Relations, Functions, and Graphs

CHAPTER OUTLINE

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CHAPTER CONNECTIONS

Viewing a function in terms of an equation, a table of values, and the related graph, often brings a clearer understanding of the relationships involved. For example, the power generated by a wind turbine is often modeled by the function $P(v) = \frac{8v^3}{125}$, where P is the power in watts and v is the wind velocity in miles per hour. While the formula enables us to predict the power generated for a given wind speed, the graph offers a visual representation of this relationship, where we note a rapid growth in power output as the wind speed increases. This application appears as Exercise 107 in Section 2.6.

Check out these other real-world connections:

- Earthquake Area (Section 2.1, Exercise 84)
- Height of an Arrow (Section 2.5, Exercise 61)
- Garbage Collected per Number of Garbage Trucks (Section 2.2, Exercise 42)
- Number of People Connected to the Internet (Section 2.3, Exercise 109)

2.1 Rectangular Coordinates; Graphing Circles and Other Relations

Learning Objectives

In Section 2.1 you will learn how to:

- ☐ **A.** Express a relation in mapping notation and ordered pair form
- ☐ **B.** Graph a relation
- ☐ **C.** Develop the equation of a circle using the distance and midpoint formulas
- ☐ **D.** Graph circles

WORTHY OF NOTE

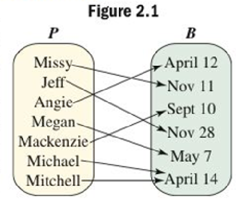
From a purely practical standpoint, we note that while it is possible for two different people to share the same birthday, it is quite impossible for the same person to have two different birthdays. Later, this observation will help us mark the difference between a relation and a function.

In everyday life, we encounter a large variety of relationships. For instance, the time it takes us to get to work is related to our average speed; the monthly cost of heating a home is related to the average outdoor temperature; and in many cases, the amount of our charitable giving is related to changes in the cost of living. In each case we say that a relation exists between the two quantities.

A. Relations, Mapping Notation, and Ordered Pairs

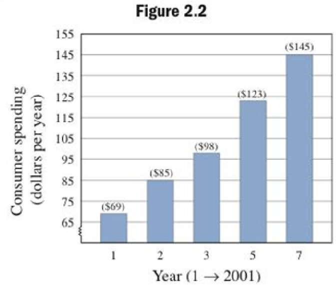
In the most general sense, a **relation** is simply a correspondence between two sets. Relations can be represented in many different ways and may even be very "unmathematical," like the one shown in Figure 2.1 between a set of people and the set of their corresponding birthdays. If P represents the set of people and B represents the set of birthdays, we say that elements of P correspond to elements of B, or the birthday relation maps elements of P to elements of B. Using what is called **mapping notation,** we might simply write $P \rightarrow B$.

Figure 2.1



The bar graph in Figure 2.2 is also an example of a relation. In the graph, each year is related to average annual consumer spending on Internet media (music downloads, Internet radio, Web-based news articles, etc.). As an alternative to mapping or a bar graph, the relation could also be represented using **ordered pairs.** For example, the ordered pair (3, 98) would indicate that in 2003, spending per person on Internet media averaged \$98 in the United States. Over a long period of time, we could collect many ordered pairs of the form (t, s), where consumer spending s *depends* on the time t. For this reason we often call the second coordinate of an ordered pair (in this case s) the **dependent variable,** with the first coordinate designated as the **independent variable.** In this form, the set of all first coordinates is called the **domain** of the relation. The set of all second coordinates is called the **range.**

Figure 2.2

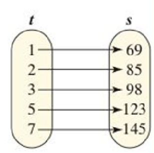


Source: 2006 Statistical Abstract of the United States

EXAMPLE 1 ► **Expressing a Relation as a Mapping and in Ordered Pair Form**

Represent the relation from Figure 2.2 in mapping notation and ordered pair form, then state its domain and range.

Solution ► Let t represent the year and s represent consumer spending. The mapping $t \rightarrow s$ gives the diagram shown. In ordered pair form we have (1, 69), (2, 85), (3, 98), (5, 123), and (7, 145). The domain is {1, 2, 3, 5, 7}, the range is {69, 85, 98, 123, 145}.



Now try Exercises 7 through 12 ►

☑ **A. You've just learned how to express a relation in mapping notation and ordered pair form**

For more on this relation, **see Exercise 81.**

Table 2.1 $y = x - 1$

x	y
−4	−5
−2	−3
0	−1
2	1
4	3

Table 2.2 $x = |y|$

x	y
2	−2
1	−1
0	0
1	1
2	2

B. The Graph of a Relation

Relations can also be stated in **equation form.** The equation $y = x - 1$ expresses a relation where each y-value is one less than the corresponding x-value (see Table 2.1). The equation $x = |y|$ expresses a relation where each x-value corresponds to the absolute value of y (see Table 2.2). In each case, the relation is the set of all ordered pairs (x, y) that create a true statement when substituted, and a few ordered pair solutions are shown in the tables for each equation.

Relations can be expressed graphically using a **rectangular coordinate system.** It consists of a horizontal number line (the x-axis) and a vertical number line (the y-axis) intersecting at their zero marks. The point of intersection is called the *origin*. The x- and y-axes create a flat, two-dimensional surface called the ***xy*-plane** and divide the plane into four regions called **quadrants.** These are labeled using a capital "Q" (for quadrant) and the Roman numerals I through IV, beginning in the upper right and moving counterclockwise (Figure 2.3). The **grid lines** shown denote the integer values on each axis and further divide the plane into a **coordinate grid,** where every point in the plane corresponds to an ordered pair. Since a point at the origin has not moved along either axis, it has coordinates (0, 0). To plot a point (x, y) means we place a dot at its location in the xy-plane. A few of the ordered pairs from $y = x - 1$ are plotted in Figure 2.4, where a noticeable pattern emerges—the points seem to lie along a straight line.

Figure 2.3

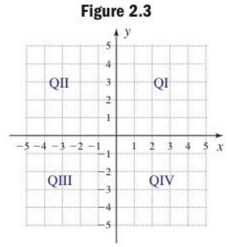
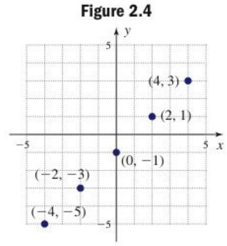


Figure 2.4



If a relation is defined by a set of ordered pairs, the graph of the relation is simply the plotted points. The graph of a relation *in equation form,* such as $y = x - 1$, is the set of *all* ordered pairs (x, y) that make the equation true. We generally use only a few select points to determine the shape of a graph, then draw a straight line or smooth curve through these points, as indicated by any patterns formed.

EXAMPLE 2 ► **Graphing Relations**

Graph the relations $y = x - 1$ and $x = |y|$ using the ordered pairs given earlier.

Solution ► For $y = x - 1$, we plot the points then connect them with a straight line (Figure 2.5). For $x = |y|$, the plotted points form a V-shaped graph made up of two half lines (Figure 2.6).

Figure 2.5

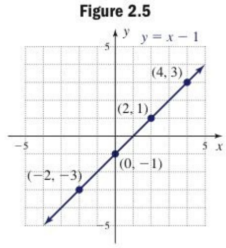
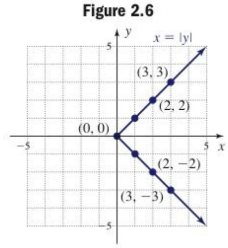


Figure 2.6



Now try Exercises 13 through 16 ►

WORTHY OF NOTE

As the graphs in Example 2 indicate, arrowheads are used where appropriate to indicate the infinite extension of a graph.

While we used only a few points to graph the relations in Example 2, they are actually made up of an *infinite number of ordered pairs* that satisfy each equation, including those that might be rational or irrational. All of these points together make these graphs **continuous,** which for our purposes means you can draw the entire graph without lifting your pencil from the paper.

Actually, a majority of graphs cannot be drawn using only a straight line or directed line segments. In these cases, we rely on a "sufficient number" of points to outline the basic shape of the graph, then connect the points with a smooth curve. As your experience with graphing increases, this "sufficient number of points" tends to get smaller as you learn to anticipate what the graph of a given relation should look like.

EXAMPLE 3 ► Graphing Relations

Graph the following relations by completing the tables given.

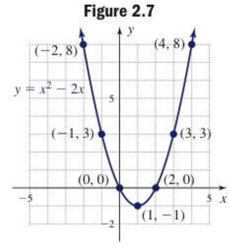
a. $y = x^2 - 2x$ **b.** $y = \sqrt{9 - x^2}$ **c.** $x = y^2$

Solution ► For each relation, we use each x-input in turn to determine the related y-output(s), if they exist. Results can be entered in a table and the ordered pairs used to draw the graph.

a. $y = x^2 - 2x$

x	y	(x, y) Ordered Pairs
−4	24	(−4, 24)
−3	15	(−3, 15)
−2	8	(−2, 8)
−1	3	(−1, 3)
0	0	(0, 0)
1	−1	(1, −1)
2	0	(2, 0)
3	3	(3, 3)
4	8	(4, 8)

Figure 2.7

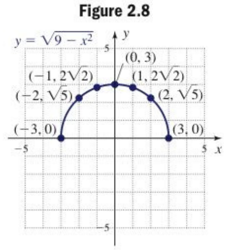


The result is a fairly common graph (Figure 2.7), called a **vertical parabola.** Although (−4, 24) and (−3, 15) cannot be plotted here, the arrowheads indicate an infinite extension of the graph, which will include these points.

b. $y = \sqrt{9 - x^2}$

x	y	(x, y) Ordered Pairs
−4	not real	—
−3	0	(−3, 0)
−2	$\sqrt{5}$	$(-2, \sqrt{5})$
−1	$2\sqrt{2}$	$(-1, 2\sqrt{2})$
0	3	(0, 3)
1	$2\sqrt{2}$	$(1, 2\sqrt{2})$
2	$\sqrt{5}$	$(2, \sqrt{5})$
3	0	(3, 0)
4	not real	—

Figure 2.8



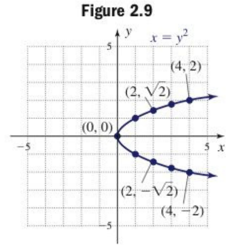
The result is the graph of a **semicircle** (Figure 2.8). The points with irrational coordinates were graphed by estimating their location. Note that when $x < -3$ or $x > 3$, the relation $y = \sqrt{9 - x^2}$ does not represent a real number and no points can be graphed. Also note that no arrowheads are used since the graph terminates at $(-3, 0)$ and $(3, 0)$.

c. Similar to $x = |y|$, the relation $x = y^2$ is defined only for $x \geq 0$ since y^2 is always nonnegative ($-1 = y^2$ has no real solutions). In addition, we reason that each positive x-value will correspond to two y-values. For example, given $x = 4$, $(4, -2)$ and $(4, 2)$ are both solutions.

$x = y^2$

x	y	(x, y) Ordered Pairs
-2	not real	—
-1	not real	—
0	0	(0, 0)
1	$-1, 1$	$(1, -1)$ and $(1, 1)$
2	$-\sqrt{2}, \sqrt{2}$	$(2, -\sqrt{2})$ and $(2, \sqrt{2})$
3	$-\sqrt{3}, \sqrt{3}$	$(3, -\sqrt{3})$ and $(3, \sqrt{3})$
4	$-2, 2$	$(4, -2)$ and $(4, 2)$

Figure 2.9



This is the graph of a **horizontal parabola** (Figure 2.9).

B. You've just learned how to graph a relation

Now try Exercises 17 through 24 ►

C. The Equation of a Circle

Using the midpoint and distance formulas, we can develop the equation of another very important relation, that of a circle. As the name suggests, the **midpoint of a line segment** is located halfway between the endpoints. On a standard number line, the midpoint of the line segment with endpoints 1 and 5 is 3, but more important, note that 3 is the **average distance** (from zero) of 1 unit and 5 units: $\frac{1 + 5}{2} = \frac{6}{2} = 3$. This observation can be extended to find the midpoint between any two points (x_1, y_1) and (x_2, y_2). We simply find the average distance between the x-coordinates and the average distance between the y-coordinates.

The Midpoint Formula

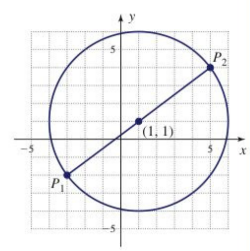
Given any line segment with endpoints $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, the midpoint M is given by

$$M: \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The midpoint formula can be used in many different ways. Here we'll use it to find the coordinates of the center of a circle.

EXAMPLE 4 ► Using the Midpoint Formula

The diameter of a circle has endpoints at $P_1 = (-3, -2)$ and $P_2 = (5, 4)$. Use the midpoint formula to find the coordinates of the center, then plot this point.



Solution ► Midpoint: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

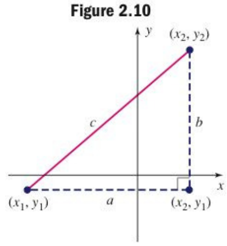
$$M: \left(\frac{-3 + 5}{2}, \frac{-2 + 4}{2}\right)$$

$$M: \left(\frac{2}{2}, \frac{2}{2}\right) = (1, 1)$$

The center is at (1, 1), which we graph directly on the diameter as shown.

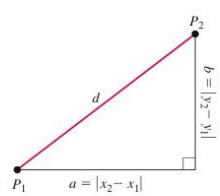
Now try Exercises 25 through 34 ►

Figure 2.10



The Distance Formula

In addition to a line segment's midpoint, we are often interested in the *length* of the segment. For any two points (x_1, y_1) and (x_2, y_2) not lying on a horizontal or vertical line, a right triangle can be formed as in Figure 2.10. Regardless of the triangle's orientation, the length of side a (the horizontal segment or base of the triangle) will have length $|x_2 - x_1|$ units, with side b (the vertical segment or height) having length $|y_2 - y_1|$ units. From the Pythagorean theorem (Section R.6), we see that $c^2 = a^2 + b^2$ corresponds to $c^2 = (|x_2 - x_1|)^2 + (|y_2 - y_1|)^2$. By taking the square root of both sides we obtain the length of the hypotenuse, *which is identical to the distance between these two points*: $c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The result is called the **distance formula,** although it's most often written using ***d*** for **d**istance, rather than c. Note the absolute value bars are dropped from the formula, since the square of any quantity is always nonnegative. This also means that *either* point can be used as the initial point in the computation.



The Distance Formula

Given any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, the straight line distance between them is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 5 ► Using the Distance Formula

Use the distance formula to find the diameter of the circle from Example 4.

Solution ► For $(x_1, y_1) = (-3, -2)$ and $(x_2, y_2) = (5, 4)$, the distance formula gives

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[5 - (-3)]^2 + [4 - (-2)]^2} \\ &= \sqrt{8^2 + 6^2} \\ &= \sqrt{100} = 10 \end{aligned}$$

The diameter of the circle is 10 units long.

Now try Exercises 35 through 38 ►

EXAMPLE 6 ▶ Determining if Three Points Form a Right Triangle

Use the distance formula to determine if the following points are the vertices of a right triangle: $(-8, 1)$, $(-2, 9)$, and $(10, 0)$

Solution ▶ We begin by finding the distance between each pair of points, then attempt to apply the Pythagorean theorem.

For $(x_1, y_1) = (-8, 1), (x_2, y_2) = (-2, 9)$:

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[-2 - (-8)]^2 + (9 - 1)^2} \\ &= \sqrt{6^2 + 8^2} \\ &= \sqrt{100} = 10 \end{aligned}$$

For $(x_2, y_2) = (-2, 9), (x_3, y_3) = (10, 0)$:

$$\begin{aligned} d &= \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} \\ &= \sqrt{[10 - (-2)]^2 + (0 - 9)^2} \\ &= \sqrt{12^2 + (-9)^2} \\ &= \sqrt{225} = 15 \end{aligned}$$

For $(x_1, y_1) = (-8, 1), (x_3, y_3) = (10, 0)$:

$$\begin{aligned} d &= \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} \\ &= \sqrt{[10 - (-8)]^2 + (0 - 1)^2} \\ &= \sqrt{18^2 + (-1)^2} \\ &= \sqrt{325} = 5\sqrt{13} \end{aligned}$$

Using the unsimplified form, we clearly see that $a^2 + b^2 = c^2$ corresponds to $(\sqrt{100})^2 + (\sqrt{225})^2 = (\sqrt{325})^2$, a true statement. Yes, the triangle is a right triangle.

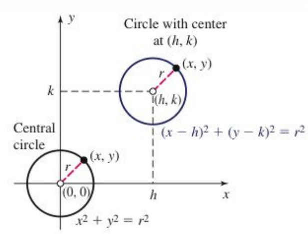
Now try Exercises 39 through 44 ▶

A circle can be defined as the set of all points in a plane that are a *fixed distance* called the **radius,** from a *fixed point* called the **center.** Since the definition involves *distance,* we can construct the general equation of a circle using the distance formula. Assume the center has coordinates (h, k), and let (x, y) represent any point on the graph. Since the distance between these points is equal to the radius r, the distance formula yields: $\sqrt{(x-h)^2 + (y-k)^2} = r$. Squaring both sides gives the equation of a circle in **standard form:** $(x-h)^2 + (y-k)^2 = r^2$.

The Equation of a Circle

A circle of radius r with center at (h, k) has the equation $(x-h)^2 + (y-k)^2 = r^2$

If $h = 0$ and $k = 0$, the circle is centered at $(0, 0)$ and the graph is a **central circle** with equation $x^2 + y^2 = r^2$. At other values for h or k, the center is at (h, k) with no change in the radius. Note that an open dot is used for the center, as it's actually a point of reference and not a part of the actual graph.



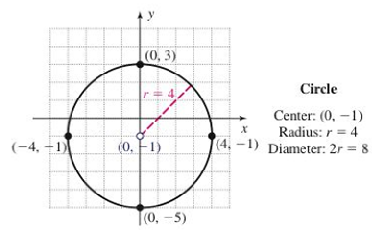
EXAMPLE 7 ▶ Finding the Equation of a Circle

Find the equation of a circle with center $(0, -1)$ and radius 4.

Solution ▶ Since the center is at $(0, -1)$ we have $h = 0$, $k = -1$, and $r = 4$. Using the standard form $(x-h)^2 + (y-k)^2 = r^2$ we obtain

$$\begin{aligned} (x-0)^2 + [y-(-1)]^2 &= 4^2 && \text{substitute 0 for } h, -1 \text{ for } k, \text{ and 4 for } r \\ x^2 + (y+1)^2 &= 16 && \text{simplify} \end{aligned}$$

The graph of $x^2 + (y + 1)^2 = 16$ is shown in the figure.



C. You've just learned how to develop the equation of a circle using the distance and midpoint formulas

Now try Exercises 45 through 62 ►

D. The Graph of a Circle

The graph of a circle can be obtained by first identifying the coordinates of the center and the length of the radius from the equation in standard form. After plotting the center point, we count a distance of r units left and right of center in the horizontal direction, and up and down from center in the vertical direction, obtaining four points on the circle. Neatly graph a circle containing these four points.

EXAMPLE 8 ► **Graphing a Circle**

Graph the circle represented by $(x - 2)^2 + (y + 3)^2 = 12$. Clearly label the center and radius.

Solution ► Comparing the given equation with the standard form, we find the center is at $(2, -3)$ and the radius is $r = 2\sqrt{3} \approx 3.5$.

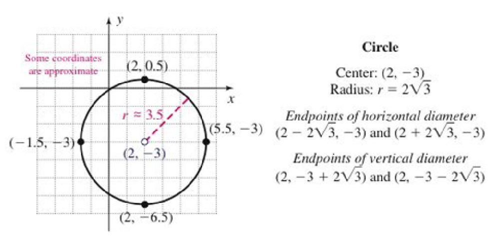
$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{standard form}$$

$$(x - 2)^2 + (y + 3)^2 = 12 \quad \text{given equation}$$

$$-h = -2 \qquad -k = 3 \qquad r^2 = 12$$

$$h = 2 \qquad k = -3 \qquad r = \sqrt{12} = 2\sqrt{3} \approx 3.5 \quad \text{radius must be positive}$$

Plot the center $(2, -3)$ and count approximately 3.5 units in the horizontal and vertical directions. Complete the circle by freehand drawing or using a compass. The graph shown is obtained.



Now try Exercises 63 through 68 ►

WORTHY OF NOTE

After writing the equation in standard form, it is possible to end up with a constant that is zero or negative. In the first case, the graph is a single point. In the second case, no graph is possible since roots of the equation will be complex numbers. These are called *degenerate cases*. **See Exercise 91.**

In Example 8, note the equation is composed of binomial squares in both x and y. By expanding the binomials and collecting like terms, we can write the equation of the circle in the general form:

$$(x - 2)^2 + (y + 3)^2 = 12 \quad \text{standard form}$$
$$x^2 - 4x + 4 + y^2 + 6y + 9 = 12 \quad \text{expand binomials}$$
$$x^2 + y^2 - 4x + 6y + 1 = 0 \quad \text{combine like terms—general form}$$

For future reference, observe the general form contains a *sum* of second-degree terms in x and y, and that *both terms have the same coefficient* (in this case, "1").

Since this form of the equation was derived by squaring binomials, it seems reasonable to assume we can go back to the standard form by creating binomial squares in x and y. This is accomplished by *completing the square*.

EXAMPLE 9 ► Finding the Center and Radius of a Circle

Find the center and radius of the circle with equation $x^2 + y^2 + 2x - 4y - 4 = 0$. Then sketch its graph and label the center and radius.

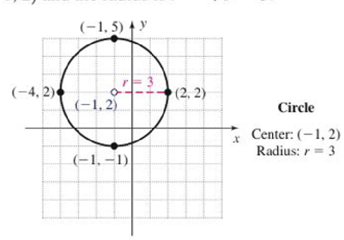
Solution ► To find the center and radius, we complete the square in both x and y.

$$x^2 + y^2 + 2x - 4y - 4 = 0 \quad \text{given equation}$$
$$(x^2 + 2x + \_\_) + (y^2 - 4y + \_\_) = 4 \quad \text{group } x\text{-terms and } y\text{-terms; add 4}$$
$$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 4 + 1 + 4 \quad \text{complete each binomial square}$$

adds 1 to left side; adds 4 to left side; add 1 + 4 to right side

$$(x + 1)^2 + (y - 2)^2 = 9 \quad \text{factor and simplify}$$

The center is at $(-1, 2)$ and the radius is $r = \sqrt{9} = 3$.



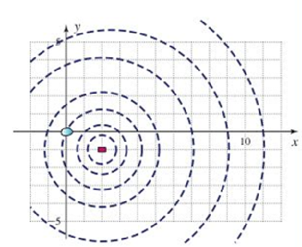
Now try Exercises 69 through 80 ►

EXAMPLE 10 ► Applying the Equation of a Circle

To aid in a study of nocturnal animals, some naturalists install a motion detector near a popular watering hole. The device has a range of 10 m in any direction. Assume the water hole has coordinates (0, 0) and the device is placed at $(2, -1)$.

a. Write the equation of the circle that models the maximum effective range of the device.

b. Use the distance formula to determine if the device will detect a badger that is approaching the water and is now at coordinates $(11, -5)$.



Solution ► **a.** Since the device is at $(2, -1)$ and the radius (or reach) of detection is 10 m, any movement in the interior of the circle defined by $(x - 2)^2 + (y + 1)^2 = 10^2$ will be detected.

b. Using the points $(2, -1)$ and $(11, -5)$ in the distance formula yields:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{distance formula}$$
$$= \sqrt{(11 - 2)^2 + [-5 - (-1)]^2} \quad \text{substitute given values}$$
$$= \sqrt{9^2 + (-4)^2} \quad \text{simplify}$$
$$= \sqrt{81 + 16} \quad \text{compute squares}$$
$$= \sqrt{97} \approx 9.85 \quad \text{result}$$

Since $9.85 < 10$, the badger is within range of the device and will be detected.

D. You've just learned how to graph circles

Now try Exercises 83 through 88 ►

TECHNOLOGY HIGHLIGHT

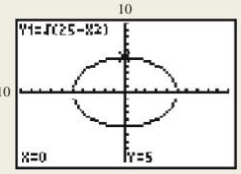
The Graph of a Circle

When using a graphing calculator to study circles, it is important to keep two things in mind. First, we must modify the equation of the circle before it can be graphed using this technology. Second, most standard viewing windows have the *x*- and *y*-values preset at $[-10, 10]$ even though the calculator screen is not square. This tends to compress the *y*-values and give a skewed image of the graph. Consider the *relation* $x^2 + y^2 = 25$, which we know is the equation of a circle centered at (0, 0) with radius $r = 5$. To enable the calculator to graph this relation, we must define it in two pieces by solving for *y*:

$$x^2 + y^2 = 25 \quad \text{original equation}$$
$$y^2 = 25 - x^2 \quad \text{isolate } y^2$$
$$y = \pm\sqrt{25 - x^2} \quad \text{solve for } y$$

Figure 2.11

10

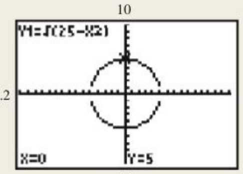


−10 10 −10

Note that we can separate this result into two parts, enabling the calculator to draw the circle: $Y_1 = \sqrt{25 - x^2}$ gives the "upper half" of the circle, and $Y_2 = -\sqrt{25 - x^2}$ gives the "lower half." Enter these on the Y= screen (note that $Y_2 = -Y_1$ can be used instead of reentering the entire expression: VARS ► ENTER). But if we graph Y_1 and Y_2 on the standard screen, the result appears more oval than circular (Figure 2.11). One way to fix this is to use the ZOOM **5:ZSquare** option, which places the tick marks equally spaced on both axes, instead of trying to force both to display points from −10 to 10 (see Figure 2.12). Although it is a much improved graph, the circle does not appear "closed" as the calculator lacks sufficient pixels to show the proper curvature. A second alternative is to manually set a "friendly" window. Using Xmin = −9.4, Xmax = 9.4, Ymin = −6.2, and Ymax = 6.2 will generate a better graph, which we can use to study the relation more closely. Note that we can jump between the upper and lower halves of the circle using the up ▲ or down ▼ arrows.

Figure 2.12

10



−15.2 15.2 −10

Exercise 1: Graph the circle defined by $x^2 + y^2 = 36$ using a friendly window, then use the TRACE feature to find the value of *y* when $x = 3.6$. Now find the value of *y* when $x = 4.8$. Explain why the values seem "interchangeable."

Exercise 2: Graph the circle defined by $(x - 3)^2 + y^2 = 16$ using a friendly window, then use the TRACE feature to find the value of the *y*-intercepts. Show you get the same intercept by computation.

2.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. If a relation is defined by a set of ordered pairs, the domain is the set of all ______ components, the range is the set of all ______ components.

2. For the equation $y = x + 5$ and the ordered pair (x, y), x is referred to as the input or ______ variable, while y is called the ______ or dependent variable.

3. A circle is defined as the set of all points that are an equal distance, called the ______, from a given point, called the ______.

4. For $x^2 + y^2 = 25$, the center of the circle is at ______ and the length of the radius is ______ units. The graph is called a ______ circle.

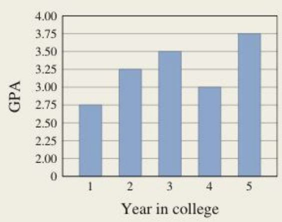
5. Discuss/Explain how to find the center and radius of the circle defined by the equation $x^2 + y^2 - 6x = 7$. How would this circle differ from the one defined by $x^2 + y^2 - 6y = 7$?

6. In Example 3b we graphed the semicircle defined by $y = \sqrt{9 - x^2}$. Discuss how you would obtain the equation of the full circle from this equation, and how the two equations are related.

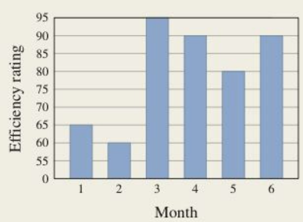
▶ DEVELOPING YOUR SKILLS

Represent each relation in mapping notation, then state the domain and range.

7.



8.



State the domain and range of each relation.

9. $\{(1, 2), (3, 4), (5, 6), (7, 8), (9, 10)\}$

10. $\{(-2, 4), (-3, -5), (-1, 3), (4, -5), (2, -3)\}$

11. $\{(4, 0), (-1, 5), (2, 4), (4, 2), (-3, 3)\}$

12. $\{(-1, 1), (0, 4), (2, -5), (-3, 4), (2, 3)\}$

Complete each table using the given equation. For Exercises 15 and 16, each input may correspond to two outputs (be sure to find both if they exist). Use these points to graph the relation.

13. $y = -\frac{2}{3}x + 1$

x	y
−6	
−3	
0	
3	
6	
8	

14. $y = -\frac{5}{4}x + 3$

x	y
−8	
−4	
0	
4	
8	
10	

15. $x + 2 = |y|$

x	y
−2	
0	
1	
3	
6	
7	

16. $|y + 1| = x$

x	y
0	
1	
3	
5	
6	
7	

17. $y = x^2 - 1$

x	y
−3	
−2	
0	
2	
3	
4	

18. $y = -x^2 + 3$

x	y
−2	
−1	
0	
1	
2	
3	

19. $y = \sqrt{25 - x^2}$

x	y
−4	
−3	
0	
2	
3	
4	

20. $y = \sqrt{169 - x^2}$

x	y
−12	
−5	
0	
3	
5	
12	

21. $x - 1 = y^2$

x	y
10	
5	
4	
2	
1.25	
1	

22. $y^2 + 2 = x$

x	y
2	
3	
4	
5	
6	
11	

23. $y = \sqrt[3]{x + 1}$

x	y
−9	
−2	
−1	
0	
4	
7	

24. $y = (x - 1)^3$

x	y
−2	
−1	
0	
1	
2	
3	

Find the midpoint of each segment with the given endpoints.

25. (1, 8), (5, −6)

26. (5, 6), (6, −8)

27. (−4.5, 9.2), (3.1, −9.8)

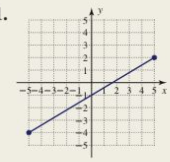
28. (5.2, 7.1), (6.3, −7.1)

29. $\left(\frac{1}{5}, -\frac{2}{3}\right), \left(-\frac{1}{10}, \frac{3}{4}\right)$

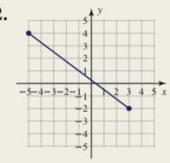
30. $\left(-\frac{3}{4}, -\frac{1}{3}\right), \left(\frac{3}{8}, \frac{5}{6}\right)$

Find the midpoint of each segment.

31.

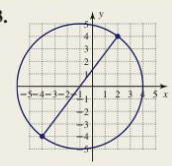


32.

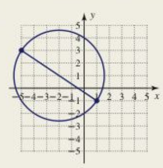


Find the center of each circle with the diameter shown.

33.



34.



35. Use the distance formula to find the length of the line segment in Exercise 31.

36. Use the distance formula to find the length of the line segment in Exercise 32.

37. Use the distance formula to find the length of the diameter for the circle in Exercise 33.

38. Use the distance formula to find the length of the diameter for the circle in Exercise 34.

In Exercises 39 to 44, three points that form the vertices of a triangle are given. Use the distance formula to determine if any of the triangles are right triangles.

39. (5, 2), (0, −3), (4, −4)

40. (7, 0), (−1, 0), (7, 4)

41. (−4, 3), (−7, −1), (3, −2)

42. (−3, 7), (2, 2), (5, 5)

43. (−3, 2), (−1, 5), (−6, 4)

44. (0, 0), (−5, 2), (2, −5)

Find the equation of a circle satisfying the conditions given, then sketch its graph.

45. center (0, 0), radius 3

46. center (0, 0), radius 6

47. center (5, 0), radius $\sqrt{3}$

48. center (0, 4), radius $\sqrt{5}$

49. center (4, −3), radius 2

50. center (3, −8), radius 9

51. center (−7, −4), radius $\sqrt{7}$

52. center $(-2, -5)$, radius $\sqrt{6}$

53. center $(1, -2)$, diameter 6

54. center $(-2, 3)$, diameter 10

55. center $(4, 5)$, diameter $4\sqrt{3}$

56. center $(5, 1)$, diameter $4\sqrt{5}$

57. center at $(7, 1)$, graph contains the point $(1, -7)$

58. center at $(-8, 3)$, graph contains the point $(-3, 15)$

59. center at $(3, 4)$, graph contains the point $(7, 9)$

60. center at $(-5, 2)$, graph contains the point $(-1, 3)$

61. diameter has endpoints $(5, 1)$ and $(5, 7)$

62. diameter has endpoints $(2, 3)$ and $(8, 3)$

Identify the center and radius of each circle, then graph. Also state the domain and range of the relation.

63. $(x - 2)^2 + (y - 3)^2 = 4$

64. $(x - 5)^2 + (y - 1)^2 = 9$

65. $(x + 1)^2 + (y - 2)^2 = 12$

66. $(x - 7)^2 + (y + 4)^2 = 20$

67. $(x + 4)^2 + y^2 = 81$

68. $x^2 + (y - 3)^2 = 49$

Write each equation in standard form to find the center and radius of the circle. Then sketch the graph.

69. $x^2 + y^2 - 10x - 12y + 4 = 0$

70. $x^2 + y^2 + 6x - 8y - 6 = 0$

71. $x^2 + y^2 - 10x + 4y + 4 = 0$

72. $x^2 + y^2 + 6x + 4y + 12 = 0$

73. $x^2 + y^2 + 6y - 5 = 0$

74. $x^2 + y^2 - 8x + 12 = 0$

75. $x^2 + y^2 + 4x + 10y + 18 = 0$

76. $x^2 + y^2 - 8x - 14y - 47 = 0$

77. $x^2 + y^2 + 14x + 12 = 0$

78. $x^2 + y^2 - 22y - 5 = 0$

79. $2x^2 + 2y^2 - 12x + 20y + 4 = 0$

80. $3x^2 + 3y^2 - 24x + 18y + 3 = 0$

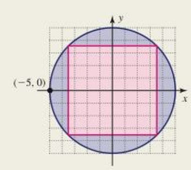
► WORKING WITH FORMULAS

81. Spending on Internet media: $s = 12.5t + 59$

The data from Example 1 is closely modeled by the formula shown, where t represents the year ($t = 0$ corresponds to the year 2000) and s represents the average amount spent per person, per year in the United States. (a) List five ordered pairs for this relation using $t = 1, 2, 3, 5, 7$. Does the model give a good approximation of the actual data? (b) According to the model, what will be the average amount spent on Internet media in the year 2008? (c) According to the model, in what year will annual spending surpass \$196? (d) Use the table to graph this relation.

82. Area of an inscribed square: $A = 2r^2$

The area of a square inscribed in a circle is found by using the formula given where r is the radius of the circle. Find the area of the inscribed square shown.

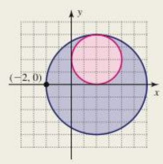


► APPLICATIONS

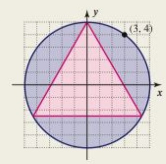
83. Radar detection: A luxury liner is located at map coordinates $(5, 12)$ and has a radar system with a range of 25 nautical miles in any direction. (a) Write the equation of the circle that models the range of the ship's radar, and (b) Use the distance formula to determine if the radar can pick up the liner's sister ship located at coordinates $(15, 36)$.

84. Earthquake range: The epicenter (point of origin) of a large earthquake was located at map coordinates $(3, 7)$, with the quake being felt up to 12 mi away. (a) Write the equation of the circle that models the range of the earthquake's effect. (b) Use the distance formula to determine if a person living at coordinates $(13, 1)$ would have felt the quake.

