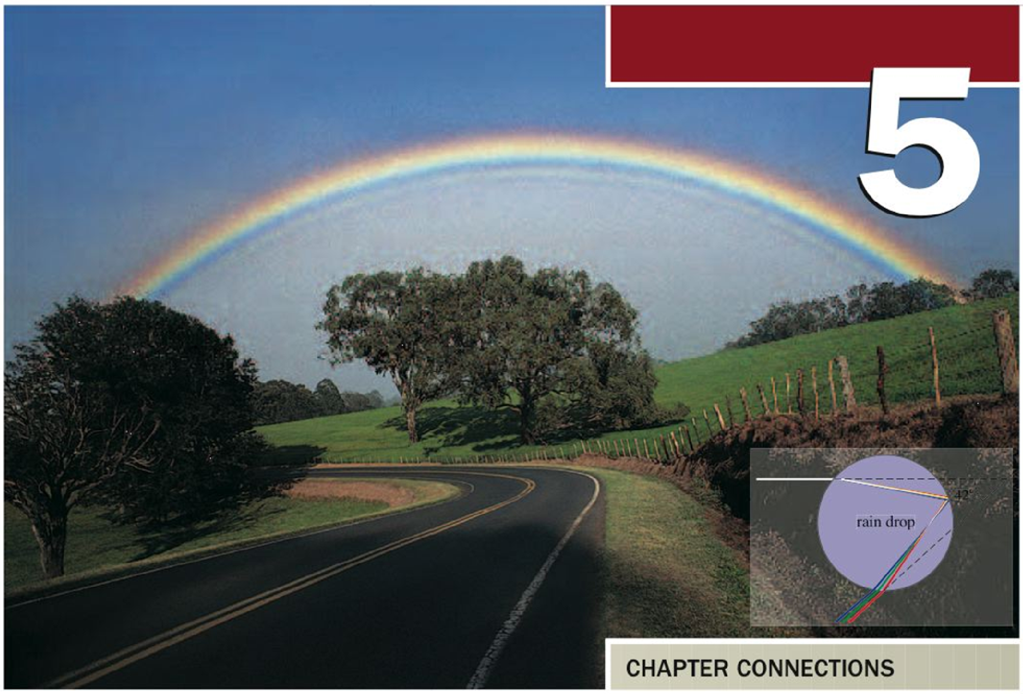


5

An Introduction to Trigonometric Functions

CHAPTER OUTLINE

CHAPTER CONNECTIONS

While rainbows have been admired for centuries for their beauty and color, understanding the physics of rainbows is of fairly recent origin. Answers to questions regarding their seven-color constitution, the order the colors appear, the circular shape of the bow, and their relationship to moisture, all have answers deeply rooted in mathematics. The relationship between light and color can be understood in terms of trigonometry, with questions regarding the apparent height of the rainbow, as well as the height of other natural and man-made phenomena, found using the trigometry of right triangles. This application appears as Exercise 85 in Section 5.2

Check out these other real-world connections:

- Determining the Area Monitored by a Motion Detector (Section 5.1, Exercise 98)
- Determining the Height of Taipei 101, World's Tallest Building (Section 5.2, Exercise 80)
- Identifying Various Colors Using Their Wavelengths (Section 5.5, Exercise 69)
- Modeling the Number of Daylight Hours for Cities at Various Latitudes (Section 5.7, Exercise 55)

5.1 Angle Measure, Special Triangles, and Special Angles

Learning Objectives

In Section 5.1 you will learn how to:

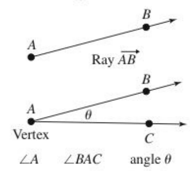
- ☐ **A.** Use the vocabulary associated with a study of angles and triangles
- ☐ **B.** Find fixed ratios of the sides of special triangles
- ☐ **C.** Use radians for angle measure and compute circular arc length and area using radians
- ☐ **D.** Convert between degrees and radians for nonstandard angles
- ☐ **E.** Solve applications involving angular velocity and linear velocity using radians

Trigonometry, like her sister science geometry, has its origins deeply rooted in the practical use of measurement and proportion. In this section, we'll look at the fundamental concepts on which trigonometry is based, which we hope will lead to a better understanding and a greater appreciation of the wonderful study that trigonometry has become.

A. Angle Measure in Degrees

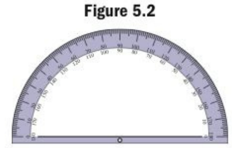
Beginning with the common notion of a straight line, a **ray** is a half line, or all points extending from a single point, in a single direction. An **angle** is the joining of two rays at a common endpoint called the **vertex.** Arrowheads are used to indicate the half lines continue forever and can be extended if necessary. Angles can be named using a single letter at the vertex, the letters from the rays forming the sides, or by a single Greek letter, with the favorites being **alpha** α, **beta** β, **gamma** γ, and **theta** θ. The symbol $\angle$ is often used to designate an angle (see Figure 5.1).

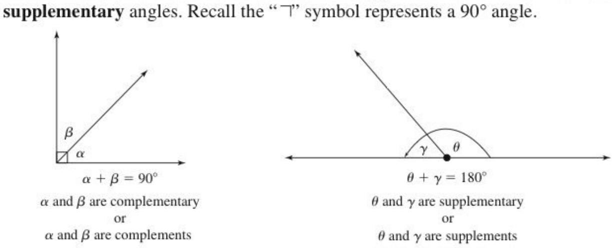
Figure 5.1



Euclid (325–265 B.C.), often thought of as the *father of geometry,* described an angle as "the inclination of one to another of two lines which meet in a plane." This *amount of inclination* gives rise to the common notion of angle measure in degrees, often measured with a semicircular **protractor** like the one shown in Figure 5.2. The notation for degrees is the ° symbol. By definition 1° is $\frac{1}{360}$ of a full rotation, so this protractor can be used to measure any angle from 0° (where the two rays are coincident), to 180° (where they form a straight line). An angle measuring 180° is called a **straight angle,** while an angle that measures 90° is called a **right angle.** Two angles that sum to 90° are said to be **complementary,** while two that sum to 180° are **supplementary** angles. Recall the "⊤" symbol represents a 90° angle.

Figure 5.2



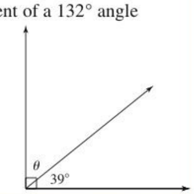


EXAMPLE 1 ► **Finding the Complement and Supplement of an Angle**

Determine the measure of each angle described.

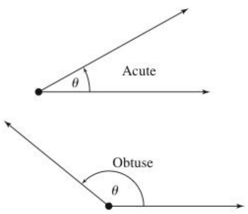
a. the complement of a 57° angle **b.** the supplement of a 132° angle

c. the measure of angle θ shown in the figure



Solution ►
a. The complement of 57° is 33° since $90 - 57 = 33° \Rightarrow 33 + 57 = 90°$.
b. The supplement of 132° is 48° since $180 - 132 = 48° \Rightarrow 48 + 132 = 180°$.
c. Since θ and 39° are complements, $\theta = 90 - 39 = 51°$.

Now try Exercises 7 through 10 ►



In the study of trigonometry, it helps to further classify the various angles we encounter. An angle greater than 0° but less than 90° is called an **acute** angle. An angle greater than 90° but less than 180° is called an **obtuse** angle. For very fine measurements, each degree is divided into 60 smaller parts called **minutes,** and each minute into 60 smaller parts called **seconds.** This means that a minute is $\frac{1}{60}$ of a degree, while a second is $\frac{1}{3600}$ of a degree. The angle whose measure is "sixty-one degrees, eighteen minutes, and forty-five seconds" is written as $61°\,18'\,45''$. The degrees-minutes-seconds (DMS) method of measuring angles is commonly used in aviation and navigation, while in other areas **decimal degrees** such as 61.3125° are preferred. You will sometimes be asked to convert between the two.

EXAMPLE 2 ▶ Converting Between Decimal Degrees and Degrees/Minutes/Seconds

Convert as indicated.

a. $61°\,18'\,45''$ to decimal degrees

b. 142.2075° to DMS

Solution ▶ **a.** Since $1' = \frac{1}{60}$ of a degree and $1'' = \frac{1}{3600}$ of a degree, we have

$$61°\,18'\,45'' = \left[61 + 18\left(\frac{1}{60}\right) + 45\left(\frac{1}{3600}\right)\right]^{\circ} = 61.3125°$$

b. For the conversion to DMS we write the fractional part separate from the whole number part to compute the number of degrees and minutes represented, then repeat the process to find the number of degrees, minutes, and seconds:

$$\begin{aligned}
142.2075° &= 142° + 0.2075° && \text{separate fractional part from the whole}\\
&= 142° + 0.2075(60)' && 0.2075° = 0.2075 \cdot 1°\text{; substitute } 60' \text{ for } 1°\\
&= 142°\,12.45' && \text{result in degrees and minutes}\\
&= 142°\,12' + 0.45' && \text{separate fractional part from the whole}\\
&= 142°\,12' + 0.45(60)'' && 0.45' = 0.45 \cdot 1'\text{; substitute } 60'' \text{ for } 1'\\
&= 142°\,12'\,27'' && \text{result in degrees, minutes, and seconds}
\end{aligned}$$

A. You've just learned how to use the vocabulary associated with a study of angles and triangles

Now try Exercises 11 through 26 ▶

B. Triangles and Properties of Triangles

A triangle is a closed plane figure with three straight sides and three angles. It is customary to name each angle using a capital letter and the side opposite the angle using the corresponding lowercase letter. Regardless of their size or orientation, triangles have the following properties.

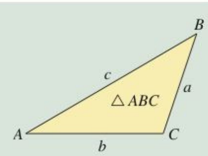
Properties of Triangles

Given triangle ABC with sides a, b, and c respectively,

I. The sum of the angles is 180°:
$A + B + C = 180°$

II. The combined length of any two sides exceeds that of the third side:
$a + b > c$, $a + c > b$, and $b + c > a$.

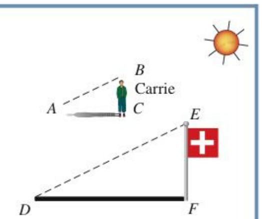
III. Larger angles are opposite larger sides:
If $C > B$, then $c > b$.



Two triangles are **similar triangles** if corresponding angles are equal, meaning for $\triangle ABC$ and $\triangle DEF$, $A = D$, $B = E$, and $C = F$. Since antiquity it's been known that *if two triangles are similar, corresponding sides are proportional* (corresponding sides are those opposite the equal angles from each triangle). This relationship, used extensively by the engineers of virtually all ancient civilizations, is very important to our study of trigonometry. Example 3 illustrates how proportions and similar triangles are often used.

EXAMPLE 3 ▶ Using Similar Triangles to Find Heights

To estimate the height of a flagpole, Carrie reasons that $\triangle ABC$ formed by her height and shadow must be similar to $\triangle DEF$ formed by the flagpole. She is 5 ft 6 in. tall and casts an 8-ft shadow, while the shadow of the flagpole measures 44 ft. How tall is the pole?



Solution ▶ Let H represent the height of the flagpole.

$$\frac{\text{Height}}{\text{Shadow Length}}: \frac{5.5}{8} = \frac{H}{44} \quad \text{original proportion, } 5'\,6'' = 5.5 \text{ ft}$$

$$8H = 242 \quad \text{cross multiply}$$

$$H = 30.25 \quad \text{result}$$

The flagpole is $30\frac{1}{4}$ ft tall (30 ft 3 in.).

Now try Exercises 27 through 34 ▶

Figure 5.3 shows Carrie standing along the shadow of the flagpole, again illustrating the proportional relationships that exist. Early mathematicians recognized the power of these relationships, realizing that if the triangles were similar and the related fixed proportions were known, they had the ability to find mountain heights, the widths of lakes, and even the ability to estimate the distance to the Sun and Moon. What was needed was an accurate and systematic method of finding these "fixed proportions" for various angles, so they could be applied more widely. In support of this search, two special triangles were used. These triangles, commonly called **45-45-90** and **30-60-90** triangles, are *special* because no estimation or interpolation is needed to find the relationships between their sides. For the first, consider an isosceles right triangle—a right triangle with two equal sides and two 45° angles (Figure 5.4). After naming the equal sides x and the hypotenuse h, we can apply the Pythagorean theorem to find a relationship between the sides and the hypotenuse in terms of x.

Figure 5.3

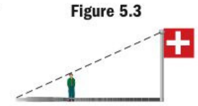
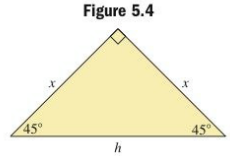


Figure 5.4



WORTHY OF NOTE

Recall that the Pythagorean theorem states that for any right triangle, the sum of the squares of the two legs, is equal to the square of the hypotenuse: $a^2 + b^2 = c^2$.

$$c^2 = a^2 + b^2 \quad \text{Pythagorean theorem}$$

$$h^2 = x^2 + x^2 \quad \text{substitute } x \text{ for } a,\ x \text{ for } b, \text{ and } h \text{ for } c$$

$$= 2x^2 \quad \text{combine like terms}$$

$$h = \sqrt{2}x \quad \text{solve for } h\ (h > 0)$$

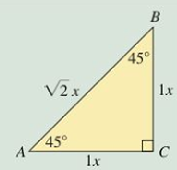
This important result is summarized in the following box.

45-45-90 Triangles

Given a 45-45-90 triangle with one side of length x, the relationship between corresponding sides is:

$$1x : 1x : \sqrt{2}\,x.$$

(1) The two legs are equal
(2) The hypotenuse is $\sqrt{2}$ times the length of either leg.



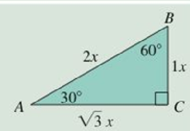
The proportional relationship for a 30-60-90 triangle is developed in **Exercise 110,** and the result is stated here.

30-60-90 Triangles

Given a 30-60-90 triangle with the shortest side of length x, the relationship between corresponding sides is:

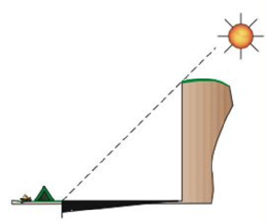
$$1x : \sqrt{3}\,x : 2x.$$

(1) The hypotenuse is 2 times the length of the shorter leg,
(2) The longer leg is $\sqrt{3}$ times the length of the shorter leg.



EXAMPLE 4 ► **Applications of 45-45-90 Triangles: The Height of a Cliff**

A group of campers has pitched their tent some distance from the base of a tall cliff. The evening's conversation turns to a discussion of the cliff's height, and they all lodge an estimate. Then one of them says, "Wait . . . how will we know who's closest?" Describe how a 45-45-90 triangle can help determine a winner.



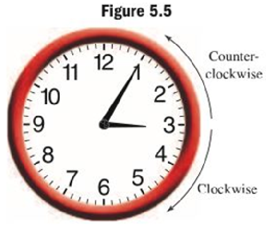
Solution ► In the morning, cut a pole equal in height to any of the campers. Then follow the shadow of the cliff as it moves, with the selected camper occasionally laying the pole at her feet and checking her shadow's length against the length of the pole. At the moment her shadow is equal to the pole's length, the sun is shining at a 45° angle and the campers can use the pole to measure the shadow cast by the cliff (by counting the number of pole lengths needed to reach it), which will be equal to its height since a 45-45-90 triangle is formed.

B. You've just learned how to find fixed ratios of the sides of special triangles

Now try Exercises 35 and 36 ►

C. Angle Measure in Radians; Arc Length and Area

Figure 5.5



As an alternative to viewing angles as "the amount of inclination" between two rays, angle measure can also be considered as the *amount of rotation* from a fixed ray called the **initial side,** to a rotated ray called the **terminal side.** This enables angle measure to be free from the context of a triangle, and allows for positive or negative angles, depending on the direction of rotation. Angles formed by a counterclockwise rotation are considered **positive angles,** and angles formed by a clockwise rotation are **negative angles** (see Figure 5.5). We can then name an angle of any size, including those greater than 360° where the amount of rotation exceeds one revolution. See Figures 5.6 through 5.10.

Figure 5.6

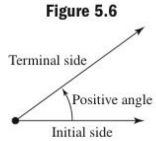


Figure 5.7

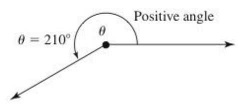


Figure 5.8

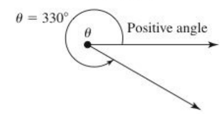


Figure 5.9

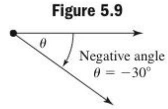


Figure 5.10

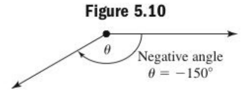
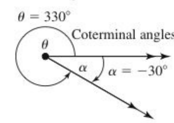


Figure 5.11



Note in Figure 5.11 that angle $\theta = 330°$ and angle $\alpha = -30°$ share the same initial and terminal sides and are called **coterminal angles.** Coterminal angles will always differ by 360°, meaning for that any integer k, angles θ and $\theta + 360k$ will be coterminal.

EXAMPLE 5 ► **Finding Coterminal Angles**

Find two positive angles and two negative angles that are coterminal with 60°.

Solution ►

For $k = -2$, $60° + 360(-2) = -660°$.
For $k = -1$, $60° + 360(-1) = -300°$.
For $k = 1$, $60° + 360(1) = 420°$.
For $k = 2$, $60° + 360(2) = 780°$.

Note that many other answers are possible.

Now try Exercises 37 through 40 ►

WORTHY OF NOTE

Using the properties of ratios, we note that since both r (radius) and s (arc length) are measured in like units, the units actually "cancel" making radians a unitless measure: $\theta = \frac{s \text{ units}}{r \text{ units}} = \frac{s}{r}$.

An angle is said to be in **standard position** in the rectangular coordinate system if its vertex is at the origin and the initial side is along the positive x-axis. In standard position, the terminal sides of 90°, 180°, 270°, and 360° angles coincide with one of the axes and are called **quadrantal angles.** To help develop these ideas further, we use a **central circle,** that is, a circle in the xy-plane with its center at the origin. A **central angle** is an angle whose vertex is at the center of the circle. For central angle θ intersecting the circle at points B and C, we say circular arc BC, denoted $\overset{\frown}{BC}$, **subtends** $\angle BAC$, as shown in Figure 5.12. The letter s is commonly used to represent arc length, and if we define **1 radian** (abbreviated *rad*) to be the measure of an angle subtended by an arc equal in length to the radius, then $\theta = 1$ rad when $s = r$ (see Figure 5.13). We can then find the radian measure of any central angle by dividing the length of the subtended arc by r: $\frac{s}{r} = \theta$ radians. Multiplying both sides by r gives a formula for the length of any arc subtended on a circle of radius r: $s = r\theta$ if θ is in radians.

Figure 5.12

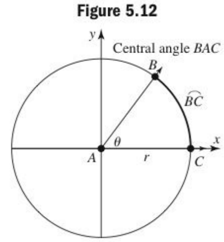
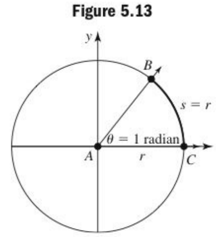


Figure 5.13



Radians

If central angle θ is subtended by an arc that is equal in length to the radius, then $\theta = 1$ *radian*.

Arc Length

If θ is a central angle in a circle of radius r, then the length of the subtended arc s is

$$s = r\theta,$$

provided θ is expressed in radians.

EXAMPLE 6 ► Using the Formula for Arc Length

If the circle in Figure 5.13 has radius $r = 10$ cm, what is the length of the arc subtended by an angle of 3.5 rad?

Solution ► Using the formula $s = r\theta$ with $r = 10$ and $\theta = 3.5$ gives

$s = 10(3.5)$ substitute 10 for r and 3.5 for θ

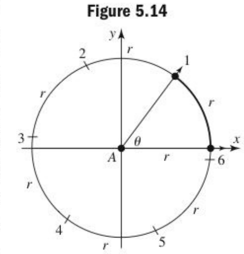
$s = 35$ result

The subtended arc has a length of 35 cm.

Now try Exercises 41 through 52 ►

Using a central angle θ measured in radians, we can also develop a formula for the **area of a circular sector** (a pie slice) using a proportion. Recall the circumference of a cicle is $C = 2\pi r$. While you may not have considered this before, note the formula can be written as $C = 2\pi \cdot r$, which implies that the radius, or an arc of length r, can be wrapped around the circumference of the circle $2\pi \approx 6.28$ times, as illustrated in Figure 5.14. This shows the radian measure of a full 360° rotation is 2π: 2π *rad* $= 360°$.This can be verified as before, using the relation θ radians $= \frac{s}{r} = \frac{2\pi r}{r} = 2\pi$. The ratio of the area of a sector to the total area will be identical to the ratio of the subtended angle to one full rotation. Using $\mathcal{A}$ to represent the area of the sector, we have $\frac{\mathcal{A}}{\pi r^2} = \frac{\theta}{2\pi}$ and solving for $\mathcal{A}$ gives $\mathcal{A} = \frac{1}{2}r^2\theta$.

Figure 5.14



Area of a Sector

If θ is a central angle in a circle of radius r, the area of the sector formed is

$$\mathcal{A} = \frac{1}{2}r^2\theta,$$

provided θ is expressed in radians.

EXAMPLE 7 ► Using the Formula for the Area of a Sector

What is the area of the circular sector formed by a central angle of $\frac{3\pi}{4}$, if the radius of the circle is 72 ft? Round to tenths.

Solution ► Using the formula $\mathcal{A} = \frac{1}{2}r^2\theta$ we have

$$\mathcal{A} = \left(\frac{1}{2}\right)(72)^2\left(\frac{3\pi}{4}\right) \quad \text{substitute 72 for } r, \frac{3\pi}{4} \text{ for } \theta$$
$$= 1944\pi \text{ ft}^2 \quad \text{result}$$

The area of this sector is approximately 6107.3 ft^2.

C. You've just learned how to use radians for angle measure and to compute circular arc length and area using radians

Now try Exercises 53 through 64 ►

D. Converting Between Degrees and Radians

In addition to its use in developing formulas for arc length and the area of a sector, the relation 2π rad $= 360°$ enables us to state the radian measures of the standard angles using a simple division. For π rad $= 180°$ we have

division by 2: $\frac{\pi}{2} = 90°$ division by 3: $\frac{\pi}{3} = 60°$

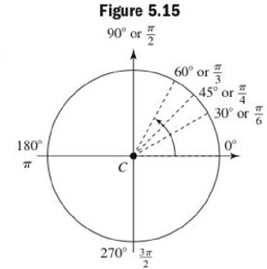
division by 4: $\frac{\pi}{4} = 45°$ division by 6: $\frac{\pi}{6} = 30°$.

WORTHY OF NOTE

We will often use the convention that unless degree measure is explicitly implied or noted with the ° symbol, radian measure is being used. In other words, $\theta = \frac{\pi}{2}$, $\theta = 2$, and $\theta = 32.76$ all indicate angles measured in radians.

See Figure 5.15. The radian measures of these standard angles play a major role in this chapter, and you are encouraged to become very familiar with them. Additional conversions can quickly be found using multiples of these four. For example, multiplying both sides of $\frac{\pi}{3} = 60°$ by two gives $\frac{2\pi}{3} = 120°$. The relationship $\pi = 180°$ also gives the factors needed for converting from degrees to radians or from radians to degrees, even if θ is a nonstandard angle. Dividing by π we have $1 = \frac{180°}{\pi}$, while division by 180° shows $1° = \frac{\pi}{180°}$. Multiplying a given angle by the appropriate conversion factor gives the equivalent measure.

Figure 5.15



Degrees/Radians Conversion Factors

To convert from radians to degrees: multiply by $\frac{180°}{\pi}$.

To convert from degrees to radians: multiply by $\frac{\pi}{180°}$.

EXAMPLE 8 ► Converting Between Radians and Degrees

Convert each angle as indicated:

a. 75° to radians. **b.** $\frac{\pi}{24}$ to degrees.

Solution ► **a.** For degrees to radians, use the conversion factor $\frac{\pi}{180°}$.

$$75° = 75° \cdot \frac{\pi}{180°} = \frac{5\pi}{12} \quad \frac{75}{180} = \frac{5}{12}$$

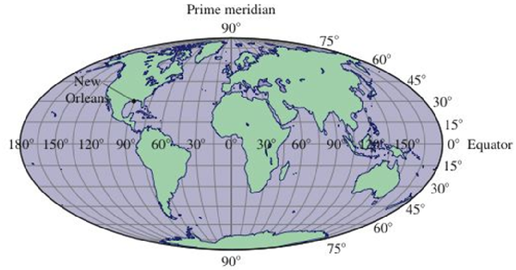
b. For radians to degrees, use the conversion factor $\frac{180°}{\pi}$:

$$\frac{\pi}{24} = \frac{\pi}{24} \cdot \left(\frac{180°}{\pi}\right) = 7.5° \qquad \frac{\pi}{\pi} = 1, \frac{180}{24} = 7.5$$

Now try Exercises 65 through 92 ►

One example where these conversions are useful is in applications involving longitude and latitude (see Figure 5.16). The **latitude** of a fixed point on the Earth's surface tells how many degrees north or south of the equator the point is, as measured from the center of the Earth. The **longitude** of a fixed point on the Earth's surface tells how many degrees east or west of the Prime Meridian (through Greenwich, England) the point is, as measured along the equator to the longitude line going through the point. For example, the city of New Orleans, Louisiana, is located at 30° N latitude, 90° W longitude (see Figure 5.16).

Figure 5.16



EXAMPLE 9 ► Applying the Arc Length Formula: Distances Between Cities

The cities of Quito, Ecuador, and Macapá, Brazil, both lie very near the equator, at a latitude of 0°. However, Quito is at approximately 78° west longitude, while Macapá is at 51° west longitude (see Figure 5.16). Assuming the Earth has a radius of 3960 mi, how far apart are these cities?

Solution ► First we note that $(78 - 51)° = 27°$ of longitude separate the two cities. Using the conversion factor $1° = \frac{\pi}{180}$ we find the equivalent radian measure is $27\left(\frac{\pi}{180}\right) = \frac{3\pi}{20}$. The arc length formula gives

$$\begin{aligned} s &= r\theta && \text{arc length formula; } \theta \text{ in radians} \\ &= 3960\left(\frac{3\pi}{20}\right) && \text{substitute 3960 for } r \text{ and } \frac{3\pi}{20} \text{ for } \theta \\ &= 594\pi && \text{result} \end{aligned}$$

Quito and Macapá are approximately 1866 mi apart (see *Worthy of Note* in the margin).

Now try Exercises 95 through 98 ►

WORTHY OF NOTE

Note that $r = 3960$ mi was used because Quito and Macapá *are both on the equator.* For other cities sharing the same longitude but not on the equator, the radius of the Earth *at that longitude* must be used. See Section 5.2, Exercise 91.

☑ D. You've just learned how to convert between degrees and radians for nonstandard angles

E. Angular and Linear Velocity

The **angular velocity** of an object is defined as the *amount of rotation* per unit time. Here, we often use the symbol ω (omega) to represent the angular velocity, and θ to represent the angle through which the terminal side has rotated, measured in radians: $\omega = \frac{\theta}{t}$. For instance, a Ferris wheel turning at 10 revolutions per minute has an angular velocity of

$$\omega = \frac{10 \text{ revolutions}}{1 \text{ min}} \qquad \omega = \frac{\theta}{t}$$

$$= \frac{10(2\pi)}{1 \text{ min}} \qquad \text{substitute } 2\pi \text{ for 1 revolution}$$

$$= \frac{20\pi \text{ rad}}{1 \text{ min}} \qquad 10(2) = 20$$

WORTHY OF NOTE

Generally speaking, the *velocity* of an object is its change in position per unit time, and can be either positive or negative. The *rate* or *speed* of an object is the magnitude of the velocity, regardless of direction.

The **linear velocity** of an object is defined as a *change of position* or *distance traveled* per unit time. In the context of angular motion, we consider the distance traveled by a point on the circumference of the Ferris wheel, *which is equivalent to the length of the resulting arc s*. This relationship is expressed as $V = \frac{s}{t}$, a formula that can be written directly in terms of the angular velocity since $s = r\theta$: $V = \frac{r\theta}{t} = r\left(\frac{\theta}{t}\right) = r\omega$.

Angular and Linear Velocity

Given a circle of radius r with point P on the circumference, and central angle θ in radians with P on the terminal side. If P moves along the circumference at a uniform rate:

1. The rate at which θ changes is called the *angular velocity* ω,

$$\omega = \frac{\theta}{t}.$$

2. The rate at which the position of P changes is called the *linear velocity V*,

$$V = \frac{r\theta}{t} \quad \Rightarrow \quad V = r\omega.$$

EXAMPLE 10 ▶ Using Angular Velocity to Determine Linear Velocity

The wheels on a racing bicycle have a radius of 13 in. How fast is the cyclist traveling in miles per hour, if the wheels are turning at 300 rpm?

Solution ▶ Note that $\omega = \frac{300 \text{ rev}}{1 \text{ min}} = \frac{300(2\pi)}{1 \text{ min}} = \frac{600\pi}{1 \text{ min}}$.

Using the formula $V = r\omega$ gives a linear velocity of

$$V = (13 \text{ in.})\frac{600\pi}{1 \text{ min}} \approx \frac{24{,}504.4 \text{ in.}}{1 \text{ min}}.$$

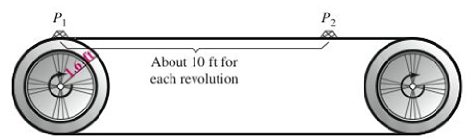
To convert this to miles per hour we convert minutes to hours (1 hr = 60 min) and inches to miles (1 mi = 5280 × 12 in.):

$$\left(\frac{24{,}504.4 \text{ in.}}{1 \text{ min}}\right)\left(\frac{60 \text{ min}}{1 \text{ hr}}\right)\left(\frac{1 \text{ mi}}{63{,}360 \text{ in.}}\right) \approx 23.2 \text{ mph}.$$

The bicycle is traveling about 23.2 mph.

Now try Exercises 99 through 102 ▶

To help understand the relationship between angular velocity and linear velocity, consider two large rollers with a radius of 1.6 ft, used to move an industrial conveyor belt. The rollers have a circumference of $C = 2\pi(1.6\text{ ft}) \approx 10.05$ ft, meaning that for each revolution of the rollers, an object on the belt will move 10.05 ft (from P_1 to P_2).



If the rollers are rotating at 20 revolutions per minute (rpm), an object on the belt (or a point on the circumference of a roller), will be moving at a rate of $20 \cdot 10.05 = 201$ ft/min (about 2.3 miles per hour). In other words,

$$\omega = \frac{20\text{ revolutions}}{1\text{ min}} \qquad \omega = \frac{\theta}{t}$$

$$= \frac{20 \cdot 2\pi}{1\text{ min}} = \frac{40\pi}{1\text{ min}} \qquad \text{substitute } 2\pi \text{ for 1 revolution}$$

$$V = r\omega \qquad \text{formula for velocity}$$

$$= (1.6\text{ ft})\frac{40\pi}{1\text{ min}} \qquad \text{substitute 1.6 ft for } r\text{, } 40\pi \text{ for } \omega$$

$$\approx 201\text{ ft per min} \qquad \text{result}$$

☑ E. You've just learned how to solve applications involving angular velocity and linear velocity using radians

5.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. __________ angles sum to 90°. Supplementary angles sum to _____°. Acute angles are _____ than 90°. Obtuse angles are _______ than 90°.

2. The expression "theta equals two degrees" is written _____ using the "°" notation. The expression, "theta equals two radians" is simply written _____.

3. The formula for arc length is $s =$ _____. The area of a sector is $A =$ _____. For both formulas, θ must be in _____.

4. If θ is not a special angle, multiply by _____ to convert radians to degrees. To convert degrees to radians, multiply by _____.

5. Discuss/Explain the difference between angular velocity and linear velocity. In particular, why does one depend on the radius while the other does not?

6. Discuss/Explain the difference between 1° and 1 radian. Exactly what is a radian? Without any conversions, explain why an angle of 4 rad terminates in QIII.

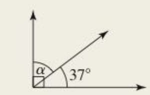
▶ DEVELOPING YOUR SKILLS

Determine the measure of the angle described.

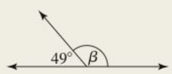
7. a. The complement of a 12.5° angle
b. The supplement of a 149.2° angle

8. a. The complement of a 62.4° angle
b. The supplement of a 74.7° angle

9. The measure of angle α



10. The measure of angle β



Convert from DMS (degree/minute/seconds) notation to decimal degrees.

11. 42°30′ **12.** 125°45′

13. 67°33′18″ **14.** 9°15′36″

15. 285°00′09″ **16.** 312°00′54″

17. 45°45′45″ **18.** 30°30′27″

Convert the angles from decimal degrees to DMS (degree/minute/sec) notation.

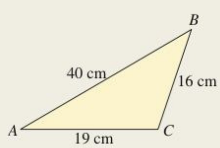
19. 20.25° **20.** 40.75°

21. 67.307° **22.** 83.516°

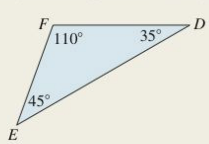
23. 275.33° **24.** 330.45°

25. 5.4525° **26.** 12.3275°

27. Is the triangle shown possible? Why/why not?

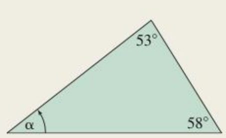


28. Is the triangle below possible? Why/why not?

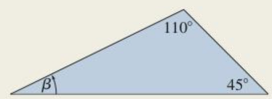


Determine the measure of the angle indicated.

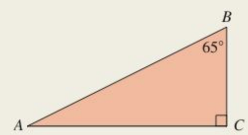
29. angle α



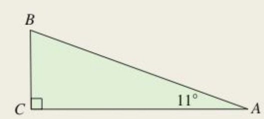
30. angle β



31. $\angle A$



32. $\angle B$

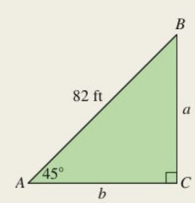


33. Similar triangles: A helicopter is hovering over a crowd of people watching a police standoff in a parking garage across the street. Stewart notices the shadow of the helicopter is lagging approximately 50 m behind a point directly below the helicopter. If he is 2 m tall and casts a shadow of 1.6 m at this time, what is the altitude of the helicopter?

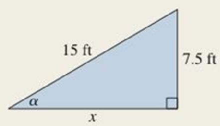
34. Similar triangles: Near Fort Macloud, Alberta (Canada), there is a famous cliff known as *Head Smashed in Buffalo Jump*. The area is now a Canadian National Park, but at one time the Native Americans hunted buffalo by steering a part of the herd over the cliff. While visiting the park late one afternoon, Denise notices that its shadow reaches 201 ft from the foot of the cliff, at the same time she is casting a shadow of 12′1″. If Denise is 5′4″. tall, what is the height of the cliff?

Solve using special triangles. Answer in both exact and approximate form.

35. Special triangles: A ladder-truck arrives at a high-rise apartment complex where a fire has broken out. If the maximum length the ladder extends is 82 ft and the angle of inclination is 45°, how high up the side of the building does the ladder reach? Assume the ladder is mounted atop a 10 ft high truck.



36. Special triangles: A heavy-duty ramp is used to winch heavy appliances from street level up to a warehouse loading dock. If the ramp is 7.5 ft high and the incline is 15 ft long, (a) what angle α does the dock make with the street? (b) How long is the base of the ramp?



Find two positive angles and two negative angles that are coterminal with the angle given. Answers may vary.

37. $\theta = 75°$
38. $\theta = 225°$
39. $\theta = -45°$
40. $\theta = -60°$

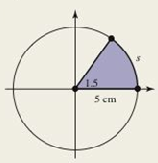
Use the formula for arc length to find the value of the unknown quantity: $s = r\theta$.

