teams can be formed?

▶ WORKING WITH FORMULAS

79. Stirling’s Formula: $n! \approx \sqrt{2\pi} \cdot (n^{n+0.5}) \cdot e^{-n}$

Values of $n!$ grow very quickly as n gets larger (13! is already in the billions). For some applications, scientists find it useful to use the approximation for $n!$ shown, called Stirling’s Formula.

a. Compute the value of 7! on your calculator, then use Stirling’s Formula with $n = 7$. By what percent does the approximate value differ from the true value?

b. Compute the value of 10! on your calculator, then use Stirling’s Formula with $n = 10$. By what percent does the approximate value differ from the true value?

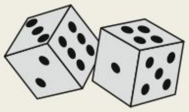
80. Factorial formulas: For $n, k \in \mathbb{W}$, where $n > k$,

$$\frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1)$$

a. Verify the formula for $n = 7$ and $k = 5$.

b. Verify the formula for $n = 9$ and $k = 6$.

▶ APPLICATIONS



81. Yahtzee: In the game of "Yahtzee"® (Milton Bradley) five dice are rolled simultaneously on the first turn in an attempt to obtain various arrangements (worth various point values). How many different arrangements are possible?

82. Twister: In the game of "Twister"® (Milton Bradley) a simple spinner is divided into four quadrants designated Left Foot (LF), Right Hand (RH), Right Foot (RF), and Left Hand (LH), with four different color possibilities in each quadrant (red, green, yellow, blue). Determine the number of possible outcomes for three spins.

83. Clue: In the game of "Clue"® (Parker Brothers) a crime is committed in one of nine rooms, with one of six implements, by one of six people. In how many different ways can the crime be committed?

Phone numbers in North America have 10 digits: a three-digit area code, a three-digit exchange number, and the four final digits that make each phone number unique. Neither area codes nor exchange numbers can start with 0 or 1. Prior to 1994 the second digit of the area code *had to be* a 0 or 1. Sixteen area codes are reserved for special services (such as 911 and 411). In 1994, the last area code was used up and the rules were changed to allow the digits 2 through 9 as the middle digit in area codes.

84. How many different area codes were possible prior to 1994?

85. How many different exchange numbers were possible prior to 1994?

86. How many different phone numbers were possible *prior to* 1994?

87. How many different phone numbers were possible *after* 1994?

Aircraft N-numbers: In the United States, private aircraft are identified by an "N-Number," which is generally the letter "N" followed by five characters and includes these restrictions: (1) the N-Number can consist of five digits, four digits followed by one letter, or three digits followed by two letters; (2) the first digit cannot be a zero; (3) to avoid confusion with the numbers zero and one, the letters O and I cannot be used; and (4) repetition of digits and letters is allowed. How many unique N-Numbers can be formed

88. that have four digits and one letter?

89. that have three digits and two letters?

90. that have five digits?

91. that have three digits, two letters with no repetitions of any kind allowed?

Seating arrangements: Eight people would like to be seated. Assuming some will have to stand, in how many ways can the seats be filled if the number of seats available is

92. eight

93. five

94. three

95. one

Seating arrangements: In how many different ways can eight people (six students and two teachers) sit in a row of eight seats if

96. the teachers must sit on the ends

97. the teachers must sit together

Television station programming: A television station needs to fill eight half-hour slots for its Tuesday evening schedule with eight programs. In how many ways can this be done if

98. there are no constraints

99. *Seinfeld* must have the 8:00 P.M. slot

100. *Seinfeld* must have the 8:00 P.M. slot and *The Drew Carey Show* must be shown at 6:00 P.M.

101. *Friends* can be aired at 7:00 or 9:00 P.M. and *Everybody Loves Raymond* can be aired at 6:00 or 8:00 P.M.

Scholarship awards: Fifteen students at Roosevelt Community College have applied for six available scholarship awards. How many ways can the awards be given if

102. there are six different awards given to six different students

103. there are six identical awards given to six different students

Committee composition: The local city council has 10 members and is trying to decide if they want to be governed by a committee of three people or by a president, vice-president, and secretary.

104. If they are to be governed by committee, how many unique committees can be formed?

105. How many different president, vice-president, and secretary possibilities are there?

106. Team rosters: A soccer team has three goalies, eight defensive players, and eight forwards on its roster. How many different starting line-ups can be formed (one goalie, three defensive players, and three forwards)?

107. e-mail addresses: A business wants to standardize the e-mail addresses of its employees. To make them easier to remember and use, they consist of two letters and two digits (followed by @esmtb.com), with zero being excluded from use as the first digit and no repetition of letters or digits allowed. Will this provide enough unique addresses for their 53,000 employees worldwide?

▶ EXTENDING THE CONCEPT

108. In Exercise 79, we learned that an approximation for $n!$ can be found using Stirling's Formula: $n! \approx \sqrt{2\pi}(n^{n+0.5})e^{-n}$. As with other approximations, mathematicians are very interested in whether the approximation gets better or worse for larger values of n (does their ratio get closer to 1 or farther from 1). Use your calculator to investigate and answer the question.

109. Verify that the following equations are true, then generalize the patterns and relationships noted to create your own equation. Afterward, write each of the four factors from Part (a) (the two combinations on each side) in expanded form and discuss/explain why the two sides are equal.

a. ${}_{10}C_3 \cdot {}_{7}C_2 = {}_{10}C_2 \cdot {}_{8}C_5$

b. ${}_{9}C_3 \cdot {}_{6}C_2 = {}_{9}C_2 \cdot {}_{7}C_4$

c. ${}_{11}C_4 \cdot {}_{7}C_5 = {}_{11}C_5 \cdot {}_{6}C_4$

d. ${}_{8}C_3 \cdot {}_{5}C_2 = {}_{8}C_2 \cdot {}_{6}C_3$

110. Tic-Tac-Toe: In the game *Tic-Tac-Toe,* players alternately write an "X" or an "O" in one of nine squares on a 3 × 3 grid. If either player gets three in a row horizontally, vertically, or diagonally, that player wins. If all nine squares are played with neither person winning, the game is a draw. Assuming "X" always goes first,

a. How many different "boards" are possible if the game ends after five plays?

b. How many different "boards" are possible if the game ends after six plays?

▶ MAINTAINING YOUR SKILLS

111. **(5.4)** Solve the given system of linear inequalities by graphing. Shade the feasible region.

$$\begin{cases} 2x + y < 6 \\ x + 2y < 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

112. **(5.2)** Given $\sin\theta = \frac{12}{13}$, determine the other five trig functions of the acute angle θ.

113. **(6.3)** Rewrite $\cos(2\alpha)\cos(3\alpha) - \sin(2\alpha)\sin(3\alpha)$ as a single expression.

114. **(7.3)** Graph the hyperbola that is defined by $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$.

11.6 Introduction to Probability

Learning Objectives

In Section 11.6 you will learn how to:

- ☐ **A.** Define an event on a sample space
- ☐ **B.** Compute elementary probabilities
- ☐ **C.** Use certain properties of probability
- ☐ **D.** Compute probabilities using quick-counting techniques
- ☐ **E.** Compute probabilities involving nonexclusive events

There are few areas of mathematics that give us a better view of the world than **probability** and **statistics.** Unlike statistics, which seeks to analyze and interpret data, probability (for our purposes) attempts to use observations and data to make statements concerning the likelihood of future events. Such predictions of what *might* happen have found widespread application in such diverse fields as politics, manufacturing, gambling, opinion polls, product life, and many others. In this section, we develop the basic elements of probability.

A. Defining an Event

In Section 11.5 we defined the following terms: experiment and sample outcome. Flipping a coin twice in succession is an *experiment,* and two sample outcomes are HH and HT. An **event *E*** is *any designated set of sample outcomes,* and is a subset of the sample space. One event might be E_1: (two heads occur), another possibility is E_2: (at least one tail occurs).

EXAMPLE 1 ► **Stating a Sample Space and Defining an Event**

Consider the experiment of rolling one standard, six-sided die (plural is dice). State the sample space S and define any two events relative to S.

Solution ► S is the set of all possible outcomes, so $S = \{1, 2, 3, 4, 5, 6\}$. Two possible events are E_1: (a 5 is rolled) and E_2: (an even number is rolled).

Now try Exercises 7 through 10 ►

☑ **A. You've just learned how to define an event on a sample space**

WORTHY OF NOTE

Our study of probability will involve only those sample spaces with events that are equally likely.

B. Elementary Probability

When rolling the die, we know the result can be any of the six equally likely outcomes in the sample space, so the chance of E_1:(a five is rolled) is $\frac{1}{6}$. Since three of the elements in S are even numbers, the chance of E_2:(an even number is rolled) is $\frac{3}{6} = \frac{1}{2}$. This suggests the following definition.

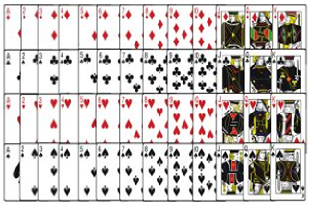
The Probability of an Event *E*

Given S is a sample space of equally likely events and E is an event relative to S, the probability of E, written $P(E)$, is computed as

$$P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ represents the number of elements in E, and $n(S)$ represents the number of elements in S.

A standard deck of playing cards consists of 52 cards divided in four groups or *suits.* There are 13 hearts (♥), 13 diamonds (◆), 13 spades (♠), and 13 clubs (♣). As you can see in the illustration, each of the 13 cards in a suit is labeled 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, and A. Also notice that 26 of the cards are red (hearts and diamonds), 26 are black (spades and clubs) and 12 of the cards are "face cards" (J, Q, K of each suit).



EXAMPLE 2 ► **Stating a Sample Space and the Probability of a Single Outcome**

A single card is drawn from a well-shuffled deck. Define S and state the probability of any single outcome. Then define E as *a King is drawn* and find $P(E)$.

Solution ► Sample space: $S = \{\text{the 52 cards}\}$. There are 52 equally likely outcomes, so the probability of any one outcome is $\frac{1}{52}$. Since S has four Kings, $P(E) = \frac{n(E)}{n(S)} = \frac{4}{52}$ or about 0.077.

Now try Exercises 11 through 14 ►

EXAMPLE 3 ► **Stating a Sample Space and the Probability of a Single Outcome**

A family of five has two girls and three boys named Sophie, Maria, Albert, Isaac, and Pythagoras. Their ages are 21, 19, 15, 13, and 9, respectively. One is to be selected randomly. Find the probability a teenager is chosen.

Solution ► The sample space is $S = \{9, 13, 15, 19, 21\}$. Three of the five are teenagers, meaning the probability is $\frac{3}{5}$, 0.6, or 60%.

Now try Exercises 15 and 16 ►

☑ **B.** You've just learned how to compute elementary probabilities

C. Properties of Probability

A study of probability necessarily includes recognizing some basic and fundamental properties. For example, when a fair die is rolled, what is $P(E)$ if E is defined as a *1, 2, 3, 4, 5,* or *6* is rolled? The event E will occur 100% of the time, since 1, 2, 3, 4, 5, 6 are the only possibilities. In symbols we write P(outcome is in the sample space) or simply $P(S) = 1$ (100%).

What percent of the time will a result *not* in the sample space occur? Since the die has only the six sides numbered 1 through 6, the probability of rolling something else is zero. In symbols, P(outcome is not in sample space) $= 0$ or simply $P(\sim S) = 0$.

WORTHY OF NOTE

In probability studies, the tilde "~" acts as a negation symbol. For any event E defined on the sample space, ~E means the event does not occur.

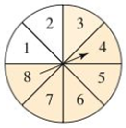
Properties of Probability

Given sample space S and any event E defined relative to S.

1. $P(S) = 1$ **2.** $P(\sim S) = 0$ **3.** $0 \le P(E) \le 1$

EXAMPLE 4 ► **Determining the Probability of an Event**

A game is played using a spinner like the one shown. Determine the probability of the following events:



E_1: A nine is spun. E_2: An integer greater than 0 and less than 9 is spun.

Solution ► The sample space consists of eight equally likely outcomes.

$$P(E_1) = \frac{0}{8} = 0 \qquad P(E_2) = \frac{8}{8} = 1.$$

Technically, E_1: A nine is spun is not an "event," since it is not in the sample space and cannot occur, while E_2 contains the entire sample space and must occur.

Now try Exercises 17 and 18 ►

Because we know $P(S) = 1$ and all sample outcomes are equally likely, the probabilities of all single events defined on the sample space must sum to 1. For the experiment of rolling a fair die, the sample space has six outcomes that are equally likely. Note that $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$, and $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$.

Probability and Sample Outcomes

Given a sample space S with n equally likely sample outcomes $s_1, s_2, s_3, \ldots, s_n$.

$$\sum_{i=1}^{n} P(s_i) = P(s_1) + P(s_2) + P(s_3) + \cdots + P(s_n) = 1$$

The **complement** of an event E is the set of sample outcomes in S not contained in E. Symbolically, $\sim E$ is the complement of E.

Probability and Complementary Events

Given sample space S and any event E defined relative to S, the complement of E, written $\sim E$, is the set of all outcomes not in E and:

1. $P(E) = 1 - P(\sim E)$ **2.** $P(E) + P(\sim E) = 1$

EXAMPLE 5 ► **Stating a Probability Using Complements**

Use complementary events to answer the following questions:

a. A single card is drawn from a well-shuffled deck. What is the probability that it is not a diamond?

b. A single letter is picked at random from the letters in the word "divisibility." What is the probability it is not an "i"?