85. Inscribed circle: Find the equation for both the red and blue circles, then find the area of the region shaded in blue.



86. Inscribed triangle: The area of an equilateral triangle inscribed in a circle is given by the formula $A = \frac{3\sqrt{3}}{4}r^2$, where r is the radius of the circle. Find the area of the equilateral triangle shown.



87. Radio broadcast range: Two radio stations may not use the same frequency if their broadcast areas *overlap.* Suppose station KXRQ has a broadcast area bounded by $x^2 + y^2 + 8x - 6y = 0$ and WLRT has a broadcast area bounded by $x^2 + y^2 - 10x + 4y = 0$. Graph the circle representing each broadcast area on the same grid to determine if both stations may broadcast on the same frequency.

88. Radio broadcast range: The emergency radio broadcast system is designed to alert the population by relaying an emergency signal to all points of the country. A signal is sent from a station whose broadcast area is bounded by $x^2 + y^2 = 2500$ (x and y in miles) and the signal is picked up and relayed by a transmitter with range $(x - 20)^2 + (y - 30)^2 = 900$. Graph the circle representing each broadcast area on the same grid to determine the greatest distance from the original station that this signal can be received. Be sure to scale the axes appropriately.

► EXTENDING THE THOUGHT

89. Although we use the word "domain" extensively in mathematics, it is also commonly seen in literature and heard in everyday conversation. Using a college-level dictionary, look up and write out the various meanings of the word, noting how closely the definitions given are related to its mathematical use.

90. Consider the following statement, then determine whether it is true or false and discuss why. *A graph will exhibit some form of symmetry if, given a point that is h units from the x-axis, k units from the y-axis, and d units from the origin, there is a second point on the graph that is a like distance from the origin and each axis.*

91. When completing the square to find the center and radius of a circle, we sometimes encounter a value for r^2 that is negative or zero. These are called **degenerate cases.** If $r^2 < 0$, no circle is possible, while if $r^2 = 0$, the "graph" of the circle is simply the point (h, k). Find the center and radius of the following circles (if possible).

a. $x^2 + y^2 - 12x + 4y + 40 = 0$

b. $x^2 + y^2 - 2x - 8y - 8 = 0$

c. $x^2 + y^2 - 6x - 10y + 35 = 0$

► MAINTAINING YOUR SKILLS

92. (1.3) Solve the absolute value inequality and write the solution in interval notation.

$$\frac{|w - 2|}{3} + \frac{1}{4} \geq \frac{5}{6}$$

93. (R.1) Give an example of each of the following:

a. a whole number that is not a natural number

b. a natural number that is not a whole number

c. a rational number that is not an integer

d. an integer that is not a rational number

e. a rational number that is not a real number

f. a real number that is not a rational number.

94. (1.5) Solve $x^2 + 13 = 6x$ using the quadratic equation. Simplify the result.

95. (1.6) Solve $1 - \sqrt{n + 3} = -n$ and check solutions by substitution. If a solution is extraneous, so state.

2.2 Graphs of Linear Equations

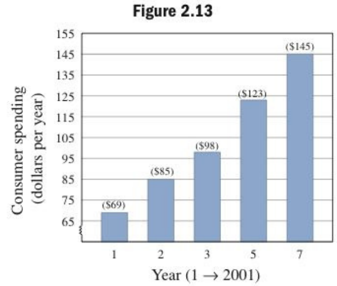
Learning Objectives

In Section 2.2 you will learn how to:

- **A.** Graph linear equations using the intercept method
- **B.** Find the slope of a line
- **C.** Graph horizontal and vertical lines
- **D.** Identify parallel and perpendicular lines
- **E.** Apply linear equations in context

In preparation for sketching graphs of other relations, we'll first consider the characteristics of linear graphs. While linear graphs are fairly simple models, they have many substantive and meaningful applications. For instance, most of us are aware that music and video downloads have been increasing in popularity since they were first introduced. A close look at Example 1 of Section 2.1 reveals that spending on music downloads and Internet radio increased from \$69 per person per year in 2001 to \$145 in 2007 (Figure 2.13). From an investor's or a producer's point of view, there is a very high interest in the questions, How fast are sales increasing? Can this relationship be modeled mathematically to help predict sales in future years? Answers to these and other questions are precisely what our study in this section is all about.

Figure 2.13



Source: 2006 SAUS

A. The Graph of a Linear Equation

A linear equation can be identified using these three tests: (1) the exponent on any variable is one, (2) no variable occurs in a denominator, and (3) no two variables are multiplied together. The equation $3y = 9$ is a linear equation in one variable, while $2x + 3y = 12$ and $y = -\frac{2}{3}x + 4$ are linear equations in two variables. In general, we have the following definition:

Linear Equations

A linear equation is one that can be written in the form

$$ax + by = c$$

where a and b are not simultaneously zero.

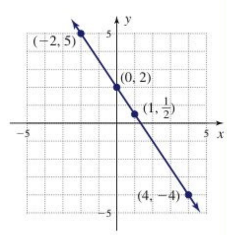
The most basic method for graphing a line is to simply plot a few points, then draw a straight line through the points.

EXAMPLE 1 ► Graphing a Linear Equation in Two Variables

Graph the equation $3x + 2y = 4$ by plotting points.

Solution ► Selecting $x = -2$, $x = 0$, $x = 1$, and $x = 4$ as inputs, we compute the related outputs and enter the ordered pairs in a table. The result is

x input	*y* output	(*x*, *y*) ordered pairs
−2	5	(−2, 5)
0	2	(0, 2)
1	0.5	$(1, \frac{1}{2})$
4	−4	(4, −4)



WORTHY OF NOTE

If you cannot draw a straight line through the plotted points, a computational error has been made. All points satisfying a linear equation *lie on a straight line.*

Now try Exercises 7 through 12 ►

Note the line in Example 1 crosses the y-axis at (0, 2), and this point is called the **y-intercept** of the line. In general, y-intercepts have the form $(0, y)$. Although difficult to see graphically, substituting 0 for y and solving for x shows the line crosses the x-axis at $(\frac{4}{3}, 0)$ and this point is called the **x-intercept.** In general, x-intercepts have the form $(x, 0)$. The x- and y-intercepts are usually easier to calculate than other points (since $y = 0$ or $x = 0$, respectively) and we often graph linear equations using only these two points. This is called the **intercept method** for graphing linear equations.

The Intercept Method

1. Substitute 0 for x and solve for y. This will give the y-intercept $(0, y)$.
2. Substitute 0 for y and solve for x. This will give the x-intercept $(x, 0)$.
3. Plot the intercepts and use them to graph a straight line.

EXAMPLE 2 ► **Graphing Lines Using the Intercept Method**

Graph $3x + 2y = 9$ using the intercept method.

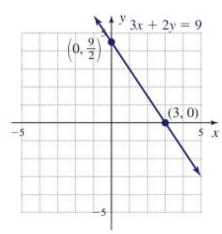
Solution ►

Substitute 0 for x (y-intercept)

$$3(0) + 2y = 9$$
$$2y = 9$$
$$y = \frac{9}{2}$$
$$\left(0, \frac{9}{2}\right)$$

Substitute 0 for y (x-intercept)

$$3x + 2(0) = 9$$
$$3x = 9$$
$$x = 3$$
$$(3, 0)$$



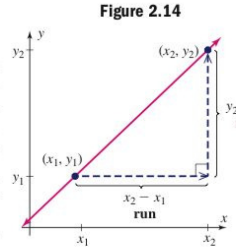
A. You've just learned how to graph linear equations using the intercept method

Now try Exercises 13 through 32 ►

B. The Slope of a Line

After the x- and y-intercepts, we next consider the **slope of a line.** We see applications of the concept in many diverse occupations, including the *grade* of a highway (trucking), the *pitch* of a roof (carpentry), the *climb* of an airplane (flying), the *drainage* of a field (landscaping), and the *slope* of a mountain (parks and recreation). While the general concept is an intuitive one, we seek to quantify the concept (assign it a numeric value) for purposes of comparison and decision making. In each of the preceding examples, slope is a measure of "steepness," as defined by the ratio $\frac{\text{vertical change}}{\text{horizontal change}}$. Using a line segment through arbitrary points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, we can create the right triangle shown in Figure 2.14. The figure illustrates that the **vertical change** or the

Figure 2.14



WORTHY OF NOTE

While the original reason that "*m*" was chosen for slope is uncertain, some have speculated that it was because in French, the verb for "to climb" is *m*onter. Others say it could be due to the "*m*odulus of slope," the word *modulus* meaning a numeric measure of a given property, in this case the inclination of a line.

change in *y* (also called the **rise**) is simply the difference in *y*-coordinates: $y_2 - y_1$. The **horizontal change** or **change in *x*** (also called the **run**) is the difference in *x*-coordinates: $x_2 - x_1$. In algebra, we typically use the letter "*m*" to represent slope, giving $m = \frac{y_2 - y_1}{x_2 - x_1}$ as the $\frac{\text{change in } y}{\text{change in } x}$. The result is called the **slope formula.**

The Slope Formula

Given two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, the slope of any nonvertical line through P_1 and P_2 is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_2 \neq x_1$.

EXAMPLE 3 ► Using the Slope Formula

Find the slope of the line through the given points.

a. (2, 1) and (8, 4) **b.** (−2, 6) and (4, 2)

Solution ► **a.** For $P_1 = (2, 1)$ and $P_2 = (8, 4)$,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{8 - 2} = \frac{3}{6} = \frac{1}{2}$$

The slope of this line is $\frac{1}{2}$.

b. For $P_1 = (-2, 6)$ and $P_2 = (4, 2)$,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 6}{4 - (-2)} = \frac{-4}{6} = \frac{-2}{3}$$

The slope of this line is $\frac{-2}{3}$.

Now try Exercises 33 through 40 ►

CAUTION ► When using the slope formula, try to avoid these common errors.

1. The order that the *x*- and *y*-coordinates are subtracted must be consistent, since $\frac{y_2 - y_1}{x_2 - x_1} \neq \frac{y_2 - y_1}{x_1 - x_2}$.
2. The vertical change (involving the *y*-values) always occurs in the numerator: $\frac{y_2 - y_1}{x_2 - x_1} \neq \frac{x_2 - x_1}{y_2 - y_1}$.
3. When x_1 or y_1 is negative, use parentheses when substituting into the formula to prevent confusing the negative sign with the subtraction operation.

Actually, the slope value does much more than quantify the slope of a line, it expresses a **rate of change** between the quantities measured along each axis. In applications of slope, the ratio $\frac{\text{change in } y}{\text{change in } x}$ is symbolized as $\frac{\Delta y}{\Delta x}$. The symbol Δ is the Greek letter **delta** and has come to represent a change in some quantity, and the notation $m = \frac{\Delta y}{\Delta x}$ is read, "slope is equal to the *change in y* over the *change in x*." Interpreting slope as a rate of change has many significant applications in college algebra and beyond.

EXAMPLE 4 ► Interpreting the Slope Formula as a Rate of Change

Jimmy works on the assembly line for an auto parts remanufacturing company. By 9:00 A.M. his group has assembled 29 carburetors. By 12:00 noon, they have completed 87 carburetors. Assuming the relationship is linear, find the slope of the line and discuss its meaning in this context.

Solution ► First write the information as ordered pairs using c to represent the carburetors assembled and t to represent time. This gives $(t_1, c_1) = (9, 29)$ and $(t_2, c_2) = (12, 87)$. The slope formula then gives:

$$\frac{\Delta c}{\Delta t} = \frac{c_2 - c_1}{t_2 - t_1} = \frac{87 - 29}{12 - 9}$$
$$= \frac{58}{3} \text{ or } 19.\overline{3}$$

Here the slope ratio measures $\frac{\text{carburetors assembled}}{\text{hours}}$, and we see that Jimmy's group can assemble 58 carburetors every 3 hr, or about $19\frac{1}{3}$ carburetors per hour.

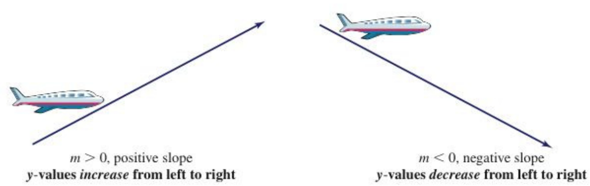
Now try Exercises 41 through 44 ►

WORTHY OF NOTE

Actually, the assignment of (t_1, c_1) to (9, 29) and (t_2, c_2) to (12, 87) was arbitrary. The slope ratio will be the same *as long as the order of subtraction is the same.* In other words, if we reverse this assignment and use $(t_1, c_1) = (12, 87)$ and $(t_2, c_2) = (9, 29)$, we have $m = \frac{29 - 87}{9 - 12} = \frac{-58}{-3} = \frac{58}{3}$.

Positive and Negative Slope

If you've ever traveled by air, you've likely heard the announcement, "Ladies and gentlemen, please return to your seats and fasten your seat belts as we begin our descent." For a time, the descent of the airplane follows a linear path, but now the *slope of the line is negative* since the altitude of the plane is decreasing. Positive and negative slopes, as well as the rate of change they represent, are important characteristics of linear graphs. In Example 3a, the slope was a positive number ($m > 0$) and the line will slope upward from left to right since the y-values are increasing. If $m < 0$, the slope of the line is negative and the line slopes downward as you move left to right since y-values are decreasing.



EXAMPLE 5 ► Applying Slope to Changes in Altitude

At a horizontal distance of 10 mi after take-off, an airline pilot receives instructions to decrease altitude from their current level of 20,000 ft. A short time later, they are 17.5 mi from the airport at an altitude of 10,000 ft. Find the slope ratio for the descent of the plane and discuss its meaning in this context. Recall that 1 mi = 5280 ft.

Solution ► Let a represent the altitude of the plane and d its horizontal distance from the airport. Converting all measures to feet, we have $(d_1, a_1) = (52{,}800, 20{,}000)$ and $(d_2, a_2) = (92{,}400, 10{,}000)$, giving

$$\frac{\Delta a}{\Delta d} = \frac{a_2 - a_1}{d_2 - d_1} = \frac{10{,}000 - 20{,}000}{92{,}400 - 52{,}800}$$
$$= \frac{-10{,}000}{39{,}600} = \frac{-25}{99}$$

Since this slope ratio measures $\frac{\Delta\text{altitude}}{\Delta\text{distance}}$, we note the plane decreased 25 ft in altitude for every 99 ft it traveled horizontally.

Now try Exercises 45 through 48 ►

B. You've just learned how to find the slope of a line

C. Horizontal Lines and Vertical Lines

Horizontal and vertical lines have a number of important applications, from finding the boundaries of a given graph, to performing certain tests on nonlinear graphs. To better understand them, consider that in *one dimension,* the graph of $x = 2$ is a single point (Figure 2.15), indicating a location on the number line 2 units from zero in the positive direction. In *two dimensions,* the equation $x = 2$ represents **all points** with an x-coordinate of 2. A few of these are graphed in Figure 2.16, but since there are an infinite number, we end up with a solid *vertical line* whose equation is $x = 2$ (Figure 2.17).

Figure 2.15

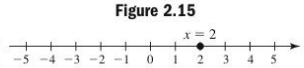


Figure 2.16

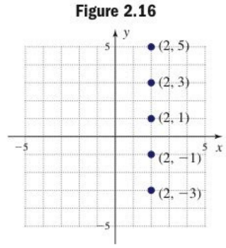
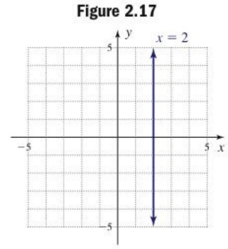


Figure 2.17



WORTHY OF NOTE

If we write the equation $x = 2$ in the form $ax + by = c$, the equation becomes $x + 0y = 2$, since the original equation has no y-variable. Notice that regardless of the value chosen for y, x will always be 2 and we end up with the set of ordered pairs $(2, y)$, which gives us a vertical line.

The same idea can be applied to horizontal lines. In *two dimensions,* the equation $y = 4$ represents *all points* with a y-coordinate of positive 4, and there are an infinite number of these as well. The result is a solid horizontal line whose equation is $y = 4$. **See Exercises 49–54.**

Vertical Lines	Horizontal Lines
The equation of a vertical line is $x = h$ where $(h, 0)$ is the x-intercept.	The equation of a horizontal line is $y = k$ where $(0, k)$ is the y-intercept.

So far, the slope formula has only been applied to lines that were nonhorizontal or nonvertical. So what *is* the slope of a horizontal line? On an intuitive level, we expect that a perfectly level highway would have an incline or slope of zero. In general, for any two points on a horizontal line, $y_2 = y_1$ and $y_2 - y_1 = 0$, giving a slope of $m = \frac{0}{x_2 - x_1} = 0$. For any two points on a vertical line, $x_2 = x_1$ and $x_2 - x_1 = 0$, making the slope ratio undefined: $m = \frac{y_2 - y_1}{0}$.

The Slope of a Vertical Line	The Slope of a Horizontal Line
The slope of any vertical line is undefined.	The slope of any horizontal line is zero.

EXAMPLE 6 ► **Calculating Slopes**

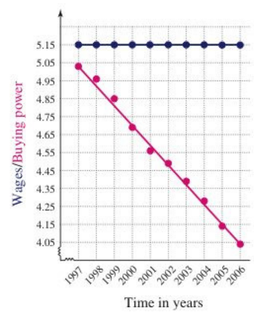
The federal minimum wage remained constant from 1997 through 2006. However, the buying power (in 1996 dollars) of these wage earners fell each year due to inflation (see Table 2.3). This decrease in buying power is approximated by the red line shown.

a. Using the data or graph, find the slope of the line segment representing the minimum wage.

b. Select two points on the line representing buying power to approximate the slope of the line segment, and explain what it means in this context.

Table 2.3

Time t (years)	Minimum wage w	Buying power p
1997	5.15	5.03
1998	5.15	4.96
1999	5.15	4.85
2000	5.15	4.69
2001	5.15	4.56
2002	5.15	4.49
2003	5.15	4.39
2004	5.15	4.28
2005	5.15	4.14
2006	5.15	4.04



Solution ► **a.** Since the minimum wage did not increase or decrease from 1997 to 2006, the line segment has slope $m = 0$.

b. The points (1997, 5.03) and (2006, 4.04) from the table appear to be on or close to the line drawn. For buying power p and time t, the slope formula yields:

$$\frac{\Delta p}{\Delta t} = \frac{p_2 - p_1}{t_2 - t_1} = \frac{4.04 - 5.03}{2006 - 1997} = \frac{-0.99}{9} = \frac{-0.11}{1}$$

The buying power of a minimum wage worker decreased by 11¢ per year during this time period.

WORTHY OF NOTE

In the context of lines, try to avoid saying that a horizontal line has "no slope," since it's unclear whether a slope of zero or an undefined slope is intended.

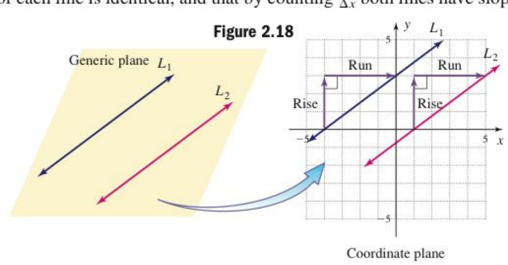
☑ **C.** You've just learned how to graph horizontal and vertical lines

Now try Exercises 55 and 56 ►

D. Parallel and Perpendicular Lines

Two lines in the same plane that never intersect are called **parallel lines.** When we place these lines on the coordinate grid, we find that "never intersect" is equivalent to saying "the lines have equal slopes but different y-intercepts." In Figure 2.18, notice the rise and run of each line is identical, and that by counting $\frac{\Delta y}{\Delta x}$ both lines have slope $m = \frac{3}{4}$.

Figure 2.18



Parallel Lines

Given L_1 and L_2 are distinct, nonvertical lines with slopes of m_1 and m_2, respectively.

1. If $m_1 = m_2$, then L_1 is parallel to L_2.

2. If L_1 is parallel to L_2, then $m_1 = m_2$.

In symbols we write $L_1 \| L_2$.

Any two vertical lines (undefined slope) are parallel.

EXAMPLE 7A ▶ Determining Whether Two Lines Are Parallel

Teladango Park has been mapped out on a rectangular coordinate system, with a ranger station at (0, 0). BJ and Kapi are at coordinates $(-24, -18)$ and have set a direct course for the pond at (11, 10). Dave and Becky are at $(-27, 1)$ and are heading straight to the lookout tower at $(-2, 21)$. Are they hiking on parallel or nonparallel courses?

Solution ▶ To respond, we compute the slope of each trek across the park.

For BJ and Kapi:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - (-18)}{11 - (-24)} = \frac{28}{35} = \frac{4}{5}$$

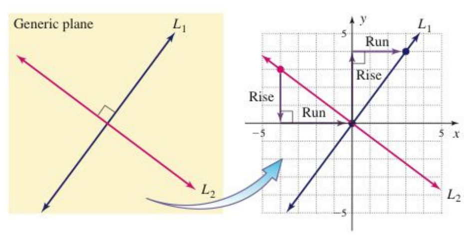
For Dave and Becky:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{21 - 1}{-2 - (-27)} = \frac{20}{25} = \frac{4}{5}$$

Since the slopes are equal, the couples are hiking on parallel courses.

Two lines in the same plane that intersect at right angles are called **perpendicular lines.** Using the coordinate grid, we note that *intersect at right angles* suggests that *their slopes are negative reciprocals.* From Figure 2.19, the ratio $\frac{\text{rise}}{\text{run}}$ for L_1 is $\frac{4}{3}$, the ratio $\frac{\text{rise}}{\text{run}}$ for L_2 is $\frac{-3}{4}$. Alternatively, we can say their **slopes have a product of −1,** since $m_1 \cdot m_2 = -1$ implies $m_1 = -\frac{1}{m_2}$.

Figure 2.19



WORTHY OF NOTE

Since $m_1 \cdot m_2 = -1$ implies $m_1 = -\frac{1}{m_2}$, we can easily find the slope of a line perpendicular to a second line whose slope is given—just find the reciprocal and make it negative. For $m_1 = -\frac{3}{7}$ $m_2 = \frac{7}{3}$, and for $m_1 = \mathbf{-5}$, $m_2 = \frac{1}{5}$.

Perpendicular Lines

Given L_1 and L_2 are distinct, nonvertical lines with slopes of m_1 and m_2, respectively.

1. If $m_1 \cdot m_2 = -1$, then L_1 is perpendicular to L_2.

2. If L_1 is perpendicular to L_2, then $m_1 \cdot m_2 = -1$.

In symbols we write $L_1 \perp L_2$.

Any vertical line (undefined slope) is perpendicular to any horizontal line (slope m = 0).

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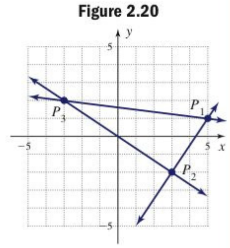
EXAMPLE 7B ► Determining Whether Two Lines Are Perpendicular

The three points $P_1 = (5, 1)$, $P_2 = (3, -2)$, and $P_3 = (-3, 2)$ form the vertices of a triangle. Use these points to draw the triangle, then use the slope formula to determine if they form a *right* triangle.

Solution ► For a right triangle to be formed, two of the lines through these points must be perpendicular (forming a right angle). From Figure 2.20, it *appears* a right triangle is formed, but we must *verify* that two of the sides are perpendicular. Using the slope formula, we have:

Figure 2.20



For P_1 and P_2

$$m_1 = \frac{-2 - 1}{3 - 5} = \frac{-3}{-2} = \frac{3}{2}$$

For P_1 and P_3

$$m_2 = \frac{2 - 1}{-3 - 5} = \frac{1}{-8}$$

For P_2 and P_3

$$m_3 = \frac{2 - (-2)}{-3 - 3} = \frac{4}{-6} = \frac{2}{-3}$$

Since $m_1 \cdot m_3 = -1$, the triangle has a right angle and must be a right triangle.

D. You've just learned how to identify parallel and perpendicular lines

Now try Exercises 57 through 68 ►

E. Applications of Linear Equations

The graph of a linear equation can be used to help solve many applied problems. If the numbers you're working with are either very small or very large, **scale the axes** appropriately. This can be done by letting each tic mark represent a smaller or larger unit so the data points given will fit on the grid. Also, many applications use only nonnegative values and although points with negative coordinates may be used to graph a line, only ordered pairs in QI can be meaningfully interpreted.

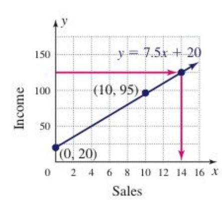
EXAMPLE 8 ► Applying a Linear Equation Model—Commission Sales

Use the information given to create a linear equation model in two variables, then graph the line and use the graph to answer the question:

A salesperson gets a daily \$20 meal allowance plus \$7.50 for every item she sells. How many sales are needed for a daily income of \$125?

Solution ► Let x represent sales and y represent income. This gives

verbal model: Daily income (y) equals \$7.5 per sale (x) + \$20 for meals
equation model: $y = 7.5x + 20$



Using $x = 0$ and $x = 10$, we find (0, 20) and (10, 95) are points on this graph. From the graph, we estimate that 14 sales are needed to generate a daily income of \$125.00. Substituting $x = 14$ into the equation verifies that (14, 125) is indeed on the graph:

E. You've just learned how to apply linear equations in context

$$\begin{aligned} y &= 7.5x + 20 \\ &= 7.5(14) + 20 \\ &= 105 + 20 \\ &= 125 \checkmark \end{aligned}$$

Now try Exercises 71 through 74 ►

TECHNOLOGY HIGHLIGHT

Linear Equations, Window Size, and Friendly Windows

To graph linear equations on the TI-84 Plus, we (1) solve the equation for the variable *y*, (2) enter the equation on the Y= screen, and (3) GRAPH the equation and adjust the WINDOW if necessary.

1. Solve the equation for *y*.
   For the equation $2x - 3y = -3$, we have

   $2x - 3y = -3$ given equation

   $-3y = -2x - 3$ subtract $2x$ from each side

   $y = \frac{2}{3}x + 1$ divide both sides by -3

2. Enter the equation on the Y= screen.
   On the Y= screen, enter $\frac{2}{3}x + 1$. Note that for some calculators parentheses are needed to group $(2 \div 3)x$, to prevent the calculator from interpreting this term as $2 \div (3x)$.

3. GRAPH the equation, adjust the WINDOW.
   Since much of our work is centered at (0, 0) on the coordinate grid, the calculator's default settings have a domain of $x \in [-10, 10]$ and a range of $y \in [-10, 10]$, as shown in Figure 2.21. This is referred to as the WINDOW size. To graph the line in this window, it is easiest to use the ZOOM key and select **6:ZStandard,** which resets the window to these default settings. The graph is shown in Figure 2.22. The Xscl and Yscl entries give the scale used on each axis, indicating that each "tic mark" represents 1 unit. Graphing calculators have many features that enable us to find ordered pairs on a line. One is the (2nd GRAPH) **(TABLE)** feature we have seen previously. We can also use the calculator's TRACE feature. As the name implies, this feature enables us to trace along the line by moving a blinking cursor using the left ◄ and right ► arrow keys. The calculator simultaneously displays the coordinates of the current location of the cursor. After pressing the TRACE button, the cursor appears automatically—usually at the *y*-intercept. Moving the cursor left and right, note the coordinates changing at the bottom of the screen. The point (3.4042553, 3.2695035) is on the line and satisfies the equation of the line. The calculator is displaying decimal values because the screen is exactly 95 pixels wide, 47 pixels to the left of the *y*-axis, and 47 pixels to the right. This means that each time you press the left or right arrow, the *x*-value changes by 1/47—which is *not* a nice round number. To TRACE through "friendlier" values, we can use the ZOOM **4:ZDecimal** feature, which sets Xmin $= -4.7$ and Xmax $= 4.7$, or **8:Zinteger,** which sets Xmin $= -47$ and Xmax $= 47$. Press ZOOM **4:ZDecimal** and the calculator will automatically regraph the line. Now when you TRACE the line, "friendly" decimal values are displayed.

Figure 2.21

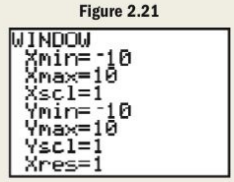
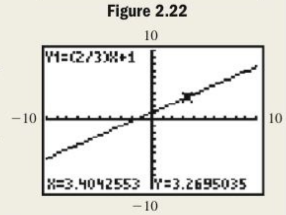


Figure 2.22



Exercise 1: Use the ZOOM **4:ZDecimal** and **TRACE** features to identify the *x*- and *y*-intercepts for $Y_1 = \frac{2}{3}x + 1$.

Exercise 2: Use the ZOOM **8:Zinteger** and **TRACE** features to graph the line $79x - 55y = 869$, then identify the *x*- and *y*-intercepts.

2.2 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. To find the x-intercept of a line, substitute ______ for y and solve for x. To find the y-intercept, substitute ______ for x and solve for y.

2. The slope formula is $m =$ ______ $=$ ______, and indicates a rate of change between the x- and y-variables.

3. If $m < 0$, the slope of the line is ______ and the line slopes ______ from left to right.

4. The slope of a horizontal line is ______, the slope of a vertical line is ______, and the slopes of two parallel lines are ______.

5. Discuss/Explain If $m_1 = 2.1$ and $m_2 = 2.01$, will the lines intersect? If $m_1 = \frac{2}{3}$ and $m_2 = -\frac{2}{3}$, are the lines perpendicular?

6. Discuss/Explain the relationship between the slope formula, the Pythagorean theorem, and the distance formula. Include several illustrations.

▶ DEVELOPING YOUR SKILLS

Create a table of values for each equation and sketch the graph.

7. $2x + 3y = 6$

x	y

8. $-3x + 5y = 10$

x	y

9. $y = \frac{3}{2}x + 4$

x	y

10. $y = \frac{5}{3}x - 3$

x	y

11. If you completed Exercise 9, verify that $(-3, -0.5)$ and $(\frac{1}{2}, \frac{19}{4})$ also satisfy the equation given. Do these points appear to be on the graph you sketched?

12. If you completed Exercise 10, verify that $(-1.5, -5.5)$ and $(\frac{11}{2}, \frac{37}{6})$ also satisfy the equation given. Do these points appear to be on the graph you sketched?

Graph the following equations using the intercept method. Plot a third point as a check.

13. $3x + y = 6$ **14.** $-2x + y = 12$

15. $5y - x = 5$ **16.** $-4y + x = 8$

17. $-5x + 2y = 6$ **18.** $3y + 4x = 9$

19. $2x - 5y = 4$ **20.** $-6x + 4y = 8$

21. $2x + 3y = -12$ **22.** $-3x - 2y = 6$

23. $y = -\frac{1}{2}x$ **24.** $y = \frac{2}{3}x$

25. $y - 25 = 50x$ **26.** $y + 30 = 60x$

27. $y = -\frac{2}{5}x - 2$ **28.** $y = \frac{3}{4}x + 2$

29. $2y - 3x = 0$ **30.** $y + 3x = 0$

31. $3y + 4x = 12$ **32.** $-2x + 5y = 8$

Compute the slope of the line through the given points, then graph the line and use $m = \frac{\Delta y}{\Delta x}$ to find two additional points on the line. Answers may vary.

33. $(3, 5), (4, 6)$ **34.** $(-2, 3), (5, 8)$

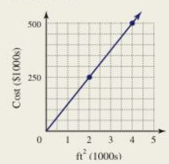
35. $(10, 3), (4, -5)$ **36.** $(-3, -1), (0, 7)$

37. $(1, -8), (-3, 7)$ **38.** $(-5, 5), (0, -5)$

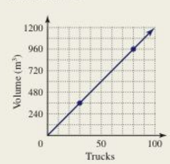
39. $(-3, 6), (4, 2)$ **40.** $(-2, -4), (-3, -1)$

41. The graph shown models the relationship between the cost of a new home and the size of the home in square feet. (a) Determine the slope of the line and interpret what the slope ratio means in this context and (b) estimate the cost of a 3000 ft^2 home.

Exercise 41



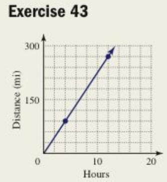
Exercise 42



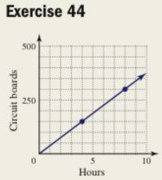
42. The graph shown models the relationship between the volume of garbage that is dumped in a landfill and the number of commercial garbage trucks that enter the site. (a) Determine the slope of the line and interpret what the slope ratio means in this context and (b) estimate the number of trucks entering the site daily if 1000 m^3 of garbage is dumped per day.

43. The graph shown models the relationship between the distance of an aircraft carrier from its home port and the number of hours since departure. (a) Determine the slope of the line and interpret what the slope ratio means in this context and (b) estimate the distance from port after 8.25 hours.

Exercise 43



Exercise 44



44. The graph shown models the relationship between the number of circuit boards that have been assembled at a factory and the number of hours since starting time. (a) Determine the slope of the line and interpret what the slope ratio means in this context and (b) estimate how many hours the factory has been running if 225 circuit boards have been assembled.

45. Height and weight: While there are many exceptions, numerous studies have shown a close relationship between an average height and average weight. Suppose a person 70 in. tall weighs 165 lb, while a person 64 in. tall weighs 142 lb. Assuming the relationship is linear, (a) find the slope of the line and discuss its meaning in this context and (b) determine how many pounds are added for each inch of height.

46. Rate of climb: Shortly after takeoff, a plane increases altitude at a constant (linear) rate. In 5 min the altitude is 10,000 feet. Fifteen minutes after takeoff, the plane has reached its cruising altitude of 32,000 ft. (a) Find the slope of the line and discuss its meaning in this context and (b) determine how long it takes the plane to climb from 12,200 feet to 25,400 feet.

47. Sewer line slope: Fascinated at how quickly the plumber was working, Ryan watched with great interest as the new sewer line was laid from the house to the main line, a distance of 48 ft. At the edge of the house, the sewer line was six in. under ground. If the plumber tied in to the main line at a depth of 18 in., what is the slope of the (sewer) line? What does this slope indicate?

48. Slope (pitch) of a roof: A contractor goes to a lumber yard to purchase some trusses (the triangular frames) for the roof of a house. Many sizes are available, so the contractor takes some measurements to ensure the roof will have the desired slope. In one case, the height of the truss (base to ridge) was 4 ft, with a width of 24 ft (eave to eave). Find the slope of the roof if these trusses are used. What does this slope indicate?

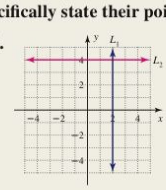
Graph each line using two or three ordered pairs that satisfy the equation.

49. $x = -3$ **50.** $y = 4$

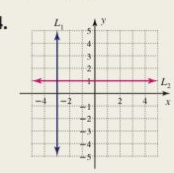
51. $x = 2$ **52.** $y = -2$

Write the equation for each line L_1 and L_2 shown. Specifically state their point of intersection.

53.



54.



55. The table given shows the total number of justices j sitting on the Supreme Court of the United States for selected time periods t (in decades), along with the number of nonmale, nonwhite justices n for the same years. (a) Use the data to graph the linear relationship between t and j, then determine the slope of the line and discuss its meaning in this context. (b) Use the data to graph the linear relationship between t and n, then determine the slope of the line and discuss its meaning.

Exercise 55

Time t ($1960 \rightarrow 0$)	Justices j	Nonwhite, nonmale n
0	9	0
10	9	1
20	9	2
30	9	3
40	9	4
50	9	5 (est)

56. The table shown gives the boiling temperature t of water as related to the altitude h. Use the data to graph the linear relationship between h and t, then determine the slope of the line and discuss its meaning in this context.