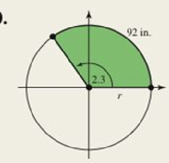
41. $\theta = 3.5$; $r = 280$ m
42. $\theta = 2.3$; $r = 129$ cm
43. $s = 2007$ mi; $r = 2676$ mi
44. $s = 4435.2$ km; $r = 12{,}320$ km
45. $\theta = \frac{3\pi}{4}$; $s = 4146.9$ yd
46. $\theta = \frac{11\pi}{6}$; $s = 28.8$ nautical miles
47. $\theta = \frac{4\pi}{3}$; $r = 2$ mi
48. $\theta = \frac{3\pi}{2}$; $r = 424$ in.
49. $s = 252.35$ ft; $r = 980$ ft
50. $s = 942.3$ mm; $r = 1800$ mm
51. $\theta = 320°$; $s = 52.5$ km
52. $\theta = 220.5°$; $s = 7627$ m

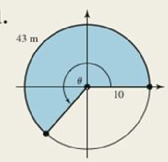
Use the formula for area of a circular sector to find the value of the unknown quantity: $A = \frac{1}{2}r^2\theta$.

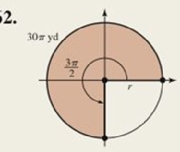
53. $\theta = 5$; $r = 6.8$ km
54. $\theta = 3$; $r = 45$ mi
55. $A = 1080$ mi^2; $r = 60$ mi
56. $A = 437.5$ cm^2; $r = 12.5$ cm
57. $\theta = \frac{7\pi}{6}$; $A = 16.5$ m^2
58. $\theta = \frac{19\pi}{12}$; $A = 753$ cm^2

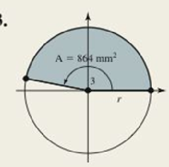
Find the angle, radius, arc length, and/or area as needed, until all values are known.

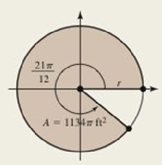
59.


60.


61.


62.


63.


64.


Convert the following degree measures to radians in exact form, without the use of a calculator.

65. $\theta = 360°$
66. $\theta = 180°$
67. $\theta = 45°$
68. $\theta = 30°$
69. $\theta = 210°$
70. $\theta = 330°$
71. $\theta = -120°$
72. $\theta = -225°$

Convert each degree measure to radians. Round to the nearest ten-thousandth.

73. $\theta = 27°$
74. $\theta = 52°$
75. $\theta = 227.9°$
76. $\theta = 154.4°$

Convert each radian measure to degrees, without the use of a calculator.

77. $\theta = \frac{\pi}{3}$
78. $\theta = \frac{\pi}{4}$
79. $\theta = \frac{\pi}{6}$
80. $\theta = \frac{\pi}{2}$
81. $\theta = \frac{2\pi}{3}$
82. $\theta = \frac{5\pi}{6}$
83. $\theta = 4\pi$
84. $\theta = 6\pi$

Convert each radian measure to degrees. Round to the nearest tenth.

85. $\theta = \frac{11\pi}{12}$

86. $\theta = \frac{17\pi}{36}$

87. $\theta = 3.2541$

88. $\theta = 1.0257$

89. $\theta = 3$

90. $\theta = 5$

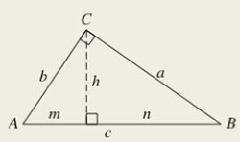
91. $\theta = -2.5$

92. $\theta = -3.7$

▶ WORKING WITH FORMULAS

93. Relationships in a right triangle: $h = \frac{ab}{c}$, $m = \frac{b^2}{c}$, **and** $n = \frac{a^2}{c}$

Given $\angle C$ is a right angle, and h is the altitude of $\triangle ABC$, then h, m, and n can all be expressed directly in terms of a, b, and c by the relationships shown here. Compute the value of h, m, and n for a right triangle with sides of 8, 15, and 17 cm.



94. The height of an equilateral triangle: $H = \frac{\sqrt{3}}{2}S$

Given an equilateral triangle with sides of length S, the height of the triangle is given by the formula shown. Once the height is known the area of the triangle can easily be found (also see Exercise 93). The Gateway Arch in St. Louis, Missouri, is actually composed of stainless steel sections that are equilateral triangles. At the base of the arch the length of the sides is 54 ft. The smallest cross section at the top of the arch has sides of 17 ft. Find the area of these cross sections.

▶ APPLICATIONS

95. Arc length: The city of Pittsburgh, Pennsylvania, is directly north of West Palm Beach, Florida. Pittsburg is at 40.3° north latitude, while West Palm Beach is at 26.4° north latitude. Assuming the Earth has a radius of 3960 mi, how far apart are these cities?

96. Arc length: Both Libreville, Gabon, and Jamame, Somalia, lie near the equator, but on opposite ends of the African continent. If Libreville is at 9.3° east longitude and Jamame is 42.5° east longitude, how wide is the continent of Africa at the equator?

97. Area of a sector: A water sprinkler is set to shoot a stream of water a distance of 12 m and rotate through an angle of 40°. (a) What is the area of the lawn it waters? (b) For $r = 12$ m, what angle is required to water twice as much area? (c) For $\theta = 40°$, what range for the water stream is required to water twice as much area?

98. Area of a sector: A motion detector can detect movement up to 25 m away through an angle of 75°. (a) What area can the motion detector monitor? (b) For $r = 25$ m, what angle is required to monitor 50% more area? (c) For $\theta = 75°$, what range is required for the detector to monitor 50% more area?

99. Riding a round-a-bout: At the park two blocks from our home, the kids' round-a-bout has a radius of 56 in. About the time the kids stop screaming, "Faster, Daddy, faster!" I estimate the round-a-bout is turning at $\frac{3}{4}$ revolutions per second. (a) What is the related angular velocity? (b) What is the linear velocity (in miles per hour) of Eli and Reno, who are "hanging on for dear life" at the rim of the round-a-bout?

100. Carnival rides: At carnivals and fairs, the *Gravity Drum* is a popular ride. People stand along the wall of a circular drum with radius 12 ft, which begins spinning very fast, pinning them against the wall. The drum is then turned on its side by an armature, with the riders screaming and squealing with delight. As the drum is raised to a near-vertical position, it is spinning at a rate of 35 rpm. (a) What is the angular velocity in radians? (b) What is the linear velocity (in miles per hour) of a person on this ride?

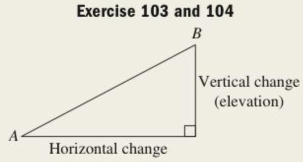
101. Speed of a winch: A winch is being used to lift a turbine off the ground so that a tractor-

trailer can back under it and load it up for transport. The winch drum has a radius of 3 in. and is turning at 20 rpm. Find (a) the angular velocity of the drum in radians, (b) the linear velocity of the turbine in feet per second as it is being raised, and (c) how long it will take to get the load to the desired height of 6 ft (ignore the fact that the cable may wind over itself on the drum).

102. Speed of a current: An instrument called a flowmeter is used to measure the speed of flowing water, like that in a river or stream. A cruder method involves placing a paddle wheel in the current, and using the wheel's radius and angular velocity to calculate the speed of water flow. If the paddle wheel has a radius of 5.6 ft and is turning at 30 rpm, find (a) the angular velocity of the wheel in radians and (b) the linear velocity of the water current in miles per hour.

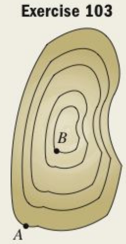
On topographical maps, each closed figure represents a fixed elevation (a vertical change) according to a given *contour interval*. The *measured distance* on the map from point *A* to point *B* indicates the horizontal distance or the horizontal change between point *A* and a location directly beneath point *B*, according to a given *scale of distances*.

Exercise 103 and 104



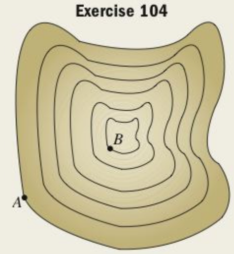
103. Special triangles: In the figure shown, the *contour interval* is 1:250 (each figure indicates a change of 250 m in elevation) and the scale of distances is 1 cm = 625 m. (a) Find the change of elevation from *A* to *B*; (b) use a proportion to find the horizontal distance between points *A* and *B* if the measured distance on the map is 1.6 cm; and (c) Draw the corresponding right triangle and use a special triangle relationship to find the length of the trail up the mountain side that connects *A* and *B*.

Exercise 103



104. Special triangles: As part of park maintenance, the 2 by 4 handrail alongside a mountain trail leading to the summit of Mount Marilyn must be replaced. In the figure, the *contour interval* is 1:200 (each figure indicates a change of 200 m in elevation) and the scale of distances is 1 cm = 400 m. (a) Find the change of elevation from *A* to *B*; (b) use a proportion to find the horizontal distance between *A* and *B* if the measured distance on the map is 4.33 cm; and (c) draw the corresponding right triangle and use a special triangle relationship to find the length needed to replace the handrail (recall that $\sqrt{3} \approx 1.732$).

Exercise 104



105. Special triangles: Two light planes are flying in formation at 100 mph, doing some reconnaissance work. At a designated instant, one pilot breaks to the left at an angle of 90° to the other plane. Assuming they keep the same altitude and continue to fly at 100 mph, use a special triangle to find the distance between them after 0.5 hr.

106. Special triangles: Two ships are cruising together on the open ocean at 10 nautical miles per hour. One of them turns to make a 90° angle with the first and increases speed, heading for port. Assuming the first ship continues traveling at 10 knots, use a special triangle to find the speed of the other ship if they are 20 mi apart after 1 hr.

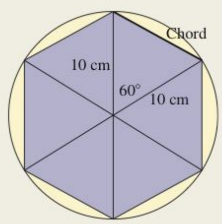
107. Angular and linear velocity: The planet Jupiter's largest moon, Ganymede, rotates around the planet at a distance of about 656,000 miles, in an orbit that is perfectly circular. If the moon completes one rotation about Jupiter in 7.15 days, (a) find the angle θ that the moon moves through in 1 day, in both degrees and radians, (b) find the angular velocity of the moon in radians per hour, and (c) find the moon's linear velocity in miles per second as it orbits Jupiter.

108. Angular and linear velocity: The planet Neptune has an orbit that is nearly circular. It orbits the Sun at a distance of 4497 million kilometers and completes one revolution every 165 yr. (a) Find the angle θ that the planet moves through in one year in both degrees and radians and (b) find the linear velocity (km/hr) as it orbits the Sun.

▶ EXTENDING THE CONCEPT

109. Many methods have been used for angle measure over the centuries, some more logical or meaningful than what is popular today. Do some research on the evolution of angle measure, and compare/contrast the benefits and limitations of each method. In particular, try to locate information on the history of degrees, radians, mils, and gradients, and identify those still in use.

110. Ancient geometers knew that a hexagon (six sides) could be inscribed in a circle by laying out six consecutive chords equal in length to the radius ($r = 10$ cm for illustration). After connecting the diagonals of the hexagon, six equilateral triangles are formed with sides of 10 cm. Use the diagram given to develop the fixed ratios for the sides of a 30-60-90 triangle. (*Hint:* Use a perpendicular bisector.)



111. The Duvall family is out on a family bicycle ride around Creve Couer Lake. The adult bikes have a pedal sprocket with a 4-in. radius, wheel sprocket with 2-in. radius, and tires with a 13-in. radius. The kids' bikes have pedal sprockets with a 2.5-in. radius, wheel sprockets with 1.5-in. radius, and tires with a 9-in. radius. (a) If adults and kids all pedal at 50 rpm, how far ahead (in yards) are the adults after 2 min? (b) If adults pedal at 50 rpm, how fast do the kids have to pedal to keep up?

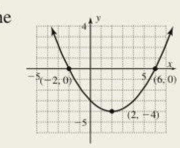
▶ MAINTAINING YOUR SKILLS

112. **(2.6)** Describe how the graph of $g(x) = -2\sqrt{x+3} - 1$ can be obtained from transformations of $y = \sqrt{x}$.

113. **(4.5)** Find the interest rate required for \$1000 to grow to \$1500 if the money is compounded monthly and remains on deposit for 5 yr.

114. **(2.2)** Given a line segment with endpoints $(-2, 3)$ and $(6, -1)$, find the equation of the line that bisects and is perpendicular to this segment.

115. **(3.1)** Find the equation of the function whose graph is shown.



5.2 The Trigonometry of Right Triangles

Learning Objectives

In Section 5.2 you will learn how to:

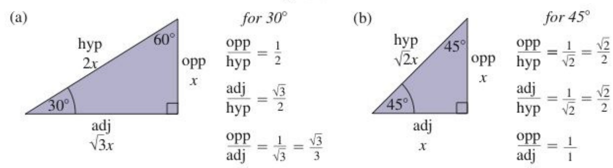
- ❑ **A.** Find values of the six trigonometric functions from their ratio definitions
- ❑ **B.** Solve a right triangle given one angle and one side
- ❑ **C.** Solve a right triangle given two sides
- ❑ **D.** Use cofunctions and complements to write equivalent expressions
- ❑ **E.** Solve applications involving angles of elevation and depression
- ❑ **F.** Solve general applications of right triangles

Over a long period of time, what began as a study of chord lengths by Hipparchus, Ptolemy, Aryabhata, and others became a systematic application of the ratios of the sides of a right triangle. In this section, we develop the sine, cosine, and tangent functions from a right triangle perspective, and explore certain relationships that exist between them. This view of the trig functions also leads to a number of significant applications.

A. Trigonometric Ratios and Their Values

In Section 5.1, we looked at applications involving 45-45-90 and 30-60-90 triangles, using the fixed ratios that exist between their sides. To apply this concept more generally using other right triangles, each side is given a specific name using its location relative to a specified angle. For the 30-60-90 triangle in Figure 5.17(a), the side **opposite (opp)** and the side **adjacent (adj)** are named with respect to the 30° angle, with the **hypotenuse (hyp)** always across from the right angle. Likewise for the 45-45-90 triangle in Figure 5.17(b).

Figure 5.17



Using these designations to define the various trig ratios, we can now develop a systematic method for applying them. Note that the x's "cancel" in each ratio, reminding us the ratios are independent of the triangle's size (if two triangles are similar, the ratio of corresponding sides is constant).

Ancient mathematicians were able to find values for the ratios corresponding to *any acute angle* in a right triangle, and realized that *naming* each ratio would be helpful. These names are $\frac{\text{opp}}{\text{hyp}} \rightarrow$ **sine,** $\frac{\text{adj}}{\text{hyp}} \rightarrow$ **cosine,** and $\frac{\text{opp}}{\text{adj}} \rightarrow$ **tangent.** Since each ratio depends on the measure of an acute angle θ, they are often referred to as **functions of an acute angle** and written in function form.

$$\text{sine } \theta = \frac{\text{opp}}{\text{hyp}} \qquad \text{cosine } \theta = \frac{\text{adj}}{\text{hyp}} \qquad \text{tangent } \theta = \frac{\text{opp}}{\text{adj}}$$

The reciprocal of these ratios, for example, $\frac{\text{hyp}}{\text{opp}}$ instead of $\frac{\text{opp}}{\text{hyp}}$, also play a significant role in this view of trigonometry, and are likewise given names:

$$\text{cosecant } \theta = \frac{\text{hyp}}{\text{opp}} \qquad \text{secant } \theta = \frac{\text{hyp}}{\text{adj}} \qquad \text{cotangent } \theta = \frac{\text{adj}}{\text{opp}}$$

The definitions hold regardless of the triangle's orientation or which of the acute angles is used.

In actual use, each function name is written in abbreviated form as $\sin \theta$, $\cos \theta$, $\tan \theta$, $\csc \theta$, $\sec \theta$, and $\cot \theta$ respectively. Note that based on these designations, we have the following reciprocal relationships:

$$\sin \theta = \frac{1}{\csc \theta} \qquad \cos \theta = \frac{1}{\sec \theta} \qquad \tan \theta = \frac{1}{\cot \theta}$$

$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$$

In general:

WORTHY OF NOTE

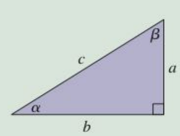
Over the years, a number of memory tools have been invented to help students recall these ratios correctly. One such tool is the acronym SOH CAH TOA, from the first letter of the function and the corresponding ratio. It is often recited as, "Sit On a Horse, Canter Away Hurriedly, To Other Adventures." Try making up a memory tool of your own.

Trigonometric Functions of an Acute Angle

$$\sin \alpha = \frac{a}{c} \qquad \sin \beta = \frac{b}{c}$$

$$\cos \alpha = \frac{b}{c} \qquad \cos \beta = \frac{a}{c}$$

$$\tan \alpha = \frac{a}{b} \qquad \tan \beta = \frac{b}{a}$$



Now that these ratios have been formally named, we can state values of all six functions given sufficient information about a right triangle.

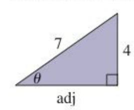
EXAMPLE 1 ► **Finding Function Values Using a Right Triangle**

Given $\sin\theta = \frac{4}{7}$, find the values of the remaining trig functions.

Solution ► For $\sin\theta = \frac{4}{7} = \frac{\text{opp}}{\text{hyp}}$, we draw a triangle with a side of 4 units opposite a designated angle θ, and label a hypotenuse of 7 (see the figure). Using the Pythagorean theorem we find the length of the adjacent side: adj $= \sqrt{7^2 - 4^2} = \sqrt{33}$. The ratios are

$$\sin\theta = \frac{4}{7} \qquad \cos\theta = \frac{\sqrt{33}}{7} \qquad \tan\theta = \frac{4}{\sqrt{33}}$$

$$\csc\theta = \frac{7}{4} \qquad \sec\theta = \frac{7}{\sqrt{33}} \qquad \cot\theta = \frac{\sqrt{33}}{4}$$

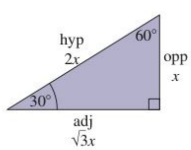


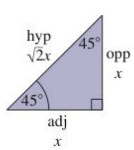
Now try Exercises 7 through 12 ►

A. You've just learned how to find values of the six trigonometric functions from their ratio definitions

Note that due to the properties of similar triangles, identical results would be obtained using any ratio of sides that is equal to $\frac{4}{7}$. In other words, $\frac{2}{3.5} = \frac{4}{7} = \frac{8}{14} = \frac{16}{28}$ and so on, will all give the same value for $\sin\theta$.

B. Solving Right Triangles Given One Angle and One Side





Example 1 gave values of the trig functions for an *unknown angle* θ. Using the special triangles, we can state the value of each trig function for 30°, 45°, and 60° based on the related ratio (see Table 5.1). These values are used extensively in a study of trigonometry and must be committed to memory.

Table 5.1

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	2	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	2	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

To **solve a right triangle** means to find the measure of all three angles and all three sides. This is accomplished using combinations of the Pythagorean theorem, the properties of triangles, and the trigonometric ratios. We will adopt the convention of naming each angle with a capital letter at the vertex or using a Greek letter on the interior. Each side is labeled using the related lowercase letter from the angle opposite. The complete solution should be organized in table form as in Example 2. Note the quantities shown in **bold** were given, and the remaining values were found using the techniques mentioned.

EXAMPLE 2 ► **Solving a Right Triangle**

Solve the triangle shown below.

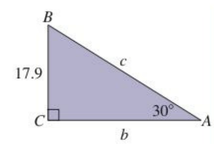
Solution ► Applying the sine ratio (since the side opposite 30° is given), we have: $\sin 30° = \frac{\text{opp}}{\text{hyp}}$.

For side c: $\sin 30° = \frac{17.9}{c}$ $\quad$ $\sin 30° = \frac{\text{opposite}}{\text{hypotenuse}}$

$c \sin 30° = 17.9$ $\quad$ multiply by c

$c = \frac{17.9}{\sin 30°}$ $\quad$ divide by $\sin 30° = \frac{1}{2}$

$= 35.8$ $\quad$ result



Using the Pythagorean theorem shows $b \approx 31$, and since $\angle A$ and $\angle B$ are complements, $B = 60°$. Note the results would have been identical if the special ratios from the 30-60-90 triangle were applied. The hypotenuse is twice the shorter side: $c = 2(17.9) = 35.8$, and the longer side is $\sqrt{3}$ times the shorter: $b = 17.9(\sqrt{3}) \approx 31$.

Angles	Sides
$A = 30°$	**$a = 17.9$**
$B = 60°$	$b \approx 31$
$C = 90°$	$c = 35.8$

Now try Exercises 13 through 16 ►

Prior to the widespread availability of handheld calculators, a table of values was used to find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for nonstandard angles. Table 5.2 shows the sine of 49° 30′ is approximately 0.7604.

Table 5.2
sin θ

θ	0′	10′	20′	30′
45°	0.7071	0.7092	0.7112	0.7133
46	0.7193	0.7214	0.7234	0.7254
47	0.7314	0.7333	0.7353	0.7373
48	0.7431	0.7451	0.7470	0.7490
49	0.7547	0.7566	0.7585	0.7604

Today these trig values are programmed into your calculator and we can retrieve them with the push of a button (or two). To find the sine of 48°, make sure your calculator is in degree MODE, then press the SIN key, 48, and ENTER. The result should be very close to 0.7431 as the table indicates.

EXAMPLE 3 ► Solving a Right Triangle

Solve the triangle shown in the figure.

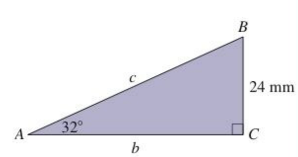
Solution ► We know $\angle B = 58°$ since $A + B = 90°$. We can find length b using the tangent function:

$\tan 32° = \frac{24}{b}$ $\quad$ $\tan 32° = \frac{\text{opp}}{\text{adj}}$

$b \tan 32° = 24$ $\quad$ multiply by b

$b = \frac{24}{\tan 32°}$ $\quad$ divide by $\tan 32°$

≈ 38.41 mm $\quad$ result



We can find the length c by simply applying the Pythagorean theorem, or by using another trig ratio and a known angle.

For side c:

$$\sin 32° = \frac{24}{c} \qquad \sin 32° = \frac{\text{opp}}{\text{hyp}}$$

$$c \sin 32° = 24 \qquad \text{multiply by } c$$

$$c = \frac{24}{\sin 32°} \qquad \text{divide by } \sin 32°$$

$$\approx 45.29 \text{ mm} \qquad \text{result}$$

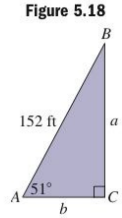
Angles	Sides
$A = 32°$	**$a = 24$**
$B = 58°$	$b \approx 38.41$
$C = 90°$	$c \approx 45.29$

The complete solution is shown in the table.

Now try Exercises 17 through 22 ►

When solving a right triangle, any of the triangle relationships can be employed: (1) angles must sum to 180°, (2) Pythagorean theorem, (3) special triangles, and (4) the trigonometric functions of an acute angle. However, the resulting equation must have only one unknown or it cannot be used. For the triangle shown in Figure 5.18, we cannot begin with the Pythagorean theorem since sides a and b are unknown, and tan 51° is unusable for the same reason. Since the hypotenuse is given, we could begin with $\cos 51° = \frac{b}{152}$ and solve for b, or with $\sin 51° = \frac{a}{152}$ and solve for a, then work out a complete solution. Verify that $a \approx 118.13$ ft and $b \approx 95.66$ ft.

Figure 5.18



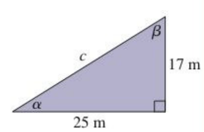
B. You've just learned how to solve a right triangle given one angle and one side

C. Solving Right Triangles Given Two Sides

The partial table for $\sin \theta$ given earlier was also used in times past to find an angle whose sine was known, meaning if $\sin \theta \approx 0.7604$, then θ must be 49.5° (see the last line of Table 5.2). The modern notation for "an angle whose sine is known" is $\theta = \mathbf{\sin^{-1} x}$ or $\theta = \mathbf{\arcsin x}$, where x is the known value for $\sin \theta$. The values for the acute angles $\theta = \sin^{-1}x$, $\theta = \cos^{-1}x$, and $\theta = \tan^{-1}x$ are also programmed into your calculator and are generally accessed using the INV or 2nd keys with the related SIN, COS, or TAN key. With these we are completely equip to find all six measures of a right triangle, given at least one side and any two other measures.

EXAMPLE 4 ► **Solving a Right Triangle**

Solve the triangle given in the figure.



Solution ► Since the hypotenuse is unknown, we cannot begin with the sine or cosine ratios. The opposite and adjacent sides for α *are* known, so we use $\tan \alpha$. For $\tan \alpha = \frac{17}{25}$ we find $\alpha = \tan^{-1}\left(\frac{17}{25}\right) \approx 34.2°$ [verify that $\tan(34.2°) = 0.6795992982 \approx \frac{17}{25}$]. Since α and β are complements, $\beta \approx 90 - 34.2 = 55.8°$. The Pythagorean theorem shows the hypotenuse is about 30.23 m.

Angles	Sides
$\alpha \approx 34.2°$	**$a = 17$**
$\beta \approx 55.8°$	**$b = 25$**
$\gamma = 90°$	$c \approx 30.23$

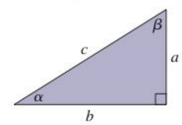
C. You've just learned how to solve a right triangle given two sides

Now try Exercises 23 through 54 ►

D. Using Cofunctions and Complements to Write Equivalent Expressions

In Figure 5.19, $\angle\alpha$ and $\angle\beta$ must be complements since we have a right triangle, and the sum of the three angles must be 180°. The complementary angles in a right triangle have a unique relationship that is often used. Specifically $\alpha + \beta = 90°$ means $\beta = 90° - \alpha$. Note that $\sin\alpha = \frac{a}{c}$ and $\cos\beta = \frac{a}{c}$. This means $\sin\alpha = \cos\beta$ or $\sin\alpha = \cos(90° - \alpha)$ by substitution. In words, "The sine of an angle is equal to the cosine of its complement." For this reason sine and cosine are called **cofunctions** (hence the name **co**sine), as are secant/cosecant, and tangent/cotangent. As a test, we use a calculator to check the statement $\sin 52.3° = \cos(90 - 52.3)°$

Figure 5.19



WORTHY OF NOTE

The word *cosine* is actually a shortened form of the words "*complement of sine*," a designation suggested by Edmund Gunter around 1620 since the sine of an angle is equal to the cosine of its complement [$\text{sine}(\theta) = \text{cosine}(90° - \theta)$].

$$\sin 52.3° \stackrel{?}{=} \cos 37.7°$$
$$0.791223533 = 0.791223533 \checkmark$$

To verify the cofunction relationship for $\sec\theta$ and $\csc\theta$, recall their reciprocal relationship to cosine and sine, respectively.

$$\sec 52.3° \stackrel{?}{=} \csc \stackrel{?}{=} 37.7°$$
$$\frac{1}{\cos 52.3°} \stackrel{?}{=} \frac{1}{\sin 37.7°}$$
$$1.635250666 = 1.635250666 \checkmark$$

The cofunction relationship for $\tan\theta$ and $\cot\theta$ can similarly be verified.

Summary of Cofunctions

sine and cosine	tangent and cotangent	secant and cosecant
$\sin\theta = \cos(90 - \theta)$	$\tan\theta = \cot(90 - \theta)$	$\sec\theta = \csc(90 - \theta)$
$\cos\theta = \sin(90 - \theta)$	$\cot\theta = \tan(90 - \theta)$	$\csc\theta = \sec(90 - \theta)$

For use in Example 5 and elsewhere in the text, note the expression $\tan^2 15°$ is simply a more convenient way of writing $(\tan 15°)^2$.

EXAMPLE 5 ► Applying the Cofunction Relationship

Given $\cot 75° = 2 - \sqrt{3}$ in exact form, find the exact value of $\tan^2 15°$ using a cofunction. Check the result using a calculator.

Solution ► Using $\cot 75° = \tan(90° - 75°) = \tan 15°$ gives

$$\begin{aligned} \cot^2 75° &= \tan^2 15° && \text{cofunctions} \\ &= (2 - \sqrt{3})^2 && \text{substitute known value} \\ &= 4 - 4\sqrt{3} + 3 && \text{square as indicated} \\ &= 7 - 4\sqrt{3} && \text{result} \end{aligned}$$

Using a calculator, we verify $\tan^2 15° \approx 0.0717967697 \approx 7 - 4\sqrt{3}$.

D. You've just learned how to use cofunctions and complements to write equivalent expressions

Now try Exercises 55 through 68 ►

E. Applications Using Angles of Elevation/Depression

While the name seems self-descriptive, in more formal terms an **angle of elevation** is defined to be the acute angle formed by a **horizontal line of orientation** (parallel to level ground) and the line of sight (see Figure 5.20). An **angle of depression** is likewise defined but involves a line of sight that is below the horizontal line of orientation (Figure 5.21).

Figure 5.20

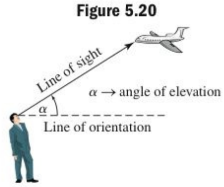
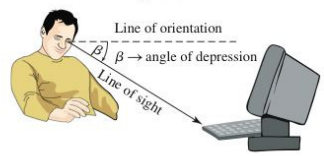


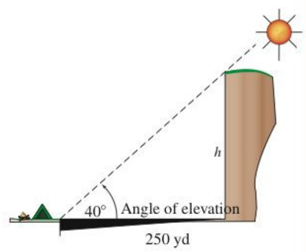
Figure 5.21



Angles of elevation/depression make distance and length computations of all sizes a relatively easy matter and are extensively used by surveyors, engineers, astronomers, and even the casual observer who is familiar with the basics of trigonometry.

EXAMPLE 6 ► **Applying Angles of Elevation**

In Example 4 from Section 5.1, a group of campers used a 45-45-90 triangle to estimate the height of a cliff. It was a time consuming process as they had to wait until mid-morning for the shadow of the cliff to make the needed 45° angle. If the campsite was 250 yd from the base of the cliff and the angle of elevation was 40° at that point, how tall is the cliff?



Solution ► As described we want to know the height of the opposite side, given the adjacent side, so we use the tangent function.

For height h: $\tan 40° = \dfrac{h}{250}$ $\tan 40° = \dfrac{\text{opp}}{\text{adj}}$

$250 \tan 40° = h$ multiply by 250

$209.8 \approx h$ result ($\tan 40° \approx 0.8391$)

The cliff is approximately 209.8 yd high (about 629 ft).

☑ **E.** You've just learned how to solve applications involving angles of elevation and depression

Now try Exercises 71 through 76 ►

F. Additional Applications of Right Triangles

In their widest and most beneficial use, the trig functions of acute angles are used with other problem-solving skills, such as drawing a diagram, labeling unknowns, working the solution out in stages, and so on. Example 7 serves to illustrate some of these combinations.

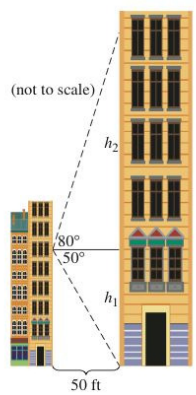
EXAMPLE 7 ► **Applying Angles of Elevation and Depression**

From his hotel room window on the sixth floor, Singh notices some window washers high above him on the hotel across the street. Curious as to their height above ground, he quickly estimates the buildings are 50 ft apart, the angle of elevation to the workers is about 80°, and the angle of depression to the base of the hotel is about 50°.

a. How high above ground is the window of Singh's hotel room?

b. How high above ground are the workers?

Solution ► **a.** Begin by drawing a diagram of the situation (see figure). To find the height of the window we'll use the tangent ratio, since the adjacent side of the angle is known, and the opposite side is the height we desire.



For the height h_1:

$$\tan 50° = \frac{h_1}{50} \quad \tan 50° = \frac{\text{opp}}{\text{adj}}$$

$$50 \tan 50° = h_1 \quad \text{solve for } h_1$$

$$59.6 \approx h_1 \quad \text{result } (\tan 50° \approx 1.1918)$$

The window is approximately 59.6 ft above ground.

b. For the height h_2:

$$\tan 80° = \frac{h_2}{50} \quad \tan 80° = \frac{\text{opp}}{\text{adj}}$$

$$50 \tan 80° = h_2 \quad \text{solve for } h_2$$

$$283.6 \approx h_2 \quad \text{result } (\tan 80° \approx 5.6713)$$

The workers are approximately 283.6 + 59.6 = 343.2 ft above ground.

Now try Exercises 77 through 80 ►

There are a number of additional, interesting applications in the exercise set.

F. You've just learned how to solve general applications of right triangles

5.2 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The phrase, "an angle whose tangent is known," is written notationally as ________.

2. Given $\sin\theta = \frac{7}{24}$, $\csc\theta =$ ________ because they are ________.

3. The sine of an angle is the ratio of the ________ side to the ________.

4. The cosine of an angle is the ratio of the ________ side to the ________.

5. Discuss/Explain exactly what is meant when you are asked to "solve a triangle." Include an illustrative example.

6. Given an acute angle and the length of the adjacent leg, which four (of the six) trig functions could be used to begin solving the triangle?

▶ DEVELOPING YOUR SKILLS

Use the function value given to determine the value of the other five trig functions of the acute angle θ. Answer in exact form (a diagram will help).

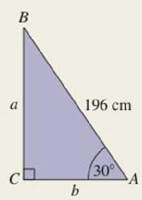
7. $\cos\theta = \frac{5}{13}$ **8.** $\sin\theta = \frac{20}{29}$

9. $\tan\theta = \frac{84}{13}$ **10.** $\sec\theta = \frac{53}{45}$

11. $\cot\theta = \frac{2}{11}$ **12.** $\cos\theta = \frac{2}{3}$

Solve each triangle using trig functions of an acute angle θ. Give a complete answer (in table form) using exact values.

13.



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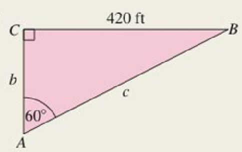

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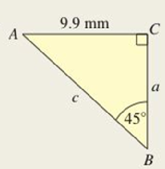

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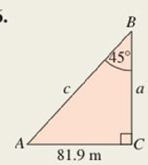
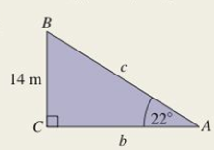


Solve the triangles shown and write answers in table form. Round sides to the nearest 100th of a unit. Verify that angles sum to 180° and that the three sides satisfy (approximately) the Pythagorean theorem.

17.



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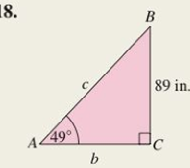
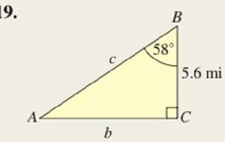


19.



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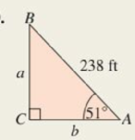

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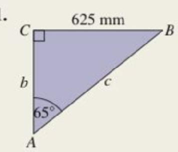

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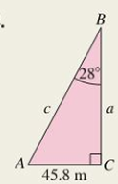


Use a calculator to find the value of each expression, rounded to four decimal places.

23. $\sin 27°$ **24.** $\cos 72°$

25. $\tan 40°$ **26.** $\cot 57.3°$

27. $\sec 40.9°$ **28.** $\csc 39°$

29. $\sin 65°$ **30.** $\tan 84.1°$

Use a calculator to find the acute angle whose corresponding ratio is given. Round to the nearest 10th of a degree. For Exercises 31 through 38, use Exercises 23 through 30 to answer.

31. $\sin A = 0.4540$ **32.** $\cos B = 0.3090$

33. $\tan\theta = 0.8391$ **34.** $\cot A = 0.6420$

35. $\sec B = 1.3230$ **36.** $\csc\beta = 1.5890$

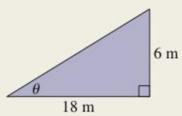
37. $\sin A = 0.9063$ **38.** $\tan B = 9.6768$

39. $\tan\alpha = 0.9896$ **40.** $\cos\alpha = 0.7408$

41. $\sin\alpha = 0.3453$ **42.** $\tan\alpha = 3.1336$

Select an appropriate function to find the angle indicated (round to 10ths of a degree).

43.



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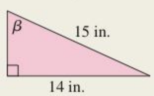

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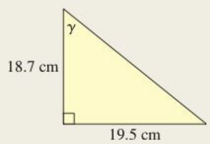

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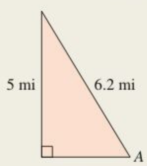

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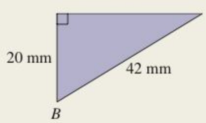
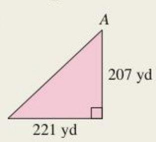
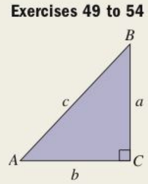


48.



Draw a right triangle *ABC* as shown, using the information given. Then select an appropriate ratio to find the side indicated. Round to the nearest 100th.

