6

Trigonometric Identities, Inverses, and Equations

CHAPTER OUTLINE

CHAPTER CONNECTIONS

Have you ever noticed that people who arrive early at a movie tend to choose seats about halfway up the theater's incline and in the middle of a row? More than likely, this is due to a phenomenon called the *optimal viewing angle,* or the angle formed by the viewer's eyes and the top and bottom of the screen. Seats located in this area maximize the viewing angle, with the measure of the angle depending on factors such as the distance from the floor to the bottom of the screen, the height of the screen, the location of a seat, and the incline of the auditorium. Here, trigonometric functions and identities play an important role. This application appears as Exercise 59 of Section 6.2.

Check out these other real-world connections:

- Finding the Viewing Angle from a Seat at the Theatre (Section 6.2, Exercise 64)
- Modeling the Range of a Projectile (Section 6.4, Exercise 109)
- Maximizing the Shooting Angle during a Break-away (Section 6.5, Exercise 95)
- Modeling the Grade of a Treadmill during a Workout Session (Section 6.7, Exercise 54)

6.1 Fundamental Identities and Families of Identities

Learning Objectives

In Section 6.1 you will learn how to:

- **A.** Use fundamental identities to help understand and recognize identity "families"
- **B.** Verify other identities using the fundamental identities and basic algebra skills
- **C.** Use fundamental identities to express a given trig function in terms of the other five
- **D.** Use counterexamples and contradictions to show an equation is not an identity

In this section we begin laying the foundation necessary to work with identities successfully. The cornerstone of this effort is a healthy respect for the fundamental identities and vital role they play. Students are strongly encouraged to do more than memorize them—they should be *internalized,* meaning they must become a natural and instinctive part of your core mathematical knowledge.

A. Fundamental Identities and Identity Families

An **identity** is an equation that is true for all elements in the domain. In trigonometry, some identities result directly from the way the trig functions are defined. For instance, since $\sin\theta = \frac{y}{r}$ and $\csc\theta = \frac{r}{y}$, $\frac{1}{\csc\theta} = \frac{y}{r}$, and the identity $\sin\theta = \frac{1}{\csc\theta}$ immediately follows. We call identities of this type *fundamental identities*. Successfully working with *other* identities will depend a great deal on your mastery of these fundamental types. For convenience, the definition of the trig functions are reviewed here, followed by the fundamental identities that result.

Given point $P(x, y)$ on the unit circle, and the central angle θ associated with P, we have $\sqrt{x^2 + y^2} = 1$ and

$$\cos\theta = x \qquad \sin\theta = y \qquad \tan\theta = \frac{y}{x};\ x \neq 0$$

$$\sec\theta = \frac{1}{x};\ x \neq 0 \qquad \csc\theta = \frac{1}{y};\ y \neq 0 \qquad \cot\theta = \frac{x}{y};\ y \neq 0$$

WORTHY OF NOTE

The word *fundamental* itself means, "a basis or foundation supporting an essential structure or function" (Merriam Webster).

Fundamental Trigonometric Identities

Reciprocal identities	Ratio identities	Pythagorean identities	Identities due to symmetry
$\sin\theta = \frac{1}{\csc\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\sin^2\theta + \cos^2\theta = 1$	$\sin(-\theta) = -\sin\theta$
$\cos\theta = \frac{1}{\sec\theta}$	$\tan\theta = \frac{\sec\theta}{\csc\theta}$	$\tan^2\theta + 1 = \sec^2\theta$	$\cos(-\theta) = \cos\theta$
$\tan\theta = \frac{1}{\cot\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$1 + \cot^2\theta = \csc^2\theta$	$\tan(-\theta) = -\tan\theta$

These identities seem to naturally separate themselves into the four groups or families listed, with each group having additional relationships that can be inferred from the definitions. For instance, since $\sin\theta$ is the reciprocal of $\csc\theta$, $\csc\theta$ must be the reciprocal of $\sin\theta$. Similar statements can be made regarding $\cos\theta$ and $\sec\theta$ as well as $\tan\theta$ and $\cot\theta$. Recognizing these additional "family members" enlarges the number of identities you can work with, and will help you use them more effectively. In particular, since they *are* reciprocals: $\sin\theta\csc\theta = 1$, $\cos\theta\sec\theta = 1$, $\tan\theta\cot\theta = 1$. **See Exercises 7 and 8.**

EXAMPLE 1 ► Identifying Families of Identities

Use algebra to write four additional identities that belong to the Pythagorean family.

Solution ► Starting with $\sin^2\theta + \cos^2\theta = 1$,

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{original identity}$$
$$\bullet\ \sin^2\theta = 1 - \cos^2\theta \quad \text{subtract } \cos^2\theta$$
$$\bullet\ \sin\theta = \pm\sqrt{1 - \cos^2\theta} \quad \text{take square root}$$
$$\sin^2\theta + \cos^2\theta = 1 \quad \text{original identity}$$
$$\bullet\ \cos^2\theta = 1 - \sin^2\theta \quad \text{subtract } \sin^2\theta$$
$$\bullet\ \cos\theta = \pm\sqrt{1 - \sin^2\theta} \quad \text{take square root}$$

For the identities involving a radical, the choice of sign will depend on the quadrant of the terminal side.

Now try Exercises 9 and 10 ►

The four additional Pythagorean identities are marked with a "•" in Example 1. The fact that each of them represents an equality gives us more options when attempting to verify or prove more complex identities. For instance, since $\cos^2\theta = 1 - \sin^2\theta$, we can replace $\cos^2\theta$ with $1 - \sin^2\theta$, or replace $1 - \sin^2\theta$ with $\cos^2\theta$, *any time they occur in an expression.* Note there are many other members of this family, since similar steps can be performed on the other Pythagorean identities. In fact, each of the fundamental identities can be similarly rewritten and there are a variety of exercises at the end of this section for practice.

A. You've just learned how to use fundamental identities to help understand and recognize identity "families"

B. Verifying an Identity Using Algebra

Note that we cannot *prove* an equation is an identity by repeatedly substituting input values and obtaining a true equation. This would be an infinite exercise and we might easily miss a value or even a range of values for which the equation is false. Instead we attempt to rewrite one side of the equation until we obtain a match with the other side, so there can be no doubt. As hinted at earlier, this is done using basic algebra skills combined with the fundamental identities and the substitution principle. For now we'll focus on verifying identities by using algebra. In Section 6.2 we'll introduce some guidelines and ideas that will help you verify a wider range of identities.

EXAMPLE 2 ► Using Algebra to Help Verify an Identity

Use the distributive property to verify that $\sin\theta(\csc\theta - \sin\theta) = \mathbf{\cos^2\theta}$ is an identity.

Solution ► Use the distributive property to simplify the left-hand side.

$$\sin\theta(\csc\theta - \sin\theta) = \sin\theta\csc\theta - \sin^2\theta \quad \text{distribute}$$
$$= 1 - \sin^2\theta \quad \text{substitute 1 for } \sin\theta\csc\theta$$
$$= \mathbf{\cos^2\theta} \quad 1 - \sin^2\theta = \cos^2\theta$$

Since we were able to transform the left-hand side into a duplicate of the right, there can be no doubt the original equation is an identity.

Now try Exercises 11 through 20 ►

Often we must *factor* an expression, rather than multiply, to begin the verification process.

EXAMPLE 3 ► Using Algebra to Help Verify an Identity

Verify that $\mathbf{1} = \cot^2x\sec^2x - \cot^2x$ is an identity.

Solution ► The left side is as simple as it gets. The terms on the right side have a common factor and we begin there.

$$\begin{aligned} \cot^2 x \sec^2 x - \cot^2 x &= \cot^2 x\,(\sec^2 x - 1) && \text{factor out } \cot^2 x \\ &= \cot^2 x \tan^2 x && \text{substitute } \tan^2 x \text{ for } \sec^2 x - 1 \\ &= (\cot x \tan x)^2 && \text{power property of exponents} \\ &= 1^2 = \mathbf{1} && \cot x \tan x = 1 \end{aligned}$$

Now try Exercises 21 through 28 ►

Examples 2 and 3 show you can begin the verification process on either the left or right side of the equation, whichever seems more convenient. Example 4 shows how the special products $(A + B)(A - B) = A^2 - B^2$ and/or $(A \pm B)^2 = A^2 \pm 2AB + B^2$ can be used in the verification process.

EXAMPLE 4 ► Using a Special Product to Help Verify an Identity

Use a special product and fundamental identities to verify that $(\sin x - \cos x)^2 = \mathbf{1 + 2\sin(-x)\cos x}$ is an identity.

Solution ► Begin by squaring the left-hand side, in hopes of using a Pythagorean identity.

$$\begin{aligned} (\sin x - \cos x)^2 &= \sin^2 x - 2\sin x\cos x + \cos^2 x && \text{binomial square} \\ &= \sin^2 x + \cos^2 x - 2\sin x \cos x && \text{rewrite terms} \\ &= 1 - 2\sin x\cos x && \text{substitute 1 for } \sin^2 x + \cos^2 x \end{aligned}$$

At this point we appear to be off by a sign, but quickly recall that sine is on odd function and $-\sin x = \sin(-x)$. By writing $1 - 2\sin x\cos x$ as $1 + 2(-\sin x)(\cos x)$, we can complete the verification:

$$\begin{aligned} &= 1 + 2(-\sin x)(\cos x) && \text{rewrite expression to obtain } -\sin x \\ &= \mathbf{1 + 2\sin(-x)\cos x} \checkmark && \text{substitute } \sin(-x) \text{ for } -\sin x \end{aligned}$$

Now try Exercises 29 through 34 ►

B. You've just learned how to verify other identities using the fundamental identities and basic algebra skills

Another common method used to verify identities is simplification by combining terms, using the model $\frac{A}{B} \pm \frac{C}{D} = \frac{AD \pm BC}{BD}$. For $\sec u = \frac{\sin^2 u}{\cos u} + \cos u$, the right-hand side immediately becomes $\frac{\sin^2 u + \cos^2 u}{\cos u}$, which gives $\frac{1}{\cos u} = \sec u$. **See Exercises 35 through 40.**

C. Writing One Function in Terms of Another

Any one of the six trigonometric functions can be written in terms of any of the other functions using fundamental identities. The process involved offers practice in working with identities, highlights how each function is related to the other, and has practical applications in verifying more complex identities.

EXAMPLE 5 ► Writing One Trig Function in Terms of Another

Write the function $\cos x$ in terms of the tangent function.

Solution ► Begin by noting these functions share "common ground" via $\sec x$, since $\sec^2 x = 1 + \tan^2 x$ and $\cos x = \frac{1}{\sec x}$. Starting with $\sec^2 x$,

$$\begin{aligned} \sec^2 x &= 1 + \tan^2 x && \text{Pythagorean identity} \\ \sec x &= \pm\sqrt{1 + \tan^2 x} && \text{square roots} \end{aligned}$$

We can now substitute $\pm\sqrt{1+\tan^2 x}$ for sec x in $\cos x = \frac{1}{\sec x}$.

$$\cos x = \frac{1}{\pm\sqrt{1+\tan^2 x}} \quad \text{substitute } \pm\sqrt{1+\tan^2 x} \text{ for sec } x$$

Now try Exercises 41 through 46 ►

WORTHY OF NOTE

It is important to note the stipulation "valid where both are defined" does not preclude a difference in the domains of each function. The result of Example 5 is indeed an identity, even though the expressions have unequal domains.

Example 5 also reminds us of a very important point — the sign we choose for the final answer is dependent on the terminal side of the angle. If the terminal side is in QI or QIV we chose the positive sign since $\cos x > 0$ in those quadrants. If the angle terminates in QII or QIII, the final answer is negative since $\cos x < 0$ in those quadrants.

Similar to our work in Chapter 5, given the value of cot t and the quadrant of t, the fundamental identities enable us to find the value of the other five functions at t. In fact, this is generally true for any given trig function and real number or angle t.

EXAMPLE 6 ► Using a Known Value and Quadrant Analysis to Find Other Function Values

Given $\cot t = \frac{-9}{40}$ with t in QIV, find the value of the other five functions at t.

Solution ► The function value $\tan t = -\frac{40}{9}$ follows immediately, since cotangent and tangent are reciprocals. The value of sec t can be found using $\sec^2 t = 1 + \tan^2 t$.

$$\sec^2 t = 1 + \tan^2 t \quad \text{Pythagorean identity}$$
$$= 1 + \left(-\frac{40}{9}\right)^2 \quad \text{substitute } -\frac{40}{9} \text{ for tan } t$$
$$= \frac{81}{81} + \frac{1600}{81} \quad \text{square } -\frac{40}{9}\text{, substitute } \frac{81}{81} \text{ for 1}$$
$$= \frac{1681}{81} \quad \text{combine terms}$$
$$\sec t = \pm\frac{41}{9} \quad \text{take square roots}$$

Since sec t is positive in QIV, we have $\sec t = \frac{41}{9}$. This automatically gives $\cos t = \frac{9}{41}$ (reciprocal identities), and we find $\sin t = -\frac{40}{41}$ using $\sin^2 t = 1 - \cos^2 t$ or the ratio identity $\tan t = \frac{\sin t}{\cos t}$ (verify).

Now try Exercises 47 through 55 ►

☑ **C. You've just learned how to use fundamental identities to express a given trig function in terms of the other five**

D. Showing an Equation Is Not an Identity

To show an equation is *not* an identity, we need only find a single value for which the functions involved are defined but the equation is *false*. This can often be done by trial and error, or even by inspection. To illustrate the process, we'll use two common misconceptions that arise in working with identities.

EXAMPLE 7 ► Showing an Equation is Not an Identity

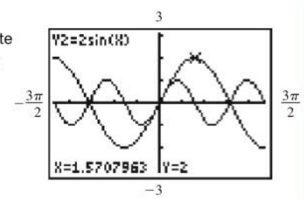
Show the equations given are *not* identities.

a. $\sin(2x) = 2\sin x$ **b.** $\cos(\alpha + \beta) = \cos\alpha + \cos\beta$

Solution ► **a.** The assumption here seems to be that we can factor out the coefficient from the argument. By inspection we note the amplitude of $\sin(2x)$ is $A = 1$, while the amplitude of $2 \sin x$ is $A = 2$. This means they cannot possibly be equal for all values of x, although they are equal for integer multiples of π. Verify they are not equivalent using $x = \frac{\pi}{6}$ or other standard values.

GRAPHICAL SUPPORT

While not a definitive method of proof, a graphing calculator can be used to investigate whether an equation is an identity. Since the left and right members of the equation must be equal for all values (where they are defined), their graphs must be identical. Graphing the functions from Example 7(a) as Y_1 and Y_2 shows the equation $\sin(2x) = 2 \sin x$ is definitely *not* an identity.



b. The assumption here is that we can distribute function values. This is similar to saying $\sqrt{x + 4} = \sqrt{x} + 2$, a statement obviously false for all values except $x = 0$. Here we'll substitute convenient values to prove the equation false, namely, $\alpha = \frac{3\pi}{4}$ and $\beta = \frac{\pi}{4}$.

$$\cos\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \quad \text{substitute } \frac{\pi}{3} \text{ for } \alpha \text{ and } \frac{\pi}{4} \text{ for } \beta$$

$$\cos \pi = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \quad \text{simplify}$$

$$-1 \neq 0 \quad \text{result is false}$$

☑ **D.** You've just learned how to use counterexamples and contradictions to show an equation is not an identity

Now try Exercises 56 through 62 ►

6.1 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Three fundamental ratio identities are $\tan\theta = \frac{?}{\cos\theta}$, $\tan\theta = \frac{?}{\csc\theta}$, and $\cot\theta = \frac{?}{\sin\theta}$.

2. When applying identities due to symmetry, $\sin(-x)\tan x =$ ________ and $\cos(-x)\cot x =$ ________.

3. To show an equation is *not an identity,* we must find at least ____ value(s) where both sides of the equation are defined, but which makes the equation ______.

4. Using a calculator we find $\sec^2 45° =$ ____ and $3\tan 45° - 1 =$ ____. We also find $\sec^2 225° =$ ____ and $3\tan 225° - 1 =$ ____. Is the equation $\sec^2\theta = 3\tan\theta - 1$ an identity?

5. Use the pattern $\frac{A}{B} \pm \frac{C}{D} = \frac{AD \pm BC}{BD}$ to add the following terms, and comment on this process versus "finding a common denominator:" $\frac{\cos x}{\sin x} - \frac{\sin x}{\sec x}$.

6. Name at least four algebraic skills that are used with the fundamental identities in order to rewrite a trigonometric expression. Use algebra to quickly rewrite $(\sin x + \cos x)^2$.

▶ DEVELOPING YOUR SKILLS

Starting with the ratio identity given, use substitution and fundamental identities to write four new identities belonging to the ratio family. Answers may vary.

7. $\tan x = \dfrac{\sin x}{\cos x}$ **8.** $\cot x = \dfrac{\cos x}{\sin x}$

Starting with the Pythagorean identity given, use algebra to write four additional identities belonging to the Pythagorean family. Answers may vary.

9. $1 + \tan^2 x = \sec^2 x$ **10.** $1 + \cot^2 x = \csc^2 x$

Verify the equation is an identity using multiplication and fundamental identities.

11. $\sin x \cot x = \cos x$ **12.** $\cos x \tan x = \sin x$

13. $\sec^2 x \cot^2 x = \csc^2 x$ **14.** $\csc^2 x \tan^2 x = \sec^2 x$

15. $\cos x(\sec x - \cos x) = \sin^2 x$

16. $\tan x(\cot x + \tan x) = \sec^2 x$

17. $\sin x(\csc x - \sin x) = \cos^2 x$

18. $\cot x(\tan x + \cot x) = \csc^2 x$

19. $\tan x(\csc x + \cot x) = \sec x + 1$

20. $\cot x(\sec x + \tan x) = \csc x + 1$

Verify the equation is an identity using factoring and fundamental identities.

21. $\tan^2 x \csc^2 x - \tan^2 x = 1$

22. $\sin^2 x \cot^2 x + \sin^2 x = 1$

23. $\dfrac{\sin x \cos x + \sin x}{\cos x + \cos^2 x} = \tan x$

24. $\dfrac{\sin x \cos x + \cos x}{\sin x + \sin^2 x} = \cot x$

25. $\dfrac{1 + \sin x}{\cos x + \cos x \sin x} = \sec x$

26. $\dfrac{1 + \cos x}{\sin x + \cos x \sin x} = \csc x$

27. $\dfrac{\sin x \tan x + \sin x}{\tan x + \tan^2 x} = \cos x$

28. $\dfrac{\cos x \cot x + \cos x}{\cot x + \cot^2 x} = \sin x$

Verify the equation is an identity using special products and fundamental identities.

29. $\dfrac{(\sin x + \cos x)^2}{\cos x} = \sec x + 2\sin x$

30. $\dfrac{(1 + \tan x)^2}{\sec x} = \sec x + 2\sin x$

31. $(1 + \sin x)[1 + \sin(-x)] = \cos^2 x$

32. $(\sec x + 1)[\sec(-x) - 1] = \tan^2 x$

33. $\dfrac{(\csc x - \cot x)(\csc x + \cot x)}{\tan x} = \cot x$

34. $\dfrac{(\sec x + \tan x)(\sec x - \tan x)}{\csc x} = \sin x$

Verify the equation is an identity using fundamental identities and $\dfrac{A}{B} \pm \dfrac{C}{D} = \dfrac{AD \pm BC}{BD}$ to combine terms.

35. $\dfrac{\cos^2 x}{\sin x} + \dfrac{\sin x}{1} = \csc x$

36. $\dfrac{\sec \alpha}{1} - \dfrac{\tan^2 \alpha}{\sec \alpha} = \cos \alpha$

37. $\dfrac{\tan x}{\csc x} - \dfrac{\sin x}{\cos x} = \dfrac{\sin x - 1}{\cot x}$

38. $\dfrac{\cot x}{\sec x} - \dfrac{\cos x}{\sin x} = \dfrac{\cos x - 1}{\tan x}$

39. $\dfrac{\sec x}{\sin x} - \dfrac{\csc x}{\sec x} = \tan x$ **40.** $\dfrac{\csc x}{\cos x} - \dfrac{\sec x}{\csc x} = \cot x$

Write the given function entirely in terms of the second function indicated.

41. $\tan x$ in terms of $\sin x$ **42.** $\tan x$ in terms of $\sec x$

43. $\sec x$ in terms of $\cot x$ **44.** $\sec x$ in terms of $\sin x$

45. $\cot x$ in terms of $\sin x$ **46.** $\cot x$ in terms of $\csc x$

For the function $f(\theta)$ and the quadrant in which θ terminates, state the value of the other five trig functions.

47. $\cos \theta = -\dfrac{20}{29}$ with θ in QII

48. $\sin \theta = \dfrac{12}{37}$ with θ in QII

49. $\tan \theta = \dfrac{15}{8}$ with θ in QIII

50. $\sec \theta = \dfrac{45}{27}$ with θ in QIV

51. $\cot \theta = \dfrac{x}{5}$ with θ in QI

52. $\csc\theta = \frac{7}{x}$ with θ in QII

53. $\sin\theta = -\frac{7}{13}$ with θ in QIII

54. $\cos\theta = \frac{23}{25}$ with θ in QIV

55. $\sec\theta = -\frac{9}{7}$ with θ in QII

Show that the following equations *are not identities.*

56. $\sin\left(\theta + \frac{\pi}{3}\right) = \sin\theta + \sin\left(\frac{\pi}{3}\right)$

57. $\cos\left(\frac{\pi}{4}\right) + \cos\theta = \cos\left(\frac{\pi}{4} + \theta\right)$

58. $\cos(2\theta) = 2\cos\theta$

59. $\tan(2\theta) = 2\tan\theta$

60. $\tan\left(\frac{\theta}{4}\right) = \frac{\tan\theta}{\tan 4}$

61. $\cos^2\theta - \sin^2\theta = -1$

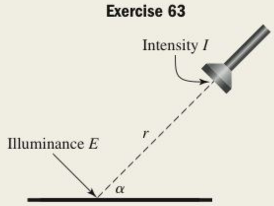
62. $\sqrt{\sin^2 x - 9} = \sin x - 3$

▶ WORKING WITH FORMULAS

63. The illuminance of a point on a surface by a source of light: $E = \frac{I\cos\theta}{r^2}$

The illuminance E (in lumens/m^2) of a point on a horizontal surface is given by the formula shown, where I is the intensity of the light source in lumens, r is the distance in meters from the light source to the point, and θ is the complement of the angle α (in degrees) made by the light source and the horizontal surface. Calculate the illuminance if $I = 800$ lumens, and the flashlight is held so that the distance r is 2 m while the angle α is 40°.

Exercise 63



64. The area of regular polygon: $A = \left(\frac{nx^2}{4}\right)\frac{\cos(\frac{\pi}{n})}{\sin(\frac{\pi}{n})}$

The area of a regular polygon is given by the formula shown, where n represents the number of sides and x is the length of each side.

a. Rewrite the formula in terms of a single trig function.

b. Verify the formula for a square with sides of 8 m.

c. Find the area of a dodecagon (12 sides) with 10-in. sides.

▶ APPLICATIONS

Writing a given expression in an alternative form is an idea used at all levels of mathematics. In future classes, it is often helpful to decompose a power into smaller powers (as in writing A^3 as $A \cdot A^2$) or to rewrite an expression using known identities so that it can be factored.

65. Show that $\cos^3 x$ can be written as $\cos x(1 - \sin^2 x)$.

66. Show that $\tan^3 x$ can be written as $\tan x(\sec^2 x - 1)$.

67. Show that $\tan x + \tan^3 x$ can be written as $\tan x(\sec^2 x)$.

68. Show that $\cot^3 x$ can be written as $\cot x(\csc^2 x - 1)$.

69. Show $\tan^2 x\sec x - 4\tan^2 x$ can be factored into $(\sec x - 4)(\sec x - 1)(\sec x + 1)$.

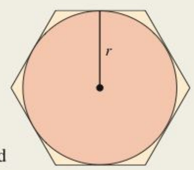
70. Show $2\sin^2 x\cos x - \sqrt{3}\sin^2 x$ can be factored into $(1 - \cos x)(1 + \cos x)(2\cos x - \sqrt{3})$.

71. Show $\cos^2 x\sin x - \cos^2 x$ can be factored into $-1(1 + \sin x)(1 - \sin x)^2$.

72. Show $2\cot^2 x\csc x + 2\sqrt{2}\cot^2 x$ can be factored into $2(\csc x + \sqrt{2})(\csc x - 1)(\csc x + 1)$.

Many applications of fundamental identities involve geometric figures, as in Exercises 73 and 74.

73. Area of a polygon: The area of a regular polygon that has been circumscribed about a circle of radius r (see figure) is given by the formula $A = nr^2\frac{\sin(\frac{\pi}{n})}{\cos(\frac{\pi}{n})}$, where n represents the number of sides. (a) Rewrite the formula in terms of a single trig function; (b) verify the formula for a square circumscribed about a circle with radius 4 m; and (c) find the area of a dodecagon (12 sides) circumscribed about the same circle.



74. Perimeter of a polygon: The perimeter of a regular polygon circumscribed about a circle of radius r is given by the formula $P = 2nr\frac{\sin(\frac{\pi}{n})}{\cos(\frac{\pi}{n})}$, where n represents the number of sides. (a) Rewrite the formula in terms of a single trig function; (b) verify the formula for a square circumscribed about a circle with radius 4 m; and (c) Find the perimeter of a dodecagon (12 sides) circumscribed about the same circle.

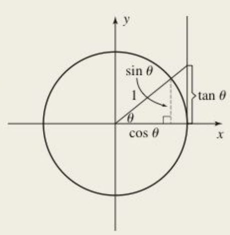
75. Angle of intersection: At their point of intersection, the angle θ between any two nonparallel lines satisfies the relationship $(m_2 - m_1)\cos\theta = \sin\theta + m_1m_2\sin\theta$, where m_1 and m_2 represent the slopes of the two lines. Rewrite the equation in terms of a single trig function.

76. Angle of intersection: Use the result of Exercise 75 to find the angle between the lines $Y_1 = \frac{2}{5}x - 3$ and $Y_2 = \frac{7}{3}x + 1$.

77. Angle of intersection: Use the result of Exercise 75 to find the angle between the lines $Y_1 = 3x - 1$ and $Y_2 = -2x + 7$.

▶ EXTENDING THE CONCEPT

78. The word *tangent* literally means "to touch," which in mathematics we take to mean *touches in only and exactly one point.* In the figure, the circle has a radius of 1 and the vertical line is "tangent" to the circle at the x-axis. The figure can be used to verify the Pythagorean identity for sine and cosine, as well as the ratio identity for tangent. Discuss/Explain how.



79. Use factoring and fundamental identities to help find the x-intercepts of f in $[0, 2\pi)$.

$$f(\theta) = -2\sin^4\theta + \sqrt{3}\sin^3\theta + 2\sin^2\theta - \sqrt{3}\sin\theta.$$

▶ MAINTAINING YOUR SKILLS

80. (4.6) Solve for x:

$$2351 = \frac{2500}{1 + e^{-1.015x}}$$

81. (5.6) Standing 265 ft from the base of the Strastosphere Tower in Las Vegas, Nevada, the angle of elevation to the top of the tower is about 77°. Approximate the height of the tower to the nearest foot.

82. (3.3) Use the rational zeroes theorem and other "tools" to find all zeroes of the function $f(x) = 2x^4 + 9x^3 - 4x^2 - 36x - 16$.

83. (5.3) Use a reference rectangle and the *rule of fourths* to sketch the graph of $y = 2\sin(2t)$ for t in $[0, 2\pi)$.

6.2 Constructing and Verifying Identities

Learning Objectives

In Section 6.2 you will learn how to:

- **A.** Create and verify a new identity
- **B.** Verify general identities

In Section 6.1, our primary goal was to illustrate how basic algebra skills are used to help rewrite trigonometric expressions. In this section, we'll sharpen and refine these skills so they can be applied more generally, as we develop the ability to verify a much wider range of identities.

A. Creating and Verifying Identities

In Example 2 of Section 6.1, we showed $\sin\theta(\csc\theta - \sin\theta) = \cos^2\theta$ was an identity by transforming the left-hand side into $\cos^2\theta$. There, the instructions were very specific: "Use the distributive property to . . ." When verifying identities, one of the biggest issues students face is that the directions are deliberately vague—because there is no single, fail-proof approach for verifying an identity. This sometimes leaves students feeling they don't know where to start, or what to do first. To help overcome this discomfort, we'll first *create an identity* by substituting fundamental identities into a given expression, then reverse these steps to get back the original expression. This return to the original illustrates the essence of verifying identities, namely, if two things are equal, one can be substituted for the other at any time. The process may seem arbitrary (actually—it *is*), and the steps could vary. But try to keep the underlying message in mind, rather than any specific steps. When working with identities, there is actually *no right place* to start, and the process begins by using the substitution principle to create an equivalent expression as you work toward the expression you're trying to match.

EXAMPLE 1 ► **Creating and Verifying an Identity**

Starting with the expression $\csc x + \cot x$, use fundamental identities to rewrite the expression and create a new identity. Then verify the identity by reversing the steps.

Solution ►

$\csc x + \cot x$ — original expression

$= \dfrac{1}{\sin x} + \dfrac{\cos x}{\sin x}$ — substitute reciprocal and ratio identities

$= \dfrac{1 + \cos x}{\sin x}$ — write as a single term

Resulting identity ► $\csc x + \cot x = \dfrac{1 + \cos x}{\sin x}$

Verify identity ► Working with the right-hand side, we reverse each step with a view toward the original expression.

$\dfrac{1 + \cos x}{\sin x} = \dfrac{1}{\sin x} + \dfrac{\cos x}{\sin x}$ — rewrite as individual terms

$= \csc x + \cot x$ — substitute reciprocal and ratio identities

Now try Exercises 7 through 9 ►

In actual practice, all you'll see is this instruction, "Verify the following is an identity: $\csc x + \cot x = \dfrac{1 + \cos x}{\sin x}$," and it will be up to you to employ the algebra and fundamental identities needed.

EXAMPLE 2 ► Creating and Verifying an Identity

Starting with the expression 2 tan x sec x, use fundamental identities to rewrite the expression and create a new identity. Then verify the identity by reversing the steps.

Solution ►

$$2\tan x\sec x \quad \text{original expression}$$
$$= 2\cdot\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} \quad \text{substitute ratio and reciprocal identities}$$
$$= \frac{2\sin x}{\cos^2 x} \quad \text{multiply}$$
$$= \frac{2\sin x}{1-\sin^2 x} \quad \text{substitute } 1-\sin^2 x \text{ for } \cos^2 x$$

Resulting identity ► $2\tan x\sec x = \dfrac{2\sin x}{1-\sin^2 x}$ identity

Verify identity ► Working with the right-hand side, we reverse each step with a view toward the original expression.

$$\frac{2\sin x}{1-\sin^2 x} = \frac{2\sin x}{\cos^2 x} \quad \text{substitute } \cos^2 x \text{ for } 1-\sin^2 x$$
$$= 2\cdot\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} \quad \text{substitute } \cos x\cdot\cos x \text{ for } \cos^2 x$$
$$= 2\tan x\sec x \quad \text{substitute ratio and reciprocal identities}$$

A. You've just learned how to create and verify a new identity

Now try Exercises 10 through 12 ►

B. Verifying Identities

We're now ready to put these ideas, and the ideas from Section 6.1, to work for us. When verifying identities we attempt to mold, change, or rewrite one side of the equality until we obtain a match with the other side. What follows is a collection of the ideas and methods we've observed so far, which we'll call the *Guidelines for Verifying Identities*. But remember, there really is no *right* place to start. Think things over for a moment, then attempt a substitution, simplification, or operation and see where it leads. If you hit a dead end, that's okay! Just back up and try something else.

WORTHY OF NOTE

When verifying identities, it is actually permissible to work on each side of the equality *independently,* in the effort to create a "match." But properties of equality can never be used, since we cannot assume an equality exists.

Guidelines for Verifying Identities

1. As a general rule, work on only one side of the identity.
   - We cannot assume the equation is true, so properties of equality cannot be applied.
   - We verify the identity by changing the form of one side until we get a match with the other.
2. Work with the more complex side, as it is easier to reduce/simplify than to "build."
3. If an expression contains more than one term, it is often helpful to combine terms using $\dfrac{A}{B}\pm\dfrac{C}{D}=\dfrac{AD\pm BC}{BD}$.
4. Converting all functions to sines and cosines can be helpful.
5. Apply other algebra skills as appropriate: distribute, factor, multiply by a conjugate, and so on.
6. *Know the fundamental identities inside out, upside down, and backward—they are the key!*

Note how these ideas are employed in Examples 3 through 5, particularly the frequent use of fundamental identities.

EXAMPLE 3 ► **Verifying an Identity**

Verify the identity: $\mathbf{\sin^2\theta \tan^2\theta} = \tan^2\theta - \sin^2\theta$.

Solution ► As a general rule, the side with the greater number of terms or the side with rational terms is considered "more complex," so we begin with the right-hand side.

$$\begin{aligned}
\tan^2\theta - \sin^2\theta &= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta && \text{substitute } \frac{\sin^2\theta}{\cos^2\theta} \text{ for } \tan^2\theta \\
&= \frac{\sin^2\theta}{1} \cdot \frac{1}{\cos^2\theta} - \sin^2\theta && \text{decompose rational term} \\
&= \sin^2\theta \sec^2\theta - \sin^2\theta && \text{substitute } \sec^2\theta \text{ for } \frac{1}{\cos^2\theta} \\
&= \sin^2\theta\,(\sec^2\theta - 1) && \text{factor out } \sin^2\theta \\
&= \mathbf{\sin^2\theta \tan^2\theta} && \text{substitute } \tan^2\theta \text{ for } \sec^2\theta - 1
\end{aligned}$$

Now try Exercises 13 through 18 ►

Example 3 involved *factoring* out a common expression. Just as often, we'll need to *multiply* numerators and denominators by a common expression, as in Example 4.

EXAMPLE 4 ► **Verifying an Identity by Multiplying Conjugates**

Verify the identity: $\dfrac{\cos t}{1 + \sec t} = \dfrac{\mathbf{1 - \cos t}}{\mathbf{\tan^2 t}}$.

Solution ► Both sides of the identity have a single term and one is really no more complex than the other. As a matter of choice we begin with the left side. Noting the denominator on the left has the term $\sec t$, with a corresponding term of $\tan^2 t$ to the right, we reason that multiplication by a conjugate might be productive.

$$\begin{aligned}
\frac{\cos t}{1 + \sec t} &= \left(\frac{\cos t}{1 + \sec t}\right)\left(\frac{1 - \sec t}{1 - \sec t}\right) && \text{multiply above and below by the conjugate} \\
&= \frac{\cos t - 1}{1 - \sec^2 t} && \text{distribute: } \cos t \sec t = 1,\ (A + B)(A - B) = A^2 - B^2 \\
&= \frac{\cos t - 1}{-\tan^2 t} && \text{substitute } -\tan^2 t \text{ for } 1 - \sec^2 t \\
& && (1 + \tan^2 t = \sec^2 t \Rightarrow 1 - \sec^2 t = -\tan^2 t) \\
&= \frac{\mathbf{1 - \cos t}}{\mathbf{\tan^2 t}} && \text{multiply above and below by } -1
\end{aligned}$$

Now try Exercises 19 through 22 ►

Example 4 highlights the need to be very familiar with families of identities. To replace $1 - \sec^2 t$, we had to use $-\tan^2 t$, not simply $\tan^2 t$, since the related Pythagorean identity is $1 + \tan^2 t = \sec^2 t$.

As noted in the *Guidelines,* combining rational terms is often helpful. At this point, students are encouraged to work with the pattern $\dfrac{A}{B} \pm \dfrac{C}{D} = \dfrac{AD \pm BC}{BD}$ as a means of combing rational terms quickly and efficiently.

EXAMPLE 5 ► **Verifying an Identity by Combining Terms**

Verify the identity: $\frac{\sec x}{\sin x} - \frac{\sin x}{\sec x} = \frac{\tan^2 x + \cos^2 x}{\tan x}$.

Solution ► We begin with the left-hand side.

$$\frac{\sec x}{\sin x} - \frac{\sin x}{\sec x} = \frac{\sec^2 x - \sin^2 x}{\sin x \sec x} \qquad \text{combine terms: } \frac{A}{B} - \frac{C}{D} = \frac{AD - BC}{BD}$$

$$= \frac{(1 + \tan^2 x) - (1 - \cos^2 x)}{\left(\frac{\sin x}{1}\right)\left(\frac{1}{\cos x}\right)} \qquad \text{substitute } 1 + \tan^2 x \text{ for } \sec^2 x,\ 1 - \cos^2 x \text{ for } \sin^2 x,\ \frac{1}{\cos x} \text{ for } \sec x,$$

$$= \frac{\tan^2 x + \cos^2 x}{\tan x} \qquad \text{simplify numerator, substitute } \tan x \text{ for } \frac{\sin x}{\cos x}$$

B. You've just learned how to verify general identities

Now try Exercises 23 through 28 ►

Identities come in an infinite variety and it would be impossible to illustrate all variations. Using the general ideas and skills presented should prepare you to verify any of those given in the exercise set, as well as those you encounter in your future studies. See **Exercises 29 through 58.**

6.2 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. If two expressions are equal, then one may be ________ for the other at any time and the result will be equivalent.
2. We verify an identity by changing the form of one side, working until we ________ the other side.
3. To verify an identity, always begin with the more __________ expression, since it is easier to ________ than to ________.
4. Converting all terms to functions of ________ and ________ may help verify an identity.
5. Discuss/Explain why you must not add, subtract, multiply, or divide both sides of the equation when verifying identities.
6. Discuss/Explain the difference between operating on both sides of an equation (see Exercise 5) and working on each side independently.

► DEVELOPING YOUR SKILLS

Using algebra and the fundamental identities, rewrite each given expression to create a new identity relationship. Then verify your identity by reversing the steps. Answers will vary.