Exercise 56

Altitude h (ft)	Boiling Temperature t (°F)
0	212.0
1000	210.2
2000	208.4
3000	206.6
4000	204.8
5000	203.0
6000	201.2

Two points on L_1 and two points on L_2 are given. Use the slope formula to determine if lines L_1 and L_2 are parallel, perpendicular, or neither.

57. L_1: $(-2, 0)$ and $(0, 6)$
L_2: $(1, 8)$ and $(0, 5)$

58. L_1: $(1, 10)$ and $(-1, 7)$
L_2: $(0, 3)$ and $(1, 5)$

59. L_1: $(-3, -4)$ and $(0, 1)$
L_2: $(0, 0)$ and $(-4, 4)$

60. L_1: $(6, 2)$ and $(8, -2)$
L_2: $(5, 1)$ and $(3, 0)$

61. L_1: $(6, 3)$ and $(8, 7)$
L_2: $(7, 2)$ and $(6, 0)$

62. L_1: $(-5, -1)$ and $(4, 4)$
L_2: $(4, -7)$ and $(8, 10)$

In Exercises 63 to 68, three points that form the vertices of a triangle are given. Use the points to draw the triangle, then use the slope formula to determine if any of the triangles are right triangles. Also see Exercises 39–44 in Section 2.1.

63. $(5, 2), (0, -3), (4, -4)$

64. $(7, 0), (-1, 0), (7, 4)$

65. $(-4, 3), (-7, -1), (3, -2)$

66. $(-3, 7), (2, 2), (5, 5)$

67. $(-3, 2), (-1, 5), (-6, 4)$

68. $(0, 0), (-5, 2), (2, -5)$

▶ WORKING WITH FORMULAS

69. Human life expectancy: $L = 0.11T + 74.2$

The average number of years that human beings live has been steadily increasing over the years due to better living conditions and improved medical care. This relationship is modeled by the formula shown, where L is the average life expectancy and T is number of years since 1980. (a) What was the life expectancy in the year 2000? (b) In what year will average life expectancy reach 77.5 yr?

70. Interest earnings: $I = \left(\frac{7}{100}\right)(5000)T$

If $5000 dollars is invested in an account paying 7% simple interest, the amount of interest earned is given by the formula shown, where I is the interest and T is the time in years. (a) How much interest is earned in 5 yr? (b) How much is earned in 10 yr? (c) Use the two points (5 yr, interest) and (10 yr, interest) to calculate the slope of this line. What do you notice?

▶ APPLICATIONS

For exercises 71 to 74, use the information given to build a linear equation model, then use the equation to respond.

71. Business depreciation: A business purchases a copier for $8500 and anticipates it will depreciate in value $1250 per year.

a. What is the copier's value after 4 yr of use?

b. How many years will it take for this copier's value to decrease to $2250?

72. Baseball card value: After purchasing an autographed baseball card for $85, its value increases by $1.50 per year.

a. What is the card's value 7 yr after purchase?

b. How many years will it take for this card's value to reach $100?

73. Water level: During a long drought, the water level in a local lake decreased at a rate of 3 in. per month. The water level before the drought was 300 in.

a. What was the water level after 9 months of drought?

b. How many months will it take for the water level to decrease to 20 ft?

74. Gas mileage: When empty, a large dump-truck gets about 15 mi per gallon. It is estimated that for each 3 tons of cargo it hauls, gas mileage decreases by $\frac{3}{4}$ mi per gallon.

a. If 10 tons of cargo is being carried, what is the truck's mileage?

b. If the truck's mileage is down to 10 mi per gallon, how much weight is it carrying?

75. Parallel/nonparallel roads: Aberville is 38 mi north and 12 mi west of Boschertown, with a straight road "farm and machinery road" (FM 1960) connecting the two cities. In the next county, Crownsburg is 30 mi north and 9.5 mi west of Dower, and these cities are likewise connected by a straight road (FM 830). If the two roads continued indefinitely in both directions, would they intersect at some point?

76. Perpendicular/nonperpendicular course headings: Two shrimp trawlers depart Charleston Harbor at the same time. One heads for the shrimping grounds located 12 mi north and 3 mi east of the harbor. The other heads for a point 2 mi south and 8 mi east of the harbor. Assuming the harbor is at (0, 0), are the routes of the trawlers perpendicular? If so, how far apart are the boats when they reach their destinations (to the nearest one-tenth mi)?

77. Cost of college: For the years 1980 to 2000, the cost of tuition and fees per semester (in constant dollars) at a public 4-yr college can be approximated by the equation $y = 144x + 621$, where y represents the cost in dollars and $x = 0$ represents the year 1980. Use the equation to find: (a) the cost of tuition and fees in 2002 and (b) the year this cost will exceed $5250.

Source: 2001 New York Times Almanac, p. 356

78. Female physicians: In 1960 only about 7% of physicians were female. Soon after, this percentage began to grow dramatically. For the years 1980 to 2002, the percentage of physicians that were female can be approximated by the equation $y = 0.72x + 11$, where y represents the percentage (as a whole number) and $x = 0$ represents the year 1980. Use the equation to find: (a) the percentage of physicians that were female in 1992 and (b) the projected year this percentage will exceed 30%.

Source: Data from the 2004 *Statistical Abstract of the United States,* Table 149

79. Decrease in smokers: For the years 1980 to 2002, the percentage of the U.S. adult population who were smokers can be approximated by the equation $y = -\frac{7}{15}x + 32$, where y represents the percentage of smokers (as a whole number) and $x = 0$ represents 1980. Use the equation to find: (a) the percentage of adults who smoked in the year 2000 and (b) the year the percentage of smokers is projected to fall below 20%.

Source: Statistical Abstract of the United States, various years

80. Temperature and cricket chirps: Biologists have found a strong relationship between temperature and the number of times a cricket chirps. This is modeled by the equation $T = \frac{N}{4} + 40$, where N is the number of times the cricket chirps per minute and T is the temperature in Fahrenheit. Use the equation to find: (a) the outdoor temperature if the cricket is chirping 48 times per minute and (b) the number of times a cricket chirps if the temperature is 70°.

▶ EXTENDING THE CONCEPT

81. If the lines $4y + 2x = -5$ and $3y + ax = -2$ are perpendicular, what is the value of a?

82. Let m_1, m_2, m_3, and m_4 be the slopes of lines L_1, L_2, L_3, and L_4, respectively. Which of the following statements is true?

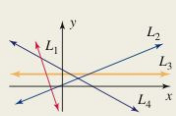
a. $m_4 < m_1 < m_3 < m_2$

b. $m_3 < m_2 < m_4 < m_1$

c. $m_3 < m_4 < m_2 < m_1$

d. $m_1 < m_3 < m_4 < m_2$

e. $m_1 < m_4 < m_3 < m_2$



83. An *arithmetic sequence* is a sequence of numbers where each successive term is found by adding a fixed constant, called the common difference d, to the preceding term. For instance 3, 7, 11, 15, . . . is an arithmetic sequence with $d = 4$. The formula for the "nth term" t_n of an arithmetic sequence is a linear equation of the form $t_n = t_1 + (n - 1)d$, where d is the common difference and t_1 is the first term of the sequence. Use the equation to find the term specified for each sequence.

a. 2, 9, 16, 23, 30, . . . ; 21st term

b. 7, 4, 1, −2, −5, . . . ; 31st term

c. 5.10, 5.25, 5.40, 5.55, . . . ; 27th term

d. $\frac{3}{2}, \frac{9}{4}, 3, \frac{15}{4}, \frac{9}{2}, \ldots$; 17th term

▶ MAINTAINING YOUR SKILLS

84. **(1.1)** Simplify the equation, then solve. Check your answer by substitution:
$3x^2 - 3 + 4x + 6 = 4x^2 - 3(x + 5)$

85. **(R.7)** Identify the following formulas:

$P = 2L + 2W$ $\quad V = LWH$
$V = \pi r^2 h$ $\quad C = 2\pi r$

86. **(1.1)** How many gallons of a 35% brine solution must be mixed with 12 gal of a 55% brine solution in order to get a 45% solution?

87. **(1.1)** Two boats leave the harbor at Lahaina, Maui, going in opposite directions. One travels at 15 mph and the other at 20 mph. How long until they are 70 mi apart?

2.3 Linear Graphs and Rates of Change

Learning Objectives

In Section 2.3 you will learn how to:

- ☐ **A.** Write a linear equation in slope-intercept form
- ☐ **B.** Use slope-intercept form to graph linear equations
- ☐ **C.** Write a linear equation in point-slope form
- ☐ **D.** Apply the slope-intercept form and point-slope form in context

The concept of slope is an important part of mathematics, because it gives us a way to measure and compare change. The value of an automobile changes with time, the circumference of a circle increases as the radius increases, and the tension in a spring grows the more it is stretched. The real world is filled with examples of how one change affects another, and slope helps us understand how these changes are related.

A. Linear Equations and Slope-Intercept Form

In Section 1.1, formulas and literal equations were written in an alternate form by solving for an object variable. The new form made using the formula more efficient. Solving for y in equations of the form $ax + by = c$ offers similar advantages to linear graphs and their applications.

EXAMPLE 1 ► Solving for y in $ax + by = c$

Solve $2y - 6x = 4$ for y, then evaluate at $x = 4$, $x = 0$, and $x = -\frac{1}{3}$.

Solution ►

$2y - 6x = 4$ given equation

$2y = 6x + 4$ add $6x$

$y = 3x + 2$ divide by 2

Since the coefficients are integers, evaluate the function mentally. Inputs are multiplied by 3, then increased by 2, yielding the ordered pairs (4, 14), (0, 2), and $(-\frac{1}{3}, 1)$.

Now try Exercises 7 through 12 ►

This form of the equation (where y has been written in terms of x) enables us to quickly identify what operations are performed on x in order to obtain y. For $y = 3x + 2$, *multiply inputs by 3, then add 2*.

EXAMPLE 2 ► Solving for y in $ax + by = c$

Solve the linear equation $3y - 2x = 6$ for y, then identify the new coefficient of x and the constant term.

Solution ►

$3y - 2x = 6$ given equation

$3y = 2x + 6$ add $2x$

$y = \frac{2}{3}x + 2$ divide by 3

The new coefficient of x is $\frac{2}{3}$ and the constant term is 2.

Now try Exercises 13 through 18 ►

WORTHY OF NOTE

In Example 2, the final form can be written $y = \frac{2}{3}x + 2$ as shown (inputs are multiplied by two-thirds, then increased by 2), or written as $y = \frac{2x}{3} + 2$ (inputs are multiplied by two, the result divided by 3 and this amount increased by 2). The two forms are equivalent.

When the coefficient of x is rational, it's helpful to select inputs that are multiples of the denominator if the context or application requires us to evaluate the equation. This enables us to perform most operations mentally. For $y = \frac{2}{3}x + 2$, possible inputs might be $x = -9, -6, 0, 3, 6$, and so on. **See Exercises 19 through 24.**

In Section 2.2, linear equations were graphed using the intercept method. When a linear equation is written with y in terms of x, we notice a powerful connection between the graph and its equation, and one that highlights the primary characteristics of a linear graph.

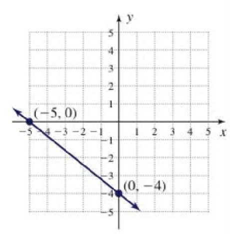
EXAMPLE 3 ► Noting Relationships between an Equation and Its Graph

Find the intercepts of $4x + 5y = -20$ and use them to graph the line. Then,

a. Use the intercepts to calculate the slope of the line, then

b. Write the equation with y in terms of x and compare the calculated slope and y-intercept to the equation in this form. Comment on what you notice.

Solution ► Substituting 0 for x in $4x + 5y = -20$, we find the y-intercept is $(0, -4)$. Substituting 0 for y gives an x-intercept of $(-5, 0)$. The graph is displayed here.



a. By calculation or counting $\frac{\Delta y}{\Delta x}$, the slope is $m = -\frac{4}{5}$.

b. Solving for y:

$4x + 5y = -20$ given equation

$5y = -4x - 20$ subtract $4x$

$y = -\frac{4}{5}x - 4$ divide by 5

The slope value seems to be the coefficient of x, while the y-intercept is the constant term.

Now try Exercises 25 through 30 ►

A. You've just learned how to write a linear equation in slope-intercept form

B. Slope-Intercept Form and the Graph of a Line

After solving a linear equation for y, an input of $x = 0$ causes the "x-term" to become zero, so the y-intercept is automatically the constant term. As Example 3 illustrates, we can also identify the slope of the line—it is the coefficient of x. In general, a linear equation of the form $y = mx + b$ is said to be in **slope-intercept form,** since the slope of the line is m and the y-intercept is $(0, b)$.

Slope-Intercept Form

For a nonvertical line whose equation is $y = \boldsymbol{m}x + \boldsymbol{b}$, the slope of the line is $\boldsymbol{m}$ and the y-intercept is $(0, \boldsymbol{b})$.

EXAMPLE 4 ► Finding the Slope-Intercept Form

Write each equation in slope-intercept form and identify the slope and y-intercept of each line.

a. $3x - 2y = 9$ **b.** $y + x = 5$ **c.** $2y = x$

Solution ►

a. $3x - 2y = 9$

$$-2y = -3x + 9$$
$$y = \frac{3}{2}x - \frac{9}{2}$$
$$m = \frac{3}{2}, b = -\frac{9}{2}$$
y-intercept $\left(0, -\frac{9}{2}\right)$

b. $y + x = 5$

$$y = -x + 5$$
$$y = -1x + 5$$
$$m = -1, b = 5$$
y-intercept $(0, 5)$

c. $2y = x$

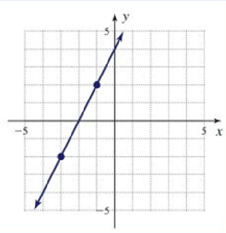
$$y = \frac{x}{2}$$
$$y = \frac{1}{2}x$$
$$m = \frac{1}{2}, b = 0$$
y-intercept $(0, 0)$

Now try Exercises 31 through 38 ►

If the slope and y-intercept of a linear equation are known or can be found, we can construct its equation by substituting these values directly into the slope-intercept form $y = mx + b$.

EXAMPLE 5 ► Finding the Equation of a Line from Its Graph

Find the slope-intercept form of the line shown.



Solution ► Using $(-3, -2)$ and $(-1, 2)$ in the slope formula, or by simply counting $\frac{\Delta y}{\Delta x}$, the slope is $m = \frac{4}{2}$ or $\frac{2}{1}$. By inspection we see the y-intercept is $(0, 4)$. Substituting $\frac{2}{1}$ for m and 4 for b in the slope-intercept form we obtain the equation $y = 2x + 4$.

Now try Exercises 39 through 44 ►

Actually, if the slope is known and we have *any* point (x, y) on the line, we can still construct the equation since the given point *must satisfy the equation of the line*. In this case, we're treating $y = mx + b$ as a simple formula, solving for b after substituting known values for m, x, and y.

EXAMPLE 6 ► Using $y = mx + b$ as a Formula

Find the equation of a line that has slope $m = \frac{4}{5}$ and contains $(-5, 2)$.

Solution ► Using $y = mx + b$ as a "formula," we have $m = \frac{4}{5}$, $x = -5$, and $y = 2$.

$y = mx + b$ slope-intercept form

$2 = \frac{4}{5}(-5) + b$ substitute $\frac{4}{5}$ for m, -5 for x, and 2 for y

$2 = -4 + b$ simplify

$6 = b$ solve for b

The equation of the line is $y = \frac{4}{5}x + 6$.

Now try Exercises 45 through 50 ►

Writing a linear equation in slope-intercept form enables us to draw its graph with a minimum of effort, since we can easily locate the *y*-intercept and a second point using $m = \frac{\Delta y}{\Delta x}$. For instance, $\frac{\Delta y}{\Delta x} = \frac{-2}{3}$ means count down 2 and right 3 from a known point.

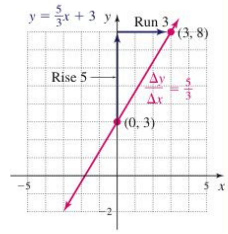
EXAMPLE 7 ► **Graphing a Line Using Slope-Intercept Form**

Write $3y - 5x = 9$ in slope-intercept form, then graph the line using the *y*-intercept and slope.

Solution ►

$$\begin{aligned} 3y - 5x &= 9 && \text{given equation} \\ 3y &= 5x + 9 && \text{isolate } y \text{ term} \\ y &= \tfrac{5}{3}x + 3 && \text{divide by 3} \end{aligned}$$

The slope is $m = \frac{5}{3}$ and the *y*-intercept is (0, 3). Plot the *y*-intercept, then use $\frac{\Delta y}{\Delta x} = \frac{5}{3}$ (up 5 and right 3—shown in blue) to find another point on the line (shown in red). Finish by drawing a line through these points.



Now try Exercises 51 through 62 ►

WORTHY OF NOTE

Noting the fraction $\frac{5}{3}$ is equal to $\frac{-5}{-3}$, we could also begin at (0, 3) and count $\frac{\Delta y}{\Delta x} = \frac{-5}{-3}$ (down 5 and left 3) to find an additional point on the line: (−3, −2). Also, for any negative slope $\frac{\Delta y}{\Delta x} = -\frac{a}{b}$, note $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$.

For a discussion of what graphing method might be most efficient for a given linear equation, **see Exercises 103 and 115.**

Parallel and Perpendicular Lines

From Section 2.2 we know parallel lines have equal slopes: $m_1 = m_2$, and perpendicular lines have slopes with a product of -1: $m_1 \cdot m_2 = -1$ or $m_1 = -\frac{1}{m_2}$. In some applications, we need to find the equation of a second line parallel or perpendicular to a given line, through a given point. Using the slope-intercept form makes this a simple four-step process.

Finding the Equation of a Line Parallel or Perpendicular to a Given Line

1. Identify the slope m_1 of the given line.
2. Find the slope m_2 of the new line using the parallel or perpendicular relationship.
3. Use m_2 with the point (x, y) in the "formula" $y = mx + b$ and solve for b.
4. The desired equation will be $y = m_2x + b$.

EXAMPLE 8 ► **Finding the Equation of a Parallel Line**

Find the equation of a line that goes through (−6, −1) and is parallel to $2x + 3y = 6$.

Solution ► Begin by writing the equation in slope-intercept form to identify the slope.

$$\begin{aligned} 2x + 3y &= 6 && \text{given line} \\ 3y &= -2x + 6 && \text{isolate } y \text{ term} \\ y &= \tfrac{-2}{3}x + 2 && \text{result} \end{aligned}$$

The original line has slope $m_1 = \frac{-2}{3}$ and this will also be the slope of any line parallel to it. Using $m_2 = \frac{-2}{3}$ with $(x, y) \to (-6, -1)$ we have

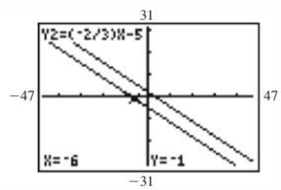
$$y = mx + b \quad \text{slope-intercept form}$$
$$-1 = \frac{-2}{3}(-6) + b \quad \text{substitute } \tfrac{-2}{3} \text{ for } m, -6 \text{ for } x, \text{ and } -1 \text{ for } y$$
$$-1 = 4 + b \quad \text{simplify}$$
$$-5 = b \quad \text{solve for } b$$

The equation of the new line is $y = \frac{-2}{3}x - 5$.

Now try Exercises 63 through 76 ►

GRAPHICAL SUPPORT

Graphing the lines from Example 8 as Y1 and Y2 on a graphing calculator, we note the lines do appear to be parallel (they actually *must* be since they have identical slopes). Using the ZOOM **8:ZInteger** feature of the TI-84 Plus we can quickly verify that Y2 indeed contains the point $(-6, -1)$.



For any nonlinear graph, a straight line drawn through two points on the graph is called a **secant line.** The slope of the secant line, and lines parallel and perpendicular to this line, play fundamental roles in the further development of the rate-of-change concept.

EXAMPLE 9 ► Finding Equations for Parallel and Perpendicular Lines

A secant line is drawn using the points $(-4, 0)$ and $(2, -2)$ on the graph of the function shown. Find the equation of a line that is:

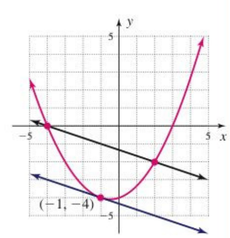
a. parallel to the secant line through $(-1, -4)$

b. perpendicular to the secant line through $(-1, -4)$.

Solution ► Either by using the slope formula or counting $\frac{\Delta y}{\Delta x}$, we find the secant line has slope $m = \frac{-2}{6} = \frac{-1}{3}$.

a. For the parallel line through $(-1, -4)$, $m_2 = \frac{-1}{3}$.

$$y = mx + b \quad \text{slope-intercept form}$$
$$-4 = \frac{-1}{3}(-1) + b \quad \text{substitute } \tfrac{-1}{3} \text{ for } m, -1 \text{ for } x, \text{ and } -4 \text{ for } y$$
$$-\frac{12}{3} = \frac{1}{3} + b \quad \text{simplify}$$
$$-\frac{13}{3} = b \quad \text{result}$$



The equation of the parallel line (in blue) is $y = \frac{-1}{3}x - \frac{13}{3}$.

WORTHY OF NOTE

The word "secant" comes from the Latin word *secare*, meaning "to cut." Hence a secant line is one that cuts through a graph, as opposed to a tangent line, which touches the graph at only one point.

b. For the line perpendicular through $(-1, -4)$, $m_2 = 3$.

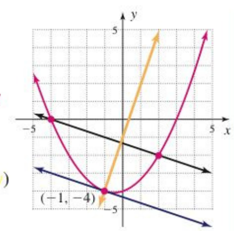
$y = mx + b$ slope-intercept form

$-4 = 3(-1) + b$ substitute 3 for m, -1 for x, and -4 for y

$-4 = -3 + b$ simplify

$-1 = b$ result

The equation of the perpendicular line (in yellow) is $y = 3x - 1$.



B. You've just learned how to use the slope-intercept form to graph linear equations

Now try Exercises 77 through 82 ►

C. Linear Equations in Point-Slope Form

As an alternative to using $y = mx + b$, we can find the equation of the line using the slope formula $\frac{y_2 - y_1}{x_2 - x_1} = m$, and the fact that *the slope of a line is constant*. For a given slope m, we can let (x_1, y_1) represent a *given* point on the line and (x, y) represent *any other point* on the line, and the formula becomes $\frac{y - y_1}{x - x_1} = m$. Isolating the "$y$" terms on one side gives a new form for the equation of a line, called the **point-slope form:**

$$\frac{y - y_1}{x - x_1} = m \quad \text{slope formula}$$

$$\frac{(x - x_1)}{1}\left(\frac{y - y_1}{x - x_1}\right) = m(x - x_1) \quad \text{multiply both sides by } (x - x_1)$$

$$y - y_1 = m(x - x_1) \quad \text{simplify} \rightarrow \text{point-slope form}$$

The Point-Slope Form of a Linear Equation

For a nonvertical line whose equation is $y - y_1 = m(x - x_1)$, the slope of the line is $\boldsymbol{m}$ and $(\boldsymbol{x_1}, \boldsymbol{y_1})$ is a point on the line.

While using $y = mx + b$ as in Example 6 may appear to be easier, both the y-intercept form and point-slope form have their own advantages and it will help to be familiar with both.

EXAMPLE 10 ► Using $y - y_1 = m(x - x_1)$ as a Formula

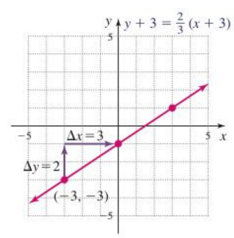
Find the equation of a line in point-slope form, if $m = \frac{2}{3}$ and $(-3, -3)$ is on the line. Then graph the line.

Solution ►

$y - y_1 = m(x - x_1)$ point-slope form

$y - (-3) = \frac{2}{3}[x - (-3)]$ substitute $\frac{2}{3}$ for m; $(-3, -3)$ for (x_1, y_1)

$y + 3 = \frac{2}{3}(x + 3)$ simplify, point-slope form



To graph the line, plot $(-3, -3)$ and use $\frac{\Delta y}{\Delta x} = \frac{2}{3}$ to find additional points on the line.

C. You've just learned how to write a linear equation in point-slope form

Now try Exercises 83 through 94 ►

D. Applications of Linear Equations

As a mathematical tool, linear equations rank among the most common, powerful, and versatile. In all cases, it's important to remember that slope represents a *rate of change*. The notation $m = \frac{\Delta y}{\Delta x}$ literally means the quantity measured along the y-axis, is changing with respect to changes in the quantity measured along the x-axis.

EXAMPLE 11 ► **Relating Temperature to Altitude**

In meteorological studies, atmospheric temperature depends on the altitude according to the formula $T = -3.5h + 58.6$, where T represents the approximate Fahrenheit temperature at height h (in thousands of feet).

a. Interpret the meaning of the slope in this context.
b. Determine the temperature at an altitude of 12,000 ft.
c. If the temperature is $-10°$F what is the approximate altitude?

Solution ► **a.** Notice that h is the input variable and T is the output. This shows $\frac{\Delta T}{\Delta h} = \frac{-3.5}{1}$, meaning the temperature drops 3.5°F for every 1000-ft increase in altitude.

b. Since height is in thousands, use $h = 12$.

$$\begin{aligned} T &= -3.5h + 58.6 && \text{original function} \\ &= -3.5(12) + 58.6 && \text{substitute 12 for } h \\ &= 16.6 && \text{result} \end{aligned}$$

At a height of 12,000 ft, the temperature is about 17°F.

c. Replacing T with -10 and solving gives

$$\begin{aligned} -10 &= -3.5h + 58.6 && \text{substitute } -10 \text{ for } T \\ -68.6 &= -3.5h && \text{simplify} \\ 19.6 &= h && \text{result} \end{aligned}$$

The temperature is $-10°$F at a height of $19.6 \times 1000 = 19{,}600$ ft.

Now try Exercises 105 and 106 ►

In some applications, the relationship is known to be linear but only a few points on the line are given. In this case, we can use two of the known data points to calculate the slope, then the point-slope form to find an equation model. One such application is *linear depreciation,* as when a government allows businesses to depreciate vehicles and equipment over time (the less a piece of equipment is worth, the less you pay in taxes).

EXAMPLE 12A ► **Using Point-Slope Form to Find an Equation Model**

Five years after purchase, the auditor of a newspaper company estimates the value of their printing press is \$60,000. Eight years after its purchase, the value of the press had depreciated to \$42,000. Find a linear equation that models this depreciation and discuss the slope and y-intercept in context.

Solution ► Since the value of the press depends on time, the ordered pairs have the form (time, value) or (t, v) where *time* is the input, and *value* is the output. This means the ordered pairs are (5, 60,000) and (8, 42,000).

$$m = \frac{v_2 - v_1}{t_2 - t_1} \quad \text{slope formula}$$
$$= \frac{42{,}000 - 60{,}000}{8 - 5} \quad (t_1, v_1) = (5, 60{,}000);\ (t_2, v_2) = (8, 42{,}000)$$
$$= \frac{-18{,}000}{3} = \frac{-6000}{1} \quad \text{simplify and reduce}$$

The slope of the line is $\frac{\Delta\text{value}}{\Delta\text{time}} = \frac{-6000}{1}$, indicating the printing press loses $6000 in value with each passing year.

$$v - v_1 = m(t - t_1) \quad \text{point-slope form}$$
$$v - 60{,}000 = -6000(t - 5) \quad \text{substitute } -6000 \text{ for } m;\ (5, 60{,}000) \text{ for } (t_1, v_1)$$
$$v - 60{,}000 = -6000t + 30{,}000 \quad \text{simplify}$$
$$v = -6000t + 90{,}000 \quad \text{solve for } v$$

The depreciation equation is $v = -6000t + 90{,}000$. The v-intercept (0, 90,000) indicates the original value (cost) of the equipment was $90,000.

WORTHY OF NOTE

Actually, it doesn't matter which of the two points are used in Example 12A. Once the point (5, 60,000) is plotted, a constant slope of $m = -6000$ will "drive" the line through (8, 42,000). If we first graph (8, 42,000), the same slope would "drive" the line through (5, 60,000). Convince yourself by reworking the problem using the other point.

Once the depreciation equation is found, it represents the (time, value) relationship for all future (and intermediate) ages of the press. In other words, we can now predict the value of the press for any given year. However, note that some equation models are valid for only a set period of time, and each model should be used with care.

EXAMPLE 12B ► Using an Equation Model to Gather Information

From Example 12A,

a. How much will the press be worth after 11 yr?
b. How many years until the value of the equipment is less than $9,000?
c. Is this equation model valid for $t = 18$ yr (why or why not)?

Solution ► **a.** Find the value v when $t = 11$:

$$v = -6000t + 90{,}000 \quad \text{equation model}$$
$$v = -6000(11) + 90{,}000 \quad \text{substitute 11 for } t$$
$$= 24{,}000 \quad \text{result (11, 24,000)}$$

After 11 yr, the printing press will only be worth $24,000.

b. ". . . value is less than $9000" means $v < 9000$:

$$v < 9000 \quad \text{value at time } t$$
$$-6000t + 90{,}000 < 9000 \quad \text{substitute } -6000t + 90{,}000 \text{ for } v$$
$$-6000t < -81{,}000 \quad \text{subtract 90,000}$$
$$t > 13.5 \quad \text{divide by } -6000, \textbf{ reverse inequality symbol}$$

After 13.5 yr, the printing press will be worth less than $9000.

c. Since substituting 18 for t gives a negative quantity, the equation model is not valid for $t = 18$. In the current context, the model is only valid while $v \geq 0$ and we note the domain of the function is $t \in [0, 15]$.

D. You've just learned how to apply the slope-intercept form and point-slope form in context

Now try Exercises 107 through 112 ►

2.3 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For the equation $y = -\frac{7}{4}x + 3$, the slope is______ and the y-intercept is ______.
2. The notation $\frac{\Delta\text{cost}}{\Delta\text{time}}$ indicates the ______ is changing in response to changes in ______.
3. Line 1 has a slope of -0.4. The slope of any line perpendicular to line 1 is ______.
4. The equation $y - y_1 = m(x - x_1)$ is called the ______ form of a line.
5. Discuss/Explain how to graph a line using only the slope and a point on the line (no equations).
6. Given $m = -\frac{3}{5}$ and $(-5, 6)$ is on the line. Compare and contrast finding the equation of the line using $y = mx + b$ versus $y - y_1 = m(x - x_1)$.

▶ DEVELOPING YOUR SKILLS

Solve each equation for y and evaluate the result using $x = -5, x = -2, x = 0, x = 1$, and $x = 3$.

7. $4x + 5y = 10$
8. $3y - 2x = 9$
9. $-0.4x + 0.2y = 1.4$
10. $-0.2x + 0.7y = -2.1$
11. $\frac{1}{3}x + \frac{1}{5}y = -1$
12. $\frac{1}{7}y - \frac{1}{3}x = 2$

For each equation, solve for y and identify the new coefficient of x and new constant term.

13. $6x - 3y = 9$
14. $9y - 4x = 18$
15. $-0.5x - 0.3y = 2.1$
16. $-0.7x + 0.6y = -2.4$
17. $\frac{5}{6}x + \frac{1}{7}y = -\frac{4}{7}$
18. $\frac{7}{12}y - \frac{4}{15}x = \frac{7}{6}$

Evaluate each equation by selecting three inputs that will result in integer values. Then graph each line.

19. $y = -\frac{4}{3}x + 5$
20. $y = \frac{5}{4}x + 1$
21. $y = -\frac{3}{2}x - 2$
22. $y = \frac{2}{5}x - 3$
23. $y = -\frac{1}{6}x + 4$
24. $y = -\frac{1}{3}x + 3$

Find the x- and y-intercepts for each line, then (a) use these two points to calculate the slope of the line, (b) write the equation with y in terms of x (solve for y) and compare the calculated slope and y-intercept to the equation from part (b). Comment on what you notice.

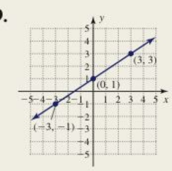
25. $3x + 4y = 12$
26. $3y - 2x = -6$
27. $2x - 5y = 10$
28. $2x + 3y = 9$
29. $4x - 5y = -15$
30. $5y + 6x = -25$

Write each equation in slope-intercept form (solve for y), then identify the slope and y-intercept.

31. $2x + 3y = 6$
32. $4y - 3x = 12$
33. $5x + 4y = 20$
34. $y + 2x = 4$
35. $x = 3y$
36. $2x = -5y$
37. $3x + 4y - 12 = 0$
38. $5y - 3x + 20 = 0$

For Exercises 39 to 50, use the slope-intercept form to state the equation of each line.

39.



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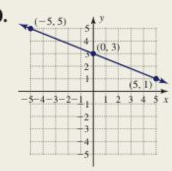

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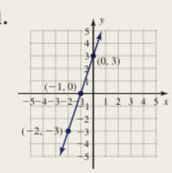
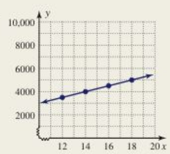


42. $m = -2$; y-intercept $(0, -3)$
43. $m = 3$; y-intercept $(0, 2)$
44. $m = -\frac{3}{2}$; y-intercept $(0, -4)$

45.



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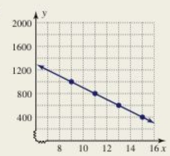

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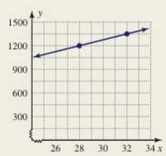


48. $m = -4$; $(-3, 2)$ is on the line

49. $m = 2$; $(5, -3)$ is on the line

50. $m = -\frac{3}{2}$; $(-4, 7)$ is on the line

Write each equation in slope-intercept form, then use the slope and intercept to graph the line.

51. $3x + 5y = 20$

52. $2y - x = 4$

53. $2x - 3y = 15$

54. $-3x + 2y = 4$

Graph each linear equation using the *y*-intercept and slope determined from each equation.

55. $y = \frac{2}{3}x + 3$

56. $y = \frac{5}{2}x - 1$

57. $y = \frac{-1}{3}x + 2$

58. $y = \frac{-4}{5}x + 2$

59. $y = 2x - 5$

60. $y = -3x + 4$

61. $y = \frac{1}{2}x - 3$

62. $y = \frac{-3}{2}x + 2$

Find the equation of the line using the information given. Write answers in slope-intercept form.

63. parallel to $2x - 5y = 10$, through the point $(-5, 2)$

64. parallel to $6x + 9y = 27$, through the point $(-3, -5)$

65. perpendicular to $5y - 3x = 9$, through the point $(6, -3)$

66. perpendicular to $x - 4y = 7$, through the point $(-5, 3)$

67. parallel to $12x + 5y = 65$, through the point $(-2, -1)$

68. parallel to $15y - 8x = 50$, through the point $(3, -4)$

69. parallel to $y = -3$, through the point $(2, 5)$

70. perpendicular to $y = -3$ through the point $(2, 5)$

Write the lines in slope-intercept form and state whether they are parallel, perpendicular, or neither.

71. $4y - 5x = 8$
$5y + 4x = -15$

72. $3y - 2x = 6$
$-2x + 3y = -3$

73. $2x - 5y = 20$
$4x - 3y = 18$

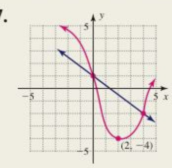
74. $5y = 11x + 135$
$11y + 5x = -77$

75. $-4x + 6y = 12$
$2x + 3y = 6$

76. $3x + 4y = 12$
$6x + 8y = 2$

A *secant line* is one that intersects a graph at two or more points. For each graph given, find the equation of the line (a) parallel and (b) perpendicular to the secant line, through the point indicated.

