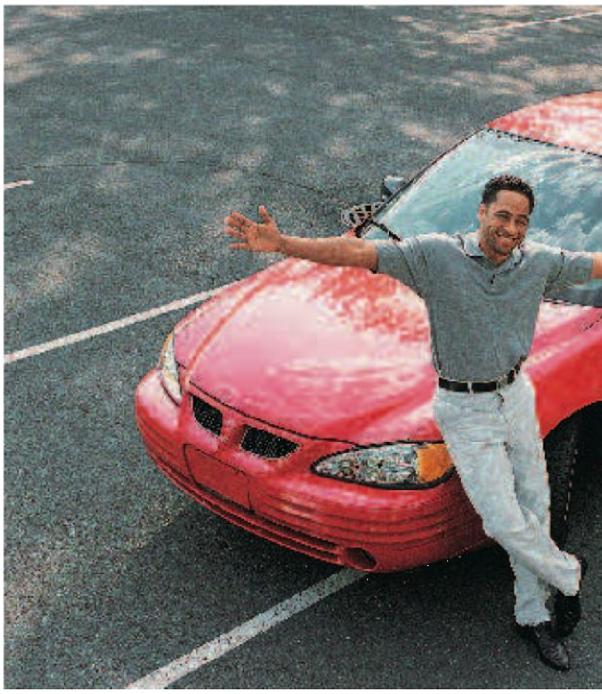

4

Exponential and Logarithmic Functions

CHAPTER OUTLINE

4.1 One-to-One and Inverse Functions 412
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4.4 Properties of Logarithms; Solving Exponential/Logarithmic Equations 451
4.5 Applications from Business, Finance, and Science 467

CHAPTER CONNECTIONS

The largest purchase that most individuals will make in their lifetime is that of a car or home. The monthly payment $\mathcal{P}$ required to amortize (pay off) the loan can be calculated using the formula

$$\mathcal{P} = \frac{AR}{1 - (1 - R)^{-12t}}$$

where A is the amount financed; t is the time in years; and $R = \frac{r}{12}$, where r is the annual rate of interest. This study of exponential and logarithmic functions will help you become a more knowledgeable consumer. This application appears as Exercise 53 in Section 4.5.

Check out these other real-world connections:

- Calculating the Effects of Inflation (Section 4.2, Exercises 87 and 88)
- Calculating the Intensity of Sound (Section 4.3, Exercises 87 to 90)
- Calculating the Proper Ventilation of a Home (Section 4.3, Exercise 97)
- Calculating Freezing Time for Water Puddles (Section 4.4, Exercise 120)

4.1 One-to-One and Inverse Functions

Learning Objectives

In Section 4.1 you will learn how to:

- ☐ **A.** Identify one-to-one functions
- ☐ **B.** Explore inverse functions using ordered pairs
- ☐ **C.** Find inverse functions using an algebraic method
- ☐ **D.** Graph a function and its inverse
- ☐ **E.** Solve applications of inverse functions

Consider the function $f(x) = 2x - 3$. If $f(x) = 7$, the equation becomes $2x - 3 = 7$, and the corresponding value of x can be found using *inverse operations*. In this section, we introduce the concept of an *inverse function,* which can be viewed as a formula for finding x-values that correspond to *any* given value of $f(x)$.

A. Identifying One-to-One Functions

The graphs of $y = 2x$ and $y = x^2$ are shown in Figures 4.1 and 4.2. The dashed, vertical lines clearly indicate both are functions, with each x-value corresponding to only one y. But the points on $y = 2x$ have one characteristic those from $y = x^2$ do not—*each y-value also corresponds to only one x* (for $y = x^2$, 4 corresponds to both -2 and 2). If each element from the range of a function corresponds to only one element of the domain, the function is said to be **one-to-one.**

Figure 4.1

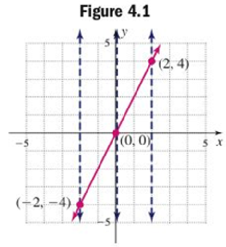
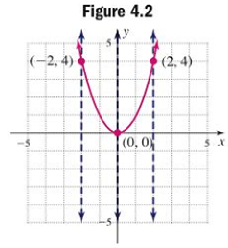


Figure 4.2



One-to-One Functions

A function f is one-to-one if every element in the range corresponds to only one element of the domain.

In symbols, if $f(x_1) = f(x_2)$ then $x_1 = x_2$, or
if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

From this definition we note the graph of a one-to-one function must not only pass a vertical line test (to show each x corresponds to only one y), but also pass a **horizontal line test** (to show each y corresponds to only one x).

Horizontal Line Test

If every horizontal line intersects the graph of a function in at most one point, the function is one-to-one.

Notice the graph of $y = 2x$ (Figure 4.3) passes the horizontal line test, while the graph of $y = x^2$ (Figure 4.4) does not.

Figure 4.3

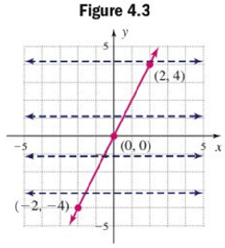
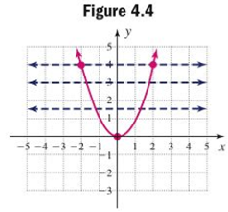
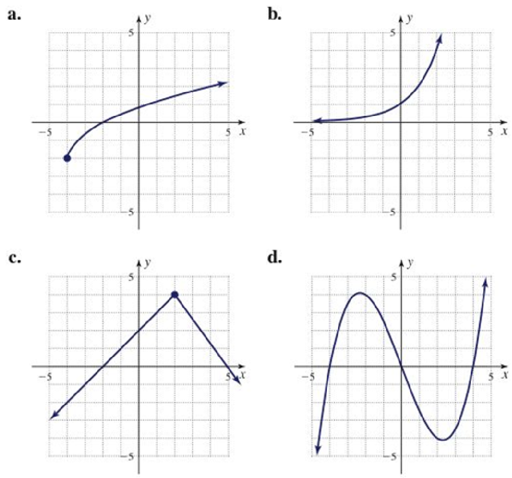


Figure 4.4



EXAMPLE 1 ► **Identifying One-to-One Functions**

Use the horizontal line test to determine whether each graph is the graph of a one-to-one function.



Solution ► A careful inspection shows all four graphs depict a function, since each passes the vertical line test. Only (a) and (b) pass the horizontal line test and are *one-to-one* functions.

Now try Exercises 7 through 28 ►

A. You've just learned how to identify a one-to-one function

If the function is given in ordered pair form, we simply check to see that no given second coordinate is paired with more than one first coordinate.

B. Inverse Functions and Ordered Pairs

Consider the function $f(x) = 2x - 3$ and the solutions shown in Table 4.1. Figure 4.5 shows this function in diagram form (in blue), and illustrates that for each element of the domain, we *multiply by 2, then subtract 3*. An **inverse function** for f is one that takes the result of these operations (elements of the range), and returns the original domain element. Figure 4.6 shows that function F achieves this by "undoing" the operations in reverse order: *add 3, then divide by 2* (in red). A table of values for $F(x)$ is shown (Table 4.2).

Table 4.1

x	$f(x)$
−3	−9
0	−3
2	1
5	7
8	13

Table 4.2

x	$F(x)$
−9	−3
−3	0
1	2
7	5
13	8

Figure 4.5

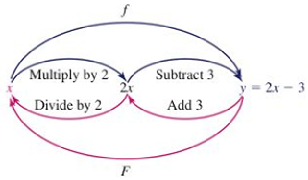
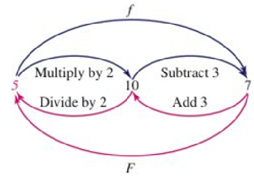


Figure 4.6



From this illustration we make the following observations regarding an inverse function, which we actually denote as $f^{-1}(x)$.

Inverse Functions

If f is a one-to-one function with ordered pairs (a, b),

1. $f^{-1}(x)$ is a one-to-one function with ordered pairs (b, a).
2. The range of f will be the domain of $f^{-1}(x)$.
3. The domain of f will be the range of $f^{-1}(x)$.

CAUTION ► The notation $f^{-1}(x)$ is simply a way of denoting an inverse function and has nothing to do with exponential properties. In particular, $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$.

EXAMPLE 2 ► **Finding the Inverse of a Function**

Find the inverse of each one-to-one function given:

a. $f(x) = \{(-4, 13), (-1, 7), (0, 5), (2, 1), (5, -5), (8, -11)\}$
b. $p(x) = -3x + 2$

Solution ► **a.** When a function is defined as a set of ordered pairs, the inverse function is found by simply interchanging the x- and y-coordinates: $f^{-1}(x) = \{(13, -4), (7, -1), (5, 0), (1, 2), (-5, 5), (-11, 8)\}$.

b. Using diagrams similar to Figures 4.5 and 4.6, we reason that $p^{-1}(x)$ will subtract 2, then divide the result by -3: $p^{-1}(x) = \frac{x - 2}{-3}$. As a test, we find that $(-2, 8)$, $(0, 2)$, and $(3, -7)$ are solutions to $p(x)$, and note that $(8, -2)$, $(2, 0)$, and $(-7, 3)$ are indeed solutions to $p^{-1}(x)$.

B. You've just learned how to explore inverse functions using ordered pairs

Now try Exercises 29 through 40 ►

C. Finding Inverse Functions Using an Algebraic Method

The fact that interchanging x- and y-values helps determine an inverse function can be generalized to develop an **algebraic method** for finding inverses. Instead of interchanging *specific* x- and y-values, we actually interchange the x- and y-*variables,* then solve the equation for y. The process is summarized here.

WORTHY OF NOTE

If a function is *not* one-to-one, no inverse function exists since interchanging the x- and y-coordinates will result in a nonfunction. For instance, interchanging the coordinates of $(-2, 4)$ and $(2, 4)$ from $y = x^2$ results in $(4, -2)$ and $(4, 2)$, and we have one x-value being mapped to two y-values, in violation of the function definition.

Finding an Inverse Function

1. Use y instead of $f(x)$.
2. Interchange x and y.
3. Solve the equation for y.
4. The result gives the inverse function: substitute $f^{-1}(x)$ for y.

In this process, it might seem like we're using the *same* y to represent two different functions. To see why there is actually no contradiction, **see Exercise 103.**

EXAMPLE 3 ► **Finding Inverse Functions Algebraically**

Use the algebraic method to find the inverse function for

a. $f(x) = \sqrt[3]{x + 5}$ **b.** $g(x) = \frac{2x}{x + 1}$

Solution ► **a.**

$f(x) = \sqrt[3]{x + 5}$ given function

$y = \sqrt[3]{x + 5}$ use y instead of $f(x)$

$x = \sqrt[3]{y + 5}$ interchange x and y

$x^3 = y + 5$ cube both sides

$x^3 - 5 = y$ solve for y

$x^3 - 5 = f^{-1}(x)$ the result is $f^{-1}(x)$

For $f(x) = \sqrt[3]{x + 5}$, $f^{-1}(x) = x^3 - 5$.

b.

$g(x) = \dfrac{2x}{x + 1}$ given function

$y = \dfrac{2x}{x + 1}$ use y instead of $f(x)$

$x = \dfrac{2y}{y + 1}$ interchange x and y

$xy + x = 2y$ multiply by $y + 1$ and distribute

$x = 2y - xy$ gather terms with y

$x = y(2 - x)$ factor

$\dfrac{x}{2 - x} = y$ solve for y

$\dfrac{x}{2 - x} = g^{-1}(x)$ the result is $g^{-1}(x)$

For $g(x) = \dfrac{2x}{x + 1}$, $g^{-1}(x) = \dfrac{x}{2 - x}$.

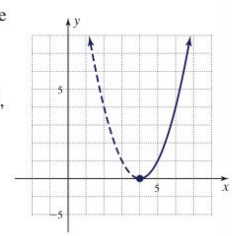
Now try Exercises 41 through 48 ►

In cases where a given function is *not* one-to-one, we can sometimes restrict the domain to create a function that *is*, and then determine an inverse. The restriction we use is arbitrary, and only requires that the result still produce all possible range values. For the most part, we simply choose a limited domain that seems convenient or reasonable.

EXAMPLE 4 ► Restricting the Domain to Create a One-to-One Function

Given $f(x) = (x - 4)^2$, restrict the domain to create a one-to-one function, then find $f^{-1}(x)$. State the domain and range of both resulting functions.

Solution ► The graph of f is a parabola, opening upward with the vertex at (4, 0). Restricting the domain to $x \geq 4$ (see figure) leaves only the "right branch" of the parabola, creating a one-to-one function without affecting the range, $y \in [0, \infty)$. For $f(x) = (x - 4)^2$, $x \geq 4$, we have



$f(x) = (x - 4)^2$ given function

$y = (x - 4)^2$ use y instead of $f(x)$

$x = (y - 4)^2$ interchange x and y

$\pm\sqrt{x} = y - 4$ take square roots

$\sqrt{x} + 4 = y$ solve for y, use $\sqrt{x}$ since $x \geq 4$

The result shows $f^{-1}(x) = \sqrt{x} + 4$, with domain $x \in [0, \infty)$ and range $y \in [4, \infty)$ (the domain of f becomes the range of f^{-1}, and the range of f becomes the domain of f^{-1}).

Now try Exercises 49 through 54 ►

While we now have the ability to *find* the inverse of a function, we still lack a definitive method of *verifying* the inverse is correct. Actually, the diagrams in Figures 4.5 and 4.6 suggest just such a method. If we use the function f itself as an input for f^{-1}, or the function f^{-1} as an input for f, the end result should simply be x, as each function "undoes" the operations of the other. From Section 2.8 this is called a composition of functions and using the notation for composition we have,

Verifying Inverse Functions

If f is a one-to-one function, then the function f^{-1} exists, where

$$(f \circ f^{-1})(x) = x \quad \text{and} \quad (f^{-1} \circ f)(x) = x$$

EXAMPLE 5 ► Finding and Verifying an Inverse Function

Use the algebraic method to find the inverse function for $f(x) = \sqrt{x + 2}$. Then verify the inverse you found is correct.

Solution ► Since the graph of f is the graph of $y = \sqrt{x}$ shifted 2 units left, we know f is one-to-one with domain $x \in [-2, \infty)$ and range $y \in [0, \infty)$. This is important since the *domain and range values will be interchanged for the inverse function*. The domain of f^{-1} will be $x \in [0, \infty)$ and its range $y \in [-2, \infty)$.

$f(x) = \sqrt{x + 2}$	given function; $x \geq -2$
$y = \sqrt{x + 2}$	use y instead of $f(x)$
$x = \sqrt{y + 2}$	interchange x and y
$x^2 = y + 2$	solve for y (square both sides)
$x^2 - 2 = y$	subtract 2
$f^{-1}(x) = x^2 - 2$	the result is $f^{-1}(x)$; D: $x \in [0, \infty)$, R: $y \in [-2, \infty)$

Verify ►

$(f \circ f^{-1})(x) = f[f^{-1}(x)]$	$f^{-1}(x)$ is an input for f
$= \sqrt{f^{-1}(x) + 2}$	f adds 2 to inputs, then takes the square root
$= \sqrt{(x^2 - 2) + 2}$	substitute $x^2 - 2$ for $f^{-1}(x)$
$= \sqrt{x^2}$	simplify
$= x$ ✓	since the domain of $f^{-1}(x)$ is $x \in [0, \infty)$

Verify ►

$(f^{-1} \circ f)(x) = f^{-1}[f(x)]$	$f(x)$ is an input for f^{-1}
$= [f(x)]^2 - 2$	f^{-1} squares inputs, then subtracts 2
$= [\sqrt{x + 2}\,]^2 - 2$	substitute $\sqrt{x + 2}$ for $f(x)$
$= x + 2 - 2$	simplify
$= x$ ✓	result

C. You've just learned how to find inverse functions using an algebraic method

Now try Exercises 55 through 80 ►

D. The Graph of a Function and Its Inverse

Graphing a function and its inverse on the same axes reveals an interesting and useful relationship—the graphs are reflections across the line $y = x$ (the identity function). Consider the function $f(x) = 2x + 3$, and its inverse $f^{-1}(x) = \frac{x - 3}{2} = \frac{1}{2}x - \frac{3}{2}$. In Figure 4.7, the points $(1, 5)$, $(0, 3)$, $(-\frac{3}{2}, 0)$, and $(-4, -5)$ from f (see Table 4.3) are graphed in blue, with the points $(5, 1)$, $(3, 0)$, $(0, -\frac{3}{2})$, and $(-5, -4)$ (see Table 4.4) from f^{-1} graphed in red (note the x- and y-values are reversed). Graphing both lines illustrates this symmetry (Figure 4.8).

Table 4.3

x	$f(x)$
1	5
0	3
$-\frac{3}{2}$	0
-4	-5

Figure 4.7

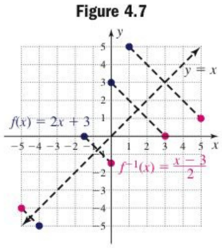


Figure 4.8

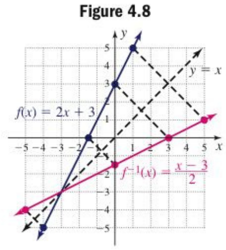


Table 4.4

x	$f^{-1}(x)$
5	1
3	0
0	$-\frac{3}{2}$
-5	-4

EXAMPLE 6 ► **Graphing a Function and Its Inverse**

In Example 5, we found the inverse function for $f(x) = \sqrt{x + 2}$ was $f^{-1}(x) = x^2 - 2, x \geq 0$. Graph these functions on the same axes and comment on how the graphs are related.

Solution ► The graph of f is a square root function with initial point $(-2, 0)$, a y-intercept of $(0, \sqrt{2})$, and an x-intercept of $(-2, 0)$ (Figure 4.9 in blue). The graph of $x^2 - 2, x \geq 0$ is the right-hand branch of a parabola, with y-intercept at $(0, -2)$ and an x-intercept at $(\sqrt{2}, 0)$ (Figure 4.9 in red).

Figure 4.9

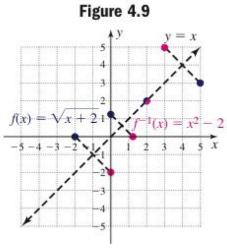
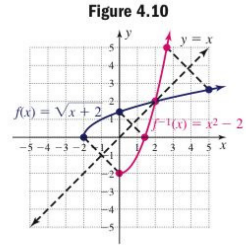


Figure 4.10



Connecting these points with a smooth curve indeed shows their graphs are symmetric to the line $y = x$ (Figure 4.10).

Now try Exercises 81 through 88 ►

EXAMPLE 7 ► **Graphing a Function and Its Inverse**

Given the graph shown in Figure 4.11, use the grid in Figure 4.12 to draw a graph of the inverse function.

Figure 4.11

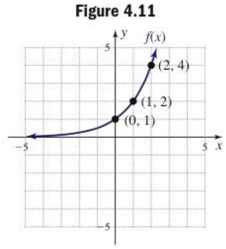
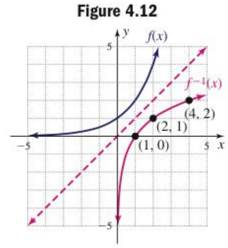


Figure 4.12



Solution ► From the graph, the domain of f appears to be $x \in \mathbb{R}$ and the range is $y \in (0, \infty)$. This means the domain of f^{-1} will be $x \in (0, \infty)$ and the range will be $y \in \mathbb{R}$. To sketch f^{-1}, draw the line $y = x$, interchange the x- and y-coordinates of the selected points, then plot these points and draw a smooth curve using the domain and range boundaries as a guide.

Now try Exercises 89 through 94 ►

A summary of important points is given here followed by their application in Example 8.

Functions and Inverse Functions

1. If the graph of a function passes the horizontal line test, the function is one-to-one.
2. If a function f is one-to-one, the function f^{-1} exists.
3. The domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.
4. For a function f and its inverse f^{-1}, $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.
5. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.

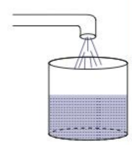
D. You've just learned how to graph a function and its inverse

E. Applications of Inverse Functions

Our final example illustrates one of the many ways that inverse functions can be applied.

EXAMPLE 8 ► Using Volume to Understand Inverse Functions

The volume of an equipoise cylinder (height equal to diameter) is given by $v(x) = 2\pi x^3$ (since $h = d = 2r$), where $v(x)$ represents the volume in units cubed and x represents the radius of the cylinder.



a. Find the volume of such a cylinder if $x = 10$ ft.

b. Find $v^{-1}(x)$, and discuss what the input and output variables represent.

c. If a volume of 1024π ft^3 is required, which formula would be easier to use to find the radius? What is this radius?

Solution ► a.

$$v(x) = 2\pi x^3 \quad \text{given function}$$
$$v(10) = 2\pi(10)^3 \quad \text{substitute 10 for } x$$
$$= 2000\pi \quad 10^3 = 1000\text{, exact form}$$

With a radius of 10 ft, the volume of the cylinder would be 2000π ft^3.

b.

$$v(x) = 2\pi x^3 \quad \text{given function}$$
$$y = 2\pi x^3 \quad \text{use } y \text{ instead of } v(x)$$
$$x = 2\pi y^3 \quad \text{interchange } x \text{ and } y$$
$$\frac{x}{2\pi} = y^3 \quad \text{solve for } y$$
$$\sqrt[3]{\frac{x}{2\pi}} = y \quad \text{result}$$

The inverse function is $v^{-1}(x) = \sqrt[3]{\frac{x}{2\pi}}$. In this case, the input x is a given volume, the output $v^{-1}(x)$ is the radius of an equipoise cylinder that will hold this volume.

c. Since the volume is known and we need the radius, using $v^{-1}(x) = \sqrt[3]{\frac{x}{2\pi}}$ would be more efficient.

$$v^{-1}(1024\pi) = \sqrt[3]{\frac{1024\pi}{2\pi}} \quad \text{substitute } 1024\pi \text{ for } x \text{ in } v^{-1}(x)$$
$$= \sqrt[3]{512} \quad \frac{2\pi}{2\pi} = 1$$
$$= 8 \quad \text{result}$$

The radius of the cylinder would be 8 ft.

E. You've just learned how to solve an application of inverse functions

Now try Exercises 97 through 102 ►

TECHNOLOGY HIGHLIGHT

Investigating Inverse Functions

Many important ideas from this section can be illustrated using a graphing calculator. To begin, enter the function $Y_1 = 2\sqrt[3]{x - 2}$ and $Y_2 = \frac{x^3}{8} + 2$ (which appear to be inverse functions) on the Y= screen, then press ZOOM **5:ZSquare**. The graphs seem to be reflections across the line $y = x$ (Figure 4.13). To verify, use the TABLE feature with inputs $x = -2, -1, 0, 1, 2$, and 3. As shown in Figure 4.14, the points $(1, -2)$, $(2, 0)$, and $(3, 2)$ are on Y_1, and the points $(-2, 1)$, $(0, 2)$, and $(2, 3)$ are all on Y_2. While this seems convincing (the x- and y-coordinates are interchanged), the technology can actually *compose the two functions* to verify an inverse relationship. Function names Y_1 and Y_2 can be accessed using the VARS and ► keys, then pressing ENTER. After entering $Y_3 = Y_1(Y_2)$ and $Y_4 = Y_2(Y_1)$ on the Y= screen, we observe whether one function "undoes" the other using the TABLE feature (Figure 4.15).

Figure 4.13

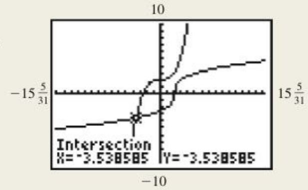
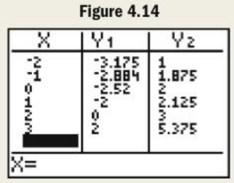


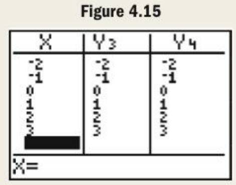
Figure 4.14



X	Y1	Y2
-2	-3.175	1
-1	-2.884	1.875
0	-2.52	2
1	-2	2.125
2	0	3
3	2	5.375

X=

Figure 4.15



X	Y3	Y4
-2	-2	-2
-1	-1	-1
0	0	0
1	1	1
2	2	2
3	3	3

X=

For the functions given, (a) find $f^{-1}(x)$, then use your calculator to verify they are inverses by (b) using ordered pairs, (c) composing the functions, and (d) showing their graphs are symmetric to $y = x$.

Exercise 1: $f(x) = 2x + 1$

Exercise 2: $g(x) = x^2 + 1; x \geq 0$

Exercise 3: $h(x) = \frac{x}{x + 1}$

4.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A function is one-to-one if each ______ coordinate corresponds to exactly ______ first coordinate.

2. If every ______ line intersects the graph of a function in at most ______ point, the function is one-to-one.

3. A certain function is defined by the ordered pairs $(-2, -11)$, $(0, -5)$, $(2, 1)$, and $(4, 19)$. The inverse function is ____________ .

4. To find f^{-1} using the algebraic method, we (1) use ______ instead of $f(x)$, (2) ______ x and y, (3) ______ for y and replace y with $f^{-1}(x)$.

5. State true or false and explain why: *To show that g is the inverse function for f, simply show that* $(f \circ g)(x) = x$. Include an example in your response.

6. Discuss/Explain why no inverse function exists for $f(x) = (x + 3)^2$ and $g(x) = \sqrt{4 - x^2}$. How would the domain of each function have to be restricted to allow for an inverse function?

▶ DEVELOPING YOUR SKILLS

Determine whether each graph given is the graph of a one-to-one function. If not, give examples of how the definition of one-to-oneness is violated.

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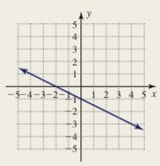

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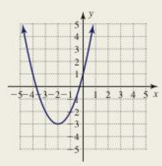

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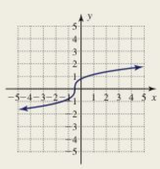

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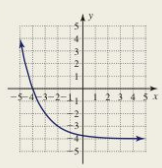

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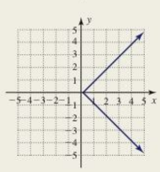

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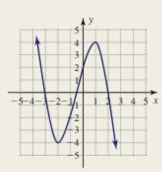

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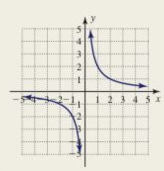

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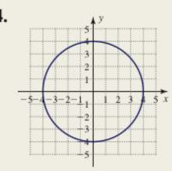

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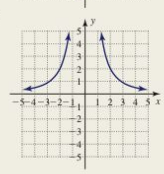


Determine whether the functions given are one-to-one. If not, state why.

16. $\{(-7, 4), (-1, 9), (0, 5), (-2, 1), (5, -5)\}$

17. $\{(9, 1), (-2, 7), (7, 4), (3, 9), (2, 7)\}$

18. $\{(-6, 1), (4, -9), (0, 11), (-2, 7), (-4, 5), (8, 1)\}$

19. $\{(-6, 2), (-3, 7), (8, 0), (12, -1), (2, -3), (1, 3)\}$

Determine if the functions given are one-to-one by noting the function family to which each belongs and mentally picturing the shape of the graph. If a function is not one-to-one, discuss how the definition of one-to-oneness is violated.

20. $f(x) = 3x - 5$

21. $g(x) = (x + 2)^3 - 1$

22. $h(x) = -|x - 4| + 3$

23. $p(t) = 3t^2 + 5$

24. $s(t) = \sqrt{2t - 1} + 5$

25. $r(t) = \sqrt[3]{t + 1} - 2$

26. $y = 3$

27. $y = -2x$

28. $y = x$

For Exercises 29 to 32, find the inverse function of the one-to-one functions given.

29. $f(x) = \{(-2, 1), (-1, 4), (0, 5), (2, 9), (5, 15)\}$

30. $g(x) = \{(-2, 30), (-1, 11), (0, 4), (1, 3), (2, 2)\}$

31. $v(x)$ is defined by the ordered pairs shown.

X	Y1	
-4	3	
-3	2	
0	1	
5	0	
12	-1	
21	-2	
32	-3	

X=32

32. $w(x)$ is defined by the ordered pairs shown.

X	Y1	
-6	4	
-5	2.5	
-2	-2	
0	-5	
3	-9.5	
4	-11	
7	-15.5	

X=7

Find the inverse function using diagrams similar to those illustrated in Example 2. Check the result using three test points.

33. $f(x) = x + 5$

34. $g(x) = x - 4$

35. $p(x) = -\frac{4}{5}x$

36. $r(x) = \frac{3}{4}x$

37. $f(x) = 4x + 3$

38. $g(x) = 5x - 2$

39. $Y_1 = \sqrt[3]{x - 4}$

40. $Y_2 = \sqrt[3]{x + 2}$

Find each function $f(x)$ given, (a) find any three ordered pair solutions (a, b), then (b) algebraically compute $f^{-1}(x)$, and (c) verify the ordered pairs (a, b) satisfy $f^{-1}(x)$.

41. $f(x) = \sqrt[3]{x - 2}$

42. $f(x) = \sqrt[3]{x + 3}$

43. $f(x) = x^3 + 1$

44. $f(x) = x^3 - 2$

45. $f(x) = \dfrac{8}{x + 2}$

46. $f(x) = \dfrac{12}{x - 1}$

47. $f(x) = \dfrac{x}{x + 1}$

48. $f(x) = \dfrac{x + 2}{1 - x}$

The functions given in Exercises 49 through 54 are not one-to-one. (a) Determine a domain restriction that preserves all range values, then state this domain and range. (b) Find the inverse function and state its domain and range.

49. $f(x) = (x + 5)^2$

50. $g(x) = x^2 + 3$

51. $v(x) = \dfrac{8}{(x - 3)^2}$

52. $V(x) = \dfrac{4}{x^2} + 2$

53. $p(x) = (x + 4)^2 - 2$

54. $q(x) = \dfrac{4}{(x - 2)^2} + 1$

For each function $f(x)$ given, prove (using a composition) that $g(x) = f^{-1}(x)$.

55. $f(x) = -2x + 5$, $g(x) = \dfrac{x - 5}{-2}$

56. $f(x) = 3x - 4$, $g(x) = \dfrac{x + 4}{3}$

57. $f(x) = \sqrt[3]{x + 5}$, $g(x) = x^3 - 5$

58. $f(x) = \sqrt[3]{x - 4}$, $g(x) = x^3 + 4$

59. $f(x) = \frac{2}{3}x - 6$, $g(x) = \frac{3}{2}x + 9$

60. $f(x) = \frac{4}{5}x + 6$, $g(x) = \frac{5}{4}x - \frac{15}{2}$

61. $f(x) = x^2 - 3$; $x \geq 0$, $g(x) = \sqrt{x + 3}$

62. $f(x) = x^2 + 8$; $x \geq 0$, $g(x) = \sqrt{x - 8}$

Find the inverse of each function $f(x)$ given, then prove (by composition) your inverse function is correct. Note the domain of f is all real numbers.

63. $f(x) = 3x - 5$

64. $f(x) = 5x + 4$

65. $f(x) = \dfrac{x - 5}{2}$

66. $f(x) = \dfrac{x + 4}{3}$

67. $f(x) = \frac{1}{2}x - 3$

68. $f(x) = \frac{2}{3}x + 1$

69. $f(x) = x^3 + 3$

70. $f(x) = x^3 - 4$

71. $f(x) = \sqrt[3]{2x + 1}$

72. $f(x) = \sqrt[3]{3x - 2}$

73. $f(x) = \dfrac{(x - 1)^3}{8}$

74. $f(x) = \dfrac{(x + 3)^3}{-27}$

Find the inverse of each function, then prove (by composition) your inverse function is correct. State the implied domain and range as you begin, and use these to state the domain and range of the inverse function.

75. $f(x) = \sqrt{3x + 2}$

76. $g(x) = \sqrt{2x - 5}$

77. $p(x) = 2\sqrt{x - 3}$

78. $q(x) = 4\sqrt{x + 1}$

79. $v(x) = x^2 + 3$; $x \geq 0$

80. $w(x) = x^2 - 1$; $x \geq 0$

Graph each function $f(x)$ and its inverse $f^{-1}(x)$ on the same grid and "dash-in" the line $y = x$. Note how the graphs are related. Then verify the "inverse function" relationship using a composition.

81. $f(x) = 4x + 1; f^{-1}(x) = \dfrac{x - 1}{4}$

82. $f(x) = 2x - 7; f^{-1}(x) = \dfrac{x + 7}{2}$

83. $f(x) = \sqrt[3]{x + 2}; f^{-1}(x) = x^3 - 2$

84. $f(x) = \sqrt[3]{x - 7}; f^{-1}(x) = x^3 + 7$

85. $f(x) = 0.2x + 1; f^{-1}(x) = 5x - 5$

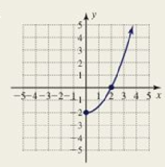
86. $f(x) = \dfrac{2}{9}x + 4; f^{-1}(x) = \dfrac{9}{2}x - 18$

87. $f(x) = (x + 2)^2; x \geq -2; f^{-1}(x) = \sqrt{x} - 2$

88. $f(x) = (x - 3)^2; x \geq 3; f^{-1}(x) = \sqrt{x} + 3$

Determine the domain and range for each function whose graph is given, and use this information to state the domain and range of the inverse function. Then sketch in the line $y = x$, estimate the location of two or more points on the graph, and use these to graph $f^{-1}(x)$ on the same grid.

89.



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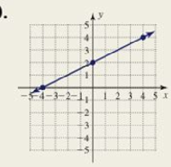

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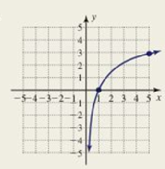

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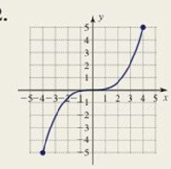

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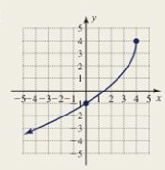

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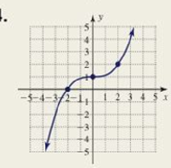


▶ WORKING WITH FORMULAS

95. The height of a projected image: $f(x) = \frac{1}{2}x - 8.5$

The height of an image projected on a screen by a projector is given by the formula shown, where $f(x)$ represents the actual height of the image on the projector (in centimeters) and x is the distance of the projector from the screen (in centimeters). (a) When the projector is 80 cm from the screen, how large is the image? (b) Show that the inverse function is $f^{-1}(x) = 2x + 17$, then input your answer from part (a) and comment on the result. What information does the inverse function give?

96. The radius of a sphere: $r(x) = \sqrt[3]{\dfrac{3x}{4\pi}}$

In generic form, the radius of a sphere is given by the formula shown, where $r(x)$ represents the radius and x represents the volume of the sphere in cubic units. (a) If a weather balloon that is roughly spherical holds 14,130 in^3 of air, what is the radius of the balloon (use $\pi \approx 3.14$)? (b) Show that the inverse function is $r^{-1}(x) = \frac{4}{3}\pi x^3$, then input your answer from part (a) and comment on the result. What information does the inverse function give?