Exercises 49 to 54



49. $\angle A = 25°$, $c = 52$ mm, find side a

50. $\angle B = 55°$, $b = 31$ ft, find side c

51. $\angle A = 32°$, $a = 1.9$ mi, find side b

52. $\angle B = 29.6°$, $c = 9.5$ yd, find side a

53. $\angle A = 62.3°$, $b = 82.5$ furlongs, find side c

54. $\angle B = 12.5°$, $a = 32.8$ km, find side b

Use a calculator to evaluate each pair of functions and comment on what you notice.

55. sin 25°, cos 65°

56. sin 57°, cos 33°

57. tan 5°, cot 85°

58. sec 40°, csc 50°

Based on your observations in Exercises 55 to 58, fill in the blank so that the functions given are equal.

59. sin 47°, cos ___

60. cos ___, sin 12°

61. cot 69°, tan ___

62. csc 17°, sec ___

Complete the following tables without referring to the text or using a calculator.

63.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sin(90-\theta)$
30°				
$\cos(90-\theta)$	$\tan(90-\theta)$	$\csc\theta$	$\sec\theta$	$\cot\theta$

64.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sin(90-\theta)$
45°				
$\cos(90-\theta)$	$\tan(90-\theta)$	$\csc\theta$	$\sec\theta$	$\cot\theta$

Evaluate the following expressions without a calculator, using the cofunction relationship and the following exact forms: $\sec 75° = \sqrt{6} + \sqrt{2}$; $\tan 75° = 2 + \sqrt{3}$.

65. $\sqrt{6}\csc 15°$

66. $\csc^2 15°$

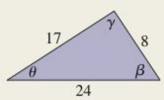
67. $\cot^2 15°$

68. $\sqrt{3}\cot 15°$

▶ WORKING WITH FORMULAS

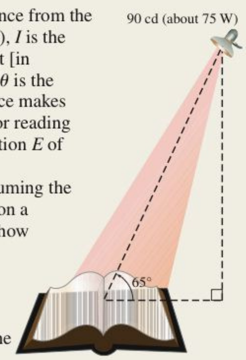
69. The sine of an angle between two sides of a triangle: $\sin\theta = \dfrac{2A}{ab}$

If the area A and two sides a and b of a triangle are known, the sine of the angle between the two sides is given by the formula shown. Find the angle θ for the triangle below given $A \approx 38.9$ units^2, and use it to solve the triangle. (*Hint:* Apply the same concept to angle γ or β.)



70. Illumination of a surface: $E = \dfrac{I\cos\theta}{d^2}$

The illumination E of a surface by a light source is a measure of the luminous flux per unit area that reaches the surface. The value of E [in lumens (lm) per square foot] is given by the formula shown, where d is the distance from the light source (in feet), I is the intensity of the light [in candelas (cd)], and θ is the angle the light source makes with the vertical. For reading a book, an illumination E of at least 18 lm/ft^2 is recommended. Assuming the open book is lying on a horizontal surface, how far away should a light source be placed if it has an intensity of 90 cd (about 75 W) and the light flux makes an angle of 65° with the book's surface (i.e., $\theta = 25°$)?



▶ APPLICATIONS

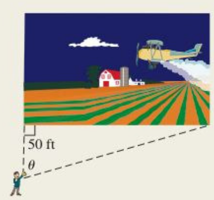
71. Angle of elevation: For a person standing 100 m from the center of the base of the Eiffel Tower, the angle of elevation to the top of the tower is 71.6°. How tall is the Eiffel Tower?

72. Angle of depression: A person standing near the top of the Eiffel Tower notices a car wreck some distance from the tower. If the angle of depression from the person's eyes to the wreck is 32°, how far away is the accident from the base of the tower? See Exercise 71.

73. Angle of elevation: In 2001, the tallest building in the world was the Petronas Tower I in Kuala Lumpur, Malaysia. For a person standing 25.9 ft from the base of the tower, the angle of elevation to the top of the tower is 89°. How tall is the Petronas tower?

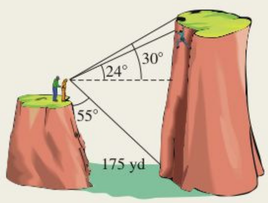
74. Angle of depression: A person standing on the top of the Petronas Tower I looks out across the city and pinpoints her residence. If the angle of depression from the person's eyes to her home is 5°, how far away (in feet and in miles) is the residence from the base of the tower? See Exercise 73.

75. Crop duster's speed: While standing near the edge of a farmer's field, Johnny watches a crop duster dust the farmer's field for insect control. Curious as to the plane's speed during each drop, Johnny attempts an estimate using the angle of rotation from one end of the field to the other, while standing 50 ft from one corner. Using a stopwatch he finds the plane makes each pass in 2.35 sec. If the angle of rotation was 83°, how fast (in miles per hour) is the plane flying as it applies the insecticide?



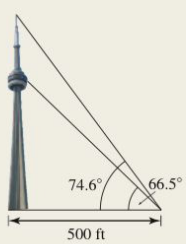
76. Train speed: While driving to their next gig, Josh and the boys get stuck in a line of cars at a railroad crossing as the gates go down. As the sleek, speedy express train approaches, Josh decides to pass the time estimating its speed. He spots a large oak tree beside the track some distance away, and figures the angle of rotation from the crossing to the tree is about 80°. If their car is 60 ft from the crossing and it takes the train 3 sec to reach the tree, how fast is the train moving in miles per hour?

77. Height of a climber: A local Outdoors Club has just hiked to the south rim of a large canyon, when they spot a climber attempting to scale the taller northern face. Knowing the distance between the sheer walls of the northern and southern faces of the canyon is approximately 175 yd, they attempt to compute the distance remaining for the climbers to reach the top of the northern rim. Using a homemade transit, they sight an angle of depression of 55° to the bottom of the north face, and angles of elevation of 24° and 30° to the climbers and top of the northern rim respectively. (a) How high is the southern rim of the canyon? (b) How high is the northern rim? (c) How much farther until the climber reaches the top?



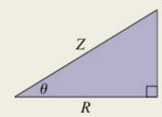
78. Observing wildlife: From her elevated observation post 300 ft away, a naturalist spots a troop of baboons high up in a tree. Using the small transit attached to her telescope, she finds the angle of depression to the bottom of this tree is 14°, while the angle of elevation to the top of the tree is 25°. The angle of elevation to the troop of baboons is 21°. Use this information to find (a) the height of the observation post, (b) the height of the baboons' tree, and (c) the height of the baboons above ground.

79. Angle of elevation: The tallest free-standing tower in the world is the CN Tower in Toronto, Canada. The tower includes a rotating restaurant high above the ground. From a distance of 500 ft the angle of elevation to the pinnacle of the tower is 74.6°. The angle of elevation to the restaurant from the same vantage point is 66.5°. How tall is the CN Tower? How far below the pinnacle of the tower is the restaurant located?



80. Angle of elevation: In August 2004, Taipei 101 captured the record as the world's tallest building, according to the Council on Tall Buildings and Urban Habitat [Source: www.ctbuh.org]. Measured at a point 108 m from its base, the angle of elevation to the top of the spire is 78°. From a distance of about 95 m, the angle of elevation to the top of the roof is also 78°. How tall is Taipei 101 from street level to the top of the spire? How tall is the spire itself?

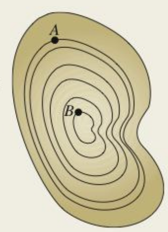
Alternating current: In AC (alternating current) applications, the relationship between measures known as the impedance (Z), resistance (R), and the phase angle (θ) can be demonstrated using a right triangle. Both the resistance and the impedance are measured in ohms (Ω).



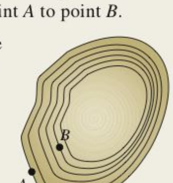
81. Find the impedance Z if the phase angle θ is 34°, and the resistance R is 320 Ω.

82. Find the phase angle θ if the impedance Z is 420 Ω, and the resistance R is 290 Ω.

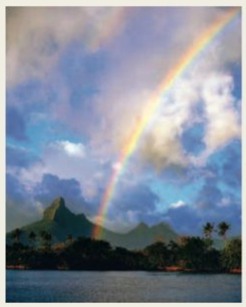
83. Contour maps: In the figure shown, the *contour interval* is 175 m (each concentric line represents an increase of 175 m in elevation), and the scale of horizontal distances is 1 cm = 500 m. (a) Find the vertical change from A to B (the increase in elevation); (b) use a proportion to find the horizontal change between points A and B if the measured distance on the map is 2.4 cm; and (c) draw the corresponding right triangle and use it to estimate the length of the trail up the mountain side that connects A and B, then use trig to compute the approximate angle of incline as the hiker climbs from point A to point B.



84. Contour maps: In the figure shown, the *contour interval* is 150 m (each concentric line represents an increase of 150 m in elevation), and the scale of horizontal distances is 1 cm = 250 m. (a) Find the vertical change from A to B (the increase in elevation); (b) use a proportion to find the horizontal change between points A and B if the measured distance on the map is 4.5 cm; and (c) draw the corresponding right triangle and use it to estimate the length of the trail up the mountain side that connects A and B, then use trig to compute the approximate angle of incline as the hiker climbs from point A to point B.

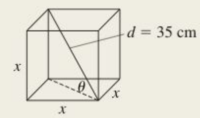


85. Height of a rainbow: While visiting the Lapahoehoe Memorial on the island of Hawaii, Bruce and Carma see a spectacularly vivid rainbow arching over the bay. Bruce speculates the rainbow is 500 ft away, while Carma estimates the angle of elevation to the highest point of the rainbow is about 42°. What was the approximate height of the rainbow?

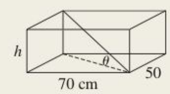


86. High-wire walking: As part of a circus act, a high-wire walker not only "walks the wire," she walks a wire that is *set at an incline of* 10° to the horizontal! If the length of the (inclined) wire is 25.39 m, (a) how much higher is the wire set at the destination pole than at the departure pole? (b) How far apart are the poles?

87. Diagonal of a cube: A cubical box has a diagonal measure of 35 cm. (a) Find the dimensions of the box and (b) the angle θ that the diagonal makes at the lower corner of the box.

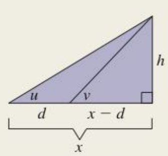


88. Diagonal of a rectangular parallelepiped: A rectangular box has a width of 50 cm and a length of 70 cm. (a) Find the height h that ensures the diagonal across the middle of the box will be 90 cm and (b) the angle θ that the diagonal makes at the lower corner of the box.

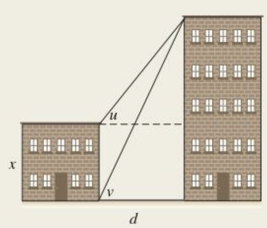


► EXTENDING THE CONCEPT

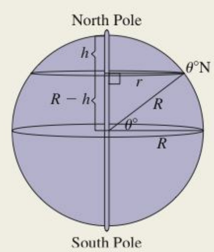
89. The formula $h = \frac{d}{\cot u - \cot v}$ can be used to calculate the height h of a building when distance x is unknown but distance d is known (see the diagram). Use the ratios for cot u and cot v to derive the formula (note x is "absent" from the formula).



90. Use the diagram given to derive a formula for the height h of the taller building in terms of the height x of the shorter building and the ratios for tan u and tan v. Then use the formula to find h given the shorter building is 75 m tall with $u = 40°$ and $v = 50°$.



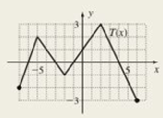
91. The radius of the Earth at the equator (0° N latitude) is approximately 3960 mi. Beijing, China, is located at 39.5° N latitude, 116° E longitude. Philadelphia, Pennsylvania, is located at the same latitude, but at 75° W longitude. (a) Use the diagram given and a cofunction relationship to find the radius r of the Earth (parallel to the equator) at this latitude; (b) use the arc length formula to compute the *shortest distance* between these two cities along this latitude; and (c) if the supersonic Concorde flew a direct flight between Beijing and Philadelphia along this latitude, approximate the flight time assuming a cruising speed of 1250 mph. Note: The shortest distance is actually traversed by heading northward, using the arc of a "great circle" that goes through these two cities.



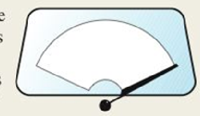
▶ MAINTAINING YOUR SKILLS

92. **(1.5)** Solve by factoring:
- **a.** $g^2 - 9g = 0$
- **b.** $g^2 - 9 = 0$
- **c.** $g^2 - 9g - 10 = 0$
- **d.** $g^2 + 9g - 10 = 0$
- **e.** $g^3 - 9g^2 - 10g + 90 = 0$

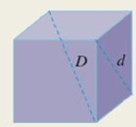
93. **(2.5)** For the graph of $T(x)$ given, (a) name the local maximums and minimums, (b) the zeroes of T, (c) intervals where $T(x)\downarrow$ and $T(x)\uparrow$, and (d) intervals where $T(x) > 0$ and $T(x) < 0$.



94. **(5.1)** The armature for the rear windshield wiper has a length of 24 in., with a rubber wiper blade that is 20 in. long. What area of my rear windshield is cleaned as the armature swings back-and-forth through an angle of 110°?



95. **(5.1)** The boxes used to ship some washing machines are perfect cubes with edges measuring 38 in. Use a special triangle to find the length of the diagonal d of one side, and the length of the interior diagonal D (through the middle of the box).



5.3 Trigonometry and the Coordinate Plane

Learning Objectives

In Section 5.3 you will learn how to:

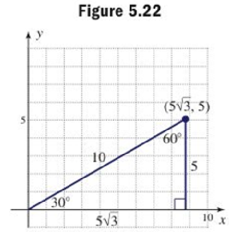
- ❑ **A.** Define the trigonometric functions using the coordinates of a point in QI
- ❑ **B.** Use reference angles to evaluate the trig functions for any angle
- ❑ **C.** Solve applications using the trig functions of any angle

This section tends to bridge the study of *static trigonometry* and the angles of a right triangle, with the study of *dynamic trigonometry* and the unit circle. This is accomplished by noting that the domain of the trig functions (unlike a triangle point of view) *need not be restricted to acute angles.* We'll soon see that the domain can be extended to include trig functions of *any* angle, a view that greatly facilitates our work in Chapter 7, where many applications involve angles greater than 90°.

A. Trigonometric Ratios and the Point $P(x, y)$

Regardless of where a right triangle is situated or how it is oriented, each trig function can be defined as a given ratio of sides with respect to a given angle. In this light, consider a 30-60-90 triangle placed in the first quadrant with the 30° angle at the origin and the longer side along the x-axis. From our previous review of similar triangles, the trig ratios will have the same value regardless of the triangle's size so for convenience, we'll use a hypotenuse of 10. This gives sides of 5, $5\sqrt{3}$, and 10, and from the diagram in Figure 5.22 we note the point (x, y) marking the vertex of the 60° angle has coordinates $(5\sqrt{3}, 5)$. Further, the diagram shows that sin 30°, cos 30°, and tan 30° can all be expressed in terms of these coordinates since $\frac{\text{opp}}{\text{hyp}} = \frac{5}{10} = \frac{y}{r}$(sine), $\frac{\text{adj}}{\text{hyp}} = \frac{5\sqrt{3}}{10} = \frac{x}{r}$(cosine), and $\frac{\text{opp}}{\text{adj}} = \frac{5}{5\sqrt{3}} = \frac{y}{x}$(tangent), where r is the length of the hypotenuse. Each result reduces to the more familiar values seen earlier: $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, and $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. This suggests we can define the six trig functions in terms of x, y, and r, where $r = \sqrt{x^2 + y^2}$.

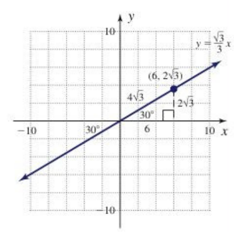
Figure 5.22



Consider that the slope of the line coincident with the hypotenuse is $\frac{\text{rise}}{\text{run}} = \frac{5}{5\sqrt{3}} = \frac{\sqrt{3}}{3}$, and since the line goes through the origin its equation must be $y = \frac{\sqrt{3}}{3}x$. Any point (x, y) on this line will be at the 60° vertex of a right triangle formed by drawing a perpendicular line from the point (x, y) to the x-axis. As Example 1 shows, we obtain the special values for sin 30°, cos 30°, and tan 30° regardless of the point chosen.

EXAMPLE 1 ► Evaluating Trig Functions Using *x*, *y*, and *r*

Pick an arbitrary point in QI that satisfies $y = \frac{\sqrt{3}}{3}x$, then draw the corresponding right triangle and evaluate sin 30°, cos 30°, and tan 30°.



Solution ► The coefficient of x has a denominator of 3, so we choose a multiple of 3 for convenience. For $x = 6$ we have $y = \frac{\sqrt{3}}{3}(6) = 2\sqrt{3}$. As seen in the figure, the point $(6, 2\sqrt{3})$ is on the line and at the vertex of the 60° angle. Evaluating the trig functions at 30°, we obtain:

$$\sin 30° = \frac{y}{r} = \frac{2\sqrt{3}}{4\sqrt{3}} = \frac{1}{2} \qquad \cos 30° = \frac{x}{r} = \frac{6}{4\sqrt{3}} = \frac{6\sqrt{3}}{4\sqrt{3}\sqrt{3}} = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{y}{x} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$

Now try Exercises 7 and 8 ►

In general, consider *any* two points (x, y) and (X, Y) on an arbitrary line $y = kx$, at corresponding distances r and R from the origin (Figure 5.23). Because the triangles formed are similar, we have $\frac{y}{x} = \frac{Y}{X}$, $\frac{x}{r} = \frac{X}{R}$, and so on, and we conclude that the value of the trig functions are indeed independent of the point chosen.

Figure 5.23

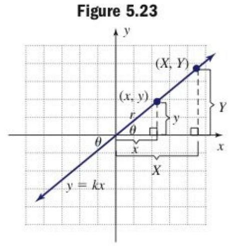
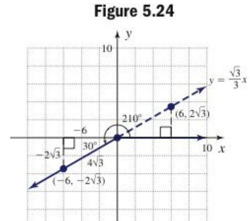


Figure 5.24



Viewing the trig functions in terms of x, y, and r produces significant results. In Figure 5.24, we note the line $y = \frac{\sqrt{3}}{3}x$ from Example 1 also extends into QIII, and *creates another* 30° *angle whose vertex is at the origin* (since vertical angles are equal). The sine, cosine, and tangent functions can still be evaluated for this angle, but in QIII both x and y are negative. If we consider the angle in QIII to be a positive rotation of 210° (180° + 30°), we can evaluate the trig functions using the values of x, y, and r from any point on the terminal side, since these are fixed by the 30° angle created and are the same as those in QI except for their sign:

$$\sin 210° = \frac{y}{r} = \frac{-2\sqrt{3}}{4\sqrt{3}} = -\frac{1}{2} \qquad \cos 210° = \frac{x}{r} = \frac{-6}{4\sqrt{3}} = -\frac{\sqrt{3}}{2} \qquad \tan 210° = \frac{y}{x} = \frac{-2\sqrt{3}}{-6} = \frac{\sqrt{3}}{3}$$

For *any* rotation θ and a point (x, y) on the terminal side, the distance r can be found using $r = \sqrt{x^2 + y^2}$ and the six trig functions likewise evaluated. Note that evaluating them correctly depends on the quadrant of the terminal side, since this will dictate the signs for x and y. Students are strongly encouraged to make these quadrant and sign observations the *first step* in any solution process. In summary, we have

Trigonometric Functions of Any Angle

Given $P(x, y)$ is any point on the terminal side of angle θ in standard position, with $r = \sqrt{x^2 + y^2}$ $(r > 0)$ the distance from the origin to (x, y). The six trigonometric functions of θ are

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x},\ x \neq 0$$

$$\csc\theta = \frac{r}{y},\ y \neq 0 \qquad \sec\theta = \frac{r}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

EXAMPLE 2 ► Evaluating Trig Functions Given the Terminal Side is on $y = mx$

Given that $P(x, y)$ is a point on the terminal side of angle θ in standard position, find the value of $\sin\theta$ and $\cos\theta$, if

a. The terminal side is in QII and coincident with the line $y = -\frac{12}{5}x$,
b. The terminal side is in QIV and coincident with the line $y = -\frac{12}{5}x$.

Solution ► **a.** Select any convenient point in QII that satisfies this equation. We select $x = -5$ since x is negative in QII, which gives $y = 12$ and the point $(-5, 12)$. Solving for r gives $r = \sqrt{(-5)^2 + (12)^2} = 13$. The ratios are

$$\sin\theta = \frac{y}{r} = \frac{12}{13} \qquad \cos\theta = \frac{x}{r} = \frac{-5}{13}$$

b. In QIV we select $x = 10$ since x is positive in QIV, giving $y = -24$ and the point $(10, -24)$. Solving for r gives $r = \sqrt{(10)^2 + (-24)^2} = 26$. The ratios are

$$\sin\theta = \frac{y}{r} = \frac{-24}{26} = -\frac{12}{13} \qquad \cos\theta = \frac{x}{r} = \frac{10}{26} = \frac{5}{13}$$

Now try Exercises 9 through 12 ►

In Example 2, note the ratios are the same in QII and QIV *except for their sign*. We will soon use this observation to great advantage.

EXAMPLE 3 ▶ Evaluating Trig Functions Given a Point P

Find the value of the six trigonometric functions given $P(-5, 5)$ is on the terminal side of angle θ in standard position.

Solution ▶ For $P(-5, 5)$ we have $x < 0$ and $y > 0$ so the terminal side is in QII. Solving for r yields $r = \sqrt{(-5)^2 + (5)^2} = \sqrt{50} = 5\sqrt{2}$. For $x = -5$, $y = 5$, and $r = 5\sqrt{2}$, we obtain

$$\sin\theta = \frac{y}{r} = \frac{5}{5\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos\theta = \frac{x}{r} = \frac{-5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \tan\theta = \frac{y}{x} = \frac{5}{-5} = -1$$

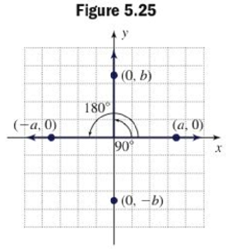
The remaining functions can be evaluated using reciprocals.

$$\csc\theta = \frac{2}{\sqrt{2}} = \sqrt{2} \qquad \sec\theta = -\frac{2}{\sqrt{2}} = -\sqrt{2} \qquad \cot\theta = -1$$

Note the connection between these results and the special values for $\theta = 45°$.

Now try Exercises 13 through 28 ▶

Figure 5.25



Now that we've defined the trig functions in terms of ratios involving x, y, and r, the question arises as to their value at the quadrantal angles. For 90° and 270°, any point on the terminal side of the angle has an *x-value* of zero, meaning tan 90°, sec 90°, tan 270°, and sec 270° are all undefined since $x = 0$ is in the denominator. Similarly, at 180° and 360°, the y-value of any point on the terminal side is zero, so cot 180°, csc 180°, cot 360°, and csc 360° are likewise undefined (see Figure 5.25).

EXAMPLE 4 ▶ Evaluating the Trig Functions for $\theta = 90°k$, k an Integer

Evaluate the six trig functions for $\theta = 270°$.

Solution ▶ Here, θ is the quadrantal angle whose terminal side separates QIII and QIV. Since the evaluation is independent of the point chosen on this side, we choose $(0, -1)$ for convenience, giving $r = 1$. For $x = 0$, $y = -1$, and $r = 1$ we obtain

$$\sin\theta = \frac{-1}{1} = -1 \qquad \cos\theta = \frac{0}{-1} = 0 \qquad \tan\theta = \frac{-1}{0}\ (\textit{undefined})$$

The remaining ratios can be evaluated using reciprocals.

$$\csc\theta = -1 \qquad \sec\theta = \frac{-1}{0}\ (\textit{undefined}) \qquad \cot\theta = \frac{0}{-1} = 0$$

Now try Exercises 29 and 30 ▶

Results for the quadrantal angles are summarized in Table 5.3.

Table 5.3

θ	$\sin\theta = \frac{y}{r}$	$\cos\theta = \frac{x}{r}$	$\tan\theta = \frac{y}{x}$	$\csc\theta = \frac{r}{y}$	$\sec\theta = \frac{r}{x}$	$\cot\theta = \frac{x}{y}$
$0° \to (1, 0)$	0	1	0	undefined	1	undefined
$90° \to (0, 1)$	1	0	undefined	1	undefined	0
$180° \to (-1, 0)$	0	−1	0	undefined	−1	undefined
$270° \to (0, -1)$	−1	0	undefined	−1	undefined	0

A. You've just learned how to define the trigonometric functions using the coordinates of a point in QI

B. Reference Angles and the Trig Functions of Any Angle

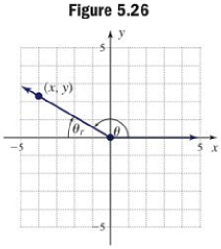
By defining a **reference angle** θ_r, we can evaluate the trigonometric functions given a point (x, y) on the terminal side of any angle.

Reference Angles

For any angle θ in standard position, the acute angle θ_r formed by the terminal side and the x-axis is called the *reference angle* for θ.

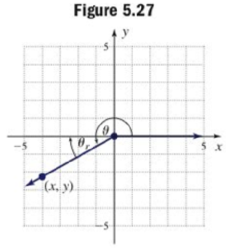
Several examples of the reference angle concept are illustrated in Figures 5.26 through 5.29 for θ in degrees, $\theta > 0$.

Figure 5.26



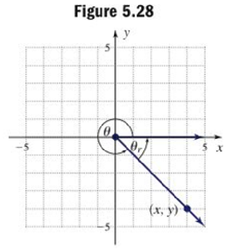
$90° < \theta < 180°$
$\theta_r = 180° - \theta$

Figure 5.27



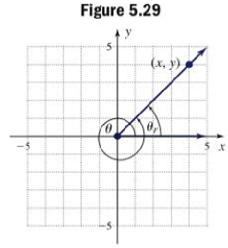
$180° < \theta < 270°$
$\theta_r = \theta - 180°$

Figure 5.28



$270° < \theta < 360°$
$\theta_r = 360° - \theta$

Figure 5.29



$360° < \theta < 450°$
$\theta_r = \theta - 360°$

EXAMPLE 5 ▶ Finding Reference Angles

Determine the reference angle for

a. 315° **b.** 150° **c.** −121° **d.** 425°

Solution ▶ Begin by mentally visualizing each angle and the quadrant where it terminates.

a. 315° is a QIV angle:
$\theta_r = 360° - 315° = 45°$

b. 150° is a QII angle:
$\theta_r = 180° - 150° = 30°$

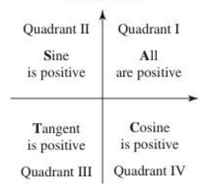
c. −121° is a QIII angle:
$\theta_r = 180° - 121° = 59°$

d. 425° is a QI angle:
$\theta_r = 425° - 360° = 65°$

Now try Exercises 31 through 42 ▶

The reference angles from Examples 5(a) and 5(b) were special angles, which means we automatically know the absolute value of the trig ratios using θ_r. The best way to remember the signs of the trig functions is to keep in mind that sine is associated with y, cosine with x, and tangent with both x and y (r is always positive). In addition, there are several mnemonic devices (memory tools) to assist you. One is to use the first letter of the function that is positive in each quadrant and create a catchy acronym. For instance **ASTC** → **A**ll **S**tudents **T**ake **C**lasses (see Figure 5.30). Note that a trig function and its reciprocal function will always have the same sign.

Figure 5.30

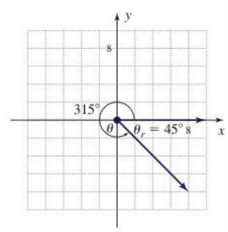


EXAMPLE 6 ► **Evaluating Trig Functions Using θ_r**

Use a reference angle to evaluate $\sin\theta$, $\cos\theta$, and $\tan\theta$ for $\theta = 315°$.

Solution ► The terminal side is in QIV where x is positive and y is negative. With $\theta_r = 45°$, we have:

$$\sin 315° = -\frac{\sqrt{2}}{2} \qquad \cos 315° = \frac{\sqrt{2}}{2}$$
$$\tan 315° = -1$$



Now try Exercises 43 through 54 ►

EXAMPLE 7 ► **Finding Function Values Using a Quadrant and Sign Analysis**

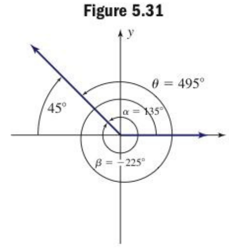
Given $\sin\theta = \frac{5}{13}$ and $\cos\theta < 0$, find the value of the other ratios.

Solution ► Always begin with a quadrant and sign analysis: $\sin\theta$ is positive in QI and QII, while $\cos\theta$ is negative in QII and QIII. Both conditions are satisfied in QII only. For $r = 13$ and $y = 5$, the Pythagorean theorem shows $x = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$. With θ in QII this gives $\cos\theta = \frac{-12}{13}$ and $\tan\theta = \frac{5}{-12}$. The reciprocal values are $\csc\theta = \frac{13}{5}$, $\sec\theta = \frac{13}{-12}$, and $\cot\theta = \frac{-12}{5}$.

Now try Exercises 55 through 62 ►

In our everyday experience, there are many actions and activities where angles greater than or equal to 360° are applied. Some common instances are a professional basketball player who "does a three-sixty" (360°) while going to the hoop, a diver who completes a "two-and-a-half" (900°) off the high board, and a skater who executes a perfect triple axel ($3\frac{1}{2}$ turns or 1260°). As these examples suggest, angles greater than 360° must still terminate on a quadrantal axis, or in one of the four quadrants, allowing a reference angle to be found and the functions to be evaluated for any angle *regardless of size.* Figure 5.31 illustrates that $\alpha = 135°$, $\beta = -225°$, and $\theta = 495°$ are all coterminal, with *each having a reference angle of* 45°.

Figure 5.31



EXAMPLE 8 ▶ Evaluating Trig Functions of Any Angle

Evaluate sin 135°, cos (−225°), and tan 495°.

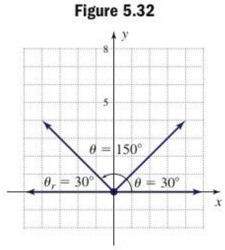
Solution ▶ The angles are coterminal and terminate in QII, where $x < 0$ and $y > 0$. With $\theta_r = 45°$ we have $\sin 135° = \frac{\sqrt{2}}{2}$, $\cos(-225°) = -\frac{\sqrt{2}}{2}$, and $\tan 495° = -1$.

Now try Exercises 63 through 74 ▶

B. You've just learned how to use reference angles to evaluate the trig functions for any angle

Since 360° is one full rotation, all angles $\theta + 360°k$ will be coterminal for any integer k. For angles with a very large magnitude, we can find the quadrant of the terminal side by subtracting as many integer multiples of 360° as needed from the angle. For $\alpha = 1908°$, $\frac{1908}{360} = 5.3$ and $1908 - 360(5) = 108°$. This angle is in QII with $\theta_r = 72°$. **See Exercises 75 through 90.**

Figure 5.32



C. Applications of the Trig Functions of Any Angle

One of the most basic uses of coterminal angles is determining all values of θ that satisfy a stated relationship. For example, by now you are aware that if $\sin\theta = \frac{1}{2}$ (positive one-half), then $\theta = 30°$ or $\theta = 150°$ (see Figure 5.32). But this is also true for all angles coterminal with these two, and we would write the solutions as $\theta = 30° + 360°k$ and $\theta = 150° + 360°k$ for all integers k.

EXAMPLE 9 ▶ Finding All Angles that Satisfy a Given Equation

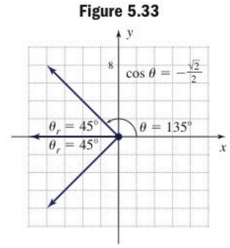
Find all angles satisfying the relationship given. Answer in degrees.

a. $\cos\theta = -\frac{\sqrt{2}}{2}$ **b.** $\tan\theta = -1.3764$

Solution ▶

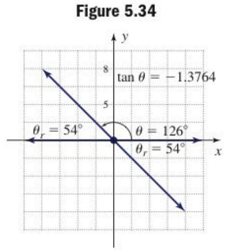
a. Cosine is negative in QII and QIII. Recognizing $\cos 45° = \frac{\sqrt{2}}{2}$, we reason $\theta_r = 45°$ and two solutions are $\theta = 135°$ from QII and $\theta = 225°$ from QIII. For all values of θ satisfying the relationship, we have $\theta = 135° + 360°k$ and $\theta = 225° + 360°k$. See Figure 5.33.

Figure 5.33



b. Tangent is negative in QII and QIV. For -1.3764 we find θ_r using a calculator: 2nd TAN ($\tan^{-1}$) -1.3764 ENTER shows $\tan^{-1}(-1.3764) \approx -54$, so $\theta_r = 54°$. Two solutions are $\theta = 180° - 54° = 126°$ from QII, and in QIV $\theta = 360° - 54° = 306°$. The result is $\theta = 126° + 360°k$ and $\theta = 306° + 360°k$. Note these can be combined into the single statement $\theta = 126° + 180°k$. See Figure 5.34.

Figure 5.34

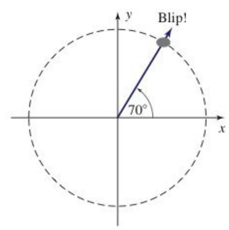


Now try Exercises 93 through 100 ▶

We close this section with an additional application of the concepts related to trigonometric functions of any angle.

EXAMPLE 10 ► Applications of Coterminal Angles: Location on Radar

A radar operator calls the captain over to her screen saying, "Sir, we have an unidentified aircraft heading 20° (20° east of due north or a standard 70° rotation). I think it's a UFO." The captain asks, "What makes you think so?" To which the sailor replies, "Because it's at 5000 ft and not moving!" Name all angles for which the UFO causes a "blip" to occur on the radar screen.



Solution ► Since radar typically sweeps out a 360° angle, a blip will occur on the screen for all angles $\theta = 70° + 360°k$, where k is an integer.

C. You've just learned how to solve applications using the trig functions of any angle

Now try Exercises 101 through 106 ►

5.3 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. An angle is in standard position if its vertex is at the ________ and the initial side is along the ________.

2. A(n) ________ angle is one where the ________ side is coincident with one of the coordinate axes.

3. Angles formed by a counterclockwise rotation are ________ angles. Angles formed by a ________ rotation are negative angles.

4. For any angle θ, its reference angle θ_r is the positive ________ angle formed by the ________ side and the nearest x-axis.

5. Discuss the similarities and differences between the trigonometry of right triangles and the trigonometry of *any* angle.

6. Let $T(x)$ represent any one of the six basic trig functions. Explain why the equation $T(x) = k$ will always have exactly two solutions in $[0, 2\pi)$ if x is not a quadrantal angle.

► DEVELOPING YOUR SKILLS

7. Draw a 30-60-90 triangle with the 60° angle at the origin and the short side along the positive x-axis. Determine the slope and equation of the line coincident with the hypotenuse, then pick any point on this line and evaluate sin 60°, cos 60°, and tan 60°. Comment on what you notice.

8. Draw a 45-45-90 triangle with a 45° angle at the origin and one side along the positive x-axis. Determine the slope and equation of the line coincident with the hypotenuse, then pick any point on this line and evaluate sin 45°, cos 45°, and tan 45. Comment on what you notice.

Graph each linear equation and state the quadrants it traverses. Then pick one point on the line from each quadrant and evaluate the functions sin θ, cos θ and tan θ using these points.

9. $y = \frac{3}{4}x$ **10.** $y = \frac{5}{12}x$

11. $y = -\frac{\sqrt{3}}{3}x$ **12.** $y = -\frac{\sqrt{3}}{2}x$

Find the value of the six trigonometric functions given $P(x, y)$ is on the terminal side of angle θ, with θ in standard position.