7. $\sec x + \tan x$
8. $(\cos x + \sin x)^2$
9. $(1 - \sin^2 x)\sec x$
10. $2 \cot x \csc x$
11. $\frac{\sin x - \sin x \cos x}{\sin^2 x}$
12. $(\cos x + \sin x)(\cos x - \sin x)$

Verify that the following equations are identities.

13. $\cos^2x\tan^2x = 1 - \cos^2x$

14. $\sin^2x\cot^2x = 1 - \sin^2x$

15. $\tan x + \cot x = \sec x\csc x$

16. $\cot x\cos x = \csc x - \sin x$

17. $\dfrac{\cos x}{\tan x} = \csc x - \sin x$

18. $\dfrac{\sin x}{\cot x} = \sec x - \cos x$

19. $\dfrac{\cos\theta}{1 - \sin\theta} = \sec\theta + \tan\theta$

20. $\dfrac{\sin\theta}{1 - \cos\theta} = \csc\theta + \cot\theta$

21. $\dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 + \sin x}$

22. $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$

23. $\dfrac{\csc x}{\cos x} - \dfrac{\cos x}{\csc x} = \dfrac{\cot^2x + \sin^2x}{\cot x}$

24. $\dfrac{1}{\cos^2x} + \dfrac{1}{\sin^2x} = \csc^2x\sec^2x$

25. $\dfrac{\sin x}{1 + \sin x} - \dfrac{\sin x}{1 - \sin x} = -2\tan^2x$

26. $\dfrac{\cos x}{1 + \cos x} - \dfrac{\cos x}{1 - \cos x} = -2\cot^2x$

27. $\dfrac{\cot x}{1 + \csc x} - \dfrac{\cot x}{1 - \csc x} = 2\sec x$

28. $\dfrac{\tan x}{1 + \sec x} - \dfrac{\tan x}{1 - \sec x} = 2\csc x$

29. $\dfrac{\sec^2x}{1 + \cot^2x} = \tan^2x$

30. $\dfrac{\csc^2x}{1 + \tan^2x} = \cot^2x$

31. $\sin^2x(\cot^2x - \csc^2x) = -\sin^2x$

32. $\cos^2x(\tan^2x - \sec^2x) = -\cos^2x$

33. $\cos x\cot x + \sin x = \csc x$

34. $\sin x\tan x + \cos x = \sec x$

35. $\dfrac{\sec x}{\cot x + \tan x} = \sin x$

36. $\dfrac{\csc x}{\cot x + \tan x} = \cos x$

37. $\dfrac{\sin x - \csc x}{\csc x} = -\cos^2x$

38. $\dfrac{\cos x - \sec x}{\sec x} = -\sin^2x$

39. $\dfrac{1}{\csc x - \sin x} = \tan x\sec x$

40. $\dfrac{1}{\sec x - \cos x} = \cot x\csc x$

41. $\dfrac{1 + \sin x}{1 - \sin x} = (\tan x + \sec x)^2$

42. $\dfrac{1 - \cos x}{1 + \cos x} = (\csc x - \cot x)^2$

43. $\dfrac{\cos x - \sin x}{1 - \tan x} = \dfrac{\cos x + \sin x}{1 + \tan x}$

44. $\dfrac{1 - \cot x}{1 + \cot x} = \dfrac{\sin x - \cos x}{\sin x + \cos x}$

45. $\dfrac{\tan^2x - \cot^2x}{\tan x - \cot x} = \csc x\sec x$

46. $\dfrac{\cot x - \tan x}{\cot^2x - \tan^2x} = \sin x\cos x$

47. $\dfrac{\cot x}{\cot x + \tan x} = 1 - \sin^2x$

48. $\dfrac{\tan x}{\cot x + \tan x} = 1 - \cos^2x$

49. $\dfrac{\sec^4x - \tan^4x}{\sec^2x + \tan^2x} = 1$

50. $\dfrac{\csc^4x - \cot^4x}{\csc^2x + \cot^2x} = 1$

51. $\dfrac{\cos^4x - \sin^4x}{\cos^2x} = 2 - \sec^2x$

52. $\dfrac{\sin^4x - \cos^4x}{\sin^2x} = 2 - \csc^2x$

53. $(\sec x + \tan x)^2 = \dfrac{(\sin x + 1)^2}{\cos^2x}$

54. $(\csc x + \cot x)^2 = \dfrac{(\cos x + 1)^2}{\sin^2x}$

55. $\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x} + \dfrac{\csc x}{\sec x} = \dfrac{\sec x + \cos x}{\sin x}$

56. $\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x} + \dfrac{\sec x}{\csc x} = \dfrac{\csc x + \sin x}{\cos x}$

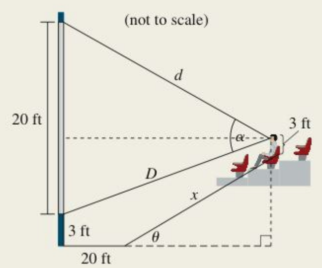
57. $\dfrac{\sin^4x - \cos^4x}{\sin^3x + \cos^3x} = \dfrac{\sin x - \cos x}{1 - \sin x\cos x}$

58. $\dfrac{\sin^4x - \cos^4x}{\sin^3x - \cos^3x} = \dfrac{\sin x + \cos x}{1 + \sin x\cos x}$

▶ WORKING WITH FORMULAS

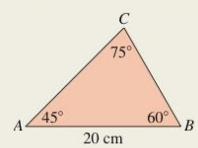
59. Distance to top of movie screen:
$$d^2 = (20 + x\cos\theta)^2 + (20 - x\sin\theta)^2$$

At a theater, the optimum viewing angle depends on a number of factors, like the height of the screen, the incline of the auditorium, the location of a seat, the height of your eyes while seated, and so on. One of the measures needed to find the "best" seat is the distance from your eyes to the top of the screen. For a theater with the dimensions shown, this distance is given by the formula here (x is the diagonal distance from the horizontal floor to your seat). (a) Show the formula is equivalent to $800 + 40x(\cos\theta - \sin\theta) + x^2$. (b) Find the distance d if $\theta = 18°$ and you are sitting in the eighth row with the rows spaced 3 ft apart.



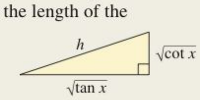
60. The area of triangle ABC: $A = \dfrac{c^2 \sin A \sin B}{2\sin C}$

If one side and three angles of a triangle are known, its area can be computed using this formula, where side c is opposite angle C. Find the area of the triangle shown in the diagram.



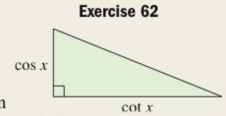
▶ APPLICATIONS

61. Pythagorean theorem: For the triangle shown, (a) find an expression for the length of the hypotenuse in terms of $\tan x$ and $\cot x$, then determine the length of the hypotenuse when $x = 1.5$ rad; (b) show the expression you found in part (a) is equivalent to $h = \sqrt{\csc x \sec x}$ and recompute the length of the hypotenuse using this expression. Did the answers match?



62. Pythagorean theorem: For the triangle shown, (a) find an expression for the area of the triangle in terms of $\cot x$ and $\cos x$, then determine its area given $x = \dfrac{\pi}{6}$; (b) show the expression you found in part (a) is equivalent to $A = \dfrac{1}{2}(\csc x - \sin x)$ and recompute the area using this expression. Did the answers match?

Exercise 62



63. Viewing distance: Referring to Exercise 59, find a formula for D—the distance from this patron's eyes to the *bottom* of the movie screen. Simplify the result using a Pythagorean identity, then find the value of D.

64. Viewing angle: Referring to Exercises 59 and 63, once d and D are known, the viewing angle α (the angle subtended by the movie screen and the viewer's eyes) can be found using the formula $\cos\alpha = \dfrac{d^2 + D^2 - 20^2}{2dD}$. Find the value of $\cos\alpha$ for this particular theater, person, and seat.

65. Intensity of light: In a study of the luminous intensity of light, the expression $\sin\alpha = \dfrac{I_1\cos\theta}{\sqrt{(I_1\cos\theta)^2 + (I_2\sin\theta)^2}}$ can occur. Simplify the equation for the moment $I_1 = I_2$.

66. Intensity of light: Referring to Exercise 65, find the angle θ given $I_1 = I_2$ and $\alpha = 60°$.

▶ EXTENDING THE CONCEPT

67. Just as the points $P(x, y)$ on the unit circle $x^2 + y^2 = 1$ are used to name the circular trigonometric functions, the points $P(x, y)$ on the unit hyperbola $x^2 - y^2 = 1$ are used to name what are called the **hyperbolic trigonometric functions.** The hyperbolic functions are used extensively in many of the applied sciences. The identities for these functions have many similarities to those for the circular functions, but also have some significant differences. Using the Internet or the resources of a library, do some research on the functions sinh t, cosh t, and tanh t, where t is any real number. In particular, see how the Pythagorean identities compare/contrast between the two forms of trigonometry.

68. Verify the identity $\dfrac{\sin^6 x - \cos^6 x}{\sin^4 x - \cos^4 x} = 1 - \sin^2 x \cos^2 x$.

69. Use factoring to show the equation is an identity: $\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x = 1$.

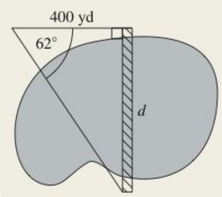
▶ MAINTAINING YOUR SKILLS

70. **(3.5)** Graph the rational function given.
$h(x) = \dfrac{x - 1}{x^2 - 4}$

71. **(5.2)** Verify that $\left(\dfrac{\sqrt{7}}{4}, \dfrac{3}{4}\right)$ is a point on the unit circle, then state the values of sin t, cos t, and tan t associated with this point.

72. **(5.7)** Use an appropriate trig ratio to find the length of the bridge needed to cross the lake shown in the figure.

Exercise 72



73. **(2.5)** Graph using transformations of a basic function:
$f(x) = -2|x - 3| + 6$

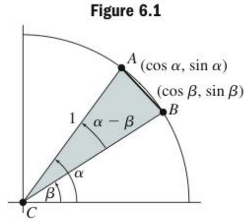
6.3 The Sum and Difference Identities

Learning Objectives

In Section 6.3 you will learn how to:

- ❑ **A.** Develop and use sum and difference identities for cosine
- ❑ **B.** Use the cofunction identities to develop the sum and difference identities for sine and tangent
- ❑ **C.** Use the sum and difference identities to verify other identities

Figure 6.1



The sum and difference formulas for sine and cosine have a long and ancient history. Originally developed to help study the motion of celestial bodies, they were used centuries later to develop more complex concepts, such as the derivatives of the trig functions, complex number theory, and the study wave motion in different mediums. These identities are also used to find exact results (in radical form) for many nonstandard angles, a result of great importance to the ancient astronomers and still of notable mathematical significance today.

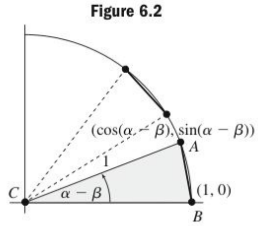
A. The Sum and Difference Identities for Cosine

On a unit circle with center C, consider the point A on the terminal side of angle α, and point B on the terminal side of angle β, as shown in Figure 6.1. Since $r = 1$, the coordinates of A and B are $(\cos\alpha, \sin\alpha)$ and $(\cos\beta, \sin\beta)$, respectively. Using the distance formula, we find that $\overline{AB}$ is equal to

$$\begin{aligned}\overline{AB} &= \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} && \text{binomial squares}\\ &= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} && \text{regroup}\\ &= \sqrt{(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}\\ &= \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} && \cos^2 u + \sin^2 u = 1\end{aligned}$$

With no loss of generality, we can rotate sector ACB clockwise, until side $\overline{CB}$ coincides with the x-axis. This creates new coordinates of $(1, 0)$ for B, and new coordinates of $(\cos(\alpha - \beta), \sin(\alpha - \beta))$ for A, *but the distance $\overline{AB}$ remains unchanged!* (see Figure 6.2). Recomputing the distance gives

Figure 6.2



$$\begin{aligned}\overline{AB} &= \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2}\\ &= \sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)}\\ &= \sqrt{[\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)] - 2\cos(\alpha - \beta) + 1}\\ &= \sqrt{2 - 2\cos(\alpha - \beta)}\end{aligned}$$

Since both expressions represent the same distance, we can set them equal to each other and solve for $\cos(\alpha - \beta)$.

$$\begin{aligned}\sqrt{2 - 2\cos(\alpha - \beta)} &= \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} && \overline{AB} = \overline{AB}\\ 2 - 2\cos(\alpha - \beta) &= 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta && \text{property of radicals}\\ -2\cos(\alpha - \beta) &= -2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta && \text{subtract 2}\\ \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta && \text{divide both sides by } -2\end{aligned}$$

The result is called the **difference identity for cosine.** The **sum identity for cosine** follows immediately, by substituting $-\beta$ for β.

$$\begin{aligned}\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta && \text{difference identity}\\ \cos(\alpha - [-\beta]) &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) && \text{substitute } -\beta \text{ for } \beta\\ \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta && \cos(-\beta) = \cos\beta;\ \sin(-\beta) = -\sin\beta\end{aligned}$$

The sum and difference identities can be used to find exact values for the trig functions of certain angles (values written in nondecimal form using radicals), simplify expressions, and to establish additional identities.

EXAMPLE 1 ► **Finding Exact Values for Non-Standard Angles**

Use the sum and difference identities for cosine to find exact values for

a. $\cos 15° = \cos(45° - 30°)$ **b.** $\cos 75° = \cos(45° + 30°)$

Check results on a calculator.

Solution ► Each involves a direct application of the related identity, and uses special values.

a.

$$\begin{aligned}\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta && \text{difference identity}\\ \cos(45° - 30°) &= \cos 45°\cos 30° + \sin 45°\sin 30° && \alpha = 45°, \beta = 30°\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) && \text{standard values}\\ \cos 15° &= \frac{\sqrt{6} + \sqrt{2}}{4} && \text{combine terms}\end{aligned}$$

To 10 decimal places, $\cos 15° = 0.9659258263$.

b.

$$\begin{aligned}\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta && \text{sum identity}\\ \cos(45° + 30°) &= \cos 45°\cos 30° - \sin 45°\sin 30° && \alpha = 45°, \beta = 30°\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) && \text{standard values}\\ \cos 75° &= \frac{\sqrt{6} - \sqrt{2}}{4} && \text{combine terms}\end{aligned}$$

To 10 decimal places, $\cos 75° = 0.2588190451$.

WORTHY OF NOTE

Be aware that $\cos(60° + 30°) \neq \cos 60° + \cos 30°$ $\left(0 \neq \frac{1}{2} + \frac{\sqrt{3}}{2}\right)$ and in general $f(a + b) \neq f(a) + f(b)$.

Now try Exercises 7 through 12 ►

These identities are listed here using the "±" and "∓" notation to avoid needless repetition. In their application, use both upper symbols or both lower symbols depending on whether you're evaluating the cosine of a sum or difference of two angles. As with the other identities, these can be rewritten to form other members of the identity family, as when they are used to consolidate a larger expression. This is shown in Example 2.

The Sum and Difference Identities for Cosine

cosine family: $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ functions repeat, signs alternate

$\cos\alpha\cos\beta \mp \sin\alpha\sin\beta = \cos(\alpha \pm \beta)$ can be used to *expand* or *contract*

EXAMPLE 2 ► Using a Sum/Difference Identity to Simplify an Expression

Write as a single expression in cosine and evaluate: $\cos 57° \cos 78° - \sin 57° \sin 78°$

Solution ► Since the functions repeat and are expressed as a difference, we use the sum identity for cosine to rewrite the difference as a single expression.

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha + \beta) \quad \text{sum identity for cosine}$$
$$\cos 57°\cos 78° - \sin 57°\sin 78° = \cos(57° + 78°) \quad \alpha = 57°, \beta = 78°$$

The expression is equal to $\cos 135° = -\frac{\sqrt{2}}{2}$.

Now try Exercises 13 through 16 ►

The sum and difference identities can be used to evaluate the cosine of the sum of two angles, even when they are not adjacent, or even expressed in terms of cosine.

EXAMPLE 3 ► Computing the Cosine of a Sum

Given $\sin\alpha = \frac{5}{13}$ with the terminal side in QI, and $\tan\beta = -\frac{24}{7}$ with the terminal side in QII. Compute the value of $\cos(\alpha + \beta)$.

Solution ► To use the sum formula we need the value of $\cos\alpha$, $\sin\alpha$, $\cos\beta$, and $\sin\beta$. Using the given information about the quadrants along with the Pythagorean theorem, we draw the triangles shown in Figures 6.3 and 6.4, yielding the values that follow.

Figure 6.3

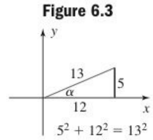
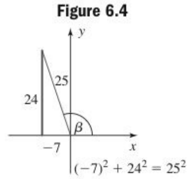


Figure 6.4



$$\cos\alpha = \frac{12}{13}\text{ (QI)},\ \sin\alpha = \frac{5}{13}\text{ (QI)},\ \cos\beta = -\frac{7}{25}\text{ (QII), and } \sin\beta = \frac{24}{25}\text{ (QII)}$$

Using $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ gives this result:

$$\cos(\alpha + \beta) = \left(\frac{12}{13}\right)\left(-\frac{7}{25}\right) - \left(\frac{5}{13}\right)\left(\frac{24}{25}\right)$$
$$= -\frac{84}{325} - \frac{120}{325}$$
$$= -\frac{204}{325}$$

Now try Exercises 17 and 18 ►

B. The Sum and Difference Identities for Sine and Tangent

A. You've just learned how to develop and use sum and difference identities for cosine

The cofunction identities were actually introduced in Section 5.1, using the complementary angles in a right triangle. In this section we'll *verify* that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

WORTHY OF NOTE

It is worth pointing out that in Example 3, if we approximate the values of α and β using tables or a calculator, we find $\alpha \approx 22.62°$ and $\beta \approx 106.26°$. Sure enough, $\cos(22.62° + 106.26°) \approx -\frac{204}{325}$!

and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$. For the first, we use the difference identity for cosine to obtain

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\ &= (0)\cos\theta + (1)\sin\theta \\ &= \sin\theta\end{aligned}$$

For the second, we use $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$, and replace θ with the real number $\frac{\pi}{2} - t$. This gives

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \text{cofunction identity for cosine}$$

$$\cos\left(\frac{\pi}{2} - \left[\frac{\pi}{2} - t\right]\right) = \sin\left(\frac{\pi}{2} - t\right) \qquad \text{replace } \theta \text{ with } \frac{\pi}{2} - t$$

$$\cos t = \sin\left(\frac{\pi}{2} - t\right) \qquad \text{result, note } \left[\frac{\pi}{2} - \left(\frac{\pi}{2} - t\right)\right] = t$$

This establishes the cofunction relationship for sine: $\sin\left(\frac{\pi}{2} - t\right) = \cos t$ for any real number t. Both identities can be written in terms of the real number t. **See Exercises 19 through 24.**

The Cofunction Identities

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t \qquad \sin\left(\frac{\pi}{2} - t\right) = \cos t$$

The sum and difference identities for sine can easily be developed using cofunction identities. Since $\sin t = \cos\left(\frac{\pi}{2} - t\right)$, we need only rename t as the sum $(\alpha + \beta)$ or the difference $(\alpha - \beta)$ and work from there.

$$\sin t = \cos\left(\frac{\pi}{2} - t\right) \qquad \text{cofunction identity}$$

$$\sin(\alpha + \beta) = \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right] \qquad \text{substitute } (\alpha + \beta) \text{ for } t$$

$$= \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right] \qquad \text{regroup argument}$$

$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta \qquad \text{apply difference identity for cosine}$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \text{result}$$

The difference identity for sine is likewise developed. The sum and difference identities for tangent can be derived using ratio identities and their derivation is left as an exercise **(see Exercise 78).**

The Sum and Difference Identities for Sine and Tangent

sine family: $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ — functions alternate, signs repeat

$\sin\alpha\cos\beta \pm \cos\alpha\sin\beta = \sin(\alpha \pm \beta)$ — can be used to *expand* or *contract*

tangent family: $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$ — signs match original in numerator; signs alternate in denominator

$\dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} = \tan(\alpha \pm \beta)$ — can be used to *expand* or *contract*

EXAMPLE 4A ► **Simplifying Expressions Using Sum/Difference Identities**

Write as a single expression in sine: $\sin(2t)\cos t + \cos(2t)\sin t$.

Solution ► Since the functions in each term alternate and the expression is written as a sum, we use the sum identity for sine:

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta) \quad \text{sum identity for sine}$$

$$\sin(2t)\cos t + \cos(2t)\sin t = \sin(2t + t) \quad \text{substitute } 2t \text{ for } \alpha \text{ and } t \text{ for } \beta$$

The expression is equal to $\sin(3t)$.

EXAMPLE 4B ► **Simplifying Expressions Using Sum/Difference Identities**

Use the sum or difference identity for tangent to find the exact value of $\tan\dfrac{11\pi}{12}$.

Solution ► Since an exact value is requested, $\dfrac{11\pi}{12}$ must be the sum or difference of two standard angles. A casual inspection reveals $\dfrac{11\pi}{12} = \dfrac{2\pi}{3} + \dfrac{\pi}{4}$. This gives

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \quad \text{sum identity for tangent}$$

$$\tan\left(\frac{2\pi}{3} + \frac{\pi}{4}\right) = \frac{\tan\left(\frac{2\pi}{3}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\left(\frac{2\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)} \quad \alpha = \frac{2\pi}{3},\ \beta = \frac{\pi}{4}$$

$$= \frac{-\sqrt{3} + 1}{1 - (-\sqrt{3})(1)} \quad \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3},\ \tan\left(\frac{\pi}{4}\right) = 1$$

$$= \frac{1 - \sqrt{3}}{1 + \sqrt{3}} \quad \text{simplify expression}$$

☑ **B.** You've just learned how to use the cofunction identities to develop the sum and difference identities for sine and tangent

Now try Exercises 25 through 54 ►

C. Verifying Other Identities

Once the sum and difference identities are established, we can simply add these to the tools we use to verify other identities.

EXAMPLE 5 ► **Verifying an Identity**

Verify that $\tan\left(\theta - \frac{\pi}{4}\right) = \frac{\tan\theta - 1}{\tan\theta + 1}$ is an identity.

Solution ► Using a direct application of the difference formula for tangent we obtain

$$\tan\left(\theta - \frac{\pi}{4}\right) = \frac{\tan\theta - \tan\frac{\pi}{4}}{1 + \tan\theta\tan\frac{\pi}{4}} \qquad \alpha = \theta, \beta = \frac{\pi}{4}$$

$$= \frac{\tan\theta - 1}{1 + \tan\theta} = \frac{\tan\theta - 1}{\tan\theta + 1} \qquad \tan\left(\frac{\pi}{4}\right) = 1$$

Now try Exercises 55 through 60 ►

EXAMPLE 6 ► **Verifying an Identity**

Verify that $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$ is an identity.

Solution ► Using the sum and difference formulas for sine we obtain

$$\sin(\alpha + \beta)\sin(\alpha - \beta) = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

$$= \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta \qquad (A+B)(A-B) = A^2 - B^2$$

$$= \sin^2\alpha\,(1 - \sin^2\beta) - (1 - \sin^2\alpha)\sin^2\beta \qquad \text{use } \cos^2 x = 1 - \sin^2 x \text{ to write the expression solely in terms of sine}$$

$$= \sin^2\alpha - \sin^2\alpha\sin^2\beta - \sin^2\beta + \sin^2\alpha\sin^2\beta \qquad \text{distribute}$$

$$= \sin^2\alpha - \sin^2\beta \qquad \text{simplify}$$

C. You've just learned how to use the sum and difference identities to verify other identities

Now try Exercises 61 through 68 ►

6.3 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Since $\tan 45° + \tan 60° > 1$, we know $\tan 45° + \tan 60° = \tan 105°$ is ____ since $\tan\theta < 0$ in ____.

2. To find an exact value for $\tan 105°$, use the sum identity for tangent with $\alpha =$ ____ and $\beta =$____.

3. For the cosine sum/difference identities, the functions ________ in each term, with the ______ sign between them.

4. For the sine sum/difference identities, the functions ________ in each term, with the ______ sign between them.

5. Discuss/Explain how we know the exact value for $\cos\frac{11\pi}{12} = \cos\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$ will be negative, prior to applying any identity.

6. Discuss/Explain why $\tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\beta - \alpha)}$ is an identity, even though the arguments of cosine have been reversed. Then verify the identity.

▶ DEVELOPING YOUR SKILLS

Find the exact value of the expression given using a sum or difference identity. Some simplifications may involve using symmetry and the formulas for negatives.

7. $\cos 105°$ **8.** $\cos 135°$

9. $\cos\left(\frac{7\pi}{12}\right)$ **10.** $\cos\left(-\frac{5\pi}{12}\right)$

Use sum/difference identities to verify that both expressions give the same result.

11. a. $\cos(45° + 30°)$ **b.** $\cos(120° - 45°)$

12. a. $\cos\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$ **b.** $\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$

Rewrite as a single expression in cosine.

13. $\cos(7\theta)\cos(2\theta) + \sin(7\theta)\sin(2\theta)$

14. $\cos\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{6}\right) - \sin\left(\frac{\theta}{3}\right)\sin\left(\frac{\theta}{6}\right)$

Find the exact value of the given expressions.

15. $\cos 183° \cos 153° + \sin 183° \sin 153°$

16. $\cos\left(\frac{7\pi}{36}\right)\cos\left(\frac{5\pi}{36}\right) - \sin\left(\frac{7\pi}{36}\right)\sin\left(\frac{5\pi}{36}\right)$

17. For $\sin\alpha = -\frac{4}{5}$ with terminal side in QIV and $\tan\beta = -\frac{5}{12}$ with terminal side in QII, find $\cos(\alpha + \beta)$.

18. For $\sin\alpha = \frac{112}{113}$ with terminal side in QII and $\sec\beta = -\frac{89}{39}$ with terminal side in QII, find $\cos(\alpha - \beta)$.

Use a cofunction identity to write an equivalent expression.

19. $\cos 57°$ **20.** $\sin 18°$ **21.** $\tan\left(\frac{5\pi}{12}\right)$

22. $\sec\left(\frac{\pi}{10}\right)$ **23.** $\sin\left(\frac{\pi}{6} - \theta\right)$ **24.** $\cos\left(\frac{\pi}{3} + \theta\right)$

Rewrite as a single expression.

25. $\sin(3x)\cos(5x) + \cos(3x)\sin(5x)$

26. $\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{3}\right) - \cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{3}\right)$

27. $\dfrac{\tan(5\theta) - \tan(2\theta)}{1 + \tan(5\theta)\tan(2\theta)}$

28. $\dfrac{\tan\left(\frac{x}{2}\right) + \tan\left(\frac{x}{8}\right)}{1 - \tan\left(\frac{x}{2}\right)\tan\left(\frac{x}{8}\right)}$

Find the exact value of the given expressions.

29. $\sin 137° \cos 47° - \cos 137° \sin 47°$

30. $\sin\left(\frac{11\pi}{24}\right)\cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{11\pi}{24}\right)\sin\left(\frac{5\pi}{24}\right)$

31. $\dfrac{\tan\left(\frac{11\pi}{21}\right) - \tan\left(\frac{4\pi}{21}\right)}{1 + \tan\left(\frac{11\pi}{21}\right)\tan\left(\frac{4\pi}{21}\right)}$

32. $\dfrac{\tan\left(\frac{3\pi}{20}\right) + \tan\left(\frac{\pi}{10}\right)}{1 - \tan\left(\frac{3\pi}{20}\right)\tan\left(\frac{\pi}{10}\right)}$

33. For $\cos\alpha = -\frac{7}{25}$ with terminal side in QII and $\cot\beta = \frac{15}{8}$ with terminal side in QIII, find

a. $\sin(\alpha + \beta)$ **b.** $\tan(\alpha + \beta)$

34. For $\csc\alpha = \frac{29}{20}$ with terminal side in QI and $\cos\beta = -\frac{12}{37}$ with terminal side in QII, find

a. $\sin(\alpha - \beta)$ **b.** $\tan(\alpha - \beta)$

Find the exact value of the expression given using a sum or difference identity. Some simplifications may involve using symmetry and the formulas for negatives.

35. $\sin 105°$ **36.** $\sin(-75°)$

37. $\sin\left(\frac{5\pi}{12}\right)$ **38.** $\sin\left(\frac{11\pi}{12}\right)$

39. $\tan 150°$ **40.** $\tan 75°$

41. $\tan\left(\frac{2\pi}{3}\right)$ **42.** $\tan\left(-\frac{\pi}{12}\right)$

Use sum/difference identities to verify that both expressions give the same result.

43. a. $\sin(45° - 30°)$ **b.** $\sin(135° - 120°)$

44. a. $\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ **b.** $\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

45. Find $\sin 255°$ given $150° + 105° = 255°$. See Exercises 7 and 35.

46. Find $\cos\left(\frac{19\pi}{12}\right)$ given $2\pi - \frac{5\pi}{12} = \frac{19\pi}{12}$. See Exercises 10 and 37.

47. Given α and β are acute angles with $\sin\alpha = \frac{12}{13}$ and $\tan\beta = \frac{35}{12}$, find

a. $\sin(\alpha + \beta)$ **b.** $\cos(\alpha - \beta)$ **c.** $\tan(\alpha + \beta)$

48. Given α and β are acute angles with $\cos\alpha = \frac{8}{17}$ and $\sec\beta = \frac{25}{7}$, find

a. $\sin(\alpha + \beta)$ **b.** $\cos(\alpha - \beta)$ **c.** $\tan(\alpha + \beta)$

49. Given α and β are obtuse angles with $\sin\alpha = \frac{28}{53}$ and $\cos\beta = -\frac{13}{85}$, find

a. $\sin(\alpha - \beta)$ **b.** $\cos(\alpha + \beta)$ **c.** $\tan(\alpha - \beta)$

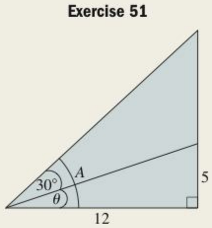
50. Given α and β are obtuse angles with $\tan\alpha = -\frac{60}{11}$ and $\sin\beta = \frac{35}{37}$, find

a. $\sin(\alpha - \beta)$ **b.** $\cos(\alpha + \beta)$ **c.** $\tan(\alpha - \beta)$

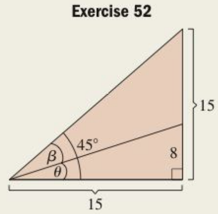
51. Use the diagram indicated to compute the following:

a. $\sin A$ **b.** $\cos A$ **c.** $\tan A$

Exercise 51



Exercise 52



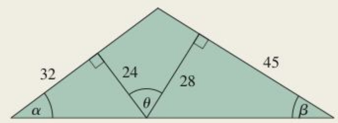
52. Use the diagram indicated to compute the following:

a. $\sin\beta$ **b.** $\cos\beta$ **c.** $\tan\beta$

53. For the figure indicated, show that $\theta = \alpha + \beta$ and compute the following:

a. $\sin\theta$ **b.** $\cos\theta$ **c.** $\tan\theta$

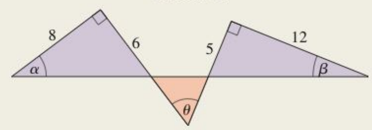
Exercise 53



54. For the figure indicated, show that $\theta = \alpha + \beta$ and compute the following:

a. $\sin\theta$ **b.** $\cos\theta$ **c.** $\tan\theta$

Exercise 54



Verify each identity.

55. $\sin(\pi - \alpha) = \sin\alpha$

56. $\cos(\pi - \alpha) = -\cos\alpha$

57. $\cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos x - \sin x)$

58. $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin x + \cos x)$

59. $\tan\left(x + \frac{\pi}{4}\right) = \frac{1 + \tan x}{1 - \tan x}$

60. $\tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - 1}{\tan x + 1}$

61. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$

62. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\sin\beta$

63. $\cos(2t) = \cos^2 t - \sin^2 t$

64. $\sin(2t) = 2\sin t\cos t$

65. $\sin(3t) = -4\sin^3 t + 3\sin t$

66. $\cos(3t) = 4\cos^3 t - 3\cos t$

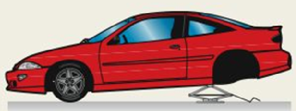
67. Use a difference identity to show $\cos\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos x + \sin x)$.

68. Use sum/difference identities to show $\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = \sqrt{2}\sin x$.

▶ WORKING WITH FORMULAS

69. Force and equilibrium: $F = \frac{Wk}{c}\tan(p - \theta)$

The force and equilibrium when a screw jack is used can be modeled by the formula shown, where p is the pitch angle of the screw, W is the weight of the load, θ is the angle of friction, with k and c being constants related to a particular jack. Simplify the formula using the difference formula for tangent given $p = \frac{\pi}{6}$ and $\theta = \frac{\pi}{4}$.



70. Brewster's law: $\tan\theta_p = \frac{n_2}{n_1}$

Brewster's law of optics states that when unpolarized light strikes a dielectric surface, the transmitted light rays and the reflected light rays are perpendicular to each other. The proof of Brewster's law involves the expression $n_1\sin\theta_p = n_2\sin\left(\frac{\pi}{2} - \theta_p\right)$. Use the difference identity for sine to verify that this expression leads to Brewster's law.

▶ APPLICATIONS

71. AC circuits: In a study of AC circuits, the equation $R = \frac{\cos s\cos t}{\omega C\sin(s + t)}$ sometimes arises. Use a sum identity and algebra to show this equation is equivalent to $R = \frac{1}{\omega C(\tan s + \tan t)}$.

72. Fluid mechanics: In studies of fluid mechanics, the equation $\gamma_1 V_1 \sin\alpha = \gamma_2 V_2 \sin(\alpha - \beta)$ sometimes arises. Use a difference identity to show that if $\gamma_1 V_1 = \gamma_2 V_2$, the equation is equivalent to $\cos\beta - \cot\alpha\sin\beta = 1$.

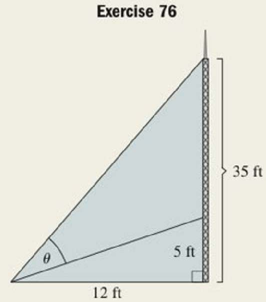
73. Art and mathematics: When working in two-point geometric perspective, artists must scale their work to fit on the paper or canvas they are using. In doing so, the equation $\frac{A}{B} = \frac{\tan\theta}{\tan(90° - \theta)}$ arises. Rewrite the expression on the right in terms of sine and cosine, then use the difference identities to show the equation can be rewritten as $\frac{A}{B} = \tan^2\theta$.

74. Traveling waves: If two waves of the same frequency, velocity, and amplitude are traveling along a string in opposite directions, they can be represented by the equations $Y_1 = A\sin(kx - \omega t)$ and $Y_2 = A\sin(kx + \omega t)$. Use the sum and difference formulas for sine to show the result $Y_R = Y_1 + Y_2$ of these waves can be expressed as $Y_R = 2A\sin(kx)\cos(\omega t)$.

75. Pressure on the eardrum: If a frequency generator is placed a certain distance from the ear, the pressure on the eardrum can be modeled by the function $P_1(t) = A\sin(2\pi ft)$, where f is the frequency and t is the time in seconds. If a second frequency generator with identical settings is placed slightly closer to the ear, its pressure on the eardrum could be represented by $P_2(t) = A\sin(2\pi ft + C)$, where C is a constant. Show that if $C = \frac{\pi}{2}$, the total pressure on the eardrum $[P_1(t) + P_2(t)]$ is $P(t) = A[\sin(2\pi ft) + \cos(2\pi ft)]$.

76. Angle between two cables: Two cables used to steady a radio tower are attached to the tower at heights of 5 ft and 35 ft, with both secured to a stake 12 ft from the tower (see figure). Find the value of $\cos\theta$, where θ is the angle between the upper and lower cables.

Exercise 76



77. Difference quotient: Given $f(x) = \sin x$, show that the difference quotient results in the expression $\sin x\dfrac{\cos h - 1}{h} + \cos x\left(\dfrac{\sin h}{h}\right)$.

78. Difference identity: Derive the difference identity for tangent using $\tan(\alpha - \beta) = \dfrac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)}$. (*Hint:* After applying the difference identities, divide the numerator and denominator by $\cos\alpha\cos\beta$.)

► EXTENDING THE CONCEPT

A family of identities called the *angle reduction formulas,* will be of use in our study of complex numbers and other areas. These formulas use the period of a function to reduce large angles to an angle in $[0, 360°)$ or $[0, 2\pi)$ having an equivalent function value: (1) $\cos(t + 2\pi k) = \cos t$; (2) $\sin(t + 2\pi k) = \sin t$. Use the reduction formulas to find values for the following functions (note the formulas can also be expressed in degrees).

79. $\cos 1665°$

80. $\cos\left(\dfrac{91\pi}{6}\right)$

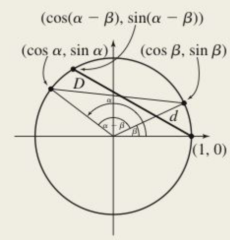
81. $\sin\left(\dfrac{41\pi}{6}\right)$

82. $\sin 2385°$

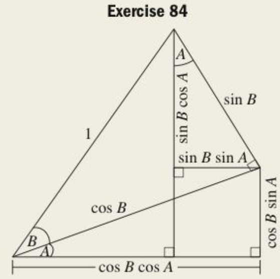
83. An alternative method of proving the difference formula for cosine uses a unit circle and the fact that equal arcs are subtended by equal chords ($D = d$ in the diagram). Using a combination of algebra, the distance formula, and a Pythagorean identity, show that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ (start by computing D^2 and d^2). Then discuss/explain how the sum identity can be found using the fact that $\beta = -(-\beta)$.

84. A proof without words: Verify the Pythagorean theorem for each right triangle in the diagram, then discuss/explain how the diagram offers a proof of the sum identities for sine and cosine. Be detailed and thorough.