Solution ►

a. Since there are 13 diamonds in a standard 52-card deck, there are 39 nondiamonds: $P(\sim D) = 1 - P(D) = 1 - \frac{13}{52} = \frac{39}{52} = 0.75$.

b. Of the 12 letters in d-i-v-i-s-i-b-i-l-i-t-y, 5 are "i's." This means $P(\sim i) = 1 - P(i)$, or $1 - \frac{5}{12} = \frac{7}{12}$. The probability of choosing a letter other than i is $0.58\overline{3}$.

WORTHY OF NOTE

Probabilities can be written in fraction form, decimal form, or as a percent. For $P(E_2)$ from Example 1, the probability is $\frac{3}{4}$, 0.75, or 75%.

Now try Exercises 19 through 22 ►

EXAMPLE 6 ► **Stating a Probability Using Complements**

Inter-Island Waterways has just opened hydrofoil service between several islands. The hydrofoil is powered by two engines, one forward and one aft, and will operate if either of its two engines is functioning. Due to testing and past experience, the company knows the probability of the aft engine failing is $P(\text{aft engine fails}) = 0.05$, the probability of the forward engine failing is $P(\text{forward engine fails}) = 0.03$, and the probability that both fail is $P(\text{both engines simultaneously fail}) = 0.012$. What is the probability the hydrofoil completes its next trip?

Solution ► Although the answer may *seem* complicated, note that $P(\text{trip is completed})$ and $P(\text{both engines simultaneously fail})$ are complements.

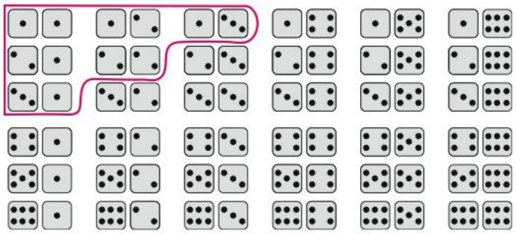
$$\begin{aligned} P(\text{trip is completed}) &= 1 - P(\text{both engines simultaneously fail}) \\ &= 1 - 0.012 \\ &= 0.988 \end{aligned}$$

There is close to a 99% probability the trip will be completed.

Now try Exercises 23 and 24 ►

The chart in Figure 11.16 shows all 36 possible outcomes (the sample space) from the experiment of rolling two fair dice.

Figure 11.16



EXAMPLE 7 ► **Stating a Probability Using Complements**

Two fair dice are rolled. What is the probability the sum of both dice is greater than or equal to 5, $P(\text{sum} \geq 5)$?

Solution ► See Figure 11.16. For $P(\text{sum} \geq 5)$ it may be easier to use complements as there are far fewer possibilities: $P(\text{sum} \geq 5) = 1 - P(\text{sum} < 5)$, which gives $1 - \frac{6}{36} = 1 - \frac{1}{6} = \frac{5}{6} = 0.8\overline{3}$.

☑ **C.** You've just learned how to use certain properties of probability

Now try Exercises 25 and 26 ►

D. Probability and Quick-Counting

Quick-counting techniques were introduced earlier to help count the number of elements in a large or more complex sample space, and the number of sample outcomes in an event.

EXAMPLE 8A ► **Stating a Probability Using Combinations**

Five cards are drawn from a shuffled 52-card deck. Calculate the probability of E_1:(*all five cards are face cards*) or E_2:(*all five cards are hearts*)?

Solution ► The sample space for both events consists of all five-card groups that can be formed from the 52 cards or ${}_{52}C_5$. For E_1 we are to select five face cards from the 12 that are available (three from each suit), or ${}_{12}C_5$. The probability of five face cards is $\frac{n(E)}{n(S)} = \frac{{}_{12}C_5}{{}_{52}C_5}$, which gives $\frac{792}{2{,}598{,}960} \approx 0.0003$. For E_2 we are to select five hearts from the 13 available, or ${}_{13}C_5$. The probability of five hearts is $\frac{n(E)}{n(S)} = \frac{{}_{13}C_5}{{}_{52}C_5}$, which is $\frac{1287}{2{,}598{,}960} \approx 0.0005$.

WORTHY OF NOTE

It seems reasonable that the probability of 5 hearts is slightly higher, as 13 of the 52 cards are hearts, while only 12 are face cards.

EXAMPLE 8B ► **Stating a Probability Using Combinations and the Fundamental Principle of Counting**

Of the 42 seniors at Jacoby High School, 23 are female and 19 are male. A group of five students is to be selected at random to attend a conference in Reno, Nevada. What is the probability the group will have exactly three females?

Solution ► The sample space consists of all five-person groups that can be formed from the 42 seniors or ${}_{42}C_5$. The event consists of selecting 3 females from the 23 available (${}_{23}C_3$) and 2 males from the 19 available (${}_{19}C_2$). Using the fundamental principle of counting $n(E) = {}_{23}C_3 \cdot {}_{19}C_2$ and the probability the group has 3 females is $\frac{n(E)}{n(S)} = \frac{{}_{23}C_3 \cdot {}_{19}C_2}{{}_{42}C_5}$, which gives $\frac{302{,}841}{850{,}668} \approx 0.356$. There is approximately a 35.6% probability the group will have exactly 3 females.

Now try Exercises 27 through 34 ►

☑ **D.** You've just learned how to compute probabilities using quick-counting techniques

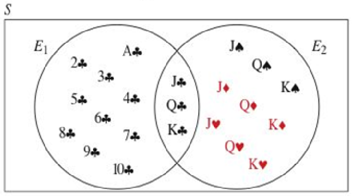
E. Probability and Nonexclusive Events

WORTHY OF NOTE

This can be verified by simply counting the elements involved: $n(E_1) = 13$ and $n(E_2) = 12$ so $n(E_1) + n(E_2) = 25$. However, there are only 22 possibilities—the J♣, Q♣, and K♣ got counted twice.

Sometimes the way events are defined causes them to share sample outcomes. Using a standard deck of playing cards once again, if we define the events E_1:(a club is drawn) and E_2:(a face card is drawn), they share the outcomes J♣, Q♣, and K♣ as shown in Figure 11.17. This overlapping region is the intersection of the events, or $E_1 \cap E_2$. If we compute $n(E_1 \cup E_2)$ as $n(E_1) + n(E_2)$ as before, this intersecting region gets counted *twice*!

Figure 11.17



In cases where the events are **nonexclusive** (not mutually exclusive), we maintain the correct count by subtracting one of the two intersections, obtaining $n(E_1 \cup E_2) = n(E_1) + n(E_2) - n(E_1 \cap E_2)$. This leads to the following calculation for the probability of nonexclusive events:

$$P(E_1 \cup E_2) = \frac{n(E_1) + n(E_2) - n(E_1 \cap E_2)}{n(S)} \quad \text{definition of probability}$$

$$= \frac{n(E_1)}{n(S)} + \frac{n(E_1)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)} \quad \text{property of rational expressions}$$

$$= P(E_1) + P(E_2) - P(E_1 \cap E_2) \quad \text{definition of probability}$$

Probability and Nonexclusive Events

Given sample space S and *nonexclusive events* E_1 and E_2 defined relative to S, the probability of E_1 *or* E_2 is given by

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

EXAMPLE 9A ► Stating the Probability of Nonexclusive Events

What is the probability that a club or a face card is drawn from a standard deck of 52 well-shuffled cards?

Solution ► As before, define the events E_1:(a club is drawn) and E_2:(a face card is drawn). Since there are 13 clubs and 12 face cards, $P(E_1) = \frac{13}{52}$ and $P(E_2) = \frac{12}{52}$. But three of the face cards are clubs, so $P(E_1 \cap E_2) = \frac{3}{52}$. This leads to

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \quad \text{nonexclusive events}$$

$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \quad \text{substitute}$$

$$= \frac{22}{52} \approx 0.423 \quad \text{combine terms}$$

There is about a 42% probability that a club or face card is drawn.

EXAMPLE 9B ► **Stating the Probability of Nonexclusive Events**

A survey of 100 voters was taken to gather information on critical issues and the demographic information collected is shown in the table. One out of the 100 voters is to be drawn at random to be interviewed on the 5 O'Clock News. What is the probability the person is a woman (W) or a Republican (R)?

	Women	Men	Totals
Republican	17	20	37
Democrat	22	17	39
Independent	8	7	15
Green Party	4	1	5
Tax Reform	2	2	4
Totals	53	47	100

Solution ► Since there are 53 women and 37 Republicans, $P(W) = 0.53$ and $P(R) = 0.37$. The table shows 17 people are both female and Republican so $P(W \cap R) = 0.17$.

$$\begin{aligned} P(W \cup R) &= P(W) + P(R) - P(W \cap R) && \text{nonexclusive events} \\ &= 0.53 + 0.37 - 0.17 && \text{substitute} \\ &= 0.73 && \text{combine} \end{aligned}$$

There is a 73% probability the person is a woman or a Republican.

☑ **E.** You've just learned how to compute probabilities involving nonexclusive events

Now try Exercises 35 through 48 ►

Two events that have no common outcomes are called **mutually exclusive** events (one excludes the other and vice versa). For example, in rolling one die, E_1:(*a 2 is rolled*) and E_2:(*an odd number is rolled*) are mutually exclusive, since 2 is not an odd number. For the probability of E_3:(*a 2 is rolled* or *an odd number is rolled*), we note that $n(E_1 \cap E_2) = 0$ and the previous formula simply reduces to $P(E_1) + P(E_2)$. See **Exercises 49 and 50.**

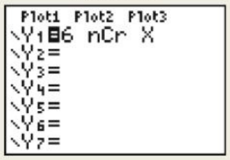
There is a large variety of additional applications in the Exercise Set. See **Exercises 53 through 68.**

TECHNOLOGY HIGHLIGHT

Principles of Quick-Counting, Combinations, and Probability

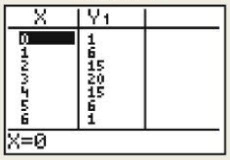
At this point you are likely using the Y= screen and tables (TBLSET, 2nd GRAPH **TABLE,** and so on) with relative ease. When probability calculations require a repeated use of permutations and combinations, these features can make the work more efficient and help to explore the patterns they generate. For choosing r children from a group of six children ($n = 6$), set the TBLSET to **AUTO,** then press Y= and enter 6 ${}_nC_r$ X as Y_1 (Figure 11.18). Access the **TABLE** (2nd GRAPH) and note that the calculator has automatically computed the value of ${}_6C_0$, ${}_6C_1$, ${}_6C_2$, . . . , ${}_6C_6$ (Figure 11.19) and the pattern of outputs is symmetric. For calculations similar to those required in Example 8B (${}_{23}C_3 \cdot {}_{19}C_2$), enter

Figure 11.18



```
Plot1 Plot2 Plot3
\Y1=6 nCr X
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure 11.19



