2.6 The Toolbox Functions and Transformations

Learning Objectives

In Section 2.6 you will learn how to:

- ❑ **A.** Identify basic characteristics of the toolbox functions
- ❑ **B.** Perform vertical/horizontal shifts of a basic graph
- ❑ **C.** Perform vertical/horizontal reflections of a basic graph
- ❑ **D.** Perform vertical stretches and compressions of a basic graph
- ❑ **E.** Perform transformations on a general function $f(x)$

Many applications of mathematics require that we select a function known to fit the context, or build a function model from the information supplied. So far we've looked extensively at linear functions, and have introduced the absolute value, squaring, square root, cubing, and cube root functions. These are the six **toolbox functions,** so called because they give us a variety of "tools" to model the real world. In the same way a study of arithmetic depends heavily on the multiplication table, a study of algebra and mathematical modeling depends (in large part) on a solid working knowledge of these functions.

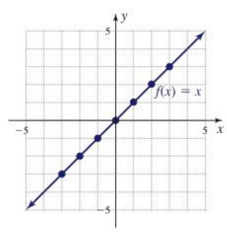
A. The Toolbox Functions

While we can accurately graph a line using only two points, most toolbox functions require more points to show all of the graph's important features. However, our work is greatly simplified in that each function belongs to a **function family,** in which all graphs from a given family share the characteristics of one basic graph, called the **parent function.** This means the number of points required for graphing will quickly decrease as we start anticipating what the graph of a given function should look like. The parent functions and their identifying characteristics are summarized here.

The Toolbox Functions

Identity function

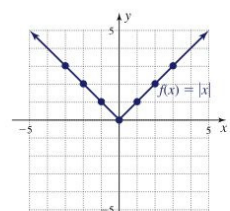
x	$f(x) = x$
−3	−3
−2	−2
−1	−1
0	0
1	1
2	2
3	3



Domain: $x \in (-\infty, \infty)$, Range: $y \in (-\infty, \infty)$
Symmetry: odd
Increasing: $x \in (-\infty, \infty)$
End behavior: down on the left/up on the right

Absolute value function

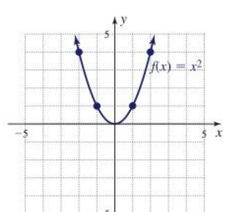
x	$f(x) = \lvert x\rvert$
−3	3
−2	2
−1	1
0	0
1	1
2	2
3	3



Domain: $x \in (-\infty, \infty)$, Range: $y \in [0, \infty)$
Symmetry: even
Decreasing: $x \in (-\infty, 0)$; Increasing: $x \in (0, \infty)$
End behavior: up on the left/up on the right
Vertex at (0, 0)

Squaring function

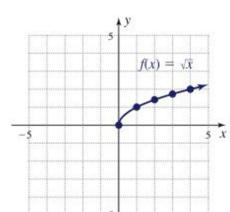
x	$f(x) = x^2$
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9



Domain: $x \in (-\infty, \infty)$, Range: $y \in [0, \infty)$
Symmetry: even
Decreasing: $x \in (-\infty, 0)$; Increasing: $x \in (0, \infty)$
End behavior: up on the left/up on the right
Vertex at (0, 0)

Square root function

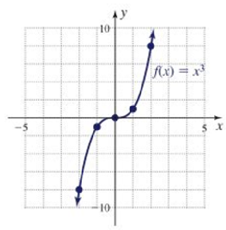
x	$f(x) = \sqrt{x}$
−2	–
−1	–
0	0
1	1
2	≈1.41
3	≈1.73
4	2



Domain: $x \in [0, \infty)$, Range: $y \in [0, \infty)$
Symmetry: neither even nor odd
Increasing: $x \in (0, \infty)$
End behavior: up on the right
Initial point at (0, 0)

Cubing function

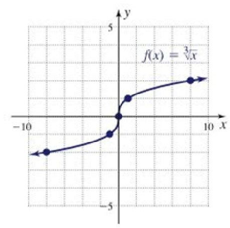
x	$f(x) = x^3$
−3	−27
−2	−8
−1	−1
0	0
1	1
2	8
3	27



Domain: $x \in (-\infty, \infty)$, Range: $y \in (-\infty, \infty)$
Symmetry: odd
Increasing: $x \in (-\infty, \infty)$
End behavior: down on the left/up on the right
Point of inflection at (0, 0)

Cube root function

x	$f(x) = \sqrt[3]{x}$
−27	−3
−8	−2
−1	−1
0	0
1	1
8	2
27	3



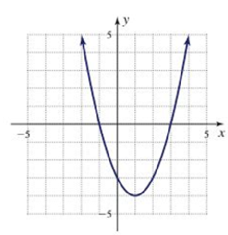
Domain: $x \in (-\infty, \infty)$, Range: $y \in (-\infty, \infty)$
Symmetry: odd
Increasing: $x \in (-\infty, \infty)$
End behavior: down on the left/up on the right
Point of inflection at (0, 0)

In applications of the toolbox functions, the parent graph may be altered and/or shifted from its original position, yet the graph will still retain its basic shape and features. The result is called a **transformation** of the parent graph. Analyzing the new graph (as in Section 2.5) will often provide the answers needed.

EXAMPLE 1 ► Identifying the Characteristics of a Transformed Graph

The graph of $f(x) = x^2 - 2x - 3$ is given. Use the graph to identify each of the features or characteristics indicated.

a. function family
b. domain and range
c. vertex
d. max or min value(s)
e. end behavior
f. x- and y-intercept(s)



Solution ► **a.** The graph is a parabola, from the squaring function family.
b. domain: $x \in (-\infty, \infty)$; range: $y \in [-4, \infty)$
c. vertex: $(1, -4)$
d. minimum value $y = -4$ at $(1, -4)$
e. end-behavior: up/up
f. y-intercept: $(0, -3)$; x-intercepts: $(-1, 0)$ and $(3, 0)$

Now try Exercises 7 through 34 ►

☑ **A. You've just learned how to identify basic characteristics of the toolbox functions**

Note that we can algebraically verify the x-intercepts by substituting 0 for $f(x)$ and solving the equation by factoring. This gives $0 = (x + 1)(x - 3)$, with solutions $x = -1$ and $x = 3$. It's also worth noting that while the parabola is no longer symmetric to the y-axis, it *is* symmetric to the vertical line $x = 1$. This line is called the **axis of symmetry** for the parabola, and will always be a vertical line that goes through the vertex.

B. Vertical and Horizontal Shifts

As we study specific transformations of a graph, try to develop a *global view* as the transformations can be applied to any function. When these are applied to the toolbox functions, we rely on characteristic features of the parent function to assist in completing the transformed graph.

Vertical Translations

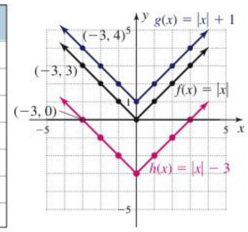
We'll first investigate vertical translations or vertical shifts of the toolbox functions, using the absolute value function to illustrate.

EXAMPLE 2 ► **Graphing Vertical Translations**

Construct a table of values for $f(x) = |x|$, $g(x) = |x| + 1$, and $h(x) = |x| - 3$ and graph the functions on the same coordinate grid. Then discuss what you observe.

Solution ► A table of values for all three functions is given, with the corresponding graphs shown in the figure.

x	$f(x) = \|x\|$	$g(x) = \|x\| + 1$	$h(x) = \|x\| - 3$
−3	3	4	0
−2	2	3	−1
−1	1	2	−2
0	0	1	−3
1	1	2	−2
2	2	3	−1
3	3	4	0



Note that outputs of $g(x)$ are one more than the outputs for $f(x)$, and that each point on the graph of f has been shifted *upward 1 unit* to form the graph of g. Similarly, each point on the graph of f has been shifted *downward 3 units* to form the graph of h. Since $h(x) = f(x) - 3$.

Now try Exercises 35 through 42 ►

We describe the transformations in Example 2 as a **vertical shift** or **vertical translation** of a basic graph. The graph of g is the graph of *f shifted up 1 unit,* and the graph of h is the graph of *f shifted down 3 units*. In general, we have the following:

Vertical Translations of a Basic Graph

Given $k > 0$ and any function whose graph is determined by $y = f(x)$,
1. The graph of $y = f(x) + k$ is the graph of $f(x)$ shifted upward k units.
2. The graph of $y = f(x) - k$ is the graph of $f(x)$ shifted downward k units.

Horizontal Translations

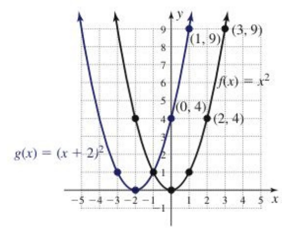
The graph of a parent function can also be shifted left or right. This happens when we *alter the inputs to the basic function,* as opposed to adding or subtracting something to the basic function itself. For $Y_1 = x^2 + 2$ note that we first square inputs, then add 2, which results in a vertical shift. For $Y_2 = (x + 2)^2$, we add 2 to *x prior to squaring* and since the input values are affected, we might anticipate the graph will shift along the x-axis—horizontally.

EXAMPLE 3 ► **Graphing Horizontal Translations**

Construct a table of values for $f(x) = x^2$ and $g(x) = (x + 2)^2$, then graph the functions on the same grid and discuss what you observe.

Solution ► Both *f* and *g* belong to the quadratic family and their graphs are parabolas. A table of values is shown along with the corresponding graphs.

x	$f(x) = x^2$	$g(x) = (x + 2)^2$
−3	9	1
−2	4	0
−1	1	1
0	0	4
1	1	9
2	4	16
3	9	25



It is apparent the graphs of *g* and *f* are identical, but the graph of *g* has been shifted horizontally 2 units left.

Now try Exercises 43 through 46 ►

We describe the transformation in Example 3 as a **horizontal shift** or **horizontal translation** of a basic graph. The graph of *g* is the graph of *f*, *shifted 2 units to the left*. Once again it seems reasonable that since *input* values were altered, the shift must be horizontal rather than vertical. From this example, we also learn the direction of the shift is **opposite the sign:** $y = (x + 2)^2$ is 2 units *to the left* of $y = x^2$. Although it may seem counterintuitive, the shift *opposite the sign* can be "seen" by locating the new *x*-intercept, which in this case is also the vertex. Substituting 0 for *y* gives $0 = (x + 2)^2$ with $x = -2$, as shown in the graph. In general, we have

Horizontal Translations of a Basic Graph

Given $h > 0$ and any function whose graph is determined by $y = f(x)$,

1. The graph of $y = f(x + h)$ is the graph of $f(x)$ shifted *to the left h* units.
2. The graph of $y = f(x - h)$ is the graph of $f(x)$ shifted *to the right h* units.

EXAMPLE 4 ► **Graphing Horizontal Translations**

Sketch the graphs of $g(x) = |x - 2|$ and $h(x) = \sqrt{x + 3}$ using a horizontal shift of the parent function and a few characteristic points (not a table of values).

Solution ► The graph of $g(x) = |x - 2|$ (Figure 2.56) is the absolute value function shifted 2 units to the right (shift the vertex and two other points from $y = |x|$). The graph of $h(x) = \sqrt{x + 3}$ (Figure 2.57) is a square root function, shifted 3 units to the left (shift the initial point and one or two points from $y = \sqrt{x}$).

Figure 2.56

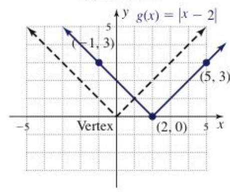
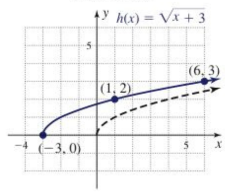


Figure 2.57



B. You've just learned how to perform vertical/horizontal shifts of a basic graph

Now try Exercises 47 through 50 ►

C. Vertical and Horizontal Reflections

The next transformation we investigate is called a **vertical reflection,** in which we compare the function $Y_1 = f(x)$ with the negative of the function: $Y_2 = -f(x)$.

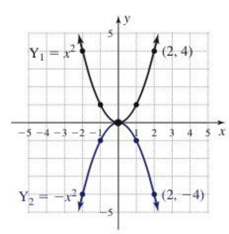
Vertical Reflections

EXAMPLE 5 ► **Graphing Vertical Reflections**

Construct a table of values for $Y_1 = x^2$ and $Y_2 = -x^2$, then graph the functions on the same grid and discuss what you observe.

Solution ► A table of values is given for both functions, along with the corresponding graphs.

x	$Y_1 = x^2$	$Y_2 = -x^2$
−2	4	−4
−1	1	−1
0	0	0
1	1	−1
2	4	−4



As you might have anticipated, the outputs for f and g differ only in sign. Each output is a **reflection** of the other, being an equal distance from the x-axis but on opposite sides.

Now try Exercises 51 and 52 ►

The vertical reflection in Example 5 is called a **reflection across the x-axis.** In general,

Vertical Reflections of a Basic Graph

For any function $y = f(x)$, the graph of $y = -f(x)$ is the graph of $f(x)$ reflected across the x-axis.

Horizontal Reflections

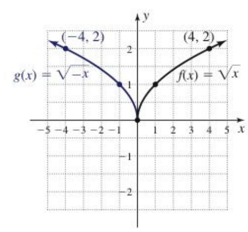
It's also possible for a graph to be reflected horizontally *across the y-axis.* Just as we noted that $f(x)$ versus $-f(x)$ resulted in a vertical reflection, $f(x)$ versus $f(-x)$ results in a horizontal reflection.

EXAMPLE 6 ▶ Graphing a Horizontal Reflection

Construct a table of values for $f(x) = \sqrt{x}$ and $g(x) = \sqrt{-x}$, then graph the functions on the same coordinate grid and discuss what you observe.

Solution ▶ A table of values is given here, along with the corresponding graphs.

x	$f(x) = \sqrt{x}$	$g(x) = \sqrt{-x}$
-4	not real	2
-2	not real	$\sqrt{2} \approx 1.41$
-1	not real	1
0	0	0
1	1	not real
2	$\sqrt{2} \approx 1.41$	not real
4	2	not real



The graph of g is the same as the graph of f, but it has been reflected across the y-axis. A study of the domain shows why—f represents a real number only for nonnegative inputs, so its graph occurs to the right of the y-axis, while g represents a real number for nonpositive inputs, so its graph occurs to the left.

Now try Exercises 53 and 54 ▶

The transformation in Example 6 is called a **horizontal reflection** of a basic graph. In general,

Horizontal Reflections of a Basic Graph

For any function $y = f(x)$, the graph of $y = f(-x)$ is the graph of $f(x)$ reflected across the y-axis.

☑ **C.** You've just learned how to perform vertical/horizontal reflections of a basic graph

D. Vertically Stretching/Compressing a Basic Graph

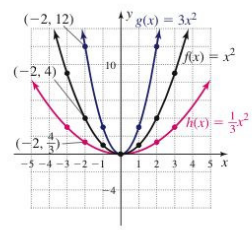
As the words "stretching" and "compressing" imply, the graph of a basic function can also become elongated or flattened after certain transformations are applied. However, even these transformations preserve the key characteristics of the graph.

EXAMPLE 7 ▶ Stretching and Compressing a Basic Graph

Construct a table of values for $f(x) = x^2$, $g(x) = 3x^2$, and $h(x) = \frac{1}{3}x^2$, then graph the functions on the same grid and discuss what you observe.

Solution ► A table of values is given for all three functions, along with the corresponding graphs.

x	$f(x) = x^2$	$g(x) = 3x^2$	$h(x) = \frac{1}{3}x^2$
−3	9	27	3
−2	4	12	$\frac{4}{3}$
−1	1	3	$\frac{1}{3}$
0	0	0	0
1	1	3	$\frac{1}{3}$
2	4	12	$\frac{4}{3}$
3	9	27	3



The outputs of g are triple those of f, making these outputs farther from the x-axis and *stretching* g upward (making the graph more narrow). The outputs of h are one-third those of f, and the graph of h is *compressed* downward, with its outputs closer to the x-axis (making the graph wider).

Now try Exercises 55 through 62 ►

WORTHY OF NOTE

In a study of trigonometry, you'll find that a basic graph can also be stretched or compressed horizontally, a phenomenon known as *frequency variations.*

The transformations in Example 7 are called **vertical stretches** or **compressions** of a basic graph. In general,

Stretches and Compressions of a Basic Graph

For any function $y = f(x)$, the graph of $y = af(x)$ is

1. the graph of $f(x)$ stretched vertically if $|a| > 1$,
2. the graph of $f(x)$ compressed vertically if $0 < |a| < 1$.

D. You've just learned how to perform vertical stretches and compressions of a basic graph

E. Transformations of a General Function

If more than one transformation is applied to a basic graph, it's helpful to use the following sequence for graphing the new function.

General Transformations of a Basic Graph

Given a function $y = f(x)$, the graph of $y = af(x \pm h) \pm k$ can be obtained by applying the following sequence of transformations:

1. horizontal shifts
2. reflections
3. stretches or compressions
4. vertical shifts

We generally use a few characteristic points to track the transformations involved, then draw the transformed graph through the new location of these points.

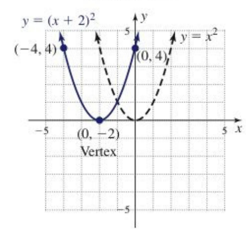
EXAMPLE 8 ► **Graphing Functions Using Transformations**

Use transformations of a parent function to sketch the graphs of

a. $g(x) = -(x + 2)^2 + 3$ **b.** $h(x) = 2\sqrt[3]{x - 2} - 1$

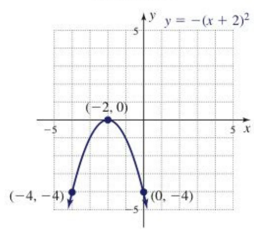
Solution ► **a.** The graph of g is a parabola, shifted left 2 units, reflected across the x-axis, and shifted up 3 units. This sequence of transformations in shown in Figures 2.58 through 2.60.

Figure 2.58



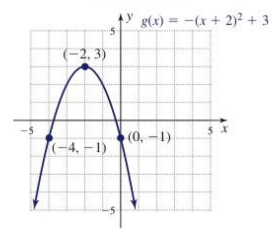
Shifted left 2 units

Figure 2.59



Reflected across the x-axis

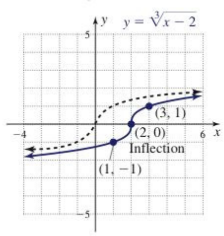
Figure 2.60



Shifted up 3

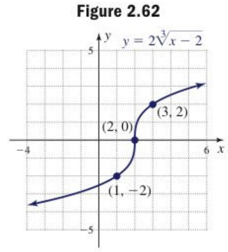
b. The graph of h is a cube root function, shifted right 2, stretched by a factor of 2, then shifted down 1. This sequence is shown in Figures 2.61 through 2.63.

Figure 2.61



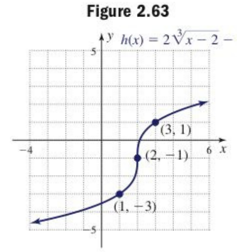
Shifted right 2

Figure 2.62



Stretched by a factor of 2

Figure 2.63



Shifted down 1

Now try Exercises 63 through 92 ►

WORTHY OF NOTE

Since the shape of the initial graph does not change when translations or reflections are applied, these are called **rigid transformations.** Stretches and compressions of a basic graph are called **nonrigid transformations,** as the graph is distended in some way.

It's important to note that the transformations can actually be applied to *any function,* even those that are new and unfamiliar. Consider the following pattern:

Parent Function	Transformation of Parent Function
quadratic: $y = x^2$	$y = -2(x - 3)^2 + 1$
absolute value: $y = \lvert x \rvert$	$y = -2\lvert x - 3 \rvert + 1$
cube root: $y = \sqrt[3]{x}$	$y = -2\sqrt[3]{x - 3} + 1$
general: $y = f(x)$	$y = -2f(x - 3) + 1$

In each case, the transformation involves a horizontal shift right 3, a vertical reflection, a vertical stretch, and a vertical shift up 1. Since the shifts are the same regardless of the initial function, we can generalize the results to any function $f(x)$.

General Function	Transformed Function
$y = f(x)$	$y = af(x \pm h) \pm k$

- a: vertical reflections, vertical stretches and compressions
- h: horizontal shift h units, opposite direction of sign
- k: vertical shift k units, same direction as sign

Also bear in mind that the graph will be reflected across the y-axis (horizontally) if x is replaced with $-x$. Use this illustration to complete Exercise 9. Remember—if the graph of a function is shifted, the *individual points* on the graph are likewise shifted.

EXAMPLE 9 ► **Graphing Transformations of a General Function**

Given the graph of $f(x)$ shown in Figure 2.64, graph $g(x) = -f(x + 1) - 2$.

Solution ► For g, the graph of f is (1) shifted horizontally 1 unit left, (2) reflected across the x-axis, and (3) shifted vertically 2 units down. The final result is shown in Figure 2.65.

Figure 2.64

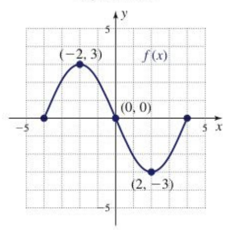
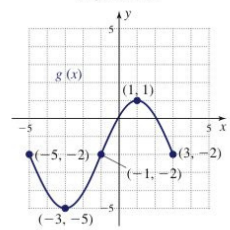


Figure 2.65



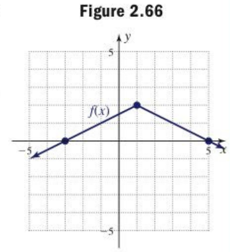
Now try Exercises 93 through 96 ►

Using the general equation $y = af(x \pm h) \pm k$, we can identify the vertex, initial point, or inflection point of any toolbox function and sketch its graph. Given the *graph* of a toolbox function, we can likewise identify these points and reconstruct its equation. We first identify the function family and the location (h, k) of the characteristic point. By selecting one other point (x, y) on the graph, we then use the general equation as a formula (substituting h, k, and the x- and y-values of the second point) to solve for a and complete the equation.

EXAMPLE 10 ► **Writing the Equation of a Function Given Its Graph**

Find the equation of the toolbox function $f(x)$ shown in Figure 2.66.

Figure 2.66



Solution ► The function f belongs to the absolute value family. The vertex (h, k) is at $(1, 2)$. For an additional point, choose the x-intercept $(-3, 0)$ and work as follows:

$y = a|x - h| + k$ — general equation

$0 = a|(-3) - 1| + 2$ — substitute 1 for h and 2 for k, substitute -3 for x and 0 for y

$0 = 4a + 2$ — simplify

$-2 = 4a$ — subtract 2

$-\frac{1}{2} = a$ — solve for a

The equation for f is $y = -\frac{1}{2}|x - 1| + 2$.

Now try Exercises 97 through 102 ►

☑ **E.** You've just learned how to perform transformations on a general function $f(x)$

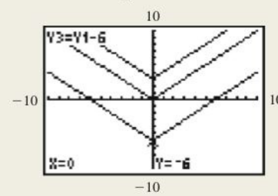
TECHNOLOGY HIGHLIGHT

Function Families

Graphing calculators are able to display a number of graphs simultaneously, making them a wonderful tool for studying families of functions. Let's begin by entering the function y = |x| [actually y = abs(x) MATH ►] as Y_1 on the Y= screen. Next, we enter different variations of the function, but always in terms of its variable name "Y_1." This enables us to simply change the basic function, and observe how the changes affect the graph. Recall that to access the function name Y_1 press VARS ► (to access the Y-VARS menu) ENTER (to access the function variables menu) and ENTER (to select Y_1). Enter the functions $Y_2 = Y_1 + 3$ and $Y_3 = Y_1 - 6$ (see Figure 2.67). Graph all three functions in the ZOOM **6:ZStandard** window. The calculator draws each graph in the order they were entered and you can always identify the functions by pressing the TRACE key and then the up arrow ▲ or down arrow ▼ keys. In the upper left corner of the window shown in Figure 2.68, the calculator identifies which function the cursor is currently on. Most importantly, note that all functions in this family maintain the same "V" shape.

Figure 2.67

Figure 2.68



Next, change Y_1 to $Y_1 = \text{abs}(x - 3)$, leaving Y_2 and Y_3 as is. What do you notice when these are graphed again?

Exercise 1: Change Y_1 to $Y_1 = \sqrt{x}$ and graph, then enter $Y_1 = \sqrt{x - 3}$ and graph once again. What do you observe? What comparisons can be made with the translations of $Y_1 = \text{abs}(x)$?

Exercise 2: Change Y_1 to $Y_1 = x^2$ and graph, then enter $Y_1 = (x - 3)^2$ and graph once again. What do you observe? What comparisons can be made with the translations of $Y_1 = \text{abs}(x)$ and $Y_1 = \sqrt{x}$?

2.6 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. After a vertical ______, points on the graph are farther from the *x*-axis. After a vertical ______, points on the graph are closer to the *x*-axis.

2. Transformations that change only the location of a graph and not its shape or form, include ______ and ______.

3. The vertex of $h(x) = 3(x + 5)^2 - 9$ is at ______ and the graph opens ______.

4. The inflection point of $f(x) = -2(x - 4)^3 + 11$ is at ______ and the end behavior is ______, ______.

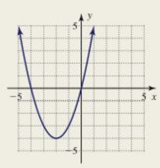
5. Given the graph of a general function $f(x)$, discuss/explain how the graph of $F(x) = -2f(x + 1) - 3$ can be obtained. If (0, 5), (6, 7), and (−9, −4) are on the graph of *f*, where do they end up on the graph of *F*?