77.



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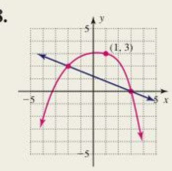

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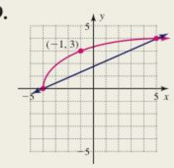

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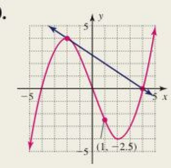

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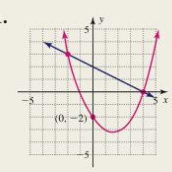

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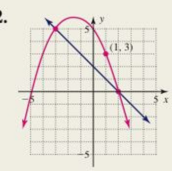


Find the equation of the line in point-slope form, then graph the line.

83. $m = 2$; $P_1 = (2, -5)$

84. $m = -1$; $P_1 = (2, -3)$

85. $P_1 = (3, -4)$, $P_2 = (11, -1)$

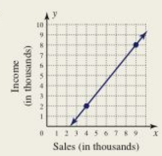
86. $P_1 = (-1, 6)$, $P_2 = (5, 1)$

87. $m = 0.5$; $P_1 = (1.8, -3.1)$

88. $m = 1.5$; $P_1 = (-0.75, -0.125)$

Find the equation of the line in point-slope form, and state the meaning of the slope in context—what information is the slope giving us?

89.



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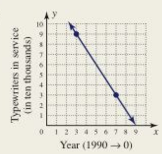

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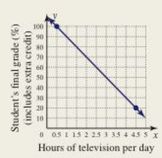

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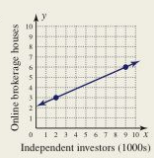

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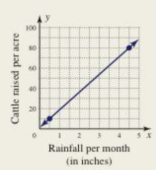

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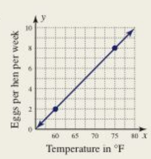
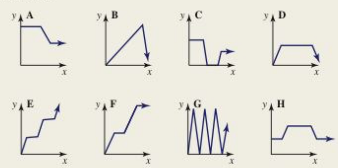


Using the concept of slope, match each description with the graph that best illustrates it. Assume time is scaled on the horizontal axes, and height, speed, or distance from the origin (as the case may be) is scaled on the vertical axis.



95. While driving today, I got stopped by a state trooper. After she warned me to slow down, I continued on my way.

96. After hitting the ball, I began trotting around the bases shouting, "Ooh, ooh, ooh!" When I saw it wasn't a home run, I began sprinting.

97. At first I ran at a steady pace, then I got tired and walked the rest of the way.

98. While on my daily walk, I had to run for a while when I was chased by a stray dog.

99. I climbed up a tree, then I jumped out.

100. I steadily swam laps at the pool yesterday.

101. I walked toward the candy machine, stared at it for a while then changed my mind and walked back.

102. For practice, the girls' track team did a series of 25-m sprints, with a brief rest in between.

▶ WORKING WITH FORMULAS

103. General linear equation: $ax + by = c$

The general equation of a line is shown here, where a, b, and c are real numbers, with a and b not simultaneously zero. Solve the equation for y and note the slope (coefficient of x) and y-intercept (constant term). Use these to find the slope and y-intercept of the following lines, without solving for y or computing points.

a. $3x + 4y = 8$ **b.** $2x + 5y = -15$

c. $5x - 6y = -12$ **d.** $3y - 5x = 9$

104. Intercept/Intercept form of a linear equation: $\frac{x}{h} + \frac{y}{k} = 1$

The x- and y-intercepts of a line can also be found by writing the equation in the form shown (with the equation set equal to 1). The x-intercept will be $(h, 0)$ and the y-intercept will be $(0, k)$. Find the x- and y-intercepts of the following lines using this method: (a) $2x + 5y = 10$, (b) $3x - 4y = -12$, and (c) $5x + 4y = 8$. How is the slope of each line related to the values of h and k?

▶ APPLICATIONS

105. Speed of sound: The speed of sound as it travels through the air depends on the temperature of the air according to the function $V = \frac{3}{5}C + 331$, where V represents the velocity of the sound waves in meters per second (m/s), at a temperature of $C°$ Celsius.

a. Interpret the meaning of the slope and y-intercept in this context.

b. Determine the speed of sound at a temperature of 20°C.

c. If the speed of sound is measured at 361 m/s, what is the temperature of the air?

106. Acceleration: A driver going down a straight highway is traveling 60 ft/sec (about 41 mph) on cruise control, when he begins accelerating at a rate of 5.2 ft/sec^2. The final velocity of the car is given by $V = \frac{26}{5}t + 60$, where V is the velocity at time t. (a) Interpret the meaning of the slope and y-intercept in this context. (b) Determine the velocity of the car after 9.4 seconds. (c) If the car is traveling at 100 ft/sec, for how long did it accelerate?

107. Investing in coins: The purchase of a "collector's item" is often made in hopes the item will increase in value. In 1998, Mark purchased a 1909-S VDB Lincoln Cent (in fair condition) for \$150. By the year 2004, its value had grown to \$190. (a) Use the relation (time since purchase, value) with $t = 0$ corresponding to 1998 to find a linear equation modeling the value of the coin. (b) Discuss what the slope and y-intercept indicate in this context. (c) How much will the penny be worth in 2009? (d) How many years after purchase will the penny's value exceed \$250? (e) If the penny is now worth \$170, how many years has Mark owned the penny?

108. Depreciation: Once a piece of equipment is put into service, its value begins to depreciate. A business purchases some computer equipment for \$18,500. At the end of a 2-yr period, the value of the equipment has decreased to \$11,500. (a) Use the relation (time since purchase, value) to find a linear equation modeling the value of the equipment. (b) Discuss what the slope and y-intercept indicate in this context. (c) What is the equipment's value after 4 yr? (d) How many years after purchase will the value decrease to \$6000? (e) Generally, companies will sell used equipment while it still has value and use the funds to purchase new equipment. According to the function, how many years will it take this equipment to depreciate in value to \$1000?

109. Internet connections: The number of households that are hooked up to the Internet (homes that are online) has been increasing steadily in recent years. In 1995, approximately 9 million homes were online. By 2001 this figure had climbed to about 51 million. (a) Use the relation (year, homes online) with $t = 0$ corresponding to 1995 to find an equation model for the number of homes online. (b) Discuss what the slope indicates in this context. (c) According to this model, in what year did the first homes begin to come online? (d) If the rate of change stays constant, how many households will be on the Internet in 2006? (e) How many years after 1995 will there be over 100 million households connected? (f) If there are 115 million households connected, what year is it?

Source: 2004 Statistical Abstract of the United States, Table 965

110. Prescription drugs: Retail sales of prescription drugs have been increasing steadily in recent years. In 1995, retail sales hit \$72 billion. By the year 2000, sales had grown to about \$146 billion. (a) Use the relation (year, retail sales of prescription drugs) with $t = 0$ corresponding to 1995 to find a linear equation modeling the growth of retail sales. (b) Discuss what the slope indicates in this context. (c) According to this model, in what year will sales reach \$250 billion? (d) According to the model, what was the value of retail prescription drug sales in 2005? (e) How many years after 1995 will retail sales exceed \$279 billion? (f) If yearly sales totaled \$294 billion, what year is it?

Source: 2004 Statistical Abstract of the United States, Table 122

111. Prison population: In 1990, the number of persons sentenced and serving time in state and federal institutions was approximately 740,000. By the year 2000, this figure had grown to nearly 1,320,000. (a) Find a linear equation with $t = 0$ corresponding to 1990 that models this data, (b) discuss the slope ratio in context, and (c) use the equation to estimate the prison population in 2007 if this trend continues.

Source: Bureau of Justice Statistics at www.ojp.usdoj.gov/bjs

112. Eating out: In 1990, Americans bought an average of 143 meals per year at restaurants. This phenomenon continued to grow in popularity and in the year 2000, the average reached 170 meals per year. (a) Find a linear equation with $t = 0$ corresponding to 1990 that models this growth, (b) discuss the slope ratio in context, and (c) use the equation to estimate the average number of times an American will eat at a restaurant in 2006 if the trend continues.

Source: The NPD Group, Inc., National Eating Trends, 2002

► EXTENDING THE CONCEPT

113. Locate and read the following article. Then turn in a one-page summary. "Linear Function Saves Carpenter's Time," Richard Crouse, *Mathematics Teacher,* Volume 83, Number 5, May 1990: pp. 400–401.

114. The general form of a linear equation is $ax + by = c$, where a and b are not simultaneously zero. (a) Find the x- and y-intercepts using the general form (substitute 0 for x, then 0 for y). Based on what you see, when does the intercept method work most efficiently? (b) Find the slope and y-intercept using the general form (solve for y). Based on what you see, when does the intercept method work most efficiently?.

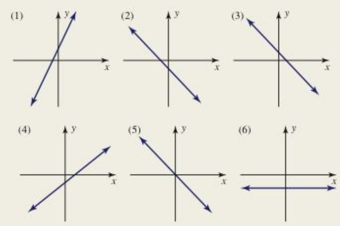
115. Match the correct graph to the conditions stated for m and b. There are more choices than graphs.

a. $m < 0, b < 0$ **b.** $m > 0, b < 0$

c. $m < 0, b > 0$ **d.** $m > 0, b > 0$

e. $m = 0, b > 0$ **f.** $m < 0, b = 0$

g. $m > 0, b = 0$ **h.** $m = 0, b < 0$



▶ MAINTAINING YOUR SKILLS

116. **(2.2)** Determine the domain:

a. $y = \sqrt{2x - 5}$

b. $y = \dfrac{5}{2x^2 + 3x - 2}$

117. **(1.5)** Solve using the quadratic formula. Answer in exact and approximate form: $3x^2 - 10x = 9$.

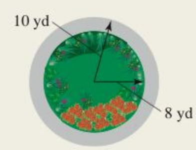
118. **(1.1)** Three equations follow. One is an identity, another is a contradiction, and a third has a solution. State which is which.

$2(x - 5) + 13 - 1 = 9 - 7 + 2x$

$2(x - 4) + 13 - 1 = 9 + 7 - 2x$

$2(x - 5) + 13 - 1 = 9 + 7 + 2x$

119. **(R.7)** Compute the area of the circular sidewalk shown here. Use your calculator's value of π and round the answer (only) to hundredths.



2.4 Functions, Function Notation, and the Graph of a Function

Learning Objectives

In Section 2.4 you will learn how to:

- ❑ **A.** Distinguish the graph of a function from that of a relation
- ❑ **B.** Determine the domain and range of a function
- ❑ **C.** Use function notation and evaluate functions
- ❑ **D.** Apply the rate-of-change concept to nonlinear functions

In this section we introduce one of the most central ideas in mathematics—the concept of a function. Functions can model the cause-and-effect relationship that is so important to using mathematics as a decision-making tool. In addition, the study will help to unify and expand on many ideas that are already familiar.

A. Functions and Relations

There is a special type of relation that merits further attention. A **function** is a relation where each element of the domain corresponds to exactly one element of the range. In other words, for each first coordinate or input value, there is only one possible second coordinate or output.

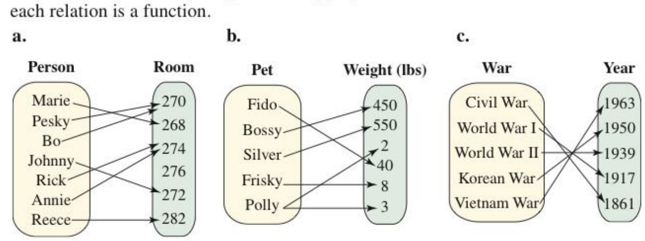
Functions

A *function* is a relation that pairs each element from the *domain* with exactly one element from the *range.*

If the relation is defined by a mapping, we need only check that each element of the domain is mapped to exactly one element of the range. This is indeed the case for the mapping $P \rightarrow B$ from Figure 2.1 (page 152), where we saw that each person corresponded to only one birthday, and that it was impossible for one person to be born on two different days. For the relation $x = |y|$ shown in Figure 2.6 (page 153), each element of the domain except zero is paired with *more than one* element of the range. The relation $x = |y|$ is *not* a function.

EXAMPLE 1 ► Determining Whether a Relation Is a Function

Three different relations are given in mapping notation below. Determine whether each relation is a function.



Solution ► Relation (a) is a function, since each person corresponds to exactly one room. This relation pairs math professors with their respective office numbers. Notice that while two people can be in one office, it is impossible for one person to physically be in two different offices. Relation (b) is not a function, since we cannot tell whether Polly the Parrot weighs 2 lb or 3 lb (one element of the domain is mapped to two elements of the range). Relation (c) is a function, where each major war is paired with the year it began.

Now try Exercises 7 through 10 ►

If the relation is defined by a set of ordered pairs or a set of individual and distinct plotted points, we need only check that no two points have the same first coordinate with a different second coordinate.

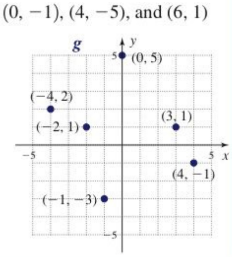
EXAMPLE 2 ► Identifying Functions

Two relations named f and g are given; f is stated as a set of ordered pairs, while g is given as a set of plotted points. Determine whether each is a function.

$$f\text{: } (-3, 0), (1, 4), (2, -5), (4, 2), (-3, -2), (3, 6), (0, -1), (4, -5), \text{and } (6, 1)$$

Solution ► The relation f is not a function, since -3 is paired with two different outputs: $(-3, 0)$ and $(-3, -2)$.

The relation g shown in the figure *is* a function. Each input corresponds to exactly one output, otherwise one point would be directly above the other and have the same first coordinate.



Now try Exercises 11 through 18 ►

WORTHY OF NOTE

The definition of a function can also be stated in ordered pair form: *A function is a set of ordered pairs (x, y), in which each first component is paired with only one second component.*

The graphs of $y = x - 1$ and $x = |y|$ from Section 2.1 offer additional insight into the definition of a function. Figure 2.23 shows the line $y = x - 1$ with emphasis on the plotted points (4, 3) and $(-3, -4)$. The vertical movement shown from the x-axis to a point on the graph illustrates *the pairing of a given x-value with one related y-value*. Note the vertical line shows *only one related y-value* ($x = 4$ is paired with only $y = 3$). Figure 2.24 gives the graph of $x = |y|$, highlighting the points (4, 4) and $(4, -4)$. The vertical movement shown here branches in two directions, associating one x-value with more than one y-value. This shows the relation $y = x - 1$ is also a function, while the relation $x = |y|$ is not.

Figure 2.23

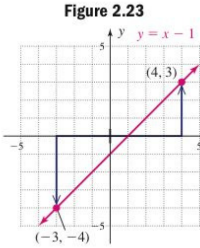
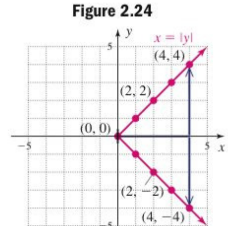


Figure 2.24



This "vertical connection" of a location on the x-axis to a point on the graph can be generalized into a **vertical line test for functions.**

Vertical Line Test

A given graph is the graph of a function, if and only if every vertical line intersects the graph in at most one point.

Applying the test to the graph in Figure 2.23 helps to illustrate that the graph of any nonvertical line is a function.

EXAMPLE 3 ▶ Using the Vertical Line Test

Use the vertical line test to determine if any of the relations shown (from Section 2.1) are functions.

Solution ▶ Visualize a vertical line on each coordinate grid (shown in solid blue), then mentally shift the line to the left and right as shown in Figures 2.25, 2.26, and 2.27 (dashed lines). In Figures 2.25 and 2.26, every vertical line intersects the graph only once, indicating both $y = x^2 - 2x$ and $y = \sqrt{9 - x^2}$ are functions. In Figure 2.27, a vertical line intersects the graph twice for any $x > 0$. The relation $x = y^2$ is not a function.

Figure 2.25

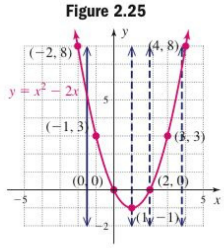


Figure 2.26

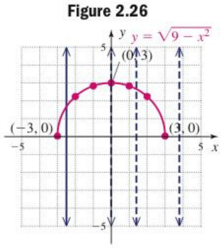
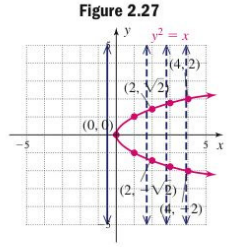


Figure 2.27



Now try Exercises 19 through 30 ▶

EXAMPLE 4 ► **Using the Vertical Line Test**

Use a table of values to graph the relations defined by

a. $y = |x|$ **b.** $y = \sqrt{x}$,

then use the vertical line test to determine whether each relation is a function.

Solution ► **a.** For $y = |x|$, using input values from $x = -4$ to $x = 4$ produces the following table and graph (Figure 2.28). Note the result is a V-shaped graph that "opens upward." The point (0, 0) of this absolute value graph is called the **vertex.** Since any vertical line will intersect the graph in at most one point, this is the graph of a function.

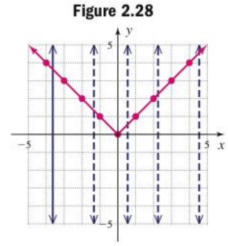
WORTHY OF NOTE

For relations and functions, a good way to view the distinction is to consider a mail carrier. It is possible for the carrier to put more than one letter into the same mailbox (more than one *x* going to the same *y*), but quite impossible for the carrier to place the same letter in two different boxes (one *x* going to two *y*'s).

$y = |x|$

x	$y = \|x\|$
−4	4
−3	3
−2	2
−1	1
0	0
1	1
2	2
3	3
4	4

Figure 2.28

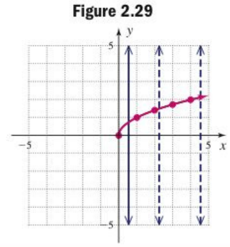


b. For $y = \sqrt{x}$, values less than zero do not produce a real number, so our graph actually begins at (0, 0) (see Figure 2.29). Completing the table for nonnegative values produces the graph shown, which appears to rise to the right and remains in the first quadrant. Since any vertical line will intersect this graph in at most one place, $y = \sqrt{x}$ is also a function.

$y = \sqrt{x}$

x	$y = \sqrt{x}$
0	0
1	1
2	$\sqrt{2} \approx 1.4$
3	$\sqrt{3} \approx 1.7$
4	2

Figure 2.29



A. You've just learned how to distinguish the graph of a function from that of a relation

Now try Exercises 31 through 34 ►

B. The Domain and Range of a Function

Vertical Boundary Lines and the Domain

In addition to its use as a graphical test for functions, a vertical line can help determine the domain of a function from its graph. For the graph of $y = \sqrt{x}$ (Figure 2.29), a vertical line will not intersect the graph until $x = 0$, and then will intersect the graph for all values $x \geq 0$ (showing the function is defined for these values). These **vertical boundary lines** indicate the domain is $x \in [0, \infty)$. For the graph of $y = |x|$ (Figure 2.28), a vertical line will intersect the graph (or its infinite extension) for *all values* of *x*, and the domain is $x \in (-\infty, \infty)$. Using vertical lines in this way also affirms the domain of $y = x - 1$ (Section 2.1, Figure 2.5) is $x \in (-\infty, \infty)$ while the domain of the relation $x = |y|$ (Section 2.1, Figure 2.6) is $x \in [0, \infty)$.

Range and Horizontal Boundary Lines

The range of a relation can be found using a **horizontal "boundary line,"** since it will associate a value on the y-axis with a point on the graph (if it exists). Simply visualize a horizontal line and move the line up or down until you determine the graph will always intersect the line, or will no longer intersect the line. This will give you the boundaries of the range. Mentally applying this idea to the graph of $y = \sqrt{x}$ (Figure 2.29) shows the range is $y \in [0, \infty)$. Although shaped very differently, a horizontal boundary line shows the range of $y = |x|$ (Figure 2.28) is also $y \in [0, \infty)$.

EXAMPLE 5 ► **Determining the Domain and Range of a Function**

Use a table of values to graph the functions defined by

a. $y = x^2$ **b.** $y = \sqrt[3]{x}$

Then use boundary lines to determine the domain and range of each.

Solution ► **a.** For $y = x^2$, it seems convenient to use inputs from $x = -3$ to $x = 3$, producing the following table and graph. Note the result is a basic parabola that "opens upward" (both ends point in the positive y direction), with a vertex at (0, 0). Figure 2.30 shows a vertical line will intersect the graph or its extension anywhere it is placed. The domain is $x \in (-\infty, \infty)$. Figure 2.31 shows a horizontal line will intersect the graph only for values of y that are greater than or equal to 0. The range is $y \in [0, \infty)$.

Squaring Function

x	$y = x^2$
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9

Figure 2.30

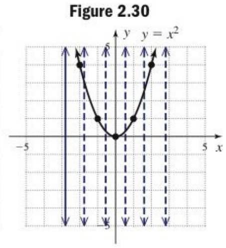
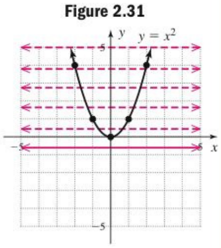


Figure 2.31



b. For $y = \sqrt[3]{x}$, we select points that are perfect cubes where possible, then a few others to round out the graph. The resulting table and graph are shown, and we notice there is a "pivot point" at (0, 0) called a **point of inflection,** and the ends of the graph point in opposite directions. Figure 2.32 shows a vertical line will intersect the graph or its extension anywhere it is placed. Figure 2.33 shows a horizontal line will likewise always intersect the graph. The domain is $x \in (-\infty, \infty)$, and the range is $y \in (-\infty, \infty)$.

Cube Root Function

x	$y = \sqrt[3]{x}$
-8	-2
-4	≈ -1.6
-1	-1
0	0
1	1
4	≈ 1.6
8	2

Figure 2.32

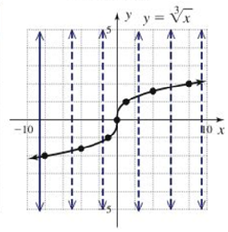
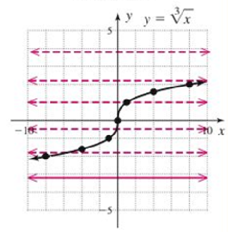


Figure 2.33



Now try Exercises 35 through 46 ►

Implied Domains

When stated in equation form, the domain of a function is implicitly given by the expression used to define it, since the expression will dictate the allowable values (Section 1.2). The **implied domain** is the set of all real numbers for which the function represents a real number. If the function involves a rational expression, the domain will exclude any input that causes a denominator of zero. If the function involves a square root expression, the domain will exclude inputs that create a negative radicand.

EXAMPLE 6 ► Determining Implied Domains

State the domain of each function using interval notation.

a. $y = \frac{3}{x + 2}$ **b.** $y = \sqrt{2x + 3}$

c. $y = \frac{x - 5}{x^2 - 9}$ **d.** $y = x^2 - 5x + 7$

Solution ►

a. By inspection, we note an x-value of -2 gives a zero denominator and must be excluded. The domain is $x \in (-\infty, -2) \cup (-2, \infty)$.

b. Since the radicand must be nonnegative, we solve the inequality $2x + 3 \geq 0$, giving $x \geq \frac{-3}{2}$. The domain is $x \in [\frac{-3}{2}, \infty)$.

c. To prevent division by zero, inputs of -3 and 3 must be excluded (set $x^2 - 9 = 0$ and solve by factoring). The domain is $x \in (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. Note that $x = 5$ *is in the domain* since $\frac{0}{16} = 0$ is defined.

d. Since squaring a number and multiplying a number by a constant are defined for all reals, the domain is $x \in (-\infty, \infty)$.

Now try Exercises 47 through 64 ►

EXAMPLE 7 ► Determining Implied Domains

Determine the domain of each function:

a. $y = \sqrt{\frac{7}{x + 3}}$ **b.** $y = \frac{2x}{\sqrt{4x + 5}}$

Solution ► **a.** For $y = \sqrt{\frac{7}{x+3}}$, we must have $\frac{7}{x+3} \geq 0$ (for the radicand) **and** $x + 3 \neq 0$ (for the denominator). Since the numerator is *always* positive, we need $x + 3 > 0$, which gives $x > -3$. The domain is $x \in (-3, \infty)$.

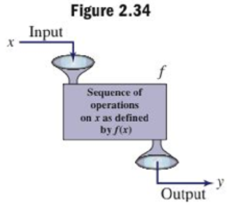
b. For $y = \frac{2x}{\sqrt{4x+5}}$, we must have $4x + 5 \geq 0$ **and** $\sqrt{4x+5} \neq 0$. This indicates $4x + 5 > 0$ or $x > -\frac{5}{4}$. The domain is $x \in (-\frac{5}{4}, \infty)$.

B. You've just learned how to determine the domain and range of a function

Now try Exercises 65 through 68 ►

C. Function Notation

Figure 2.34

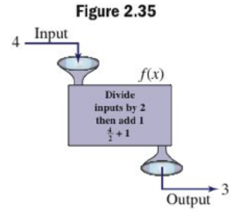


In our study of functions, you've likely noticed that the relationship between input and output values is an important one. To highlight this fact, think of a function as a simple machine, which can *process inputs* using a stated sequence of operations, then deliver a single output. The inputs are x-values, a program we'll name f performs the operations on x, and y is the resulting output (see Figure 2.34). Once again we see that "the value of y depends on the value of x," or simply "y is a function of x." Notationally, we write "y is a function of x" as $y = f(x)$ using **function notation.** You are already familiar with letting a variable represent a number. Here we do something quite different, as the letter f is used to represent *a sequence of operations to be performed on x*. Consider the function $y = \frac{x}{2} + 1$, which we'll now write as $f(x) = \frac{x}{2} + 1$ [since $y = f(x)$]. In words the function says, "divide inputs by 2, then add 1." To evaluate the function at $x = 4$ (Figure 2.35) we have:

input 4 — input 4

$$f(x) = \frac{x}{2} + 1$$
$$f(4) = \frac{4}{2} + 1$$
$$= 2 + 1$$
$$= 3$$

Figure 2.35



Instead of saying, ". . . when $x = 4$, the value of the function is 3," we simply say "f of 4 is 3," or write $f(4) = 3$. Note that the ordered pair (4, 3) is equivalent to $(4, f(4))$.

CAUTION ► Although $f(x)$ is the favored notation for a "function of x," other letters can also be used. For example, $g(x)$ and $h(x)$ also denote functions of x, where g and h represent a different sequence of operations on the x-inputs. It is also important to remember that these represent *function values* and not the product of two variables: $f(x) \neq f \cdot (x)$.

EXAMPLE 8 ► **Evaluating a Function**

Given $f(x) = -2x^2 + 4x$, find

a. $f(-2)$ **b.** $f\left(\frac{3}{2}\right)$ **c.** $f(2a)$ **d.** $f(a + 1)$

Solution ►

a.
$$f(x) = -2x^2 + 4x$$
$$f(-2) = -2(-2)^2 + 4(-2) = -8 + (-8) = -16$$

b.
$$f(x) = -2x^2 + 4x$$
$$f\left(\frac{3}{2}\right) = -2\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) = -\frac{9}{2} + 6 = \frac{3}{2}$$

c.
$$f(x) = -2x^2 + 4x$$
$$f(2a) = -2(2a)^2 + 4(2a) = -2(4a^2) + 8a = -8a^2 + 8a$$

d.
$$f(x) = -2x^2 + 4x$$
$$f(a + 1) = -2(a + 1)^2 + 4(a + 1) = -2(a^2 + 2a + 1) + 4a + 4 = -2a^2 - 4a - 2 + 4a + 4 = -2a^2 + 2$$

Now try Exercises 69 through 84 ►

Graphs are an important part of studying functions, and learning to read and interpret them correctly is a high priority. A graph highlights and emphasizes the all-important input/output relationship that defines a function. In this study, we hope to firmly establish that the following statements are synonymous:

1. $f(-2) = 5$
2. $(-2, f(-2)) = (-2, 5)$
3. $(-2, 5)$ is on the graph of f, and
4. When $x = -2, f(x) = 5$

EXAMPLE 9A ► **Reading a Graph**

For the functions $f(x)$ and $g(x)$ whose graphs are shown in Figures 2.36 and 2.37

a. State the domain of the function.

b. Evaluate the function at $x = 2$.

c. Determine the value(s) of x for which $y = 3$.

d. State the range of the function.

Figure 2.36

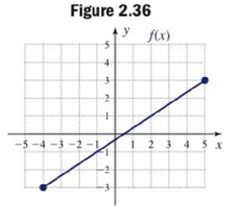
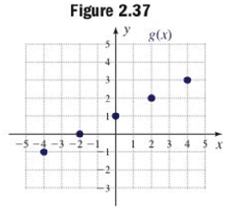


Figure 2.37



Solution ► For $f(x)$,

a. The graph is a continuous line segment with endpoints at $(-4, -3)$ and $(5, 3)$, so we state the domain in interval notation. Using a vertical boundary line we note the smallest input is -4 and the largest is 5. The domain is $x \in [-4, 5]$.

b. The graph shows an input of $x = 2$ corresponds to $y = 1$: $f(2) = 1$ since $(2, 1)$ is a point on the graph.

c. For $f(x) = 3$ (or $y = 3$) the input value must be $x = 5$ since $(5, 3)$ is the point on the graph.

d. Using a horizontal boundary line, the smallest output value is -3 and the largest is 3. The range is $y \in [-3, 3]$.

For $g(x)$,

a. Since the graph is pointwise defined, we state the domain as the set of first coordinates: $D = \{-4, -2, 0, 2, 4\}$.

b. An input of $x = 2$ corresponds to $y = 2$: $g(2) = 2$ since $(2, 2)$ is on the graph.

c. For $g(x) = 3$ (or $y = 3$) the input value must be $x = 4$, since $(4, 3)$ is a point on the graph.

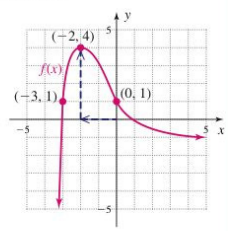
d. The range is the set of all second coordinates: $R = \{-1, 0, 1, 2, 3\}$.

EXAMPLE 9B ▶ Reading a Graph

Use the graph of $f(x)$ given to answer the following questions:

a. What is the value of $f(-2)$?

b. What value(s) of x satisfy $f(x) = 1$?



Solution ▶ **a.** The notation $f(-2)$ says to find the value of the function f when $x = -2$. Expressed graphically, we go to $x = -2$, locate the corresponding point on the graph of f (blue arrows), and find that $f(-2) = 4$.

b. For $f(x) = 1$, we're looking for x-inputs that result in an output of $y = 1$ [since $y = f(x)$]. From the graph, we note there are two points with a y-coordinate of 1, namely, $(-3, 1)$ and $(0, 1)$. This shows $f(-3) = 1$, $f(0) = 1$, and the required x-values are $x = -3$ and $x = 0$.

Now try Exercises 85 through 90 ▶

In many applications involving functions, the domain and range can be determined by the context or situation given.

EXAMPLE 10 ▶ Determining the Domain and Range from the Context

Paul's 1993 Voyager has a 20-gal tank and gets 18 mpg. The number of miles he can drive (his range) depends on how much gas is in the tank. As a function we have $M(g) = 18g$, where $M(g)$ represents the total distance in miles and g represents the gallons of gas in the tank. Find the domain and range.

Solution ▶ Begin evaluating at $x = 0$, since the tank cannot hold less than zero gallons. On a full tank the maximum range of the van is $20 \cdot 18 = 360$ miles or $M(g) \in [0, 360]$. Because of the tank's size, the domain is $g \in [0, 20]$.

Now try Exercises 94 through 101 ▶

C. You've just learned how to use function notation and evaluate functions

D. Average Rates of Change

As noted in Section 2.3, one of the defining characteristics of a linear function is that the rate of change $m = \frac{\Delta y}{\Delta x}$ is constant. For nonlinear functions the rate of change is not constant, but we can use a related concept called the **average rate of change** to study these functions.

Average Rate of Change

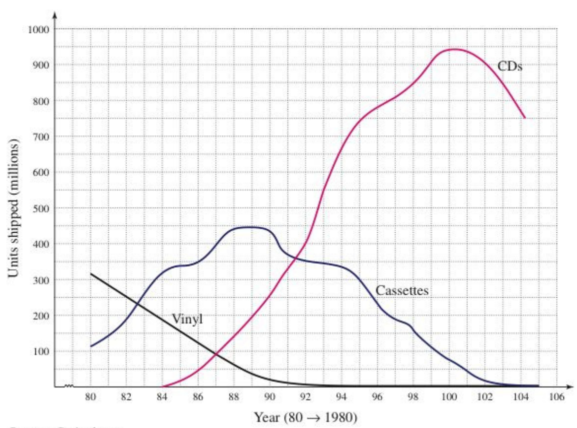
For a function that is smooth and continuous on the interval containing x_1 and x_2, the average rate of change between x_1 and x_2 is given by

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

which is the slope of the secant line through (x_1, y_1) and (x_2, y_2)

EXAMPLE 11 ► Calculating Average Rates of Change

The graph shown displays the number of units shipped of vinyl records, cassette tapes, and CDs for the period 1980 to 2005.



Source: Swivel.com

Units shipped in millions

Year	Vinyl	Cassette	CDs
1980	323	110	0
1982	244	182	0
1984	205	332	6
1986	125	345	53
1988	72	450	150
1990	12	442	287
1992	2	366	408
1994	2	345	662
1996	3	225	779
1998	3	159	847
2000	2	76	942
2004	1	5	767
2005	1	3	705

a. Find the average rate of change in CDs shipped and in cassettes shipped from 1994 to 1998. What do you notice?

b. Does it appear that the rate of increase in CDs shipped was greater from 1986 to 1992, or from 1992 to 1996? Compute the average rate of change for each period and comment on what you find.

Solution ► Using 1980 as year zero ($1980 \rightarrow 0$), we have the following:

a. **CDs**

1994: (14, 662), 1998: (18, 847)

$$\frac{\Delta y}{\Delta x} = \frac{847 - 662}{18 - 14} = \frac{185}{4} = 46.25$$

Cassettes

1994: (14, 345), 1998: (18, 159)

$$\frac{\Delta y}{\Delta x} = \frac{159 - 345}{18 - 14} = -\frac{186}{4} = -46.5$$

The decrease in the number of cassettes shipped was roughly equal to the increase in the number of CDs shipped (about 46,000,000 per year).

b. From the graph, the secant line for 1992 to 1996 appears to have a greater slope.

1986–1992 CDs	**1992–1996 CDs**
1986: (6, 53), 1992: (12, 408)	1992: (12, 408), 1996: (16, 779)
$\frac{\Delta y}{\Delta x} = \frac{408 - 53}{12 - 6}$	$\frac{\Delta y}{\Delta x} = \frac{779 - 408}{16 - 12}$
$= \frac{355}{6}$	$= \frac{371}{4}$
$= 59.1\overline{6}$	$= 92.75$

For 1986 to 1992: $m \approx 59.2$; for 1992 to 1996: $m = 92.75$, a growth rate much higher than the earlier period.