```
  X   | Y1   |
0     | 1
1     | 6
2     | 15
3     | 20
4     | 15
5     | 6
6     | 1
X=0
```

$Y_1 = 23\,{}_nC_r\,X$, $Y_2 = 19\,{}_nC_r\,(X - 1)$, and $Y_3 = Y_1 \cdot Y_2$, or any variation of these. Use these ideas to work the following exercises.

Exercise 1: Use your calculator to display the values of ${}_5C_0, {}_5C_1, \ldots, {}_5C_5$. Is the result a pattern similar to that for ${}_6C_0, {}_6C_1, {}_6C_2, \ldots, {}_6C_6$? Repeat for ${}_7C_r$. Why are the "middle values" repeated for $n = 7$ and $n = 5$, but not for $n = 6$?

Exercise 2: A committee consists of 10 Republicans and eight Democrats. In how many ways can a committee of four Republicans and three Democrats be formed?

11.6 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

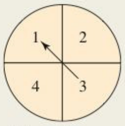
1. Given a sample space S and an event E defined relative to S, $P(E) = \dfrac{\_\_\_\_}{n(S)}$.
2. In elementary probability, we consider all events in the sample space to be ______ likely.
3. Given a sample space S and an event E defined relative to S: $\_\_\_\_ \le P(E) \le \_\_\_\_$, $P(S) = \_\_\_\_$, and $P({\sim}S) = \_\_\_\_$.
4. The ______ of an event E is the set of sample outcomes in S which are not contained in E.
5. Discuss/Explain the difference between mutually exclusive events and nonmutually exclusive events. Give an example of each.
6. A single die is rolled. With no calculations, explain why the probability of rolling an even number is greater than rolling a number greater than four.

▶ DEVELOPING YOUR SKILLS

State the sample space S and the probability of a single outcome. Then define any two events E relative to S (many answers possible).

7. Two fair coins (heads and tails) are flipped.
8. The simple spinner shown is spun.

Exercise 8



9. The head coaches for six little league teams (the Patriots, Cougars, Angels, Sharks, Eagles, and Stars) have gathered to discuss new changes in the rule book. One of them is randomly chosen to ask the first question.
10. Experts on the planets Mercury, Venus, Mars, Jupiter, Saturn, Uranus, Neptune, and the dwarf planet Pluto have gathered at a space exploration conference. One group of experts is selected at random to speak first.

Find $P(E)$ for the events defined.

11. Nine index cards numbered 1 through 9 are shuffled and placed in an envelope, then one of the cards is randomly drawn. Define event E as *the number drawn is even.*
12. Eight flash cards used for studying basic geometric shapes are shuffled and one of the cards is drawn at random. The eight cards include information on circles, squares, rectangles, kites, trapezoids, parallelograms, pentagons, and triangles. Define event E as *a quadrilateral is drawn.*
13. One card is drawn at random from a standard deck of 52 cards. What is the probability of
    - **a.** drawing a Jack
    - **b.** drawing a spade
    - **c.** drawing a black card
    - **d.** drawing a red three
14. Pinochle is a card game played with a deck of 48 cards consisting of 2 Aces, 2 Kings, 2 Queens, 2 Jacks, 2 Tens, and 2 Nines in each of the four

standard suits [hearts (♥), diamonds (♦), spades (♠), and clubs (♣)]. If one card is drawn at random from this deck, what is the probability of

a. drawing an Ace **b.** drawing a club

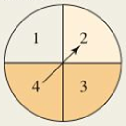
c. drawing a red card

d. drawing a face card (Jack, Queen, King)

15. A group of finalists on a game show consists of three males and five females. Hank has a score of 520 points, with Harry and Hester having 490 and 475 points, respectively. Madeline has 532 points, with Mackenzie, Morgan, Maggie, and Melanie having 495, 480, 472, and 470 points, respectively. One of the contestants is randomly selected to start the final round. Define E_1 as *Hester is chosen,* E_2 as *a female is chosen,* and E_3 as *a contestant with less than 500 points is chosen.* Find the probability of each event.

16. Soccer coach Maddox needs to fill the last spot on his starting roster for the opening day of the season and has to choose between three forwards and five defenders. The forwards have jersey numbers 5, 12, and 17, while the defenders have jersey numbers 7, 10, 11, 14, and 18. Define E_1 as *a forward is chosen,* E_2 as *a defender is chosen,* and E_3 as *a player whose jersey number is greater than 10 is chosen.* Find the probability of each event.

17. A game is played using a spinner like the one shown. For each spin,



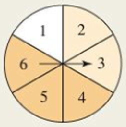
a. What is the probability the arrow lands in a shaded region?

b. What is the probability your spin is less than 5?

c. What is the probability you spin a 2?

d. What is the probability the arrow lands on a prime number?

18. A game is played using a spinner like the one shown here. For each spin,



a. What is the probability the arrow lands in a lightly shaded region?

b. What is the probability your spin is greater than 2?

c. What is the probability the arrow lands in a shaded region?

d. What is the probability you spin a 5?

Use the complementary events to complete Exercises 19 through 22.

19. One card is drawn from a standard deck of 52. What is the probability it is not a club?

20. Four standard dice are rolled. What is the probability the sum is less than 24?

21. A single digit is randomly selected from among the digits of 10!. What is the probability the digit is not a 2?

22. A corporation will be moving its offices to Los Angeles, Miami, Atlanta, Dallas, or Phoenix. If the site is randomly selected, what is the probability Dallas is not chosen?

23. A large manufacturing plant can remain at full production as long as one of its two generators is functioning. Due to past experience and the age difference between the systems, the plant manager estimates the probability of the main generator failing is 0.05, the probability of the secondary generator failing is 0.01, and the probability of both failing is 0.009. What is the probability the plant remains in full production today?

24. A fire station gets an emergency call from a shopping mall in the mid-afternoon. From a study of traffic patterns, Chief Nozawa knows the probability the most direct route is clogged with traffic is 0.07, while the probability of the secondary route being clogged is 0.05. The probability both are clogged is 0.02. What is the probability they can respond to the call unimpeded using one of these routes?

25. Two fair dice are rolled (see Figure 11.16). What is the probability of

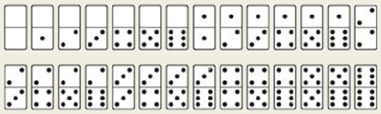
a. a sum less than four

b. a sum less than eleven

c. the sum is not nine

d. a roll is not a "double" (both dice the same)

"Double-six" dominos is a game played with the 28 numbered tiles shown in the diagram.



26. The 28 dominos are placed in a bag, shuffled, and then one domino is randomly drawn. What is the probability the total number of dots on the domino

a. is three or less

b. is greater than three

c. does not have a blank half

d. is not a "double" (both sides the same)

Find $P(E)$ given the values for $n(E)$ and $n(S)$ shown.

27. $n(E) = {}_6C_3 \cdot {}_4C_2$; $n(S) = {}_{10}C_5$

28. $n(E) = {}_{12}C_9 \cdot {}_8C_7$; $n(S) = {}_{20}C_{16}$

29. $n(E) = {}_9C_6 \cdot {}_5C_3$; $n(S) = {}_{14}C_9$

30. $n(E) = {}_7C_6 \cdot {}_3C_2$; $n(S) = {}_{10}C_8$

31. Five cards are drawn from a well-shuffled, standard deck of 52 cards. Which has the greater probability: (a) all five cards are red or (b) all five cards are numbered cards? How much greater?

32. Five cards are drawn from a well-shuffled pinochle deck of 48 cards (see Exercise 14). Which has the greater probability (a) all five cards are face cards (King, Queen, or Jack) or (b) all five cards are black? How much greater?

33. A dietetics class has 24 students. Of these, 9 are vegetarians and 15 are not. The instructor receives enough funding to send six students to a conference. If the students are selected randomly, what is the probability the group will have

- **a.** exactly two vegetarians
- **b.** exactly four nonvegetarians
- **c.** at least three vegetarians

34. A large law firm has a support staff of 15 employees: six paralegals and nine legal assistants. Due to recent changes in the law, the firm wants to send five of them to a forum on the new changes. If the selection is done randomly, what is the probability the group will have

- **a.** exactly three paralegals
- **b.** exactly two legal assistants
- **c.** at least two paralegals

Find the probability indicated using the information given.

35. Given $P(E_1) = 0.7$, $P(E_2) = 0.5$, and $P(E_1 \cap E_2) = 0.3$, compute $P(E_1 \cup E_2)$.

36. Given $P(E_1) = 0.6$, $P(E_2) = 0.3$, and $P(E_1 \cap E_2) = 0.2$, compute $P(E_1 \cup E_2)$.

37. Given $P(E_1) = \frac{3}{8}$, $P(E_2) = \frac{3}{4}$, and $P(E_1 \cup E_2) = \frac{15}{18}$; compute $P(E_1 \cap E_2)$.

38. Given $P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{3}{5}$, and $P(E_1 \cup E_2) = \frac{17}{20}$; compute $P(E_1 \cap E_2)$.

39. Given $P(E_1 \cup E_2) = 0.72$, $P(E_2) = 0.56$, and $P(E_1 \cap E_2) = 0.43$; compute $P(E_1)$.

40. Given $P(E_1 \cup E_2) = 0.85$, $P(E_1) = 0.4$, and $P(E_1 \cap E_2) = 0.21$; compute $P(E_2)$.

41. Two fair dice are rolled. What is the probability the sum of the dice is

- **a.** a multiple of 3 and an odd number
- **b.** a sum greater than 5 and a 3 on one die
- **c.** an even number and a number greater than 9
- **d.** an odd number and a number less than 10

42. *Eight Ball* is a game played on a pool table with 15 balls numbered 1 through 15 and a cue ball that is solid white. Of the 15 numbered balls, 8 are a solid (nonwhite) color and numbered 1 through 8, and seven are striped balls numbered 9 through 15. The fifteen numbered pool balls (no cueball) are placed in a large bowl and mixed, then one is drawn out. What is the probability of drawing

- **a.** the eight ball
- **b.** a number greater than fifteen
- **c.** an even number
- **d.** a multiple of three
- **e.** a solid color and an even number
- **f.** a striped ball and an odd number
- **g.** an even number and a number divisible by three
- **h.** an odd number and a number divisible by 4

43. A survey of 50 veterans was taken to gather information on their service career and what life is like out of the military. A breakdown of those surveyed is shown in the table. One out of the 50 will be selected at random for an interview and a biographical sketch. What is the probability the person chosen is

	Women	Men	Totals
Private	6	9	15
Corporal	10	8	18
Sergeant	4	5	9
Lieutenant	2	1	3
Captain	2	3	5
Totals	24	26	50

- **a.** a woman and a sergeant
- **b.** a man and a private
- **c.** a private and a sergeant
- **d.** a woman and an officer
- **e.** a person in the military

44. Referring to Exercise 43, what is the probability the person chosen is

- **a.** a woman or a sergeant
- **b.** a man or a private
- **c.** a woman or a man
- **d.** a woman or an officer
- **e.** a captain or a lieutenant

A computer is asked to randomly generate a three-digit number. What is the probability the

45. ten's digit is odd or the one's digit is even

46. first digit is prime and the number is a multiple of 10

A computer is asked to randomly generate a four-digit number. What is the probability the number is

47. at least 4000 or a multiple of 5

48. less than 7000 and an odd number

49. Two fair dice are rolled. What is the probability of rolling

- **a.** boxcars (a sum of 12) or snake eyes (a sum of 2)
- **b.** a sum of 7 or a sum of 11
- **c.** an even-numbered sum or a prime sum
- **d.** an odd-numbered sum or a sum that is a multiple of 4
- **e.** a sum of 15 or a multiple of 12
- **f.** a sum that is a prime number

50. Suppose all 16 balls from a game of pool (see Exercise 42) are placed in a large leather bag and mixed, then one is drawn out. Consider the cue ball as "0." What is the probability of drawing

- **a.** a striped ball
- **b.** a solid-colored ball
- **c.** a polka-dotted ball
- **d.** the cue ball
- **e.** the cue ball or the eight ball
- **f.** a striped ball or a number less than five
- **g.** a solid color or a number greater than 12
- **h.** an odd number or a number divisible by 4

► WORKING WITH FORMULAS

51. Games involving a fair spinner (with numbers 1 through 4): $P(n) = (\frac{1}{4})^n$

Games that involve moving pieces around a board using a fair spinner are fairly common. If a fair spinner has the numbers 1 through 4, the probability that any one number is spun n times in succession is given by the formula shown, where n represents the number of spins. What is the probability (a) the first player spins a two? (b) all four players spin a two? (c) Discuss the graph of $P(n)$ and explain the connection between the graph and the probability of consistently spinning a two.

52. Games involving a fair coin (heads and tails): $P(n) = (\frac{1}{2})^n$

When a fair coin is flipped, the probability that heads (or tails) is flipped n times in a row is given by the formula shown, where n represents the number of flips. What is the probability (a) the first flip is heads? (b) the first four flips are heads? (c) Discuss the graph of $P(n)$ and explain the connection between the graph and the probability of consistently flipping heads.

► APPLICATIONS