6. Discuss/Explain why the shift of $f(x) = x^2 + 3$ is a *vertical shift* of 3 units in the *positive* direction, while the shift of $g(x) = (x + 3)^2$ is a *horizontal shift* 3 units in the *negative* direction. Include several examples linked to a table of values.

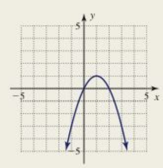
▶ DEVELOPING YOUR SKILLS

By carefully inspecting each graph given, (a) indentify the function family; (b) describe or identify the end behavior, vertex, axis of symmetry, and *x*- and *y*-intercepts; and (c) determine the domain and range. Assume required features have integer values.

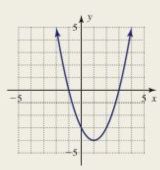
7. $f(x) = x^2 + 4x$



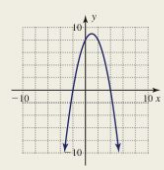
8. $g(x) = -x^2 + 2x$



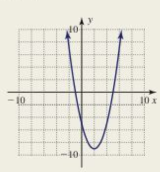
9. $p(x) = x^2 - 2x - 3$



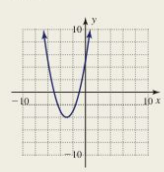
10. $q(x) = -x^2 + 2x + 8$



11. $f(x) = x^2 - 4x - 5$

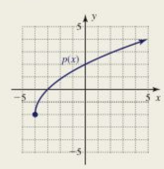


12. $g(x) = x^2 + 6x + 5$

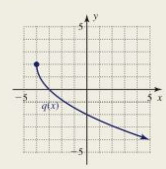


For each graph given, (a) identify the function family; (b) describe or identify the end behavior, initial point, and *x*- and *y*-intercepts; and (c) determine the domain and range. Assume required features have integer values.

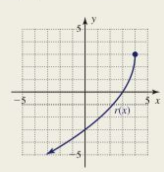
13. $p(x) = 2\sqrt{x + 4} - 2$



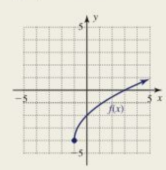
14. $q(x) = -2\sqrt{x + 4} + 2$



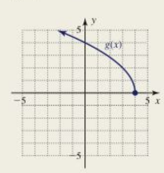
15. $r(x) = -3\sqrt{4 - x} + 3$



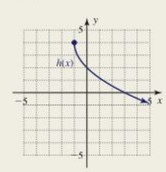
16. $f(x) = 2\sqrt{x + 1} - 4$



17. $g(x) = 2\sqrt{4 - x}$

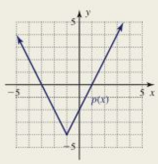


18. $h(x) = -2\sqrt{x + 1} + 4$

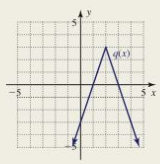


For each graph given, (a) indentify the function family; (b) describe or identify the end behavior, vertex, axis of symmetry, and x- and y-intercepts; and (c) determine the domain and range. Assume required features have integer values.

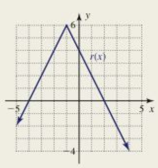
19. $p(x) = 2|x + 1| - 4$



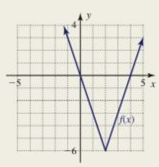
20. $q(x) = -3|x - 2| + 3$



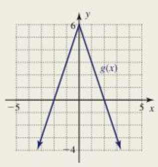
21. $r(x) = -2|x + 1| + 6$



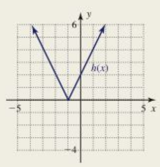
22. $f(x) = 3|x - 2| - 6$



23. $g(x) = -3|x| + 6$

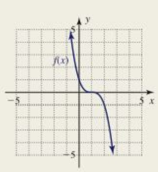


24. $h(x) = 2|x + 1|$

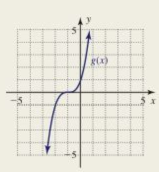


For each graph given, (a) identify the function family; (b) describe or identify the end behavior, inflection point, and x- and y-intercepts; and (c) determine the domain and range. Assume required features have integer values. Be sure to note the scaling of each axis.

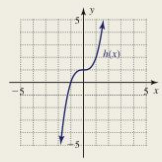
25. $f(x) = -(x - 1)^3$



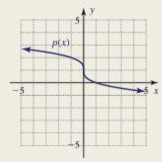
26. $g(x) = (x + 1)^3$



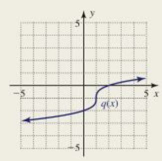
27. $h(x) = x^3 + 1$



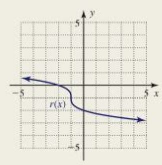
28. $p(x) = -\sqrt[3]{x} + 1$



29. $q(x) = \sqrt[3]{x - 1} - 1$

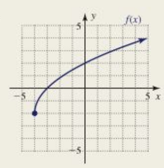


30. $r(x) = -\sqrt[3]{x + 1} - 1$

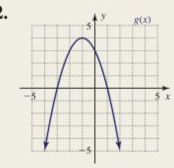


For Exercises 31–34, identify and state the characteristic features of each graph, including (as applicable) the function family, domain, range, intercepts, vertex, point of inflection, and end behavior.

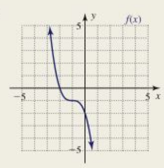
31.



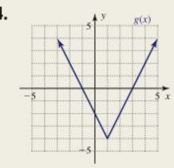
32.



33.



34.



Use a table of values to graph the functions given on the same grid. Comment on what you observe.

35. $f(x) = \sqrt{x}$, $g(x) = \sqrt{x} + 2$, $h(x) = \sqrt{x} - 3$

36. $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{x} - 3$, $h(x) = \sqrt[3]{x} + 1$

37. $p(x) = |x|$, $q(x) = |x| - 5$, $r(x) = |x| + 2$

38. $p(x) = x^2$, $q(x) = x^2 - 4$, $r(x) = x^2 + 1$

Sketch each graph using transformations of a parent function (without a table of values).

39. $f(x) = x^3 - 2$ **40.** $g(x) = \sqrt{x} - 4$

41. $h(x) = x^2 + 3$ **42.** $Y_1 = |x| - 3$

Use a table of values to graph the functions given on the same grid. Comment on what you observe.

43. $p(x) = x^2$, $q(x) = (x + 3)^2$

44. $f(x) = \sqrt{x}$, $g(x) = \sqrt{x + 4}$

45. $Y_1 = |x|$, $Y_2 = |x - 1|$

46. $h(x) = x^3$, $H(x) = (x - 2)^3$

Sketch each graph using transformations of a parent function (without a table of values).

47. $p(x) = (x - 3)^2$

48. $Y_1 = \sqrt{x - 1}$

49. $h(x) = |x + 3|$

50. $f(x) = \sqrt[3]{x + 2}$

51. $g(x) = -|x|$

52. $Y_2 = -\sqrt{x}$

53. $f(x) = \sqrt[3]{-x}$

54. $g(x) = (-x)^3$

Use a table of values to graph the functions given on the same grid. Comment on what you observe.

55. $p(x) = x^2$, $q(x) = 2x^2$, $r(x) = \frac{1}{2}x^2$

56. $f(x) = \sqrt{-x}$, $g(x) = 4\sqrt{-x}$, $h(x) = \frac{1}{4}\sqrt{-x}$

57. $Y_1 = |x|$, $Y_2 = 3|x|$, $Y_3 = \frac{1}{3}|x|$

58. $u(x) = x^3$, $v(x) = 2x^3$, $w(x) = \frac{1}{5}x^3$

Sketch each graph using transformations of a parent function (without a table of values).

59. $f(x) = 4\sqrt[3]{x}$

60. $g(x) = -2|x|$

61. $p(x) = \frac{1}{3}x^3$

62. $q(x) = \frac{3}{4}\sqrt{x}$

Use the characteristics of each function family to match a given function to its corresponding graph. The graphs are not scaled—make your selection based on a careful comparison.

63. $f(x) = \frac{1}{2}x^3$

64. $f(x) = \frac{-2}{3}x + 2$

65. $f(x) = -(x - 3)^2 + 2$

66. $f(x) = -\sqrt[3]{x - 1} - 1$

67. $f(x) = |x + 4| + 1$

68. $f(x) = -\sqrt{x + 6}$

69. $f(x) = -\sqrt{x + 6} - 1$

70. $f(x) = x + 1$

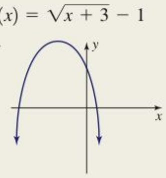
71. $f(x) = (x - 4)^2 - 3$

72. $f(x) = |x - 2| - 5$

73. $f(x) = \sqrt{x + 3} - 1$

74. $f(x) = -(x + 3)^2 + 5$

a.



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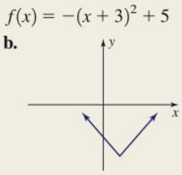

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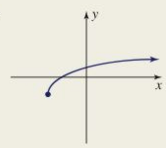

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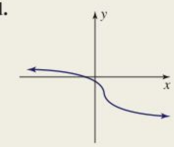

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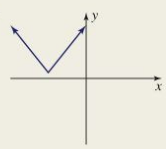

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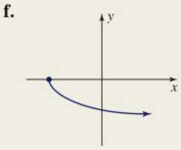

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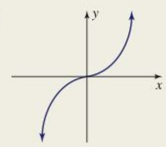

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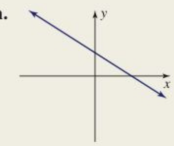
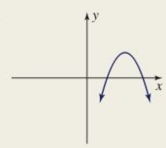


i.



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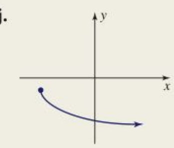
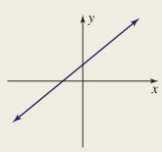
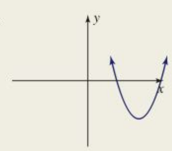


k.



l.



Graph each function using shifts of a parent function and a few characteristic points. *Clearly state and indicate the transformations used* and identify the location of all vertices, initial points, and/or inflection points.

75. $f(x) = \sqrt{x + 2} - 1$

76. $g(x) = \sqrt{x - 3} + 2$

77. $h(x) = -(x + 3)^2 - 2$

78. $H(x) = -(x - 2)^2 + 5$

79. $p(x) = (x + 3)^3 - 1$

80. $q(x) = (x - 2)^3 + 1$

81. $Y_1 = \sqrt[3]{x + 1} - 2$

82. $Y_2 = \sqrt[3]{x - 3} + 1$

83. $f(x) = -|x + 3| - 2$

84. $g(x) = -|x - 4| - 2$

85. $h(x) = -2(x + 1)^2 - 3$

86. $H(x) = \frac{1}{2}|x + 2| - 3$

87. $p(x) = -\frac{1}{3}(x + 2)^3 - 1$

88. $q(x) = 5\sqrt[3]{x + 1} + 2$

89. $Y_1 = -2\sqrt{-x - 1} + 3$

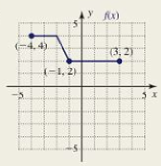
90. $Y_2 = 3\sqrt{-x + 2} - 1$

91. $h(x) = \frac{1}{5}(x - 3)^2 + 1$

92. $H(x) = -2|x - 3| + 4$

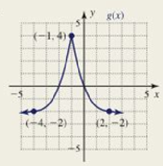
Apply the transformations indicated for the graph of the general functions given.

93.



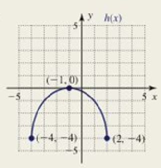
a. $f(x - 2)$
b. $-f(x) - 3$
c. $\frac{1}{2}f(x + 1)$
d. $f(-x) + 1$

94.



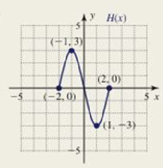
a. $g(x) - 2$
b. $-g(x) + 3$
c. $2g(x + 1)$
d. $\frac{1}{2}g(x - 1) + 2$

95.



a. $h(x) + 3$
b. $-h(x - 2)$
c. $h(x - 2) - 1$
d. $\frac{1}{4}h(x) + 5$

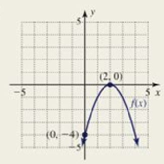
96.



a. $H(x - 3)$
b. $-H(x) + 1$
c. $2H(x - 3)$
d. $\frac{1}{3}H(x - 2) + 1$

Use the graph given and the points indicated to determine the equation of the function shown using the general form $y = af(x \pm h) \pm k$.

97.



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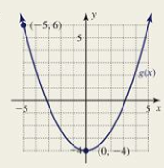

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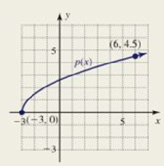

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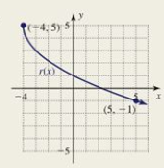

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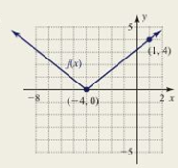

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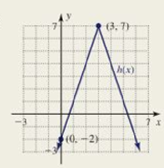


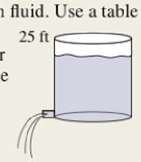
▶ WORKING WITH FORMULAS

103. Volume of a sphere: $V(r) = \frac{4}{3}\pi r^3$

The volume of a sphere is given by the function shown, where $V(r)$ is the volume in cubic units and r is the radius. Note this function belongs to the *cubic family* of functions. Approximate the value of $\frac{4}{3}\pi$ to one decimal place, then graph the function on the interval [0, 3]. From your *graph*, estimate the volume of a sphere with radius 2.5 in. Then compute the actual volume. Are the results close?

104. Fluid motion: $V(h) = -4\sqrt{h} + 20$

Suppose the velocity of a fluid flowing from an open tank (no top) through an opening in its side is given by the function shown, where $V(h)$ is the velocity of the fluid (in feet per second) at water height h (in feet). Note this function belongs to the *square root family* of functions. An open tank is 25 ft deep and filled to the brim with fluid. Use a table of values to graph the function on the interval [0, 25]. From your graph, estimate the velocity of the fluid when the water level is 7 ft, then find the actual velocity. Are the answers close? If the fluid velocity is 5 ft/sec, how high is the water in the tank?



▶ APPLICATIONS

105. Gravity, distance, time: After being released, the time it takes an object to fall x ft is given by the function $T(x) = \frac{1}{4}\sqrt{x}$, where $T(x)$ is in seconds. Describe the transformation applied to obtain the graph of T from the graph of $y = \sqrt{x}$, then sketch the graph of T for $x \in [0, 100]$. How long would it take an object to hit the ground if it were dropped from a height of 81 ft?

106. Stopping distance: In certain weather conditions, accident investigators will use the function $v(x) = 4.9\sqrt{x}$ to estimate the speed of a car (in miles per hour) that has been involved in an accident, based on the length of the skid marks x (in feet). Describe the transformation applied to obtain the graph of v from the graph of $y = \sqrt{x}$, then sketch the graph of v for $x \in [0, 400]$. If the skid marks were 225 ft long, how fast was the car traveling? Is this point on your graph?

107. Wind power: The power P generated by a certain wind turbine is given by the function $P(v) = \frac{8}{125}v^3$ where $P(v)$ is the power in watts at wind velocity v (in miles per hour). (a) Describe the transformation applied to obtain the graph of P from the graph of $y = v^3$, then sketch the graph of P for $v \in [0, 25]$ (scale the axes appropriately). (b) How much power is being generated when the wind is blowing at 15 mph? (c) Calculate the rate of change $\frac{\Delta P}{\Delta v}$ in the intervals [8, 10] and [28, 30]. What do you notice?

108. Wind power: If the power P (in watts) being generated by a wind turbine is known, the velocity of the wind can be determined using the function $v(P) = \left(\frac{5}{2}\right)\sqrt[3]{P}$. Describe the transformation applied to obtain the graph of v from the graph of $y = \sqrt[3]{P}$, then sketch the graph of v for $P \in [0, 512]$ (scale the axes appropriately). How fast is the wind blowing if 343W of power is being generated?

109. Acceleration due to gravity: The *distance* a ball rolls down an inclined plane is given by the function $d(t) = 2t^2$, where $d(t)$ represents the distance in feet after t sec. (a) Describe the transformation applied to obtain the graph of d from the graph of $y = t^2$, then sketch the graph of d for $t \in [0, 3]$. (b) How far has the ball rolled after 2.5 sec? (c) Calculate the rate of change $\frac{\Delta d}{\Delta t}$ in the intervals [1, 1.5] and [3, 3.5]. What do you notice?

110. Acceleration due to gravity: The *velocity* of a steel ball bearing as it rolls down an inclined plane is given by the function $v(t) = 4t$, where $v(t)$ represents the velocity in feet per second after t sec. Describe the transformation applied to obtain the graph of v from the graph of $y = t$, then sketch the graph of v for $t \in [0, 3]$. What is the velocity of the ball bearing after 2.5 sec?

▶ EXTENDING THE CONCEPT

111. Carefully graph the functions $f(x) = |x|$ and $g(x) = 2\sqrt{x}$ on the same coordinate grid. From the graph, in what interval is the graph of $g(x)$ *above* the graph of $f(x)$? Pick a number (call it h) from this interval and substitute it in both functions. Is $g(h) > f(h)$? In what interval is the graph of $g(x)$ below the graph of $f(x)$? Pick a number from this interval (call it k) and substitute it in both functions. Is $g(k) < f(k)$?

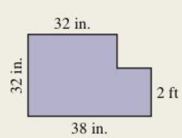
112. Sketch the graph of $f(x) = -2|x - 3| + 8$ using transformations of the parent function, then determine the area of the region in quadrant I that is beneath the graph and bounded by the vertical lines $x = 0$ and $x = 6$.

113. Sketch the graph of $f(x) = x^2 - 4$, then sketch the graph of $F(x) = |x^2 - 4|$ using your intuition and the meaning of absolute value (not a table of values). What happens to the graph?

▶ MAINTAINING YOUR SKILLS

114. (2.1) Find the distance between the points $(-13, 9)$ and $(7, -12)$, and the slope of the line containing these points.

115. (R.7) Find the perimeter and area of the figure shown (note the units).



116. (1.1) Solve for x: $\frac{2}{3}x + \frac{1}{4} = \frac{1}{2}x - \frac{7}{12}$.

117. (2.5) Without graphing, state intervals where $f(x)\uparrow$ and $f(x)\downarrow$ for $f(x) = (x - 4)^2 + 3$.

2.7 Piecewise-Defined Functions

Learning Objectives

In Section 2.7 you will learn how to:

- ☐ **A.** State the equation and domain of a piecewise-defined function
- ☐ **B.** Graph functions that are piecewise-defined
- ☐ **C.** Solve applications involving piecewise-defined functions

Most of the functions we've studied thus far have been smooth and continuous. Although "smooth" and "continuous" are defined more formally in advanced courses, for our purposes *smooth* simply means the graph has no sharp turns or jagged edges, and *continuous* means you can draw the entire graph without lifting your pencil. In this section, we study a special class of functions, called **piecewise-defined functions,** whose graphs may be various combinations of smooth/not smooth and continuous/not continuous. The absolute value function is one example (see Exercise 31). Such functions have a tremendous number of applications in the real world.

A. The Domain of a Piecewise-Defined Function

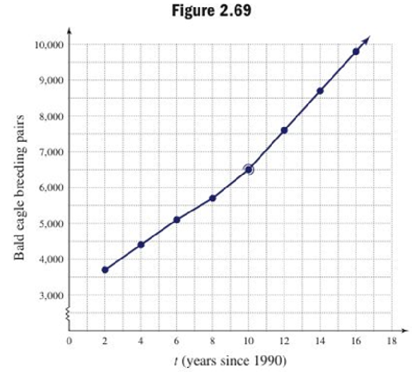
For the years 1990 to 2000, the American bald eagle remained on the nation's endangered species list, although the number of breeding pairs was growing slowly. After 2000, the population of eagles grew at a much faster rate, and they were removed from the list soon afterward. From Table 2.5 and plotted points modeling this growth (see Figure 2.69), we observe that a linear model would fit the period from 1992 to 2000 very well, but a line with greater slope would be needed for the years 2000 to 2006 and (perhaps) beyond.

Table 2.5

Year	Bald Eagle Breeding Pairs	Year	Bald Eagle Breeding Pairs
2	3700	10	6500
4	4400	12	7600
6	5100	14	8700
8	5700	16	9800

Source: www.fws.gov/midwest/eagle/population
1990 corresponds to year 0.

Figure 2.69



WORTHY OF NOTE

For the years 1992 to 2000, we can estimate the growth in breeding pairs $\frac{\Delta\text{pairs}}{\Delta\text{time}}$ using the points (2, 3700) and (10, 6500) in the slope formula. The result is $\frac{350}{1}$, or 350 pairs per year. For 2000 to 2006, using (10, 6500) and (16, 9800) shows the rate of growth is significantly larger: $\frac{\Delta\text{pairs}}{\Delta\text{years}} = \frac{550}{1}$ or 550 pairs per year.

The combination of these two lines would be a single function that modeled the population of breeding pairs from 1990 to 2006, but it would be *defined in two pieces*. This is an example of a **piecewise-defined function.**

The notation for these functions is a large "left brace" indicating the equations it groups are part of a single function. Using selected data points and techniques from Section 2.3, we find equations that could represent each piece are $p(t) = 350t + 3000$

WORTHY OF NOTE

In Figure 2.69, note that we indicated the exclusion of $t = 10$ from the second piece of the function using an open half-circle.

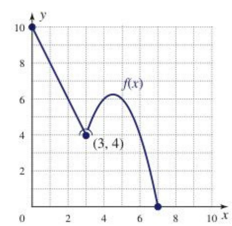
for $0 \le t \le 10$ and $p(t) = 550t + 1000$ for $t > 10$, where $p(t)$ is the number of breeding pairs in year t. The complete function is then written:

function name — function pieces — domain of each piece

$$p(t) = \begin{cases} 350t + 3000 & 2 \le t \le 10 \\ 550t + 1000 & t > 10 \end{cases}$$

EXAMPLE 1 ▶ Writing the Equation and Domain of a Piecewise-Defined Function

The linear piece of the function shown has an equation of $y = -2x + 10$. The equation of the quadratic piece is $y = -x^2 + 9x - 14$. Write the related piecewise-defined function, and state the domain of each piece by inspecting the graph.



Solution ▶ From the graph we note the linear portion is defined between 0 and 3, with these endpoints included as indicated by the closed dots. The domain here is $0 \le x \le 3$. The quadratic portion begins at $x = 3$ *but does not include 3*, as indicated by the half-circle notation. The equation is

function name — function pieces — domain

$$f(x) = \begin{cases} -2x + 10 & 0 \le x \le 3 \\ -x^2 + 9x - 14 & 3 < x \le 7 \end{cases}$$

A. You've just learned how to state the equation and domain of a piecewise-defined function

Now try Exercises 7 and 8 ▶

Piecewise-defined functions can be composed of more than two pieces, and can involve functions of many kinds.

B. Graphing Piecewise-Defined Functions

As with other functions, piecewise-defined functions can be graphed by simply plotting points. Careful attention must be paid to the domain of each piece, both to evaluate the function correctly and to consider the inclusion/exclusion of endpoints. In addition, try to keep the transformations of a basic function in mind, as this will often help graph the function more efficiently.

EXAMPLE 2 ▶ Graphing a Piecewise-Defined Function

Graph the function by plotting points, then state its domain and range:

$$h(x) = \begin{cases} -x - 2 & -5 \le x < -1 \\ 2\sqrt{x + 1} - 1 & x \ge -1 \end{cases}$$

Solution ▶ The first piece of h is a line with negative slope, while the second is a transformed square root function. Using the endpoints of each domain specified and a few additional points, we obtain the following:

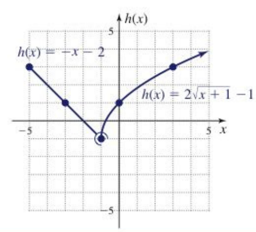
For $h(x) = -x - 2$, $-5 \le x < -1$,

x	$h(x)$
−5	3
−3	1
−1	−1

For $h(x) = 2\sqrt{x + 1} - 1$, $x \ge -1$,

x	$h(x)$
−1	−1
0	1
3	3

After plotting the points from the first piece, we connect them with a line segment noting the left endpoint is included, while the right endpoint is not (indicated using a semicircle around the point). Then we plot the points from the second piece and draw a square root graph, noting the left endpoint here *is* included, and the graph rises to the right. From the graph we note the complete domain of h is $x \in [-5, \infty)$, and the range is $y \in [-1, \infty)$.



Now try Exercises 9 through 14 ►

As an alternative to plotting points, we can graph each piece of the function using transformations of a basic graph, then erase those parts that are outside of the corresponding domain. Repeat this procedure for each piece of the function. One interesting and highly instructive aspect of these functions is the opportunity to investigate restrictions on their domain and the ranges that result.

Piecewise and Continuous Functions

EXAMPLE 3 ► Graphing a Piecewise-Defined Function

Graph the function and state its domain and range:

$$f(x) = \begin{cases} -(x-3)^2 + 12 & 0 < x \leq 6 \\ 3 & x > 6 \end{cases}$$

Solution ► The first piece of f is a basic parabola, shifted three units right, reflected across the x-axis (opening downward), and shifted 12 units up. The vertex is at (3, 12) and the axis of symmetry is $x = 3$, producing the following graphs.