Exercise 83



Exercise 84



► MAINTAINING YOUR SKILLS

85. (5.3/5.4) State the period of the functions given:

a. $y = 3\sin\left(\dfrac{\pi}{8}x - \dfrac{\pi}{3}\right)$

b. $y = 4\tan\left(2x + \dfrac{\pi}{4}\right)$

86. (2.7) Graph the piecewise-defined function given:

$$f(x) = \begin{cases} 3 & x < -1 \\ x^2 & -1 \le x \le 1 \\ x & x > 1 \end{cases}$$

87. (5.2) Clarence the Clown is about to be shot from a circus cannon to a safety net on the other side of the main tent. If the cannon is 30 ft long and must be aimed at 40° for Clarence to hit the net, the end of the cannon must be how high from ground level?

88. (2.3) Find the equation of the line parallel to $2x + 5y = -10$, containing the point $(5, -2)$. Write your answer in standard form.

6.4 The Double-Angle, Half-Angle, and Product-to-Sum Identities

Learning Objectives

In Section 6.4 you will learn how to:

- ☐ **A.** Derive and use the double-angle identities for cosine, tangent, and sine
- ☐ **B.** Develop and use the power reduction and half-angle identities
- ☐ **C.** Derive and use the product-to-sum and sum-to-product identities
- ☐ **D.** Solve applications using these identities

The derivation of the sum and difference identities in Section 6.3 was a "watershed event" in the study of identities. By making various substitutions, they lead us very naturally to many new identity families, giving us a heightened ability to simplify expressions, solve equations, find exact values, and model real-world phenomena. In fact, many of the identities are applied in very practical ways, as in a study of projectile motion and the conic sections (Chapter 10). In addition, one of the most profound principles discovered in the eighteenth and nineteenth centuries was that electricity, light, and sound could all be studied using sinusoidal waves. These waves often interact with each other, creating the phenomena known as reflection, diffraction, superposition, interference, standing waves, and others. The product-to-sum and sum-to-product identities play a fundamental role in the investigation and study of these phenomena.

A. The Double-Angle Identities

The double-angle identities for sine, cosine, and tangent can all be derived using the related sum identities with two equal angles ($\alpha = \beta$). We'll illustrate the process here for the cosine of twice an angle.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \text{sum identity for cosine}$$
$$\cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \quad \text{assume } \alpha = \beta \text{ and substitute } \alpha \text{ for } \beta$$
$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \quad \text{simplify—double-angle identity for cosine}$$

Using the Pythagorean identity $\cos^2\alpha + \sin^2\alpha = 1$, we can easily find two additional members of this family, which are often quite useful. For $\cos^2\alpha = 1 - \sin^2\alpha$ we have

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \quad \text{double-angle identity for cosine}$$
$$= (1 - \sin^2\alpha) - \sin^2\alpha \quad \text{substitute } 1 - \sin^2\alpha \text{ for } \cos^2\alpha$$
$$\cos(2\alpha) = 1 - 2\sin^2\alpha \quad \text{double-angle in terms of sine}$$

Using $\sin^2\alpha = 1 - \cos^2\alpha$ we obtain an additional form:

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \quad \text{double-angle identity for cosine}$$
$$= \cos^2\alpha - (1 - \cos^2\alpha) \quad \text{substitute } 1 - \cos^2\alpha \text{ for } \sin^2\alpha$$
$$\cos(2\alpha) = 2\cos^2\alpha - 1 \quad \text{double-angle in terms of cosine}$$

The derivations of $\sin(2\alpha)$ and $\tan(2\alpha)$ are likewise developed and are asked for in **Exercise 103.** The double-angle identities are collected here for your convenience.

The Double-Angle Identities

cosine: $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$
$= 1 - 2\sin^2\alpha$
$= 2\cos^2\alpha - 1$

sine: $\sin(2\alpha) = 2\sin\alpha\cos\alpha$

tangent: $\tan(2\alpha) = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

EXAMPLE 1 ► Using a Double-Angle Identity to Find Function Values

Given $\sin\alpha = \frac{5}{8}$, find the value of $\cos(2\alpha)$.

Solution ► Using the double-angle identity for cosine in terms of sine, we find

$$\begin{aligned}\cos(2\alpha) &= 1 - 2\sin^2\alpha && \text{double-angle in terms of sine}\\ &= 1 - 2\left(\frac{5}{8}\right)^2 && \text{substitute } \tfrac{5}{8} \text{ for } \sin\alpha\\ &= 1 - \frac{25}{32} && 2\left(\tfrac{5}{8}\right)^2 = \tfrac{25}{32}\\ &= \frac{7}{32} && \text{result}\end{aligned}$$

If $\sin\alpha = \frac{5}{8}$, then $\cos(2\alpha) = \frac{7}{32}$.

Now try Exercises 7 through 20 ►

Like the fundamental identities, the double-angle identities can be used to verify or develop others. In Example 2, we explore one of many **multiple-angle identities,** verifying that $\cos(3\theta)$ can be rewritten as $4\cos^3\theta - 3\cos\theta$ (in terms of powers of $\cos\theta$).

EXAMPLE 2 ► Verifying a Multiple Angle Identity

Verify that $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ is an identity.

Solution ► Use the sum identity for cosine, with $\alpha = 2\theta$ and $\beta = \theta$. Note that our goal is an expression using cosines only, with no multiple angles.

$$\begin{aligned}\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta && \text{sum identity for cosine}\\ \cos(2\theta + \theta) &= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta && \text{substitute } 2\theta \text{ for } \alpha \text{ and } \theta \text{ for } \beta\\ \cos(3\theta) &= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta && \text{substitute for } \cos(2\theta) \text{ and } \sin(2\theta)\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta && \text{multiply}\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta) && \text{substitute } 1 - \cos^2\theta \text{ for } \sin^2\theta\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta && \text{multiply}\\ &= 4\cos^3\theta - 3\cos\theta && \text{combine terms}\end{aligned}$$

Now try Exercises 21 and 22 ►

EXAMPLE 3 ► Using a Double-Angle Formula to Find Exact Values

Find the exact value of sin 22.5° cos 22.5°.

Solution ► A product of sines and cosines having the same argument hints at the double-angle identity for sine. Using $\sin(2\alpha) = 2\sin\alpha\cos\alpha$ and dividing by 2 gives

$$\sin\alpha\cos\alpha = \frac{\sin(2\alpha)}{2} \quad \text{double-angle identity for sine}$$
$$\sin 22.5^\circ \cos 22.5^\circ = \frac{\sin(2[22.5^\circ])}{2} \quad \text{replace } \alpha \text{ with } 22.5^\circ$$
$$= \frac{\sin 45^\circ}{2} \quad \text{multiply}$$
$$= \frac{\frac{\sqrt{2}}{2}}{2} = \frac{\sqrt{2}}{4} \quad \sin 45^\circ = \frac{\sqrt{2}}{2}$$

A. You've just learned how to derive and use the double-angle identities for cosine, tangent, and sine

Now try Exercises 23 through 30 ►

B. The Power Reduction and Half-Angle Identities

Expressions having a trigonometric function raised to a power occur quite frequently in various applications. We can rewrite even powers of these trig functions in terms of an expression containing only cosine to the power 1, using what are called the **power reduction identities.** This makes the expression easier to use and evaluate. It can legitimately be argued that the power reduction identities are actually members of the double-angle family, as all three are a direct consequence. To find identities for $\cos^2 x$ and $\sin^2 x$, we solve the related double-angle identity involving $\cos(2x)$.

$$1 - 2\sin^2\alpha = \cos(2\alpha) \quad \cos(2\alpha) \text{ in terms of sine}$$
$$-2\sin^2\alpha = \cos(2\alpha) - 1 \quad \text{subtract 1, then divide by } -2$$
$$\sin^2\alpha = \frac{1-\cos(2\alpha)}{2} \quad \text{power reduction identity for sine}$$

Using the same approach for $\cos^2\alpha$ gives $\cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$. The identity for $\tan^2\alpha$ can be derived from $\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$ **(see Exercise 104),** but in this case it's easier to use the identity $\tan^2 u = \frac{\sin^2 u}{\cos^2 u}$. The result is $\frac{1-\cos(2\alpha)}{1+\cos(2\alpha)}$.

The Power Reduction Identities

$$\cos^2\alpha = \frac{1+\cos(2\alpha)}{2} \qquad \sin^2\alpha = \frac{1-\cos(2\alpha)}{2} \qquad \tan^2\alpha = \frac{1-\cos(2\alpha)}{1+\cos(2\alpha)}$$

EXAMPLE 4 ► Using a Power Reduction Formula

Write $8\sin^4 x$ in terms of an expression containing only cosines to the power 1.

Solution ►

$$8\sin^4 x = 8(\sin^2 x)^2 \quad \text{original expression}$$
$$= 8\left[\frac{1-\cos(2x)}{2}\right]^2 \quad \text{substitute } \frac{1-\cos(2x)}{2} \text{ for } \sin^2 x$$
$$= 2[1 - 2\cos(2x) + \cos^2(2x)] \quad \text{multiply}$$
$$= 2\left[1 - 2\cos(2x) + \frac{1+\cos(4x)}{2}\right] \quad \text{substitute } \frac{1+\cos(4x)}{2} \text{ for } \cos^2(2x)$$
$$= 2 - 4\cos(2x) + 1 + \cos(4x) \quad \text{multiply}$$
$$= 3 - 4\cos(2x) + \cos(4x) \quad \text{result}$$

Now try Exercises 31 through 36 ►

The half-angle identities follow directly from those above, using algebra and a simple change of variable. For $\cos^2\alpha = \frac{1 + \cos(2\alpha)}{2}$, we first take square roots and obtain $\cos\alpha = \pm\sqrt{\frac{1 + \cos(2\alpha)}{2}}$. Using the substitution $u = 2\alpha$ gives $\alpha = \frac{u}{2}$, and making these substitutions results in the half-angle identity for cosine: $\cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 + \cos u}{2}}$, where the radical's sign depends on the quadrant in which $\frac{u}{2}$ terminates. Using the same substitution for sine gives $\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 - \cos u}{2}}$, and for the tangent identity, $\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 - \cos u}{1 + \cos u}}$. In the case of $\tan\left(\frac{u}{2}\right)$, we can actually develop identities that are free of radicals by rationalizing the denominator or numerator. We'll illustrate the former, leaving the latter as an exercise **(see Exercise 102).**

$$\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{(1 - \cos u)(1 - \cos u)}{(1 + \cos u)(1 - \cos u)}} \qquad \text{multiply by the conjugate}$$

$$= \pm\sqrt{\frac{(1 - \cos u)^2}{1 - \cos^2 u}} \qquad \text{rewrite}$$

$$= \pm\sqrt{\frac{(1 - \cos u)^2}{\sin^2 u}} \qquad \text{Pythagorean identity}$$

$$= \pm\left|\frac{1 - \cos u}{\sin u}\right| \qquad \sqrt{x^2} = |x|$$

Since $1 - \cos u > 0$ and $\sin u$ has the same sign as $\tan\left(\frac{u}{2}\right)$ for all u in its domain, the relationship can simply be written $\tan\left(\frac{u}{2}\right) = \frac{1 - \cos u}{\sin u}$.

The Half-Angle Identities

$$\cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 + \cos u}{2}} \qquad \sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 - \cos u}{2}} \qquad \tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 - \cos u}{1 + \cos u}}$$

$$\tan\left(\frac{u}{2}\right) = \frac{1 - \cos u}{\sin u} \qquad \tan\left(\frac{u}{2}\right) = \frac{\sin u}{1 + \cos u}$$

EXAMPLE 5 ► **Using Half-Angle Formulas to Find Exact Values**

Use the half-angle identities to find exact values for (a) sin 15° and (b) tan 15°.

Solution ► Noting that 15° is one-half the standard angle 30°, we can find each value by applying the respective half-angle identity with $u = 30°$ in Quadrant I.

a. $\sin\left(\frac{30}{2}\right) = \sqrt{\frac{1 - \cos 30}{2}}$

$= \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}}$

$\sin 15° = \frac{\sqrt{2 - \sqrt{3}}}{2}$

b. $\tan\left(\frac{30}{2}\right) = \frac{1 - \cos 30}{\sin 30}$

$\tan 15° = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3}$

Now try Exercises 37 through 48 ►

EXAMPLE 6 ► Using Half-Angle Formulas to Find Exact Values

For $\cos\theta = -\frac{7}{25}$ and θ in QIII, find exact values of $\sin\left(\frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2}\right)$.

Solution ► With θ in QIII $\rightarrow \pi < \theta < \frac{3\pi}{2}$, we know $\frac{\theta}{2}$ must be in QII $\rightarrow \frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$ and we choose our signs accordingly: $\sin\left(\frac{\theta}{2}\right) > 0$ and $\cos\left(\frac{\theta}{2}\right) < 0$.

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1 - \cos\theta}{2}} = \sqrt{\frac{1 - \left(-\frac{7}{25}\right)}{2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1 + \cos\theta}{2}} = -\sqrt{\frac{1 + \left(-\frac{7}{25}\right)}{2}} = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$$

☑ **B.** You've just learned how to develop and use the power reduction and half-angle identities

Now try Exercises 49 through 64 ►

C. The Product-to-Sum Identities

As mentioned in the introduction, the product-to-sum and sum-to-product identities are of immense importance to the study of any phenomenon that travels in waves, like light and sound. In fact, the tones you hear as you dial a telephone are actually the sum of two sound waves interacting with each other. Each derivation of a product-to-sum identity is very similar **(see Exercise 105),** and we illustrate by deriving the identity for $\cos\alpha\cos\beta$. Beginning with the sum and difference identities for cosine, we have

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta) \quad \text{cosine of a difference}$$

$$+\ \cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha + \beta) \quad \text{cosine of a sum}$$

$$2\cos\alpha\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta) \quad \text{combine equations}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \quad \text{divide by 2}$$

The identities from this family are listed here.

The Product-to-Sum Identities

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \qquad \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] \qquad \cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

EXAMPLE 7 ▶ Rewriting a Product as an Equivalent Sum Using Identities

Write the product $2\cos(27t)\cos(15t)$ as the sum of two cosine functions.

Solution ▶ This is a direct application of the product-to-sum identity, with $\alpha = 27t$ and $\beta = 15t$.

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \quad \text{product-to-sum identity}$$

$$2\cos(27t)\cos(15t) = 2\left(\frac{1}{2}\right)[\cos(27t - 15t) + \cos(27t + 15t)] \quad \text{substitute}$$

$$= \cos(12t) + \cos(42t) \quad \text{result}$$

Now try Exercises 65 through 73 ▶

There are times we find it necessary to "work in the other direction," writing a sum of two trig functions as a product. This family of identities can be derived from the product-to-sum identities using a change of variable. We'll illustrate the process for $\sin u + \sin v$. You are asked for the derivation of $\cos u + \cos v$ in **Exercise 106.** To begin, we use $2\alpha = u + v$ and $2\beta = u - v$. This creates the sum $2\alpha + 2\beta = 2u$ and the difference $2\alpha - 2\beta = 2v$, yielding $\alpha + \beta = u$ and $\alpha - \beta = v$, respectively. Dividing the original expressions by 2 gives $\alpha = \frac{u + v}{2}$ and $\beta = \frac{u - v}{2}$, which all together make the derivation a matter of direct substitution. Using these values in any product-to-sum identity gives the related sum-to-product, as shown here.

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] \quad \text{product-to-sum identity (sum of sines)}$$

$$\sin\left(\frac{u + v}{2}\right)\cos\left(\frac{u - v}{2}\right) = \frac{1}{2}(\sin u + \sin v) \quad \text{substitute } \frac{u+v}{2} \text{ for } \alpha, \frac{u-v}{2} \text{ for } \beta, \text{ substitute } u \text{ for } \alpha + \beta \text{ and } v \text{ for } \alpha - \beta$$

$$2\sin\left(\frac{u + v}{2}\right)\cos\left(\frac{u - v}{2}\right) = \sin u + \sin v \quad \text{multiply by 2}$$

The sum-to-product identities follow.

The Sum-to-Product Identities

$$\cos u + \cos v = 2\cos\left(\frac{u + v}{2}\right)\cos\left(\frac{u - v}{2}\right) \qquad \sin u + \sin v = 2\sin\left(\frac{u + v}{2}\right)\cos\left(\frac{u - v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u + v}{2}\right)\sin\left(\frac{u - v}{2}\right) \qquad \cos u - \cos v = -2\sin\left(\frac{u + v}{2}\right)\sin\left(\frac{u - v}{2}\right)$$

EXAMPLE 8 ► **Rewriting a Sum as an Equivalent Product Using Identities**

Given $y_1 = \sin(12\pi t)$ and $y_2 = \sin(10\pi t)$, express $y_1 + y_2$ as a product of trigonometric functions.

Solution ► This is a direct application of the sum-to-product identity $\sin u + \sin v$, with $u = 12\pi t$ and $v = 10\pi t$.

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \quad \text{sum-to-product identity}$$

$$\sin(12\pi t) + \sin(10\pi t) = 2\sin\left(\frac{12\pi t + 10\pi t}{2}\right)\cos\left(\frac{12\pi t - 10\pi t}{2}\right) \quad \text{substitute } 12\pi t \text{ for } u \text{ and } 10\pi t \text{ for } v$$

$$= 2\sin(11\pi t)\cos(\pi t) \quad \text{substitute}$$

C. You've just learned how to derive and use the product-to-sum and sum-to-product identities

Now try Exercises 74 through 82 ►

For a mixed variety of identities, **see Exercises 83–100.**

D. Applications of Identities

In more advanced mathematics courses, rewriting an expression using identities enables the extension or completion of a task that would otherwise be very difficult (or even impossible). In addition, there are a number of practical applications in the physical sciences.

Projectile Motion

A projectile is any object that is thrown, shot, kicked, dropped, or otherwise given an initial velocity, but lacking a continuing source of propulsion. If air resistance is ignored, the range of the projectile depends only on its initial velocity v and the angle θ at which it is propelled. This phenomenon is modeled by the function $r(\theta) = \frac{1}{16}v^2\sin\theta\cos\theta$.

EXAMPLE 9 ► **Using Identities to Solve an Application**

a. Use an identity to show $r(\theta) = \frac{1}{16}v^2\sin\theta\cos\theta$ is equivalent to $r(\theta) = \frac{1}{32}v^2\sin(2\theta)$.

b. If the projectile is thrown with an initial velocity of $v = 96$ ft/sec, how far will it travel if $\theta = 15°$?

c. From the result of part (a), determine what angle θ will give the maximum range for the projectile.

Solution ► **a.** Note that we can use a double-angle identity if we rewrite the coefficient. Writing $\frac{1}{16}$ as $2\left(\frac{1}{32}\right)$ and commuting the factors gives

$$r(\theta) = \left(\frac{1}{32}\right)v^2(2\sin\theta\cos\theta) = \left(\frac{1}{32}\right)v^2\sin(2\theta).$$

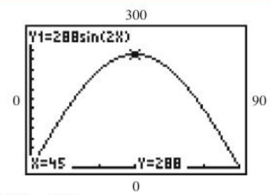
b. With $v = 96$ ft/sec and $\theta = 15°$, the formula gives $r(15°) = \left(\frac{1}{32}\right)(96)^2\sin 30°$. Evaluating the result shows the projectile travels a horizontal distance of 144 ft.

c. For any initial velocity v, $r(\theta)$ will be maximized when $\sin(2\theta)$ is a maximum. This occurs when $\sin(2\theta) = 1$, meaning $2\theta = 90°$ and $\theta = 45°$. The maximum range is achieved when the projectile is released at an angle of 45°.

Now try Exercises 109 and 110 ►

GRAPHICAL SUPPORT

The result in Example 9(c) can be verified graphically by assuming an initial velocity of 96 ft/sec and entering the function $r(\theta) = \frac{1}{32}(96)^2\sin(2\theta) = 288\sin(2\theta)$ as Y_1 on a graphing calculator. With an amplitude of 288 and results confined to the first quadrant, we set an appropriate window, graph the function, and use the 2nd TRACE **(CALC 4:maximum)** feature. As shown in the figure, the max occurs at $\theta = 45°$.

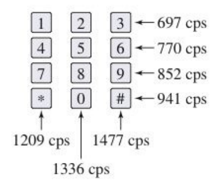


Sound Waves

Each tone you hear on a touch-tone phone is actually the combination of precisely two sound waves with different frequencies (frequency f is defined as $f = \frac{B}{2\pi}$). This is why the tones you hear sound identical, regardless of what phone you use. The sum-to-product and product-to-sum formulas help us to understand, study, and use sound in very powerful and practical ways, like sending faxes and using other electronic media.

EXAMPLE 10 ► Using an Identity to Solve an Application

On a touch-tone phone, the sound created by pressing 5 is produced by combining a sound wave with frequency 1336 cycles/sec, with another wave having frequency 770 cycles/sec. Their respective equations are $y_1 = \cos(2\pi\, 1336t)$ and $y_2 = \cos(2\pi\, 770t)$, with the resultant wave being $y = y_1 + y_2$ or $y = \cos(2672\pi t) + \cos(1540\pi t)$. Rewrite this sum as a product.



Solution ► This is a direct application of the sum-to-product identity, with $u = 2672\pi t$ and $v = 1540\pi t$. Computing one-half the sum/difference of u and v gives $\frac{2672\pi t + 1540\pi t}{2} = 2106\pi t$ and $\frac{2672\pi t - 1540\pi t}{2} = 566\pi t$.

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \quad \text{sum-to-product identity}$$

$$\cos(2672\pi t) + \cos(1540\pi t) = 2\cos(2106\pi t)\cos(566\pi t) \quad \text{substitute } 2672\pi t \text{ for } u \text{ and } 1540\pi t \text{ for } v$$

Now try Exercises 111 and 112 ►

Note we can identify the button pressed when the wave is written as a sum. If we have only the resulting wave (written as a product), the product-to-sum formula must be used to identify which button was pressed.

Additional applications requiring the use of identities can be found in **Exercises 113 through 117.**

D. You've just learned how to solve applications using identities

6.4 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The double-angle identities can be derived using the ______ identities with $\alpha = \beta$. For $\cos(2\theta)$ we expand $\cos(\alpha + \beta)$ using ______.

2. If θ is in QIII then $180° < \theta < 270°$ and $\frac{\theta}{2}$ must be in ______ since ______ $< \frac{\theta}{2} <$ ______.

3. Multiple-angle identities can be derived using the sum and difference identities. For $\sin(3x)$ use sin (______ + ______).

4. For the half-angle identities the sign preceding the radical depends on the ______ in which $\frac{u}{2}$ ______.

5. Explain/Discuss how the three different identities for $\tan\left(\frac{u}{2}\right)$ are related. Verify that $\frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$.

6. In Example 6, we were given $\cos\theta = -\frac{7}{25}$ and θ in QIII. Discuss how the result would differ if we stipulate that θ is in QII instead.

▶ DEVELOPING YOUR SKILLS

Find exact values for $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ using the information given.

7. $\sin\theta = \frac{5}{13}$; θ in QII

8. $\cos\theta = -\frac{21}{29}$; θ in QII

9. $\cos\theta = -\frac{9}{41}$; θ in QII

10. $\sin\theta = -\frac{63}{65}$; θ in QIII

11. $\tan\theta = \frac{13}{84}$; θ in QIII

12. $\sec\theta = \frac{53}{28}$; θ in QI

13. $\sin\theta = \frac{48}{73}$; $\cos\theta < 0$

14. $\cos\theta = -\frac{8}{17}$; $\tan\theta > 0$

15. $\csc\theta = \frac{5}{3}$; $\sec\theta < 0$

16. $\cot\theta = -\frac{80}{39}$; $\cos\theta > 0$

Find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ using the information given.

17. $\sin(2\theta) = \frac{24}{25}$; 2θ in QII

18. $\sin(2\theta) = -\frac{240}{289}$; 2θ in QIII

19. $\cos(2\theta) = -\frac{41}{841}$; 2θ in QII

20. $\cos(2\theta) = \frac{120}{169}$; 2θ in QIV

21. Verify the following identity: $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

22. Verify the following identity: $\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1$

Use a double-angle identity to find exact values for the following expressions.

23. $\cos 75° \sin 75°$

24. $\cos^2 15° - \sin^2 15°$

25. $1 - 2\sin^2\left(\frac{\pi}{8}\right)$

26. $2\cos^2\left(\frac{\pi}{12}\right) - 1$

27. $\frac{2\tan 22.5°}{1 - \tan^2 22.5°}$

28. $\frac{2\tan\left(\frac{\pi}{12}\right)}{1 - \tan^2\left(\frac{\pi}{12}\right)}$

29. Use a double-angle identity to rewrite $9\sin(3x)\cos(3x)$ as a single function. [*Hint:* $9 = \frac{9}{2}(2)$.]

30. Use a double-angle identity to rewrite $2.5 - 5\sin^2 x$ as a single term. [*Hint:* Factor out a constant.]

Rewrite in terms of an expression containing only cosines to the power 1.

31. $\sin^2 x\cos^2 x$ **32.** $\sin^4 x\cos^2 x$

33. $3\cos^4 x$ **34.** $\cos^4 x\sin^4 x$

35. $2\sin^6 x$ **36.** $4\cos^6 x$

Use a half-angle identity to find exact values for sin θ, cos θ, and tan θ for the given value of θ.

37. $\theta = 22.5°$ **38.** $\theta = 75°$

39. $\theta = \frac{\pi}{12}$ **40.** $\theta = \frac{5\pi}{12}$

41. $\theta = 67.5°$ **42.** $\theta = 112.5°$

43. $\theta = \frac{3\pi}{8}$ **44.** $\theta = \frac{11\pi}{12}$

Use the results of Exercises 37–40 and a half-angle identity to find the exact value.

45. $\sin 11.25°$ **46.** $\tan 37.5°$

47. $\sin\left(\frac{\pi}{24}\right)$ **48.** $\cos\left(\frac{5\pi}{24}\right)$

Use a half-angle identity to rewrite each expression as a single, nonradical function.

49. $\sqrt{\frac{1+\cos 30°}{2}}$ **50.** $\sqrt{\frac{1-\cos 45°}{2}}$

51. $\sqrt{\frac{1-\cos(4\theta)}{1+\cos(4\theta)}}$ **52.** $\frac{1-\cos(6x)}{\sin(6x)}$

53. $\frac{\sin(2x)}{1+\cos(2x)}$ **54.** $\frac{\sqrt{2(1+\cos x)}}{1+\cos x}$

Find exact values for $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$ using the information given.

55. $\sin\theta = \frac{12}{13}$; θ is obtuse

56. $\cos\theta = -\frac{8}{17}$; θ is obtuse

57. $\cos\theta = -\frac{4}{5}$; θ in QII

58. $\sin\theta = -\frac{7}{25}$; θ in QIII

59. $\tan\theta = -\frac{35}{12}$; θ in QII

60. $\sec\theta = -\frac{65}{33}$; θ in QIII

61. $\sin\theta = \frac{15}{113}$; θ is acute

62. $\cos\theta = \frac{48}{73}$; θ is acute

63. $\cot\theta = \frac{21}{20}$; $\pi < \theta < \frac{3\pi}{2}$

64. $\csc\theta = \frac{41}{9}$; $\frac{\pi}{2} < \theta < \pi$

Write each product as a sum using the product-to-sum identities.

65. $\sin(-4\theta)\sin(8\theta)$ **66.** $\cos(15\alpha)\sin(-3\alpha)$

67. $2\cos\left(\frac{7t}{2}\right)\cos\left(\frac{3t}{2}\right)$ **68.** $2\sin\left(\frac{5t}{2}\right)\sin\left(\frac{9t}{2}\right)$

69. $2\cos(1979\pi t)\cos(439\pi t)$

70. $2\cos(2150\pi t)\cos(268\pi t)$

Find the exact value using product-to-sum identities.

71. $2\cos 15°\sin 135°$

72. $\sin\left(\frac{7\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$ **73.** $\sin\left(\frac{7\pi}{12}\right)\sin\left(-\frac{\pi}{12}\right)$

Write each sum as a product using the sum-to-product identities.

74. $\cos(9h) + \cos(4h)$ **75.** $\sin(14k) + \sin(41k)$

76. $\sin\left(\frac{11x}{8}\right) - \sin\left(\frac{5x}{8}\right)$ **77.** $\cos\left(\frac{7x}{6}\right) - \cos\left(\frac{5x}{6}\right)$

78. $\cos(697\pi t) + \cos(1447\pi t)$

79. $\cos(852\pi t) + \cos(1209\pi t)$

Find the exact value using sum-to-product identities.

80. $\cos 75° + \cos 15°$

81. $\sin\left(\frac{17\pi}{12}\right) - \sin\left(\frac{13\pi}{12}\right)$

82. $\sin\left(\frac{11\pi}{12}\right) + \sin\left(\frac{7\pi}{12}\right)$

Verify the following identities.

83. $\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x} = \tan(2x)$

84. $\frac{1-2\sin^2 x}{2\sin x\cos x} = \cot(2x)$

85. $(\sin x + \cos x)^2 = 1 + \sin(2x)$

86. $(\sin^2 x - 1)^2 = \sin^4 x + \cos(2x)$

87. $\cos(8\theta) = \cos^2(4\theta) - \sin^2(4\theta)$

88. $\sin(4x) = 4\sin x\cos x(1 - 2\sin^2 x)$

89. $\dfrac{\cos(2\theta)}{\sin^2\theta} = \cot^2\theta - 1$

90. $\csc^2\theta - 2 = \dfrac{\cos(2\theta)}{\sin^2\theta}$

91. $\tan(2\theta) = \dfrac{2}{\cot\theta - \tan\theta}$

92. $\cot\theta - \tan\theta = \dfrac{2\cos(2\theta)}{\sin(2\theta)}$

93. $\tan x + \cot x = 2\csc(2x)$

94. $\csc(2x) = \dfrac{1}{2}\csc x\sec x$

95. $\cos^2\left(\dfrac{x}{2}\right) - \sin^2\left(\dfrac{x}{2}\right) = \cos x$

96. $1 - 2\sin^2\left(\dfrac{x}{4}\right) = \cos\left(\dfrac{x}{2}\right)$

97. $1 - \sin^2(2\theta) = 1 - 4\sin^2\theta + 4\sin^4\theta$

98. $2\cos^2\left(\dfrac{x}{2}\right) - 1 = \cos x$

99. $\dfrac{\sin(120\pi t) + \sin(80\pi t)}{\cos(120\pi t) - \cos(80\pi t)} = -\cot(20\pi t)$

100. $\dfrac{\sin m + \sin n}{\cos m + \cos n} = \tan\left(\dfrac{m+n}{2}\right)$

101. Show $\sin^2\alpha + (1 - \cos\alpha)^2 = \left[2\sin\left(\dfrac{\alpha}{2}\right)\right]^2$.

102. Show that $\tan\left(\dfrac{u}{2}\right) = \pm\sqrt{\dfrac{1-\cos u}{1+\cos u}}$ is equivalent to $\dfrac{\sin u}{1+\cos u}$ by rationalizing the numerator.

103. Derive the identity for $\sin(2\alpha)$ and $\tan(2\alpha)$ using $\sin(\alpha + \beta)$ and $\tan(\alpha + \beta)$, where $\alpha = \beta$.

104. Derive the identity for $\tan^2(\alpha)$ using $\tan(2\alpha) = \dfrac{2\tan(\alpha)}{1 - \tan^2(\alpha)}$. *Hint:* Solve for $\tan^2\alpha$ and work in terms of sines and cosines.

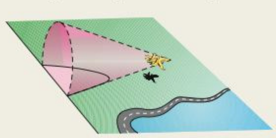
105. Derive the product-to-sum identity for $\sin\alpha\sin\beta$.

106. Derive the sum-to-product identity for $\cos u + \cos v$.

► WORKING WITH FORMULAS

107. Supersonic speeds, the sound barrier, and Mach numbers: $\mathcal{M} = \csc\left(\dfrac{\theta}{2}\right)$

The speed of sound varies with temperature and altitude. At 32°F, sound travels about 742 mi/hr at sea level. A jet-plane flying faster than the speed of sound (called supersonic speed) has "broken the sound barrier." The plane projects three-dimensional sound waves about the nose of the craft that form the shape of a cone. The cone intersects the Earth along a hyperbolic path, with a sonic boom being heard by anyone along this path. The ratio of the plane's speed to the speed of sound is called its Mach number $\mathcal{M}$, meaning a plane flying at $\mathcal{M} = 3.2$ is traveling 3.2 times the speed of sound. This Mach number can be determined using the formula given here, where θ is the vertex angle of the cone described. For the following exercises, use the formula to find $\mathcal{M}$ or θ as required. For parts (a) and (b), answer in exact form (using a half-angle identity) and approximate form.



a. $\theta = 30°$ **b.** $\theta = 45°$ **c.** $\mathcal{M} = 2$

108. Malus's law: $I = I_0\cos^2\theta$

When a beam of plane-polarized light with intensity I_0 hits an analyzer, the intensity I of the transmitted beam of light can be found using the formula shown, where θ is the angle formed between the transmission axes of the polarizer and the analyzer. Find the intensity of the beam when $\theta = 15°$ and $I_0 = 300$ candelas (cd). Answer in exact form (using a power reduction identity) and approximate form.

▶ APPLICATIONS

Range of a projectile: Exercises 109 and 110 refer to Example 9. In Example 9, we noted that the range of a projectile was maximized at $\theta = 45°$. If $\theta > 45°$ or $\theta < 45°$, the projectile falls short of its maximum potential distance. In Exercises 109 and 110 assume that the projectile has an initial velocity of 96 ft/sec.

109. Compute how many feet short of maximum the projectile falls if (a) $\theta = 22.5°$ and (b) $\theta = 67.5°$. Answer in both exact and approximate form.

110. Use a calculator to compute how many feet short of maximum the projectile falls if (a) $\theta = 40°$ and $\theta = 50°$ and (b) $\theta = 37.5°$ and $\theta = 52.5°$. Do you see a pattern? Discuss/explain what you notice and experiment with other values to confirm your observations.

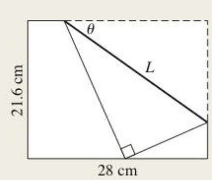
Touch-tone phones: The diagram given in Example 10 shows the various frequencies used to create the tones for a touch-tone phone. One button is randomly pressed and the resultant wave is modeled by $y(t)$ shown. Use a product-to-sum identity to write the expression as a sum and determine the button pressed.

111. $y(t) = 2\cos(2150\pi t)\cos(268\pi t)$

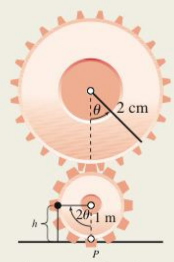
112. $y(t) = 2\cos(1906\pi t)\cos(512\pi t)$

113. Clock angles: Kirkland City has a large clock atop city hall, with a minute hand that is 3 ft long. Claire and Monica independently attempt to devise a function that will track the distance between the tip of the minute hand at t minutes between the hours, and the tip of the minute hand when it is in the vertical position as shown. Claire finds the function $d(t) = \left|6\sin\left(\frac{\pi t}{60}\right)\right|$, while Monica devises $d(t) = \sqrt{18\left[1 - \cos\left(\frac{\pi t}{30}\right)\right]}$. Use the identities from this section to show the functions are equivalent.

114. Origami: The Japanese art of origami involves the repeated folding of a single piece of paper to create various art forms. When the upper right corner of a rectangular 21.6-cm by 28-cm piece of paper is folded down until the corner is flush with the other side, the length L of the fold is related to the angle θ by $L = \frac{10.8}{\sin\theta\cos^2\theta}$. (a) Show this is equivalent to $L = \frac{21.6\sec\theta}{\sin(2\theta)}$, (b) find the length of the fold if $\theta = 30°$, and (c) find the angle θ if $L = 28.8$ cm.



115. Machine gears: A machine part involves two gears. The first has a radius of 2 cm and the second a radius of 1 cm, so the smaller gear turns twice as fast as the larger gear. Let θ represent the angle of rotation in the larger gear, measured from a vertical and downward starting position. Let P be a point on the circumference of the smaller gear, starting at the vertical and downward position. Four engineers working on an improved design for this component devise functions that track the height of point P above the horizontal plane shown, for a rotation of $\theta°$ by the larger gear. The functions they develop are: Engineer A: $f(\theta) = \sin(2\theta - 90°) + 1$; Engineer B: $g(\theta) = 2\sin^2\theta$; Engineer C: $k(\theta) = 1 + \sin^2\theta - \cos^2\theta$; and Engineer D: $h(\theta) = 1 - \cos(2\theta)$. Use any of the identities you've learned so far to show these four functions are equivalent.



116. Working with identities: Compute the value of sin 15° two ways, first using the half-angle identity for sine, and second using the difference identity for sine. (a) Find a decimal approximation for each to show the results are equivalent and (b) verify algebraically that they are equivalent. (*Hint:* Square both sides.)

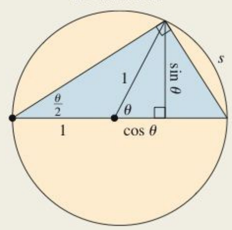
117. Working with identities: Compute the value of cos 15° two ways, first using the half-angle identity for cosine, and second using the difference identity for cosine. (a) Find a decimal approximation for each to show the results are equivalent and (b) verify algebraically that they are equivalent. (*Hint:* Square both sides.)

▶ EXTENDING THE CONCEPT

118. Can you find three distinct, real numbers whose sum is equal to their product? A little known fact from trigonometry stipulates that for any triangle, the sum of the tangents of the angles is equal to the products of their tangents. Use a calculator to test this statement, recalling the three angles must sum to 180°. Our website at www.mhhe.com/coburn shows a method that enables you to verify the statement using tangents that are all rational values.

119. A proof without words: From elementary geometry we have the following: (a) an angle inscribed in a semicircle is a right angle; and (b) the measure of an inscribed angle (vertex on the circumference) is one-half the measure of its intercepted arc. Discuss/explain how the unit-circle diagram offers a proof that $\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1 + \cos x}$. Be detailed and thorough.