53. To improve customer service, a company tracks the number of minutes a caller is "on hold" and waiting for a customer service representative. The table shows the probability that a caller will wait m minutes. Based on the table, what is the probability a caller waits

- **a.** at least 2 minutes
- **b.** less than 2 minutes
- **c.** 4 minutes or less
- **d.** over 4 minutes
- **e.** less than 2 or more than 4 minutes
- **f.** 3 or more minutes

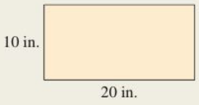
Wait Time (minutes m)	Probability
0	0.07
$0 < m < 1$	0.28
$1 \le m < 2$	0.32
$2 \le m < 3$	0.25
$3 \le m < 4$	0.08

54. To study the impact of technology on American families, a researcher first determines the probability that a family has n computers at home. Based on the table, what is the probability a home

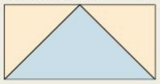
a. has at least one computer
b. has two or more computers
c. has less than four computers
d. has five computers
e. has one, two, or three computers
f. does not have two computers

Number of Computers	Probability
0	9%
1	51%
2	28%
3	9%
4	3%

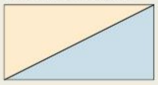
55. Jolene is an experienced markswoman and is able to hit a 10 in. by 20 in. target 100% of the time at a range of 100 yd. Assuming the probability she hits a target is related to its area, what is the probability she hits the shaded portions shown?



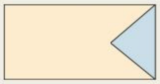
a. isosceles triangle



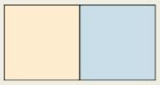
b. right triangle



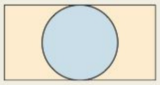
c. equilateral triangle



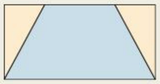
56. **a.** square



b. circle



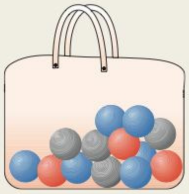
c. isosceles trapezoid with $b = \frac{B}{2}$



57. A circular dartboard has a total radius of 8 in., with circular bands that are 2 in. wide, as shown. You are skilled enough to hit this board 100% of the time so you always score at least two points each time you throw a dart. Assuming the probabilities are related to area, on the next dart that you throw what is the probability you

a. score at least a 4? **b.** score at least a 6?
c. hit the bull's-eye? **d.** score exactly 4 points?

58. Three red balls, six blue balls, and four white balls are placed in a bag. What is the probability the first ball you draw out is



a. red **b.** blue
c. not white
d. purple
e. red or white
f. red and white

59. Three red balls, six blue balls, and four white balls are placed in a bag, then two are drawn out and placed in a rack. What is the probability the balls drawn are

a. first red, second blue
b. first blue, second red
c. both white
d. first blue, second not red
e. first white, second not blue
f. first not red, second not blue

60. Consider the 210 discrete points found in the first and second quadrants where $-10 \le x \le 10$, $1 \le y \le 10$, and x and y are integers. The coordinates of each point is written on a slip of paper and placed in a box. One of the slips is then randomly drawn. What is the probability the point (x, y) drawn

a. is on the graph of $y = |x|$
b. is on the graph of $y = 2|x|$
c. is on the graph of $y = 0.5|x|$
d. has coordinates $(x, y > -2)$
e. has coordinates $(x \le 5, y > -2)$
f. is between the branches of $y = x^2$

61. Your instructor surprises you with a True/False quiz for which you are totally unprepared and must guess randomly. What is the probability you pass the quiz with an 80% or better if there are

a. three questions **b.** four questions
c. five questions

62. A robot is sent out to disarm a timed explosive device by randomly changing some switches from a neutral position to a *positive flow* or *negative flow* position. The problem is, the switches are independent and unmarked, and it is unknown which direction is positive and which direction is negative. The bomb is harmless if a majority of the switches yield a positive flow. All switches must be thrown. What is the probability the device is disarmed if there are

a. three switches **b.** four switches

c. five switches

63. A survey of 100 retirees was taken to gather information concerning how they viewed the Vietnam War back in the early 1970s. A breakdown of those surveyed is shown in the table. One out of the hundred will be selected at random for a personal, taped interview. What is the probability the person chosen had a

a. career of any kind and opposed the war

b. medical career and supported the war

c. military career and opposed the war

d. legal or business career and opposed the war

e. academic or medical career and supported the war

Career	Support	Opposed	T
Military	9	3	12
Medical	8	16	24
Legal	15	12	27
Business	18	6	24
Academics	3	10	13
Totals	53	47	100

64. Referring to Exercise 63, what is the probability the person chosen

a. had a career of any kind or opposed the war

b. had a medical career or supported the war

c. supported the war or had a military career

d. had a medical or a legal career

e. supported or opposed the war

65. The Board of Directors for a large hospital has 15 members. There are six doctors of nephrology (kidneys), five doctors of gastroenterology (stomach and intestines), and four doctors of endocrinology (hormones and glands). Eight of them will be selected to visit the nation's premier hospitals on a 3-week, expenses-paid tour. What is the probability the group of eight selected consists of exactly

a. four nephrologists and four gastroenterologists

b. three endocrinologists and five nephrologists

66. A support group for hodophobics (an irrational fear of travel) has 32 members. There are 15 aviophobics (fear of air travel), eight siderodrophobics (fear of train travel), and nine thalassophobics (fear of ocean travel) in the group. Twelve of them will be randomly selected to participate in a new therapy. What is the probability the group of 12 selected consists of exactly

a. two aviophobics, six siderodrophobics, and four thalassophobics

b. five thalassophobics, four aviophobics, and three siderodrophobics

67. A trained chimpanzee is given a box containing eight wooden cubes with the letters p, a, r, a, l, l, e, l printed on them (one letter per block). Assuming the chimp can't read or spell, what is the probability he draws the eight blocks in order and actually forms the word "parallel"?

68. A number is called a "perfect number" if the sum of its proper factors is equal to the number itself. Six is the first perfect number since the sum of its proper factors is six: $1 + 2 + 3 = 6$. Twenty-eight is the second since: $1 + 2 + 4 + 7 + 14 = 28$. A young child is given a box containing eight wooden blocks with the following numbers (one per block) printed on them: four 3's, two 5's, one 0, and one 6. What is the probability she draws the eight blocks in order and forms the fifth perfect number: 33,550,336?

▶ EXTENDING THE CONCEPT

69. The function $f(x) = (\frac{1}{2})^x$ gives the probability that x number of flips will all result in heads (or tails). Compute the probability that 20 flips results in *20 heads in a row,* then use the Internet or some other resource to find the probability of winning a state lottery. Which is more likely to happen (which has the greater probability)? Were you surprised?

70. Recall that a function is a relation in which each element of the domain is paired with only one element from the range. Is the relation defined by $C(x) = {}_nC_x$ (n is a constant) a function? To investigate, plot the points generated by $C(x) = {}_6C_x$ for $x = 0$ to $x = 6$ and answer the following questions:

a. Is the resulting graph continuous or discrete (made up of distinct points)?

b. Does the resulting graph pass the vertical line test?

c. Discuss the features of the relation and its graph, including the domain, range, maximum or minimum values, and symmetries observed.

▶ MAINTAINING YOUR SKILLS

71. **(5.3)** Given csc $\theta = 3$ and $\cos\theta < 0$, find the values of the remaining five trig functions of θ.

72. **(4.4)** Complete the following logarithmic properties:

$\log_b b =$ __ $\qquad \log_b 1 =$ __

$\log_b b^n =$ __ $\qquad b^{\log_b n} =$ __

73. **(6.4)** Find exact values for $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\cos\theta = -\frac{21}{29}$ and θ is in Quadrant II.

74. **(8.3)** A rubber ball is dropped from a height of 25 ft onto a hard surface. With each bounce, it rebounds 60% of the height from which it last fell. Use sequences/series to find (a) the height of the sixth bounce, (b) the total distance traveled up to the sixth bounce, and (c) the distance the ball will travel before coming to rest.

11.7 The Binomial Theorem

Learning Objectives

In Section 11.7 you will learn how to:

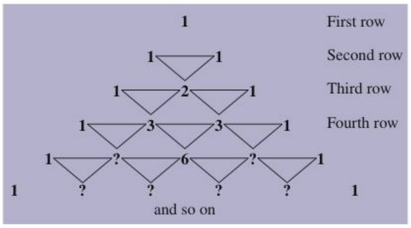
- ☐ **A.** Use Pascal's triangle to find $(a + b)^n$
- ☐ **B.** Find binomial coefficients using $\binom{n}{k}$ notation
- ☐ **C.** Use the binomial theorem to find $(a + b)^n$
- ☐ **D.** Find a specific term of a binomial expansion

Strictly speaking, a binomial is a polynomial with two terms. This limits us to terms with real number coefficients and whole number powers on variables. In this section, we will loosely regard a binomial as the sum or difference of *any* two terms. Hence $3x^2 - y^4$, $\sqrt{x} + 4$, $x + \frac{1}{x}$, and $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ are all "binomials." Our goal is to develop an ability to raise a binomial to any natural number power, with the results having important applications in genetics, probability, polynomial theory, and other areas. The tool used for this purpose is called the *binomial theorem*.

A. Binomial Powers and Pascal's Triangle

Much of our mathematical understanding comes from a study of patterns. One area where the study of patterns has been particularly fruitful is **Pascal's triangle** (Figure 11.20), named after the French scientist Blaise Pascal (although the triangle was well known before his time). It begins with a "1" at the vertex of the triangle, with 1's extending diagonally downward to the left and right as shown. The entries on the interior of the triangle are found by adding the two entries directly above and to the left and right of each new position.

Figure 11.20



There are a variety of patterns hidden within the triangle. In this section, we'll use the *horizontal rows* of the triangle to help us raise a binomial to various powers. To begin, recall that $(a + b)^0 = 1$ and $(a + b)^1 = 1a + 1b$ (unit coefficients are included for emphasis). In our earlier work, we saw that a binomial square (a binomial raised to the second power) always followed the pattern $(a + b)^2 = 1a^2 + 2ab + 1b^2$. Observe the overall pattern that is developing as we include $(a + b)^3$:

$$\begin{array}{lcl} (a+b)^0 & 1 & \text{row 1} \\ (a+b)^1 & 1a + 1b & \text{row 2} \\ (a+b)^2 & 1a^2 + 2ab + 1b^2 & \text{row 3} \\ (a+b)^3 & 1a^3 + 3a^2b + 3ab^2 + 1b^3 & \text{row 4} \end{array}$$

Apparently the coefficients of $(a + b)^n$ will occur in row $n + 1$ of Pascal's triangle. Also observe that in each term of the expansion, the exponent of the first term a *decreases by 1* as the exponent on the second term b *increases by 1*, keeping the degree of each term constant (recall the degree of a term with more than one variable is the sum of the exponents).

$$\underset{\substack{3+0 \\ \text{degree 3}}}{1a^3b^0} + \underset{\substack{2+1 \\ \text{degree 3}}}{3a^2b^1} + \underset{\substack{1+2 \\ \text{degree 3}}}{3a^1b^2} + \underset{\substack{0+3 \\ \text{degree 3}}}{1a^0b^3}$$

These observations help us to quickly expand a binomial power.

EXAMPLE 1 ► Expanding a Binomial Using Pascal's Triangle

Use Pascal's triangle and the patterns noted to expand $(x + \frac{1}{2})^4$.

Solution ► Working step-by-step we have

1. The coefficients will be in the fifth row of Pascal's triangle.

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

2. The exponents on x begin at 4 and *decrease*, while the exponents on $\frac{1}{2}$ begin at 0 and *increase*.

$$1x^4\left(\frac{1}{2}\right)^0 + 4x^3\left(\frac{1}{2}\right)^1 + 6x^2\left(\frac{1}{2}\right)^2 + 4x^1\left(\frac{1}{2}\right)^3 + 1x^0\left(\frac{1}{2}\right)^4$$

3. Simplify each term.

The result is $x^4 + 2x^3 + \frac{3}{2}x^2 + \frac{1}{2}x + \frac{1}{16}$.

Now try Exercises 7 through 10 ►

If the exercise involves a difference rather than a sum, we simply rewrite the expression using algebraic addition and proceed as before.

EXAMPLE 2 ► Raising a Complex Number to a Power Using Pascal's Triangle

Use Pascal's triangle and the patterns noted to compute $(3 - 2i)^5$.

Solution ► Begin by rewriting $(3 - 2i)^5$ as $[3 + (-2i)]^5$.

1. The coefficients will be in the sixth row of Pascal's triangle.

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

2. The exponents on 3 begin at 5 and *decrease*, while the exponents on $(-2i)$ begin at 0 and *increase*.

$$\mathbf{1}(3^5)(-2i)^0 + \mathbf{5}(3^4)(-2i)^1 + \mathbf{10}(3^3)(-2i)^2 + \mathbf{10}(3^2)(-2i)^3 + \mathbf{5}(3^1)(-2i)^4 + \mathbf{1}(3^0)(-2i)^5$$

3. Simplify each term.

$$243 - 810i - 1080 + 720i + 240 - 32i$$

The result is $-597 - 122i$.

Now try Exercises 11 and 12 ►

☑ **A. You've just learned how to use Pascal's triangle to find $(a + b)^n$**

Expanding Binomial Powers $(a + b)^n$

1. The coefficients will be in row $n + 1$ of Pascal's triangle.
2. The exponents on *the first term* begin at n and *decrease,* while the exponents on *the second term* begin at 0 and *increase.*