▶ APPLICATIONS

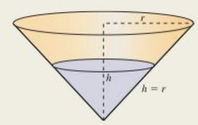
97. Temperature and altitude: The temperature (in degrees Fahrenheit) at a given altitude can be approximated by the function $f(x) = -\frac{7}{2}x + 59$, where $f(x)$ represents the temperature and x represents the altitude in thousands of feet. (a) What is the approximate temperature at an altitude of 35,000 ft (normal cruising altitude for commercial airliners)? (b) Find $f^{-1}(x)$, and state what the independent and dependent variables represent. (c) If the temperature outside a weather balloon is -18°F, what is the approximate altitude of the balloon?

98. Fines for speeding: In some localities, there is a set formula to determine the amount of a fine for exceeding posted speed limits. Suppose the amount of the fine for exceeding a 50 mph speed limit was given by the function $f(x) = 12x - 560$ $(x > 50)$ where $f(x)$ represents the fine in dollars for a speed of x mph. (a) What is the fine for traveling 65 mph through this speed zone? (b) Find $f^{-1}(x)$, and state what the independent and dependent variables represent. (c) If a fine of \$172 were assessed, how fast was the driver going through this speed zone?

99. Effect of gravity: Due to the effect of gravity, the distance an object has fallen after being dropped is given by the function $f(x) = 16x^2; x \geq 0$, where $f(x)$ represents the distance in feet after x sec. (a) How far has the object fallen 3 sec after it has been dropped? (b) Find $f^{-1}(x)$, and state what the independent and dependent variables represent. (c) If the object is dropped from a height of 784 ft, how many seconds until it hits the ground (stops falling)?

100. Area and radius: In generic form, the area of a circle is given by $f(x) = \pi x^2$, where $f(x)$ represents the area in square units for a circle with radius x. (a) A pet dog is tethered to a stake in the backyard. If the tether is 10 ft long, how much area does the dog have to roam (use $\pi \approx 3.14$)? (b) Find $f^{-1}(x)$, and state what the independent and dependent variables represent. (c) If the owners want to allow the dog 1256 ft^2 of area to live and roam, how long a tether should be used?

101. Volume of a cone: In generic form, the volume of an equipoise cone (height equal to radius) is given by $f(x) = \frac{1}{3}\pi x^3$, where $f(x)$ represents the volume in units^3 and x represents the height of the cone. (a) Find the volume of such a cone if $r = 30$ ft (use $\pi \approx 3.14$). (b) Find $f^{-1}(x)$, and state what the independent and dependent variables represent. (c) If the volume of water in the cone is 763.02 ft^3, how deep is the water at its deepest point?



102. Wind power: The power delivered by a certain wind-powered generator can be modeled by the function $f(x) = \frac{x^3}{2500}$, where $f(x)$ is the horsepower (hp) delivered by the generator and x represents the speed of the wind in miles per hour. (a) Use the model to determine how much horsepower is generated by a 30 mph wind. (b) The person monitoring the output of the generators (wind generators are usually erected in large numbers) would like a function that gives the wind speed based on the horsepower readings on the gauges in the monitoring station. For this purpose, find $f^{-1}(x)$ and state what the independent and dependent variables represent. (c) If gauges show 25.6 hp is being generated, how fast is the wind blowing?

▶ EXTENDING THE CONCEPT

103. For a deeper understanding of the algebraic method for finding an inverse, suppose a function f is defined as $f(x)$: $\{(x, y) | y = 3x - 6\}$. We can then define the inverse as f^{-1}: $\{(x, y) | x = 3y - 6\}$, having interchanged x and y in the equation portion. The equation for f^{-1} is not in standard form, but (x, y) still represents all ordered pairs satisfying either equation. Solving for y gives f^{-1}: $\left\{(x, y) | y = \frac{x}{3} + 2\right\}$, and demonstrates the role of steps 2, 3, and 4 of the method. (a) Find five ordered pairs that satisfy the equation for f, then (b) interchange their coordinates and show they satisfy the equation for f^{-1}.

104. The function $f(x) = \frac{1}{x}$ is one of the few functions that is its own inverse. This means the ordered pairs (a, b) and (b, a) must satisfy both f and f^{-1}. (a) Find f^{-1} using the algebraic method to verify that $f(x) = f^{-1}(x) = \frac{1}{x}$. (b) Graph the function $f(x) = \frac{1}{x}$ using a table of integers from -4 to 4. Note that for any ordered pair (a, b) on f, the ordered pair (b, a) is also on f. (c) State where the graph of $y = x$ will intersect the graph of this function and discuss why.

105. By inspection, which of the following is the inverse function for $f(x) = \frac{2}{3}\left(x - \frac{1}{2}\right)^5 + \frac{4}{5}$?

a. $f^{-1}(x) = \sqrt[5]{\frac{1}{2}\left(x - \frac{2}{3}\right)} - \frac{4}{5}$

b. $f^{-1}(x) = \frac{3}{2}\sqrt[5]{(x - 2)} - \frac{5}{4}$

c. $f^{-1}(x) = \frac{3}{2}\sqrt[5]{\left(x + \frac{1}{2}\right)} - \frac{5}{4}$

d. $f^{-1}(x) = \sqrt[5]{\frac{3}{2}\left(x - \frac{4}{5}\right)} + \frac{1}{2}$

106. Suppose a function is defined as $f(x) =$ *the exponent that goes on 9 to obtain x*. For example, $f(81) = 2$ since 2 is the exponent that goes on 9 to obtain 81, and $f(3) = \frac{1}{2}$ since $\frac{1}{2}$ is the exponent that goes on 9 to obtain 3. Determine the value of each of the following:

a. $f(1)$ **b.** $f(729)$ **c.** $f^{-1}(2)$ **d.** $f^{-1}\left(\frac{1}{2}\right)$

▶ MAINTAINING YOUR SKILLS

107. **(2.5)** Given $f(x) = x^2 - x - 2$, solve the inequality $f(x) \leq 0$ using the x-intercepts and end behavior of the graph.

108. **(2.4)** For the function $y = 2\sqrt{x + 3}$, find the average rate of change between $x = 1$ and $x = 2$, and between $x = 4$ and $x = 5$. Which is greater? Why?

109. **(R.7)** Write as many of the following formulas as you can from memory:

a. perimeter of a rectangle

b. area of a circle

c. volume of a cylinder

d. volume of a cone

e. circumference of a circle

f. area of a triangle

g. area of a trapezoid

h. volume of a sphere

i. Pythagorean theorem

110. **(1.3)** Solve the following cubic equations by factoring:

a. $x^3 - 5x = 0$

b. $x^3 - 7x^2 - 4x + 28 = 0$

c. $x^3 - 3x^2 = 0$

d. $x^3 - 3x^2 - 4x = 0$

4.2 Exponential Functions

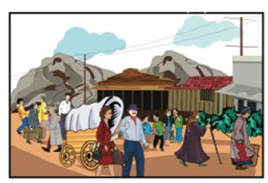
Learning Objectives

In Section 4.2 you will learn how to:

- **A.** Evaluate an exponential function
- **B.** Graph general exponential functions
- **C.** Graph base-*e* exponential functions
- **D.** Solve exponential equations and applications

Demographics is the statistical study of human populations. In this section, we introduce the family of *exponential functions,* which are widely used to model population growth or decline with additional applications in science, engineering, and many other fields. As with other functions, we begin with a study of the graph and its characteristics.

A. Evaluating Exponential Functions



In the boomtowns of the old west, it was not uncommon for a town to double in size every year (at least for a time) as the lure of gold drew more and more people westward. When this type of growth is modeled using mathematics, exponents play a lead role. Suppose the town of Goldsboro had 1000 residents when gold was first discovered. After 1 yr the population doubled to **2**000 residents. The next year it doubled again to **4**000, then again to **8**000, then to **16**,000 and so on. You probably recognize the digits in blue as powers of two (indicating the population is *doubling*), with each one multiplied by 1000 (the initial population). This suggests we can model the relationship using

$$P(x) = 1000 \cdot 2^x$$

where $P(x)$ is the population after x yr. Further, we can evaluate this function, called an **exponential function,** for *fractional parts of a year* using rational exponents. The population of Goldsboro one-and-a-half years after the gold rush was

$$\begin{aligned} P\left(\frac{3}{2}\right) &= 1000 \cdot 2^{\frac{3}{2}} \\ &= 1000 \cdot (\sqrt{2})^3 \\ &\approx 2828 \text{ people} \end{aligned}$$

WORTHY OF NOTE

To properly understand the exponential function and its graph requires that we evaluate $f(x) = 2^x$ even when x is *irrational*. For example, what does $2^{\sqrt{5}}$ mean? While the technical details require calculus, it can be shown that successive approximations of $2^{\sqrt{5}}$ as in $2^{2.2360}$, $2^{2.23606}$, $2^{2.23236067}$, . . . approach a unique real number, and $f(x) = 2^x$ exists for all real numbers x.

In general, exponential functions are defined as follows.

Exponential Functions

For $b > 0$, $b \neq 1$, and all real numbers x,

$$f(x) = b^x$$

defines the base b exponential function.

Limiting b to positive values ensures that outputs will be real numbers, and the restriction $b \neq 1$ is needed since $y = 1^x$ is a constant function (1 raised to *any* power is still 1). Specifically note the domain of an exponential function is *all real numbers,* and that all of the familiar properties of exponents still hold. A summary of these properties follows. For a complete review, see Section R.3.

Exponential Properties

For real numbers a, b, m, and n, with $a, b > 0$,

$$b^m \cdot b^n = b^{m+n} \qquad \frac{b^m}{b^n} = b^{m-n} \qquad (b^m)^n = b^{mn}$$

$$(ab)^n = a^n \cdot b^n \qquad b^{-n} = \frac{1}{b^n} \qquad \left(\frac{b}{a}\right)^{-n} = \left(\frac{a}{b}\right)^n$$

EXAMPLE 1 ► **Evaluating Exponential Functions**

Evaluate each exponential function for $x = 2$, $x = -1$, $x = \frac{1}{2}$, and $x = \pi$. Use a calculator for $x = \pi$, rounding to five decimal places.

a. $f(x) = 4^x$ **b.** $g(x) = \left(\frac{4}{9}\right)^x$

Solution ► **a.** For $f(x) = 4^x$,

$$f(2) = 4^2 = 16$$
$$f(-1) = 4^{-1} = \frac{1}{4}$$
$$f\left(\frac{1}{2}\right) = 4^{\frac{1}{2}} = \sqrt{4} = 2$$
$$f(\pi) = 4^{\pi} \approx 77.88023$$

b. For $g(x) = \left(\frac{4}{9}\right)^x$,

$$g(2) = \left(\frac{4}{9}\right)^2 = \frac{16}{81}$$
$$g(-1) = \left(\frac{4}{9}\right)^{-1} = \frac{9}{4}$$
$$g\left(\frac{1}{2}\right) = \left(\frac{4}{9}\right)^{\frac{1}{2}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$
$$g(\pi) = \left(\frac{4}{9}\right)^{\pi} \approx 0.07827$$

☑ **A.** You've just learned how to evaluate an exponential function

Now try Exercises 7 through 12 ►

B. Graphing Exponential Functions

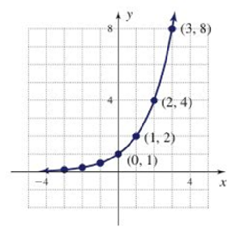
To gain a better understanding of exponential functions, we'll graph examples of $y = b^x$ and note some of the characteristic features. Since $b \neq 1$, it seems reasonable that we graph one exponential function where $b > 1$ and one where $0 < b < 1$.

EXAMPLE 2 ► **Graphing Exponential Functions with $b > 1$**

Graph $y = 2^x$ using a table of values.

Solution ► To get an idea of the graph's shape we'll use integer values from -3 to 3 in our table, then draw the graph as a continuous curve, since the function is defined for all real numbers.

x	$y = 2^x$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



Now try Exercises 13 and 14 ►

WORTHY OF NOTE

As in Example 2, functions that are increasing for all $x \in D$ are said to be **monotonically increasing** or simply **monotonic functions.** The function in Example 3 is monotonically decreasing.

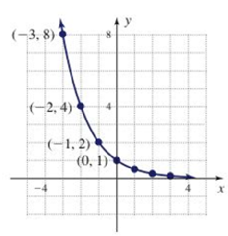
Several important observations can now be made. First note the x-axis (the line $y = 0$) is a horizontal asymptote for the function, because as $x \to -\infty$, $y \to 0$. Second, the function is increasing over its entire domain, giving the function a range of $y \in (0, \infty)$.

EXAMPLE 3 ► **Graphing Exponential Functions with $0 < b < 1$**

Graph $y = (\frac{1}{2})^x$ using a table of values.

Solution ► Using properties of exponents, we can write $(\frac{1}{2})^x$ as $(\frac{2}{1})^{-x} = 2^{-x}$. Again using integers from -3 to 3, we plot the ordered pairs and draw a continuous curve.

x	$y = 2^{-x}$
-3	$2^{-(-3)} = 2^3 = 8$
-2	$2^{-(-2)} = 2^2 = 4$
-1	$2^{-(-1)} = 2^1 = 2$
0	$2^0 = 1$
1	$2^{-1} = \frac{1}{2}$
2	$2^{-2} = \frac{1}{4}$
3	$2^{-3} = \frac{1}{8}$



Now try Exercises 15 and 16 ►

We note this graph is also asymptotic to the x-axis, but *decreasing on its domain.* In addition, both $y = 2^x$ and $y = 2^{-x} = (\frac{1}{2})^x$ are one-to-one, and have a y-intercept of (0, 1)—which we expect since any base to the zero power is 1. Finally, observe that $y = b^{-x}$ is *a reflection of* $y = b^x$ *across the y-axis,* a property that suggests these basic graphs might also be transformed in other ways, as were the toolbox functions. The characteristics of exponential functions are summarized here:

$f(x) = b^x$, $b > 0$ and $b \neq 1$

- one-to-one function
- domain: $x \in \mathbb{R}$
- increasing if $b > 1$
- y-intercept $(0, 1)$
- range: $y \in (0, \infty)$
- decreasing if $0 < b < 1$
- asymptotic to the x-axis (the line $y = 0$)

Figure 4.16

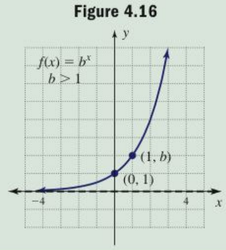
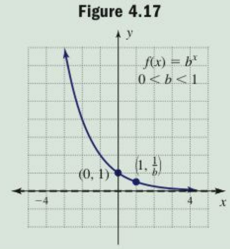


Figure 4.17



WORTHY OF NOTE

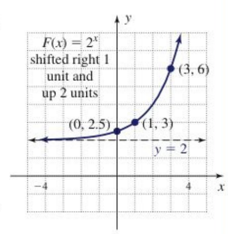
When an exponential function is increasing, it can be referred to as a "growth function." When decreasing, it is often called a "decay function." Each of the graphs shown in Figures 4.16 and 4.17 should now be added to your repertoire of basic functions, to be sketched from memory and analyzed or used as needed.

Just as the graph of a quadratic function maintains its parabolic shape regardless of the transformations applied, exponential functions will also maintain their general shape and features. Any sum or difference applied to the basic function ($y = b^x \pm k$ vs. $y = b^x$) will cause a vertical shift in the same direction as the sign, and any change to input values ($y = b^{x+h}$ vs. $y = b^x$) will cause a horizontal shift in a direction opposite the sign.

EXAMPLE 4 ▶ Graphing Exponential Functions Using Transformations

Graph $F(x) = 2^{x-1} + 2$ using transformations of the basic function (not by simply plotting points). Clearly name the parent function and state what transformations are applied.

Solution ▶ The graph of F is that of the basic function $y = 2^x$ shifted 1 unit right and 2 units up. With this in mind the horizontal asymptote shifts from $y = 0$ to $y = 2$ and $(0, 1)$ shifts to $(1, 3)$. The y-intercept of F is at $(0, 2.5)$ and to help sketch a more accurate graph, the additional point $(3, 6)$ is used: $F(3) = 6$.



☑ B. You've just learned how to graph general exponential functions

Now try Exercises 17 through 38 ▶

C. The Base-e Exponential Function: $f(x) = e^x$

In nature, exponential growth occurs when the rate of change in a population's growth, is in constant proportion to its current size. Using the rate of change notation, $\frac{\Delta P}{\Delta t} = kP$, where k is a constant. For the city of Goldsboro, we know the population at time t is given by $P(t) = 1000 \cdot 2^t$, but have no information on this value of k **(see Exercise 96).** We can actually rewrite this function, and other exponential functions, using a base that gives the value of k directly and without having to apply the difference quotient. This new base is an irrational number, symbolized by the letter e and defined as follows.

WORTHY OF NOTE

Just as the ratio of a circle's circumference to its diameter is an irrational number symbolized by π, the irrational number that results from $\left(1 + \frac{1}{x}\right)^x$ for infinitely large x is symbolized by e. Writing exponential functions in terms of e simplifies many calculations in advanced courses, and offers additional advantages in applications of exponential functions.

The Number e

For $x > 0$,

$$\text{as } x \to \infty, \left(1 + \frac{1}{x}\right)^x \to e$$

In words, e is the number that $\left(1 + \frac{1}{x}\right)^x$ approaches as x becomes infinitely large.

It has been proven that as x grows without bound, $\left(1 + \frac{1}{x}\right)^x$ indeed approaches the unique, irrational number that we have named e **(also see Exercise 97).** Table 4.5 gives approximate values of the expression for selected values of x, and shows $e \approx 2.71828$ to five decimal places.

Table 4.5

x	$(1 + \frac{1}{x})^x$
1	2
10	2.59
100	2.705
1000	2.7169
10,000	2.71815
100,000	2.718268
1,000,000	2.7182804
10,000,000	2.71828169

The result is the base-e **exponential function:** $\boldsymbol{f(x) = e^x}$, also called the **natural exponential function.** Instead of having to enter a decimal approximation when computing with e, most calculators have an "e^x" key, usually as the 2nd function for the key marked LN. To find the value of e^2, use the keystrokes 2nd LN 2) ENTER, and the calculator display should read 7.389056099. Note the calculator supplies the left parenthesis for the exponent, and you must supply the right.

EXAMPLE 5 ► Evaluating the Natural Exponential Function

Use a calculator to evaluate $f(x) = e^x$ for the values of x given. Round to six decimal places.

a. $f(3)$ **b.** $f(1)$ **c.** $f(0)$ **d.** $f(\frac{1}{2})$

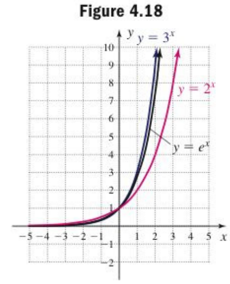
Solution ► **a.** $f(3) = e^3 \approx 20.085537$ **b.** $f(1) = e^1 \approx 2.718282$

c. $f(0) = e^0 = 1$ (exactly) **d.** $f(\frac{1}{2}) = e^{\frac{1}{2}} \approx 1.648721$

Now try Exercises 39 through 46 ►

Although e is an irrational number, the graph of $y = e^x$ behaves in exactly the same way and has the same characteristics as other exponential graphs. Figure 4.18 shows this graph on the same grid as $y = 2^x$ and $y = 3^x$. As we might expect, all three graphs are increasing, have an asymptote at $y = 0$, and contain the point (0, 1), with the graph of $y = e^x$ "between" the other two. The domain for all three functions, as with all basic exponential functions, is $x \in (-\infty, \infty)$ with range $y \in (0, \infty)$. The same transformations applied earlier can also be applied to the graph of $y = e^x$.

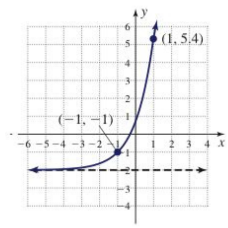
Figure 4.18



EXAMPLE 6 ▶ Graphing Exponential Functions Using a Transformation

Graph $f(x) = e^{x+1} - 2$ using transformations of $y = e^x$. Clearly state the transformations applied.

Solution ▶ The graph of f is the same as $y = e^x$, shifted 1 unit left and 2 units down. The point (0, 1) becomes $(-1, -1)$, and the horizontal asymptote becomes $y = -2$. As the basic shape of the graph is known, we compute $f(1) \approx 5.4$, and complete the graph as shown.



C. You've just learned how to graph base-*e* exponential functions

Now try Exercises 47 through 52 ▶

D. Solving Exponential Equations Using the Uniqueness Property

Since exponential functions are one-to-one, we can solve equations where each side is an exponential term with the identical base. This is because one-to-oneness guarantees a unique solution to the equation.

WORTHY OF NOTE

Exponential functions are very different from the power functions studied earlier. For power functions, the base is variable and the exponent is constant: $y = x^b$, while for exponential functions the *exponent is a variable* and the *base is constant*: $y = b^x$.

Exponential Equations and the Uniqueness Property

For all real numbers m, n, and b, where $b > 0$ and $b \neq 1$,

If $b^m = b^n$,

then $m = n$.

Equal bases imply exponents are equal.

The equation $2^x = 32$ can be written as $2^x = 2^5$, and we note $x = 5$ is a solution. Although $3^x = 32$ can be written as $3^x = 2^5$, the bases are not alike and the solution to this equation must wait until additional tools are developed in Section 4.4.

EXAMPLE 7 ▶ Solving Exponential Equations

Solve the exponential equations using the uniqueness property.

a. $3^{2x-1} = 81$ **b.** $25^{-2x} = 125^{x+7}$

c. $\left(\frac{1}{6}\right)^{-3x-2} = 36^{x+1}$ **d.** $e^x e^2 = \dfrac{e^4}{e^{x+1}}$

Solution ▶ a.

$$3^{2x-1} = 81 \quad \text{given}$$
$$3^{2x-1} = 3^4 \quad \text{rewrite using base 3}$$
$$\Rightarrow 2x - 1 = 4 \quad \text{uniqueness property}$$
$$x = \frac{5}{2} \quad \text{solve for } x$$

Check ▶

$$3^{2x-1} = 81 \quad \text{given}$$
$$3^{2(\frac{5}{2})-1} = 81 \quad \text{substitute } \tfrac{5}{2} \text{ for } x$$
$$3^{5-1} = 81 \quad \text{simplify}$$
$$3^4 = 81 \quad \text{result checks}$$
$$81 = 81$$

The remaining checks are left to the student.

b.

$$\begin{aligned} 25^{-2x} &= 125^{x+7} && \text{given} \\ (5^2)^{-2x} &= (5^3)^{x+7} && \text{rewrite using base 5} \\ 5^{-4x} &= 5^{3x+21} && \text{power property of exponents} \\ \Rightarrow -4x &= 3x + 21 && \text{uniqueness property} \\ x &= -3 && \text{solve for } x \end{aligned}$$

c.

$$\begin{aligned} \left(\frac{1}{6}\right)^{-3x-2} &= 36^{x+1} && \text{given} \\ (6^{-1})^{-3x-2} &= (6^2)^{x+1} && \text{rewrite using base 6} \\ 6^{3x+2} &= 6^{2x+2} && \text{power property of exponents} \\ \Rightarrow 3x + 2 &= 2x + 2 && \text{uniqueness property} \\ x &= 0 && \text{solve for } x \end{aligned}$$

d.

$$\begin{aligned} e^x e^2 &= \frac{e^4}{e^{x+1}} && \text{given} \\ e^{x+2} &= e^{4-(x+1)} && \text{product property; quotient property} \\ e^{x+2} &= e^{3-x} && \text{simplify} \\ \Rightarrow x + 2 &= 3 - x && \text{uniqueness property} \\ 2x &= 1 && \text{add } x\text{, subtract 2} \\ x &= \frac{1}{2} && \text{solve for } x \end{aligned}$$

Now try Exercises 53 through 72 ►

One very practical application of the natural exponential function involves **Newton's law of cooling.** This law or formula models the temperature of an object as it cools down, as when a pizza is removed from the oven and placed on the kitchen counter. The function model is

$$T(x) = T_R + (T_0 - T_R)e^{kx}, k < 0$$

where T_0 represents the initial temperature of the object, T_R represents the temperature of the room or surrounding medium, $T(x)$ is the temperature of the object x min later, and k is the cooling rate as determined by the nature and physical properties of the object.

EXAMPLE 8 ► Applying an Exponential Function—Newton's Law of Cooling

A pizza is taken from a 425°F oven and placed on the counter to cool. If the temperature in the kitchen is 75°F, and the cooling rate for this type of pizza is $k = -0.35$,

a. What is the temperature (to the nearest degree) of the pizza 2 min later?

b. To the nearest minute, how long until the pizza has cooled to a temperature below 90°F?

c. If Zack and Raef like to eat their pizza at a temperature of about 110°F, how many minutes should they wait to "dig in"?

Solution ► Begin by substituting the given values to obtain the equation model:

$$\begin{aligned} T(x) &= T_R + (T_0 - T_R)e^{kx} && \text{general equation model} \\ &= 75 + (425 - 75)e^{-0.35x} && \text{substitute 75 for } T_R\text{, 425 for } T_0 \text{ and } -0.35 \text{ for } k \\ &= 75 + 350e^{-0.35x} && \text{simplify} \end{aligned}$$

For part (a) we simply find $T(2)$:

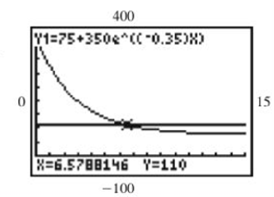
a.

$$\begin{aligned} T(2) &= 75 + 350e^{-0.35(2)} && \text{substitute 2 for } x \\ &\approx 249 && \text{result} \end{aligned}$$

Two minutes later, the temperature of the pizza is near 249°.

b. Using the TABLE feature of a graphing calculator shows the pizza reaches a temperature of just under 90° after 9 min: $T(9) \approx 90°F$.

c. We elect to use the intersection of graphs method (see the *Technology Highlight* on page 432). After setting an appropriate window, we enter $Y_1 = 75 + 350e^{-0.35x}$ and $Y_2 = 110$, then press 2nd CALC option **5: intersect.** After pressing ENTER three times, the coordinates of the point of intersection appear at the bottom of the screen: $x \approx 6.6$, $y = 110$. It appears the boys should wait about $6\frac{1}{2}$ min for the pizza to cool.



Now try Exercises 75 and 76 ►

EXAMPLE 9 ► Applications of Exponential Functions—Depreciation

For insurance purposes, it is estimated that large household appliances lose $\frac{1}{5}$ of their value each year. The current value can then be modeled by the function $V(t) = V_0(\frac{4}{5})^t$, where V_0 is the initial value and $V(t)$ represents the value after t years. How many years does it take a washing machine that cost \$625 new, to depreciate to a value of \$256?

Solution ► For this exercise, $V_0 = \$625$ and $V(t) = \$256$. The formula yields

$$V(t) = V_0\left(\frac{4}{5}\right)^t \quad \text{given}$$

$$256 = 625\left(\frac{4}{5}\right)^t \quad \text{substitute known values}$$

$$\frac{256}{625} = \left(\frac{4}{5}\right)^t \quad \text{divide by 625}$$

$$\left(\frac{4}{5}\right)^4 = \left(\frac{4}{5}\right)^t \quad \text{equate bases } \frac{256}{625} = \left(\frac{4}{5}\right)^4$$

$$\Rightarrow 4 = t \quad \text{Uniqueness Property}$$

After 4 yr, the washing machine's value has dropped to \$256.

Now try Exercises 77 through 90 ►

D. You've just learned how to solve exponential equations and applications

TECHNOLOGY HIGHLIGHT

Solving Exponential Equations Graphically

In this section, we showed that the exponential function $f(x) = b^x$ was defined for all real numbers. This is important because it establishes that equations like $2^x = 7$ must have a solution, even if x is not rational. In fact, since $2^2 = 4$ and $2^3 = 8$, the following inequalities indicate the solution must be between 2 and 3

$$4 < 7 < 8 \quad \text{7 is between 4 and 8}$$

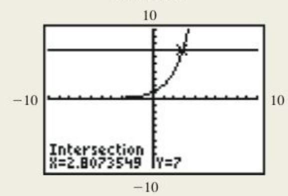
$$2^2 < 2^x < 2^3 \quad \text{replace 4 with } 2^2\text{, 8 with } 2^3$$

$$2 < x < 3 \quad x \text{ must be between 2 and 3}$$

—continued

Until we develop an inverse for exponential functions, we are unable to solve many of these equations in exact form. We can, however, get a very close approximation using a graphing calculator. For the equation $2^x = 7$, enter $Y_1 = 2^x$ and $Y_2 = 7$ on the Y= screen. Then press ZOOM 6 to graph both functions (see Figure 4.19). To find the point of intersection, press 2nd TRACE **(CALC)** and select option **5: intersect** and press ENTER *three* times (to identify the intersecting functions and bypass "Guess"). The *x*- and *y*-coordinates of the point of intersection will appear at the bottom of the screen, with the *x*-coordinate being the solution. As you can see, *x* is indeed between 2 and 3. Solve the following equations. First estimate the answer by bounding it between two integers, then solve the equation graphically. Adjust the viewing window as needed.

Figure 4.19



Exercise 1: $3^x = 22$

Exercise 2: $2^x = 0.125$

Exercise 3: $e^{x-1} = 9$

Exercise 4: $e^{0.5x} = 0.1x^3$

4.2 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. An exponential function is one of the form $y =$ ______, where ______ > 0, ______ $\neq 1$, and ______ is any real number.

2. The domain of $y = b^x$ is all ___ ______, and the range is $y \in$ _____. Further, as $x \to -\infty$, y _____.

3. For exponential functions of the form $y = ab^x$, the *y*-intercept is (0, ______), since $b^0 =$ ______ for any real number *b*.

4. If each side of an equation can be written as an exponential term with the same base, the equation can be solved using the ______ ______.

5. State true or false and explain why: $y = b^x$ is always increasing if $0 < b < 1$.

6. Discuss/Explain the statement, "For $k > 0$, the *y*-intercept of $y = ab^x + k$ is $(0, a + k)$."

▶ DEVELOPING YOUR SKILLS

Use a calculator (as needed) to evaluate each function as indicated. Round answers to thousandths.

7. $P(t) = 2500 \cdot 4^t$; $t = 2, t = \frac{1}{2}, t = \frac{3}{2}, t = \sqrt{3}$

8. $Q(t) = 5000 \cdot 8^t$; $t = 2, t = \frac{1}{3}, t = \frac{5}{3}, t = 5$

9. $f(x) = 0.5 \cdot 10^x$; $x = 3, x = \frac{1}{2}, x = \frac{2}{3}, x = \sqrt{7}$

10. $g(x) = 0.8 \cdot 5^x$; $x = 4, x = \frac{1}{4}, x = \frac{4}{5}, x = \pi$

11. $V(n) = 10{,}000(\frac{2}{3})^n$; $n = 0, n = 4, n = 4.7, n = 5$

12. $W(m) = 3300(\frac{4}{5})^m$; $m = 0, m = 5, m = 7.2, m = 10$

Graph each function using a table of values and integer inputs between −3 and 3. Clearly label the *y*-intercept and one additional point, then indicate whether the function is increasing or decreasing.

13. $y = 3^x$ **14.** $y = 4^x$

15. $y = (\frac{1}{3})^x$ **16.** $y = (\frac{1}{4})^x$

Graph each of the following functions by *translating the basic function* $y = b^x$, sketching the asymptote, and strategically plotting a few points to round out the graph. Clearly state the basic function and what shifts are applied.

17. $y = 3^x + 2$ **18.** $y = 3^x - 3$

19. $y = 3^{x+3}$ **20.** $y = 3^{x-2}$

21. $y = 2^{-x}$ **22.** $y = 3^{-x}$

23. $y = 2^{-x} + 3$ **24.** $y = 3^{-x} - 2$

25. $y = 2^{x+1} - 3$ **26.** $y = 3^{x-2} + 1$

27. $y = (\frac{1}{3})^x + 1$ **28.** $y = (\frac{1}{3})^x - 4$

29. $y = (\frac{1}{3})^{x-2}$ **30.** $y = (\frac{1}{3})^{x+2}$

31. $f(x) = (\frac{1}{3})^x - 2$ **32.** $g(x) = (\frac{1}{3})^x + 2$

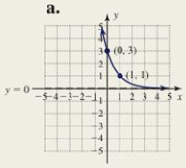
Match each graph to the correct exponential equation.

33. $y = 5^{-x}$ **34.** $y = 4^{-x}$

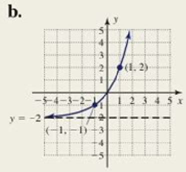
35. $y = 3^{-x+1}$ **36.** $y = 3^{-x} + 1$

37. $y = 2^{x+1} - 2$ **38.** $y = 2^{x+2} - 1$

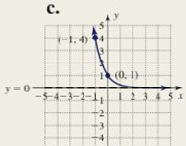
a.



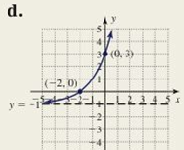
b.



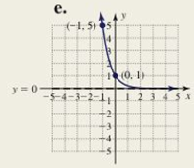
c.



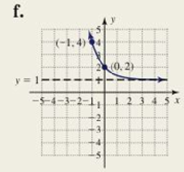
d.



e.



f.



Use a calculator to evaluate each expression, rounded to six decimal places.

39. e^1 **40.** e^0

41. e^2 **42.** e^5

43. $e^{1.5}$ **44.** $e^{-3.2}$

45. $e^{\sqrt{2}}$ **46.** e^{π}

Graph each exponential function.

47. $f(x) = e^{x+3} - 2$ **48.** $g(x) = e^{x-2} + 1$

49. $r(t) = -e^t + 2$ **50.** $s(t) = -e^{t+2}$

51. $p(x) = e^{-x+2} - 1$ **52.** $q(x) = e^{-x-1} + 2$

Solve each exponential equation and check your answer by substituting into the original equation.

53. $10^x = 1000$ **54.** $144 = 12^x$

55. $25^x = 125$ **56.** $81 = 27^x$

57. $8^{x+2} = 32$ **58.** $9^{x-1} = 27$

59. $32^x = 16^{x+1}$ **60.** $100^{x+2} = 1000^x$

61. $(\frac{1}{5})^x = 125$ **62.** $(\frac{1}{4})^x = 64$

63. $(\frac{1}{3})^{2x} = 9^{x-6}$ **64.** $(\frac{1}{2})^{3x} = 8^{x-2}$

65. $(\frac{1}{9})^{x-5} = 3^{3x}$ **66.** $2^{-2x} = (\frac{1}{32})^{x-3}$

67. $25^{3x} = 125^{x-2}$ **68.** $27^{2x+4} = 9^{4x}$

69. $\frac{e^4}{e^{2-x}} = e^3 e$ **70.** $e^x(e^x + e) = \frac{e^x + e^{3x}}{e^{-x}}$

71. $(e^{2x-4})^3 = \frac{e^{x+5}}{e^2}$ **72.** $e^x e^{x+3} = (e^{x+2})^3$

▶ WORKING WITH FORMULAS

73. The growth of a bacteria population: $P(t) = 1000 \cdot 3^t$

If the initial population of a common bacterium is 1000 and the population triples every day, its population is given by the formula shown, where $P(t)$ is the total population after t days. (a) Find the total population 12 hr, 1 day, $1\frac{1}{2}$ days, and 2 days later. (b) Do the outputs show the population is tripling every 24 hr (1 day)? (c) Explain why this is an increasing function. (d) Graph the function using an appropriate scale.