D. You've just learned how to apply the rate-of-change concept to nonlinear functions

Now try Exercises 102 and 103 ▶

2.4 EXERCISES

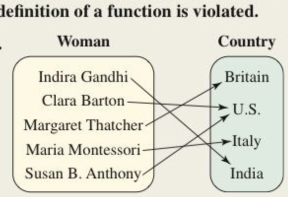
▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. If a relation is given in ordered pair form, we state the domain by listing all of the _______ coordinates in a set.
2. A relation is a function if each element of the _______ is paired with _______ _______ element of the range.
3. The set of output values for a function is called the _______ of the function.
4. Write using function notation: The function f evaluated at 3 is negative 5: _______
5. Discuss/Explain why the relation $y = x^2$ is a function, while the relation $x = y^2$ is not. Justify your response using graphs, ordered pairs, and so on.
6. Discuss/Explain the process of finding the domain and range of a function given its graph, using vertical and horizontal boundary lines. Include a few illustrative examples.

▶ DEVELOPING YOUR SKILLS

Determine whether the mappings shown represent functions or nonfunctions. If a nonfunction, explain how the definition of a function is violated.

7.


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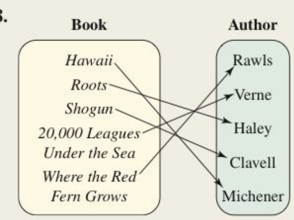

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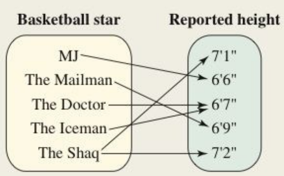

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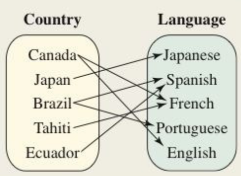


Determine whether the relations indicated represent functions or nonfunctions. If the relation is a nonfunction, explain how the definition of a function is violated.

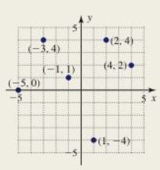
11. $(-3, 0), (1, 4), (2, -5), (4, 2), (-5, 6), (3, 6), (0, -1), (4, -5)$, and $(6, 1)$

12. $(-7, -5), (-5, 3), (4, 0), (-3, -5), (1, -6), (0, 9), (2, -8), (3, -2)$, and $(-5, 7)$

13. $(9, -10), (-7, 6), (6, -10), (4, -1), (2, -2), (1, 8), (0, -2), (-2, -7)$, and $(-6, 4)$

14. $(1, -81), (-2, 64), (-3, 49), (5, -36), (-8, 25), (13, -16), (-21, 9), (34, -4)$, and $(-55, 1)$

15.



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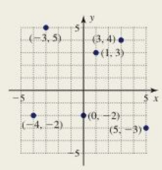

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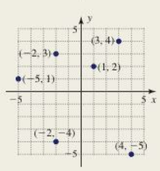

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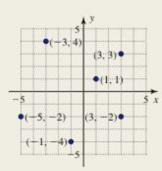
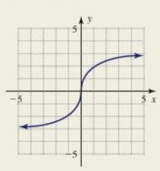


Determine whether or not the relations given represent a function. If not, explain how the definition of a function is violated.

19.



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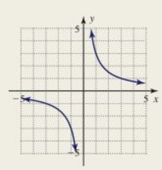

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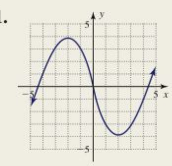

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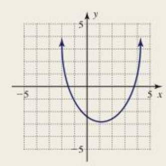

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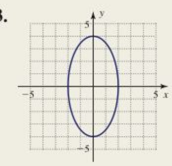

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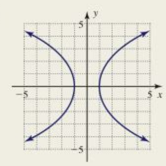

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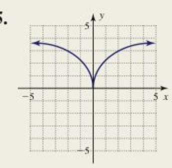

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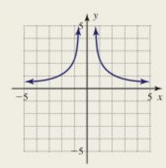

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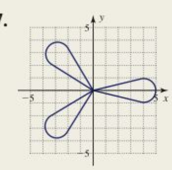

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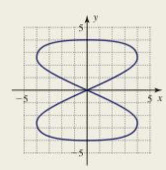

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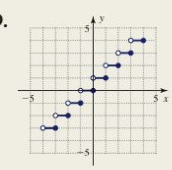

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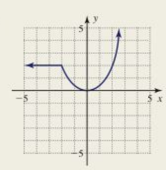


Graph each relation using a table, then use the vertical line test to determine if the relation is a function.

31. $y = x$

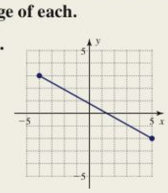
32. $y = \sqrt[3]{x}$

33. $y = (x + 2)^2$

34. $x = |y - 2|$

Determine whether or not the relations indicated represent a function, then determine the domain and range of each.

35.



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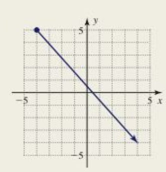

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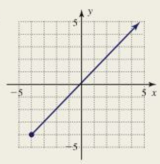

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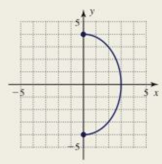

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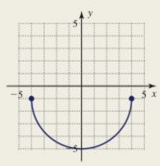

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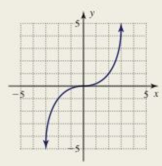

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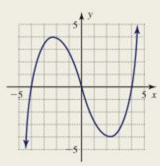

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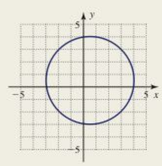

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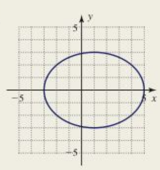

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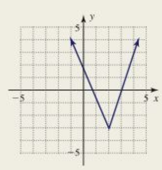

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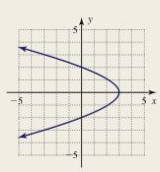

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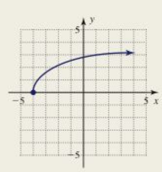


Determine the domain of the following functions.

47. $f(x) = \frac{3}{x - 5}$ **48.** $g(x) = \frac{-2}{3 + x}$

49. $h(a) = \sqrt{3a + 5}$ **50.** $p(a) = \sqrt{5a - 2}$

51. $v(x) = \frac{x + 2}{x^2 - 25}$ **52.** $w(x) = \frac{x - 4}{x^2 - 49}$

53. $u = \frac{v - 5}{v^2 - 18}$ **54.** $p = \frac{q + 7}{q^2 - 12}$

55. $y = \frac{17}{25}x + 123$ **56.** $y = \frac{11}{19}x - 89$

57. $m = n^2 - 3n - 10$ **58.** $s = t^2 - 3t - 10$

59. $y = 2|x| + 1$ **60.** $y = |x - 2| + 3$

61. $y_1 = \frac{x}{x^2 - 3x - 10}$ **62.** $y_2 = \frac{x - 4}{x^2 + 2x - 15}$

63. $y = \frac{\sqrt{x - 2}}{2x - 5}$ **64.** $y = \frac{\sqrt{x + 1}}{3x + 2}$

65. $f(x) = \sqrt{\frac{5}{x - 2}}$ **66.** $g(x) = \sqrt{\frac{-4}{3 - x}}$

67. $h(x) = \frac{-2}{\sqrt{4 + x}}$ **68.** $p(x) = \frac{-7}{\sqrt{5 - x}}$

Determine the value of $f(-6)$, $f(\frac{3}{2})$, $f(2c)$, and $f(c + 1)$, then simplify as much as possible.

69. $f(x) = \frac{1}{2}x + 3$ **70.** $f(x) = \frac{2}{3}x - 5$

71. $f(x) = 3x^2 - 4x$ **72.** $f(x) = 2x^2 + 3x$

Determine the value of $h(3)$, $h(-\frac{2}{3})$, $h(3a)$, and $h(a - 2)$, then simplify as much as possible.

73. $h(x) = \frac{3}{x}$ **74.** $h(x) = \frac{2}{x^2}$

75. $h(x) = \frac{5|x|}{x}$ **76.** $h(x) = \frac{4|x|}{x}$

Determine the value of $g(4)$, $g(\frac{3}{2})$, $g(2c)$, and $g(c + 3)$, then simplify as much as possible.

77. $g(r) = 2\pi r$ **78.** $g(r) = 2\pi rh$

79. $g(r) = \pi r^2$ **80.** $g(r) = \pi r^2 h$

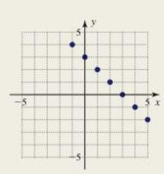
Determine the value of $p(5)$, $p(\frac{3}{2})$, $p(3a)$, and $p(a - 1)$, then simplify as much as possible.

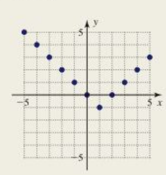
81. $p(x) = \sqrt{2x + 3}$ **82.** $p(x) = \sqrt{4x - 1}$

83. $p(x) = \frac{3x^2 - 5}{x^2}$ **84.** $p(x) = \frac{2x^2 + 3}{x^2}$

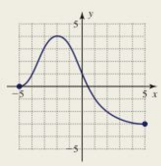
Use the graph of each function given to (a) state the domain, (b) state the range, (c) evaluate $f(2)$, and (d) find the value(s) x for which $f(x) = k$ (k a constant). Assume all results are integer-valued.

85. $k = 4$ **86.** $k = 3$

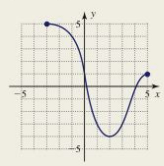




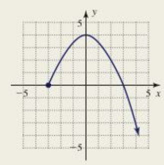
87. $k = 1$



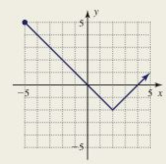
88. $k = -3$



89. $k = 2$



90. $k = -1$



▶ WORKING WITH FORMULAS

91. Ideal weight for males: $W(H) = \frac{9}{2}H - 151$

The ideal weight for an adult male can be modeled by the function shown, where W is his weight in pounds and H is his height in inches. (a) Find the ideal weight for a male who is 75 in. tall. (b) If I am 72 in. tall and weigh 210 lb, how much weight should I lose?

92. Celsius to Fahrenheit conversions: $C = \frac{5}{9}(F - 32)$

The relationship between Fahrenheit degrees and degrees Celsius is modeled by the function shown. (a) What is the Celsius temperature if °F = 41? (b) Use the formula to solve for F in terms of C, then substitute the result from part (a). What do you notice?

93. Pick's theorem: $A = \frac{1}{2}B + I - 1$

Picks theorem is an interesting yet little known formula for computing the area of a polygon drawn in the Cartesian coordinate system. The formula can be applied as long as the vertices of the polygon are lattice points (both x and y are integers). If B represents the number of lattice points lying directly on the boundary of the polygon (including the vertices), and I represents the number of points in the interior, the area of the polygon is given by the formula shown. Use some graph paper to carefully draw a triangle with vertices at $(-3, 1)$, $(3, 9)$, and $(7, 6)$, then use Pick's theorem to compute the triangle's area.

▶ APPLICATIONS

94. Gas mileage: John's old '87 LeBaron has a 15-gal gas tank and gets 23 mpg. The number of miles he can drive is a function of how much gas is in the tank. (a) Write this relationship in equation form and (b) determine the domain and range of the function in this context.

95. Gas mileage: Jackie has a gas-powered model boat with a 5-oz gas tank. The boat will run for 2.5 min on each ounce. The number of minutes she can operate the boat is a function of how much gas is in the tank. (a) Write this relationship in equation form and (b) determine the domain and range of the function in this context.

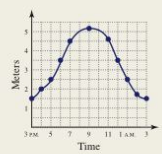
96. Volume of a cube: The volume of a cube depends on the length of the sides. In other words, volume is a function of the sides: $V(s) = s^3$. (a) In practical terms, what is the domain of this function? (b) Evaluate $V(6.25)$ and (c) evaluate the function for $s = 2x^2$.

97. Volume of a cylinder: For a fixed radius of 10 cm, the volume of a cylinder depends on its height. In other words, volume is a function of height: $V(h) = 100\pi h$. (a) In practical terms, what is the domain of this function? (b) Evaluate $V(7.5)$ and (c) evaluate the function for $h = \frac{8}{\pi}$.

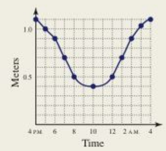
98. Rental charges: Temporary Transportation Inc. rents cars (local rentals only) for a flat fee of \$19.50 and an hourly charge of \$12.50. This means that cost is a function of the hours the car is rented plus the flat fee. (a) Write this relationship in equation form; (b) find the cost if the car is rented for 3.5 hr; (c) determine how long the car was rented if the bill came to \$119.75; and (d) determine the domain and range of the function in this context, if your budget limits you to paying a maximum of \$150 for the rental.

99. Cost of a service call: Paul's Plumbing charges a flat fee of \$50 per service call plus an hourly rate of \$42.50. This means that cost is a function of the hours the job takes to complete plus the flat fee. (a) Write this relationship in equation form; (b) find the cost of a service call that takes $2\frac{1}{2}$ hr; (c) find the number of hours the job took if the charge came to \$262.50; and (d) determine the domain and range of the function in this context, if your insurance company has agreed to pay for all charges over \$500 for the service call.

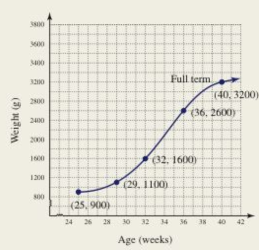
100. Predicting tides: The graph shown approximates the height of the tides at Fair Haven, New Brunswick, for a 12-hr period. (a) Is this the graph of a function? Why? (b) Approximately what time did high tide occur? (c) How high is the tide at 6 P.M.? (d) What time(s) will the tide be 2.5 m?



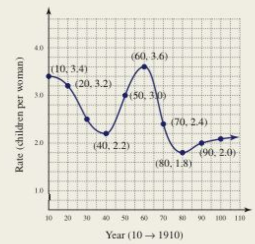
101. Predicting tides: The graph shown approximates the height of the tides at Apia, Western Samoa, for a 12-hr period. (a) Is this the graph of a function? Why? (b) Approximately what time did low tide occur? (c) How high is the tide at 2 A.M.? (d) What time(s) will the tide be 0.7 m?



102. Weight of a fetus: The growth rate of a fetus in the mother's womb (by weight in grams) is modeled by the graph shown here, beginning with the 25th week of gestation. (a) Calculate the average rate of change (slope of the secant line) between the 25th week and the 29th week. Is the slope of the secant line positive or negative? Discuss what the slope means in this context. (b) Is the fetus gaining weight faster between the 25th and 29th week, or between the 32nd and 36th week? Compare the slopes of both secant lines and discuss.



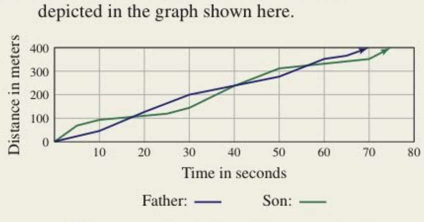
103. Fertility rates: Over the years, fertility rates for women in the United States (average number of children per woman) have varied a great deal, though in the twenty-first century they've begun to level out. The graph shown models this fertility rate for most of the twentieth century. (a) Calculate the average rate of change from the years 1920 to 1940. Is the slope of the secant line positive or negative? Discuss what the slope means in this context. (b) Calculate the average rate of change from the year 1940 to 1950. Is the slope of the secant line positive or negative? Discuss what the slope means in this context. (c) Was the fertility rate increasing faster from 1940 to 1950, or from 1980 to 1990? Compare the slope of both secant lines and comment.



Source: Statistical History of the United States from Colonial Times to Present

▶ EXTENDING THE CONCEPT

104. A father challenges his son to a 400-m race, depicted in the graph shown here.



a. Who won and what was the approximate winning time?

b. Approximately how many meters behind was the second place finisher?

c. Estimate the number of seconds the father was in the lead in this race.

d. How many times during the race were the father and son tied?

105. Sketch the graph of $f(x) = x$, then discuss how you could use this graph to obtain the graph of $F(x) = |x|$ without computing additional points. What would the graph of $g(x) = \frac{|x|}{x}$ look like?

106. Sketch the graph of $f(x) = x^2 - 4$, then discuss how you could use this graph to obtain the graph of $F(x) = |x^2 - 4|$ without computing additional points. Determine what the graph of $g(x) = \frac{|x^2 - 4|}{x^2 - 4}$ would look like.

107. If the equation of a function is given, the domain is implicitly defined by input values that generate real-valued outputs. But unless the graph is given or can be easily sketched, we must attempt to find the range analytically *by solving for x in terms of y*. We should note that sometimes this is an easy task, while at other times it is virtually impossible and we must rely on other methods. For the following functions, determine the implicit domain and find the range by solving for x in terms of y. **a.** $y = \frac{x - 3}{x + 2}$ **b.** $y = x^2 - 3$

► MAINTAINING YOUR SKILLS

108. **(2.2)** Which line has a steeper slope, the line through $(-5, 3)$ and $(2, 6)$, or the line through $(0, -4)$ and $(9, 4)$?

109. **(R.6)** Compute the sum and product indicated:
a. $\sqrt{24} + 6\sqrt{54} - \sqrt{6}$
b. $(2 + \sqrt{3})(2 - \sqrt{3})$

110. **(1.5)** Solve the equation using the quadratic formula, then check the result(s) using substitution:
$x^2 - 4x + 1 = 0$

111. **(R.4)** Factor the following polynomials completely:
a. $x^3 - 3x^2 - 25x + 75$
b. $2x^2 - 13x - 24$
c. $8x^3 - 125$

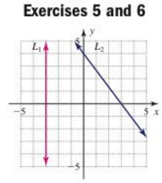
MID-CHAPTER CHECK

1. Sketch the graph of the line $4x - 3y = 12$. Plot and label at least three points.

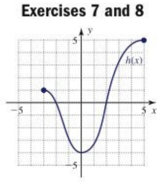
2. Find the slope of the line passing through the given points: $(-3, 8)$ and $(4, -10)$.

3. In 2002, Data.com lost \$2 million. In 2003, they lost \$0.5 million. Will the slope of the line through these points be positive or negative? Why? Calculate the slope. Were you correct? Write the slope as a unit rate and explain what it means in this context.

Exercises 5 and 6



Exercises 7 and 8



4. Sketch the line passing through $(1, 4)$ with slope $m = \frac{-2}{3}$ (plot and label at least two points). Then find the equation of the line *perpendicular to this line* through $(1, 4)$.

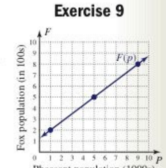
5. Write the equation for line L_1 shown. Is this the graph of a function? Discuss why or why not.

6. Write the equation for line L_2 shown. Is this the graph of a function? Discuss why or why not.

7. For the graph of function $h(x)$ shown, (a) determine the value of $h(2)$; (b) state the domain; (c) determine the value of x for which $h(x) = -3$; and (d) state the range.

8. Judging from the appearance of the graph alone, compare the average rate of change from $x = 1$ to $x = 2$ to the rate of change from $x = 4$ to $x = 5$. Which rate of change is larger? How is that demonstrated graphically?

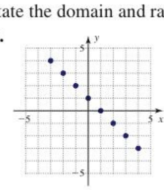
Exercise 9



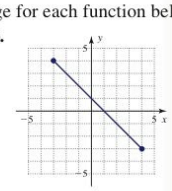
9. Find a linear function that models the graph of $F(p)$ given. Explain the slope of the line in this context, then use your model to predict the fox population when the pheasant population is 20,000.

10. State the domain and range for each function below.

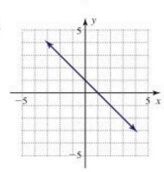
a.



b.



c.



REINFORCING BASIC CONCEPTS

The Various Forms of a Linear Equation

In a study of mathematics, getting a glimpse of the "big picture" can be an enormous help. Learning mathematics is like building a skyscraper: The final height of the skyscraper ultimately depends on the strength of the foundation and quality of the frame supporting each new floor as it is built. Our work with linear functions and their graphs, while having a number of useful applications, is actually the foundation on which *much of your future work will be built*. The study of quadratic and polynomial functions and their applications all have their roots in linear equations. For this reason, it's important that you gain a certain fluency with linear functions—even to a point where things come to you effortlessly and automatically. This level of performance requires a strong desire and a sustained effort. We begin by reviewing the basic facts a student MUST know to reach this level. MUST is an acronym for memorize, understand, synthesize, and teach others. Don't be satisfied until you've done all four. Given points (x_1, y_1) and (x_2, y_2):

Forms and Formulas

slope formula	point-slope form	slope-intercept form	standard form
$m = \frac{y_2 - y_1}{x_2 - x_1}$	$y - y_1 = m(x - x_1)$	$y = mx + b$	$Ax + By = C$
given any two points on the line	given slope m and any point (x_1, y_1)	given slope m and y-intercept $(0, b)$	also used in linear systems (Chapter 6)

Characteristics of Lines

y-intercept	x-intercept	increasing	decreasing
$(0, y)$	$(x, 0)$	$m > 0$	$m < 0$
let $x = 0$, solve for y	let $y = 0$, solve for x	line slants upward from left to right	line slants downward from left to right

Practice for Speed and Accuracy

For the two points given, (a) compute the slope of the line and state whether the line is increasing or decreasing; (b) find the equation of the line using point-slope form; (c) write the equation in slope-intercept form; (d) write the equation in standard form; and (e) find the x- and y-intercepts and graph the line.

1. $P_1(0, 5)$; $P_2(6, 7)$
2. $P_1(3, 2)$; $P_2(0, 9)$
3. $P_1(3, 2)$; $P_2(9, 5)$
4. $P_1(-5, -4)$; $P_2(3, 2)$
5. $P_1(-2, 5)$; $P_2(6, -1)$
6. $P_1(2, -7)$; $P_2(-8, -2)$

2.5 Analyzing the Graph of a Function

Learning Objectives

In Section 2.5 you will learn how to:

- ❑ **A.** Determine whether a function is even, odd, or neither
- ❑ **B.** Determine intervals where a function is positive or negative
- ❑ **C.** Determine where a function is increasing or decreasing
- ❑ **D.** Identify the maximum and minimum values of a function
- ❑ **E.** Develop a formula to calculate rates of change for any function

In this section, we'll consolidate and refine many of the ideas we've encountered related to functions. When functions and graphs are applied as real-world models, we create a numeric and visual representation that enables an informed response to questions involving *maximum* efficiency, *positive* returns, *increasing* costs, and other relationships that can have a great impact on our lives.

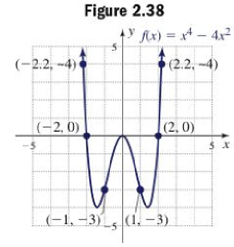
A. Graphs and Symmetry

While the domain and range of a function will remain dominant themes in our study, for the moment we turn our attention to other characteristics of a function's graph. We begin with the concept of symmetry.

Symmetry with Respect to the y-Axis

Consider the graph of $f(x) = x^4 - 4x^2$ shown in Figure 2.38, where the portion of the graph to the left of the y-axis appears to be a mirror image of the portion to the right. A function is **symmetric to the y-axis** if, given any point (x, y) on the graph, the point $(-x, y)$ is also on the graph. We note that $(-1, -3)$ is on the graph, as is $(1, -3)$, and that $(-2, 0)$ is an x-intercept of the graph, as is $(2, 0)$. Functions that are symmetric to the y-axis are also known as **even functions** and in general we have:

Figure 2.38



Even Functions: y-Axis Symmetry

A function f is an *even function* if and only if, for each point (x, y) on the graph of f, the point $(-x, y)$ is also on the graph. *In function notation*

$$f(-x) = f(x)$$

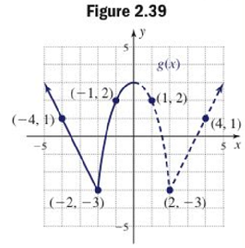
Symmetry can be a great help in graphing new functions, enabling us to plot fewer points, and to complete the graph using properties of symmetry.

EXAMPLE 1 ▶ Graphing an Even Function Using Symmetry

a. The function $g(x)$ in Figure 2.39 is known to be even. Draw the complete graph (only the left half is shown).

b. Show that $h(x) = x^{\frac{2}{3}}$ is an even function using the arbitrary value $x = k$ [show $h(-k) = h(k)$], then sketch the complete graph using $h(0)$, $h(1)$, $h(8)$, and y-axis symmetry.

Figure 2.39



Solution ▶ **a.** To complete the graph of g (see Figure 2.39) use the points $(-4, 1)$, $(-2, -3)$, $(-1, 2)$, and y-axis symmetry to find additional points. The corresponding ordered pairs are $(4, 1)$, $(2, -3)$, and $(1, 2)$, which we use to help draw a "mirror image" of the partial graph given.

b. To prove that $h(x) = x^{\frac{2}{3}}$ is an even function, we must show $h(-k) = h(k)$ for any constant k. After writing $x^{\frac{2}{3}}$ as $[x^2]^{\frac{1}{3}}$, we have:

$h(-k) \stackrel{?}{=} h(k)$ first step of proof

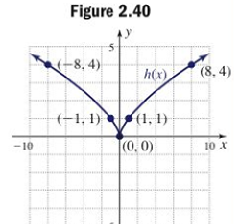
$[(-k)^2]^{\frac{1}{3}} \stackrel{?}{=} [(k)^2]^{\frac{1}{3}}$ evaluate $h(-k)$ and $h(k)$

$\sqrt[3]{(-k)^2} \stackrel{?}{=} \sqrt[3]{(k)^2}$ radical form

$\sqrt[3]{k^2} = \sqrt[3]{k^2}$ ✓ result: $(-k)^2 = k^2$

Using $h(0) = 0$, $h(1) = 1$, and $h(8) = 4$ with y-axis symmetry produces the graph shown in Figure 2.40.

Figure 2.40



WORTHY OF NOTE

The proof can also be demonstrated by writing $x^{\frac{2}{3}}$ as $(x^{\frac{1}{3}})^2$, and you are asked to complete this proof in Exercise 82.

Now try Exercises 7 through 12 ▶

Symmetry with Respect to the Origin

Another common form of symmetry is known as **symmetry to the origin.** As the name implies, the graph is somehow "centered" at (0, 0). This form of symmetry is easy to see for closed figures with their center at (0, 0), like certain polygons, circles, and ellipses (these will exhibit both y-axis symmetry *and* symmetry to the origin). Note the relation graphed in Figure 2.41 contains the points $(-3, 3)$ and $(3, -3)$, along with $(-1, -4)$ and $(1, 4)$. But the function $f(x)$ in Figure 2.42 also contains these points and is, in the same sense, symmetric to the origin (the paired points are on opposite sides of the x- and y-axes, and a like distance from the origin).

Figure 2.41

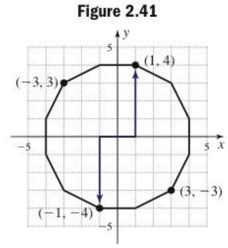
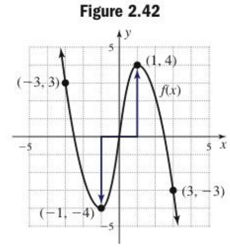


Figure 2.42



Functions symmetric to the origin are known as **odd functions** and in general we have:

Odd Functions: Symmetry about the Origin

A function f is an *odd function* if and only if, for each point (x, y) on the graph of f, the point $(-x, -y)$ is also on the graph. *In function notation*

$$f(-x) = -f(x)$$

EXAMPLE 2 ► Graphing an Odd Function Using Symmetry

a. In Figure 2.43, the function $g(x)$ given is known to be *odd*. Draw the complete graph (only the left half is shown).

b. Show that $h(x) = x^3 - 4x$ is an odd function using the arbitrary value $x = k$ [show $h(-x) = -h(x)$], then sketch the graph using $h(-2)$, $h(-1)$, $h(0)$, and odd symmetry.

Solution ► **a.** To complete the graph of g, use the points $(-6, 3)$, $(-4, 0)$, and $(-2, 2)$ and odd symmetry to find additional points. The corresponding ordered pairs are $(6, -3)$, $(4, 0)$, and $(2, -2)$, which we use to help draw a "mirror image" of the partial graph given (see Figure 2.43).

WORTHY OF NOTE

While the graph of an even function may or may not include the point (0, 0), the graph of an odd function will *always* contain this point.

Figure 2.43

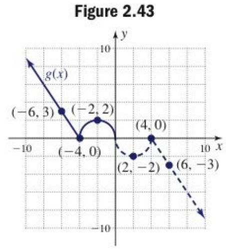
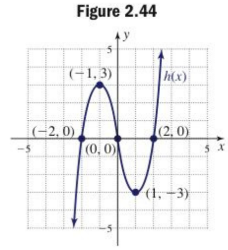


Figure 2.44



b. To prove that $h(x) = x^3 - 4x$ is an odd function, we must show that $h(-k) = -h(k)$.

$$h(-k) \stackrel{?}{=} -h(k)$$
$$(-k)^3 - 4(-k) \stackrel{?}{=} -[k^3 - 4k]$$
$$-k^3 + 4k = -k^3 + 4k \checkmark$$

Using $h(-2) = 0$, $h(-1) = 3$, and $h(0) = 0$ with symmetry about the origin produces the graph shown in Figure 2.44.

A. You've just learned how to determine whether a function is even, odd, or neither

Now try Exercises 13 through 24 ►

B. Intervals Where a Function Is Positive or Negative

Consider the graph of $f(x) = x^2 - 4$ shown in Figure 2.45, which has x-intercepts at $(-2, 0)$ and $(2, 0)$. Since x-intercepts have the form $(x, 0)$ they are also called the **zeroes** of the function (the x-input causes an output of 0). Just as zero on the number line separates negative numbers from positive numbers, the zeroes of a function that crosses the x-axis separate x-intervals where a function is negative from x-intervals where the function is positive. Noting that outputs (y-values) are positive in Quadrants I and II, $f(x) > 0$ in intervals where its graph is *above the x-axis*. Conversely, $f(x) < 0$ in x-intervals where its graph is *below the x-axis*. To illustrate, compare the graph of f in Figure 2.45, with that of g in Figure 2.46.

Figure 2.45

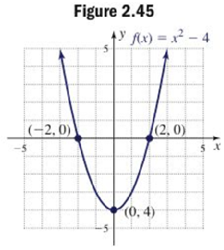
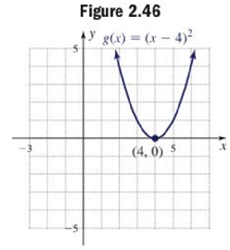


Figure 2.46



WORTHY OF NOTE

These observations form the basis for studying polynomials of higher degree, where we extend the idea to factors of the form $(x - r)^n$ in a study of **roots of multiplicity** (also see the *Calculator Exploration and Discovery* feature in this chapter).

The graph of f is a parabola, with x-intercepts of $(-2, 0)$ and $(2, 0)$. Using our previous observations, we note $f(x) \geq 0$ for $x \in (-\infty, -2] \cup [2, \infty)$ and $f(x) < 0$ for $x \in (-2, 2)$. The graph of g is also a parabola, but is entirely above or on the x-axis, showing $g(x) \geq 0$ for $x \in \mathbb{R}$. The difference is that zeroes coming from factors of the form $(x - r)$ (with degree 1) allow the graph to cross the x-axis. The zeroes of f came from $(x + 2)(x - 2) = 0$. Zeroes that come from factors of the form $(x - r)^2$ (with degree 2) cause the graph to "bounce" off the x-axis since all outputs must be nonnegative. The zero of g came from $(x - 4)^2 = 0$.

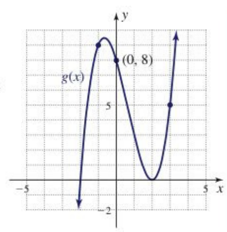
EXAMPLE 3 ► Solving an Inequality Using a Graph

Use the graph of $g(x) = x^3 - 2x^2 - 4x + 8$ given to solve the inequalities

a. $g(x) \geq 0$

b. $g(x) < 0$

Solution ► From the graph, the zeroes of g (x-intercepts) occur at $(-2, 0)$ and $(2, 0)$. a) For $g(x) \geq 0$, the graph must be on or above the x-axis, meaning the solution is $x \in [-2, \infty)$. b) For $g(x) < 0$, the graph must be below the x-axis, and the solution is $x \in (-\infty, -2)$. As we might have anticipated from the graph, factoring by grouping gives $g(x) = (x + 2)(x - 2)^2$, with the graph crossing the x-axis at -2, and bouncing off the x-axis (intersects without crossing) at $x = 2$.



Now try Exercises 25 through 28 ►

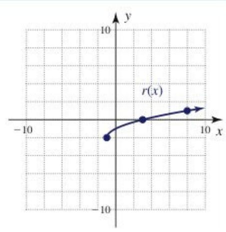
Even if the function is not a polynomial, the zeroes can still be used to find x-intervals where the function is positive or negative.

EXAMPLE 4 ► Solving an Inequality Using a Graph

For the graph of $r(x) = \sqrt{x + 1} - 2$ shown, solve

a. $r(x) \leq 0$

b. $r(x) > 0$



Solution ►

a. The only zero of r is at $(3, 0)$. The graph is on or below the x-axis for $x \in [-1, 3]$, so $r(x) \leq 0$ in this interval.

b. The graph is above the x-axis for $x \in (3, \infty)$, and $r(x) > 0$ in this interval.

B. You've just learned how to determine intervals where a function is positive or negative

Now try Exercises 29 through 32 ►

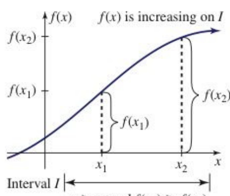
C. Intervals Where a Function Is Increasing or Decreasing

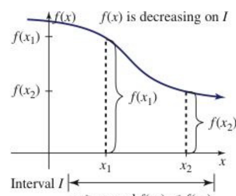
In our study of linear graphs, we said a graph was increasing if it "rose" when viewed from left to right. More generally, we say the graph of a function is increasing *on a given interval* if larger and larger x-values produce larger and larger y-values. This suggests the following tests for intervals where a function is increasing or decreasing.

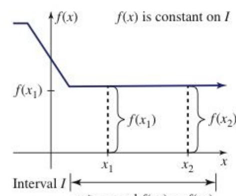
Increasing and Decreasing Functions

Given an interval I that is a subset of the domain, with x_1 and x_2 in I and $x_2 > x_1$,

1. A function is increasing on I if $f(x_2) > f(x_1)$ for all x_1 and x_2 in I (larger inputs produce larger outputs).
2. A function is decreasing on I if $f(x_2) < f(x_1)$ for all x_1 and x_2 in I (larger inputs produce smaller outputs).
3. A function is constant on I if $f(x_2) = f(x_1)$ for all x_1 and x_2 in I (larger inputs produce identical outputs).







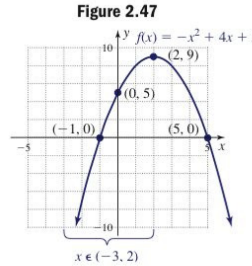
WORTHY OF NOTE

Questions about the behavior of a function are asked with respect to the *y* outputs: where is the *function* positive, where is the *function* increasing, etc. Due to the input/output, cause/effect nature of functions, the response is given in terms of *x*, that is, what is *causing* outputs to be negative, or to be decreasing.

Consider the graph of $f(x) = -x^2 + 4x + 5$ in Figure 2.47. Since the graph opens downward with the vertex at (2, 9), the function must increase until it reaches this maximum value at $x = 2$, and decrease thereafter. Notationally we'll write this as $f(x)\uparrow$ for $x \in (-\infty, 2)$ and $f(x)\downarrow$ for $x \in (2, \infty)$. Using the interval $(-3, 2)$ shown, we see that any larger input value from the interval will indeed produce a larger output value, and $f(x)\uparrow$ on the interval. For instance,

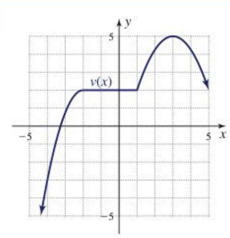
$$1 > -2 \qquad x_2 > x_1$$
$$\text{and} \qquad \text{and}$$
$$f(1) > f(-2) \qquad f(x_2) > f(x_1)$$
$$8 > -7$$

Figure 2.47



EXAMPLE 5 ► **Finding Intervals Where a Function Is Increasing or Decreasing**

Use the graph of $v(x)$ given to name the interval(s) where v is increasing, decreasing, or constant.