3. For any binomial difference $(a - b)^n$, rewrite the base as $[a + (-b)]^n$ using algebraic addition and proceed as before, then simplify each term.

B. Binomial Coefficients and Factorials

Pascal's triangle can easily be used to find the coefficients of $(a + b)^n$, as long as the exponent is relatively small. If we needed to expand $(a + b)^{25}$, writing out the first 26 rows of the triangle would be rather tedious. To overcome this limitation, we introduce a *formula* for the binomial coefficients that enables us to find the coefficients of any expansion.

The Binomial Coefficients

For natural numbers n and r where $n \geq r$, the expression $\binom{n}{r}$, read "n choose r," is called the **binomial coefficient** and evaluated as:

$$\binom{n}{r} = \frac{n!}{r!(n - r)!}$$

In Example 1, we found the coefficients of $(a + b)^4$ using the fifth or $(n + 1)$st row of Pascal's triangle. In Example 3, these coefficients are found using the formula for binomial coefficients.

EXAMPLE 3 ► Computing Binomial Coefficients

Evaluate $\binom{n}{r} = \frac{n!}{r!(n - r)!}$ as indicated:

a. $\binom{4}{1}$ **b.** $\binom{4}{2}$ **c.** $\binom{4}{3}$

Solution ► **a.** $\binom{4}{1} = \frac{4!}{1!(4 - 1)!} = \frac{4 \cdot 3!}{1!3!} = 4$

b. $\binom{4}{2} = \frac{4!}{2!(4 - 2)!} = \frac{4 \cdot 3 \cdot 2!}{2!2!} = \frac{4 \cdot 3}{2} = 6$

c. $\binom{4}{3} = \frac{4!}{3!(4 - 3)!} = \frac{4 \cdot 3!}{3!1!} = 4$

Now try Exercises 13 through 20 ►

Note $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, and $\binom{4}{3} = 4$ give the *interior entries* in the fifth row of Pascal's triangle: 1 4 6 4 1. For consistency and symmetry, we define $0! = 1$, which enables the formula to generate all entries of the triangle, including the "1's."

$$\binom{4}{0} = \frac{4!}{0!(4 - 0)!} \quad \text{apply formula} \qquad \binom{4}{4} = \frac{4!}{4!(4 - 4)!} = \frac{4!}{4! \cdot 0!} \quad \text{apply formula}$$

$$= \frac{4!}{1 \cdot 4!} = 1 \quad 0! = 1 \qquad = \frac{4!}{4! \cdot 1} = 1 \quad 0! = 1$$

The formula for $\binom{n}{r}$ with $0 \le r \le n$ now gives all coefficients in the $(n + 1)$st row. For $n = 5$, we have

$$\begin{array}{cccccc} \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} \\ 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

EXAMPLE 4 ► **Computing Binomial Coefficients**

Compute the binomial coefficients:

a. $\binom{9}{0}$ **b.** $\binom{9}{1}$ **c.** $\binom{6}{5}$ **d.** $\binom{6}{6}$

Solution ►

a. $$\binom{9}{0} = \frac{9!}{0!(9-0)!} = \frac{9!}{9!} = 1$$

b. $$\binom{9}{1} = \frac{9!}{1!(9-1)!} = \frac{9!}{8!} = 9$$

c. $$\binom{6}{5} = \frac{6!}{5!(6-5)!} = \frac{6!}{5!} = 6$$

d. $$\binom{6}{6} = \frac{6!}{6!(6-6)!} = \frac{6!}{6!} = 1$$

Now try Exercises 21 through 24 ►

☑ **B. You've just learned how to find binomial coefficients using $\binom{n}{k}$ notation**

You may have noticed that the formula for $\binom{n}{r}$ is identical to that of ${}_nC_r$, and both yield like results for given values of n and r. For future use, it will help to commit the general results from Example 4 to memory: $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{n-1} = n$, and $\binom{n}{n} = 1$.

C. The Binomial Theorem

Using $\binom{n}{r}$ notation and the observations made regarding binomial powers, we can now state the **binomial theorem.**

Binomial Theorem

For any binomial $(a + b)$ and natural number n,

$$(a + b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n$$

The theorem can also be stated in summation form as

$$(a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r$$

The expansion actually looks overly impressive in this form, and it helps to summarize the process in words, as we did earlier. The exponents on the first term a begin at n and decrease, while the exponents on the second term b begin at 0 and increase, keeping the degree of each term constant. The $\binom{n}{r}$ notation simply gives the coefficients of each term. As a final note, observe that the r in $\binom{n}{r}$ gives the exponent on b.

EXAMPLE 5 ► **Expanding a Binomial Using the Binomial Theorem**

Expand $(a + b)^6$ using the binomial theorem.

Solution ►

$$(a+b)^6 = \binom{6}{0}a^6b^0 + \binom{6}{1}a^5b^1 + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 + \binom{6}{6}a^0b^6$$

$$= \frac{6!}{0!6!}a^6 + \frac{6!}{1!5!}a^5b^1 + \frac{6!}{2!4!}a^4b^2 + \frac{6!}{3!3!}a^3b^3 + \frac{6!}{4!2!}a^2b^4 + \frac{6!}{5!1!}a^1b^5 + \frac{6!}{6!0!}b^6$$

$$= 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6$$

Now try Exercises 25 through 32 ►

EXAMPLE 6 ► **Using the Binomial Theorem to Find the Initial Terms of an Expansion**

Find the first three terms of $(2x + y^2)^{10}$.

Solution ► Use the binomial theorem with $a = 2x$, $b = y^2$, and $n = 10$.

$$(2x + y^2)^{10} = \binom{10}{0}(2x)^{10}(y^2)^0 + \binom{10}{1}(2x)^9(y^2)^1 + \binom{10}{2}(2x)^8(y^2)^2 + \cdots \quad \text{first three terms}$$

$$= (1)1024x^{10} + (10)512x^9y^2 + \frac{10!}{2!8!}256x^8y^4 + \cdots \quad \binom{10}{0} = 1, \binom{10}{1} = 10$$

$$= 1024x^{10} + 5120x^9y^2 + (45)256x^8y^4 + \cdots \quad \frac{10!}{2!8!} = 45$$

$$= 1024x^{10} + 5120x^9y^2 + 11{,}520x^8y^4 + \cdots \quad \text{result}$$

☑ **C.** You've just learned how to use the binomial theorem to find $(a + b)^n$

Now try Exercises 33 through 36 ►

D. Finding a Specific Term of the Binomial Expansion

In some applications of the binomial theorem, our main interest is a *specific term* of the expansion, rather than the expansion as a whole. To find a specified term, it helps to consider that the expansion of $(a + b)^n$ has $n + 1$ terms: $(a + b)^0$ has one term, $(a + b)^1$ has two terms, $(a + b)^2$ has three terms, and so on. Because the notation $\binom{n}{r}$ always begins at $r = 0$ for the first term, the value of r will be *1 less than the term we are seeking*. In other words, for the seventh term of $(a + b)^9$, we use $r = 6$.

The *k*th Term of a Binomial Expansion

For the binomial expansion $(a + b)^n$, the kth term is given by

$$\binom{n}{r}a^{n-r}b^r, \text{ where } r = k - 1.$$

EXAMPLE 7 ► **Finding a Specific Term of a Binomial Expansion**

Find the eighth term in the expansion of $(x + 2y)^{12}$.

Solution ► By comparing $(x + 2y)^{12}$ to $(a + b)^n$ we have $a = x$, $b = 2y$, and $n = 12$. Since we want the eighth term, $k = 8 \rightarrow r = 7$. The eighth term of the expansion is

$$\begin{aligned}\binom{12}{7}x^5(2y)^7 &= \frac{12!}{7!5!}128x^5y^7 && 2^7 = 128\\ &= (792)(128x^5y^7) && \tbinom{12}{7} = 792\\ &= 101{,}376x^5y^7 && \text{result}\end{aligned}$$

Now try Exercises 37 through 42 ►

One application of the binomial theorem involves a **binomial experiment** and **binomial probability.** For binomial probabilities, the following must be true: (1) The experiment must have only two possible outcomes, typically called success and failure, and (2) if the experiment has n trials, the probability of success must be constant for all n trials. If the probability of success for each trial is p, the formula $\boldsymbol{\binom{n}{k}(1 - p)^{n-k}p^k}$ gives the probability that exactly k trials will be successful.

Binomial Probability

Given a binomial experiment with n trials, where the probability for success in each trial is p. The probability that exactly k trials are successful is given by

$$\binom{n}{k}(1 - p)^{n-k}\,p^k.$$

EXAMPLE 8 ► **Applying the Binomial Theorem–Binomial Probability**

Paula Rodrigues has a free-throw shooting average of 85%. On the last play of the game, with her team behind by three points, she is fouled at the three-point line, and is awarded two additional free throws via technical fouls on the opposing coach (a total of five free-throws). What is the probability she makes *at least three* (meaning they at least tie the game)?

Solution ► Here we have $p = 0.85$, $1 - p = 0.15$, and $n = 5$. The key idea is to recognize the phrase *at least three* means "3 or 4 or 5." So $P(\text{at least } 3) = P(3 \cup 4 \cup 5)$.

$$\begin{aligned}P(\text{at least } 3) &= P(3 \cup 4 \cup 5) && \text{"or" implies a union}\\ &= P(3) + P(4) + P(5) && \text{sum of probabilities (mutually exclusive events)}\\ &= \binom{5}{3}(0.15)^2(0.85)^3 + \binom{5}{4}(0.15)^1(0.85)^4 + \binom{5}{5}(0.15)^0(0.85)^5\\ &\approx 0.1382 + 0.3915 + 0.4437\\ &= 0.9734\end{aligned}$$

Paula's team has an excellent chance (≈97.3%) of at least tying the game.

☑ **D.** You've just learned how to find a specific term of a binomial expansion

New try Exercises 45 and 46 ►

11.7 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. In any binomial expansion, there is always ______ more term than the power applied.
2. In all terms in the expanded form of $(a + b)^n$, the exponents on a and b must sum to ______.
3. To expand a binomial *difference* such as $(a - 2b)^5$, we rewrite the binomial as ______ and proceed as before.
4. In a binomial experiment with n trials, the probability there are exactly k successes is given by the formula ______.
5. Discuss why the expansion of $(a + b)^n$ has $n + 1$ terms.
6. For any defined binomial experiment, discuss the relationships between the phrases, "exactly k success," and "at least k successes."

▶ DEVELOPING YOUR SKILLS

Use *Pascal's triangle* and the patterns explored to write each expansion.

7. $(x + y)^5$ **8.** $(a + b)^6$ **9.** $(2x + 3)^4$

10. $(x^2 + \frac{1}{3})^3$ **11.** $(1 - 2i)^5$ **12.** $(2 - 5i)^4$

Evaluate each of the following

13. $\binom{7}{4}$ **14.** $\binom{8}{2}$ **15.** $\binom{5}{3}$

16. $\binom{9}{5}$ **17.** $\binom{20}{17}$ **18.** $\binom{30}{26}$

19. $\binom{40}{3}$ **20.** $\binom{45}{3}$ **21.** $\binom{6}{0}$

22. $\binom{5}{0}$ **23.** $\binom{15}{15}$ **24.** $\binom{10}{10}$

Use the *binomial theorem* to expand each expression. Write the general form first, then simplify.

25. $(c + d)^5$ **26.** $(v + w)^4$ **27.** $(a - b)^6$

28. $(x - y)^7$ **29.** $(2x - 3)^4$ **30.** $(a - 2b)^5$

31. $(1 - 2i)^3$ **32.** $(2 + \sqrt{3}i)^5$

Use the *binomial theorem* to write the first three terms.

33. $(x + 2y)^9$ **34.** $(3p + q)^8$ **35.** $(v^2 - \frac{1}{2}w)^{12}$

36. $(\frac{1}{2}a - b^2)^{10}$

Find the indicated term for each binomial expansion.

37. $(x + y)^7$; 4th term **38.** $(m + n)^6$; 5th term

39. $(p - 2)^8$; 7th term **40.** $(a - 3)^{14}$; 10th term

41. $(2x + y)^{12}$; 11th term **42.** $(3n + m)^9$; 6th term

▶ WORKING WITH FORMULAS

43. Binomial probability: $P(k) = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k}$

The theoretical probability of getting exactly k heads in n flips of a fair coin is given by the formula above. What is the probability that you would get 5 heads in 10 flips of the coin?

44. Binomial probability: $P(k) = \binom{n}{k}\left(\frac{1}{5}\right)^k\left(\frac{4}{5}\right)^{n-k}$

A multiple choice test has five options per question. The probability of guessing correctly k times out of n questions is found using the formula shown. What is the probability a person scores a 70% by guessing randomly (7 out of 10 questions correct)?

▶ APPLICATIONS

45. Batting averages: Tony Gwynn (San Diego Padres) had a lifetime batting average of 0.347, ranking him as one of the greatest hitters of all time. Suppose he came to bat five times in any given game.

- **a.** What is the probability that he will get exactly three hits?
- **b.** What is the probability that he will get at least three hits?

46. Pollution testing: Erin suspects that a nearby iron smelter is contaminating the drinking water over a large area. A statistical study reveals that 83% of the wells in this area are likely contaminated. If the figure is accurate, find the probability that if another 10 wells are tested

- **a.** exactly 8 are contaminated
- **b.** at least 8 are contaminated

47. Late rental returns: The manager of Victor's DVD Rentals knows that 6% of all DVDs rented are returned *late*. Of the eight videos rented in the last hour, what is the probability that

- **a.** exactly five are returned on time
- **b.** exactly six are returned on time
- **c.** at least six are returned on time
- **d.** none of them will be returned late

48. Opinion polls: From past experience, a research firm knows that 20% of telephone respondents will agree to answer an opinion poll. If 20 people are contacted by phone, what is the probability that

- **a.** exactly 18 refuse to be polled
- **b.** exactly 19 refuse to be polled
- **c.** at least 18 refuse to be polled
- **d.** none of them agree to be polled

▶ EXTENDING THE CONCEPT

49. Prior to calculators and computers, the binomial theorem was frequently used to approximate the value of compound interest given by the expression $\left(1 + \frac{r}{n}\right)^{nt}$ by expanding the first three terms. For example, if the interest rate were 8% ($r = 0.08$) and the interest was compounded quarterly ($n = 4$) for 5 yr ($t = 5$), we have $(1 + \frac{0.08}{4})^{(4)(5)} = (1 + 0.02)^{20}$. The first three terms of the expansion give a value of: $1 + 20(0.02) + 190(0.0004) = 1.476$.

- **a.** Calculate the percent error:
$$\%\text{error} = \frac{\text{approximate value}}{\text{actual value}}$$
- **b.** What is the percent error if only two terms are used.

50. If you sum the entries in each row of Pascal's triangle, a pattern emerges. Find a formula that generalizes the result for any row of the triangle, and use it to find the sum of the entries in the 12th row of the triangle.

51. Show that $\binom{n}{k} = \binom{n}{n-k}$ for $n = 6$ and $k \le 6$.