1. Graph first piece of f (Figure 2.70).
2. Erase portion outside domain of $0 < x \leq 6$ (Figure 2.71).

Figure 2.70

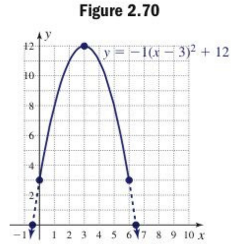
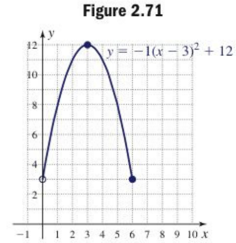


Figure 2.71



The second function is simply a horizontal line through (0, 3).

3. Graph second piece of f (Figure 2.72).

4. Erase portion outside domain of $x > 6$ (Figure 2.73).

Figure 2.72

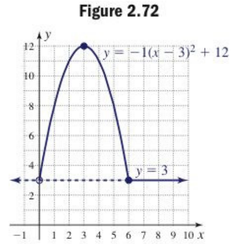
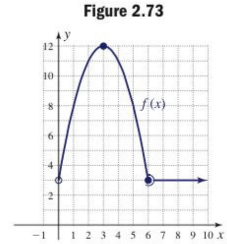


Figure 2.73



The domain of f is $x \in (0, \infty)$, and the corresponding range is $y \in [3, 12]$.

Now try Exercises 15 through 18 ►

Piecewise and Discontinuous Functions

Notice that although the function in Example 3 was piecewise-defined, the graph was actually continuous—we could draw the entire graph without lifting our pencil. Piecewise graphs also come in the *discontinuous* variety, which makes the domain and range issues all the more important.

EXAMPLE 4 ► Graphing a Discontinuous Piecewise-Defined Function

Graph $g(x)$ and state the domain and range:

$$g(x) = \begin{cases} -\frac{1}{2}x + 6 & 0 \le x \le 4 \\ -|x - 6| + 10 & 4 < x \le 9 \end{cases}$$

Solution ► The first piece of g is a line, with y-intercept (0, 6) and slope $\frac{\Delta y}{\Delta x} = -\frac{1}{2}$.

1. Graph first piece of g (Figure 2.74).

2. Erase portion outside domain of $0 \le x \le 4$ (Figure 2.75).

Figure 2.74

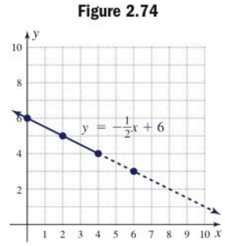
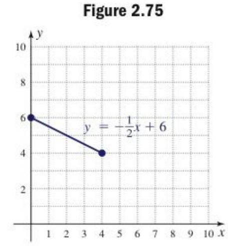


Figure 2.75



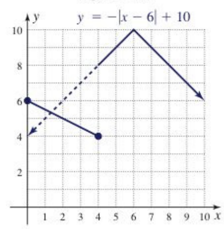
The second is an absolute value function, shifted right 6 units, reflected across the x-axis, then shifted up 10 units.

WORTHY OF NOTE

As you graph piecewise-defined functions, keep in mind that they *are* functions and the end result must pass the vertical line test. This is especially important when we are drawing each piece as a complete graph, then erasing portions outside the effective domain.

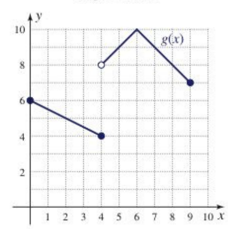
3. Graph second piece of g (Figure 2.76).

Figure 2.76



4. Erase portion outside domain of $4 < x \le 9$ (Figure 2.77).

Figure 2.77



Note that the left endpoint of the absolute value portion is not included (this piece is not defined at $x = 4$), signified by the open dot. The result is a discontinuous graph, as there is no way to draw the graph other than by jumping the pencil from where one piece ends to where the next begins. Using a vertical boundary line, we note the domain of g includes all values between 0 and 9 inclusive: $x \in [0, 9]$. Using a horizontal boundary line shows the smallest y-value is 4 and the largest is 10, but no range values exist between 6 and 7. The range is $y \in [4, 6] \cup [7, 10]$.

Now try Exercises 19 through 22 ►

EXAMPLE 5 ► Graphing a Discontinuous Function

The given piecewise-defined function is not continuous. Graph $h(x)$ to see why, then comment on what could be done to make it continuous.

$$h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & x \ne 2 \\ 1 & x = 2 \end{cases}$$

Solution ► The first piece of h is unfamiliar to us, so we elect to graph it by plotting points, noting $x = 2$ is outside the domain. This produces the table shown in Figure 2.78. After connecting the points, the graph of h turns out to be a straight line, but with no corresponding y-value for $x = 2$. This leaves a "hole" in the graph at (2, 4), as designated by the open dot.

Figure 2.78

x	$h(x)$
−4	−2
−2	0
0	2
2	—
4	6

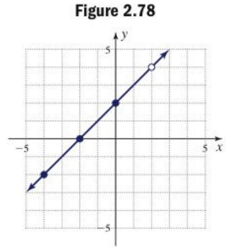
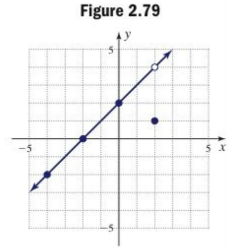


Figure 2.79



The second piece is point-wise defined, and its graph is simply the point (2, 1) shown in Figure 2.79. It's interesting to note that while the domain of h is all real numbers (h *is* defined at all points), the range is $y \in (-\infty, 4) \cup (4, \infty)$ as the function never takes on the value $y = 4$. In order for h to be continuous, we would need to redefine the second piece as $y = 4$ when $x = 2$.

Now try Exercises 23 through 26 ►

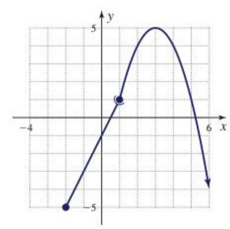
WORTHY OF NOTE

The discontinuity illustrated here is called a **removable discontinuity,** as the discontinuity can be removed by redefining a piece of the function. Note that after factoring the first piece, the denominator is a factor of the numerator, and writing the result in lowest terms gives $h(x) = \frac{(x+2)(x-2)}{x-2} = x + 2, x \ne 2$. This is precisely the equation of the line in Figure 2.78 $[h(x) = x + 2]$.

To develop these concepts more fully, it will help to practice finding the equation of a piecewise-defined function *given its graph,* a process similar to that of Example 10 in Section 2.6.

EXAMPLE 6 ▶ Determining the Equation of a Piecewise-Defined Function

Determine the equation of the piecewise-defined function shown, including the domain for each piece.



Solution ▶ By counting $\frac{\Delta y}{\Delta x}$ from $(-2, -5)$ to $(1, 1)$, we find the linear portion has slope $m = 2$, and the y-intercept must be $(0, -1)$. The equation of the line is $y = 2x - 1$. The second piece appears to be a parabola with vertex (h, k) at $(3, 5)$. Using this vertex with the point $(1, 1)$ in the general form $y = a(x - h)^2 + k$ gives

$$\begin{aligned} y &= a(x-h)^2 + k && \text{general form} \\ 1 &= a(1-3)^2 + 5 && \text{substitute 1 for } x\text{, 1 for } y\text{, 3 for } h\text{, 5 for } k \\ -4 &= a(-2)^2 && \text{simplify; subtract 5} \\ -4 &= 4a && (-2)^2 = 4 \\ -1 &= a && \text{divide by 4} \end{aligned}$$

The equation of the parabola is $y = -(x - 3)^2 + 5$. Considering the domains shown in the figure, the equation of this piecewise-defined function must be

$$p(x) = \begin{cases} 2x - 1 & -2 \le x \le 1 \\ -(x-3)^2 + 5 & x > 1 \end{cases}$$

B. You've just learned how to graph functions that are piecewise-defined

Now try Exercises 27 through 30 ▶

C. Applications of Piecewise-Defined Functions

The number of applications for piecewise-defined functions is practically limitless. It is actually fairly rare for a single function to accurately model a situation over a long period of time. Laws change, spending habits change, and technology can bring abrupt alterations in many areas of our lives. To accurately model these changes often requires a piecewise-defined function.

EXAMPLE 7 ▶ Modeling with a Piecewise-Defined Function

For the first half of the twentieth century, per capita spending on police protection can be modeled by $S(t) = 0.54t + 12$, where $S(t)$ represents per capita spending on police protection in year t (1900 corresponds to year 0). After 1950, perhaps due to the growth of American cities, this spending greatly increased: $S(t) = 3.65t - 144$. Write these as a piecewise-defined function $S(t)$, state the domain for each piece, then graph the function. According to this model, how much was spent (per capita) on police protection in 2000? How much will be spent in 2010?

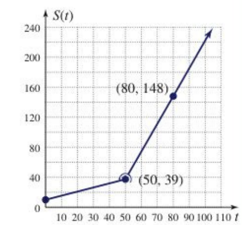
Source: Data taken from the *Statistical Abstract of the United States* for various years.

Solution ►

function name — function pieces — effective domain

$$S(t) = \begin{cases} 0.54t + 12 & 0 \le t \le 50 \\ 3.65t - 144 & t > 50 \end{cases}$$

Since both pieces are linear, we can graph each part using two points. For the first function, $S(0) = 12$ and $S(50) = 39$. For the second function $S(50) \approx 39$ and $S(80) = 148$. The graph for each piece is shown in the figure. Evaluating S at $t = 100$:



$$\begin{aligned} S(t) &= 3.65t - 144 \\ S(100) &= 3.65(100) - 144 \\ &= 365 - 144 \\ &= 221 \end{aligned}$$

About \$221 per capita was spent on police protection in the year 2000. For 2010, the model indicates that \$257.50 per capita will be spent: $S(110) = 257.5$.

Now try Exercises 33 through 44 ►

Step Functions

The last group of piecewise-defined functions we'll explore are the **step functions,** so called because the pieces of the function form a series of horizontal steps. These functions find frequent application in the way consumers are charged for services, and have a number of applications in number theory. Perhaps the most common is called the **greatest integer function,** though recently its alternative name, **floor function,** has gained popularity (see Figure 2.80). This is in large part due to an improvement in notation and as a better contrast to **ceiling functions.** The floor function of a real number x, denoted $f(x) = \lfloor x \rfloor$ or $[\![x]\!]$ (we will use the first), is the largest integer less than or equal to x. For instance, $\lfloor 5.9 \rfloor = 5$, $\lfloor 7 \rfloor = 7$, and $\lfloor -3.4 \rfloor = -4$.

In contrast, the ceiling function $C(x) = \lceil x \rceil$ is the smallest integer greater than or equal to x, meaning $\lceil 5.9 \rceil = 6$, $\lceil 7 \rceil = 7$, and $\lceil -3.4 \rceil = -3$ (see Figure 2.81). In simple terms, for any noninteger value on the number line, the floor function returns the integer to the left, while the ceiling function returns the integer to the right. A graph of each function is shown.

Figure 2.80

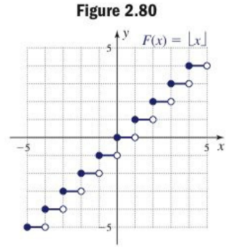
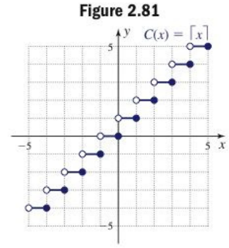


Figure 2.81

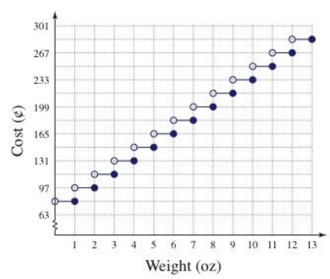


One common application of floor functions is the price of theater admission, where children 12 and under receive a discounted price. Right up until the day they're 13, they qualify for the lower price: $\lfloor 12\frac{364}{365} \rfloor = 12$. Applications of ceiling functions would include how phone companies charge for the minutes used (charging the 12-min rate for a phone call that only lasted 11.3 min: $\lceil 11.3 \rceil = 12$), and postage rates, as in Example 8.

EXAMPLE 8 ▶ Modeling Using a Step Function

As of May 2007, the first-class postage rate for large envelopes sent through the U.S. mail was 80¢ for the first ounce, then an additional 17¢ per ounce thereafter, up to 13 ounces. Graph the function and state its domain and range. Use the graph to state the cost of mailing a report weighing (a) 7.5 oz, (b) 8 oz, and (c) 8.1 oz in a large envelope.

Solution ▶ The 80¢ charge applies to letters weighing between 0 oz and 1 oz. Zero is not included since we have to mail *something,* but 1 is included since a large envelope and its contents weighing exactly one ounce still costs 80¢. The graph will be a horizontal line segment.



The function is defined for all weights between 0 and 13 oz, excluding zero and including 13: $x \in (0, 13]$. The range consists of single outputs corresponding to the step intervals: $R \in \{80, 97, 114, \ldots, 267, 284\}$.

a. The cost of mailing a 7.5-oz report is 199¢.

b. The cost of mailing an 8.0-oz report is still 199¢.

c. The cost of mailing an 8.1-oz report is $199 + 17 = 216$¢, since this brings you up to the next step.

C. You've just learned how to solve applications involving piecewise-defined functions

Now try Exercises 45 through 48 ▶

TECHNOLOGY HIGHLIGHT

Piecewise-Defined Functions

Most graphing calculators are able to graph piecewise-defined functions. Consider the function f shown here:

$$f(x) = \begin{cases} x + 2 & x < 2 \\ (x - 4)^2 + 3 & x \geq 2 \end{cases}$$

Both "pieces" are well known—the first is a line with slope $m = 1$ and y-intercept $(0, 2)$. The second is a parabola that opens upward, shifted 4 units to the right and 3 units up. If we attempt to graph $f(x)$ using $Y_1 = x + 2$ and $Y_2 = (x - 4)^2 + 3$ as they stand, the resulting graph may be difficult to analyze because the pieces conflict and intersect (Figure 2.82). To graph the functions we must indicate the domain for each piece, separated by a slash and enclosed in parentheses. For instance, for the first piece we enter $Y_1 = x + 2/(x < 2)$, and for the second, $Y_2 = (x - 4)^2 + 3/(x \geq 2)$ (Figure 2.83). The slash looks like (is) the division symbol, but in this context, the calculator interprets it as a means of separating the function from the domain. The inequality symbols are accessed using the 2nd MATH (**TEST**) keys. The graph is shown on Figure 2.84, where we see the function is linear for $x \in (-\infty, 2)$ and quadratic for $x \in [2, \infty)$. How does the calculator remind us the function is defined only for $x = 2$ on the second piece? Using the 2nd GRAPH **(TABLE)** feature reveals the calculator will give an **ERR:** (ERROR) message for inputs outside of its domain (Figure 2.85).

Figure 2.82

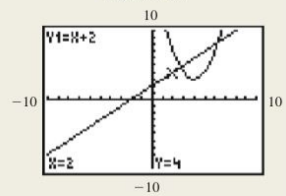
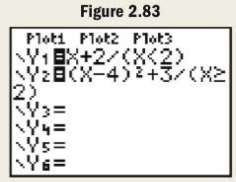


Figure 2.83



We can also use the calculator to investigate endpoints of the domain. For instance, we know that $Y_1 = x + 2$ is not defined for $x = 2$, but what about numbers very close to 2? Go to 2nd WINDOW **(TBLSET)** and place the calculator in the Indpnt: Auto ASK mode. With both Y_1 and Y_2 enabled, use the 2nd GRAPH **(TABLE)** feature to evaluate the functions at numbers very near 2. Use $x = 1.9, 1.99, 1.999$, and so on.

Figure 2.84

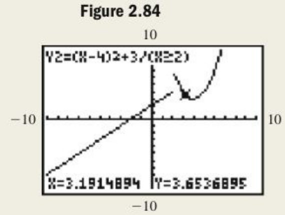
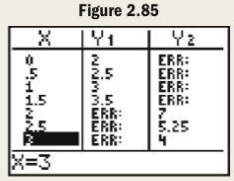


Figure 2.85



X	Y1	Y2
0	2	ERR:
.5	2.5	ERR:
1	3	ERR:
1.5	3.5	ERR:
2	ERR:	7
2.5	ERR:	5.25
3	ERR:	4

X=3

Exercise 1: What appears to be happening to the output values for Y_1? What about Y_2?

Exercise 2: What do you notice about the output values when 1.99999 is entered? Use the right arrow key ► to move the cursor into columns Y_1 and Y_2. Comment on what you think the calculator is doing. Will Y_1 ever really have an output equal to 4?

2.7 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A function whose entire graph can be drawn without lifting your pencil is called a _______ function.

2. The input values for which each part of a piecewise function is defined is the _______ of the function.

3. A graph is called _______ if it has no sharp turns or jagged edges.

4. When graphing $2x + 3$ over a domain of $x > 0$, we leave an ______ dot at (0, 3).

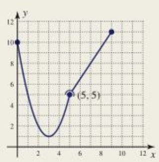
5. Discuss/Explain how to determine if a piecewise-defined function is continuous, without having to graph the function. Illustrate with an example.

6. Discuss/Explain how it is possible for the domain of a function to be defined for all real numbers, but have a range that is defined on more than one interval. Construct an illustrative example.

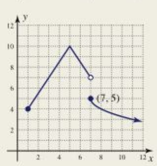
▶ DEVELOPING YOUR SKILLS

For Exercises 7 and 8, (a) use the correct notation to write them as a single piecewise-defined function, state the domain for each piece by inspecting the graph, and (b) state the range of the function.

7. $Y_1 = x^2 - 6x + 10$; $Y_2 = \frac{3}{2}x - \frac{5}{2}$



8. $Y_1 = -1.5|x - 5| + 10$; $Y_2 = -\sqrt{x - 7} + 5$



Evaluate each piecewise-defined function as indicated (if possible).

9. $h(x) = \begin{cases} -2 & x < -2 \\ |x| & -2 \le x < 3 \\ 5 & x \ge 3 \end{cases}$

$h(-5)$, $h(-2)$, $h(-\frac{1}{2})$, $h(0)$, $h(2.999)$, and $h(3)$

10. $H(x) = \begin{cases} 2x + 3 & x < 0 \\ x^2 + 1 & 0 \le x < 2 \\ 5 & x > 2 \end{cases}$

$H(-3)$, $H(-\frac{3}{2})$, $H(-0.001)$, $H(1)$, $H(2)$, and $H(3)$

11. $p(x) = \begin{cases} 5 & x < -3 \\ x^2 - 4 & -3 \le x \le 3 \\ 2x + 1 & x > 3 \end{cases}$

$p(-5)$, $p(-3)$, $p(-2)$, $p(0)$, $p(3)$, and $p(5)$

12. $q(x) = \begin{cases} -x - 3 & x < -1 \\ 2 & -1 \le x < 2 \\ -\frac{1}{2}x^2 + 3x - 2 & x \ge 2 \end{cases}$

$q(-3)$, $q(-1)$, $q(0)$, $q(1.999)$, $q(2)$, and $q(4)$

Graph each piecewise-defined function by plotting points, then state its domain and range.

13. $p(x) = \begin{cases} x + 2 & -6 \le x \le 2 \\ 2|x - 4| & x > 2 \end{cases}$

14. $q(x) = \begin{cases} \sqrt{x + 4} & -4 \le x \le 0 \\ |x - 2| & 0 < x \le 7 \end{cases}$

Graph each piecewise-defined function and state its domain and range. Use transformations of the toolbox functions where possible.

15. $g(x) = \begin{cases} -(x - 1)^2 + 5 & -2 \le x \le 4 \\ 2x - 12 & x > 4 \end{cases}$

16. $h(x) = \begin{cases} \frac{1}{2}x + 1 & x \le 0 \\ (x - 2)^2 - 3 & 0 < x \le 5 \end{cases}$

17. $p(x) = \begin{cases} \frac{1}{2}x + 1 & x \ne 4 \\ 2 & x = 4 \end{cases}$

18. $q(x) = \begin{cases} \frac{1}{2}(x - 1)^3 - 1 & x \ne 3 \\ -2 & x = 3 \end{cases}$

19. $H(x) = \begin{cases} -x + 3 & x < 1 \\ -|x - 5| + 6 & 1 \le x < 9 \end{cases}$

20. $w(x) = \begin{cases} \sqrt[3]{x + 1} & x < 1 \\ (x - 3)^2 - 2 & 1 \le x \le 6 \end{cases}$

21. $f(x) = \begin{cases} -x - 3 & x < -3 \\ 9 - x^2 & -3 \le x < 2 \\ 4 & x \ge 2 \end{cases}$

22. $h(x) = \begin{cases} -\frac{1}{2}x - 1 & x < -3 \\ -|x| + 5 & -3 \le x \le 5 \\ 3\sqrt{x - 5} & x > 5 \end{cases}$

Each of the following functions has a pointwise discontinuity. Graph the first piece of each function, then find the value of *c* so that a continuous function results.

23. $f(x) = \begin{cases} \dfrac{x^2 - 9}{x + 3} & x \neq -3 \\ c & x = -3 \end{cases}$

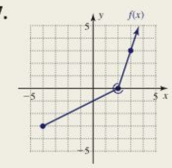
24. $f(x) = \begin{cases} \dfrac{x^2 - 3x - 10}{x - 5} & x \neq 5 \\ c & x = 5 \end{cases}$

25. $f(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1} & x \neq 1 \\ c & x = 1 \end{cases}$

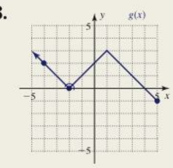
26. $f(x) = \begin{cases} \dfrac{4x - x^3}{x + 2} & x \neq -2 \\ c & x = -2 \end{cases}$

Determine the equation of each piecewise-defined function shown, including the domain for each piece. Assume all pieces are toolbox functions.

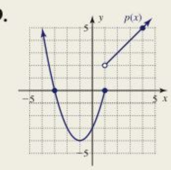
27.



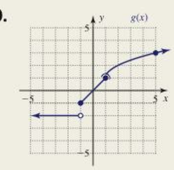
28.



29.



30.



▶ WORKING WITH FORMULAS

31. Definition of absolute value: $|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$

The absolute value function can be stated as a piecewise-defined function, a technique that is sometimes useful in graphing variations of the function or solving absolute value equations and inequalities. How does this definition ensure that the absolute value of a number is always positive? Use this definition to help sketch the graph of $f(x) = \frac{|x|}{x}$. Discuss what you notice.

32. Sand dune function:

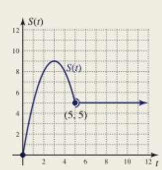
$$f(x) = \begin{cases} -|x - 2| + 1 & 1 \leq x < 3 \\ -|x - 4| + 1 & 3 \leq x < 5 \\ -|x - 2k| + 1 & 2k - 1 \leq x < 2k + 1, \text{ for } k \in N \end{cases}$$

There are a number of interesting graphs that can be created using piecewise-defined functions, and these functions have been the basis for more than one piece of modern art. (a) Use the descriptive name and the pieces given to graph the function *f*. Is the function accurately named? (b) Use any combination of the toolbox functions to explore your own creativity by creating a piecewise-defined function with some interesting or appealing characteristics.

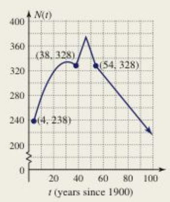
▶ APPLICATIONS

For Exercises 33 and 34, a. write the information given as a piecewise-defined function, and state the domain for each piece by inspecting the graph. b. Give the range of each.

33. Due to heavy advertising, initial sales of the Lynx Digital Camera grew very rapidly, but started to decline once the advertising blitz was over. During the advertising campaign, sales were modeled by the function $S(t) = -t^2 + 6t$, where $S(t)$ represents hundreds of sales in month t. However, as Lynx Inc. had hoped, the new product secured a foothold in the market and sales leveled out at a steady 500 sales per month.



34. From the turn of the twentieth century, the number of newspapers (per thousand population) grew rapidly until the 1930s, when the growth slowed down and then declined. The years 1940 to 1946 saw a "spike" in growth, but the years 1947 to 1954 saw an almost equal decline. Since 1954 the number has continued to decline, but at a slower rate. The number of papers



N per thousand population for each period, respectively, can be approximated by
$N_1(t) = -0.13t^2 + 8.1t + 208$,
$N_2(t) = -5.75|t - 46| + 374$, and
$N_3(t) = -2.45t + 460$.

Source: Data from the *Statistical Abstract of the United States*, various years; data from *The First Measured Century, The AEI Press*, Caplow, Hicks, and Wattenberg, 2001.

35. The percentage of American households that own publicly traded stocks began rising in the early 1950s, peaked in 1970, then began to decline until 1980 when there was a dramatic increase due to easy access over the Internet, an improved economy, and other factors. This phenomenon is modeled by the function $P(t)$, where $P(t)$ represents the percentage of households owning stock in year t, with 1950 corresponding to year 0.

$$P(t) = \begin{cases} -0.03t^2 + 1.28t + 1.68 & 0 \le t \le 30 \\ 1.89t - 43.5 & t > 30 \end{cases}$$

a. According to this model, what percentage of American households held stock in the years 1955, 1965, 1975, 1985, and 1995? If this pattern continues, what percentage held stock in 2005?

b. Why is there a discrepancy in the outputs of each piece of the function for the year 1980 ($t = 30$)? According to how the function is defined, which output should be used?

Source: 2004 *Statistical Abstract of the United States*, Table 1204; various other years.