13. $(8, 15)$ **14.** $(7, 24)$

15. $(-20, 21)$ **16.** $(-3, -1)$

17. $(7.5, -7.5)$ **18.** $(9, -9)$

19. $(4\sqrt{3}, 4)$ **20.** $(-6, 6\sqrt{3})$

21. $(2, 8)$ **22.** $(6, -15)$

23. $(-3.75, -2.5)$ **24.** $(6.75, 9)$

25. $\left(-\frac{5}{9}, \frac{2}{3}\right)$ **26.** $\left(\frac{3}{4}, -\frac{7}{16}\right)$

27. $\left(\frac{1}{4}, -\frac{\sqrt{5}}{2}\right)$ **28.** $\left(-\frac{\sqrt{3}}{5}, \frac{22}{25}\right)$

29. Evaluate the six trig functions in terms of x, y, and r for $\theta = 90°$.

30. Evaluate the six trig functions in terms of x, y, and r for $\theta = 180°$.

Name the reference angle θ_r for the angle θ given.

31. $\theta = 120°$ **32.** $\theta = 210°$

33. $\theta = 135°$ **34.** $\theta = 315°$

35. $\theta = -45°$ **36.** $\theta = -240°$

37. $\theta = 112°$ **38.** $\theta = 179°$

39. $\theta = 500°$ **40.** $\theta = 750°$

41. $\theta = -168.4°$ **42.** $\theta = -328.2°$

State the quadrant of the terminal side of θ, using the information given.

43. $\sin\theta > 0$, $\cos\theta < 0$

44. $\cos\theta < 0$, $\tan\theta < 0$

45. $\tan\theta < 0$, $\sin\theta > 0$

46. $\sec\theta > 0$, $\tan\theta > 0$

Find the exact value of $\sin\theta$, $\cos\theta$, and $\tan\theta$ using reference angles.

47. $\theta = 330°$ **48.** $\theta = 390°$

49. $\theta = -45°$ **50.** $\theta = -120°$

51. $\theta = 240°$ **52.** $\theta = 315°$

53. $\theta = -150°$ **54.** $\theta = -210°$

For the information given, find the values of x, y, and r. Clearly indicate the quadrant of the terminal side of θ, then state the values of the six trig functions of θ.

55. $\cos\theta = \frac{4}{5}$ and $\sin\theta < 0$

56. $\tan\theta = -\frac{12}{5}$ and $\cos\theta > 0$

57. $\csc\theta = -\frac{37}{35}$ and $\tan\theta > 0$

58. $\sin\theta = -\frac{20}{29}$ and $\cot\theta < 0$

59. $\csc\theta = 3$ and $\cos\theta > 0$

60. $\csc\theta = -2$ and $\cos\theta > 0$

61. $\sin\theta = -\frac{7}{8}$ and $\sec\theta < 0$

62. $\cos\theta = \frac{5}{12}$ and $\sin\theta < 0$

Find two positive and two negative angles that are coterminal with the angle given. Answers will vary.

63. $52°$ **64.** $12°$

65. $87.5°$ **66.** $22.8°$

67. $225°$ **68.** $175°$

69. $-107°$ **70.** $-215°$

Evaluate in exact form as indicated.

71. $\sin 120°$, $\cos(-240°)$, $\tan 480°$

72. $\sin 225°$, $\cos 585°$, $\tan(-495°)$

73. $\sin(-30°)$, $\cos(-390°)$, $\tan 690°$

74. $\sin 210°$, $\cos 570°$, $\tan(-150°)$

Find the exact value of $\sin\theta$, $\cos\theta$, and $\tan\theta$ using reference angles.

75. $\theta = 600°$ **76.** $\theta = 480°$

77. $\theta = -840°$ **78.** $\theta = -930°$

79. $\theta = 570°$ **80.** $\theta = 495°$

81. $\theta = -1230°$ **82.** $\theta = 3270°$

For each exercise, state the quadrant of the terminal side and the sign of the function in that quadrant. Then evaluate the expression using a calculator. Round to four decimal places.

83. $\sin 719°$ **84.** $\cos 528°$

85. $\tan(-419°)$ **86.** $\sec(-621°)$

87. $\csc 681°$ **88.** $\tan 995°$

89. $\cos 805°$ **90.** $\sin 772°$

▶ WORKING WITH FORMULAS

91. The area of a parallelogram: $A = ab \sin\theta$

The area of a parallelogram is given by the formula shown, where a and b are the lengths of the sides and θ is the angle between them. Use the formula to complete the following: (a) find the area of a parallelogram with sides $a = 9$ and $b = 21$ given $\theta = 50°$. (b) What is the smallest integer value of θ where the area is greater than 150 units^2? (c) State what happens when $\theta = 90°$. (d) How can you find the area of a triangle using this formula?

92. The angle between two intersecting lines: $\tan\theta = \dfrac{m_2 - m_1}{1 + m_2 m_1}$

Given line 1 and line 2 with slopes m_1 and m_2, respectively, the angle between the two lines is given by the formula shown. Find the angle θ if the equation of line 1 is $y_1 = \frac{3}{4}x + 2$ and line 2 has equation $y_2 = -\frac{2}{3}x + 5$.

▶ APPLICATIONS

Find all angles satisfying the stated relationship. For standard angles, express your answer in exact form. For nonstandard values, use a calculator and round function values to tenths.

93. $\cos\theta = \dfrac{1}{2}$ **94.** $\sin\theta = \dfrac{\sqrt{2}}{2}$

95. $\sin\theta = -\dfrac{\sqrt{3}}{2}$ **96.** $\tan\theta = -\dfrac{\sqrt{3}}{1}$

97. $\sin\theta = 0.8754$ **98.** $\cos\theta = 0.2378$

99. $\tan\theta = -2.3512$ **100.** $\cos\theta = -0.0562$

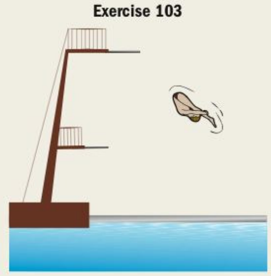
101. Nonacute angles: At a recent carnival, one of the games on the midway was played using a large spinner that turns clockwise. On Jorge's spin the number 25 began at the 12 o'clock (top/center) position, returned to this position five times during the spin and stopped at the 3 o'clock position. What angle θ did the spinner spin through? Name all angles that are coterminal with θ.

Exercise 101

102. Nonacute angles: One of the four blades on a ceiling fan has a decal on it and begins at a designated "12 o'clock" position. Turning the switch on and then immediately off, causes the blade to make over three complete, counterclockwise rotations, with the blade stopping at the 8 o'clock position. What angle θ did the blade turn through? Name all angles that are coterminal with θ.

103. High dives: As part of a diving competition, David executes a perfect reverse two-and-a-half flip. Does he enter the water feet first or head first? Through what angle did he turn from takeoff until the moment he entered the water?

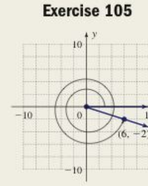
Exercise 103



104. Gymnastics: While working out on a trampoline, Charlene does three complete, forward flips and then belly-flops on the trampoline before returning to the upright position. What angle did she turn through from the start of this maneuver to the moment she belly-flops?

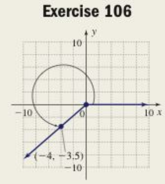
105. Spiral of Archimedes: The graph shown is called the spiral of Archimedes. Through what angle θ has the spiral turned, given the spiral terminates at $(6, -2)$ as indicated?

Exercise 105



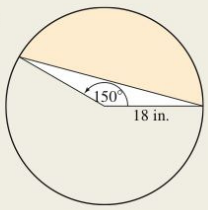
106. Involute of a circle: The graph shown is called the involute of a circle. Through what angle θ has the involute turned, given the graph terminates at $(-4, -3.5)$ as indicated?

Exercise 106

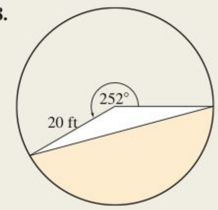


Area bounded by chord and circumference: Find the area of the shaded region, rounded to the nearest 100th. Note the area of a triangle is one-half the area of a parallelogram (see Exercise 91).

107.



108.



▶ EXTENDING THE CONCEPT

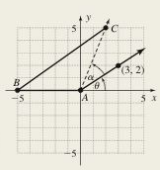
109. In an elementary study of trigonometry, the hands of a clock are often studied because of the angle relationship that exists between the hands. For example, at 3 o'clock, the angle between the two hands is a right angle and measures 90°.

a. What is the angle between the two hands at 1 o'clock? 2 o'clock? Explain why.

b. What is the angle between the two hands at 6:30? 7:00? 7:30? Explain why.

c. Name four times at which the hands will form a 45° angle.

110. In the diagram shown, the indicated ray is of arbitrary length. (a) Through what additional angle α would the ray have to be rotated to create triangle *ABC*? (b) What will be the length of side *AC* once the triangle is complete?



111. Referring to Exercise 102, suppose the fan blade had a radius of 20 in. and is turning at a rate of 12 revolutions per second. (a) Find the angle the blade turns through in 3 sec. (b) Find the circumference of the circle traced out by the tip of the blade. (c) Find the total distance traveled by the blade tip in 10 sec. (d) Find the speed, in miles per hour, that the tip of the blade is traveling.

▶ MAINTAINING YOUR SKILLS

112. (5.1) For emissions testing, automobiles are held stationary while a heavy roller installed in the floor allows the wheels to turn freely. If the large wheels of a customized pickup have a radius of 18 in. and are turning at 300 revolutions per minute, what speed is the odometer of the truck reading in miles per hour?

113. (5.2) Jazon is standing 117 ft from the base of the Washington Monument in Washington, D.C. If his eyes are 5 ft above level ground and he must hold his head at a 78° angle from horizontal to see the top of the monument (the angle of elevation of 78°), estimate the height of the monument. Answer to the nearest tenth of a foot.

114. (4.4) Solve for t. Answer in both exact and approximate form:

$$-250 = -150e^{-0.05t} - 202.$$

115. (2.3) Find the equation of the line perpendicular to $4x - 5y = 15$ that contains the point $(4, -3)$.

5.4 Unit Circles and the Trigonometry of Real Numbers

Learning Objectives

In Section 5.4 you will learn how to:

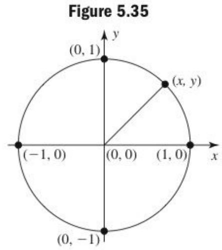
- **A.** Locate points on a unit circle and use symmetry to locate other points
- **B.** Use special triangles to find points on a unit circle and locate other points using symmetry
- **C.** Define the six trig functions in terms of a point on the unit circle
- **D.** Define the six trig functions in terms of a real number t
- **E.** Find the real number t corresponding to given values of sin t, cos t, and tan t

In this section, we introduce the **trigonometry of real numbers,** a view of trigonometry that can exist free of its historical roots in a study of right triangles. In fact, the ultimate value of these functions is not in their classical study, but in the implications and applications that follow from understanding them as functions of a real number, rather than simply as functions of a given angle.

A. The Unit Circle

A circle is defined as the set of all points in a plane that are a *fixed distance* called the **radius** from a *fixed point* called the **center.** Since the definition involves distance, we can construct the general equation of a circle using the distance formula. Assume the center has coordinates (h, k) and let (x, y) represent any point on the graph. Since the distance between these points is the radius r, the distance formula yields $\sqrt{(x-h)^2 + (y-k)^2} = r$. Squaring both sides gives $(x-h)^2 + (y-k)^2 = r^2$. For central circles both h and k are zero, and the result is the equation for a **central circle** of radius r: $x^2 + y^2 = r^2 (r > 0)$. The **unit circle** is defined as a central circle with radius 1 unit: $x^2 + y^2 = 1$. As such, the figure can easily be graphed by drawing a circle through the four **quadrantal points** $(1, 0)$, $(-1, 0)$, $(0, 1)$, and $(0, -1)$ as in Figure 5.35. To find other points on the circle, we simply select any value of x, where $|x| < 1$, then substitute and solve for y; or any value of y, where $|y| < 1$, then solve for x.

Figure 5.35



EXAMPLE 1 ► **Finding Points on a Unit Circle**

Find a point on the unit circle given $y = \frac{1}{2}$ with (x, y) in QII.

Solution ► Using the equation of a unit circle, we have

$$x^2 + y^2 = 1 \quad \text{unit circle equation}$$

$$x^2 + \left(\frac{1}{2}\right)^2 = 1 \quad \text{substitute } \frac{1}{2} \text{ for } y$$

$$x^2 + \frac{1}{4} = 1 \quad \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$x^2 = \frac{3}{4} \quad \text{subtract } \frac{1}{4}$$

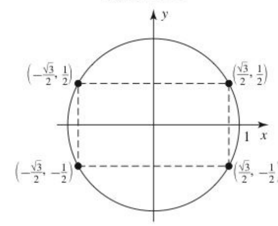
$$x = \pm\frac{\sqrt{3}}{2} \quad \text{result}$$

With (x, y) in QII, we choose $x = -\frac{\sqrt{3}}{2}$. The point is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Now try Exercises 7 through 18 ►

Additional points on the unit circle can be found using symmetry. The simplest examples come from the quadrantal points, where $(1, 0)$ and $(-1, 0)$ are on opposite sides of the y-axis, and $(0, 1)$ and $(0, -1)$ are on opposite sides of the x-axis. In general, if a and b are positive real numbers and (a, b) is on the unit circle, then $(-a, b)$, $(a, -b)$, and $(-a, -b)$ are also on the circle *because a circle is symmetric to both axes* *and the origin*! For the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ from Example 1, three other points are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ in QIII, $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ in QIV, and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ in QI. See Figure 5.36.

Figure 5.36

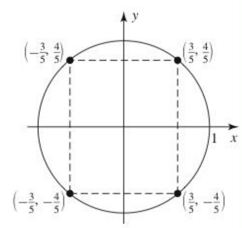


EXAMPLE 2 ▶ Using Symmetry to Locate Points on a Unit Circle

Name the quadrant containing $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ and verify it's on a unit circle. Then use symmetry to find three other points on the circle.

Solution ▶ Since both coordinates are negative, $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ is in QIII. Substituting into the equation for a unit circle yields

$$x^2 + y^2 = 1 \quad \text{unit circle equation}$$
$$\left(\frac{-3}{5}\right)^2 + \left(\frac{-4}{5}\right)^2 \stackrel{?}{=} 1 \quad \text{substitute } \tfrac{-3}{5} \text{ for } x \text{ and } \tfrac{-4}{5} \text{ for } y$$
$$\frac{9}{25} + \frac{16}{25} \stackrel{?}{=} 1 \quad \text{simplify}$$
$$\frac{25}{25} = 1 \quad \text{result checks}$$



Since $\left(\frac{-3}{5}, \frac{-4}{5}\right)$ is on the unit circle, $\left(\frac{3}{5}, \frac{-4}{5}\right)$, $\left(\frac{-3}{5}, \frac{4}{5}\right)$, and $\left(\frac{3}{5}, \frac{4}{5}\right)$ are also on the circle due to symmetry (see figure).

Now try Exercises 19 through 26 ▶

☑ **A.** You've just learned how to locate points on a unit circle and use symmetry to locate other points

B. Special Triangles and the Unit Circle

The special triangles from Section 5.1 can also be used to find points on a unit circle. As usually written, the triangles state a proportional relationship between their sides after assigning a value of 1 to the shortest side. However, precisely due to this proportional relationship, *we can divide all sides by the length of the hypotenuse,* giving *it* a length of 1 unit (see Figures 5.37 and 5.38).

Figure 5.37

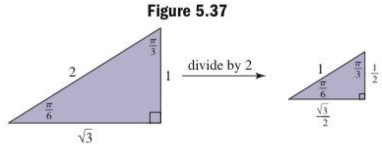
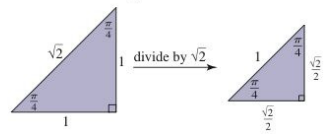


Figure 5.38



We then place the triangle within the unit circle, and reflect it from quadrant to quadrant to find additional points. We use *the sides* of the triangle to determine the absolute value of each coordinate, and *the quadrant* to give each coordinate the appropriate sign. Note the angles in these special triangles are now expressed in radians.

EXAMPLE 3 ▶ Using a Special Triangle and Symmetry to Locate Points on a Unit Circle

Use the $\frac{\pi}{4}:\frac{\pi}{4}:\frac{\pi}{2}$ triangle from Figure 5.38 to find four points on the unit circle.

Solution ▶ Begin by superimposing the triangle in QI, noting it gives the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ shown in Figure 5.39. By reflecting the triangle into QII, we find the additional point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on this circle. Realizing we can simply apply the circle's remaining symmetries, we obtain the two additional points $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ shown in Figure 5.40.

Figure 5.39

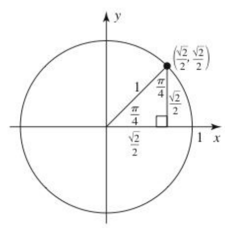
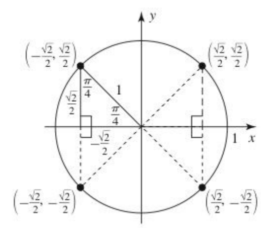
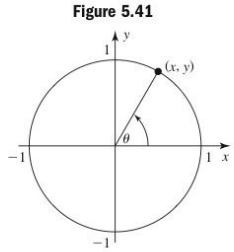


Figure 5.40



Now try Exercises 27 and 28 ▶

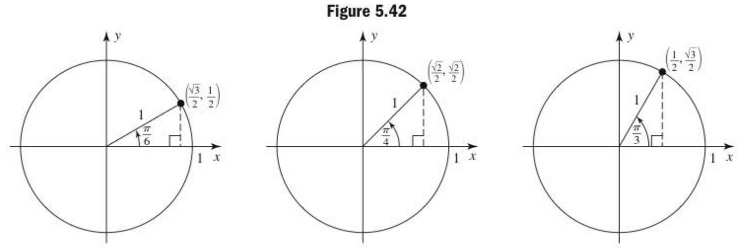
Figure 5.41



Applying the same idea to a $\frac{\pi}{6}:\frac{\pi}{3}:\frac{\pi}{2}$ triangle would give the points $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, *the same points we found in Example 1.*

When a central angle θ is viewed as a rotation, each rotation can be associated with a unique point (x, y) on the terminal side, where it intersects the unit circle (see Figure 5.41). For the quadrantal angles $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π, we associate the points $(0, 1)$, $(-1, 0)$, $(0, -1)$, and $(0, 0)$, respectively. When this rotation results in a special angle θ, the association can be found using a special triangle in a manner similar to Example 3. Figure 5.42 shows we associate the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ with $\theta = \frac{\pi}{6}$,

Figure 5.42

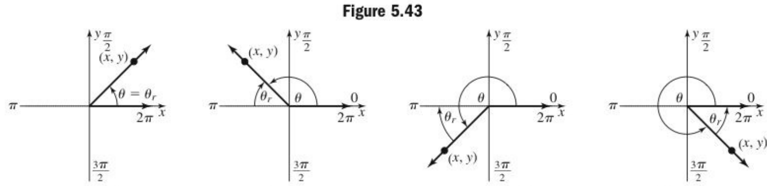


$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ with $\theta = \frac{\pi}{4}$, and by reorienting the $\frac{\pi}{6}:\frac{\pi}{3}:\frac{\pi}{2}$ triangle, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is associated with a rotation of $\theta = \frac{\pi}{3}$. For standard rotations from $\theta = 0$ to $\theta = \frac{\pi}{2}$ we have the following:

Rotation θ	**0**	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Associated point (x, y)	(0, 0)	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	(0, 1)

Each non-quadrantal point gives rise to three others using the symmetry of the circle.
Recall that for any angle θ in standard position, the acute angle θ_r formed by the terminal side and the x-axis is called the *reference angle* for θ. Several examples of reference angles are shown in Figure 5.43 for θ in radians, $\theta > 0$.

Figure 5.43



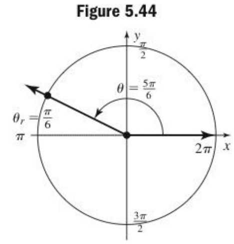
Due to the symmetries of the circle, reference angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ serve to fix the absolute value of the coordinates for x and y, and we simply *use the appropriate sign for each coordinate* (r is always positive). As before, this depends solely on the quadrant of the terminal side.

EXAMPLE 4 ► **Finding Points on a Unit Circle Associated with a Rotation θ**

Determine the reference angle for each rotation given, then find the associated point (x, y) on the unit circle.

a. $\theta = \frac{5\pi}{6}$ **b.** $\theta = \frac{4\pi}{3}$ **c.** $\theta = \frac{7\pi}{4}$

Figure 5.44



Solution ► **a.** A rotation of $\frac{5\pi}{6}$ terminates in QII:
$\theta_r = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$. The associated point is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ since $x < 0$ in QII. See Figure 5.44.

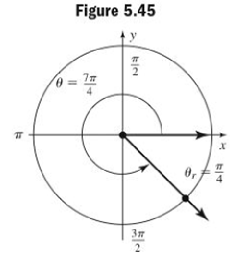
b. A rotation of $\frac{4\pi}{3}$ terminates in QIII:

$\theta_r = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$. The associated point is $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ since $x < 0$ and $y < 0$ in QIII.

c. A rotation of $\frac{7\pi}{4}$ terminates in QIV:

$\theta_r = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$. The associated point is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ since $y < 0$ in QIV. See Figure 5.45.

Figure 5.45



B. You've just learned how to use special triangles to find points on a unit circle and locate other points using symmetry

Now try Exercises 29 through 36 ►

C. Trigonometric Functions and Points on the Unit Circle

We can now define the six trigonometric functions in terms of a point (x, y) on the unit circle, with the use of right triangles fading from view. For this reason they are sometimes called the **circular functions.**

The Circular Functions

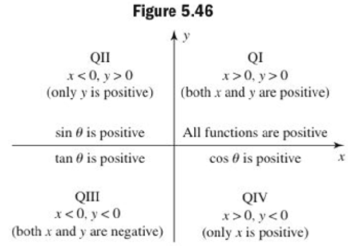
For any rotation θ and point $P(x, y)$ on the unit circle associated with θ,

$\cos\theta = x$ $\sin\theta = y$ $\tan\theta = \frac{y}{x}; x \neq 0$

$\sec\theta = \frac{1}{x}; x \neq 0$ $\csc\theta = \frac{1}{y}; y \neq 0$ $\cot\theta = \frac{x}{y}; y \neq 0$

Note that once $\sin\theta$, $\cos\theta$, and $\tan\theta$ are known, the values of $\csc\theta$, $\sec\theta$, and $\cot\theta$ follow automatically since a number and its reciprocal always have the same sign. See Figure 5.46.

Figure 5.46

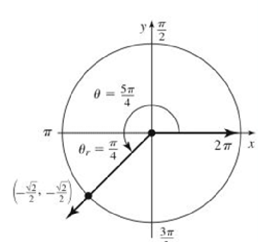


EXAMPLE 5 ► Evaluating Trig Functions for a Rotation θ

Evaluate the six trig functions for $\theta = \frac{5\pi}{4}$.

Solution ► A rotation of $\frac{5\pi}{4}$ terminates in QIII, so

$\theta_r = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$. The associated point is $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ since $x < 0$ and $y < 0$ in QIII.



This yields

$$\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} \qquad \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} \qquad \tan\left(\frac{5\pi}{4}\right) = 1$$

Noting the reciprocal of $-\frac{\sqrt{2}}{2}$ is $-\sqrt{2}$ after rationalizing, we have

$$\sec\left(\frac{5\pi}{4}\right) = -\sqrt{2} \qquad \csc\left(\frac{5\pi}{4}\right) = -\sqrt{2} \qquad \cot\left(\frac{5\pi}{4}\right) = 1$$

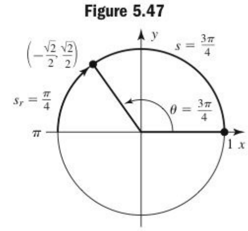
C. You've just learned how to define the six trig functions in terms of a point on the unit circle

Now try Exercises 37 through 40 ►

D. The Trigonometry of Real Numbers

Defining the trig functions in terms of a point on the unit circle is precisely what we needed to work with them as functions of real numbers. This is because when $r = 1$ and θ is in radians, *the length of the subtended arc is numerically the same as the measure of the angle:* $s = (1)\theta \Rightarrow s = \theta$! This means we can view any function of θ as a like function of arc length s, where $s \in \mathbb{R}$ (see the *Reinforcing Basic Concepts* feature following this section). As a compromise the variable t is commonly used, with t representing *either* the amount of rotation *or* the length of the arc. As such we will assume t is a unitless quantity, although there are other reasons for this assumption. In Figure 5.47, a rotation of $\theta = \frac{3\pi}{4}$ is subtended by an arc length of $s = \frac{3\pi}{4}$ (about 2.356 units). The reference angle for θ is $\frac{\pi}{4}$, which we will now refer to as a **reference arc.** As you work through the remaining examples and the exercises that follow, it will often help to draw a quick sketch similar to that in Figure 5.47 to determine the quadrant of the terminal side, the reference arc, and the sign of each function.

Figure 5.47

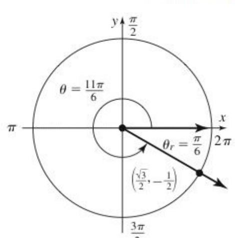


EXAMPLE 6 ► **Evaluating Trig Functions for a Real Number *t***

Evaluate the six trig functions for the given value of t.

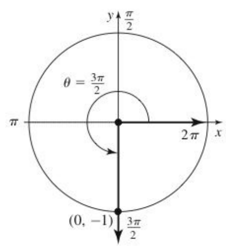
a. $t = \frac{11\pi}{6}$ **b.** $t = \frac{3\pi}{2}$

Solution ► **a.** For $t = \frac{11\pi}{6}$, the arc terminates in QIV where $x > 0$ and $y < 0$. The reference arc is $\frac{\pi}{6}$, and from our previous work we know the corresponding point (x, y) is $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. This gives



$$\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2} \qquad \tan\left(\frac{11\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

$$\sec\left(\frac{11\pi}{6}\right) = \frac{2\sqrt{3}}{3} \qquad \csc\left(\frac{11\pi}{6}\right) = -2 \qquad \cot\left(\frac{11\pi}{6}\right) = -\sqrt{3}$$



b. $t = \frac{3\pi}{2}$ is a quadrantal angle and the associated point is $(0, -1)$.

This yields

$$\cos\left(\frac{3\pi}{2}\right) = 0 \qquad \sin\left(\frac{3\pi}{2}\right) = -1 \qquad \tan\left(\frac{3\pi}{2}\right) = \text{undefined}$$

$$\sec\left(\frac{3\pi}{2}\right) = \text{undefined} \qquad \csc\left(\frac{3\pi}{2}\right) = -1 \qquad \cot\left(\frac{3\pi}{2}\right) = 0$$

Now try Exercises 41 through 44 ►

As Example 6(b) indicates, as functions of a real number the concept of domain comes into play. From their definition it is apparent there are no restrictions on the domain of cosine and sine, but the domains of the other functions must be restricted to exclude division by zero. For functions with x in the denominator, we cast out the odd multiples of $\frac{\pi}{2}$, since the x-coordinate of the related quadrantal points is zero: $\frac{\pi}{2} \to (0, 1)$, $\frac{3\pi}{2} \to (0, -1)$, and so on. The excluded values can be stated as $t \neq \frac{\pi}{2} + \pi k$ for all integers k. For functions with y in the denominator, we cast out all multiples of π ($t \neq \pi k$ for all integers k) since the y-coordinate of these points is zero: $0 \to (1, 0)$, $\pi \to (-1, 0)$, $2\pi \to (1, 0)$, and so on.

The Domains of the Trig Functions as Functions of a Real Number

For $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, the domains of the trig functions are:

$\cos t = x$	$\sin t = y$	$\tan t = \frac{y}{x};\ x \neq 0$
$t \in \mathbb{R}$	$t \in \mathbb{R}$	$t \neq \frac{\pi}{2} + \pi k$
$\sec t = \frac{1}{x};\ x \neq 0$	$\csc t = \frac{1}{y};\ y \neq 0$	$\cot t = \frac{x}{y};\ y \neq 0$
$t \neq \frac{\pi}{2} + \pi k$	$t \neq \pi k$	$t \neq \pi k$

For a given point (x, y) on the unit circle associated with the real number t, the value of each function at t can still be determined even if t is unknown.

EXAMPLE 7 ► **Finding Function Values Given a Point on the Unit Circle**

Given $\left(\frac{-7}{25}, \frac{24}{25}\right)$ is a point on the unit circle corresponding to a real number t, find the value of all six trig functions of t.

Solution ► Using the definitions from the previous box we have $\cos t = \frac{-7}{25}$, $\sin t = \frac{24}{25}$, and $\tan t = \frac{\sin t}{\cos t} = \frac{24}{-7}$. The values of the reciprocal functions are then $\sec t = \frac{25}{-7}$, $\csc t = \frac{25}{24}$, and $\cot t = \frac{-7}{24}$.

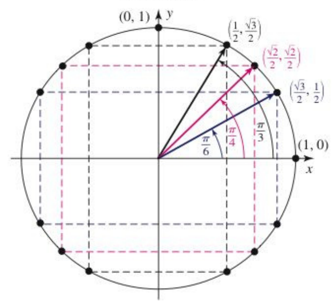
Now try Exercises 45 through 70 ►

D. You've just learned how to define the six trig functions in terms of a real number *t*

E. Finding a Real Number *t* Whose Function Value Is Known

In Example 7, we were able to determine the values of the trig functions even though t was unknown. In many cases, however, we need to *find* the value of t. For instance, what is the value of t given $\cos t = -\frac{\sqrt{3}}{2}$ with t in QII? Exercises of this type fall into two broad categories: (1) you recognize the given number as one of the special values: $\pm\left\{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}, \sqrt{3}, 1\right\}$; or (2) you don't. If you recognize a special value, you can often name the real number t after a careful consideration of the related quadrant and required sign.

Figure 5.48



The diagram in Figure 5.48 reviews these special values for $0 \le t \le \frac{\pi}{2}$ but remember—all other special values can be found using reference arcs and the symmetry of the circle.

EXAMPLE 8 ► Finding *t* for Given Values and Conditions

Find the value of t that corresponds to the given function values.

a. $\cos t = -\frac{\sqrt{2}}{2}$; t in QII **b.** $\tan t = \sqrt{3}$; t in QIII

Solution ► **a.** The cosine function is negative in QII and QIII, where $x < 0$. We recognize $-\frac{\sqrt{2}}{2}$ as a standard value for sine and cosine, related to certain multiples of $t = \frac{\pi}{4}$. In QII, we have $t = \frac{3\pi}{4}$.

b. The tangent function is positive in QI and QIII, where x and y have like signs. We recognize $\sqrt{3}$ as a standard value for tangent and cotangent, related to certain multiples of $t = \frac{\pi}{3}$. For tangent in QIII, we have $t = \frac{4\pi}{3}$.

Now try Exercises 71 through 94 ►

If the given function value is not one of the special values, properties of the inverse trigonometric functions must be used to find the associated value of t. The inverse functions are developed in Section 6.5.

Using radian measure and the unit circle is much more than a simple convenience to trigonometry and its applications. Whether the unit is 1 cm, 1 m, 1 km, or even 1 light-year, using 1 unit designations serves to simplify a great many practical applications, including those involving the arc length formula, $s = r\theta$. **See Exercises 97 through 104.**

The following table summarizes the relationship between a special arc t (t in QI) and the value of each trig function at t. Due to the frequent use of these relationships, students are encouraged to commit them to memory.

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	undefined	1	undefined
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	2	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	2	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	undefined	1	undefined	0

E. You've just learned how to find the real number t corresponding to given values of $\sin t$, $\cos t$, and $\tan t$

5.4 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A central circle is symmetric to the ____ axis, the ____ axis and to the ____.

2. Since $(\frac{5}{13}, -\frac{12}{13})$ is on the unit circle, the point ______ in QII is also on the circle.

3. On a unit circle, $\cos t =$ ____, $\sin t =$ ____, and $\tan t =$ ____; while $\frac{1}{x} =$ ____, $\frac{1}{y} =$ ____, and $\frac{x}{y} =$ ____.

4. On a unit circle with θ in radians, the length of a(n) ________ is numerically the same as the measure of the ______, since for $s = r\theta$, $s = \theta$ when $r = 1$.

5. Discuss/Explain how knowing only one point on the unit circle, actually gives the location of four points. Why is this helpful to a study of the circular functions?

6. A student is asked to find t using a calculator, given $\sin t \approx 0.5592$ with t in QII. The answer submitted is $t = \sin^{-1} 0.5592 \approx 34°$. Discuss/Explain why this answer is not correct. What is the correct response?

▶ DEVELOPING YOUR SKILLS

Given the point is on a unit circle, complete the ordered pair (x, y) for the quadrant indicated. For Exercises 7 to 14, answer in radical form as needed. For Exercises 15 to 18, round results to four decimal places.

7. $(x, -0.8)$; QIII

8. $(-0.6, y)$; QII

9. $\left(\frac{5}{13}, y\right)$; QIV

10. $\left(x, -\frac{8}{17}\right)$; QIV

11. $\left(\frac{\sqrt{11}}{6}, y\right)$; QI

12. $\left(x, -\frac{\sqrt{13}}{7}\right)$; QIII

13. $\left(-\frac{\sqrt{11}}{4}, y\right)$; QII

14. $\left(x, \frac{\sqrt{6}}{5}\right)$; QI

15. $(x, -0.2137)$; QIII

16. $(0.9909, y)$; QIV

17. $(x, 0.1198)$; QII

18. $(0.5449, y)$; QI

Verify the point given is on a unit circle, then use symmetry to find three more points on the circle. Results for Exercises 19 to 22 are exact, results for Exercises 23 to 26 are approximate.

19. $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

20. $\left(\frac{\sqrt{7}}{4}, -\frac{3}{4}\right)$

21. $\left(\frac{\sqrt{11}}{6}, -\frac{5}{6}\right)$

22. $\left(-\frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}\right)$

23. (0.3325, 0.9431) **24.** (0.7707, −0.6372)

25. (0.9937, −0.1121) **26.** (−0.2029, 0.9792)

27. Use a $\frac{\pi}{6}$: $\frac{\pi}{3}$: $\frac{\pi}{2}$ triangle with a hypotenuse of length 1 to verify that $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is a point on the unit circle.

28. Use the results from Exercise 27 to find three additional points on the circle and name the quadrant of each point.

Find the reference angle associated with each rotation, then find the associated point (x, y) on the unit circle.

29. $\theta = \frac{5\pi}{4}$ **30.** $\theta = \frac{5\pi}{3}$

31. $\theta = -\frac{5\pi}{6}$ **32.** $\theta = -\frac{7\pi}{4}$

33. $\theta = \frac{11\pi}{4}$ **34.** $\theta = \frac{11\pi}{3}$

35. $\theta = \frac{25\pi}{6}$ **36.** $\theta = \frac{39\pi}{4}$

Without the use of a calculator, state the exact value of the trig functions for the given angle. A diagram may help.

37. a. $\sin\left(\frac{\pi}{4}\right)$ **b.** $\sin\left(\frac{3\pi}{4}\right)$
c. $\sin\left(\frac{5\pi}{4}\right)$ **d.** $\sin\left(\frac{7\pi}{4}\right)$
e. $\sin\left(\frac{9\pi}{4}\right)$ **f.** $\sin\left(-\frac{\pi}{4}\right)$
g. $\sin\left(-\frac{5\pi}{4}\right)$ **h.** $\sin\left(-\frac{11\pi}{4}\right)$

38. a. $\tan\left(\frac{\pi}{3}\right)$ **b.** $\tan\left(\frac{2\pi}{3}\right)$
c. $\tan\left(\frac{4\pi}{3}\right)$ **d.** $\tan\left(\frac{5\pi}{3}\right)$
e. $\tan\left(\frac{7\pi}{3}\right)$ **f.** $\tan\left(-\frac{\pi}{3}\right)$
g. $\tan\left(-\frac{4\pi}{3}\right)$ **h.** $\tan\left(-\frac{10\pi}{3}\right)$

39. a. $\cos \pi$ **b.** $\cos 0$
c. $\cos\left(\frac{\pi}{2}\right)$ **d.** $\cos\left(\frac{3\pi}{2}\right)$

40. a. $\sin \pi$ **b.** $\sin 0$
c. $\sin\left(\frac{\pi}{2}\right)$ **d.** $\sin\left(\frac{3\pi}{2}\right)$

Use the symmetry of the circle and reference arcs as needed to state the exact value of the trig functions for the given real number, without the use of a calculator. A diagram may help.