74. Games involving a spinner with numbers 1 through 4: $P(x) = (\frac{1}{4})^x$

Games that involve moving pieces around a board using a fair spinner are fairly common. If the spinner has the numbers 1 through 4, the probability that any one number is spun repeatedly is given by the formula shown, where x represents the number of spins and $P(x)$ represents the probability the same number results x times. (a) What is the probability that the first player spins a 2? (b) What is the probability that all four players spin a 2? (c) Explain why this is a decreasing function.

▶ APPLICATIONS

Use Newton's law of cooling to complete Exercises 75 and 76: $T(x) = T_R + (T_0 - T_R)e^{kx}$.

75. Cold party drinks: Janae was late getting ready for the party, and the liters of soft drinks she bought were still at room temperature (73°F) with guests due to arrive in 15 min. If she puts these in her freezer at −10°F, will the drinks be cold enough (35°F) for her guests? Assume $k \approx -0.031$.

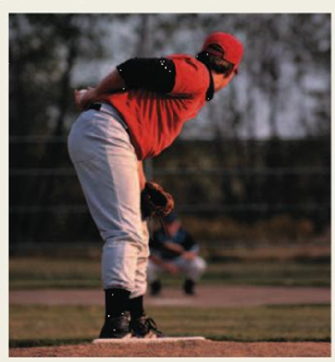
76. Warm party drinks: Newton's law of cooling applies equally well if the "cooling is negative," meaning the object is taken from a colder medium and placed in a warmer one. If a can of soft drink is taken from a 35°F cooler and placed in a room where the temperature is 75°F, how long will it take the drink to warm to 65°F? Assume $k \approx -0.031$.

77. Depreciation: The financial analyst for a large construction firm estimates that its heavy equipment loses one-fifth of its value each year. The current value of the equipment is then modeled by the function $V(t) = V_0(\frac{4}{5})^t$, where V_0 represents the initial value, t is in years, and $V(t)$ represents the value after t years. (a) How much is a large earthmover worth after 1 yr if it cost $125 thousand new? (b) How many years does it take for the earthmover to depreciate to a value of $64 thousand?

78. Depreciation: Photocopiers have become a critical part of the operation of many businesses, and due to their heavy use they can depreciate in value very quickly. If a copier loses $\frac{3}{8}$ of its value each year, the current value of the copier can be modeled by the function $V(t) = V_0(\frac{5}{8})^t$, where V_0 represents the initial value, t is in years, and $V(t)$ represents the value after t yr. (a) How much is this copier worth after one year if it cost $64 thousand new? (b) How many years does it take for the copier to depreciate to a value of $25 thousand?

79. Depreciation: Margaret Madison, DDS, estimates that her dental equipment loses one-sixth of its value each year. (a) Determine the value of an x-ray machine after 5 yr if it cost $216 thousand new, and (b) determine how long until the machine is worth less than $125 thousand.

80. Exponential decay: The groundskeeper of a local high school estimates that due to heavy usage by the baseball and softball teams, the pitcher's mound loses one-fifth of its height every month. (a) Determine the height of the mound after 3 months if it was 25 cm to begin, and (b) determine how long until the pitcher's mound is less than 16 cm high (meaning it must be rebuilt).



81. **Exponential growth:** Similar to a small town doubling in size after a discovery of gold, a business that develops a product in high demand has the potential for doubling its revenue each year for a number of years. The revenue would be modeled by the function $R(t) = R_0 2^t$, where R_0 represents the initial revenue, and $R(t)$ represents the revenue after t years. (a) How much revenue is being generated after 4 yr, if the company's initial revenue was \$2.5 million? (b) How many years does it take for the business to be generating \$320 million in revenue?

82. **Exponential growth:** If a company's revenue grows at a rate of 150% per year (rather than doubling as in Exercise 81), the revenue would be modeled by the function $R(t) = R_0(\frac{3}{2})^t$, where R_0 represents the initial revenue, and $R(t)$ represents the revenue after t years. (a) How much revenue is being generated after 3 yr, if the company's initial revenue was \$256 thousand? (b) How long until the business is generating \$1944 thousand in revenue? (*Hint:* Reduce the fraction.)

Photochromatic sunglasses: Sunglasses that darken in sunlight (photochromatic sunglasses) contain millions of molecules of a substance known as *silver halide*. The molecules are transparent indoors in the absence of ultraviolent (UV) light. Outdoors, UV light from the sun causes the molecules to change shape, darkening the lenses in response to the intensity of the UV light. For certain lenses, the function $T(x) = 0.85^x$ models the transparency of the lenses (as a percentage) based on a UV index x. Find the transparency (to the nearest percent), if the lenses are exposed to

83. sunlight with a UV index of 7 (a high exposure).

84. sunlight with a UV index of 5.5 (a moderate exposure).

85. Given that a UV index of 11 is very high and most individuals should stay indoors, what is the minimum transparency percentage for these lenses?

86. Use trial-and-error to determine the UV index when the lenses are 50% transparent.

Modeling inflation: Assuming the rate of inflation is 5% per year, the predicted price of an item can be modeled by the function $P(t) = P_0(1.05)^t$, where P_0 represents the initial price of the item and t is in years. Use this information to solve Exercises 87 and 88.

87. What will the price of a new car be in the year 2010, if it cost \$20,000 in the year 2000?

88. What will the price of a gallon of milk be in the year 2010, if it cost \$2.95 in the year 2000? Round to the nearest cent.

Modeling radioactive decay: The half-life of a radioactive substance is the time required for half an initial amount of the substance to disappear through decay. The amount of the substance remaining is given by the formula $Q(t) = Q_0(\frac{1}{2})^{\frac{t}{h}}$, where h is the half-life, t represents the elapsed time, and $Q(t)$ represents the amount that remains (t and h must have the same unit of time). Use this information to solve Exercises 89 and 90.

89. Some isotopes of the substance known as thorium have a half-life of only 8 min. (a) If 64 grams are initially present, how many grams (g) of the substance remain after 24 min? (b) How many minutes until only 1 gram (g) of the substance remains?

90. Some isotopes of sodium have a half-life of about 16 hr. (a) If 128 g are initially present, how many grams of the substance remain after 2 days (48 hr)? (b) How many hours until only 1 g of the substance remains?

▶ EXTENDING THE CONCEPT

91. The formula $f(x) = (\frac{1}{2})^x$ gives the probability that "x" number of flips result in heads (or tails). First determine the probability that 20 flips results in *20 heads in a row.* Then use the Internet or some other resource to determine the probability of winning a state lottery (expressed as a decimal). Which has the greater probability? Were you surprised?

92. If $10^{2x} = 25$, what is the value of 10^{-x}?

93. If $5^{3x} = 27$, what is the value of 5^{2x}?

94. If $3^{0.5x} = 5$, what is the value of 3^{x+1}?

95. If $\left(\frac{1}{2}\right)^{x+1} = \frac{1}{3}$, what is the value of $\left(\frac{1}{2}\right)^{-x}$?

The growth rate constant that governs an exponential function was introduced on page 427.

96. In later sections, we will easily be able to find the growth constant k for Goldsboro, where

$P(t) = 1000 \cdot 2^t$. For now we'll approximate its value using the rate of change formula on a very small interval of the domain. From the definition of an exponential function, $\frac{\Delta P}{\Delta t} = kP(t)$. Since k is constant, we can choose any value of t, say $t = 4$. For $h = 0.0001$, we have

$$\frac{1000 \cdot 2^{4+0.0001} - 1000 \cdot 2^4}{0.0001} = k \cdot P(4)$$

(a) Use the equation shown to solve for k (round to thousandths). (b) Show that k is constant by completing the same exercise for $t = 2$ and $t = 6$. (c) Verify that $P(t) = 1000 \cdot 2^t$ and $P(t) = 1000e^{kt}$ give approximately the same results.

97. As we analyze the expression $\left(1 + \frac{1}{x}\right)^x$, we notice a battle (of sorts) takes place between the base $\left(1 + \frac{1}{x}\right)$ and the exponent x. As $x \to \infty$, $\frac{1}{x}$ becomes infinitely small, but the exponent becomes infinitely large. So what happens? The answer is best understood by computing a series of *average rates of change,* using the intervals given here. Using the tools of Calculus, it can be shown that this rate of change becomes infinitely small, and that the "battle" ends at the irrational number e. In other words, e is an upper bound on the value of this expression, regardless of how large x becomes.

a. Use a calculator to find the average rate of change for $y = \left(1 + \frac{1}{x}\right)^x$ in these intervals: $[1, 1.01]$, $[4, 4.01]$, $[10, 10.01]$, and $[20, 20.01]$. What do you notice?

b. What is the smallest integer value for x that gives the value of e correct to four decimal places?

c. Use a graphing calculator to graph this function on a window size of $x \in [0, 25]$ and $y \in [0, 3]$. Does the graph seem to support the statements above?

▶ MAINTAINING YOUR SKILLS

98. **(2.2)** Given $f(x) = 2x^2 - 3x$, determine:
$f(-1)$, $f(\frac{1}{3})$, $f(a)$, $f(a + h)$

99. **(3.3)** Graph $g(x) = \sqrt{x + 2} - 1$ using a shift of the parent function. Then state the domain and range of g.

100. **(1.3)** Solve the following equations:

a. $-2\sqrt{x - 3} + 7 = 21$

b. $\frac{9}{x + 3} + 3 = \frac{12}{x - 3}$

101. **(R.7)** Identify each formula:

a. $\frac{4}{3}\pi r^3$

b. $\frac{1}{2}bh$

c. lwh

d. $a^2 + b^2 = c^2$

4.3 Logarithms and Logarithmic Functions

Learning Objectives

In Section 4.3 you will learn how to:

- ❑ **A.** Write exponential equations in logarithmic form
- ❑ **B.** Find common logarithms and natural logarithms
- ❑ **C.** Graph logarithmic functions
- ❑ **D.** Find the domain of a logarithmic function
- ❑ **E.** Solve applications of logarithmic functions

A **transcendental function** is one whose solutions are beyond or *transcend* the methods applied to polynomial functions. The exponential function and its inverse, called the logarithmic function, are transcendental functions. In this section, we'll use the concept of an inverse to develop an understanding of the logarithmic function, which has numerous applications that include measuring pH levels, sound and earthquake intensities, barometric pressure, and other natural phenomena.

A. Exponential Equations and Logarithmic Form

While exponential functions have a large number of significant applications, we can't appreciate their full value until we develop the inverse function. Without it, we're unable to solve all but the simplest equations, of the type encountered in Section 4.2. Using the fact that $f(x) = b^x$ is one-to-one, we have the following:

1. The function $f^{-1}(x)$ must exist.
2. We can graph $f^{-1}(x)$ by interchanging the x- and y-coordinates of points from $f(x)$.
3. The domain of $f(x)$ will become the range of $f^{-1}(x)$.
4. The range of $f(x)$ will become the domain of $f^{-1}(x)$.
5. The graph of $f^{-1}(x)$ will be a reflection of $f(x)$ across the line $y = x$.

Table 4.6 contains selected values for $f(x) = 2^x$. The values for $f^{-1}(x)$ in Table 4.7 were found by interchanging x- and y-coordinates. Both functions were then graphed using these values.

Table 4.6

$f(x)$: $y = 2^x$

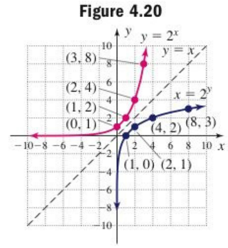
x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

Table 4.7

$f^{-1}(x)$: $x = 2^y$

x	(x)
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Figure 4.20



The interchange of x and y and the graphs in Figure 4.20 show that $f^{-1}(x)$ has an x-intercept of $(1, 0)$, a vertical asymptote at $x = 0$, a domain of $x \in (0, \infty)$, and a range of $y \in (-\infty, \infty)$. To find *an equation* for $f^{-1}(x)$, we'll attempt to use the algebraic approach employed previously. For $f(x) = 2^x$,

1. use y instead of $f(x)$: $y = 2^x$.
2. interchange x and y: $x = 2^y$.

At this point we have an *implicit* equation for the inverse function, but no algebraic operations that enable us to solve *explicitly* for y in terms of x. Instead, we write $x = 2^y$ in function form by noting that "y is the exponent that goes on base 2 to obtain x."

In the language of mathematics, this phrase is represented by $y = \log_2 x$ and is called a **logarithmic function** with base 2. For $y = b^x$, $x = b^y \rightarrow y = \log_b x$ is the inverse function, and is read, "y is the logarithm base b of x." For this new function, we must always keep in mind what y *represents*—y is an exponent. In fact, *y is the exponent that goes on base b to obtain x*: $y = \log_b x$.

Logarithmic Functions

For positive numbers x and b, with $b \neq 1$,

$$y = \log_b x \text{ if and only if } x = b^y$$

The function $f(x) = \log_b x$ is a logarithmic function with base b. The expression $\log_b x$ is simply called a logarithm, and represents the exponent on b that yields x.

Finally, note the equations $x = b^y$ and $y = \log_b x$ are equivalent. We say that $x = b^y$ is the **exponential form** of the equation, whereas $y = \log_b x$ is written in **logarithmic form.**

EXAMPLE 1 ► **Converting from Logarithmic Form to Exponential Form**

Write each equation in words, then in exponential form.

a. $3 = \log_2 8$ **b.** $1 = \log_{10} 10$ **c.** $0 = \log_e 1$ **d.** $-2 = \log_3(\frac{1}{9})$

Solution ►

a. $3 = \log_2 8 \rightarrow 3$ is the exponent on base 2 for 8: $2^3 = 8$.
b. $1 = \log_{10} 10 \rightarrow 1$ is the exponent on base 10 for 10: $10^1 = 10$.
c. $0 = \log_e 1 \rightarrow 0$ is the exponent on base e for 1: $e^0 = 1$.
d. $-2 = \log_3(\frac{1}{9}) \rightarrow -2$ is the exponent on base 3 for $\frac{1}{9}$: $3^{-2} = \frac{1}{9}$.

Now try Exercises 7 through 22 ►

To convert from exponential form to logarithmic form, note the exponent on the base and read from there. For $5^3 = 125$, "3 is the exponent that goes on base 5 for 125," or *3 is the logarithm base 5 of 125*: $3 = \log_5 125$.

EXAMPLE 2 ► **Converting from Exponential Form to Logarithmic Form**

Write each equation in words, then in logarithmic form.

a. $10^3 = 1000$ **b.** $2^{-1} = \frac{1}{2}$ **c.** $e^2 \approx 7.389$ **d.** $9^{\frac{3}{2}} = 27$

Solution ►

a. $10^3 = 1000 \rightarrow 3$ is the exponent on base 10 for 1000, or
3 is the logarithm base 10 of 1000: $3 = \log_{10} 1000$.
b. $2^{-1} = \frac{1}{2} \rightarrow -1$ is the exponent on base 2 for $\frac{1}{2}$, or
-1 is the logarithm base 2 of $\frac{1}{2}$: $-1 = \log_2(\frac{1}{2})$.
c. $e^2 \approx 7.389 \rightarrow 2$ is the exponent on base e for 7.389, or
2 is the logarithm base e of 7.389: $2 \approx \log_e 7.389$.
d. $9^{\frac{3}{2}} = 27 \rightarrow \frac{3}{2}$ is the exponent on base 9 for 27, or
$\frac{3}{2}$ is the logarithm base 9 of 27: $\frac{3}{2} = \log_9 27$.

A. You've just learned how to write exponential equations in logarithmic form

Now try Exercises 23 through 38 ►

B. Finding Common Logarithms and Natural Logarithms

Of all possible bases for $\log_b x$, the most common are base 10 (likely due to our base-10 number system), and base e (due to the advantages it offers in advanced courses). The expression $\log_{10} x$ is called a **common logarithm,** and we simply write $\log x$ for $\log_{10} x$. The expression $\log_e x$ is called a **natural logarithm,** and is written in abbreviated form as $\ln x$.

Some logarithms are easy to evaluate. For example, $\log 100 = 2$ since $10^2 = 100$, and $\log \frac{1}{100} = -2$ since $10^{-2} = \frac{1}{100}$. But what about the expressions log 850 and ln 4? Because logarithmic functions are continuous on their domains, a value exists for log 850 and the equation $10^x = 850$ must have a solution. Further, the inequalities

$$\log 100 < \log 850 < \log 1000$$
$$2 < \log 850 < 3$$

tell us that log 850 must be between 2 and 3. Fortunately, modern calculators can compute base-10 and base-e logarithms instantly, often with nine-decimal-place accuracy. For log 850, press LOG, then input 850 and press ENTER. The display should read 2.929418926. We can also use the calculator to verify $10^{2.929418926} = 850$ (see Figure 4.21). For ln 4, press the LN key, then input 4 and press ENTER to obtain 1.386294361. Figure 4.22 verifies that $e^{1.386294361} = 4$.

WORTHY OF NOTE

We do something similar with square roots. Technically, the "square root of x" should be written $\sqrt[2]{x}$. However, square roots are so common we often leave off the two, assuming that if no index is written, an index of two is intended.

Figure 4.21

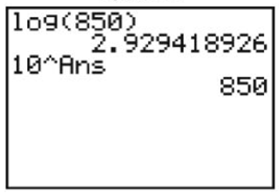
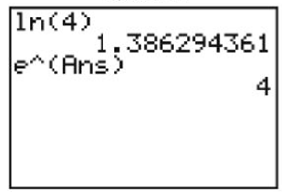


Figure 4.22



EXAMPLE 3 ► **Finding the Value of a Logarithm**

Determine the value of each logarithm without using a calculator:

a. $\log_2 8$ **b.** $\log_5(\frac{1}{25})$ **c.** $\log_e e$ **d.** $\log_{10}\sqrt{10}$

Solution ►

a. $\log_2 8$ represents the exponent on 2 for 8: $\log_2 8 = 3$, since $2^3 = 8$.

b. $\log_5(\frac{1}{25})$ represents the exponent on 5 for $\frac{1}{25}$: $\log_5 \frac{1}{25} = -2$, since $5^{-2} = \frac{1}{25}$.

c. $\log_e e$ represents the exponent on e for e: $\log_e e = 1$, since $e^1 = e$.

d. $\log_{10}\sqrt{10}$ represents the exponent on 10 for $\sqrt{10}$: $\log_{10}\sqrt{10} = \frac{1}{2}$, since $10^{\frac{1}{2}} = \sqrt{10}$.

Now try Exercises 39 through 50 ►

EXAMPLE 4 ► **Using a Calculator to Find Logarithms**

Use a calculator to evaluate each logarithmic expression. Verify the result.

a. $\log 1857$ **b.** $\log 0.258$ **c.** $\ln 3.592$

Solution ►

a. $\log 1857 = 3.268811904$,
$10^{3.268811904} = 1857$ ✓

b. $\log 0.258 = -0.588380294$,
$10^{-0.588380294} = 0.258$ ✓

c. $\ln 3.592 \approx 1.27870915$
$e^{1.27870915} \approx 3.592$ ✓

☑ **B.** You've just learned how to find common logarithms and natural logarithms

Now try Exercises 51 through 58 ►

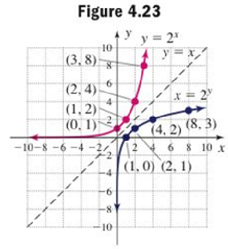
C. Graphing Logarithmic Functions

For convenience and ease of calculation, our first examples of logarithmic graphs are done using base-2 logarithms. However, the basic shape of a logarithmic graph remains unchanged regardless of the base used, and transformations can be applied to $y = \log_b(x)$ for any value of b. For $y = a\log(x \pm h) \pm k$, a continues to govern stretches, compressions, and vertical reflections, the graph will shift horizontally h units opposite the sign, and shift k units vertically in the same direction as the sign. Our earlier graph of $y = \log_2 x$ was completed using $x = 2^y$ as the inverse function for $y = 2^x$ (Figure 4.20). For reference, the graph is repeated in Figure 4.23.

WORTHY OF NOTE

As with the basic graphs we studied in Section 2.6, logarithmic graphs maintain the same characteristics when transformations are applied, and these graphs *should be added to your collection of basic functions,* ready for recall or analysis as the situation requires.

Figure 4.23



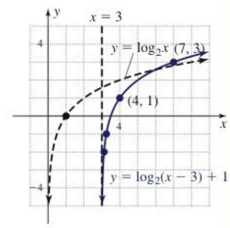
EXAMPLE 5 ► Graphing Logarithmic Functions Using Transformations

Graph $f(x) = \log_2(x - 3) + 1$ using transformations of $y = \log_2 x$ (not by simply plotting points). Clearly state what transformations are applied.

Solution ► The graph of f is the same as that of $y = \log_2 x$, shifted 3 units right and 1 unit up. The vertical asymptote will be at $x = 3$ and the point $(1, 0)$ from the basic graph becomes $(1 + 3, 0 + 1) = (4, 1)$. Knowing the graph's basic shape, we compute one additional point using $x = 7$:

$$\begin{aligned} f(7) &= \log_2(7 - 3) + 1 \\ &= \log_2 4 + 1 \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

Figure 4.24



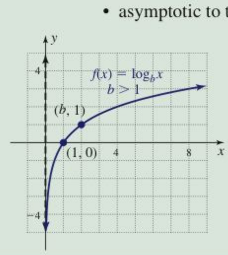
The point (7, 3) is on the graph, shown in Figure 4.24.

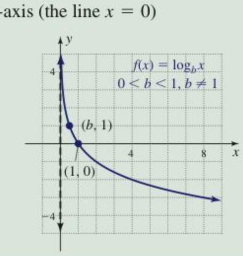
Now try Exercises 59 through 62 ►

As with the exponential functions, much can be learned from graphs of logarithmic functions and a summary of important characteristics is given here.

$f(x) = \log_b x$, $b > 0$ and $b \neq 1$

- one-to-one function
- domain: $x \in (0, \infty)$
- increasing if $b > 1$
- x-intercept $(1, 0)$
- range: $y \in \mathbb{R}$
- decreasing if $0 < b < 1$
- asymptotic to the y-axis (the line $x = 0$)



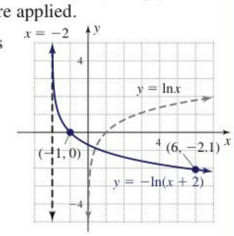


EXAMPLE 6 ► Graphing Logarithmic Functions Using Transformations

Graph $g(x) = -\ln(x + 2)$ using transformations of $y = \ln x$ (not by simply plotting points). Clearly state what transformations are applied.

Solution ► The graph of g is the same as $y = \ln x$, shifted 2 units left, then reflected across the x-axis. The vertical asymptote will be at $x = -2$, and the point $(1, 0)$ from the basic function becomes $(-1, 0)$. To complete the graph we compute $f(6)$:

$$\begin{aligned} f(6) &= -\ln(6 + 2) \\ &= -\ln 8 \\ &\approx -2.1 \text{ (using a calculator).} \end{aligned}$$



The point (6, −2.1) is on the graph shown in the figure.

Now try Exercises 63 through 72 ►

WORTHY OF NOTE

Accurate graphs can actually be drawn for logarithms of *any base* using what is called the base-change formula, introduced in Section 4.4.

C. You've just learned how to graph logarithmic functions

D. Finding the Domain of a Logarithmic Function

Examples 5 and 6 illustrate how the domain of a logarithmic function can change when certain transformations are applied. Since the domain consists of *positive* real numbers, the argument of a logarithmic function must be greater than zero. This means finding the domain often consists of solving various inequalities, which can be done using the skills acquired in Sections 2.5 and 3.7.

EXAMPLE 7 ▶ Finding the Domain of a Logarithmic Function

Determine the domain of each function.

a. $p(x) = \log_2(2x + 3)$ **b.** $q(x) = \log_5(x^2 - 2x)$

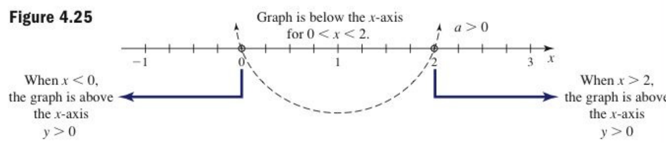
c. $r(x) = \log\left(\frac{3 - x}{x + 3}\right)$ **d.** $f(x) = \ln|x - 2|$

Solution ▶ Begin by writing the argument of each logarithmic function as a greater than inequality.

a. Solving $2x + 3 > 0$ for x gives $x > -\frac{3}{2}$, and the domain of p is $x \in (-\frac{3}{2}, \infty)$.

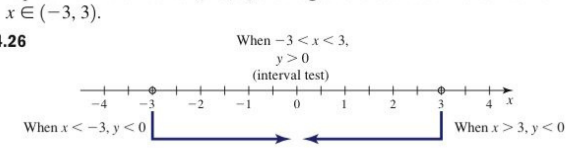
b. For $x^2 - 2x > 0$, we note $y = x^2 - 2x$ is a parabola, opening upward, with zeroes at $x = 0$ and $x = 2$ (see Figure 4.25). This means $x^2 - 2x$ will be positive for $x < 0$ and $x > 2$. The domain of q is $x \in (-\infty, 0) \cup (2, \infty)$.

Figure 4.25



c. For $\frac{3 - x}{x + 3} > 0$, we note $y = \frac{3 - x}{x + 3}$ has a zero at $x = 3$, and a vertical asymptote at $x = -3$. Outputs are positive when $x = 0$ (see Figure 4.26), so y is positive in the interval $(-3, 3)$ and negative elsewhere. The domain of r is $x \in (-3, 3)$.

Figure 4.26



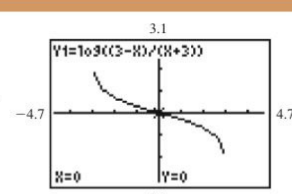
d. For $|x - 2| > 0$, we note $y = |x - 2|$ is the graph of $y = |x|$ shifted 2 units right, with its vertex at (2, 0). The graph is positive for all x, except at $x = 2$. The domain of f is $x \in (-\infty, 2) \cup (2, \infty)$.

☑ **D.** You've just learned how to find the domain of a logarithmic function

Now try Exercises 73 through 78 ▶

GRAPHICAL SUPPORT

The domain for $r(x) = \log_{10}\left(\frac{3 - x}{x + 3}\right)$ from Example 6c can be confirmed using the LOG key on a graphing calculator. Use the key to enter the equation as Y_1 on the Y= screen, then graph the function using the ZOOM **4:ZDecimal** option. Both the graph and TABLE feature help to confirm the domain is $x \in (-3, 3)$.



E. Applications of Logarithms

As we use mathematics to model the real world, there are times when the range of outcomes is so large that using a linear scale would be hard to manage. For example, compared to a whisper—the scream of a jet engine may be up to *ten billion times* louder. Similar ranges exist in the measurement of earthquakes, light, acidity, and voltage. In lieu of a linear scale, logarithms are used where each whole number increase in magnitude represents a tenfold increase in the intensity. For earthquake intensities (a measure of the wave energy produced by the quake), units called **magnitudes** (or **Richter values**) are used. Earthquakes with a magnitude of 6.5 or more often cause significant damage, while the slightest earthquakes have magnitudes near 1 and are barely perceptible. The magnitude of the intensity $M(I)$ is given by $M(I) = \log\left(\frac{I}{I_0}\right)$ where I is the measured intensity and I_0 represents a minimum or **reference intensity.** The value of I is often given as a multiple of this reference intensity.

WORTHY OF NOTE

The **decibel** (dB) is the reference unit for sound, and is based on the faintest sound a person can hear, called the **threshold of audibility.** It is a base-10 logarithmic scale, meaning a sound 10 times more intense is one decibel louder.

EXAMPLE 8A ► Finding the Magnitude of an Earthquake

Find the magnitude of an earthquake (rounded to hundredths) with the intensities given.

a. $I = 4000I_0$ **b.** $I = 8{,}252{,}000I_0$

Solution ► **a.**

$$M(I) = \log\left(\frac{I}{I_0}\right) \quad \text{magnitude equation}$$

$$M(4000I_0) = \log\left(\frac{4000I_0}{I_0}\right) \quad \text{substitute } 4000I_0 \text{ for } I$$

$$= \log 4000 \quad \text{simplify}$$

$$\approx 3.60 \quad \text{result}$$

The earthquake had a magnitude of 3.6.

b.

$$M(I) = \log\left(\frac{I}{I_0}\right) \quad \text{magnitude equation}$$

$$M(8{,}252{,}000I_0) = \log\left(\frac{8252000I_0}{I_0}\right) \quad \text{substitute } 8{,}252{,}000\ I_0 \text{ for } I$$

$$= \log 8{,}252{,}000 \quad \text{simplify}$$

$$\approx 6.92 \quad \text{result}$$

The earthquake had a magnitude of about 6.92.

EXAMPLE 8B ► Comparing Earthquake Intensity to the Reference Intensity

How many times more intense than the reference intensity I_0 is an earthquake with a magnitude of 6.7?

Solution ►

$$M(I) = \log\left(\frac{I}{I_0}\right) \quad \text{magnitude equation}$$

$$6.7 = \log\left(\frac{I}{I_0}\right) \quad \text{substitute 6.7 for } M(I)$$

$$10^{6.7} = \left(\frac{I}{I_0}\right) \quad \text{exponential form}$$

$$I = 10^{6.7}I_0 \quad \text{solve for } I$$

$$I = 5{,}011{,}872I_0 \quad 10^{6.7} \approx 5{,}011{,}872$$

An earthquake of magnitude 6.7 is over 5 million times more intense than the reference intensity.

EXAMPLE 8C ► Comparing Earthquake Intensities

The Great San Francisco Earthquake of 1906 left over 800 dead, did \$80,000,000 in damage (see photo), and had an estimated magnitude of 7.7. The 2004 Indian Ocean earthquake, which had a magnitude of approximately 9.2, triggered a series of deadly tsunamis and was responsible for nearly 300,000 casualties. How much more intense was the 2004 quake?

Solution ► To find the intensity of each quake, substitute the given magnitude for $M(I)$ and solve for I:

$M(I) = \log\left(\frac{I}{I_0}\right)$	magnitude equation	$M(I) = \log\left(\frac{I}{I_0}\right)$
$7.7 = \log\left(\frac{I}{I_0}\right)$	substitute for $M(I)$	$9.2 = \log\left(\frac{I}{I_0}\right)$
$10^{7.7} = \left(\frac{I}{I_0}\right)$	exponential form	$10^{9.2} = \left(\frac{I}{I_0}\right)$
$10^{7.7}I_0 = I$	solve for I	$10^{9.2}I_0 = I$

Using these intensities, we find that the Indian Ocean quake was $\frac{10^{9.2}}{10^{7.7}} = 10^{1.5} \approx 31.6$ times more intense.

Now try Exercises 81 through 90 ►

A second application of logarithmic functions involves the relationship between altitude and barometric pressure. The altitude or height above sea level can be determined by the formula $H = (30T + 8000)\ln\left(\frac{P_0}{P}\right)$, where H is the altitude in meters for a temperature T in degrees Celsius, P is the barometric pressure at a given altitude in units called **centimeters of mercury** (cmHg), and P_0 is the barometric pressure at sea level: 76 cmHg.

EXAMPLE 9 ► Using Logarithms to Determine Altitude

Hikers at the summit of Mt. Shasta in northern California take a pressure reading of 45.1 cmHg at a temperature of 9°C. How high is Mt. Shasta?

Solution ► For this exercise, $P_0 = 76$, $P = 45.1$, and $T = 9$. The formula yields

$$H = (30T + 8000)\ln\left(\frac{P_0}{P}\right) \quad \text{given formula}$$

$$= [30(9) + 8000]\ln\left(\frac{76}{45.1}\right) \quad \text{substitute given values}$$

$$= 8270\ln\left(\frac{76}{45.1}\right) \quad \text{simplify}$$

$$\approx 4316 \quad \text{result}$$

Mt. Shasta is about 4316 m high.

Now try Exercises 91 through 94 ►

Our final application shows the versatility of logarithmic functions, and their value as a real-world model. Large advertising agencies are well aware that after a new ad campaign, sales will increase rapidly as more people become aware of the product. Continued advertising will give the new product additional market share, but once the "newness" wears off and the competition begins responding, sales tend to taper off—regardless of any additional amount spent on ads. This phenomenon can be modeled by the function

$$S(d) = k + a \ln d,$$

where $S(d)$ is the number of expected sales after d dollars are spent, and a and k are constants related to product type and market size (**see Exercises 95 and 96).**

EXAMPLE 10 ► Using Logarithms for Marketing Strategies

Market research has shown that sales of the MusicMaster, a new system for downloading and playing music, can be approximated by the equation $S(d) = 2500 + 250 \ln d$, where $S(d)$ is the number of sales after d thousand dollars is spent on advertising.

a. What sales volume is expected if the advertising budget is $40,000?

b. If the company needs to sell 3500 units to begin making a profit, how much should be spent on advertising?

c. To gain a firm hold on market share, the company is willing to continue spending on advertising up to a point where only 3 additional sales are gained for each $1000 spent, in other words, $\frac{\Delta S}{\Delta d} = \frac{3}{1}$. Verify that spending between $83,200 and $83,300 puts them very close to this goal.

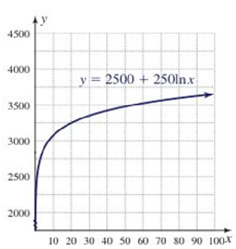
Solution ► **a.** For sales volume, we simply evaluate the function for $d = 40$ (d in thousands):

$$\begin{aligned} S(d) &= 2500 + 250 \ln d && \text{given equation} \\ S(40) &= 2500 + 250 \ln 40 && \text{substitute 40 for } d \\ &\approx 2500 + 922 && 250 \ln 40 \approx 922 \\ &= 3422 \end{aligned}$$

Spending $40,000 on advertising will generate approximately 3422 sales.

b. To find the advertising budget needed, we substitute number of sales and solve for d.

$$\begin{aligned} S(d) &= 2500 + 250 \ln d && \text{given equation} \\ 3500 &= 2500 + 250 \ln d && \text{substitute 2500 for } S(d) \\ 1000 &= 250 \ln d && \text{subtract 2500} \\ 4 &= \ln d && \text{divide by 250} \\ e^4 &= d && \text{exponential form} \\ 54.598 &\approx d && e^4 \approx 54.598 \end{aligned}$$



About $54,600 should be spent in order to sell 3500 units.

c. To verify, we calculate the average rate of change on the interval [83.2, 83.3].

$$\frac{\Delta S}{\Delta d} = \frac{S(d_2) - S(d_1)}{d_2 - d_1} \quad \text{formula for average rate of change}$$

$$= \frac{S(83.3) - S(83.2)}{83.3 - 83.2} \quad \text{substitute 83.3 for } d_2 \text{ and 83.2 for } d_1$$

$$\approx \frac{3605.6 - 3605.3}{0.1} \quad \text{evaluate } S(83.3) \text{ and } S(83.2)$$

$$= 3$$

The average rate of change in this interval is very close to $\frac{3}{1}$.