Exercise 119



120. Using $\theta = 30°$ and repeatedly applying the half-angle identity for cosine, show that cos 3.75° is equal to $\frac{\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}{2}$. Verify the result using a calculator, then use the patterns noted to write the value of cos 1.875° in closed form (also verify this result). As θ becomes very small, what appears to be happening to the value of $\cos \theta$?

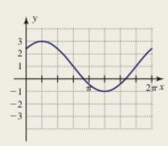
▶ MAINTAINING YOUR SKILLS

121. **(3.3)** Use the rational roots theorem to find all zeroes of $x^4 + x^3 - 8x^2 - 6x + 12 = 0$.

122. **(5.1)** The hypotenuse of a certain right triangle is twice the shortest side. Solve the triangle.

123. **(5.3)** Verify that $\left(\frac{16}{65}, \frac{63}{65}\right)$ is on the unit circle, then find $\tan \theta$ and $\sec \theta$ to verify $1 + \tan^2\theta = \sec^2\theta$.

124. **(5.5)** Write the equation of the function graphed in terms of a sine function of the form $y = A \sin(Bx + C) + D$.



MID-CHAPTER CHECK

1. Verify the identity using a multiplication:
$\sin x(\csc x - \sin x) = \cos^2 x$

2. Verify the identity by factoring:
$\cos^2 x - \cot^2 x = -\cos^2 x \cot^2 x$

3. Verify the identity by combining terms:
$\frac{2 \sin x}{\sec x} - \frac{\cos x}{\csc x} = \cos x \sin x$

4. Show the equation given is not an identity.
$1 + \sec^2 x = \tan^2 x$

5. Verify each identity.

a. $\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x \cos x$

b. $\frac{1 + \sec x}{\csc x} - \frac{1 + \cos x}{\cot x} = 0$

6. Verify each identity.

a. $\frac{\sec^2 x - \tan^2 x}{\sec^2 x} = \cos^2 x$

b. $\frac{\cot x - \tan x}{\csc x \sec x} = \cos^2 x - \sin^2 x$

7. Given α and β are obtuse angles with $\sin \alpha = \frac{56}{65}$ and $\tan \beta = -\frac{80}{39}$, find

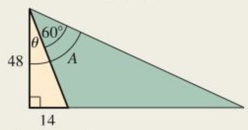
a. $\sin(\alpha - \beta)$

b. $\cos(\alpha + \beta)$

c. $\tan(\alpha - \beta)$

8. Use the diagram shown to compute $\sin A$, $\cos A$, and $\tan A$.

Exercise 8



9. Given $\cos\theta = -\frac{15}{17}$ and θ in QII, find exact values of $\sin\left(\frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2}\right)$.

10. Given $\sin\alpha = -\frac{7}{25}$ with α in QIII, find the value of $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$.

REINFORCING BASIC CONCEPTS

Identities—Connections and Relationships

It is a well-known fact that information is retained longer and used more effectively when it is organized, sequential, and connected. In this *Strengthening Core Skills (SCS),* we attempt to do just that with our study of identities. In flowchart form we'll show that the entire range of identities has only two tiers, and that the fundamental identities and the sum and difference identities are really the keys to the entire range of identities. Beginning with the right triangle definition of sine, cosine, and tangent, the **reciprocal identities** and **ratio identities** are more semantic (word related) than mathematical, and the **Pythagorean identities** follow naturally from the properties of right triangles. These form the first tier.

Basic Definitions

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

Fundamental Identities

defined → **Reciprocal Identities**

defined → **Ratio Identities**

derived → **Pythagorean Identities**

Reciprocal Identities

$$\frac{1}{\sin\theta} = \csc\theta \qquad \frac{1}{\cos\theta} = \sec\theta \qquad \frac{1}{\tan\theta} = \cot\theta$$

Ratio Identities

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \qquad \frac{\cos\theta}{\sin\theta} = \cot\theta \qquad \frac{\sec\theta}{\csc\theta} = \tan\theta$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$\tan^2\theta + 1 = \sec^2\theta$ (divide by $\cos^2\theta$) $\qquad$ $1 + \cot^2\theta = \csc^2\theta$ (divide by $\sin^2\theta$)

The reciprocal and ratio identities are actually *defined,* while the Pythagorean identities are *derived* from these two families. In addition, the identity $\sin^2\theta + \cos^2\theta = 1$ is the only Pythagorean identity we actually need to memorize; the other two follow by division of $\cos^2\theta$ and $\sin^2\theta$ as indicated.

In virtually the same way, the sum and difference identities for sine and cosine are the only identities that need to be memorized, as all other identities in the second tier flow from these.

Sum/Difference Identities

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

Double-Angle Identities
use $\alpha = \beta$ in sum identities

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$
$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$$\cos(2\alpha) = 2\cos^2\alpha - 1$$
(use $\sin^2\alpha = 1 - \cos^2\alpha$)

$$\cos(2\alpha) = 1 - 2\sin^2\alpha$$
(use $\cos^2\alpha = 1 - \sin^2\alpha$)

Power Reduction Identities
solve for $\cos^2\alpha$, $\sin^2\alpha$ in related $\cos(2\alpha)$ identity

$$\cos^2\alpha = \frac{1 + \cos(2\alpha)}{2}$$
$$\sin^2\alpha = \frac{1 - \cos(2\alpha)}{2}$$

Half-Angle Identities
solve for $\cos\alpha$, $\sin\alpha$ and use $\alpha = u/2$ in the power reduction identities

$$\cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 + \cos u}{2}}$$
$$\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 - \cos u}{2}}$$

Product-to-Sum Identities
combine various sum/difference identities

see Section 6.4

see Section 6.4

Exercise 1: Starting with the identity $\sin^2\alpha + \cos^2\alpha = 1$, derive the other two Pythagorean identities.

Exercise 2: Starting with the identity $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$, derive the double-angle identities for cosine.

6.5 The Inverse Trig Functions and Their Applications

Learning Objectives

In Section 6.5 you will learn how to:

- **A.** Find and graph the inverse sine function and evaluate related expressions
- **B.** Find and graph the inverse cosine and tangent functions and evaluate related expressions
- **C.** Apply the definition and notation of inverse trig functions to simplify compositions
- **D.** Find and graph inverse functions for sec *x*, csc *x*, and cot *x*
- **E.** Solve applications involving inverse functions

While we usually associate the number π with the features of a circle, it also occurs in some "interesting" places, such as the study of normal (bell) curves, Bessel functions, Stirling's formula, Fourier series, Laplace transforms, and infinite series. In much the same way, the trigonometric functions are surprisingly versatile, finding their way into a study of complex numbers and vectors, the simplification of algebraic expressions, and finding the area under certain curves—applications that are hugely important in a continuing study of mathematics. As you'll see, a study of the inverse trig functions helps support these fascinating applications.

A. The Inverse Sine Function

In Section 4.1 we established that only one-to-one functions have an inverse. All six trig functions fail the horizontal line test and are not one-to-one as given. However, by suitably restricting the domain, a one-to-one function can be defined that makes finding an inverse possible. For the sine function, it seems natural to choose the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ since it is centrally located and the sine function attains all possible range values in this interval. A graph of $y = \sin x$ is shown in Figure 6.5, with the portion corresponding to this interval colored in red. Note the range is still $[-1, 1]$ (Figure 6.6).

Figure 6.5

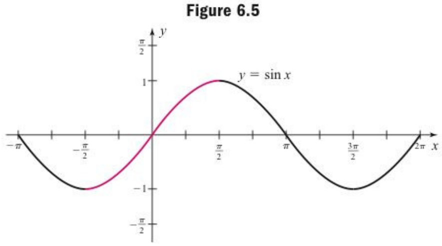


Figure 6.6

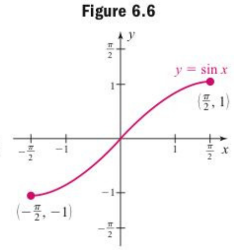
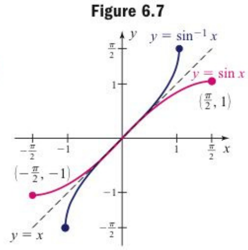


Figure 6.7



WORTHY OF NOTE

In Example 4 of Section 4.1, we noted that by suitably restricting the domain of $y = x^2$, a one-to-one function could be defined that made finding an inverse function possible. Specifically, for $f(x) = x^2; x \geq 0, f^{-1}(x) = \sqrt{x}$.

We can obtain an implicit equation for the inverse of $y = \sin x$ by interchanging x- and y-values, obtaining $x = \sin y$. By accepted convention, the *explicit* form of the inverse sine function is written $y = \sin^{-1}x$ or $y = \arcsin x$. Since domain and range values have been interchanged, the domain of $y = \sin^{-1}x$ is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The graph of $y = \sin^{-1}x$ can be found by reflecting the portion in red across the line $y = x$ and using the endpoints of the domain and range (see Figure 6.7).

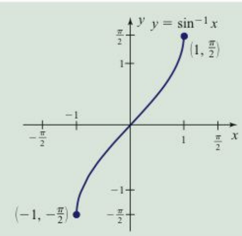
The Inverse Sine Function

For $y = \sin x$ with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range $[-1, 1]$,
the inverse sine function is

$$\mathbf{y = \sin^{-1}x \text{ or } y = \arcsin x,}$$

with domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$y = \sin^{-1}x \text{ if and only if } \sin y = x$$



From the implicit form $x = \sin y$, we learn to interpret the inverse function as, "y is the number or angle whose sine is x." Learning to read and interpret the explicit form in this way will be helpful. That is, $y = \sin^{-1}x$ means "y is the number or angle whose sine is x."

$$\mathbf{y = \sin^{-1}x \Leftrightarrow x = \sin y \qquad x = \sin y \Leftrightarrow y = \sin^{-1}x}$$

EXAMPLE 1 ► Evaluating $y = \sin^{-1}x$ Using Special Values

Evaluate the inverse sine function for the values given:

a. $y = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **b.** $y = \arcsin\left(-\frac{1}{2}\right)$ **c.** $y = \sin^{-1}2$

Solution ► For x in $[-1, 1]$ and y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

a. $y = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$: y is the number or angle whose sine is $\frac{\sqrt{3}}{2}$

$\Rightarrow \sin y = \frac{\sqrt{3}}{2}$, so $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

b. $y = \arcsin\left(-\frac{1}{2}\right)$: y is the arc or angle whose sine is $-\frac{1}{2}$
$\Rightarrow \sin y = -\frac{1}{2}$, so $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

c. $y = \sin^{-1}(2)$: y is the number or angle whose sine is 2
$\Rightarrow \sin y = 2$. Since 2 is not in $[-1, 1]$, $\sin^{-1}(2)$ is undefined.

Now try Exercises 7 through 12 ►

Table 6.1

x	$\sin x$
$-\frac{\pi}{2}$	-1
$-\frac{\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$-\frac{\pi}{6}$	$-\frac{1}{2}$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

In Examples 1a and 1b, note that the equations $\sin y = \frac{\sqrt{3}}{2}$ and $\sin y = -\frac{1}{2}$ each have an infinite number of solutions, but only one solution in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

When x is one of the standard values $\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1\text{, and so on}\right)$, $y = \sin^{-1}x$ can be evaluated by reading a standard table "in reverse." For $y = \arcsin(-1)$, we locate the number -1 in the right-hand column of Table 6.1, and note the "number or angle whose sine is -1," is $-\frac{\pi}{2}$. If x is between -1 and 1 but is not a standard value, we can use the $\sin^{-1}$ function on a calculator, which is most often the 2nd or INV function for SIN.

EXAMPLE 2 ► Evaluating $y = \sin^{-1}x$ Using a Calculator

Evaluate each inverse sine function twice. First in radians rounded to four decimal places, then in degrees to the nearest tenth.

a. $y = \sin^{-1}0.8492$ **b.** $y = \arcsin(-0.2317)$

Solution ► For x in $[-1, 1]$, we evaluate $y = \sin^{-1}x$.

a. $y = \sin^{-1}0.8492$: With the calculator in radian MODE, use the keystrokes 2nd SIN 0.8492) ENTER. We find $\sin^{-1}(0.8492) \approx 1.0145$ radians. In degree MODE, the same sequence of keystrokes gives $\sin^{-1}(0.8492) \approx 58.1°$ (note that 1.0145 rad $\approx 58.1°$).

b. $y = \arcsin(-0.2317)$: In radian MODE, we find $\sin^{-1}(-0.2317) \approx -0.2338$ rad. In degree MODE, $\sin^{-1}(-0.2317) \approx -13.4°$.

Now try Exercises 13 through 16 ►

WORTHY OF NOTE

The $\sin^{-1}x$ notation for the inverse sine function is a carryover from the $f^{-1}(x)$ notation for a general inverse function, and likewise has nothing to do with the reciprocal of the function. The arcsin x notation derives from our work in radians on the unit circle, where $y = \arcsin x$ can be interpreted as "y is an arc whose sine is x."

From our work in Section 4.1, we know that if f and g are inverses, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. This suggests the following properties.

Inverse Function Properties for Sine

For $f(x) = \sin x$ and $g(x) = \sin^{-1}x$:

I. $(f \circ g)(x) = \sin(\sin^{-1}x) = x$ for x in $[-1, 1]$

and

II. $(g \circ f)(x) = \sin^{-1}(\sin x) = x$ for x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

EXAMPLE 3 ► **Evaluating Expressions Using Inverse Function Properties**

Evaluate each expression and verify the result on a calculator.

a. $\sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$ **b.** $\arcsin\left[\sin\left(\frac{\pi}{4}\right)\right]$ **c.** $\sin^{-1}\left[\sin\left(\frac{5\pi}{6}\right)\right]$

Solution ► **a.** $\sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right] = \frac{1}{2}$, since $\frac{1}{2}$ is in $[-1, 1]$ Property I

b. $\arcsin\left[\sin\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{4}$, since $\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ Property II

c. $\sin^{-1}\left[\sin\left(\frac{5\pi}{6}\right)\right] \neq \frac{5\pi}{6}$, since $\frac{5\pi}{6}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

This doesn't mean the expression cannot be evaluated, only that we cannot use Property II. Since $\sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right)$, $\sin^{-1}\left[\left(\sin\frac{5\pi}{6}\right)\right] = \sin^{-1}\left[\sin\left(\frac{\pi}{6}\right)\right] = \frac{\pi}{6}$.

The calculator verification for each is shown in Figures 6.8 and 6.9. Note $\frac{\pi}{6} \approx 0.5236$ and $\frac{\pi}{4} \approx 0.7854$.

Figure 6.8
Parts (a) and (b)

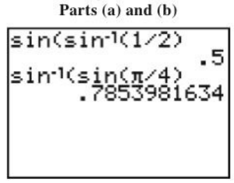
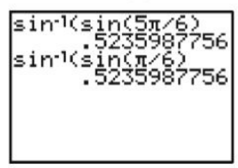


Figure 6.9
Part (c)



A. You've just learned how to find and graph the inverse sine function and evaluate related expressions

Now try Exercises 17 through 24 ►

B. The Inverse Cosine and Inverse Tangent Functions

Like the sine function, the cosine function is not one-to-one and its domain must also be restricted to develop an inverse function. For convenience we choose the interval $x \in [0, \pi]$ since it is again somewhat central and takes on all of its range values in this interval. A graph of the cosine function, with the interval corresponding to this interval shown in red, is given in Figure 6.10. Note the range is still $[-1, 1]$ (Figure 6.11).

Figure 6.10

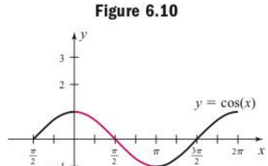


Figure 6.11

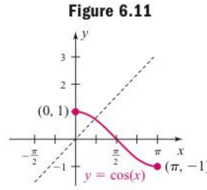
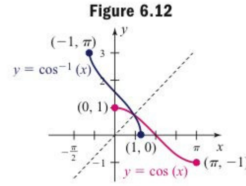


Figure 6.12



For the implicit equation of inverse cosine, $y = \cos x$ becomes $x = \cos y$, with the corresponding explicit forms being $y = \cos^{-1}x$ or $y = \arccos x$. By reflecting the graph of $y = \cos x$ across the line $y = x$, we obtain the graph of $y = \cos^{-1}x$ shown in Figure 6.12.

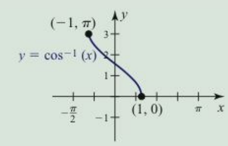
The Inverse Cosine Function

For $y = \cos x$ with domain $[0, \pi]$ and range $[-1, 1]$, the inverse cosine function is

$$\mathbf{y = \cos^{-1}x \text{ or } y = \arccos x}$$

with domain $[-1, 1]$ and range $[0, \pi]$.

$$y = \cos^{-1}x \text{ if and only if } \cos y = x$$



EXAMPLE 4 ► Evaluating $y = \cos^{-1}x$ Using Special Values

Evaluate the inverse cosine for the values given:

a. $y = \cos^{-1}0$ **b.** $y = \arccos\left(-\frac{\sqrt{3}}{2}\right)$ **c.** $y = \cos^{-1}\pi$

Solution ► For x in $[-1, 1]$ and y in $[0, \pi]$,

a. $y = \cos^{-1}0$: y is the number or angle whose cosine is $0 \Rightarrow \cos y = 0$. This shows $\cos^{-1}0 = \frac{\pi}{2}$.

b. $y = \arccos\left(-\frac{\sqrt{3}}{2}\right)$: y is the arc or angle whose cosine is $-\frac{\sqrt{3}}{2} \Rightarrow \cos y = -\frac{\sqrt{3}}{2}$. This shows $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

c. $y = \cos^{-1}\pi$: y is the number or angle whose cosine is $\pi \Rightarrow \cos y = \pi$. Since $\pi \notin [-1, 1]$, $\cos^{-1}\pi$ is undefined.

Now try Exercises 25 through 34 ►

Knowing that $y = \cos x$ and $y = \cos^{-1}x$ are inverse functions enables us to state inverse function properties similar to those for sine.

Inverse Function Properties for Cosine

For $f(x) = \cos x$ and $g(x) = \cos^{-1}x$:

I. $(f \circ g)(x) = \cos(\cos^{-1}x) = x$ for x in $[-1, 1]$

and

II. $(g \circ f)(x) = \cos^{-1}(\cos x) = x$ for x in $[0, \pi]$

EXAMPLE 5 ► Evaluating Expressions Using Inverse Function Properties

Evaluate each expression.

a. $\cos[\cos^{-1}(0.73)]$ **b.** $\arccos\left[\cos\left(\frac{\pi}{12}\right)\right]$ **c.** $\cos^{-1}\left[\cos\left(\frac{4\pi}{3}\right)\right]$

Solution ► **a.** $\cos[\cos^{-1}(0.73)] = 0.73$, since 0.73 is in $[-1, 1]$ Property I

b. $\arccos\left[\cos\left(\frac{\pi}{12}\right)\right] = \frac{\pi}{12}$, since $\frac{\pi}{12}$ is in $[0, \pi]$ Property II

c. $\cos^{-1}\left[\cos\left(\frac{4\pi}{3}\right)\right] \neq \frac{4\pi}{3}$, since $\frac{4\pi}{3}$ is not in $[0, \pi]$.

This expression cannot be evaluated using Property II. Since $\cos\left(\frac{4\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$, $\cos^{-1}\left[\cos\left(\frac{4\pi}{3}\right)\right] = \cos^{-1}\left[\cos\left(\frac{2\pi}{3}\right)\right] = \frac{2\pi}{3}$.

The results can also be verified using a calculator.

Now try Exercises 35 through 42 ►

For the tangent function, we likewise restrict the domain to obtain a one-to-one function, with the most common choice being $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The corresponding range is $\mathbb{R}$. The *implicit* equation for the inverse tangent function is $x = \tan y$ with the explicit forms $y = \tan^{-1}x$ or $y = \arctan x$. With the domain and range interchanged, the domain of $y = \tan^{-1}x$ is $\mathbb{R}$, and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The graph of $y = \tan x$ for x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is shown in red (Figure 6.13), with the inverse function $y = \tan^{-1}x$ shown in blue (Figure 6.14).

Figure 6.13

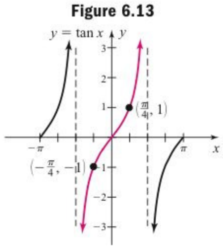
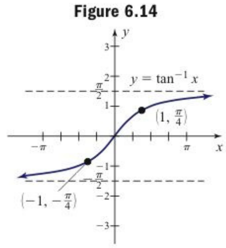


Figure 6.14



The Inverse Tangent Function	Inverse Function Properties for Tangent
For $y = \tan x$ with domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and range $\mathbb{R}$, the inverse tangent function is **$y = \tan^{-1}x$ or $y = \arctan x$,** with domain $\mathbb{R}$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. $y = \tan^{-1}x$ if and only if $\tan y = x$	For $f(x) = \tan x$ and $g(x) = \tan^{-1}x$: **I.** $(f \circ g)(x) = \tan(\tan^{-1}x) = x$ for x in $\mathbb{R}$ and **II.** $(g \circ f)(x) = \tan^{-1}(\tan x) = x$ for x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

EXAMPLE 6 ► Evaluating Expressions Involving Inverse Tangent

Evaluate each expression.

a. $\tan^{-1}(-\sqrt{3})$ **b.** $\arctan[\tan(-0.89)]$

Solution ► For x in $\mathbb{R}$ and y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

a. $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$, since $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$

b. $\arctan[\tan(-0.89)] = -0.89$, since -0.89 is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ Property II

B. You've just learned how to find and graph the inverse cosine and tangent functions and evaluate related expressions

Now try Exercises 43 through 52 ►

C. Using the Inverse Trig Functions to Evaluate Compositions

In the context of angle measure, the expression $y = \sin^{-1}\left(-\frac{1}{2}\right)$ represents an angle—the *angle* y whose sine is $-\frac{1}{2}$. It seems natural to ask, "What happens if we take the tangent of this angle?" In other words, what does the expression $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$ mean? Similarly, if $y = \cos\left(\frac{\pi}{3}\right)$ represents a real number between -1 and 1, how do we compute $\sin^{-1}\left[\cos\left(\frac{\pi}{3}\right)\right]$? Expressions like these occur in many fields of study.

EXAMPLE 7 ► Simplifying Expressions Involving Inverse Trig Functions

Simplify each expression:

a. $\tan\left[\arcsin\left(-\frac{1}{2}\right)\right]$ **b.** $\sin^{-1}\left[\cos\left(\frac{\pi}{3}\right)\right]$

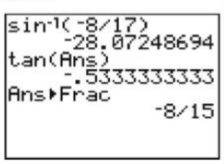
Solution ► **a.** In Example 1 we found $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$. Substituting $-\frac{\pi}{6}$ for $\arcsin\left(-\frac{1}{2}\right)$ gives $\tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$, showing $\tan\left[\arcsin\left(-\frac{1}{2}\right)\right] = -\frac{\sqrt{3}}{3}$.

b. For $\sin^{-1}\left[\cos\left(\frac{\pi}{3}\right)\right]$, we begin with the inner function $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. Substituting $\frac{1}{2}$ for $\cos\left(\frac{\pi}{3}\right)$ gives $\sin^{-1}\left(\frac{1}{2}\right)$. With the appropriate checks satisfied we have $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, showing $\sin^{-1}\left[\cos\left(\frac{\pi}{3}\right)\right] = \frac{\pi}{6}$.

Now try Exercises 53 through 64 ►

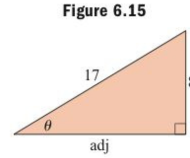
WORTHY OF NOTE

To verify the result of Example 8, we can actually find the value of $\sin^{-1}\left(-\frac{8}{17}\right)$ on a calculator, then take the tangent of the result. See the figure.



If the argument is not a special value and we need the answer in exact form, we can draw the triangle described by the inner expression using the definition of the trigonometric functions as ratios. In other words, for either y or $\theta = \sin^{-1}\left(\frac{8}{17}\right)$, we draw a triangle with hypotenuse 17 and side 8 opposite θ to model the statement, "an angle whose sine is $\frac{8}{17} = \frac{\text{opp}}{\text{hyp}}$," (see Figure 6.15). Using the Pythagorean theorem, we find the adjacent side is 15 and can now name any of the other trig functions.

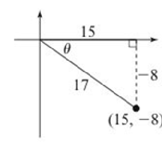
Figure 6.15



EXAMPLE 8 ► **Using a Diagram to Evaluate an Expression Involving Inverse Trig Functions**

Evaluate the expression $\tan\left[\sin^{-1}\left(-\frac{8}{17}\right)\right]$.

Solution ► The expression $\tan\left[\sin^{-1}\left(-\frac{8}{17}\right)\right]$ is equivalent to $\tan\theta$, where $\theta = \sin^{-1}\left(-\frac{8}{17}\right)$ with θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (QIV or QI). For $\sin\theta = -\frac{8}{17}$ ($\sin\theta < 0$), θ must be in QIII or QIV. To satisfy both, θ must be in QIV. From the figure we note $\tan\theta = -\frac{8}{15}$, showing $\tan\left[\sin^{-1}\left(-\frac{8}{17}\right)\right] = -\frac{8}{15}$.



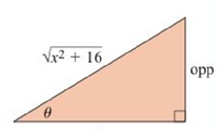
Now try Exercises 65 through 72 ►

These ideas apply even when one side of the triangle is unknown. In other words, we can still draw a triangle for $\theta = \cos^{-1}\left(\frac{x}{\sqrt{x^2+16}}\right)$, since "$\theta$ is an angle whose cosine is $\frac{x}{\sqrt{x^2+16}} = \frac{\text{adj}}{\text{hyp}}$."

EXAMPLE 9 ► **Using a Diagram to Evaluate an Expression Involving Inverse Trig Functions**

Evaluate the expression $\tan\left[\cos^{-1}\left(\frac{x}{\sqrt{x^2+16}}\right)\right]$. Assume $x > 0$ and the inverse function is defined for the expression given.

Solution ► Rewrite $\tan\left[\cos^{-1}\left(\frac{x}{\sqrt{x^2+16}}\right)\right]$ as $\tan\theta$, where $\theta = \cos^{-1}\left(\frac{x}{\sqrt{x^2+16}}\right)$. Draw a triangle with side x adjacent to θ and a hypotenuse of $\sqrt{x^2+16}$. The Pythagorean theorem gives $x^2 + \text{opp}^2 = (\sqrt{x^2+16})^2$, which leads to $\text{opp}^2 = (x^2+16) - x^2$ giving $\text{opp} = \sqrt{16} = 4$. This shows $\tan\theta = \tan\left[\cos^{-1}\left(\frac{x}{\sqrt{x^2+16}}\right)\right] = \frac{4}{x}$ (see the figure).



C. You've just learned how to apply the definition and notation of inverse trig functions to simplify compositions

Now try Exercises 73 through 76 ►

D. The Inverse Functions for Secant, Cosecant, and Cotangent

As with the other functions, we restrict the domain of the secant, cosecant, and cotangent functions to obtain a one-to-one function that is invertible (an inverse can be found). Once again the choice is arbitrary, and some domains are easier to work with than others in more advanced mathematics. For $y = \sec x$, we've chosen the "most

Figure 6.16
$y = \sec x$

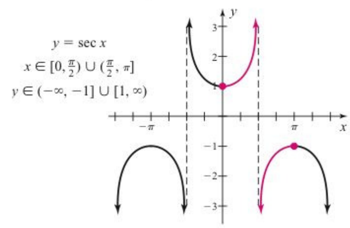
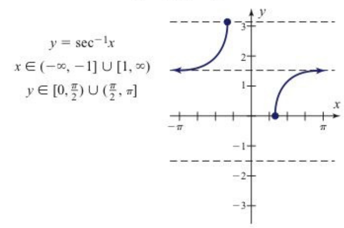


Figure 6.17
$y = \sec^{-1} x$



intuitive" restriction, one that seems more centrally located (nearer the origin). The graph of $y = \sec x$ is reproduced here, along with its inverse function (see Figures 6.16 and 6.17). The domain, range, and graphs of the functions $y = \csc^{-1}x$ and $y = \cot^{-1}x$ are asked for in the Exercises **(see Exercise 100).**

The functions $y = \sec^{-1}x$, $y = \csc^{-1}x$, and $y = \cot^{-1}x$ can be evaluated by noting their relationship to $y = \cos^{-1}x$, $y = \sin^{-1}x$, and $y = \tan^{-1}x$, respectively. For $y = \sec^{-1}x$, we have

$$\sec y = x \quad \text{definition of inverse function}$$
$$\frac{1}{\sec y} = \frac{1}{x} \quad \text{property of reciprocals}$$
$$\cos y = \frac{1}{x} \quad \text{reciprocal ratio}$$
$$y = \cos^{-1}\left(\frac{1}{x}\right) \quad \text{rewrite using inverse function notation}$$
$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right) \quad \text{substitute } \sec^{-1}x \text{ for } y$$

In other words, to find the value of $y = \sec^{-1}x$, evaluate $y = \cos^{-1}\left(\frac{1}{x}\right)$, $|x| \ge 1$. Similarly, the expression $\csc^{-1}x$ can be evaluated using $\sin^{-1}\left(\frac{1}{x}\right)$, $|x| \ge 1$. The expression $\cot^{-1}x$ can likewise be evaluated using an inverse tangent function: $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$.

WORTHY OF NOTE

While the domains of $y = \cot^{-1}x$ and $y = \tan^{-1}x$ both include all real numbers, evaluating $\cot^{-1}x$ using $\tan^{-1}\left(\frac{1}{x}\right)$ involves the restriction $x \ne 0$. To maintain consistency, the equation $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$ is often used. The graph of $y = \frac{\pi}{2} - \tan^{-1}x$ is that of $y = \tan^{-1}x$ reflected across the x-axis and shifted $\frac{\pi}{2}$ units up, with the result identical to the graph of $y = \cot^{-1}x$.

EXAMPLE 10 ► Evaluating an Inverse Trig Function

Evaluate using a calculator only if necessary:

a. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **b.** $\cot^{-1}\left(\frac{\pi}{12}\right)$

Solution ► **a.** From our previous discussion, for $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$, we evaluate $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. Since this is a standard value, no calculator is needed and the result is 30°.

b. For $\cot^{-1}\left(\frac{\pi}{12}\right)$, find $\tan^{-1}\left(\frac{12}{\pi}\right)$ on a calculator:

$$\cot^{-1}\left(\frac{\pi}{12}\right) = \tan^{-1}\left(\frac{12}{\pi}\right) \approx 1.3147.$$

D. You've just learned how to find and graph inverse functions for sec *x*, csc *x*, and cot *x*

Now try Exercises 77 through 86 ►

A summary of the highlights from this section follows.

Summary of Inverse Function Properties and Compositions

1. For $\sin x$ and $\sin^{-1}x$,
   $\sin(\sin^{-1}x) = x$, for any x in the interval $[-1, 1]$
   $\sin^{-1}(\sin x) = x$, for any x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2. For $\cos x$ and $\cos^{-1}x$,
   $\cos(\cos^{-1}x) = x$, for any x in the interval $[-1, 1]$
   $\cos^{-1}(\cos x) = x$, for any x in the interval $[0, \pi]$
3. For $\tan x$ and $\tan^{-1}x$,
   $\tan(\tan^{-1}x) = x$, for any real number x
   $\tan^{-1}(\tan x) = x$, for any x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4. To evaluate $\sec^{-1}x$, use $\cos^{-1}\left(\frac{1}{x}\right)$, $|x| \geq 1$,
   $\csc^{-1}x$, use $\sin^{-1}\left(\frac{1}{x}\right)$, $|x| \geq 1$
   $\cot^{-1}x$, use $\frac{\pi}{2} - \tan^{-1}x$, for all real numbers x

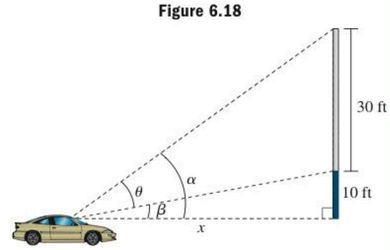
E. Applications of Inverse Trig Functions

We close this section with one example of the many ways that inverse functions can be applied.

EXAMPLE 11 ► Using Inverse Trig Functions to Find Viewing Angles

Believe it or not, the drive-in movie theaters that were so popular in the 1950s are making a comeback! If you arrive early, you can park in one of the coveted "center spots," but if you arrive late, you might have to park very close and strain your neck to watch the movie. Surprisingly, the maximum viewing angle (not the most comfortable viewing angle in this case) is actually very close to the front. Assume the base of a 30-ft screen is 10 ft above eye level (see Figure 6.18).

Figure 6.18



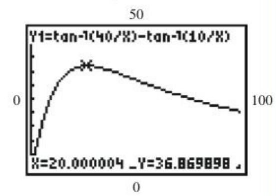
a. Use the inverse function concept to find expressions for angle α and angle β.
b. Use the result of Part (a) to find an expression for the *viewing angle* θ.
c. Use a calculator to find the viewing angle θ (to tenths of a degree) for distances of 15, 25, 35, and 45 ft, then to determine the distance x (to tenths of a foot) that maximizes the viewing angle.

Solution ► a. The side opposite β is 10 ft, and we want to know x — the adjacent side. This suggests we use $\tan\beta = \frac{10}{x}$, giving $\beta = \tan^{-1}\left(\frac{10}{x}\right)$. In the same way, we find that $\alpha = \tan^{-1}\left(\frac{40}{x}\right)$.

b. From the diagram we note that $\theta = \alpha - \beta$, and substituting for α and β directly gives $\theta = \tan^{-1}\left(\frac{40}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)$.

c. After we enter $Y_1 = \tan^{-1}\left(\frac{40}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)$, a graphing calculator gives approximate viewing angles of 35.8 °, 36.2°, 32.9°, and 29.1°, for $x = 15$, 25, 35, and 45 ft, respectively. From these data, we note the distance x that makes θ a maximum must be between 15 and 35 ft, and using 2nd TRACE (CALC) **4:maximum** shows θ is a maximum of 36.9° at a distance of 20 ft from the screen (see Figure 6.19).

Figure 6.19



E. You've just learned how to solve applications involving inverse functions

Now try Exercises 89 through 95 ►

TECHNOLOGY HIGHLIGHT

More on Inverse Functions

The domain and range of the inverse functions for sine, cosine, and tangent are preprogrammed into most graphing calculators, making them an ideal tool for reinforcing the concepts involved. In particular, $\sin x = y$ implies that $\sin^{-1}y = x$ only if $-90° \le y \le 90°$ and $-1 \le x \le 1$. For a stark reminder of this fact we'll use the TABLE feature of the grapher. Begin by using the TBLSET screen (2nd WINDOW) to set TblStart = 90 with ΔTbl = −30. After placing the calculator in degree MODE, go to the Y= screen and input $Y_1 = \sin x$, $Y_2 = \sin^{-1}x$, and $Y_3 = Y_2(Y_1)$ (the composition $Y_2 \circ Y_1$). Then disable Y_2 [turn it off—Y_3 will read it anyway) so that both Y_1 and Y_3 will be displayed simultaneously on the TABLE screen. Pressing 2nd GRAPH brings up the TABLE shown in Figure 6.20, where we note the inputs are standard angles, the outputs in Y_1 are the (expected) standard values, and the outputs in Y_3 return the original standard values. Now scroll upward until 180° is at the top of the X column (Figure 6.21), and note that Y_3 continues to return standard angles from the interval $[-90°, 90°]$—a stark reminder that while the expression $\sin 150° = 0.5$, $\sin^{-1}(\sin 150°) \ne 150°$. Once again we note that while $\sin^{-1}(\sin 150°)$ can be evaluated, it cannot be evaluated directly using the inverse function properties. Use these ideas to complete the following exercises.

Figure 6.20

X	Y1	Y3
90	1	90
60	.86603	60
30	.5	30
0	0	0
-30	-.5	-30
-60	-.866	-60
-90	-1	-90

X=90

Figure 6.21

X	Y1	Y3
180	0	0
150	.5	30
120	.86603	60
90	1	90
60	.86603	60
30	.5	30
0	0	0

X=180

Exercise 1: Go through an exercise similar to the one here using $Y_1 = \cos x$ and $Y_2 = \cos^{-1}x$. Remember to modify the TBLSET to accommodate the restricted domain for cosine.

Exercise 2: Complete parts (a) and (b) using the TABLE from Exercise 1. Complete parts (c) and (d) without a calculator.

a. $\cos^{-1}(\cos 150°)$ **b.** $\cos^{-1}(\cos 210°)$

c. $\cos^{-1}(\cos 120°)$ **d.** $\cos^{-1}(\cos 240°)$

6.5 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. All six trigonometric functions fail the ________ ________ test and therefore are not ________ to ________.
2. The two most common ways of writing the inverse function for $y = \sin x$ are ________ and ________.
3. The domain for the inverse sine function is ________ and the range is ________.
4. The domain for the inverse cosine function is ________ and the range is ________.
5. Most calculators do not have a key for evaluating an expression like $\sec^{-1}5$. Explain how it is done using the COS key.
6. Discuss/Explain what is meant by the *implicit form* of an inverse function and the *explicit form*. Give algebraic and trigonometric examples.

▶ DEVELOPING YOUR SKILLS

The tables here show values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for $\theta \in [-180° \text{ to } 210°]$. The restricted domain used to develop the inverse functions is shaded. Use the information from these tables to complete the exercises that follow.