52. The *derived polynomial* of $f(x)$ is $f(x + h)$ or the original polynomial evaluated at $x + h$. Use Pascal's triangle or the binomial theorem to find the derived polynomial for $f(x) = x^3 + 3x^2 + 5x - 11$. Simplify the result completely.

▶ MAINTAINING YOUR SKILLS

53. (2.7) Graph the function shown and find $f(3)$: $f(x) = \begin{cases} x + 2 & x \le 2 \\ (x - 4)^2 & x > 2 \end{cases}$

54. (5.4) Given the point $(-0.6, y)$ is a point on the unit circle in the third quadrant, find y.

55. (3.4) Graph the function $g(x) = x^3 - x^2 - 6x$. Clearly indicate all intercepts and intervals where $g(x) > 0$.

56. (6.5) Evaluate $\arcsin\left[\sin\left(\frac{5\pi}{6}\right)\right]$.

SUMMARY AND CONCEPT REVIEW

SECTION 11.1 Sequences and Series

KEY CONCEPTS

- A *finite sequence* is a function a_n whose domain is the set of natural numbers from 1 to n.
- The terms of the sequence are labeled $a_1, a_2, a_3, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_{n-2}, a_{n-1}, a_n$.
- The expression a_n, which defines the sequence (generates the terms in order), is called the nth term.
- An *infinite sequence* is a function whose domain is the set of natural numbers.
- When each term of a sequence is larger than the preceding term, it is called an increasing sequence.
- When each term of a sequence is smaller than the preceding term, it is called a decreasing sequence.
- When successive terms of a sequence alternate in sign, it is called an alternating sequence.
- When the terms of a sequence are generated using previous term(s), it is called a recursive sequence.
- Sequences are sometimes defined using factorials, which are the product of a given natural number with all natural numbers that precede it: $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$.
- Given the sequence $a_1, a_2, a_3, a_4, \dots, a_n$ the sum is called a finite series and is denoted S_n.
- $S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$. The sum of the first n terms is called a partial sum.
- In sigma notation, the expression $\sum_{i=1}^{k} a_i = a_1 + a_2 + \cdots + a_k$ represents a finite series.
- When sigma notation is used, the letter "i" is called the index of summation.

EXERCISES

Write the first four terms that are defined and the value of a_{10}.

1. $a_n = 5n - 4$

2. $a_n = \dfrac{n+1}{n^2+1}$

Find the general term a_n for each sequence, and the value of a_6.

3. $1, 16, 81, 256, \dots$

4. $-17, -14, -11, -8, \dots$

Find the eighth partial sum (S_8).

5. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

6. $-21, -19, -17, \dots$

Evaluate each sum.

7. $\sum_{n=1}^{7} n^2$

8. $\sum_{n=1}^{5} (3n - 2)$

Write the first five terms that are defined.

9. $a_n = \dfrac{n!}{(n-2)!}$

10. $\begin{cases} a_1 = \frac{1}{2} \\ a_{n+1} = 2a_n - \frac{1}{4} \end{cases}$

Write as a single summation and evaluate.

11. $\sum_{n=1}^{7} n^2 + \sum_{n=1}^{7} (3n - 2)$

SECTION 11.2 Arithmetic Sequences

KEY CONCEPTS

- In an arithmetic sequence, successive terms are found by adding a fixed constant to the preceding term.
- In a sequence, if there exists a number d, called the common difference, such that $a_{k+1} - a_k = d$, then the sequence is arithmetic. Alternatively, $a_{k+1} = a_k + d$ for $k \geq 1$.

- The nth term n of an arithmetic sequence is given by $a_n = a_1 + (n - 1)d$, where a_1 is the first term and d is the common difference.
- If the initial term is unknown or is not a_1 the nth term can be written $a_n = a_k + (n - k)d$, where the subscript of the term a_k and the coefficient of d sum to n.
- For an arithmetic sequence with first term a_1, the nth partial sum (the sum of the first n terms) is given by $S_n = \dfrac{n(a_1 + a_n)}{2}$.

EXERCISES

Find the general term (a_n) for each arithmetic sequence. Then find the indicated term.

12. $2, 5, 8, 11, \ldots$; find a_{40}

13. $3, 1, -1, -3, \ldots$; find a_{35}

Find the sum of each series.

14. $-1 + 3 + 7 + 11 + \cdots + 75$

15. $1 + 4 + 7 + 10 + \cdots + 88$

16. $3 + 6 + 9 + 12 + \cdots; S_{20}$

17. $1 + \frac{3}{4} + \frac{1}{2} + \frac{1}{4} + \cdots; S_{15}$

18. $\sum_{n=1}^{25} (3n - 4)$

19. $\sum_{n=1}^{40} (4n - 1)$

SECTION 11.3 Geometric Sequences

KEY CONCEPTS

- In a geometric sequence, successive terms are found by multiplying the preceding term by a nonzero constant.
- In other words, if there exists a number r, called the common ratio, such that $\dfrac{a_{k+1}}{a_k} = r$, then the sequence is geometric. Alternatively, we can write $a_{k+1} = a_k r$ for $k \geq 1$.
- The nth term a_n of a geometric sequence is given by $a_n = a_1 r^{n-1}$, where a_1 is the first term and a_n represents the general term of a finite sequence.
- If the initial term is unknown or is not a_1, the nth term can be written $a_n = a_k r^{n-k}$, where the subscript of the term a_k and the exponent on r sum to n.
- The nth partial sum of a geometric sequence is $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$.
- If $|r| < 1$, the sum of an infinite geometric series is $S_\infty = \dfrac{a_1}{1 - r}$.

EXERCISES

Find the indicated term for each geometric sequence.

20. $a_1 = 5, r = 3$; find a_7

21. $a_1 = 4, r = \sqrt{2}$; find a_7

22. $a_1 = \sqrt{7}, r = \sqrt{7}$; find a_8

Find the indicated sum, if it exists.

23. $16 - 8 + 4 - \cdots$ find S_7

24. $2 + 6 + 18 + \cdots$; find S_8

25. $\frac{4}{5} + \frac{2}{5} + \frac{1}{5} + \frac{1}{10} + \cdots$; find S_{12}

26. $4 + 8 + 12 + 24 + \cdots$

27. $5 + 0.5 + 0.05 + 0.005 + \cdots$

28. $6 - 3 + \frac{3}{2} - \frac{3}{4} + \cdots$

29. $\sum_{n=1}^{8} 5\left(\frac{2}{3}\right)^n$

30. $\sum_{n=1}^{\infty} 12\left(\frac{4}{3}\right)^n$

31. $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^n$

32. Charlene began to work for Grayson Natural Gas in January of 1990 with an annual salary of \$26,000. Her contract calls for a \$1220 raise each year. Use a sequence/series to compute her salary after nine years, and her total earnings up to and including that year. (*Hint:* For $a_1 = 26{,}000$, her salary after *9 yrs* will be what term of the sequence?)

33. Sumpter reservoir contains 121,500 ft^3 of water and is being drained in the following way. Each day one-third of the water is *drained* (and not replaced). Use a sequence/series to compute how much water *remains in the pond* after 7 days.

34. Credit-hours taught at Cody Community College have been increasing at 7% per year since it opened in 2000 and taught 1225 credit-hours. For the new faculty, the college needs to predict the number of credit-hours that will be taught in 2009. Use a sequence/series to compute the credit-hours for 2009 and to find the total number of credit hours taught through the 2009 school year.

SECTION 11.4 Mathematical Induction

KEY CONCEPTS

- Functions written in subscript notation can be evaluated, graphed, and composed with other functions.
- A sum formula involving only natural numbers n as inputs can be proven valid using a proof by induction. Given that S_n represents a sum formula involving natural numbers, if (1) S_1 is true and (2) $S_k + a_{k+1} = S_{k+1}$, then S_n must be true for all natural numbers.
- Proof by induction can also be used to validate other relationships, using a more general statement of the principle. Let S_n be a statement involving the natural numbers n. If (1) S_1 is true (S_n for $n = 1$) and (2) the truth of S_k implies that S_{k+1} is also true, then S_n must be true for all natural numbers n.

EXERCISES

Use the principle of mathematical induction to prove the indicated sum formula is true for all natural numbers n.

35. $1 + 2 + 3 + 4 + 5 + \cdots + n$; $a_n = n$ and $S_n = \dfrac{n(n+1)}{2}$.

36. $1 + 4 + 9 + 16 + 25 + 36 + \cdots + n^2$; $a_n = n^2$ and $S_n = \dfrac{n(n+1)(2n+1)}{6}$.

Use the principle of mathematical induction to prove that each statement is true for all natural numbers n.

37. $4^n \geq 3n + 1$

38. $6 \cdot 7^{n-1} \leq 7^n - 1$

39. $3^n - 1$ is divisible by 2

SECTION 11.5 Counting Techniques

KEY CONCEPTS

- An experiment is any task that can be repeated and has a well-defined set of possible outcomes.
- Each repetition of an experiment is called a trial.
- Any potential outcome of an experiment is called a sample outcome.
- The set of all sample outcomes is called the sample space.
- An experiment with N (equally likely) sample outcomes that is repeated t times, has a sample space with N^t elements.
- If a sample outcome can be used more than once, the counting is said to be with repetition. If a sample outcome can be used only once the counting is said to be without repetition.
- The fundamental principle of counting states: If there are p possibilities for a first task, q possibilities for the second, and r possibilities for the third, the total number of ways the experiment can be completed is pqr. This fundamental principle can be extended to include any number of tasks.
- If the elements of a sample space have precedence or priority (order or rank is important), the number of elements is counted using a permutation, denoted ${}_nP_r$ and read, "the distinguishable permutations of n objects taken r at a time."
- To expand ${}_nP_r$, we can write out the first r factors of $n!$ or use the formula ${}_nP_r = \dfrac{n!}{(n-r)!}$.
- If any of the sample outcomes are identical, certain permutations will be nondistinguishable. In a set containing n elements where one element is repeated p times, another is repeated q times, and another r times ($p + q + r \leq n$), the number of distinguishable permutations is given by $\dfrac{{}_nP_n}{p!q!r!} = \dfrac{n!}{p!q!r!}$.

- If the elements of a set have no rank, order, or precedence (as in a committee of colleagues) permutations with the same elements are considered identical. The result is the number of combinations, ${}_nC_r = \frac{n!}{r!(n-r)!}$.

EXERCISES

40. Three slips of paper with the letters A, B, and C are placed in a box and randomly drawn one at a time. Show all possible ways they can be drawn using a tree diagram.

41. The combination for a certain bicycle lock consists of three digits. How many combinations are possible if (a) repetition of digits is not allowed and (b) repetition of digits is allowed.

42. Jethro has three work shirts, four pairs of work pants, and two pairs of work shoes. How many different ways can he dress himself (shirt, pants, shoes) for a day's work?

43. From a field of 12 contestants in a pet show, three cats are chosen at random to be photographed for a publicity poster. In how many different ways can the cats be chosen?

44. How many subsets can be formed from the elements of this set: {◘, ⊡, □, ▣, ⧈}?

45. Compute the following values by hand, showing all work:

a. 7! **b.** ${}_7P_4$ **c.** ${}_7C_4$

46. Six horses are competing in a race at the McClintock Ranch. Assuming there are no ties, (a) how many different ways can the horses finish the race? (b) How many different ways can the horses finish first, second, and third place? (c) How many finishes are possible if it is well known that Nellie-the-Nag will finish last and Sea Biscuit will finish first?

47. How many distinguishable permutations can be formed from the letters in the word "tomorrow"?

48. Quality Construction Company has 12 equally talented employees. (a) How many ways can a three-member crew be formed to complete a small job? (b) If the company is in need of a Foreman, Assistant Foreman, and Crew Chief, in how many ways can the positions be filled?

SECTION 11.6 Introduction to Probability

KEY CONCEPTS

- An event E is any designated set of sample outcomes.
- Given S is a sample space of equally likely sample outcomes and E is an event relative to S, the probability of E, written $P(E)$, is computed as $P(E) = \frac{n(E)}{n(S)}$, where $n(E)$ represents the number of elements in E, and $n(S)$ represents the number of elements in S.
- The complement of an event E is the set of sample outcomes in S, but not in E and is denoted $\sim E$.
- Given sample space S and any event E defined relative to S:
  (1) $P(\sim S) = 0$, (2) $0 \le P(E) \le 1$, (3) $P(S) = 1$, (4) $P(E) = 1 - P(\sim E)$, and
  (5) $P(E) + P(\sim E) = 1$.
- Two events that have no outcomes in common are said to be mutually exclusive.
- If two events are not mutually exclusive, $P(E_1 \text{ or } E_2) \rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$.
- If two events are mutually exclusive, $P(E_1 \text{ or } E_2) \rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

EXERCISES

49. One card is drawn from a standard deck. What is the probability the card is a ten or a face card?

50. One card is drawn from a standard deck. What is the probability the card is a Queen or a face card?

51. One die is rolled. What is the probability the result is not a three?

52. Given $P(E_1) = \frac{3}{8}$, $P(E_2) = \frac{3}{4}$, and $P(E_1 \cup E_2) = \frac{5}{6}$, compute $P(E_1 \cap E_2)$.

53. Find $P(E)$ given that $n(E) = {}_7C_4 \cdot {}_5C_3$ and $n(S) = {}_{12}C_7$.

54. To determine if more physicians should be hired, a medical clinic tracks the number of days between a patient's request for an appointment and the actual appointment date. The table given shows the probability that a patient must wait "*d*" days. Based on the table, what is the probability a patient must wait

a. at least 20 days
b. less than 20 days
c. 40 days or less
d. over 40 days
e. less than 40 and more than 10 days