E. You've just learned how to solve applications of logarithmic functions

Now try Exercises 95 and 96 ►

4.3 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A logarithmic function is of the form $y =$ ______, where _____ > 0, _____ $\neq 1$ and inputs are ______ than zero.

2. The range of $y = \log_b x$ is all ______ ______, and the domain is $x \in$ ______. Further, as $x \to 0$, $y \to$ ____.

3. For logarithmic functions of the form $y = \log_b x$, the x-intercept is ______, since $\log_b 1 =$ ______.

4. The function $y = \log_b x$ is an increasing function if ______, and a decreasing function if ______.

5. What number does the expression $\log_2 32$ represent? Discuss/Explain how $\log_2 32 = \log_2 2^5$ justifies this fact.

6. Explain how the graph of $Y = \log_b(x - 3)$ can be obtained from $y = \log_b x$. Where is the "new" x-intercept? Where is the new asymptote?

► DEVELOPING YOUR SKILLS

Write each equation in exponential form.

7. $3 = \log_2 8$ **8.** $2 = \log_3 9$

9. $-1 = \log_7 \frac{1}{7}$ **10.** $-3 = \log_e \frac{1}{e^3}$

11. $0 = \log_9 1$ **12.** $0 = \log_e 1$

13. $\frac{1}{3} = \log_8 2$ **14.** $\frac{1}{2} = \log_{81} 9$

15. $1 = \log_2 2$ **16.** $1 = \log_e e$

17. $\log_7 49 = 2$ **18.** $\log_4 16 = 2$

19. $\log_{10} 100 = 2$ **20.** $\log_{10} 10{,}000 = 4$

21. $\log_e(54.598) \approx 4$ **22.** $\log_{10} 0.001 = -3$

Write each equation in logarithmic form.

23. $4^3 = 64$ **24.** $e^3 \approx 20.086$

25. $3^{-2} = \frac{1}{9}$ **26.** $2^{-3} = \frac{1}{8}$

27. $e^0 = 1$ **28.** $8^0 = 1$

29. $(\frac{1}{3})^{-3} = 27$ **30.** $(\frac{1}{5})^{-2} = 25$

31. $10^3 = 1000$ **32.** $e^1 = e$

33. $10^{-2} = \frac{1}{100}$ **34.** $10^{-5} = \frac{1}{100{,}000}$

35. $4^{\frac{3}{2}} = 8$ **36.** $e^{\frac{3}{4}} \approx 2.117$

37. $4^{\frac{-3}{2}} = \frac{1}{8}$ **38.** $27^{\frac{-2}{3}} = \frac{1}{9}$

Determine the value of each logarithm without using a calculator.

39. $\log_4 4$ **40.** $\log_9 9$

41. $\log_{11} 121$ **42.** $\log_{12} 144$

43. $\log_e e$ **44.** $\log_e e^2$

45. $\log_4 2$ **46.** $\log_{81} 9$

47. $\log_7 \frac{1}{49}$ **48.** $\log_9 \frac{1}{81}$

49. $\log_e \dfrac{1}{e^2}$ **50.** $\log_e \dfrac{1}{\sqrt{e}}$

Use a calculator to evaluate each expression, rounded to four decimal places.

51. log 50 **52.** log 47

53. ln 1.6 **54.** ln 0.75

55. ln 225 **56.** ln 381

57. $\log \sqrt{37}$ **58.** $\log 4\pi$

Graph each function *using transformations* of $y = \log_b x$ and strategically plotting a few points. Clearly state the transformations applied.

59. $f(x) = \log_2 x + 3$ **60.** $g(x) = \log_2(x - 2)$

61. $h(x) = \log_2(x - 2) + 3$ **62.** $p(x) = \log_3 x - 2$

63. $q(x) = \ln(x + 1)$ **64.** $r(x) = \ln(x + 1) - 2$

65. $Y_1 = -\ln(x + 1)$ **66.** $Y_2 = -\ln x + 2$

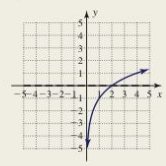
Use the transformation equation $y = af(x \pm h) \pm k$ and the asymptotes and intercept(s) of the parent function to match each equation to one of the graphs given.

67. $y = \log_b(x + 2)$ **68.** $y = 2\log_b x$

69. $y = 1 - \log_b x$ **70.** $y = \log_b x - 1$

71. $y = \log_b x + 2$ **72.** $y = -\log_b x$

I.



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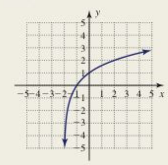

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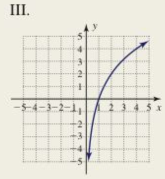

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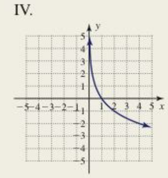

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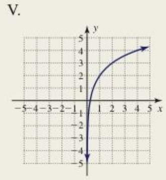

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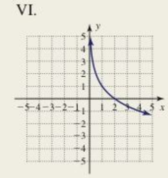


Determine the domain of the following functions.

73. $y = \log_6\left(\dfrac{x + 1}{x - 3}\right)$ **74.** $y = \ln\left(\dfrac{x - 2}{x + 3}\right)$

75. $y = \log_5 \sqrt{2x - 3}$ **76.** $y = \ln\sqrt{5 - 3x}$

77. $y = \log(9 - x^2)$ **78.** $y = \ln(9x - x^2)$

▶ WORKING THE FORMULAS

79. pH level: $f(x) = -\log_{10} x$

The pH level of a solution indicates the concentration of hydrogen (H^+) ions in a unit called *moles per liter.* The pH level $f(x)$ is given by the formula shown, where x is the ion concentration (given in scientific notation). A solution with pH < 7 is called an acid (lemon juice: pH ≈ 2), and a solution with pH > 7 is called a base (household ammonia: pH ≈ 11). Use the formula to determine the pH level of tomato juice if $x = 7.94 \times 10^{-5}$ moles per liter. Is this an acid or base solution?

80. Time required for an investment to double: $T(r) = \dfrac{\log 2}{\log(1 + r)}$

The time required for an investment to double in value is given by the formula shown, where r represents the interest rate (expressed as a decimal) and $T(r)$ gives the years required. How long would it take an investment to double if the interest rate were (a) 5%, (b) 8%, (c) 12%?

▶ APPLICATIONS

Earthquake intensity: Use the information provided in Example 8 to answer the following.

81. Find the value of $M(I)$ given
a. $I = 50{,}000I_0$ and **b.** $I = 75{,}000I_0$.

82. Find the intensity I of the earthquake given
a. $M(I) = 3.2$ and **b.** $M(I) = 8.1$.

83. Earthquake intensity: On June 25, 1989, an earthquake with magnitude 6.2 shook the southeast side of the Island of Hawaii (near Kalapana), causing some \$1,000,000 in damage. On October 15, 2006, an earthquake measuring 6.7 on the Richter scale shook the northwest side of the island, causing over \$100,000,000 in damage. How much more intense was the 2006 quake?

84. Earthquake intensity: The most intense earthquake of the modern era occurred in Chile on May 22, 1960, and measured 9.5 on the Richter scale. How many times more intense was this earthquake, than the quake that hit Northern Sumatra (Indonesia) on March 28, 2005, and measured 8.7?

Brightness of a star: The brightness or intensity I of a star as perceived by the naked eye is measured in units called *magnitudes.* The brightest stars have magnitude 1 $[M(I) = 1]$ and the dimmest have magnitude 6 $[M(I) = 6]$. The magnitude of a star is given by the equation $M(I) = 6 - 2.5 \cdot \log\left(\frac{I}{I_0}\right)$, where I is the actual intensity of light from the star and I_0 is the faintest light visible to the human eye, called the reference intensity. The intensity I is often given as a multiple of this reference intensity.

85. Find the value of $M(I)$ given
a. $I = 27I_0$ and **b.** $I = 85I_0$.

86. Find the intensity I of a star given
a. $M(I) = 1.6$ and **b.** $M(I) = 5.2$.

Intensity of sound: The intensity of sound as perceived by the human ear is measured in units called decibels (dB). The loudest sounds that can be withstood without damage to the eardrum are in the 120- to 130-dB range, while a whisper may measure in the 15- to 20-dB range. Decibel measure is given by the equation $D(I) = 10\log\left(\frac{I}{I_0}\right)$, where I is the actual intensity of the sound and I_0 is the faintest sound perceptible by the human ear—called the reference intensity. The intensity I is often given as a multiple of this reference intensity, but often the constant 10^{-16} (watts per cm^2; W/cm^2) is used as the threshold of audibility.

87. Find the value of $D(I)$ given
a. $I = 10^{-14}$ and **b.** $I = 10^{-4}$.

88. Find the intensity I of the sound given
a. $D(I) = 83$ and **b.** $D(I) = 125$.

89. Sound intensity of a hair dryer: Every morning (it seems), Jose is awakened by the mind-jarring, ear-jamming sound of his daughter's hair dryer (75 dB). He knew he was exaggerating, but told her (many times) of how it reminded him of his railroad days, when the air compressor for the pneumatic tools was running (110 dB). In fact, how many times more intense was the sound of the air compressor compared to the sound of the hair dryer?

90. Sound intensity of a busy street: The decibel level of noisy, downtown traffic has been estimated at 87 dB, while the laughter and banter at a loud party might be in the 60 dB range. How many times more intense is the sound of the downtown traffic?

The *barometric equation* $H = (30T + 8000)\ln\left(\frac{P_0}{P}\right)$ was discussed in Example 9.

91. Temperature and atmospheric pressure: Determine the height of Mount McKinley (Alaska), if the temperature at the summit is $-10°$C, with a barometric reading of 34 cmHg.

92. Temperature and atmospheric pressure: A large passenger plane is flying cross-country. The instruments on board show an air temperature of 3°C, with a barometric pressure of 22 cmHg. What is the altitude of the plane?

93. Altitude and atmospheric pressure: By definition, a mountain pass is a low point between two mountains. Passes may be very short with steep slopes, or as large as a valley between two peaks. Perhaps the highest drivable pass in the world is the Semo La pass in central Tibet. At its highest elevation, a temperature reading of 8°C was taken, along with a barometer reading of 39.3 cmHg. (a) Approximately how high is the Semo La pass? (b) While traveling up to this pass, an elevation marker is seen. If the barometer reading was 47.1 cmHg at a temperature of 12°C, what height did the marker give?

94. Altitude and atmospheric pressure: Hikers on Mt. Everest take successive readings of 35 cmHg at 5°C and 30 cmHg at −10°C. (a) How far up the mountain are they at each reading? (b) Approximate the height of Mt. Everest if the temperature at the summit is −27°C and the barometric pressure is 22.2 cmHg.

95. Marketing budgets: An advertising agency has determined the number of items sold by a certain client is modeled by the equation $N(A) = 1500 + 315 \ln A$, where $N(A)$ represents the number of sales after spending A thousands of dollars on advertising. Determine the approximate number of items sold on an advertising budget of (a) \$10,000; (b) \$50,000. (c) Use the TABLE feature of a calculator to estimate how large a budget is needed (to the nearest \$500 dollars) to sell 3000 items. (d) This company is willing to continue advertising as long as eight additional sales are gained for every \$1000 spent: $\frac{\Delta N}{\Delta A} = \frac{8}{1}$. Show this occurs by spending between \$39,300 to \$39,400.

96. Sports promotions: The accountants for a major boxing promoter have determined that the number of pay-per-view subscriptions sold to their championship bouts can be modeled by the function $N(d) = 15{,}000 + 5850 \ln d$, where $N(d)$ represents the number of subscriptions sold after spending d thousand dollars on promotional activities. Determine the number of subscriptions sold if (a) \$50,000 and (b) \$100,000 is spent. (c) Use the TABLE feature of a calculator to estimate how much should be spent (to the nearest \$1000 dollars) to sell over 50,000 subscriptions. (d) This promoter is willing to continue promotional spending as long as 14 additional subscriptions are sold for every \$1000 spent: $\frac{\Delta N}{\Delta d} = \frac{14}{1}$. Show this occurs by spending between \$417,800 and \$417,900.

97. Home ventilation: In the construction of new housing, there is considerable emphasis placed on correct ventilation. If too little outdoor air enters a home, pollutants can sometimes accumulate to levels that pose a health risk. For homes of various sizes, ventilation requirements have been established and are based on floor area and the number of bedrooms. For a three-bedroom home, the relationship can be modeled by the function $C(x) = 42 \ln x - 270$, where $C(x)$ represents the number of cubic feet of air per minute (cfm) that should be exchanged with outside air in a home with floor area x (in square feet). (a) How many cfm of exchanged air are needed for a three-bedroom home with a floor area of 2500 ft^2? (b) If a three-bedroom home is being mechanically ventilated by a system with 40 cfm capacity, what is the square footage of the home, assuming it is built to code?

98. Runway takeoff distance: Many will remember the August 27, 2006, crash of a commuter jet at Lexington's Blue Grass Airport, that was mistakenly trying to take off on a runway that was just too short. Forty-nine lives were lost. The minimum required length of a runway depends on the maximum allowable takeoff weight (mtw) of a specific plane. This relationship can be approximated by the function $L(x) = 2085 \ln x - 14{,}900$, where $L(x)$ represents the required length of a runway in feet, for a plane with x mtw in pounds.

a. The Airbus-320 has a 169,750 lb mtw. What minimum runway length is required for takeoff?

b. A Learjet 30 model requires a runway of 5550 ft to takeoff safely. What is its mtw?

Memory retention: Under certain conditions, a person's retention of random facts can be modeled by the equation $P(x) = 95 - 14 \log_2 x$, where $P(x)$ is the percentage of those facts retained after x number of days. Find the percentage of facts a person might retain after:

99. **a.** 1 day **b.** 4 days **c.** 16 days

100. **a.** 32 days **b.** 64 days **c.** 78 days ≈7%

101. pH level: Use the formula given in Exercise 79 to determine the pH level of black coffee if $x = 5.1 \times 10^{-5}$ moles per liter. Is black coffee considered an acid or base solution?

102. The length of time required for an amount of money to *triple* is given by the formula $T(r) = \frac{\log 3}{\log(1 + r)}$ (refer to Exercise 80). Construct a table of values to help estimate what interest rate is needed for an investment to triple in nine years.

▶ EXTENDING THE CONCEPT

103. Many texts and reference books give estimates of the noise level (in decibels dB) of common sounds. Through reading and research, try to locate or approximate where the following sounds would fall along this scale. In addition, determine at what point pain or ear damage begins to occur.

a. threshold of audibility **b.** lawn mower
c. whisper **d.** loud rock concert
e. lively party **f.** jet engine

104. Determine the value of x that makes the equation true: $\log_3[\log_3(\log_3 x)] = 0$.

105. Find the value of each expression without using a calculator.

a. $\log_{64}\frac{1}{16}$ **b.** $\log_{\frac{4}{9}}\frac{27}{8}$ **c.** $\log_{0.25}32$

106. Suppose you and I represent two different numbers. Is the following cryptogram true or false? *The log of me base me is one and the log of you base you is one, but the log of you base me is equal to the log of me base you turned upside down.*

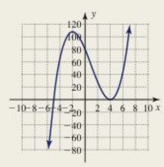
▶ MAINTAINING YOUR SKILLS

107. **(3.3)** Graph $g(x) = \sqrt[3]{x + 2} - 1$ by shifting the parent function. Then state the domain and range of g.

108. **(R.4)** Factor the following expressions:

a. $x^3 - 8$ **b.** $a^2 - 49$
c. $n^2 - 10n + 25$ **d.** $2b^2 - 7b + 6$

109. **(3.4/3.7)** For the graph shown, write the solution set for $f(x) < 0$. Then write the equation of the graph in factored form and in polynomial form.



110. **(2.2)** A function $f(x)$ is defined by the ordered pairs shown in the table. Is the function (a) linear? (b) increasing? Justify your answers.

x	y
−10	0
−9	−2
−8	−8
−6	−18
−5	−50
−4	−72

MID-CHAPTER CHECK

1. Write the following in logarithmic form.

a. $27^{\frac{2}{3}} = 9$ **b.** $81^{\frac{5}{4}} = 243$

2. Write the following in exponential form.

a. $\log_8 32 = \frac{5}{3}$ **b.** $\log_{1296}6 = 0.25$

3. Solve each equation for the unknown:

a. $4^{2x} = 32^{x-1}$ **b.** $(\frac{1}{3})^{4b} = 9^{2b-5}$

4. Solve each equation for the unknown:

a. $\log_{27}x = \frac{1}{3}$ **b.** $\log_b 125 = 3$

5. The homes in a popular neighborhood are growing in value according to the formula $V(t) = V_0(\frac{9}{8})^t$, where t is the time in years, V_0 is the purchase price of the home, and $V(t)$ is the current value of the home. (a) In 3 yr, how much will a $50,000 home be worth? (b) Use the TABLE feature of your calculator to estimate how many years (to the nearest year) until the home doubles in value.

6. The graph of the function $f(x) = 5^x$ has been shifted right 3 units, up 2 units, and stretched by a factor of 4. What is the equation of the resulting function?

7. State the domain and range for $f(x) = \sqrt{x - 3} + 1$, then find $f^{-1}(x)$ and state its domain and range. Verify the inverse relationship using composition.

8. Write the following equations in logarithmic form, then verify the result on a calculator.

a. $81 = 3^4$ **b.** $e^4 \approx 54.598$

9. Write the following equations in exponential form, then verify the result on a calculator.

a. $\frac{2}{3} = \log_{27}9$ **b.** $1.4 \approx \ln 4.0552$

10. On August 15, 2007, an earthquake measuring 8.0 on the Richter scale struck coastal Peru. On October 17, 1989, right before Game 3 of the World Series between the Oakland A's and the San Francisco Giants, the Loma Prieta earthquake, measuring 7.1 on the Richter scale, struck the San Francisco Bay area. How much more intense was the Peruvian earthquake?

REINFORCING BASIC CONCEPTS

Linear and Logarithm Scales

The use of logarithmic scales as a tool of measurement is primarily due to the range of values for the phenomenon being measured. For instance, time is generally measured on a linear scale, and for short periods a linear scale is appropriate. For the time line in Figure 4.27, each tick-mark *represents 1 unit,* and the time line can display a period of 10 yr. However, the scale would be useless in a study of world history or geology. If we scale the number line logarithmically, each tick-mark *represents a power of 10* (Figure 4.28) and a scale of the same length can now display a time period of 10 billion years.

Figure 4.27

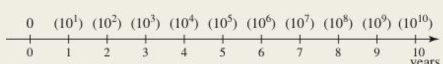


Figure 4.28

In much the same way, logarithmic measures are needed in a study of sound and earthquake intensity, as the scream of a jet engine is over 1 billion times more intense than the threshold of hearing, and the most destructive earthquakes are billions of times stronger than the slightest earth movement that can be felt. Figures 4.29 and 4.30 show logarithmic scales for measuring sound in decibels (1 bel = 10 decibels) and earthquake intensity in Richter values (or magnitudes).

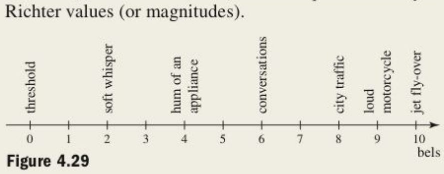


Figure 4.29

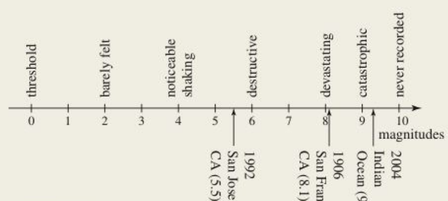


Figure 4.30

As you view these scales, remember that each unit increase represents a power of 10. For instance, the 1906 San Francisco earthquake was $8.1 - 5.5 = 2.6$ magnitudes greater than the San Jose quake of 1992, meaning it was $10^{2.6} \approx$ *398 times more intense*. Use this information to complete the following exercises. Determine how many times more intense the first sound is compared to the second.

Exercise 1: jet engine: 14 bels
rock concert: 11.8 bels

Exercise 2: pneumatic hammer: 11.2 bels
heavy lawn mower: 8.5 bels

Exercise 3: train horn: 7.5 bels
soft music: 3.4 bels

Determine how many times more intense the first quake was compared to the second.

Exercise 4: Great Chilean quake (1960): magnitude 9.5
Kobe, Japan, quake (1995): magnitude 6.9

Exercise 5: Northern Sumatra (2004): magnitude 9.1
Southern, Greece (2008): magnitude 4.5

4.4 Properties of Logarithms; Solving Exponential/Logarithmic Equations

Learning Objectives

In Section 4.4 you will learn how to:

- ❑ **A.** Solve logarithmic equations using the fundamental properties of logarithms
- ❑ **B.** Apply the product, quotient, and power properties of logarithms
- ❑ **C.** Solve general logarithmic and exponential equations
- ❑ **D.** Solve applications involving logistic, exponential, and logarithmic functions

In this section, we develop the ability to solve logarithmic and exponential equations of any base. A **logarithmic equation** has at least one term that involves the logarithm of a variable. Likewise, an **exponential equation** has at least one term that involves a variable exponent. In the same way that we might *square both sides* or *divide both sides* of an equation in the solution process, we'll show that we can also *exponentiate both sides* or *take logarithms of both sides* to help obtain a solution.

A. Solving Equations Using the Fundamental Properties of Logarithms

In Section 4.3, we converted expressions from exponential form to logarithmic form using the basic definition: $x = b^y \Leftrightarrow y = \log_b x$. This relationship reveals the following four properties:

Fundamental Properties of Logarithms

For any base $b > 0$, $b \neq 1$,

I. $\log_b b = 1$, since $b^1 = b$

II. $\log_b 1 = 0$, since $b^0 = 1$

III. $\log_b b^x = x$, since $b^x = b^x$ (exponential form)

IV. $b^{\log_b x} = x$, since $\log_b x = \log_b x$ (logarithmic form)

To see the verification of Property IV more clearly, again note that for $y = \log_b x$, $b^y = x$ is the exponential form, and substituting $\log_b x$ for y yields $b^{\log_b x} = x$. Also note that Properties III and IV demonstrate that $y = \log_b x$ and $y = b^x$ are inverse functions. In common language, "a base-*b* logarithm *undoes* a base-*b* exponential," and "a base-*b* exponential *undoes* a base-*b* logarithm." For $f(x) = \log_b x$ and $f^{-1}(x) = b^x$, using a composition verifies the inverse relationship:

$$\begin{aligned}(f \circ f^{-1})(x) &= f[f^{-1}(x)] \\ &= \log_b b^x \\ &= x\end{aligned} \qquad \begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}[f(x)] \\ &= b^{\log_b x} \\ &= x\end{aligned}$$

These properties can be used to solve basic equations involving logarithms and exponentials.

EXAMPLE 1 ► Solving Basic Logarithmic Equations

Solve each equation by applying fundamental properties. Answer in exact form and approximate form using a calculator (round to 1000ths).

a. $\ln x = 2$ **b.** $-0.52 = \log x$

Solution ►

a. $\ln x = 2$ given

$e^{\ln x} = e^2$ exponentiate both sides

$x = e^2$ Property IV

≈ 7.389 result

b. $-0.52 = \log x$ given

$10^{-0.52} = 10^{\log x}$ exponentiate both sides

$10^{-0.52} = x$ Property IV

$0.302 \approx x$ result

Now try Exercises 7 through 10 ►

Note that exponentiating both sides of the equation produced the same result as simply writing the original equation in exponential form, and either approach can be used.

EXAMPLE 2 ► Solving Basic Exponential Equations

Solve each equation by applying fundamental properties. Answer in exact form and approximate form using a calculator (round to 1000ths).

a. $e^x = 167$ **b.** $10^x = 8.223$

Solution ► **a.**

$e^x = 167$ given

$\ln e^x = \ln 167$ take natural log of both sides

$x = \ln 167$ Property III

$x \approx 5.118$ result

b.

$10^x = 8.223$ given

$\log 10^x = \log 8.223$ take common log of both sides

$x = \log 8.223$ Property III

$x \approx 0.915$ result

Now try Exercises 11 through 14 ►

Similar to our previous observation, taking the logarithm of both sides produced the same result as writing the original equation in logarithmic form, and either approach can be used.

If an equation has a single logarithmic or exponential term (base 10 or base e), the equation can be solved by isolating this term and applying one of the fundamental properties.

EXAMPLE 3 ► Solving Exponential Equations

Solve each equation. Write answers in exact form and approximate form to four decimal places.

a. $10^x - 29 = 51$ **b.** $3e^{x+1} - 5 = 7$

Solution ► **a.**

$10^x - 29 = 51$ given

$10^x = 80$ add 29

Since the left-hand side is base 10, we apply a common logarithm.

$\log 10^x = \log 80$ take the common log of both sides

$x = \log 80$ Property III (exact form)

≈ 1.9031 approximate form

b.

$3e^{x+1} - 5 = 7$ given

$3e^{x+1} = 12$ add 5

$e^{x+1} = 4$ divide by 3

Since the left-hand side is base e, we apply a natural logarithm.

$\ln e^{x+1} = \ln 4$ take the natural log of both sides

$x + 1 = \ln 4$ Property III

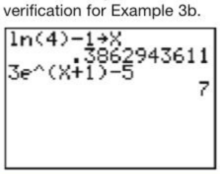
$x = \ln 4 - 1$ solve for x (exact form)

≈ 0.3863 approximate form

Now try Exercises 15 through 20 ►

WORTHY OF NOTE

To check solutions using a calculator, we can [STO▸] (store) the exact result in storage location [x, T, θ, n] (the function variable x) and simply enter the original equation on the home screen. The figure shows this verification for Example 3b.



EXAMPLE 4 ► Solving Logarithmic Equations

Solve each equation. Write answers in exact form and approximate form to four decimal places.

a. $2\log(7x) + 1 = 4$ **b.** $-4\ln(x+1) - 5 = 7$

Solution ►

a.

$2\log(7x) + 1 = 4$ given

$2\log(7x) = 3$ subtract 1

$\log(7x) = \frac{3}{2}$ divide by 2

$7x = 10^{\frac{3}{2}}$ exponential form

$x = \frac{10^{\frac{3}{2}}}{7}$ divide by 7 (exact form)

≈ 4.5175 approximate form

b.

$-4\ln(x+1) - 5 = 7$ given

$-4\ln(x+1) = 12$ add 5

$\ln(x+1) = -3$ divide by -4

$x + 1 = e^{-3}$ exponential form

$x = e^{-3} - 1$ subtract 1 (exact form)

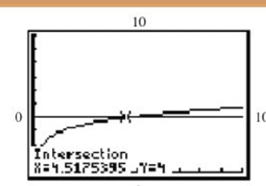
≈ -0.9502 approximate form

A. You've just learned how to solve logarithmic equations using the fundamental properties of logarithms

Now try Exercises 21 through 26 ►

GRAPHICAL SUPPORT

Solutions can be also checked using the intersection of graphs method (Technology Highlight, page 432). For Example 4a, enter $Y_1 = 2\log(7x) + 1$ and $Y_2 = 4$ on the Y= screen. From the domain of the function and the expected answer, we set a window that includes only Quadrant I. Use the keystrokes 2nd TRACE (CALC) **5:intersect**, and identify each graph by pressing ENTER 3 times. The calculator will find the point of intersection and display it at the bottom of the screen.



B. The Product, Quotient, and Power Properties of Logarithms

Generally speaking, equation solving involves simplifying the equation, isolating a variable term on one side, and applying an inverse to solve for the unknown. For logarithmic equations such as $\log x + \log(x + 3) = 1$, we must find a way to combine the terms on the left, before we can work toward a solution. This requires a further exploration of logarithmic properties.

Due to the close connection between exponents and logarithms, their properties are very similar. To illustrate, we'll use terms that can all be written in the form 2^x, and write the equations $8 \cdot 4 = 32$, $\frac{8}{4} = 2$, and $8^2 = 64$ in both exponential form and logarithmic form.

The exponents from a product are added:	exponential form:	$2^3 \cdot 2^2 = 2^{3+2}$
	logarithmic form:	$\log_2(8 \cdot 4) = \log_2 8 + \log_2 4$

The exponents from a quotient are subtracted:

exponential form: $\frac{2^3}{2^2} = 2^{3-2}$

logarithmic form: $\log_2\left(\frac{8}{4}\right) = \log_2 8 - \log_2 4$

The exponents from a power are multiplied:

exponential form: $(2^3)^2 = 2^{3 \cdot 2}$

logarithmic form: $(\log_2 8)^2 = 2 \cdot \log_2 8$

Each illustration can be generalized and applied with any base b.

Properties of Logarithms

Give M, N, and $b \neq 1$ are *positive* real numbers, and *any* real number p.

Product Property	Quotient Property	Power Property
$\log_b(MN) = \log_b M + \log_b N$	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$	$\log_b M^p = p\log_b M$
The log of a product is a sum of logarithms.	The log of a quotient is a difference of logarithms.	The log of a quantity to a power is the power times the log of the quantity.

For a detailed verification of these properties, see Appendix I.

CAUTION ► It's very important that you read and understand these properties correctly. In particular, note that $\log_b(M + N) \neq \log_b M + \log_b N$, and $\log_b\left(\frac{M}{N}\right) \neq \frac{\log_b M}{\log_b N}$. In the first case, it might help to compare the statement with $f(x + 3)$, which represents a horizontal shift of the graph 3 units left, and in particular, $f(x + 3) \neq f(x) + f(3)$.

In many cases, these properties are applied to consolidate logarithmic terms in preparation for equation solving.

EXAMPLE 5 ► Rewriting Expressions Using Logarithmic Properties

Use the properties of logarithms to write each expression as a single term.

a. $\log_2 7 + \log_2 5$ **b.** $2\ln x + \ln(x + 6)$ **c.** $\ln(x + 2) - \ln x$

Solution ►

a. $\log_2 7 + \log_2 5 = \log_2(7 \cdot 5)$ product property
$= \log_2 35$ simplify

b. $2\ln x + \ln(x + 6) = \ln x^2 + \ln(x + 6)$ power property
$= \ln[x^2(x + 6)]$ product property
$= \ln[x^3 + 6x^2]$ simplify

c. $\ln(x + 2) - \ln x = \ln\left(\frac{x + 2}{x}\right)$ quotient property

Now try Exercises 27 through 42 ►

EXAMPLE 6 ► Rewriting Logarithmic Expressions Using the Power Property

Use the power property of logarithms to rewrite each term as a product.

a. $\ln 5^x$ **b.** $\log 32^{x+2}$ **c.** $\log \sqrt{x}$

Solution ►
a. $\ln 5^x = x \ln 5$ power property
b. $\log 32^{x+2} = (x+2)\log 32$ power property (note use of parentheses)
c. $\log\sqrt{x} = \log x^{\frac{1}{2}}$ write radical using a rational exponent
$= \frac{1}{2}\log x$ power property

Now try Exercises 43 through 50 ►

⚠ **CAUTION** ► Note from Example 6b that parentheses *must be used* whenever the exponent is a sum or difference. There is a huge difference between $(x+2)\log 32$ and $x + 2\log 32$.

In other cases, these properties help rewrite an expression so that certain procedures can be applied more easily. Example 7 actually lays the foundation for more advanced work in mathematics.

EXAMPLE 7 ► **Rewriting Expressions Using Logarithmic Properties**

Use the properties of logarithms to write the following expressions as a sum or difference of simple logarithmic terms.

a. $\log(x^2z)$ **b.** $\ln\sqrt{\frac{x}{x+5}}$ **c.** $\ln\left[\frac{e\sqrt{x^2+1}}{(2x+5)^3}\right]$

Solution ►
a. $\log(x^2z) = \log x^2 + \log z$ product property
$= 2\log x + \log z$ power property

b. $\ln\sqrt{\frac{x}{x+5}} = \ln\left(\frac{x}{x+5}\right)^{\frac{1}{2}}$ write radical using a rational exponent
$= \frac{1}{2}\ln\left(\frac{x}{x+5}\right)$ power property
$= \frac{1}{2}[\ln x - \ln(x+5)]$ quotient property

c. $\ln\left[\frac{e\sqrt{x^2+1}}{(2x+5)^3}\right] = \ln\left[\frac{e(x^2+1)^{\frac{1}{2}}}{(2x+5)^3}\right]$ write radical using a rational exponent
$= \ln[e(x^2+1)^{\frac{1}{2}}] - \ln(2x+5)^3$ quotient property
$= \ln e + \ln(x^2+1)^{\frac{1}{2}} - \ln(2x+3)^3$ product property
$= 1 + \frac{1}{2}\ln(x^2+1) - 3\ln(2x+3)$ power property

Now try Exercises 51 through 60 ►

Although base-10 and base-e logarithms dominate the mathematical landscape, there are many practical applications that use other bases. Fortunately, a formula exists that will convert any given base into either base 10 or base e. It's called the **change-of-base formula.**

Change-of-Base Formula

For the positive real numbers M, a, and b, with $a, b \neq 1$,

$\log_b M = \dfrac{\log M}{\log b}$	$\log_b M = \dfrac{\ln M}{\ln b}$	$\log_b M = \dfrac{\log_a M}{\log_a b}$
base 10	base e	arbitrary base a

Proof of the Change-of-Base Formula:

For $y = \log_b M$, we have $b^y = M$ in exponential form. It follows that

$$\log_a(b^y) = \log_a M \quad \text{take base-}a\text{ logarithm of both sides}$$
$$y \log_a b = \log_a M \quad \text{power property of logarithms}$$
$$y = \frac{\log_a M}{\log_a b} \quad \text{divide by } \log_a b$$
$$\log_b M = \frac{\log_a M}{\log_a b} \quad \text{substitute } \log_b M \text{ for } y$$

EXAMPLE 8 ► **Using the Change-of-Base Formula to Evaluate Expressions**

Find the value of each expression using the change-of-base formula. Answer in exact form and approximate form using nine digits, then *verify the result* using the original base.

a. $\log_3 29$ **b.** $\log_5 3.6$

Solution ►

a. $\log_3 29 = \dfrac{\log 29}{\log 3}$
$= 3.065044752$

Check: $3^{3.065044752} = 29$ ✓

b. $\log_5 3.6 = \dfrac{\log 3.6}{\log 5}$
$= 0.795888947$

Check: $5^{0.795888947} = 3.6$ ✓

Now try Exercises 61 through 72 ►

B. You've just learned how to apply the product, quotient, and power properties of logarithms

The change-of-base formula can also be used to study and graph logarithmic functions of *any* base. For $y = \log_b x$, the right-hand expression is simply rewritten using the formula and the equivalent function is $y = \dfrac{\log x}{\log b}$. The new function can then be evaluated as in Example 8, or used to study the graph of $y = \log_b x$ for any base b.