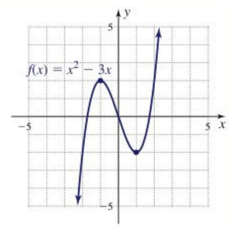
Solution ► From left to right, the graph of v increases until leveling off at $(-2, 2)$, then it remains constant until reaching $(1, 2)$. The graph then increases once again until reaching a peak at $(3, 5)$ and decreases thereafter. The result is $v(x)\uparrow$ for $x \in (-\infty, -2) \cup (1, 3)$, $v(x)\downarrow$ for $x \in (3, \infty)$, and $v(x)$ is constant for $x \in (-2, 1)$.

Now try Exercises 33 through 36 ►

Notice the graph of *f* in Figure 2.47 and the graph of *v* in Example 5 have something in common. It appears that both the far left and far right branches of each graph point downward (in the negative *y*-direction). We say that the **end behavior** of both graphs is identical, which is the term used to describe what happens to a graph as $|x|$ becomes very large. For $x > 0$, we say a graph is, "up on the right" or "down on the right," depending on the direction the "end" is pointing. For $x < 0$, we say the graph is "up on the left" or "down on the left," as the case may be.

EXAMPLE 6 ► **Describing the End Behavior of a Graph**

The graph of $f(x) = x^3 - 3x$ is shown. Use the graph to name intervals where f is increasing or decreasing, and comment on the end-behavior of the graph.



Solution ► From the graph we observe that: $f(x)\uparrow$ for $x \in (-\infty, -1) \cup (1, \infty)$, and $f(x)\downarrow$ for $x \in (-1, 1)$. The end behavior of the graph is down on the left, up on the right (down/up).

C. You've just learned how to determine where a function is increasing or decreasing

Now try Exercises 37 through 40 ►

D. More on Maximum and Minimum Values

The y-coordinate of the vertex of a parabola where $a < 0$, and the y-coordinate of "peaks" from other graphs are called **maximum values.** A **global maximum** (also called an *absolute* maximum) names the largest range value over the entire domain. A local **maximum** (also called a *relative* maximum) gives the largest range value in a specified interval; and an **endpoint maximum** can occur at an endpoint of the domain. The same can be said for the corresponding minimum values.

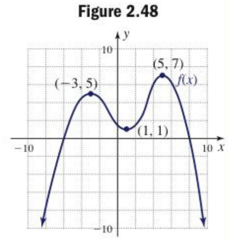
We will soon develop the ability to locate maximum and minimum values for quadratic and other functions. In future courses, methods are developed to help locate maximum and minimum values for almost *any* function. For now, our work will rely chiefly on a function's graph.

EXAMPLE 7 ► **Analyzing Characteristics of a Graph**

Analyze the graph of function f shown in Figure 2.48. Include specific mention of

a. domain and range,
b. intervals where f is increasing or decreasing,
c. maximum (max) and minimum (min) values,
d. intervals where $f(x) \geq 0$ and $f(x) < 0$,
e. whether the function is even, odd, or neither.

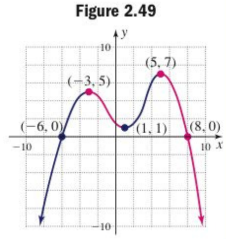
Figure 2.48



Solution ►

a. Using vertical and horizontal boundary lines show the domain is $x \in \mathbb{R}$, with range: $y \in (-\infty, 7]$.
b. $f(x)\uparrow$ for $x \in (-\infty, -3) \cup (1, 5)$ shown in **blue** in Figure 2.49, and $f(x)\downarrow$ for $x \in (-3, 1) \cup (5, \infty)$ as shown in **red.**
c. From Part (b) we find that $y = 5$ at $(-3, 5)$ and $y = 7$ at $(5, 7)$ are local maximums, with a local minimum of $y = 1$ at $(1, 1)$. The point $(5, 7)$ is also a global maximum (there is no global minimum).
d. $f(x) \geq 0$ for $x \in [-6, 8]$; $f(x) < 0$ for $x \in (-\infty, -6) \cup (8, \infty)$
e. The function is neither even nor odd.

Figure 2.49



D. You've just learned how to identify the maximum and minimum values of a function

Now try Exercises 41 through 48 ►

The ideas presented here can be applied to functions of all kinds, including rational functions, piecewise-defined functions, step functions, and so on. There is a wide variety of applications in **Exercises 51 through 58.**

E. Rates of Change and the Difference Quotient

We complete our study of graphs by revisiting the concept of average rates of change. In many business, scientific, and economic applications, it is this attribute of a function that draws the most attention. In Section 2.4 we computed average rates of change by selecting two points from a graph, and computing the slope of the secant line: $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. With a simple change of notation, we can *use the function's equation* rather than relying on a graph. Note that y_2 corresponds to the function evaluated at x_2: $y_2 = f(x_2)$. Likewise, $y_1 = f(x_1)$. Substituting these into the slope formula yields $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$, giving the average rate of change between x_1 and x_2 *for any function f* (assuming the function is smooth and continuous between x_1 and x_2).

Average Rate of Change

For a function f and $[x_1, x_2]$ a subset of the domain, the average rate of change between x_1 and x_2 is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \; x_1 \neq x_2$$

Average Rates of Change Applied to Projectile Velocity

A projectile is any object that is thrown, shot, or cast upward, with no continuing source of propulsion. The object's height (in feet) after t sec is modeled by the function $h(t) = -16t^2 + vt + k$, where v is the initial velocity of the projectile, and k is the height of the object at contact. For instance, if a soccer ball is kicked upward from ground level ($k = 0$) with an initial speed of 64 ft/sec, the height of the ball t sec later is $h(t) = -16t^2 + 64t$. From Section 2.5, we recognize the graph will be a parabola and evaluating the function for $t = 0$ to 4 produces Table 2.4 and the graph shown in Figure 2.50. Experience tells us the ball is traveling at a faster rate immediately after being kicked, as compared to when it nears its maximum height where it momentarily stops, then begins its descent. In other words, the rate of change $\frac{\Delta\text{height}}{\Delta\text{time}}$ has a larger value at any time prior to reaching its maximum height. To quantify this we'll compute the average rate of change between $t = 0.5$ and $t = 1$, and compare it to the average rate of change between $t = 1$ and $t = 1.5$.

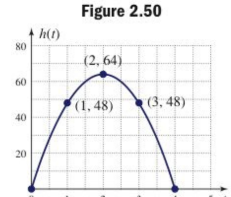
WORTHY OF NOTE

Keep in mind the graph of h represents the relationship between the soccer ball's height in feet and the elapsed time t. It does not model the actual path of the ball.

Table 2.4

Time in seconds	Height in feet
0	0
1	48
2	64
3	48
4	0

Figure 2.50



EXAMPLE 8 ► Calculating Average Rates of Change

For the projectile function $h(t) = -16t^2 + 64t$, find

a. the average rate of change for $t \in [0.5, 1]$

b. the average rate of change for $t \in [1, 1.5]$.

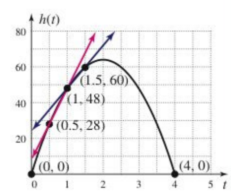
Then graph the secant lines representing these average rates of change and comment.

Solution ► Using the given intervals in the formula $\frac{\Delta h}{\Delta t} = \frac{h(t_2) - h(t_1)}{t_2 - t_1}$ yields

a. $\frac{\Delta h}{\Delta t} = \frac{h(1) - h(0.5)}{1 - (0.5)} = \frac{48 - 28}{0.5} = 40$

b. $\frac{\Delta h}{\Delta t} = \frac{h(1.5) - h(1)}{1.5 - 1} = \frac{60 - 48}{0.5} = 24$

For $t \in [0.5, 1]$, the average rate of change is $\frac{40}{1}$, meaning the height of the ball is increasing at an average rate of 40 ft/sec. For $t \in [1, 1.5]$, the average rate of change has slowed to $\frac{24}{1}$, and the soccer ball's height is increasing at only 24 ft/sec. The secant lines representing these rates of change are shown in the figure, where we note the line from the first interval (in **red**), has a steeper slope than the line from the second interval (in **blue**).



Now try Exercises 59 through 64 ►

The approach in Example 8 works very well, but requires us to recalculate $\frac{\Delta y}{\Delta x}$ for each new interval. Using a slightly different approach, we can develop a general *formula* for the average rate of change. This is done by selecting a point $x_1 = x$ from the domain, then a point $x_2 = x + h$ that is very close to x. Here, $h \neq 0$ is assumed to be a small, arbitrary constant, meaning the interval $[x, x + h]$ is very small as well. Substituting $x + h$ for x_2 and x for x_1 in the rate of change formula gives $\frac{\Delta y}{\Delta x} = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}$. The result is called the **difference quotient** and represents the average rate of change between x and $x + h$, or equivalently, the slope of the secant line for this interval.

The Difference Quotient

For a function $f(x)$ and constant $h \neq 0$, if f is smooth and continuous on the interval containing x and $x + h$,

$$\frac{f(x + h) - f(x)}{h}$$

is the difference quotient for f.

Note the formula has three parts: (1) the function f evaluated at $x + h \rightarrow f(x + h)$, (2) the function f itself, and (3) the constant h. For convenience, the expression $f(x + h)$ can be evaluated and simplified prior to its use in the difference quotient.

$$\frac{\overset{(1)}{f(x+h)} - \overset{(2)}{f(x)}}{\underset{(3)}{h}}$$

EXAMPLE 9 ► **Computing a Difference Quotient and Average Rates of Change**

For $f(x) = x^2 - 4x$,

a. Compute the difference quotient.

b. Find the average rate of change in the intervals [1.9, 2.0] and [3.6, 3.7].

c. Sketch the graph of f along with the secant lines and comment on what you notice.

Solution ► **a.** For $f(x) = x^2 - 4x$, $f(x + h) = (x + h)^2 - 4(x + h)$
$= x^2 + 2xh + h^2 - 4x - 4h$

Using this result in the difference quotient yields,

$$\frac{f(x+h) - f(x)}{h} = \frac{(x^2 + 2xh + h^2 - 4x - 4h) - (x^2 - 4x)}{h} \quad \text{substitute into the difference quotient}$$

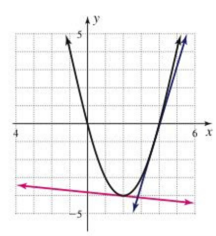
$$= \frac{x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x}{h} \quad \text{eliminate parentheses}$$

$$= \frac{2xh + h^2 - 4h}{h} \quad \text{combine like terms}$$

$$= \frac{h(2x + h - 4)}{h} \quad \text{factor out } h$$

$$= 2x - 4 + h \quad \text{result}$$

b. For the interval [1.9, 2.0], $x = 1.9$ and $h = 0.1$. The slope of the secant line is $\frac{\Delta y}{\Delta x} = 2(1.9) - 4 + 0.1 = -0.1$. For the interval [3.6, 3.7], $x = 3.6$ and $h = 0.1$. The slope of this secant line is $\frac{\Delta y}{\Delta x} = 2(3.6) - 4 + 0.1 = 3.3$.



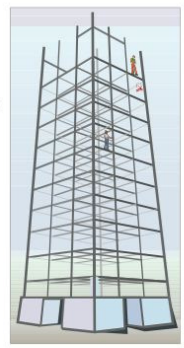
c. After sketching the graph of f and the secant lines from each interval (see the figure), we note the slope of the first line (in **red**) is negative and very near zero, while the slope of the second (in **blue**) is positive and very steep.

Now try Exercises 65 through 76 ►

You might be familiar with Galileo Galilei and his studies of gravity. According to popular history, he demonstrated that unequal weights will fall equal distances in equal time periods, by dropping cannonballs from the upper floors of the Leaning Tower of Pisa. Neglecting air resistance, this distance an object falls is modeled by the function $d(t) = 16t^2$, where $d(t)$ represents the distance fallen after t sec. Due to the effects of gravity, the velocity of the object increases as it falls. In other words, the velocity or the average rate of change $\frac{\Delta \textbf{distance}}{\Delta \textbf{time}}$ is a nonconstant (increasing) rate of change. We can analyze this rate of change using the difference quotient.

EXAMPLE 10 ► Applying the Difference Quotient in Context

A construction worker drops a heavy wrench from atop the girder of new skyscraper. Use the function $d(t) = 16t^2$ to



a. Compute the distance the wrench has fallen after 2 sec and after 7 sec.

b. Find a formula for the velocity of the wrench (average rate of change in distance per unit time).

c. Use the formula to find the rate of change in the intervals [2, 2.01] and [7, 7.01].

d. Graph the function and the secant lines representing the average rate of change. Comment on what you notice.

Solution ► **a.** Substituting $t = 2$ and $t = 7$ in the given function yields

$$d(2) = 16(2)^2 \qquad d(7) = 16(7)^2 \quad \text{evaluate } d(t) = 16t^2$$
$$= 16(4) \qquad = 16(49) \quad \text{square input}$$
$$= 64 \qquad = 784 \quad \text{multiply}$$

After 2 sec, the wrench has fallen 64 ft; after 7 sec, the wrench has fallen 784 ft.

b. For $d(t) = 16t^2$, $d(t + h) = 16(t + h)^2$, which we compute separately.

$$d(t + h) = 16(t + h)^2 \quad \text{substitute } t + h \text{ for } t$$
$$= 16(t^2 + 2th + h^2) \quad \text{square binomial}$$
$$= 16t^2 + 32th + 16h^2 \quad \text{distribute 16}$$

Using this result in the difference quotient yields

$$\frac{d(t + h) - d(t)}{h} = \frac{(16t^2 + 32th + 16h^2) - 16t^2}{h} \quad \text{substitute into the difference quotient}$$
$$= \frac{16t^2 + 32th + 16h^2 - 16t^2}{h} \quad \text{eliminate parentheses}$$
$$= \frac{32th + 16h^2}{h} \quad \text{combine like terms}$$
$$= \frac{h(32t + 16h)}{h} \quad \text{factor out } h \text{ and simplify}$$
$$= 32t + 16h \quad \text{result}$$

For any number of seconds t and h a small increment of time thereafter, the velocity of the wrench is modeled by $\frac{\Delta\textbf{distance}}{\Delta\textbf{time}} = \frac{\mathbf{32t + 16h}}{\mathbf{1}}$.

c. For the interval $[t, t + h] = [2, 2.01]$, $t = 2$ and $h = 0.01$:

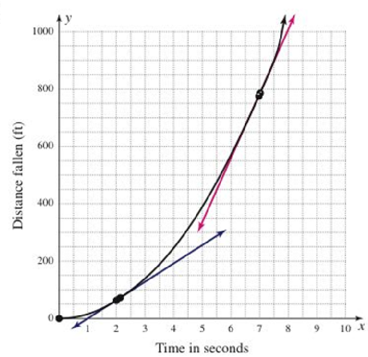
$$\frac{\Delta\text{distance}}{\Delta\text{time}} = \frac{32(2) + 16(0.01)}{1} \quad \text{substitute 2 for } t \text{ and 0.01 for } h$$
$$= 64 + 0.16 = 64.16$$

Two seconds after being dropped, the velocity of the wrench is approximately 64.16 ft/sec. For the interval $[t, t + h] = [7, 7.01]$, $t = 7$ and $h = 0.01$:

$$\frac{\Delta\text{distance}}{\Delta\text{time}} = \frac{32(7) + 16(0.01)}{1} \quad \text{substitute 7 for } t \text{ and 0.01 for } h$$
$$= 224 + 0.16 = 224.16$$

Seven seconds after being dropped, the velocity of the wrench is approximately 224.16 ft/sec (about 153 mph).

d.



The velocity increases with time, as indicated by the steepness of each secant line.

E. You've learned how to develop a formula to calculate rates of change for any function

Now try Exercises 77 and 78 ►

TECHNOLOGY HIGHLIGHT

Locating Zeroes, Maximums, and Minimums

Graphically, the **zeroes** of a function appear as x-intercepts with coordinates $(x, 0)$. An estimate for these zeroes can easily be found using a graphing calculator. To illustrate, enter the function $y = x^2 - 8x + 9$ on the Y= screen and graph it using the standard window (ZOOM 6). We access the option for finding zeroes by pressing 2nd TRACE (**CALC**), which displays the screen shown in Figure 2.51. Pressing the number "2" selects **2:zero** and returns you to the graph, where you're asked to enter a "Left Bound." The calculator is asking you to narrow the area it has to search. Select any number conveniently to the left of the x-intercept you're interested in. For this graph, we entered a left bound of "0" (press ENTER). The calculator marks this choice with a "▶" marker (pointing to the right), then asks you to enter a "Right Bound." Select any value to the right of the x-intercept, but be sure the value you enter *bounds only one intercept* (see Figure 2.52). For this graph, a choice of 10 would include both x-intercepts, while a choice of 3 would bound only the intercept on the left. After entering 3, the calculator asks for a "Guess." This option is used when there is more than one zero in the interval, and most of the time we'll bypass this option by pressing ENTER again. The calculator then finds the zero in the selected interval (if it exists), with the coordinates displayed at the bottom of the screen (Figure 2.53).

The maximum and minimum values of a function are located in the same way. Enter $y = x^3 - 3x - 2$ on the Y= screen and graph the function. As seen in Figure 2.54, it appears a local maximum occurs near $x = -1$. To check, we access the **CALC 4:maximum** option, which returns you to the graph and asks you for a *Left Bound*, a *Right Bound*, and a *Guess* as before. After entering a left bound of "−3" and a right bound of "0," and

Figure 2.51

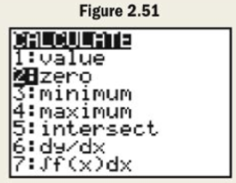


Figure 2.52

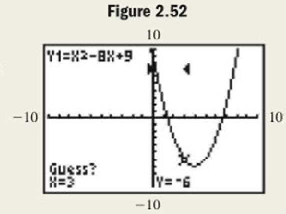


Figure 2.53

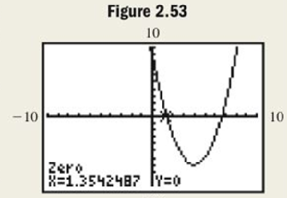


Figure 2.54

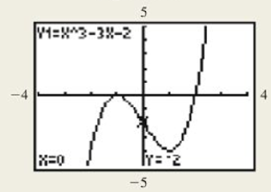
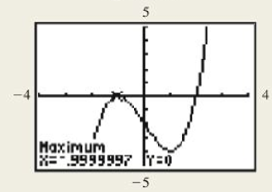


Figure 2.55



bypassing the Guess option (note the "▶" and "◀" markers), the calculator locates the maximum you selected, and again displays the coordinates. Due to the algorithm used by the calculator to find these values, a decimal number is sometimes displayed, even if the actual value is an integer (see Figure 2.55).

Use a calculator to find all zeroes and to locate the local maximum and minimum values. Round to the nearest hundredth as needed.

Exercise 1: $y = 2x^2 + 4x - 5$

Exercise 2: $y = w^3 - 3w + 1$

Exercise 3: $y = x^2 - 8x + 9$

Exercise 4: $y = x^3 - 2x^2 - 4x + 8$

Exercise 5: $y = x^4 - 5x^2 - 2x$

Exercise 6: $y = x\sqrt{x + 4}$

2.5 EXERCISES

▶ CONCEPTS AND VOCABULARY

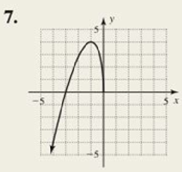
Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The graph of a polynomial will cross through the x-axis at zeroes of ______ factors of degree 1, and ______ off the x-axis at the zeroes from linear factors of degree 2.
2. If $f(-x) = f(x)$ for all x in the domain, we say that f is an ______ function and symmetric to the ______ axis. If $f(-x) = -f(x)$, the function is ______ and symmetric to the ______.
3. If $f(x_2) > f(x_1)$ for $x_1 < x_2$ for all x in a given interval, the function is ______ in the interval.
4. If $f(c) \geq f(x)$ for all x in a specified interval, we say that $f(c)$ is a local ______ for this interval.
5. Discuss/Explain the following statement and give an example of the conclusion it makes. "If a function f is decreasing to the left of $(c, f(c))$ and increasing to the right of $(c, f(c))$, then $f(c)$ is either a local or a global minimum."
6. Without referring to notes or textbook, list as many features/attributes as you can that are related to analyzing the graph of a function. Include details on how to locate or determine each attribute.

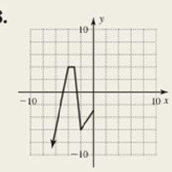
▶ DEVELOPING YOUR SKILLS

The following functions are known to be even. Complete each graph using symmetry.

7.



8.



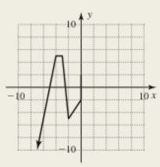
Determine whether the following functions are even: $f(-k) = f(k)$.

9. $f(x) = -7|x| + 3x^2 + 5$ **10.** $p(x) = 2x^4 - 6x + 1$

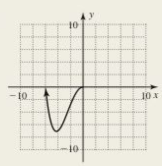
11. $g(x) = \frac{1}{3}x^4 - 5x^2 + 1$ **12.** $q(x) = \frac{1}{x^2} - |x|$

The following functions are known to be odd. Complete each graph using symmetry.

13.



14.



Determine whether the following functions are odd: $f(-k) = -f(k)$.

15. $f(x) = 4\sqrt[3]{x} - x$ **16.** $g(x) = \frac{1}{2}x^3 - 6x$

17. $p(x) = 3x^3 - 5x^2 + 1$ **18.** $q(x) = \frac{1}{x} - x$

Determine whether the following functions are even, odd, or neither.

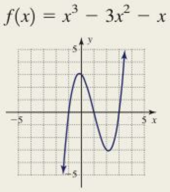
19. $w(x) = x^3 - x^2$ **20.** $q(x) = \frac{3}{4}x^2 + 3|x|$

21. $p(x) = 2\sqrt[3]{x} - \frac{1}{4}x^3$ **22.** $g(x) = x^3 + 7x$

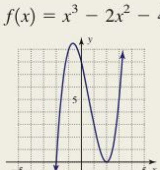
23. $v(x) = x^3 + 3|x|$ **24.** $f(x) = x^4 + 7x^2 - 30$

Use the graphs given to solve the inequalities indicated. Write all answers in interval notation.

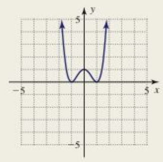
25. $f(x) = x^3 - 3x^2 - x + 3; f(x) \geq 0$



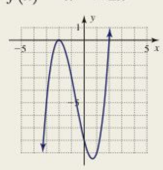
26. $f(x) = x^3 - 2x^2 - 4x + 8; f(x) > 0$



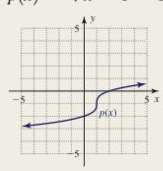
27. $f(x) = x^4 - 2x^2 + 1; f(x) > 0$



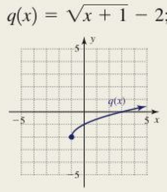
28. $f(x) = x^3 + 2x^2 - 4x - 8; f(x) \geq 0$



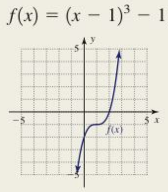
29. $p(x) = \sqrt[3]{x - 1} - 1; p(x) \geq 0$



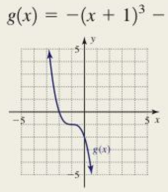
30. $q(x) = \sqrt{x + 1} - 2; q(x) > 0$



31. $f(x) = (x - 1)^3 - 1; f(x) \leq 0$

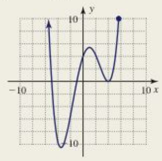


32. $g(x) = -(x + 1)^3 - 1; g(x) < 0$

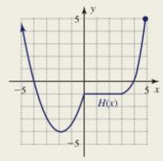


Name the interval(s) where the following functions are increasing, decreasing, or constant. Write answers using interval notation. Assume all endpoints have integer values.

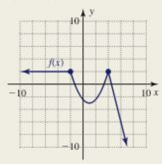
33. $y = V(x)$



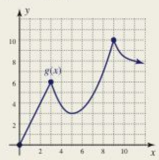
34. $y = H(x)$



35. $y = f(x)$

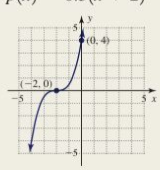


36. $y = g(x)$

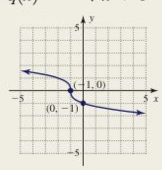


For Exercises 37 through 40, determine (a) interval(s) where the function is increasing, decreasing or constant, and (b) comment on the end behavior.

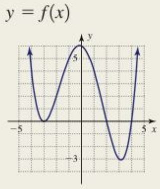
37. $p(x) = 0.5(x + 2)^3$



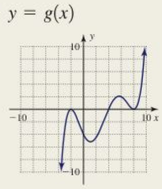
38. $q(x) = -\sqrt[3]{x + 1}$



39. $y = f(x)$

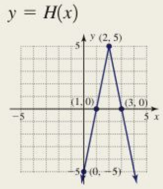


40. $y = g(x)$

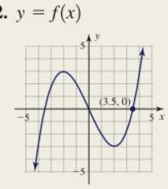


For Exercises 41 through 48, determine the following (answer in interval notation as appropriate): (a) domain and range of the function; (b) zeroes of the function; (c) interval(s) where the function is greater than or equal to zero, or less than or equal to zero; (d) interval(s) where the function is increasing, decreasing, or constant; and (e) location of any local max or min value(s).

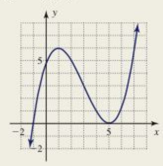
41. $y = H(x)$



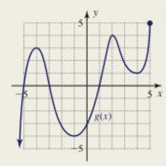
42. $y = f(x)$



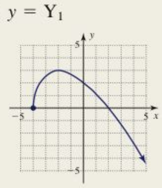
43. $y = g(x)$



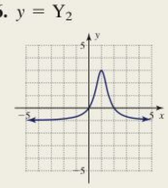
44. $y = h(x)$



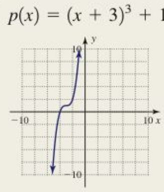
45. $y = Y_1$



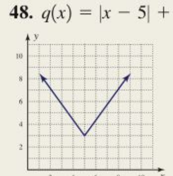
46. $y = Y_2$



47. $p(x) = (x + 3)^3 + 1$



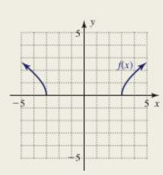
48. $q(x) = |x - 5| + 3$



▶ WORKING WITH FORMULAS

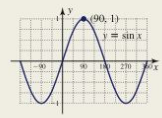
49. Conic sections—hyperbola: $y = \frac{1}{3}\sqrt{4x^2 - 36}$

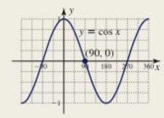
While the conic sections are not covered in detail until later in the course, we've already developed a number of tools that will help us understand these relations and their graphs. The equation here gives the "upper branches" of a hyperbola, as shown in the figure. Find the following by analyzing the equation: (a) the domain and range; (b) the zeroes of the relation; (c) interval(s) where y is increasing or decreasing; and (d) whether the relation is even, odd, or neither.



50. Trigonometric graphs: $y = \sin(x)$ and $y = \cos(x)$

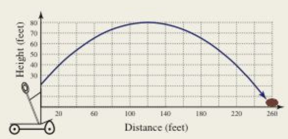
The trigonometric functions are also studied at some future time, but we can apply the same tools to analyze the graphs of these functions as well. The graphs of $y = \sin x$ and $y = \cos x$ are given, graphed over the interval $x \in [-180, 360]$ degrees. Use them to find (a) the range of the functions; (b) the zeroes of the functions; (c) interval(s) where y is increasing/decreasing; (d) location of minimum/maximum values; and (e) whether each relation is even, odd, or neither.





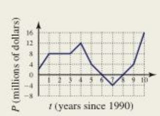
▶ APPLICATIONS

51. Catapults and projectiles: Catapults have a long and interesting history that dates back to ancient times, when they were used to launch javelins, rocks, and other projectiles. The diagram given illustrates the path of the projectile after release, which follows a parabolic arc. Use the graph to determine the following:



a. State the domain and range of the projectile.
b. What is the maximum height of the projectile?
c. How far from the catapult did the projectile reach its maximum height?
d. Did the projectile clear the castle wall, which was 40 ft high and 210 ft away?
e. On what interval was the height of the projectile increasing?
f. On what interval was the height of the projectile decreasing?

52. Profit and loss: The profit of DeBartolo Construction Inc. is illustrated by the graph shown. Use the graph to estimate the point(s) or the interval(s) for which the profit P was:

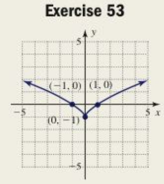


a. increasing
b. decreasing
c. constant
d. a maximum
e. a minimum
f. positive
g. negative
h. zero

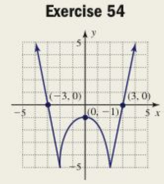
53. Functions and rational exponents: The graph of $f(x) = x^{\frac{2}{3}} - 1$ is shown. Use the graph to find:

a. domain and range of the function
b. zeroes of the function
c. interval(s) where $f(x) \geq 0$ or $f(x) < 0$
d. interval(s) where $f(x)$ is increasing, decreasing, or constant
e. location of any max or min value(s)

Exercise 53



Exercise 54



54. Analyzing a graph: Given $h(x) = |x^2 - 4| - 5$, whose graph is shown, use the graph to find:

a. domain and range of the function
b. zeroes of the function
c. interval(s) where $h(x) \geq 0$ or $h(x) \leq 0$
d. interval(s) where $f(x)$ is increasing, decreasing, or constant
e. location of any max or min value(s)

55. Analyzing interest rates: The graph shown approximates the average annual interest rates on 30-yr fixed mortgages, rounded to the nearest $\frac{1}{4}\%$. Use the graph to estimate the following (write all answers in interval notation).

a. domain and range

b. interval(s) where $I(t)$ is increasing, decreasing, or constant

c. location of the maximum and minimum values

d. the one-year period with the greatest rate of increase and the one-year period with the greatest rate of decrease

Source: 1998 Wall Street Journal Almanac, p. 446; 2004 Statistical Abstract of the United States, Table 1178

56. Analyzing the deficit: The following graph approximates the federal deficit of the United States. Use the graph to estimate the following (write answers in interval notation).

a. the domain and range

b. interval(s) where $D(t)$ is increasing, decreasing, or constant

c. the location of the maximum and minimum values

d. the one-year period with the greatest rate of increase, and the one-year period with the greatest rate of decrease

Source: 2005 Statistical Abstract of the United States, Table 461

57. Constructing a graph: Draw the function f that has the following characteristics, then state the zeroes and the location of all maximum and minimum values. [*Hint:* Write them as $(c, f(c))$.]

a. Domain: $x \in (-10, \infty)$

b. Range: $y \in (-6, \infty)$

c. $f(0) = 0; f(4) = 0$

d. $f(x)\uparrow$ for $x \in (-10, -6) \cup (-2, 2) \cup (4, \infty)$

e. $f(x)\downarrow$ for $x \in (-6, -2) \cup (2, 4)$

f. $f(x) \geq 0$ for $x \in [-8, -4] \cup [0, \infty)$

g. $f(x) < 0$ for $x \in (-\infty, -8) \cup (-4, 0)$

58. Constructing a graph: Draw the function g that has the following characteristics, then state the zeroes and the location of all maximum and minimum values. [*Hint:* Write them as $(c, g(c))$.]

a. Domain: $x \in (-\infty, 8)$

b. Range: $y \in [-6, \infty)$

c. $g(0) = 4.5; g(6) = 0$

d. $g(x)\uparrow$ for $x \in (-6, 3) \cup (6, 8)$

e. $g(x)\downarrow$ for $x \in (-\infty, -6) \cup (3, 6)$

f. $g(x) \geq 0$ for $x \in (-\infty, -9] \cup [-3, 8)$

g. $g(x) < 0$ for $x \in (-9, -3)$

For Exercises 59 to 64, use the formula for the average rate of change $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

59. Average rate of change: For $f(x) = x^3$, (a) calculate the average rate of change for the interval $x = -2$ and $x = -1$ and (b) calculate the average rate of change for the interval $x = 1$ and $x = 2$. (c) What do you notice about the answers from parts (a) and (b)? (d) Sketch the graph of this function along with the lines representing these average rates of change and comment on what you notice.

60. Average rate of change: Knowing the general shape of the graph for $f(x) = \sqrt[3]{x}$, (a) is the average rate of change greater between $x = 0$ and $x = 1$ or between $x = 7$ and $x = 8$? Why? (b) Calculate the rate of change for these intervals and verify your response. (c) Approximately how many times greater is the rate of change?

61. Height of an arrow: If an arrow is shot vertically from a bow with an initial speed of 192 ft/sec, the height of the arrow can be modeled by the function $h(t) = -16t^2 + 192t$, where $h(t)$ represents the height of the arrow after t sec (assume the arrow was shot from ground level).

a. What is the arrow's height at $t = 1$ sec?

b. What is the arrow's height at $t = 2$ sec?

c. What is the average rate of change from $t = 1$ to $t = 2$?

d. What is the rate of change from $t = 10$ to $t = 11$? Why is it the same as (c) except for the sign?

62. Height of a water rocket: Although they have been around for decades, water rockets continue to be a popular toy. A plastic rocket is filled with water and then pressurized using a handheld pump. The rocket is then released and off it goes! If the rocket has an initial velocity of 96 ft/sec, the height of the rocket can be modeled by the function $h(t) = -16t^2 + 96t$, where $h(t)$ represents the height of the rocket after t sec (assume the rocket was shot from ground level).

a. Find the rocket's height at $t = 1$ and $t = 2$ sec.

b. Find the rocket's height at $t = 3$ sec.

c. Would you expect the average rate of change to be greater between $t = 1$ and $t = 2$, or between $t = 2$ and $t = 3$? Why?

d. Calculate each rate of change and discuss your answer.

63. Velocity of a falling object: The impact velocity of an object dropped from a height is modeled by $v = \sqrt{2gs}$, where v is the velocity in feet per second (ignoring air resistance), g is the acceleration due to gravity (32 ft/sec^2 near the Earth's surface), and s is the height from which the object is dropped.

a. Find the velocity at $s = 5$ ft and $s = 10$ ft.

b. Find the velocity at $s = 15$ ft and $s = 20$ ft.

c. Would you expect the average rate of change to be greater between $s = 5$ and $s = 10$, or between $s = 15$ and $s = 20$?

d. Calculate each rate of change and discuss your answer.

64. Temperature drop: One day in November, the town of Coldwater was hit by a sudden winter storm that caused temperatures to plummet. During the storm, the temperature T (in degrees Fahrenheit) could be modeled by the function $T(h) = 0.8h^2 - 16h + 60$, where h is the number of hours since the storm began. Graph the function and use this information to answer the following questions.

a. What was the temperature as the storm began?

b. How many hours until the temperature dropped below zero degrees?

c. How many hours did the temperature remain below zero?

d. What was the coldest temperature recorded during this storm?

Compute and simplify the difference quotient $\frac{f(x + h) - f(x)}{h}$ for each function given.