36. America's dependency on foreign oil has always been a "hot" political topic, with the amount of imported oil fluctuating over the years due to political climate, public awareness, the economy, and other factors. The amount of crude oil imported can be approximated by the function given, where $A(t)$ represents the number of barrels imported in year t (in billions), with 1980 corresponding to year 0.

$$A(t) = \begin{cases} 0.047t^2 - 0.38t + 1.9 & 0 \le t < 8 \\ -0.075t^2 + 1.495t - 5.265 & 8 \le t \le 11 \\ 0.133t + 0.685 & t > 11 \end{cases}$$

a. Use $A(t)$ to estimate the number of barrels imported in the years 1983, 1989, 1995, and 2005.

b. What was the minimum number of barrels imported between 1980 and 1988?

Source: 2004 *Statistical Abstract of the United States*, Table 897; various other years.

37. Energy rationing: In certain areas of the United States, power blackouts have forced some counties to ration electricity. Suppose the cost is \$0.09 per kilowatt (kW) for the first 1000 kW a household uses. After 1000 kW, the cost increases to 0.18 per kW: Write these charges for electricity in the form of a piecewise-defined function $C(h)$, where $C(h)$ is the cost for h kilowatt hours. State the domain for each piece. Then sketch the graph and determine the cost for 1200 kW.

38. Water rationing: Many southwestern states have a limited water supply, and some state governments try to control consumption by manipulating the cost of water usage. Suppose for the first 5000 gal a household uses per month, the charge is \$0.05 per gallon. Once 5000 gal is used the charge doubles to \$0.10 per gallon. Write these charges for water usage in the form of a piecewise-defined function $C(w)$, where $C(w)$ is the cost for w gallons of water and state the domain for each piece. Then sketch the graph and determine the cost to a household that used 9500 gal of water during a very hot summer month.

39. Pricing for natural gas: A local gas company charges \$0.75 per therm for natural gas, up to 25 therms. Once the 25 therms has been exceeded, the charge doubles to \$1.50 per therm due to limited supply and great demand. Write these charges for natural gas consumption in the form of a piecewise-defined function $C(t)$, where $C(t)$ is the charge for t therms and state the domain for each piece. Then sketch the graph and determine the cost to a household that used 45 therms during a very cold winter month.

40. Multiple births: The number of multiple births has steadily increased in the United States during the twentieth century and beyond. Between 1985 and 1995 the number of twin births could be modeled by the function $T(x) = -0.21x^2 + 6.1x + 52$, where x is the

number of years since 1980 and T is in thousands. After 1995, the incidence of twins becomes more linear, with $T(x) = 4.53x + 28.3$ serving as a better model. Write the piecewise-defined function modeling the incidence of twins for these years, including the domain of each piece. Then sketch the graph and use the function to estimate the incidence of twins in 1990, 2000, and 2005. If this trend continues, how many sets of twins will be born in 2010?

Source: National Vital Statistics Report, Vol. 50, No. 5, February 12, 2002

41. U.S. military expenditures: Except for the year 1991 when military spending was cut drastically, the amount spent by the U.S. government on national defense and veterans' benefits rose steadily from 1980 to 1992. These expenditures can be modeled by the function $S(t) = -1.35t^2 + 31.9t + 152$, where $S(t)$ is in billions of dollars and 1980 corresponds to $t = 0$.

Source: 1992 *Statistical Abstract of the United States,* Table 525

From 1992 to 1996 this spending declined, then began to rise in the following years. From 1992 to 2002, military-related spending can be modeled by $S(t) = 2.5t^2 - 80.6t + 950$.

Source: 2004 *Statistical Abstract of the United States,* Table 492

Write $S(t)$ as a single piecewise-defined function, stating the domain for each piece. Then sketch the graph and use the function to find the projected amount the United States will spend on its military in 2005, 2008, and 2010 if this trend continues.

42. Amusement arcades: At a local amusement center, the owner has the SkeeBall machines programmed to reward very high scores. For scores of 200 or less, the function $T(x) = \frac{x}{10}$ models the number of tickets awarded (rounded to the nearest whole). For scores over 200, the number of tickets is modeled by $T(x) = 0.001x^2 - 0.3x + 40$. Write these equation models of the number of tickets awarded in the form of a piecewise-defined function and state the domain for each piece. Then sketch the graph and find the number of tickets awarded to a person who scores 390 points.

43. Phone service charges: When it comes to phone service, a large number of calling plans are available. Under one plan, the first 30 min of any phone call costs only 3.3¢ per minute. The charge increases to 7¢ per minute thereafter. Write this information in the form of a piecewise-defined function and state the domain for each piece. Then sketch the graph and find the cost of a 46-min phone call.

44. Overtime wages: Tara works on an assembly line, putting together computer monitors. She is paid \$9.50 per hour for regular time (0, 40 hr], \$14.25 for overtime (40, 48 hr], and when demand for computers is high, \$19.00 for double-overtime (48, 84 hr]. Write this information in the form of a simplified piecewise-defined function, and state the domain for each piece. Then sketch the graph and find the gross amount of Tara's check for the week she put in 54 hr.

45. Admission prices: At Wet Willy's Water World, infants under 2 are free, then admission is charged according to age. Children 2 and older but less than 13 pay \$2, teenagers 13 and older but less than 20 pay \$5, adults 20 and older but less than 65 pay \$7, and senior citizens 65 and older get in at the teenage rate. Write this information in the form of a piecewise-defined function and state the domain for each piece. Then sketch the graph and find the cost of admission for a family of nine which includes: one grandparent (70), two adults (44/45), 3 teenagers, 2 children, and one infant.

46. Demographics: One common use of the floor function $y = \lfloor x \rfloor$ is the reporting of ages. As of 2007, the record for longest living human is 122 yr, 164 days for the life of Jeanne Calment, formerly of France. While she actually lived $x = 122\frac{164}{365}$ years, ages are normally reported using the floor function, or the greatest integer number of years less than or equal to the actual age: $\lfloor 122\frac{164}{365} \rfloor = 122$ years. (a) Write a function $A(t)$ that gives a person's age, where $A(t)$ is the reported age at time t. (b) State the domain of the function (be sure to consider Madame Calment's record). Report the age of a person who has been living for (c) 36 years; (d) 36 years, 364 days; (e) 37 years; and (f) 37 years, 1 day.

47. Postage rates: The postal charge function from Example 8 is simply a transformation of the basic ceiling function $y = \lceil x \rceil$. Using the ideas from Section 2.6, (a) write the postal charges as a step function $C(w)$, where $C(w)$ is the cost of mailing a large envelope weighing w ounces, and (b) state the domain of the function. Then use the function to find the cost of mailing reports weighing: (c) 0.7 oz, (d) 5.1 oz, (e) 5.9 oz; (f) 6 oz, and (g) 6.1 oz.

48. Cell phone charges: A national cell phone company advertises that calls of 1 min or less do not count toward monthly usage. Calls lasting longer than 1 min are calculated normally using a ceiling function, meaning a call of 1 min, 1 sec will be counted as a 2-min call. Using the ideas from Section 2.6, (a) write the cell phone charges as a piecewise-defined function $C(m)$, where $C(m)$ is the cost of a call lasting m minutes, and include the domain of the function. Then (b) graph the function, and (c) use the graph or function to determine if a cell phone subscriber has exceeded the 30 free minutes granted by her calling plan for calls lasting 2 min 3 sec, 13 min 46 sec, 1 min 5 sec, 3 min 59 sec, 8 min 2 sec. (d) What was the actual usage in minutes and seconds?

49. Combined absolute value graphs: Carefully graph the function $h(x) = |x - 2| - |x + 3|$ using a table of values over the interval $x \in [-5, 5]$. Is the function continuous? Write this function in piecewise-defined form and state the domain for each piece.

50. Combined absolute value graphs: Carefully graph the function $H(x) = |x - 2| + |x + 3|$ using a table of values over the interval $x \in [-5, 5]$. Is the function continuous? Write this function in piecewise-defined form and state the domain for each piece.

▶ EXTENDING THE CONCEPT

51. You've heard it said, "*any number divided by itself is one*." Consider the functions $Y_1 = \frac{x+2}{x+2}$, and $Y_2 = \frac{|x+2|}{x+2}$. Are these functions continuous?

52. Find a linear function $h(x)$ that will make the function shown a *continuous* function. Be sure to include its domain.

$$f(x) = \begin{cases} x^2 & x < 1 \\ h(x) & \\ 2x + 3 & x > 3 \end{cases}$$

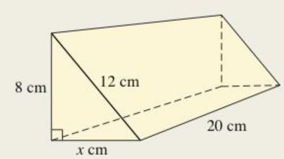
▶ MAINTAINING YOUR SKILLS

53. **(1.3)** Solve: $\dfrac{3}{x - 2} + 1 = \dfrac{30}{x^2 - 4}$.

54. **(R.5)** Compute the following and write the result in lowest terms:

$$\frac{x^3 + 3x^2 - 4x - 12}{x - 3} \cdot \frac{2x - 6}{x^2 + 5x + 6} \div (3x - 6)$$

55. **(R.7)** For the figure shown, (a) find the length of the missing side, (b) state the area of the triangular base, and (c) compute the volume of the prism.



56. **(2.4)** Find the equation of the line perpendicular to $3x + 4y = 8$, and through the point $(0, -2)$. Write the result in slope-intercept form.

2.8 The Algebra and Composition of Functions

Learning Objectives

In Section 2.8 you will learn how to:

- ☐ **A.** Compute a sum or difference of functions and determine the domain of the result
- ☐ **B.** Compute a product or quotient of functions and determine the domain
- ☐ **C.** Compose two functions and determine the domain; decompose a function
- ☐ **D.** Interpret operations on functions graphically
- ☐ **E.** Apply the algebra and composition of functions in context

In Section 2.5, we created new functions *graphically* by applying transformations to basic functions. In this section, we'll use two (or more) functions to create new functions *algebraically*. Previous courses often contain material on the sum, difference, product, and quotient of polynomials. Here we'll combine these functions with the basic operations, noting the result is also a function that can be evaluated, graphed, and analyzed. We call these basic operations on functions the **algebra of functions.**

A. Sums and Differences of Functions

This section introduces the notation used for basic operations on functions. Here we'll note the result is also a function whose domain depends on the original functions. In general, if f and g are functions *with overlapping domains,* $f(x) + g(x) = (f + g)(x)$ and $f(x) - g(x) = (f - g)(x)$.

Sums and Differences of Functions

For functions f and g with domains P and Q respectively, the sum and difference of f and g are defined by:

	Domain of result
$(f + g)(x) = f(x) + g(x)$	$P \cap Q$
$(f - g)(x) = f(x) - g(x)$	$P \cap Q$

EXAMPLE 1A ▶ Evaluating a Difference of Functions

Given $f(x) = x^2 - 5x$ and $g(x) = 2x - 9$,

a. Determine the domain of $h(x) = (f - g)(x)$. **b.** Find $h(3)$ using the definition.

Solution ▶ **a.** Since the domain of both f and g is $\mathbb{R}$, their intersection is $\mathbb{R}$, so the domain of h is also $\mathbb{R}$.

b.

$$\begin{aligned} h(x) &= (f - g)(x) && \text{given difference} \\ &= f(x) - g(x) && \text{by definition} \\ h(3) &= f(3) - g(3) && \text{substitute 3 for } x \\ &= [(3)^2 - 5(3)] - [2(3) - 9] && \text{evaluate} \\ &= [9 - 15] - [6 - 9] && \text{multiply} \\ &= -6 - [-3] && \text{subtract} \\ &= -3 && \text{result} \end{aligned}$$

If the function h is to be graphed or evaluated numerous times, it helps to compute a *new function rule* for h, rather than repeatedly apply the definition.

EXAMPLE 1B ▶ For the functions f, g, and h, as defined in Example 1A,

a. Find a new function rule for h. **b.** Use the result to find $h(3)$.

Solution ▶ **a.**

$$\begin{aligned} h(x) &= (f - g)(x) && \text{given difference} \\ &= f(x) - g(x) && \text{by definition} \\ &= (x^2 - 5x) - (2x - 9) && \text{replace } f(x) \text{ with } (x^2 - 5x) \text{ and } g(x) \text{ with } (2x - 9) \\ &= x^2 - 7x + 9 && \text{distribute and combine like terms} \end{aligned}$$

b. $h(3) = (3)^2 - 7(3) + 9$ — substitute 3 for x

$= 9 - 21 + 9$ — multiply

$= -3$ — result

Notice the result from Part (b) is identical to that in Example 1A.

Now try Exercises 7 through 10 ►

CAUTION ► From Example 1A, note the importance of using grouping symbols with the algebra of functions. Without them, we could easily confuse the signs of g when computing the difference. Also, note that any operation applied to the functions f and g simply results in an *expression* representing a new function rule for h, and is not an *equation* that needs to be factored or solved.

EXAMPLE 2 ► Evaluating a Sum of Functions

For $f(x) = x^2$ and $g(x) = \sqrt{x - 2}$,

a. Determine the domain of $h(x) = (f + g)(x)$.

b. Find a new function rule for h.

c. Evaluate $h(3)$.

d. Evaluate $h(-1)$.

Solution ► **a.** The domain of f is $\mathbb{R}$, while the domain of g is $x \in [2, \infty)$. Since their intersection is $[2, \infty)$, this is the domain of the new function h.

b. $h(x) = (f + g)(x)$ — given sum

$= f(x) + g(x)$ — by definition

$= x^2 + \sqrt{x - 2}$ — substitute x^2 for $f(x)$ and $\sqrt{x-2}$ for $g(x)$ (no other simplifications possible)

c. $h(3) = (3)^2 + \sqrt{3 - 2}$ — substitute 3 for x

$= 10$ — result

d. $x = -1$ is outside the domain of h.

Now try Exercises 11 through 14 ►

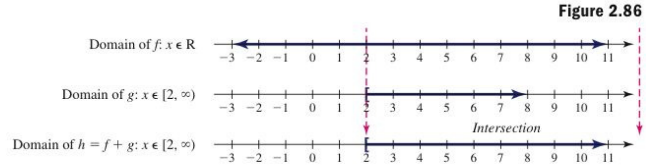
WORTHY OF NOTE

If we *did* try to evaluate $h(-1)$, the result would be $1 + \sqrt{-3}$, which is not a real number. While it's true we could write $1 + \sqrt{-3}$ as $1 + i\sqrt{3}$ and consider it an "answer," our study here focuses on real numbers and the graphs of functions in a coordinate system where x and y are both real.

A. You've just learned how to compute a sum or difference of functions and determine the domain of the result

This "intersection of domains" is illustrated in Figure 2.86 using ideas from Section 1.2.

Figure 2.86



B. Products and Quotients of Functions

The product and quotient of two functions is defined in a manner similar to that for sums and differences. For example, if f and g are functions *with overlapping domains,* $(f \cdot g)(x) = f(x) \cdot g(x)$ and $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. As you might expect, for quotients we must stipulate $g(x) \neq 0$.

Products and Quotients of Functions

For functions f and g with domains P and Q, respectively, the product and quotient of f and g are defined by:

	Domain of result
$(f \cdot g)(x) = f(x) \cdot g(x)$	$P \cap Q$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	$P \cap Q$, for all $g(x) \neq 0$

EXAMPLE 3 ► Computing a Product of Functions

Given $f(x) = \sqrt{1 + x}$ and $g(x) = \sqrt{3 - x}$,

a. Determine the domian of $h(x) = (f \cdot g)(x)$.

b. Find a new function rule for h.

c. Use the result from part (b) to evaluate $h(2)$ and $h(4)$.

Solution ► **a.** The domain of f is $x \in [-1, \infty)$ and the domain of g is $x \in (-\infty, 3]$. The intersection of these domains gives $x \in [-1, 3]$, which is the domain for h.

b.
$h(x) = (f \cdot g)(x)$ — given product
$= f(x) \cdot g(x)$ — by definition
$= \sqrt{1 + x} \cdot \sqrt{3 - x}$ — substitute $\sqrt{1 + x}$ for f and $\sqrt{3 + x}$ for g
$= \sqrt{3 + 2x - x^2}$ — combine using properties of radicals

c.
$h(2) = \sqrt{3 + 2(2) - (2)^2}$ — substitute 2 for x
$= \sqrt{3} \approx 1.732$ — result
$h(4) = \sqrt{3 + 2(4) - (4)^2}$ — substitute 4 for x
$= \sqrt{-5}$ — *not a real number*

The second result of Part (c) is not surprising, since $x = 4$ is not in the domain of h [meaning $h(4)$ is not defined for this function].

Now try Exercises 15 through 18 ►

In future sections, we use polynomial division as a tool for factoring, an aid to graphing, and to determine whether two expressions are equivalent. Understanding the notation and domain issues related to division will strengthen our ability in these areas.

EXAMPLE 4 ► Computing a Quotient of Functions

Given $f(x) = x^3 - 3x^2 + 2x - 6$ and $g(x) = x - 3$,

a. Determine the domain of $h(x) = \left(\frac{f}{g}\right)(x)$.

b. Find a new function rule for h.

c. Use the result from part (b) to evaluate $h(3)$ and $h(0)$.

Solution ► **a.** While the domain of both f and g is $\mathbb{R}$ and their intersection is also $\mathbb{R}$, we know from the definition (and past experience) *that $g(x)$ cannot be zero*. The domain of h is $x \in (-\infty, 3) \cup (3, \infty)$.

b.
$h(x) = \left(\frac{f}{g}\right)(x)$ — given quotient
$= \frac{f(x)}{g(x)}$ — by definition
$= \frac{x^3 - 3x^2 + 2x - 6}{x - 3}$ — replace f with $x^3 - 3x^2 + 2x - 6$ and g with $x - 3$

c. Recall that $x = 3$ is not in the domain of h. For $h(0)$ we have:

$$h(0) = \frac{(0)^3 - 3(0)^2 + 2(0) - 6}{(0) - 3} \quad \text{replace } x \text{ with } 0$$

$$= \frac{-6}{-3} = 2 \qquad h(0) = 2$$

Now try Exercises 19 through 34 ►

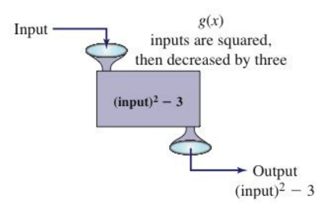
B. You've just learned how to compute a product or quotient of functions and determine the domain

From our work with rational expressions in Section R.5, the expression that defines h can be simplified: $\frac{x^3 - 3x^2 + 2x - 6}{x - 3} = \frac{x^2(x - 3) + 2(x - 3)}{x - 3} = \frac{(x^2 + 2)\cancel{(x - 3)}}{\cancel{x - 3}} = x^2 + 2$. But from the original expression, h is not defined if $g(x) = 3$, *even if the result for h is a polynomial*. In this case, we write the simplified form as $h(x) = x^2 + 2, x \neq 3$.

For additional practice with the algebra of functions, **see Exercises 35 through 46.**

C. Composition of Functions

The composition of functions is best understood by studying the "input/output" nature of a function. Consider $g(x) = x^2 - 3$. For $g(x)$ we might say, "inputs are squared, then decreased by three." In diagram form we have:



In many respects, a function box can be regarded as a very simple machine, running a simple program. It doesn't matter what the input is, this machine is going to *square the input then subtract three*.

EXAMPLE 5 ► **Evaluating a Function**

For $g(x) = x^2 - 3$, find

a. $g(-5)$
b. $g(5t)$
c. $g(t - 4)$

Solution ► **a.** $g(x) = x^2 - 3$ original function

input -5

$g(-5) = (-5)^2 - 3$ square input, then subtract 3

$= 25 - 3$ simplify

$= 22$ result

b. $g(x) = x^2 - 3$ original function

input $5t$

$g(5t) = (5t)^2 - 3$ square input, then subtract 3

$= 25t^2 - 3$ result

WORTHY OF NOTE

It's important to note that t and $t - 4$ are two different, distinct values—the number represented by t, and a number four less than t. Examples would be 7 and 3, 12 and 8, as well as -10 and -14. There should be nothing awkward or unusual about evaluating $g(t)$ versus evaluating $g(t - 4)$ as in Example 5c.

c. $g(x) = x^2 - 3$ original function

input $t - 4$

$g(t - 4) = (t - 4)^2 - 3$ square input, then subtract 3

$= t^2 - 8t + 16 - 3$ expand binomial

$= t^2 - 8t + 13$ result

Now try Exercises 47 and 48 ►

When the input value is itself a function (rather than a single number or variable), this process is called the **composition of functions.** The evaluation method is exactly the same, we are simply using a function input. Using a general function $g(x)$ and a function diagram as before, we illustrate the process in Figure 2.87.

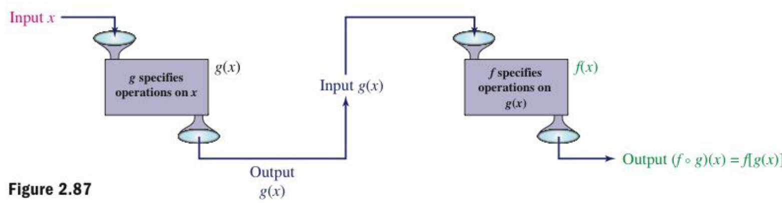


Figure 2.87

The notation used for the composition of f with g is an open dot "∘" placed between them, and is read, "f composed with g." The notation $(f \circ g)(x)$ indicates that $g(x)$ is an input for f: $(f \circ g)(x) = f[g(x)]$. If the order is reversed, as in $(g \circ f)(x)$, $f(x)$ becomes the input for g: $(g \circ f)(x) = g[f(x)]$. Figure 2.87 also helps us determine the domain of a composite function, in that the first function g can operate only if x is a valid input for g, and the second function f can operate only if $g(x)$ is a valid input for f. In other words, $(f \circ g)(x)$ is defined for *all x in the domain of g, such that g(x) is in the domain of f.*

CAUTION ► Try not to confuse the new "open dot" notation for the *composition* of functions, with the multiplication dot used to indicate the *product* of two functions: $(f \cdot g)(x) = (fg)(x)$ or the product of f with g; $(f \circ g)(x) = f[g(x)]$ or f composed with g.

The Composition of Functions

Given two functions f and g, the composition of f with g is defined by

$$(f \circ g)(x) = f[g(x)]$$

The domain of the composition is all x in the domain of g for which $g(x)$ is in the domain of f.

In Figure 2.88, these ideas are displayed using mapping notation, as we consider the simple case where $g(x) = x$ and $f(x) = \sqrt{x}$.

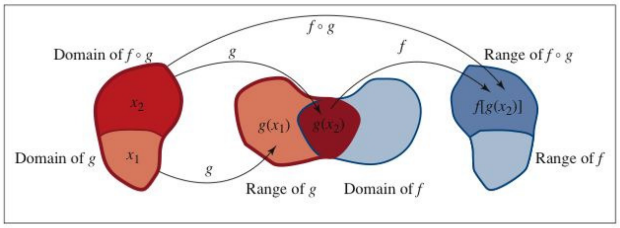


Figure 2.88

The domain of g (all real numbers) is shown within the red border, with g taking the negative inputs represented by x_1 (light red), to a like-colored portion of the range—the negative outputs $g(x_1)$. The nonnegative inputs represented by x_2 (dark red) are also mapped to a like-colored portion of the range—the nonnegative outputs $g(x_2)$. While the range of g is also all real numbers, function f can only use the nonnegative inputs represented by $g(x_2)$. This restricts the domain of $(f \circ g)(x)$ to only the inputs from g, where $g(x)$ is in the domain of f.

EXAMPLE 6 ▶ Finding a Composition of Functions

Given $f(x) = \sqrt{x - 4}$ and $g(x) = 3x + 2$, find

a. $(f \circ g)(x)$

b. $(g \circ f)(x)$

Also determine the domain for each.

Solution ▶ **a.** $f(x) = \sqrt{x - 4}$ says "decrease inputs by 4, and take the square root of the result."

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] && g(x) \text{ is an input for } f\\ &= \sqrt{g(x) - 4} && \text{decrease input by 4, and take the square root of the result}\\ &= \sqrt{(3x + 2) - 4} && \text{substitute } 3x + 2 \text{ for } g(x)\\ &= \sqrt{3x - 2} && \text{result}\end{aligned}$$

While g is defined for all real numbers, f is defined only for nonnegative numbers. Since $f[g(x)] = \sqrt{3x - 2}$, we need $3x - 2 \geq 0$, $x \geq \frac{2}{3}$. In interval notation, the domain of $(f \circ g)(x)$ is $x \in [\frac{2}{3}, \infty)$.

b. The function g says "inputs are multiplied by 3, then increased by 2."

$$\begin{aligned}(g \circ f)(x) &= g[f(x)] && f(x) \text{ is an input for } g\\ &= 3f(x) + 2 && \text{multiply input by 3, then increase by 2}\\ &= 3\sqrt{x - 4} + 2 && \text{substitute } \sqrt{x-4} \text{ for } f(x)\end{aligned}$$

For $g[f(x)]$, g can accept any real number input, but f can supply only those where $x \geq 4$. The domain of $(g \circ f)(x)$ is $x \in [4, \infty)$.

Now try Exercises 49 through 58 ▶

WORTHY OF NOTE

Example 6 shows that $(f \circ g)(x)$ is generally not equal to $(g \circ f)(x)$. On those occasions when they *are* equal, the functions have a unique relationship that we'll study in Section 4.1.

EXAMPLE 7 ► Finding a Composition of Functions

For $f(x) = \dfrac{3x}{x-1}$ and $g(x) = \dfrac{2}{x}$, analyze the domain of

a. $(f \circ g)(x)$.
b. $(g \circ f)(x)$.
c. Find the actual compositions and comment.