41. a. $\cos\left(\frac{\pi}{6}\right)$ **b.** $\cos\left(\frac{5\pi}{6}\right)$
c. $\cos\left(\frac{7\pi}{6}\right)$ **d.** $\cos\left(\frac{11\pi}{6}\right)$
e. $\cos\left(\frac{13\pi}{6}\right)$ **f.** $\cos\left(-\frac{\pi}{6}\right)$
g. $\cos\left(-\frac{5\pi}{6}\right)$ **h.** $\cos\left(-\frac{23\pi}{6}\right)$

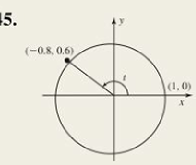
42. a. $\csc\left(\frac{\pi}{6}\right)$ **b.** $\csc\left(\frac{5\pi}{6}\right)$
c. $\csc\left(\frac{7\pi}{6}\right)$ **d.** $\csc\left(\frac{11\pi}{6}\right)$
e. $\csc\left(\frac{13\pi}{6}\right)$ **f.** $\csc\left(-\frac{\pi}{6}\right)$
g. $\csc\left(-\frac{11\pi}{6}\right)$ **h.** $\csc\left(-\frac{17\pi}{6}\right)$

43. a. $\tan \pi$ **b.** $\tan 0$
c. $\tan\left(\frac{\pi}{2}\right)$ **d.** $\tan\left(\frac{3\pi}{2}\right)$

44. a. $\cot \pi$ **b.** $\cot 0$
c. $\cot\left(\frac{\pi}{2}\right)$ **d.** $\cot\left(\frac{3\pi}{2}\right)$

Given (x, y) is a point on a unit circle corresponding to t, find the value of all six circular functions of t.

45.



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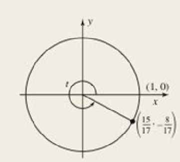

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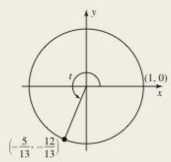

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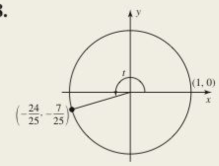

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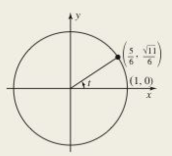

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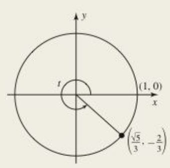


51. $\left(-\frac{2}{5}, \frac{\sqrt{21}}{5}\right)$ **52.** $\left(\frac{\sqrt{7}}{4}, -\frac{3}{4}\right)$

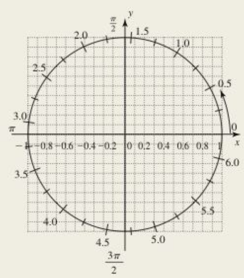
53. $\left(-\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$ **54.** $\left(-\frac{2\sqrt{6}}{5}, -\frac{1}{5}\right)$

55. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **56.** $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

57. $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **58.** $\left(\frac{\sqrt{2}}{3}, -\frac{\sqrt{7}}{3}\right)$

On a unit circle, the real number t can represent either the amount of rotation or the *length of the arc* when we associate t with a point (x, y) on the circle. In the circle diagram shown, the real number t in radians is marked off along the circumference. For Exercises 59 through 70, name the quadrant in which t terminates and use the figure to estimate function values to one decimal place (use a straightedge). Check results using a calculator.

Exercises 59 to 70



59. sin 0.75 **60.** cos 2.75

61. cos 5.5 **62.** sin 4.0

63. tan 0.8 **64.** sec 3.75

65. csc 2.0 **66.** cot 0.5

67. $\cos\left(\frac{5\pi}{8}\right)$ **68.** $\sin\left(\frac{5\pi}{8}\right)$

69. $\tan\left(\frac{8\pi}{5}\right)$ **70.** $\sec\left(\frac{8\pi}{5}\right)$

Without using a calculator, find the value of t in $[0, 2\pi)$ that corresponds to the following functions.

71. $\sin t = \frac{\sqrt{3}}{2}$; t in QII

72. $\cos t = \frac{1}{2}$; t in QIV

73. $\cos t = -\frac{\sqrt{3}}{2}$; t in QIII

74. $\sin t = -\frac{1}{2}$; t in QIV

75. $\tan t = -\sqrt{3}$; t in QII

76. $\sec t = -2$; t in QIII

77. $\sin t = 1$; t is quadrantal

78. $\cos t = -1$; t is quadrantal

Without using a calculator, find the two values of t (where possible) in $[0, 2\pi)$ that make each equation true.

79. $\sec t = -\sqrt{2}$ **80.** $\csc t = -\frac{2}{\sqrt{3}}$

81. $\tan t$ undefined **82.** $\csc t$ undefined

83. $\cos t = -\frac{\sqrt{2}}{2}$ **84.** $\sin t = \frac{\sqrt{2}}{2}$

85. $\sin t = 0$ **86.** $\cos t = -1$

87. Given $(\frac{3}{4}, -\frac{4}{5})$ is a point on the unit circle that corresponds to t. Find the coordinates of the point corresponding to (a) $-t$ and (b) $t + \pi$.

88. Given $(-\frac{7}{25}, \frac{24}{25})$ is a point on the unit circle that corresponds to t. Find the coordinates of the point corresponding to (a) $-t + \pi$ and (b) $t - \pi$.

Find an additional value of t in $[0, 2\pi)$ that makes the equation true.

89. $\sin 0.8 \approx 0.7174$

90. $\cos 2.12 \approx -0.5220$

91. $\cos 4.5 \approx -0.2108$

92. $\sin 5.23 \approx -0.8690$

93. $\tan 0.4 \approx 0.4228$

94. $\sec 5.7 \approx 1.1980$

▶ WORKING WITH FORMULAS

95. From Pythagorean triples to points on the unit circle: $(x, y, r) \rightarrow \left(\frac{x}{r}, \frac{y}{r}, 1\right)$

While not strictly a "formula," dividing a Pythagorean triple by r is a simple algorithm for rewriting any Pythagorean triple as a triple with hypotenuse 1. This enables us to identify certain points on a unit circle, and to evaluate the six trig functions of the related acute angle. Rewrite each triple as a triple with hypotenuse 1, verify $\left(\frac{x}{r}, \frac{y}{r}\right)$ is a point on the unit circle, and evaluate the six trig functions using this point.

a. (5, 12, 13) **b.** (7, 24, 25)

c. (12, 35, 37) **d.** (9, 40, 41)

96. The sine and cosine of $(2k + 1)\frac{\pi}{4}$; $k \in \mathbb{Z}$

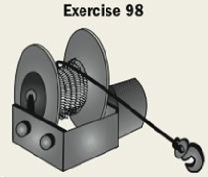
In the solution to Example 8(a), we mentioned $\pm\frac{\sqrt{2}}{2}$ were standard values for sine and cosine, "related to certain multiples of $\frac{\pi}{4}$." Actually, we meant "odd multiples of $\frac{\pi}{4}$." The odd multiples of $\frac{\pi}{4}$ are given by the "formula" shown, where k is any integer. (a) What multiples of $\frac{\pi}{4}$ are generated by $k = -3, -2, -1, 0, 1, 2, 3$? (b) Find similar formulas for Example 8(b), where $\sqrt{3}$ is a standard value for tangent and cotangent, "related to certain multiples of $\frac{\pi}{6}$."

▶ APPLICATIONS

97. Laying new sod: When new sod is laid, a heavy roller is used to press the sod down to ensure good contact with the ground beneath. The radius of the roller is 1 ft. (a) Through what angle (in radians) has the roller turned after being pulled across 5 ft of yard? (b) What angle must the roller turn through to press a length of 30 ft?

98. Cable winch: A large winch with a radius of 1 ft winds in 3 ft of cable. (a) Through what angle (in radians) has it turned? (b) What angle must it turn through in order to winch in 12.5 ft of cable?

Exercise 98



99. Wiring an apartment: In the wiring of an apartment complex, electrical wire is being pulled from a spool with radius 1 decimeter

(1 dm = 10 cm). (a) What length (in decimeters) is removed as the spool turns through 5 rad? (b) How many decimeters are removed in one complete turn ($t = 2\pi$) of the spool?

100. Barrel races: In the barrel races popular at some family reunions, contestants stand on a hard rubber barrel with a radius of 1 cubit (1 cubit = 18 in.), and try to "walk the barrel" from the start line to the finish line without falling. (a) What distance (in cubits) is traveled as the barrel is walked through an angle of 4.5 rad? (b) If the race is 25 cubits long, through what angle will the winning barrel walker walk the barrel?

Interplanetary measurement: In the year 1905, astronomers began using astronomical units or AU to study the distances between the celestial bodies of our solar system. One AU represents the average distance between the Earth and the Sun, which is about 93 million miles. Pluto is roughly 39.24 AU from the Sun.

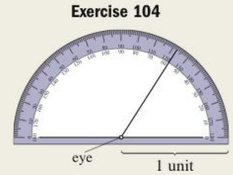
101. If the Earth travels through an angle of 2.5 rad about the Sun, (a) what distance in astronomical units (AU) has it traveled? (b) How many AU does it take for one complete orbit around the Sun?

102. If you include the dwarf planet Pluto, Jupiter is the middle (fifth of nine) planet from the Sun. Suppose astronomers had decided to use *its* average distance from the Sun as 1 AU. In this case, 1 AU would be 480 million miles. If Jupiter travels through an angle of 4 rad about the Sun, (a) what distance in the "new" astronomical units (AU) has it traveled? (b) How many of the new AU does it take to complete one-half an orbit about the Sun? (c) What distance in the new AU is the dwarf planet Pluto from the Sun?

103. Compact disk circumference: A standard compact disk has a radius of 6 cm. Call this length "1 unit." Mark a starting point on any large surface, then carefully roll the compact disk along this line without slippage, through one full revolution (2π rad) and mark this spot. Take an accurate measurement of the resulting line segment. Is the result close to 2π "units" ($2\pi \times 6$ cm)?

104. Verifying $s = r\theta$: On a protractor, carefully measure the distance from the middle of the protractor's eye to the edge of the protractor along the 0° mark, to the nearest half-millimeter. Call this length "1 unit." Then use a ruler to draw a straight line on a blank sheet of paper, and with the protractor on edge, start the zero degree mark at one end of the line, carefully roll the protractor until it reaches 1 radian (57.3°), and mark this spot. Now measure the length of the line segment created. Is it very close to 1 "unit" long?

Exercise 104



► EXTENDING THE CONCEPT

105. In this section, we discussed the *domain* of the circular functions, but said very little about their *range*. Review the concepts presented here and determine the range of $y = \cos t$ and $y = \sin t$. In other words, what are the smallest and largest output values we can expect?

106. Since $\tan t = \frac{\sin t}{\cos t}$, what can you say about the range of the tangent function?

Use the radian grid given with Exercises 59–70 to answer Exercises 107 and 108.

107. Given $\cos(2t) = -0.6$ with the terminal side of the arc in QII, (a) what is the value of $2t$? (b) What quadrant is t in? (c) What is the value of $\cos t$? (d) Does $\cos(2t) = 2\cos t$?

108. Given $\sin(2t) = -0.8$ with the terminal side of the arc in QIII, (a) what is the value of $2t$? (b) What quadrant is t in? (c) What is the value of $\sin t$? (d) Does $\sin(2t) = 2\sin t$?

▶ MAINTAINING YOUR SKILLS

109. **(2.1)** Given the points $(-3, -4)$ and $(5, 2)$ find
  a. the distance between them
  b. the midpoint between them
  c. the slope of the line through them.

110. **(4.3)** Use a calculator to find the value of each expression, then explain the results.
  a. $\log 2 + \log 5 =$ ______
  b. $\log 20 - \log 2 =$ ______

111. **(1.3)** Solve each equation:
  a. $2|x + 1| - 3 = 7$
  b. $2\sqrt{x + 1} - 3 = 7$

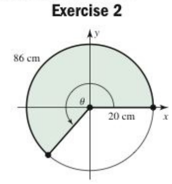
112. **(3.2)** Use the rational zeroes theorem to solve the equation completely, given $x = -3$ is one root.

$$x^4 + x^3 - 3x^2 + 3x - 18 = 0$$

MID-CHAPTER CHECK

1. The city of Las Vegas, Nevada, is located at 36°06′36″ north latitude, 115°04′48″ west longitude. (a) Convert both measures to decimal degrees. (b) If the radius of the Earth is 3960 mi, how far north of the equator is Las Vegas?

Exercise 2

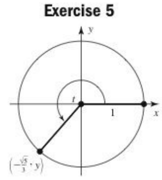


2. Find the angle subtended by the arc shown in the figure, then determine the area of the sector.

3. Evaluate without using a calculator: (a) cot 60° and (b) $\sin\left(\frac{7\pi}{4}\right)$.

4. Evaluate using a calculator: (a) $\sec\left(\frac{\pi}{12}\right)$ and (b) tan 83.6°.

Exercise 5

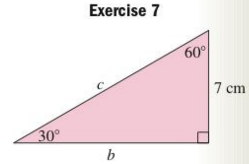


5. Complete the ordered pair indicated on the unit circle in the figure and find the value of all six trigonometric functions at this point.

6. For the point on the unit circle in Exercise 5, find the related angle t in both degrees (to tenths) and radians (to ten-thousandths).

7. Use the special triangle to state the length of side b and hypotenuse c.

Exercise 7



8. From a distance of 325 ft, the angle of elevation from eye level to the top of the world's tallest tree is 48°. If the person taking the sighting is 6 ft tall, how tall is the tree to the nearest foot?

9. On a unit circle, if arc t has length 5.94, (a) in what quadrant does it terminate? (b) What is its reference arc? (c) Of $\sin t$, $\cos t$, and $\tan t$, which are negative for this value of t?

10. At a high school gym, sightings are taken from the basketball half-court line to help determine the height of the backboard. The angle of elevation to the top of the backboard is 18°, while the angle of elevation to the bottom of the backboard is 13.4°. If the half-court line is 40 ft away, how tall is the backboard? Answer in feet and inches to the nearest inch.

REINFORCING BASIC CONCEPTS

Trigonometry of the Real Numbers and the Wrapping Function

The circular functions are sometimes discussed in terms of what is called a *wrapping function,* in which the real number line is literally wrapped around the unit circle. This approach can help illustrate how the trig functions can be seen as functions of the real numbers, and apart from any reference to a right triangle. Figure 5.49 shows (1) a unit circle with the location of certain points on the circumference clearly marked and (2) a number line that has been marked in multiples of $\frac{\pi}{12}$ to coincide with the length of the special arcs (integers are shown in the background). Figure 5.50 shows this same number line wrapped counterclockwise around the unit circle in the positive direction. Note how the resulting diagram confirms that an arc of length $t = \frac{\pi}{4}$ is associated with the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on the unit circle: $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$; while an arc of length of $t = \frac{5\pi}{6}$ is associated with the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$: $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\sin\frac{5\pi}{6} = \frac{1}{2}$. Use this information to complete the exercises given.

Figure 5.49

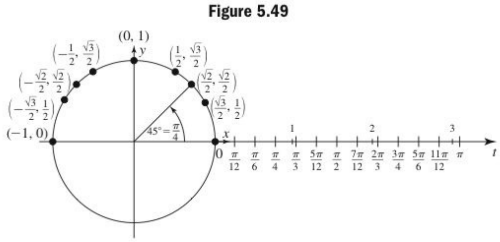
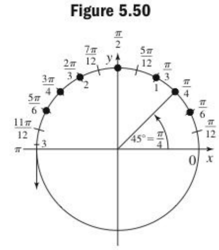


Figure 5.50



1. What is the ordered pair associated with an arc length of $t = \frac{2\pi}{3}$? What is the value of cos t? sin t?

2. What arc length t is associated with the ordered pair $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$? Is cos t positive or negative? Why?

3. If we continued to wrap this number line all the way around the circle, in what quadrant would an arc length of $t = \frac{11\pi}{6}$ terminate? Would sin t be positive or negative?

4. Suppose we wrapped a number line with negative values clockwise around the unit circle. In what quadrant would an arc length of $t = -\frac{5\pi}{3}$ terminate? What is cos t? sin t? What positive rotation terminates at the same point?

5.5 Graphs of the Sine and Cosine Functions; Cosecant and Secant Functions

Learning Objectives

In Section 5.5 you will learn how to:

- ❑ **A.** Graph $f(t) = \sin t$ using special values and symmetry
- ❑ **B.** Graph $f(t) = \cos t$ using special values and symmetry
- ❑ **C.** Graph sine and cosine functions with various amplitudes and periods
- ❑ **D.** Investigate graphs of the reciprocal functions $f(t) = \csc(Bt)$ and $f(t) = \sec(Bt)$
- ❑ **E.** Write the equation for a given graph

As with the graphs of other functions, trigonometric graphs contribute a great deal toward the understanding of each trig function and its applications. For now, our primary interest is the general shape of each basic graph and some of the transformations that can be applied. We will also learn to analyze each graph, and to capitalize on the features that enable us to apply the functions as real-world models.

A. Graphing $f(t) = \sin t$

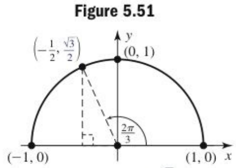
Consider the following table of values (Table 5.4) for sin *t* and the special angles in *QI*.

Table 5.4

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

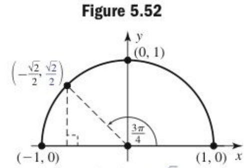
Observe that in this interval, sine values are increasing from 0 to 1. From $\frac{\pi}{2}$ to π (QII), special values taken from the unit circle show sine values are decreasing from 1 to 0, *but through the same output values as in QI*. See Figures 5.51 through 5.53.

Figure 5.51



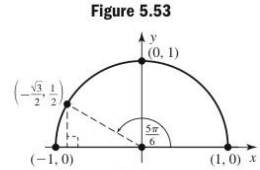
$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

Figure 5.52



$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Figure 5.53



$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$

With this information we can extend our table of values through π, noting that $\sin \pi = 0$ (see Table 5.5).

Table 5.5

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Using the symmetry of the circle and the fact that *y* is negative in QIII and QIV, we can complete the table for values between π and 2π.

EXAMPLE 1 ► Finding Function Values Using Symmetry

Use the symmetry of the unit circle and reference arcs of special values to complete Table 5.6. Recall that *y* is negative in QIII and QIV.

Table 5.6

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin t$									

Solution ► Symmetry shows that for any odd multiple of $t = \frac{\pi}{4}$, $\sin t = \pm\frac{\sqrt{2}}{2}$ depending on the quadrant of the terminal side. Similarly, for any reference arc of $\frac{\pi}{6}$, $\sin t = \pm\frac{1}{2}$, while any reference arc of $\frac{\pi}{3}$ will give $\sin t = \pm\frac{\sqrt{3}}{2}$. The completed table is shown in Table 5.7.

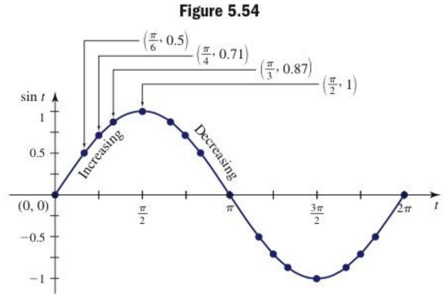
Table 5.7

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin t$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0

Now try Exercises 7 and 8 ►

Noting that $\frac{1}{2} = 0.5$, $\frac{\sqrt{2}}{2} \approx 0.71$, and $\frac{\sqrt{3}}{2} \approx 0.87$, we plot these points and connect them with a smooth curve to graph $y = \sin t$ in the interval $[0, 2\pi]$. The first five plotted points are labeled in Figure 5.54.

Figure 5.54



Expanding the table from 2π to 4π using reference arcs and the unit circle shows that function values begin to repeat. For example, $\sin\left(\frac{13\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right)$ since $\theta_r = \frac{\pi}{6}$; $\sin\left(\frac{9\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$ since $\theta_r = \frac{\pi}{4}$, and so on. Functions that cycle through a set pattern of values are said to be **periodic functions.**

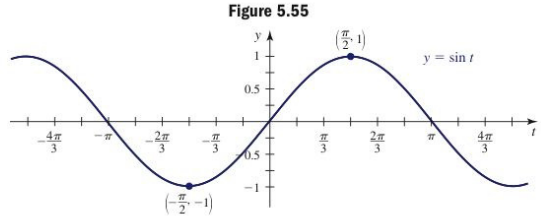
Periodic Functions

A function f is said to be periodic if there is a positive number P such that $f(t + P) = f(t)$ for all t in the domain. The smallest number P for which this occurs is called the **period** of f.

For the sine function we have $\sin t = \sin(t + 2\pi)$, as in $\sin\left(\frac{13\pi}{6}\right) = \sin\left(\frac{\pi}{6} + 2\pi\right)$ and $\sin\left(\frac{9\pi}{4}\right) = \sin\left(\frac{\pi}{4} + 2\pi\right)$, with the idea extending to all other real numbers t: $\sin t = \sin(t + 2\pi k)$ for all integers k. The sine function is periodic with period $P = 2\pi$.

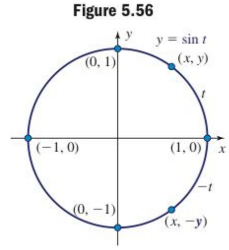
Although we initially focused on positive values of t in $[0, 2\pi]$, $t < 0$ and $k < 0$ are certainly possibilities and we note the graph of $y = \sin t$ extends infinitely in both directions (see Figure 5.55).

Figure 5.55



Finally, both the graph and the unit circle confirm that the range of $y = \sin t$ is $[-1, 1]$, and that $y = \sin t$ is an odd function. In particular, the graph shows $\sin\left(-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right)$, and the unit circle shows (Figure 5.56) $\mathbf{\sin t} = y$, and $\sin(-t) = \mathbf{-y}$, from which we obtain $\sin(-t) = -\sin t$ by substitution. As a handy reference, the following box summarizes the main characteristics of $y = \sin t$.

Figure 5.56



Characteristics of $f(t) = \sin t$

For all real numbers t and integers k,

Domain	**Range**	**Period**
$(-\infty, \infty)$	$[-1, 1]$	2π
Symmetry	**Maximum value**	**Minimum value**
odd	$\sin t = 1$	$\sin t = -1$
$\sin(-t) = -\sin t$	at $t = \frac{\pi}{2} + 2\pi k$	at $t = \frac{3\pi}{2} + 2\pi k$
Increasing	**Decreasing**	**Zeroes**
$\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$	$t = k\pi$

EXAMPLE 2 ► Using the Period of sin *t* to Find Function Values

Use the characteristics of $f(t) = \sin t$ to match the given value of t to the correct value of $\sin t$.

a. $t = \left(\frac{\pi}{4} + 8\pi\right)$ **b.** $t = -\frac{\pi}{6}$ **c.** $t = \frac{17\pi}{2}$ **d.** $t = 21\pi$ **e.** $t = \frac{11\pi}{2}$

I. $\sin t = 1$ **II.** $\sin t = -\frac{1}{2}$ **III.** $\sin t = -1$ **IV.** $\sin t = \frac{\sqrt{2}}{2}$ **V.** $\sin t = 0$

Solution ► **a.** Since $\sin\left(\frac{\pi}{4} + 8\pi\right) = \sin\frac{\pi}{4}$, the correct match is (IV).

b. Since $\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6}$, the correct match is (II).

c. Since $\sin\left(\frac{17\pi}{2}\right) = \sin\left(\frac{\pi}{2} + 8\pi\right) = \sin\frac{\pi}{2}$, the correct match is (I).

d. Since $\sin(21\pi) = \sin(\pi + 20\pi) = \sin\pi$, the correct match is (V).

e. Since $\sin\left(\frac{11\pi}{2}\right) = \sin\left(\frac{3\pi}{2} + 4\pi\right) = \sin\left(\frac{3\pi}{2}\right)$, the correct match is (III).

Now try Exercises 9 and 10 ►

Many of the transformations applied to algebraic graphs can also be applied to trigonometric graphs. These transformations may stretch, reflect, or translate the graph, but it will still retain its basic shape. In numerous applications it will help if you're able to draw a quick, accurate sketch of the transformations involving $f(t) = \sin t$. To assist this effort, we'll begin with the interval $[0, 2\pi]$, combine the characteristics just listed with some simple geometry, and offer the following four-step process. Steps I through IV are illustrated in Figures 5.57 through 5.60.

Step I: Draw the y-axis, mark zero halfway up, with -1 and 1 an equal distance from this zero. Then draw an extended t-axis and tick mark 2π to the extreme right (Figure 5.57).

Step II: On the t-axis, mark halfway between 0 and 2π and label it "π," mark halfway between π on either side and label the marks $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Halfway between these you can draw additional tick marks to represent the remaining multiples of $\frac{\pi}{4}$ (Figure 5.58).

Step III: Next, lightly draw a rectangular frame, which we'll call the **reference rectangle,** $P = 2\pi$ units wide and 2 units tall, centered on the t-axis and with the y-axis along one side (Figure 5.59).

Step IV: Knowing $y = \sin t$ is positive and increasing in QI, that the range is $[-1, 1]$, that the zeroes are 0, π, and 2π, and that maximum and minimum values *occur halfway between the zeroes* (since there is no horizontal shift), we can draw a reliable graph of $y = \sin t$ by partitioning the rectangle into four equal parts to locate these values (note **bold** tick-marks). We will call this partitioning of the reference rectangle the **rule of fourths,** since we are then scaling the t-axis in increments of $\frac{P}{4}$ (Figure 5.60).

Figure 5.57

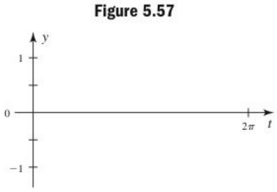


Figure 5.58

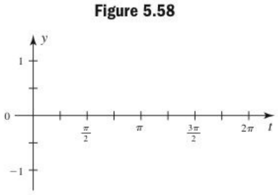


Figure 5.59

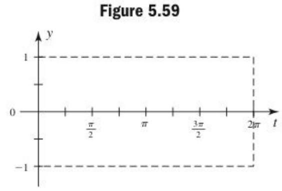
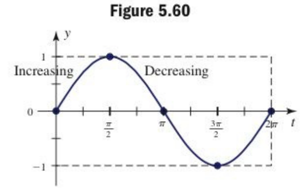


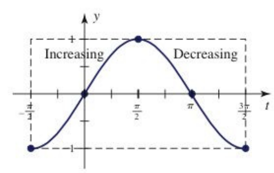
Figure 5.60



EXAMPLE 3 ► **Graphing $y = \sin t$ Using a Reference Rectangle**

Use steps I through IV to draw a sketch of $y = \sin t$ for the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Solution ► Start by completing steps I and II, then extend the t-axis to include $-\frac{\pi}{2}$. Beginning at $-\frac{\pi}{2}$, draw a reference rectangle 2π units wide and 2 units tall, centered on the x-axis $\left(\text{ending at } \frac{3\pi}{2}\right)$. After applying the rule of fourths, we note the zeroes occur at $t = 0$ and $t = \pi$, with the max/min values spaced equally between and on either side. Plot these points and connect them with a smooth curve (see the figure).



A. You've just learned how to graph $f(t) = \sin t$ using special values and symmetry

Now try Exercises 11 and 12 ►

B. Graphing $f(t) = \cos t$

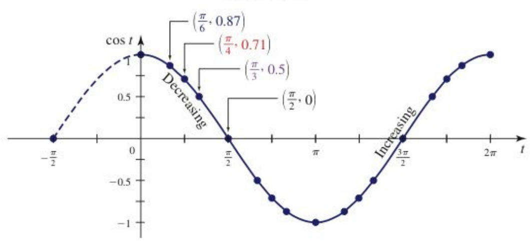
With the graph of $f(t) = \sin t$ established, sketching the graph of $f(t) = \cos t$ is a very natural next step. First, note that when $t = 0$, $\cos t = 1$ so the graph of $y = \cos t$ will begin at (0, 1) in the interval $[0, 2\pi]$. Second, we've seen $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$, $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$ and $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$ are all points on the unit circle since they satisfy $x^2 + y^2 = 1$. Since $\cos t = x$ and $\sin t = y$, the equation $\cos^2 t + \sin^2 t = 1$ can be obtained by direct substitution. This means if $\sin t = \pm\frac{1}{2}$, then $\cos t = \pm\frac{\sqrt{3}}{2}$ and vice versa, with the signs taken from the appropriate quadrant. The table of values for cosine then becomes a simple variation of the table for sine, as shown in Table 5.8 for $t \in [0, \pi]$.

Table 5.8

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin t$	0	$\frac{1}{2} = 0.5$	$\frac{\sqrt{2}}{2} \approx 0.71$	$\frac{\sqrt{3}}{2} \approx 0.87$	1	$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{\sqrt{2}}{2} \approx 0.71$	$\frac{1}{2} = 0.5$	0
$\cos t$	**1**	$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{\sqrt{2}}{2} \approx 0.71$	$\frac{1}{2} = 0.5$	**0**	$-\frac{1}{2} = -0.5$	$-\frac{\sqrt{2}}{2} \approx -0.71$	$-\frac{\sqrt{3}}{2} \approx -0.87$	**−1**

The same values can be taken from the unit circle, but this view requires much less effort and easily extends to values of t in $[\pi, 2\pi]$. Using the points from Table 5.8 and its extension through $[\pi, 2\pi]$, we can draw the graph of $y = \cos t$ in $[0, 2\pi]$ and identify where the function is increasing and decreasing in this interval. See Figure 5.61.

Figure 5.61



The function is decreasing for t in $(0, \pi)$, and increasing for t in $(\pi, 2\pi)$. The end result appears to be the graph of $y = \sin t$ shifted to the left $\frac{\pi}{2}$ units, a fact more easily seen if we extend the graph to $-\frac{\pi}{2}$ as shown. This is in fact the case, and is a relationship we will later prove in Chapter 6. Like $y = \sin t$, the function $y = \cos t$ is periodic with period $P = 2\pi$, with the graph extending infinitely in both directions.

Finally, we note that cosine is an **even function,** meaning $\cos(-t) = \cos t$ for all t in the domain. For instance, $\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$ (see Figure 5.61). Here is a summary of important characteristics of the cosine function.

Characteristics of $f(t) = \cos t$

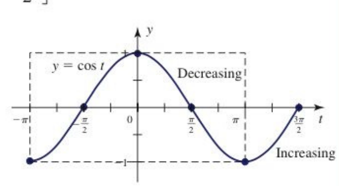
For all real numbers t and integers k,

Domain	Range	Period
$(-\infty, \infty)$	$[-1, 1]$	2π
Symmetry	**Maximum value**	**Minimum value**
even	$\cos t = 1$	$\cos t = -1$
$\cos(-t) = \cos t$	at $t = 2\pi k$	at $t = \pi + 2\pi k$
Increasing	**Decreasing**	**Zeroes**
$(\pi, 2\pi)$	$(0, \pi)$	$t = \frac{\pi}{2} + \pi k$

EXAMPLE 4 ► **Graphing $y = \cos t$ Using a Reference Rectangle**

Draw a sketch of $y = \cos t$ for t in $\left[-\pi, \frac{3\pi}{2}\right]$.

Solution ► After completing steps I and II, extend the negative x-axis to include $-\pi$. Beginning at $-\pi$, draw a reference rectangle 2π units wide and 2 units tall, centered on the x-axis. After applying the rule of fourths, we note the zeroes will occur at $t = -\pi/2$ and $t = \pi/2$, with the max/min values spaced equally between these zeroes and on either side (at $t = -\pi$, $t = 0$, and $t = \pi$). Finally, we extend the graph to include $3\pi/2$.



B. You've just learned how to graph $f(t) = \cos t$ using special values and symmetry

Now try Exercises 13 and 14 ►

WORTHY OF NOTE

Note that the equations $y = A \sin t$ and $y = A \cos t$ both indicate y is a function of t, with no reference to the unit circle definitions $\cos t = x$ and $\sin t = y$.

C. Graphing $y = A\sin(Bt)$ and $y = A\cos(Bt)$

In many applications, trig functions have maximum and minimum values other than 1 and -1, and periods other than 2π. For instance, in tropical regions the maximum and minimum temperatures may vary by no more than 20°, while for desert regions this difference may be 40° or more. This variation is modeled by the *amplitude* of sine and cosine functions.

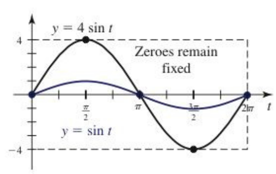
Amplitude and the Coefficient A (assume $B = 1$)

For functions of the form $y = A \sin t$ and $y = A \cos t$, let M represent the *M*aximum value and m the *m*inimum value of the functions. Then the quantity $\frac{M + m}{2}$ gives the **average value** of the function, while $\frac{M - m}{2}$ gives the **amplitude** of the function. Amplitude is the maximum displacement from the average value in the positive or negative direction. It is represented by $|A|$, with A playing a role similar to that seen for algebraic graphs $[Af(t)$ vertically stretches or compresses the graph of f, and reflects it across the t-axis if $A < 0]$. Graphs of the form $y = \sin t$ (and $y = \cos t$) can quickly be sketched with any amplitude by noting (1) the *zeroes of the function remain fixed* since $\sin t = 0$ implies $A \sin t = 0$, and (2) the *maximum and minimum values are A and $-A$*, respectively, since $\sin t = 1$ or -1 implies $A \sin t = A$ or $-A$. Note this implies the reference rectangle will be $2A$ units tall and P units wide. Connecting the points that result with a smooth curve will complete the graph.

EXAMPLE 5 ► **Graphing $y = A \sin t$ Where $A \neq 1$**

Draw a sketch of $y = 4 \sin t$ in the interval $[0, 2\pi]$.

Solution ► With an amplitude of $|A| = 4$, the reference rectangle will be $2(4) = 8$ units tall, by 2π units wide, centered on the x-axis. Using the rule of fourths, the zeroes are still $t = 0$, $t = \pi$, and $t = 2\pi$, with the max/min values spaced equally between. The maximum value is $4\sin\left(\frac{\pi}{2}\right) = 4(1) = 4$, with a minimum value of $4\sin\left(\frac{3\pi}{2}\right) = 4(-1) = -4$. Connecting these points with a "sine curve" gives the graph shown ($y = \sin t$ is also shown for comparison).



Now try Exercises 15 through 20 ►

Period and the Coefficient B

While basic sine and cosine functions have a period of 2π, in many applications the period may be very long (tsunami's) or very short (electromagnetic waves). For the equations $y = A \sin(Bt)$ and $y = A \cos(Bt)$, the period depends on the value of B. To see why, consider the function $y = \cos(2t)$ and Table 5.9. Multiplying input values by 2 means each cycle will be completed twice as fast. The table shows that $y = \cos(2t)$ completes a full cycle in $[0, \pi]$, giving a period of $P = \pi$ (Figure 5.62, red graph).

Table 5.9

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2t$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos(2t)$	1	0	-1	0	1

Dividing input values by 2 (or multiplying by $\frac{1}{2}$) will cause the function to complete a cycle only half as fast, doubling the time required to complete a full cycle. Table 5.10 shows $y = \cos\left(\frac{1}{2}t\right)$ completes only one-half cycle in 2π (Figure 5.62, blue graph).

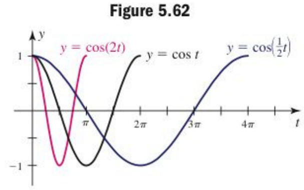
Table 5.10
(values in blue are approximate)

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\frac{1}{2}t$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
$\cos\left(\frac{1}{2}t\right)$	1	0.92	$\frac{\sqrt{2}}{2}$	0.38	0	-0.38	$-\frac{\sqrt{2}}{2}$	-0.92	-1

The graphs of $y = \cos t$, $y = \cos(2t)$, and $y = \cos\left(\frac{1}{2}t\right)$ shown in Figure 5.62 clearly illustrate this relationship and how the value of B affects the period of a graph.