65. $f(x) = 2x - 3$

66. $g(x) = 4x + 1$

67. $h(x) = x^2 + 3$

68. $p(x) = x^2 - 2$

69. $q(x) = x^2 + 2x - 3$

70. $r(x) = x^2 - 5x + 2$

71. $f(x) = \frac{2}{x}$

72. $g(x) = \frac{-3}{x}$

Use the difference quotient to find: (a) a rate of change formula for the functions given and (b)/(c) calculate the rate of change in the intervals shown. Then (d) sketch the graph of each function along with the secant lines and comment on what you notice.

73. $g(x) = x^2 + 2x$
$[-3.0, -2.9]$, $[0.50, 0.51]$

74. $h(x) = x^2 - 6x$
$[1.9, 2.0]$, $[5.0, 5.01]$

75. $g(x) = x^3 + 1$
$[-2.1, -2]$, $[0.40, 0.41]$

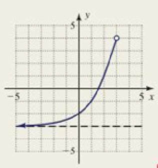
76. $r(x) = \sqrt{x}$ (*Hint*: Rationalize the numerator.)
$[1, 1.1]$, $[4, 4.1]$

77. The distance that a person can see depends on how high they're standing above level ground. On a clear day, the distance is approximated by the function $d(h) = 1.5\sqrt{h}$, where $d(h)$ represents the viewing distance (in miles) at height h (in feet). Find the average rate of change in the intervals (a) [9, 9.01] and (b) [225, 225.01]. Then (c) graph the function along with the lines representing the average rates of change and comment on what you notice.

78. A special magnifying lens is crafted and installed in an overhead projector. When the projector is x ft from the screen, the size $P(x)$ of the projected image is x^2. Find the average rate of change for $P(x) = x^2$ in the intervals (a) [1, 1.01] and (b) [4, 4.01]. Then (c) graph the function along with the lines representing the average rates of change and comment on what you notice.

▶ EXTENDING THE THOUGHT

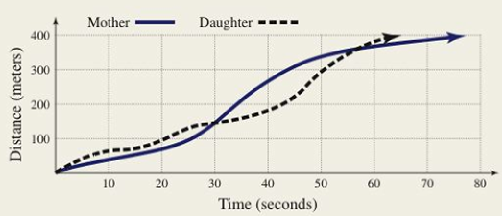
79. Does the function shown have a maximum value? Does it have a minimum value? Discuss/explain/justify why or why not.



80. The graph drawn here depicts a 400-m race between a mother and her daughter. Analyze the graph to answer questions (a) through (f).

a. Who wins the race, the mother or daughter?

b. By approximately how many meters?



c. By approximately how many seconds?

d. Who was leading at $t = 40$ seconds?

e. During the race, how many seconds was the daughter in the lead?

f. During the race, how many seconds was the mother in the lead?

81. Draw a general function $f(x)$ that has a local *maximum* at $(a, f(a))$ and a local *minimum* at $(b, f(b))$ but with $f(a) < f(b)$.

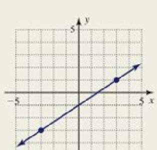
82. Verify that $h(x) = x^{\frac{2}{3}}$ is an even function, by first rewriting h as $h(x) = (x^{\frac{1}{3}})^2$.

▶ MAINTAINING YOUR SKILLS

83. **(1.5)** Solve the given quadratic equation three different ways: (a) factoring, (b) completing the square, and (c) using the quadratic formula: $x^2 - 8x - 20 = 0$

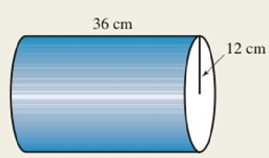
84. **(R.5)** Find the (a) sum and (b) product of the rational expressions $\frac{3}{x+2}$ and $\frac{3}{2-x}$.

85. **(2.3)** Write the equation of the line shown, in the form $y = mx + b$.



Exercise 85

86. **(R.7)** Find the surface area and volume of the cylinder shown.



2.6 The Toolbox Functions and Transformations

Learning Objectives

In Section 2.6 you will learn how to:

- ❑ **A.** Identify basic characteristics of the toolbox functions
- ❑ **B.** Perform vertical/horizontal shifts of a basic graph
- ❑ **C.** Perform vertical/horizontal reflections of a basic graph
- ❑ **D.** Perform vertical stretches and compressions of a basic graph
- ❑ **E.** Perform transformations on a general function $f(x)$

Many applications of mathematics require that we select a function known to fit the context, or build a function model from the information supplied. So far we've looked extensively at linear functions, and have introduced the absolute value, squaring, square root, cubing, and cube root functions. These are the six **toolbox functions,** so called because they give us a variety of "tools" to model the real world. In the same way a study of arithmetic depends heavily on the multiplication table, a study of algebra and mathematical modeling depends (in large part) on a solid working knowledge of these functions.

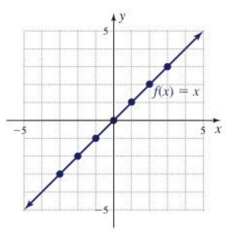
A. The Toolbox Functions

While we can accurately graph a line using only two points, most toolbox functions require more points to show all of the graph's important features. However, our work is greatly simplified in that each function belongs to a **function family,** in which all graphs from a given family share the characteristics of one basic graph, called the **parent function.** This means the number of points required for graphing will quickly decrease as we start anticipating what the graph of a given function should look like. The parent functions and their identifying characteristics are summarized here.

The Toolbox Functions

Identity function

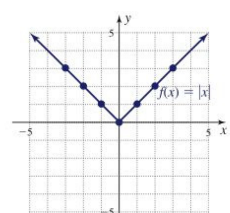
x	$f(x) = x$
−3	−3
−2	−2
−1	−1
0	0
1	1
2	2
3	3



Domain: $x \in (-\infty, \infty)$, Range: $y \in (-\infty, \infty)$
Symmetry: odd
Increasing: $x \in (-\infty, \infty)$
End behavior: down on the left/up on the right

Absolute value function

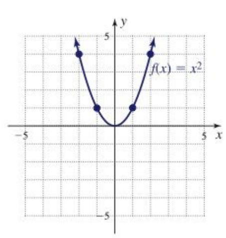
x	$f(x) = \|x\|$
−3	3
−2	2
−1	1
0	0
1	1
2	2
3	3



Domain: $x \in (-\infty, \infty)$, Range: $y \in [0, \infty)$
Symmetry: even
Decreasing: $x \in (-\infty, 0)$; Increasing: $x \in (0, \infty)$
End behavior: up on the left/up on the right
Vertex at (0, 0)

Squaring function

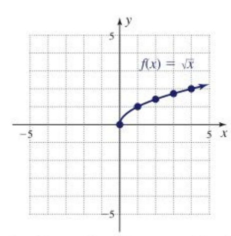
x	$f(x) = x^2$
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9



Domain: $x \in (-\infty, \infty)$, Range: $y \in [0, \infty)$
Symmetry: even
Decreasing: $x \in (-\infty, 0)$; Increasing: $x \in (0, \infty)$
End behavior: up on the left/up on the right
Vertex at (0, 0)

Square root function

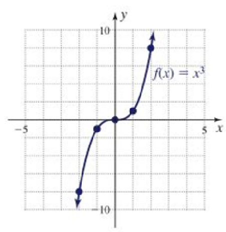
x	$f(x) = \sqrt{x}$
−2	–
−1	–
0	0
1	1
2	≈1.41
3	≈1.73
4	2



Domain: $x \in [0, \infty)$, Range: $y \in [0, \infty)$
Symmetry: neither even nor odd
Increasing: $x \in (0, \infty)$
End behavior: up on the right
Initial point at (0, 0)

Cubing function

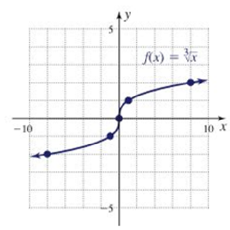
x	$f(x) = x^3$
−3	−27
−2	−8
−1	−1
0	0
1	1
2	8
3	27



Domain: $x \in (-\infty, \infty)$, Range: $y \in (-\infty, \infty)$
Symmetry: odd
Increasing: $x \in (-\infty, \infty)$
End behavior: down on the left/up on the right
Point of inflection at (0, 0)

Cube root function

x	$f(x) = \sqrt[3]{x}$
−27	−3
−8	−2
−1	−1
0	0
1	1
8	2
27	3



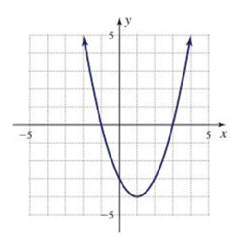
Domain: $x \in (-\infty, \infty)$, Range: $y \in (-\infty, \infty)$
Symmetry: odd
Increasing: $x \in (-\infty, \infty)$
End behavior: down on the left/up on the right
Point of inflection at (0, 0)

In applications of the toolbox functions, the parent graph may be altered and/or shifted from its original position, yet the graph will still retain its basic shape and features. The result is called a **transformation** of the parent graph. Analyzing the new graph (as in Section 2.5) will often provide the answers needed.

EXAMPLE 1 ► Identifying the Characteristics of a Transformed Graph

The graph of $f(x) = x^2 - 2x - 3$ is given. Use the graph to identify each of the features or characteristics indicated.

a. function family
b. domain and range
c. vertex
d. max or min value(s)
e. end behavior
f. x- and y-intercept(s)



Solution ►
a. The graph is a parabola, from the squaring function family.
b. domain: $x \in (-\infty, \infty)$; range: $y \in [-4, \infty)$
c. vertex: $(1, -4)$
d. minimum value $y = -4$ at $(1, -4)$
e. end-behavior: up/up
f. y-intercept: $(0, -3)$; x-intercepts: $(-1, 0)$ and $(3, 0)$

Now try Exercises 7 through 34 ►

A. You've just learned how to identify basic characteristics of the toolbox functions

Note that we can algebraically verify the x-intercepts by substituting 0 for $f(x)$ and solving the equation by factoring. This gives $0 = (x + 1)(x - 3)$, with solutions $x = -1$ and $x = 3$. It's also worth noting that while the parabola is no longer symmetric to the y-axis, it *is* symmetric to the vertical line $x = 1$. This line is called the **axis of symmetry** for the parabola, and will always be a vertical line that goes through the vertex.

B. Vertical and Horizontal Shifts

As we study specific transformations of a graph, try to develop a *global view* as the transformations can be applied to any function. When these are applied to the toolbox functions, we rely on characteristic features of the parent function to assist in completing the transformed graph.

Vertical Translations

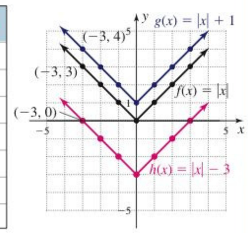
We'll first investigate vertical translations or vertical shifts of the toolbox functions, using the absolute value function to illustrate.

EXAMPLE 2 ▶ Graphing Vertical Translations

Construct a table of values for $f(x) = |x|$, $g(x) = |x| + 1$, and $h(x) = |x| - 3$ and graph the functions on the same coordinate grid. Then discuss what you observe.

Solution ▶ A table of values for all three functions is given, with the corresponding graphs shown in the figure.

x	$f(x) = \|x\|$	$g(x) = \|x\| + 1$	$h(x) = \|x\| - 3$
−3	3	4	0
−2	2	3	−1
−1	1	2	−2
0	0	1	−3
1	1	2	−2
2	2	3	−1
3	3	4	0



Note that outputs of $g(x)$ are one more than the outputs for $f(x)$, and that each point on the graph of f has been shifted *upward 1 unit* to form the graph of g. Similarly, each point on the graph of f has been shifted *downward 3 units* to form the graph of h. Since $h(x) = f(x) - 3$.

Now try Exercises 35 through 42 ▶

We describe the transformations in Example 2 as a **vertical shift** or **vertical translation** of a basic graph. The graph of g is the graph of *f shifted up 1 unit,* and the graph of h is the graph of *f shifted down 3 units*. In general, we have the following:

Vertical Translations of a Basic Graph

Given $k > 0$ and any function whose graph is determined by $y = f(x)$,

1. The graph of $y = f(x) + k$ is the graph of $f(x)$ shifted upward k units.
2. The graph of $y = f(x) - k$ is the graph of $f(x)$ shifted downward k units.

Horizontal Translations

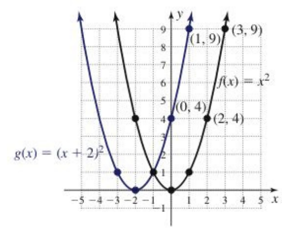
The graph of a parent function can also be shifted left or right. This happens when we *alter the inputs to the basic function,* as opposed to adding or subtracting something to the basic function itself. For $Y_1 = x^2 + 2$ note that we first square inputs, then add 2, which results in a vertical shift. For $Y_2 = (x + 2)^2$, we add 2 to *x prior to squaring* and since the input values are affected, we might anticipate the graph will shift along the x-axis—horizontally.

EXAMPLE 3 ▶ Graphing Horizontal Translations

Construct a table of values for $f(x) = x^2$ and $g(x) = (x + 2)^2$, then graph the functions on the same grid and discuss what you observe.

Solution ▶ Both f and g belong to the quadratic family and their graphs are parabolas. A table of values is shown along with the corresponding graphs.

x	$f(x) = x^2$	$g(x) = (x + 2)^2$
−3	9	1
−2	4	0
−1	1	1
0	0	4
1	1	9
2	4	16
3	9	25



It is apparent the graphs of g and f are identical, but the graph of g has been shifted horizontally 2 units left.

Now try Exercises 43 through 46 ▶

We describe the transformation in Example 3 as a **horizontal shift** or **horizontal translation** of a basic graph. The graph of g is the graph of f, *shifted 2 units to the left*. Once again it seems reasonable that since *input* values were altered, the shift must be horizontal rather than vertical. From this example, we also learn the direction of the shift is **opposite the sign:** $y = (x + 2)^2$ is 2 units *to the left* of $y = x^2$. Although it may seem counterintuitive, the shift *opposite the sign* can be "seen" by locating the new x-intercept, which in this case is also the vertex. Substituting 0 for y gives $0 = (x + 2)^2$ with $x = -2$, as shown in the graph. In general, we have

Horizontal Translations of a Basic Graph

Given $h > 0$ and any function whose graph is determined by $y = f(x)$,

1. The graph of $y = f(x + h)$ is the graph of $f(x)$ shifted *to the left h* units.
2. The graph of $y = f(x - h)$ is the graph of $f(x)$ shifted *to the right h* units.

EXAMPLE 4 ▶ Graphing Horizontal Translations

Sketch the graphs of $g(x) = |x - 2|$ and $h(x) = \sqrt{x + 3}$ using a horizontal shift of the parent function and a few characteristic points (not a table of values).

Solution ▶ The graph of $g(x) = |x - 2|$ (Figure 2.56) is the absolute value function shifted 2 units to the right (shift the vertex and two other points from $y = |x|$). The graph of $h(x) = \sqrt{x + 3}$ (Figure 2.57) is a square root function, shifted 3 units to the left (shift the initial point and one or two points from $y = \sqrt{x}$).

Figure 2.56

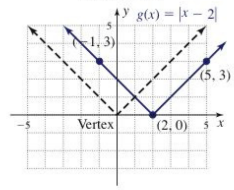
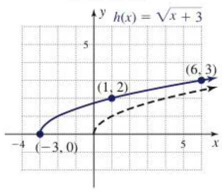


Figure 2.57



B. You've just learned how to perform vertical/horizontal shifts of a basic graph

Now try Exercises 47 through 50 ►

C. Vertical and Horizontal Reflections

The next transformation we investigate is called a **vertical reflection,** in which we compare the function $Y_1 = f(x)$ with the negative of the function: $Y_2 = -f(x)$.

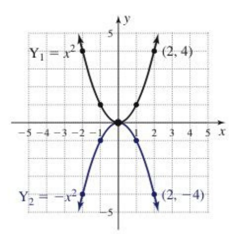
Vertical Reflections

EXAMPLE 5 ► **Graphing Vertical Reflections**

Construct a table of values for $Y_1 = x^2$ and $Y_2 = -x^2$, then graph the functions on the same grid and discuss what you observe.

Solution ► A table of values is given for both functions, along with the corresponding graphs.

x	$Y_1 = x^2$	$Y_2 = -x^2$
−2	4	−4
−1	1	−1
0	0	0
1	1	−1
2	4	−4



As you might have anticipated, the outputs for f and g differ only in sign. Each output is a **reflection** of the other, being an equal distance from the x-axis but on opposite sides.

Now try Exercises 51 and 52 ►

The vertical reflection in Example 5 is called a **reflection across the x-axis.** In general,

Vertical Reflections of a Basic Graph

For any function $y = f(x)$, the graph of $y = -f(x)$ is the graph of $f(x)$ reflected across the x-axis.

Horizontal Reflections

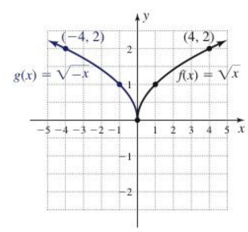
It's also possible for a graph to be reflected horizontally *across the y-axis.* Just as we noted that $f(x)$ versus $-f(x)$ resulted in a vertical reflection, $f(x)$ versus $f(-x)$ results in a horizontal reflection.

EXAMPLE 6 ► **Graphing a Horizontal Reflection**

Construct a table of values for $f(x) = \sqrt{x}$ and $g(x) = \sqrt{-x}$, then graph the functions on the same coordinate grid and discuss what you observe.

Solution ► A table of values is given here, along with the corresponding graphs.

x	$f(x) = \sqrt{x}$	$g(x) = \sqrt{-x}$
-4	not real	2
-2	not real	$\sqrt{2} \approx 1.41$
-1	not real	1
0	0	0
1	1	not real
2	$\sqrt{2} \approx 1.41$	not real
4	2	not real



The graph of g is the same as the graph of f, but it has been reflected across the y-axis. A study of the domain shows why—f represents a real number only for nonnegative inputs, so its graph occurs to the right of the y-axis, while g represents a real number for nonpositive inputs, so its graph occurs to the left.

Now try Exercises 53 and 54 ►

The transformation in Example 6 is called a **horizontal reflection** of a basic graph. In general,

Horizontal Reflections of a Basic Graph

For any function $y = f(x)$, the graph of $y = f(-x)$ is the graph of $f(x)$ reflected across the y-axis.

C. You've just learned how to perform vertical/horizontal reflections of a basic graph

D. Vertically Stretching/Compressing a Basic Graph

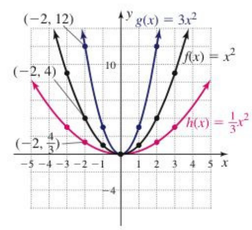
As the words "stretching" and "compressing" imply, the graph of a basic function can also become elongated or flattened after certain transformations are applied. However, even these transformations preserve the key characteristics of the graph.

EXAMPLE 7 ► **Stretching and Compressing a Basic Graph**

Construct a table of values for $f(x) = x^2$, $g(x) = 3x^2$, and $h(x) = \frac{1}{3}x^2$, then graph the functions on the same grid and discuss what you observe.

Solution ► A table of values is given for all three functions, along with the corresponding graphs.

x	$f(x) = x^2$	$g(x) = 3x^2$	$h(x) = \frac{1}{3}x^2$
−3	9	27	3
−2	4	12	$\frac{4}{3}$
−1	1	3	$\frac{1}{3}$
0	0	0	0
1	1	3	$\frac{1}{3}$
2	4	12	$\frac{4}{3}$
3	9	27	3



The outputs of g are triple those of f, making these outputs farther from the x-axis and *stretching* g upward (making the graph more narrow). The outputs of h are one-third those of f, and the graph of h is *compressed* downward, with its outputs closer to the x-axis (making the graph wider).

Now try Exercises 55 through 62 ►

WORTHY OF NOTE

In a study of trigonometry, you'll find that a basic graph can also be stretched or compressed horizontally, a phenomenon known as *frequency variations.*

The transformations in Example 7 are called **vertical stretches** or **compressions** of a basic graph. In general,

Stretches and Compressions of a Basic Graph

For any function $y = f(x)$, the graph of $y = af(x)$ is

1. the graph of $f(x)$ stretched vertically if $|a| > 1$,
2. the graph of $f(x)$ compressed vertically if $0 < |a| < 1$.

D. You've just learned how to perform vertical stretches and compressions of a basic graph

E. Transformations of a General Function

If more than one transformation is applied to a basic graph, it's helpful to use the following sequence for graphing the new function.

General Transformations of a Basic Graph

Given a function $y = f(x)$, the graph of $y = af(x \pm h) \pm k$ can be obtained by applying the following sequence of transformations:

1. horizontal shifts
2. reflections
3. stretches or compressions
4. vertical shifts

We generally use a few characteristic points to track the transformations involved, then draw the transformed graph through the new location of these points.

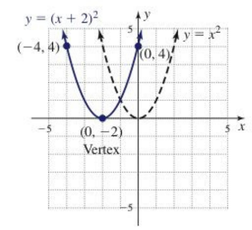
EXAMPLE 8 ► Graphing Functions Using Transformations

Use transformations of a parent function to sketch the graphs of

a. $g(x) = -(x + 2)^2 + 3$ **b.** $h(x) = 2\sqrt[3]{x - 2} - 1$

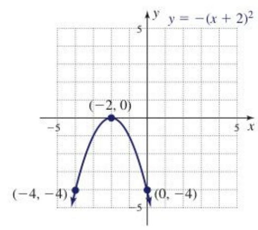
Solution ► **a.** The graph of g is a parabola, shifted left 2 units, reflected across the x-axis, and shifted up 3 units. This sequence of transformations in shown in Figures 2.58 through 2.60.

Figure 2.58



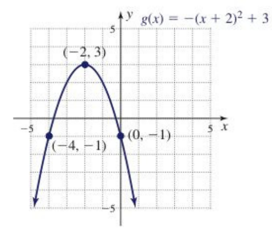
Shifted left 2 units

Figure 2.59



Reflected across the x-axis

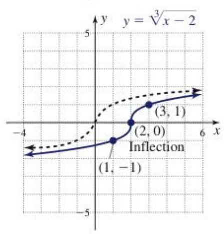
Figure 2.60



Shifted up 3

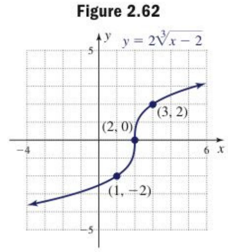
b. The graph of h is a cube root function, shifted right 2, stretched by a factor of 2, then shifted down 1. This sequence is shown in Figures 2.61 through 2.63.

Figure 2.61



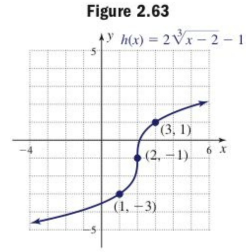
Shifted right 2

Figure 2.62



Stretched by a factor of 2

Figure 2.63



Shifted down 1

Now try Exercises 63 through 92 ►

WORTHY OF NOTE

Since the shape of the initial graph does not change when translations or reflections are applied, these are called **rigid transformations.** Stretches and compressions of a basic graph are called **nonrigid transformations,** as the graph is distended in some way.

It's important to note that the transformations can actually be applied to *any function,* even those that are new and unfamiliar. Consider the following pattern:

Parent Function	Transformation of Parent Function
quadratic: $y = x^2$	$y = -2(x - 3)^2 + 1$
absolute value: $y = \lvert x\rvert$	$y = -2\lvert x - 3\rvert + 1$
cube root: $y = \sqrt[3]{x}$	$y = -2\sqrt[3]{x - 3} + 1$
general: $y = f(x)$	$y = -2f(x - 3) + 1$

In each case, the transformation involves a horizontal shift right 3, a vertical reflection, a vertical stretch, and a vertical shift up 1. Since the shifts are the same regardless of the initial function, we can generalize the results to any function $f(x)$.

General Function	Transformed Function
$y = f(x)$	$y = af(x \pm h) \pm k$

- a: vertical reflections; vertical stretches and compressions
- h: horizontal shift h units, opposite direction of sign
- k: vertical shift k units, same direction as sign

Also bear in mind that the graph will be reflected across the y-axis (horizontally) if x is replaced with $-x$. Use this illustration to complete Exercise 9. Remember—if the graph of a function is shifted, the *individual points* on the graph are likewise shifted.

EXAMPLE 9 ► **Graphing Transformations of a General Function**

Given the graph of $f(x)$ shown in Figure 2.64, graph $g(x) = -f(x + 1) - 2$.

Solution ► For g, the graph of f is (1) shifted horizontally 1 unit left, (2) reflected across the x-axis, and (3) shifted vertically 2 units down. The final result is shown in Figure 2.65.

Figure 2.64

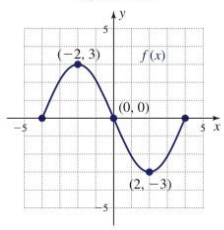
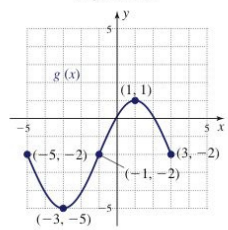


Figure 2.65



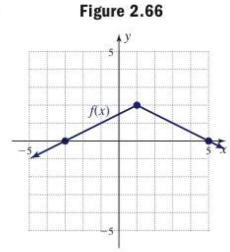
Now try Exercises 93 through 96 ►

Using the general equation $y = af(x \pm h) \pm k$, we can identify the vertex, initial point, or inflection point of any toolbox function and sketch its graph. Given the *graph* of a toolbox function, we can likewise identify these points and reconstruct its equation. We first identify the function family and the location (h, k) of the characteristic point. By selecting one other point (x, y) on the graph, we then use the general equation as a formula (substituting h, k, and the x- and y-values of the second point) to solve for a and complete the equation.

EXAMPLE 10 ► **Writing the Equation of a Function Given Its Graph**

Find the equation of the toolbox function $f(x)$ shown in Figure 2.66.

Figure 2.66



Solution ► The function f belongs to the absolute value family. The vertex (h, k) is at $(1, 2)$. For an additional point, choose the x-intercept $(-3, 0)$ and work as follows:

$y = a\|x - h\| + k$	general equation
$0 = a\|(-3) - 1\| + 2$	substitute 1 for h and 2 for k, substitute −3 for x and 0 for y
$0 = 4a + 2$	simplify
$-2 = 4a$	subtract 2
$-\frac{1}{2} = a$	solve for a

The equation for f is $y = -\frac{1}{2}|x - 1| + 2$.

Now try Exercises 97 through 102 ►

☑ **E.** You've just learned how to perform transformations on a general function $f(x)$

TECHNOLOGY HIGHLIGHT

Function Families

Graphing calculators are able to display a number of graphs simultaneously, making them a wonderful tool for studying families of functions. Let's begin by entering the function y = |x| [actually y = abs(x) [MATH] [►]] as Y_1 on the [Y=] screen. Next, we enter different variations of the function, but always in terms of its variable name "Y_1." This enables us to simply change the basic function, and observe how the changes affect the graph. Recall that to access the function name Y_1 press [VARS] [►] (to access the Y-VARS menu) [ENTER] (to access the function variables menu) and [ENTER] (to select Y_1). Enter the functions $Y_2 = Y_1 + 3$ and $Y_3 = Y_1 - 6$ (see Figure 2.67). Graph all three functions in the [ZOOM] **6:ZStandard** window. The calculator draws each graph in the order they were entered and you can always identify the functions by pressing the [TRACE] key and then the up arrow [▲] or down arrow [▼] keys. In the upper left corner of the window shown in Figure 2.68, the calculator identifies which function the cursor is currently on. Most importantly, note that all functions in this family maintain the same "V" shape.

Figure 2.67

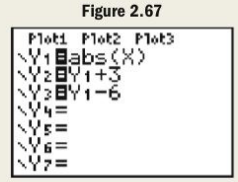
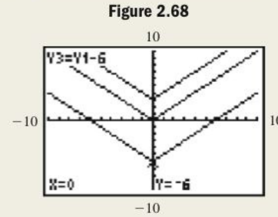


Figure 2.68



Next, change Y_1 to $Y_1 = \text{abs}(x - 3)$, leaving Y_2 and Y_3 as is. What do you notice when these are graphed again?

Exercise 1: Change Y_1 to $Y_1 = \sqrt{x}$ and graph, then enter $Y_1 = \sqrt{x - 3}$ and graph once again. What do you observe? What comparisons can be made with the translations of $Y_1 = \text{abs}(x)$?

Exercise 2: Change Y_1 to $Y_1 = x^2$ and graph, then enter $Y_1 = (x - 3)^2$ and graph once again. What do you observe? What comparisons can be made with the translations of $Y_1 = \text{abs}(x)$ and $Y_1 = \sqrt{x}$?

2.6 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. After a vertical ______, points on the graph are farther from the *x*-axis. After a vertical ______, points on the graph are closer to the *x*-axis.

2. Transformations that change only the location of a graph and not its shape or form, include ______ and ______.

3. The vertex of $h(x) = 3(x + 5)^2 - 9$ is at ______ and the graph opens ______.

4. The inflection point of $f(x) = -2(x - 4)^3 + 11$ is at _______ and the end behavior is _______, _______.

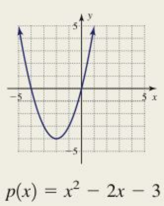
5. Given the graph of a general function $f(x)$, discuss/explain how the graph of $F(x) = -2f(x + 1) - 3$ can be obtained. If (0, 5), (6, 7), and (−9, −4) are on the graph of f, where do they end up on the graph of F?

6. Discuss/Explain why the shift of $f(x) = x^2 + 3$ is a *vertical shift* of 3 units in the *positive* direction, while the shift of $g(x) = (x + 3)^2$ is a *horizontal shift* 3 units in the *negative* direction. Include several examples linked to a table of values.

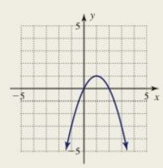
▶ DEVELOPING YOUR SKILLS

By carefully inspecting each graph given, (a) indentify the function family; (b) describe or identify the end behavior, vertex, axis of symmetry, and x- and y-intercepts; and (c) determine the domain and range. Assume required features have integer values.

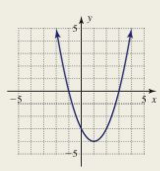
7. $f(x) = x^2 + 4x$



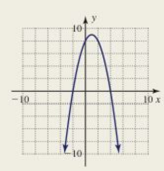
8. $g(x) = -x^2 + 2x$



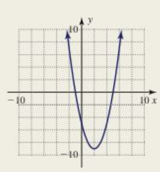
9. $p(x) = x^2 - 2x - 3$



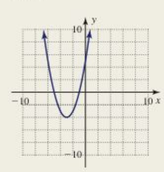
10. $q(x) = -x^2 + 2x + 8$



11. $f(x) = x^2 - 4x - 5$

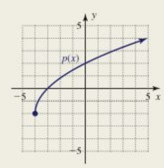


12. $g(x) = x^2 + 6x + 5$

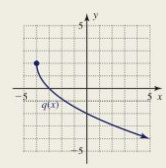


For each graph given, (a) indentify the function family; (b) describe or identify the end behavior, initial point, and x- and y-intercepts; and (c) determine the domain and range. Assume required features have integer values.

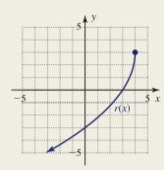
13. $p(x) = 2\sqrt{x + 4} - 2$



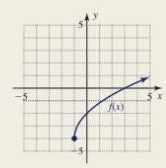
14. $q(x) = -2\sqrt{x + 4} + 2$



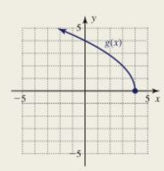
15. $r(x) = -3\sqrt{4 - x} + 3$



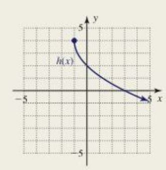
16. $f(x) = 2\sqrt{x + 1} - 4$



17. $g(x) = 2\sqrt{4 - x}$

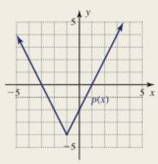


18. $h(x) = -2\sqrt{x + 1} + 4$

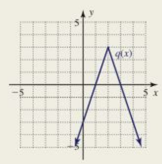


For each graph given, (a) indentify the function family; (b) describe or identify the end behavior, vertex, axis of symmetry, and *x*- and *y*-intercepts; and (c) determine the domain and range. Assume required features have integer values.

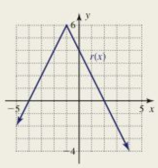
19. $p(x) = 2|x + 1| - 4$



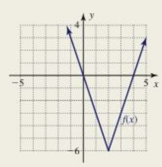
20. $q(x) = -3|x - 2| + 3$



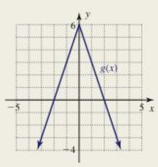
21. $r(x) = -2|x + 1| + 6$



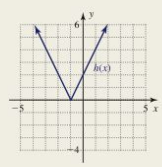
22. $f(x) = 3|x - 2| - 6$



23. $g(x) = -3|x| + 6$

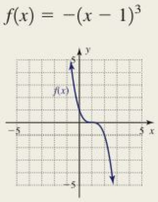


24. $h(x) = 2|x + 1|$

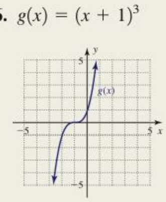


For each graph given, (a) indentify the function family; (b) describe or identify the end behavior, inflection point, and *x*- and *y*-intercepts; and (c) determine the domain and range. Assume required features have integer values. Be sure to note the scaling of each axis.

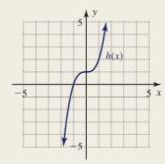
25. $f(x) = -(x - 1)^3$



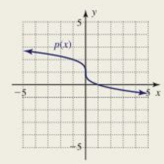
26. $g(x) = (x + 1)^3$



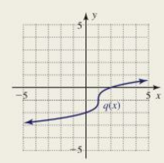
27. $h(x) = x^3 + 1$



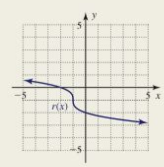
28. $p(x) = -\sqrt[3]{x} + 1$



29. $q(x) = \sqrt[3]{x - 1} - 1$

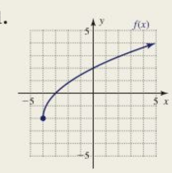


30. $r(x) = -\sqrt[3]{x + 1} - 1$

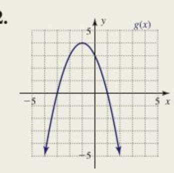


For Exercises 31–34, identify and state the characteristic features of each graph, including (as applicable) the function family, domain, range, intercepts, vertex, point of inflection, and end behavior.

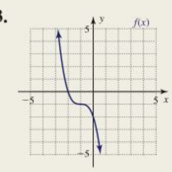
31.



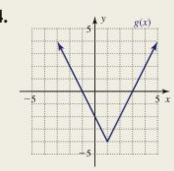
32.



33.



34.



Use a table of values to graph the functions given on the same grid. Comment on what you observe.

35. $f(x) = \sqrt{x}$, $g(x) = \sqrt{x} + 2$, $h(x) = \sqrt{x} - 3$

36. $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{x} - 3$, $h(x) = \sqrt[3]{x} + 1$

37. $p(x) = |x|$, $q(x) = |x| - 5$, $r(x) = |x| + 2$

38. $p(x) = x^2$, $q(x) = x^2 - 4$, $r(x) = x^2 + 1$

Sketch each graph using transformations of a parent function (without a table of values).

39. $f(x) = x^3 - 2$

40. $g(x) = \sqrt{x} - 4$

41. $h(x) = x^2 + 3$

42. $Y_1 = |x| - 3$

Use a table of values to graph the functions given on the same grid. Comment on what you observe.