f. 30 or more days

Wait (days *d*)	Probability
0	0.002
$0 < d < 10$	0.07
$10 \le d < 20$	0.32
$20 \le d < 30$	0.43
$30 \le d < 40$	0.178

SECTION 11.7 The Binomial Theorem

KEY CONCEPTS

- To expand $(a + b)^n$ for n of "moderate size," we can use Pascal's triangle and observed patterns.
- For any natural numbers n and r, where $n \ge r$, the expression $\binom{n}{r}$ (read "n choose r") is called the *binomial coefficient* and evaluated as $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.
- If n is large, it is more efficient to expand using the binomial coefficients and binomial theorem.
- The following binomial coefficients are useful/common and should be committed to memory:
$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{n-1} = n \qquad \binom{n}{n} = 1$$
- We define $0! = 1$; for example $\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{1}{0!} = \frac{1}{1} = 1$.
- The binomial theorem: $(a + b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n$.
- The kth term of $(a + b)^n$ can be found using the formula $\binom{n}{r}a^{n-r}b^r$, where $r = k - 1$.

EXERCISES

55. Evaluate each of the following:
a. $\binom{7}{5}$ **b.** $\binom{8}{3}$

56. Use Pascal's triangle to expand the binomials:
a. $(x - y)^4$ **b.** $(1 + 2i)^5$

Use the binomial theorem to:

57. Write the first four terms of
a. $(a + \sqrt{3})^8$ **b.** $(5a + 2b)^7$

58. Find the indicated term of each expansion.
a. $(x + 2y)^7$; fourth **b.** $(2a - b)^{14}$; 10th

MIXED REVIEW

1. Identify each sequence as arithmetic, geometric, or neither. If neither, try to identify the pattern that forms the sequence.
   - **a.** 120, 163, 206, 249, . . .
   - **b.** 4, 4, 4, 4, 4, 4, . . .
   - **c.** 1, 2, 6, 24, 120, 720, 5040, . . .
   - **d.** 2.00, 1.95, 1.90, 1.85, . . .
   - **e.** $\frac{5}{8}, \frac{5}{64}, \frac{5}{512}, \frac{5}{4096}, \ldots$
   - **f.** $-5.5, 6.05, -6.655, 7.3205, \ldots$
   - **g.** $0.\overline{1}, 0.\overline{2}, 0.\overline{3}, 0.\overline{4}, \ldots$
   - **h.** 525, 551.25, 578.8125, . . .
   - **i.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$

2. Compute by hand (show your work).

a. 10! **b.** $\frac{10!}{6!}$ **c.** ${}_{10}P_4$

d. ${}_{10}P_9$ **e.** ${}_{10}C_6$ **f.** ${}_{10}C_4$

3. The call letters for a television station must consist of four letters and begin with either a K or a W. How many distinct call letters are possible if repeating any letter is not allowed?

4. Given $a_1 = 9$ and $r = \frac{1}{3}$, write out the first five terms and the 15th term.

5. Given $a_1 = 0.1$ and $r = 5$, write out the first five terms and the 15th term.

6. One card is drawn from a well-shuffled deck of standard cards. What is the probability the card is a Queen or an Ace?

7. Two fair dice are rolled. What is the probability the result is not doubles (doubles = same number on both die)?

8. A house in a Boston suburb cost $185,000 in 1985. Each year its value increased by 8%. If this appreciation were to continue, find the value of the house in 2005 and 2015 using a sequence.

9. Evaluate each sum using summation formulas.

a. $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ **b.** $\sum_{n=1}^{10}(9 + 2n)$

c. $\sum_{n=1}^{5}12n + \sum_{n=1}^{5}(-5) + \sum_{n=1}^{5}n^2$

10. Expand each binomial using the binomial theorem. Simplify each term.

a. $(2x + 5)^5$ **b.** $(1 - 2i)^4$

11. For $(a + b)^n$, determine:

a. the first three terms for $n = 20$

b. the last three terms for $n = 20$

c. the fifth term for $n = 35$

d. the fifth term for $n = 35$, $a = 0.2$, and $b = 0.8$

12. On average, bears older than 3 yr old increase their weight by 0.87% per day from July to November. If a bear weighed 110 kg on June 30th: (a) identify the type of sequence that gives the bear's weight each day; (b) find the general term for the sequence; and (c) find the bear's weight on July 1, July 2, July 3, July 31, August 31, and September 30.

13. Use a proof by induction to show that $3 + 6 + 9 + \cdots + 3n = \frac{3n(n + 1)}{2}$.

14. The owner of an arts and crafts store makes specialty key rings by placing five colored beads on a nylon cord and tying it to the ring that will hold the keys. If there are eight different colors to choose from, (a) how many distinguishable key rings are possible if no colors are repeated? (b) How many distinguishable key rings are possible if a repetition of colors is allowed?

15. Donell bought 15 raffle tickets from the Inner City Children's Music School, and five tickets from the Arbor Day Everyday raffle. The Music School sold a total of 2000 tickets and the Arbor Day foundation sold 550 tickets. For E_1: Donell wins the Music School raffle and E_2: Donell wins the Arbor Day raffle, find $P(E_1 \text{ or } E_2)$.

Find the sum if it exists.

16. $\frac{1}{3} + \frac{2}{3} + 1 + \frac{4}{3} + \cdots + \frac{20}{3}$

17. $0.36 + 0.0036 + 0.000036 + 0.00000036 + \cdots$

Find the first five terms of the sequences in Exercises 18 and 19.

18. $a_n = \frac{12!}{(12 - n)!}$ **19.** $\begin{cases} a_1 = 10 \\ a_{n+1} = a_n\left(\frac{1}{5}\right) \end{cases}$

20. A random survey of 200 college students produces the data shown. One student from this group is randomly chosen for an interview. Use the data to find

a. P(student works more than 10 hr)

b. P(student takes less than 13 credit-hours)

c. P(student works more than 20 hr and takes more than 12 credit-hours)

d. P(student works between 11 and 20 hr or takes 6 to 12 credit-hours)

	0-10 hr	11-20 hr	Over 20 hr	Total
1-5 credits	3	7	10	20
6-12 credits	21	55	48	124
over 13 credits	8	28	20	56
Total:	32	90	78	200

PRACTICE TEST

1. The general term of a sequence is given. Find the first four terms, the 8th term, and the 12th term.

a. $a_n = \dfrac{2n}{n+3}$ **b.** $a_n = \dfrac{(n+2)!}{n!}$

c. $a_n = \begin{cases} a_1 = 3 \\ a_{n+1} = \sqrt{(a_n)^2 - 1} \end{cases}$

2. Expand each series and evaluate.

a. $\sum_{k=2}^{6}(2k^2 - 3)$ **b.** $\sum_{j=2}^{6}(-1)^j\left(\dfrac{j}{j+1}\right)$

c. $\sum_{j=1}^{5}(-2)\left(\dfrac{3}{4}\right)^j$ **d.** $\sum_{k=1}^{\infty}7\left(\dfrac{1}{2}\right)^k$

3. Identify the first term and the common difference or common ratio. Then find the general term a_n.

a. 7, 4, 1, −2, . . . **b.** −8, −6, −4, −2, . . .

c. 4, −8, 16, −32, . . . **d.** 10, 4, $\frac{8}{5}$, $\frac{16}{25}$, . . .

4. Find the indicated value for each sequence.

a. $a_1 = 4, d = 5$; find a_{40}

b. $a_1 = 2, a_n = -22, d = -3$; find n

c. $a_1 = 24, r = \frac{1}{2}$; find a_6

d. $a_1 = -2, a_n = 486, r = -3$; find n

5. Find the sum of each series.

a. $7 + 10 + 13 + \cdots + 100$

b. $\sum_{k=1}^{37}(3k + 2)$

c. For $4 - 12 + 36 - 108 + \cdots$, find S_7

d. $6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$

6. Each swing of a pendulum (in one direction) is 95% of the previous one. If the first swing is 12 ft, (a) find the length of the seventh swing and (b) determine the distance traveled by the pendulum for the first seven swings.

7. A rare coin that cost \$3000 appreciates in value 7% per year. Find the value of after 12 yr.

8. A car that costs \$50,000 decreases in value by 15% per year. Find the value of the car after 5 yr.

9. Use mathematical induction to prove that for $a_n = 5n - 3$, the sum formula $S_n = \dfrac{5n^2 - n}{2}$ is true for all natural numbers n.

10. Use the principle of mathematical induction to prove that S_n: $2 \cdot 3^{n-1} \le 3^n - 1$ is true for all natural numbers n.

11. Three colored balls (Aqua, Brown, and Creme) are to be drawn without replacement from a bag. List all possible ways they can be drawn using (a) a tree diagram and (b) an organized list.

12. Suppose that license plates for motorcycles must consist of three numbers followed by two letters. How many license plates are possible if zero and "Z" cannot be used and no repetition is allowed?

13. How many subsets can be formed from the elements in this set: {📪, 📫, 📬, 📭, 📨, ✉}.

14. Compute the following values by hand, showing all work: (a) 6! (b) ${}_6P_3$ (c) ${}_6C_3$

15. An English major has built a collection of rare books that includes two identical copies of *The Canterbury Tales* (Chaucer), three identical copies of *Romeo and Juliet* (Shakespeare), four identical copies of *Faustus* (Marlowe), and four identical copies of *The Faerie Queen* (Spenser). If these books are to be arranged on a shelf, how many distinguishable permutations are possible?

16. A company specializes in marketing various *cornucopia* (traditionally a curved horn overflowing with fruit, vegetables, gourds, and ears of grain) for Thanksgiving table settings. The company has seven fruit, six vegetable, five gourd, and four grain varieties available. If two from each group (without repetition) are used to fill the horn, how many different cornucopia are possible?

17. Use Pascal's triangle to expand/simplify:

a. $(x - 2y)^4$ **b.** $(1 + i)^4$

18. Use the binomial theorem to write the first three terms of (a) $(x + \sqrt{2})^{10}$ and (b) $(a - 2b^3)^8$.

19. Michael and Mitchell are attempting to make a nonstop, 100-mi trip on a tandem bicycle. The probability that Michael cannot continue pedaling for the entire trip is 0.02. The probability that Mitchell cannot continue pedaling for the entire trip is 0.018. The probability that neither one can pedal the entire trip is 0.011. What is the probability that they complete the trip?

20. The spinner shown is spun once. What is the probability of spinning

a. a striped wedge

b. a shaded wedge

c. a clear wedge

d. an even number

e. a two or an odd number

f. a number greater than nine

g. a shaded wedge *or* a number greater than 12

h. a shaded wedge *and* a number greater than 12

21. To improve customer service, a cable company tracks the number of days a customer must wait until their cable service is installed. The table shows the probability that a customer must wait d days. Based on the table, what is the probability a customer waits

Wait (days d)	Probability
0	0.02
$0 < d < 1$	0.30
$1 \le d < 2$	0.60
$2 \le d < 3$	0.05
$3 \le d < 4$	0.03

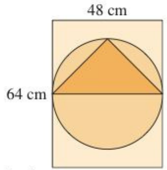
a. at least 2 days **b.** less than 2 days

c. 4 days or less **d.** over 4 days

e. less than 2 or at least 3 days

f. three or more days

22. An experienced archer can hit the rectangular target shown 100% of the time at a range of 75 m. Assuming the probability the target is hit is related to its area, what is the probability the archer hits within the



a. triangle **b.** circle

c. circle but outside the triangle

d. lower half-circle

e. rectangle but outside the circle

f. lower half-rectangle, outside the circle

23. A survey of 100 union workers was taken to register concerns to be raised at the next bargaining session. A breakdown of those surveyed is shown in the table in the right column. One out of the hundred will be selected at random for a personal interview. What is the probability the person chosen is a

Expertise Level	Women	Men	Total
Apprentice	16	18	34
Technician	15	13	28
Craftsman	9	9	18
Journeyman	7	6	13
Contractor	3	4	7
Totals	50	50	100

a. woman or a craftsman

b. man or a contractor

c. man and a technician

d. journeyman or an apprentice

24. Cheddar is a 12-year-old male box turtle. Provolone is an 8-year-old female box turtle. The probability that Cheddar will live another 8 yr is 0.85. The probability that Provolone will live another 8 yr is 0.95. Find the probability that

a. both turtles live for another 8 yr

b. neither turtle lives for another 8 yr

c. at least one of them will live another 8 yr

25. Use a proof by induction to show that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$. That is, prove $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

CALCULATOR EXPLORATION AND DISCOVERY

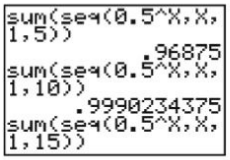
Infinite Series, Finite Results

Although there were many earlier flirtations with infinite processes, it may have been the paradoxes of Zeno of Elea (~450 B.C.) that crystallized certain questions that simultaneously frustrated and fascinated early mathematicians. The first paradox, called the dichotomy paradox, can be summarized by the following question: How can one ever finish a race, seeing that one-half the distance must first be traversed, then one-half the remaining distance, then one-half the distance that then remains, and so on an infinite number of times? Although we easily accept that races can be finished, the subtleties involved in this question stymied mathematicians for centuries and were not satisfactorily resolved until the eighteenth century. In modern notation, Zeno's first paradox says $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots < 1$. This is a geometric series with $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$.

Illustration 1 ▶ For the geometric sequence with $a_1 = \frac{1}{2}$ and $r = \dfrac{1}{2}$, the nth term is $a_n = \dfrac{1}{2^n}$. Use the "**sum(**" and "**seq(**" features of your calculator to compute S_5, S_{10}, and S_{15} (see *Technology Highlight* from Section 11.1). Does the sum appear to be approaching some "limiting value"? If so, what is this value? Now compute S_{20}, S_{25}, and S_{30}. Does there still appear to be a limit to the sum? What happens when you have the calculator compute S_{35}?

Solution ▶ CLEAR the calculator and enter sum(seq (0.5^X, X, 1, 5)) on the home screen. Pressing ENTER gives

Figure 11.21