To find the period for arbitrary values of B, the formula $P = \frac{2\pi}{|B|}$ is used. Note for $y = \cos(2t)$, $B = 2$ and $P = \frac{2\pi}{2} = \pi$, as shown. For $y = \cos\left(\frac{1}{2}t\right)$, $|B| = \frac{1}{2}$, and $P = \frac{2\pi}{1/2} = 4\pi$.

Figure 5.62



Period Formula for Sine and Cosine

For B a real number and functions $y = A\sin(Bt)$ and $y = A\cos(Bt)$,

$$P = \frac{2\pi}{|B|}.$$

To sketch these functions for periods other than 2π, we still use a reference rectangle of height $2A$ and length P, then break the enclosed t-axis in four equal parts to help draw the graph. In general, if the period is "very large" one full cycle is appropriate for the graph. If the period is very small, graph at least two cycles.

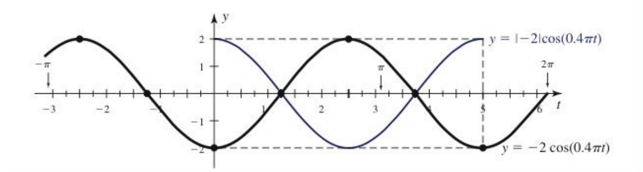
Note the value of B in Example 6 includes a factor of π. This actually happens quite frequently in applications of the trig functions.

EXAMPLE 6 ▶ Graphing $y = A\cos(Bt)$, Where $A, B \neq 1$

Draw a sketch of $y = -2\cos(0.4\pi t)$ for t in $[-\pi, 2\pi]$.

Solution ▶ The amplitude is $|A| = 2$, so the reference rectangle will be $2(2) = 4$ units high. Since $A < 0$ the *graph will be vertically reflected across the t-axis*. The period is $P = \frac{2\pi}{0.4\pi} = 5$ (note the factors of π reduce to 1), so the reference rectangle will be 5 units in length. Breaking the t-axis into four parts within the frame (rule of fourths) gives $\left(\frac{1}{4}\right)5 = \frac{5}{4}$ units, indicating that we should scale the t-axis in multiples of $\frac{1}{4}$. Note the zeroes occur at $\frac{5}{4}$ and $\frac{15}{4}$, with a maximum value at $\frac{10}{4}$. In cases where the π factor reduces, we scale the t-axis as a "standard" number line, and *estimate the location of multiples of π*. For practical reasons, we first draw the unreflected graph (shown in blue) for guidance in drawing the reflected graph, which is then extended to fit the given interval.

C. You've just learned how to graph sine and cosine functions with various amplitudes and periods



Now try Exercises 21 through 32 ▶

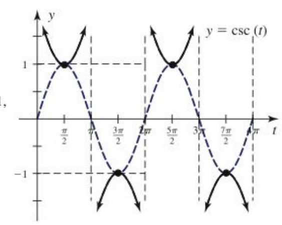
D. Graphs of $y = \csc(Bt)$ and $y = \sec(Bt)$

The graphs of these reciprocal functions follow quite naturally from the graphs of $y = A\sin(Bt)$ and $y = A\cos(Bt)$, by using these observations: (1) you cannot divide by zero, (2) the reciprocal of a very small number is a very large number (and vice versa), and (3) the reciprocal of ±1 is ±1. Just as with rational functions, division by zero creates a vertical asymptote, so the graph of $y = \csc t = \frac{1}{\sin t}$ will have a vertical asymptote at every point where $\sin t = 0$. This occurs at $t = \pi k$, where k is an integer $(\ldots -2\pi, -\pi, 0, \pi, 2\pi, \ldots)$. Further, when $\csc(Bt) = \pm1$, $\sin(Bt) = \pm1$ since the reciprocal of 1 and -1 are still 1 and -1, respectively. Finally, due to observation 2, the graph of the cosecant function will be increasing when the sine function is decreasing, and decreasing when the sine function is increasing. In most cases, we graph $y = \csc(Bt)$ by drawing a sketch of $y = \sin(Bt)$, then using these observations as demonstrated in Example 7. In doing so, we discover that the period of the cosecant function is also 2π and that $y = \csc(Bt)$ is an odd function.

EXAMPLE 7 ▶ Graphing a Cosecant Function

Graph the function $y = \csc t$ for $t \in [0, 4\pi]$.

Solution ▶ The related sine function is $y = \sin t$, which means we'll draw a rectangular frame $2A = 2$ units high. The period is $P = \frac{2\pi}{1} = 2$, so the reference frame will be 2π units in length. Breaking the t-axis into four parts within the frame means each tick mark will be $\left(\frac{1}{4}\right)\left(\frac{2\pi}{1}\right) = \frac{\pi}{2}$ units apart, with the asymptotes occurring at 0, π, and 2π. A partial table and the resulting graph are shown.



t	$\sin t$	$\csc t$
0	0	$\frac{1}{0} \to$ undefined
$\frac{\pi}{6}$	$\frac{1}{2} = 0.5$	$\frac{2}{1} = 2$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.71$	$\frac{2}{\sqrt{2}} \approx 1.41$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{2}{\sqrt{3}} \approx 1.15$
$\frac{\pi}{2}$	1	1

Now try Exercises 33 and 34 ►

D. You've just learned how to investigate graphs of the reciprocal functions $f(t) = \csc(Bt)$ and $f(t) = \sec(Bt)$

Similar observations can be made regarding $y = \sec(Bt)$ and its relationship to $y = \cos(Bt)$ (see **Exercises 8, 35, and 36**). The most important characteristics of the cosecant and secant functions are summarized in the following box. For these functions, there is no discussion of amplitude, and no mention is made of their zeroes since neither graph intersects the t-axis.

Characteristics of $f(t) = \csc t$ and $f(t) = \sec t$

For all real numbers t and integers k,

$y = \csc t$

Domain	Range	Asymptotes
$t \neq k\pi$	$(-\infty, -1] \cup [1, \infty)$	$t = k\pi$

Period	Symmetry
2π	odd $\csc(-t) = -\csc t$

$y = \sec t$

Domain	Range	Asymptotes
$t \neq \frac{\pi}{2} + \pi k$	$(-\infty, -1] \cup [1, \infty)$	$t = \frac{\pi}{2} + \pi k$

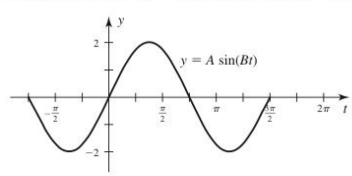
Period	Symmetry
2π	even $\sec(-t) = \sec t$

E. Writing Equations from Graphs

Mathematical concepts are best reinforced by working with them in both "forward and reverse." Where graphs are concerned, this means we should attempt to find the equation of a given graph, rather than only using an equation to sketch the graph. Exercises of this type require that you become very familiar with the graph's basic characteristics and how each is expressed as part of the equation.

EXAMPLE 8 ► Determining the Equation of a Given Graph

The graph shown here is of the form $y = A\sin(Bt)$. Find the value of A and B.



Solution ► By inspection, the graph has an amplitude of $A = 2$ and a period of $P = \frac{3\pi}{2}$. To find B we used the period formula $P = \frac{2\pi}{|B|}$, substituting $\frac{3\pi}{2}$ for P and solving.

$$P = \frac{2\pi}{|B|} \quad \text{period formula}$$

$$\frac{3\pi}{2} = \frac{2\pi}{B} \quad \text{substitute } \tfrac{3\pi}{2} \text{ for } P;\ B > 0$$

$$3\pi B = 4\pi \quad \text{multiply by } 2B$$

$$B = \frac{4}{3} \quad \text{solve for } B$$

The result is $B = \frac{4}{3}$, which gives us the equation $y = 2\sin\left(\frac{4}{3}t\right)$.

E. You've just learned how to write the equation for a given graph

Now try Exercises 37 through 58 ►

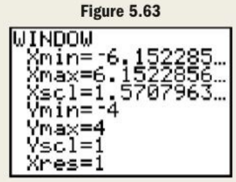
There are a number of interesting applications of this "graph to equation" process in the exercise set. **See Exercises 61 to 72.**

TECHNOLOGY HIGHLIGHT

Exploring Amplitudes and Periods

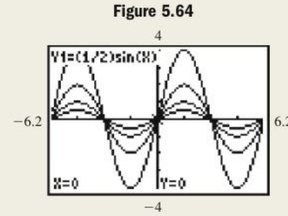
In practice, trig applications offer an immense range of coefficients, creating amplitudes that are sometimes very large and sometimes extremely small, as well as periods ranging from nanoseconds, to many years. This *Technology Highlight* is designed to help you use the calculator more effectively in the study of these functions. To begin, we note that many calculators offer a preset ZOOM option that automatically sets a window size convenient to many trig graphs. The resulting WINDOW after pressing ZOOM **7:ZTrig** on a TI-84 Plus is shown in Figure 5.63 for a calculator set in **Radian** MODE.

Figure 5.63



In Section 5.3 we noted that a change in amplitude will not change the location of the zeroes or max/min values. On the Y= screen, enter $Y_1 = \frac{1}{2}\sin x$, $Y_2 = \sin x$, $Y_3 = 2\sin x$, and $Y_4 = 4\sin x$, then use ZOOM **7:ZTrig** to graph the functions. As you see in Figure 5.64, each graph rises to the expected amplitude at the expected location, while "holding on" to the zeroes.

Figure 5.64



To explore concepts related to the coefficient B and the period of a trig function, enter $Y_1 = \sin\left(\frac{1}{2}x\right)$ and $Y_2 = \sin(2x)$ on the Y= screen and graph using ZOOM **7:ZTrig.** While the result is "acceptable," the graphs are difficult to read and compare, so we manually change the window size to obtain a better view (Figure 5.65).

A true test of effective calculator use comes when the amplitude or period is a very large or very small number. For instance, the tone you hear while pressing "5" on your telephone is actually a combination of the tones modeled by $Y_1 = \sin[2\pi(770)t]$ and $Y_2 = \sin[2\pi(1336)t]$. Graphing these functions requires a careful analysis of the period, otherwise the graph can appear garbled, misleading, or difficult to read —try graphing Y_1 on the ZOOM **7:ZTrig** or ZOOM **6:ZStandard** screens (see Figure 5.66). First note $A = 1$, and $P = \frac{2\pi}{2\pi 770}$ or $\frac{1}{770}$. With a period this short, even graphing the function from Xmin = −1 to Xmax = 1 gives a distorted graph. Setting Xmin to −1/770, Xmax to 1/770, and Xscl to (1/770)/10 gives the graph in Figure 5.67, which can be used to investigate characteristics of the function.

Figure 5.65

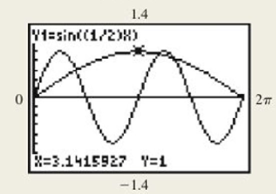


Figure 5.66

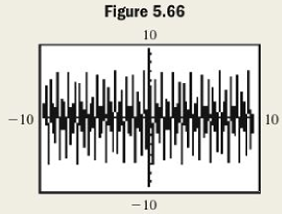
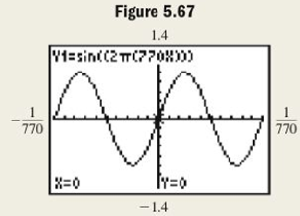


Figure 5.67



Exercise 1: Graph the second tone $Y_2 = \sin[2\pi(1336)t]$ and find its value at $t = 0.00025$ sec.

Exercise 2: Graph the function $Y_1 = 950\sin(0.005t)$ on a "friendly" window and find the value at $x = 550$.

5.5 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For the sine function, output values are ______ in the interval $\left[0, \frac{\pi}{2}\right]$.

2. For the cosine function, output values are ______ in the interval $\left[0, \frac{\pi}{2}\right]$.

3. For the sine and cosine functions, the domain is ______ and the range is ______.

4. The amplitude of sine and cosine is defined to be the maximum ______ from the ______ value in the positive and negative directions.

5. Discuss/Describe the four-step process outlined in this section for the graphing of basic trig functions. Include a worked-out example and a detailed explanation.

6. Discuss/Explain how you would determine the domain and range of $y = \sec x$. Where is this function undefined? Why? Graph $y = 2\sec(2t)$ using $y = 2\cos(2t)$. What do you notice?

▶ DEVELOPING YOUR SKILLS

7. Use the symmetry of the unit circle and reference arcs of standard values to complete a table of values for $y = \cos t$ in the interval $t \in [\pi, 2\pi]$.

8. Use the standard values for $y = \cos t$ for $t \in [\pi, 2\pi]$ to create a table of values for $y = \sec t$ on the same interval.

Use the characteristics of $f(t) = \sin t$ to match the given value of t (a through e) to the correct value of $\sin t$ (I through V).

9. a. $t = \left(\frac{\pi}{6} + 10\pi\right)$ **b.** $t = -\frac{\pi}{4}$

c. $t = \frac{-15\pi}{4}$ **d.** $t = 13\pi$

e. $t = \frac{21\pi}{2}$ **I.** $\sin t = 0$

II. $\sin t = \frac{1}{2}$ **III.** $\sin t = 1$

IV. $\sin t = \frac{\sqrt{2}}{2}$ **V.** $\sin t = -\frac{\sqrt{2}}{2}$

10. a. $t = \left(\frac{\pi}{4} - 12\pi\right)$ **b.** $t = \frac{11\pi}{6}$

c. $t = \frac{23\pi}{2}$ **d.** $t = -19\pi$

e. $t = -\frac{25\pi}{4}$ **I.** $\sin t = -\frac{1}{2}$

II. $\sin t = -\frac{\sqrt{2}}{2}$ **III.** $\sin t = 0$

IV. $\sin t = \frac{\sqrt{2}}{2}$ **V.** $\sin t = -1$

Use steps I through IV given in this section to draw a sketch of each graph.

11. $y = \sin t$ for $t \in \left[-\frac{3\pi}{2}, \frac{\pi}{2}\right]$

12. $y = \sin t$ for $t \in [-\pi, \pi]$

13. $y = \cos t$ for $t \in \left[-\frac{\pi}{2}, 2\pi\right]$

14. $y = \cos t$ for $t \in \left[-\frac{\pi}{2}, \frac{5\pi}{2}\right]$

Use a reference rectangle and the *rule of fourths* to draw an accurate sketch of the following functions through two complete cycles—one where $t > 0$, and one where $t < 0$. Clearly state the amplitude and period as you begin.

15. $y = 3\sin t$ **16.** $y = 4\sin t$

17. $y = -2\cos t$ **18.** $y = -3\cos t$

19. $y = \frac{1}{2}\sin t$ **20.** $y = \frac{3}{4}\sin t$

21. $y = -\sin(2t)$ **22.** $y = -\cos(2t)$

23. $y = 0.8\cos(2t)$ **24.** $y = 1.7\sin(4t)$

25. $f(t) = 4\cos\left(\frac{1}{2}t\right)$ **26.** $y = -3\cos\left(\frac{3}{4}t\right)$

27. $f(t) = 3\sin(4\pi t)$ **28.** $g(t) = 5\cos(8\pi t)$

29. $y = 4\sin\left(\frac{5\pi}{3}t\right)$ **30.** $y = 2.5\cos\left(\frac{2\pi}{5}t\right)$

31. $f(t) = 2\sin(256\pi t)$ **32.** $g(t) = 3\cos(184\pi t)$

Draw the graph of each function by first sketching the related sine and cosine graphs, and applying the observations made in this section.

33. $y = 3\csc t$ **34.** $g(t) = 2\csc(4t)$

35. $y = 2\sec t$ **36.** $f(t) = 3\sec(2t)$

Clearly state the amplitude and period of each function, then match it with the corresponding graph.

37. $y = -2\cos(4t)$ **38.** $y = 2\sin(4t)$

39. $y = 3\sin(2t)$ **40.** $y = -3\cos(2t)$

41. $y = 2\csc\left(\frac{1}{2}t\right)$ **42.** $y = 2\sec\left(\frac{1}{4}t\right)$

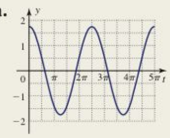
43. $f(t) = \frac{3}{4}\cos(0.4t)$ **44.** $g(t) = \frac{7}{4}\cos(0.8t)$

45. $y = \sec(8\pi t)$ **46.** $y = \csc(12\pi t)$

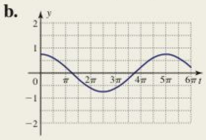
47. $y = 4\sin(144\pi t)$

48. $y = 4\cos(72\pi t)$

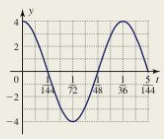
a.



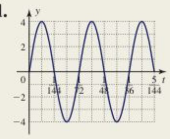
b.



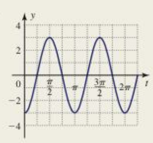
c.



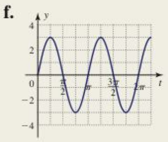
d.



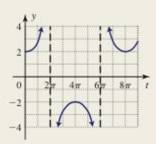
e.



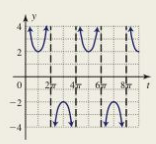
f.



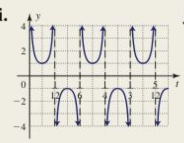
g.



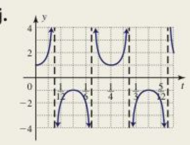
h.



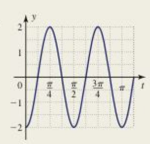
i.



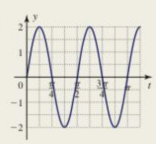
j.



k.

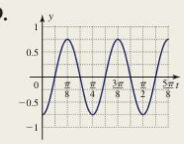


l.

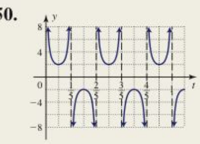


The graphs shown are of the form $y = A\cos(Bt)$ or $y = A\csc(Bt)$. Use the characteristics illustrated for each graph to determine its equation.

49.



50.



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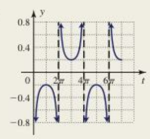
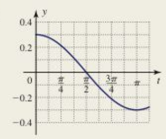
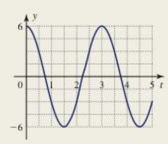


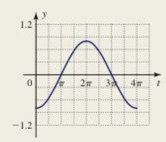
52.



53.

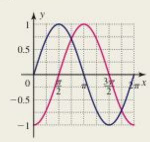


54.

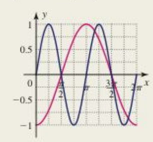


Match each graph to its equation, then graphically estimate the points of intersection. Confirm or contradict your estimate(s) by substituting the values into the given equations using a calculator.

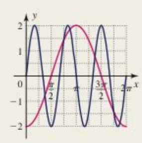
55. $y = -\cos x$; $y = \sin x$



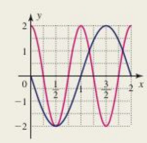
56. $y = -\cos x$; $y = \sin(2x)$



57. $y = -2\cos x$; $y = 2\sin(3x)$



58. $y = 2\cos(2\pi x)$; $y = -2\sin(\pi x)$

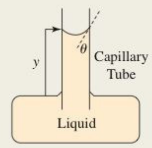


▶ WORKING WITH FORMULAS

59. The Pythagorean theorem in trigonometric form: $\sin^2\theta + \cos^2\theta = 1$

The formula shown is commonly known as a Pythagorean identity and is introduced more formally in Chapter 6. It is derived by noting that on a unit circle, $\cos t = x$ and $\sin t = y$, while $x^2 + y^2 = 1$. Given that $\sin t = \frac{15}{113}$, use the formula to find the value of $\cos t$ in Quadrant I. What is the Pythagorean triple associated with these values of x and y?

60. Hydrostatics, surface tension, and contact angles: $y = \dfrac{2\gamma\cos\theta}{kr}$

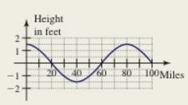


The height that a liquid will rise in a capillary tube is given by the formula shown, where r is the radius of the tube, θ is the contact angle of the liquid (the meniscus), γ is the surface tension of the liquid-vapor film, and k is a constant that depends on the weight-density of the liquid. How high will the liquid rise given that the surface tension $\gamma = 0.2706$, the tube has radius $r = 0.2$ cm, the contact angle $\theta = 22.5°$, and $k = 1.25$?

▶ APPLICATIONS

Tidal waves: Tsunamis, also known as tidal waves, are ocean waves produced by earthquakes or other upheavals in the Earth's crust and can move through the water undetected for hundreds of miles at great speed. While traveling in the open ocean, these waves can be represented by a sine graph with a very long wavelength (period) and a very small amplitude. Tsunami waves only attain a monstrous size as they approach the shore, and represent a very different phenomenon than the ocean swells created by heavy winds over an extended period of time.

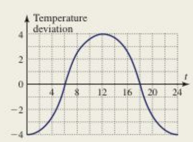
61. A graph modeling a tsunami wave is given in the figure. (a) What is the height of the tsunami wave (from crest to trough)? Note that $h = 0$ is considered the level of a calm ocean. (b) What is the tsunami's wavelength? (c) Find the equation for this wave.



62. A heavy wind is kicking up ocean swells approximately 10 ft high (from crest to trough), with wavelengths of 250 ft. (a) Find the equation that models these swells. (b) Graph the equation. (c) Determine the height of a wave measured 200 ft from the trough of the previous wave.

Sinusoidal models: The sine and cosine functions are of great importance to meteorological studies, as when modeling the temperature based on the time of day, the illumination of the Moon as it goes through its phases, or even the prediction of tidal motion.

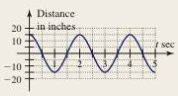
63. The graph given shows the deviation from the average daily temperature for the hours of a given day, with $t = 0$ corresponding to 6 A.M. (a) Use the graph to determine the related equation. (b) Use the equation to find the deviation at $t = 11$ (5 P.M.) and confirm that this point is on the graph. (c) If the average temperature for this day was 72°, what was the temperature at midnight?



64. The equation $y = 7\sin\left(\frac{\pi}{6}t\right)$ models the height of the tide along a certain coastal area, as compared to average sea level. Assuming $t = 0$ is midnight, (a) graph this function over a 12-hr period. (b) What will the height of the tide be at 5 A.M.? (c) Is the tide rising or falling at this time?

Sinusoidal movements: Many animals exhibit a wavelike motion in their movements, as in the tail of a shark as it swims in a straight line or the wingtips of a large bird in flight. Such movements can be modeled by a sine or cosine function and will vary depending on the animal's size, speed, and other factors.

65. The graph shown models the position of a shark's tail at time t, as measured to the left (negative) and right (positive) of a straight line along its length. (a) Use the graph to determine the related equation. (b) Is the tail to the right, left, or at center when $t = 6.5$ sec? How far? (c) Would you say the shark is "swimming leisurely," or "chasing its prey"? Justify your answer.

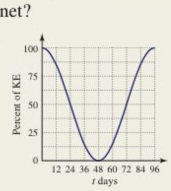


66. The State Fish of Hawaii is the *humuhumunukunukuapua'a*, a small colorful fish found abundantly in coastal waters. Suppose the tail motion of an adult fish is modeled by the equation $d(t) = \sin(15\pi t)$ with $d(t)$ representing the position of the fish's tail at time t, as measured in inches to the left (negative) or right (positive) of a straight line along its length. (a) Graph the equation over two periods. (b) Is the tail to the left or right of center at $t = 2.7$ sec? How far? (c) Would you say this fish is "swimming leisurely," or "running for cover"? Justify your answer.

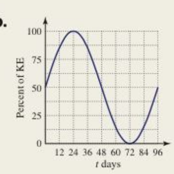
Kinetic energy: The kinetic energy a planet possesses as it orbits the Sun can be modeled by a cosine function. When the planet is at its apogee (greatest distance from the Sun), its kinetic energy is at its lowest point as it slows down and "turns around" to head back toward the Sun. The kinetic energy is at its highest when the planet "whips around the Sun" to begin a new orbit.

67. Two graphs are given here. (a) Which of the graphs could represent the kinetic energy of a planet orbiting the Sun if the planet is at its perigee (closest distance to the Sun) when $t = 0$? (b) For what value(s) of t does this planet possess 62.5% of its maximum kinetic energy with the kinetic energy increasing? (c) What is the orbital period of this planet?

a.

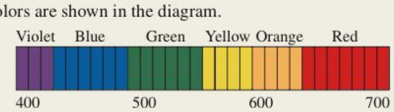


b.



68. The *potential energy* of the planet is the antipode of its kinetic energy, meaning when kinetic energy is at 100%, the potential energy is 0%, and when kinetic energy is at 0% the potential energy is at 100%. (a) How is the graph of the kinetic energy related to the graph of the potential energy? In other words, what transformation could be applied to the kinetic energy graph to obtain the potential energy graph? (b) If the kinetic energy is at 62.5% and increasing [as in Graph 67(b)], what can be said about the potential energy in the planet's orbit at this time?

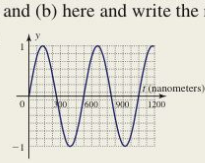
Visible light: One of the narrowest bands in the electromagnetic spectrum is the region involving visible light. The wavelengths (periods) of visible light vary from 400 nanometers (purple/violet colors) to 700 nanometers (bright red). The approximate wavelengths of the other colors are shown in the diagram.



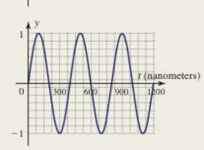
69. The equations for the colors in this spectrum have the form $y = \sin(\gamma t)$, where $\frac{2\pi}{\gamma}$ gives the length of the sine wave. (a) What color is represented by the equation $y = \sin\left(\frac{\pi}{240}t\right)$? (b) What color is represented by the equation $y = \sin\left(\frac{\pi}{310}t\right)$?

70. Name the color represented by each of the graphs (a) and (b) here and write the related equation.

a.



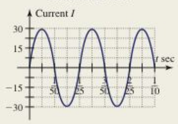
b.



Alternating current: Surprisingly, even characteristics of the electric current supplied to your home can be modeled by sine or cosine functions. For alternating current (AC), the amount of current I (in amps) at time t can be modeled by $I = A\sin(\omega t)$, where A represents the maximum current that is produced, and ω is related to the frequency at which the generators turn to produce the current.

71. Find the equation of the household current modeled by the graph, then use the equation to determine I when $t = 0.045$ sec. Verify that the resulting ordered pair is on the graph.

Exercise 71



72. If the *voltage* produced by an AC circuit is modeled by the equation $E = 155\sin(120\pi t)$, (a) what is the period and amplitude of the related graph? (b) What voltage is produced when $t = 0.2$?

▶ EXTENDING THE CONCEPT

73. For $y = A\sin(Bx)$ and $y = A\cos(Bx)$, the expression $\frac{M+m}{2}$ gives the average value of the function, where M and m represent the maximum and minimum values, respectively. What was the average value of every function graphed in this section? Compute a table of values for $y = 2\sin t + 3$, and note its maximum and minimum values. What is the average value of this function? What transformation has been applied to change the average value of the function? Can you name the average value of $y = -2\cos t + 1$ by inspection?

74. To understand where the period formula $P = \frac{2\pi}{B}$ came from, consider that if $B = 1$, the graph of $y = \sin(Bt) = \sin(1t)$ completes one cycle from $1t = 0$ to $1t = 2\pi$. If $B \neq 1$, $y = \sin(Bt)$ completes one cycle from $Bt = 0$ to $Bt = 2\pi$. Discuss how this observation validates the period formula.

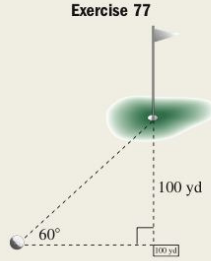
75. The tone you hear when pressing the digit "9" on your telephone is actually a combination of two separate tones, which can be modeled by the functions $f(t) = \sin[2\pi(852)t]$ and $g(t) = \sin[2\pi(1477)t]$. Which of the two functions has the shortest period? By carefully scaling the axes, graph the function having the shorter period using the steps I through IV discussed in this section.

▶ MAINTAINING YOUR SKILLS

76. **(5.2)** Given $\sin 1.12 \approx 0.9$, find an additional value of t in $[0, 2\pi)$ that makes the equation $\sin t \approx 0.9$ true.

77. **(5.1)** Use a special triangle to calculate the distance from the ball to the pin on the seventh hole, given the ball is in a straight line with the 100-yd plate, as shown in the figure.

Exercise 77



78. **(5.1)** Invercargill, New Zealand, is at 46°14′24″ south latitude. If the Earth has a radius of 3960 mi, how far is Invercargill from the equator?

79. **(1.4)** Given $z_1 = 1 + i$ and $z_2 = 2 - 5i$, compute the following:

a. $z_1 + z_2$

b. $z_1 - z_2$

c. $z_1 z_2$

d. $\frac{z_2}{z_1}$

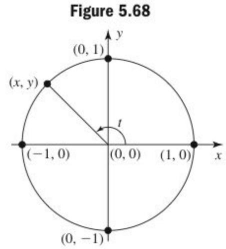
5.6 Graphs of Tangent and Cotangent Functions

Learning Objectives

In Section 5.6 you will learn how to:

- ❑ **A.** Graph $y = \tan t$ using asymptotes, zeroes, and the ratio $\frac{\sin t}{\cos t}$
- ❑ **B.** Graph $y = \cot t$ using asymptotes, zeroes, and the ratio $\frac{\cos t}{\sin t}$
- ❑ **C.** Identify and discuss important characteristics of $y = \tan t$ and $y = \cot t$
- ❑ **D.** Graph $y = A\tan(Bt)$ and $y = A\cot(Bt)$ with various values of A and B
- ❑ **E.** Solve applications of $y = \tan t$ and $y = \cot t$

Figure 5.68



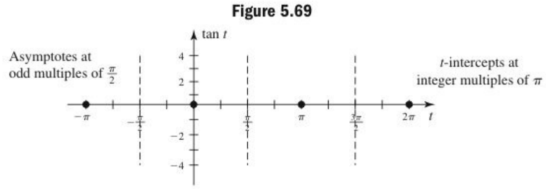
$\tan t = \frac{y}{x}$

Unlike the other four trig functions, tangent and cotangent have no maximum or minimum value on any open interval of their domain. However, it is precisely this unique feature that adds to their value as mathematical models. Collectively, the six functions give scientists the tools they need to study, explore, and investigate a wide range of phenomena, extending our understanding of the world around us.

A. The Graph of $y = \tan t$

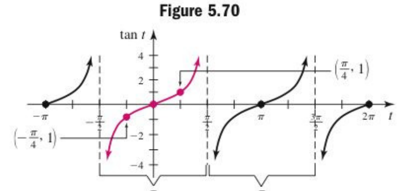
Like the secant and cosecant functions, tangent is defined in terms of a ratio, creating asymptotic behavior at the zeroes of the denominator. In terms of the unit circle, $\tan t = \frac{y}{x}$, which means in $[-\pi, 2\pi]$, vertical asymptotes occur at $t = -\frac{\pi}{2}$, $t = \frac{\pi}{2}$, and $\frac{3\pi}{2}$, since the x-coordinate on the unit circle is zero (see Figure 5.68). We further note $\tan t = 0$ when the y-coordinate is zero, so the function will have t-intercepts at $t = -\pi, 0, \pi$, and 2π in the same interval. This produces the framework for graphing the tangent function shown in Figure 5.69.

Figure 5.69



Knowing the graph must go through these zeroes and approach the asymptotes, we are left with determining the *direction of the approach.* This can be discovered by noting that in QI, the y-coordinates of points on the unit circle start at 0 and increase, while the x-values start at 1 and decrease. This means the ratio $\frac{y}{x}$ defining $\tan t$ is increasing, and in fact becomes infinitely large as t gets very close to $\frac{\pi}{2}$. A similar observation can be made for a negative rotation of t in QIV. Using the additional points provided by $\tan\left(-\frac{\pi}{4}\right) = -1$ and $\tan\left(\frac{\pi}{4}\right) = 1$, we find the graph of $\tan t$ is increasing throughout the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and that the function has a period of π. We also note $y = \tan t$ is an odd function (symmetric about the origin), since $\tan(-t) = -\tan t$ as evidenced by the two points just computed. The completed graph is shown in Figure 5.70 with the primary interval in red.

Figure 5.70



The graph can also be developed by noting $\sin t = y$, $\cos t = x$, and $\tan t = \frac{y}{x}$. This gives $\tan t = \frac{\sin t}{\cos t}$ by direct substitution and we can quickly complete a table of values for tan *t*, as shown in Example 1. These and other relationships between the trig functions will be fully explored in Chapter 6.

EXAMPLE 1 ► Constructing a Table of Values for $f(t) = \tan t$

Complete Table 5.11 shown for $\tan t = \frac{y}{x}$ using the values given for sin *t* and cos *t*, then graph the function by plotting points.

Table 5.11

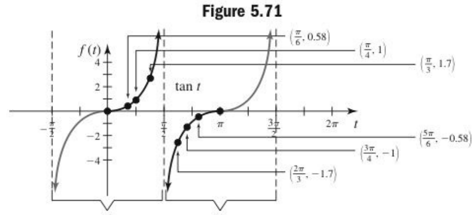
t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin t = y$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos t = x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan t = \frac{y}{x}$									

Solution ► For the noninteger values of *x* and *y*, the "twos will cancel" each time we compute $\frac{y}{x}$. This means we can simply list the ratio of numerators. The resulting points are shown in Table 5.12, along with the plotted points. The graph shown in Figure 5.71 was completed using symmetry and the previous observations.

Table 5.12

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin t = y$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos t = x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan t = \frac{y}{x}$	0	$\frac{1}{\sqrt{3}} \approx 0.58$	1	$\sqrt{3} \approx 1.7$	undefined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

Figure 5.71



Now try Exercises 7 and 8 ►

A. You've just learned how to graph $y = \tan t$ using asymptotes, zeroes, and the ratio $\frac{\sin t}{\cos t}$

Additional values can be found using a calculator as needed. For future use and reference, it will help to recognize the approximate decimal equivalent of all special values and radian angles. In particular, note that $\sqrt{3} \approx 1.73$ and $\frac{1}{\sqrt{3}} \approx 0.58$. **See Exercises 9 through 14.**

B. The Graph of $y = \cot t$

Since the cotangent function is also defined in terms of a ratio, it too displays asymptotic behavior at the zeroes of the denominator, with t-intercepts at the zeroes of the numerator. Like the tangent function, $\cot t = \frac{x}{y}$ can be written in terms of $\cos t = x$ and $\sin t = y$: $\cot t = \frac{\cos t}{\sin t}$, and the graph obtained by plotting points.

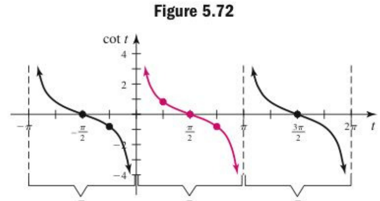
EXAMPLE 2 ► **Constructing a Table of Values for $f(t) = \cot t$**

Complete a table of values for $\cot t = \frac{x}{y}$ for t in $[0, \pi]$ using its ratio relationship with $\cos t$ and $\sin t$. Use the results to graph the function for t in $(-\pi, 2\pi)$.

Solution ► The completed table is shown here. In this interval, the cotangent function has asymptotes at 0 and π since $y = 0$ at these points, and has a t-intercept at $\frac{\pi}{2}$ since $x = 0$. The graph shown in Figure 5.72 was completed using the period $P = \pi$.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin t = y$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos t = x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\cot t = \frac{x}{y}$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	undefined

Figure 5.72



Now try Exercises 15 and 16 ►

B. You've just learned how to graph $y = \cot t$ using asymptotes, zeroes, and the ratio $\frac{\cos t}{\sin t}$

C. Characteristics of $y = \tan t$ and $y = \cot t$

The most important characteristics of the tangent and cotangent functions are summarized in the following box. There is no discussion of amplitude, maximum, or minimum values, since maximum or minimum values do not exist. For future use and reference, perhaps the most significant characteristic distinguishing tan t from cot t is that *tan t increases,* while *cot t decreases* over their respective domains. Also note that due to symmetry, the zeroes of each function are always located halfway between the asymptotes.