$y = \sin\theta$

θ	$\sin\theta$	θ	$\sin\theta$
−180°	0	30°	$\frac{1}{2}$
−150°	$-\frac{1}{2}$	60°	$\frac{\sqrt{3}}{2}$
−120°	$-\frac{\sqrt{3}}{2}$	90°	1
−90°	−1	120°	$\frac{\sqrt{3}}{2}$
−60°	$-\frac{\sqrt{3}}{2}$	150°	$\frac{1}{2}$
−30°	$-\frac{1}{2}$	180°	0
0	0	210°	$-\frac{1}{2}$

$y = \cos\theta$

θ	$\cos\theta$	θ	$\cos\theta$
−180°	−1	30°	$\frac{\sqrt{3}}{2}$
−150°	$-\frac{\sqrt{3}}{2}$	60°	$\frac{1}{2}$
−120°	$-\frac{1}{2}$	90°	0
−90°	0	120°	$-\frac{1}{2}$
−60°	$\frac{1}{2}$	150°	$-\frac{\sqrt{3}}{2}$
−30°	$\frac{\sqrt{3}}{2}$	180°	−1
0	1	210°	$-\frac{\sqrt{3}}{2}$

$y = \tan\theta$

θ	$\tan\theta$	θ	$\tan\theta$
−180°	0	30°	$\frac{\sqrt{3}}{3}$
−150°	$\frac{\sqrt{3}}{3}$	60°	$\sqrt{3}$
−120°	$\sqrt{3}$	90°	—
−90°	—	120°	$-\sqrt{3}$
−60°	$-\sqrt{3}$	150°	$-\frac{\sqrt{3}}{3}$
−30°	$-\frac{\sqrt{3}}{3}$	180°	0
0	0	210°	$\sqrt{3}$

Use the preceding tables to fill in each blank (principal values only).

7.

$\sin 0 = 0$	$\sin^{-1}0 = $ ____
$\sin\left(\frac{\pi}{6}\right) = $ ____	$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$
$\sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}$	$\sin^{-1}\left(-\frac{1}{2}\right) = $ ____
$\sin\left(-\frac{\pi}{2}\right) = -1$	$\sin^{-1}(-1) = $ ____

8.

$\sin\pi = 0$	$\sin^{-1}0 = $ ____
$\sin 120° = \frac{\sqrt{3}}{2}$	$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = $ ____
$\sin(-60°) = -\frac{\sqrt{3}}{2}$	$\arcsin\left(-\frac{\sqrt{3}}{2}\right) = $ ____
$\sin 180° = $ ____	$\arcsin 0 = 0$

Evaluate without the aid of calculators or tables, *keeping the domain and range of each function in mind.* Answer in radians.

9. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ **10.** $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

11. $\sin^{-1}1$ **12.** $\arcsin\left(-\frac{1}{2}\right)$

Evaluate using a calculator, *keeping the domain and range of each function in mind.* Answer in radians to the nearest ten-thousandth *and* in degrees to the nearest tenth.

13. arcsin 0.8892 **14.** $\arcsin\left(\frac{7}{8}\right)$

15. $\sin^{-1}\left(\frac{1}{\sqrt{7}}\right)$ **16.** $\sin^{-1}\left(\frac{1-\sqrt{5}}{2}\right)$

Evaluate each expression.

17. $\sin\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right]$ **18.** $\sin\left[\arcsin\left(\frac{\sqrt{3}}{2}\right)\right]$

19. $\arcsin\left[\sin\left(\frac{\pi}{3}\right)\right]$ **20.** $\sin^{-1}(\sin 30°)$

21. $\sin^{-1}(\sin 135°)$ **22.** $\arcsin\left[\sin\left(\frac{-2\pi}{3}\right)\right]$

23. $\sin(\sin^{-1} 0.8205)$ **24.** $\sin\left[\arcsin\left(\frac{3}{5}\right)\right]$

Use the tables given prior to Exercise 7 to fill in each blank (principal values only).

25.

$\cos 0 = 1$	$\cos^{-1}1 =$ ____
$\cos\left(\frac{\pi}{6}\right) =$ ____	$\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$
$\cos 120° = -\frac{1}{2}$	$\arccos\left(-\frac{1}{2}\right) =$ ____
$\cos \pi = -1$	$\cos^{-1}(-1) =$ ____

26.

$\cos(-60°) = \frac{1}{2}$	$\cos^{-1}\left(\frac{1}{2}\right) =$ ____
$\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) =$ ____
$\cos(-120°) =$ ____	$\arccos\left(-\frac{1}{2}\right) = 120°$
$\cos(2\pi) = 1$	$\cos^{-1}1 =$ ____

Evaluate without the aid of calculators or tables. Answer in radians.

27. $\cos^{-1}\left(\frac{1}{2}\right)$ **28.** $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

29. $\cos^{-1}(-1)$ **30.** $\arccos(0)$

Evaluate using a calculator. Answer in radians to the nearest ten-thousandth, degrees to the nearest tenth.

31. arccos 0.1352 **32.** $\arccos\left(\frac{4}{7}\right)$

33. $\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)$ **34.** $\cos^{-1}\left(\frac{\sqrt{6}-1}{5}\right)$

Evaluate each expression.

35. $\arccos\left[\cos\left(\frac{\pi}{4}\right)\right]$ **36.** $\cos^{-1}(\cos 60°)$

37. $\cos(\cos^{-1} 0.5560)$ **38.** $\cos\left[\arccos\left(-\frac{8}{17}\right)\right]$

39. $\cos\left[\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right]$ **40.** $\cos\left[\arccos\left(\frac{\sqrt{3}}{2}\right)\right]$

41. $\cos^{-1}\left[\cos\left(\frac{5\pi}{4}\right)\right]$ **42.** $\arccos(\cos 44.2°)$

Use the tables presented before Exercise 7 to fill in each blank. Convert from radians to degrees as needed.

43.

$\tan 0 = 0$	$\tan^{-1}0 =$ ____
$\tan\left(-\frac{\pi}{3}\right) =$ ____	$\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
$\tan 30° = \frac{\sqrt{3}}{3}$	$\arctan\left(\frac{\sqrt{3}}{3}\right) =$ ____
$\tan\left(\frac{\pi}{3}\right) =$ ____	$\tan^{-1}(\sqrt{3}) =$ ____

44.

$\tan(-150°) = \frac{\sqrt{3}}{3}$	$\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) =$ ____
$\tan \pi = 0$	$\tan^{-1}0 =$ ____
$\tan 120° = -\sqrt{3}$	$\arctan(-\sqrt{3}) =$ ____
$\tan\left(\frac{\pi}{4}\right) =$ ____	$\arctan 1 = \frac{\pi}{4}$

Evaluate without the aid of calculators or tables.

45. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ **46.** $\arctan(-1)$

47. $\arctan(\sqrt{3})$ **48.** $\tan^{-1}0$

Evaluate using a calculator, *keeping the domain and range of each function in mind.* Answer in radians to the nearest ten-thousandth *and* in degrees to the nearest tenth.

49. $\tan^{-1}(-2.05)$

50. $\tan^{-1}(0.3267)$

51. $\arctan\left(\frac{29}{21}\right)$

52. $\arctan(-\sqrt{6})$

Simplify each expression without using a calculator.

53. $\sin^{-1}\left[\cos\left(\frac{2\pi}{3}\right)\right]$

54. $\cos^{-1}\left[\sin\left(-\frac{\pi}{3}\right)\right]$

55. $\tan\left[\arccos\left(\frac{\sqrt{3}}{2}\right)\right]$

56. $\sec\left[\arcsin\left(\frac{1}{2}\right)\right]$

57. $\csc\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right]$

58. $\cot\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$

59. $\arccos[\sin(-30°)]$

60. $\arcsin(\cos 135°)$

Explain why the following expressions are not defined.

61. $\tan(\sin^{-1}1)$

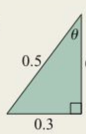
62. $\cot(\arccos 1)$

63. $\sin^{-1}\left[\csc\left(\frac{\pi}{4}\right)\right]$

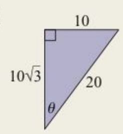
64. $\cos^{-1}\left[\sec\left(\frac{2\pi}{3}\right)\right]$

Use the diagrams below to write the value of: (a) sin θ, (b) cos θ, and (c) tan θ.

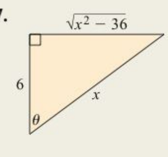
65.



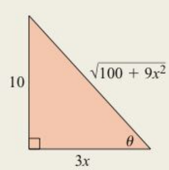
66.



67.



68.



Evaluate each expression by drawing a right triangle and labeling the sides.

69. $\sin\left[\cos^{-1}\left(-\frac{7}{25}\right)\right]$

70. $\cos\left[\sin^{-1}\left(-\frac{11}{61}\right)\right]$

71. $\sin\left[\tan^{-1}\left(\frac{\sqrt{5}}{2}\right)\right]$

72. $\tan\left[\cos^{-1}\left(\frac{\sqrt{23}}{12}\right)\right]$

73. $\cot\left[\arcsin\left(\frac{3x}{5}\right)\right]$

74. $\tan\left[\operatorname{arcsec}\left(\frac{5}{2x}\right)\right]$

75. $\cos\left[\sin^{-1}\left(\frac{x}{\sqrt{12+x^2}}\right)\right]$

76. $\tan\left[\sec^{-1}\left(\frac{\sqrt{9+x^2}}{x}\right)\right]$

Use the tables given prior to Exercise 7 to help fill in each blank.

77.

$\sec 0 = 1$	$\sec^{-1}1 =$ ___
$\sec\left(\frac{\pi}{3}\right) =$ ___	$\operatorname{arcsec} 2 = \frac{\pi}{3}$
$\sec(-30°) = \frac{2}{\sqrt{3}}$	$\operatorname{arcsec}\left(\frac{2}{\sqrt{3}}\right) =$ ___
$\sec(\pi) =$ ___	$\sec^{-1}(-1) = \pi$

78.

$\sec(-60°) = 2$	$\operatorname{arcsec} 2 =$ ___
$\sec\left(\frac{7\pi}{6}\right) = -\frac{2}{\sqrt{3}}$	$\operatorname{arcsec}\left(-\frac{2}{\sqrt{3}}\right) =$ ___
$\sec(-360°) = 1$	$\operatorname{arcsec} 1 =$ ___
$\sec(60°) =$ ___	$\sec^{-1} 2 = 60°$

Evaluate using a calculator only as necessary.

79. $\operatorname{arccsc} 2$

80. $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

81. $\cot^{-1}\sqrt{3}$

82. $\operatorname{arccot}(-1)$

83. $\operatorname{arcsec} 5.789$

84. $\cot^{-1}\left(-\frac{\sqrt{7}}{2}\right)$

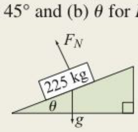
85. $\sec^{-1}\sqrt{7}$

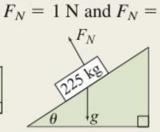
86. $\operatorname{arccsc} 2.9875$

► WORKING WITH FORMULAS

87. The force normal to an object on an inclined plane: $F_N = mg\cos\theta$

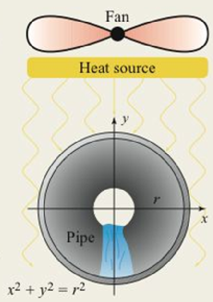
When an object is on an inclined plane, the **normal force** is the force acting perpendicular to the plane and away from the force of gravity, and is measured in a unit called **newtons (N).** The magnitude of this force depends on the angle of incline of the plane according to the formula above, where m is the mass of the object in kilograms and g is the force of gravity (9.8 m/sec²). Given $m = 225$ g, find (a) F_N for $\theta = 15°$ and $\theta = 45°$ and (b) θ for $F_N = 1$ N and $F_N = 2$ N.





88. Heat flow on a cylindrical pipe: $T = (T_0 - T_R)\sin\left(\frac{y}{\sqrt{x^2 + y^2}}\right) + T_R;\ y \geq 0$

When a circular pipe is exposed to a fan-driven source of heat, the temperature of the air reaching the pipe is greatest at the point nearest to the source (see diagram). As you move around the circumference of the pipe away from the source, the temperature of the air reaching the pipe gradually decreases.



One possible model of this phenomenon is given by the formula shown, where T is the temperature of the air at a point (x, y) on the circumference of a pipe with outer radius $r = \sqrt{x^2 + y^2}$, T_0 is the temperature of the air at the source, and T_R is the surrounding room temperature. Assuming $T_0 = 220°$F, $T_R = 72°$ and $r = 5$ cm: (a) Find the temperature of the air at the points (0, 5), (3, 4), (4, 3), (4.58, 2), and (4.9, 1).
(b) Why is the temperature decreasing for this sequence of points? (c) Simplify the formula using $r = 5$ and use it to find two points on the pipe's circumference where the temperature of the air is 113°.

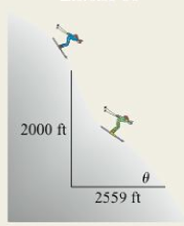
▶ APPLICATIONS

89. Snowcone dimensions: *Made in the Shade Snowcones* sells a colossal size cone that uses a conical cup holding 20 oz of ice and liquid. The cup is 20 cm tall and has a radius of 5.35 cm. Find the angle θ formed by a cross-section of the cup.

Exercise 89

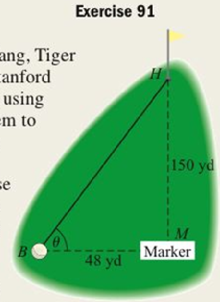
90. Avalanche conditions: Winter avalanches occur for many reasons, one being the slope of the mountain. Avalanches seem to occur most often for slopes between 35° and 60° (snow gradually slides off steeper slopes). The slopes at a local ski resort have an average rise of 2000 ft for each horizontal run of 2559 ft. Is this resort prone to avalanches? Find the angle θ and respond.

Exercise 90



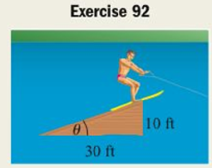
91. Distance to hole: A popular story on the PGA Tour has Gerry Yang, Tiger Woods' teammate at Stanford and occasional caddie, using the Pythagorean theorem to find the distance Tiger needed to reach a particular hole. Suppose you notice a marker in the ground stating that the straight line distance from the marker to the hole (H) is 150 yd. If your ball B is 48 yd from the marker (M) and angle BMH is a right angle, determine the angle θ and *your* straight line distance from the hole.

Exercise 91



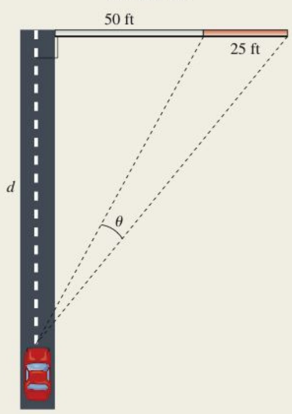
92. Ski jumps: At a waterskiing contest on a large lake, skiers use a ramp rising out of the water that is 30 ft long and 10 ft high at the high end. What angle θ does the ramp make with the lake?

Exercise 92



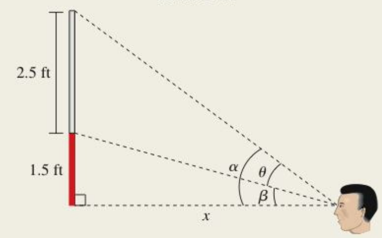
93. Viewing angles for advertising: A 25-ft-wide billboard is erected perpendicular to a straight highway, with the closer edge 50 ft away (see figure). Assume the advertisement on the billboard is most easily read when the viewing angle is 10.5° or more. (a) Use inverse functions to find an expression for the viewing angle θ. (b) Use a calculator to help determine the distance d (to tenths of a foot) for which the viewing angle is greater than 10.5°. (c) What distance d maximizes this viewing angle?

Exercise 93



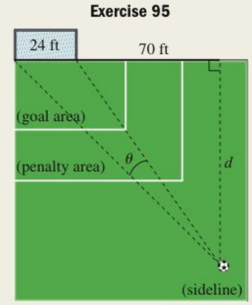
94. Viewing angles at an art show: At an art show, a painting 2.5 ft in height is hung on a wall so that its base is 1.5 ft above the eye level of an average viewer (see figure). (a) Use inverse functions to find expressions for angles α and β. (b) Use the result to find an expression for the *viewing angle* θ. (c) Use a calculator to help determine the distance x (to tenths of a foot) that maximizes this viewing angle.

Exercise 94



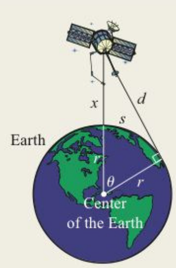
95. Shooting angles and shots on goal: A soccer player is on a breakaway and is dribbling just inside the right sideline toward the opposing goal (see figure). As the defense closes in, she has just a few seconds to decide when to shoot. (a) Use inverse functions to find an expression for the shooting angle θ. (b) Use a calculator to help determine the distance d (to tenths of a foot) that will maximize the shooting angle for the dimensions shown.

Exercise 95



▶ EXTENDING THE CONCEPT

Consider a satellite orbiting at an altitude of x mi above the Earth. The distance d from the satellite to the horizon and the length s of the corresponding arc of the Earth are shown in the diagram.

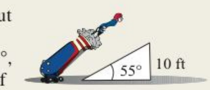


96. To find the distance d we use the formula $d = \sqrt{2rx + x^2}$. (a) Show how this formula was developed using the Pythagorean theorem. (b) Find a formula for the angle θ in terms of r and x, then a formula for the arc length s.

97. If the Earth has a radius of 3960 mi and the satellite is orbiting at an altitude of 150 mi, (a) what is the measure of angle θ? (b) how much longer is d than s?

A projectile is any object that is shot, thrown, slung, or otherwise projected and has no continuing source of propulsion. The horizontal and vertical position of the projectile depends on its initial velocity, angle of projection, and height of release (air resistance is neglected). The horizontal position of the projectile is given by $x = v_0 \cos\theta\, t$, while its vertical position is modeled by $y = y_0 + v_0 \sin\theta\, t - 16t^2$, where y_0 is the height it is projected from, θ is the projection angle, and t is the elapsed time in seconds.

98. A circus clown is shot out of a specially made cannon at an angle of 55°, with an initial velocity of 85 ft/sec, and the end of the cannon is 10 ft high.

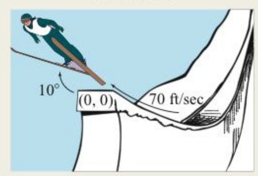


a. Find the position of the safety net (distance from the cannon and height from the ground) if the clown hits the net after 4.3 sec.

b. Find the angle at which the clown was shot if the initial velocity was 75 ft/sec and the clown hits a net which is placed 175.5 ft away after 3.5 sec.

99. A winter ski jumper leaves the ski-jump with an initial velocity of 70 ft/sec at an angle of 10°. Assume the jump-off point has coordinates (0, 0).

Exercise 99



a. What is the horizontal position of the skier after 6 sec?

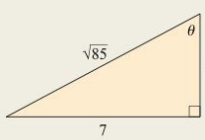
b. What is the vertical position of the skier after 6 sec?

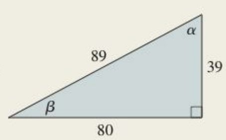
c. What diagonal distance (down the mountain side) was traveled if the skier touched down after being airborne for 6 sec?

100. Suppose the domain of $y = \csc x$ was restricted to $x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, and the domain of $y = \cot x$ to $x \in (0, \pi)$. **(a)** Would these functions then be one-to-one? **(b)** What are the corresponding ranges? **(c)** State the domain and range of $y = \csc^{-1}x$ and $y = \cot^{-1}x$. **(d)** Graph each function.

▶ MAINTAINING YOUR SKILLS

101. (6.4) Use the triangle given with a double-angle identity to find the exact value of $\sin(2\theta)$.





102. (6.3) Use the triangle given with a sum identity to find the exact value of $\sin(\alpha + \beta)$.

103. (3.7) Solve the inequality $f(x) \le 0$ using zeroes and end behavior given $f(x) = x^3 - 9x$.

104. (2.3) In 2000, Space Tourists Inc. sold 28 low-orbit travel packages. By 2005, yearly sales of the low-orbit package had grown to 105. Assuming the growth is linear, (a) find the equation that models this growth ($2000 \rightarrow t = 0$), (b) discuss the meaning of the slope in this context, and (c) use the equation to project the number of packages that will be sold in 2010.

6.6 Solving Basic Trig Equations

Learning Objectives

In Section 6.6 you will learn how to:

- ☐ **A.** Use a graph to gain information about principal roots, roots in $[0, 2\pi)$, and roots in $\mathbb{R}$
- ☐ **B.** Use inverse functions to solve trig equations for the principal root
- ☐ **C.** Solve trig equations for roots in $[0, 2\pi)$ or $[0, 360°)$
- ☐ **D.** Solve trig equations for roots in $\mathbb{R}$
- ☐ **E.** Solve trig equations using fundamental identities
- ☐ **F.** Solve trig equations using graphing technology

In this section, we'll take the elements of basic equation solving and use them to help solve **trig equations,** or equations containing trigonometric functions. All of the algebraic techniques previously used can be applied to these equations, including the properties of equality and all forms of factoring (common terms, difference of squares, etc.). As with polynomial equations, we continue to be concerned with the *number of solutions* as well as with the *solutions themselves,* but there is one major difference. There is no "algebra" that can transform a function like $\sin x = \frac{1}{2}$ into $x =$ *solution*. For that we rely on the inverse trig functions from Section 6.5.

A. The Principal Root, Roots in [0, 2π), and Real Roots

In a study of polynomial equations, making a connection between the degree of an equation, its graph, and its possible roots, helped give insights as to the number, location, and nature of the roots. Similarly, keeping graphs of basic trig functions *constantly* in mind helps you gain information regarding the solutions to trig equations. When solving trig equations, we refer to the solution found using $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ as the **principal root.** You will alternatively be asked to find (1) the principal root, (2) solutions in $[0, 2\pi)$ or $[0°, 360°)$, or (3) solutions from the set of real numbers $\mathbb{R}$. For convenience, graphs of the basic sine, cosine, and tangent functions are repeated in Figures 6.22 through 6.24. Take a mental snapshot of them and keep them close at hand.

Figure 6.22

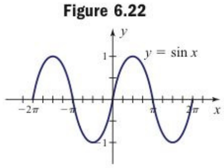


Figure 6.23

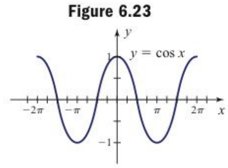
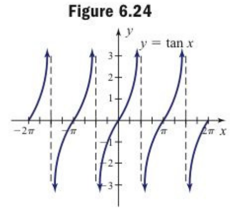


Figure 6.24



EXAMPLE 1 ► Visualizing Solutions Graphically

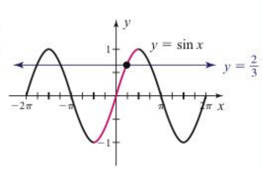
Consider the equation $\sin x = \frac{2}{3}$. Using a graph of $y = \sin x$ and $y = \frac{2}{3}$,

a. State the quadrant of the principal root.

b. State the number of roots in $[0, 2\pi)$ and their quadrants.

c. Comment on the number of real roots.

Solution ► We begin by drawing a quick sketch of $y = \sin x$ and $y = \frac{2}{3}$, noting that solutions will occur where the graphs intersect.



a. The sketch shows the principal root occurs between 0 and $\frac{\pi}{2}$ in QI.

b. For $[0, 2\pi)$ we note the graphs intersect twice and there will be two solutions in this interval.

c. Since the graphs of $y = \sin x$ and $y = \frac{2}{3}$ extend infinitely in both directions, they will intersect an infinite number of times—*but at regular intervals!* Once a root is found, adding integer multiples of 2π (the period of sine) to this root will give the location of additional roots.

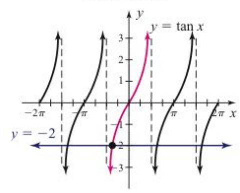
Now try Exercises 7 through 10 ►

WORTHY OF NOTE

Note that we refer to $\left(0, \frac{\pi}{2}\right)$ as Quadrant I or QI, regardless of whether we're discussing the unit circle or the graph of the function. In Example 1b, the solutions correspond to those found in QI and QII on the unit circle, where sin x is also positive.

When this process is applied to the equation $\tan x = -2$, the graph shows the principal root occurs between $-\frac{\pi}{2}$ and 0 in QIV (see Figure 6.25). In the interval $[0, 2\pi)$ the graphs intersect twice, in QII and QIV where $\tan x$ is negative (graphically—below the x-axis). As in Example 1, the graphs continue infinitely and will intersect an infinite number of times—*but again at regular intervals!* Once a root is found, adding integer multiples of π (the period of tangent) to this root will give the location of other roots.

Figure 6.25



A. You've just learned how to use a graph to gain information about principal roots, roots in $[0, 2\pi)$, and roots in $\mathbb{R}$

B. Inverse Functions and Principal Roots

To solve equations having a single variable term, the basic goal is to isolate the variable term and apply the inverse function or operation. This is true for algebraic equations like $2x - 1 = 0$, $2\sqrt{x} - 1 = 0$, or $2x^2 - 1 = 0$, and for trig equations like $2 \sin x - 1 = 0$. In each case we would add 1 to both sides, divide by 2, then apply the appropriate inverse. When the inverse trig functions are applied, the result is only the principal root and other solutions may exist depending on the interval under consideration.

EXAMPLE 2 ► **Finding Principal Roots**

Find the principal root of $\sqrt{3} \tan x - 1 = 0$.

Solution ► We begin by isolating the variable term, then apply the inverse function.

$$\sqrt{3} \tan x - 1 = 0 \quad \text{given equation}$$

$$\tan x = \frac{1}{\sqrt{3}} \quad \text{add 1 and divide by } \sqrt{3}$$

$$\tan^{-1}(\tan x) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad \text{apply inverse tangent to both sides}$$

$$x = \frac{\pi}{6} \quad \text{result (exact form)}$$

B. You've just learned how to use inverse functions to solve trig equations for the principal root

Now try Exercises 11 through 28 ►

Table 6.2

θ	$\sin \theta$	$\cos \theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
π	0	-1

Equations like the one in Example 2 demonstrate the need to be *very* familiar with the functions of a special angle. They are frequently used in equations and applications to ensure results don't get so messy they obscure the main ideas. For convenience, the values of $\sin \theta$ and $\cos \theta$ are repeated in Table 6.2 for $x \in [0, \pi]$. Using symmetry and the appropriate sign, the table can easily be extended to all values in $[0, 2\pi)$. Using the reciprocal and ratio relationships, values for the other trig functions can also be found.

C. Solving Trig Equations for Roots in $[0, 2\pi)$ or $[0°, 360°)$

To find multiple solutions to a trig equation, we simply take the reference angle of the principal root, and *use this angle to find all solutions* within a specified range. A mental image of the graph still guides us, and the standard table of values (also held in memory) allows for a quick solution to many equations.

EXAMPLE 3 ► **Finding Solutions in $[0, 2\pi)$**

For $2\cos\theta + \sqrt{2} = 0$, find all solutions in $[0, 2\pi)$.

Solution ► Isolate the variable term, then apply the inverse function.

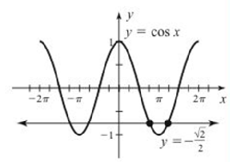
$$2\cos\theta + \sqrt{2} = 0 \quad \text{given equation}$$

$$\cos\theta = -\frac{\sqrt{2}}{2} \quad \text{subtract } \sqrt{2} \text{ and divide by 2}$$

$$\cos^{-1}(\cos\theta) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) \quad \text{apply inverse cosine to both sides}$$

$$\theta = \frac{3\pi}{4} \quad \text{result}$$

With $\frac{3\pi}{4}$ as the principal root, we know $\theta_r = \frac{\pi}{4}$. Since $\cos x$ is negative in QII and QIII, the second solution is $\frac{5\pi}{4}$. The second solution could also have been found from memory, recognition, or symmetry on the unit circle. Our (mental) graph verifies these are the only solutions in $[0, 2\pi)$.



WORTHY OF NOTE

Note how the graph of a trig function displays the information regarding *quadrants*. From the graph of $y = \cos x$ we "read" that cosine is negative in QII and QIII [the lower "hump" of the graph is below the x-axis in $(\pi/2, 3\pi/2)$] and positive in QI and QIV [the graph is above the x-axis in the intervals $(0, \pi/2)$ and $(3\pi/2, 2\pi)$].

Now try Exercises 29 through 34 ►

EXAMPLE 4 ► **Finding Solutions in $[0, 2\pi)$**

For $\tan^2 x - 1 = 0$, find all solutions in $[0, 2\pi)$.

Solution ► As with the other equations having a single variable term, we try to isolate this term or attempt a solution by factoring.

$$\tan^2 x - 1 = 0 \quad \text{given equation}$$

$$\sqrt{\tan^2 x} = \pm\sqrt{1} \quad \text{add 1 to both sides and take square roots}$$

$$\tan x = \pm 1 \quad \text{result}$$

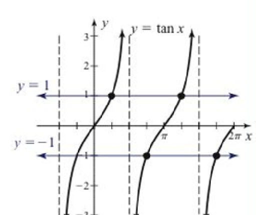
The algebra gives $\tan x = 1$ or $\tan x = -1$ and we solve each equation independently.

$$\tan x = 1 \qquad\qquad \tan x = -1$$

$$\tan^{-1}(\tan x) = \tan^{-1}(1) \qquad \tan^{-1}(\tan x) = \tan^{-1}(-1) \quad \text{apply inverse tangent}$$

$$x = \frac{\pi}{4} \qquad\qquad x = -\frac{\pi}{4} \quad \text{principal roots}$$

Of the principal roots, only $x = \frac{\pi}{4}$ is in the specified interval. With $\tan x$ positive in QI and QIII, a second solution is $\frac{5\pi}{4}$. While $x = -\frac{\pi}{4}$ is not in the interval, we still use it as a reference angle in QII and QIV (for $\tan x = -1$) and find the solutions $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$. The four solutions are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$, which is supported by the graph shown.



Now try Exercises 35 through 42 ►

For any trig function that is not equal to a standard value, we can use a calculator to approximate the principal root or leave the result in exact form, and apply the same ideas to this root to find all solutions in the interval.

EXAMPLE 5 ▶ Finding Solutions in [0, 360°)

Find all solutions in $[0°, 360°)$ for $3\cos^2\theta + \cos\theta - 2 = 0$.

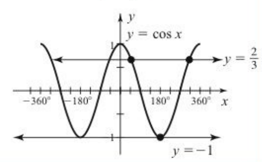
Solution ▶ Use a u-substitution to simplify the equation and help select an appropriate strategy. For $u = \cos\theta$, the equation becomes $3u^2 + u - 2 = 0$ and factoring seems the best approach. The factored form is $(u + 1)(3u - 2) = 0$, with solutions $u = -1$ and $u = \frac{2}{3}$. Re-substituting $\cos\theta$ for u gives

$$\cos\theta = -1 \qquad \cos\theta = \frac{2}{3} \qquad \text{equations from factored form}$$

$$\cos^{-1}(\cos\theta) = \cos^{-1}(-1) \qquad \cos^{-1}(\cos\theta) = \cos^{-1}\left(\frac{2}{3}\right) \qquad \text{apply inverse cosine}$$

$$\theta = 180° \qquad \theta \approx 48.2° \qquad \text{principal roots}$$

Both principal roots are in the specified interval. The first is quadrantal, the second was found using a calculator and is approximately 48.2°. With cos x positive in QI and QIV, a second solution is $(360 - 48.2)° = 311.8°$. The three solutions are 48.2°, 180°, and 311.8° although only $\theta = 180°$ is exact.



☑ C. You've just learned how to solve trig equations for roots in $[0, 2\pi)$ or $[0, 360°)$

Now try Exercises 43 through 50 ▶

D. Solving Trig Equations for All Real Roots (ℝ)

As we noted, the intersections of a trig function with a horizontal line occur at regular, *predictable* intervals. This makes finding solutions from the set of real numbers a simple matter of extending the solutions we found in $[0, 2\pi)$ or $[0°, 360°)$. To illustrate, consider the solutions to Example 3. For $2\cos\theta + \sqrt{2} = 0$, we found the solutions $\theta = \frac{3\pi}{4}$ and $\theta = \frac{5\pi}{4}$. For solutions in ℝ, we note the "predictable interval" between roots *is identical to the period of the function.* This means all real solutions will be represented by $\theta = \frac{3\pi}{4} + 2\pi k$ and $\theta = \frac{5\pi}{4} + 2\pi k$, $k \in \mathbb{Z}$ (k is an integer). Both are illustrated in Figures 6.26 and 6.27 with the primary solution indicated with a "*."

Figure 6.26

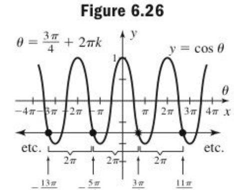
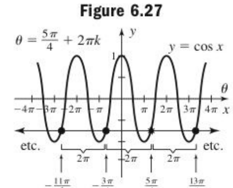
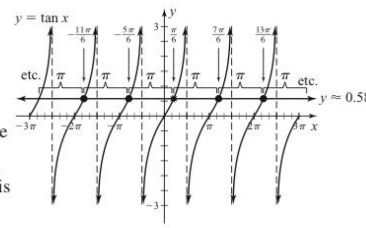


Figure 6.27



EXAMPLE 6 ► **Finding Solutions in ℝ**

Find all real solutions to $\sqrt{3}\tan x - 1 = 0$.



Solution ► In Example 2 we found the principal root was $x = \frac{\pi}{6}$. Since the tangent function has a period of π, adding integer multiples of π to this root will identify all solutions: $x = \frac{\pi}{6} + \pi k, k \in \mathbb{Z}$, as illustrated here.

Now try Exercises 51 through 56 ►

These fundamental ideas can be extended to many different situations. When asked to find *all real solutions,* be sure you find all roots in a stipulated interval before naming solutions by applying the period of the function. For instance, $\cos x = 0$ has two solutions in $[0, 2\pi)$ $\left[x = \frac{\pi}{2} \text{ and } x = \frac{3\pi}{2}\right]$, which we can quickly extend to find all real roots. But using $x = \cos^{-1}0$ or a calculator limits us to the single (principal) root $x = \frac{\pi}{2}$, and we'd miss all solutions stemming from $\frac{3\pi}{2}$. Note that solutions involving multiples of an angle (or fractional parts of an angle) should likewise be "handled with care," as in Example 7.

EXAMPLE 7 ► **Finding Solutions in ℝ**

Find all real solutions to $2\sin(2x)\cos x - \cos x = 0$.

Solution ► Since we have a common factor of $\cos x$, we begin by rewriting the equation as $\cos x[2\sin(2x) - 1] = 0$ and solve using the zero factor property. The resulting equations are $\cos x = 0$ and $2\sin(2x) - 1 = 0 \rightarrow \sin(2x) = \frac{1}{2}$.

$$\cos x = 0 \qquad \sin(2x) = \frac{1}{2} \qquad \text{equations from factored form}$$

In $[0, 2\pi)$, $\cos x = 0$ has solutions $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, giving $x = \frac{\pi}{2} + 2\pi k$ and $x = \frac{3\pi}{2} + 2\pi k$ as solutions in ℝ. Note these can actually be combined and written as $x = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$. The solution process for $\sin(2x) = \frac{1}{2}$ yields $2x = \frac{\pi}{6}$ and $2x = \frac{5\pi}{6}$. Since we seek all real roots, *we first extend each solution by* $2\pi k$ before dividing by 2, otherwise multiple solutions would be overlooked.

$$2x = \frac{\pi}{6} + 2\pi k \qquad 2x = \frac{5\pi}{6} + 2\pi k \qquad \text{solutions from } \sin(2x) = \tfrac{1}{2};\ k \in \mathbb{Z}$$

$$x = \frac{\pi}{12} + \pi k \qquad x = \frac{5\pi}{12} + \pi k \qquad \text{divide by 2}$$

Now try Exercises 57 through 66 ►

WORTHY OF NOTE

When solving trig equations that involve arguments other than a single variable, a *u*-substitution is sometimes used. For Example 7, substituting *u* for 2*x* gives the equation $\sin u = \frac{1}{2}$, making it "easier to see" that $u = \frac{\pi}{6}$ (since $\frac{1}{2}$ is a special value), and therefore $2x = \frac{\pi}{6}$ and $x = \frac{\pi}{12}$.

☑ **D.** You've just learned how to solve trig equations for roots in ℝ

E. Trig Equations and Trig Identities

In the process of solving trig equations, we sometimes employ fundamental identities to help simplify an equation, or to make factoring or some other method possible.

EXAMPLE 8 ▶ Solving Trig Equations Using an Identity

Find all solutions in $[0°, 360°)$ for $\cos(2\theta) + \sin^2\theta - 3\cos\theta = 1$.

Solution ▶ With a mixture of functions, exponents, and arguments, the equation is almost impossible to solve as it stands. But we can eliminate the sine function using the identity $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, leaving a quadratic equation in $\cos x$.

$$\cos(2\theta) + \sin^2\theta - 3\cos\theta = 1 \quad \text{given equation}$$
$$\cos^2\theta - \sin^2\theta + \sin^2\theta - 3\cos\theta = 1 \quad \text{substitute } \cos^2\theta - \sin^2\theta \text{ for } \cos(2\theta)$$
$$\cos^2\theta - 3\cos\theta = 1 \quad \text{combine like terms}$$
$$\cos^2\theta - 3\cos\theta - 1 = 0 \quad \text{subtract 1}$$

Let's substitute u for $\cos\theta$ to give us a simpler view of the equation. This gives $u^2 - 3u - 1 = 0$, which is clearly not factorable over the integers. Using the quadratic formula with $a = 1,\ b = -3,$ and $c = -1$ gives

$$u = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)} \quad \text{quadratic formula in } u$$
$$= \frac{3 \pm \sqrt{13}}{2} \quad \text{simplified}$$

To four decimal places we have $u = 3.3028$ and $u = -0.3028$. To answer in terms of the original variable we re-substitute $\cos\theta$ for u, realizing that $\cos\theta \approx 3.3028$ has no solution, so solutions in $[0°, 360°)$ must be provided by $\cos\theta \approx -0.3028$ and occur in QII and QIII. The solutions are $\theta = \cos^{-1}(-0.3028) = 107.6°$ and $360° - 107.6° = 252.4°$ to the nearest tenth of a degree.

☑ **E.** You've just learned how to solve trig equations using fundamental identities

Now try Exercises 67 through 82 ▶

F. Trig Equations and Graphing Technology

A majority of the trig equations you'll encounter in your studies can be solved using the ideas and methods presented here. But there are some equations that cannot be solved using standard methods because they mix polynomial functions (linear, quadratic, and so on) that can be solved using algebraic methods, with what are called **transcendental functions** (trigonometric, logarithmic, and so on). By definition, transcendental functions are those that *transcend* the reach of standard algebraic methods. These kinds of equations serve to highlight the value of graphing and calculating technology to today's problem solvers.