C. Solving Logarithmic Equations

One of the most common mistakes in solving exponential and logarithmic equations is to apply the inverse function too early—before the equation has been simplified. In addition, since the domain of $y = \log_b x$ is $x > 0$, logarithmic equations can sometimes produce **extraneous roots,** and checking all answers is a good practice. We'll illustrate by solving the equation mentioned earlier: $\log x + \log(x + 3) = 1$.

EXAMPLE 9 ► **Solving a Logarithmic Equation**

Solve for x and check your answer: $\log x + \log(x + 3) = 1$.

Solution ►

$\log x + \log(x + 3) = 1$ — original equation

$\log[x(x + 3)] = 1$ — product property

$x^2 + 3x = 10^1$ — exponential form, distribute x

$x^2 + 3x - 10 = 0$ — set equal to 0

$(x + 5)(x - 2) = 0$ — factor

$x = -5$ or $x = 2$ — result

Check: The "solution" $x = -5$ is outside the domain and is discarded. For $x = 2$,

$\log x + \log(x + 3) = 1$ — original equation

$\log 2 + \log(2 + 3) = 1$ — substitute 2 for x

$\log 2 + \log 5 = 1$ — simplify

$\log(2 \cdot 5) = 1$ — product property

$\log 10 = 1$ — Property I

Now try Exercises 73 through 80 ►

As an alternative check, you could also use a calculator to verify $\log 2 + \log 5 = 1$ directly.

If the simplified form of an equation yields a logarithmic term on both sides, the **uniqueness property of logarithms** provides an efficient way to work toward a solution. Since logarithmic functions are one-to-one, we have

The Uniqueness Property of Logarithms

For positive real numbers m, n, and $b \neq 1$,

$$\text{If } \log_b m = \log_b n, \quad \text{then } m = n$$

Equal bases imply equal arguments.

EXAMPLE 10 ► Solving Logarithmic Equations Using the Uniqueness Property

Solve each equation using the uniqueness property.

a. $\log(x + 2) = \log 7 + \log x$ **b.** $\ln 87 - \ln x = \ln 29$

Solution ►

a. $\log(x + 2) = \log 7 + \log x$

$\log(x + 2) = \log 7x$ — properties of logarithms

$x + 2 = 7x$ — uniqueness property

$2 = 6x$ — solve for x

$\frac{1}{3} = x$ — result

b. $\ln 87 - \ln x = \ln 29$

$\ln\left(\frac{87}{x}\right) = \ln 29$

$\frac{87}{x} = 29$

$87 = 29x$

$3 = x$

The checks are left to the student.

Now try Exercises 81 through 86 ►

WORTHY OF NOTE

The uniqueness property can also be viewed as exponentiating both sides using the appropriate base, then applying Property IV.

Often the solution may depend on using a variety of algebraic skills in addition to logarithmic or exponential properties.

EXAMPLE 11 ► Solving Logarithmic Equations

Solve each equation and check your answers.

a. $\ln(x + 7) - 2\ln 5 = 0.9$ **b.** $\log(x + 12) - \log x = \log(x + 9)$

Solution ► **a.**

$\ln(x+7) - 2\ln 5 = 0.9$	given
$\ln(x+7) - \ln 5^2 = 0.9$	power property
$\ln\left(\frac{x+7}{25}\right) = 0.9$	quotient property
$\frac{x+7}{25} = e^{0.9}$	exponential form
$x + 7 = 25e^{0.9}$	clear denominator
$x = 25e^{0.9} - 7$	solve for x (exact form)
≈ 54.49	approximate form (to 100ths)

Check:

$\ln(x+7) - 2\ln 5 = 0.9$	original equation
$\ln(54.49+7) - 2\ln 5 \approx 0.9$	substitute 54.49 for x
$\ln 61.49 - 2\ln 5 \approx 0.9$	simplify
$0.9 \approx 0.9$ ✓	result checks

b.

$\log(x+12) - \log x = \log(x+9)$	given equation
$\log\left(\frac{x+12}{x}\right) = \log(x+9)$	quotient property
$\frac{x+12}{x} = x + 9$	uniqueness property
$x + 12 = x^2 + 9x$	clear denominator
$0 = x^2 + 8x - 12$	set equal to 0

The equation is not factorable, and the quadratic formula must be used.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	quadratic formula
$= \frac{-8 \pm \sqrt{(8)^2 - 4(1)(-12)}}{2(1)}$	substitute 1 for a, 8 for b, -12 for c
$= \frac{-8 \pm \sqrt{112}}{2} = \frac{-8 \pm 4\sqrt{7}}{2}$	simplify
$= -4 \pm 2\sqrt{7}$	result

Substitution shows $x = -4 + 2\sqrt{7}$ ($x \approx 1.29150$) checks, but substituting $-4 - 2\sqrt{7}$ for x gives $\log(2.7085) - \log(-9.2915) = \log(-0.2915)$ and two of the three terms do not represent real numbers ($x = -4 - 2\sqrt{7}$ is an extraneous root).

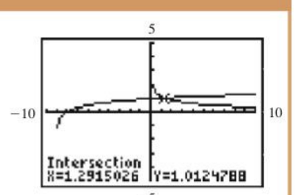
WORTHY OF NOTE

If all digits given by your calculator are used in the check, a calculator will generally produce "exact" answers. Try using the solution $x = 54.49007778$ in Example 11a by substituting directly, or by storing the result of the original computation and using your home screen.

Now try Exercises 87 through 102 ►

GRAPHICAL SUPPORT

Logarithmic equations can also be checked using the intersection of graphs method. For Example 11b, we first enter $\log(x + 12) - \log x$ as Y_1 and $\log(x + 9)$ as Y_2 on the Y= screen. Using 2nd TRACE **(CALC) 5:intersect**, we find the graphs intersect at $x = 1.2915026$, and that *this is the only solution* (knowing the graph's basic shape, we conclude they cannot intersect again).



CAUTION ► Be careful not to dismiss or discard a possible solution simply because it's negative. For the equation $\log(-6 - x) = 1$, $x = -16$ is the solution (the domain here allows negative numbers: $-6 - x > 0$ yields $x < -6$ as the domain). In general, when a logarithmic equation has multiple solutions, all solutions should be checked.

Solving exponential equations likewise involves isolating an exponential term on one side, or writing the equation where exponential terms of like base occur on each side. The latter case can be solved using the uniqueness property. If the exponential base is neither 10 nor *e*, logarithms of either base can be used along with the Power Property to solve the equation.

EXAMPLE 12 ► Solving an Exponential Equation Using Base 10 or Base *e*

Solve the exponential equation. Answer in both exact form, and approximate form to four decimal places: $4^{3x} - 1 = 8$

Solution ► $4^{3x} - 1 = 8$ given equation

$4^{3x} = 9$ add 1

The left-hand side is neither base 10 or base *e*, so the choice is arbitrary. Here we chose base 10 to solve.

$\log 4^{3x} = \log 9$ take logarithm base 10 of both sides

$3x \log 4 = \log 9$ power property

$x = \frac{\log 9}{3 \log 4}$ divide by 3 log 4 (exact form)

$x \approx 0.5283$ approximate form

Now try Exercises 103 through 106 ►

WORTHY OF NOTE

The equation $\log 4^{3x} = \log 9$ from Example 12, can actually be solved using the change-of-base property, by taking logarithms base 4 of both sides.

$\log_4 4^{3x} = \log_4 9$ logarithms base 4

$3x = \frac{\log 9}{\log 4}$ Property III; change-of-base property

$x = \frac{\log 9}{3 \log 4}$ divide by 3

In some cases, two exponential terms with *unlike* bases may be involved. Here again, either common logs or natural logs can be used, but be sure to distinguish between constant terms like *ln 5* and variable terms like *x ln 5*. As with all equations, the goal is to isolate the *variable terms* on one side, with all constant terms on the other.

EXAMPLE 13 ► Solving an Exponential Equation with Unlike Bases

Solve the exponential equation $5^{x+1} = 6^{2x}$.

Solution ► $5^{x+1} = 6^{2x}$ original equation

Begin by taking the natural log of both sides:

$\ln(5^{x+1}) = \ln(6^{2x})$ apply base-*e* logarithms

$(x + 1)\ln 5 = 2x \ln 6$ power property

$x \ln 5 + \ln 5 = 2x \ln 6$ distribute

$\ln 5 = 2x \ln 6 - x \ln 5$ variable terms to one side

$\ln 5 = x(2 \ln 6 - \ln 5)$ factor out *x*

$\frac{\ln 5}{2 \ln 6 - \ln 5} = x$ solve for *x* (exact form)

$0.8153 \approx x$ approximate form

The solution can be checked on a calculator.

Now try Exercises 107 through 110 ►

C. You've just learned how to solve general logarithmic and exponential equations

D. Applications of Logistic, Exponential, and Logarithmic Functions

Applications of exponential and logarithmic functions take many different forms and it would be impossible to illustrate them all. As you work through the exercises, try to adopt a "big picture" approach, applying the general principles illustrated here to other applications. Some may have been introduced in previous sections. The difference here is that we can now *solve for the independent variable,* instead of simply evaluating the relationships.

In applications involving the **logistic growth** of animal populations, the initial stage of growth is virtually exponential, but due to limitations on food, space, or other resources, growth slows and at some point it reaches a limit. In business, the same principle applies to the logistic growth of sales or profits, due to market saturation. In these cases, the exponential term appears in the denominator of a quotient, and we "clear denominators" to begin the solution process.

EXAMPLE 14 ► **Solving a Logistics Equation**

A small business makes a new discovery and begins an aggressive advertising campaign, confident they can capture 66% of the market in a short period of time. They anticipate their market share will be modeled by the function $M(t) = \frac{66}{1 + 10e^{-0.05t}}$, where $M(t)$ represents the percentage after t days. Use this function to answer the following.

a. What was the company's initial market share ($t = 0$)? What was their market share 30 days later?

b. How long will it take the company to reach a 60% market share?

Solution ► **a.**

$$M(t) = \frac{66}{1 + 10e^{-0.05t}} \quad \text{given}$$

$$M(0) = \frac{66}{1 + 10e^{-0.05(0)}} \quad \text{substitute 0 for } t$$

$$= \frac{66}{11} \quad \text{simplify}$$

$$= 6 \quad \text{result}$$

The company originally had only a 6% market share.

$$M(30) = \frac{66}{1 + 10e^{-0.05(30)}} \quad \text{substitute 30 for } t$$

$$= \frac{66}{1 + 10e^{-1.5}} \quad \text{simplify}$$

$$\approx 20.4 \quad \text{result}$$

After 30 days, they held a 20.4% market share.

b. For Part b, we replace $M(t)$ with 60 and solve for t.

$$60 = \frac{66}{1 + 10e^{-0.05t}} \quad \text{given}$$

$$60(1 + 10e^{-0.05t}) = 66 \quad \text{multiply by } 1 + 10e^{-0.05t}$$

$$1 + 10e^{-0.05t} = 1.1 \quad \text{divide by 60}$$

$$10e^{-0.05t} = 0.1 \quad \text{subtract 1}$$
$$e^{-0.05t} = 0.01 \quad \text{divide by 10}$$
$$\ln e^{-0.05t} = \ln 0.01 \quad \text{apply base-}e\text{ logarithms}$$
$$-0.05t = \ln 0.01 \quad \text{Property III}$$
$$t = \frac{\ln 0.01}{-0.05} \quad \text{solve for } t \text{ (exact form)}$$
$$\approx 92 \quad \text{approximate form}$$

According to this model, the company will reach a 60% market share in about 92 days.

Now try Exercises 111 through 116 ►

Earlier we used the barometric equation $H = (30T + 8000)\ln\left(\frac{P_0}{P}\right)$ to find an altitude H, given a temperature and the atmospheric (barometric) pressure in centimeters of mercury (cmHg). Using the tools from this section, we are now able to find the atmospheric pressure for a given altitude and temperature.

EXAMPLE 15 ► **Using Logarithms to Determine Atmospheric Pressure**

Suppose a group of climbers has just scaled Mt. Rainier, the highest mountain of the Cascade Range in western Washington State. If the mountain is about 4395 m high and the temperature at the summit is −22.5°C, what is the atmospheric pressure at this altitude? The pressure at sea level is $P_0 = 76$ cmHg.

Solution ►

$$H = (30T + 8000)\ln\left(\frac{P_0}{P}\right) \quad \text{given}$$
$$4395 = [30(-22.5) + 8000]\ln\left(\frac{76}{P}\right) \quad \text{substitute 4395 for } H\text{, 76 for } P_0\text{, and } -22.5 \text{ for } T$$
$$4395 = 7325\ln\left(\frac{76}{P}\right) \quad \text{simplify}$$
$$0.6 = \ln\left(\frac{76}{P}\right) \quad \text{divide by 7325}$$
$$e^{0.6} = \frac{76}{P} \quad \text{exponential form}$$
$$Pe^{0.6} = 76 \quad \text{multiply by } P$$
$$P = \frac{76}{e^{0.6}} \quad \text{divide by } e^{0.6} \text{ (exact form)}$$
$$\approx 41.7 \quad \text{approximate form}$$

Under these conditions and at this altitude, the atmospheric pressure would be 41.7 cmHg.

Now try Exercises 117 through 120 ►

D. You've just learned how to solve applications involving logistic, exponential, and logarithmic functions

4.4 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For $e^{-0.02x+1} = 10$, the solution process is most efficient if we apply a base _______ logarithm to both sides.
2. To solve $3 \ln x - \ln(x + 3) = 0$, we can combine terms using the _______ property, or add $\ln(x + 3)$ to both sides and use the _______ property.
3. Since logarithmic functions are not defined for all real numbers, we should check all "solutions" for _______ roots.
4. The statement $\log_e 10 = \frac{\log 10}{\log e}$ is an example of the _______ -of- _______ property.
5. Solve the equation here, giving a step-by-step discussion of the solution process: $\ln(4x + 3) + \ln(2) = 3.2$
6. Describe the difference between *evaluating* the equation below given $x = 9.7$ and *solving* the equation given $y = 9.7$: $y = 3 \log_2(x - 1.7) - 2.3$.

▶ DEVELOPING YOUR SKILLS

Solve each equation by applying fundamental properties. Round to thousandths.

7. $\ln x = 3.4$
8. $\ln x = \frac{1}{2}$
9. $\log x = \frac{1}{4}$
10. $\log x = 1.6$
11. $e^x = 9.025$
12. $e^x = 0.343$
13. $10^x = 18.197$
14. $10^x = 0.024$

Solve each equation. Write answers in exact form and in approximate form to four decimal places.

15. $4e^{x-2} + 5 = 70$
16. $2 - 3e^{0.4x} = -7$
17. $10^{x+5} - 228 = -150$
18. $10^{2x} + 27 = 190$
19. $-150 = 290.8 - 190e^{-0.75x}$
20. $250e^{0.05x+1} + 175 = 1175$

Solve each equation. Write answers in exact form and in approximate form to four decimal places.

21. $3 \ln(x + 4) - 5 = 3$
22. $-15 = -8 \ln(3x) + 7$
23. $-1.5 = 2 \log(5 - x) - 4$
24. $-4 \log(2x) + 9 = 3.6$
25. $\frac{1}{2}\ln(2x + 5) + 3 = 3.2$
26. $\frac{3}{4}\ln(4x) - 6.9 = -5.1$

Use properties of logarithms to write each expression as a single term.

27. $\ln(2x) + \ln(x - 7)$
28. $\ln(x + 2) + \ln(3x)$
29. $\log(x + 1) + \log(x - 1)$
30. $\log(x - 3) + \log(x + 3)$
31. $\log_3 28 - \log_3 7$
32. $\log_6 30 - \log_6 10$
33. $\log x - \log(x + 1)$
34. $\log(x - 2) - \log x$
35. $\ln(x - 5) - \ln x$
36. $\ln(x + 3) - \ln(x - 1)$
37. $\ln(x^2 - 4) - \ln(x + 2)$
38. $\ln(x^2 - 25) - \ln(x + 5)$
39. $\log_2 7 + \log_2 6$
40. $\log_9 2 + \log_9 15$
41. $\log_5(x^2 - 2x) + \log_5 x^{-1}$
42. $\log_3(3x^2 + 5x) - \log_3 x$

Use the power property of logarithms to rewrite each term as the product of a constant and a logarithmic term.

43. $\log 8^{x+2}$
44. $\log 15^{x-3}$
45. $\ln 5^{2x-1}$
46. $\ln 10^{3x+2}$
47. $\log \sqrt{22}$
48. $\log \sqrt[3]{34}$
49. $\log_5 81$
50. $\log_7 121$

Use the properties of logarithms to write the following expressions as a sum or difference of simple logarithmic terms.

51. $\log(a^3b)$

52. $\log(m^2n)$

53. $\ln(x\sqrt[4]{y})$

54. $\ln(\sqrt[3]{pq})$

55. $\ln\left(\frac{x^2}{y}\right)$

56. $\ln\left(\frac{m^2}{n^3}\right)$

57. $\log\left(\sqrt{\frac{x-2}{x}}\right)$

58. $\log\left(\sqrt[3]{\frac{3-v}{2v}}\right)$

59. $\ln\left(\frac{7x\sqrt{3-4x}}{2(x-1)^3}\right)$

60. $\ln\left(\frac{x^4\sqrt{x^2-4}}{\sqrt[3]{x^2+5}}\right)$

Evaluate each expression using the change-of-base formula and either base 10 or base *e*. Answer in exact form and in approximate form using nine decimal places, then verify the result using the original base.

61. $\log_7 60$

62. $\log_8 92$

63. $\log_5 152$

64. $\log_6 200$

65. $\log_3 1.73205$

66. $\log_2 1.41421$

67. $\log_{0.5} 0.125$

68. $\log_{0.2} 0.008$

Use the change-of-base formula to write an equivalent function, then evaluate the function as indicated (round to four decimal places). Investigate and discuss any patterns you notice in the output values, then determine the next input that will continue the pattern.

69. $f(x) = \log_3 x$; $f(5)$, $f(15)$, $f(45)$

70. $g(x) = \log_2 x$; $g(5)$, $g(10)$, $g(20)$

71. $h(x) = \log_9 x$; $h(2)$, $h(4)$, $h(8)$

72. $H(x) = \log_\pi x$; $H(\sqrt{2})$, $H(2)$, $H(\sqrt{2^3})$

Solve each equation and check your answers.

73. $\log 4 + \log(x - 7) = 2$

74. $\log 5 + \log(x - 9) = 1$

75. $\log(2x - 5) - \log 78 = -1$

76. $\log(4 - 3x) - \log 145 = -2$

77. $\log(x - 15) - 2 = -\log x$

78. $\log x - 1 = -\log(x - 9)$

79. $\log(2x + 1) = 1 - \log x$

80. $\log(3x - 13) = 2 - \log x$

Solve each equation using the uniqueness property of logarithms.

81. $\log(5x + 2) = \log 2$

82. $\log(2x - 3) = \log 3$

83. $\log_4(x + 2) - \log_4 3 = \log_4(x - 1)$

84. $\log_3(x + 6) - \log_3 x = \log_3 5$

85. $\ln(8x - 4) = \ln 2 + \ln x$

86. $\ln(x - 1) + \ln 6 = \ln(3x)$

Solve each logarithmic equation using any appropriate method. Clearly identify any extraneous roots. If there are no solutions, so state.

87. $\log(2x - 1) + \log 5 = 1$

88. $\log(x - 7) + \log 3 = 2$

89. $\log_2(9) + \log_2(x + 3) = 3$

90. $\log_3(x - 4) + \log_3(7) = 2$

91. $\ln(x + 7) + \ln 9 = 2$

92. $\ln 5 + \ln(x - 2) = 1$

93. $\log(x + 8) + \log x = \log(x + 18)$

94. $\log(x + 14) - \log x = \log(x + 6)$

95. $\ln(2x + 1) = 3 + \ln 6$

96. $\ln 21 = 1 + \ln(x - 2)$

97. $\log(-x - 1) = \log(5x) - \log x$

98. $\log(1 - x) + \log x = \log(x + 4)$

99. $\ln(2t + 7) = \ln 3 - \ln(t + 1)$

100. $\ln 6 - \ln(5 - r) = \ln(r + 2)$

101. $\log(x - 1) - \log x = \log(x - 3)$

102. $\ln x + \ln(x - 2) = \ln 4$

103. $7^{x+2} = 231$

104. $6^{x+2} = 3589$

105. $5^{3x-2} = 128{,}965$

106. $9^{5x-3} = 78{,}462$

107. $2^{x+1} = 3^x$

108. $7^x = 4^{2x-1}$

109. $5^{2x+1} = 9^{x+1}$

110. $\left(\frac{1}{5}\right)^{x-1} = \left(\frac{1}{2}\right)^{3-x}$

111. $\frac{250}{1 + 4e^{-0.06x}} = 200$

112. $\frac{80}{1 + 15e^{-0.06x}} = 50$

▶ WORKING WITH FORMULAS

113. Logistic growth: $P(t) = \dfrac{C}{1 + ae^{-kt}}$

For populations that exhibit logistic growth, the population at time t is modeled by the function shown, where C is the carrying capacity of the population (the maximum population that can be supported over a long period of time), k is the growth constant, and $a = \frac{C - P(0)}{P(0)}$. Solve the formula for t, then use the result to find the value of t given $C = 450$, $a = 8$, $P = 400$, and $k = 0.075$.

114. Forensics—estimating time of death: $h = -3.9 \cdot \ln\left(\dfrac{T - T_R}{T_0 - T_R}\right)$

Using the formula shown, a forensic expert can compute the approximate time of death for a person found recently expired, where T is the body temperature when it was found, T_R is the (constant) temperature of the room, T_0 is the body temperature at the time of death ($T_0 = 98.6$°F), and h is the number of hours since death. If the body was discovered at 9:00 A.M. with a temperature of 86.2°F, in a room at 73°F, at approximately what time did the person expire? (Note this formula is a version of Newton's law of cooling.)

▶ APPLICATIONS

115. Stocking a lake: A farmer wants to stock a private lake on his property with catfish. A specialist studies the area and depth of the lake, along with other factors, and determines it can support a maximum population of around 750 fish, with growth modeled by the function $P(t) = \dfrac{750}{1 + 24e^{-0.075t}}$, where $P(t)$ gives the current population after t months. (a) How many catfish did the farmer initially put in the lake? (b) How many months until the population reaches 300 fish?

116. Increasing sales: After expanding their area of operations, a manufacturer of small storage buildings believes the larger area can support sales of 40 units per month. After increasing the advertising budget and enlarging the sales force, sales are expected to grow according to the model $S(t) = \dfrac{40}{1 + 1.5e^{-0.08t}}$, where $S(t)$ is the expected number of sales after t months. (a) How many sales were being made each month, prior to the expansion? (b) How many months until sales reach 25 units per month?

Use the *barometric equation* $H = (30T + 8000)\ln\left(\dfrac{P_0}{P}\right)$ for exercises 117 and 118. Recall that $P_0 = 76$ cmHg.

117. Altitude and temperature: A sophisticated spy plane is cruising at an altitude of 18,250 m. If the temperature at this altitude is −75°C, what is the barometric pressure?

118. Altitude and temperature: A large weather balloon is released and takes altitude, pressure, and temperature readings as it climbs, and radios the information back to Earth. What is the pressure reading at an altitude of 5000 m, given the temperature is −18°C?

Use *Newton's law of cooling* $T = T_R + (T_0 - T_R)e^{kh}$ to complete Exercises 119 and 120. Recall that water freezes at 32°F and use $k = -0.012$. Refer to Section 4.2, page 430 as needed.

119. Making popsicles: On a hot summer day, Sean and his friends mix some Kool-Aid® and decide to freeze it in an ice tray to make popsicles. If the water used for the Kool-Aid® was 75°F and the freezer has a temperature of −20°F, how long will they have to wait to enjoy the treat?

120. Freezing time: Suppose the current temperature in Esconabe, Michigan, was 47°F when a 5°F arctic cold front moved over the state. How long would it take a puddle of water to freeze over?

Depreciation/appreciation: As time passes, the value of certain items decrease (appliances, automobiles, etc.), while the value of other items increase (collectibles, real estate, etc.). The time T in years for an item to reach a future value can be modeled by the formula $T = k\ln\left(\dfrac{V_n}{V_f}\right)$, where V_n is the purchase price when new, V_f is its future value, and k is a constant that depends on the item.

121. Automobile depreciation: If a new car is purchased for \$28,500, find its value 3 yr later if $k = 5$.

122. Home appreciation: If a new home in an "upscale" neighborhood is purchased for \$130,000, find its value 12 yr later if $k = -16$.

Drug absorption: The time required for a certain percentage of a drug to be *absorbed* by the body depends on the drug's absorption rate. This can be modeled by the function $T(p) = \frac{-\ln p}{k}$, where p represents the percent of the drug that *remains unabsorbed* (expressed as a decimal), k is the absorption rate of the drug, and $T(p)$ represents the elapsed time.

123. For a drug with an absorption rate of 7.2%, (a) find the time required (to the nearest hour) for the body to *absorb* 35% of the drug, and (b) find the percent of this drug (to the nearest half percent) that remains unabsorbed after 24 hr.

124. For a drug with an absorption rate of 5.7%, (a) find the time required (to the nearest hour) for the body to *absorb* 50% of the drug, and (b) find the percent of this drug (to the nearest half percent) that remains unabsorbed after 24 hr.

Spaceship velocity: In space travel, the change in the velocity of a spaceship V_s (in km/sec) depends on the mass of the ship M_s (in tons), the mass of the fuel which has been burned M_f (in tons) and the escape velocity of the exhaust V_e (in km/sec). Disregarding frictional forces, these are related by the equation $V_s = V_e \ln\left(\frac{M_s}{M_s - M_f}\right)$.

125. For the Jupiter VII rocket, find the mass of the fuel M_f that has been burned if $V_s = 6$ km/sec when $V_e = 8$ km/sec, and the ship's mass is 100 tons.

126. For the Neptune X satellite booster, find the mass of the ship M_s if $M_f = 75$ tons of fuel has been burned when $V_s = 8$ km/sec and $V_e = 10$ km/sec.

Learning curve: The job performance of a new employee when learning a repetitive task (as on an assembly line) improves very quickly at first, then grows more slowly over time. This can be modeled by the function $P(t) = a + b \ln t$, where a and b are constants that depend on the type of task and the training of the employee.

127. The number of toy planes an employee can assemble from its component parts depends on the length of time the employee has been working. This output is modeled by $P(t) = 5.9 + 12.6 \ln t$, where $P(t)$ is the number of planes assembled daily after working t days. (a) How many planes is an employee making after 5 days on the job? (b) How many days until the employee is able to assemble 34 planes per day?

128. The number of circuit boards an associate can assemble from its component parts depends on the length of time the associate has been working. This output is modeled by $B(t) = 1 + 2.3 \ln t$, where $B(t)$ is the number of boards assembled daily after working t days. (a) How many boards is an employee completing after 9 days on the job? (b) How long will it take until the employee is able to complete 10 boards per day?

▶ EXTENDING THE CONCEPT

Use prime factors, properties of logs, and the values given to evaluate each expression without a calculator. Check each result using the change-of-base formula:

129. $\log_3 4 = 1.2619$ and $\log_3 5 = 1.4649$:

a. $\log_3 20$

b. $\log_3 \frac{4}{5}$

c. $\log_3 25$

130. $\log_5 2 \approx 0.4307$ and $\log_5 3 \approx 0.6826$:

a. $\log_5 \frac{9}{2}$

b. $\log_5 216$

c. $\log_5 \sqrt[3]{6}$

131. Match each equation with the most appropriate solution strategy, and justify/discuss why.

a. $e^{x+1} = 25$ ______ apply base-10 logarithm to both sides

b. $\log(2x + 3) = \log 53$ ______ rewrite and apply uniqueness property for exponentials

c. $\log(x^2 - 3x) = 2$ ______ apply uniqueness property for logarithms

d. $10^{2x} = 97$ ______ apply either base-10 or base-e logarithm

e. $2^{5x-3} = 32$ ______ apply base-e logarithm

f. $7^{x+2} = 23$ ______ write in exponential form

Solve the following equations. Note that equations Exercises 132 and 133 are in quadratic form.

132. $2e^{2x} - 7e^{x} = 15$

133. $3e^{2x} - 4e^{x} - 7 = -3$

134. $\log_2(x + 5) = \log_4(21x + 1)$

135. Show that $g(x) = f^{-1}(x)$ by composing the functions.

a. $f(x) = 3^{x-2}$; $g(x) = \log_3 x + 2$

b. $f(x) = e^{x-1}$; $g(x) = \ln x + 1$

136. Use the algebraic method to find the inverse function.

a. $f(x) = 2^{x+1}$ **b.** $y = 2\ln(x - 3)$

137. Use properties of logarithms and/or exponents to show

a. $y = 2^x$ is equivalent to $y = e^{x \ln 2}$.

b. $y = b^x$ is equivalent to $y = e^{rx}$, where $r = \ln b$.

138. To understand the formula for the half-life of radioactive material, consider that for each time increment, a constant proportion of mass m is lost. In symbols; $m(t + 1) - m(t) = -km(t)$. (a) Solve for $m(t + 1)$ and factor the right-hand side. (b) Evaluate the new equation for $t = 0, 1, 2,$ and 3, to show that $m(t) = m(0)(1 - k)^t$. (c) For any half-life h, we have $m(h) = m(0)(1 - k)^h = \frac{1}{2}m(0)$. Solve for $1 - k$, raise both sides to the power t, and substitute to show $m(t) = m(0)(\frac{1}{2})^{\frac{t}{h}}$.

139. Use test values for p and q to demonstrate that the following relationships are *false,* then state the correct property and use the same test value to verify the property.

a. $\ln(pq) = \ln p \ln q$

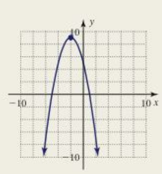
b. $\ln\left(\frac{p}{q}\right) = \frac{\ln p}{\ln q}$

c. $\ln p + \ln q = \ln(p + q)$

140. Verify that $\ln x = (\ln 10)(\log x)$, and discuss *why* they're equal. Then use the relationship to find the value of $\ln e$, $\ln 10$, and $\ln 2$.

▶ MAINTAINING YOUR SKILLS

141. **(2.4)** Match the graph shown with its correct equation, without actually graphing the function.



a. $y = x^2 + 4x - 5$

b. $y = -x^2 - 4x + 5$

c. $y = -x^2 + 4x + 5$

d. $y = x^2 - 4x - 5$

142. **(3.3)** State the domain and range of the functions.

a. $y = \sqrt{2x + 3}$ **b.** $y = |x + 2| - 3$

143. **(4.6)** Graph the function $r(x) = \frac{x^2 - 4}{x - 1}$. Label all intercepts and asymptotes.

144. **(3.6)** Suppose the maximum load (in tons) that can be supported by a cylindrical post varies directly with its diameter raised to the fourth power and inversely as the square of its height. A post 8 ft high and 2 ft in diameter can support 6 tons. How many tons can be supported by a post 12 ft high and 3 ft in diameter?

4.5 Applications from Business, Finance, and Science

Learning Objectives

In Section 4.5 you will learn how to:

- **A.** Calculate simple interest and compound interest
- **B.** Calculate interest compounded continuously
- **C.** Solve applications of annuities and amortization
- **D.** Solve applications of exponential growth and decay

Would you pay $750,000 for a home worth only $250,000? Surprisingly, when a conventional mortgage is repaid over 30 years, this is not at all rare. Over time, the accumulated interest on the mortgage is easily more than two or three times the original value of the house. In this section we explore how interest is paid or charged, and look at other applications of exponential and logarithmic functions from business, finance, as well as the physical and social sciences.

A. Simple and Compound Interest

Simple interest is an amount of interest that is computed only once during the lifetime of an investment (or loan). In the world of finance, the initial deposit or base amount is referred to as the **principal p,** the **interest rate r** is given as a percentage and stated as an annual rate, with the term of the investment or loan most often given as *time t* in years. Simple interest is merely an application of the basic percent equation, with the additional element of time coming into play: *interest* = *principal* × *rate* × *time,* or $I = prt$. To find the total amount A that has accumulated (for deposits) or is due (for loans) after t years, we merely add the accumulated interest to the initial principal: $A = p + prt$.

WORTHY OF NOTE

If a loan is kept for only a certain number of months, weeks, or days, the time t should be stated as a fractional part of a year so the time period for the rate (years) matches the time period over which the loan is repaid.

Simple Interest Formula

If principal p is deposited or borrowed at interest rate r for a period of t years, the simple interest on this account will be

$$I = prt$$

The total amount A accumulated or due after this period will be:

$$A = p + prt \qquad \text{or} \qquad A = p(1 + rt)$$

EXAMPLE 1 ▶ Solving an Application of Simple Interest

Many finance companies offer what have become known as *PayDay Loans*—a small $50 loan to help people get by until payday, usually no longer than 2 weeks. If the cost of this service is $12.50, determine the annual rate of interest charged by these companies.

Solution ▶ The interest charge is $12.50, the initial principal is $50.00, and the time period is 2 weeks or $\frac{2}{52} = \frac{1}{26}$ of a year. The simple interest formula yields

$$I = prt \qquad \text{simple interest formula}$$

$$12.50 = 50r\left(\frac{1}{26}\right) \qquad \text{substitute \$12.50 for } I\text{, \$50.00 for } p\text{, and } \tfrac{1}{26} \text{ for } t$$

$$6.5 = r \qquad \text{solve for } r$$

The annual interest rate on these loans is a whopping 650%!