Characteristics of $f(t) = \tan t$ and $f(t) = \cot t$

For all real numbers t and integers k,

$y = \tan t$			$y = \cot t$		
Domain	**Range**	**Asymptotes**	**Domain**	**Range**	**Asymptotes**
$t \neq \frac{\pi}{2} + \pi k$	$(-\infty, \infty)$	$t = \frac{\pi}{2} + \pi k$	$t \neq k\pi$	$(-\infty, \infty)$	$t = k\pi$
Period	**Behavior**	**Symmetry**	**Period**	**Behavior**	**Symmetry**
π	increasing	odd $\tan(-t) = -\tan t$	π	decreasing	odd $\cot(-t) = -\cot t$

EXAMPLE 3 ► Using the Period of $f(t) = \tan t$ to Find Additional Points

Given $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, what can you say about $\tan\left(\frac{7\pi}{6}\right)$, $\tan\left(\frac{13\pi}{6}\right)$, and $\tan\left(-\frac{5\pi}{6}\right)$?

Solution ► Each value of t differs from $\frac{\pi}{6}$ by a multiple of π: $\tan\left(\frac{7\pi}{6}\right) = \tan\left(\frac{\pi}{6} + \pi\right)$, $\tan\left(\frac{13\pi}{6}\right) = \tan\left(\frac{\pi}{6} + 2\pi\right)$ and $\tan\left(-\frac{5\pi}{6}\right) = \tan\left(\frac{\pi}{6} - \pi\right)$. Since the period of the tangent function is $P = \pi$, all of these expressions have a value of $\frac{1}{\sqrt{3}}$.

Now try Exercises 17 through 22 ►

Since the tangent function is more common than the cotangent, many needed calculations will first be done using the tangent function and its properties, then reciprocated. For instance, to evaluate $\cot\left(-\frac{\pi}{6}\right)$ we reason that cot t is an odd function, so $\cot\left(-\frac{\pi}{6}\right) = -\cot\left(\frac{\pi}{6}\right)$. Since cotangent is the reciprocal of tangent and $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, $-\cot\left(\frac{\pi}{6}\right) = -\sqrt{3}$. **See Exercises 23 and 24.**

C. You've just learned how to identify and discuss important characteristics of $y = \tan t$ and $y = \cot t$

D. Graphing $y = A\tan(Bt)$ and $y = A\cot(Bt)$

The Coefficient A: Vertical Stretches and Compressions

For the tangent and cotangent functions, the role of coefficient A is best seen through an analogy from basic algebra (the concept of amplitude is foreign to these functions). Consider the graph of $y = x^3$ (Figure 5.73). Comparing the parent function $y = x^3$ with functions $y = Ax^3$, the graph is stretched vertically if $|A| > 1$ (see Figure 5.74) and compressed if $0 < |A| < 1$. In the latter case the graph becomes very "flat" near the zeroes, as shown in Figure 5.75.

Figure 5.73
$y = x^3$

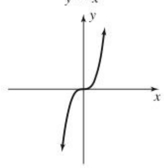


Figure 5.74
$y = 4x^3; A = 4$

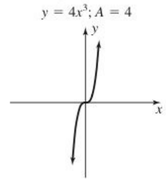
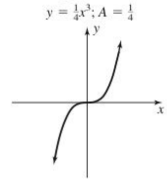


Figure 5.75
$y = \frac{1}{4}x^3; A = \frac{1}{4}$

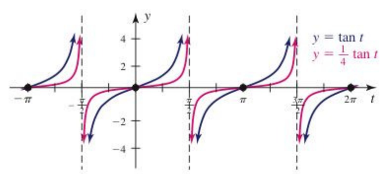


While ***cubic functions are not asymptotic,*** they are a good illustration of A's effect on the tangent and cotangent functions. Fractional values of A ($|A| < 1$) compress the graph, flattening it out near its zeroes. Numerically, this is because a fractional part of a small quantity is an even smaller quantity. For instance, compare $\tan\left(\frac{\pi}{6}\right)$ with $\frac{1}{4}\tan\left(\frac{\pi}{6}\right)$. To two decimal places, $\tan\left(\frac{\pi}{6}\right) = 0.57$, while $\frac{1}{4}\tan\left(\frac{\pi}{6}\right) = 0.14$, so the graph must be "nearer the t-axis" at this value.

EXAMPLE 4 ► **Comparing the Graph of $f(t) = \tan t$ and $g(t) = A \tan t$**

Draw a "comparative sketch" of $y = \tan t$ and $y = \frac{1}{4}\tan t$ on the same axis and discuss similarities and differences. Use the interval $[-\pi, 2\pi]$.

Solution ► Both graphs will maintain their essential features (zeroes, asymptotes, period, increasing, and so on). However, the graph of $y = \frac{1}{4}\tan t$ is vertically compressed, causing it to flatten out near its zeroes and changing how the graph approaches its asymptotes in each interval.



Now try Exercises 25 through 28 ►

The Coefficient *B*: The Period of Tangent and Cotangent

Like the other trig functions, the value of B has a material impact on the period of the function, and with the same effect. The graph of $y = \cot(2t)$ completes a cycle twice as fast as $y = \cot t$ $\left(P = \frac{\pi}{2} \text{ versus } P = \pi\right)$, while $y = \cot\left(\frac{1}{2}t\right)$ completes a cycle one-half as fast ($P = 2\pi$ versus $P = \pi$).

This reasoning leads us to a **period formula** for tangent and cotangent, namely, $P = \frac{\pi}{|B|}$, where B is the coefficient of the input variable.

WORTHY OF NOTE

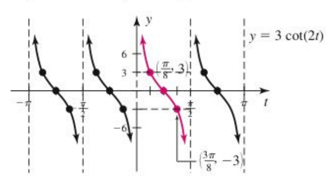
It may be easier to interpret the phrase "twice as fast" as $2P = \pi$ and "one-half as fast" as $\frac{1}{2}P = \pi$. In each case, solving for P gives the correct interval for the period of the new function.

Similar to the four-step process used to graph sine and cosine functions, we can graph tangent and cotangent functions using a rectangle $P = \frac{\pi}{B}$ units in length and $2A$ units high, centered on the primary interval. After dividing the length of the rectangle into fourths, the t-intercept will always be the halfway point, with y-values of $|A|$ occurring at the $\frac{1}{4}$ and $\frac{3}{4}$ marks. See Example 5.

EXAMPLE 5 ▶ Graphing $y = A\cot(Bt)$ for $A, B, \neq 1$

Sketch the graph of $y = 3\cot(2t)$ over the interval $[-\pi, \pi]$.

Solution ▶ For $y = 3\cot(2t)$, $|A| = 3$ which results in a vertical stretch, and $|B| = 2$ which gives a period of $\frac{\pi}{2}$. The function is still undefined at $t = 0$ and is asymptotic there, then at all integer multiples of $P = \frac{\pi}{2}$. We also know the graph is decreasing, with zeroes of the function halfway between the asymptotes. The inputs $t = \frac{\pi}{8}$ and $t = \frac{3\pi}{8}$ $\left(\text{the } \frac{1}{4} \text{ and } \frac{3}{4} \text{ marks between 0 and } \frac{\pi}{2}\right)$ yield the points $\left(\frac{\pi}{8}, 3\right)$ and $\left(\frac{3\pi}{8}, -3\right)$, which we'll use along with the period and symmetry of the function to complete the graph:

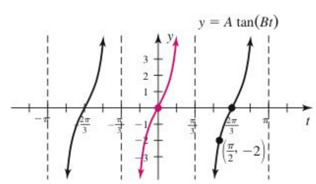


Now try Exercises 29 through 40 ▶

As with the trig functions from Section 5.3, it is possible to determine the equation of a tangent or cotangent function from a given graph. Where previously we used the amplitude, period, and max/min values to obtain our equation, here we first determine the period of the function by calculating the "distance" between asymptotes, then choose any convenient point on the graph (other than a t-intercept) and substitute in the equation to solve for A.

EXAMPLE 6 ▶ Constructing the Equation for a Given Graph

Find the equation of the graph, given it's of the form $y = A\tan(Bt)$.



Solution ► Using the primary interval and the asymptotes at $t = -\frac{\pi}{3}$ and $t = \frac{\pi}{3}$, we find the period is $P = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}$. To find the value of B we substitute $\frac{2\pi}{3}$ for P in $P = \frac{\pi}{B}$ and find $B = \frac{3}{2}$ (verify). This gives the equation $y = A\tan\left(\frac{3}{2}t\right)$.

To find A, we take the point $\left(\frac{\pi}{2}, -2\right)$ shown, and use $t = \frac{\pi}{2}$ with $y = -2$ to solve for A:

$$y = A\tan\left(\frac{3}{2}t\right) \quad \text{substitute } \tfrac{3}{2} \text{ for } B$$

$$-2 = A\tan\left[\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right)\right] \quad \text{substitute } -2 \text{ for } y \text{ and } \tfrac{\pi}{2} \text{ for } t$$

$$-2 = A\tan\left(\frac{3\pi}{4}\right) \quad \text{multiply}$$

$$A = \frac{-2}{\tan\left(\frac{3\pi}{4}\right)} \quad \text{solve for } A$$

$$= 2 \quad \text{result}$$

The equation of the graph is $y = 2\tan(\frac{3}{2}t)$.

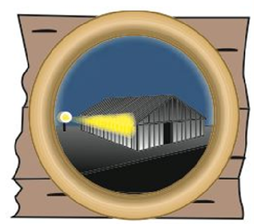
D. You've just learned how to graph $y = A\tan(Bt)$ and $y = A\cot(Bt)$ with various values of A and B

Now try Exercises 41 through 46 ►

E. Applications of Tangent and Cotangent Functions

We end this section with one example of how tangent and cotangent functions can be applied. Numerous others can be found in the exercise set.

EXAMPLE 7 ► Applications of $y = A\tan(Bt)$: Modeling the Movement of a Light Beam



One evening, in port during a *Semester at Sea,* Richard is debating a project choice for his Precalculus class. Looking out his porthole, he notices a revolving light turning at a constant speed near the corner of a long warehouse. The light throws its beam along the length of the warehouse, then disappears into the air, and then returns time and time again. Suddenly—Richard has his project. He notes the time it takes the beam to traverse the warehouse wall is very close to 4 sec, and in the morning he measures the wall's length at 127.26 m. His project? Modeling the distance of the beam from the corner of the warehouse as a function of time using a tangent function. Can you help?

Solution ► The equation model will have the form $D(t) = A\tan(Bt)$, where $D(t)$ is the distance (in meters) of the beam from the corner after t sec. The distance along the wall is measured in positive values so we're using only $\frac{1}{2}$ the period of the function, giving $\frac{1}{2}P = 4$ (the beam "disappears" at $t = 4$) so $P = 8$. Substitution in the period formula gives $B = \frac{\pi}{8}$ and the equation $D = A\tan\left(\frac{\pi}{8}t\right)$.

Knowing the beam travels 127.26 m in about 4 sec (when it disappears into infinity), we'll use $t = 3.9$ and $D = 127.26$ in order to solve for A and complete our equation model (see note following this example).

$$A\tan\left(\frac{\pi}{8}t\right) = D \quad \text{equation model}$$

$$A\tan\left[\frac{\pi}{8}(3.9)\right] = 127.26 \quad \text{substitute 127.26 for } D \text{ and 3.9 for } t$$

$$A = \frac{127.26}{\tan\left[\frac{\pi}{8}(3.9)\right]} \quad \text{solve for } A$$

$$\approx 5 \quad \text{result}$$

One equation approximating the distance of the beam from the corner of the warehouse is $D(t) = 5\tan\left(\frac{\pi}{8}t\right)$.

Now try Exercises 49 through 52 ►

E. You've just learned how to solve applications of $y = \tan t$ and $y = \cot t$

For Example 7, we should note the choice of 3.9 for t was arbitrary, and while we obtained an "acceptable" model, different values of A would be generated for other choices. For instance, $t = 3.95$ gives $A \approx 2.5$, while $t = 3.99$ gives $A \approx 0.5$. The true value of A depends on the distance of the light from the corner of the warehouse wall. In any case, it's interesting to note that at $t = 2$ sec (one-half the time it takes the beam to disappear), the beam has traveled only 5m from the corner of the building: $D(2) = 5\tan\left(\frac{\pi}{4}\right) = 5$ m. Although the light is rotating at a constant angular speed, the speed of the beam along the wall increases *dramatically* as t gets close to 4 sec.

TECHNOLOGY HIGHLIGHT

Zeroes, Asymptotes, and the Tangent/Cotangent Functions

In this *Technology Highlight* we'll explore the tangent and cotangent functions from the perspective of their ratio definition. While we could easily use $Y_1 = \tan x$ to generate and explore the graph, we would miss an opportunity to note the many important connections that emerge from a ratio definition perspective. To begin, enter $Y_1 = \sin x$, $Y_2 = \cos x$, and $Y_3 = \frac{Y_1}{Y_2}$, as shown in Figure 5.76 [recall that function variables are accessed using VARS ► **(Y-VARS)** ENTER **(1:Function)]**. Note that Y_2 has been disabled by overlaying the cursor on the equal sign and pressing ENTER. In addition, note the slash next to Y_1 is more **bold** than the other slashes. The TI-84 Plus offers options that help distinguish between graphs when more than one is being displayed, and we selected a **bold** line for Y_1 by moving the cursor to the far left position and repeatedly pressing ENTER until the desired option appeared. Pressing ZOOM **7:ZTrig** at this point produces the screen shown in Figure 5.77, where we note that tan x is zero everywhere that sin x

Figure 5.76

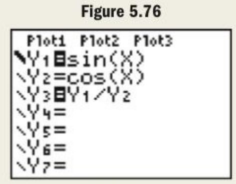
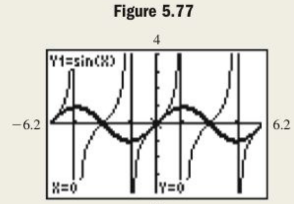
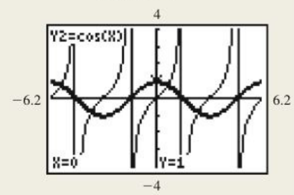


Figure 5.77



is zero. This is hardly surprising since $\tan x = \frac{\sin x}{\cos x}$, but is a point that is often overlooked. Going back to the Y= screen and disabling Y_1 while enabling Y_2 will produce the graph shown in Figure 5.78.

Figure 5.78



Exercise 1: What do you notice about the zeroes of cos *x* as they relate to the graph of $Y_3 = \tan x$?

Exercise 2: Go to the Y= screen and change Y_3 from $\frac{Y_1}{Y_2}$ (tangent) to $\frac{Y_2}{Y_1}$ (cotangent), then repeat the previous investigation regarding $y = \sin x$ and $y = \cos x$.

5.6 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The period of $y = \tan t$ and $y = \cot t$ is ________. To find the period of $y = \tan(Bt)$ and $y = \cot(Bt)$, the formula _________ is used.
2. The function $y = \tan t$ is ___________ everywhere it is defined. The function $y = \cot t$ is ___________ everywhere it is defined.
3. Tan *t* and cot *t* are _______ functions, so $f(-t) =$ ___________. If $\tan\left(-\frac{11\pi}{12}\right) \approx 0.268$, then $\tan\left(\frac{11\pi}{12}\right) \approx$ _________.
4. The asymptotes of $y =$ _________ are located at odd multiples of $\frac{\pi}{2}$. The asymptotes of $y =$ _________ are located at integer multiples of π.
5. Discuss/Explain how you can obtain a table of values for $y = \cot t$ (a) given the values for $y = \sin t$ and $y = \cos t$, and (b) given the values for $y = \tan t$.
6. Explain/Discuss how the zeroes of $y = \sin t$ and $y = \cos t$ are related to the graphs of $y = \tan t$ and $y = \cot t$. How can these relationships help graph functions of the form $y = A\tan(Bt)$ and $y = A\cot(Bt)$?

▶ DEVELOPING YOUR SKILLS

Use the values given for sin *t* and cos *t* to complete the tables.

7.

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\sin t = y$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\cos t = x$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\tan t = \frac{y}{x}$					

8.

	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin t = y$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\cos t = x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\tan t = \frac{y}{x}$					

9. Without reference to a text or calculator, attempt to name the decimal equivalent of the following values to one decimal place.

$$\frac{\pi}{2} \quad \frac{\pi}{4} \quad \frac{\pi}{6} \quad \sqrt{2} \quad \frac{\sqrt{2}}{2} \quad \frac{2}{\sqrt{3}}$$

10. Without reference to a text or calculator, attempt to name the decimal equivalent of the following values to one decimal place.

$$\frac{\pi}{3} \quad \pi \quad \frac{3\pi}{2} \quad \sqrt{3} \quad \frac{\sqrt{3}}{2} \quad \frac{1}{\sqrt{3}}$$

11. State the value of each expression without the use of a calculator.

a. $\tan\left(-\frac{\pi}{4}\right)$ **b.** $\cot\left(\frac{\pi}{6}\right)$

c. $\cot\left(\frac{3\pi}{4}\right)$ **d.** $\tan\left(\frac{\pi}{3}\right)$

12. State the value of each expression without the use of a calculator.

a. $\cot\left(\frac{\pi}{2}\right)$ **b.** $\tan \pi$

c. $\tan\left(-\frac{5\pi}{4}\right)$ **d.** $\cot\left(-\frac{5\pi}{6}\right)$

13. State the value of t without the use of a calculator, given $t \in [0, 2\pi)$ terminates in the quadrant indicated.

a. $\tan t = -1$, t in QIV

b. $\cot t = \sqrt{3}$, t in QIII

c. $\cot t = -\frac{1}{\sqrt{3}}$, t in QIV

d. $\tan t = -1$, t in QII

14. State the value of t without the use of a calculator, given $t \in [0, 2\pi)$ terminates in the quadrant indicated.

a. $\cot t = 1$, t in QI

b. $\tan t = -\sqrt{3}$, t in QII

c. $\tan t = \frac{1}{\sqrt{3}}$, t in QI

d. $\cot t = 1$, t in QIII

Use the values given for sin *t* and cos *t* to complete the tables.

15.

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\sin t = y$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\cos t = x$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\cot t = \frac{x}{y}$					

16.

	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin t = y$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\cos t = x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cot t = \frac{x}{y}$					

17. Given $t = \frac{11\pi}{24}$ is a solution to $\tan t \approx 7.6$, use the period of the function to name three additional solutions. Check your answer using a calculator.

18. Given $t = \frac{7\pi}{24}$ is a solution to $\cot t \approx 0.77$, use the period of the function to name three additional solutions. Check your answer using a calculator.

19. Given $t \approx 1.5$ is a solution to $\cot t = 0.07$, use the period of the function to name three additional solutions. Check your answers using a calculator.

20. Given $t \approx 1.25$ is a solution to $\tan t = 3$, use the period of the function to name three additional solutions. Check your answers using a calculator.

Verify the value shown for *t* is a solution to the equation given, then use the period of the function to name all real roots. Check two of these roots on a calculator.

21. $t = \frac{\pi}{10}$; $\tan t \approx 0.3249$

22. $t = -\frac{\pi}{16}$; $\tan t \approx -0.1989$

23. $t = \frac{\pi}{12}$; $\cot t = 2 + \sqrt{3}$

24. $t = \frac{5\pi}{12}$; $\cot t = 2 - \sqrt{3}$

Graph each function over the interval indicated, noting the period, asymptotes, zeroes, and value of A. Include a comparative sketch of $y = \tan t$ or $y = \cot t$ as indicated.

25. $f(t) = 2\tan t;\ [-2\pi, 2\pi]$

26. $g(t) = \frac{1}{2}\tan t;\ [-2\pi, 2\pi]$

27. $h(t) = 3\cot t;\ [-2\pi, 2\pi]$

28. $r(t) = \frac{1}{4}\cot t;\ [-2\pi, 2\pi]$

Graph each function over the interval indicated, noting the period, asymptotes, zeroes, and value of A and B.

29. $y = \tan(2t);\ \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

30. $y = \tan\left(\frac{1}{4}t\right);\ [-4\pi, 4\pi]$

31. $y = \cot(4t);\ \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

32. $y = \cot\left(\frac{1}{2}t\right);\ [-2\pi, 2\pi]$

33. $y = 2\tan(4t);\ \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

34. $y = 4\tan\left(\frac{1}{2}t\right);\ [-2\pi, 2\pi]$

35. $y = 5\cot\left(\frac{1}{3}t\right);\ [-3\pi, 3\pi]$

36. $y = \frac{1}{2}\cot(2t);\ \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

37. $y = 3\tan(2\pi t);\ \left[-\frac{1}{2}, \frac{1}{2}\right]$

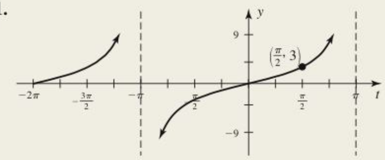
38. $y = 4\tan\left(\frac{\pi}{2}t\right);\ [-2, 2]$

39. $f(t) = 2\cot(\pi t);\ [-1, 1]$

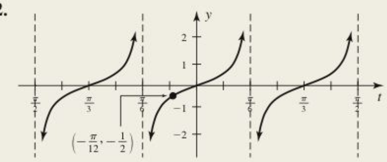
40. $p(t) = \frac{1}{2}\cot\left(\frac{\pi}{4}t\right);\ [-4, 4]$

Find the equation of each graph, given it is of the form $y = A\tan(Bt)$.

41.

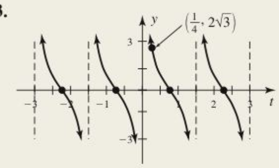


42.

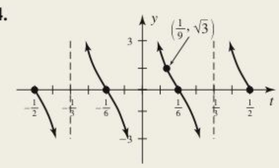


Find the equation of each graph, given it is of the form $y = A\cot(Bt)$.

43.



44.



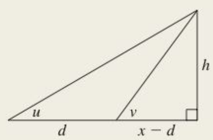
45. Given that $t = -\frac{\pi}{8}$ and $t = -\frac{3\pi}{8}$ are solutions to $\cot(3t) = \tan t$, use a graphing calculator to find two additional solutions in $[0, 2\pi]$.

46. Given $t = \frac{1}{6}$ is a solution to $\tan(2\pi t) = \cot(\pi t)$, use a graphing calculator to find two additional solutions in $[-1, 1]$.

▶ WORKING WITH FORMULAS

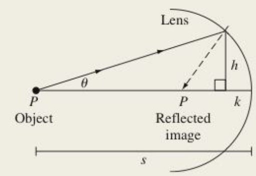
47. The height of an object calculated from a distance: $h = \dfrac{d}{\cot u - \cot v}$

The height h of a tall structure can be computed using two angles of elevation measured some distance apart along a straight line with the object. This height is given by the formula shown, where d is the distance between the two points from which angles u and v were measured. Find the height h of a building if $u = 40°$, $v = 65°$, and $d = 100$ ft.



48. Position of an image reflected from a spherical lens: $\tan\theta = \dfrac{h}{s - k}$

The equation shown is used to help locate the position of an image reflected by a spherical mirror, where s is the distance of the object from the lens along a horizontal axis, θ is the angle of elevation from this axis, h is the altitude of the right triangle indicated, and k is distance from the lens to the foot of altitude h. Find the distance k given $h = 3$ mm, $\theta = \dfrac{\pi}{24}$, and that the object is 24 mm from the lens.



▶ APPLICATIONS

Tangent function data models: Model the data in Exercises 49 and 50 using the function $y = A\tan(Bx)$. State the period of the function, the location of the asymptotes, the value of A, and name the point (x, y) used to calculate A (answers may vary). Use your equation model to evaluate the function at $x = -2$ and $x = 2$. What observations can you make? Also see Exercise 58.

49.

Input	Output	Input	Output
−6	−∞	1	1.4
−5	−20	2	3
−4	−9.7	3	5.2
−3	−5.2	4	9.7
−2	−3	5	20
−1	−1.4	6	∞
0	0		

50.

Input	Output	Input	Output
−3	−∞	0.5	6.4
−2.5	−91.3	1	13.7
−2	−44.3	1.5	23.7
−1.5	−23.7	2	44.3
−1	−13.7	2.5	91.3
−0.5	−6.4	3	∞
0	0		

51. As part of a lab setup, a laser pen is made to swivel on a large protractor as illustrated in the figure. For their lab project, students are asked to take the instrument to one end of a long hallway and measure the distance of the projected beam relative to the angle the pen is being held, and collect the data in a table. Use the data to find a function of the form $y = A\tan(B\theta)$. State the period of the function, the location of the asymptotes, the value of A, and name the point (θ, y) you used to calculate A (answers may vary). Based on the result, can you approximate the length of the laser pen? Note that in degrees, the period formula for tangent is $P = \dfrac{180°}{B}$.

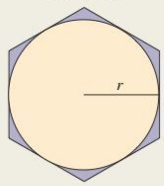
Exercise 51

θ (degrees)	Distance (cm)
0	0
10	2.1
20	4.4
30	6.9
40	10.1
50	14.3
60	20.8
70	33.0
80	68.1
89	687.5

52. Use the equation model obtained in Exercise 51 to compare the values given by the equation with the actual data. As a percentage, what was the largest deviation between the two?

53. Circumscribed polygons: The *perimeter* of a regular polygon circumscribed about a circle of radius r is given by $P = 2nr\tan\left(\frac{\pi}{n}\right)$, where n is the number of sides ($n \geq 3$) and r is the radius of the circle. Given $r = 10$ cm, (a) What is the circumference of the circle? (b) What is the circumference of the polygon when $n = 4$? Why? (c) Calculate the perimeter of the polygon for $n = 10, 20, 30$, and 100. What do you notice?

Exercise 53



54. Circumscribed polygons: The *area* of a regular polygon circumscribed about a circle of radius r is given by $A = nr^2\tan\left(\frac{\pi}{n}\right)$, where n is the number of sides ($n \geq 3$) and r is the radius of the circle. Given $r = 10$ cm,

a. What is the area of the circle?

b. What is the area of the polygon when $n = 4$? Why?

c. Calculate the area of the polygon for $n = 10, 20, 30$, and 100. What do you notice?

Coefficients of friction: Pulling someone on a sled is much easier during the winter than in the summer, due to a phenomenon known as the *coefficient of friction*. The friction between the sled's skids and the snow is much lower than the friction between the skids and the dry ground or pavement. Basically, the coefficient of friction is defined by the relationship $\mu = \tan\theta$, where θ is the angle at which a block composed of one material will slide down an inclined plane made of another material, with a constant velocity. Coefficients of friction have been established experimentally for many materials and a short list is shown here.

Material	Coefficient
steel on steel	0.74
copper on glass	0.53
glass on glass	0.94
copper on steel	0.68
wood on wood	0.5

55. Graph the function $\mu = \tan\theta$, with θ in degrees over the interval $[0°, 60°]$ and use the graph to estimate solutions to the following. Confirm or contradict your estimates using a calculator.

a. A block of copper is placed on a sheet of steel, which is slowly inclined. Is the block of copper moving when the angle of inclination is 30°? At what angle of inclination will the copper block be moving with a constant velocity down the incline?

b. A block of copper is placed on a sheet of cast-iron. As the cast-iron sheet is slowly inclined, the copper block begins sliding at a constant velocity when the angle of inclination is approximately 46.5°. What is the coefficient of friction for copper on cast-iron?

c. Why do you suppose coefficients of friction greater than $\mu = 2.5$ are extremely rare? Give an example of two materials that likely have a high μ-value.

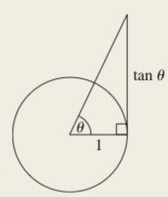
56. Graph the function $\mu = \tan\theta$ with θ in radians over the interval $\left[0, \frac{5\pi}{12}\right]$ and use the graph to estimate solutions to the following. Confirm or contradict your estimates using a calculator.

a. A block of glass is placed on a sheet of glass, which is slowly inclined. Is the block of glass moving when the angle of inclination is $\frac{\pi}{4}$? What is the smallest angle of inclination for which the glass block will be moving with a constant velocity down the incline (rounded to four decimal places)?

b. A block of Teflon is placed on a sheet of steel. As the steel sheet is slowly inclined, the Teflon block begins sliding at a constant velocity when the angle of inclination is approximately 0.04. What is the coefficient of friction for Teflon on steel?

c. Why do you suppose coefficients of friction less than $\mu = 0.04$ are extremely rare for two solid materials? Give an example of two materials that likely have a very low μ value.

57. Tangent lines: The actual definition of the word *tangent* comes from the Latin *tangere*, meaning "to touch." In mathematics, a tangent line touches the graph of a circle at only one point and function values for $\tan\theta$ are obtained from the length of the line segment tangent to a unit circle.



a. What is the length of the line segment when $\theta = 80°$?

b. If the line segment is 16.35 units long, what is the value of θ?

c. Can the line segment ever be greater than 100 units long? Why or why not?

d. How does your answer to (c) relate to the asymptotic behavior of the graph?

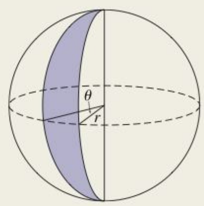
▶ EXTENDING THE CONCEPT

58. Rework Exercises 49 and 50, obtaining a new equation for the data using a different ordered pair to compute the value of A. What do you notice? Try yet another ordered pair and calculate A once again for another equation Y_2. Complete a table of values using the given inputs, with the outputs of the three equations generated (original, Y_1, and Y_2). Does any one equation seem to model the data better than the others? Are all of the equation models "acceptable"? Please comment.

59. Regarding Example 7, we can use the standard distance/rate/time formula $D = RT$ to compute the average velocity of the beam of light along the wall in any interval of time: $R = \frac{D}{T}$. For example, using $D(t) = 5\tan\left(\frac{\pi}{8}t\right)$, the average velocity in the interval [0, 2] is $\frac{D(2) - D(0)}{2 - 0} = 2.5$ m/sec. Calculate the average velocity of the beam in the time intervals [2, 3], [3, 3.5], and [3.5, 3.8] sec. What do you notice? How would the average velocity of the beam in the interval [3.9, 3.99] sec compare?

▶ MAINTAINING YOUR SKILLS

60. **(5.1)** A lune is a section of surface area on a sphere, which is subtended by an angle θ at the circumference. For θ *in radians,* the surface area of a lune is $A = 2r^2\theta$, where r is the radius of the sphere. Find the area of a lune on the surface of the Earth which is subtended by an angle of 15°. Assume the radius of the Earth is 6373 km.



61. **(3.4/3.5)** Find the y-intercept, x-intercept(s), and all asymptotes of each function, but do not graph.

a. $h(x) = \frac{3x^2 - 9x}{2x^2 - 8}$ **b.** $t(x) = \frac{x + 1}{x^2 - 4x}$ **c.** $p(x) = \frac{x^2 - 1}{x + 2}$

62. **(5.2)** State the points on the unit circle that correspond to $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{4}, \frac{3\pi}{2}$, and 2π. What is the value of $\tan\left(\frac{\pi}{2}\right)$? Why?

63. **(4.1)** The radioactive element potassium-42 is sometimes used as a tracer in certain biological experiments, and its decay can be modeled by the formula $Q(t) = Q_0e^{-0.055t}$, where $Q(t)$ is the amount that remains after t hours. If 15 grams (g) of potassium-42 are initially present, how many hours until only 10 g remain?

5.7 Transformations and Applications of Trigonometric Graphs

Learning Objectives

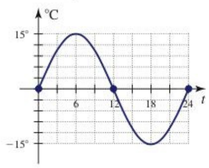
In Section 5.7 you will learn how to:

- ❑ **A.** Apply vertical translations in context
- ❑ **B.** Apply horizontal translations in context
- ❑ **C.** Solve applications involving harmonic motion

From your algebra experience, you may remember beginning with a study of linear graphs, then moving on to quadratic graphs and their characteristics. By combining and extending the knowledge you gained, you were able to investigate and understand a variety of polynomial graphs—along with some powerful applications. A study of trigonometry follows a similar pattern, and by "combining and extending" our understanding of the basic trig graphs, we'll look at some powerful applications in *this* section.

A. Vertical Translations: $y = A\sin(Bt) + D$

Figure 5.79



On any given day, outdoor temperatures tend to follow a **sinusoidal pattern,** or a pattern that can be modeled by a sine function. As the sun rises, the morning temperature begins to warm and rise until reaching its high in the late afternoon, then begins to cool during the early evening and nighttime hours until falling to its nighttime low just prior to sunrise. Next morning, the cycle begins again. In the northern latitudes where the winters are very cold, it's not unreasonable to assume an average daily temperature of 0°C (32°F), and a temperature graph in degrees Celsius that looks like the one in Figure 5.79. For the moment, we'll assume that $t = 0$ corresponds to 12:00 noon. Note that $|A| = 15$ and $P = 24$, yielding $24 = \frac{2\pi}{B}$ or $B = \frac{\pi}{12}$.

If you live in a more temperate area, the daily temperatures still follow a sinusoidal pattern, but the average temperature could be much higher. This is an example of a **vertical shift,** and is the role D plays in the equation $y = A\sin(Bt) + D$. All other aspects of a graph remain the same; it is simply shifted D units up if $D > 0$ and D units down if $D < 0$. As in Section 5.3, for maximum value M and minimum value m, $\frac{M - m}{2}$ gives the amplitude A of a sine curve, while $\frac{M + m}{2}$ gives the **average value** D.

EXAMPLE 1 ► **Modeling Temperature Using a Sine Function**

On a fine day in Galveston, Texas, the high temperature might be about 85°F with an overnight low of 61°F.

a. Find a sinusoidal equation model for the daily temperature.

b. Sketch the graph.

c. Approximate what time(s) of day the temperature is 65°F. Assume $t = 0$ corresponds to 12:00 noon.

Solution ► **a.** We first note the period is still $P = 24$, so $B = \frac{\pi}{12}$, and the equation model will have the form $y = A\sin\left(\frac{\pi}{12}t\right) + D$. Using $\frac{M + m}{2} = \frac{85 + 61}{2}$, we find the *average value* $D = 73$, with amplitude $A = \frac{85 - 61}{2} = 12$. The resulting equation is $y = 12\sin\left(\frac{\pi}{12}t\right) + 73$.

b. To sketch the graph, use a reference rectangle $2A = 24$ units tall and $P = 24$ units wide, along with the *rule of fourths* to locate zeroes and max/min values (see Figure 5.80). Then lightly sketch a sine curve through these points and within the rectangle as shown. This is the graph of $y = 12\sin\left(\frac{\pi}{12}t\right) + 0$.

Using an appropriate scale, shift the rectangle and plotted points vertically upward 73 units and carefully draw the finished graph through the points and within the rectangle (see Figure 5.81).

Figure 5.80

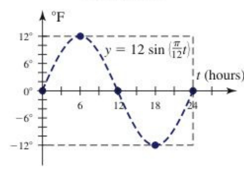
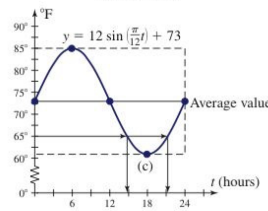


Figure 5.81



WORTHY OF NOTE

Recall from Section 5.5 that transformations of any function $y = f(x)$ remain consistent regardless of the function f used. For the sine function, the transformation $y = af(x \pm h) \pm k$ is more commonly written $y = A\sin(t \pm C) \pm D$, and $|A|$ gives a vertical stretch or compression, C is a horizontal shift opposite the sign, and D is a vertical shift, as seen in Example 1.

This gives the graph of $y = 12\sin\left(\frac{\pi}{12}t\right) + 73$. Note the brokenline notation "≷" in Figure 5.81 indicates that certain values along an axis are unused (in this case, we skipped 0° to 60°), and we began scaling the axis with the values needed.

c. As indicated in Figure 5.81, the temperature hits 65° twice, at about 15 and 21 hr after 12:00 noon, or at 3:00 A.M. and 9:00 A.M. Verify by computing $f(15)$ and $f(21)$.

Now try Exercises 7 through 18 ►

Sinusoidal graphs actually include both sine and cosine graphs, the difference being that sine graphs begin at the average value, while cosine graphs begin at the maximum value. Sometimes it's more advantageous to use one over the other, but equivalent forms can easily be found. In Example 2, a cosine function is used to model an animal population that fluctuates sinusoidally due to changes in food supplies.