EXAMPLE 9 ▶ Solving Trig Equations Using Technology

Use a graphing calculator in radian mode to find all real roots of $2\sin x + \frac{3x}{5} - 2 = 0$. Round solutions to four decimal places.

Solution ▶ When using graphing technology our initial concern is the size of the viewing window. After carefully entering the equation on the Y= screen, we note the term $2\sin x$ will never be larger than 2 or less than -2 for any real number x. On the other hand, the term $\frac{3x}{5}$ becomes larger for larger values of x, which would seem to cause $2\sin x + \frac{3x}{5}$ to "grow" as x gets larger. We conclude the standard window is a good place to start, and the resulting graph is shown in Figure 6.28.

Figure 6.28

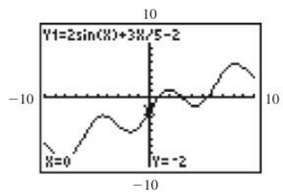
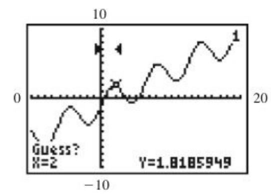


Figure 6.29



From this screen it appears there are three real roots, but to be sure none are hidden to the right, we extend the Xmax value to 20 (Figure 6.29). Using 2nd TRACE CALC **2:zero,** we follow the prompts and enter a left bound of 0 (a number to the left of the zero) and a right bound of 2 (a number to the right of the zero—see Figure 6.29). If you can visually approximate the root, the calculator prompts you for a GUESS, otherwise just bypass the request by pressing ENTER. The smallest root is approximately $x = 0.8435$. Repeating this sequence we find the other roots are $x \approx 3.0593$ and $x \approx 5.5541$.

F. You've just learned how to solve trig equations using graphing technology

Now try Exercises 83 through 88 ►

TECHNOLOGY HIGHLIGHT

Solving Equations Graphically

Some equations are very difficult to solve analytically, and even with the use of a graphing calculator, a strong combination of analytical skills with technical skills is required to state the solution set. Consider the equation $5\sin\left(\frac{1}{2}x\right) + 5 = \cot\left(\frac{1}{2}x\right)$ and solutions in $[-2\pi, 2\pi)$. There appears to be no quick analytical solution, and the first attempt at a graphical solution holds some hidden surprises. Enter $Y_1 = 5\sin\left(\frac{1}{2}x\right) + 5$ and $Y_2 = \frac{1}{\tan(\frac{1}{2}x)}$ on the Y= screen. Pressing ZOOM **7:ZTrig** gives the screen in Figure 6.30, where we note there are at least two and possibly three solutions, depending on how the sine graph intersects the cotangent graph. We are also uncertain as to whether the graphs intersect again between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Increasing the maximum Y-value to Ymax = 8 shows they do indeed. But once again, are there now three or four solutions? In situations like this it may be helpful to use the **zeroes method** for solving graphically. On the Y= screen, disable Y_1 and Y_2 and enter Y_3 as $Y_1 - Y_2$. Pressing ZOOM **7:ZTrig** at this point clearly shows that there are four solutions in this interval (Figure 6.31), which can easily be found using 2nd CALC **2:zero:** $x \approx -5.7543, -4.0094, -3.1416$, and 0.3390. Use these ideas to find solutions in $[-2\pi, 2\pi)$ for the exercises that follow.

Figure 6.30

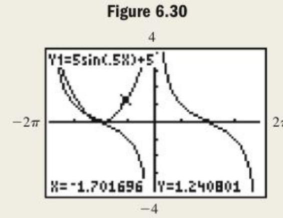
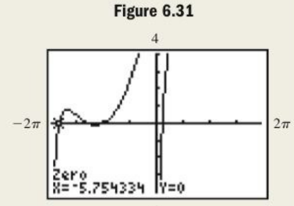


Figure 6.31



Exercise 1: $(1 + \sin x)^2 + \cos(2x) = 4\cos x(1 + \sin x)$

Exercise 2: $4\sin x = 2\cos^2\left(\frac{x}{2}\right)$

6.6 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if necessary.

1. For simple equations, a mental graph will tell us the quadrant of the ______ root, the number of roots in ______, and show a pattern for all ______ roots.

2. Solving trig equations is similar to solving algebraic equations, in that we first ______ the variable term, then apply the appropriate ______ function.

3. For $\sin x = \frac{\sqrt{2}}{2}$ the principal root is ______, solutions in $[0, 2\pi)$ are ______ and ______, and an expression for all real roots is ______ and ______; $k \in \mathbb{Z}$.

4. For $\tan x = -1$, the principal root is ______, solutions in $[0, 2\pi)$ are ______ and ______, and an expression for all real roots is ______.

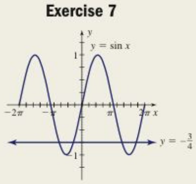
5. Discuss/Explain/Illustrate why $\tan x = \frac{3}{4}$ and $y = \cos x$ have two solutions in $[0, 2\pi)$, even though the period of $y = \tan x$ is π, while the period of $y = \cos x$ is 2π.

6. The equation $\sin^2 x = \frac{1}{2}$ has four solutions in $[0, 2\pi)$. Explain how these solutions can be viewed as the vertices of a square inscribed in the unit circle.

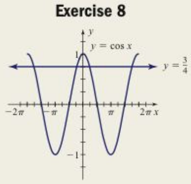
▶ DEVELOPING YOUR SKILLS

7. For the equation $\sin x = -\frac{3}{4}$ and the graphs of $y = \sin x$ and $y = -\frac{3}{4}$ given, state (a) the quadrant of the principal root and (b) the number of roots in $[0, 2\pi)$.

Exercise 7



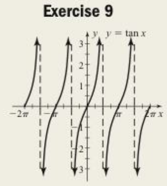
Exercise 8



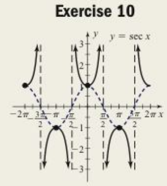
8. For the equation $\cos x = \frac{3}{4}$ and the graphs of $y = \cos x$ and $y = \frac{3}{4}$ given, state (a) the quadrant of the principal root and (b) the number of roots in $[0, 2\pi)$.

9. Given the graph $y = \tan x$ shown here, draw the horizontal line $y = -1.5$ and then for $\tan x = -1.5$ state (a) the quadrant of the principal root and (b) the number of roots in $[0, 2\pi)$.

Exercise 9



Exercise 10



10. Given the graph of $y = \sec x$ shown, draw the horizontal line $y = \frac{5}{4}$ and then for $\sec x = \frac{5}{4}$, state (a) the quadrant of the principal root and (b) the number of roots in $[0, 2\pi)$.

11. The table that follows shows θ in multiples of $\frac{\pi}{6}$ between 0 and $\frac{4\pi}{3}$, with the values for $\sin\theta$ given. Complete the table without a calculator or references using your knowledge of the unit circle, the signs of $f(\theta)$ in each quadrant, memory/recognition, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, and so on.

Exercise 11

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0		
$\frac{\pi}{6}$	$\frac{1}{2}$		
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$		
$\frac{\pi}{2}$	1		
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$		
$\frac{5\pi}{6}$	$\frac{1}{2}$		
π	0		
$\frac{7\pi}{6}$	$-\frac{1}{2}$		
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$		

Exercise 12

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0		1	
$\frac{\pi}{4}$		$\frac{\sqrt{2}}{2}$	
$\frac{\pi}{2}$		0	
$\frac{3\pi}{4}$		$-\frac{\sqrt{2}}{2}$	
π		-1	
$\frac{5\pi}{4}$		$-\frac{\sqrt{2}}{2}$	
$\frac{3\pi}{2}$		0	
$\frac{7\pi}{4}$		$\frac{\sqrt{2}}{2}$	
2π		1	

12. The table shows θ in multiples of $\frac{\pi}{4}$ between 0 and 2π, with the values for $\cos\theta$ given. Complete the table without a calculator or references using your knowledge of the unit circle, the signs of $f(\theta)$ in each quadrant, memory/recognition, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, and so on.

Find the principal root of each equation.

13. $2\cos x = \sqrt{2}$ **14.** $2\sin x = -1$

15. $-4\sin x = 2\sqrt{2}$ **16.** $-4\cos x = 2\sqrt{3}$

17. $\sqrt{3}\tan x = 1$ **18.** $-2\sqrt{3}\tan x = 2$

19. $2\sqrt{3}\sin x = -3$ **20.** $-3\sqrt{2}\csc x = 6$

21. $-6\cos x = 6$ **22.** $4\sec x = -8$

23. $\frac{7}{8}\cos x = \frac{7}{16}$ **24.** $-\frac{5}{3}\sin x = \frac{5}{6}$

25. $2 = 4\sin\theta$ **26.** $\pi\tan x = 0$

27. $-5\sqrt{3} = 10\cos\theta$ **28.** $4\sqrt{3} = 4\tan\theta$

Find all solutions in $[0, 2\pi)$.

29. $9\sin x - 3.5 = 1$ **30.** $6.2\cos x + 4 = 7.1$

31. $8\tan x + 7\sqrt{3} = -\sqrt{3}$

32. $\frac{1}{2}\sec x - \frac{3}{4} = -\frac{7}{4}$ **33.** $\frac{2}{3}\cot x - \frac{5}{6} = -\frac{3}{2}$

34. $-110\sin x = -55\sqrt{3}$ **35.** $4\cos^2 x = 3$

36. $4\sin^2 x = 1$ **37.** $-7\tan^2 x = -21$

38. $3\sec^2 x = 6$ **39.** $-4\csc^2 x = -8$

40. $6\sqrt{3}\cos^2 x = 3\sqrt{3}$ **41.** $4\sqrt{2}\sin^2 x = 4\sqrt{2}$

42. $\frac{2}{3}\cos^2 x + \frac{5}{6} = \frac{4}{3}$

Solve the following equations by factoring. State all real solutions in radians using the exact form where possible and rounded to four decimal places if the result is not a standard value.

43. $3\cos^2\theta + 14\cos\theta - 5 = 0$

44. $6\tan^2\theta - 2\sqrt{3}\tan\theta = 0$

45. $2\cos x\sin x - \cos x = 0$

46. $2\sin^2 x + 7\sin x = 4$ **47.** $\sec^2 x - 6\sec x = 16$

48. $2\cos^3 x + \cos^2 x = 0$ **49.** $4\sin^2 x - 1 = 0$

50. $4\cos^2 x - 3 = 0$

Find all real solutions. Note that identities are not required to solve these exercises.

51. $-2\sin x = \sqrt{2}$ **52.** $2\cos x = 1$

53. $-4\cos x = 2\sqrt{2}$ **54.** $4\sin x = 2\sqrt{3}$

55. $\sqrt{3}\tan x = -\sqrt{3}$ **56.** $2\sqrt{3}\tan x = 2$

57. $6\cos(2x) = -3$ **58.** $2\sin(3x) = -\sqrt{2}$

59. $\sqrt{3}\tan(2x) = -\sqrt{3}$ **60.** $2\sqrt{3}\tan(3x) = 6$

61. $-2\sqrt{3}\cos\left(\frac{1}{3}x\right) = 2\sqrt{3}$

62. $-8\sin\left(\frac{1}{2}x\right) = -4\sqrt{3}$

63. $\sqrt{2}\cos x\sin(2x) - 3\cos x = 0$

64. $\sqrt{3}\sin x\tan(2x) - \sin x = 0$

65. $\cos(3x)\csc(2x) - 2\cos(3x) = 0$

66. $\sqrt{3}\sin(2x)\sec(2x) - 2\sin(2x) = 0$

Solve each equation using calculator and inverse trig functions to determine the principal root (not by graphing). Clearly state (a) the principal root and (b) all real roots.

67. $3\cos x = 1$ **68.** $5\sin x = -2$

69. $\sqrt{2}\sec x + 3 = 7$ **70.** $\sqrt{3}\csc x + 2 = 11$

71. $\frac{1}{2}\sin(2\theta) = \frac{1}{3}$ **72.** $\frac{2}{5}\cos(2\theta) = \frac{1}{4}$

73. $-5\cos(2\theta) - 1 = 0$ **74.** $6\sin(2\theta) - 3 = 2$

Solve the following equations using an identity. State all real solutions in radians using the exact form where possible and rounded to four decimal places if the result is not a standard value.

75. $\cos^2 x - \sin^2 x = \frac{1}{2}$

76. $4\sin^2 x - 4\cos^2 x = 2\sqrt{3}$

77. $2\cos\left(\frac{1}{2}x\right)\cos x - 2\sin\left(\frac{1}{2}x\right)\sin x = 1$

78. $\sqrt{2}\sin(2x)\cos(3x) + \sqrt{2}\sin(3x)\cos(2x) = 1$

79. $(\cos\theta + \sin\theta)^2 = 1$

80. $(\cos\theta + \sin\theta)^2 = 2$

81. $\cos(2\theta) + 2\sin^2\theta - 3\sin\theta = 0$

82. $3\sin(2\theta) - \cos^2(2\theta) - 1 = 0$

Find all roots in $[0, 2\pi)$ using a graphing calculator. State answers in radians rounded to four decimal places.

83. $5\cos x - x = 3$

84. $3\sin x + x = 4$

85. $\cos^2(2x) + x = 3$

86. $\sin^2(2x) + 2x = 1$

87. $x^2 + \sin(2x) = 1$

88. $\cos(2x) - x^2 = -5$

▶ WORKING WITH FORMULAS

89. Range of a projectile: $R = \frac{5}{49}v^2\sin(2\theta)$

The distance a projectile travels is called its range and is modeled by the formula shown, where R is the range in meters, v is the initial velocity in meters per second, and θ is the angle of release. Two friends are standing 16 m apart playing catch. If the first throw has an initial velocity of 15 m/sec, what *two* angles will insure the ball travels the 16 m between the friends?

90. Fine-tuning a golf swing:
(club head to shoulder)^2 = (club length)^2 + (arm length)^2 − 2 (club length)(arm length)cos θ

A golf pro is taking specific measurements on a client's swing to help improve her game. If the angle θ is too small, the ball is hit late and "too thin" (you *top the ball*). If θ is too large, the ball is hit early and "too fat" (you *scoop the ball*). Approximate the angle θ formed by the club and the extended (left) arm using the given measurements and formula shown.

▶ APPLICATIONS

Acceleration due to gravity: When a steel ball is released down an inclined plane, the rate of the ball's acceleration depends on the angle of incline. The acceleration can be approximated by the formula $A(\theta) = 9.8\sin\theta$, where θ is in degrees and the acceleration is measured in meters per second/per second. To the nearest tenth of a degree,

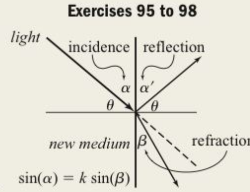
91. What angle produces an acceleration of 0 m/sec^2 when the ball is released? Explain why this is reasonable.

92. What angle produces an acceleration of 9.8 m/sec^2? What does this tell you about the acceleration due to gravity?

93. What angle produces an acceleration of 5 m/sec^2? Will the angle be larger or smaller for an acceleration of 4.5 m/sec^2?

94. Will an angle producing an acceleration of 2.5 m/sec^2 be one-half the angle required for an acceleration of 5 m/sec^2? Explore and discuss.

Exercises 95 to 98



Snell's law states that when a ray of light passes from one medium into another, the sine of the angle of incidence α *varies directly with* the sine of the angle of refraction β (see the figure). This phenomenon is modeled by the formula $\sin\alpha = k\sin\beta$, where k is called the **index of refraction.** Note the angle θ is the angle at which the light strikes the surface, so that $\alpha = 90° - \theta$. Use this information to work Exercises 95 to 98.

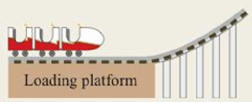
95. A ray of light passes from air into water, striking the water at an angle of 55°. Find the angle of incidence α and the angle of refraction β, if the index of refraction for water is $k = 1.33$.

96. A ray of light passes from air into a diamond, striking the surface at an angle of 75°. Find the angle of incidence α and the angle of refraction β, if the index of refraction for a diamond is $k = 2.42$.

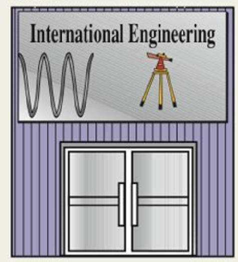
97. Find the index of refraction for ethyl alcohol if a beam of light strikes the surface of this medium at an angle of 40° and produces an angle of refraction $\beta = 34.3°$. Use this index to find the angle of incidence if a second beam of light created an angle of refraction measuring 15°.

98. Find the index of refraction for rutile (a type of mineral) if a beam of light strikes the surface of this medium at an angle of 30° and produces an angle of refraction $\beta = 18.7°$. Use this index to find the angle of incidence if a second beam of light created an angle of refraction measuring 10°.

99. Roller coaster design: As part of a science fair project, Hadra builds a scale model of a roller coaster using the equation $y = 5\sin\left(\frac{1}{2}x\right) + 7$, where y is the height of the model in inches and x is the distance from the "loading platform" in inches. (a) How high is the platform? (b) What distances from the platform does the model attain a height of 9.5 in.?

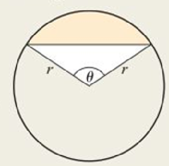


100. Company logo: Part of the logo for an engineering firm was modeled by a cosine function. The logo was then manufactured in steel and installed on the entrance marquee of the home office. The position and size of the logo is modeled by the function $y = 9\cos x + 15$, where y is the height of the graph above the base of the marquee in inches and x represents the distance from the edge of the marquee. Assume the graph begins flush with the edge. (a) How far above the base is the beginning of the cosine graph? (b) What distances from the edge does the graph attain a height of 19.5 in.?

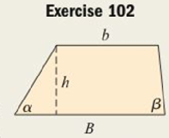


Geometry applications: Solve Exercises 101 and 102 graphically using a calculator. For Exercise 101, give θ in radians rounded to four decimal places. For Exercise 102, answer in degrees to the nearest tenth of a degree.

101. The area of a circular segment (the shaded portion shown) is given by the formula $A = \frac{1}{2}r^2(\theta - \sin\theta)$, where θ is in radians. If the circle has a radius of 10 cm, find the angle θ that gives an area of 12 cm^2.



Exercise 102



102. The perimeter of a trapezoid with parallel sides B and b, altitude h, and base angles α and β is given by the formula $P = B + b + h(\csc\alpha + \csc\beta)$. If $b = 30$ m, $B = 40$ m, $h = 10$ m, and $\alpha = 45°$, find the angle β that gives a perimeter of 105 m.

▶ EXTENDING THE CONCEPT

103. Find all real solutions to $5\cos x - x = -x$ in two ways. First use a calculator with $Y_1 = 5\cos x - x$ and $Y_2 = -x$ to determine the regular intervals between points of intersection. Second, simplify by adding x to both sides, and draw a quick sketch of the result to locate x-intercepts. Explain why both methods give the same result, even though the first presents you with a very different graph.

104. Once the fundamental ideas of solving a given family of equations is understood and practiced, a student usually begins to generalize them—making the numbers or symbols used in the equation irrelevant. (a) Use the inverse sine function to find the principal root of $y = A\sin(Bx - C) + D$, by solving for x in terms of y, A, B, C, and D. (b) Solve the following equation using the techniques addressed in this section, and then using the "formula" from Part (a). Do the results agree?

$$5 = 2\sin\left(\frac{1}{2}x + \frac{\pi}{4}\right) + 3.$$

▶ MAINTAINING YOUR SKILLS

105. **(1.4)** Use a substitution to show that $x = 2 + i$ is a zero of $f(x) = x^2 - 4x + 5$.

106. **(3.1)** Currently, tickets to productions of the Shakespeare Community Theater cost $10.00, with an average attendance of 250 people. Due to market research, the theater director believes that for each $0.50 reduction in price, 25 more people will attend. What ticket price will maximize the theater's revenue? What will the average attendance projected to become at that price?

107. **(6.5)** Evaluate without using a calculator:

a. $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$ **b.** $\sin[\tan^{-1}(-1)]$

108. **(5.1)** The largest Ferris wheel in the world, located in Yokohama, Japan, has a radius of 50 m. To the nearest hundredth of a meter, how far does a seat on the rim travel as the wheel turns through $\theta = 292.5°$?

6.7 General Trig Equations and Applications

Learning Objectives

In Section 6.7 you will learn how to:

- ❑ **A.** Use additional algebraic techniques to solve trig equations
- ❑ **B.** Solve trig equations using multiple angle, sum and difference, and sum-to-product identities
- ❑ **C.** Solve trig equations of the form $A\sin(Bx + C) + D = k$
- ❑ **D.** Use a combination of skills to model and solve a variety of applications

At this point you're likely beginning to understand the true value of trigonometry to the scientific world. Essentially, any phenomenon that is cyclic or periodic is beyond the reach of polynomial (and other) functions, and may require trig for an accurate understanding. And while there is an abundance of trig applications in oceanography, astronomy, meteorology, geology, zoology, and engineering, their value is not limited to the hard sciences. There are also rich applications in business and economics, and a growing number of modern artists are creating works based on attributes of the trig functions. In this section, we try to place some of these applications within your reach, with the exercise set offering an appealing variety from many of these fields.

A. Trig Equations and Algebraic Methods

We begin this section with a follow-up to Section 6.6, by introducing trig equations that require slightly more sophisticated methods to work out a solution.

EXAMPLE 1 ► Solving a Trig Equation by Squaring Both Sides

Find all solutions in $[0, 2\pi)$: $\sec x + \tan x = \sqrt{3}$.

Solution ► Our first instinct might be to rewrite the equation in terms of sine and cosine, but that simply leads to a similar equation that still has two different functions $[\sqrt{3}\cos x - \sin x = 1]$. Instead, we *square both sides* and see if the Pythagorean identity $1 + \tan^2 x = \sec^2 x$ will be of use. Prior to squaring, we separate the functions on opposite sides to avoid the mixed term $2\tan x \sec x$.

$$\sec x + \tan x = \sqrt{3} \quad \text{given equation}$$
$$(\sec x)^2 = (\sqrt{3} - \tan x)^2 \quad \text{subtract } \tan x \text{ and square}$$
$$\sec^2 x = 3 - 2\sqrt{3}\tan x + \tan^2 x \quad \text{result}$$

Since $\sec^2 x = 1 + \tan^2 x$, we substitute directly and obtain an equation in tangent alone.

$$1 + \tan^2 x = 3 - 2\sqrt{3}\tan x + \tan^2 x \quad \text{substitute } 1 + \tan^2 x \text{ for } \sec^2 x$$
$$-2 = -2\sqrt{3}\tan x \quad \text{simplify}$$
$$\frac{1}{\sqrt{3}} = \tan x \quad \text{solve for } \tan x$$

$\tan x > 0$ in QI and QIII

The proposed solutions are $x = \frac{\pi}{6}$ [QI] and $\frac{7\pi}{6}$ [QIII]. Since squaring an equation sometimes introduces extraneous roots, both should be checked in the original equation. The check shows only $x = \frac{\pi}{6}$ is a solution.

Now try Exercises 7 through 12 ►

Here is one additional example that uses a factoring strategy commonly employed when an equation has more than three terms.

EXAMPLE 2 ► **Solving a Trig Equation by Factoring**

Find all solutions in $[0°, 360°)$: $8\sin^2\theta\cos\theta - 2\cos\theta - 4\sin^2\theta + 1 = 0$.

Solution ► The four terms in the equation share no common factors, so we attempt to factor by grouping. We could factor $2\cos\theta$ from the first two terms but instead elect to group the $\sin^2\theta$ terms and begin there.

$$8\sin^2\theta\cos\theta - 2\cos\theta - 4\sin^2\theta + 1 = 0 \quad \text{given equation}$$
$$(8\sin^2\theta\cos\theta - 4\sin^2\theta) - (2\cos\theta - 1) = 0 \quad \text{rearrange and group terms}$$
$$4\sin^2\theta(2\cos\theta - 1) - 1(2\cos\theta - 1) = 0 \quad \text{remove common factors}$$
$$(2\cos\theta - 1)(4\sin^2\theta - 1) = 0 \quad \text{remove common binomial factors}$$

Using the zero factor property, we write two equations and solve each independently.

$$2\cos\theta - 1 = 0 \qquad 4\sin^2\theta - 1 = 0 \quad \text{resulting equations}$$
$$2\cos\theta = 1 \qquad \sin^2\theta = \frac{1}{4} \quad \text{isolate variable term}$$
$$\cos\theta = \frac{1}{2} \qquad \sin\theta = \pm\frac{1}{2} \quad \text{solve}$$

$\cos\theta > 0$ in QI and QIV
$\theta = 60°, 300°$

$\sin\theta > 0$ in QI and QII
$\sin\theta < 0$ in QIII and QIV
$\theta = 30°, 150°, 210°, 330°$ solutions

A. You've just learned how to use additional algebraic techniques to solve trig equations

Initially factoring $2\cos\theta$ from the first two terms and proceeding from there would have produced the same result.

Now try Exercises 13 through 16 ►

B. Solving Trig Equations Using Various Identities

To solve equations effectively, a student should strive to develop *all* of the necessary "tools." Certainly the underlying concepts and graphical connections are of primary importance, as are the related algebraic skills. But to solve *trig* equations effectively we must also have a ready command of commonly used identities. Observe how Example 3 combines a double-angle identity with factoring by grouping.

EXAMPLE 3 ► **Using Identities and Algebra to Solve a Trig Equation**

Find all solutions in $[0, 2\pi)$: $3\sin(2x) + 2\sin x - 3\cos x = 1$. Round solutions to four decimal places as necessary.

Solution ► Noting that one of the terms involves a double angle, we attempt to replace that term to make factoring a possibility. Using the double identity for sine, we have

$$3(2\sin x\cos x) + 2\sin x - 3\cos x = 1$$ substitute $2\sin x\cos x$ for $\sin(2x)$

$$(6\sin x\cos x + 2\sin x) - (3\cos x + 1) = 0$$ set equal zero and group terms

$$2\sin x(3\cos x + 1) - 1(3\cos x + 1) = 0$$ factor using $3\cos x + 1$

$$(3\cos x + 1)(2\sin x - 1) = 0$$ common binomial factor

Use the zero factor property to solve each equation independently.

$3\cos x + 1 = 0$	$2\sin x - 1 = 0$	resulting equations
$\cos x = -\frac{1}{3}$	$\sin x = \frac{1}{2}$	isolate variable term
$\cos x < 0$ in QII and QIII	$\sin x > 0$ in QI and QII	
$x \approx 1.9106, 4.3726$	$x = \frac{\pi}{6}, \frac{5\pi}{6}$	solutions

Should you prefer the exact form, the solutions from the cosine equation could be written as $x = \cos^{-1}\left(-\frac{1}{3}\right)$ and $x = 2\pi - \cos^{-1}\left(-\frac{1}{3}\right)$.

B. You've just learned how to solve trig equations using multiple angle, sum and difference, and sum-to-product identities

Now try Exercises 17 through 26 ►

C. Solving Equations of the Form $A\sin(Bx \pm C) \pm D = k$

You may remember equations of this form from Section 5.7. They actually occur quite frequently in the investigation of many natural phenomena and in the modeling of data from a periodic or seasonal context. Solving these equations requires a good combination of algebra skills with the fundamentals of trig.

EXAMPLE 4 ► Solving Equations That Involve Transformations

Given $f(x) = 160\sin\left(\frac{\pi}{3}x + \frac{\pi}{3}\right) + 320$ and $x \in [0, 2\pi)$, for what real numbers x is $f(x)$ less than 240?

Solution ► We reason that to find values where $f(x) < 240$, we should begin by finding values where $f(x) = 240$. The result is

$$160\sin\left(\frac{\pi}{3}x + \frac{\pi}{3}\right) + 320 = 240$$ equation

$$\sin\left(\frac{\pi}{3}x + \frac{\pi}{3}\right) = -0.5$$ subtract 320 and divide by 160; isolate variable term

At this point we elect to use a u-substitution for $\left(\frac{\pi}{3}x + \frac{\pi}{3}\right) = \frac{\pi}{3}(x + 1)$ to obtain a "clearer view."

$$\sin u = -0.5$$ substitute u for $\frac{\pi}{3}(x + 1)$

$\sin u < 0$ in QIII and QIV

$$u = \frac{7\pi}{6} \qquad u = \frac{11\pi}{6}$$ solutions in u

To complete the solution we re-substitute $\frac{\pi}{3}(x + 1)$ for u and solve.

$\frac{\pi}{3}(x + 1) = \frac{7\pi}{6}$	$\frac{\pi}{3}(x + 1) = \frac{11\pi}{6}$	re-substitute $\frac{\pi}{3}(x + 1)$ for u
$x + 1 = \frac{7}{2}$	$x + 1 = \frac{11}{2}$	multiply both sides by $\frac{3}{\pi}$
$x = 2.5$	$x = 4.5$	solutions

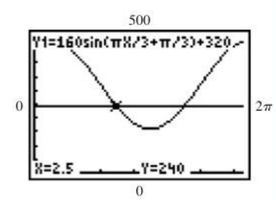
C. You've just learned how to solve trig equations of the form $A\sin(Bx + C) + D = k$

We now know $f(x) = 240$ when $x = 2.5$ and $x = 4.5$ but when will $f(x)$ be *less than* 240? By analyzing the equation, we find the function has period of $P = \frac{2\pi}{\frac{\pi}{3}} = 6$ and is shifted to the left $\frac{\pi}{3}$ units. This would indicate the graph peaks early in the interval $[0, 2\pi)$ with a "valley" in the interior. We conclude $f(x) < 240$ in the interval $(2.5, 4.5)$.

Now try Exercises 27 through 30 ►

GRAPHICAL SUPPORT

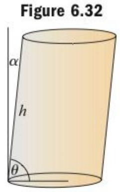
Support for the result in Example 4 can be obtained by graphing the equation over the specified interval. Enter $Y_1 = 160\sin\left(\frac{\pi}{3}x + \frac{\pi}{3}\right) + 320$ on the Y= screen, then $Y_2 = 240$. After locating points of intersection, we note the graphs indeed verify that in the interval $[0, 2\pi)$, $f(x) < 240$ for $x \in (2.5, 4.5)$.



There is a mixed variety of equation types in **Exercises 31 through 40.**

D. Applications Using Trigonometric Equations

Figure 6.32



Using characteristics of the trig functions, we can often generalize and extend many of the formulas that are familiar to you. For example, the formulas for the volume of a right circular cylinder and a right circular cone are well known, but what about the volume of a nonright figure (see Figure 6.32)? Here, trigonometry provides the answer, as the most general volume formula is $V = V_0 \sin\theta$, where V_0 is a "standard" volume formula and θ is the complement of angle of deflection **(see Exercises 43 and 44).**

As for other applications, consider the following from the environmental sciences. Natural scientists are very interested in the discharge rate of major rivers, as this gives an indication of rainfall over the inland area served by the river. In addition, the discharge rate has a large impact on the freshwater and saltwater zones found at the river's estuary (where it empties into the sea).

EXAMPLE 5 ► Solving an Equation Modeling the Discharge Rate of a River

For May through December, the discharge rate of the Ganges River (Bangladesh) can be modeled by $D(t) = 16{,}580\sin\left(\frac{\pi}{3}t - \frac{2\pi}{3}\right) + 17{,}760$ where $t = 1$ represents May 1, and $D(t)$ is the discharge rate in m^3/sec.

Source: Global River Discharge Database Project; www.rivdis.sr.unh.edu.

a. What is the discharge rate in mid-October?

b. For what months (within this interval) is the discharge rate over 26,050 m^3/sec?

Solution ► **a.** To find the discharge rate in mid-October we simply evaluate the function at $t = 6.5$:

$$D(t) = 16{,}580\sin\left(\frac{\pi}{3}t - \frac{2\pi}{3}\right) + 17{,}760 \quad \text{given function}$$

$$D(6.5) = 16{,}580\sin\left[\frac{\pi}{3}(6.5) - \frac{2\pi}{3}\right] + 17{,}760 \quad \text{substitute 6.5 for } t$$

$$= 1180 \quad \text{compute result on a calculator}$$

In mid-October the discharge rate is 1180 m^3/sec.

b. We first find when the rate is *equal* to 26,050 m^3/sec: $D(t) = 26{,}050$.

$$26{,}050 = 16{,}580\sin\left(\frac{\pi}{3}t - \frac{2\pi}{3}\right) + 17{,}760 \quad \text{substitute 26,050 for } D(t)$$

$$0.5 = \sin\left(\frac{\pi}{3}t - \frac{2\pi}{3}\right) \quad \text{subtract 17,760; divide by 16,580}$$

Using a u-substitution for $\left(\frac{\pi}{3}t - \frac{2\pi}{3}\right)$ we obtain the equation

$0.5 = \sin u$

$\sin u > 0$ in QI and QII

$$u = \frac{\pi}{6} \qquad u = \frac{5\pi}{6} \quad \text{solutions in } u$$

To complete the solution we re-substitute $\left(\frac{\pi}{3}t - \frac{2\pi}{3}\right) = \frac{\pi}{3}(t - 2)$ for u and solve.

$$\frac{\pi}{3}(t - 2) = \frac{\pi}{6} \qquad \frac{\pi}{3}(t - 2) = \frac{5\pi}{6} \quad \text{re-substitute } \frac{\pi}{3}(t-2) \text{ for } u$$

$$t - 2 = 0.5 \qquad t - 2 = 2.5 \quad \text{multiply both sides by } \frac{3}{\pi}$$

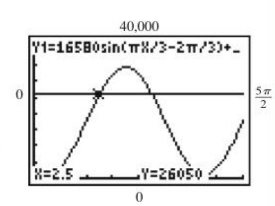
$$t = 2.5 \qquad t = 4.5 \quad \text{solutions}$$

The Ganges River will have a flow rate of over 26,050 m^3/sec between mid-June (2.5) and mid-August (4.5).

Now try Exercises 45 through 48 ►

GRAPHICAL SUPPORT

To obtain a graphical view of the solution to Example 5, enter $Y_1 = 16{,}580\sin\left(\frac{\pi}{3}t - \frac{2\pi}{3}\right) + 17{,}760$ on the Y= screen, then $Y_2 = 26{,}050$. To set an appropriate window, note the amplitude is 16,580 and that the graph has been vertically shifted by 17,760. Also note the x-axis represents months 5 through 12. After locating points of intersection, we note the graphs verify that in the interval [1, 9] $D(t) > 26{,}050$ for $t \in (2.5, 4.5)$.



D. You've just learned how to use a combination of skills to model and solve a variety of applications

There is a variety of additional exercises in the Exercise Set. **See Exercises 49 through 54.**

6.7 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The three Pythagorean identities are ___________, ___________, and ___________.

2. When an equation contains two functions from a Pythagorean identity, sometimes _______ both sides will lead to a solution.

3. One strategy to solve equations with four terms and no common factors is _______ by _______.

4. To combine two sine or cosine terms with different arguments, we can use the _______ to _______ formulas.

5. Regarding Example 5, discuss/explain how to determine the months of the year the discharge rate is *under* 26,050 m³/sec, using the solution set given.

6. Regarding Example 6, discuss/explain how to determine the months of the year the revenue projection is *under* $1250 using the solution set given.

DEVELOPING YOUR SKILLS

Solve each equation in $[0, 2\pi)$ using the method indicated. Round nonstandard values to four decimal places.

• Squaring both sides

7. $\sin x + \cos x = \frac{\sqrt{6}}{2}$

8. $\cot x - \csc x = \sqrt{3}$

9. $\tan x - \sec x = -1$

10. $\sin x + \cos x = \sqrt{2}$

11. $\cos x + \sin x = \frac{4}{3}$

12. $\sec x + \tan x = 2$

• Factor by grouping

13. $\cot x \csc x - 2\cot x - \csc x + 2 = 0$

14. $4\sin x\cos x - 2\sqrt{3}\sin x - 2\cos x + \sqrt{3} = 0$

15. $3\tan^2 x\cos x - 3\cos x + 2 = 2\tan^2 x$

16. $4\sqrt{3}\sin^2 x\sec x - \sqrt{3}\sec x + 2 = 8\sin^2 x$

• Using identities

17. $\frac{1 + \cot^2 x}{\cot^2 x} = 2$

18. $\frac{1 + \tan^2 x}{\tan^2 x} = \frac{4}{3}$

19. $3\cos(2x) + 7\sin x - 5 = 0$

20. $3\cos(2x) - \cos x + 1 = 0$

21. $2\sin^2\left(\frac{x}{2}\right) - 3\cos\left(\frac{x}{2}\right) = 0$

22. $2\cos^2\left(\frac{x}{3}\right) + 3\sin\left(\frac{x}{3}\right) - 3 = 0$

23. $\cos(3x) + \cos(5x)\cos(2x) + \sin(5x)\sin(2x) - 1 = 0$

24. $\sin(7x)\cos(4x) + \sin(5x) - \cos(7x)\sin(4x) + \cos x = 0$

25. $\sec^4 x - 2\sec^2 x\tan^2 x + \tan^4 x = \tan^2 x$

26. $\tan^4 x - 2\sec^2 x\tan^2 x + \sec^4 x = \cot^2 x$

State the period P of each function and find all solutions in $[0, P)$. Round to four decimal places as needed.

27. $250\sin\left(\frac{\pi}{6}x + \frac{\pi}{3}\right) - 125 = 0$

28. $-75\sqrt{2}\sec\left(\frac{\pi}{4}x + \frac{\pi}{6}\right) + 150 = 0$

29. $1235\cos\left(\frac{\pi}{12}x - \frac{\pi}{4}\right) + 772 = 1750$

30. $-0.075\sin\left(\frac{\pi}{2}x + \frac{\pi}{3}\right) - 0.023 = -0.068$

• Using any appropriate method to solve.