Now try Exercises 7 through 16 ▶

Compound Interest

Many financial institutions pay **compound interest** on deposits they receive, which is interest paid on previously accumulated interest. The most common compounding periods are yearly, semiannually (two times per year), quarterly (four times per year), monthly (12 times per year), and daily (365 times per year). Applications of compound interest typically involve exponential functions. For convenience, consider $1000 in principal, deposited at 8% for 3 yr. The simple interest calculation shows \$240 in interest is earned and there will be \$1240 in the account: $A = 1000[1 + (0.08)(3)] = \$1240$. If the interest is *compounded each year* ($t = 1$) instead of once at the start of the 3-yr period, the interest calculation shows

$$A_1 = 1000(1 + 0.08) = 1080 \text{ in the account at the end of year 1,}$$
$$A_2 = 1080(1 + 0.08) = 1166.40 \text{ in the account at the end of year 2,}$$
$$A_3 = 1166.40(1 + 0.08) \approx 1259.71 \text{ in the account at the end of year 3.}$$

The account has earned an additional \$19.71 interest. More importantly, notice that we're multiplying by $(1 + 0.08)$ each compounding period, meaning results can be computed more efficiently by simply applying the factor $(1 + 0.08)^t$ to the initial principal p. For example,

$$A_3 = 1000(1 + 0.08)^3 \approx \$1259.71.$$

In general, for interest compounded yearly the **accumulated value** is $A = p(1 + r)^t$. Notice that solving this equation for p will tell us the amount we need to deposit *now,* in order to accumulate A dollars in t years: $p = \frac{A}{(1 + r)^t}$. This is called the **present value equation.**

Interest Compounded Annually

If a principal p is deposited at interest rate r and compounded yearly for a period of t yr, the ***accumulated value*** is

$$A = p(1 + r)^t$$

If an accumulated value A is desired after t yr, and the money is deposited at interest rate r and compounded yearly, the *present value* is

$$p = \frac{A}{(1 + r)^t}$$

EXAMPLE 2 ▶ Finding the Doubling Time of an Investment

An initial deposit of \$1000 is made into an account paying 6% compounded yearly. How long will it take for the money to double?

Solution ▶ Using the formula for interest compounded yearly we have

$A = p(1 + r)^t$	given
$2000 = 1000(1 + 0.06)^t$	substitute 2000 for A, 1000 for p, and 0.06 for r
$2 = 1.06^t$	isolate variable term
$\ln 2 = t \ln 1.06$	apply base-e logarithms; power property
$\frac{\ln 2}{\ln 1.06} = t$	solve for t
$11.9 \approx t$	approximate form

The money will double in just under 12 yr.

Now try Exercises 17 through 22 ▶

If interest is compounded monthly (12 times each year), the bank will divide the interest rate by 12 (the number of compoundings), but then pay you interest 12 times per year (interest is *compounded*). The net effect is an increased gain in the interest you earn, and the final compound interest formula takes this form:

$$\text{total amount} = \text{principal}\left(1 + \frac{\text{interest rate}}{\text{compoundings per year}}\right)^{(\text{years} \times \text{compoundings per year})}$$

Compounded Interest Formula

If principal p is deposited at interest rate r and compounded n times per year for a period of t yr, the *accumulated value* will be:

$$A = p\left(1 + \frac{r}{n}\right)^{nt}$$

EXAMPLE 3 ▶ Solving an Application of Compound Interest

Macalyn won $150,000 in the Missouri lottery and decides to invest the money for retirement in 20 yr. Of all the options available here, which one will produce the most money for retirement?

a. A certificate of deposit paying 5.4% compounded yearly.
b. A money market certificate paying 5.35% compounded semiannually.
c. A bank account paying 5.25% compounded quarterly.
d. A bond issue paying 5.2% compounded daily.

Solution ▶

a. $A = \$150{,}000\left(1 + \frac{0.054}{1}\right)^{(20\times1)}$
$\approx \$429{,}440.97$

b. $A = \$150{,}000\left(1 + \frac{0.0535}{2}\right)^{(20\times2)}$
$\approx \$431{,}200.96$

c. $A = \$150{,}000\left(1 + \frac{0.0525}{4}\right)^{(20\times4)}$
$\approx \$425{,}729.59$

d. $A = \$150{,}000\left(1 + \frac{0.052}{365}\right)^{(20\times365)}$
$\approx \$424{,}351.12$

The best choice is (b), semiannual compounding at 5.35% for 20 yr.

A. You've just learned how to calculate simple interest and compound interest

Now try Exercises 23 through 30 ▶

B. Interest Compounded Continuously

It seems natural to wonder what happens to the interest accumulation as n (the number of compounding periods) becomes very large. It appears the interest rate becomes very small (because we're dividing by n), but the exponent becomes very large (since we're multiplying by n). To see the result of this interplay more clearly, it will help to rewrite the compound interest formula $A = p(1 + \frac{r}{n})^{nt}$ using the substitution $n = xr$. This gives $\frac{r}{n} = \frac{1}{x}$, and by direct substitution (xr for n and $\frac{1}{x}$ for $\frac{r}{n}$) we obtain the form

$$A = p\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt}$$

by regrouping. This allows for a more careful study of the "denominator versus exponent" relationship using $(1 + \frac{1}{x})^x$, *the same expression we used in Section 4.2 to define the number e* (also **see Section 4.2 Exercise 97**). Once again, note what happens as $x \to \infty$ (meaning the number of compounding periods increase without bound).

x	1	10	100	1000	10,000	100,000	1,000,000
$\left(1+\frac{1}{x}\right)^x$	2	2.56374	2.70481	2.71692	2.71815	2.71827	2.71828

As before, we have, as $x \to \infty$, $(1 + \frac{1}{x})^x \to e$. The net result of this investigation is a formula for **interest compounded continuously,** derived by replacing $(1 + \frac{1}{x})^x$ with the number e in the formula for compound interest, where

$$A = p\left[\left(1 + \frac{1}{x}\right)^x\right]^{rt} = pe^{rt}$$

Interest Compounded Continuously

If a principal p is deposited at interest rate r and compounded continuously for a period of t years, the *accumulated value* will be

$$A = pe^{rt}$$

EXAMPLE 4 ► Solving an Application of Interest Compounded Continuously

Jaimin has $10,000 to invest and wants to have at least $25,000 in the account in 10 yr for his daughter's college education fund. If the account pays interest compounded continuously, what interest rate is required?

Solution ► In this case, $P = \$10{,}000$, $A = \$25{,}000$, and $t = 10$.

$A = pe^{rt}$	given
$25{,}000 = 10{,}000e^{10r}$	substitute 25,000 for *A*, 10,000 for *p*, and 10 for *t*
$2.5 = e^{10r}$	isolate variable term
$\ln 2.5 = 10r \ln e$	apply base-*e* logarithms ($\ln e = 1$); power property
$\frac{\ln 2.5}{10} = r$	solve for *r*
$0.092 \approx r$	approximate form

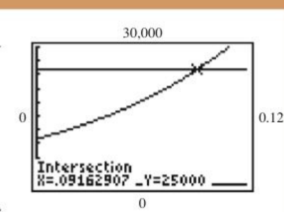
Jaimin will need an interest rate of about 9.2% to meet his goal.

Now try Exercises 31 through 40 ►

B. You've just learned how to calculate interest compounded continuously

GRAPHICAL SUPPORT

To check the result from Example 4, use $Y_1 = 10{,}000e^{10x}$ and $Y_2 = 25{,}000$, then look for their point of intersection. We need only set an appropriate window size to ensure the answer will appear in the viewing window. Since 25,000 is the goal, $y \in [0, 30{,}000]$ seems reasonable for *y*. Although 12% interest ($x = 0.12$) is too good to be true, $x \in [0, 0.12]$ leaves a nice frame for the *x*-values. Verify that the calculator's answer is equal to $\frac{\ln 2.5}{10}$.



WORTHY OF NOTE

It is often assumed that the first payment into an annuity is made *at the end of a compounding period,* and hence earns no interest. This is why the first \$100 deposit is not multiplied by the interest factor. These terms are actually the terms of a **geometric sequence,** which we will study later in Section 8.3.

C. Applications Involving Annuities and Amortization

Our previous calculations for simple and compound interest involved a single (lump) deposit (the principal) that accumulated interest over time. Many savings and investment plans involve a regular schedule of deposits (monthly, quarterly, or annual deposits) over the life of the investment. Such an investment plan is called an **annuity.**

Suppose that for 4 yr, \$100 is deposited annually into an account paying 8% compounded yearly. Using the compound interest formula we can track the accumulated value A in the account:

$$A = 100 + 100(1.08)^1 + 100(1.08)^2 + 100(1.08)^3$$

To develop an annuity formula, we multiply the annuity equation by 1.08, then subtract the original equation. This leaves only the first and last terms, since the other (interior) terms add to zero:

$$1.08A = 100(1.08) + 100(1.08)^2 + 100(1.08)^3 + 100(1.08)^4 \quad \text{multiply by 1.08}$$
$$-A = -[100 + 100(1.08)^1 + 100(1.08)^2 + 100(1.08)^3] \quad \text{original equation}$$
$$1.08A - A = 100(1.08)^4 - 100 \quad \text{subtract ("interior terms" sum to zero)}$$
$$0.08A = 100[(1.08)^4 - 1] \quad \text{factor out 100}$$
$$A = \frac{100[(1.08)^4 - 1]}{0.08} \quad \text{solve for } A$$

This result can be generalized for any periodic payment $\mathcal{P}$, interest rate r, number of compounding periods n, and number of years t. This would give

$$A = \frac{\mathcal{P}\left[\left(1 + \frac{r}{n}\right)^{nt} - 1\right]}{\frac{r}{n}}$$

The formula can be made less formidable using $R = \frac{r}{n}$, where R is the interest rate per compounding period.

Accumulated Value of an Annuity

If a periodic payment $\mathcal{P}$ is deposited n times per year at an *annual interest rate r* with interest compounded n times per year for t years, the accumulated value is given by

$$A = \frac{\mathcal{P}}{R}[(1 + R)^{nt} - 1], \text{ where } R = \frac{r}{n}$$

This is also referred to as the **future value** of the account.

EXAMPLE 5 ▶ Solving an Application of Annuities

Since he was a young child, Fitisemanu's parents have been depositing \$50 each month into an annuity that pays 6% annually and is compounded monthly. If the account is now worth \$9875, how long has it been open?

Solution ▶ In this case $\mathcal{P} = 50$, $r = 0.06$, $n = 12$, $R = 0.005$, and $A = 9875$. The formula gives

$$A = \frac{\mathcal{P}}{R}[(1 + R)^{nt} - 1] \quad \text{future value formula}$$
$$9875 = \frac{50}{0.005}[(1.005)^{(12)(t)} - 1] \quad \text{substitute 9875 for } A\text{, 50 for } \mathcal{P}\text{, 0.005 for } R\text{, and 12 for } n$$
$$1.9875 = 1.005^{12t} \quad \text{simplify and isolate variable term}$$

$$\ln(1.9875) = 12t(\ln 1.005) \quad \text{apply base-}e\text{ logarithms; power property}$$
$$\frac{\ln(1.9875)}{12\ln(1.005)} = t \quad \text{solve for } t \text{ (exact form)}$$
$$11.5 \approx t \quad \text{approximate form}$$

The account has been open approximately 11.5 yr.

Now try Exercises 41 through 44 ►

The periodic payment required to meet a future goal or obligation can be computed by solving for $\mathcal{P}$ in the future value formula: $\mathcal{P} = \frac{AR}{[(1+R)^{nt} - 1]}$. In this form, $\mathcal{P}$ is referred to as a **sinking fund.**

EXAMPLE 6 ► Solving an Application of Sinking Funds

Sheila is determined to stay out of debt and decides to save $20,000 to pay cash for a new car in 4 yr. The best investment vehicle she can find pays 9% compounded monthly. If $300 is the most she can invest each month, can she meet her "4-yr" goal?

Solution ► Here we have $\mathcal{P} = 300$, $A = 20{,}000$, $r = 0.09$, $n = 12$, and $R = 0.0075$. The sinking fund formula gives

$$\mathcal{P} = \frac{AR}{[(1+R)^{nt} - 1]} \quad \text{sinking fund}$$
$$300 = \frac{(20{,}000)(0.0075)}{(1.0075)^{12t} - 1} \quad \text{substitute 300 for } \mathcal{P}\text{, 20,000 for } A\text{, 0.0075 for } R\text{, and 12 for } n$$
$$300(1.0075^{12t} - 1) = 150 \quad \text{multiply in numerator, clear denominators}$$
$$1.0075^{12t} = 1.5 \quad \text{isolate variable term}$$
$$12t\ln(1.0075) = \ln 1.5 \quad \text{apply base-}e\text{ logarithms; power property}$$
$$t = \frac{\ln(1.5)}{12\ln(1.0075)} \quad \text{solve for } t \text{ (exact form)}$$
$$\approx 4.5 \quad \text{approximate form}$$

No. She is close, but misses her original 4-yr goal.

Now try Exercises 45 and 46 ►

☑ **C.** You've just learned how to solve applications of annuities and amortization

For Example 6, we could have substituted 4 for t and left $\mathcal{P}$ unknown, to see if a payment of $300 per month would be sufficient. You can verify the result would be $\mathcal{P} \approx \$347.70$, which is what Sheila would need to invest to meet her 4-yr goal exactly.

For additional practice with the formulas for interest earned or paid, the *Working with Formulas* portion of this Exercise Set has been expanded. See **Exercises 47 through 54.**

WORTHY OF NOTE

Notice the formula for exponential growth is virtually identical to the formula for interest compounded continuously. In fact, both are based on the same principles. If we let $A(t)$ represent the amount in an account after t years and A_0 represent the initial deposit (instead of P), we have: $A(t) = A_0e^{rt}$ versus $Q(t) = Q_0e^{rt}$ and the two cannot be distinguished.

D. Applications Involving Exponential Growth and Decay

Closely related to interest compounded continuously are applications of **exponential growth** and **exponential decay.** If Q (quantity) and t (time) are variables, then Q grows exponentially as a function of t if $Q(t) = Q_0e^{rt}$ for positive constants Q_0 and r. Careful studies have shown that population growth, whether it be humans, bats, or bacteria, can be modeled by these "base-e" exponential growth functions. If $Q(t) = Q_0e^{-rt}$, then we say Q decreases or **decays exponentially** over time. The constant r determines how rapidly a quantity grows or decays and is known as the **growth rate** or **decay rate** constant.

EXAMPLE 7 ► Solving an Application of Exponential Growth

Because fruit flies multiply very quickly, they are often used in a study of genetics. Given the necessary space and food supply, a certain population of fruit flies is known to double every 12 days. If there were 100 flies to begin, find (a) the growth rate r and (b) the number of days until the population reaches 2000 flies.

Solution ► **a.** Using the formula for exponential growth with $Q_0 = 100$, $t = 12$, and $Q(t) = 200$, we can solve for the growth rate r.

$$Q(t) = Q_0e^{rt} \quad \text{exponential growth function}$$
$$200 = 100e^{12r} \quad \text{substitute 200 for } Q(t)\text{, 100 for } Q_0\text{, and 12 for } t$$
$$2 = e^{12r} \quad \text{isolate variable term}$$
$$\ln 2 = 12r \ln e \quad \text{apply base-}e\text{ logarithms; power property}$$
$$\frac{\ln 2}{12} = r \quad \text{solve for } r \text{ (exact form)}$$
$$0.05776 \approx r \quad \text{approximate form}$$

The growth rate is approximately 5.78%.

b. To find the number of days until the fly population reaches 2000, we substitute 0.05776 for r in the exponential growth function.

$$Q(t) = Q_0e^{rt} \quad \text{exponential growth function}$$
$$2000 = 100e^{0.05776t} \quad \text{substitute 2000 for } Q(t)\text{, 100 for } Q_0\text{, and 0.05776 for } r$$
$$20 = e^{0.05776t} \quad \text{isolate variable term}$$
$$\ln 20 = 0.05776t \ln e \quad \text{apply base-}e\text{ logarithms; power property}$$
$$\frac{\ln 20}{0.05776} = t \quad \text{solve for } t \text{ (exact form)}$$
$$51.87 \approx t \quad \text{approximate form}$$

The fruit fly population will reach 2000 on day 51.

WORTHY OF NOTE

Many population growth models assume an unlimited supply of resources, nutrients, and room for growth. When this is not the case, a logistic growth model often results. See the *Modeling with Technology* feature following this chapter.

Now try Exercises 55 and 56 ►

Perhaps the best-known examples of exponential decay involve radioactivity. Ever since the end of World War II, common citizens have been aware of the existence of **radioactive elements** and the power of atomic energy. Today, hundreds of additional applications have been found for these materials, from areas as diverse as biological research, radiology, medicine, and archeology. Radioactive elements decay of their own accord by emitting radiation. The rate of decay is measured using the **half-life** of the substance, which is the time required for a mass of radioactive material to decay until only one-half of its original mass remains. This half-life is used to find the rate of decay r, first mentioned in Section 4.4. In general, if h represents the half-life of the substance, one-half the initial amount remains when $t = h$.

$$Q(t) = Q_0e^{-rt} \quad \text{exponential decay function}$$
$$\frac{1}{2}Q_0 = Q_0e^{-rh} \quad \text{substitute } \tfrac{1}{2}Q_0 \text{ for } Q(t)\text{, } h \text{ for } t$$
$$\frac{1}{2} = \frac{1}{e^{rh}} \quad \text{divide by } Q_0\text{; rewrite expression}$$
$$2 = e^{rh} \quad \text{property of ratios}$$
$$\ln 2 = rh \ln e \quad \text{apply base-}e\text{ logarithms; power property}$$
$$\frac{\ln 2}{h} = r \quad \text{solve for } r$$

Radioactive Rate of Decay

If h represents the half-life of a radioactive substance per unit time, the nominal rate of decay per a like unit of time is given by

$$r = \frac{\ln 2}{h}$$

The rate of decay for known radioactive elements varies greatly. For example, the element carbon-14 has a half-life of about 5730 yr, while the element lead-211 has a half-life of only about 3.5 min. Radioactive elements can be detected in extremely small amounts. If a drug is "labeled" (mixed with) a radioactive element and injected into a living organism, its passage through the organism can be traced and information on the health of internal organs can be obtained.

EXAMPLE 8 ▶ Solving a Rate of Decay Application

The radioactive element potassium-42 is often used in biological experiments, since it has a half-life of only about 12.4 hr. How much of a 2-g sample will remain after 18 hr and 45 min?

Solution ▶ To begin we must find the nominal rate of decay r and use this value in the exponential decay function.

$$r = \frac{\ln 2}{h} \quad \text{radioactive rate of decay}$$

$$r = \frac{\ln 2}{12.4} \quad \text{substitute 12.4 for } h$$

$$r \approx 0.056 \quad \text{result}$$

The rate of decay is approximately 5.6%. To determine how much of the sample remains after 18.75 hr, we use $r = 0.056$ in the decay function and evaluate it at $t = 18.75$.

$$Q(t) = Q_0e^{-rt} \quad \text{exponential decay function}$$

$$Q(18.75) = 2e^{(-0.056)(18.75)} \quad \text{substitute 2 for } Q_0\text{, 0.056 for } r\text{, and 18.75 for } t$$

$$Q(18.75) \approx 0.7 \quad \text{evaluate}$$

After 18 hr and 45 min, only 0.7 g of potassium-42 will remain.

D. You've just learned how to solve applications of exponential growth and decay

Now try Exercises 57 through 62 ▶

TECHNOLOGY HIGHLIGHT

Exploring Compound Interest

The graphing calculator is an excellent tool for exploring mathematical relationships, particularly when many variables work simultaneously to produce a single result. For example, the formula $A = P(1 + \frac{r}{n})^{nt}$ has five different unknowns. In Example 2, we asked how long it would take \$1000 to double if it were compounded yearly at 6% ($n = 1$, $r = 0.06$). What if we deposited \$5000 instead of \$1000? Compounded daily instead of quarterly? Or invested at 12% rather than 10%? There are many ways a graphing calculator can be used to answer such questions. In this exercise, we make use of the calculator's "alpha constants." Most graphing calculators can use any of the 26 letters of the English alphabet (and even a few other symbols) to store constant values. We can use them to write a formula

—continued

Figure 4.31

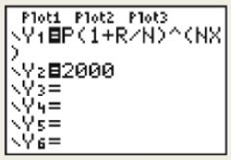


Figure 4.32

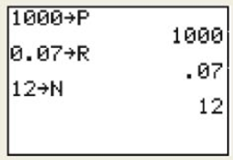
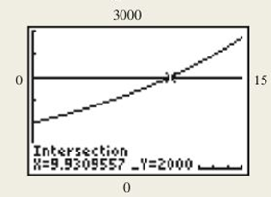


Figure 4.33



on the [Y=] screen, then change any constant on the home screen to see how other values are affected. On the TI-84 Plus, these alpha constants are shown in green and accessed by pressing the [ALPHA] key. Suppose we wanted to study the relationship between an interest rate *r* and the time *t* required for a deposit to double. Using Y_1 in place of *A* as output variable, and *x* in place of *t*, enter $A = P\left(1 + \frac{r}{n}\right)^{nt}$ as Y_1 on the [Y=] screen (Figure 4.31). Let's start with a deposit of \$1000 at 7% interest compounded monthly. The keystrokes are: 1000 [STO→] [ALPHA] [8] [ENTER], 0.07 [STO→] [ALPHA] [×] [ENTER], and 12 [STO→] [ALPHA] [LOG] [ENTER] (Figure 4.32). After setting an appropriate window size (perhaps Xmax = 15 and Ymax = 3000), and entering $Y_2 = 2000$ we can graph both functions and use the intersection of graphs method to find the doubling time. This produces the result in Figure 4.33, where we note it will take about 9.9 yr. Return to the home screen ([2nd] [MODE]), change the interest rate to 10%, and graph the functions again. This time the point of intersection is just less than 7 (yr). Experiment with other rates and compounding periods.

Exercise 1: With $P = \$1000$, and $r = 0.08$, investigate the "doubling time" for interest compounded quarterly, monthly, daily, and hourly.

Exercise 2: With $P = \$1000$, investigate "doubling time" for rates of 6%, 8%, 10%, and 12%, and $n = 4$, $n = 12$, and $n = 365$. Which had a more significant impact, more compounding periods, or a greater interest rate?

Exercise 3: Will a larger principal cause the money to double faster? Investigate and respond.

4.5 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. ______ interest is interest paid to you on previously accumulated interest.
2. The formula for interest compounded ______ is $A = pe^{rt}$, where *e* is approximately ______.
3. Given constants Q_0 and *r*, and that *Q* decays exponentially as a function of *t*, the equation model is $Q(t) =$ ______.
4. Investment plans calling for regularly scheduled deposits are called ______. The annuity formula gives the ______ value of the account.
5. Explain/Describe the difference between the future value and present value of an annuity. Include an example.
6. Describe/Explain how you would find the rate of growth *r*, given that a population of ants grew from 250 to 3000 in 6 weeks.

▶ DEVELOPING YOUR SKILLS

For simple interest accounts, the interest earned or due depends on the principal p, interest rate r, and the time t in years according to the formula $I = prt$.

7. Find p given $I = \$229.50$, $r = 6.25\%$, and $t = 9$ months.

8. Find r given $I = \$1928.75$, $p = \$8500$, and $t = 3.75$ yr.

9. Larry came up a little short one month at bill-paying time and had to take out a title loan on his car at Check Casher's, Inc. He borrowed $260, and 3 weeks later he paid off the note for $297.50. What was the annual interest rate on this title loan? (*Hint:* How much *interest* was charged?)

10. Angela has $750 in a passbook savings account that pays 2.5% simple interest. How long will it take the account balance to hit the $1000 mark at this rate of interest, if she makes no further deposits? (*Hint:* How much *interest* will be paid?)

For simple interest accounts, the amount A accumulated or due depends on the principal p, interest rate r, and the time t in years according to the formula $A = p(1 + rt)$.

11. Find p given $A = \$2500$, $r = 6.25\%$, and $t = 31$ months.

12. Find r given $A = \$15{,}800$, $p = \$10{,}000$, and $t = 3.75$ yr.

13. Olivette Custom Auto Service borrowed $120,000 at 4.75% simple interest to expand their facility from three service bays to four. If they repaid $149,925, what was the term of the loan?

14. Healthy U sells nutritional supplements and borrows $50,000 to expand their product line. When the note is due 3 yr later, they repay the lender $62,500. If it was a simple interest note, what was the annual interest rate?

15. Simple interest: The owner of Paul's Pawn Shop loans Larry $200.00 using his Toro riding mower as collateral. Thirteen weeks later Larry comes back to get his mower out of pawn and pays Paul $240.00. What was the annual simple interest rate on this loan?

16. Simple interest: To open business in a new strip mall, Laurie's Custom Card Shoppe borrows $50,000 from a group of investors at 4.55% simple interest. Business booms and blossoms, enabling Laurie to repay the loan fairly quickly. If Laurie repays $62,500, how long did it take?

For accounts where interest is compounded annually, the amount A accumulated or due depends on the principal p, interest rate r, and the time t in years according to the formula $A = p(1 + r)^t$.

17. Find t given $A = \$48{,}428$, $p = \$38{,}000$, and $r = 6.25\%$.

18. Find p given $A = \$30{,}146$, $r = 5.3\%$, and $t = 7$ yr.

19. How long would it take $1525 to triple if invested at 7.1%?

20. What interest rate will ensure a $747.26 deposit will be worth $1000 in 5 yr?

For accounts where interest is compounded annually, the principal P needed to ensure an amount A has been accumulated in the time period t when deposited at interest rate r is given by the formula $P = \frac{A}{(1 + r)^t}$.

21. The Stringers need to make a $10,000 balloon payment in 5 yr. How much should be invested now at 5.75%, so that the money will be available?

22. Morgan is 8 yr old. If her mother wants to have $25,000 for Morgan's first year of college (in 10 yr), how much should be invested now if the account pays a 6.375% fixed rate?

For compound interest accounts, the amount A accumulated or due depends on the principal p, interest rate r, number of compoundings per year n, and the time t in years according to the formula $A = p\left(1 + \frac{r}{n}\right)^{nt}$.

23. Find t given $A = \$129{,}500$, $p = \$90{,}000$, and $r = 7.125\%$ compounded weekly.

24. Find r given $A = \$95{,}375$, $p = \$65{,}750$, and $t = 15$ yr with interest compounded monthly.

25. How long would it take a $5000 deposit to double, if invested at a 9.25% rate and compounded daily?

26. What principal should be deposited at 8.375% compounded monthly to ensure the account will be worth $20,000 in 10 yr?

27. Compound interest: As a curiosity, David decides to invest $10 in an account paying 10% interest compounded 10 times per year for 10 yr. Is that enough time for the $10 to triple in value?

28. Compound interest: As a follow-up experiment (see Exercise 27), David invests $10 in an account paying 12% interest compounded 10 times per year for 10 yr, and another \$10 in an account paying 10% interest compounded 12 times per year for 10 yr. Which produces the better investment—more compounding periods or a higher interest rate?

29. **Compound interest:** Due to demand, Donovan's Dairy (Wisconsin, USA) plans to double its size in 4 yr and will need \$250,000 to begin development. If they invest \$175,000 in an account that pays 8.75% compounded semiannually, (a) will there be sufficient funds to break ground in 4 yr? (b) If not, find the *minimum interest rate* that will allow the dairy to meet its 4-yr goal.

30. **Compound interest:** To celebrate the birth of a new daughter, Helyn invests 6000 Swiss francs in a college savings plan to pay for her daughter's first year of college in 18 yr. She estimates that 25,000 francs will be needed. If the account pays 7.2% compounded daily, (a) will she meet her investment goal? (b) If not, find the *minimum rate of interest* that will enable her to meet this 18-yr goal.

For accounts where interest is compounded continuously, the amount A accumulated or due depends on the principal p, interest rate r, and the time t in years according to the formula $A = pe^{rt}$.

31. Find t given $A = \$2500$, $p = \$1750$, and $r = 4.5\%$.

32. Find r given $A = \$325{,}000$, $p = \$250{,}000$, and $t = 10$ yr.

33. How long would it take \$5000 to double if it is invested at 9.25%? Compare the result to Exercise 25.

34. What principal should be deposited at 8.375% to ensure the account will be worth \$20,000 in 10 yr? Compare the result to Exercise 26.

35. **Interest compounded continuously:** Valance wants to build an addition to his home outside Madrid (Spain) so he can watch over and help his parents in their old age. He hopes to have 20,000 euros put aside for this purpose within 5 yr. If he invests 12,500 euros in an account paying 8.6% interest compounded continuously, (a) will he meet his investment goal? (b) If not, find the *minimum rate of interest* that will enable him to meet this 5-yr goal.

36. **Interest compounded continuously:** Minh-Ho just inherited her father's farm near Mito (Japan), which badly needs a new barn. The estimated cost of the barn is 8,465,000 yen and she would like to begin construction in 4 yr. If she invests 6,250,000 yen in an account paying 6.5% interest compounded continuously, (a) will she meet her investment goal? (b) If not, find the *minimum rate of interest* that will enable her to meet this 4-yr goal.

37. **Interest compounded continuously:** William and Mary buy a small cottage in Dovershire (England), where they hope to move after retiring in 7 yr. The cottage needs about 20,000 euros worth of improvements to make it the retirement home they desire. If they invest 12,000 euros in an account paying 5.5% interest compounded continuously, (a) will they have enough to make the repairs? (b) If not, find the *minimum amount they need to deposit* that will enable them to meet this goal in 7 yr.

38. **Interest compounded continuously:** After living in Oslo (Norway) for 20 years, Zirkcyt and Shybrt decide to move inland to help operate the family ski resort. They hope to make the move in 6 yr, after they have put aside 140,000 kroner. If they invest 85,000 kroner in an account paying 6.9% interest compounded continuously, (a) will they meet their 140,000 kroner goal? (b) If not, find the *minimum amount they need to deposit* that will allow them to meet this goal in 6 yr.

The length of time T (in years) required for an initial principal P to grow to an amount A at a given interest rate r is given by $T = \frac{1}{r}\ln\left(\frac{A}{P}\right)$.

39. **Investment growth:** A small business is planning to build a new \$350,000 facility in 8 yr. If they deposit \$200,000 in an account that pays 5% interest compounded continuously, will they have enough for the new facility in 8 yr? If not, what amount should be invested on these terms to meet the goal?

40. **Investment growth:** After the twins were born, Sasan deposited \$25,000 in an account paying 7.5% compounded continuously, with the goal of having \$120,000 available for their college education 20 yr later. Will Sasan meet the 20-yr goal? If not, what amount should be invested on these terms to meet the goal?

Ordinary annuities: If a periodic payment $\mathcal{P}$ is deposited n times per year, with annual interest rate r also compounded n times per year for t years, the future value of the account is given by $A = \frac{\mathcal{P}[(1 + R)^{nt} - 1]}{R}$, where $R = \frac{r}{n}$ (if the rate is 9% compounded monthly, $R = \frac{0.09}{12} = 0.0075$).

41. **Saving for a rainy day:** How long would it take Jasmine to save \$10,000 if she deposits \$90/month at an annual rate of 7.75% compounded monthly?

42. Saving for a sunny day: What quarterly investment amount is required to ensure that Larry can save \$4700 in 4 yr at an annual rate of 8.5% compounded quarterly?

43. Saving for college: At the birth of their first child, Latasha and Terrance opened an annuity account and have been depositing \$50 per month in the account ever since. If the account is now worth \$30,000 and the interest on the account is 6.2% compounded monthly, how old is the child?

44. Saving for a bequest: When Cherie (Brandon's first granddaughter) was born, he purchased an annuity account for her and stipulated that she should receive the funds (in trust, if necessary) upon his death. The quarterly annuity payments were \$250 and interest on the account was 7.6% compounded quarterly. The account balance of \$17,500 was recently given to Cherie. How much longer did Brandon live?

45. Saving for a down payment: Tae-Hon is tired of renting and decides that within the next 5 yr he must save \$22,500 for the down payment on a home. He finds an investment company that offers 8.5% interest compounded monthly and begins depositing \$250 each month in the account. (a) Is this monthly amount sufficient to help him meet his 5 yr goal? (b) If not, find the *minimum amount he needs to deposit each month* that will allow him to meet his goal in 5 yr.

46. Saving to open a business: Madeline feels trapped in her current job and decides to save \$75,000 over the next 7 yr to open up a Harley Davidson franchise. To this end, she invests \$145 every week in an account paying $7\frac{1}{2}\%$ interest compounded weekly. (a) Is this weekly amount sufficient to help her meet the seven-year goal? (b) If not, find the *minimum amount she needs to deposit each week* that will allow her to meet this goal in 7 yr?

▶ WORKING WITH FORMULAS

Solve for the indicated unknowns.

47. $A = p + prt$
- **a.** solve for t
- **b.** solve for p

48. $A = p(1 + r)^t$
- **a.** solve for t
- **b.** solve for r

49. $A = P\left(1 + \frac{r}{n}\right)^{nt}$
- **a.** solve for r
- **b.** solve for t

50. $A = pe^{rt}$
- **a.** solve for p
- **b.** solve for r

51. $Q(t) = Q_0e^{rt}$
- **a.** solve for Q_0
- **b.** solve for t

52. $p = \dfrac{AR}{[(1 + R)^{nt} - 1]}$
- **a.** solve for A
- **b.** solve for n

53. Amount of a mortgage payment:

$$\mathcal{P} = \frac{AR}{1 - (1 + R)^{-nt}}$$

The mortgage payment required to pay off (or amortize) a loan is given by the formula shown, where $\mathcal{P}$ is the payment amount, A is the original amount of the loan, t is the time in years, r is the annual interest rate, n is the number of payments per year, and $R = \frac{r}{n}$. Find the *monthly payment* required to amortize a \$125,000 home, if the interest rate is 5.5%/year and the home is financed over 30 yr.

54. Total interest paid on a home mortgage:

$$I = \left[\frac{prt}{1 - \left(\frac{1}{1 + 0.08\overline{3}r}\right)^{12t}}\right] - p$$

The total interest I paid in t years on a home mortgage of p dollars is given by the formula shown, where r is the interest rate on the loan (note that $0.08\overline{3} = \frac{1}{12}$). If the original mortgage was \$198,000 at an interest rate of 6.5%, (a) how much interest has been paid in 10 yr? (b) Use a table of values to determine how many years it will take for the interest paid to exceed the amount of the original mortgage.

▶ APPLICATIONS

55. Exponential growth: As part of a lab experiment, Luamata needs to grow a culture of 200,000 bacteria, which are known to double in number in 12 hr. If he begins with 1000 bacteria, (a) find the growth rate r and (b) find how many hours it takes for the culture to produce the 200,000 bacteria.

56. Exponential growth: After the wolf population was decimated due to overhunting, the rabbit population in the Boluhti Game Reserve began to double every 6 months. If there were an estimated 120 rabbits to begin, (a) find the growth rate r and (b) find the number of months required for the population to reach 2500.

57. Radioactive decay: The radioactive element iodine-131 has a half-life of 8 days and is often used to help diagnose patients with thyroid problems. If a certain thyroid procedure requires 0.5 g and is scheduled to take place in 3 days, what is the minimum amount that must be on hand now (to the nearest hundredth of a gram)?

58. Radioactive decay: The radioactive element sodium-24 has a half-life of 15 hr and is used to help locate obstructions in blood flow. If the procedure requires 0.75 g and is scheduled to take place in 2 days (48 hr), what minimum amount must be on hand *now* (to the nearest hundredth of a gram)?

59. Radioactive decay: The radioactive element americium-241 has a half-life of 432 yr and although extremely small amounts are used (about 0.0002 g), it is the most vital component of standard household smoke detectors. How many years will it take a 10-g mass of americium-241 to decay to 2.7 g?