Solution ► **a.** $(f \circ g)(x)$: For g to be defined, $x \neq 0$ is our first restriction. Once $g(x)$ is used as the input, we have $f[g(x)] = \dfrac{3g(x)}{g(x)-1}$, and additionally note that $g(x)$ cannot equal 1. This means $\dfrac{2}{x} \neq 1$, so $x \neq 2$. The domain of $f \circ g$ is $\{x \mid x \neq 0, x \neq 2\}$.

b. $(g \circ f)(x)$: For f to be defined, $x \neq 1$ is our first restriction. Once $f(x)$ is used as the input, we have $g[f(x)] = \dfrac{2}{f(x)}$, and additionally note that $f(x)$ cannot be 0. This means $\dfrac{3x}{x-1} \neq 0$, so $x \neq 0$. The domain of $(g \circ f)(x)$ is $\{x \mid x \neq 0, x \neq 1\}$.

c. For $(f \circ g)(x)$:

$$f[g(x)] = \frac{3g(x)}{g(x)-1} \quad \text{composition of } f \text{ with } g$$

$$= \frac{\left(\frac{3}{1}\right)\left(\frac{2}{x}\right)}{\left(\frac{2}{x}\right)-1} \quad \text{substitute } \tfrac{2}{x} \text{ for } g(x)$$

$$= \frac{\frac{6}{x}}{\frac{2-x}{x}} = \frac{6}{x}\cdot\frac{x}{2-x} \quad \text{simplify denominator; invert and multiply}$$

$$= \frac{6}{2-x} \quad \text{result}$$

Notice the function rule for $(f \circ g)(x)$ has an implied domain of $x \neq 2$, but does not show that g (the inner function) is undefined when $x = 0$ (see Part a). The domain of $(f \circ g)(x)$ is actually $x \neq 2$ **and** $x \neq 0$.

For $(g \circ f)(x)$ we have:

$$g[f(x)] = \frac{2}{f(x)} \quad \text{composition of } g \text{ with } f$$

$$\frac{2}{f(x)} = \frac{2}{\frac{3x}{x-1}} \quad \text{substitute } \tfrac{3x}{x-1} \text{ for } f(x)$$

$$= \frac{2}{1}\cdot\frac{x-1}{3x} \quad \text{invert and multiply}$$

$$= \frac{2(x-1)}{3x} \quad \text{result}$$

Similarly, the function rule for $(g \circ f)(x)$ has an implied domain of $x \neq 0$, but does not show that f (the inner function) is undefined when $x = 1$ (see Part a). The domain of $(g \circ f)(x)$ is actually $x \neq 0$ **and** $x \neq 1$.

Now try Exercises 59 through 64 ►

WORTHY OF NOTE

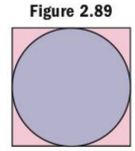
As Example 7 illustrates, the domain of $h(x) = (f \circ g)(x)$ *cannot simply be taken from the new function rule for h*. It *must* be determined from the functions composed to obtain *h*.

To further explore concepts related to the domain of a composition, **see Exercises 92 through 94.**

Decomposing a Composite Function

Based on Figure 2.89, would you say that the circle is inside the square or the square is inside the circle? The decomposition of a composite function is related to a similar question, as we ask ourselves what function (of the composition) is on the "inside"—the input quantity—and what function is on the "outside." For instance, consider $h(x) = \sqrt{x - 4}$, where we see that $x - 4$ is "inside" the radical. Letting $g(x) = x - 4$ and $f(x) = \sqrt{x}$, we have $h(x) = (f \circ g)(x)$ or $f[g(x)]$.

Figure 2.89



WORTHY OF NOTE

The decomposition of a function is not unique and can often be done in many different ways.

EXAMPLE 8 ► **Decomposing a Composite Function**

Given $h(x) = (\sqrt[3]{x} + 1)^2 - 3$, identify two functions f and g so that $(f \circ g)(x) = h(x)$, then check by composing the functions to obtain $h(x)$.

Solution ► Noting that $\sqrt[3]{x} + 1$ is inside the squaring function, we assign $g(x)$ as this inner function: $g(x) = \sqrt[3]{x} + 1$. The outer function is the squaring function decreased by 3, so $f(x) = x^2 - 3$.

Check: $(f \circ g)(x) = f[g(x)]$ — $g(x)$ is an input for f

$= [g(x)]^2 - 3$ — f squares inputs, then decreases the result by 3

$= [\sqrt[3]{x} + 1]^2 - 3$ — substitute $\sqrt[3]{x} + 1$ for $g(x)$

$= h(x)$ ✓

Now try Exercises 65 through 68 ►

✔ **C.** You've just learned how to compose two functions and determine the domain, and decompose a function

D. A Graphical View of Operations on Functions

The algebra and composition of functions also has an instructive *graphical interpretation,* in which values for $f(k)$ and $g(k)$ are read from a graph (k is a given constant), with operations like $(f + g)(k) = f(k) + g(k)$ then computed and lodged. Once the value of $g(k)$ is known, $(f \circ g)(k) = f[g(k)]$ is likewise interpreted and computed (also **see Exercise 95**).

EXAMPLE 9 ► **Interpreting Operations on Functions Graphically**

Use the graph given to find the value of each expression:

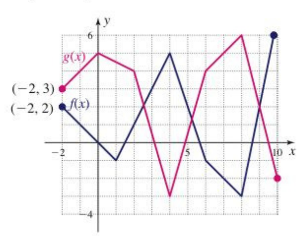
a. $(f + g)(-2)$

b. $(f \circ g)(7)$

c. $(g - f)(6)$

d. $\left(\dfrac{g}{f}\right)(8)$

e. $(f \cdot g)(4)$



Solution ► Since the needed input values for this example are $x = -2, 4, 6, 7,$ and 8, we begin by reading the value of $f(x)$ and $g(x)$ at each point. From the graph, we note that $f(-2) = 2$ and $g(-2) = 3$. The other values are likewise found and appear in the table. For $(f + g)(-2)$ we have:

x	$f(x)$	$g(x)$
−2	2	3
4	5	−3
6	−1	4
7	−2	5
8	−3	6

a. $(f + g)(-2) = f(-2) + g(-2)$ definition
$= 2 + 3$ substitute 2 for $f(-2)$ and 3 for $g(-2)$
$= 5$ result

b. $(f \circ g)(7) = f[g(7)]$ definition
$= f(5)$ substitute 5 for $g(7)$
$= 2$ result read from graph: $f(5) = 2$

With some practice, the computations can be done mentally and we have

c. $(g - f)(6) = g(6) - f(6)$
$= 4 - (-1) = 5$

d. $\left(\frac{g}{f}\right)(8) = \frac{g(8)}{f(8)}$
$= \frac{6}{-3} = -2$

e. $(f \cdot g)(4) = f(4) \cdot g(4)$
$= 5(-3) = -15$

D. You've just learned how to interpret operations on functions graphically

Now try Exercises 69 through 78 ►

E. Applications of the Algebra and Composition of Functions

The algebra of functions plays an important role in the business world. For example, the cost to manufacture an item, the revenue a company brings in, and the profit a company earns are all functions of the number of items made and sold. Further, we know a company "breaks even" (making \$0 profit) when the difference between their revenue R and their cost C, is zero.

EXAMPLE 10 ► Applying Operations on Functions in Context

The fixed costs to publish *Relativity Made Simple* (by N.O. Way) is \$2500, and the variable cost is \$4.50 per book. Marketing studies indicate the best selling price for the book is \$9.50 per copy.

a. Find the cost, revenue, and profit functions for this book.
b. Determine how many copies must be sold for the company to break even.

Solution ► **a.** Let x represent the number of books published and sold. The cost of publishing is \$4.50 per copy, plus fixed costs (labor, storage, etc.) of \$2500. The cost function is $C(x) = 4.50x + 2500$. If the company charges \$9.50 per book, the revenue function will be $R(x) = 9.50x$. Since profit equals revenue minus costs,

$P(x) = R(x) - C(x)$
$= 9.50x - (4.50x + 2500)$ substitute $9.50x$ for R and $4.50x + 2500$ for C
$= 9.50x - 4.50x - 2500$ distribute
$= 5x - 2500$ result

The profit function is $P(x) = 5x - 2500$.

b. When a company "breaks even," the profit is zero: $P(x) = 0$.

$P(x) = 5x - 2500$ profit function

$0 = 5x - 2500$ substitute 0 for $P(x)$

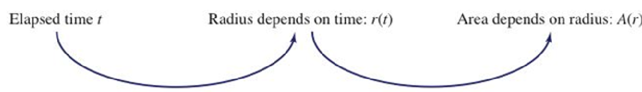
$2500 = 5x$ add 2500

$500 = x$ divide by 5

In order for the company to break even, 500 copies must be sold.

Now try Exercises 81 through 84 ►

Suppose that due to a collision, an oil tanker is spewing oil into the open ocean. The oil is spreading outward in a shape that is roughly circular, with the radius of the circle modeled by the function $r(t) = 2\sqrt{t}$, where t is the time in minutes and r is measured in feet. How could we determine the *area* of the oil slick in terms of t? As you can see, the radius depends on the time and the area depends on the radius. In diagram form we have:



It is possible to create a direct relationship between the elapsed time and the area of the circular spill using a composition of functions.

EXAMPLE 11 ► Applying a Composition in Context

Given $r(t) = 2\sqrt{t}$ and $A(r) = \pi r^2$,

a. Write A directly as a function of t by computing $(A \circ r)(t)$.

b. Find the area of the oil spill after 30 min.

Solution ► **a.** The function A squares inputs, then multiplies by π.

$(A \circ r)(t) = A[r(t)]$ $r(t)$ is the input for A

$= [r(t)]^2 \cdot \pi$ square input, multiply by π

$= [2\sqrt{t}]^2 \cdot \pi$ substitute $2\sqrt{t}$ for $r(t)$

$= 4\pi t$ result

Since the result contains no variable r, we can now compute the area of the spill directly, given the elapsed time t (in minutes): $A(t) = 4\pi t$.

b. To find the area after 30 min, use $t = 30$.

$A(t) = 4\pi t$ composite function

$A(30) = 4\pi(30)$ substitute 30 for t

$= 120\pi$ simplify

≈ 377 result (rounded to the nearest unit)

After 30 min, the area of the spill is approximately 377 ft^2.

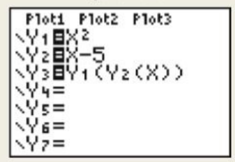
Now try Exercises 85 through 90 ►

☑ **E.** You've just learned how to apply the algebra and composition of functions in context

TECHNOLOGY HIGHLIGHT

Composite Functions

Figure 2.90



The graphing calculator is truly an amazing tool when it comes to studying composite functions. Using this powerful tool, composite functions can be graphed, evaluated, and investigated with ease. To begin, enter the functions $y = x^2$ and $y = x - 5$ as Y_1 and Y_2 on the Y= screen. Enter the composition $(Y_1 \circ Y_2)(x)$ as $Y_3 = Y_1(Y_2(X))$, as shown in Figure 2.90 [in our standard notation we have $f(x) = x^2$, $g(x) = x - 5$, and $h(x) = (f + g)(x) = f[g(x)]$. On the TI-84 *Plus*, we access the function variables Y_1, Y_2, Y_3, and so on by pressing VARS ► ENTER and selecting the function desired. Pressing ZOOM **6:ZStandard** will graph all three functions in the standard window. Let's look at the relationship between Y_1 and Y_3. Deactivate Y_2 and regraph Y_1 and Y_3. What do you notice about the graphs? Y_3 is the same as the graph of Y_1, but shifted 5 units to the right! Does this have any connection to $Y_2 = x - 5$? Try changing Y_2 to $Y_2 = x + 4$, then regraph Y_1 and Y_3. Use what you notice to complete the following exercises and continue the exploration.

Exercise 1: Change Y_1 to $Y_1 = \sqrt{x}$, then experiment by changing Y_2 to $x + 3$, then to $x - 6$. Did you notice anything similar? What would happen if we changed Y_2 to $Y_2 = x + 7$?

Exercise 2: Change Y_1 to $Y_1 = x^3$, then experiment by changing Y_2 to $x + 5$, then to $x - 1$. Did the same "shift" occur? What would happen if we changed Y_1 to $Y_1 = |x|$?

2.8 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. Given function f with domain A and function g with domain B, the sum $f(x) + g(x)$ can also be written ________. The domain of the result is ________.

2. For the product $h(x) = f(x) \cdot g(x)$, $h(5)$ can be found by evaluating f and g then multiplying the result, or multiplying $f \cdot g$ and evaluating the result. Notationally these are written ________ and ________.

3. When combining functions f and g using basic operations, the domain of the result is the ________ of the domains of f and g. For division, we further stipulate that ________ cannot equal zero.

4. When evaluating functions, if the input value is a function itself, the process is called the ________ of functions. The notation $(f \circ g)(x)$ indicates that ________ is the input value for ________, which we can also write as ________.

5. For $f(x) = 2x^3 - 50x$ and $g(x) = x - 5$, discuss/explain why the domain of $h(x) = \left(\frac{f}{g}\right)(x)$ must exclude $x = 5$, even though the resulting quotient is the polynomial $2x^2 + 10x$.

6. For $f(x) = \sqrt{2x + 7}$ and $g(x) = \frac{2}{x - 1}$, discuss/explain how the domain of $h(x) = (f \circ g)(x)$ is determined. In particular, why is $h(1)$ not defined even though $f(1) = 3$?

▶ DEVELOPING YOUR SKILLS

7. Given $f(x) = 2x^2 - x - 3$ and $g(x) = x^2 + 5x$, (a) determine the domain for $h(x) = f(x) - g(x)$ and (b) find $h(-2)$ using the definition.

8. Given $f(x) = 2x^2 - 18$ and $g(x) = -3x - 7$, (a) determine the domain for $h(x) = f(x) + g(x)$ and (b) find $h(5)$ using the definition.

9. For the functions f, g, and h, as defined in Exercise 7, (a) find a new function rule for h, and (b) use the result to find $h(-2)$. (c) How does the result compare to that of Exercise 7?

10. For the functions f, g, and h as defined in Exercise 8, (a) find a new function rule for h, and (b) use the result to find $h(5)$. (c) How does the result compare to that in Exercise 8?

11. For $f(x) = \sqrt{x-3}$ and $g(x) = 2x^3 - 54$, (a) determine the domain of $h(x) = (f + g)(x)$, (b) find a new function rule for h, and (c) evaluate $h(4)$ and $h(2)$, if possible.

12. For $f(x) = 4x^2 - 2x + 3$ and $g(x) = \sqrt{2x-5}$, (a) determine the domain of $h(x) = (f - g)(x)$, (b) find a new function rule for h, and (c) evaluate $h(7)$ and $h(2)$, if possible.

13. For $p(x) = \sqrt{x+5}$ and $q(x) = \sqrt{3-x}$, (a) determine the domain of $r(x) = (p + q)(x)$, (b) find a new function rule for r, and (c) evaluate $r(2)$ and $r(4)$, if possible.

14. For $p(x) = \sqrt{6-x}$ and $q(x) = \sqrt{x+2}$, (a) determine the domain of $r(x) = (p - q)(x)$, (b) find a new function rule for r, and (c) evaluate $r(-3)$ and $r(2)$, if possible.

15. For $f(x) = \sqrt{x+4}$ and $g(x) = 2x + 3$, (a) determine the domain of $h(x) = (f \cdot g)(x)$, (b) find a new function rule for h, and (c) evaluate $h(-4)$ and $h(21)$, if possible.

16. For $f(x) = -3x + 5$ and $g(x) = \sqrt{x-7}$, (a) determine the domain of $h(x) = (f \cdot g)(x)$, (b) find a new function rule for h, and (c) evaluate $h(8)$ and $h(11)$, if possible.

17. For $p(x) = \sqrt{x+1}$ and $q(x) = \sqrt{7-x}$, (a) determine the domain of $r(x) = (p \cdot q)(x)$, (b) find a new function rule for r, and (c) evaluate $r(15)$ and $r(3)$, if possible.

18. For $p(x) = \sqrt{4-x}$ and $q(x) = \sqrt{x+4}$, (a) determine the domain of $r(x) = (p \cdot q)(x)$, (b) find a new function rule for r, and (c) evaluate $r(-5)$ and $r(-3)$, if possible.

For the functions f and g given, (a) determine the domain of $h(x) = \left(\frac{f}{g}\right)(x)$ and (b) find a new function rule for h in simplified form (if possible), noting the domain restrictions along side.

19. $f(x) = x^2 - 16$ and $g(x) = x + 4$

20. $f(x) = x^2 - 49$ and $g(x) = x - 7$

21. $f(x) = x^3 + 4x^2 - 2x - 8$ and $g(x) = x + 4$

22. $f(x) = x^3 - 5x^2 + 2x - 10$ and $g(x) = x - 5$

23. $f(x) = x^3 - 7x^2 + 6x$ and $g(x) = x - 1$

24. $f(x) = x^3 - 1$ and $g(x) = x - 1$

25. $f(x) = x + 1$ and $g(x) = x - 5$

26. $f(x) = x + 3$ and $g(x) = x - 7$

For the functions p and q given, (a) determine the domain of $r(x) = \left(\frac{p}{q}\right)(x)$, (b) find a new function rule for r, and (c) use it to evaluate $r(6)$ and $r(-6)$, if possible.

27. $p(x) = 2x - 3$ and $q(x) = \sqrt{-2-x}$

28. $p(x) = 1 - x$ and $q(x) = \sqrt{3-x}$

29. $p(x) = x - 5$ and $q(x) = \sqrt{x-5}$

30. $p(x) = x + 2$ and $q(x) = \sqrt{x+3}$

31. $p(x) = x^2 - 36$ and $q(x) = \sqrt{2x+13}$

32. $p(x) = x^2 - 6x$ and $q(x) = \sqrt{7+3x}$

For the functions f and g given, (a) find a new function rule for $h(x) = \left(\frac{f}{g}\right)(x)$ in simplified form. (b) If $h(x)$ were the original function, what would be its domain? (c) Since we know $h(x) = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, what additional values are excluded from the domain of h?

33. $f(x) = \frac{6x}{x-3}$ and $g(x) = \frac{3x}{x+2}$

34. $f(x) = \frac{4x}{x+1}$ and $g(x) = \frac{2x}{x-2}$

For each pair of functions f and g given, determine the sum, difference, product, and quotient of f and g, then determine the domain in each case.

35. $f(x) = 2x + 3$ and $g(x) = x - 2$

36. $f(x) = x - 5$ and $g(x) = 2x - 3$

37. $f(x) = x^2 + 7$ and $g(x) = 3x - 2$

38. $f(x) = x^2 - 3x$ and $g(x) = x + 4$

39. $f(x) = x^2 + 2x - 3$ and $g(x) = x - 1$

40. $f(x) = x^2 - 2x - 15$ and $g(x) = x + 3$

41. $f(x) = 3x + 1$ and $g(x) = \sqrt{x - 3}$

42. $f(x) = x + 2$ and $g(x) = \sqrt{x + 6}$

43. $f(x) = 2x^2$ and $g(x) = \sqrt{x + 1}$

44. $f(x) = x^2 + 2$ and $g(x) = \sqrt{x - 5}$

45. $f(x) = \dfrac{2}{x - 3}$ and $g(x) = \dfrac{5}{x + 2}$

46. $f(x) = \dfrac{4}{x - 3}$ and $g(x) = \dfrac{1}{x + 5}$

47. Given $f(x) = x^2 - 5x - 14$, find $f(-2), f(7)$, $f(2a)$, and $f(a - 2)$.

48. Given $g(x) = x^3 - 9x$, find $g(-3)$, $g(2)$, $g(3t)$, and $g(t + 1)$.

For each pair of functions below, find (a) $h(x) = (f \circ g)(x)$ and (b) $H(x) = (g \circ f)(x)$, and (c) determine the domain of each result.

49. $f(x) = \sqrt{x + 3}$ and $g(x) = 2x - 5$

50. $f(x) = x + 3$ and $g(x) = \sqrt{9 - x^2}$

51. $f(x) = \sqrt{x - 3}$ and $g(x) = 3x + 4$

52. $f(x) = \sqrt{x + 5}$ and $g(x) = 4x - 1$

53. $f(x) = x^2 - 3x$ and $g(x) = x + 2$

54. $f(x) = 2x^2 - 1$ and $g(x) = 3x + 2$

55. $f(x) = x^2 + x - 4$ and $g(x) = x + 3$

56. $f(x) = x^2 - 4x + 2$ and $g(x) = x - 2$

57. $f(x) = |x| - 5$ and $g(x) = -3x + 1$

58. $f(x) = |x - 2|$ and $g(x) = 3x - 5$

For the functions $f(x)$ and $g(x)$ given, analyze the domain of (a) $(f \circ g)(x)$ and (b) $(g \circ f)(x)$, then (c) find the actual compositions and comment.

59. $f(x) = \dfrac{2x}{x + 3}$ and $g(x) = \dfrac{5}{x}$

60. $f(x) = \dfrac{-3}{x}$ and $g(x) = \dfrac{x}{x - 2}$

61. $f(x) = \dfrac{4}{x}$ and $g(x) = \dfrac{1}{x - 5}$

62. $f(x) = \dfrac{3}{x}$ and $g(x) = \dfrac{1}{x - 2}$

63. For $f(x) = x^2 - 8$, $g(x) = x + 2$, and $h(x) = (f \circ g)(x)$, find $h(5)$ in two ways:
a. $(f \circ g)(5)$ **b.** $f[g(5)]$

64. For $p(x) = x^2 - 8$, $q(x) = x + 2$, and $H(x) = (p \circ q)(x)$, find $H(-2)$ in two ways:
a. $(p \circ q)(-2)$ **b.** $p[q(-2)]$

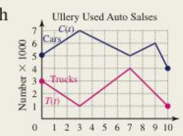
65. For $h(x) = (\sqrt{x - 2} + 1)^3 - 5$, find two functions f and g such that $(f \circ g)(x) = h(x)$.

66. For $H(x) = \sqrt[3]{x^2 - 5} + 2$, find two functions p and q such that $(p \circ q)(x) = h(x)$.

67. Given $f(x) = 2x - 1$, $g(x) = x^2 - 1$, and $h(x) = x + 4$, find $p(x) = f[g([h(x)])]$ and $q(x) = g[f([h(x)])]$.

68. Given $f(x) = 2x + 3$ and $g(x) = \dfrac{x - 3}{2}$, find (a) $(f \circ f)(x)$, (b) $(g \circ g)(x)$, (c) $(f \circ g)(x)$, and (d) $(g \circ f)(x)$.

69. Reading a graph: The graph given shows the number of cars $C(t)$ and trucks $T(t)$ sold by Ullery Used Autos for the years 2000 to 2010. Use the graph to estimate the number of



Exercise 69

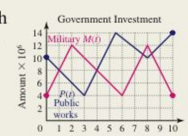
a. cars sold in 2005: $C(5)$

b. trucks sold in 2008: $T(8)$

c. vehicles sold in 2009: $C(9) + T(9)$

d. In function notation, how would you determine how many more cars than trucks were sold in 2009? What was the actual number?

70. Reading a graph: The graph given shows a government's investment in its military $M(t)$ over time, versus its investment in public works $P(t)$, in millions of dollars. Use the graph to estimate the amount of investment in



Exercise 70

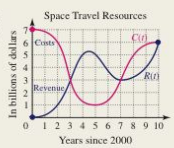
a. the military in 2002: $M(2)$

b. public works in 2005: $P(5)$

c. public works and the military in 2009: $M(9) + P(9)$

d. In function notation, how would you determine how much more will be invested in public works than the military in 2010? What is the actual number?

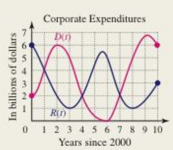
71. Reading a graph: The graph given shows the revenue $R(t)$ and operating costs $C(t)$ of Space Travel Resources (STR), for the years 2000 to 2010. Use the graph to find the



Exercise 71

a. revenue in 2002: $R(2)$
b. costs in 2008: $C(8)$
c. years STR broke even: $R(t) = C(t)$
d. years costs exceeded revenue: $C(t) > R(t)$
e. years STR made a profit: $R(t) > C(t)$
f. For the year 2005, use function notation to write the profit equation for STR. What was their profit?

72. Reading a graph: The graph given shows a large corporation's investment in research and development $R(t)$ over time, and the amount paid to investors as dividends $D(t)$, in billions of dollars. Use the graph to find the

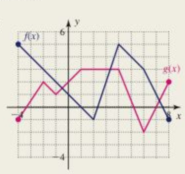


Exercise 72

a. dividend payments in 2002: $D(2)$
b. investment in 2006: $R(6)$
c. years where $R(t) = D(t)$
d. years where $R(t) > D(t)$
e. years where $R(t) < D(t)$
f. Use function notation to write an equation for the total expenditures of the corporation in year t. What was the total for 2010?

73. Reading a graph: Use the given graph to find the result of the operations indicated.

Note $f(-4) = 5$, $g(-4) = -1$, and so on.