31. $\cos x - \sin x = \frac{\sqrt{2}}{2}$

32. $5\sec^2 x - 2\tan x - 8 = 0$

33. $\frac{1 - \cos^2 x}{\tan^2 x} = \frac{\sqrt{3}}{2}$

34. $5\csc^2 x - 5\cot x - 5 = 0$

35. $\csc x + \cot x = 1$

36. $\frac{1 - \sin^2 x}{\cot^2 x} = \frac{\sqrt{2}}{2}$

37. $\sec x \cos\left(\frac{\pi}{2} - x\right) = -1$

38. $\sin\left(\frac{\pi}{2} - x\right)\csc x = \sqrt{3}$

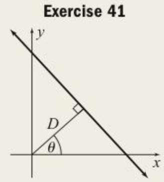
39. $\sec^2 x \tan\left(\frac{\pi}{2} - x\right) = 4$

40. $2\tan\left(\frac{\pi}{2} - x\right)\sin^2 x = \frac{\sqrt{3}}{2}$

► WORKING WITH FORMULAS

41. The equation of a line in trigonometric form:
$y = \frac{D - x\cos\theta}{\sin\theta}$

Exercise 41



The trigonometric form of a linear equation is given by the formula shown, where D is the perpendicular distance from the origin to the line and θ is the angle between the perpendicular segment and the x-axis. For each pair of perpendicular lines given, (a) find the point (a, b) of their intersection; (b) compute the distance $D = \sqrt{a^2 + b^2}$ and the angle $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, and give the equation of the line in trigonometric form; and (c) use the GRAPH or the 2nd GRAPH TABLE feature of a graphing calculator to verify that both equations name the same line.

I. L_1: $y = -x + 5$
L_2: $y = x$

II. L_1: $y = -\frac{1}{2}x + 5$
L_2: $y = 2x$

III. L_1: $y = -\frac{\sqrt{3}}{3}x + \frac{4\sqrt{3}}{3}$
L_2: $y = \sqrt{3}x$

42. Rewriting $y = a\cos x + b\sin x$ as a single function: $y = k\sin(x + \theta)$

Linear terms of sine and cosine can be rewritten as a single function using the formula shown, where $k = \sqrt{a^2 + b^2}$ and $\theta = \sin^{-1}\left(\frac{a}{k}\right)$. Rewrite the equations given using these relationships and verify they are equivalent using the GRAPH or the 2nd GRAPH TABLE feature of a graphing calculator:

a. $y = 2\cos x + 2\sqrt{3}\sin x$

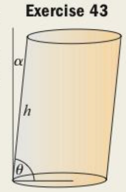
b. $y = 4\cos x + 3\sin x$

The ability to rewrite a trigonometric equation in simpler form has a tremendous number of applications in graphing, equation solving, working with identities, and solving applications.

► APPLICATIONS

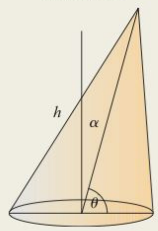
43. Volume of a cylinder: The volume of a cylinder is given by the formula $V = \pi r^2 h \sin\theta$, where r is the radius and h is the height of the cylinder, and θ is the indicated complement of the angle of deflection α. Note that when $\theta = \frac{\pi}{2}$, the formula becomes that of a right circular cylinder (if $\theta \neq \frac{\pi}{2}$, then h is called the *slant height or lateral height* of the cylinder). An old farm silo is built in the form of a right circular cylinder with a radius of 10 ft and a height of 25 ft. After an earthquake, the silo became tilted with an angle of deflection $\alpha = 5°$. (a) Find the volume of the silo before the earthquake. (b) Find the volume of the silo after the earthquake. (c) What angle θ is required to bring the original volume of the silo down 2%?

Exercise 43



44. Volume of a cone: The volume of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h \sin\theta$, where r is the radius and h is the height of the cone, and θ is the indicated complement of the angle of deflection α. Note that when $\theta = \frac{\pi}{2}$, the formula becomes that of a right circular cone (if $\theta \neq \frac{\pi}{2}$, then h is called the *slant height or lateral height* of the cone). As part of a sculpture exhibit, an artist is constructing three such structures each with a radius of 2 m and a slant height of 3 m. (a) Find the volume of the sculptures if the angle of deflection is $\alpha = 15°$. (b) What angle θ was used if the volume of each sculpture is 12 m^3?

Exercise 44



45. River discharge rate: For June through February, the discharge rate of the La Corcovada River (Venezuela) can be modeled by the function $D(t) = 36\sin\left(\frac{\pi}{4}t - \frac{9}{4}\right) + 44$, where t represents the months of the year with $t = 1$ corresponding to June, and D(t) is the discharge rate in cubic meters per second. (a) What is the discharge rate in mid-September? (b) For what months of the year is the discharge rate over 50 m^3/sec?
Source: Global River Discharge Database Project; www.rivdis.sr.unh.edu.

46. River discharge rate: For February through June, the average monthly discharge of the Point Wolfe River (Canada) can be modeled by the function $D(t) = 4.6\sin\left(\frac{\pi}{2}t + 3\right) + 7.4$, where t represents the months of the year with $t = 1$ corresponding to February, and D(t) is the discharge rate in cubic meters/second. (a) What is the discharge rate in mid-March ($t = 2.5$)? (b) For what months of the year is the discharge rate less than 7.5 m^3/sec?
Source: Global River Discharge Database Project; www.rivdis.sr.unh.edu.

47. Seasonal sales: Hank's Heating Oil is a very seasonal enterprise, with sales in the winter far exceeding sales in the summer. Monthly sales for the company can be modeled by $S(x) = 1600\cos\left(\frac{\pi}{6}x - \frac{\pi}{12}\right) + 5100$, where $S(x)$ is the average sales in month x ($x = 1 \rightarrow$ January). (a) What is the average sales amount for July? (b) For what months of the year are sales less than \$4000?

48. Seasonal income: As a roofing company employee, Mark's income fluctuates with the seasons and the availability of work. For the past several years his average monthly income could be approximated by the function $I(m) = 2100\sin\left(\frac{\pi}{6}m - \frac{\pi}{2}\right) + 3520$, where $I(m)$ represents income in month m ($m = 1 \rightarrow$ January). (a) What is Mark's average monthly income in October? (b) For what months of the year is his average monthly income over \$4500?

49. Seasonal ice thickness: The average thickness of the ice covering an arctic lake can be modeled by the function $T(x) = 9\cos\left(\frac{\pi}{6}x\right) + 15$, where $T(x)$ is the average thickness in month x ($x = 1 \rightarrow$ January). (a) How thick is the ice in mid-March? (b) For what months of the year is the ice at most 10.5 in. thick?

50. Seasonal temperatures: The function $T(x) = 19\sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 53$ models the average monthly temperature of the water in a mountain stream, where $T(x)$ is the temperature (°F) of the water in month x ($x = 1 \rightarrow$ January). (a) What is the temperature of the water in October? (b) What two months are most likely to give a temperature reading of 62°F? (c) For what months of the year is the temperature below 50°F?

51. Coffee sales: Coffee sales fluctuate with the weather, with a great deal more coffee sold in the winter than in the summer. For Joe's Diner, assume the function $G(x) = 21\cos\left(\frac{2\pi}{365}x + \frac{\pi}{2}\right) + 29$ models daily coffee sales (for non-leap years), where $G(x)$ is the number of gallons sold and x represents the days of the year ($x = 1 \rightarrow$ January 1). (a) How many gallons are projected to be sold on March 21? (b) For what days of the year are more than 40 gal of coffee sold?

52. Park attendance: Attendance at a popular state park varies with the weather, with a great deal more visitors coming in during the summer months. Assume daily attendance at the park can be modeled by the function $V(x) = 437\cos\left(\frac{2\pi}{365}x - \pi\right) + 545$ (for non-leap years), where $V(x)$ gives the number of visitors on day x ($x = 1 \rightarrow$ January 1). (a) Approximately how many people visited the park on November 1 ($11 \times 30.5 = 335.5$)? (b) For what days of the year are there more than 900 visitors?

53. **Exercise routine:** As part of his yearly physical, Manu Tuiosamoa's heart rate is closely monitored during a 12-min, cardiovascular exercise routine. His heart rate in beats per minute (bpm) is modeled by the function $B(x) = 58\cos\left(\frac{\pi}{6}x + \pi\right) + 126$ where x represents the duration of the workout in minutes. (a) What was his resting heart rate? (b) What was his heart rate 5 min into the workout? (c) At what times during the workout was his heart rate over 170 bpm?

54. **Exercise routine:** As part of her workout routine, Sara Lee programs her treadmill to begin at a slight initial grade (angle of incline), gradually increase to a maximum grade, then gradually decrease back to the original grade. For the duration of her workout, the grade is modeled by the function $G(x) = 3\cos\left(\frac{\pi}{5}x - \pi\right) + 4$, where $G(x)$ is the percent grade x minutes after the workout has begun. (a) What is the initial grade for her workout? (b) What is the grade at $x = 4$ min? (c) At $G(x) = 4.9\%$, how long has she been working out? (d) What is the duration of the treadmill workout?

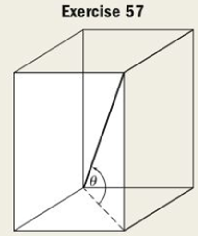
► EXTENDING THE CONCEPT

55. As we saw in Chapter 6, cosine is the cofunction of sine and each can be expressed in terms of the other: $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$. This implies that either function can be used to model the phenomenon described in this section by adjusting the phase shift. By experimentation, (a) find a model using cosine that will produce results identical to the sine function in Exercise 50 and (b) find a model using sine that will produce results identical to the cosine function in Exercise 51.

56. Use multiple identities to find all real solutions for the equation given: $\sin(5x) + \sin(2x)\cos x + \cos(2x)\sin x = 0$.

57. A rectangular parallelepiped with square ends has 12 edges and six surfaces. If the sum of all edges is 176 cm and the total surface area is 1288 cm^2, find (a) the length of the diagonal of the parallelepiped (shown in bold) and (b) the angle the diagonal makes with the base (two answers are possible).

Exercise 57



► MAINTAINING YOUR SKILLS

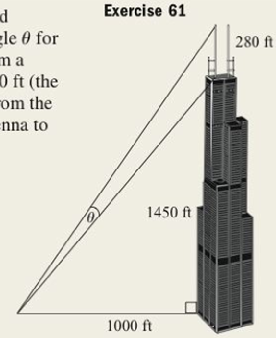
58. **(5.7)** Find $f(\theta)$ for all six trig functions, given $P(-51, 68)$ is on the terminal side.

59. **(3.4)** Sketch the graph of f by locating its zeroes and using end behavior: $f(x) = x^4 - 3x^3 + 4x$.

60. **(4.3)** Use a calculator and the change-of-base formula to find the value of $\log_5 279$.

61. **(5.6)** The Sears Tower in Chicago, Illinois, remains one of the tallest structures in the world. The top of the roof reaches 1450 ft above the street below and the antenna extends an additional 280 ft into the air. Find the viewing angle θ for the antenna from a distance of 1000 ft (the angle formed from the base of the antenna to its top).

Exercise 61



SUMMARY AND CONCEPT REVIEW

SECTION 6.1 Fundamental Identities and Families of Identities

KEY CONCEPTS

- The fundamental identities include the *reciprocal, ratio, and Pythagorean identities.*
- A given identity can algebraically be rewritten to obtain other identities in an identity "family."
- Standard algebraic skills like distribution, factoring, combining terms, and special products play an important role in working with identities.
- The pattern $\frac{A}{B} \pm \frac{C}{D} = \frac{AD \pm BC}{BD}$ gives an efficient method for combining rational terms.
- Using fundamental identities, a given trig function can be expressed in terms of any other trig function.
- Once the value of a given trig function is known, the value of the other five can be uniquely determined using fundamental identities, *if the quadrant of the terminal side is known*.
- To show an equation is not an identity, find any one value where the expressions are defined but the equation is false, or graph both functions on a calculator to see if the graphs are identical.

EXERCISES

Verify using the method specified and fundamental identities.

1. multiplication
$\sin x(\csc x - \sin x) = \cos^2 x$

2. factoring
$\frac{\tan^2 x \csc x + \csc x}{\sec^2 x} = \csc x$

3. special products
$\frac{(\sec x - \tan x)(\sec x + \tan x)}{\csc x} = \sin x$

4. combine terms using
$\frac{A}{B} \pm \frac{C}{D} = \frac{AD \pm BC}{BD}$
$\frac{\sec^2 x}{\csc x} - \sin x = \frac{\tan^2 x}{\csc x}$

Find the value of all six trigonometric functions using the information given.

5. $\cos\theta = -\frac{12}{37}$; θ in QIII

6. $\sec\theta = \frac{25}{23}$; θ in QIV

SECTION 6.2 Constructing and Verifying Identities

KEY CONCEPTS

- The steps used to verify an identity must be reversible.
- If two expressions are equal, one may be substituted for the other and the result will be equivalent.
- To verify an identity we mold, change, substitute, and rewrite one side until we "match" the other side.
- Verifying identities often involves a combination of algebraic skills with the fundamental trig identities.

A collection and summary of the *Guidelines for Verifying Identities* can be found on page 625.

EXERCISES

Rewrite each expression to create a new identity, then verify the identity by reversing the steps.

7. $\csc x + \cot x$

8. $\frac{\cos x - \sin x \cos x}{\cos^2 x}$

Verify that each equation is an identity.

9. $\frac{\csc^2 x(1 - \cos^2 x)}{\tan^2 x} = \cot^2 x$

10. $\frac{\cot x}{\sec x} - \frac{\csc x}{\tan x} = \cot x(\cos x - \csc x)$

11. $\dfrac{\sin^4 x - \cos^4 x}{\sin x \cos x} = \tan x - \cot x$

12. $\dfrac{(\sin x + \cos x)^2}{\sin x \cos x} = \csc x \sec x + 2$

SECTION 6.3 The Sum and Difference Identities

KEY CONCEPTS

The sum and difference identities can be used to

- Find exact values for nonstandard angles that are a sum or difference of two standard angles.
- Verify the cofunction identities and to rewrite a given function in terms of its cofunction.
- Find coterminal angles in $[0, 360°)$ for very large angles (the angle reduction formulas).
- Evaluate the difference quotient for $\sin x$, $\cos x$, and $\tan x$.
- Rewrite a sum as a single expression: $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$.

The sum and difference identities for sine and cosine can be remembered by noting

- For $\cos(\alpha \pm \beta)$, the function repeats and the signs alternate: $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- For $\sin(\alpha \pm \beta)$ the signs repeat and the functions alternate: $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

EXERCISES

Find exact values for the following expressions using sum and difference formulas.

13. a. $\cos 75°$ **b.** $\tan\left(\dfrac{\pi}{12}\right)$

14. a. $\tan 15°$ **b.** $\sin\left(-\dfrac{\pi}{12}\right)$

Evaluate exactly using sum and difference formulas.

15. a. $\cos 109°\cos 71° - \sin 109°\sin 71°$ **b.** $\sin 139°\cos 19° - \cos 139°\sin 19°$

Rewrite as a single expression using sum and difference formulas.

16. a. $\cos(3x)\cos(-2x) - \sin(3x)\sin(-2x)$ **b.** $\sin\left(\dfrac{x}{4}\right)\cos\left(\dfrac{3x}{8}\right) + \cos\left(\dfrac{x}{4}\right)\sin\left(\dfrac{3x}{8}\right)$

Evaluate exactly using sum and difference formulas, by reducing the angle to an angle in $[0, 360°)$ or $[0, 2\pi)$.

17. a. $\cos 1170°$ **b.** $\sin\left(\dfrac{57\pi}{4}\right)$

Use a cofunction identity to write an equivalent expression for the one given.

18. a. $\cos\left(\dfrac{x}{8}\right)$ **b.** $\sin\left(x - \dfrac{\pi}{12}\right)$

19. Verify that both expressions yield the same result using sum and difference formulas. $\tan 15° = \tan(45° - 30°)$ and $\tan 15° = \tan(135° - 120°)$.

20. Use sum and difference formulas to verify the following identity.
$\cos\left(x + \dfrac{\pi}{6}\right) + \cos\left(x - \dfrac{\pi}{6}\right) = \sqrt{3}\cos x$

SECTION 6.4 The Double-Angle, Half-Angle, and Product-to-Sum Identities

KEY CONCEPTS

- When multiple angle identities (identities involving $n\theta$) are used to find exact values, the terminal side of θ must be determined so the appropriate sign can be used.
- The power reduction identities for $\cos^2 x$ and $\sin^2 x$ are closely related to the double-angle identities, and can be derived directly from $\cos(2x) = 2\cos^2 x - 1$ and $\cos(2x) = 1 - 2\sin^2 x$.

- The half-angle identities can be developed from the power reduction identities by using a change of variable and taking square roots. The sign is then chosen based on the quadrant of the half angle.
- The product-to-sum and sum-to-product identities can be derived using the sum and difference formulas, and have important applications in many areas of science.

EXERCISES

Find exact values for $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ using the information given.

21. a. $\cos\theta = \frac{13}{85}$; θ in QIV **b.** $\csc\theta = -\frac{29}{20}$; θ in QIII

Find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ using the information given.

22. a. $\cos(2\theta) = -\frac{41}{841}$; θ in QII **b.** $\sin(2\theta) = -\frac{336}{625}$; θ in QII

Find exact values using the appropriate double-angle identity.

23. a. $\cos^2 22.5° - \sin^2 22.5°$ **b.** $1 - 2\sin^2\left(\frac{\pi}{12}\right)$

Find exact values for $\sin\theta$ and $\cos\theta$ using the appropriate half-angle identity.

24. a. $\theta = 67.5°$ **b.** $\theta = \frac{5\pi}{8}$

Find exact values for $\sin\left(\frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2}\right)$ using the given information.

25. a. $\cos\theta = \frac{24}{25}$; $0° < \theta < 360°$; θ in QIV **b.** $\csc\theta = -\frac{65}{33}$; $-90° < \theta < 0$; θ in QIV

26. Verify the equation is an identity.
$$\frac{\cos(3\alpha) - \cos\alpha}{\cos(3\alpha) + \cos\alpha} = \frac{2\tan^2\alpha}{\sec^2\alpha - 2}$$

27. Solve using a sum-to-product formula.
$\cos(3x) + \cos x = 0$

28. The area of an isosceles triangle (two equal sides) is given by the formula $A = x^2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$, where the equal sides have length x and the vertex angle measures $\theta°$. (a) Use this formula and the half-angle identities to find the area of an isosceles triangle with vertex angle $\theta = 30°$ and equal sides of 12 cm. (b) Use substitution and a double-angle identity to verify that $x^2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{1}{2}x^2\sin\theta$, then recompute the triangle's area. Do the results match?

SECTION 6.5 The Inverse Trig Functions and Their Applications

KEY CONCEPTS

- In order to create one-to-one functions, the domains of $y = \sin t$, $y = \cos t$, and $y = \tan t$ are restricted as follows: (a) $y = \sin t, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; (b) $y = \cos t, t \in [0, \pi]$; and (c) $y = \tan t; t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- For $y = \sin x$, the inverse function is given implicitly as $x = \sin y$ and explicitly as $y = \sin^{-1}x$ or $y = \arcsin x$.
- The expression $y = \sin^{-1}x$ is read, "y is the angle or real number whose sine is x." The other inverse functions are similarly read/understood.
- For $y = \cos x$, the inverse function is given implicitly as $x = \cos y$ and explicitly as $y = \cos^{-1}x$ or $y = \arccos x$.
- For $y = \tan x$, the inverse function is given implicitly as $x = \tan y$ and explicitly as $y = \tan^{-1}x$ or $y = \arctan x$.

- The domains of $y = \sec x$, $y = \csc x$, and $y = \cot x$ are likewise restricted to create one-to-one functions: (a) $y = \sec t$; $t \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$; (b) $y = \csc t$, $t \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$; and (c) $y = \cot t$, $t \in (0, \pi)$.
- In some applications, inverse functions occur in a composition with other trig functions, with the expression best evaluated by drawing a diagram using the ratio definition of the trig functions.
- To evaluate $y = \sec^{-1}t$, we use $y = \cos^{-1}\left(\frac{1}{t}\right)$; for $y = \cot^{-1}t$, use $\tan^{-1}\left(\frac{1}{t}\right)$; and so on.
- Trigonometric substitutions can be used to simplify certain algebraic expressions.

EXERCISES

Evaluate without the aid of calculators or tables. State answers in both radians and degrees in exact form.

29. $y = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ **30.** $y = \csc^{-1}2$ **31.** $y = \arccos\left(-\frac{\sqrt{3}}{2}\right)$

Evaluate the following using a calculator, *keeping the domain and range of each function in mind.* Answer in radians to the nearest ten-thousandth *and* in degrees to the nearest tenth. Some may be undefined.

32. $y = \tan^{-1}4.3165$ **33.** $y = \sin^{-1}0.8892$ **34.** $f(x) = \arccos\left(\frac{7}{8}\right)$

Evaluate the following without the aid of a calculator. Some may be undefined.

35. $\sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$ **36.** $\operatorname{arcsec}\left[\sec\left(\frac{\pi}{4}\right)\right]$ **37.** $\cos(\cos^{-1}2)$

Evaluate the following using a calculator. Some may be undefined.

38. $\sin^{-1}(\sin 1.0245)$ **39.** $\arccos[\cos(-60°)]$ **40.** $\cot^{-1}\left[\cot\left(\frac{11\pi}{4}\right)\right]$

Evaluate each expression by drawing a right triangle and labeling the sides.

41. $\sin\left[\cos^{-1}\left(\frac{12}{37}\right)\right]$ **42.** $\tan\left[\operatorname{arcsec}\left(\frac{7}{3x}\right)\right]$ **43.** $\cot\left[\sin^{-1}\left(\frac{x}{\sqrt{81 + x^2}}\right)\right]$

Use an inverse function to solve the following equations for θ in terms of x.

44. $x = 5\cos\theta$ **45.** $7\sqrt{3}\sec\theta = x$ **46.** $x = 4\sin\left(\theta - \frac{\pi}{6}\right)$

SECTION 6.6 Solving Basic Trig Equations

KEY CONCEPTS

- When solving trig equations, we often consider either the principal root, roots in $[0, 2\pi)$, or all real roots.
- Keeping the graph of each function in mind helps to determine the desired solution set.
- After isolating the trigonometric term containing the variable, we solve by applying the appropriate inverse function, realizing the result is only the principal root.
- Once the principal root is found, roots in $[0, 2\pi)$ or all real roots can be found using reference angles and the period of the function under consideration.
- Trig identities can be used to obtain an equation that can be solved by factoring or other solution methods.

EXERCISES

Solve each equation without the aid of a calculator (all solutions are standard values). Clearly state (a) the principal root; (b) all solutions in the interval $[0, 2\pi)$; and (c) all real roots.

47. $2\sin x = \sqrt{2}$ **48.** $3\sec x = -6$ **49.** $8\tan x + 7\sqrt{3} = -\sqrt{3}$

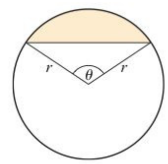
Solve using a calculator and the inverse trig functions (not by graphing). Clearly state (a) the principal root; (b) solutions in $[0, 2\pi)$; and (c) all real roots. Answer in radians to the nearest ten-thousandth as needed.

50. $9 \cos x = 4$

51. $\frac{2}{5}\sin(2\theta) = \frac{1}{4}$

52. $\sqrt{2} \csc x + 3 = 7$

53. The area of a circular segment (the shaded portion shown in the diagram) is given by the formula $A = \frac{1}{2}r^2(\theta - \sin\theta)$, where θ is in radians. If the circle has a radius of 10 cm, find the angle θ that gives an area of 12 cm^2.



SECTION 6.7 General Trig Equations and Applications

KEY CONCEPTS

- In addition to the basic solution methods from Section 6.6, additional strategies include squaring both sides, factoring by grouping, and using the full range of identities to simplify an equation.
- Many applications result in equations of the form $A\sin(Bx + C) + D = k$. To solve, isolate the factor $\sin(Bx + C)$ (subtract D and divide by A), then apply the inverse function.
- Once the principal root is found, roots in $[0, 2\pi)$ or all real roots can be found using reference angles and the period of the function under consideration.

EXERCISES

Find solutions in $[0, 2\pi)$ using the method indicated. Round nonstandard values to four decimal places.

54. squaring both sides

$\sin x + \cos x = \frac{\sqrt{6}}{2}$

55. using identities

$3\cos(2x) + 7\sin x - 5 = 0$

56. factor by grouping

$4\sin x\cos x - 2\sqrt{3}\sin x - 2\cos x + \sqrt{3} = 0$

57. using any appropriate method

$\csc x + \cot x = 1$

State the period P of each function and find all solutions in $[0, P)$. Round to four decimal places as needed.

58. $-750\sin\left(\frac{\pi}{6}x + \frac{\pi}{2}\right) + 120 = 0$

59. $80\cos\left(\frac{\pi}{3}x + \frac{\pi}{4}\right) - 40\sqrt{2} = 0$

60. The revenue earned by Waipahu Joe's Tanning Lotions fluctuates with the seasons, with a great deal more lotion sold in the summer than in the winter. The function $R(x) = 15\sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 30$ models the monthly sales of lotion nationwide, where $R(x)$ is the revenue in thousands of dollars and x represents the months of the year ($x = 1 \rightarrow$ Jan). (a) How much revenue is projected for July? (b) For what months of the year does revenue exceed \$37,000?

MIXED REVIEW

Find the value of all six trig functions using the information given.

1. $\csc\theta = \frac{\sqrt{117}}{6}$; θ in QII **2.** $\tan^{-1}\left(\frac{4}{3}\right) = \theta$

Find the exact value of each expression using a sum or difference identity.

3. $\tan 255°$ **4.** $\cos\left(\frac{19\pi}{12}\right)$

Evaluate each expression by drawing a right triangle and labeling the sides appropriately.

5. $\tan\left[\operatorname{arccsc}\left(\frac{10}{x}\right)\right]$ **6.** $\sin\left[\sec^{-1}\left(\frac{\sqrt{64 + x^2}}{x}\right)\right]$

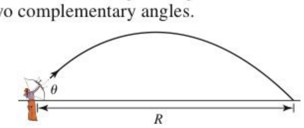
7. Solve for x in the interval $[0, 2\pi)$. Round to four decimal places as needed:

$-100\sin\left(\frac{\pi}{4}x - \frac{\pi}{6}\right) + 80 = 100$

8. The horizontal distance R that an object will travel when it is projected at angle θ with initial velocity v is given by the equation $R = \frac{1}{16}v^2 \sin\theta\cos\theta$.

a. Use an identity to show this equation can be written as $R = \frac{1}{32}v^2 \sin(2\theta)$.

b. Use this equation to show why the horizontal distance traveled by the object is the same for any two complementary angles.



9. The profits of Red-Bud Nursery can be modeled by a sinusoid, with profit peaking twice each year. Given profits reach a yearly low of \$4000 in mid-January (month 1.5), and a yearly high of \$14,000 in mid-April (month 4.5). (a) Construct an equation for their yearly profits. (b) Use the model to find their profits for August. (c) Name the other month at which profit peaks.

10. Find the exact value of $2\cos^2\left(\frac{\pi}{12}\right) - 1$ using an appropriate identity.

Verify the following identity.

11. $\dfrac{1 - \cos^2\theta + \sin^2\theta}{\tan^2\theta} = 1 + \cos(2\theta)$

12. Find exact values for $\sin\left(\frac{x}{2}\right)$ and $\cos\left(\frac{x}{2}\right)$ using the information given.

a. $\sin x = \frac{-6}{7.5}$; $540° < x < 630°$

b. $\sec x = \frac{11.7}{4.5}$; $0 < x < \frac{\pi}{2}$

Evaluate without the aid of calculator or tables. Answer in both radians and degrees.

13. $y = \text{arcsec}(-\sqrt{2})$ **14.** $y = \tan^{-1}\sqrt{3}$

15. Verify the following identities *using a sum formula.*

a. $\sin(2x) = 2\sin x\cos x$

b. $\cos(2x) = \cos^2 x - \sin^2 x$

Use an inverse function to solve each equation for θ in terms of x.

16. $\frac{x}{10} = \tan\theta$ **17.** $2\sqrt{2}\csc\left(\theta - \frac{\pi}{4}\right) = x$

18. Find the value of each expression using sum-to-product and half-angle identities (without using a calculator).

a. $\sin 172.5° - \sin 52.5°$

b. $\cos 172.5° + \cos 52.5°$

19. Given $100\sin t = 70$, use a calculator to find (a) the principal root, (b) all solutions in $[0, 2\pi]$, and (c) all real solutions. Round to the nearest ten-thousandth.

20. Use the product-to-sum formulas to find the exact value of

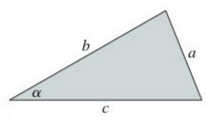
a. $\sin\left(\frac{13\pi}{24}\right)\cos\left(\frac{7\pi}{24}\right)$ **b.** $\sin\left(\frac{13\pi}{24}\right)\sin\left(\frac{7\pi}{24}\right)$

PRACTICE TEST

Verify each identity using fundamental identities and the method specified.

1. special products $\dfrac{(\csc x - \cot x)(\csc x + \cot x)}{\sec x} = \cos x$
2. factoring $\dfrac{\sin^3 x - \cos^3 x}{1 + \cos x \sin x} = \sin x - \cos x$
3. Find the value of all six trigonometric functions given $\cos\theta = \frac{48}{73}$; θ in QIV
4. Find the exact value of tan 15° using a sum or difference formula.
5. Rewrite as a single expression and evaluate: $\cos 81° \cos 36° + \sin 81° \sin 36°$
6. Evaluate cos 1935° exactly using an angle reduction formula.
7. Use sum and difference formulas to verify $\sin\left(x + \frac{\pi}{4}\right) - \sin\left(x - \frac{\pi}{4}\right) = \sqrt{2}\cos x$.
8. Find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ given $\cos(2\theta) = -\frac{161}{289}$; θ in QI
9. Use a double-angle identity to evaluate $2\cos^2 75° - 1$.
10. Find exact values for $\sin\left(\frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2}\right)$ given $\tan\theta = \frac{12}{35}$; θ in QI

11. The area of a triangle is given geometrically as $A = \frac{1}{2}\text{base} \cdot \text{height}$. The trigonometric formula for the triangle's area is $A = \frac{1}{2}bc\sin\alpha$, where α is the angle formed by the sides b and c. In a certain triangle, $b = 8$, $c = 10$, and $\alpha = 22.5°$. Use the formula for A given here and a half-angle identity to find the area of the triangle in exact form.



12. The equation $Ax^2 + Bxy + Cy^2 = 0$ can be written in an alternative form that makes it easier to graph. This is done by eliminating the mixed xy-term using the relation $\tan(2\theta) = \frac{B}{A - C}$ to find θ. We can then find values for $\sin\theta$ and $\cos\theta$, which are used in a conversion formula. Find $\sin\theta$ and $\cos\theta$ for $17x^2 + 5\sqrt{3}xy + 2y^2 = 0$, assuming 2θ in QI.

13. Evaluate without the aid of calculators or tables.

a. $y = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **b.** $y = \sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$

c. $y = \arccos(\cos 30°)$

14. Evaluate the following. Use a calculator for part (a), give exact answers for part (b), and find the value of the expression in part (c) without using a calculator. Some may be undefined.

a. $y = \sin^{-1}0.7528$ **b.** $y = \arctan(\tan 78.5°)$

c. $y = \sec^{-1}\left[\sec\left(\frac{7\pi}{24}\right)\right]$

Evaluate the expressions by drawing a right triangle and labeling the sides.

15. $\cos\left[\tan^{-1}\left(\frac{56}{33}\right)\right]$

16. $\cot\left[\cos^{-1}\left(\frac{x}{\sqrt{25 + x^2}}\right)\right]$

17. Solve without the aid of a calculator (all solutions are standard values). Clearly state (a) the principal root, (b) all solutions in the interval $[0, 2\pi)$, and (c) all real roots.

I. $8\cos x = -4\sqrt{2}$ **II.** $\sqrt{3}\sec x + 2 = 4$

18. Solve each equation using a calculator and inverse trig functions to find the principal root (not by graphing). Then state (a) the principal root, (b) all solutions in the interval $[0, 2\pi)$, and (c) all real roots.

I. $\frac{2}{3}\sin(2x) = \frac{1}{4}$ **II.** $-3\cos(2x) - 0.8 = 0$

19. Solve the equations graphically in the indicated interval using a graphing calculator. State answers in radians rounded to the nearest ten-thousandth.

a. $3\cos(2x - 1) = \sin x;\ x \in [-\pi, \pi]$

b. $2\sqrt{x} - 1 = 3\cos^2 x;\ x \in [0, 2\pi)$

20. Solve the following equations for $x \in [0, 2\pi)$ using a combination of identities and/or factoring. State solutions in radians using the exact form where possible.

a. $2\sin x\sin(2x) + \sin(2x) = 0$

b. $(\cos x + \sin x)^2 = \frac{1}{2}$

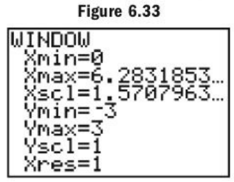
CALCULATOR EXPLORATION AND DISCOVERY

Seeing the Beats as the Beats Go On

When two sound waves of slightly different frequencies are combined, the resultant wave varies periodically in amplitude over time. These amplitude pulsations are called **beats.** In this *Exploration and Discovery,* we'll look at ways to "see" the beats more clearly on a graphing calculator, by representing sound waves very simplistically as $Y_1 = \cos(mt)$ and $Y_2 = \cos(nt)$ and noting a relationship between m, n, and the number of beats in $[0, 2\pi]$. Using a sum-to-product formula, we can represent the resultant wave as a single term. For $Y_1 = \cos(12t)$ and $Y_2 = \cos(8t)$ the result is

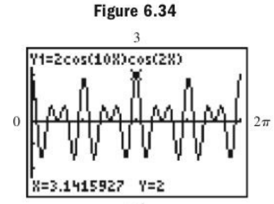
$$\cos(12t) + \cos(8t) = 2\cos\left(\frac{12t + 8t}{2}\right)\cos\left(\frac{12t - 8t}{2}\right)$$
$$= 2\cos(10t)\cos(2t)$$

Figure 6.33



The window used and resulting graph are shown in Figures 6.33 and 6.34, and it appears that "silence" occurs four times in this interval—where the graph of the combined waves is

Figure 6.34



tangent to (bounces off of) the x-axis. This indicates a total of four beats. Note the number of beats is equal to the difference $m - n$: $12 - 8 = 4$. Further experimentation will show this is not a coincidence, and this enables us to construct two additional functions that will *frame these pulsations* and make them easier to see. Since the maximum amplitude of the resulting wave is 2, we use functions of the form $\pm 2\cos\left(\frac{k}{2}x\right)$ to construct the frame, where k is the number of beats in the interval $(m - n = k)$. For $Y_1 = \cos(12t)$ and $Y_2 = \cos(8t)$, we have $k = \frac{12 - 8}{2} = 2$ and the functions we use will be $Y_2 = 2\cos(2x)$ and $Y_3 = -2\cos(2x)$ as shown in Figure 6.35. The result is shown in Figure 6.36, where the frame clearly shows the four beats or more precisely, the four moments of silence.

Figure 6.35

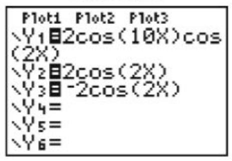


Figure 6.36

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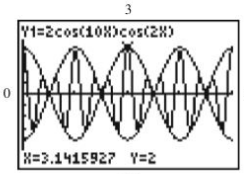
0  2π

−3

For each exercise, (a) express the sum $Y_1 + Y_2$ as a product, (b) graph Y_R on a graphing calculator for $x \in [0, 2\pi]$ and identify the number of beats in this interval, and (c) determine what value of k in $\pm 2\cos\left(\frac{k}{2}x\right)$ would be used to frame the resultant Y_R, then enter these as Y_2 and Y_3 to check the result.

Exercise 1: $Y_1 = \cos(14t)$; $Y_2 = \cos(8t)$

Exercise 2: $Y_1 = \cos(12t)$; $Y_2 = \cos(9t)$

Exercise 3: $Y_1 = \cos(14t)$; $Y_2 = \cos(6t)$

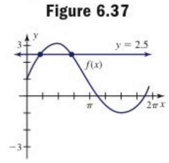
Exercise 4: $Y_1 = \cos(11t)$; $Y_2 = \cos(10t)$

STRENGTHENING CORE SKILLS

Trigonometric Equations and Inequalities

The ability to draw a quick graph of the trigonometric functions is a tremendous help in understanding equations and inequalities. A basic sketch can help reveal the number of solutions in $[0, 2\pi)$ and the quadrant of each solution. For nonstandard angles, the value given by the inverse function can then be used as a basis for stating the solution set for all real numbers. We'll illustrate the process using a few simple examples, then generalize our observations to solve more realistic applications. Consider the function $f(x) = 2\sin x + 1$, a sine wave with amplitude 2, and a vertical translation of $+1$. To find intervals in $[0, 2\pi)$ where $f(x) > 2.5$, we reason that f has a maximum of $3 = 2(1) + 1$ and a minimum of $-1 = 2(-1) + 1$, since $-1 \le \sin x \le 1$. With no phase shift and a standard period of 2π, we can easily draw a quick sketch of f by vertically translating x-intercepts and max/min points 1 unit up. After drawing the line $y = 2.5$ (see Figure 6.37), it appears there are two intersections in the interval, one in QI and one in QII. More importantly, it is clear that $f(x) > 2.5$ *between these two solutions*. Substituting 2.5 for $f(x)$ in $f(x) = 2\sin x + 1$, we solve for $\sin x$ to obtain $\sin x = 0.75$, which we use to state the solution in exact form: $f(x) > 2.5$ for $x \in (\sin^{-1}0.75, \pi - \sin^{-1}0.75)$. In approximate form the solution interval is $x \in (0.85, 2.29)$.

Figure 6.37

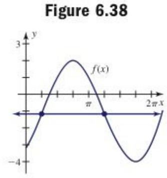


If the function involves a horizontal shift, the graphical analysis will reveal which intervals should be chosen to satisfy the given inequality.

Illustration 1 ► Given $g(x) = 3\sin\left(x - \frac{\pi}{4}\right) - 1$, solve $g(x) \le -1.2$ for $x \in [0, 2\pi)$.