60. Radioactive decay: Carbon-14 is a radioactive compound that occurs naturally in all living organisms, with the amount in the organism constantly renewed. After death, no new carbon-14 is acquired and the amount in the organism begins to decay exponentially. If the half-life of carbon-14 is 5730 yr, how old is a mummy having only 30% of the normal amount of carbon-14?

Carbon-14 dating: If the percentage p of carbon-14 that remains in a fossil can be determined, the formula $T = -8267 \ln p$ can be used to estimate the number of years T since the organism died.

61. Dating the Lascaux Cave Dwellers: Bits of charcoal from Lascaux Cave (home of the prehistoric Lascaux Cave Paintings) were used to estimate that the fire had burned some 17,255 yr ago. What percent of the original amount of carbon-14 remained in the bits of charcoal?

62. Dating Stonehenge: Using organic fragments found near Stonehenge (England), scientists were able to determine that the organism that produced the fragments lived about 3925 yr ago. What percent of the original amount of carbon-14 remained in the organism?

▶ EXTENDING THE CONCEPT

63. Many claim that inheritance taxes are put in place simply to prevent a massive accumulation of wealth by a select few. Suppose that in 1890, your great-grandfather deposited \$10,000 in an account paying 6.2% compounded continuously. If the account were to pass to you untaxed, what would it be worth in 2010? Do some research on the inheritance tax laws in your state. In particular, what amounts can be inherited untaxed (i.e., before the inheritance tax kicks in)?

64. In Section 4.2, we noted that one important characteristic of exponential functions is their rate of growth is in constant proportion to the population at time t: $\frac{\Delta P}{\Delta t} = kP$. This rate of growth can also be applied to finance and biological models, as well as the growth of tumors, and is of great value in studying these applications. In Exercise 96 of Section 4.2, we computed the value of k for the Goldsboro model ($P = 1000 \cdot 2^t$) using the difference quotient. If we rewrite this model in terms of base e ($P = 1000 \cdot e^{kt}$), the value of k is given directly. The following sequence shows how this is done, and you are asked to supply the reason or justification for each step.

$P = b^t$	base-b exponential
$\ln P = \ln b^t$	______________
$\ln P = t \ln b$	______________
$P = e^{t \ln b}$	______________
$P = e^{kt}$	$k = \ln b$

The last step shows the growth rate constant is equal to the natural log of the given base b: $k = \ln b$.

a. Use this result to verify the growth rate constant for Goldsboro is 0.6931472.

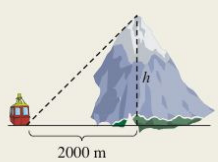
b. After the Great Oklahoma Land Run of 1890, the population of the state grew rapidly for the next 2 decades. For this time period, population growth could be approximated by $P = 260(1.10^t)$. Find the growth rate constant for this model, and use it to write the base-e population equation. Use the TABLE feature of a graphing calculator to verify that the equations are equivalent.

65. If you have not already completed Exercise 30, please do so. For *this* exercise, *solve the compound interest equation for r* to find the exact rate of interest that will allow Helyn to meet her 18-yr goal.

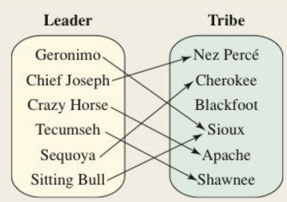
66. If you have not already completed Exercise 43, please do so. Suppose the final balance of the account was \$35,100 with interest again being compounded monthly. For *this* exercise, use a graphing calculator to find r, the exact rate of interest the account would have been earning.

► MAINTAINING YOUR SKILLS

67. **(2.1)** In an effort to boost tourism, a trolley car is being built to carry sightseers from a strip mall to the top of Mt. Vernon, 1580-m high. Approximately how long will the trolley cables be?



68. **(2.2)** Is the following relation a function? If not, state how the definition of a function is violated.



69. **(4.3)** A polynomial is known to have the zeroes $x = 3$, $x = -1$, and $x = 1 + 2i$. Find the equation of the polynomial, given it has degree 4 and a y-intercept of $(0, -15)$.

70. **(2.2/3.8)** Name the toolbox functions that are (a) one-to-one, (b) even, (c) increasing for $x \in R$, and (d) asymptotic.

SUMMARY AND CONCEPT REVIEW

SECTION 4.1 One-to-One and Inverse Functions

KEY CONCEPTS

- A function is one-to-one if each element of the range corresponds to a unique element of the domain.
- If every horizontal line intersects the graph of a function in at most one point, the function is one-to-one.
- If *f* is a one-to-one function with ordered pairs (a, b), then the inverse of *f* exists and is that one-to-one function f^{-1} with ordered pairs of the form (b, a).
- The range of *f* becomes the domain of f^{-1}, and the domain of *f* becomes the range of f^{-1}.
- To find f^{-1} using the algebraic method:
  1. Use *y* instead of $f(x)$.
  2. Interchange *x* and *y*.
  3. Solve the equation for *y*.
  4. Substitute $f^{-1}(x)$ for *y*.
- If *f* is a one-to-one function, the inverse f^{-1} exists, where $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.
- The graphs of *f* and f^{-1} are symmetric with respect to the identity function $y = x$.

EXERCISES

Determine whether the functions given are one-to-one by noting the function family to which each belongs and mentally picturing the shape of the graph.

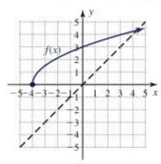
1. $h(x) = -|x - 2| + 3$ **2.** $p(x) = 2x^2 + 7$ **3.** $s(x) = \sqrt{x - 1} + 5$

Find the inverse of each function given. Then show using composition that your inverse function is correct. State any necessary restrictions.

4. $f(x) = -3x + 2$ **5.** $f(x) = x^2 - 2, x \geq 0$ **6.** $f(x) = \sqrt{x - 1}$

Determine the domain and range for each function whose graph is given, and use this information to state the domain and range of the inverse function. Then sketch in the line $y = x$, estimate the location of three points on the graph, and use these to graph $f^{-1}(x)$ on the same grid.

7.



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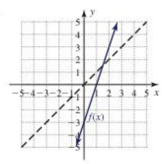

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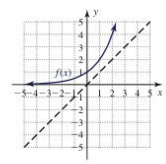


10. Fines for overdue material: Some libraries have set fees and penalties to discourage patrons from holding borrowed materials for an extended period. Suppose the fine for overdue DVDs is given by the function $f(t) = 0.15t + 2$, where $f(t)$ is the amount of the fine t days after it is due. (a) What is the fine for keeping a DVD seven (7) extra days? (b) Find $f^{-1}(t)$, then input your answer from part (a) and comment on the result. (c) If a fine of \$3.80 was assessed, how many days was the DVD overdue?

SECTION 4.2 Exponential Functions

KEY CONCEPTS

- An exponential function is defined as $f(x) = b^x$, where $b > 0, b \neq 1$, and b, x are real numbers.
- The natural exponential function is $f(x) = e^x$, where $e \approx 2.71828182846$.
- For exponential functions, we have
  - one-to-one function
  - y-intercept $(0, 1)$
  - domain: $x \in \mathbb{R}$
  - range: $y \in (0, \infty)$
  - increasing if $b > 1$
  - decreasing if $0 < b < 1$
  - asymptotic to x-axis
- The graph of $y = b^{x \pm h} \pm k$ is a translation of the basic graph of $y = b^x$, horizontally h units opposite the sign and vertically k units in the same direction as the sign.
- If an equation can be written with like bases on each side, we solve it using the uniqueness property: If $b^m = b^n$, then $m = n$ (equal bases imply equal exponents).
- All previous properties of exponents also apply to exponential functions.

EXERCISES

Graph each function using *transformations of the basic function,* then strategically plot a few points to check your work and round out the graph. Draw and label the asymptote.

11. $y = 2^x + 3$ **12.** $y = 2^{-x} - 1$ **13.** $y = -e^{x+1} - 2$

Solve using the uniqueness property.

14. $3^{2x-1} = 27$ **15.** $4^x = \frac{1}{16}$ **16.** $e^x \cdot e^{x+1} = e^6$

17. A ballast machine is purchased new for \$142,000 by the AT & SF Railroad. The machine loses 15% of its value each year and must be replaced when its value drops below \$20,000. How many years will the machine be in service?

SECTION 4.3 Logarithms and Logarithmic Functions

KEY CONCEPTS

- A logarithm is an exponent. For $x, b > 0$, and $b \neq 1$, the expression $\log_b x$ represents the exponent that goes on base b to obtain x: If $y = \log_b x$, then $b^y = x \Rightarrow b^{\log_b x} = x$ (by substitution).

- The equations $x = b^y$ and $y = \log_b x$ are equivalent. We say $x = b^y$ is the *exponential* form and $y = \log_b x$ is the *logarithmic* form of the equation.
- The value of $\log_b x$ can sometimes be determined by writing the expression in exponential form. If $b = 10$ or $b = e$, the value of $\log_b x$ can be found directly using a calculator.
- A logarithmic *function* is defined as $f(x) = \log_b x$, where $x, b > 0$, and $b \neq 1$.
  - $y = \log_{10} x = \log x$ is called a *common* logarithmic function.
  - $y = \log_e x = \ln x$ is called a *natural* logarithmic function.
- For $f(x) = \log_b x$ as defined we have
  - one-to-one function
  - x-intercept $(1, 0)$
  - domain: $x \in (0, \infty)$
  - range: $y \in \mathbb{R}$
  - increasing if $b > 1$
  - decreasing if $0 < b < 1$
  - asymptotic to y-axis
- The graph of $y = \log_b(x \pm h) \pm k$ is a translation of the graph of $y = \log_b x$, horizontally h units opposite the sign and vertically k units in the same direction as the sign.

EXERCISES

Write each expression in *exponential* form.

18. $\log_3 9 = 2$ **19.** $\log_5 \frac{1}{125} = -3$ **20.** $\ln 43 \approx 3.7612$

Write each expression in *logarithmic* form.

21. $5^2 = 25$ **22.** $e^{-0.25} \approx 0.7788$ **23.** $3^4 = 81$

Find the value of each expression without using a calculator.

24. $\log_2 32$ **25.** $\ln(\frac{1}{e})$ **26.** $\log_9 3$

Graph each function using *transformations of the basic function,* then strategically plot a few points to check your work and round out the graph. Draw and label the asymptote.

27. $f(x) = \log_2 x$ **28.** $f(x) = \log_2(x + 3)$ **29.** $f(x) = 2 + \ln(x - 1)$

Find the domain of the following functions.

30. $f(x) = \ln(x^2 - 6x)$ **31.** $g(x) = \log \sqrt{2x + 3}$

32. The magnitude of an earthquake is given by $M(I) = \log\frac{I}{I_0}$, where I is the intensity and I_0 is the reference intensity. (a) Find $M(I)$ given $I = 62{,}000I_0$ and (b) find the intensity I given $M(I) = 7.3$.

SECTION 4.4 Properties of Logarithms; Solving Exponential and Logarithmic Equations

KEY CONCEPTS

- The basic definition of a logarithm gives rise to the following properties: For any base $b > 0$, $b \neq 1$,
  1. $\log_b b = 1$ (since $b^1 = b$)
  2. $\log_b 1 = 0$ (since $b^0 = 1$)
  3. $\log_b b^x = x$ (since $b^x = b^x$)
  4. $b^{\log_b x} = x$
- Since a logarithm is an exponent, they have properties that parallel those of exponents.

Product Property	Quotient Property	Power Property
like base and multiplication, add exponents:	like base and division, subtract exponents:	exponent raised to a power, multiply exponents:
$\log_b(MN) = \log_b M + \log_b N$	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$	$\log_b M^p = p\log_b M$

- The logarithmic properties can be used to expand an expression: $\log(2x) = \log 2 + \log x$.
- The logarithmic properties can be used to contract an expression: $\ln(2x) - \ln(x + 3) = \ln\left(\frac{2x}{x + 3}\right)$.

- To evaluate logarithms with bases other than 10 or e, use the change-of-base formula:

$$\log_b M = \frac{\log M}{\log b} = \frac{\ln M}{\ln b}$$

- If an equation can be written with like bases on each side, we solve it using the uniqueness property: if $\log_b m = \log_b n$, then $m = n$ (equal bases imply equal arguments).
- If a single exponential or logarithmic term can be isolated on one side, then for any base b:

$$\text{If } b^x = k, \text{ then } x = \frac{\log k}{\log b} \qquad \text{If } \log_b x = k, \text{ then } x = b^k.$$

EXERCISES

33. Solve each equation by applying fundamental properties.

a. $\ln x = 32$ **b.** $\log x = 2.38$ **c.** $e^x = 9.8$ **d.** $10^x = \sqrt{7}$

34. Solve each equation. Write answers in exact form and in approximate form to four decimal places.

a. $15 = 7 + 2e^{0.5x}$ **b.** $10^{0.2x} = 19$ **c.** $-2\log(3x) + 1 = -5$ **d.** $-2\ln x + 1 = 6.5$

35. Use the product or quotient property of logarithms to write each sum or difference as a single term.

a. $\ln 7 + \ln 6$ **b.** $\log_9 2 + \log_9 15$ **c.** $\ln(x + 3) - \ln(x - 1)$ **d.** $\log x + \log(x + 1)$

36. Use the power property of logarithms to rewrite each term as a product.

a. $\log_5 9^2$ **b.** $\log_7 4^2$ **c.** $\ln 5^{2x-1}$ **d.** $\ln 10^{3x+2}$

37. Use the properties of logarithms to write the following expressions as a sum or difference of simple logarithmic terms.

a. $\ln(x\sqrt[4]{y})$ **b.** $\ln(\sqrt[3]{pq})$ **c.** $\log\left(\frac{\sqrt[3]{x^5 \cdot y^4}}{\sqrt{x^5y^3}}\right)$ **d.** $\log\left(\frac{4\sqrt[3]{p^5q^4}}{\sqrt{p^3q^2}}\right)$

38. Evaluate using a change-of-base formula. Answer in exact form and approximate form to thousandths.

a. $\log_6 45$ **b.** $\log_3 128$ **c.** $\ln_2 124$ **d.** $\ln_5 0.42$

Solve each equation.

39. $2^x = 7$ **40.** $3^{x+1} = 5$ **41.** $e^{x-2} = 3^x$

42. $\ln(x + 1) = 2$ **43.** $\log x + \log(x - 3) = 1$ **44.** $\log_{25}(x + 2) - \log_{25}(x - 3) = \frac{1}{2}$

45. The rate of decay for radioactive material is related to its half-life by the formula $R(h) = \frac{\ln 2}{h}$, where h represents the half-life of the material and $R(h)$ is the rate of decay expressed as a decimal. The element radon-222 has a half-life of approximately 3.9 days. (a) Find its rate of decay to the nearest hundredth of a percent. (b) Find the half-life of thorium-234 if its rate of decay is 2.89% per day.

46. The *barometric equation* $H = (30T + 8000)\ln(\frac{P_0}{P})$ relates the altitude H to atmospheric pressure P, where $P_0 = 76$ cmHg. Find the atmospheric pressure at the summit of Mount Pico de Orizaba (Mexico), whose summit is at 5657 m. Assume the temperature at the summit is $T = 12$°C.

SECTION 4.5 Applications from Investment, Finance, and Physical Science

KEY CONCEPTS

- Simple interest: $I = prt$; p is the initial principal, r is the interest rate per year, and t is the time in years.
- Amount in an account after t years: $A = p + prt$ or $A = p(1 + rt)$.
- Interest compounded n times per year: $A = p\left(1 + \frac{r}{n}\right)^{nt}$; p is the initial principal, r is the interest rate per year, t is the time in years, and n is the times per year interest is compounded.
- Interest compounded continuously: $A = pe^{rt}$; p is the initial principal, r is the interest rate per year, and t is the time in years.

- If a loan or savings plan calls for a regular schedule of deposits, the plan is called an annuity.
- For periodic payment $\mathcal{P}$, deposited or paid n times per year, at annual interest rate r, with interest compounded or calculated n times per year for t years, and $R = \frac{r}{n}$:
  - The accumulated value of the account is $A = \frac{\mathcal{P}}{R}[(1 + R)^{nt} - 1]$.
  - The payment required to meet a future goal is $\mathcal{P} = \frac{AR}{[(1 + R)^{nt} - 1]}$.
  - The payment required to amortize an amount A is $\mathcal{P} = \frac{AR}{1 - (1 + R)^{-nt}}$.
  - The general formulas for exponential growth and decay are $Q(t) = Q_0e^{rt}$ and $Q(t) = Q_0e^{-rt}$, respectively.

EXERCISES

Solve each application.

47. Jeffery borrows \$600.00 from his dad, who decides it's best to charge him interest. Three months later Jeff repays the loan plus interest, a total of \$627.75. What was the annual interest rate on the loan?

48. To save money for her first car, Cheryl invests the \$7500 she inherited in an account paying 7.8% interest compounded monthly. She hopes to buy the car in 6 yr and needs \$12,000. Is this possible?

49. To save up for the vacation of a lifetime, Al-Harwi decides to save \$15,000 over the next 4 yr. For this purpose he invests \$260 every month in an account paying $7\frac{1}{2}\%$ interest compounded monthly. (a) Is this monthly amount sufficient to meet the four-year goal? (b) If not, find the *minimum amount he needs to deposit each month* that will allow him to meet this goal in 4 yr.

50. Eighty prairie dogs are released in a wilderness area in an effort to repopulate the species. Five years later a statistical survey reveals the population has reached 1250 dogs. Assuming the growth was exponential, approximate the growth rate to the nearest tenth of a percent.

MIXED REVIEW

1. Evaluate each expression using the change-of-base formula.

a. $\log_2 30$ **b.** $\log_{0.25} 8$

c. $\log_8 2$

2. Solve each equation using the uniqueness property.

a. $10^{4x-5} = 1000$ **b.** $5^{3x-1} = \sqrt{5}$

c. $2^x \cdot 2^{0.5x} = 64$

3. Use the power property of logarithms to rewrite each expression as a product.

a. $\log_{10} 20^2$ **b.** $\log 10^{0.05x}$

c. $\ln 2^{x-3}$

Graph each of the following functions by shifting the basic function, then strategically plotting a few points to check your work and round out the graph. Graph and label the asymptote.

4. $y = -e^x + 15$ **5.** $y = 5 \cdot 2^{-x}$

6. $y = \ln(x + 5) + 7$ **7.** $y = \log_2(-x) - 4$

8. Use the properties of logarithms to write the following expressions as a sum or difference of simple logarithmic terms.

a. $\ln\left(\frac{x^3}{2y}\right)$ **b.** $\log(10a\sqrt[3]{a^2b})$

c. $\log_2\left(\frac{8x^4\sqrt{x}}{3\sqrt{y}}\right)$

9. Write the following expressions in exponential form.

a. $\log_5 625 = 4$ **b.** $\ln 0.15x = 0.45$

c. $\log(0.1 \times 10^8) = 7$

10. Write the following expressions in logarithmic form.

a. $343^{1/3} = 7$ **b.** $256^{3/4} = 64$

c. $2^{-3} = \frac{1}{8}$

11. For $g(x) = \sqrt{x - 1} + 2$, (a) state the domain and range, (b) find $g^{-1}(x)$ and state its domain and range, and (c) compute at least three ordered pairs (a, b) for g and show the order pairs (b, a) are solutions to g^{-1}.

Solve the following equations. State answers in exact form.

12. $\log_5(4x + 7) = 0$ **13.** $10^{x-4} = 200$

14. $e^{x+1} = 3^x$

15. $\log_2(2x - 5) + \log_2(x - 2) = 4$

16. $\log(3x - 4) - \log(x - 2) = 1$

Solve each application.

17. The magnitude of an earthquake is given by $M(I) = \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the quake and I_0 is the reference intensity 2×10^{11} (energy released from the smallest detectable quake). On October 23, 2004, the Niigata region of Japan was hit by an earthquake that registered 6.5 on the Richter scale. Find the intensity of this earthquake by solving the following equation for I: $6.5 = \log\left(\frac{I}{2 \times 10^{11}}\right)$.

18. Serene is planning to buy a house. She has $6500 to invest in a certificate of deposit that compounds interest quarterly at an annual rate of 4.4%. (a) Find how long it will take for this account to grow to the $12,500 she will need for a 10% down payment for a $125,000 house. Round to the nearest tenth of a year. (b) Suppose instead of investing an initial $6500, Serene deposits $500 a quarter in an account paying 4% each quarter. Find how long it will take for this account to grow to $12,500. Round to the nearest tenth of a year.

19. British artist Simon Thomas designs sculptures he calls hypercones. These sculptures involve rings of exponentially decreasing radii rotated through space. For one sculpture, the radii follow the model $r(n) = 2(0.8)^n$, where n counts the rings (outer-most first) and $r(n)$ is radii in meters. Find the radii of the six largest rings in the sculpture. Round to the nearest hundredth of a meter.

Source: http://www.plus.maths.org/issue8/features/art/

20. Ms. Chan-Chiu works for MediaMax, a small business that helps other companies purchase advertising in publications. Her model for the benefits of advertising is $P(a) = 1000(1.07)^a$, where P represents the number of potential customers reached when a dollars (in thousands) are invested in advertising.

a. Use this model to predict (to the nearest thousand) how many potential customers will be reached when $50,000 is invested in advertising.

b. Use this model to determine how much money a company should expect to invest in advertising (to the nearest thousand), if it wants to reach 100,000 potential customers.

PRACTICE TEST

1. Write the expression $\log_3 81 = 4$ in exponential form.

2. Write the expression $25^{1/2} = 5$ in logarithmic form.

3. Write the expression $\log_b\left(\frac{\sqrt{x^5}y^3}{z}\right)$ as a sum or difference of logarithmic terms.

4. Write the expression $\log_b m + \left(\frac{3}{2}\right)\log_b n - \frac{1}{2}\log_b p$ as a single logarithm.

Solve for x using the uniqueness property.

5. $5^{x-7} = 125$

6. $2 \cdot 4^{3x} = \frac{8^x}{16}$

Given $\log_a 3 \approx 0.48$ and $\log_a 5 \approx 1.72$, evaluate the following without the use of a calculator:

7. $\log_a 45$

8. $\log_a 0.6$

Graph using transformations of the parent function. Verify answers using a graphing calculator.

9. $g(x) = -2^{x-1} + 3$

10. $h(x) = \log_2(x - 2) + 1$

11. Use the change-of-base formula to evaluate. Verify results using a calculator.

a. $\log_3 100$

b. $\log_6 0.235$

12. State the domain and range of $f(x) = (x - 2)^2 - 3$ and determine if f is a one-to-one function. If so, find its inverse. If not, restrict the domain of f to create a one-to-one function, then find the inverse of this new function, including the domain and range.

Solve each equation.

13. $3^{x-1} = 89$

14. $\log_5 x + \log_5(x + 4) = 1$

15. A copier is purchased new for \$8000. The machine loses 18% of its value each year and must be replaced when its value drops below \$3000. How many years will the machine be in service?

16. How long would it take \$1000 to double if invested at 8% annual interest compounded daily?

17. The number of ounces of unrefined platinum drawn from a mine is modeled by $Q(t) = -2600 + 1900 \ln(t)$, where $Q(t)$ represents the number of ounces mined in t months. How many months did it take for the number of ounces mined to exceed 3000?

18. Septashi can invest his savings in an account paying 7% compounded semi-annually, or in an account paying 6.8% compounded daily. Which is the better investment?

19. Jacob decides to save \$4000 over the next 5 yr so that he can present his wife with a new diamond ring for their 20th anniversary. He invests \$50 every month in an account paying $8\frac{1}{4}\%$ interest compounded monthly. (a) Is this amount sufficient to meet the 5-yr goal? (b) If not, find the *minimum amount he needs to save monthly* that will enable him to meet this goal.

20. Chaucer is a typical Welsh Corgi puppy. During his first year of life, his weight very closely follows the model $W(t) = 6.79 \ln t - 11.97$, where $W(t)$ is his weight in pounds after t weeks and $8 \le t \le 52$.

a. How much will Chaucer weigh when he is 6 months old (to the nearest one-tenth pound)?

b. To the nearest week, how old is Chaucer when he weighs 8 lb?

CALCULATOR EXPLORATION AND DISCOVERY

Investigating Logistic Equations

As we saw in Section 4.4, logistics models have the form $P(t) = \frac{c}{1 + ae^{-bt}}$, where a, b, and c are constants and $P(t)$ represents the population at time t. For populations modeled by a logistics curve (sometimes called an "S" curve) growth is very rapid at first (like an exponential function), but this growth begins to slow down and level off due to various factors. This *Calculator Exploration and Discovery* is designed to investigate the effects that a, b, and c have on the resulting graph.

I. From our earlier observation, as t becomes larger and larger, the term ae^{-bt} becomes smaller and smaller (approaching 0) because it is a decreasing function: as $t \to \infty$, $ae^{-bt} \to 0$. If we allow that the term eventually becomes so small it can be disregarded, what remains is $P(t) = \frac{c}{1}$ or c. This is why c is called the capacity constant and the population can get no larger than c. In Figure 4.34, the graph of $P(t) = \frac{1000}{1 + 50e^{-1x}}$ ($a = 50$, $b = 1$, and $c = 1000$) is shown using a lighter line, while the graph of $P(t) = \frac{750}{1 + 50e^{-1x}}$ ($a = 50$, $b = 1$, and $c = 750$), is given in bold. The window size is indicated in Figure 4.35.

Figure 4.34

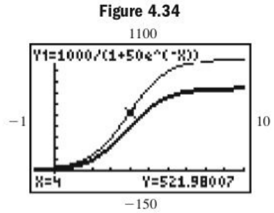
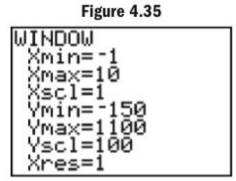


Figure 4.35



Also note that if a is held constant, smaller values of c cause the "interior" of the S curve to grow at a slower rate than larger values, a concept studied in some detail in a Calculus I class.

II. If $t = 0$, $ae^{-bt} = ae^0 = a$, and we note the ratio $P(0) = \frac{c}{1 + a}$ represents the *initial population.* This also means for constant values of c, larger values of a make the ratio $\frac{c}{1 + a}$ smaller; while smaller values of a make the ratio $\frac{c}{1 + a}$ larger. From this we conclude that a primarily affects the initial population. For the screens shown next, $P(t) = \dfrac{1000}{1 + 50e^{-1x}}$ (from I) is graphed using a lighter line. For comparison, the graph of $P(t) = \dfrac{1000}{1 + 5e^{-1x}}$ ($a = 5, b = 1$, and $c = 1000$) is shown in bold in Figure 4.36, while the graph of $P(t) = \dfrac{1000}{1 + 500e^{-1x}}$ ($a = 500, b = 1$, and $c = 1000$) is shown in bold in Figure 4.37.

Figure 4.36

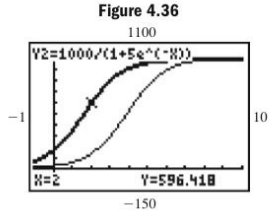
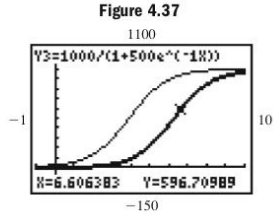


Figure 4.37



Note that changes in a appear to have no effect on the rate of growth in the interior of the S curve.

III. As for the value of b, we might expect that it affects the rate of growth in much the same way as the growth rate r does for exponential functions $Q(t) = Q_0e^{-rt}$. Sure enough, we note from the graphs shown that b has no effect on the initial value or the eventual capacity, but causes the population to approach this capacity more quickly for larger values of b, and more slowly for smaller values of b. For the screens shown, $P(t) = \dfrac{1000}{1 + 50e^{-1x}}$ ($a = 50$, $b = 1$, and $c = 1000$) is graphed using a lighter line. For comparison, the graph of $P(t) = \dfrac{1000}{1 + 50e^{-1.2x}}$ ($a = 50$, $b = 1.2$, and $c = 1000$) is shown in bold in Figure 4.38, while the graph of $P(t) = \dfrac{1000}{1 + 50e^{-0.8x}}$ ($a = 50, b = 0.8$, and $c = 1000$) is shown in bold in Figure 4.39.

Figure 4.38

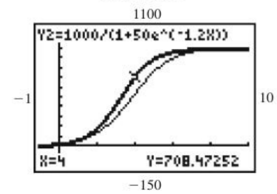
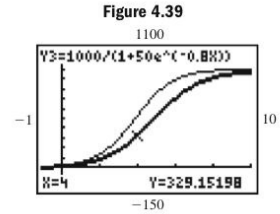


Figure 4.39



The following exercises are based on the population of an ant colony, modeled by the logistic function $P(t) = \dfrac{2500}{1 + 25e^{-0.5x}}$. Respond to Exercises 1 through 6 without the use of a calculator.

Exercise 1: Identify the values of a, b, and c for this logistics curve.

Exercise 2: What was the approximate initial population of the colony?

Exercise 3: Which gives a larger initial population: (a) $c = 2500$ and $a = 25$ or (b) $c = 3000$ and $a = 15$?

Exercise 4: What is the maximum population capacity for this colony?

Exercise 5: Would the population of the colony surpass 2000 more quickly if $b = 0.6$ or if $b = 0.4$?

Exercise 6: Which causes a slower population growth: (a) $c = 2000$ and $a = 25$ or (b) $c = 3000$ and $a = 25$?

Exercise 7: Verify your responses to Exercises 2 through 6 using a graphing calculator.

STRENGTHENING CORE SKILLS

Understanding Properties of Logarithms

To effectively use the properties of logarithms as a mathematical tool, a student must attain some degree of comfort and fluency in their application. Otherwise we are resigned to using them as a template or formula, leaving little room for growth or insight. This feature is divided into two parts. The first is designed to promote an understanding of the product and quotient properties of logarithms, which play a role in the solution of logarithmic and exponential equations.

We begin by looking at some logarithmic expressions that are obviously true:

$\log_2 2 = 1 \qquad \log_2 4 = 2 \qquad \log_2 8 = 3$

$\log_2 16 = 4 \qquad \log_2 32 = 5 \qquad \log_2 64 = 6$

Next, we view the same expressions with their value *understood mentally,* illustrated by the numbers in the background, rather than expressly written.

$\log_2 2 \quad \log_2 4 \quad \log_2 8 \quad \log_2 16 \quad \log_2 32 \quad \log_2 64$

This will make the product and quotient properties of equality much easier to "see." Recall the product property states: $\log_b M + \log_b N = \log_b(MN)$ and the quotient property states: $\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$. Consider the following.

$\log_2 4 + \log_2 8 = \log_2 32$ which is the same as saying $\log_2 4 + \log_2 8 = \log_2(4 \cdot 8)$ (since $4 \cdot 8 = 32$)

$\log_b M + \log_b N = \log_b(MN)$

$\log_2 64 - \log_2 32 = \log_2 2$ which is the same as saying $\log_2 64 - \log_2 32 = \log_2(\frac{64}{32})$ (since $\frac{64}{32} = 2$)

$\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$

Exercise 1: Repeat this exercise using logarithms of base 3 and various sums and differences.

Exercise 2: Use the basic concept behind these exercises to combine these expressions: (a) $\log(x) + \log(x + 3)$, (b) $\ln(x + 2) + \ln(x - 2)$, and (c) $\log(x) - \log(x + 3)$.

The second part is similar to the first, but highlights the power property: $\log_b M^x = x \log_b M$. For instance, knowing that $\log_2 64 = 6$, $\log_2 8 = 3$, and $\log_2 2 = 1$, consider the following:

$\log_2 8$ can be written as $\log_2 2^3$ (since $2^3 = 8$). Applying the power property gives $3 \cdot \log_2 2 = 3$.

$\log_2 64$ can be written as $\log_2 2^6$ (since $2^6 = 64$). Applying the power property gives $6 \cdot \log_2 2 = 6$.

$$\log_b M^x = x \log_b M$$

Exercise 3: Repeat this exercise using logarithms of base 3 and various powers.

Exercise 4: Use the basic concept behind these exercises to rewrite each expression as a product: (a) $\log 3^x$, (b) $\ln x^5$, and (c) $\ln 2^{3x-1}$.

CUMULATIVE REVIEW CHAPTERS 1–4

Use the quadratic formula to solve for x.

1. $x^2 - 4x + 53 = 0$ **2.** $6x^2 + 19x = 36$

3. Use substitution to show that $4 + 5i$ is a zero of $f(x) = x^2 - 8x + 41$.

4. Graph using transformations of a basic function: $y = 2\sqrt{x + 2} - 3$.

5. Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and comment on what you notice: $f(x) = x^3 - 2$; $g(x) = \sqrt[3]{x + 2}$.

6. State the domain of $h(x)$ in interval notation: $h(x) = \dfrac{\sqrt{x + 3}}{x^2 + 6x + 8}$.

7. According to the 2002 *National Vital Statistics Report* (Vol. 50, No. 5, page 19) there were 3100 sets of triplets born in the United States in 1991, and 6740 sets of triplets born in 1999. Assuming the relationship (year, sets of triplets) is linear: (a) find the equation of the line, (b) explain the meaning of the slope in this context, and (c) use the equation to estimate the number of sets born in 1996, and to project the number of sets that will be born in 2007 if this trend continues.

8. State the following geometric formulas:

a. area of a circle

b. Pythagorean theorem

c. perimeter of a rectangle

d. area of a trapezoid

9. Graph the following piecewise-defined function and state its domain, range, and intervals where it is increasing and decreasing.

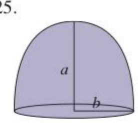
$$h(x) = \begin{cases} -4 & -10 \le x < -2 \\ -x^2 & -2 \le x < 3 \\ 3x - 18 & x \ge 3 \end{cases}$$

10. Solve the inequality and write the solution in interval notation: $\frac{2x + 1}{x - 3} \ge 0$.

11. Use the rational roots theorem to find all zeroes of $f(x) = x^4 - 3x^3 - 12x^2 + 52x - 48$.

12. Given $f(c) = \frac{9}{5}c + 32$, find k, where $k = f(25)$. Then find the inverse function using the algebraic method, and verify that $f^{-1}(k) = 25$.

13. Solve the formula $V = \frac{1}{2}\pi b^2 a$ (the volume of a paraboloid) for the variable b.



14. Use the *Guidelines for Graphing* to graph

a. $p(x) = x^3 - 4x^2 + x + 6$.

b. $r(x) = \frac{5x^2}{x^2 + 4}$.

15. For $f(x) = \frac{2x + 3}{5}$, (a) find f^{-1}, (b) graph both functions and verify they are symmetric to the line $y = x$, and (c) show they are inverses using composition.

16. Solve for x: $10 = -2e^{-0.05x} + 25$.

17. Solve for x: $\ln(x + 3) + \ln(x - 2) = \ln(24)$.

18. Once in orbit, satellites are often powered by radioactive isotopes. From the natural process of radioactive decay, the power output declines over a period of time. For an initial amount of 50 g, suppose the power output is modeled by the function $p(t) = 50e^{-0.002t}$, where $p(t)$ is the power output in watts, t days after the satellite has been put into service. (a) Approximately how much power remains 6 *months* later? (b) How many *years* until only one-fourth of the original power remains?