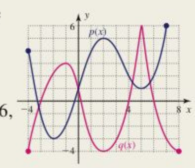


Exercise 73

a. $(f + g)(-4)$
b. $(f \cdot g)(1)$
c. $(f - g)(4)$
d. $(f + g)(0)$
e. $\left(\frac{f}{g}\right)(2)$
f. $(f \cdot g)(-2)$
g. $(g \cdot f)(2)$
h. $(f - g)(-1)$
i. $(f + g)(8)$
j. $\left(\frac{f}{g}\right)(7)$
k. $(g \circ f)(4)$
l. $(f \circ g)(4)$

74. Reading a graph: Use the given graph to find the result of the operations indicated.

Note $p(-1) = 3$, $q(5) = 6$, and so on.

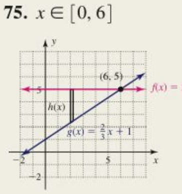


Exercise 74

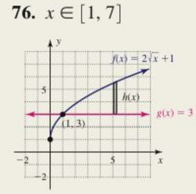
a. $(p + q)(-4)$
b. $(p \cdot q)(1)$
c. $(p - q)(4)$
d. $(p + q)(0)$
e. $\left(\frac{p}{q}\right)(5)$
f. $(p \cdot q)(-2)$
g. $(q \cdot p)(2)$
h. $(p - q)(-1)$
i. $(p + q)(7)$
j. $\left(\frac{p}{q}\right)(6)$
k. $(q \circ p)(4)$
l. $(p \circ q)(-1)$

Some advanced applications require that we use the algebra of functions to find a function rule for the vertical distance between two graphs. For $f(x) = 3$ and $g(x) = -2$ (two horizontal lines), we "see" this vertical distance is 5 units, or in function form: $d(x) = f(x) - g(x) = 3 - (-2) = 5$ units. However, $d(x) = f(x) - g(x)$ also serves as a *general formula* for the vertical distance between two curves (even those that are not horizontal lines), so long as $f(x) > g(x)$ in a chosen interval. Find a function rule in simplified form, for the vertical distance $h(x)$ between the graphs of f and g shown, for the interval indicated.

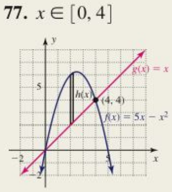
75. $x \in [0, 6]$



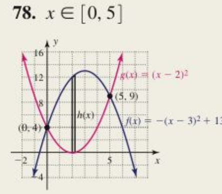
76. $x \in [1, 7]$



77. $x \in [0, 4]$



78. $x \in [0, 5]$



▶ WORKING WITH FORMULAS

79. Surface area of a cylinder: $A = 2\pi rh + 2\pi r^2$

If the height of a cylinder is fixed at 20 cm, the formula becomes $A = 40\pi r + 2\pi r^2$. Write this formula in factored form and find two functions $f(r)$ and $g(r)$ such that $A(r) = (f \cdot g)(r)$. Then find $A(5)$ by direct calculation and also by computing the product of $f(5)$ and $g(5)$, then comment on the results.

80. Compound annual growth: $A(r) = P(1 + r)^t$

The amount of money A in a savings account t yr after an initial investment of P dollars depends on the interest rate r. If $1000 is invested for 5 yr, find $f(r)$ and $g(r)$ such that $A(r) = (f \circ g)(r)$.

▶ APPLICATIONS

81. Boat manufacturing: Giaro Boats manufactures a popular recreational vessel, the *Revolution*. To plan for expanded production and increased labor costs, the company carefully tracks current costs and income. The fixed cost to produce this boat is $108,000 and the variable costs are $28,000 per boat. If the *Revolution* sells for $40,000, (a) find the profit function and (b) determine how many boats must be sold for the company to break even.

82. Non-profit publications: Adobe Hope, a nonprofit agency, publishes the weekly newsletter *Community Options*. In doing so, they provide useful information to the surrounding area while giving high school dropouts valuable work experience. The fixed cost for publishing the newsletter is $900 per week, with a variable cost of $0.25 per newsletter. If the newsletter is sold for $1.50 per copy, (a) find the profit function for the newsletter, (b) determine how many newsletters must be sold to break even, and (c) determine how much money will be returned to the community if 1000 newsletters are sold (to preserve their status as a nonprofit organization).

83. Cost, revenue, and profit: Suppose the total cost of manufacturing a certain computer component can be modeled by the function $C(n) = 0.1n^2$, where n is the number of components made and $C(n)$ is in dollars. If each component is sold at a price of $11.45, the revenue is modeled by $R(n) = 11.45n$. Use this information to complete the following.

a. Find the function that represents the total profit made from sales of the components.

b. How much profit is earned if 12 components are made and sold?

c. How much profit is earned if 60 components are made and sold?

d. Explain why the company is making a "negative profit" after the 114th component is made and sold.

84. Cost, revenue, and profit: For a certain manufacturer, revenue has been increasing but so has the cost of materials and the cost of employee benefits. Suppose revenue can be modeled by $R(t) = 10\sqrt{t}$, the cost of materials by $M(t) = 2t + 1$, and the cost of benefits by $C(t) = 0.1t^2 + 2$, where t represents the number of months since operations began and outputs are in thousands of dollars. Use this information to complete the following.

a. Find the function that represents the total manufacturing costs.

b. Find the function that represents how much more the operating costs are than the cost of materials.

c. What was the cost of operations in the 10th month after operations began?

d. How much less were the operating costs than the cost of materials in the 10th month?

e. Find the function that represents the profit earned by this company.

f. Find the amount of profit earned in the 5th month and 10th month. Discuss each result.

85. International shoe sizes: Peering inside her athletic shoes, Morgan notes the following shoe sizes: *US 8.5, UK 6, EUR 40*. The function that relates the U.S. sizes to the European (EUR) sizes is $g(x) = 2x + 23$ where x represents the U.S. size and $g(x)$ represents the EUR size. The function that relates European sizes to sizes in the United Kingdom (UK) is $f(x) = 0.5x - 14$ where x represents the EUR size and $f(x)$ represents the UK size. Find the function $h(x)$ that relates the U.S. measurement directly to the UK measurement by finding $h(x) = (f \circ g)(x)$. Find The UK size for a shoe that has a U.S. size of 13.

86. Currency conversion: On a trip to Europe, Megan had to convert American dollars to euros using the

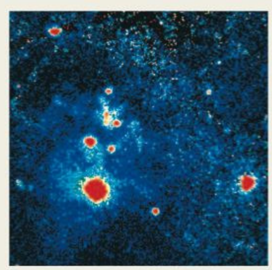
function $E(x) = 1.12x$, where x represents the number of dollars and $E(x)$ is the equivalent number of euros. Later, she converts her euros to Japanese yen using the function $Y(x) = 1061x$, where x represents the number of euros and $Y(x)$ represents the equivalent number of yen. (a) Convert 100 U.S. dollars to euros. (b) Convert the answer from part (a) into Japanese yen. (c) Express yen as a function of dollars by finding $M(x) = (Y \circ E)(x)$, then use $M(x)$ to convert 100 dollars directly to yen. Do parts (b) and (c) agree?
Source: 2005 *World Almanac*, p. 231

87. Currency conversion: While traveling in the Far East, Timi must convert U.S. dollars to Thai baht using the function $T(x) = 41.6x$, where x represents the number of dollars and $T(x)$ is the equivalent number of baht. Later she needs to convert her baht to Malaysian ringgit using the function $R(x) = 10.9x$. (a) Convert 100 dollars to baht. (b) Convert the result from part (a) to ringgit. (c) Express ringgit as a function of dollars using $M(x) = (R \circ T)(x)$, then use $M(x)$ to convert 100 dollars to ringgit directly. Do parts (b) and (c) agree?
Source: 2005 *World Almanac*, p. 231

88. Spread of a fire: Due to a lightning strike, a forest fire begins to burn and is spreading outward in a shape that is roughly circular. The radius of the circle is modeled by the function $r(t) = 2t$, where t is the time in minutes and r is measured in meters. (a) Write a function for the area burned by the fire directly as a function of t by computing $(A \circ r)(t)$. (b) Find the area of the circular burn after 60 min.

89. Radius of a ripple: As Mark drops firecrackers into a lake one 4th of July, each "pop" caused a circular ripple that expanded with time. The radius of the circle is a function of time t. Suppose the function is $r(t) = 3t$, where t is in seconds and r is in feet. (a) Find the radius of the circle after 2 sec. (b) Find the area of the circle after 2 sec. (c) Express the area as a function of time by finding $A(t) = (A \circ r)(t)$ and use $A(t)$ to find the area of the circle after 2 sec. Do the answers agree?

90. Expanding supernova: The surface area of a star goes through an expansion phase prior to going *supernova*. As the star begins expanding, the radius becomes a function of time. Suppose this function is $r(t) = 1.05t$, where t is in days and $r(t)$ is in gigameters (Gm). (a) Find the radius of the star two days after the expansion phase begins. (b) Find the surface area after two days. (c) Express the surface area as a function of time by finding $h(t) = (S \circ r)(t)$, then use $h(t)$ to compute the surface area after two days directly. Do the answers agree?



▶ EXTENDING THE CONCEPT

91. In a certain country, the function $C(x) = 0.0345x^4 - 0.8996x^3 + 7.5383x^2 - 21.7215x + 40$ approximates the number of Conservatives in the senate for the years 1995 to 2007, where $x = 0$ corresponds to 1995. The function $L(x) = -0.0345x^4 + 0.8996x^3 - 7.5383x^2 + 21.7215x + 10$ gives the number of Liberals for these years. Use this information to answer the following. (a) During what years did the Conservatives control the senate? (b) What was the greatest difference between the number of seats held by each faction in any one year? In what year did this occur? (c) What was the minimum number of seats held by the Conservatives? In what year? (d) Assuming no independent or third-party candidates are elected, what information does the function $T(x) = C(x) + L(x)$ give us? What information does $t(x) = |C(x) - L(x)|$ give us?

92. Given $f(x) = x^3 + 2$ and $g(x) = \sqrt[3]{x - 2}$, graph each function on the same axes by plotting the points that correspond to integer inputs for $x \in [-3, 3]$. Do you notice anything? Next, find $h(x) = (f \circ g)(x)$ and $H(x) = (g \circ f)(x)$. What happened? Look closely at the functions f and g to see how they are related. Can you come up with two additional functions where the same thing occurs?

93. Given $f(x) = \sqrt{1 - x}$ and $g(x) = \sqrt{x - 2}$, what can you say about the domain of $(f + g)(x)$? Enter the functions as Y_1 and Y_2 on a graphing calculator, then enter $Y_3 = Y_1 + Y_2$. See if you can determine why the calculator gives an error message for Y_3, regardless of the input.

94. Given $f(x) = \frac{1}{x^2 - 4}$, $g(x) = \sqrt{x + 1}$, and $h(x) = (f \circ g)(x)$, (a) find the new function rule for h and (b) determine the implied domain of h. Does this *implied* domain include $x = 2$, $x = -2$, and $x = -3$ as valid inputs? (c) Determine the actual domain for $h(x) = (f \circ g)(x)$ and discuss the result.

95. Instead of calculating the result of an operation on two functions at a *specific point* as in Exercises 69–74, we can actually *graph the function* that results from the operation. This skill, called the **addition of ordinates,** is widely applied in a study of tides and other areas. For $f(x) = (x - 3)^2 + 2$ and $g(x) = 4|x - 3| - 5$, complete a table of values like the one shown for $x \in [-2, 8]$. For the last column, remember that $(f - g)(x) = f(x) - g(x)$, and use this relation to complete the column. Finally, use the ordered pairs $(x, (f - g)(x))$ to graph the new function. Is the new function smooth? Is the new function continuous?

Exercise 95

x	$f(x)$	$g(x)$	$(f - g)(x)$
−2			
−1			
0			
1			
2			
3			
4			
5			
6			
7			
8			

▶ MAINTAINING YOUR SKILLS

96. **(1.4)** Find the sum and product of the complex numbers $2 + 3i$ and $2 - 3i$.

97. **(2.4)** Draw a sketch of the functions (a) $f(x) = \sqrt{x}$, (b) $g(x) = \sqrt[3]{x}$, and (c) $h(x) = |x|$ *from memory*.

98. **(1.5)** Use the quadratic formula to solve $2x^2 - 3x + 4 = 0$.

99. **(2.3)** Find the equation of the line perpendicular to $-2x + 3y = 9$, that also goes through the origin.

SUMMARY AND CONCEPT REVIEW

SECTION 2.1 Rectangular Coordinates; Graphing Circles and Other Relations

KEY CONCEPTS

- A relation is a collection of ordered pairs (x, y) and can be given in set or equation form.
- As a set of ordered pairs, the domain of the relation is the set of all first coordinates, and the range is the set of all corresponding second coordinates.
- A relation can be expressed in mapping notation $x \rightarrow y$, indicating an element from the domain is mapped to (corresponds to or is associated with) an element from the range.
- The graph of a relation in equation form is the set of all ordered pairs (x, y) that satisfy the equation. We plot a sufficient number of points and connect them with a straight line or smooth curve, depending on the pattern formed.
- The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- The distance between the points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The equation of a circle centered at (h, k) with radius r is $(x - h)^2 + (y - k)^2 = r^2$.

EXERCISES

1. Represent the relation in mapping notation, then state the domain and range.
$\{(-7, 3), (-4, -2), (5, 1), (-7, 0), (3, -2), (0, 8)\}$

2. Graph the relation $y = \sqrt{25 - x^2}$ by completing the table, then state the domain and range of the relation.

x	y
−5	
−4	
−2	
0	
2	
4	
5	

Mr. Northeast and Mr. Southwest live in Coordinate County and are good friends. Mr. Northeast lives at *19 East 25 North* or (19, 25), while Mr. Southwest lives at *14 West and 31 South* or (−14, −31). If the streets in Coordinate County are laid out in one mile squares,

3. Use the distance formula to find how far apart they live.

4. If they agree to meet halfway between their homes, what are the coordinates of their meeting place?

5. Sketch the graph of $x^2 + y^2 = 16$.

6. Sketch the graph of $x^2 + y^2 + 6x + 4y + 9 = 0$. Clearly state the center and radius.

7. Find the equation of the circle whose diameter has the endpoints (−3, 0) and (0, 4).

SECTION 2.2 Graphs of Linear Equations

KEY CONCEPTS

- A linear equation can be written in the form $ax + by = c$, where a and b are not simultaneously equal to 0.
- The slope of the line through (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$, where $x_1 \neq x_2$.
- Other designations for slope are $m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{\text{vertical change}}{\text{horizontal change}}$.
- Lines with positive slope ($m > 0$) rise from left to right; lines with negative slope ($m < 0$) fall from left to right.
- The equation of a horizontal line is $y = k$; the slope is $m = 0$.
- The equation of a vertical line is $x = h$; the slope is undefined.
- Lines can be graphed using the intercept method. First determine $(x, 0)$ (substitute 0 for y and solve for x), then $(0, y)$ (substitute 0 for x and solve for y). Then draw a straight line through these points.
- Parallel lines have equal slopes ($m_1 = m_2$); perpendicular lines have slopes that are negative reciprocals ($m_1 = -\frac{1}{m_2}$ or $m_1 \cdot m_2 = -1$).

EXERCISES

8. Plot the points and determine the slope, then use the ratio $\frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$ to find an additional point on the line:
a. (−4, 3) and (5, −2) and **b.** (3, 4) and (−6, 1).

9. Use the slope formula to determine if lines L_1 and L_2 are parallel, perpendicular, or neither:
a. L_1: (−2, 0) and (0, 6); L_2: (1, 8) and (0, 5)
b. L_1: (1, 10) and (−1, 7): L_2: (−2, −1) and (1, −3)

10. Graph each equation by plotting points: (a) $y = 3x - 2$ and (b) $y = -\frac{3}{2}x + 1$.

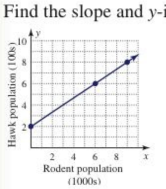
11. Find the intercepts for each line and sketch the graph: (a) $2x + 3y = 6$ and (b) $y = \frac{4}{3}x - 2$.

12. Identify each line as either horizontal, vertical, or neither, and graph each line.

a. $x = 5$ **b.** $y = -4$ **c.** $2y + x = 5$

13. Determine if the triangle with the vertices given is a right triangle: $(-5, -4)$, $(7, 2)$, $(0, 16)$.

14. Find the slope and y-intercept of the line shown and discuss the slope ratio in this context.



SECTION 2.3 Linear Graphs and Rates of Change

KEY CONCEPTS

- The equation of a nonvertical line in slope-intercept form is $y = mx + b$ or $f(x) = mx + b$. The slope of the line is m and the y-intercept is $(0, b)$.
- To graph a line given its equation in slope-intercept form, plot the y-intercept, then use the slope ratio $m = \frac{\Delta y}{\Delta x}$ to find a second point, and draw a line through these points.
- If the slope m and a point (x_1, y_1) on the line are known, the equation of the line can be written in point-slope form: $y - y_1 = m(x - x_1)$.
- A secant line is the straight line drawn through two points on a nonlinear graph.
- The notation $m = \frac{\Delta y}{\Delta x}$ literally means the quantity measured along the y-axis is changing with respect to changes in the quantity measured along the x-axis.
- The average rate of change on the interval containing x_1 and x_2 is the slope of the secant line through (x_1, y_1) and (x_2, y_2), or $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

EXERCISES

15. Write each equation in slope-intercept form, then identify the slope and y-intercept.

a. $4x + 3y - 12 = 0$ **b.** $5x - 3y = 15$

16. Graph each equation using the slope and y-intercept.

a. $f(x) = -\frac{2}{3}x + 1$ **b.** $h(x) = \frac{5}{2}x - 3$

17. Graph the line with the given slope through the given point.

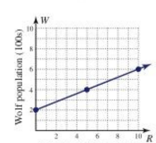
a. $m = \frac{2}{3}$; $(1, 4)$ **b.** $m = -\frac{1}{2}$; $(-2, 3)$

18. What are the equations of the horizontal line and the vertical line passing through $(-2, 5)$? Which line is the point $(7, 5)$ on?

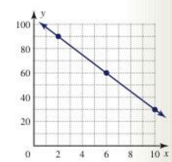
19. Find the equation of the line passing through $(1, 2)$ and $(-3, 5)$. Write your final answer in slope-intercept form.

20. Find the equation for the line that is parallel to $4x - 3y = 12$ and passes through the point $(3, 4)$. Write your final answer in slope-intercept form.

21. Determine the slope and y-intercept of the line shown. Then write the equation of the line in slope-intercept form and interpret the slope ratio $m = \frac{\Delta W}{\Delta R}$ in the context of this exercise.



22. For the graph given, (a) find the equation of the line in point-slope form, (b) use the equation to predict the x- and y-intercepts, (c) write the equation in slope-intercept form, and (d) find y when $x = 20$, and the value of x for which $y = 15$.



SECTION 2.4 Functions, Function Notation, and the Graph of a Function

KEY CONCEPTS

- A function is a relation, rule, or equation that pairs each element from the domain with exactly one element of the range.
- The vertical line test says that if every vertical line crosses the graph of a relation in at most one point, the relation is a function.
- On a graph, vertical boundary lines can be used to identify the domain, or the set of "allowable inputs" for a function.
- On a graph, horizontal boundary lines can be used to identify the range, or the set of y-values (outputs) generated by the function.
- When a function is stated as an equation, the implied domain is the set of x-values that yield real number outputs.
- x-values that cause a denominator of zero or that cause the radicand of a square root expression to be negative must be excluded from the domain.
- *The phrase "y is a function of x,"* is written as $y = f(x)$. This notation enables us to evaluate functions while tracking corresponding x- and y-values.

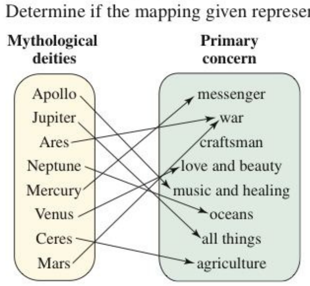
EXERCISES

23. State the implied domain of each function:

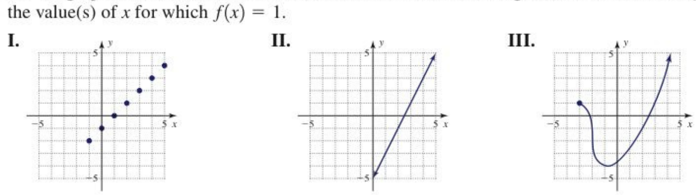
a. $f(x) = \sqrt{4x + 5}$ **b.** $g(x) = \dfrac{x - 4}{x^2 - x - 6}$

24. Determine $h(-2)$, $h(-\frac{2}{3})$, and $h(3a)$ for $h(x) = 2x^2 - 3x$.

25. Determine if the mapping given represents a function. If not, explain how the definition of a function is violated.



26. For the graph of each function shown, (a) state the domain and range, (b) find the value of $f(2)$, and (c) determine the value(s) of x for which $f(x) = 1$.



SECTION 2.5 Analyzing the Graph of a Function

KEY CONCEPTS

- A function f is even (symmetric to the y-axis), if and only if when a point (x, y) is on the graph, then $(-x, y)$ is also on the graph. In function notation: $f(-x) = f(x)$.
- A function f is odd (symmetric to the origin), if and only if when a point (x, y) is on the graph, then $(-x, -y)$ is also on the graph. In function notation: $f(-x) = -f(x)$.

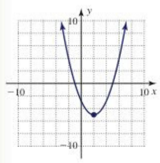
Intuitive descriptions of the characteristics of a graph are given here. The formal definitions can be found within Section 2.5.

- A function is *increasing* in an interval if the graph rises from left to right (larger inputs produce larger outputs).
- A function is *decreasing* in an interval if the graph falls from left to right (larger inputs produce smaller outputs).
- A function is *positive* in an interval if the graph is above the x-axis in that interval.
- A function is *negative* in an interval if the graph is below the x-axis in that interval.
- A function is *constant* in an interval if the graph is parallel to the x-axis in that interval.
- A maximum value can be a *local* maximum, or *global* maximum. An *endpoint* maximum can occur at the endpoints of the domain. Similar statements can be made for minimum values.
- For any function f, the average rate of change in the interval $[x_1, x_2]$ is $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.
- The difference quotient for a function $f(x)$ is $\dfrac{f(x + h) - f(x)}{h}$.

EXERCISES

State the domain and range for each function $f(x)$ given. Then state the intervals where f is increasing or decreasing and intervals where f is positive or negative. Assume all endpoints have integer values.

27.



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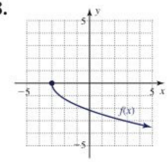

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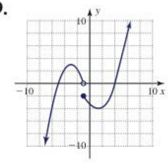


30. Determine which of the following are even $[f(-k) = f(k)]$, odd $[f(-k) = -f(k)]$, or neither.

a. $f(x) = 2x^5 - \sqrt[3]{x}$ **b.** $g(x) = x^4 - \dfrac{\sqrt[3]{x}}{x}$

c. $p(x) = |3x| - x^3$ **d.** $q(x) = \dfrac{x^2 - |x|}{x}$

31a. Given $f(x) = \sqrt{x + 4}$, find the average rate of change in the interval $[-3, 5]$. What does the result confirm about the graph of this toolbox function?

31b. Use the difference quotient to find a rate of change formula for the function given, then calculate the rate of change for the interval indicated: $j(x) = x^2 - x$; $[2.00, 2.01]$.

32. Draw the function f that has all of the following characteristics, then name the zeroes of the function and the location of all maximum and minimum values. [*Hint:* Write them in the form $(c, f(c))$.]

a. Domain: $x \in [-6, 10)$ **b.** Range: $y \in (-8, 6)$

c. $f(0) = 0$ **d.** $f(x)\downarrow$ for $x \in (-6, -3) \cup (3, 7.5)$

e. $f(x)\uparrow$ for $x \in (-3, 3) \cup (7.5, 10)$ **f.** $f(x) < 0$ for $x \in (-6, 0) \cup (6, 9)$

g. $f(x) > 0$ for $x \in (0, 6) \cup (9, 10)$

SECTION 2.6 The Toolbox Functions and Transformations

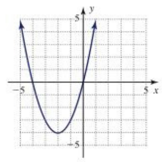
KEY CONCEPTS

- The *toolbox functions* and graphs commonly used in mathematics are
  - the identity function $f(x) = x$
  - squaring function: $f(x) = x^2$, parabola
  - square root function: $f(x) = \sqrt{x}$
  - absolute value function: $f(x) = |x|$
  - cubing function: $f(x) = x^3$
  - cube root function: $f(x) = \sqrt[3]{x}$
- For a basic or parent function $y = f(x)$, the general equation of the transformed function is $y = af(x \pm h) \pm k$. For any function $y = f(x)$ and $h, k > 0$,
  - the graph of $y = f(x) + k$ is the graph of $y = f(x)$ shifted upward k units
  - the graph of $y = f(x) - k$ is the graph of $y = f(x)$ shifted downward k units
  - the graph of $y = f(x + h)$ is the graph of $y = f(x)$ shifted left h units
  - the graph of $y = f(x - h)$ is the graph of $y = f(x)$ shifted right h units
  - the graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected across the x-axis
  - the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis
  - $y = af(x)$ results in a vertical stretch when $a > 1$
  - $y = af(x)$ results in a vertical compression when $0 < a < 1$
- Transformations are applied in the following order: (1) horizontal shifts, (2) reflections, (3) stretches or compressions, and (4) vertical shifts.

EXERCISES

Identify the function family for each graph given, then (a) describe the end behavior; (b) name the x- and y-intercepts; (c) identify the vertex, initial point, or point of inflection (as applicable); and (d) state the domain and range.

33.