```
sum(seq(0.5^X,X,
1,5))
             .96875
sum(seq(0.5^X,X,
1,10))
       .9990234375
sum(seq(0.5^X,X,
1,15))
```

$S_5 = 0.96875$ (Figure 11.21). Press 2nd ENTER to recall the expression and overwrite the 5, changing it to a 10. Pressing ENTER shows $S_{10} = 0.9990234375$. Repeating these steps gives $S_{15} = 0.9999694824$, and it seems that "1" may be a limiting value. Our conjecture receives further support as S_{20}, S_{25}, and S_{30} are closer and closer to 1, but do not exceed it.

Note that the sum of additional terms will create a longer string of 9's. That the sum of an infinite number of these terms *is* **1** can be understood by converting the repeating decimal $0.\overline{9}$ to its fractional form (as shown). For $x = 0.\overline{9}$, $10x = 9.\overline{9}$ and it follows that

$$\begin{aligned} 10x &= 9.\overline{9} \\ \underline{-x} &\underline{= -0.\overline{9}} \\ 9x &= 9 \\ x &= 1 \end{aligned}$$

For a geometric sequence, the result of an infinite sum can be verified using $S_\infty = \frac{a_1}{1 - r}$. However, there are many nongeometric, infinite series that also have a limiting value. In some cases these require many, many more terms before the limiting value can be observed.

Use a calculator to write the first five terms and to find S_5, S_{10}, and S_{15}. Decide if the sum appears to be approaching some limiting value, then compute S_{20} and S_{25}. Do these sums support your conjecture?

Exercise 1: $a_1 = \frac{1}{3}$ and $r = \frac{1}{3}$

Exercise 2: $a_1 = 0.2$ and $r = 0.2$

Exercise 3: $a_n = \dfrac{1}{(n - 1)!}$

Additional Insight: Zeno's first paradox can also be "resolved" by observing that the "half-steps" needed to complete the race require increasingly shorter (infinitesimally short) amounts of time. Eventually the race is complete.

STRENGTHENING CORE SKILLS

Probability, Quick-Counting, and Card Games

The card game known as *Five Card Stud* is often played for fun and relaxation, using toothpicks, beans, or pocket change as players attempt to develop a winning "hand" from the five cards dealt. The various "hands" are given here with the higher value hands listed first (e.g., a full house is a better/higher hand than a flush).

Five Card Hand	Description	Probability of Being Dealt
royal flush	five cards of the same suit in sequence from Ace to 10	0.000 001 540
straight flush	any five cards of the same suit in sequence (exclude royal)	0.000 013 900
four of a kind	four cards of the same rank, any fifth card	
full house	three cards of the same rank, with one pair	
flush	five cards of the same suit, no sequence required	0.001 970
straight	five cards in sequence, regardless of suit	
three of a kind	three cards of the same rank, any two other cards	
two pairs	two cards of the one rank, two of another rank, one other card	0.047 500
one pair	two cards of the same rank, any three others	0.422 600

For this study, we will consider the hands that are based on suit (the flushes) and the sample space to be five cards dealt from a deck of 52, or ${}_{52}C_5$.

A flush consists of five cards in the same suit, a straight consists of five cards in sequence. Let's consider that an Ace can be used as either a high card (as in 10, J, Q, K, A) or a low card (as in A, 2, 3, 4, 5). Since the dominant characteristic of a flush is its *suit,* we first consider choosing one suit of the four, then the number of ways that the straight can be formed (if needed).

Illustration 1 ▶ What is the probability of being dealt a royal flush?

Solution ▶ For a royal flush, all cards must be of one suit. Since there are four suits, it can be chosen in ${}_4C_1$ ways. A royal flush must have the cards A, K, Q, J, and 10 and once the suit has been decided, it can happen in only this (one) way or ${}_1C_1$. This means $P\,(\text{royal flush}) = \dfrac{{}_4C_1 \cdot {}_1C_1}{{}_{52}C_5} \approx 0.000\,001\,540.$

Illustration 2 ▶ What is the probability of being dealt a straight flush?

Solution ▶ Once again all cards must be of one suit, which can be chosen in ${}_4C_1$ ways. A straight flush is any five cards in sequence and once the suit has been decided, this can happen in 10 ways (Ace on down, King on down, Queen on down, and so on). By the FCP, there are ${}_4C_1 \cdot {}_{10}C_1 = 40$ ways this can happen, but *four of these will be royal flushes that are of a higher value* and must be subtracted from this total. So in the intended context we have $P(\text{straight flush}) = \dfrac{{}_4C_1 \cdot {}_{10}C_1 - 4}{{}_{52}C_5} \approx 0.000\,013\,900$

Using these examples, determine the probability of being dealt

Exercise 1: a simple flush (no royal or straight flushes)

Exercise 2: three cards of the same suit and any two other (nonsuit) cards

Exercise 3: four cards of the same suit and any one other (nonsuit) card

Exercise 4: a flush having no face cards

CUMULATIVE REVIEW CHAPTERS 1–11

1. Robot Moe is assembling memory cards for computers. At 9:00 A.M., 52 cards had been assembled. At 11:00 A.M., a total of 98 had been made. Assuming the production rate is linear

- **a.** Find the slope of this line and explain what it means in this context.
- **b.** Determine how many boards Moe can assemble in an eight-hour day.
- **c.** Find a linear equation model for this data.
- **d.** Determine the approximate time that Moe began work this morning.

2. When using a calculator to find $\sin 120°$, you get $\frac{\sqrt{3}}{2}$, yet $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \neq 120°$. Explain why.

3. Complete this table of special values for $y = \cos x$ without using a calculator.

Table for Exercise 3

x	y
0	
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	
$\frac{\pi}{2}$	
$\frac{2\pi}{3}$	
$\frac{5\pi}{6}$	
π	

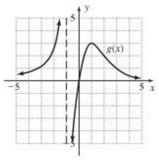
4. Sketch the graph of $y = \sqrt{x + 4} - 3$ using transformations of a parent function. Label the x- and y-intercepts and state what transformations were used.

5. Solve using the quadratic formula: $3x^2 + 5x - 7 = 0$. State your answer in exact and approximate form.

6. The orbit of Venus around the Sun is nearly circular, with a radius of 67 million miles. The planet completes one revolution in about 225 days. Calculate the planet's (a) angular velocity in radians per hour and (b) the planet's orbital velocity in miles per hour.

7. For the graph of $g(x)$ shown, state where

- **a.** $g(x) = 0$
- **b.** $g(x) < 0$
- **c.** $g(x) > 0$
- **d.** $g(x)\uparrow$
- **e.** $g(x)\downarrow$
- **f.** local max
- **g.** local min
- **h.** $g(x) = 2$
- **i.** $g(4)$
- **j.** $g(-1)$
- **k.** as $x \to -1^+$, $g(x) \to$ ______
- **l.** as $x \to \infty$, $g(x) \to$ ______
- **m.** the domain of $g(x)$



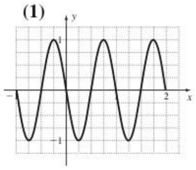
8. Match each equation to its corresponding graph.

a. $y = \sin(\pi x)$ **b.** $y = \sin(\pi x - \pi)$

c. $y = \sin(2\pi x - \pi)$ **d.** $y = \sin\left(\pi x - \frac{\pi}{2}\right)$

e. $y = \sin(2\pi x)$ **f.** $y = \sin\left(\pi x + \frac{\pi}{2}\right)$

(1)



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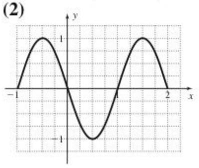

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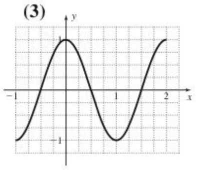

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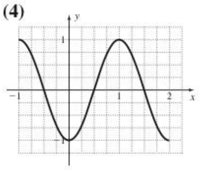

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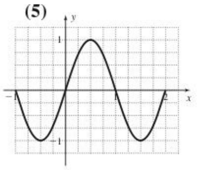
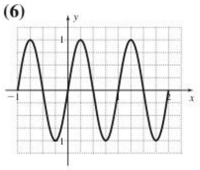


(6)



9. Graph the piecewise function and state the domain and range.

$$y = \begin{cases} -2 & -3 \le x \le -1 \\ x & -1 < x < 2 \\ x^2 & 2 \le x \le 3 \end{cases}$$

10. For $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = -12\mathbf{i} + 8\mathbf{j}$, find the resultant vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$, then use the dot product to compute the angle between $\mathbf{u}$ and $\mathbf{w}$.

11. Compute the difference quotient for each function given.

a. $f(x) = 2x^2 - 3x$ **b.** $h(x) = \dfrac{1}{x - 2}$

12. Graph the polynomial function given. Clearly indicate all intercepts. $f(x) = x^3 + x^2 - 4x - 4$

13. Graph the rational function $h(x) = \dfrac{2x^2 - 8}{x^2 - 1}$. Clearly indicate all asymptotes and intercepts.

14. Write each expression in logarithmic form:

a. $x = 10^y$ **b.** $\dfrac{1}{81} = 3^{-4}$

15. Write each expression in exponential form:

a. $3 = \log_x(125)$ **b.** $\ln(2x - 1) = 5$

16. What interest rate is required to ensure that \$2000 will double in 10 yr if interest is compounded continuously?

17. Solve for x.

a. $e^{2x-1} = 217$ **b.** $\log(3x - 2) + 1 = 4$

18. Solve using matrices and row reduction:

$$\begin{cases} 2a + 3b - 6c = 15 \\ 4a - 6b + 5c = 35 \\ 3a + 2b - 5c = 24 \end{cases}$$

19. Solve using a calculator and inverse matrices.

$$\begin{cases} 0.7x + 1.2y - 3.2z = -32.5 \\ 1.5x - 2.7y + 0.8z = -7.5 \\ 2.8x + 1.9y - 2.1z = 1.5 \end{cases}$$

20. Find the equation of the hyperbola with foci at $(-6, 0)$ and $(6, 0)$ and vertices at $(-4, 0)$ and $(4, 0)$.

21. Write $x^2 + 4y^2 - 24y + 6x + 29 = 0$ by completing the square, then identify the center, vertices, and foci.

22. Use properties of sequences to determine a_{20} and S_{20}.

a. 262144, 65536, 16384, 4096, . . .

b. $\frac{7}{8}, \frac{27}{40}, \frac{19}{40}, \frac{11}{40}, \ldots$

23. Use the difference identity for cosine to

a. verify that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ and

b. find the value of cos 15° in exact form.

24. Caleb's grandparents live in a small town that lies 125 mi away at a heading of 110°. Having just received his pilot's license, he sets out on a heading of 110° in a rented plane, traveling at 125 mph (total flight time 1 hr). Unfortunately, he forgets to account for the wind, which is coming from the northeast at 20 mph on a heading of 190°. (a) If Caleb starts out at coordinates (0, 0), what are the coordinates of his grandparent's town? (b) What are the vector coordinates of the plane 1 hr later? (c) How many miles is he actually away from his grandparent's town?

25. Empty 55-gal drums are stacked at a storage facility in the form of a pyramid with 52 barrels in the bottom row, 51 barrels in the next row, and so on, until there are 10 barrels in the top row. Use properties of sequences to determine how many barrels are in this stack.

26. Three \$20 bills, six \$50 bills, and four \$100 bills are placed in a large box and mixed thoroughly, then two bills are drawn out and placed in a savings account. What is the probability the bills drawn are

a. first \$20, second \$50
b. first \$50, second \$20
c. both \$100
d. first \$100, second not \$20
e. first \$100, second not \$50
f. first not \$20, second \$20

27. The manager of Tom's Tool and Equipment Rentals knows that 4% of all tools rented are returned late. Of the 12 tools rented in the last hour, what is the probability that

a. exactly ten will be returned on time
b. at least eleven will be returned on time
c. at least ten will be returned on time
d. none of them will be returned on time

28. Use a proof by induction to show $3 + 7 + 11 + 15 + \cdots + (4n - 1) = n(2n + 1)$ for all natural numbers n.

29. State the three double angle formulas for cosine. If $\cos(2\theta) = \frac{1}{2}$, what is the value of $\sin\theta$?

30. A park ranger tracks the number of campers at a remote national park from January ($m = 1$) to December ($m = 12$) and collects the following data (month, number of campers): (3, 6), (5, 110), (7, 134), and (9, 78). Assuming the data is quadratic ($y = ax^2 + bx + c$), (a) select any of the three points and create a system of three equations in three variables to obtain a parabolic equation model for the data and (b) determine the month that brought the maximum number of campers. (c) What was this maximum number? (d) How many campers might be expected in September? (e) Based on your model, what month(s) is the park apparently closed to campers (number of campers is zero or negative)?