EXAMPLE 2 ► Modeling Population Fluctuations Using a Cosine Function

The population of a certain animal species can be modeled by the function $P(t) = 1200\cos\left(\frac{\pi}{5}t\right) + 9000$, where $P(t)$ represents the population in year t. Use the model to

a. Find the period of the function.
b. Graph the function over one period.
c. Find the maximum and minimum values.
d. Estimate the number of years the population is less than 8000.

Solution ► **a.** Since $B = \frac{\pi}{5}$, the period is $P = \frac{2\pi}{\pi/5} = 10$, meaning the population of this species fluctuates over a 10-yr cycle.

b. Use a reference rectangle ($2A = 2400$ by $P = 10$ units) and the *rule of fourths* to locate zeroes and max/min values, then sketch the unshifted graph $y = 1200\cos\left(\frac{\pi}{5}t\right)$. With $P = 10$, these occur at $t = 0, 2.5, 5, 7.5$, and 10 (see Figure 5.82). Shift this graph upward 9000 units (using an appropriate scale) to obtain the graph of $P(t)$ shown in Figure 5.83.

Figure 5.82

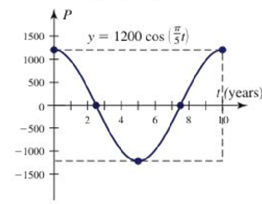
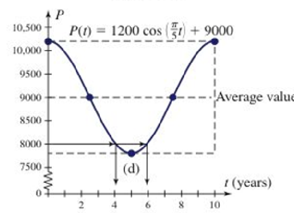


Figure 5.83



A. You've just learned how to apply vertical translations in context

c. The maximum value is $9000 + 1200 = 10{,}200$ and the minimum value is $9000 - 1200 = 7800$.

d. As determined from the graph, the population drops below 8000 animals for approximately 2 yr. Verify by computing $P(4)$ and $P(6)$.

Now try Exercises 19 and 20 ►

B. Horizontal Translations: $y = A\sin(Bt + C) + D$

In some cases, scientists would rather "benchmark" their study of sinusoidal phenomena by placing the average value at $t = 0$ instead of a maximum value (as in Example 2), or by placing the maximum or minimum value at $t = 0$ instead of the average value (as in Example 1). Rather than make additional studies or recompute using available data, *we can simply shift these graphs using a horizontal translation*. To help understand how, consider the graph of $y = x^2$. The graph is a parabola, concave up, with a vertex at the origin. Comparing this function with $y_1 = (x - 3)^2$ and $y_2 = (x + 3)^2$, we note y_1 is simply the parent graph shifted 3 units right, and y_2 is the parent graph shifted 3 units left ("opposite the sign"). See Figures 5.84 through 5.86.

While ***quadratic functions have no maximum value if $A > 0$,*** these graphs are a good reminder of how a basic graph can be horizontally shifted. We simply *replace the independent variable x with $(x \pm h)$ or t with $(t \pm h)$,* where h is the desired shift and the sign is chosen depending on the direction of the shift.

Figure 5.84 $y = x^2$

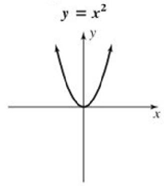


Figure 5.85 $y_1 = (x - 3)^2$

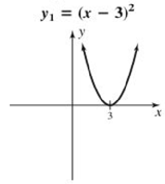
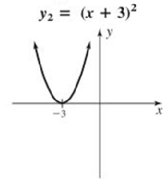
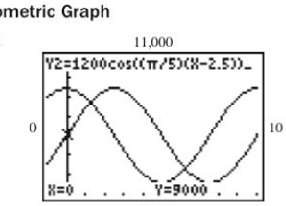


Figure 5.86 $y_2 = (x + 3)^2$



EXAMPLE 3 ► Investigating Horizontal Shifts of a Trigonometric Graph

Use a horizontal translation to shift the graph from Example 2 so that the average population begins at $t = 0$. Verify the result on a graphing calculator, then find a sine function that gives the same graph as the shifted cosine function.



Solution ► For $P(t) = 1200\cos\left(\frac{\pi}{5}t\right) + 9000$ from Example 2, the average value first occurs at $t = 2.5$. For the average value to occur at $t = 0$, we must shift the graph to the right 2.5 units. Replacing t with $(t - 2.5)$ gives $P(t) = 1200\cos\left[\frac{\pi}{5}(t - 2.5)\right] + 9000$. A graphing calculator shows the desired result is obtained (see figure). The new graph appears to be a sine function with the same amplitude and period, and the equation is $y = 1200\sin\left(\frac{\pi}{5}t\right) + 9000$.

Now try Exercises 21 and 22 ►

WORTHY OF NOTE

When the function $P(t) = 1200\cos\left[\frac{\pi}{5}(t - 2.5)\right] + 9000$ is written in standard form as $P(t) = 1200\cos\left[\frac{\pi}{5}t - \frac{\pi}{2}\right] + 9000$, we can easily see why they are equivalent to $P(t) = 1200\sin\left(\frac{\pi}{5}t\right) + 9000$. Using the cofunction relationship, $\cos\left[\frac{\pi}{5}t - \frac{\pi}{2}\right] = \sin\left(\frac{\pi}{5}t\right)$.

Equations like $P(t) = 1200\cos\left[\frac{\pi}{5}(t - 2.5)\right] + 9000$ from Example 3 are said to be written in **shifted form,** since we can easily tell the magnitude and direction of the shift. To obtain the **standard form** we *distribute the value of B*: $P(t) = 1200\cos\left(\frac{\pi}{5}t - \frac{\pi}{2}\right) + 9000$. In general, the *standard form* of a sinusoidal equation (using *either* a cosine or sine function) is written $y = A\sin(Bt \pm C) + D$, with the *shifted form* found by factoring out B from $Bt \pm C$:

$$y = A\sin(Bt \pm C) + D \rightarrow y = A\sin\left[B\left(t \pm \frac{C}{B}\right)\right] + D$$

In either case, C gives what is known as the **phase angle** of the function, and is used in a study of AC circuits and other areas, to discuss how far a given function is "out of phase" with a reference function. In the latter case, $\frac{C}{B}$ is simply the horizontal shift (or phase shift) of the function and gives the magnitude and direction of this shift (opposite the sign).

Characteristics of Sinusoidal Models

Transformations of the graph of $y = \sin t$ are written as $y = A\sin(Bt)$, where

1. $|A|$ gives the *amplitude* of the graph, or the maximum displacement from the average value.
2. B is related to the *period* P of the graph according to the ratio $P = \frac{2\pi}{B}$ (the interval required for one complete cycle). Translations of $y = A\sin(Bt)$ can be written as follows:

Standard form	Shifted form
$y = A\sin(Bt \pm C) + D$	$y = A\sin\left[B\left(t \pm \frac{C}{B}\right)\right] + D$

3. In either case, C is called the *phase angle* of the graph, while $\pm\frac{C}{B}$ gives the magnitude and direction of the *horizontal shift* (opposite the given sign).
4. D gives the *vertical shift* of the graph, and the location of the average value. The shift will be in the same direction as the given sign.

Knowing where each cycle begins and ends is a helpful part of sketching a graph of the equation model. The **primary interval** for a sinusoidal graph can be found by solving the inequality $0 \le Bt \pm C < 2\pi$, with the reference rectangle and *rule of fourths* giving the zeroes, max/min values, and a sketch of the graph in this interval. The graph can then be extended in either direction, and shifted vertically as needed.

EXAMPLE 4 ▶ Analyzing the Transformation of a Trig Function

Identify the amplitude, period, horizontal shift, vertical shift (average value), and endpoints of the primary interval.

$$y = 2.5\sin\left(\frac{\pi}{4}t + \frac{3\pi}{4}\right) + 6$$

Solution ▶ The equation gives an amplitude of $|A| = 2.5$, with an average value of $D = 6$. The maximum value will be $y = 2.5(1) + 6 = 8.5$, with a minimum of $y = 2.5(-1) + 6 = 3.5$. With $B = \frac{\pi}{4}$, the period is $P = \frac{2\pi}{\pi/4} = 8$. To find the horizontal shift, we factor out $\frac{\pi}{4}$ to write the equation in shifted form: $\left(\frac{\pi}{4}t + \frac{3\pi}{4}\right) = \frac{\pi}{4}(t + 3)$. The horizontal shift is 3 units left. For the endpoints of the primary interval we solve $0 \le \frac{\pi}{4}(t + 3) < 2\pi$, which gives $-3 \le t < 5$.

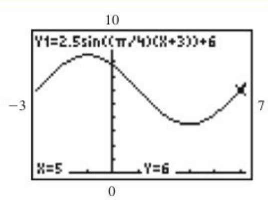
Now try Exercises 23 through 34 ▶

WORTHY OF NOTE

It's important that you don't confuse the standard form with the shifted form. Each has a place and purpose, but the horizontal shift can be identified only by focusing on the change in an independent variable. Even though the equations $y = 4(x + 3)^2$ and $y = (2x + 6)^2$ are equivalent, only the first explicitly shows that $y = 4x^2$ has been shifted three units left. Likewise $y = \sin[2(t + 3)]$ and $y = \sin(2t + 6)$ are equivalent, but only the first explicitly gives the horizontal shift (three units left). Applications involving a horizontal shift come in an infinite variety, and the shifts are generally not uniform or standard.

GRAPHICAL SUPPORT

The analysis of $y = 2.5\sin\left[\frac{\pi}{4}(t + 3)\right] + 6$ from Example 4 can be verified on a graphing calculator. Enter the function as Y_1 on the Y= screen and set an appropriate window size using the information gathered. Press the TRACE key and -3 ENTER and the calculator gives the average value $y = 6$ as output. Repeating this for $x = 5$ shows one complete cycle has been completed.

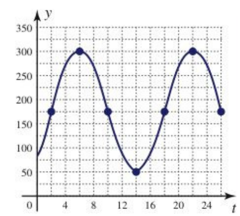


To help gain a better understanding of sinusoidal functions, their graphs, and the role the coefficients A, B, C, and D play, it's often helpful to reconstruct the equation of a given graph.

EXAMPLE 5 ▶ Determining the Equation of a Trig Function from Its Graph

Determine the equation of the given graph using a sine function.

Solution ► From the graph it is apparent the maximum value is 300, with a minimum of 50. This gives a value of $\frac{300 + 50}{2} = 175$ for D and $\frac{300 - 50}{2} = 125$ for A. The graph completes one cycle from $t = 2$ to $t = 18$, showing $P = 18 - 2 = 16$ and $B = \frac{\pi}{8}$. The average value first occurs at $t = 2$, so the basic graph has been shifted to the right 2 units. The equation is $y = 125\sin\left[\frac{\pi}{8}(t - 2)\right] + 175$.



B. You've just learned how to apply horizontal translations in context

Now try Exercises 35 through 44 ►

C. Simple Harmonic Motion: $y = A\sin(Bt)$ or $y = A\cos(Bt)$

The periodic motion of springs, tides, sound, and other phenomena all exhibit what is known as **harmonic motion,** which can be modeled using sinusoidal functions.

Harmonic Models—Springs

Consider a spring hanging from a beam with a weight attached to one end. When the weight is at rest, we say it is in **equilibrium,** or has zero displacement from center. Stretching the spring and then releasing it causes the weight to "bounce up and down," with its displacement from center neatly modeled over time by a sine wave (see Figure 5.87).

Figure 5.87

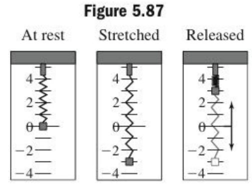
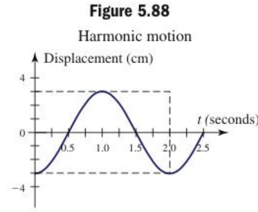


Figure 5.88



For objects in harmonic *motion* (there are other harmonic models), the input variable t is always a time unit (seconds, minutes, days, etc.), so in addition to the period of the sinusoid, we are very interested in its **frequency**—the number of cycles it completes per unit time (see Figure 5.88). Since the period gives the time required to complete one cycle, the frequency f is given by $f = \frac{1}{P} = \frac{B}{2\pi}$.

EXAMPLE 6 ► Applications of Sine and Cosine: Harmonic Motion

For the harmonic motion modeled by the sinusoid in Figure 5.88,

a. Find an equation of the form $y = A\cos(Bt)$.
b. Determine the frequency.
c. Use the equation to find the position of the weight at $t = 1.8$ sec.

Solution ► **a.** By inspection the graph has an amplitude $|A| = 3$ and a period $P = 2$. After substitution into $P = \frac{2\pi}{B}$, we obtain $B = \pi$ and the equation $y = -3\cos(\pi t)$.

b. Frequency is the reciprocal of the period so $f = \frac{1}{2}$, showing one-half a cycle is completed each second (as the graph indicates).

c. Evaluating the model at $t = 1.8$ gives $y = -3\cos[\pi(1.8)] \approx -2.43$, meaning the weight is 2.43 cm below the equilibrium point at this time.

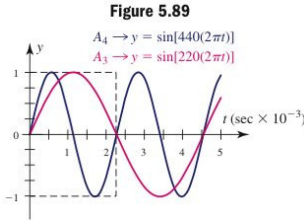
Now try Exercises 47 through 50 ►

Harmonic Models—Sound Waves

A second example of harmonic motion is the production of sound. For the purposes of this study, we'll look at musical notes. The vibration of matter produces a **pressure wave** or **sound energy,** which in turn vibrates the eardrum. Through the intricate structure of the middle ear, this sound energy is converted into mechanical energy and sent to the inner ear where it is converted to nerve impulses and transmitted to the brain. If the sound wave has a high frequency, the eardrum vibrates with greater frequency, which the brain interprets as a "high-pitched" sound. The *intensity* of the sound wave can also be transmitted to the brain via these mechanisms, and if the arriving sound wave has a high amplitude, the eardrum vibrates more forcefully and the sound is interpreted as "loud" by the brain. These characteristics are neatly modeled using $y = A\sin(Bt)$. For the moment we will focus on the frequency, keeping the amplitude constant at $A = 1$.

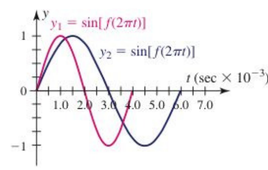
The musical note known as A_4 or "the A above middle C" is produced with a frequency of 440 vibrations per second, or 440 hertz (Hz) (this is the note most often used in the tuning of pianos and other musical instruments). For any given note, the same note one octave higher will have double the frequency, and the same note one octave lower will have one-half the frequency. In addition, with $f = \frac{1}{P}$ the value of $B = 2\pi\left(\frac{1}{P}\right)$ can always be expressed as $B = 2\pi f$, so A_4 has the equation $y = \sin[440(2\pi t)]$ (after rearranging the factors). The same note one octave lower is A_3 and has the equation $y = \sin[220(2\pi t)]$, with one-half the frequency. To draw the representative graphs, we must scale the t-axis in very small increments (seconds $\times 10^{-3}$) since $P = \frac{1}{440} \approx 0.0023$ for A_4, and $P = \frac{1}{220} \approx 0.0045$ for A_3. Both are graphed in Figure 5.89, where we see that the higher note completes two cycles in the same interval that the lower note completes one.

Figure 5.89



EXAMPLE 7 ► **Applications of Sine and Cosine: Sound Frequencies**

The table here gives the frequencies for three octaves of the 12 "chromatic" notes with frequencies between 110 Hz and 840 Hz. Two of the 36 notes are graphed in the figure. Which two?



Frequency by Octave			
Note	**Octave 3**	**Octave 4**	**Octave 5**
A	110.00	220.00	440.00
A#	116.54	233.08	466.16
B	123.48	**246.96**	493.92
C	130.82	261.64	523.28
C#	138.60	277.20	554.40
D	146.84	293.68	587.36
D#	155.56	311.12	622.24
E	**164.82**	329.24	659.28
F	174.62	349.24	698.48
F#	185.00	370.00	740.00
G	196.00	392.00	784.00
G#	207.66	415.32	830.64

Solution ► Since amplitudes are equal, the only difference is the frequency and period of the notes. It appears that y_1 has a period of about 0.004 sec, giving a frequency of $\frac{1}{0.004} = 250$ Hz—very likely a B_4 (in bold). The graph of y_2 has a period of about 0.006, for a frequency of $\frac{1}{0.006} \approx 167$ Hz—probably an E_3 (also in bold).

C. You've just learned how to solve applications involving harmonic motion

Now try Exercises 51 through 54 ►

TECHNOLOGY HIGHLIGHT

Locating Zeroes, Roots, and *x*-Intercepts

As you know, the zeroes of a function are *input* values that cause an *output* of zero. Graphically, these show up as *x*-intercepts and once a function is graphed they can be located (if they exist) using the 2nd CALC **2:zero** feature. This feature is similar to the **3:minimum** and **4:maximum** features, in that we have the calculator search a specified interval by giving a **left bound** and a **right bound.** To illustrate, enter $Y_1 = 3\sin\left(\frac{\pi}{2}x\right) - 1$ on the Y= screen and graph it using the ZOOM **7:ZTrig** option. The resulting graph shows there are six zeroes in this interval and we'll locate the first negative root. Knowing the **7:Trig** option uses tick marks that are spaced every $\frac{\pi}{2}$ units, this root is in the interval $\left(-\pi, -\frac{\pi}{2}\right)$. After pressing 2nd CALC **2:zero** the calculator returns you to the graph, and requests a "Left Bound," (see Figure 5.90). We enter $-\pi$ (press ENTER) and the calculator marks this choice with a "►" marker (pointing to the right), then asks for a "Right Bound." After entering $-\frac{\pi}{2}$, the calculator marks this with a "◄" marker and asks for a "Guess." Bypass this option by pressing ENTER once again (see Figure 5.91). The calculator searches the interval until it locates a zero (Figure 5.92) or displays an error message indicating it was unable to comply (no zeroes in the interval). Use these ideas to locate the zeroes of the following functions in $[0, \pi]$.

Figure 5.90

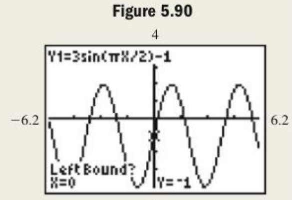


Figure 5.91

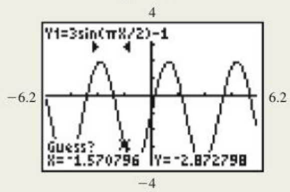
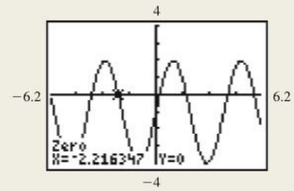


Figure 5.92



Exercise 1: $y = -2\cos(\pi t) + 1$

Exercise 2: $y = 0.5\sin[\pi(t - 2)]$

Exercise 3: $y = \frac{3}{2}\tan(2x) - 1$

Exercise 4: $y = x^3 - \cos x$

5.7 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A sinusoidal wave is one that can be modeled by functions of the form ______________ or ______________.

2. The graph of $y = \sin x + k$ is the graph of $y = \sin x$ shifted __________ k units. The graph of $y = \sin(x - h)$ is the graph of $y = \sin x$ shifted __________ h units.

3. To find the primary interval of a sinusoidal graph, solve the inequality ____________.

4. Given the period P, the frequency is __________, and given the frequency f, the value of B is __________.

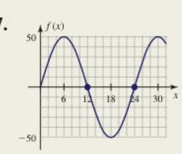
5. Explain/Discuss the difference between the *standard form* of a sinusoidal equation, and the *shifted form*. How do you obtain one from the other? For what benefit?

6. Write out a step-by-step procedure for sketching the graph of $y = 30\sin\left(\frac{\pi}{2}t - \frac{1}{2}\right) + 10$. Include use of the reference rectangle, primary interval, zeroes, max/mins, and so on. Be complete and thorough.

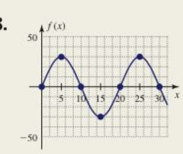
▶ DEVELOPING YOUR SKILLS

Use the graphs given to (a) state the amplitude A and period P of the function; (b) estimate the value at $x = 14$; and (c) estimate the interval in $[0, P]$ where $f(x) \geq 20$.

7.

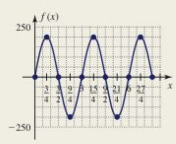


8.

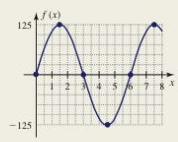


Use the graphs given to (a) state the amplitude A and period P of the function; (b) estimate the value at $x = 2$; and (c) estimate the interval in $[0, P]$, where $f(x) \leq -100$.

9.



10.



Use the information given to write a sinusoidal equation and sketch its graph. Recall $B = \frac{2\pi}{P}$.

11. Max: 100, min: 20, $P = 30$

12. Max: 95, min: 40, $P = 24$

13. Max: 20, min: 4, $P = 360$

14. Max: 12,000, min: 6500, $P = 10$

Use the information given to write a sinusoidal equation, sketch its graph, and answer the question posed.

15. In Geneva, Switzerland, the daily temperature in January ranges from an average high of 39°F to an average low of 29°F. (a) Find a sinusoidal equation model for the daily temperature; (b) sketch the graph; and (c) approximate the time(s) each January day the temperature reaches the freezing point (32°F). Assume $t = 0$ corresponds to noon.
Source: 2004 Statistical Abstract of the United States, Table 1331.

16. In Nairobi, Kenya, the daily temperature in January ranges from an average high of 77°F to an average low of 58°F. (a) Find a sinusoidal equation model for the daily temperature; (b) sketch the graph; and (c) approximate the time(s) each January day the temperature reaches a comfortable 72°F. Assume $t = 0$ corresponds to noon.
Source: 2004 Statistical Abstract of the United States, Table 1331.

17. In Oslo, Norway, the number of hours of daylight reaches a low of 6 hr in January, and a high of nearly 18.8 hr in July. (a) Find a sinusoidal equation model for the number of daylight hours each month; (b) sketch the graph; and (c) approximate the number of *days* each year there are more than 15 hr of daylight. Use 1 month ≈ 30.5 days. Assume $t = 0$ corresponds to January 1.
Source: www.visitnorway.com/templates.

18. In Vancouver, British Columbia, the number of hours of daylight reaches a low of 8.3 hr in January, and a high of nearly 16.2 hr in July. (a) Find a sinusoidal equation model for the number of daylight hours each month; (b) sketch the graph; and (c) approximate the number of *days* each year there are more than 15 hr of daylight. Use 1 month ≈ 30.5 days. Assume $t = 0$ corresponds to January 1.
Source: www.bcpassport.com/vital/temp.

19. Recent studies seem to indicate the population of North American porcupine (*Erethizon dorsatum*) varies sinusoidally with the solar (sunspot) cycle due to its effects on Earth's ecosystems. Suppose the population of this species in a certain locality is modeled by the function $P(t) = 250\cos\left(\frac{2\pi}{11}t\right) + 950$, where $P(t)$ represents the population of porcupines in year t. Use the model to (a) find the period of the function; (b) graph the function over one period; (c) find the maximum and minimum values; and (d) estimate the number of years the population is less than 740 animals.
Source: Ilya Klvana, McGill University (Montreal), Master of Science thesis paper, November 2002.

20. The population of mosquitoes in a given area is primarily influenced by precipitation, humidity, and temperature. In tropical regions, these tend to fluctuate sinusoidally in the course of a year. Using trap counts and statistical projections, fairly accurate estimates of a mosquito population can be obtained. Suppose the population in a certain region was modeled by the function $P(t) = 50\cos\left(\frac{\pi}{26}t\right) + 950$, where $P(t)$ was the mosquito population (in thousands) in week t of the year. Use the model to (a) find the period of the function; (b) graph the function over one period; (c) find the maximum and minimum population values; and (d) estimate the number of weeks the population is less than 915,000.

21. Use a horizontal translation to shift the graph from Exercise 19 so that the average population of the North American porcupine begins at $t = 0$. Verify results on a graphing calculator, then find a sine function that gives the same graph as the shifted cosine function.

22. Use a horizontal translation to shift the graph from Exercise 20 so that the average population of mosquitoes begins at $t = 0$. Verify results on a graphing calculator, then find a sine function that gives the same graph as the shifted cosine function.

Identify the amplitude (*A*), period (*P*), horizontal shift (HS), vertical shift (VS), and endpoints of the primary interval (PI) for each function given.

23. $y = 120\sin\left[\frac{\pi}{12}(t - 6)\right]$

24. $y = 560\sin\left[\frac{\pi}{4}(t + 4)\right]$

25. $h(t) = \sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$

26. $r(t) = \sin\left(\frac{\pi}{10}t - \frac{2\pi}{5}\right)$

27. $y = \sin\left(\frac{\pi}{4}t - \frac{\pi}{6}\right)$

28. $y = \sin\left(\frac{\pi}{3}t + \frac{5\pi}{12}\right)$

29. $f(t) = 24.5\sin\left[\frac{\pi}{10}(t - 2.5)\right] + 15.5$

30. $g(t) = 40.6\sin\left[\frac{\pi}{6}(t - 4)\right] + 13.4$

31. $g(t) = 28\sin\left(\frac{\pi}{6}t - \frac{5\pi}{12}\right) + 92$

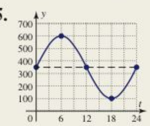
32. $f(t) = 90\sin\left(\frac{\pi}{10}t - \frac{\pi}{5}\right) + 120$

33. $y = 2500\sin\left(\frac{\pi}{4}t + \frac{\pi}{12}\right) + 3150$

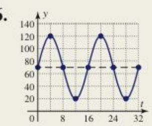
34. $y = 1450\sin\left(\frac{3\pi}{4}t + \frac{\pi}{8}\right) + 2050$

Find the equation of the graph given. Write answers in the form $y = A\sin(Bt + C) + D$.

35.



36.



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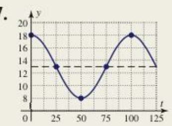

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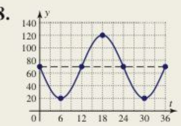

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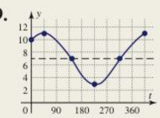
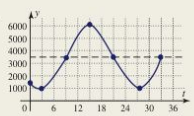


40.



Sketch one complete period of each function.

41. $f(t) = 25\sin\left[\frac{\pi}{4}(t - 2)\right] + 55$

42. $g(t) = 24.5\sin\left[\frac{\pi}{10}(t - 2.5)\right] + 15.5$

43. $h(t) = 3\sin(4t - \pi)$

44. $p(t) = -2\cos\left(3t - \frac{\pi}{2}\right)$

► WORKING WITH FORMULAS

45. The relationship between the coefficient *B*, the frequency *f*, and the period *P*

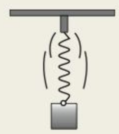
In many applications of trigonometric functions, the equation $y = A\sin(Bt)$ is written as $y = A\sin[(2\pi f)t]$, where $B = 2\pi f$. Justify the new equation using $f = \frac{1}{P}$ and $P = \frac{2\pi}{B}$. In other words, explain how $A\sin(Bt)$ becomes $A\sin[(2\pi f)t]$, as though you were trying to help another student with the ideas involved.

46. Number of daylight hours:
$$D(t) = \frac{K}{2}\sin\left[\frac{2\pi}{365}(t - 79)\right] + 12$$

The number of daylight hours for a particular day of the year is modeled by the formula given, where $D(t)$ is the number of daylight hours on day *t* of the year and *K* is a constant related to the total variation of daylight hours, latitude of the location, and other factors. For the city of Reykjavik, Iceland, $K \approx 17$, while for Detroit, Michigan, $K \approx 6$. How many hours of daylight will each city receive on June 30 (the 182nd day of the year)?

▶ APPLICATIONS

47. Harmonic motion: A weight on the end of a spring is oscillating in harmonic motion. The equation model for the oscillations is $d(t) = 6\sin\left(\frac{\pi}{2}t\right)$, where d is the distance (in centimeters) from the equilibrium point in t sec.



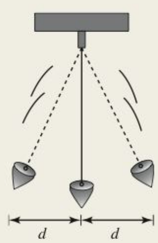
a. What is the period of the motion? What is the frequency of the motion?

b. What is the displacement from equilibrium at $t = 2.5$? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

c. What is the displacement from equilibrium at $t = 3.5$? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

d. How far does the weight move between $t = 1$ and $t = 1.5$ sec? What is the average velocity for this interval? Do you expect a greater or lesser velocity for $t = 1.75$ to $t = 2$? Explain why.

48. Harmonic motion: The bob on the end of a 24-in. pendulum is oscillating in harmonic motion. The equation model for the oscillations is $d(t) = 20\cos(4t)$, where d is the distance (in inches) from the equilibrium point, t sec after being released from one side.



a. What is the period of the motion? What is the frequency of the motion?

b. What is the displacement from equilibrium at $t = 0.25$ sec? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

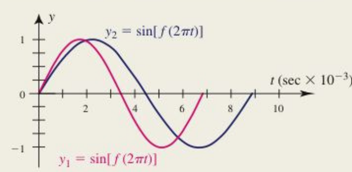
c. What is the displacement from equilibrium at $t = 1.3$ sec? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

d. How far does the bob move between $t = 0.25$ and $t = 0.35$ sec? What is its average velocity for this interval? Do you expect a greater velocity for the interval $t = 0.55$ to $t = 0.6$? Explain why.

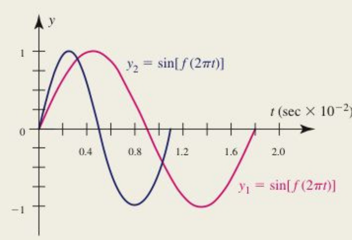
49. Harmonic motion: A simple pendulum 36 in. in length is oscillating in harmonic motion. The bob at the end of the pendulum swings through an arc of 30 in. (from the far left to the far right, or one-half cycle) in about 0.8 sec. What is the equation model for this harmonic motion?

50. Harmonic motion: As part of a study of wave motion, the motion of a floater is observed as a series of uniform ripples of water move beneath it. By careful observation, it is noted that the floater bobs up and down through a distance of 2.5 cm every $\frac{1}{3}$ sec. What is the equation model for this harmonic motion?

51. Sound waves: Two of the musical notes from the chart on page 595 are graphed in the figure. Use the graphs given to determine which two.



52. Sound waves: Two chromatic notes *not on the chart from page 595* are graphed in the figure. Use the graphs and the discussion regarding octaves to determine which two. Note the scale of the t-axis *has been changed* to hundredths of a second.



Sound waves: Use the chart on page 595 to write the equation for each note in the form $y = \sin[f(2\pi t)]$ and clearly state the period of each note.

53. notes D_3 and G_4

54. the notes A_5 and $C\#_3$

Daylight hours model: Solve using a graphing calculator and the formula given in Exercise 46.

55. For the city of Caracas, Venezuela, $K \approx 1.3$, while for Tokyo, Japan, $K \approx 4.8$.

a. How many hours of daylight will each city receive on January 15th (the 15th day of the year)?

b. Graph the equations modeling the hours of daylight on the same screen. Then determine (i) what days of the year these two cities will have the same number of hours of daylight, and (ii) the number of days each year that each city receives 11.5 hr or less of daylight.

56. For the city of Houston, Texas, $K \approx 3.8$, while for Pocatello, Idaho, $K \approx 6.2$.

a. How many hours of daylight will each city receive on December 15 (the 349th day of the year)?

b. Graph the equations modeling the hours of daylight on the same screen. Then determine (i) how many days each year Pocatello receives more daylight than Houston, and (ii) the number of days each year that each city receives 13.5 hr or more of daylight.

▶ EXTENDING THE CONCEPT

57. The formulas we use in mathematics can sometimes seem very mysterious. We know they "work," and we can graph and evaluate them—but where did they come from? Consider the formula for the number of daylight hours from Exercise 46: $D(t) = \frac{K}{2}\sin\left[\frac{2\pi}{365}(t - 79)\right] + 12.$

a. We know that the addition of 12 represents a vertical shift, but what does a vertical shift of 12 mean *in this context*?

b. We also know the factor $(t - 79)$ represents a phase shift of 79 to the right. But what does a horizontal (phase) shift of 79 mean *in this context*?

c. Finally, the coefficient $\frac{K}{2}$ represents a change in amplitude, but what does a change of amplitude mean *in this context*? Why is the coefficient bigger for the northern latitudes?

58. Use a graphing calculator to graph the equation $f(x) = \frac{3x}{2} - 2\sin(2x) - 1.5$.

a. Determine the interval between each peak of the graph. What do you notice?

b. Graph $g(x) = \frac{3x}{2} - 1.5$ on the same screen and comment on what you observe.

c. What would the graph of $f(x) = -\frac{3x}{2} + 2\sin(2x) + 1.5$ look like? What is the x-intercept?

▶ MAINTAINING YOUR SKILLS

59. **(5.1)** In what quadrant does the arc $t = 3.7$ terminate? What is the reference arc?

60. **(3.1)** Given $f(x) = -3(x + 1)^2 - 4$, name the vertex and solve the inequality $f(x) > 0$.

61. **(1.4)** Compute the sum, difference, product and quotient of $-1 + i\sqrt{5}$ and $-1 - i\sqrt{5}$.

62. **(5.3/5.4)** Sketch the graph of (a) $y = \cos t$ in the interval $[0, 2\pi)$ and (b) $y = \tan t$ in the interval $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

SUMMARY AND CONCEPT REVIEW

SECTION 5.1 Angle Measure, Special Triangles, and Special Angles

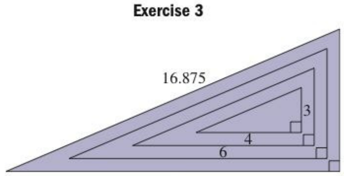
KEY CONCEPTS

- An angle is defined as the joining of two rays at a common endpoint called the vertex.
- An angle in standard position has its vertex at the origin and its initial side on the positive x-axis.
- Two angles in standard position are coterminal if they have the same terminal side.
- A counterclockwise rotation gives a positive angle, a clockwise rotation gives a negative angle.
- One degree (1°) is defined to be $\frac{1}{360}$ of a full revolution. One (1) radian is the measure of a central angle subtended by an arc equal in length to the radius.
- Degrees can be divided into a smaller unit called minutes: $1° = 60'$; minutes can be divided into a smaller unit called seconds: $1' = 60''$. This implies $1° = 3600''$.
- Two angles are complementary if they sum to 90° and supplementary if they sum to 180°.
- Properties of triangles: (I) the sum of the angles is 180°; (II) the combined length of any two sides must exceed that of the third side and; (III) larger angles are opposite larger sides.
- Given two triangles, if all three corresponding angles are equal, the triangles are said to be similar. If two triangles are similar, then corresponding sides are in proportion.
- In a 45-45-90 triangle, the sides are in the proportion $1x: 1x: \sqrt{2}x$.
- In a 30-60-90 triangle, the sides are in the proportion $1x: \sqrt{3}x: 2x$.
- The formula for arc length: $s = r\theta$; area of a circular sector: $A = \frac{1}{2}r^2\theta$, θ in radians.
- To convert degree measure to radians, multiply by $\frac{\pi}{180°}$; for radians to degrees, multiply by $\frac{180°}{\pi}$.
- Special angle conversions: $30° = \frac{\pi}{6}$, $45° = \frac{\pi}{4}$, $60° = \frac{\pi}{3}$, $90° = \frac{\pi}{2}$.
- Angular velocity is a rate of rotation per unit time: $\omega = \frac{\theta}{t}$.
- Linear velocity is a change in position per unit time: $V = \frac{\theta r}{t}$ or $V = r\omega$.

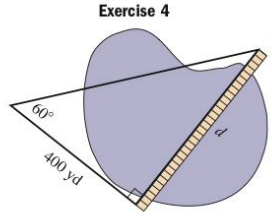
EXERCISES

1. Convert 147°36′48″ to decimal degrees.
2. Convert 32.87° to degrees, minutes, and seconds.
3. All of the right triangles given are similar. Find the dimensions of the largest triangle.

Exercise 3



Exercise 4



4. Use special angles/special triangles to find the length of the bridge needed to cross the lake shown in the figure.
5. Convert to degrees: $\frac{2\pi}{3}$.
6. Convert to radians: 210°.
7. Find the arc length if $r = 5$ and $\theta = 57°$.
8. Evaluate without using a calculator: $\sin\left(\frac{7\pi}{6}\right)$.