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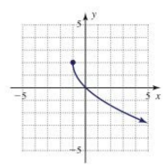

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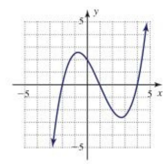

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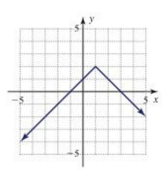

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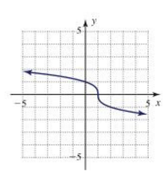


Identify each function as belonging to the linear, quadratic, square root, cubic, cube root, or absolute value family. Then sketch the graph using shifts of a parent function and a few characteristic points.

38. $f(x) = -(x + 2)^2 - 5$

39. $f(x) = 2|x + 3|$

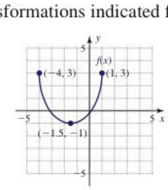
40. $f(x) = x^3 - 1$

41. $f(x) = \sqrt{x - 5} + 2$

42. $f(x) = \sqrt[3]{x + 2}$

43. Apply the transformations indicated for the graph of $f(x)$ given.

a. $f(x - 2)$
b. $-f(x) + 4$
c. $\frac{1}{2}f(x)$



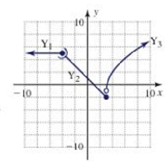
SECTION 2.7 Piecewise-Defined Functions

KEY CONCEPTS

- Each piece of a piecewise-defined function has a domain over which that piece is defined.
- To evaluate a piecewise-defined function, identify the domain interval containg the input value, then use the piece of the function corresponding to this interval.
- To graph a piecewise-defined function you can plot points, or graph each piece in its entirety, then erase portions of the graph outside the domain indicated for each piece.
- If the graph of a function can be drawn without lifting your pencil from the paper, the function is continuous.
- A pointwise discontinuity is said to be removable because we can redefine the function to "fill the hole."
- Step functions are discontinuous and formed by a series of horizontal steps.
- The floor function $\lfloor x \rfloor$ gives the first integer less than or equal to x.
- The ceiling function $\lceil x \rceil$ is the first integer greater than or equal to x.

EXERCISES

44. For the graph and functions given, (a) use the correct notation to write the relation as a single piecewise-defined function, stating the effective domain for each piece by inspecting the graph; and (b) state the range of the function: $Y_1 = 5$, $Y_2 = -x + 1$, $Y_3 = 3\sqrt{x-3} - 1$.



45. Use a table of values as needed to graph $h(x)$, then state its domain and range. If the function has a pointwise discontinuity, state how the second piece could be redefined so that a continuous function results.

$$h(x) = \begin{cases} \dfrac{x^2 - 2x - 15}{x + 3} & x \neq -3 \\ -6 & x = -3 \end{cases}$$

46. Evaluate the piecewise-defined function $p(x)$: $p(-4)$, $p(-2)$, $p(2.5)$, $p(2.99)$, $p(3)$, and $p(3.5)$

$$p(x) = \begin{cases} -4 & x < -2 \\ -|x| - 2 & -2 \le x < 3 \\ 3\sqrt{x} - 9 & x \ge 3 \end{cases}$$

47. Sketch the graph of the function and state its domain and range. Use transformations of the toolbox functions where possible.

$$q(x) = \begin{cases} 2\sqrt{-x-3} - 4 & x \le -3 \\ -2|x| + 2 & -3 < x < 3 \\ 2\sqrt{x-3} - 4 & x \ge 3 \end{cases}$$

48. Many home improvement outlets now rent flatbed trucks in support of customers that purchase large items. The cost is \$20 per hour for the first 2 hr, \$30 for the next 2 hr, then \$40 for each hour afterward. Write this information as a piecewise-defined function, then sketch its graph. What is the total cost to rent this truck for 5 hr?

SECTION 2.8 The Algebra and Composition of Functions

KEY CONCEPTS

- The notation used to represent the basic operations on two functions is
  - $(f + g)(x) = f(x) + g(x)$
  - $(f - g)(x) = f(x) - g(x)$
  - $(f \cdot g)(x) = f(x) \cdot g(x)$
  - $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$; $g(x) \neq 0$
- The result of these operations is a new function $h(x)$. The domain of h is the intersection of domains for f and g, excluding values that make $g(x) = 0$ for $h(x) = \left(\dfrac{f}{g}\right)(x)$.

- The composition of two functions is written $(f \circ g)(x) = f[g(x)]$ (g is an input for f).
- The domain of $f \circ g$ is all x in the domain of g, such that $g(x)$ is in the domain of f.
- To evaluate $(f \circ g)(2)$, we find $(f \circ g)(x)$ then substitute $x = 2$. Alternatively, we can find $g(2) = k$, then find $f(k)$.
- A composite function $h(x) = (f \circ g)(x)$ can be "decomposed" into individual functions by identifying functions f and g such that $(f \circ g)(x) = h(x)$. The decomposition is not unique.

EXERCISES

For $f(x) = x^2 + 4x$ and $g(x) = 3x - 2$, find the following:

49. $(f + g)(a)$ **50.** $(f \cdot g)(3)$ **51.** the domain of $\left(\frac{f}{g}\right)(x)$

Given $p(x) = 4x - 3$, $q(x) = x^2 + 2x$, and $r(x) = \frac{x + 3}{4}$ find:

52. $(p \circ q)(x)$ **53.** $(q \circ p)(3)$ **54.** $(p \circ r)(x)$ and $(r \circ p)(x)$

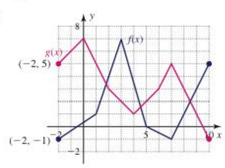
For each function here, find functions $f(x)$ and $g(x)$ such that $h(x) = f[g(x)]$:

55. $h(x) = \sqrt{3x - 2} + 1$ **56.** $h(x) = x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 10$

57. A stone is thrown into a pond causing a circular ripple to move outward from the point of entry. The radius of the circle is modeled by $r(t) = 2t + 3$, where t is the time in seconds. Find a function that will give the area of the circle directly as a function of time. In other words, find $A(t)$.

58. Use the graph given to find the value of each expression:

a. $(f + g)(-2)$
b. $(g \circ f)(5)$
c. $(g - f)(7)$
d. $\left(\frac{g}{f}\right)(10)$
e. $(f \cdot g)(3)$

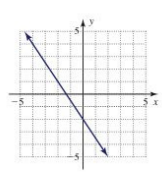


MIXED REVIEW

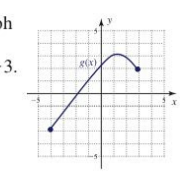
1. Write the given equation in slope-intercept form: $4x + 3y = 12$
2. Find the equation of the line perpendicular to $x - 2y = 8$ that passes through (1, 3).
3. Find the implied domain of:

   a. $f(x)\dfrac{x+1}{x^2 - 5x + 4}$ **b.** $g(x) = \dfrac{1}{\sqrt{2x - 3}}$
4. Given $p(x) = -x^2 + 3x - 1$, find

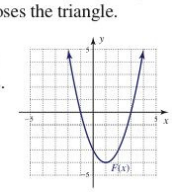
   a. $p\left(\dfrac{-1}{3}\right)$ **b.** $p(3a)$ **c.** $p(a - 1)$
5. State the equation of the line shown, in slope-intercept form.



6. For the function g whose graph is given, find (a) domain, (b) $g(2)$, and (c) k if $g(k) = -3$.

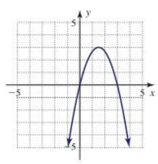


7. The following three points form a right triangle: $(-3, 7)$, $(2, 2)$ and $(5, 5)$. Use the distance formula to help determine which point is at the vertex of the right angle. Then find the equation of the smallest circle, centered at that point, that encloses the triangle.
8. Discuss the end behavior of $F(x)$ and name the vertex, axis of symmetry, and all intercepts.



9. Graph by plotting the y-intercept, then counting $m = \frac{\Delta y}{\Delta x}$ to find additional points:
$y = \frac{3}{5}x - 2$

10. Solve the inequality using the graph provided:
$f(x) = 4x - \frac{4}{3}x^2$; $f(x) < 0$.



11. a. Graph the function $p(x) = -2x^2 + 8x$. By observing the graph, is the average rate of change positive or negative in the interval $[-2, -1]$? Why? Do you expect the rate of change in [1, 2] to be greater or less than the rate of change in $[-2, -1]$? Calculate the average rate of change in each interval and comment.

b. If \$1000 is deposited in an account paying 7% interest compounded continuously, the function model is $A(t) = 1000e^{0.07t}$. Use the average rate of change formula to determine if the amount of interest added to the account exceeds \$200 per year $\left(\frac{\Delta A}{\Delta t} > 200\right)$ in the 10th, 15th, or 20th year. Use the intervals [10, 10.01], [15, 15.01], and [20, 20.01].

Given $f(x) = \frac{3}{x^2 - 1}$ and $g(x) = 3x - 2$, find

12. $\frac{g}{f}\left(\frac{1}{2}\right)$

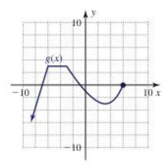
13. $(f \circ g)(x)$ and its domain

14. Sketch the function h as defined.
$$h(x) = \begin{cases} 5 & 0 \le x < 8 \\ x - 3 & 8 \le x \le 15 \\ -2x + 40 & x > 15 \end{cases}$$

15. Given $f(x) = x^2 + 1$ and $g(x) = 3x - 2$, calculate the difference quotient for each function and use the results to estimate the value of x for which their rates of change are equal.

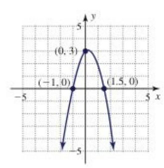
16. Identify the function family for the function $g(x) = -2|x + 3| + 4$. Then sketch the graph using transformations of a parent function and a few characteristic points.

17. For the graph shown, determine
a. the domain and range of g,
b. intervals where g is increasing, decreasing, or constant,
c. intervals where g is positive or negative,
d. any maximum or minimum values for g.



18. Draw a function f that has the following characteristics, then write the zeroes of the function and the location of all maximum and minimum values.
a. domain: $x \in [0, 30]$
b. range: $y \in [-10, 12]$
c. $f(2) = f(10) = 0$
d. $f(x)\downarrow$ from $x \in (0, 5) \cup (15, 20)$
e. $f(x)\uparrow$ for $x \in (5, 15)$
f. $f(x) < 0$ for $x \in (2, 10)$
g. $f(x) > 0$ for $x \in (0, 2) \cup (10, 30)$
h. $f(x) = 5$ for $x \in [20, 30]$

19. Find the equation of the function $f(x)$ whose graph is given.



20. Since 1975, the number of deaths in the United States due to heart disease has been declining at a rate that is close to linear. Find an equation model if there were 431 thousand deaths in 1975 and 257 thousand deaths in 2000 (let x represent years since 1975 and $f(x)$ deaths in thousands). How many deaths due to heart disease does the model predict for 2008?

Source: 2004 Statistical Abstract of the United States, Table 102

PRACTICE TEST

1. Two relations here are functions and two are not. Identify the nonfunctions (justify your response).
a. $x = y^2 + 2y$ **b.** $y = \sqrt{5 - 2x}$
c. $|y| + 1 = x$ **d.** $y = x^2 + 2x$

2. Determine if the lines are parallel, perpendicular, or neither:
L_1: $2x + 5y = -15$ and L_2: $y = \frac{2}{5}x + 7$.

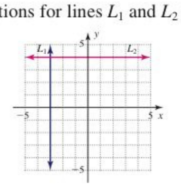
3. Graph the line using the slope and y-intercept: $x + 4y = 8$

4. Find the center and radius of the circle defined by $x^2 + y^2 - 4x + 6y - 3 = 0$, then sketch its graph.

5. Find the equation of the line parallel to $6x + 5y = 3$, containing the point $(2, -2)$. Answer in slope-intercept form.

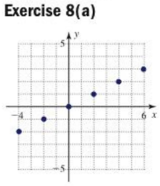
6. My partner and I are at coordinates $(-20, 15)$ on a map. If our destination is at coordinates $(35, -12)$, (a) what are the coordinates of the rest station located halfway to our destination? (b) How far away is our destination? Assume that each unit is 1 mi.

7. Write the equations for lines L_1 and L_2 shown.

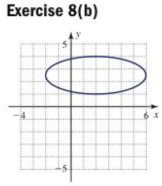


8. State the domain and range for the relations shown on graphs 8(a) and 8(b).

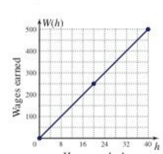
Exercise 8(a)



Exercise 8(b)

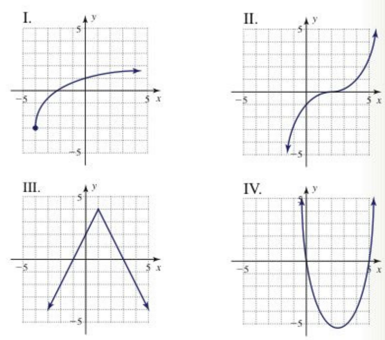


9. For the linear function shown,
a. Determine the value of $W(24)$ from the graph.
b. What input h will give an output of $W(h) = 375$?
c. Find a linear function for the graph.



d. What does the slope indicate in this context?
e. State the domain and range of h.

10. Each function graphed here is from a toolbox function family. For each graph, (a) identify the function family, (b) state the domain and range, (c) identify x- and y-intercepts, (d) discuss the end behavior, and (e) solve the inequality $f(x) > 0$, and (f) solve $f(x) < 0$.



11. Given $f(x) = \dfrac{2 - x^2}{x^2}$, evaluate and simplify:
(a) $f(\frac{2}{3})$ (b) $f(a + 3)$ (c) $f(1 + 2i)$

12. Given $f(x) = x^2 + 2$ and $g(x) = \sqrt{3x - 1}$, determine $(f \circ g)(x)$ and its domain.

13. Monthly sales volume for a successful new company is modeled by $S(t) = 2t^2 - 3t$, where $S(t)$ represents sales volume in thousands in month t ($t = 0$ corresponds to January 1). (a) Would you expect the average rate of change from May to June to be greater than that from June to July? Why? (b) Calculate the rates of change in these intervals to verify your answer. (c) Calculate the difference quotient for $S(t)$ and use it to estimate the sales volume rate of change after 10, 18, and 24 months.

Sketch each graph using a transformation.

14. $f(x) = |x - 2| + 3$

15. $g(x) = -(x + 3)^2 - 2$

16. A snowball increases in size as it rolls downhill. The snowball is roughly spherical with a radius that can be modeled by the function $r(t) = \sqrt{t}$, where t

is time in seconds and r is measured in inches. The volume of the snowball is given by the function $V(r) = \frac{4}{3}\pi r^3$. Use a composition to (a) write V directly as a function of t and (b) find the volume of the snowball after 9 sec.

17. Determine the following from the graph shown.

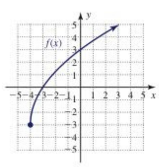
a. the domain and range

b. estimate the value of $f(-1)$

c. interval(s) where $f(x)$ is negative or positive

d. interval(s) where $f(x)$ is increasing, decreasing, or constant

e. an equation for $f(x)$



18. Given $h(x) = \begin{cases} 4 & x < -2 \\ 2x & -2 \le x \le 2 \\ x^2 & x > 2 \end{cases}$

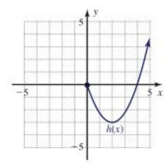
a. Find $h(-3)$, $h(-2)$, and $h(\frac{5}{2})$

b. Sketch the graph of h. Label important points.

For the function $h(x)$ whose partial graph is given,

19. complete the graph if h is known to be even.

20. complete the graph if h is known to be odd.



CALCULATOR EXPLORATION AND DISCOVERY

Using a Simple Program to Explore Transformations

In Section 2.6 we studied transformations of the toolbox functions. On page 231, an organized sequence for applying these transformations was given. Since the transformations are identical regardless of the function used, a simple program is an efficient way to explore these transformations further. As a good programming practice, clear all functions on the Y= screen, and preset the graphing window to ZOOM 6:ZStandard. Begin by pressing the PRGM key, then the ► key twice to name the new program. At the prompt, enter TRANSFRM. The specific functions we use for programming are all accessed in sub menus of the PRGM key. Recall that the relational operators (=, <, >, ≤, ≥) are accessed using 2nd MATH (TEST).

PROGRAM: TRANSFRM

```
:ClrHome
:FnOff 1,2,3,4,5,6,7,8,9
:Disp "FUNCTION FAMILY"
:Disp "1:SQUARING"
:Disp "2:SQUARE ROOT"
:Disp "3:ABSOLUTE VALUE"
:Disp "4:CUBING"
:Disp "5:CUBE ROOT"
:Input T
:If T=1:"X^2"→Y1
:If T=2:"√(X)"→Y1
:If T=3:"abs(X)"→Y1
:If T=4:"X^3"→Y1
:If T=5:"X^(1/3)"→Y1
:DispGraph:Pause
:ClrHome
:Disp "HORIZONTAL SHIFT"
:Disp "ENTER 0 IF NONE"
:Prompt H
:"Y1(X + H)"→Y2
:DispGraph
:FnOff 1
:DispGraph:Pause
:ClrHome
:Disp "STRETCH FACTOR A"
:Disp "(A>0)"
:Disp "ENTER 1 IF A=1"
:Input A
:"A*Y2"→Y3
:DispGraph
:FnOff 2
:DispGraph:Pause
```

```
:ClrHome
:Disp "REFLECTIONS?"
:Disp "0:NONE"
:Disp "1:ACROSS X-AXIS"
:Disp "2:ACROSS Y-AXIS"
:Disp "3:ACROSS BOTH"
:Input B
:If B=0:"Y3"→Y4
:If B=1:"-Y3"→Y4
:If B=2:"Y3(-X)"→Y4
:If B=3:"-Y3(-X)"→Y4
:DispGraph
:FnOff 3
:DispGraph:Pause
:ClrHome
:Disp "VERTICAL SHIFT"
:Disp "ENTER 0 IF NONE"
:Prompt V
:"Y4 + V"→Y5
:DispGraph
:FnOff 4
:DispGraph:Pause
:Stop
```

Enter the TRANSFRM program into your calculator. Note that as you are writing or editing a program:

1. The "FnOff" command is located at VARS Y–VARS 4:On/Off.
2. The "ClrHome" command is located at PRGM CTL 8.
3. The "Pause" command is located at PRGM I/O 8.

All other needed commands are visible as Options 1 through 7 on the CTL and I/O menus.

Exercise 1: Use the TRANSFRM program to apply the following transformations to $y = x^2$: (1) shift left 4 units, (2) stretch by a factor of 5, (3) reflect across the x-axis, (4) shift up 6 units. What is the equation of the final graph? Where is the vertex located?

Exercise 2: Use TRANSFRM to graph the function $y = -4\sqrt[3]{x - 2} + 3$. Where is the point of inflection? Estimate the y-intercept from the graph, then compare the estimate to the computed value.

CUMULATIVE REVIEW CHAPTERS 1–2

1. Perform the division by factoring the numerator: $(x^3 - 5x^2 + 2x - 10) \div (x - 5)$.

2. Find the solution set for: $2 - x < 5$ **and** $3x + 2 < 8$.

3. The area of a circle is 69 cm^2. Find the circumference of the same circle.

4. The surface area of a cylinder is $A = 2\pi r^2 + 2\pi rh$. Write r in terms of A and h (solve for r).

5. Solve for x: $-2(3 - x) + 5x = 4(x + 1) - 7$.

6. Evaluate without using a calculator: $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$.

7. Find the slope of each line:

a. through the points: $(-4, 7)$ and $(2, 5)$.

b. a line with equation $3x - 5y = 20$.

8. Graph using transformations of a parent function.

a. $f(x) = \sqrt{x - 2} + 3$.

b. $f(x) = -|x + 2| - 3$.

9. Graph the line passing through $(-3, 2)$ with a slope of $m = \frac{1}{2}$, then state its equation.

10. Show that $x = 1 + 5i$ is a solution to $x^2 - 2x + 26 = 0$.

11. Given $f(x) = 3x^2 - 6x$ and $g(x) = x - 2$ find: $(f \cdot g)(x)$, $(f \div g)(x)$, and $(g \circ f)(-2)$.

12. Graph by plotting the y-intercept, then counting $m = \frac{\Delta y}{\Delta x}$ to find additional points: $y = \frac{1}{3}x - 2$

13. Graph the piecewise defined function $f(x) = \begin{cases} x^2 - 4 & x < 2 \\ x - 1 & 2 \le x \le 8 \end{cases}$ and determine the following:

a. the domain and range

b. the value of $f(-3), f(-1), f(1), f(2)$, and $f(3)$

c. the zeroes of the function

d. interval(s) where $f(x)$ is negative/positive

e. location of any max/min values

f. interval(s) where $f(x)$ is increasing/decreasing

14. Given $f(x) = x^2$ and $g(x) = x^3$, use the formula for average rate of change to determine which of these functions is increasing faster in the intervals:
a. $[0.5, 0.6]$ **b.** $[1.5, 1.6]$.

15. Add the rational expressions:

a. $\frac{-2}{x^2 - 3x - 10} + \frac{1}{x + 2}$

b. $\frac{b^2}{4a^2} - \frac{c}{a}$

16. Simplify the radical expressions:

a. $\frac{-10 + \sqrt{72}}{4}$ **b.** $\frac{1}{\sqrt{2}}$

17. Determine which of the following statements are false, and state why.

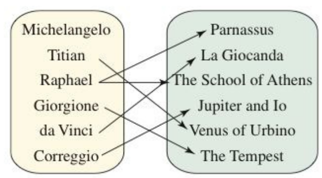
a. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{W} \subset \mathbb{Q} \subset \mathbb{R}$

b. $\mathbb{W} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

c. $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

d. $\mathbb{N} \subset \mathbb{R} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{W}$

18. Determine if the following relation is a function. If not, how is the definition of a function violated?



19. Solve by completing the square. Answer in both exact and approximate form: $2x^2 + 49 = -20x$

20. Solve using the quadratic formula. If solutions are complex, write them in $a + bi$ form.
$2x^2 + 20x = -51$

21. The *National Geographic Atlas of the World* is a very large, rectangular book with an almost inexhaustible panoply of information about the world we live in. The length of the front cover is 16 cm more than its width, and the area of the cover is 1457 cm^2. Use this information to write an equation model, then use the quadratic formula to determine the length and width of the Atlas.

22. Compute as indicated:

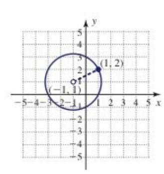
a. $(2 + 5i)^2$ **b.** $\frac{1 - 2i}{1 + 2i}$

23. Solve by factoring:

a. $6x^2 - 7x = 20$

b. $x^3 + 5x^2 - 15 = 3x$

24. A theorem from elementary geometry states, *"A line tangent to a circle is perpendicular to the radius at the point of tangency."* Find the equation of the tangent line for the circle and radius shown.



25. A triangle has its vertices at $(-4, 5)$, $(4, -1)$, and $(0, 8)$. Find the perimeter of the triangle and determine whether or not it is a *right* triangle.

Modeling With Technology I Linear and Quadratic Equation Models

Learning Objectives

In this section you will learn how to:

- **A.** Draw a scatter-plot and identify positive and negative associations
- **B.** Use a scatter-plot to identify linear and nonlinear associations
- **C.** Use linear regression to find the line of best fit
- **D.** Use quadratic regression to find the parabola of best fit

Collecting and analyzing data is a tremendously important mathematical endeavor, having applications throughout business, industry, science, and government. The link between classroom mathematics and real-world mathematics is called a **regression,** in which we attempt to find an equation that will act as a model for the raw data. In this section, we focus on linear and quadratic equation models.

A. Scatter-Plots and Positive/Negative Association

In this section, we continue our study of ordered pairs and functions, but this time using data collected from various sources or from observed real-world relationships. You can hardly pick up a newspaper or magazine without noticing it contains a large volume of data presented in graphs, charts, and tables. In addition, there are many simple experiments or activities that enable you to collect your own data. We begin analyzing the collected data using a **scatter-plot,** which is simply a graph of all of the ordered pairs in a data set. Often, real data (sometimes called **raw data**) is not very "well behaved" and the points may be somewhat scattered—the reason for the name.

Positive and Negative Associations

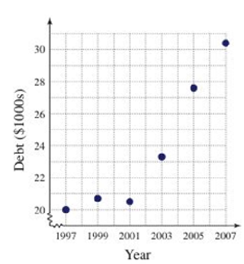
Earlier we noted that lines with positive slope rise from left to right, while lines with negative slope fall from left to right. We can extend this idea to the data from a scatter-plot. The data points in Example 1A seem to *rise* as you move from left to right, with larger input values generally resulting in larger outputs. In this case, we say there is a **positive association** between the variables. If the data seems to decrease or fall as you move left to right, we say there is a **negative association.**

EXAMPLE 1A ► **Drawing a Scatter-Plot and Observing Associations**

The ratio of the federal debt to the total population is known as the *per capita debt*. The per capita debt of the United States is shown in the table for the odd-numbered years from 1997 to 2007. Draw a scatter-plot of the data and state whether the association is positive or negative.

Source: Data from the Bureau of Public Debt at www.publicdebt.treas.gov

Year	Per Capita Debt ($1000s)
1997	20.0
1999	20.7
2001	20.5
2003	23.3
2005	27.6
2007	30.4

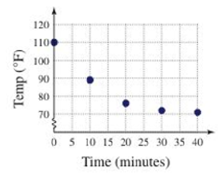


Solution ► Since the amount of debt depends on the year, *year* is the input x and *per capita debt* is the output y. Scale the x-axis from 1997 to 2007 and the y-axis from 20 to 30 to comfortably fit the data (the "squiggly lines," near the 20 and 1997 in the graph are used to show that some initial values have been skipped). The graph indicates a positive association between the variables, meaning the debt is generally *increasing* as time goes on.