Solution ► Plot the x-intercepts and maximum/minimum values for a standard sine wave with amplitude 3, then shift these points $\frac{\pi}{4}$ units to the right. Then shift each point one unit down and draw a sine wave through the points (see Figure 6.38). This sketch along with the graph of $y = -1.2$ is sufficient to reveal that solutions to $g(x) = -1.2$ occur in QI and QIII, with solutions to $g(x) \le -1.2$ *outside this interval*. Substituting -1.2 for $g(x)$ and isolating the sine function we obtain $\sin\left(x - \frac{\pi}{4}\right) = -\frac{1}{15}$, then $x = \sin^{-1}\left(-\frac{1}{15}\right) + \frac{\pi}{4}$ after taking the inverse sine of both sides. This is the QI solution, with $x = \left[\pi - \sin^{-1}\left(-\frac{1}{15}\right)\right] + \frac{\pi}{4}$ being the solution in

Figure 6.38



QIII. In approximate form the solution interval is $x \in [0, 0.72] \cup [3.99, 2\pi]$.

The basic ideas remain the same regardless of the complexity of the equation. Remember—our current goal is not a supremely accurate graph, just a sketch that will guide us to the solution using the inverse functions and the correct quadrants. Perhaps that greatest challenge is recalling that when $B \neq 1$, the horizontal shift is $-\frac{C}{B}$, but other than this a fairly accurate sketch can quickly be obtained.

Practice with these ideas by solving the following inequalities within the intervals specified.

Exercise 1: $f(x) = 3\sin x + 2; f(x) > 3.7; x \in [0, 2\pi)$

Exercise 2: $g(x) = 4\sin\left(x - \frac{\pi}{3}\right) - 1;$
$g(x) \leq -2; x \in [0, 2\pi)$

Exercise 3: $h(x) = 125\sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 175;$
$h(x) \leq 150; x \in [0, 12)$

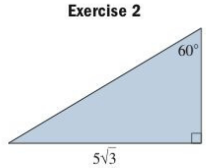
Exercise 4: $f(x) = 15{,}750\sin\left(\frac{2\pi}{360}x - \frac{\pi}{4}\right) + 19{,}250;$
$f(x) > 25{,}250; x \in [0, 360)$

CUMULATIVE REVIEW CHAPTERS 1–6

1. Find $f(\theta)$ for all six trig functions, given $P(-13, 84)$ is on the terminal side with θ in QII.

2. Find the lengths of the missing sides.

Exercise 2



3. Verify that $x = 2 + \sqrt{3}$ is a zero of $g(x) = x^2 - 4x + 1$.

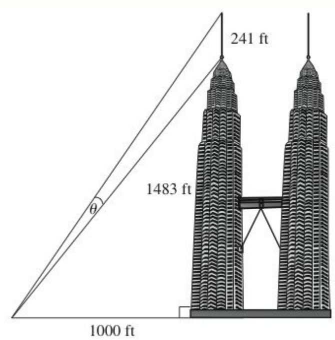
4. Determine the domain of $r(x) = \sqrt{9 - x^2}$. Answer in interval notation.

5. Standing 5 mi (26,400 ft) from the base of Mount Logan (Yukon) the angle of elevation to the summit is 36° 56′. How much taller is Mount McKinley (Alaska) which stands at 20,320 ft high?

6. Use the *Guidelines for Graphing Polynomial Functions* to sketch the graph of $f(x) = x^3 + 3x^2 - 4$.

7. Use the *Guidelines for Graphing Rational Functions* to sketch the graph of $h(x) = \dfrac{x - 1}{x^2 - 4}$

8. The Petronas Towers in Malaysia are two of the tallest structures in the world. The top of the roof reaches 1483 ft above the street below and the stainless steel pinnacles extend an additional 241 ft into the air (see figure). Find the viewing angle θ for the pinnacles from a distance of 1000 ft (the angle formed from the base of the antennae to its top).



9. A wheel with radius 45 cm is turning at 5 revolutions per second. Find the linear velocity of a point on the rim in kilometers per hour, rounded to the nearest 10th of a kilometer.

10. Solve for x: $2(x - 3)^{\frac{3}{4}} + 1 = 55$.

11. Solve for x: $-3\left|x - \frac{1}{2}\right| + 5 \geq -10$

12. The Earth has a radius of 3960 mi. Tokyo, Japan, is located at 35.4° N latitude, very near the 139° E latitude line. Adelaide, Australia, is at 34.6° S latitude, and also very near 139° E latitude. How many miles separate the two cities?

13. Since 1970, sulphur dioxide emissions in the United States have been decreasing at a nearly linear rate. In 1970, about 31 million tons were emitted into the atmosphere. In 2000, the amount had decreased to approximately 16 million tons. (a) Find a linear equation that models sulphur dioxide emissions. (b) Discuss the meaning of the slope ratio in this context. (c) Use the equation model to estimate the emissions in 1985, and project the emission for 2010.
Source: 2004 Statistical Abstract of the United States, Table 360.

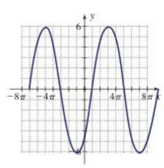
14. List the three Pythagorean identities and three identities equivalent to $\cos(2\theta)$.

15. For $f(x) = 325\cos\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 168$, what values of x in $[0, 2\pi)$ satisfy $f(x) > 330.5$?

16. Write as a single logarithmic expression in simplest form: $\log(x^2 - 9) + \log(x + 1) - \log(x^2 - 2x - 3)$.

17. After doing some market research, the manager of a sporting goods store finds that when a four-pack of premium tennis balls are priced at \$9 per pack, 20 packs per day are sold. For each decrease of \$0.25, 1 additional pack per day will be sold. Find the price at which four-packs of tennis balls should be sold in order to maximize the store's revenue on this item.

18. Write the equation of the function whose graph is given, in terms of a sine function.



19. Verify that the following is an identity: $\dfrac{\cos x + 1}{\tan^2 x} = \dfrac{\cos x}{\sec x - 1}$

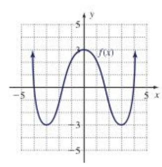
20. The graph of a function $f(x)$ is shown. Given the zeroes are $x = \pm 4$ and $x = \pm\sqrt{2}$, estimate the following:

a. the domain and range of the function

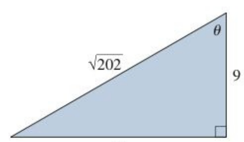
b. intervals where $f(x) > 0$ and $f(x) \le 0$

c. intervals where $f(x)\downarrow$ and $f(x)\uparrow$

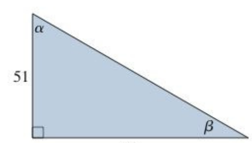
d. name the location of all local maximums and minimums



21. Use the triangle shown to find the exact value of $\sin(2\theta)$.



22. Use the triangle shown to find the exact value of $\sin(\alpha + \beta)$.



23. The amount of waste product released by a manufacturing company varies according to its production schedule, which is much heavier during the summer months and lighter in the winter. Waste product amount reaches a maximum of 32.5 tons in the month of July, and falls to a minimum of 21.7 tons in January ($t = 1$). (a) Use this information to build a sinusoidal equation that models the amount of waste produced each month. (b) During what months of the year does output exceed 30 tons?

24. At what interest rate will \$2500 grow to \$3500 if it's left on deposit for 6 yr and interest is compounded continuously?

25. Identify each geometric formula:

a. $y = \pi r^2 h$ **b.** $y = LWH$

c. $y = 2\pi r$ **d.** $y = \frac{1}{2}bh$

Modeling With Technology III Trigonometric Equation Models

Learning Objectives

In MWT III you will learn how to:

- ❑ **A.** Create a trigonometric model from critical points or data
- ❑ **B.** Create a sinusoidal model from data using regression

In the most common use of the word, a cycle is any series of events or operations that occur in a predictable pattern and return to a starting point. This includes things as diverse as the wash cycle on a washing machine and the powers of i. There are a number of common events that occur in *sinusoidal* cycles, or events that can be modeled by a sine wave. As in Section 5.7, these include monthly average temperatures, monthly average daylight hours, and harmonic motion, among many others. Less well-known applications include alternating current, biorhythm theory, and animal populations that fluctuate over a known period of years. In this section, we develop two methods for creating a sinusoidal model. The first uses information about the critical points (where the cycle reaches its maximum or minimum values), the second involves computing the equation of best fit (a regression equation) for a set of data.

A. Critical Points and Sinusoidal Models

Although future courses will define them more precisely, we will consider **critical points** to be *inputs* where a function attains a minimum or maximum value. If an event or phenomenon is known to behave sinusoidally (regularly fluctuating between a maximum and minimum), we can create an acceptable model of the form $y = A\sin(Bx + C) + D$ given these **critical points** (x, y) and the period. For instance, many weather patterns have a period of 12 months. Using the formula $P = \frac{2\pi}{B}$, we find $B = \frac{2\pi}{P}$ and substituting 12 for P gives $B = \frac{\pi}{6}$ (always the case for phenomena with a 12-month cycle). The maximum value of $A\sin(Bx + C) + D$ will always occur when $\sin(Bx + C) = 1$, and the minimum at $\sin(Bx + C) = -1$, giving this system of equations: max value $M = A(1) + D$ and min value $m = A(-1) + D$. Solving the system for A and D gives $A = \frac{M - m}{2}$ and $D = \frac{M + m}{2}$ as before. To find C, assume the maximum and minimum values occur at (x_2, M) and (x_1, m), respectively. We can substitute the values computed for A, B, and D in $y = A\sin(Bx + C) + D$, along with either (x_2, M) or (x_1, m), and solve for C. Using the minimum value (x_1, m), where $x = x_1$ and $y = m$, we have

$$y = A\sin(Bx + C) + D \quad \text{sinusoidal equation model}$$
$$m = A\sin(Bx_1 + C) + D \quad \text{substitute } m \text{ for } y \text{ and } x_1 \text{ for } x$$
$$\frac{m - D}{A} = \sin(Bx_1 + C) \quad \text{isolate sine function}$$

Fortunately, for sine models constructed from critical points we have $\frac{y - D}{A} \to \frac{m - D}{A}$, which is always equal to -1 **(see Exercise 27).** This gives a simple result for C, since $-1 = \sin(Bx_1 + C)$ leads to $\frac{3\pi}{2} = Bx_1 + C$ or $C = \frac{3\pi}{2} - Bx_1$. **See Exercises 1 through 6** for practice with these ideas.

EXAMPLE 1 ► Developing a Model for Polar Ice Cap Area from Critical Points

When the Spirit and Odyssey Rovers landed on Mars (January 2004), there was a renewed public interest in studying the planet. Of particular interest were the polar ice caps, which are now thought to hold frozen water, especially the northern cap. The Martian ice caps expand and contract with the seasons, just as they do here on Earth but there are about 687 days in a Martian year, making each Martian "month" just over 57 days long (1 Martian day ≈ 1 Earth day). At its smallest size, the northern ice cap covers an area of roughly 0.17 million square miles. At the height of winter, the cap covers about 3.7 million square miles (an area about the size of the 50 United States). Suppose these occur at the beginning of month 4 ($x = 4$) and month 10 ($x = 10$) respectively.

a. Use this information to create a sinusoidal model of the form $f(x) = A\sin(Bx + C) + D$.

b. Use the model to predict the area of the ice cap in the eighth Martian month.

c. Use a graphing calculator to determine the number of months the cap covers less than 1 million mi^2.

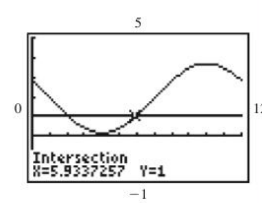
Solution ► **a.** Assuming a "12-month" weather pattern, $P = 12$ and $B = \frac{\pi}{6}$. The maximum and minimum points are (10, 3.7) and (4, 0.17). Using this information, $D = \frac{3.7 + 0.17}{2} = 1.935$ and $A = \frac{3.7 - 0.17}{2} = 1.765$. Using $C = \frac{3\pi}{2} - Bx_1$, gives $C = \frac{3\pi}{2} - \frac{\pi}{6}(4) = \frac{5\pi}{6}$. The equation model is $f(x) = 1.765\sin\left(\frac{\pi}{6}x + \frac{5\pi}{6}\right) + 1.935$, where $f(x)$ represents millions of square miles in month x.

b. For the size of the cap in month 8 we evaluate the function at $x = 8$.

$$f(8) = 1.765\sin\left[\frac{\pi}{6}(8) + \frac{5\pi}{6}\right] + 1.935 \quad \text{substitute 8 for } x$$
$$= 2.8175 \quad \text{result}$$

In month 8, the polar ice cap will cover about 2,817,500 mi^2.

c. Of the many options available, we opt to solve by locating the points where $Y_1 = 1.765\sin\left(\frac{\pi}{6}x + \frac{5\pi}{6}\right) + 1.935$ and $Y_2 = 1$ intersect. After entering the functions on the Y= screen, we set $x \in [0, 12]$ and $y \in [-1, 5]$ for a window with a frame around the output values. Press 2nd TRACE (**CALC**) **5:intersect** to find the intersection points. To four decimal places they occur at $x = 2.0663$ and $x = 5.9337$. The ice cap at the northern pole of Mars has an area of less than 1 million mi^2 from early in the second month to late in the fifth month. The second intersection is shown in the figure.



Now try Exercises 7 and 8 ►

While this form of "equation building" can't match the accuracy of a regression model (computed from a larger set of data), it does lend insight as to how sinusoidal functions work. The equation will always contain the maximum and minimum values, and using the period of the phenomena, we can create a smooth sine wave that "fills in the blanks" between these critical points.

EXAMPLE 2 ► Developing a Model of Wildlife Population from Critical Points

Naturalists have found that many animal populations, such as the arctic lynx, some species of fox, and certain rabbit breeds, tend to fluctuate sinusoidally over 10-year periods. Suppose that an extended study of a lynx population began in 2000, and in the third year of the study, the population had fallen to a minimum of 2500. In the eighth year the population hit a maximum of 9500.

a. Use this information to create a sinusoidal model of the form $P(x) = A\sin(Bx + C) + D$.

b. Use the model to predict the lynx population in the year 2006.

c. Use a graphing calculator to determine the number of years the lynx population is above 8000 in a 10-year period.

Solution ► **a.** Since $P = 10$, we have $B = \frac{2\pi}{10} = \frac{\pi}{5}$. Using 2000 as year zero, the minimum and maximum populations occur at (3, 2500) and (8, 9500). From the information given, $D = \frac{9500 + 2500}{2} = 6000$, and $A = \frac{9500 - 2500}{2} = 3500$. Using the minimum value we have $C = \frac{3\pi}{2} - \frac{\pi}{5}(3) = \frac{9\pi}{10}$, giving an equation model of $P(x) = 3500\sin\left(\frac{\pi}{5}x + \frac{9\pi}{10}\right) + 6000$, where $P(x)$ represents the lynx population in year x.

b. For the population in 2006 we evaluate the function at $x = 6$.

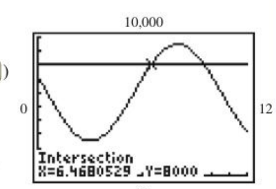
$$P(x) = 3500\sin\left(\frac{\pi}{5}x + \frac{9\pi}{10}\right) + 6000 \quad \text{sinusoidal function model}$$

$$P(6) = 3500\sin\left[\frac{\pi}{5}(6) + \frac{9\pi}{10}\right] + 6000 \quad \text{substitute 6 for } x$$

$$\approx 7082 \quad \text{result}$$

In 2006, the lynx population was about 7082.

c. Using a graphing calculator and the functions $Y_1 = 3500\sin\left(\frac{\pi}{5}x + \frac{9\pi}{10}\right) + 6000$ and $Y_2 = 8000$, we attempt to find points of intersection. Enter the functions (press Y=) and set a viewing window (we used $x \in [0, 12]$ and $y \in [0, 10{,}000]$). Press 2nd TRACE **(CALC) 5:intersect** to find where Y_1 and Y_2 intersect. To four decimal places this occurs at $x = 6.4681$ and $x = 9.5319$. The lynx population exceeded 8000 for roughly 3 yr. The first intersection is shown.



Now try Exercises 9 and 10 ►

This type of equation building isn't limited to the sine function, in fact there are many situations where a sine model cannot be applied. Consider the length of the shadows cast by a flagpole or radio tower as the Sun makes its way across the sky. The shadow's length follows a regular pattern (shortening then lengthening) and "always returns to a starting point," yet when the Sun is low in the sky the shadow becomes (theoretically) infinitely long, unlike the output values from a sine function. In this case, an equation involving tan x might provide a good model, although the data will vary greatly depending on latitude. We'll attempt to model the data using $y = A\tan(Bx \pm C)$, with the D-term absent since a vertical shift in this context has no meaning. Recall that the period of the tangent function is $P = \frac{\pi}{|B|}$ and that $\pm\frac{C}{B}$ gives the magnitude and direction of the horizontal shift, in a direction opposite the sign.

EXAMPLE 3 ► Using Data to Develop a Function Model for Shadow Length

The data given tracks the length of a gnomon's shadow for the 12 daylight hours at a certain location near the equator (positive and negative values indicate lengths before noon and after noon respectively). Assume $t = 0$ represents 6:00 A.M.

Hour of the Day	Length (cm)	Hour of the Day	Length (cm)
0	∞	7	−2.1
1	29.9	8	−4.6
2	13.9	9	−8.0
3	8.0	10	−13.9
4	4.6	11	−29.9
5	2.1	12	−∞
6	0		

a. Use the data to find an equation model of the form $L(t) = A\tan(Bt \pm C)$.

b. Graph the function and scatter-plot.

c. Find the shadow's length at 4:30 P.M.

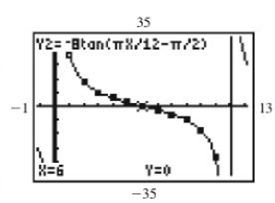
d. If the shadow is 6.1 cm long, what time in the morning is it?

Solution ► **a.** We begin by noting this phenomenon has a period of $P = 12$. Using the period formula for tangent we solve for B: $P = \frac{\pi}{B}$ gives $12 = \frac{\pi}{B}$, so $B = \frac{\pi}{12}$. Since we want (6, 0) to be the "center" of the function [instead of (0, 0)], we desire a horizontal shift 6 units to the right. Using the ratio $\frac{C}{B}\left(\text{with } B = \frac{\pi}{12}\right)$ gives $-6 = \frac{12C}{\pi}$ so $C = -\frac{\pi}{2}$. To find A we use the equation built so far: $L(t) = A\tan\left(\frac{\pi}{12}t - \frac{\pi}{2}\right)$, and *any data point* to solve for A. Using (3, 8) we obtain $8 = A\tan\left(\frac{\pi}{12}(3) - \frac{\pi}{2}\right)$:

$$8 = A\tan\left(-\frac{\pi}{4}\right) \quad \text{simplify}$$

$$-8 = A \quad \text{solve for } A\text{: } \tan\left(-\frac{\pi}{4}\right) = -1$$

The equation model is $L(t) = -8\tan\left(\frac{\pi}{12}t - \frac{\pi}{2}\right)$.



WORTHY OF NOTE

A gnomon is the protruding feature of a sundial, casting the shadow used to estimate the time of day (see photo).

b. The scatter-plot and graph are shown in the figure.

c. 4:30 P.M. indicates $t = 10.5$. Evaluating $L(10.5)$ gives

$$L(t) = -8\tan\left(\frac{\pi}{12}t - \frac{\pi}{2}\right) \quad \text{function model}$$

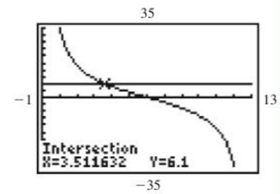
$$L(10.5) = -8\tan\left[\frac{\pi}{12}(10.5) - \frac{\pi}{2}\right] \quad \text{substitute 10.5 for } t$$

$$= -8\tan\left(\frac{3\pi}{8}\right) \quad \text{simplify}$$

$$\approx -19.31 \quad \text{result}$$

At 4:30 P.M., the shadow has a length of $|-19.31| = 19.31$ cm.

d. Substituting 6.1 for $L(t)$ and solving for t graphically gives the graph shown, where we note the day is about 3.5 hr old—it is about 9:30 A.M.



A. You've just learned how to create a trigonometric model from critical points or data

Now try Exercises 11 and 14 ►

B. Data and Sinusoidal Regression

Most graphing calculators are programmed to handle numerous forms of polynomial and nonpolynomial regression, including **sinusoidal regression.** The sequence of steps used is the same regardless of the form chosen. **Exercises 15 through 18** offer further practice with regression fundamentals. Example 4 illustrates their use in context.

EXAMPLE 4 ► **Calculating a Regression Equation for Seasonal Temperatures**

The data shown give the record high temperature for selected months in Bismarck, North Dakota.

Month (Jan → 1)	Temp. (°F)	Month (Jan → 1)	Temp. (°F)
1	63	9	105
3	81	11	79
5	98	12	65
7	109		

a. Use the data to draw a scatter-plot, then find a sinusoidal regression model and graph both on the same screen.

b. Use the equation model to estimate the record high temperatures for months 2, 6, and 8.

c. Determine what month gives the largest difference between the actual data and the computed results.

Source: NOAA Comparative Climate Data 2004.

Solution ► **a.** Entering the data and running the regression (in radian mode) results in the coefficients shown in Figure MWT III.1. After entering the equation in Y_1 and pressing ZOOM **9:Zoom Stat** we obtain the graph shown in Figure MWT III.2 (indicated window settings have been rounded).

Figure MWT III.1

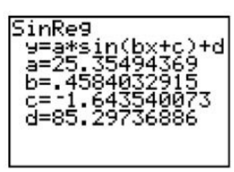
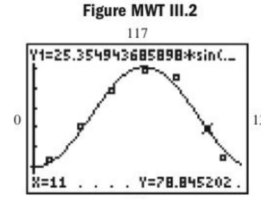


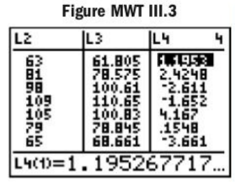
Figure MWT III.2



b. Using $x = 2$, $x = 6$, and $x = 8$ as inputs, projects record high temperatures of 68.5°, 108.0°, and 108.1°, respectively.

c. In the header of L3, use Y_1(L1) ENTER to evaluate the regression model using the inputs from L1, and place the results in L3. Entering L2 − L3 in the header of L4 gives the results shown in Figure MWT III.3 and we note the largest difference occurs in September—about 4°.

Figure MWT III.3



Now try Exercises 19 and 20 ►

Weather patterns differ a great deal depending on the locality. For example, the annual rainfall received by Seattle, Washington, far exceeds that received by Cheyenne, Wyoming. Our final example compares the two amounts and notes an interesting fact about the relationship.

EXAMPLE 5 ► **Calculating a Regression Model for Seasonal Rainfall**

The average monthly rainfall (in inches) for Cheyenne, Wyoming, and Seattle, Washington, is shown in the table.

Month (Jan. → 1)	WY Rain	WA Rain
1	0.45	5.13
2	0.44	4.18
3	1.05	3.75
4	1.55	2.59
5	2.48	1.77
6	2.12	1.49
7	2.26	0.79
8	1.82	1.02
9	1.43	1.63
10	0.75	3.19
11	0.64	5.90
12	0.46	5.62

a. Use the data to find a sinusoidal regression model for the average monthly rainfall in each city. Enter or paste the equation for Cheyenne in Y_1 and the equation for Seattle in Y_2.

b. Graph both equations on the same screen (without the scatter-plots) and use TRACE or 2nd TRACE **(CALC) 5:intersect** to help estimate the number of months Cheyenne receives more rainfall than Seattle.

Source: NOAA Comparative Climate Data 2004.

Solution ► **a.** Setting the calculator in **Float 0 1 2 3 4 5 6 7 8 9** MODE and running sinusoidal regressions gives the equations shown in Figure MWT III.4.

b. Both graphs are shown in Figure MWT III.5. Using the TRACE feature, we find the graphs intersect at approximately (4.7, 2.0) and (8.4, 1.7). While Cheyenne receives far less rainfall each year, it actually receives more rain than Seattle for about $8.4 - 4.7 = 3.7$ months of the year.

Figure MWT III.4

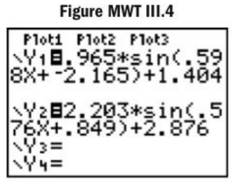
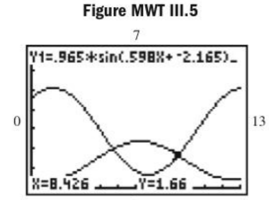


Figure MWT III.5



B. You've just learned how to create a sinusoidal model from data using regression

Now try Exercises 21 through 24 ►

MODELING WITH TECHNOLOGY EXERCISES

Find the sinusoidal equation for the information as given.

1. minimum value at (9, 25); maximum value at (3, 75); period: 12 min

2. minimum value at (4.5, 35); maximum value at (1.5, 121); period: 6 yr

3. minimum value at (15, 3); maximum value at (3, 7.5); period: 24 hr

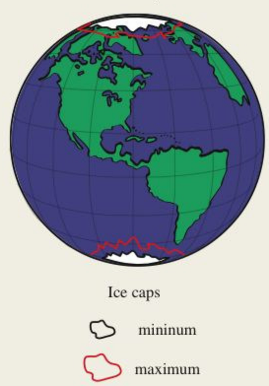
4. minimum value at (3, 3.6); maximum value at (7, 12); period: 8 hr

5. minimum value at (5, 279); maximum value at (11, 1285); period: 12 yr

6. minimum value at (6, 8280); maximum value at (22, 23,126); period: 32 yr

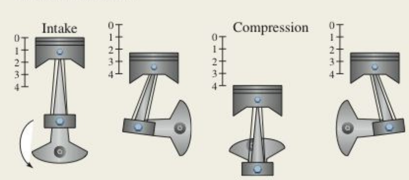
7. Record monthly temperatures: The U.S. National Oceanic and Atmospheric Administration (NOAA) keeps temperature records for most major U.S. cities. For Phoenix, Arizona, they list an average high temperature of 65.0°F for the month of January (month 1) and an average high temperature of 104.2°F for July (month 7). Assuming January and July are the coolest and warmest months of the year, (a) build a sinusoidal function model for temperatures in Phoenix, and (b) use the model to find the average high temperature in September. (c) If a person has a tremendous aversion to temperatures over 95°, during what months should they plan to vacation elsewhere?

8. Seasonal size of polar ice caps: Much like the polar ice cap on Mars, the sea ice that surrounds the continent of Antarctica (the Earth's southern polar cap) varies seasonally, from about 8 million mi^2 in September to about 1 million mi^2 in March. Use this information to (a) build a sinusoidal equation that models the advance and retreat of the sea ice, and (b) determine the size of the ice sheet in May. (c) Find the months of the year that the sea ice covers more than 6.75 million mi^2.



9. Body temperature cycles: A phenomenon is said to be *circadian* if it occurs in 24-hr cycles. A person's body temperature is circadian, since there are normally small, sinusoidal variations in body temperature from a low of 98.2°F to a high of 99°F throughout a 24-hr day. Use this information to (a) build the circadian equation for a person's body temperature, given $t = 0$ corresponds to midnight and that a person usually reaches their minimum temperature at 5 A.M.; (b) find the time(s) during a day when a person reaches "normal" body temperature (98.6°); and (c) find the number of hours each day that body temperature is 98.4°F or less.

10. Position of engine piston: For an internal combustion engine, the position of a piston in the cylinder can be modeled by a sinusoidal function. For a particular engine size and idle speed, the piston head is 0 in. from the top of the cylinder (the minimum value) when $t = 0$ at the beginning of the intake stroke, and reaches a maximum distance of 4 in. from the top of the cylinder (the maximum value) when $t = \frac{1}{48}$ sec at the beginning of the compression stroke. Following the compression stroke is the power stroke ($t = \frac{2}{48}$), the exhaust stroke ($t = \frac{3}{48}$), and the intake stroke ($t = \frac{4}{48}$), after which it all begins again. Given the period of a four-stroke engine under these conditions is $P = \frac{1}{24}$ second, (a) find the sinusoidal equation modeling the position of the piston, and (b) find the distance of the piston from the top of the cylinder at $t = \frac{1}{9}$ sec. Which stroke is the engine in at this moment?



Data and tangent functions: Use the data given to find an equation model of the form $f(x) = A\tan(Bx + C)$. Then graph the function and scatter plot to help find (a) the output for $x = 2.5$, and (b) the value of x where $f(x) = 16$.

11.

x	*y*	*x*	*y*
0	∞	7	−1.4
1	20	8	−3
2	9.7	9	−5.2
3	5.2	10	−9.7
4	3	11	−20
5	1.4	12	−∞
6	0		

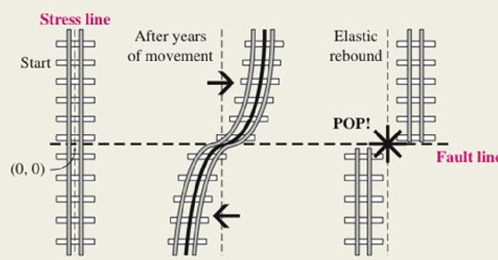
12.

x	*y*	*x*	*y*
0	∞	7	6.4
1	−91.3	8	13.7
2	−44.3	9	23.7
3	−23.7	10	44.3
4	−13.7	11	91.3
5	−6.4	12	∞
6	0		

13. Distance and apparent height: While driving toward a Midwestern town on a long, flat stretch of highway, I decide to pass the time by measuring the apparent height of the tallest building in the downtown area as I approach. At the time the idea occurred to me, the buildings were barely visible. Three miles later I hold a 30-cm ruler up to my eyes at arm's length, and the apparent height of the tallest building is 1 cm. After three more miles the apparent height is 1.8 cm. Measurements are taken every 3 mi until I reach town and are shown in the table (assume I was 24 mi from the parking garage when I began this activity). (a) Use the data to come up with a tangent function model of the building's apparent height after traveling a distance of x mi closer. (b) What was the apparent height of the building at after I had driven 19 mi? (c) How many miles had I driven when the apparent height of the building took up all 30 cm of my ruler?

Distance Traveled (mi)	Height (cm)
0	0
3	1
6	1.8
9	2.8
12	4.2
15	6.3
18	10
21	21
24	∞

14. Earthquakes and elastic rebound: The **theory of elastic rebound** has been used by seismologists to study the cause of earthquakes. As seen in the figure, the Earth's crust is stretched to a breaking point by the slow movement of one tectonic plate in a direction opposite the other along a fault line, and when the rock snaps—each half violently rebounds to its original alignment causing the Earth to quake.



x	*y*	*x*	*y*
−4.5	−61	1	2.1
−4	−26	2	6.8
−3	−14.8	3	15.3
−2	−7.2	4	25.4
−1	−1.9	4.5	59
0	0		

Suppose the *misalignment* of these plates through the stress and twist of crustal movement can be modeled by a tangent graph, where x represents the horizontal distance from the original stress line, and y represents the vertical distance from the fault line. Assume a "period" of 10.2 m. (a) Use the data from the table on page 141 to come up with a trigonometric model of the deformed stress line. (b) At a point 4.8 m along the fault line, what is the distance to the deformed stress line (moving parallel to the original stress line)? (c) At what point along the fault line is the vertical distance to the deformed stress line 50 m?

Data and sinusoidal regression models: For the following sets of data (a) find a sinusoidal regression equation using your calculator; (b) construct an equation manually using the period and maximum/minimum values;

and (c) graph both on the same screen, then use a TABLE to find the largest difference between output values.

15.

Day of Month	Output
1	15
4	41
7	69
10	91
13	100
16	90
19	63
22	29
25	5
28	2
31	18

16.

Day of Month	Output
1	179
4	201
7	195
10	172
13	145
16	120
19	100
22	103
25	124
28	160
31	188

17.

Month (Jan. = 1)	Output
1	16
2	19
3	21
4	22
5	21
6	19
7	16
8	13
9	11
10	10
11	11
12	13

18.

Month (Jan. = 1)	Output
1	86
2	96
3	99
4	95
5	83
6	72
7	56
8	48
9	43
10	49
11	58
12	73

19. Record monthly temperatures: The highest temperature of record for the even months of the year are given in the table for the city of Pittsburgh, Pennsylvania. (a) Use the data to draw a scatter-plot, then find a sinusoidal regression model and graph both on the same screen. (b) Use the equation to estimate the record high temperature for the odd-numbered months. (c) What month shows the largest difference between the actual data and the computed results?

Month (Jan. → 1)	High Temp. (°F)
2	76
4	89
6	98
8	100
10	87
12	74

Source: *2004 Statistical Abstract of the United States*, Table 378.

20. River discharge rate: The average discharge rate of the Alabama River is given in the table for the odd-numbered months of the year. (a) Use the data to draw a scatter-plot, then find a sinusoidal regression model and graph both on the same screen. (b) Use the equation to estimate the flow rate for the even-numbered months. (c) Use the graph and equation to estimate the number of days per year the flow rate is below 500 m^3/sec.

Month (Jan. → 1)	Rate (m^3/sec)
1	1569
3	1781
5	1333
7	401
9	261
11	678

Source: Global River Discharge Database Project; www.rivdis.sr.unh.edu.

21. Average monthly rainfall: The average monthly rainfall (in inches) for Reno, Nevada, is shown in the table. (a) Use the data to find a sinusoidal regression model for the monthly rainfall. (b) Graph this equation model and the rainfall equation model for Cheyenne, Wyoming (from Example 5), on the same screen, and estimate the number of months that Reno gets more rainfall than Cheyenne.

Source: NOAA Comparative Climate Data 2004.

Month (Jan → 1)	Reno Rainfall	Month (Jan → 1)	Reno Rainfall
1	1.06	7	0.24
2	1.06	8	0.27
3	0.86	9	0.45
4	0.35	10	0.42
5	0.62	11	0.80
6	0.47	12	0.88

22. Hours of daylight by month: The number of daylight hours per month (as measured on the 15th of each month) is shown in the table for the cities of Beaumont, Texas, and Minneapolis, Minnesota. (a) Use the data to find a sinusoidal regression model of the daylight hours for each city. (b) Graph both equations on the same screen and use the graphs to estimate the *number of days* each year that Beaumont receives more daylight than Minneapolis (use 1 month = 30.5 days).

Month (Jan → 1)	TX Sunlight	MN Sunlight
1	10.4	9.1
2	11.2	10.4
3	12.0	11.8
4	12.9	13.5
5	14.4	16.2
6	14.1	15.7
7	13.9	15.2
8	13.3	14.2
9	12.4	12.6
10	11.5	11.0
11	10.7	9.6
12	10.2	8.7

Source: www.encarta.msn.com/media_701500905/Hours_of_Daylight_by_Latitude.html.

23. Illumination of the moon's surface: The table given indicates the percent of the Moon that is illuminated for the days of a particular month, at a given latitude. (a) Use a graphing calculator to find a sinusoidal regression model. (b) Use the model to determine what percent of the Moon is illuminated on day 20. (c) Use the maximum and minimum values with the period and an appropriate horizontal shift to create your own model of the data. How do the values for A, B, C, and D compare?

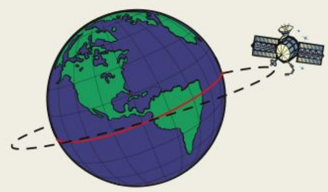
Day	% Illum.	Day	% Illum.
1	28	19	34
4	55	22	9
7	82	25	0
10	99	28	9
13	94	31	30
16	68		

24. Connections between weather and mood: The mood of persons with SAD syndrome (seasonal affective disorder) often depends on the weather. Victims of SAD are typically more despondent in rainy weather than when the Sun is out, and more comfortable in the daylight hours than at night. The table shows the average number of daylight hours for Vancouver, British Columbia, for 12 months of a year. (a) Use a calculator to find a sinusoidal regression model. (b) Use the model to estimate the *number of days* per year (use 1 month $\approx$ 30.5 days) with more than 14 hr of daylight. (c) Use the maximum and minimum values with the period and an appropriate horizontal shift to *create* a model of the data. How do the values for A, B, C, and D compare?

Source: Vancouver Climate at www.bcpassport.com/vital.

Month	Hours	Month	Hours
1	8.3	7	16.2
2	9.4	8	15.1
3	11.0	9	13.5
4	12.9	10	11.7
5	14.6	11	9.9
6	15.9	12	8.5

25. Orbiting distance north or south of the equator: $D(t) = A\cos(Bt)$ Unless a satellite is placed in a strict equatorial orbit, its distance north or south of the equator will vary according to the sinusoidal



model shown, where $D(t)$ is the distance t min after entering orbit. Negative values indicate it is south of the equator, and the distance D is actually a two-dimensional distance, as seen from a vantage point in outer space. The value of B depends on the speed of the satellite and the time it takes to complete one orbit, while $|A|$ represents the maximum distance from the equator. (a) Find the equation model for a satellite whose maximum distance north of the equator is 2000 miles and that completes one orbit every 2 hr ($P = 120$). (b) How many minutes after entering orbit is the satellite directly above the equator $[D(t) = 0]$? (c) Is the satellite north or south of the equator 257 min after entering orbit? How far north or south?

26. Biorhythm theory: $P(d) = 50\sin(Bd) + 50$ Advocates of biorhythm theory believe that human beings are influenced by certain biological cycles that begin at birth, have different periods, and continue throughout life. The classical cycles and their periods are physical potential (23 days), emotional potential (28 days), and intellectual potential (33 days). On any given day of life, the percent of potential in these three areas is purported to be modeled by the function shown, where $P(d)$ is the percent of available potential on day d of life. Find the value of B for each of the physical, emotional, and intellectual potentials and use it to see what the theory has to say about your potential today. Use day $d = 365.25(\text{age}) +$ days since last birthday.

27. Verifying the amplitude formula: For the equations from Examples 1 and 2, use the minimum value (x, m) to show that $\frac{y - D}{A} = \frac{m - D}{A}$ is equal to -1. Then verify this relationship in general by substituting $\frac{M - m}{2}$ for A, $\frac{M + m}{2}$ for D.