43. $p(x) = x^2, \quad q(x) = (x + 3)^2$

44. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x + 4}$

45. $Y_1 = |x|, \quad Y_2 = |x - 1|$

46. $h(x) = x^3, \quad H(x) = (x - 2)^3$

Sketch each graph using transformations of a parent function (without a table of values).

47. $p(x) = (x - 3)^2$ **48.** $Y_1 = \sqrt{x - 1}$

49. $h(x) = |x + 3|$ **50.** $f(x) = \sqrt[3]{x + 2}$

51. $g(x) = -|x|$ **52.** $Y_2 = -\sqrt{x}$

53. $f(x) = \sqrt[3]{-x}$ **54.** $g(x) = (-x)^3$

Use a table of values to graph the functions given on the same grid. Comment on what you observe.

55. $p(x) = x^2, \quad q(x) = 2x^2, \quad r(x) = \frac{1}{2}x^2$

56. $f(x) = \sqrt{-x}, \quad g(x) = 4\sqrt{-x}, \quad h(x) = \frac{1}{4}\sqrt{-x}$

57. $Y_1 = |x|, \quad Y_2 = 3|x|, \quad Y_3 = \frac{1}{3}|x|$

58. $u(x) = x^3, \quad v(x) = 2x^3, \quad w(x) = \frac{1}{5}x^3$

Sketch each graph using transformations of a parent function (without a table of values).

59. $f(x) = 4\sqrt[3]{x}$ **60.** $g(x) = -2|x|$

61. $p(x) = \frac{1}{3}x^3$ **62.** $q(x) = \frac{3}{4}\sqrt{x}$

Use the characteristics of each function family to match a given function to its corresponding graph. The graphs are not scaled—make your selection based on a careful comparison.

63. $f(x) = \frac{1}{2}x^3$ **64.** $f(x) = \frac{-2}{3}x + 2$

65. $f(x) = -(x - 3)^2 + 2$ **66.** $f(x) = -\sqrt[3]{x - 1} - 1$

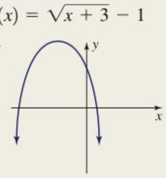
67. $f(x) = |x + 4| + 1$ **68.** $f(x) = -\sqrt{x + 6}$

69. $f(x) = -\sqrt{x + 6} - 1$ **70.** $f(x) = x + 1$

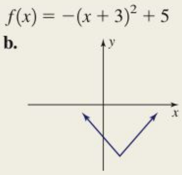
71. $f(x) = (x - 4)^2 - 3$ **72.** $f(x) = |x - 2| - 5$

73. $f(x) = \sqrt{x + 3} - 1$ **74.** $f(x) = -(x + 3)^2 + 5$

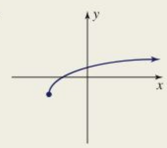
a.



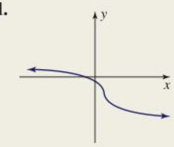
b.



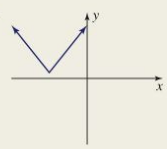
c.



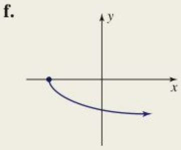
d.



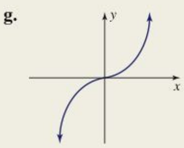
e.



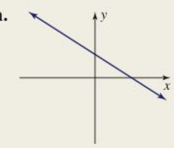
f.



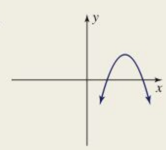
g.



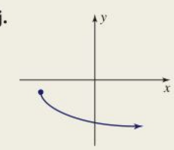
h.



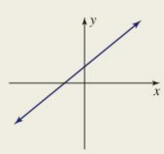
i.



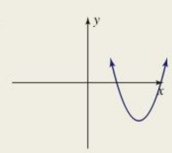
j.



k.



l.



Graph each function using shifts of a parent function and a few characteristic points. *Clearly state and indicate the transformations used* and identify the location of all vertices, initial points, and/or inflection points.

75. $f(x) = \sqrt{x + 2} - 1$ **76.** $g(x) = \sqrt{x - 3} + 2$

77. $h(x) = -(x + 3)^2 - 2$ **78.** $H(x) = -(x - 2)^2 + 5$

79. $p(x) = (x + 3)^3 - 1$ **80.** $q(x) = (x - 2)^3 + 1$

81. $Y_1 = \sqrt[3]{x + 1} - 2$ **82.** $Y_2 = \sqrt[3]{x - 3} + 1$

83. $f(x) = -|x + 3| - 2$ **84.** $g(x) = -|x - 4| - 2$

85. $h(x) = -2(x + 1)^2 - 3$ **86.** $H(x) = \frac{1}{2}|x + 2| - 3$

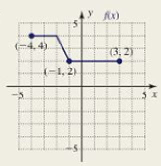
87. $p(x) = -\frac{1}{3}(x + 2)^3 - 1$ **88.** $q(x) = 5\sqrt[3]{x + 1} + 2$

89. $Y_1 = -2\sqrt{-x - 1} + 3$ **90.** $Y_2 = 3\sqrt{-x + 2} - 1$

91. $h(x) = \frac{1}{5}(x - 3)^2 + 1$ **92.** $H(x) = -2|x - 3| + 4$

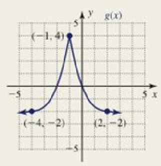
Apply the transformations indicated for the graph of the general functions given.

93.



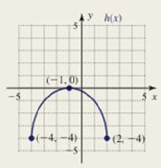
a. $f(x - 2)$
b. $-f(x) - 3$
c. $\frac{1}{2}f(x + 1)$
d. $f(-x) + 1$

94.



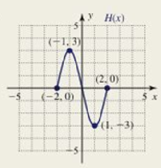
a. $g(x) - 2$
b. $-g(x) + 3$
c. $2g(x + 1)$
d. $\frac{1}{2}g(x - 1) + 2$

95.



a. $h(x) + 3$
b. $-h(x - 2)$
c. $h(x - 2) - 1$
d. $\frac{1}{4}h(x) + 5$

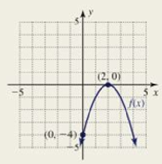
96.



a. $H(x - 3)$
b. $-H(x) + 1$
c. $2H(x - 3)$
d. $\frac{1}{3}H(x - 2) + 1$

Use the graph given and the points indicated to determine the equation of the function shown using the general form $y = af(x \pm h) \pm k$.

97.



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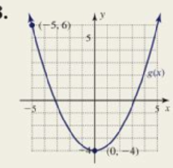

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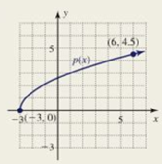
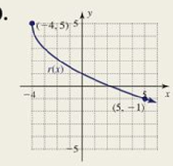


100.



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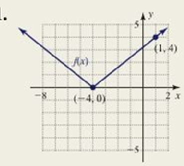

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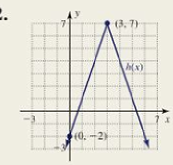


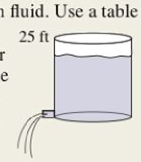
▶ WORKING WITH FORMULAS

103. Volume of a sphere: $V(r) = \frac{4}{3}\pi r^3$

The volume of a sphere is given by the function shown, where $V(r)$ is the volume in cubic units and r is the radius. Note this function belongs to the *cubic family* of functions. Approximate the value of $\frac{4}{3}\pi$ to one decimal place, then graph the function on the interval [0, 3]. From your *graph,* estimate the volume of a sphere with radius 2.5 in. Then compute the actual volume. Are the results close?

104. Fluid motion: $V(h) = -4\sqrt{h} + 20$

Suppose the velocity of a fluid flowing from an open tank (no top) through an opening in its side is given by the function shown, where $V(h)$ is the velocity of the fluid (in feet per second) at water height h (in feet). Note this function belongs to the *square root family* of functions. An open tank is 25 ft deep and filled to the brim with fluid. Use a table of values to graph the function on the interval [0, 25]. From your graph, estimate the velocity of the fluid when the water level is 7 ft, then find the actual velocity. Are the answers close? If the fluid velocity is 5 ft/sec, how high is the water in the tank?



▶ APPLICATIONS

105. Gravity, distance, time: After being released, the time it takes an object to fall x ft is given by the function $T(x) = \frac{1}{4}\sqrt{x}$, where $T(x)$ is in seconds. Describe the transformation applied to obtain the graph of T from the graph of $y = \sqrt{x}$, then sketch the graph of T for $x \in [0, 100]$. How long would it take an object to hit the ground if it were dropped from a height of 81 ft?

106. Stopping distance: In certain weather conditions, accident investigators will use the function $v(x) = 4.9\sqrt{x}$ to estimate the speed of a car (in miles per hour) that has been involved in an accident, based on the length of the skid marks x (in feet). Describe the transformation applied to obtain the graph of v from the graph of $y = \sqrt{x}$, then sketch the graph of v for $x \in [0, 400]$. If the skid marks were 225 ft long, how fast was the car traveling? Is this point on your graph?

107. Wind power: The power P generated by a certain wind turbine is given by the function $P(v) = \frac{8}{125}v^3$ where $P(v)$ is the power in watts at wind velocity v (in miles per hour). (a) Describe the transformation applied to obtain the graph of P from the graph of $y = v^3$, then sketch the graph of P for $v \in [0, 25]$ (scale the axes appropriately). (b) How much power is being generated when the wind is blowing at 15 mph? (c) Calculate the rate of change $\frac{\Delta P}{\Delta v}$ in the intervals [8, 10] and [28, 30]. What do you notice?

108. Wind power: If the power P (in watts) being generated by a wind turbine is known, the velocity of the wind can be determined using the function $v(P) = (\frac{5}{2})\sqrt[3]{P}$. Describe the transformation applied to obtain the graph of v from the graph of $y = \sqrt[3]{P}$, then sketch the graph of v for $P \in [0, 512]$ (scale the axes appropriately). How fast is the wind blowing if 343W of power is being generated?

109. Acceleration due to gravity: The *distance* a ball rolls down an inclined plane is given by the function $d(t) = 2t^2$, where $d(t)$ represents the distance in feet after t sec. (a) Describe the transformation applied to obtain the graph of d from the graph of $y = t^2$, then sketch the graph of d for $t \in [0, 3]$. (b) How far has the ball rolled after 2.5 sec? (c) Calculate the rate of change $\frac{\Delta d}{\Delta t}$ in the intervals [1, 1.5] and [3, 3.5]. What do you notice?

110. Acceleration due to gravity: The *velocity* of a steel ball bearing as it rolls down an inclined plane is given by the function $v(t) = 4t$, where $v(t)$ represents the velocity in feet per second after t sec. Describe the transformation applied to obtain the graph of v from the graph of $y = t$, then sketch the graph of v for $t \in [0, 3]$. What is the velocity of the ball bearing after 2.5 sec?

▶ EXTENDING THE CONCEPT

111. Carefully graph the functions $f(x) = |x|$ and $g(x) = 2\sqrt{x}$ on the same coordinate grid. From the graph, in what interval is the graph of $g(x)$ *above* the graph of $f(x)$? Pick a number (call it h) from this interval and substitute it in both functions. Is $g(h) > f(h)$? In what interval is the graph of $g(x)$ below the graph of $f(x)$? Pick a number from this interval (call it k) and substitute it in both functions. Is $g(k) < f(k)$?

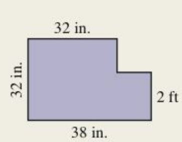
112. Sketch the graph of $f(x) = -2|x - 3| + 8$ using transformations of the parent function, then determine the area of the region in quadrant I that is beneath the graph and bounded by the vertical lines $x = 0$ and $x = 6$.

113. Sketch the graph of $f(x) = x^2 - 4$, then sketch the graph of $F(x) = |x^2 - 4|$ using your intuition and the meaning of absolute value (not a table of values). What happens to the graph?

▶ MAINTAINING YOUR SKILLS

114. (2.1) Find the distance between the points $(-13, 9)$ and $(7, -12)$, and the slope of the line containing these points.

115. (R.7) Find the perimeter and area of the figure shown (note the units).



116. (1.1) Solve for x: $\frac{2}{3}x + \frac{1}{4} = \frac{1}{2}x - \frac{7}{12}$.

117. (2.5) Without graphing, state intervals where $f(x)\uparrow$ and $f(x)\downarrow$ for $f(x) = (x - 4)^2 + 3$.

2.7 Piecewise-Defined Functions

Learning Objectives

In Section 2.7 you will learn how to:

- ☐ **A.** State the equation and domain of a piecewise-defined function
- ☐ **B.** Graph functions that are piecewise-defined
- ☐ **C.** Solve applications involving piecewise-defined functions

Most of the functions we've studied thus far have been smooth and continuous. Although "smooth" and "continuous" are defined more formally in advanced courses, for our purposes *smooth* simply means the graph has no sharp turns or jagged edges, and *continuous* means you can draw the entire graph without lifting your pencil. In this section, we study a special class of functions, called **piecewise-defined functions,** whose graphs may be various combinations of smooth/not smooth and continuous/not continuous. The absolute value function is one example (see Exercise 31). Such functions have a tremendous number of applications in the real world.

A. The Domain of a Piecewise-Defined Function

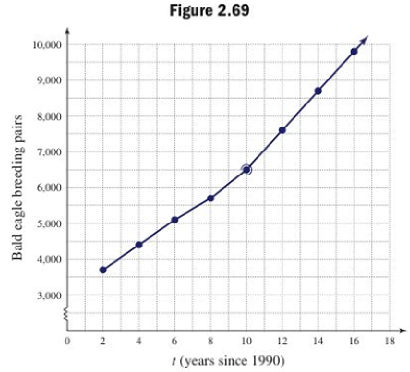
For the years 1990 to 2000, the American bald eagle remained on the nation's endangered species list, although the number of breeding pairs was growing slowly. After 2000, the population of eagles grew at a much faster rate, and they were removed from the list soon afterward. From Table 2.5 and plotted points modeling this growth (see Figure 2.69), we observe that a linear model would fit the period from 1992 to 2000 very well, but a line with greater slope would be needed for the years 2000 to 2006 and (perhaps) beyond.

Table 2.5

Year	Bald Eagle Breeding Pairs	Year	Bald Eagle Breeding Pairs
2	3700	10	6500
4	4400	12	7600
6	5100	14	8700
8	5700	16	9800

Source: www.fws.gov/midwest/eagle/population
1990 corresponds to year 0.

Figure 2.69



WORTHY OF NOTE

For the years 1992 to 2000, we can estimate the growth in breeding pairs $\frac{\Delta\text{pairs}}{\Delta\text{time}}$ using the points (2, 3700) and (10, 6500) in the slope formula. The result is $\frac{350}{1}$, or 350 pairs per year. For 2000 to 2006, using (10, 6500) and (16, 9800) shows the rate of growth is significantly larger: $\frac{\Delta\text{pairs}}{\Delta\text{years}} = \frac{550}{1}$ or 550 pairs per year.

The combination of these two lines would be a single function that modeled the population of breeding pairs from 1990 to 2006, but it would be *defined in two pieces*. This is an example of a **piecewise-defined function.**

The notation for these functions is a large "left brace" indicating the equations it groups are part of a single function. Using selected data points and techniques from Section 2.3, we find equations that could represent each piece are $p(t) = 350t + 3000$

WORTHY OF NOTE

In Figure 2.69, note that we indicated the exclusion of $t = 10$ from the second piece of the function using an open half-circle.

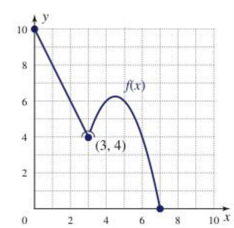
for $0 \le t \le 10$ and $p(t) = 550t + 1000$ for $t > 10$, where $p(t)$ is the number of breeding pairs in year t. The complete function is then written:

function name — function pieces — domain of each piece

$$p(t) = \begin{cases} 350t + 3000 & 2 \le t \le 10 \\ 550t + 1000 & t > 10 \end{cases}$$

EXAMPLE 1 ► Writing the Equation and Domain of a Piecewise-Defined Function

The linear piece of the function shown has an equation of $y = -2x + 10$. The equation of the quadratic piece is $y = -x^2 + 9x - 14$. Write the related piecewise-defined function, and state the domain of each piece by inspecting the graph.



Solution ► From the graph we note the linear portion is defined between 0 and 3, with these endpoints included as indicated by the closed dots. The domain here is $0 \le x \le 3$. The quadratic portion begins at $x = 3$ *but does not include 3*, as indicated by the half-circle notation. The equation is

function name — function pieces — domain

$$f(x) = \begin{cases} -2x + 10 & 0 \le x \le 3 \\ -x^2 + 9x - 14 & 3 < x \le 7 \end{cases}$$

A. You've just learned how to state the equation and domain of a piecewise-defined function

Now try Exercises 7 and 8 ►

Piecewise-defined functions can be composed of more than two pieces, and can involve functions of many kinds.

B. Graphing Piecewise-Defined Functions

As with other functions, piecewise-defined functions can be graphed by simply plotting points. Careful attention must be paid to the domain of each piece, both to evaluate the function correctly and to consider the inclusion/exclusion of endpoints. In addition, try to keep the transformations of a basic function in mind, as this will often help graph the function more efficiently.

EXAMPLE 2 ► Graphing a Piecewise-Defined Function

Graph the function by plotting points, then state its domain and range:

$$h(x) = \begin{cases} -x - 2 & -5 \le x < -1 \\ 2\sqrt{x + 1} - 1 & x \ge -1 \end{cases}$$

Solution ► The first piece of h is a line with negative slope, while the second is a transformed square root function. Using the endpoints of each domain specified and a few additional points, we obtain the following:

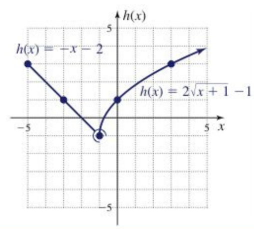
For $h(x) = -x - 2$, $-5 \le x < -1$,

x	$h(x)$
−5	3
−3	1
−1	−1

For $h(x) = 2\sqrt{x + 1} - 1$, $x \ge -1$,

x	$h(x)$
−1	−1
0	1
3	3

After plotting the points from the first piece, we connect them with a line segment noting the left endpoint is included, while the right endpoint is not (indicated using a semicircle around the point). Then we plot the points from the second piece and draw a square root graph, noting the left endpoint here *is* included, and the graph rises to the right. From the graph we note the complete domain of h is $x \in [-5, \infty)$, and the range is $y \in [-1, \infty)$.



Now try Exercises 9 through 14 ►

As an alternative to plotting points, we can graph each piece of the function using transformations of a basic graph, then erase those parts that are outside of the corresponding domain. Repeat this procedure for each piece of the function. One interesting and highly instructive aspect of these functions is the opportunity to investigate restrictions on their domain and the ranges that result.

Piecewise and Continuous Functions

EXAMPLE 3 ► Graphing a Piecewise-Defined Function

Graph the function and state its domain and range:

$$f(x) = \begin{cases} -(x-3)^2 + 12 & 0 < x \le 6 \\ 3 & x > 6 \end{cases}$$

Solution ► The first piece of f is a basic parabola, shifted three units right, reflected across the x-axis (opening downward), and shifted 12 units up. The vertex is at (3, 12) and the axis of symmetry is $x = 3$, producing the following graphs.

1. Graph first piece of f (Figure 2.70).
2. Erase portion outside domain of $0 < x \le 6$ (Figure 2.71).

Figure 2.70

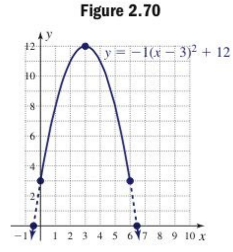
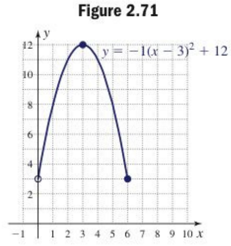


Figure 2.71



The second function is simply a horizontal line through (0, 3).

3. Graph second piece of f (Figure 2.72).

4. Erase portion outside domain of $x > 6$ (Figure 2.73).

Figure 2.72

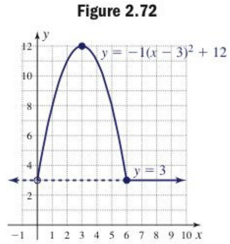
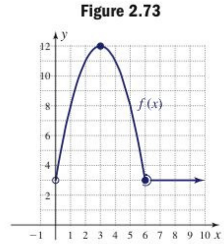


Figure 2.73



The domain of f is $x \in (0, \infty)$, and the corresponding range is $y \in [3, 12]$.

Now try Exercises 15 through 18 ►

Piecewise and Discontinuous Functions

Notice that although the function in Example 3 was piecewise-defined, the graph was actually continuous—we could draw the entire graph without lifting our pencil. Piecewise graphs also come in the *discontinuous* variety, which makes the domain and range issues all the more important.

EXAMPLE 4 ► Graphing a Discontinuous Piecewise-Defined Function

Graph $g(x)$ and state the domain and range:

$$g(x) = \begin{cases} -\frac{1}{2}x + 6 & 0 \le x \le 4 \\ -|x - 6| + 10 & 4 < x \le 9 \end{cases}$$

Solution ► The first piece of g is a line, with y-intercept (0, 6) and slope $\frac{\Delta y}{\Delta x} = -\frac{1}{2}$.

1. Graph first piece of g (Figure 2.74).

2. Erase portion outside domain of $0 \le x \le 4$ (Figure 2.75).

Figure 2.74

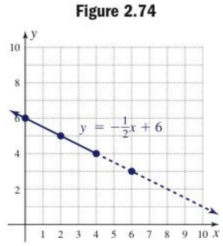
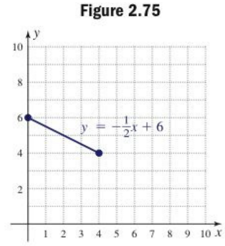


Figure 2.75



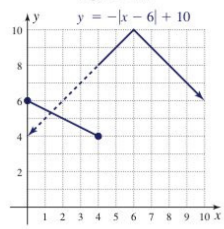
The second is an absolute value function, shifted right 6 units, reflected across the x-axis, then shifted up 10 units.

WORTHY OF NOTE

As you graph piecewise-defined functions, keep in mind that they *are* functions and the end result must pass the vertical line test. This is especially important when we are drawing each piece as a complete graph, then erasing portions outside the effective domain.

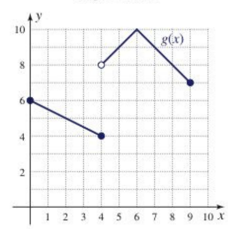
3. Graph second piece of g (Figure 2.76).

Figure 2.76



4. Erase portion outside domain of $4 < x \leq 9$ (Figure 2.77).

Figure 2.77



Note that the left endpoint of the absolute value portion is not included (this piece is not defined at $x = 4$), signified by the open dot. The result is a discontinuous graph, as there is no way to draw the graph other than by jumping the pencil from where one piece ends to where the next begins. Using a vertical boundary line, we note the domain of g includes all values between 0 and 9 inclusive: $x \in [0, 9]$. Using a horizontal boundary line shows the smallest y-value is 4 and the largest is 10, but no range values exist between 6 and 7. The range is $y \in [4, 6] \cup [7, 10]$.

Now try Exercises 19 through 22 ►

EXAMPLE 5 ► Graphing a Discontinuous Function

The given piecewise-defined function is not continuous. Graph $h(x)$ to see why, then comment on what could be done to make it continuous.

$$h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & x \neq 2 \\ 1 & x = 2 \end{cases}$$

Solution ► The first piece of h is unfamiliar to us, so we elect to graph it by plotting points, noting $x = 2$ is outside the domain. This produces the table shown in Figure 2.78. After connecting the points, the graph of h turns out to be a straight line, but with no corresponding y-value for $x = 2$. This leaves a "hole" in the graph at (2, 4), as designated by the open dot.

Figure 2.78

x	$h(x)$
−4	−2
−2	0
0	2
2	—
4	6

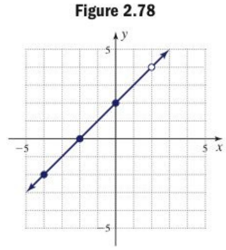
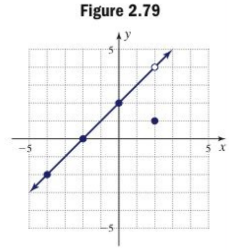


Figure 2.79



WORTHY OF NOTE

The discontinuity illustrated here is called a **removable discontinuity,** as the discontinuity can be removed by redefining a piece of the function. Note that after factoring the first piece, the denominator is a factor of the numerator, and writing the result in lowest terms gives $h(x) = \frac{(x + 2)(x - 2)}{x - 2} = x + 2, x \neq 2$. This is precisely the equation of the line in Figure 2.78 $[h(x) = x + 2]$.

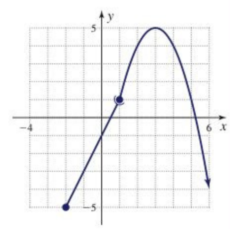
The second piece is point-wise defined, and its graph is simply the point (2, 1) shown in Figure 2.79. It's interesting to note that while the domain of h is all real numbers (h *is* defined at all points), the range is $y \in (-\infty, 4) \cup (4, \infty)$ as the function never takes on the value $y = 4$. In order for h to be continuous, we would need to redefine the second piece as $y = 4$ when $x = 2$.

Now try Exercises 23 through 26 ►

To develop these concepts more fully, it will help to practice finding the equation of a piecewise-defined function *given its graph,* a process similar to that of Example 10 in Section 2.6.

EXAMPLE 6 ▶ Determining the Equation of a Piecewise-Defined Function

Determine the equation of the piecewise-defined function shown, including the domain for each piece.



Solution ▶ By counting $\frac{\Delta y}{\Delta x}$ from $(-2, -5)$ to $(1, 1)$, we find the linear portion has slope $m = 2$, and the y-intercept must be $(0, -1)$. The equation of the line is $y = 2x - 1$. The second piece appears to be a parabola with vertex (h, k) at $(3, 5)$. Using this vertex with the point $(1, 1)$ in the general form $y = a(x - h)^2 + k$ gives

$$
\begin{aligned}
y &= a(x - h)^2 + k && \text{general form} \\
1 &= a(1 - 3)^2 + 5 && \text{substitute 1 for } x\text{, 1 for } y\text{, 3 for } h\text{, 5 for } k \\
-4 &= a(-2)^2 && \text{simplify; subtract 5} \\
-4 &= 4a && (-2)^2 = 4 \\
-1 &= a && \text{divide by 4}
\end{aligned}
$$

The equation of the parabola is $y = -(x - 3)^2 + 5$. Considering the domains shown in the figure, the equation of this piecewise-defined function must be

$$
p(x) = \begin{cases} 2x - 1 & -2 \le x \le 1 \\ -(x - 3)^2 + 5 & x > 1 \end{cases}
$$

B. You've just learned how to graph functions that are piecewise-defined

Now try Exercises 27 through 30 ▶

C. Applications of Piecewise-Defined Functions

The number of applications for piecewise-defined functions is practically limitless. It is actually fairly rare for a single function to accurately model a situation over a long period of time. Laws change, spending habits change, and technology can bring abrupt alterations in many areas of our lives. To accurately model these changes often requires a piecewise-defined function.

EXAMPLE 7 ▶ Modeling with a Piecewise-Defined Function

For the first half of the twentieth century, per capita spending on police protection can be modeled by $S(t) = 0.54t + 12$, where $S(t)$ represents per capita spending on police protection in year t (1900 corresponds to year 0). After 1950, perhaps due to the growth of American cities, this spending greatly increased: $S(t) = 3.65t - 144$. Write these as a piecewise-defined function $S(t)$, state the domain for each piece, then graph the function. According to this model, how much was spent (per capita) on police protection in 2000? How much will be spent in 2010?

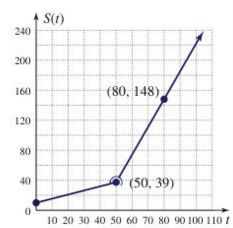
Source: Data taken from the *Statistical Abstract of the United States* for various years.

Solution ►

function name / function pieces / effective domain

$$S(t) = \begin{cases} 0.54t + 12 & 0 \le t \le 50 \\ 3.65t - 144 & t > 50 \end{cases}$$

Since both pieces are linear, we can graph each part using two points. For the first function, $S(0) = 12$ and $S(50) = 39$. For the second function $S(50) \approx 39$ and $S(80) = 148$. The graph for each piece is shown in the figure. Evaluating S at $t = 100$:



$$\begin{aligned} S(t) &= 3.65t - 144 \\ S(100) &= 3.65(100) - 144 \\ &= 365 - 144 \\ &= 221 \end{aligned}$$

About \$221 per capita was spent on police protection in the year 2000. For 2010, the model indicates that \$257.50 per capita will be spent: $S(110) = 257.5$.

Now try Exercises 33 through 44 ►

Step Functions

The last group of piecewise-defined functions we'll explore are the **step functions,** so called because the pieces of the function form a series of horizontal steps. These functions find frequent application in the way consumers are charged for services, and have a number of applications in number theory. Perhaps the most common is called the **greatest integer function,** though recently its alternative name, **floor function,** has gained popularity (see Figure 2.80). This is in large part due to an improvement in notation and as a better contrast to **ceiling functions.** The floor function of a real number x, denoted $f(x) = \lfloor x \rfloor$ or $[\![x]\!]$ (we will use the first), is the largest integer less than or equal to x. For instance, $\lfloor 5.9 \rfloor = 5$, $\lfloor 7 \rfloor = 7$, and $\lfloor -3.4 \rfloor = -4$.

In contrast, the ceiling function $C(x) = \lceil x \rceil$ is the smallest integer greater than or equal to x, meaning $\lceil 5.9 \rceil = 6$, $\lceil 7 \rceil = 7$, and $\lceil -3.4 \rceil = -3$ (see Figure 2.81). In simple terms, for any noninteger value on the number line, the floor function returns the integer to the left, while the ceiling function returns the integer to the right. A graph of each function is shown.

Figure 2.80

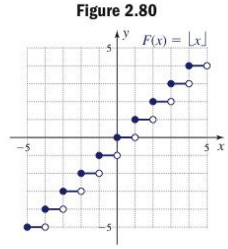
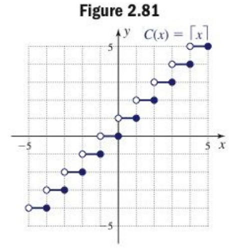


Figure 2.81

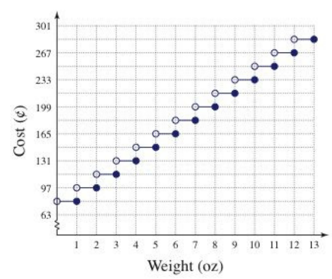


One common application of floor functions is the price of theater admission, where children 12 and under receive a discounted price. Right up until the day they're 13, they qualify for the lower price: $\lfloor 12\frac{364}{365} \rfloor = 12$. Applications of ceiling functions would include how phone companies charge for the minutes used (charging the 12-min rate for a phone call that only lasted 11.3 min: $\lceil 11.3 \rceil = 12$), and postage rates, as in Example 8.

EXAMPLE 8 ▶ Modeling Using a Step Function

As of May 2007, the first-class postage rate for large envelopes sent through the U.S. mail was 80¢ for the first ounce, then an additional 17¢ per ounce thereafter, up to 13 ounces. Graph the function and state its domain and range. Use the graph to state the cost of mailing a report weighing (a) 7.5 oz, (b) 8 oz, and (c) 8.1 oz in a large envelope.

Solution ▶ The 80¢ charge applies to letters weighing between 0 oz and 1 oz. Zero is not included since we have to mail *something,* but 1 is included since a large envelope and its contents weighing exactly one ounce still costs 80¢. The graph will be a horizontal line segment.



The function is defined for all weights between 0 and 13 oz, excluding zero and including 13: $x \in (0, 13]$. The range consists of single outputs corresponding to the step intervals: $R \in \{80, 97, 114, \ldots, 267, 284\}$.

a. The cost of mailing a 7.5-oz report is 199¢.

b. The cost of mailing an 8.0-oz report is still 199¢.

c. The cost of mailing an 8.1-oz report is $199 + 17 = 216$¢, since this brings you up to the next step.

☑ **C.** You've just learned how to solve applications involving piecewise-defined functions

Now try Exercises 45 through 48 ▶

TECHNOLOGY HIGHLIGHT

Piecewise-Defined Functions

Most graphing calculators are able to graph piecewise-defined functions. Consider the function f shown here:

$$f(x) = \begin{cases} x + 2 & x < 2 \\ (x - 4)^2 + 3 & x \geq 2 \end{cases}$$

Both "pieces" are well known—the first is a line with slope $m = 1$ and y-intercept (0, 2). The second is a parabola that opens upward, shifted 4 units to the right and 3 units up. If we attempt to graph $f(x)$ using $Y_1 = x + 2$ and $Y_2 = (x - 4)^2 + 3$ as they stand, the resulting graph may be difficult to analyze because the pieces conflict and intersect (Figure 2.82). To graph the functions we must indicate the domain for each piece, separated by a slash and enclosed in parentheses. For instance, for the first piece we enter $Y_1 = x + 2/(x < 2)$, and for the second, $Y_2 = (x - 4)^2 + 3/(x \geq 2)$ (Figure 2.83). The slash looks like (is) the division symbol, but in this context, the calculator interprets it as a means of separating the function from the domain. The inequality symbols are accessed using the 2nd MATH (**TEST**) keys. The graph is shown on Figure 2.84, where we see the function is linear for $x \in (-\infty, 2)$ and quadratic for $x \in [2, \infty)$. How does the calculator remind us the function is defined only for $x = 2$ on the second piece? Using the 2nd GRAPH **(TABLE)** feature reveals the calculator will give an **ERR:** (ERROR) message for inputs outside of its domain (Figure 2.85).

Figure 2.82

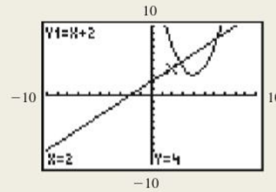
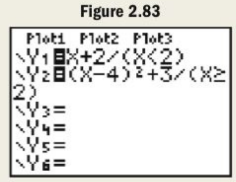


Figure 2.83



We can also use the calculator to investigate endpoints of the domain. For instance, we know that $Y_1 = x + 2$ is not defined for $x = 2$, but what about numbers very close to 2? Go to 2nd WINDOW **(TBLSET)** and place the calculator in the Indpnt: Auto ASK mode. With both Y_1 and Y_2 enabled, use the 2nd GRAPH **(TABLE)** feature to evaluate the functions at numbers very near 2. Use $x = 1.9, 1.99, 1.999$, and so on.

Figure 2.84

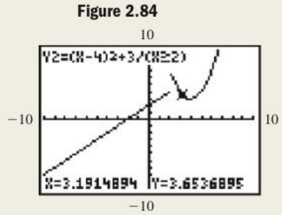
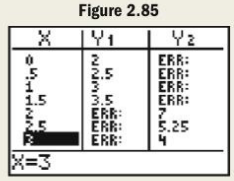


Figure 2.85



Exercise 1: What appears to be happening to the output values for Y_1? What about Y_2?

Exercise 2: What do you notice about the output values when 1.99999 is entered? Use the right arrow key ► to move the cursor into columns Y_1 and Y_2. Comment on what you think the calculator is doing. Will Y_1 ever really have an output equal to 4?

2.7 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A function whose entire graph can be drawn without lifting your pencil is called a ______ function.
2. The input values for which each part of a piecewise function is defined is the ______ of the function.
3. A graph is called ______ if it has no sharp turns or jagged edges.