EXAMPLE 1B ► **Drawing a Scatter-Plot and Observing Associations**

A cup of coffee is placed on a table and allowed to cool. The temperature of the coffee is measured every 10 min and the data are shown in the table. Draw the scatter-plot and state whether the association is positive or negative.

Elapsed Time (minutes)	Temperature (°F)
0	110
10	89
20	76
30	72
40	71



Solution ► Since temperature depends on cooling time, *time* is the input x and *temperature* is the output y. Scale the x-axis from 0 to 40 and the y-axis from 70 to 110 to comfortably fit the data. As you see in the figure, there is a negative association between the variables, meaning the temperature *decreases* over time.

A. You've just learned how to draw a scatter-plot and identify positive and negative associations

Now try Exercises 1 and 2 ►

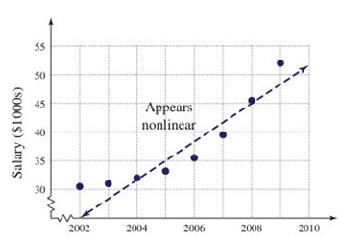
B. Scatter-Plots and Linear/Nonlinear Associations

The data in Example 1A had a positive association, while the association in Example 1B was negative. But the data from these examples differ in another important way. In Example 1A, the data seem to cluster about an imaginary line. This indicates a linear equation model might be a good approximation for the data, and we say there is a **linear association** between the variables. The data in Example 1B could not accurately be modeled using a straight line, and we say the variables *time* and *cooling temperature* exhibit a **nonlinear association.**

EXAMPLE 2 ► **Drawing a Scatter-Plot and Observing Associations**

A college professor tracked her annual salary for 2002 to 2009 and the data are shown in the table. Draw the scatter-plot and determine if there is a linear or nonlinear association between the variables. Also state whether the association is positive, negative, or cannot be determined.

Year	Salary ($1000s)
2002	30.5
2003	31
2004	32
2005	33.2
2006	35.5
2007	39.5
2008	45.5
2009	52



Solution ► Since salary earned depends on a given year, *year* is the input *x* and *salary* is the output *y*. Scale the *x*-axis from 1996 to 2005, and the *y*-axis from 30 to 55 to comfortably fit the data. A line doesn't seem to model the data very well, and the association appears to be nonlinear. The data rises from left to right, indicating a positive association between the variables. This makes good sense, since we expect our salaries to increase over time.

B. You've just learned how to use a scatter-plot to identify linear and nonlinear associations

Now try Exercises 3 and 4 ►

C. Linear Regression and the Line of Best Fit

There is actually a sophisticated method for calculating the equation of a line that best fits a data set, called the **regression line.** The method minimizes the vertical distance between all data points and the line itself, making it the unique **line of best fit.** Most graphing calculators have the ability to perform this calculation quickly. The process involves these steps: (1) clearing old data, (2) entering new data; (3) displaying the data; (4) calculating the regression line; and (5) displaying and using the regression line. We'll illustrate by finding the regression line for the data shown in Table MWT 1.1, which gives the men's 400-m freestyle gold medal times (in seconds) for the 1980 through the 2004 Olympics, with 1980→0.

Table MWT 1.1

Year (x) (1980→0)	Time (y) (sec)
0	231
4	231
8	227
12	225
16	228
20	221
24	223

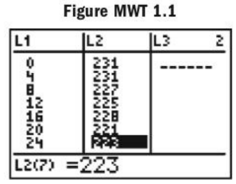
Step 1: Clear Old Data

To prepare for the new data, we first clear out any old data. Press the STAT key and select option **4:ClrList.** This places the **ClrList** command on the home screen. We tell the calculator which lists to clear by pressing 2nd 1 to indicate List1 (L1), then enter a comma using the , key, and continue entering other lists we want to clear: 2nd 2 , 2nd 3 ENTER will clear List1 (L1), List2 (L2), and List3 (L3).

Step 2: Enter New Data

Press the STAT key and select option **1:Edit.** Move the cursor to the first position of List1, and simply enter the data from the first column of Table MWT 1.1 in order: 0 ENTER 4 ENTER 8 ENTER, and so on. Then use the right arrow ► to navigate to List2, and enter the data from the second column: 231 ENTER 231 ENTER 227 ENTER, and so on. When finished, you should obtain the screen shown in Figure MWT 1.1.

Figure MWT 1.1



Step 3: Display the Data

With the data held in these lists, we can now display the related ordered pairs on the coordinate grid. First press the Y= key and CLEAR any existing equations. Then press 2nd Y= to access the "**STATPLOTS**" screen. With the cursor on **1:Plot1,** press ENTER and be sure the options shown in Figure MWT 1.2 are highlighted. If you need to make any changes, navigate the cursor to the desired option and press ENTER. Note the data in L1 ranges from 0 to 24, while the data in L2 ranges from 221 to 231. This means an appropriate viewing window might be [0, 30] for the *x*-values, and [200, 250] for the *y*-values. Press the WINDOW key and set up the window accordingly. After you're finished, pressing the GRAPH key should produce the graph shown in Figure MWT 1.3.

Figure MWT 1.2

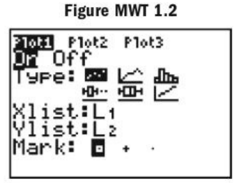
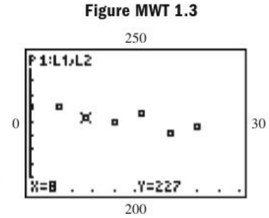


Figure MWT 1.3



WORTHY OF NOTE

As a rule of thumb, the tic marks for Xscl can be set by mentally estimating $\frac{|Xmax| + |Xmin|}{10}$ and using a convenient number in the neighborhood of the result (the same goes for Yscl). As an alternative to manually setting the window, the ZOOM **9:ZoomStat** feature can be used.

WORTHY OF NOTE

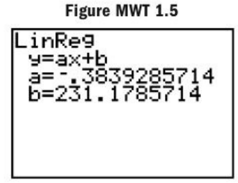
If the input variable is a unit of time, particularly the time in years, we often **scale the data** to avoid working with large numbers. For instance, if the data involved the cost of attending a major sporting event for the years 1980 to 2000, we would say 1980 corresponds to 0 and use input values of 0 to 20 (subtracting the smallest value from itself and all other values has the effect of scaling down the data). This is easily done on a graphing calculator. Simply enter the four-digit years in L1, then with the cursor in the header of L1—use the keystrokes 2nd 1 (L1) − 1980 ENTER and the data in this list automatically adjusts.

Step 4: Calculate the Regression Equation

To have the calculator compute the regression equation, press the STAT and ► keys to move the cursor over to the **CALC** options (see Figure MWT 1.4). Since it appears the data is best modeled by a linear equation, we choose option **4:LinReg(ax + b).** Pressing the number 4 places this option on the home screen, and pressing ENTER computes the values of a and b (the calculator automatically uses the values in L1 and L2 unless instructed otherwise). Rounded to hundredths, the linear regression model is $y = -0.38x + 231.18$ (Figure MWT 1.5).

Figure MWT 1.4

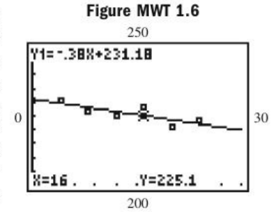
Figure MWT 1.5



Step 5: Display and Use the Results

Although graphing calculators have the ability to paste the regression equation directly into Y_1 on the Y= screen, for now we'll enter $Y_1 = -0.38x + 231.18$ by hand. Afterward, pressing the GRAPH key will plot the data points (if Plot1 is still active) and graph the line. Your display screen should now look like the one in Figure MWT 1.6. The regression line is the best estimator for the set of data as a whole, but there will still be some difference between the values it generates and the values from the set of raw data (the output in Figure MWT 1.6 shows the estimated time for the 1996 Olympics is 225.1 sec, while the actual time was 228 sec).

Figure MWT 1.6



EXAMPLE 3 ► **Using Regression to Model Employee Performance**

Riverside Electronics reviews employee performance semiannually, and awards increases in their hourly rate of pay based on the review. The table shows Thomas' hourly wage for the last 4 yr (eight reviews). Find the regression equation for the data and use it to project his hourly wage for the year 2011, after his fourteenth review.

Year (x)	Wage (y)
(2004) 1	$9.58
2	$9.75
(2005) 3	$10.54
4	$11.41
(2006) 5	$11.60
6	$11.91
(2007) 7	$12.11
8	$13.02

Solution ► Following the prescribed sequence produces the equation $y = 0.48x + 9.09$. For $x = 14$ we obtain $y = 0.48(14) + 9.09$ or a wage of $15.81. According to this model, Thomas will be earning $15.81 per hour in 2011.

☑ **C. You've just learned how to use a linear regression to find the line of best fit**

Now try Exercises 9 through 14 ►

D. Quadratic Regression and the Parabola of Best Fit

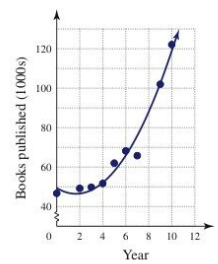
Once the data have been entered, graphing calculators have the ability to find many different regression equations. The choice of regression depends on the context of the data, patterns formed by the scatter-plot, and/or some foreknowledge of how the data are related. Earlier we focused on linear regression equations. We now turn our attention to quadratic regression equations.

EXAMPLE 4A ► Drawing a Scatter-Plot to Sketch a Best-Fit Curve

Since 1990, the number of *new* books published each year has been growing at a rate that can be approximated by a quadratic function. The table shows the number of books published in the United States for selected years. Draw a scatter-plot and sketch an estimated parabola of best fit by hand.

Source: 1998, 2000, 2002, and 2004 *Statistical Abstract of the United States.*

Year (1990→0)	Books Published (1000s)
0	46.7
2	49.2
3	49.8
4	51.7
5	62.0
6	68.2
7	65.8
9	102.0
10	122.1



Solution ► Begin by drawing the scatter-plot, being sure to scale the axes appropriately. The data appear to form a quadratic pattern, and we sketch a parabola that seems to best fit the data (see graph).

The regression abilities of a graphing calculator can be used to find a **parabola of best fit** and the steps are identical to those for linear regression.

EXAMPLE 4B ► Calculating a Nonlinear Regression Model from a Data Set

Use the data from Example 4A to calculate a quadratic regression equation, then display the data and graph. How well does the equation match the data?

Solution ► Begain by entering the data in L1 and L2 as shown in Figure MWT 1.7. Press [2nd] [Y=] to be sure that Plot 1 is still active and is using L1 and L2 with the desired point type. Set the window size to comfortably fit the data (see Figure MWT 1.9—window size is indicated along the perimeter). Finally, press [STAT] and the right arrow [►] to overlay the **CALC** option. The quadratic regression option is number **5:QuadReg.** Pressing [5] places this option directly on the home screen. Lists L1 and L2 are the default lists, so pressing [ENTER] will have the calculator compute the regression equation for the data in L1 and L2. After "chewing on the data" for a short while, the calculator returns the regression equation in the form shown in Figure MWT 1.8. To maintain a higher degree of accuracy, we can actually paste the entire regression equation in Y_1. Recall the last

Figure MWT 1.7

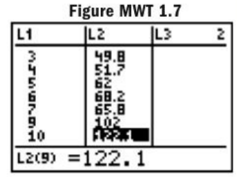
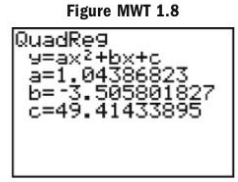


Figure MWT 1.8

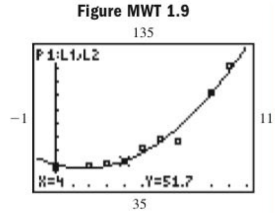


WORTHY OF NOTE

The TI-84 Plus can round all coefficients to any desired number of decimal places. For three decimal places, press [MODE] and change the **Float** setting to "3." Also, be aware that there are additional methods for pasting the equation in Y1.

operation using 2nd ENTER, and **QuadReg** should (re)appear. Then enter the function Y_1 after the QuadReg option by pressing VARS ► (**Y-Vars**) and ENTER (**1:Function**) and ENTER (Y_1). After pressing ENTER once again, the full equation is automatically pasted in Y_1. To compare this equation model with the data, simply press GRAPH and both the graph and plotted data will appear. The graph and data seem to match very well (Figure MWT 1.9).

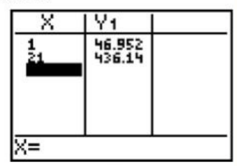
Figure MWT 1.9



EXAMPLE 4C ► Using a Regression Model to Predict Trends

Use the equation from Example 4B to answer the following questions: According to the function model, how many new books were published in 1991? If this trend continues, how many new books will be published in 2011?

Solution ► Since the year 1990 corresponds to 0 in this data set, we use an input value of 1 for 1991, and an input of 21 for 2011. Accessing the table (2nd GRAPH) feature and inputting 1 and 21 gives the screen shown. Approximately 47,000 new books were published in 1991, and about 436,000 will be published in the year 2011.



D. You just learned how to use quadratic regression to find the parabola of best fit

Now try Exercises 15 through 18 ►

MODELING WITH TECHNOLOGY EXERCISES

► DEVELOPING YOUR SKILLS

1. For mail with a high priority, "Express Mail" offers next day delivery by 12:00 noon to most destinations, 365 days of the year. The service was first offered by the U.S. Postal Service in the early 1980s and has been growing in use ever since. The cost of the service (in cents) for selected years is shown in the table. Draw a scatter-plot of the data, then decide if the association is positive, negative, or cannot be determined.

x	*y*
1981	935
1985	1075
1988	1200
1991	1395
1995	1500
1999	1575
2002	1785

Source: 2004 *Statistical Abstract of the United States*

2. After the Surgeon General's first warning in 1964, cigarette consumption began a steady decline as advertising was banned from television and radio, and public awareness of the dangers of cigarette smoking grew. The percentage of the U.S. adult population who considered themselves smokers is shown in the table for selected years. Draw a scatter-plot of the data, then decide if the association is positive, negative, or cannot be determined.

x	*y*
1965	42.4
1974	37.1
1979	33.5
1985	29.9
1990	25.3
1995	24.6
2000	23.1
2002	22.4

Source: 1998 *Wall Street Journal Almanac* and 2004 *Statistical Abstract of the United States*, Table 188

3. Since the 1970s women have made tremendous gains in the political arena, with more and more female candidates running and winning seats in the U.S. Senate and U.S. Congress. The number of women candidates for the U.S. Congress is shown in the table for selected years. Draw a scatter-plot of the data and then decide (a) if the association is linear or nonlinear and (b) if the association is positive or negative.

x	y
1972	32
1978	46
1984	65
1992	106
1998	121
2004	141

Source: Center for American Women and Politics at www.cawp.rutgers.edu/Facts3.html

4. The number of shares traded on the New York Stock Exchange experienced dramatic change in the 1990s as more and more individual investors gained access to the stock market via the Internet and online brokerage houses. The volume is shown in the table for 2002, and the odd numbered years from 1991 to 2001 (in billions of shares). Draw a scatter-plot of the data then decide (a) if the association is linear or nonlinear; and (b) if the association is positive, negative, or cannot be determined.

x	y
1991	46
1993	67
1995	88
1997	134
1999	206
2001	311
2002	369

Source: 2000 and 2004 *Statistical Abstract of the United States,* Table 1202

The data sets in Exercises 5 and 6 are known to be linear.

5. The total value of the goods and services produced by a nation is called its gross domestic product or GDP. The *GDP per capita* is the ratio of the GDP for a given year to the population that year, and is one of many indicators of economic health. The GDP per capita (in $1000s) for the United States is shown in the table for selected years. (a) Draw a scatter-plot using a scale that appropriately fits the data; (b) sketch an estimated line of best fit and decide if the association is positive or negative, then (c) approximate the slope of the line.

x $1970 \to 0$	y
0	5.1
5	7.6
10	12.3
15	17.7
20	23.3
25	27.7
30	35.0
33	37.8

Source: 2004 *Statistical Abstract of the United States,* Tables 2 and 641

6. Real estate brokers carefully track sales of new homes looking for trends in location, price, size, and other factors. The table relates the average selling price within a price range (homes in the $120,000 to $140,000 range are represented by the $130,000 figure), to the number of new homes sold by Homestead Realty in 2004. (a) Draw a scatter-plot using a scale that appropriately fits the data; (b) sketch an estimated line of best fit and decide if the association is positive or negative, then (c) approximate the slope of the line.

Price	Sales
130's	126
150's	95
170's	103
190's	75
210's	44
230's	59
250's	21

7. In most areas of the country, law enforcement has become a major concern. The number of law enforcement officers employed by the federal government and having the authority to carry firearms and make arrests is shown in the table for selected years. (a) Draw a scatter-plot using a scale that appropriately fits the data. (b) Sketch an estimated line of best fit and decide if the association is positive or negative. (c) Choose two points on or near the estimated line of best fit, and use them to find an equation model and predict the number of federal law enforcement officers in 1995 and the projected number for 2011. Answers may vary.

x $(1990 \to 0)$	y (1000s)
3	68.8
6	74.5
8	83.1
10	88.5
14	93.4

Source: U.S. Bureau of Justice, Statistics at www.ojp.usdoj.gov/bjs/fedle.htm

8. Due to atmospheric pressure, the temperature at which water will boil varies predictably with the altitude. Using special equipment designed to duplicate atmospheric pressure, a lab experiment is set up to study this relationship for altitudes up to 8000 ft. The set of data collected is shown to the right, with the boiling temperature y in degrees Fahrenheit, depending on the altitude x in feet. (a) Draw a scatter-plot using a scale that appropriately fits the data. (b) Sketch an estimated line of best fit and decide if the association is

x	y
−1000	213.8
0	212.0
1000	210.2
2000	208.4
3000	206.5
4000	204.7
5000	202.9
6000	201.0
7000	199.2
8000	197.4

positive or negative. (c) Choose two points on or near the estimated line of best fit, and use them to find an equation model and predict the boiling point of water on the summit of Mt. Hood in Washington State (11,239 ft height), and along the shore of the Dead Sea (approximately 1312 ft below sea level). Answers may vary.

▶ APPLICATIONS

Use the regression capabilities of a graphing calculator to complete Exercises 9 through 18.

9. Height versus wingspan: Leonardo da Vinci's famous diagram is an illustration of how the human body comes in predictable proportions. One such comparison is a person's wingspan to their height. Careful measurements were taken on eight students and the set of data is shown here. Using the data, (a) draw the scatter-plot; (b) determine whether the association is linear or nonlinear; (c) determine whether the association is positive or negative; and (d) find the regression equation and use it to predict the height of a student with a wingspan of 65 in.

Height (x)	Wingspan (y)
61	60.5
61.5	62.5
54.5	54.5
73	71.5
67.5	66
51	50.75
57.5	54
52	51.5

10. Patent applications: Every year the United States Patent and Trademark Office (USPTO) receives thousands of applications from scientists and inventors. The table given shows the number of appplications received for the odd years from 1993 to 2003 (1990 → 0). Use the data to (a) draw the scatter-plot; (b) determine whether the association is linear or nonlinear; (c) determine whether the association is positive or negative; and (d) find the regression equation and use it to predict the number of applications that will be received in 2011.

Source: United States Patent and Trademark Office at www.uspto.gov/web

Year (1990→0)	Applications (1000s)
3	188.0
5	236.7
7	237.0
9	278.3
11	344.7
13	355.4

11. Patents issued: An increase in the number of patent applications (see Exercise 10), typically brings an increase in the number of patents issued, though many applications are denied due to improper filing, lack of scientific support, and other reasons. The table given shows the number of patents issued for the odd years from 1993 to 2003 (1999 → 0). Use the data to (a) draw the scatter-plot; (b) determine whether the association is linear or nonlinear; (c) determine whether the association is positive or negative; and (d) find the regression equation and use it to predict the number of applications that will be approved in 2011. Which is increasing faster, the number of patent applications or the number of patents issued? How can you tell for sure?

Source: United States Patent and Trademark Office at www.uspto.gov/web

Year (1990→0)	Patents (1000s)
3	107.3
5	114.2
7	122.9
9	159.2
11	187.8
13	189.6

12. High jump records: In the sport of track and field, the high jumper is an unusual athlete. They seem to defy gravity as they launch their bodies over the high bar. The winning height at the summer Olympics (to the nearest unit) has steadily increased over time, as shown in the table for selected years. Using the data, (a) draw the scatter-plot, (b) determine whether the association is linear or nonlinear, (c) determine whether the association is positive or negative, and (d) find the regression equation using $t = 0$ corresponding to 1900 and predict the winning height for the 2004 and 2008 Olympics. How close did the model come to the actual heights?

Source: athens2004.com

Year (x)	Height (in.) (y)
0	75
12	76
24	78
36	80
56	84
68	88
80	93
88	94
92	92
96	94
100	93

13. Females/males in the workforce: Over the last 4 decades, the percentage of the female population in the workforce has been increasing at a fairly steady rate. At the same time, the percentage of the male population in the workforce has been declining. The set of data is shown in the tables. Using the data, (a) draw scatter-plots for both data sets, (b) determine whether the associations are linear or nonlinear, (c) determine whether the associations are positive or negative, and (d) determine if the percentage of females in the workforce is increasing faster than the percentage of males is decreasing. Discuss/Explain how you can tell for sure.

Source: 1998 *Wall Street Journal Almanac,* p. 316

Exercise 13 (women)

Year (x) (1950→0)	Percent
5	36
10	38
15	39
20	43
25	46
30	52
35	55
40	58
45	59
50	60

Exercise 13 (men)

Year (x) (1950→0)	Percent
5	85
10	83
15	81
20	80
25	78
30	77
35	76
40	76
45	75
50	73

14. Height versus male shoe size: While it seems reasonable that taller people should have larger feet, there is actually a wide variation in the relationship between height and shoe size. The data in the table show the height (in inches) compared to the shoe size worn for a random sample of 12 male chemistry students. Using the data, (a) draw the scatter-plot, (b) determine whether the association is linear or nonlinear, (c) determine whether the association is positive or negative, and (d) find the regression equation and use it to predict the shoe size of a man 80 in. tall and another that is 60 in. tall.

Height	Shoe Size
66	8
69	10
72	9
75	14
74	12
73	10.5
71	10
69.5	11.5
66.5	8.5
73	11
75	14
65.5	9

15. Plastic money: The total amount of business transacted using credit cards has been changing rapidly over the last 15 to 20 years. The total volume (in billions of dollars) is shown in the table for selected years. (a) Use a graphing calculator to draw a scatter-plot of the data and decide on an appropriate form of regression. (b) Calculate a regression equation with $x = 0$ corresponding to 1990 and display the scatter-plot and graph on the same screen. (c) According to the equation model, how many billions of dollars were transacted in 2003? How much will be transacted in the year 2011?

Source: Statistical Abstract of the United States, various years

x (1990→0)	y
1	481
2	539
4	731
7	1080
8	1157
9	1291
10	1458
12	1638

16. Homeschool education: Since the early 1980s the number of parents electing to homeschool their children has been steadily increasing. Estimates for the number of children homeschooled (in 1000s) are given in the table for selected years.

x (1985→0)	y
0	183
3	225
5	301
7	470
8	588
9	735
10	800
11	920
12	1100

(a) Use a graphing calculator to draw a scatter-plot of the data and decide on an appropriate form of regression. (b) Calculate a regression equation with $x = 0$ corresponding to 1985 and display the scatter-plot and graph on the same screen. (c) According to the equation model, how many children were homeschooled in 1991? If growth continues at the same rate, how many children will be homeschooled in 2010?

Source: National Home Education Research Institute

▶ EXTENDING THE CONCEPT

17. The height of a projectile: $h(t) = -\frac{1}{2}gt^2 + vt$

The height of a projectile thrown upward from ground level depends primarily on two things—the object's initial velocity and the acceleration due to gravity. This is modeled by the formula shown, where $h(t)$ represents the height of the object at time t, v represents the initial velocity, and g represents the acceleration due to gravity. Suppose an astronaut on one of the inner planets threw a surface rock upward and used hand-held radar to collect the data shown. Given that on Mercury $g = 12$ ft/sec^2, Venus $g = 29$ ft/sec^2, and Earth $g = 32$ ft/sec^2, (a) use your calculator to find an appropriate regression model for the data, (b) use the model to determine the initial velocity of the object, and (c) name the planet on which the astronaut is standing.

Time	Height
1	75.5
2	122
3	139.5
4	128
5	87.5
6	18

18. In his book *Gulliver's Travels,* Jonathan Swift describes how the Lilliputians were able to measure Gulliver for new clothes, even though he was a giant compared to them. According to the text, "Then they measured my right thumb, and desired no more . . . for by mathematical computation, once around the thumb is twice around the wrist, and so on to the neck and waist." Is it true that once around the neck is twice around the waist? Find at least 10 willing subjects and take measurements of their necks and waists in millimeters. Arrange the data in ordered pair form (circumference of neck, circumference of waist). Draw the scatter-plot for this data. Does the association appear to be linear? Find the equation of the best fit line for this set of data. What is the slope of this line? Is the slope near $m = 2$?