19. Simon and Christine own a sport wagon and a minivan. The sport wagon has a power curve that is closely modeled by $H(r) = 123 \ln r - 897$, where $H(r)$ is the horsepower at r rpm, with $2200 \le r \le 5600$. The power curve for the minivan is $h(r) = 193 \ln r - 1464$, for $2600 < r < 5800$.

a. How much horsepower is generated by each engine at 3000 rpm?

b. At what rpm are the engines generating the same horsepower?

c. If Christine wants the maximum horsepower available, which vehicle should she drive? What is the maximum horsepower?

20. Wilson's disease is a hereditary disease that causes the body to retain copper. Radioactive copper, ^{64}Cu, has been used extensively to study and understand this disease. ^{64}Cu has a relatively short half-life of 12.7 hr. How many hours will it take for a 5-g mass of ^{64}Cu to decay to 1 g?

Modeling with Technology II Exponential, Logarithmic, and Other Regression Models

Learning Objectives

In this feature you will learn how to:

- **A.** Choose an appropriate form of regression for a set of data
- **B.** Use a calculator to obtain exponential and logarithmic regression models
- **C.** Determine when a logistics model is appropriate and apply logistics models to a set of data
- **D.** Use a regression model to answer questions and solve applications

WORTHY OF NOTE

For more information on the use of residuals, see the *Calculator Exploration and Discovery* feature on Residuals at www.mhhe.com/coburn

The basic concepts involved in calculating a regression equation were presented in *Modeling with Technology I*. In this section, we extend these concepts to data sets that are best modeled by power, exponential, logarithmic, or logistic functions. All data sets, while contextual and accurate, *have been carefully chosen* to provide a maximum focus on regression fundamentals and related mathematical concepts. In reality, data sets are often not so "well-behaved" and many require sophisticated statistical tests before any conclusions can be drawn.

A. Choosing an Appropriate Form of Regression

Most graphing calculators have the ability to perform several forms of regression, and selecting which of these to use is a critical issue. When various forms are applied to a given data set, some are easily discounted due to a poor fit. Others may fit very well for only a portion of the data, while still others may compete for being the "best-fit" equation. In a statistical study of regression, an in-depth look at the correlation coefficient (r), the coefficient of determination (r^2 or R^2), and a study of **residuals** are used to help make an appropriate choice. For our purposes, the correct or best choice will generally depend on two things: (1) how well the graph appears to fit the scatter-plot, and (2) the context or situation that generated the data, coupled with a dose of common sense.

As we've noted previously, the final choice of regression can rarely be based on the scatter-plot alone, although relying on the basic characteristics and end behavior of certain graphs can be helpful **(see Exercise 58).** With an awareness of the toolbox functions, polynomial graphs, and applications of exponential and logarithmic functions, the context of the data can aid a decision.

EXAMPLE 1 ► **Choosing an Appropriate Form of Regression**

Suppose a set of data is generated from each context given. Use common sense, previous experience, or your own knowledge base to state whether a linear, quadratic, logarithmic, exponential, or power regression might be most appropriate. Justify your answers.

a. population growth of the United States since 1800

b. the distance covered by a jogger running at a constant speed

c. height of a baseball t seconds after it's thrown

d. the time it takes for a cup of hot coffee to cool to room temperature

Solution ►

a. From examples in Section 4.5 and elsewhere, we've seen that animal and human populations tend to grow exponentially over time. Here, an exponential model is likely most appropriate.

b. Since the jogger is moving at a constant speed, the rate-of-change $\frac{\Delta\text{distance}}{\Delta\text{time}}$ is constant and a linear model would be most appropriate.

c. As seen in numerous places throughout the text, the height of a projectile is modeled by the equation $h(t) = -16t^2 + vt + k$, where $h(t)$ is the height after t seconds. Here, a quadratic model would be most appropriate.

d. Many have had the experience of pouring a cup of hot chocolate, coffee, or tea, only to leave it on the counter as they turn their attention to other things. The hot drink seems to cool quickly at first, then slowly approach room temperature. This experience, perhaps coupled with our awareness of *Newton's law of cooling,* shows a logarithmic or exponential model might be appropriate here.

A. You've just learned how to choose an appropriate form of regression for a set of data

Now try Exercises 1 through 14 ►

B. Exponential and Logarithmic Regression Models

We now focus our attention on regression models that involve exponential and logarithmic functions. Recall the process of developing a regression equation involves these five stages: (1) clearing old data, (2) entering new data, (3) displaying the data, (4) calculating the regression equation, and (5) displaying and using the regression graph and equation.

EXAMPLE 2 ► **Calculating an Exponential Regression Model**

The number of centenarians (people who are 100 years of age or older) has been climbing steadily over the last half century. The table shows the number of centenarians (per million population) for selected years. Use the data and a graphing calculator to draw the scatter-plot, then use the scatter-plot and context to decide on an appropriate form of regression.

Source: Data from 2004 *Statistical Abstract of the United States*, Table 14; various other years

Year "t" ($1950 \rightarrow 0$)	Number "N" (per million)
0	16
10	18
20	25
30	74
40	115
50	262

Solution ► After clearing any existing data in the data lists, enter the input values (years since 1950) in L1 and the output values (number of centenarians per million population) in L2 (Figure MWT II.1). For the viewing window, scale the x-axis (years since 1950) from -10 to 70 and the y-axis (number per million) from -50 to 300 to comfortably fit the data and allow room for the coordinates to be shown at the bottom of the screen (Figure MWT II.2). The scatter-plot rules out a linear model. While a quadratic model may fit the data, we expect that the correct model should exhibit asymptotic behavior since extremely few people lived to be 100 years of age prior to dramatic advances in hygiene, diet, and medical care. This would lead us toward an exponential equation model. The keystrokes STAT ► brings up the **CALC** menu, with **ExpReg** (exponential regression) being option "0." The option can be selected by simply pressing "0," or by using the up arrow ▲ or down arrow ▼ to scroll to **0:ExpReg** then pressing ENTER. The exponential model seems to fit the data very well (Figures MWT II.3 and MWT II.4). To four decimal places the equation model is $y = (11.5090)1.0607^x$.

Figure MWT II.1

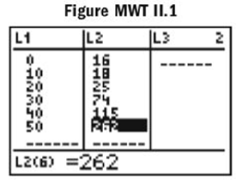


Figure MWT II.2

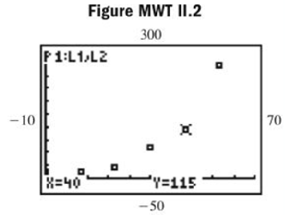


Figure MWT II.3

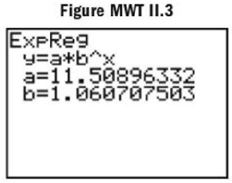
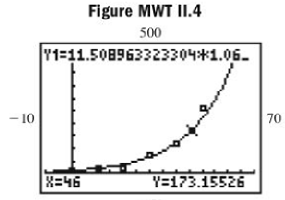


Figure MWT II.4



Now try Exercises 15 and 16 ►

EXAMPLE 3 ► Calculating a Logarithmic Regression Model

One measure used in studies related to infant growth, nutrition, and development, is the relation between the circumference of a child's head and their age. The table to the right shows the average circumference of a female child's head for ages 0 to 36 months. Use the data and a graphing calculator to draw the scatter-plot, then use the scatter-plot and context to decide on an appropriate form of regression.

Source: *National Center for Health Statistics*

Age a (months)	Circumference C (cm)
0	34.8
6	43.0
12	45.2
18	46.5
24	47.5
30	48.2
36	48.6

Solution ► After clearing any existing data, enter the child's age (in months) as L1 and the circumference of the head (in cm) as L2. For the viewing window, scale the x-axis from -5 to 50 and the y-axis from 25 to 60 to comfortably fit the data (Figure MWT II.5). The scatter-plot again rules out a linear model, and the context rules out a polynomial model due to end-behavior. As we expect the circumference of the head to continue increasing slightly for many more months, it appears a logarithmic model may be the best fit. Note that since ln (0) is undefined, $a = 0.1$ was used to represent the age at birth (rather than $a = 0$), prior to running the regression. The **LnReg** (logarithmic regression) option is option 9, and the keystrokes STAT ► **(CALC) 9** ENTER gives the equation shown in Figure MWT II.6, which fits the data very well (Figure MWT II.7).

WORTHY OF NOTE

For applications involving exponential growth and logarithmic functions, it helps to remember that while both basic functions are increasing, a logarithmic function increases at a much slower rate.

Figure MWT II.5

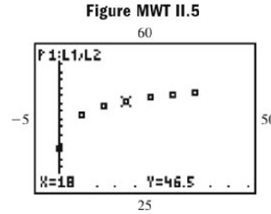


Figure MWT II.6

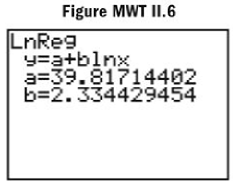
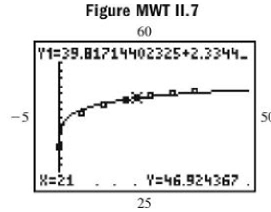


Figure MWT II.7



B. You've just learned how to use a calculator to obtain exponential and logarithmic regression models

Now try Exercises 17 and 18 ►

C. Logistics Equations and Regression Models

Many population growth models assume an unlimited supply of resources, nutrients, and room for growth, resulting in an exponential growth model. When resources become scarce or room for further expansion is limited, the result is often a **logistic growth model.** At first, growth is very rapid (like an exponential function), but this growth begins to taper off and slow down as nutrients are used up, living space becomes restricted, or due to other factors. Surprisingly, this type of growth can take many forms, including population growth, the spread of a disease, the growth of a tumor, or the spread of a stain in fabric. Specific logistic equations were encountered in Section 4.4. The general equation model for logistic growth is

Logistic Growth

Given constants a, b, and c, the logistic growth $P(t)$ of a population depends on time t according to the model

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

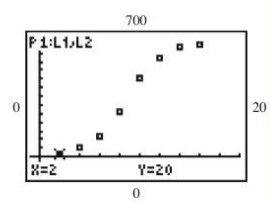
The constant c is called the **carrying capacity** of the population, in that as $t \to \infty$, $P(t) \to c$. In words, as the elapsed time becomes very large, the population will approach (but not exceed) c.

EXAMPLE 4 ► **Calculating a Logistic Regression Model**

Yeast cultures have a number of applications that are a great benefit to civilization and have been an object of study for centuries. A certain strain of yeast is grown in a lab, with its population checked at 2-hr intervals, and the data gathered are given in the table. Use the data and a graphing calculator to draw a scatter-plot, and decide on an appropriate form of regression. If a logistic regression is the best model, attempt to estimate the capacity coefficient c prior to using your calculator to find the regression equation. How close were you to the actual value?

Elapsed Time (hours)	Population (100s)
2	20
4	50
6	122
8	260
10	450
12	570
14	630
16	650

Solution ► After clearing the data lists, enter the input values (elapsed time) in L1 and the output values (population) in L2. For the viewing window, scale the t-axis from 0 to 20 and the P-axis from 0 to 700 to comfortably fit the data. From the context and scatter-plot, it's apparent the data are best modeled by a logistic function. Noting that Ymax = 700 and the data seem to level off near the top of the window, a good estimate for c would be about 675. Using logistic regression on the home screen (option **B:Logistic**), we obtain the equation $Y_1 = \frac{663}{1 + 123.9e^{-0.553x}}$ (rounded).



WORTHY OF NOTE

Notice that calculating a logistic regression model takes a few seconds longer than for other forms.

☑ **C. You've just learned how to determine when a logistic model is appropriate and apply logistics models to a set of data**

Now try Exercises 19 and 20 ►

When a regression equation is used to gather information, many of the equation solving skills from prior sections are employed. **Exercises 21 through 28** offer a variety of these equations for practice and warm-up.

D. Applications of Regression

Once the equation model for a data set has been obtained, it can be used to **interpolate** or approximate values that might occur *between* those given in the data set. It can also be used to **extrapolate** or predict future values. In this case, the investigation extends *beyond* the values from the data set, and is based on the assumption that projected trends will continue for an extended period of time.

Regardless of the regression applied, interpolation and extrapolation involve substituting a given or known value, then solving for the remaining unknown. We'll demonstrate here using the regression model from Example 3. The exercise set offers a large variety of regression applications, including some power regressions and additional applications of linear and quadratic regression.

EXAMPLE 5 ▶ Using a Regression Equation to Interpolate or Extrapolate Information

Use the regression equation from Example 3 to answer the following questions:

a. What is the average circumference of a female child's head, if the child is 21 months old?

b. According to the equation model, what will the average circumference be when the child turns $3\frac{1}{2}$ years old?

c. If the circumference of the child's head is 46.9 cm, about how old is the child?

Solution ▶ **a.** Using function notation we have $C(a) \approx 39.8171 + 2.3344\ln(a)$. Substituting 21 for a gives:

$$C(21) \approx 39.8171 + 2.3344\ln(21) \quad \text{substitute 21 for } a$$
$$\approx 46.9 \quad \text{result}$$

The circumference is approximately 46.9 cm.

b. Substituting 3.5 yr × 12 = 42 months for a gives:

$$C(42) \approx 39.8171 + 2.3344\ln(42) \quad \text{substitute 42 for } a$$
$$\approx 48.5 \quad \text{result}$$

The circumference will be approximately 48.5 cm.

c. For part (c) we're given the circumference C and are asked to find the age a in which this circumference (46.9) occurs. Substituting 46.9 for $C(a)$ we obtain:

$$46.9 = 39.8171 + 2.3344\ln(a) \quad \text{substitute 46.9 for } C(a)$$
$$\frac{7.0829}{2.3344} = \ln(a) \quad \text{subtract 39.8171, then divide by 2.3344}$$
$$e^{\frac{7.0859}{2.3344}} = a \quad \text{write in exponential form}$$
$$20.8 \approx a \quad \text{result}$$

The child must be about 21 months old.

Now try Exercises 29 through 32 ▶

WORTHY OF NOTE

When extrapolating from a set of data, care and common sense must be used or results can be very misleading. For example, while the Olympic record for the 100-m dash has been steadily declining since the first Olympic Games, it would be foolish to think it will ever be run in 0 sec.

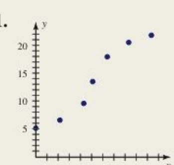
D. You've just learned how to use a regression model to answer questions and solve applications

MODELING WITH TECHNOLOGY EXERCISES

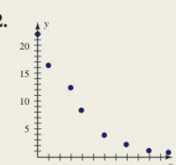
▶ DEVELOPING YOUR SKILLS

Match each scatter-plot given with one of the following: (a) likely linear, (b) likely quadratic, (c) likely exponential, (d) likely logarithmic, (e) likely logistic, or (f) none of these.

1.



2.



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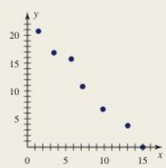

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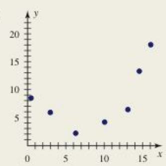

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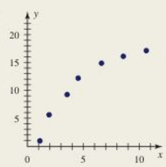
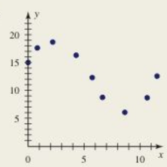


6.


For Exercises 7 to 14, suppose a set of data is generated from the context indicated. Use common sense, previous experience, or your own knowledge base to state whether a linear, quadratic, logarithmic, exponential, power, or logistic regression might be most appropriate. Justify your answers.

7. total revenue and number of units sold
8. page count in a book and total number of words
9. years on the job and annual salary
10. population growth with unlimited resources
11. population growth with limited resources
12. elapsed time and the height of a projectile
13. the cost of a gallon of milk over time
14. elapsed time and radioactive decay

Discuss why an exponential model could be an appropriate form of regression for each data set, then find the regression equation.

15. Radioactive Studies

Time in Hours	Grams of Material
0.1	1.0
1	0.6
2	0.3
3	0.2
4	0.1
5	0.06

16. Rabbit Population

Month	Population (in hundreds)
0	2.5
3	5.0
6	6.1
9	12.3
12	17.8
15	30.2

Discuss why a logarithmic model could be an appropriate form of regression for each data set, then find the regression equation.

17. Total number of sales compared to the amount spent on advertising

Advertising Costs ($1000s)	Total Number of Sales
1	125
5	437
10	652
15	710
20	770
25	848
30	858
35	864

18. Cumulative weight of diamonds extracted from a diamond mine

Time (months)	Weight (carats)
1	500
3	1748
6	2263
9	2610
12	3158
15	3501
18	3689
21	3810

19. **Spread of disease:** Estimates of the cumulative number of SARS (sudden acute respiratory syndrome) cases reported in Hong-Kong during the spring of 2003 are shown in the table, with day 0 corresponding to February 20. (a) Use the data to draw a scatter-plot, then use the context and scatter-plot to decide on the best form of regression. (b) If a logistic model seems best, attempt to estimate the carrying capacity c, then (c) use your calculator to find the regression equation.

Days After Outbreak	Cumulative Total
0	100
14	560
21	870
35	1390
56	1660
70	1710
84	1750

Source: Center for Disease Control @ www.cdc.gov/ncidod/EID/vol9no12.

20. **Cable television subscribers:** The percentage of American households having cable television is given in the table for select years from 1976 to 2004. (a) Use the data to draw a scatter-plot, then use the context and scatter-plot to decide on the best form of regression. (b) If a logistic model seems best, attempt to estimate the carrying capacity c, then (c) use your calculator to find the regression equation (use $1976 \to 0$).

Year $1976 \to 0$	Percentage with Cable TV
0	16
4	22.6
8	43.7
12	53.8
16	61.5
20	66.7
24	68
28	70

Source: Data pooled from the 2001 *New York Times Almanac*, p. 393; 2004 *Statistical Abstract of the United States*, Table 1120; various other years.

The applications in this section require solving equations similar to those that follow. Solve each equation.

21. $96.35 = (9.4)1.6^x$
22. $(3.7)2.9^x = 1253.93$
23. $4.8x^{2.5} = 468.75$
24. $4375 = 1.4x^{-1.25}$
25. $52 = 63.9 - 6.8 \ln x$
26. $498.53 + 18.2 \ln x = 595.9$
27. $52 = \dfrac{67}{1 + 20e^{-0.62x}}$
28. $\dfrac{975}{1 + 82.3e^{-0.423x}} = 890$

▶ APPLICATIONS

Answer the questions using the given data and the related regression equation. All extrapolations assume the mathematical model will continue to represent future trends.

29. Weight loss: Harold needed to lose weight and started on a new diet and exercise regimen. The number of pounds he's lost since the diet began is given in the table. Draw the scatter-plot, decide on an appropriate form of regression, and find an equation that models the data.

Time (days)	Pounds Lost
10	2
20	14
30	20
40	23
50	25.5
60	27.6
70	29.2
80	30.7

a. What was Harold's total weight loss after 15 days?
b. Approximately how many days did it take to lose a total of 18 pounds?
c. According to the model, what is the projected weight loss for 100 days?

30. Depletion of resources: The longer an area is mined for gold, the more difficult and expensive it gets to obtain. The cumulative total of the ounces produced by a particular mine is shown in the table. Draw the scatter-plot, use the scatter-plot and context to determine whether an exponential or logarithmic model is more appropriate, then find an equation that models the data.

Time (months)	Ounces Mined
5	275
10	1890
15	2610
20	3158
25	3501
30	3789
35	4109
40	4309

a. What was his total number of ounces mined after 18 months?
b. About how many months did it take to mine a total of 4000 oz?
c. According to the model, what is the projected total after 50 months?

31. Number of U.S. post offices: Due in large part to the ease of travel and increased use of telephones, e-mail and instant messaging, the number of post offices in the United States has been on the decline since the twentieth century. The data given show number of post offices (in thousands) for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation (use $1900 \to 0$).

Year ($1900 \to 0$)	Offices (1000s)
1	77
20	52
40	43
60	37
80	32
100	28

Source: Statistical Abstract of the United States; The First Measured Century

a. Approximately how many post offices were there in 1915?
b. In what year did the number of post offices drop below 34,000?
c. According to the model, how many post offices will there be in the year 2010?

32. Automobile value: While it is well known that most cars decrease in value over time, what is the equation model for this decline? Use the data given to draw a scatter-plot, then use the context and scatter-plot to find the regression equation.

Age of Car	Value of Car
1	19,500
2	16,950
4	12,420
6	11,350
8	8,375
10	7,935
12	6,900

a. What was the car's value after 7.5 years?
b. About how old is the car if its current value is \$8150?
c. Using the model, how old is the car when value $\le$ \$3000?

33. Female physicians: The number of females practicing medicine as MDs is given in the table for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation.

Year ($1980 \to 0$)	Number (in 1000s)
0	48.7
5	74.8
10	96.1
13	117.2
14	124.9
15	140.1
16	148.3

Source: Statistical Abstract of the United States.

a. What was the approximate number of female MDs in 1988?

b. Approximately how many female MDs will there be in 2005?

c. In what year did the number of female MDs exceed 100,000?

34. Telephone use: The number of telephone calls per capita has been rising dramatically since the invention of the telephone in 1876. The table shows the number of phone calls per capita per year for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation.

Year $(1900 \to 0)$	Number (per capita/ per year)
0	38
20	180
40	260
60	590
80	1250
97	2325

Source: The First Measured Century by Theodore Caplow, Louis Hicks, and Ben J. Wattenberg, The AEI Press, Washington, D.C., 2001.

a. What was the approximate number of calls per capita in 1970?

b. Approximately how many calls per capita will there be in 2005?

c. In what year did the number of calls per capita exceed 1800?

35. Milk production: Since 1980, the number of family farms with milk cows for commercial production has been decreasing. Use the data from the table given to draw a scatter-plot, then use the context and scatter-plot to find the regression equation.

Year $(1980 \to 0)$	Number (in 1000s)
0	334
5	269
10	193
15	140
17	124
18	117
19	111

Source: Statistical Abstract of the United States, 2000.

a. What was the approximate number of farms with milk cows in 1993?

b. Approximately how many farms will have milk cows in 2004?

c. In what year did this number of farms drop below 150 thousand?

36. Froth height—carbonated beverages: The height of the froth on carbonated drinks and other beverages can be manipulated by the ingredients used in making the beverage and lends itself very well to the modeling process. The data in the table given show the froth height of a certain beverage as a function of time, after the froth has reached a maximum height. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation.

Time (seconds)	Height of Froth (in.)
0	0.90
2	0.65
4	0.40
6	0.21
8	0.15
10	0.12
12	0.08

a. What was the approximate height of the froth after 6.5 sec?

b. How long does it take for the height of the froth to reach one-half of its maximum height?

c. According to the model, how many seconds until the froth height is 0.02 in.?

37. Chicken production: In 1980, the production of chickens in the United States was about 392 million. In the next decade, the demand for chicken first dropped, then rose dramatically. The number of chickens produced is given in the table to the right for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation.

Year $(1980 \to 0)$	Number (millions)
0	392
5	370
9	356
14	386
16	393
17	410
18	424

Source: Statistical Abstract of the United States, 2000.

a. What was the approximate number of chickens produced in 1987?

b. Approximately how many chickens will be produced in 2004?

c. According to the model, for what years was the production of chickens below 365 million?

38. Veterans in civilian life: The number of military veterans in civilian life fluctuates with the number of persons inducted into the military (higher in times of war) and the passing of time. The number of living veterans is given in the table for selected years from 1950 to 1999. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation.

Year $(1950 \to 0)$	Number (millions)
0	19.1
10	22.5
20	27.6
30	28.6
40	27
48	25.1
49	24.6

Source: Statistical Abstract of the United States, 2000.

a. What was the approximate number of living military veterans in 1995?

b. Approximately how many living veterans will there be in 2006?

c. According to the model, in what years did the number of veterans exceed 26 million?

39. Use of debit cards: Since 1990, the use of debit cards to obtain cash and pay for purchases has become very common. The number of debit cards nationwide is given in the table for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation.

Year (1990 → 0)	Number of Cards (millions)
0	164
5	201
8	217
10	230

Source: Statistical Abstract of the United States, 2000.

a. Approximately how many debit cards were there in 1999?

b. Approximately how many debit cards will there be in 2005?

c. In what year did the number of debit cards exceed 300 million?

40. Quiz grade versus study time: To determine the value of doing homework, a student in college algebra records the time spent by classmates in preparation for a quiz the next day. Then she records their scores, which are shown in the table. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation. According to the model, what grade can I expect if I study for 120 min?

x (min study)	y (score)
45	70
30	63
10	59
20	67
60	73
70	85
90	82
75	90

41. Population of coastal areas: The percentage of the U.S. population that can be categorized as living in *Pacific coastal areas* (minimum of 15% of the state's land area is a coastal watershed) has been growing steadily for decades, as indicated by the data given for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation. According to the model, what is the predicted percentage of the population living in Pacific coastal areas in 2005 and 2010?

Year	Percentage
1970	22.8
1980	27.0
1990	33.2
1995	35.2
2000	37.8
2001	38.5
2002	38.9
2003	39.4

Source: 2004 Statistical Abstract of the United States, Table 23.

42. Water depth and pressure: As anyone who's been swimming knows, the deeper you dive, the more pressure you feel on your body and eardrums. This pressure (in pounds per square inch or psi) is shown in the table for selected depths. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation. According to the model, what pressure can be expected at a depth of 100 ft?

Depth (ft)	Pressure (psi)
15	6.94
25	11.85
35	15.64
45	19.58
55	24.35
65	28.27
75	32.68

43. Personal debt-load: The data given tracks the total amount of debt carried by a family over a 6-month period. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation. According to the model, how much debt will the family have by the end of December? When will their debt-load exceed $10,000?

Month (x)	Debt (y)
(start)	$0
Jan	471
Feb	1105
March	1513
April	1921
May	2498
June	3129

44. Use of debit cards: Since 1990, the dollar volume of business transacted using debit cards has been growing. The volume of business nationwide is given in the table to the right for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation.

Year (1990 → 0)	Volume (billions)
0	12
5	62
8	239
10	423

Source: Statistical Abstract of the United States, 2004.

a. In 1993, what was the approximate dollar volume of business transacted with debit cards?

b. Approximately how much dollar volume of business was transacted in 1997?

c. In what year did the volume of business transacted using debit cards exceed 1000 billion?

45. Musical notes: The table shown gives the frequency (vibrations per second for each of the twelve notes in a selected octave) from the standard chromatic scale. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation.

#	Note	Frequency
1	A	110.00
2	A#	116.54
3	B	123.48
4	C	130.82
5	C#	138.60
6	D	146.84
7	D#	155.56
8	E	164.82
9	F	174.62
10	F#	185.00
11	G	196.00
12	G#	207.66

a. What is the frequency of the "A" note that is an octave higher than the one shown? [*Hint:* The names repeat every 12 notes (one octave), so this would be the 13th note in this sequence.]

b. If the frequency is 370.00 what note is being played?

c. What pattern do you notice for the F#'s in each octave (the 10th, 22nd, 34th, and 46th notes in sequence)? Does the pattern hold for all notes?

46. Basketball salaries: In 1970, the average player salary for a professional basketball player was about \$43,000. Since that time player salaries have risen dramatically. The average player salary for a professional player is given in the table to the right for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation.

Year (1970 $\rightarrow$ 0)	Salary (\$1000s)
0	43
10	260
15	325
20	750
25	1900
27	2200
28	2600

Source: Wall Street Journal Almanac.

a. What was the approximate salary for a player in 1993?

b. Approximately how much will the average salary be in 2005?

c. In what year did the average salary exceed \$5,000,000?

47. Cost of cable service: The average monthly cost of cable TV has been rising steadily since it became very popular in the early 1980s. The data given shows the average monthly rate for selected years (1980 $\rightarrow$ 0). Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation. According to the model, what will be the cost of cable service in 2010? 2015?

Year (1980 $\rightarrow$ 0)	Monthly Charge
0	\$7.69
5	\$9.73
10	\$16.78
20	\$23.07
25	\$30.70

Source: 2004–2005 Statistical Abstract of the United States, page 725, Table 1138.

48. Research and development expenditures: The development of new products, improved health care, greater scientific achievement, and other advances is fueled by huge investments in research and development (R & D). Since 1960, total R & D expenditures in the United States have shown a distinct pattern of growth, and the data are given in the table for selected years from 1960 to 1999. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation. According to the model, what was spent on R & D in 1992? In what year did expenditures for R & D exceed 450 billion?

Year (1960 $\rightarrow$ 0)	R & D (billion \$)
0	13.7
5	20.3
10	26.3
15	35.7
20	63.3
25	114.7
30	152.0
35	183.2
39	247.0

49. Business start-up costs: As many new businesses open, they experience a period where little or no profit is realized due to start-up expenses, equipment purchases, and so on. The data given shows the profit of a new company for the first 6 months of business. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation. According to the model, what is the first month that a profit will be earned?

Month	Profit (\$1000s)
1	−5
2	−13
3	−18
4	−20
5	−21
6	−19

50. Low birth weight: For many years, the association between low birth weight (less than 2500 g or about 5.5 lb) and a mother's age has been well documented. The data given are grouped by age and give the percent of total births with low birth weight.

Source: National Vital Statistics Report, Vol. 50, No. 5, February 12, 2002.

Ages	Percent
15–19	8.5
20–24	6.5
25–29	5.2
30–34	5
35–39	6
40–44	8
45–54	10

a. Using the median age of each group, use the data to draw a scatter-plot and decide on an appropriate form of regression.

b. Find a regression equation that models the data. According to the model, what percent of births will have a low birth weight if the mother was 58 years old?

51. Growth of cell phone use: The tremendous surge in cell phone use that began in the early nineties has continued unabated into the new century. The total number of subscriptions is shown in the table for selected years, with $1990 \to 0$ and the number of subscriptions in millions. Use the data to draw a scatter-plot. Does the data seem to follow an exponential or logistic pattern? Find the regression equation. According to the model, how many subscriptions were there in 1997? How many subscriptions does your model project for 2005? 2010? In what year will the subscriptions exceed 220 million?

Year ($1990 \to 0$)	Subscriptions (millions)
0	5.3
3	16.0
6	44.0
8	69.2
12	140.0
13	158.7

Source: 2000/2004 *Statistical Abstracts of the United States,* Tables 919/1144.

52. Absorption rates of fabric: Using time lapse photography, the spread of a liquid is tracked in one-fifth of a second intervals, as a small amount of liquid is dropped on a piece of fabric. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation. To the nearest hundredth of a second, how long will it take the stain to reach a size of 15 mm?

Time (sec)	Size (mm)
0.2	0.39
0.4	1.27
0.6	3.90
0.8	10.60
1.0	21.50
1.2	31.30
1.4	36.30
1.6	38.10
1.8	39.00

53. Planetary orbits: The table shown gives the time required for the first five planets to make one complete revolution around the Sun (in years), along with the average orbital radius of the planet in astronomical units (1 AU = 92.96 million miles). Use a graphing calculator to draw the scatter-plot, then use the scatter-plot, the context, and any previous experience to decide whether a polynomial, exponential, logarithmic, or power regression is most appropriate. Then (a) find the regression equation and use it to estimate the average orbital radius of Saturn, given it orbits the Sun every 29.46 yr, and (b) estimate how many years it takes Uranus to orbit the Sun, given it has an average orbital radius of 19.2 AU.

Planet	Years	Radius
Mercury	0.24	0.39
Venus	0.62	0.72
Earth	1.00	1.00
Mars	1.88	1.52
Jupiter	11.86	5.20

54. Ocean temperatures: The temperature of ocean water depends on several factors, including salinity, latitude, depth, and density. However, between depths of 125 m and 2000 m, ocean temperatures are relatively predictable, as indicated by the data shown for tropical oceans in the table. Use a graphing calculator to draw the scatter-plot, then use the scatter-plot, the context, and any previous experience to decide whether a polynomial, exponential, logarithmic, or power regression is most appropriate (end behavior rules out linear and quadratic models as possibilities).

Depth (meters)	Temp (°C)
125	13.0
250	9.0
500	6.0
750	5.0
1000	4.4
1250	3.8
1500	3.1
1750	2.8
2000	2.5

Source: UCLA at www.msc.ucla.oceanglobe/pdf/thermo_plot_lab

a. Find the regression equation and use it to estimate the water temperature at a depth of 2850 m.

b. If the model were still valid at greater depths, what is the ocean temperature at the bottom of the Marianas Trench, some 10,900 m below sea level?

55. Predator/prey model: In the wild, some rodent populations vary inversely with the number of predators in the area. Over a period of time, a conservation team does an extensive study on this relationship and gathers the data shown. Draw a scatter-plot of the data and (a) find a regression equation that models the data. According to the model, (b) if there are 150 predators in the area, what is the rodent population? (c) How many predators are in the area if studies show a rodent population of 3000 animals?

Predators	Rodents
10	5100
20	2500
30	1600
40	1200
50	950
60	775
70	660
80	575
90	500
100	450

56. Children and AIDS: Largely due to research, education, prevention, and better health care, estimates of the number of AIDS (acquired immune deficiency syndrome) cases diagnosed in children less than 13 yr of age have been declining. Data for the years 1995 through 2002 is given in the table.

Source: National Center for Disease Control and Prevention.

Years Since 1990	Cases
5	686
6	518
7	328
8	238
9	183
10	118
11	110
12	92

a. Use the data to draw a scatter-plot and decide on an appropriate form of regression.

b. Find a regression equation that models the data. According to the model, how many cases of AIDS in children are projected for 2010?

c. In what year did the number of cases fall below 50?

57. Growth rates of children: After reading a report from The National Center for Health Statistics regarding the growth of children from age 0 to 36 months, Maryann decides to track the relationships (length in inches, weight in pounds) and (age in months, circumference of head in centimeters) for her newborn child, a beautiful baby girl—Morgan.

a. Use the (length, weight) data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation. According to the model, how much will Morgan weigh when she reaches a height (length) of 39 in.? What will her length be when she weighs 28 lb?

b. Use the (age, circumference) data to draw a scatter-plot, then use the context and scatter-plot to find the regression equation. According to the model, what is the circumference of Morgan's head when she is 27 months old? How old will she be when the circumference of her head is 50 cm?

Exercise 57a

Length (in.)	Weight (lb)
17.5	5.50
21	10.75
25.5	16.25
28.5	19.00
33	25.25

Exercise 57b

Age (months)	Circumference (cm)
1	38.0
6	44.0
12	46.5
18	48.0
21	48.3

58. Correlation coefficients: Although correlation coefficients can be very helpful, other factors must also be considered when selecting the most appropriate equation model for a set of data. To see why, use the data given to (a) find a linear regression equation and note its correlation coefficient, and (b) find an exponential regression equation and note its correlation coefficient. What do you notice? Without knowing the context of the data, would you be able to tell which model might be more suitable? (c) Use your calculator to graph the scatter-plot and both functions. Which function appears to be a better fit?